Essays on Contracts, Mechanisms and Information Revelation

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Sina Litterscheid
aus Bad Honnef

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Dekan:	Prof. Dr. Klaus Sandmann
Erstreferent:	Prof. Dr. Dezsö Szalay
Zweitreferent:	Prof. Dr. Daniel Krähmer

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Contents

[1]This chapter is based on the paper "Sequential, multidimensional screening", Litterscheid and Szalay 2014.

Introduction

In 2001, Akerlof, Spence and Stiglitz won the Nobel prize for their work on adverse selection, signalling and screening. The prize was in recognition of their foundational contribution to information economics, a revolution in economic research that brought the underlying idea of information asymmetries to the heart of many emerging fields of economic research (Stiglitz 2000); for instance, economics of privacy, auctions with information revelation and mechanism design. This dissertation contributes to these three areas of microeconomic research.

Chapter 1.[2] The first chapter is a contribution to the literature on the economics of privacy. During the last decade, an increasing number of economists have researched the economics of privacy. This economic literature reports an apparent dichotomy between a high degree of privacy concerns across the US population and a low degree of data protecting actions (see Acquisti 2004, Acquisti and Grosklags 2005 for an overview). This dichotomy has been called the 'privacy paradox'. In a natural environment with demand uncertainty and customer entry, I identify customer entry as a new explanation for the behavior of firms and the privacy paradox.

I investigate a two-period model with two monopolists and two buyers. One monopolist sells her good 1 only in period 1 and one monopolist sells her good 2 only in period 2. In period 1, one buyer demands good 1 and then goes on to demand good 2 with positive probability. In period 2, players learn whether this buyer has demand for good 2, and

[2]This chapter is based on the paper "On the Value of Purchase Histories - Type-dependent Demand Uncertainty and Consumer Entry", Litterscheid 2014.

there is a second buyer with demand for good 2. Seller 1's purchase history contains her customer's purchases and name/identity. I am interested in the first monopolist's incentives to sell information about her customer's characteristics to the monopolist of a second good and whether seller 1 prefers a disclosure or a confidential policy. I provide conditions for the parameters so that the first monopolist prefers the disclosure policy and profitably sells the purchase history to seller 2. Given that a second buyer enters, seller 2 is willing to pay more for buyer 1's purchase history than she would have been willing to pay if she had expected no other buyer to enter. The reason is that the purchase history, containing the buyer's identity, enables seller 2 to distinguish between the two buyers and to make targeted offers. In other words, the intuition for my main result lies in the new additional value of the purchase history. Consumer entry allows me to evaluate a value of the purchase history that stems from the second seller's ability to identify and target the customer. This additional value is generated by the new entrant since the optimal offer is distorted if the seller cannot distinguish between the customers.

Chapter 2.[3] The second chapter is a contribution to the literature on public information revelation prior to an auction. A typical example is a situation where the owner of a company announces the sale of this company (target) via an auction (takeover auction). All bidders share a common interest in the quality of the target, e.g. the target's future cash flows. The potential bidders are asymmetrically and imperfectly informed about the target's quality. Potential bidders are also heterogenous and have some additional private interest in the company, e.g. potential synergies that arise when the buyer merges with the target. Before the auction, the seller can open her books and disclose private and common value information. Private value information that drives synergies may arise in many areas, for example in procurement, research and development, production, human resources, sales and marketing etc. Common value information is related to quality, e.g. cash flow forecast. While one potential bidder's strength is his marketing environment, another potential bidder

[3]This chapter is based on the paper "Revealing Independent Private Value Information When Bidders Have Interdependent Values", Litterscheid 2014.

may have technological know-how that helps to decrease production costs (see Szech 2011 for a similar argument or Gärtner and Schmutzler 2009). The seminal paper that inspired most of the related research is Milgrom and Weber 1982a who showed that a seller prefers public disclosure of affiliated information in an interdependent value auction setting. This is the so-called linkage principle. The main question I address in this chapter is whether the seller also prefers public disclosure of private value information over concealing her information.

I restrict attention to disclosure of private value information prior to an interdependent value second-price auction with two bidders who hold preliminary private information about the good. To investigate the main research question and to disentangle the effect of public common value information from public private value information, I assume that the seller does not hold common value information. The key aspect is the extent to which disclosure affects the bidders' bidding strategies in equilibrium. Unlike Milgrom and Weber 1982a, the disclosed information affects bidders idiosyncratically allowing to enhance the bidders' exposition to the winner's curse. I find that the linkage principle (see Milgrom and Weber 1982a) holds if the seller's information is sufficiently informative, but it does not hold if the information contains little information.

Chapter 3.[4] The third chapter is a contribution to several branches of the literature on mechanism design: literature on optimal contracts in a principal-agent model with asymmetric information about the agent's type, literature on sequential screening, and literature on multi-dimensional screening. The principal is the buyer and the agent is the seller.

Together with Dezsö Szalay, I analyze a screening problem where the agent produces an object consisting of multiple items and has a multi-dimensional type that he learns over time. The principal would like to buy this object from the agent and contracts with an agent to trade a bundle of services. Moreover, the agent has private information about the costs of producing one item in the bundle from the outset and privately learns the cost of producing the other item later on. When the principal and the agent write the contract

[4] This chapter is based on the paper "Sequential, multidimensional screening", Litterscheid and Szalay 2014.

after the agent knows part of his information but before he perfectly knows his cost type, then the known part of his cost type is called his ex-ante type and the other type is called his ex-post type. The optimal sequential mechanism or optimal contracting is dynamic and consists of a menu of n submenus each of which contains m contracts; where n is the number of ex-ante types and m is the number of ex-post types. Principal and agent get together both at the outset, when the agent picks one of the n submenus, and later on, when the agent knows his ex-post type and picks one of the m contracts of the submenu he selected. Only afterwards is the object produced and the agent paid. The seminal paper of the sequential screening literature that considers the same type of dynamic contracting is Courty and Li 2000. Our work differs from the current literature in that our allocation problem is two-dimensional and that we allow for interdependencies, substitutionality or complementarity between the two dimensions of the object. This two-dimensional screening problem lacks structure and thus is potentially very complicated to solve. To derive an explicit solution, we consider a simplified situation and restrict the agent's type to the realization of a vector of two binary random variables. We provide a solution method to derive the optimal contract and a characterization of the optimal contract. We find that the distortions of the optimal two-dimensional allocation depends on the strength of complementarity/substitutionality of the two components of the object. For mild complements or substitutes, a simple solution procedure picks up the optimum. For substitutes or strong complements upward distortions are possible. Thus, we provide a natural setting in which upward distortions may arise as a feature of the optimal mechanism.

On the Value of Purchase Histories - Type-dependent Demand Uncertainty and Consumer Entry

1.1 Introduction

The ability to predict a customer's valuation and future demand has high economic value because it may enable a monopolist to reduce a customer's information rent. There is ample evidence for synergies firms generate by sharing information about their customers. For instance, there is evidence that hospitals profit from exchanging information with each other (Miller and Tucker 2009). Profit-oriented companies such as Google, Facebook or Amazon collect huge data sets about their customers. Google and Facebook then sell the service of behavior-based/targeted advertisement to other companies.

During the last decade, an increasing number of economists have researched the economics of privacy. This economic literature reports an apparent dichotomy between a high degree of privacy concerns across the US population and a low degree of data protecting actions (see Acquisti 2004, Acquisti and Grosklags 2005 for an overview). This dichotomy has been called the 'privacy paradox'.

So, on the one hand there are firms that collect and sell large amounts of data about customers and on the other hand there is the privacy paradox. One important question in this context is how the two motivating phenomena fit together (Taylor 2004). To answer this question, most relevant papers analyze a seller's privacy policy in a variant of a simple two period model and compare the optimality of two privacy policies, the confidential policy and the disclosure policy, from the sellers' perspectives. Selling customer purchase histories is forbidden by the confidential policy and allowed by the disclosure policy. The confidential policy does not allow the seller(s) to exchange the information a buyer has revealed about himself. The disclosure policy allows the seller(s) to exchange, and to sell, personalized information, but introduces the ratchet effect.[1] To justify the privacy paradox, environments or conditions that imply that the seller prefers the disclosure policy have to be found.

Most papers on the economics of privacy find that a confidential policy outperforms the disclosure policy when customers are rational and positively correlated (see e.g. Taylor 2004, Dodds 2003, Calzolari and Pavan 2006; for a survey, see Fudenberg and Villas-Boas 2006, 2012, Hui and Png 2006, Zhan and Rajamani 2008).[2] The main challenge is to enlarge the contractual space so that there is a contract that sets both, sellers and buyers, better off. In the spirit of Fudenberg and Tirole 1983, Dodds 2003 finds that the principal's joint surplus is higher under the disclosure policy than under the confidential policy if the principal's discount factor is sufficiently higher than the worker's discount factor, but he does not characterize the contract explicitly. Calzolari and Pavan 2006 provide conditions so that in the presence of negatively correlated valuations and changing support, the seller benefits from a disclosure policy. The intuition in this setting is that there are countervailing incentives. The first seller may also profit from disclosure in the case of direct externalities on seller 1's

[1] The ratchet effect is present in models where the buyer has a persistent type and the seller has perfect memory but no commitment power to long term contracts (see e.g. Fudenberg and Tirole 1983). The ratchet effect describes the idea that a buyer who has persistent information cannot undo the revelation of his private information once he has revealed it. Since he will never again receive any information rent for revealed information, he might refrain from potentially revealing actions such as purchasing a good. This might inhibit trade and lower the seller's expected revenue. Fudenberg and Tirole 1983 consider sequential bargaining without commitment in a two period model between a seller and a buyer.

[2] For a broader overview of the economic literature on privacy, we recommend a survey by Hui and Png 2006. See also Zhan and Rajamani 2008. For a general overview of the economic literature on behavior-based pricing, see Fudenberg and Villas-Boas 2006. For a recent contribution, overview and a discussion of different types of behavior-based pricing models, see Fudenberg and Villas-Boas 2012. The literature on privacy policies is related to the literature on dynamic pricing (see e.g. Baron and Besanko 1984), which shows that the optimal long-term contract implements a sequence of the solution to the short-term contracting problem.

payoff (Calzolari and Pavan 2006).

I depart from the assumptions of these related papers in the following dimensions. First, I assume that there is a customer with uncertain, type-dependent future demand. Second, a new customer enters in the second period. Third, I restrict attention to persistent valuations (as true for the examples from the introduction). In particular, two monopolistic sellers (in this chapter either called monopolist or seller) trade sequentially with two buyers: One incumbent (in this chapter also called buyer 1) and one entrant (in this chapter also called buyer 2). The incumbent customer has unit demand for the first monopolist's good 1 in period 1 and with positive probability a unit demand for the second monopolist's good 2 in period 2. The entrant buyer has a unit demand for the second monopolist's good 2 in period 2. In my model the incumbent customer's type determines his time-persistent valuation and his probability to demand one unit of the good 2. The incumbent privately knows his type at the outset of the game, information that seller 2 does not have but could gain from seller 1. So, the first seller's purchase history can be informative about her customer's type and enables seller 2 to distinguish the incumbent from the new buyer.

This chapter provides a new explanation for the privacy paradox and extends existing results to a very natural setting with persistent valuation, type-dependent demand uncertainty, and customer entry. To the best of my knowledge, the paper on which this chapter is based is the first paper addressing the privacy paradox and considering a dynamic pricing model with persistent valuation, type-dependent demand uncertainty, and customer entry. In the presence of demand uncertainty for good 2, the second monopolist updates her belief about the incumbent buyer's true valuation conditional on the event that the buyer has positive demand for the object. A typical example for such preferences with demand uncertainty is a customer's status preference. Some customers have a higher probability to buy further status goods in the future. One can find many more applications for preferences that have an underlying persistent type but demand uncertainty: Add-on products, applications for mobile devices, insurances, media and newspapers, portfolio management, health care, schooling, etc.

My main insight concerning the privacy policy of the first monopolist is that she sometimes prefers the disclosure policy. I find that the first seller prefers to sell the purchase history at a strictly positive price if the second seller cannot identify the buyers and is sufficiently more pessimistic about her incumbent's type than she is about the entrant's type.

Why is the purchase history more valuable if a new customer enters seller 2's market? When the new customer, buyer 2, enters the market and the first monopolist's former customer, buyer 2, comes to the second monopolist to buy good 2, then the second monopolist cannot distinguish the two customers. The purchase history of the first monopolist's former customer provides two types of information. First, it provides the second monopolist with information about the valuation of the first monopolist's former customer. Second, it informs about the identity of the first monopolist's former customer. The latter type of information implies that the second monopolist then can distinguish the two customers if she buys the purchase history from the first monopolist. This purchase history provides her with some additional value that would not be present without the entry of the buyer 2. One can find conditions under which the first monopolist's total revenue from committing to a disclosure policy, which is the sum of first period profits and the price of the purchase history, exceeds her total revenue under the confidential policy, which is equal to her first period profits.

Section 2 presents the main assumptions of the model with customer entry and my approach to derive the main result. Section 3 presents the analysis of the model. Subsection 3.1 considers seller 2's contracting problem if she bought the purchase history of the first monopolist's customer. Subsection 3.2 considers seller 2's contracting problem if she did not buy the purchase history. Subsection 3.3 presents seller 1's offer of the purchase history to seller 2. Subsection 3.4 presents seller 1's contracting problem under the confidential policy. Subsection 3.5 presents the last step of the analysis, seller 1's contracting problem, and my main result. Section 4 presents the conclusion. Proofs are relegated to Appendix 1.

1.2 Model and Approach

1.2.1 The Model

I consider a two-period bargaining model. There are two sellers (in this chapter, always female, i.e. in the "she" form) and two buyers (in this chapter, always male, i.e. in the "he" form). Seller 1 sells good 1 and seller 2 sells good 2. One buyer has unit demand for good 1 in period 1 and lives with certainty in period 1. I will often refer to him by calling him buyer 1. His valuation of one unit of either of the two goods is determined by his persistent type $i \in \{A, B\}$. If his type is A (B), then his valuation, θ_1, is θ_A (θ_B) and his probability to have unit demand for good 2 in period 2 is δ_A (δ_B); $\theta_A > \theta_B$ and $\delta_A \in [0,1]$ and $\delta_B \in [0,1]$. Buyer 2 has only unit demand for good 2 in period 2. I assume without loss of generality that he enters at the beginning of period 2. Buyer 2's type is his valuation $\theta_2 \in \{\theta_A, \theta_B\}$.

Nature draws both buyers' types at the beginning of the game. The type of any of the two buyers is the respective buyer's private information; that is, none of the other players, including sellers 1 and 2, can observe his type. Buyer 1 learns his type when at the beginning of period 1. Buyer 2 privately learns his type at the beginning of period 2, when he enters the market. It is common knowledge that buyer 1's type i is a binary random variable with probability $\alpha \equiv P(i = A)$ and $(1 - \alpha) \equiv P(i = B)$. Similarly, with probability β buyer 2's type is θ_A and with probability $1 - \beta$ his type is θ_B. From the other players' perspectives, buyer 2's type θ_2 is a binary random variable with probability $\beta \equiv P(\theta_2 = \theta_A)$ and $(1 - \beta) \equiv P(\theta_2 = \theta_B)$.

Payment p denotes the price set by seller 1 for x units of good 1. Let t denote the price set by seller 2 for y units of good 2. x and y can be chosen from the unit interval. Then x denotes a buyer's consumption of good 1 and y denotes the buyer's consumption of good 2. A buyer's utility of purchasing good 1 (or 2) with probability x (or y) at price p (or t) is quasilinear in the payment $x\theta - p$ (or $y\theta - t$). Let P denote the price for seller 1's customer information. Both sellers' valuations and production costs are normalized to 0, which is common knowledge. Seller 2's willingness to pay is denoted by WTP and is the additional expected payoff that she can earn by making use of the information contained in the purchase history. I assume that seller 1 has full bargaining power with respect to this additional expected payoff. Seller 1's offers are publicly observable to all players. Seller 1 can generate revenue p from trade with the buyer and P from trade with seller 2. Seller 2 can generate revenue t from trade with each buyer. She may set different prices for the incumbent and the entrant if she can distinguish them. She can only distinguish them under disclosure.

Like Taylor 2004 I assume that seller 1 possesses a device that saves the buyer's purchase decision, and which she cannot manipulate. In my setting, the purchase history contains the buyer's identity and his purchase decisions[3].

Seller 1 can commit to a privacy policy, which is either a disclosure policy or a confidential policy as in Taylor 2004. The confidential policy does not allow seller 1 to use the information that she has learnt about her customers. She commits in particular to not selling the purchase history. The disclosure policy allows seller 1 to choose to sell the purchase history to seller 2.

The exact timing of the game is the following:

Period 1:

1. Nature selects buyer 1's type. Buyer 1 enters and learns his type. Seller 1 commits to a privacy policy and makes offer to buyer 1. If offer accepted: Buyer 1 receives good and pays price.

2. Only under disclosure policy: Seller 1 offers purchase history to seller 2. If seller 2 accepts seller 1's offer, then seller 2 receives the purchase history and pays price. If seller 2 rejects seller 1's offer, then seller 2 does not receive the purchase history and pays nothing.

[3] I will assume that the list contains reports instead of purchase decisions, since I restrict attention to direct mechanisms.

Period 2:

3. Nature draws demand for good 2 of buyer 1 and type of buyer 2. Buyer 2 enters and learns his type. Seller 2 makes offer to each of her customers. If seller 2's offer accepted by a customer, then customer receives good and pays price.

It is helpful to explicitly describe the buyers' demand for seller 2's good in detail. At the beginning of period 2 nature draws buyer 1's demand. With probability δ_i buyer 1's demand is 1 and with $1 - \delta_i$ it is 0. Buyer 2's demand is 1. Thus, there are either one or two buyers active in period 2. If buyer 1's demand is 0, then seller 2 faces a single customer. If buyer 1's demand is 1, then seller 2 faces two customers.

Seller 1's strategy consists of several actions: a decision on the privacy policy, an offer to buyer 1 under the confidential policy, an offer to buyer 1 under the disclosure policy and the price P for the purchase history under the disclosure policy. I can formulate seller 1's offer to seller 2 in short form if I allow her to choose $P = \infty$, implying that she does not want to sell the purchase history.

Keep in mind that seller 2's belief when she buys the purchase history is different from her posterior belief conditional on the buyer's actual purchase decision. Seller 2's strategy consists of several actions: her reply to seller 1' offer, an offer to buyer 2 if she can identify her, an offer to buyer 1 if he has unit demand and she can identify him and an offer to both buyers if she cannot distinguish between them.[4]

1.2.2 The Approach

Before I begin the analysis, I briefly outline my approach. I apply the Perfect Bayesian equilibrium solution concept (Fudenberg and Tirole 1991). The main goal of the analysis is to derive sufficient conditions for a Perfect Bayesian equilibria (PBE) in which the seller chooses the privacy policy and her expected revenue exceeds the total payoff from committing to a confidential policy. I solve the game backwards. In order to solve for the sellers' optimal

[4]Note that seller 2 can only distinguish the buyers if she purchases buyer 1's purchase history.

offers at each of their information sets, I apply the adequate revelation principle, which allows attention to be restricted to the set of direct mechanisms when solving for seller 1's and seller 2's optimal offers. Then seller 1's customer's purchase history contains buyer 1's reported type and his identity.

The proof of my main result is done in five main steps.

First, I consider seller 2's decision problem in period 2 after seller 2 bought the purchase history, state her optimal play after she bought the purchase history given the purchase history contains full information about the valuation of seller 1's customer and derive seller 2's expected revenue provided she did not buy the purchase history.

The second step is the analogue to step 1 for the case when seller 2 did not buy the purchase history.

Third, I derive the price for the purchase history if the purchase history of the first monopolist's customer fully reveals the customer's type to seller 2.

Fourth, I derive seller 1's expected revenue from committing to the confidential policy.

Fifth, I need to show in a final step that there are conditions under which seller 1 prefers the disclosure policy. I do this by showing that the sum of the revenue from selling good 1 to her customer and the revenue from selling the purchase history to seller 2 exceeds the revenue from committing to the confidential policy, which I derived in step 4. In particular, I derive a lower bound for seller 1's expected revenue and show that this lower bound can be higher than her revenue under the confidential policy. In order to do so, I consider a particular mechanism of seller 1. I show that this mechanism is incentive compatible and individual rational; that is, the mechanism induces buyer 1 to fully reveal his valuation and to participate. I provide conditions under which seller 1's total expected revenue from offering this mechanism under the disclosure policy exceeds her revenue from selling to the customer under the confidential policy. Since the buyer's behavior is consistent with the postulated beliefs, this is a PBE unless seller 1 can generate higher expected revenue by offering another mechanism that is also incentive compatible and individual rational. I conclude that seller 1 strictly prefers the disclosure policy under the provided conditions. Note that in principle there could be another incentive-compatible and individual rational mechanism that seller 1 would prefer under the disclosure policy.

1.3 Analysis

1.3.1 Seller 2's Contracting Problem After She Bought the Purchase History

In this subsection, I consider the second monopolist's contracting problem after she bought the purchase history from seller 1. If she bought the purchase history from seller 1, then she has the possibility to identify the customers. I have to distinguish two branches of the game tree. In the first case she faces two buyers and in the second case only one buyer demands her good. The former case occurs with probability $\delta_A \alpha + (1-\alpha)\,\delta_B$ and the latter case occurs with probability $(1 - (\delta_A \alpha + (1-\alpha)\,\delta_B))$.

If she faces only one customer, then she knows that this is buyer 2 which implies that the purchase history does not contain any valuable information. Therefore seller 2's optimal offer to buyer 2 is independent of buyer 1's purchase history in the latter case.

Since the posterior about the buyer is always valuable, her optimal offer to the new customer solves

$$\max_{y_A; y_B, t_A, t_B} \beta t_A + (1-\beta)\, t_B \tag{1.1}$$

subject to

$$\theta_j y_j - t_j \geq 0, \tag{1.2}$$

$$\theta_j y_j - t_j \geq \theta_j y_i - t_i, \tag{1.3}$$

$$y_j \in [0,1] \tag{1.4}$$

for $i, j \in \{A, B\}$,$j \neq i$,

where constraint (1.2) is the individual rationality condition of type j of a buyer, constraint (1.3) is the incentive compatibility condition of type j of a buyer and constraint (1.4) is the feasibility condition since the allocation is restricted by the unit demand of a buyer.

Constraint (1.4) must be imposed because the buyer has unit demand in my setting.

If seller 2 observes that the incumbent customer has unit demand, then her optimal offer to buyer 1, seller 1's former customer, conditions on the information in the purchase history and is a function of seller 2's posterior. Let s_i denote the probability with which the incumbent buyer with type i reports type A to seller 1, $i \in \{A, B\}$, i.e. $1 - s_i$ is the probability that the incumbent buyer of type i reports type B. By Bayes' rule, seller 2's belief that buyer 1's type is A conditional on report A and positive demand for good 2 is equal to

$$\mu_A(s_A, s_B) = \frac{s_A \alpha \delta_A}{s_A \alpha \delta_A + s_B (1-\alpha) \delta_B}. \tag{1.5}$$

Analogously, seller 2's belief that buyer 1's type is A conditional on report B and positive demand for good 2 is equal to

$$\mu_B(s_A, s_B) = \frac{(1-s_A) \alpha \delta_A}{(1-s_A) \alpha \delta_A + (1-s_B)(1-\alpha) \delta_B}. \tag{1.6}$$

Then her optimal offer to buyer 1 who reported k, $k \in \{A, B\}$, solves

$$\max_{y_A, y_B, t_A, t_B} \mu_k(s_A, s_B) t_A + (1 - \mu_k(s_A, s_B)) t_B \tag{1.7}$$

subject to

$$(1.2), (1.3), (1.4)$$

for $i, j \in \{A, B\}$,$j \neq i$.

The purchase history can have positive value only in the former setting. In order to derive seller 2's willingness to pay for the purchase history, I can restrict attention to the branch of the game with two buyers in period 2.

Proposition 1.3.1 *If seller 1's customer reported A to seller 1 with probability 1 if he has type A and with probability 0 if he has type B, then seller 2's posterior beliefs are $\mu_A(1,0) = 1$ and $\mu_B(1,0) = 0$. If seller 1's customer demands good 2, then seller 2's expected revenue* *conditional on the report h is equal to*

$$\begin{cases} \max(\beta\theta_A, \theta_B) + \theta_A, & \textit{if } h = A \\ \max(\beta\theta_A, \theta_B) + \theta_B, & \textit{if } h = B \end{cases}$$

Proof. In Appendix 1. ■

If the first monopolist's buyer fully reveals her type to seller 1, then the second monopolist can perfectly discriminate this customer. Moreover the purchase history has another value, which is the value from being able to distinguish the two customers.

1.3.2 Seller 2's Contracting Problem If She Did Not Buy the Purchase History

In this subsection, I discuss seller 2's contracting problem if she cannot condition on seller 1's purchase history.

It is obvious that seller 2 knows that her customer was not the customer of seller 1, if she has only one customer. This event occurs with probability $(1 - (\alpha\delta_A + (1-\alpha)\delta_B))$. The purchase history provides no valuable information about buyer 2 in this situation. Therefore the purchase history is valuable only with probability $\delta_A\alpha + \delta_B(1-\alpha)$.

Next, I consider the case with two customers. In principle, her posterior belief that buyer 1's type is A conditional on the event that he has positive demand for good 2 is given by

$$\lambda \equiv \frac{\alpha\delta_A}{\alpha\delta_A + (1-\alpha)\delta_B}.$$

Note that $\lambda \geq \alpha$ if and only if $\delta_A \geq \delta_B$. Therefore the probability that seller 2 will be serving buyer 1 who has a valuation θ_A is given by

$$\alpha\delta_A.$$

However, she cannot distinguish the two customers and only knows that one of the two customers must be seller 1's former customer. Her belief that a customer's type is A is then equal to $\frac{1}{2}\lambda + \frac{1}{2}\beta$. Her optimal offer to buyer 1 solves

$$\max_{y_A, y_B, t_A, t_B} \left(\frac{1}{2}\lambda + \frac{1}{2}\beta\right) t_A + \left(1 - \left(\frac{1}{2}\lambda + \frac{1}{2}\beta\right)\right) t_B \tag{1.8}$$

subject to

$$(1.2), (1.3), (1.4)$$

for $i, j \in \{A, B\}, j \neq i$.

Proposition 1.3.2 *From the perspective of stage 2 at period 1, seller 2's expected revenue under the confidential policy is equal to*

$$\left\{ \begin{array}{c} (1 - (\alpha\delta_A + (1-\alpha)\delta_B)) \max(\beta\theta_A, \theta_B) \\ + (\alpha\delta_A + (1-\alpha)\delta_B) 2 \max\left(\left(\frac{1}{2}\lambda + \frac{1}{2}\beta\right)\theta_A, \theta_B\right) \end{array} \right\}.$$

Proof. In Appendix 1. ■

1.3.3 Seller 1's Optimal Offer to Seller 2 Under the Disclosure Policy

After trading with the buyer, seller 1 maximizes her revenue from selling the purchase history at a price P to seller 2 and has to respect that seller 2 rejects any price above her willingness to pay (WTP). WTP is a function of seller 2's posteriors, since seller 2's expected revenue after having purchased the purchase history is a function of her posterior about buyer 1. From the perspective of stage 2 at period 1, WTP is the difference between seller 2's expected revenue conditional on the information provided by the purchase history and seller 2's expected revenue without this information.

From the perspective of stage 2 at period 1, seller 2's expected revenue conditional on the information provided by the purchase history is equal to the sum of the expected revenue from selling to buyer 2 and the expected revenue from selling to buyer 1. The expected revenue from selling to buyer 2 depends on her belief about buyer 2's type, β. The expected revenue from selling to buyer 1 depends on her belief about buyer 1: the posterior belief about buyer 1's type and about buyer 1's type-dependent probability to demand good 2. From the perspective of stage 2 at period 1, seller 2's belief that buyer 1 will have positive demand is equal to

$$\alpha\delta_A + (1-\alpha)\,\delta_B.$$

From the perspective of stage 2 at period 1, seller 2's belief that buyer 1 sends a report A conditional on positive demand is given by

$$\frac{s_A\alpha\delta_A + s_B(1-\alpha)\,\delta_B}{\alpha\delta_A + (1-\alpha)\,\delta_B}. \tag{1.9}$$

The posterior belief that buyer 1 has type A, conditional on positive demand and report h, $h \in \{A, B\}$, is given by (1.5) and (1.6). Therefore, from the perspective of seller 2 at the beginning of stage 2, the probability that buyer 1 has type A, sent report A to seller 1 and will have demand for good 2 is given by

$$s_A\alpha\delta_A.$$

Seller 1 maximizes the price for the purchase history P subject to

$$P \leq WTP\left(\mu_A\left(s_A, s_B\right), \mu_B\left(s_A, s_B\right)\right). \tag{1.10}$$

Since seller 1 has the full bargaining power with respect to the additional information rent that her purchase history provides to seller 2, seller 1 can set the price equal to seller 2's willingness to pay for the purchase history. Then seller 2's optimal price is a function of the buyer's reporting behavior in equilibrium and the posterior

$$P^*\left(s_A, s_B, \mu_A\left(s_A, s_B\right), \mu_B\left(s_A, s_B\right)\right) = WTP\left(s_A, s_B, \mu_A\left(s_A, s_B\right), \mu_B\left(s_A, s_B\right)\right).$$

I would like to consider the case where the value of the purchase history reaches its upper bound and derive seller 2's WTP. Suppose buyer 1 reports his type truthfully to seller 1, i.e. buyer 1 reports A with probability $s_A = 1$ and B with probability $s_B = 0$. Substitution into (1.5) and (1.6) gives that seller 2's posteriors are $\mu_A(1,0) = 1$ and $\mu_B(1,0) = 0$. Then seller 2 will be able to perfectly screen buyer 1 provided she buys the purchase history. The purchase history also provides seller 2 with the identity of the buyers.

Proposition 1.3.3 *$s_A = 1$ and $s_B = 0$. At stage 2 of period 1, seller 1 offers the purchase history to seller 1 for a price equal to*

$$\begin{aligned} & P^*(1,0,1,0) \\ = \quad & (\delta_A \alpha + \delta_B (1-\alpha)) \left(\begin{array}{c} \max(\beta\theta_A, \theta_B) + \lambda\theta_A \\ + (1-\lambda)\theta_B - \max((\beta+\lambda)\theta_A, 2\theta_B) \end{array} \right). \end{aligned} \tag{1.11}$$

Proof. In Appendix 1. ■

1.3.4 Seller 1's Contracting Problem Under the Confidential Policy

If seller 1 commits to the confidential policy, then her optimal offer is a myopic decision. I can apply the standard revelation principle. Her optimal offer to her customer solves

$$\max_{x_A, x_B, p_A, p_B} \alpha p_A + (1-\alpha) p_B \tag{1.12}$$

subject to

$$\theta_j x_j - p_j \geq 0, \tag{1.13}$$

$$\theta_j x_j - p_j \geq \theta_j x_i - p_i, \tag{1.14}$$

$$x_j \in [0,1] \tag{1.15}$$

for $i, j \in \{A, B\}, j \neq i$.

Proposition 1.3.4 *Seller 1's revenue from committing to the confidential policy is equal to* $\max(\alpha\theta_A, \theta_B)$.

Proof. In Appendix 1. ■

This result implies the following threshold (1.16), which is very important for the derivation of my main result.

Corollary 1.3.1 *Seller 1 chooses the disclosure policy if and only if the expected revenue exceeds*

$$\max\left(\alpha\theta_A, \theta_B\right). \tag{1.16}$$

In the next section, I will consider a mechanism that is implementable with reporting strategies $s_A = 1$ and $s_B = 0$. I will provide conditions so that seller 1's expected revenue under disclosure policy exceeds this threshold (1.16).

1.3.5 Seller 1's Contracting Problem Under the Disclosure Policy

In this section, I state seller 1's optimal mechanism under the disclosure policy. Clearly, the solution to this problem is the same as if seller 1 sold also good 2 but had no commitment power to the prices for good 2. I can solve the hypothetical game in which seller 1 sells both goods and has perfect memory but cannot write long-term contracts. This hypothetical game can be solved by applying the revelation principle by Bester and Strausz 2001.

Assumption 1.3.1 $\alpha\theta_A > \theta_B$.

By the revelation principle of Bester and Strausz 2001, the optimal mechanism under the disclosure policy satisfies feasibility, individual rationality and incentive compatibility, sequential rationality, and Bayes' rule. Therefore seller 1 takes into account that her choice of a mechanism affects the optimal mechanism of seller 2 via the sale of the purchase history and seller 2's updated posteriors μ_A and μ_B.

Before I state seller 1's contracting problem, I make one simplifying assumption. A buyer of type B can never profit since he never receives a positive rent. However a buyer of type A may profit from rejecting seller 1's offer. Therefore I assume $\mu_{off}\theta_A \geq \theta_B$, where μ_{off} denotes a seller's off path posterior about the buyer conditional on the event that the buyer does not participate in mechanism 1 or rejects seller 1's offer.

Note that I restrict attention to the case in which seller 1 always sells the purchase history under the disclosure policy. Seller 1's optimal mechanism $\left(\left(x_A^*, p_A^*\right), \left(x_B^*, p_B^*\right)\right)$ under the disclosure policy solves

$$\max_{x_A, x_B, p_A, p_B, s_A, s_B} \left(\begin{array}{c} \left(s_A \alpha + s_B \left(1-\alpha\right)\right) p_A + \left(\left(1-s_A\right)\alpha + \left(1-s_B\right)\left(1-\alpha\right)\right) p_B \\ +P^* \left(s_A, s_B, \mu_A \left(s_A, s_B\right), \mu_B \left(s_A, s_B\right)\right) \end{array} \right) \tag{1.17}$$

subject to

$$x_A \theta_A - p_A + \delta \left(y_{AA}^* \theta_A - t_{AA}^*\right) \geq 0, \tag{1.18}$$

$$x_B \theta_B - p_B + \delta \left(y_{BB}^* \theta_B - t_{BB}^*\right) \geq 0, \tag{1.19}$$

$$\begin{array}{c} x_A \theta_A - p_A + \delta_A \left(y_{AA}^* \theta_A - t_{AA}^*\right) \\ \geq x_B \theta_A - p_B + \delta_A \left(y_{BA}^* \theta_A - t_{BA}^*\right) \end{array}, \tag{1.20}$$

$$\begin{array}{c} x_B \theta_B - p_B + \delta_B \left(y_{BB}^* \theta_B - t_{BB}^*\right) \\ \geq x_A \theta_B - p_A + \delta_B \left(y_{AB}^* \theta_B - t_{AB}^*\right) \end{array}, \tag{1.21}$$

$$s_A \in \begin{array}{cc} \{1\} & \text{if } (1.20) \text{ is slack,} \\ (0,1] & \text{if } (1.20) \text{ binds} \end{array}, \tag{1.22}$$

$$s_B \in \begin{array}{cc} \{0\} & \text{if } (1.21) \text{ is slack,} \\ [0,1) & \text{if } (1.21) \text{ binds} \end{array}, \tag{1.23}$$

$$(1.5), \ (1.6),$$

$$x_i \in [0,1], i \in \{A, B\}. \tag{1.24}$$

Note that $\left(\left(y_{jA}^*, t_{jA}^*\right), \left(y_{jB}^*, t_{jB}^*\right)\right)$ is seller 2's equilibrium offer conditional on having purchased the purchase history of buyer 1 who reported j. In the PBE, seller 1's conjecture about seller 2's optimal offer in period 2 to the customer who reports h is correct. Constraints (1.18) and (1.19) are the individual rationality constraints of types A and B, respectively.

Constraints (1.20) and (1.21) are the incentive compatibility constraints of type A and B, respectively. Constraints (1.22) and (1.23) are consistency conditions that make sure that a buyer with type A or B, respectively, lies about his type only if the respective incentive compatibility constraint binds, which implies that he is indifferent between reporting A or B. Moreover, conditions (1.22) and (1.23) imply that a buyer does not lie with probability 1. As explained above, conditions (1.5) and (1.6) define seller 2's posterior beliefs conditional on observing the buyer's report and given that the buyer reports A with probabilities s_A and s_B. Condition (1.24) is the technological feasibility, i.e. a seller cannot sell more to a buyer than he demands. Moreover, x can be interpreted as a probability.

Lemma 1.3.1 *If seller 1 offers a perfectly separating mechanism so that $s_A = 1$ and $s_B = 0$, then the expected price of the purchase history, $P^*(1, 0, \mu_A(1, 0), \mu_B(1, 0))$, is given by*

$$\begin{array}{ll} \{(1-\alpha)\,\delta_B\theta_B\} & \text{if } (\beta+\lambda)\,\theta_A \geq 2\theta_B \text{ and } \beta\theta_A \geq \theta_B \\ \{\alpha\delta_A(\theta_B - \beta\theta_A) + (1-\alpha)\,\delta_B(2\theta_B - \beta\theta_A)\} & \text{if } (\beta+\lambda)\,\theta_A \geq 2\theta_B \text{ and } \beta\theta_A \leq \theta_B \\ \left\{\begin{array}{c} \alpha\delta_A(\beta\theta_A + \theta_A - 2\theta_B) \\ +(1-\alpha)\,\delta_B(\beta\theta_A - \theta_B) \end{array}\right\} & \text{if } (\beta+\lambda)\,\theta_A \leq 2\theta_B \text{ and } \beta\theta_A \geq \theta_B \\ \{\alpha\delta_A(\theta_A - \theta_B)\} & \text{if } (\beta+\lambda)\,\theta_A \leq 2\theta_B \text{ and } \beta\theta_A \leq \theta_B \end{array}. \quad (1.25)$$

Proof. In Appendix 1. ■

By corollary 1, seller 1's expected revenue is equal to $\alpha\theta_A$ if she chooses the confidential policy.

Theorem 1.3.1 *Suppose assumption 1 holds. Then seller 1 strictly prefers the disclosure policy if parameters $\beta, \delta_A, \delta_B, \theta_A, \theta_B$ satisfy either one of the following parameter regimes:*

I) $(\beta+\lambda)\,\theta_A \geq 2\theta_B$ and $\beta\theta_A \geq \theta_B$ and $\theta_B - \lambda\theta_A > 0$ or

II) $(\beta+\lambda)\,\theta_A \leq 2\theta_B$ and $\beta\theta_A > \theta_B$.

Proof. In Appendix 1. ■

Theorem 1.3.1 is the main result of the chapter; it constitutes the last step that is needed to show that firms could profit from a disclosure policy simply because of two reasons;

personalized information is valuable to a company if there is another clientele. This result explains the firms' behavior to collect customer information and not to commit to strict privacy policies when customers are rational.

Theorem 1.3.1 shows that a firm can profit from a privacy policy that allows the sale of the customer information when the buyer of the purchase history has a slightly different set of customers. Assuming that seller 2 has a slightly different clientele than seller 1 is natural, when seller 1 and seller 2 offer different products.

In the case of type-dependent demand uncertainty and customer entry, the identity of a buyer himself may be valuable to seller 2. Therefore, type-dependent uncertainty differs from the assumption of discounting. The main difference in terms of trade-offs is that seller 2 may have a positive value for the pure knowledge of buyer 1's identity.

Note that $\beta \leq \lambda$ implies that the parameter regime characterized by conditions I) of Theorem 1.3.1 is empty; $\beta \leq \lambda$ implies also that the parameter regime characterized by conditions I) of Theorem 1.3.1 is empty. The sufficient conditions I and II can be interpreted as follows: seller 2 is optimistic about her other clientele and pessimistic about seller 1's clientele. Theorem 1.3.1 shows that under conditions I and II there must be an equilibrium in which seller 1's customer reveals information about his type. Conditions I and II show that the value of the purchase history depends on seller 2's belief about the other customer, buyer 2. Hence, the purchase history is valuable for seller 1, because informs seller 2 about the identity of seller 1's customer.

1.4 Discussion and Conclusion

As in Taylor 2004, the model integrates different dimensions of privacy (information): a customer's purchases and his identity. Taylor's major explanation for firms that collect and sell large amounts of data about customers and the privacy paradox is that customers do not understand the ratchet effect. Therefore customers reveal their preferences without receiving any information rent. However, I derive an explanation for a setting with rational customers.

The model differs from Taylor 2004 in two dimensions: I allow for type-dependent demand uncertainty and customer entry in period 2.

The intuition for my main result lies in the new additional value of the purchase history. This additional value is generated by the new entrant, since the optimal offer is distorted if the seller cannot distinguish the customers. This is the case when the second monopolist is sufficiently more optimistic about his other clientele than about seller 1's clientele.

Revealing Independent Private Value Information When Bidders Have Interdependent Values

2.1 Introduction

2.1.1 Motivation and Main Findings

Suppose the owner of a company announces the sale of her company (target) via an auction (takeover auction). All bidders share a common interest in the quality of the target, e.g. the target's future cash flows. The potential bidders are asymmetrically and imperfectly informed about the target's quality. Besides, potential bidders are heterogenous and have some additional private interest in the company, e.g. because of the potential synergies that arise when the bidder merges with the target.

It has been shown that the information structure is very important for the outcome of an auction (see e.g. Milgrom and Weber 1982a or Bergemann and Pesendorfer 2007). If the seller has more information about the target than the bidders and she can choose how much of her information she wants to publish, then her incentives to disclose depend on the effect of disclosure. In the presence of informational externalities, public disclosure of information may have a variety of effects. First, public disclosure of the seller's information has a direct informational effect on a bidder's estimate of his valuation, which induces him to adjust his valuation. Second, it may induce a linkage principle if it reduces the winner's curse of a bidder by providing more information about the common value component (Milgrom and Weber 1982a). Third, public disclosure of information about all bidders' private values may induce strategic effects as a reaction to the relative asymmetries among the bidders. This chapter assumes that the seller's information only contains private value information; I make this assumption in order to evaluate the seller's incentives to publicly disclose private value information prior to a second-price auction.[1] To the best of my knowledge, the underlying paper to this chapter is the first to evaluate the third effect of disclosed information about the target's private value characteristics in the presence of informational externalities.

Private value information that drives synergies may arise in many areas. For example in procurement, research and development, production, human resources, sales and marketing etc. While one potential bidder's strength is his marketing environment, another potential bidder may have technological know-how that helps to decrease production costs (see Szech 2011 for a similar argument or Gärtner and Schmutzler 2009). Other typical examples for goods that have interdependent value/common and private value character are financial assets or houses (Bulow and Klemperer 2002).[2] Jehiel, Meyer-ter-Vehn, Moldovanu and Zame 2006 argue that valuations are interdependent for reasons that are related to the market structure and the companies' relationships with each other.

In this context, I evaluate the seller's incentives to disclose information prior to a second-price auction in the following simple model (a variant of the model of Milgrom and Weber 1982a). The seller has a private source of information containing information about the bidders' private values. There are two bidders with interdependent values. The bidders have a preliminary private signal, which they learn at the beginning of the game, about the good's common and private value. The game has two periods. In period 1 the seller chooses a disclosure policy, either full disclosure or no disclosure. If the seller chose to disclose the

[1] Although I assume that the seller's information does not contain common value information, there can be a positive linkage between the seller's information and the seller's expected revenue.

[2] Restricting attention to common values is overly restrictive since a bidder's valuation depends not only on the good's quality, prestige value or resale value (Milgrom and Weber 1982a) but also on the buyer's preference for the good (see e.g. Myerson 1981). Similarly, the assumption of private values has been criticized as being overly restrictive (Jehiel, Meyer-ter-Vehn, Moldovanu, Zame 2006). Therefore I consider a setting where bidders have interdependent values.

information, then she discloses the information in period 2 and both bidders learn all of the seller's signals. All signals are independently distributed. Otherwise no bidder learns the seller's information. Afterwards, the second-price auction with two bidders takes place. Since the bidders' information about the common value is incomplete, there are informational exernalities between the bidders. Note that I abstract from allocative externalities[3].

To evaluate the third effect of disclosure, I assume that once the seller discloses information publicly all bidders update their beliefs about each other's valuation in the same way. In particular, I assume that bidders observe how the good's published characteristics affect each bidder's valuation. This assumption implies that bidders have only one-dimensional private information[4] and can strategically adjust their strategies to their estimate about their rival's synergies.

I characterize the seller-optimal Bayesian Nash equilibrium of the game. First, I solve for the bidders' equilibrium bidding strategies and then for the seller's equilibrium disclosure policy. Note that the findings of this chapter are also very relevant for the English auction since every last stage of an English auction is strategically equivalent to a second-price auction with two bidders.

The main insight of this chapter is that the linkage principle holds if the seller's information is sufficiently informative about the bidders' private value information. To derive this result, I characterize the seller-optimal equilibrium. I show that there are two types of seller-optimal equilibria where the bidders' bidding strategies are of linear form and continuous and strictly increasing in the bidders' preliminary information. First, there exists an equilibrium in which the seller publicly discloses her information if the seller's information has a high impact. Second, if the seller possesses information that has a low impact on the bidders' valuations, then an equilibrium exists in which the seller conceals the information. I

[3] We follow the definition of interdependent values of Jehiel and Moldovanu 2001 but rule out allocative externalities.

[4] The seminal papers on efficient mechanisms where bidders have multidimensional private information are Maskin 1992 and Jehiel and Moldovanu 2001. We rule out multidimensional private information and allow bidders to condition their bid strategies on the public information. As a result, efficient equilibria after disclosure of private value information may exist.

also discuss conditions under which each of these two types of equilibria is the seller-optimal equilibrium in one of two mutually exclusive parameter regimes.

Before being able to give an intuition for the main insight, I discuss the effect of public disclosure of the seller's information on the bidders' beliefs. If the seller discloses her information, then each bidder updates the estimate of his own valuation and his rival's valuation, i.e. of both bidders' private values for the good. Most importantly, bidders may perceive each other as asymmetric since the seller discloses several independent signals that idiosyncratically affect the bidders' private values. A bidder with a private value advantage is said to be strong, and a bidder with a private value disadvantage is said to be weak.

Next, I discuss the assumption of interdependent values. Since there are informational externalities between the bidders, the bidders are exposed to the winner's curse conditional on winning. Conditional on losing, bidders are not exposed to the winner's curse. Compared with the exposition to the winner's curse in the auction without disclosure, in the auction with disclosure, bidders can be asymmetric. The weak bidder's exposition to the winner's curse conditional on winning is stronger and the strong bidder's exposition to the winner's curse is weaker than in a symmetric setting.

The strength of the effect of disclosure on the bidders' exposition to the winner's curse depends on the size of the informational externalities and the importance of the seller's information for the bidders' valuations. If the seller's independent private value information has a low importance for the bidders' valuations, then the effect of the informational externalities is low; that is, a bidder's exposition to the winner's curse changes only slightly. In this case, the seller's incentives are similar to the incentives in an independent private value setting. In contrast, one can show that the independent private value information has a high impact on a bidder's exposition to the winner's curse if the effect of the informational externalities is high.

Bidders shade their bids when they are exposed to a winner's curse. When a bidder's exposition to the winner's curse changes, then the bidder's bid shading behavior changes. If the bidders turn out to be asymmetric after disclosure, then the bidders adjust their bidding strategies to the disclosed information in the following way. While the strong bidder increases his bid conditional on winning, the weak bidder shades his bid conditional on winning. However, if a bidder knows that he will lose, he is willing to bid up to an amount so that he is sure that he loses. Whether a weak bidder may win in the equilibrium depends on the degree of informational externality and the informativeness/importance of the seller's information.

The strategic effect of disclosure on the bidders' bidding strategies is weak if the informational externality and the seller's information are of low importance. Basically, the strong bidder increases his bid conditional on winning and the weak bidder decreases it. In comparison to the auction when no information is disclosed, the strong bidder wins more often and the weak bidder loses more often. Overall the seller's expected revenue decreases. Intuitively, the setting and equilibrium behavior resembles very much the independent private value setting with the main difference that bidders shade their bids to account for the winner's curse.

The strategic effect is strong if the informational externality and the seller's information are very high. The weak bidder has to shade his bid conditional on winning so much that he bids something negative. In equilibrium, the weak bidder never wins. Conditional on losing, the weak bidder is willing to bid at least his minimal valuation, or some bid that is adjusted for his rival's advantage. This effect introduces a linkage between the seller's information and the price paid in the auction. Therefore the seller profits from disclosing her information, or, in other words, the linkage principle holds if the information is very important and the informational externalities are high.

I also discuss an interesting concept to evaluate the effect of information disclosure: the *allocation effect*. Board 2009 defines it as the effect of disclosed information on the revenue triggered by the change of the allocation/winning bidder. The allocation effect is a consequence of the individual bidder's adaptation of his bidding strategies to the disclosed information. Board finds that the allocation effect of public private information on the expected revenue is always negative. I find that this allocation effect is positive for some realizations of the bidders' valuations and negative for others when bidders have informational externalities.

2.1.2 Related Literature

Numerous papers consider disclosure of information prior to auctions, but, again, to the best of my knowledge, the paper underlying to this chapter is the first one to consider disclosure of private value information in the presence of informational externalities. Some of the papers consider public disclosure of information, and the seminal paper (Milgrom and Weber 1982a) mainly analyzes optimal disclosure of common value information in different standard auctions. The authors find a revenue-ranking in the presence of affiliated signals and show that public disclosure is optimal. They rule out disclosure of private value information and asymmetric disclosure. Mares and Harstad 2003 relax the implicit assumption of symmetric and public disclosure in first-price and second-price auctions. For special valuation functions, they show that asymmetric or private disclosure can improve revenue under some circumstances. Larson 2009 addresses how disclosure of independent information about common values has no effect on the seller's expected revenue when bidders have preliminary private information. Larson rules out disclosure of private value information. Board 2009 considers public disclosure of private value information but rules out informational externalities.

My setting lies between Milgrom and Weber 1982a and Board 2009. Milgrom and Weber show that the linkage principle holds for the disclosure of affiliated common value signals in a second-price auction. Board 2009 shows that the linkage principle fails to hold in an independent private value second-price auction with two bidders. Notice that Board 2009 considers independently distributed signals, which is a special case of affiliated signals. I consider an interdependent value second-price auction with independent signals. The seller can disclose private value signals, as in Board 2009.

Another branch of the literature considers private disclosure of information. Mares and Harstad 2003 show that private disclosure of common value information may be better than public disclosure of this information. Ganuza and Penalva 2010 consider optimal costly disclosure, but rule out preliminary information and informational externalities. Szech 2011 considers costly disclosure of several private value information packages before an auction with entry fees but rules out preliminary information and informational externalities.

Other papers apply a mechanism design approach to related questions. Esö and Szentes 2007 address the question of optimal disclosure in an auction with preliminary information, but rule out informational externalities. Bergemann and Pesendorfer 2007 and Bergemann and Wambach 2013 consider the optimal information structure in an auction and employ a mechanism design approach to analyze this question, but rule out informational externalities. Gershkov 2009 considers the disclosure of common value information, but rules out informational externalities of private information. Skreta 2009 considers optimal information disclosure in an auction when the seller is informed about her information. She shows that disclosure is irrelevant in a private value setting. Otherwise, it is optimal not to disclose information.

Further strongly related literature analyses auctions with informational externalities (e.g. Jehiel, Moldovanu and Stacchetti 1999). The seminal papers on the efficiency of auctions with informational externalities are Maskin 1992, Dasgupta and Maskin 2000 and Jehiel and Moldovanu 2001, which provides general results for general mechanisms, such as the impossibility of ex-post implementation of efficient allocations with multi-dimensional signals. The main difference to this chapter is the information structure, which is the reason why the good may be allocated efficiently in an equilibrium in which the seller discloses her information. For more recent contributions, see Birulin 2003 or de Frutos and Pechlivanos 2006. For a general overview of the literature, I refer the reader to Jehiel and Moldovanu 2006.

This chapter also relates to the papers on *almost common value auctions*, which are auctions with informational externalities where valuations are additive in the bidders' private information (see among others Bikhchandani 1988, Klemperer 1998, Bulow and Klemperer 2002, Levin and Kagel 2005). I basically analyze a symmetric almost common value setting (Bulow and Klemperer 2002) with an independent private value perturbation. My model differs from that literature in that I analyze the effect of disclosure.

Since publishing private value information implies that bidders potentially perceive each other as asymmetric, the literature on asymmetric auctions is also related. Asymmetries can prevail in different ways, one bidder may be (more) informed and the other (less) uninformed (see e.g. Milgrom and Weber 1982b, Harstad and Levin 1985 and Einy et al 2002). Since the literature on asymmetric auctions is too large to be covered here, I refer the reader to Rothkopf and Harstad 1994.

The remainder of the chapter is organized as follows. Section 2 presents the model and the approach. Section 3 derives the characterization of two types of equilibria. Section 4 compares the two types of equilibria and discusses the seller-optimal equilibrium. Section 5 concludes.

2.2 Model and Approach

2.2.1 The Model

I consider a game with three players, one seller and two potential buyers. Bidders are ex-ante symmetric with respect to the valuation structure and information structure. The seller intends to sell a single, indivisible object to which she attaches a value zero. The bidders have interdependent values, but in a slightly different way than in Milgrom and Weber 1982a. Bidder i has the following valuation

$$V_i(t_i, t_j, z) = at_i + bt_j + \alpha z_i,\ a > b \geq 0,\ i, j \in \{1, 2\}, j \neq i, \tag{2.1}$$

where a, b and α are the weights with which the bidder's private signal, the opponent's signal t_j and one of the seller's signal z_i enter the bidder's valuation. Note that my specification of bidder i's valuation (2.1) is not symmetric in (t_i, t_j), but in (t_i, t_j, z_i).[5]

Bidder i's valuation, (2.1), can be rewritten as the sum of the good's private value

[5] In contrast to our specification, Milgrom and Weber 1982a assume that the valuation is symmetric in $t_i.t_{-i}$, i.e. $V_i(t_i, t_j, z) = V(t_i, t_j, z)$. Also Board 2009 assumes that valuations are given by $v(t_i, z)$ for all bidders i. See also Krishna 2009 for the definition of symmetric valuations.

component (PV) and the good's common value component (CV)[6]

$$\underbrace{(a-b)\,t_i+\alpha z_i}_{PV}+\underbrace{b\,(t_1+t_2)}_{CV}.$$

It is common knowledge that T_1 and T_2, with typical realizations t_1 and t_2, are independently distributed by F on $\left[\underline{t},\bar{t}\right]$ with associated density function f. Let $\mathbb{E}\left[T\right]$ denote the mean of random variable T_i, $i\in\{1,2\}$. I define $T_{2:2}\equiv\min\left(T_1,T_2\right)$.

The realization of T_i, t_i, is bidder i's private information at the beginning of the game. From the other players' perspectives, bidder i's signal is a random variable that is distributed by T_i's true distribution.

The seller possesses information that enters the bidders' idiosyncratic valuation shocks z_1 and z_2, but she cannot interpret z_1, z_2. α can be interpreted as the marginal impact of the seller's information Z. It is common knowledge that Z_1 and Z_2 are identically and independently distributed binary random variables with typical realizations z_1 and z_2; $z_i\in\{z_h,z_l\}$ with $z_h>z_l$ with $\lambda\equiv P\left(z_i=z_h\right)$. Let $\mathbb{E}\left[Z\right]$ denote the mean of random variable Z_i, $i\in\{1,2\}$. I define $Z_{1:2}\equiv\max\left(Z_1,Z_2\right)$ and $Z_{2:2}\equiv\min\left(Z_1,Z_2\right)$.

At the outset, the realizations of Z_1 and Z_2 are unobservable to all players. As long as the seller does not disclose Z, from the perspective of all players, her information is a vector of random variables with commonly known distributions. The seller can commit to disclose the information or conceal it. Once the seller discloses Z, the bidders learn the realizations z_1 and z_2, but the seller does not. A bidder's valuation of not participating is zero.

The auction format is a second-price auction, which is strategically equivalent to the English auction in the case of two bidders who have interdependent valuations (Milgrom and Weber 1982a, Maskin 1992). The second-price auction here has the same rules as in Maskin 2001, 2003. The winner is the bidder who submitted the highest bid. The winner

[6] This preference structure is a subcase of the one defined in Myerson 1981 or Jehiel and Moldovanu 2001. Myerson 1981 argued that a preference structure usually features *preference uncertainty* and *quality uncertainty*. We refer to these two forms by distinguishing private and common value components. The case of independent private values does not feature quality uncertainty whereas the case of pure common values does not feature preference uncertainty.

receives the good and pays his rival's bid. The loser pays nothing. If only one bidder participates, then he gets the good and pays nothing. Milgrom and Weber 1982a showed that these two formats are not strategically equivalent for more than two bidders who have interdependent valuations. See also Maskin 1992 for a discussion.

The exact timing of the game is the following.

1. Nature draws T_1, T_2, Z_1 and Z_2. Bidders learn their private signals. The seller commits to a disclosure policy. Bidders observe the announced disclosure policy and decide whether to participate or not.

2. If the seller committed to full disclosure, she discloses Z. The bidders observe z_1 and z_2. Bidders announce bids. The seller's good is allocated to the bidder with the highest bid. He then pays the loser's bid. The loser receives nothing and pays nothing.

   If the seller committed to concealing Z, she conceals it. Bidders announce bids. The seller's good is allocated to the bidder with the highest bid, who then pays the loser's bid. The loser receives nothing and pays nothing.

The seller's strategy is her disclosure policy, which is either full disclosure or no disclosure/concealment. D denotes the full disclosure policy, and N denotes the no disclosure policy. In this paper, I use "disclosure" and "full disclosure" as synonyms. I do not need commitment, since the seller cannot observe the realization of Z_1 and Z_2 at any time in the game.

Since the seller has no private information and she can either conceal or fully reveal z_1 and z_2 to the bidders, bidder i's information set at the auction stage is equal to his observable information, which is denoted by $h_i^N = \{t_i\}$ after concealment, disclosure policy N, and $h_i^D = \{t_i, z_i, z_j\}$ after full disclosure, disclosure policy D. β_i^N is a mapping from $\left[\underline{t}, \bar{t}\right]$ to the set of positive real numbers. β_i^D is a mapping from $\left[\underline{t}, \bar{t}\right] \times \{z_l, z_h\}^2$ to the set of positive real numbers. I denote the bid at information set $\{t_i\}$ in the auction after concealment by $\beta_i^N(t_i)$

and the bid in the auction at the information set $\{t_i, z_i, z_j\}$ after the seller's disclosure of Z by $\beta_i^D(t_i, z_i, z_j)$.

A bidder's expected utility in equilibrium must exceed 0 irrespective of his information set, h_i, since the bidder can always bid 0 which promises a payoff of 0.

2.2.2 The Approach

I consider the set of Bayesian Nash equilibria (see Fudenberg and Tirole 1991) in pure strategies. The seller's strategy and the bidders' bidding strategies must be mutually best responses. It is known that even in asymmetric second-price auctions there may be multiple equilibria (Krishna 2009). I restrict attention to the equilibria in linear bidding strategies that are continuous and strictly increasing in the bidders' preliminary private information.

In principle, a unique equilibrium may not exist, but I am merely interested in the seller-optimal equilibrium and what level of expected revenue the seller may realize in the second price-auction when she can choose her revelation policy. Thus, I focus on seller-optimal equilibria and address whether the linkage principle may hold for this type of equilibrium.

I also discuss which equilibria survive the elimination of ex-post weakly dominated strategies. For interdependent values, Chung and Ely 2001 define ex-post weak dominance.

Definition 2.2.1 *(Ely and Chung 2001) Let $\hat{\Sigma}_j \subset \Sigma_j$ be a subset of strategy profiles for the opponents of j. Strategy $\beta_i \in \Sigma_i$ ex-post weakly dominates strategy $\hat{\beta}_i$ against $\hat{\Sigma}_j$ if for every information set profile and every $\beta_j \in \hat{\Sigma}_j$*

$$\pi_i\left(\beta_i(h_i), \beta_j(h_j), h_i, h_j\right) \geq \pi_i\left(\hat{\beta}_i(h_i), \beta_j(h_j), h_i, h_j\right)$$

with strict inequality for at least one $\beta_j \in \hat{\Sigma}_j$ and t.

In equilibrium, the seller commits to the disclosure policy that maximizes her expected revenue. Bidder i's equilibrium bidding strategy is given by

$$\beta_i^* \equiv \left(\left\{ \beta_i^{N,*}\left(h_i^N\right) \right\}_{h_i^N \in \left[\underline{t}, \bar{t}\right]}, \left\{ \beta_i^{D,*}\left(h_i^D\right) \right\}_{h_i^D \in \left[\underline{t}, \bar{t}\right] \times \{z_l, z_h\}^2} \right), i \in \{1, 2\},$$

To derive the seller's decision, I compare the expected revenues from disclosure and no disclosure.

First, I analyze the auction in the benchmark setting, in which the seller does not disclose. In this case, I derive the revenue-maximizing strategy, which is not always the symmetric strategy although bidders are symmetric. Sometimes the seller's expected revenue is higher if the bidders bid their minimal valuation.

Then, I characterize an equilibrium in which bidders' bidding behaviors are similar to that in the benchmark setting whenever they are symmetric. However, if the seller discloses information such that one bidder is advantaged and the other is disadvantaged, then it is unclear whether the bidders play bids that constitute corner solutions or interior solutions to their maximization problems. Remember that I call bidder i *strong* and bidder j *weak* if $z_i = z_h$ and $z_j = z_l$. Because of the potential asymmetry of the bidders, the analysis of the revenue-maximizing bidding strategies when the seller discloses her information is a bit more involved.

Therefore I split up the characterization of the revenue-maximizing equilibrium in two steps. In a first step, I analyze the bidding behavior after the seller chose no disclosure. Second, I characterize the equilibrium in which bidders play the unique interior solution to their maximization problems. The following partition of the parameter regime is useful. If the realizations of Z_1 and Z_2 are not identical, then I distinguish regime A and regime B:

- Regime A is defined by $\frac{(a-b)\left(\bar{t}-\underline{t}\right)}{(z_h - z_l)} > \alpha$. In this regime, the seller's information is uninformative about/not important for the bidders' valuations.
- Regime B is defined by $\frac{(a-b)\left(\bar{t}-\underline{t}\right)}{(z_h - z_l)} \leq \alpha$. In this regime, the seller's information is very informative about/ important for the bidders' valuations.

I can show that the interior solution for the auction after disclosure exists only if parameters lie in regime A and that the interior solution is unique. Moreover, it turns out that the equilibrium in which bidders play revenue-maximizing corner solutions if $z_1 \neq z_2$ only survives the elimination of ex-post weakly dominated strategies for parameters in regime B. Therefore I first analyze the equilibrium where bidders play the interior solution if the seller chose disclosure, and then I analyze the other equilibrium with the corner solution.

Last I have to compare the expected revenues and show that the seller receives a higher expected revenue in the former equilibrium than in the latter equilibrium.

Note that the discussion of the efficient allocation is relegated to Appendix 2.B.

2.3 Analysis

2.3.1 Benchmark: No Disclosure

Consider bidder i's maximization problem at stage 2 when the seller committed to conceal her information. Let $\sigma_i^N\left(h_i^N\right)$ denote bidder i's reply to bidder j's bidding strategy featuring $\beta_j^N\left(h_j^N\right)$. Then bidder i's best reply to bidder j's bidding strategy when i observe h_i^N solves the following problem:

$$\max_{\sigma_i^N\left(h_i^N\right)} \mathbb{E}_{T_j}\left[at_i + bT_j + \alpha\mathbb{E}\left[Z\right] - \beta_j^{N,*}\left(h_j^N\right) \middle| \beta_i^N\left(h_i^N\right) \geq \beta_j^{N,*}\left(h_j^N\right), h_i^N\right].$$

$\beta_i^{N,*}\left(h_i^N\right)$ can be an interior solution or a corner solution, where one bidder wins with certainty. The first derivative is given by

$$\frac{\partial\beta_j^{N,*^{-1}}\left(\sigma_i^N\left(h_i^N\right)\right)}{\partial\sigma_i^N\left(h_i^N\right)}\left(at_i + b\beta_j^{N,*^{-1}}\left(\sigma_i^N\left(h_i^N\right)\right) + \alpha\mathbb{E}\left[Z\right] - \sigma_i^N\left(h_i^N\right)\right) f\left(\beta_j^{N,*^{-1}}\left(\sigma_i^N\left(h_i^N\right)\right)\right). \tag{2.2}$$

Let $\sigma_i^{BR}\left(\beta_j^N\left(h_j^N\right)\right)$ denote bidder i's best reply bidder j's bidding strategy such that $\beta_j^N\left(h_j^N\right)$ by , $i \in \{1, 2\}$.

Proposition 2.3.1 *Assume that the seller concealed her information.*

In the class of equilibria where a bidder's strategy has a linear form, is continuous, strictly increasing in the bidder's type and is not weakly dominated, the equilibrium bidding strategies feature:

$$\begin{aligned} \beta_i^{N,*}(t_i) &= (a+b)\,t_i + \alpha \mathbb{E}[Z] \\ \beta_j^{N,*}(t_j) &= (a+b)\,t_j + \alpha \mathbb{E}[Z] \\ i,j &\in \{1,2\}, i \neq j. \end{aligned} \tag{2.3}$$

The seller's expected revenue is equal to

$$\mathbb{E}\left[R^N\left(\beta_i^{N,*}\left(h_i^N\right), \beta_j^{N,*}\left(h_j^N\right)\right)\right] = (a+b)\,\mathbb{E}[T_{2:2}] + \alpha \mathbb{E}[Z] \tag{2.4}$$

Proof. In Appendix 2.A. ■

I briefly outline the proof. First, I derive the unique interior solution, which solves bidder i's first-order condition, i.e. the reply of bidder i to bidder j's bidding strategy such that (2.2) is equal to 0. Since I am interested in equilibria in linear strategies, I will suppose bidder j's bid is of the linear form

$$\beta_j^{N,*}\left(h_j^N\right) = x_j t_j + y_j$$

and then I show that bidder i's best reply is

$$\sigma_i^{BR}\left(\beta_j^{N,*}\left(h_j^N\right)\right) = \frac{x_j}{x_j - b}\left(a t_j - b\frac{y_j}{x_j} + \alpha \mathbb{E}[Z]\right).$$

Then I show that bidder j's best reply to $\sigma_i^{BR}\left(\beta_j^{N,*}\left(h_j^N\right)\right)$ is

$$\sigma_j^{BR}\left(\sigma_i^{BR}\left(\beta_j^{N,*}\left(h_j^N\right)\right)\right) = \frac{a x_j}{(a-b)\,x_j + b^2}\left(a t_j + \frac{b^2}{a x_j} y_j + \frac{a-b}{a}\alpha \mathbb{E}[Z]\right),$$

which must be equal to his bid in equilibrium so that

$$\sigma_j^{BR}\left(\sigma_i^{BR}\left(\beta_j^{N,*}\left(h_j^N\right)\right)\right)=\beta_j^{N,*}\left(h_j^N\right).$$

This linear system has a unique solution for x_j and y_j, namely

$$x_j=a+b$$

and

$$y_j=\alpha\mathbb{E}\left[Z\right].$$

Substitution into the best reply functions gives that at stage 2 the unique interior solutions in linear form to the bidders' maximization problems, which are mutually best replies provided the seller conceals her information are given by

$$\begin{aligned}\sigma_j^{BR}\left(\sigma_i^{BR}\left(\beta_j^{N,*}\left(h_j^N\right)\right)\right) &= \beta_j^{N,*}\left(h_j^N\right)=(a+b)\,t_j+\alpha\mathbb{E}\left[Z\right], \\ \sigma_i^{BR}\left(\beta_j^{N,*}\left(h_j^N\right)\right) &= (a+b)\,t_i+\alpha\mathbb{E}\left[Z\right].\end{aligned} \tag{2.5}$$

It can be shown by substitution that the bidders' expected payoffs are positive.

Moreover, there are corner solutions $\beta_i^N\left(h_i^N\right),\beta_j^N\left(h_j^N\right)$ such that bidder i wins with probability 1, i.e.

$$\beta_i^N\left(h_i^N\right)\geq\beta_j^N\left(h_j^N\right)$$

for all h_i^N and h_j^N. It is relatively easy to see that this corner solution can be ruled out as an equilibrium.

Bidder i's expected utility is positive, i.e.

$$\mathbb{E}_{T_j}\left[at_i+bT_j+\alpha\mathbb{E}\left[Z\right]-\beta_j^{N,*}\left(h_j^N\right)\right]\geq 0.$$

if bidders play

$$
\begin{aligned}
\beta_i^N\left(h_i^N\right) &= a\bar{t} + bt_i + \alpha\mathbb{E}[Z] \\
\beta_j^N\left(h_j^N\right) &= a\underline{t} + bt_i + \alpha\mathbb{E}[Z].
\end{aligned}
$$

However, bidder i's candidate equilibrium strategy $a\underline{t} + bt_i + \alpha\mathbb{E}[Z]$ is weakly dominated by the strategy to bid his true minimal expected valuation $at_i + b\underline{t} + \alpha\mathbb{E}[Z]$ since

$$
at_i + b\underline{t} + \alpha\mathbb{E}[Z] > a\underline{t} + bt_i + \alpha\mathbb{E}[Z],
$$

which is true for all t_i.

It remains to be shown that the candidate equilibrium $at_i + b\underline{t} + \alpha\mathbb{E}[Z]$ and $a\bar{t} + bt_j + \alpha\mathbb{E}[Z]$, $i \neq j, i, j \in \{1, 2\}$ cannot be an equilibrium in increasing and continuous strategies. The reason is that bidder j always wins, but receives negative payoff for some of his types.

In the subsequent part of the chapter I write R^N if I want to refer to (2.4).

2.3.2 Equilibrium I

In this subsection, I discuss an equilibrium in which the bidding strategies are interior solutions to the bidders' maximization problems at each information set. It can be shown that this type of equilibrium exists only if parameters lie in regime A, i.e. with $\alpha \leq \frac{(a-b)\left(\bar{t}-\underline{t}\right)}{z_h - z_l}$. In this equilibrium the seller conceals her information. The main intuition is that the weak bidder's exposition to the winner's curse conditional on winning is larger than it would be in the case without disclosure. Therefore the weak bidder has to shade his bid much more.

Auction Stage After Disclosure in Equilibrium I

If the seller discloses Z, then bidder i's information set is represented by $h_i^D = \{t_i, z_i, z_j\}$, $i, j \in \{1, 2\}$, $j \neq i$. I solve the auction after disclosure for a given information set profile $\left(h_1^D, h_2^D\right) = (\{t_1, z_1, z_2\}, \{t_2, z_2, z_1\})$. Then bidder i's maximization problem after disclosure is given by

$$\max_{\beta_i\left(h_i^D\right)} \mathbb{E}_{T_j}\left[\left(at_i+bt_j+\alpha z_i-\beta_j^{D*}\left(t_j,z_i,z_j\right)\right)\middle|\,\beta_i\left(h_i^D\right)>\beta^{D*}\left(h_j^D\right),h_i^D\right] \tag{2.6}$$

if his type is t_i and bidder j bids equilibrium strategy $\beta^{D*}\left(t_j,z_i,z_j\right)\ \forall t_j$.

$\beta_i^{D,*}\left(h_i^D\right)$ can be an interior solution or a corner solution, where one bidder wins with certainty. The first derivative is given by

$$\frac{\partial\beta_j^{D,*^{-1}}\left(\beta_i^D\left(h_i^D\right)\right)}{\partial\beta_i^D\left(h_i^D\right)}\left(at_i+b\beta_j^{D,*^{-1}}\left(\beta_i^D\left(h_i^D\right)\right)+\alpha z_i-\beta_i^D\left(h_i^D\right)\right)f\left(\beta_j^{D,*^{-1}}\left(\beta_i^D\left(h_i^D\right)\right)\right). \tag{2.7}$$

Bidder i solves (2.13) for each information set h_i^D provided that his opponent plays the equilibrium candidate.

Proposition 2.3.2 *Assume* $(a+b)\,\underline{t}+\alpha\frac{az_l-bz_h}{a-b}\geq 0$ *and* $\alpha<\frac{(a-b)\left(\bar{t}-\underline{t}\right)}{(z_h-z_l)}$. *There exists an equilibrium of the game in which the bidders play*

$$\beta_i^{N,*}\left(t_i\right)=\mathbb{E}_{Z_i}\left[v_i(t_i,t_i,Z_i)\right],i,i\in\{1,2\}$$

after the seller committed to no disclosure and

$$\beta_i^{D,*}\left(t_i,z_i,z_j\right)=v_i(t_i,t_i,z_i)-\alpha\frac{b\left(z_j-z_i\right)}{a-b},\quad j\neq i,i,j\in\{1,2\} \tag{2.8}$$

after the seller committed to disclosure.

Proof. In Appendix 2.A. ■

For some parameters, there is an equilibrium where a bidder plays an interior solution to his maximization problem. Clearly, the intuition for the bidders' bidding strategies after disclosure is identical to the intuition in the benchmark setting for the bidders' symmetric bidding strategies in Proposition 2.3.1. Therefore I focus on the equilibrium strategies after disclosure, for which I sketch the main part of the proof.

Suppose a bidder plays a linear strategy $\hat{\beta}_j = w_j t_j + x_j z_j + y_j z_i + e_j$. Let σ_i denote bidder i's reply to this linear strategy and $\sigma_i^{BR}\left(\hat{\beta}_j\right)$ i's best reply to $\hat{\beta}_j$, $i, j \in \{1, 2\}, j \neq i$. If bidder j play $\hat{\beta}_j$, then the first-order condition of bidder i's maximization problem is satisfied if and only if

$$\frac{\partial \hat{\beta}_j^{-1}(\sigma_i)}{\partial \sigma_i}\left(at_i + \frac{b}{w_j}(\sigma_i - x_j z_j - y_j z_i - e_j) + \alpha z_i - \sigma_i\right) f\left(\hat{\beta}_j^{-1}(\sigma_i)\right) = 0$$

which implies that bidder i's best reply to $\hat{\beta}_j$ is equal to

$$\sigma_i^{BR}\left(\hat{\beta}_j\right) = \frac{aw_j}{w_j - b} t_i + z_i \left(\frac{\alpha w_j - y_j b}{w_j - b}\right) - \frac{b x_j}{w_j - b} z_j - \frac{b e_j}{w_j - b}.$$

Then the first-order condition of bidder j's maximization problem is satisfied if

$$\frac{\partial \sigma_i^{BR^{-1}}(\sigma_j)}{\partial \sigma_j}\left(at_j + \frac{bw_j - b^2}{aw_j}\begin{pmatrix} \sigma_j + \frac{bx_j}{w_j - b} z_j + \frac{be_j}{w_j - b} \\ -z_i\left(\frac{\alpha w_j - y_j b}{w_j - b}\right)\end{pmatrix} + \alpha z_j - \sigma_j\right) f\left(\sigma_i^{BR^{-1}}(\sigma_j)\right) = 0$$

which implies that bidder j's best reply $\sigma_j^{BR}\left(\sigma_i^{BR}\left(\hat{\beta}_j\right)\right)$ to bidder i's best reply $\sigma_i^{BR}\left(\hat{\beta}_j\right)$ against bidder j's linear strategy is given by

$$\begin{aligned}\sigma_j^{BR}\left(\sigma_i^{BR}\left(\hat{\beta}_j\right)\right) &= \frac{a^2 w_j}{(a-b) w_j + b^2} t_j + \frac{b^2 x_j + a w_j \alpha}{(a-b) w_j + b^2} z_j \\ &\quad + \frac{b^2 e_j}{(a-b) w_j + b^2} - z_i \frac{\alpha b w_j - y_j b^2}{(a-b) w_j + b^2}\end{aligned}$$

The strategies are mutually best replies if $\sigma_j^{BR}\left(\sigma_i^{BR}\left(\hat{\beta}_j\right)\right)=\hat{\beta}_j$ which implies

$$\begin{aligned}
\frac{b^2}{(a-b)\,w_j+b^2}e_j &= e_j \iff (a-b)\,w_j=1 \text{ or } e_j=0,\\
\frac{a^2w_j}{(a-b)\,w_j+b^2} &= w_j \iff (a+b)=w_j \text{ for } a\neq b \text{ ot } w_j=0,\\
\frac{b^2x_j+aw_j\alpha}{(a-b)\,w_j+b^2} &= x_j \iff \frac{a\alpha}{a-b}=x_j,\\
-\frac{b\alpha w_j-y_jb^2}{(a-b)\,w_j+b^2} &= y_j \iff -\frac{b\alpha}{a-b}=y_j.
\end{aligned}$$

Note that $w_j>0$, since $\hat{\beta}_j$ is increasing and continuous in t_j. This implies $e_j=0$. Then $\hat{\beta}_j=\sigma_j^{BR}\left(\sigma_i^{BR}\left(\hat{\beta}_j\right)\right)=(a+b)\,t_j+\frac{a}{a-b}\alpha z_j-\frac{b}{a-b}\alpha z_i$.Substitution into $\sigma_i^{BR}\left(\hat{\beta}_j\right)$ gives $\sigma_i^{BR}\left(\hat{\beta}_j\right)=\frac{a(a+b)}{(a+b)-b}t_i+z_i\left(\frac{\alpha(a+b)+\frac{b\alpha}{a-b}b}{(a+b)-b}\right)-\frac{b\frac{a\alpha}{a-b}}{(a+b)-b}z_j$, which simplifies to $\sigma_i^{BR}\left(\hat{\beta}_j\right)=(a+b)\,t_i+\alpha z_i\frac{a}{a-b}-\frac{b}{a-b}\alpha z_j$. Hence the equilibrium in linear strategies that are increasing and continuous in the bidders' preliminary private information is unique and symmetric. Assumption $(a+b)\,\underline{t}+\alpha\frac{az_l-bz_h}{a-b}\geq 0$ guarantees that the bids are nonnegative.

Then the seller's expected revenue from disclosure is equal to

$$\begin{aligned}
&\mathbb{E}\left[R_I^D\left(\beta_1^{D,*}\left(h_1^D\right),\beta_2^{D,*}\left(h_2^D\right)\right)\right]\\
&=\mathbb{E}_{Z_i,Z_j}\left[\sum_{\substack{i=1\\j\neq i}}^2\int_{\left\{t_i,t_j|\beta_i^{D,**}(h_i^D)\leq\beta_j^{D,**}(h_j^D)\right\}}\left[v_i(t_i,t_i,Z_i)-\alpha\tfrac{b}{a-b}(Z_j-Z_i)\right]dF\left(t_i,t_j\right)\right],
\end{aligned}\tag{2.9}$$

In the subsequent part of the section, I will write R_I^D referring to (2.9).

If the parameters lie in regime B i.e. $\alpha<\frac{(a-b)\left(\bar{t}-\underline{t}\right)}{z_h-z_l}$, then the strong bidder would always win with this strategy. In this parameter regime, the derived strategies do not characterize an interior solution.

The Seller's Incentives in Equilibrium I

I have seen that when bidders bid according to (2.8) after disclosure, then both bidders win with positive probability in the auction following disclosure, provided that the parameter values are in regime A.

For regime A, I can rewrite the seller's expected revenue after disclosure, (2.9) as

$$R_I^D = \left\{ \begin{array}{c} \sum_{\substack{i=1 \\ j\neq i}}^{2} \lambda(1-\lambda) \left[\begin{array}{l} \int\limits_{\left\{t_i,t_j | t_j > t_i + \alpha\frac{z_h - z_l}{a-b}\right\}} \left((a+b)\,t_i + \alpha z_h - \alpha\frac{b(z_l - z_h)}{a-b}\right) dF(t_i,t_j) \\ + \int\limits_{\left\{t_i,t_j | t_j < t_i + \alpha\frac{z_h - z_l}{a-b}\right\}} \left((a+b)\,t_j + \alpha z_l - \alpha\frac{b(z_h - z_l)}{a-b}\right) dF(t_i,t_j) \end{array} \right] \\ + \lambda^2 \sum_{\substack{i=1 \\ j\neq i}}^{2} \int_{\{t_i,t_j | t_i \le t_j\}} \left[(a+b)\,t_i + \alpha z_h\right] dF(t_i,t_j) \\ + (1-\lambda)^2 \sum_{\substack{i=1 \\ j\neq i}}^{2} \int_{\{t_i,t_j | t_i \le t_j\}} \left[(a+b)\,t_i + \alpha z_l\right] dF(t_i,t_j) \end{array} \right\}. \tag{2.10}$$

If R_I^D exceeds the seller's revenue without disclosure, R^N, then the seller commits to publicly disclosing Z at the beginning of the game.

Theorem 2.3.1 *Assume* $(a+b)\,\underline{t} + \alpha\frac{az_h - bz_l}{a-b} \geq 0$ *and* $0 < \alpha < \frac{(a-b)\left(\bar{t}-\underline{t}\right)}{z_h - z_l}$. *Suppose bidders bid according to* (2.8). *Then the seller's expected gain from publicly disclosing* Z *is equal to*

$$W_I = R_I^D - R^N \tag{2.11}$$

$$= \lambda(1-\lambda)(a+b) \cdot \left\{ \begin{array}{c} \sum_{\substack{i=1 \\ j\neq i}}^{2} \int_{\underline{t}}^{\bar{t}-\alpha\frac{z_h - z_l}{a-b}} \int\limits_{t_i + \alpha\frac{z_h - z_l}{a-b}}^{\bar{t}} (t_i - t_j)\, dF(t_j)\, dF(t_i) \\ + \frac{2(z_h - z_l)}{a-b} \left(\begin{array}{c} \frac{a-b}{z_h - z_l}\left(\mathbb{E}[T] - \mathbb{E}[T_{2:2}]\right) \\ +\alpha \left[\begin{array}{c} \int_{\underline{t}}^{\bar{t}-\alpha\frac{z_h - z_l}{a-b}} F(t_i)\, f\left(t_i + \alpha\frac{z_h - z_l}{a-b}\right) dt_i \\ -\frac{1}{2} \end{array} \right] \end{array} \right) \end{array} \right\}. \tag{2.12}$$

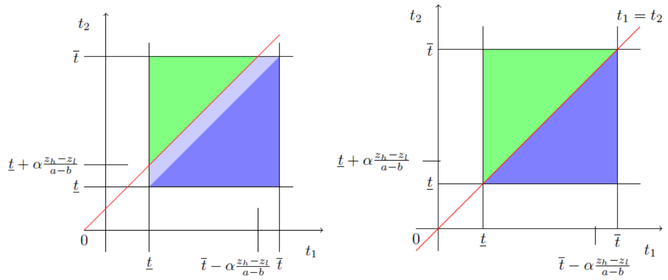
Then the seller prefers to conceal Z.

Proof. In Appendix 2.A. ■

This result shows that the seller prefers no disclosure if her information is very uninformative (in regime A, i.e. $\alpha < \frac{(a-b)\left(\bar{t}-\underline{t}\right)}{z_h - z_l}$).

The Allocation Effect of Disclosure in Equilibrium I

The allocation effect is the change of the revenue in reaction to the information. There is no allocation effect if the public information does not change the winning bidder, i.e. if $z_1 = z_2$ (see Board 2009 for a similar argument). If $z_1 \neq z_2$, then the strong bidder bids more aggressively than the weak bidder and the weak bidder bids less than his expected value conditional on winning, because he fears the winner's curse much more. The effect of public information here is that the strong bidder wins more often than the weak bidder and that the weak bidder bids less than under disclosure.



Allocation with Disclosure. Allocation without disclosure.

Figure 2.1.

Figure 2.1 illustrates the allocation effect of disclosure compared to the auction after no disclosure for $z_1 = z_h$ and $z_2 = z_l$. The equilibrium allocation is efficient. For allocations above the red line in the left graph of figure 2.1 bidder 2 has the highest valuation and wins. For all allocations below the red line, he has the lowest allocation and loses. One can easily see that the strong bidder, bidder 1, wins more often. The reason is that bidder 1 is the strong bidder who bids more than in the auction without disclosed information. The weak bidder shades his bid much more. Together this implies that bidder 2 loses more often than under no disclosure. On average the bids decrease.

Discussion of Equilibrium I

Clearly, the unique best reply to the strategy of player j in the equilibrium with the interior solution for $\alpha < \frac{(a-b)(\bar{t}-\underline{t})}{z_h - z_l}$ is playing the symmetric strategy. This equilibrium behavior in the auction after no disclosure does not constitute an interior solution for $\alpha > \frac{(a-b)(\bar{t}-\underline{t})}{z_h - z_l}$. Furthermore, it can be shown that there are other Bayesian equilibria in which the seller's expected revenue is at least weakly higher than R^N. Therefore I can ignore this solution candidate for regime B.

One can show that bidder i's equilibrium bidding strategy is not weakly dominated by any bidding strategy that features to bid his true minimal valuation by a strategy adjusted to $at_i + b\underline{t} + \alpha z_l$ for $z_i < z_j$ or by $at_i + b\underline{t} + \alpha z_h$ for $z_i > z_j$ respectively.

2.3.3 Equilibrium II

In this subsection, I derive a Bayesian Nash equilibrium of the game where bidders bidding strategies involve corner solutions if $z_1 \neq z_2$ after the seller disclosed her information. In this corner solution the advantaged bidder wins with certainty if $z_1 \neq z_2$ and the seller discloses if the informational externalities are sufficiently high. The characterization will show that in the class of equilibria that I consider this equilibrium only exists if the parameters of the model lie in regime B. For this, we must check robustness of the equilibrium strategies to the elimination of weakly dominated strategies.

Since it can be shown for parameter regime A, $\alpha < \frac{(a-b)(\bar{t}-\underline{t})}{z_h - z_l}$, that the seller conceals her information in the revenue-optimal equilibrium involving such corner solutions, I can restrict attention to regime B with $\alpha \geq \frac{(a-b)(\bar{t}-\underline{t})}{z_h - z_l}$.

The Auction Stage After Disclosure in Equilibrium II

Assume $\alpha \geq \frac{(a-b)(\bar{t}-\underline{t})}{z_h - z_l}$. For parameter regime A the equilibria are ruled out by the elimination of ex-post weakly dominated strategies, which I will show later.

If the seller discloses Z, then bidder i's information set is represented by $h_i^D = \{t_i, z_i, z_j\}$,

$i, j \in \{1,2\}$, $j \neq i$. Bidder i's maximization problem after disclosure is given by

$$\max_{\beta_i(h_i^D)} \mathbb{E}_{T_j}\left[\left(at_i + bt_j + \alpha z_i - \beta_j^{D*}(t_j, z_i, z_j)\right)\middle|\, \beta_i\left(h_i^D\right) > \beta^{D*}\left(h_j^D\right), h_i^D\right] \tag{2.13}$$

if his type is t_i and bidder j bids equilibrium strategy $\beta^{D,*}(t_j, z_i, z_j)\ \forall t_j$.

$\beta_i^{D,*}\left(h_i^D\right)$ can be an interior solution or a corner solution, where one bidder wins with certainty. The first derivative is given by

$$\frac{\partial \beta_j^{D,*^{-1}}\left(\beta_i^D\left(h_i^D\right)\right)}{\partial \beta_i^D\left(h_i^D\right)}\left(at_i + b\beta_j^{D,*^{-1}}\left(\beta_i^D\left(h_i^D\right)\right) + \alpha z_i - \beta_i^D\left(h_i^D\right)\right) f\left(\beta_j^{D,*^{-1}}\left(\beta_i^D\left(h_i^D\right)\right)\right). \tag{2.14}$$

Denote a best reply of bidder i to some bid $\beta_j^D\left(h_j^D\right)$ by $\sigma_i^{BR}\left(\beta_j^D\left(h_j^D\right)\right)$, $i \in \{1,2\}$.

Let γ be some real number between $\underline{t}$ and $\overline{t}$.

Proposition 2.3.3 *Assume* $\alpha \geq \frac{(a-b)\left(\overline{t}-\underline{t}\right)}{z_h - z_l}$. *If the seller discloses her information, then there exists an equilibrium in which the bidders' equilibrium bidding strategies* β_1^* *and* β_2^* *satisfy*

$$\begin{aligned}
\beta_i^{D,*}\left(h_i^D\right) &= \begin{cases} (a+b)\,t_i + \alpha z_i & \text{if } z_i = z_j \\ a\underline{t} + bt_i + \alpha z_h & \text{if } z_i < z_j \\ (a-b)\,\underline{t} + b\overline{t} + bt_i + \alpha z_h & \text{if } z_i > z_j \end{cases}, \forall t_i \in \left[\underline{t}, \overline{t}\right], \\
\beta_j^{D,*}\left(h_j^D\right) &= \begin{cases} (a+b)\,t_j + \alpha z_j & \text{if } z_i = z_j \\ a\underline{t} + bt_j + \alpha z_h & \text{if } z_i > z_j \\ (a-b)\,\underline{t} + b\overline{t} + bt_j + \alpha z_h & \text{if } z_i > z_j \end{cases}, \forall t_j \in \left[\underline{t}, \overline{t}\right], \\
i, j &\in \{1,2\}, j \neq i.
\end{aligned} \tag{2.15}$$

The seller's expected revenue from concealing is given by (2.4). *The seller's expected revenue from disclosing is given by*

$$\begin{aligned}
&\mathbb{E}\left[R^D\left(\beta_1^{D,*}\left(h_1^D\right), \beta_2^{D,*}\left(h_2^D\right)\right)\right] \\
&= (1-2\lambda(1-\lambda))(a+b)\,\mathbb{E}\left[T_{2:2}\right] + 2\lambda(1-\lambda)\left(a\underline{t} + b\mathbb{E}[T]\right) + \alpha\mathbb{E}\left[Z_{1:2}\right]
\end{aligned}. \tag{2.16}$$

Proof. In Appendix 2.A. ■

This equilibrium where the bidders' bidding strategies constitute corner solutions whenever $z_1 \neq z_2$ provided the seller disclosed her information exists if the parameters lie in parameter regime B. The only difference to the auction after without disclosed information is that z_i and z_j will be known to both bidders if the seller publicly discloses her information. The solution for the case $z_1 = z_2$ are derived in a fashion similar to the solution to the bidders' problems in the auction following no disclosure, since both bidders are symmetric in these two situations. In the case of $z_i = z_j$, the bidders are symmetric. I provided an in-depth intuition for the bidding strategies in the benchmark case in the previous section, which is the reason why I do not provide an intuition for the equilibrium bids in case the bidders are symmetric here.

However, I provide an intuition for the bidders' equilibrium bidding behaviors in case of $z_i \neq z_j$ provided the seller disclosed. Consider without loss of generality the case $z_i = z_l$ and $z_j = z_h$, where bidders bid for $\beta_i^D(t_i, z_l, z_h) = a\underline{t} + bt_i + \alpha z_h, \beta_j^D(t_j, z_h, z_l) = (a-b)\underline{t} + b\bar{t} + bt_j + \alpha z_h$. Bidder j is the strong bidder and always wins since

$$a\underline{t} + bt_i + \alpha z_h \leq (a-b)\underline{t} + b\bar{t} + bt_j + \alpha z_h \ \forall t_i, t_j$$

It is relatively easy to see that bidder j's expected utility must be positive, i.e.

$$\mathbb{E}_{T_i}\left[at_j + bT_i + \alpha z_h - (a\underline{t} + bt_i + \alpha z_h)\right] \geq 0 \ \forall t_j.$$

Moreover, bidder j's expected revenue for a slightly higher expected bid of bidder i is negative for $t_j \in \left[\underline{t}, \underline{t} + \frac{\epsilon}{a}\right)$ with arbitrarily small $\epsilon > 0$, since

$$\begin{aligned} \mathbb{E}_{T_i}\left[at_j + bT_i + \alpha z_h - (a\underline{t} + bt_i + \alpha z_h + \epsilon)\right] &< 0 \\ &\Longleftrightarrow \\ a\left(t_j - \underline{t} - \frac{\epsilon}{a}\right) &< 0. \end{aligned}$$

This implies that there is no other solution of linear form such that one bidder wins for certainty, so that the seller would gain more.

The weak bidder i can bid anything as long as he loses without having to fear the potential winner's curse. By deviating from $(a-b)\underline{t}+b\bar{t}+bt_j+\alpha z_h$ to a higher bid bidder j cannot change his expected probability of winning. By decreasing his bid, he can only decrease his probability of winning and thereby decrease his expected payoff. By deviating upwards, bidder i can only decrease his expected revenue, by winning with positive probability and paying $(a-b)\underline{t}+b\bar{t}+bt_j+\alpha z_h$, which exceeds his expected valuation. If bidder i deviates downwards, he still loses and receives payoff 0. Therefore none of the bidders has an incentive to deviate in the case $z_i = z_l$ and $z_j = z_h$, which holds for all $i, j \in \{1,2\}, i \neq j$.

In the subsequent part of the section I write R_{II}^D if I want to refer to (2.16).

The Seller's Incentives to Disclose in Equilibrium II

The seller's expected net utility from publicly disclosing Z, which I denote by W_{II} for $\alpha \geq \frac{(a-b)(\bar{t}-\underline{t})}{z_h - z_l}$, is equal to the difference between the expected revenue with disclosure and without disclosure

$$W_{II} = R_{II}^D - R^N.$$

Substitution gives

$$W_{II} = 2\lambda(1-\lambda)\left(\begin{array}{c} b\left(\frac{(\mathbb{E}[T]-\mathbb{E}[T_{2:2}])}{(\mathbb{E}[T_{2:2}]-\underline{t})}-\frac{a}{b}\right)(\mathbb{E}[T_{2:2}]-\underline{t}) \\ +\frac{\alpha(z_h-z_l)}{2} \end{array}\right)$$

if $\alpha \geq \frac{(a-b)(\bar{t}-\underline{t})}{z_h - z_l}$. If R_{II}^D exceeds the seller's revenue without disclosure, R^N, then the seller commits to publicly disclosing Z at the beginning of the game.

Theorem 2.3.2 *If* $\alpha \geq \frac{(a-b)(\bar{t}-\underline{t})}{z_h - z_l}$ *and bidders play strategies satisfying* (2.3) *and* (2.15), *then the seller publicly discloses her information if and only if*

$$\alpha \geq \max\left(2\left(a - b\frac{(\mathbb{E}[T]-\mathbb{E}[T_{2:2}])}{(\mathbb{E}[T_{2:2}]-\underline{t})}\right)\frac{(\mathbb{E}[T_{2:2}]-\underline{t})}{(z_h-z_l)}, \frac{(a-b)(\bar{t}-\underline{t})}{z_h-z_l}\right) \tag{2.17}$$

holds true; otherwise she conceals her information.

Proof. In Appendix 2.A. ■

Condition (2.17) says that the seller discloses if the impact of her information on the bidders' valuations is sufficiently important.

The shape of the distribution influences $(\mathbb{E}[T] - \mathbb{E}[T_{2:2}])$ and $(\mathbb{E}[T_{2:2}] - \underline{t})$.

Example 2.3.1 [7] *For instance, if $t \sim t^k$ on $[0,1]$, $k > 0$, with associated density $f(t) > 0$ for all $t \in [0,1]$, then*

$$\frac{\mathbb{E}[T] - \mathbb{E}[T_{2:2}]}{\mathbb{E}[T_{2:2}] - \underline{t}} = \frac{1}{2k}.$$

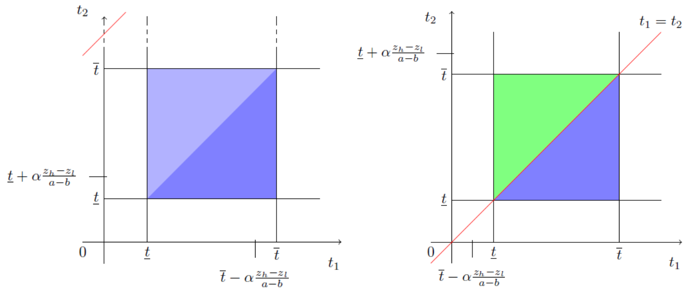
For the uniform distribution, i.e. for $k = 1$, condition (2.17) reduces to

$$\alpha \geq \frac{(a-b)}{z_h - z_l}$$

The Allocation Effect of Disclosure in Equilibrium II

There is no allocation effect if the realizations satisfy $z_1 = z_2$. Therefore I only discuss the potentially positive effects for the case $z_1 \neq z_2$. In these cases the strong bidder always wins. From the characterization of the efficient allocation, I know that the auction is efficient in regime B.

[7] Levin and Kagel 2005 mention the power distribution as an example for which it can be shown that an auctioneer may profit from having an advantage bidder with valuation $v(t_1, t_2) + \alpha z$, $z > 0$ and a regular bidder with $v(t_1, t_2)$ instead of having two regular bidders with valuations $v(t_1, t_2)$ and $v(t_2, t_1)$.



Allocation with disclosure.

Allocation without disclosure.

Figure 2.2.

Figure 2.2 (regime B) depicts the allocations in equilibrium I in case of asymmetric realizations of $z_1 = z_h$ and $z_2 = z_l$. In both graphs of Figure 2.2, the red line separates the equilibrium allocation conditional on the realizations of t_1 and t_2. For (t_1, t_2) below the red line in the blue area, bidder 1 wins the second-price auction, and for (t_1, t_2) above the red line in the green area, bidder 2 wins the second-price auction. The left graph illustrates the equilibrium allocation of the second-price auction after disclosure, and the right graph shows the equilibrium allocation of the second-price auction without disclosure. If no information is disclosed, then a bidder wins if he is the bidder with the highest preliminary private signal (right graph). If the information is disclosed, then bidder 1, the strong bidder wins (left graph). The light blue area in the left graph marks the allocation effect. Comparing the equilibrium allocations, one easily sees that disclosure induces an allocation effect for all type profiles such that $t_2 > t_1$, since bidder 2 would receive the good in the auction after no disclosure.

Disclosure induces the weak bidder to lose more often (always). As a consequence, the weak bidder loses having the higher preliminary signal and the seller's expected revenue is a function of $\mathbb{E}[T]$ instead of $\mathbb{E}[\min(T_1, T_2)]$ whenever $z_1 \neq z_2$, where $\mathbb{E}[T] > \mathbb{E}[\min(T_1, T_2)]$.

This is a positive allocation effect on the seller's expected revenue. Moreover, a comparison of the seller's expected revenues from disclosure and no disclosure shows the higher b, relative to a, the higher the positive allocation effect; that is the higher the informational externality of the winning bidder's private information, relative to the losing bidder's private information, the more profitable is disclosure.

The winning bidder does not receive an information rent with respect to the seller's information, since the losing bidder always bids at least the winning bidder's idiosyncratic shock. Since the losing bidder does not want to be subject to the winner's curse, he must shade his bid sufficiently. The higher α, the higher the losing bidder's bid.

Discussion of Equilibrium II

Proposition 2.3.3 and Theorem 2.3.2 describe equilibrium II. I characterize an equilibrium, where bidders bid the seller-optimal corner solution for regime B. In this section I show that this equilibrium is not robust to the elimination of ex-post weakly dominated strategies if the parameters of the model satisfy regime A.

Suppose the bidder wins with $\beta_i^* \in \left[v\left(t_i, \underline{t}, z_i\right), v\left(t_i, \bar{t}, z_i\right)\right]$ against β_j^* with probability 1 and that he bids $\hat{\beta}_i \neq \beta_j^*$. We can show that the following holds true. If, for a given information set, bidder i's equilibrium bid lies in the interval between his minimal valuation, $v\left(t_i, \underline{t}, z_i\right)$, and his maximal valuation, $v\left(t_i, \bar{t}, z_i\right)$, i.e. in $\beta_i^* \in \left[v\left(t_i, \underline{t}, z_i\right), v\left(t_i, \bar{t}, z_i\right)\right]$, then the effect of a deviation always depends on his rival's true type t_j.

First, I show that $\beta_i^{D,*}\left(t_i, z_h, z_l\right) \in \left[v\left(t_i, \underline{t}, z_h\right), v\left(t_i, \bar{t}, z_h\right)\right]$. Two conditions must hold

$$
\begin{aligned}
(a-b)\, \underline{t} + b\bar{t} + bt_i + \alpha z_h &\leq at_i + b\bar{t} + \alpha z_h \forall t_i \\
&\Longleftrightarrow \\
(a-b)\, \underline{t} &\leq (a-b)\, t_i \forall t_i
\end{aligned}
$$

and

$$\begin{aligned}(a-b)\underline{t}+b\overline{t}+bt_i+\alpha z_h &\geq at_i+b\underline{t}+\alpha z_h \forall t_i \\ &\Longleftrightarrow \\ b\left(\overline{t}-\underline{t}\right) &\geq (a-b)\left(t_i-\underline{t}\right)\forall t_i.\end{aligned}$$

The former condition is true. The latter condition holds true if

$$2b > a.$$

Next, I show that $\beta_i^{D,*}(t_i, z_l, z_h) \in \left[v(t_i, \underline{t}, z_l), v\left(t_i, \overline{t}, z_l\right)\right]$. Two conditions must hold

$$\begin{aligned}a\underline{t}+bt_i+\alpha z_h &\leq at_i+b\overline{t}+\alpha z_l \forall t_i \\ &\Longleftrightarrow \\ \alpha &\leq \frac{(a-b)\,t_i+b\overline{t}-a\underline{t}}{(z_h-z_l)}\forall t_i\end{aligned}$$

and

$$\begin{aligned}a\underline{t}+bt_i+\alpha z_h &\geq at_i+b\underline{t}+\alpha z_l \forall t_i \\ &\Longleftrightarrow \\ \alpha &\geq \frac{(a-b)\left(t_i-\underline{t}\right)}{(z_h-z_l)}\forall t_i.\end{aligned}$$

These conditions hold true for α such that

$$\alpha \in \left[\frac{(a-b)\left(\overline{t}-\underline{t}\right)}{(z_h-z_l)}, \frac{b\left(\overline{t}-\underline{t}\right)}{(z_h-z_l)}\right], \tag{2.18}$$

which implies again that $2b > a$ is a necessary condition.

Therefore the seller-optimal equilibrium in the considered class of equilibria is unique if it exists.

2.4 Disclosure and the Seller-optimal Equilibrium

In this section, I would like to discuss the revenue-maximizing equilibrium. Together with Example 2.3.1 the discussions of the equilibria about the robustness to the elimination of ex-post weakly dominated strategies imply that there is a nonempty parameter regime such that there exists and equilibrium in which the seller discloses her information.

In the equilibrium of regime B, the seller discloses if her information is sufficiently informative, provided the equilibrium exists.

Equilibrium I exists if $(a+b)\,\underline{t} + \alpha\,(az_l - bz_h) \geq 0$ and $\alpha < \frac{(a-b)\left(\bar{t}-\underline{t}\right)}{(z_h-z_l)}$. The seller always conceals her information. If $\underline{t} = 0$, then the former condition is equivalent to $a\frac{z_l}{z_h} \geq b$. Equilibrium II is the unique seller-optimal equilibrium of the considered class of equilibria if conditions (2.17) and (2.18) hold true. The reason is that the seller prefers disclosure over no disclosure if and only if (2.17), which implies that the expected revenue is strictly higher than in the first equilibrium. Moreover, if (2.17) is not true and $\alpha < \frac{(a-b)\left(\bar{t}-\underline{t}\right)}{(z_h-z_l)}$ holds true, then the seller conceals the information in both equilibria and the seller's expected revenue is identical in both equilibria.

We considered the robustness to the elimination of ex-post weakly dominated strategies. The second equilibrium is robust to the elimination of ex-post weakly dominated strategies if $\frac{(a-b)\left(\bar{t}-\underline{t}\right)}{(z_h-z_l)} \leq \alpha \leq \frac{b\left(\bar{t}-\underline{t}\right)}{(z_h-z_l)}$ implying that the informational externality must be sufficiently strong, $b > \frac{a}{2}$. Therefore the first equilibrium is unique in the considered class of equilibria if $\alpha < \frac{(a-b)\left(\bar{t}-\underline{t}\right)}{(z_h-z_l)}$, provided it exists.

2.5 Conclusion

I have studied the seller's incentives to publicly disclose a set of signals about the bidders' private valuations precedent to a second-price auction with two bidders. I assume that bidders have preliminary information with informational externalities on other bidders. An important aspect is how disclosure affects the bidders' bidding strategies.

I characterize the revenue-maximizing/seller-optimal Bayesian Nash equilibrium of the second-price auction with two bidders. The seller publicly discloses her information if the impact of her information is very high, meaning it is informative about the bidders' valuations.

The individual bidder's exposition to the winner's curse conditional on winning with positive probability is stronger for weak bidders after disclosure of the information. Only if the weak bidder loses with certainty is he willing to bid more carelessly. Disclosure may have two effects: an informational effect and a strategic effect.

My result that the linkage principle holds if the seller's information is sufficiently informative about the bidders' private value information relates to the linkage principle for positively affiliated signals in Milgrom and Weber 1982a. The common intuition is that the linkage principle holds whenever the public information reduces the bidder's winner's curse. In my model, this reduction is not a direct effect of the public information but an indirect effect, because it only occurs endogenously in the equilibrium. I show that the linkage principle may hold if the public information has a sufficiently high impact on the bidders' valuations, and in equilibrium the advantaged winner wins whenever bidders are asymmetric.

I consider the case of a two bidders second-price auction. For the case of two bidders the second-price auction is equivalent to the English auction, but not for more than two bidders (Milgrom and Weber 1982a and Maskin 1992). However, the English auction with two bidders occurs at the final stage of each English auction. Moreover, Perry, Wolfstätter and Zamir 2000 show that a two-stages auction with two bidders at the second stage is equivalent to the English auction of Milgrom and Weber 1982a. They report the use of such an auction mechanism for the privatization of an Italian conglomerate. A recent paper proposed the indicative auction mechanism (Ye 2007). In the numerical examples provided by Ye 2007 for the case with initial private value knowledge and additional common value component, the optimal number of bidders in the second stage is two. Variants of the English auction have been reported to be the oldest and probably most commonly used systems (Cassady 1967, Krishna 2009).

Sequential, Multi-dimensional Screening[1]

3.1 Introduction

3.1.1 Motivation

Consider a landowner contemplating to construct a house on her land. A constructor is contacted to build the house. The plans for the house are relatively complex; a variety of decisions have to be taken. To fix ideas, suppose there are two broad issues relating to the exterior and the interior design. The landowner's ideas for the exterior design are relatively standard, but her tastes for interior designs are quite particular. As a result, the constructor knows the costs of completing the exterior parts, but he only has a vague idea about the costs relating to the interior parts. However, he will learn these costs as time goes by.

This is a natural situation, in particular in large scale procurement environments, but the situation arises equally naturally in the context of price discrimination. Consumers thinking about buying a (new generation) smart phone know quite well how they value the services and applications they have already been consuming on their old phones. However, they may only have a vague idea about their valuation for new applications. Broadly speaking, a customer switching from a standard mobile phone to a smart phone knows how many calls he needs to make, but only time will tell how much data he will download with the phone.

[1] This chapter is based on the paper "Sequential, multidimensional screening", Litterscheid and Szalay 2014.

Moreover, casual evidence suggests that firms respond to this information structure.[2]

This chapter is based on a paper, in which we would like to advance our understanding of contracting solutions in these types of environments. We study a two-period model where a principal contracts with an agent to trade a *bundle* of services. At the beginning of the game, the agent has private information about the costs of producing one item in the bundle. He privately learns the costs of producing the other item in the second period. Optimal contracting is dynamic; principal and agent get together both at the outset of the game as well as later on, i.e. in period 2, when more information is available. At the beginning of period 1, the agent decides whether or not he will eventually deliver the bundle of services, but the precise terms of the contract may still be left open at this time. At the second get-together, the remaining details of the contract are specified. The services are produced when all information is available and the agent is paid when all services have been produced.

The literature has analyzed problems that share some but not all the ingredients of our problem. In a nutshell, our problem is a convex combination of a two-dimensional screening problem with a two-dimensional allocation à la Armstrong and Rochet 1999 and a sequential screening problem à la Courty and Li 2000. The main difference to Armstrong and Rochet 1999 is the sequential information structure. The main difference to Courty and Li 2000 is that we assume that the object has two components and that the agent's ex-ante type informs about the cost of production of both components; in particular, the agent's ex-ante type contains perfect information about the costs of producing the first component and imperfect information about the costs of producing the second good. To the best of our knowledge, the underlying paper to this chapter is the first to takes this approach. We

[2] E.g., service provider Orange UK offers a choice of pay as you go services and monthly plans. Moreover, selecting into one of these plans limits the options to choose from later on. In particular, conditional on the selected plan a consumer has the possibility to buy one out of a given variety of additional bundles of services.

Specific examples of such options include plans Dolphin and Monkey http://www.best-mobile-contracts.co.uk/networks/orange.html. Dolphin offers different bundles of classical services with modern services, i.e. texts, minutes and data volume. Monkey offered bundles of free music and texts.

The key properties for our purposes are that a *bundle* of services is traded and that the *choice* from options *is made sequentially.*

describe the relevant literature in much greater depth below.

We raise and answer the following questions. What constraints does incentive compatibility impose in this environment; put differently, what is the set of implementable allocations in the present context? Moreover, what are the qualitative properties of an optimal allocation from the principal's perspective? Is it natural to expect the classical downward distortions in economic activity due to asymmetric information? Can the principal profit from starting construction works on her house before all information is available?

3.1.2 Main Findings

To answer the first question, we provide a detailed analysis of the set of binding incentive and participation constraints. It is instructive to analyze the principal's design problem in two steps. In the first step, the allocation is taken as given and we search for the least cost way of implementing the given allocation. In the second step, we optimize over the allocations.

Since our sequential screening problem involves a two-dimensional allocation, the problem of implementing given allocations is rich. One needs to derive the rent-minimizing transfer payments as a function of the allocations, which is equivalent to identifying the set of binding constraints. If the allocation is one-dimensional, then the sequential screening problem is regular and easily solved. However, if the allocation is multi-dimensional, then, as is well known, the set of binding constraints at the optimum changes with the allocation.

Since the problem involves a sequential information structure, the agent's information rent depends on his optimal deviations off the equilibrium path. One needs to understand systematically which deviations are most tempting for the agent as a function of the allocation that the principal wishes to implement. The revelation principle for multi-stage games (Myerson 1986) keeps silent about the agent's optimal deviations off the equilibrium path.

However, the timing of the agent's learning process can simplify the buyer's problem if we put structure on the information structure; that is, we can derive a relaxed problem by assuming that the agent's cost parameters are positively correlated, which is a common assumption in the sequential screening literature. We identify for any incentive compatible allocation two constraints that must be binding in any optimal contract. In particular, the agent with a high cost of constructing the exterior parts of the house is indifferent between participating and not participating; the agent with the low cost of exterior construction is indifferent between reporting this parameter truthfully or not and moreover obtains a rent. The level of this rent depends on what the agent with an initial low cost realization would report in the second round of communication, had he falsely reported his cost of constructing the exterior parts as high.

Then substituting for the expected transfer payments, we consider a relaxed problem where we assume that the incentive compatibility constraint of the inefficient ex-ante type holds and show at the end that the solution to the relaxed problem solves the original problem. It is straightforward to derive the rent-minimizing transfer payments and optimal deviations off the equilibrium path. That is, we characterize the set of binding incentive compatibility constraints for the relaxed problem and the agent's optimal deviations off the equilibrium path as a function of the allocation.

Substituting for the ex-post transfer payments and the agent's information rents, we can optimize the relaxed problem with respect to the allocation. The buyer's optimal allocation depends on his preferences over the allocations and the agent's information rent as a function of the allocation.

Our main qualitative finding is that the optimal mechanism can induce overproduction if the two items of the object are either weak substitutes or strong complementarities. To the best of our knowledge, this is a new reason for overproduction; that is, the optimal allocation can feature overproduction if, from the buyer's perspective, the two different components of an object in a sequential screening problem are either substitutes or strong complements.

To understand the economics behind our main qualitative result, one needs to understand the trade-offs that are introduced by the interaction between the different dimensions of the object. The interactions determine the buyer's preferences over the agent's cost types.

For instance, consider the case when the two dimensions of the object are strong complements. In this case, the buyer has strong preferences for a bundle of homogenous goods/an object with components of homogenous quality; that is, the marginal value of increasing the quality of one component is increasing in the quality of the other component. Therefore the buyer would like to design a mechanism that features homogenous allocations, even for heterogenous costs of producing the two goods. In a perfect world he would do so. However, the buyer must pay the seller's efficient ex-ante type some information rent to induce incentive compatible reports. To minimize the information rent of the seller, the buyer can distort the allocations of the seller's inefficient ex-ante type for a given transfer payment. Then he chooses the optimal distortions by choosing the rent-minimizing pattern of binding constraints given the agent's optimal deviation off the equilibrium path. This choice is influenced by the positive correlation of the cost parameters. Consider the distortions of the second good for which the agent learns his production costs later. Relaxing one ex-post type's incentive constraint by downward distorting his allocation means tightening another ex-post type's incentive compatibility constraint. The tightened incentive constraint can be slightly relaxed by slightly upward distorting the other type's allocation. The buyer can save more information rent by downward distorting the second allocation of the most inefficient ex-post type for two reasons: the quantity of the inefficient ex-ante type is downward distorted, and the ex-ante and the ex-post type are positively correlated.

Our main result also depends on the agent's incentives to report his ex-post type truthfully, once he lied about his ex-ante type. Depending on how sensitive the allocation variables respond to information that arrives late, the agent's best deviation features truthtelling or lying after a false report in the first round of communication.[3] The solution depends crucially on the nature and strength of interactions between the items in the bundle in the principal's

[3] It is important to point out that an appropriate version of the revelation principle (Myerson (1986)) applies in our environment. However, the principle implies only that the agent finds it optimal to announce both parameters truthfully, which implies in particular, that he will report the second parameter truthfully once he has been truthful about the first parameter. The dynamic literature has termed this behavior truthfulness on equilibrium path. The revelation principle has no implications whatsoever on what the agent does off equilibrium path, that is after a first period lie. Clearly, the agent has by definition an incentive to remain on equilibrium path. However, the utility loss to the principal to ensure such remaining on path depends on the agent's best alternative to truthtelling. Therefore, the off-path behavior becomes a crucial ingredient to the analysis.

payoff function. Consider the case of strong complements. For strong complements, the optimal allocation triggers a second period lie after a first period lie, because the buyer's preferences are shaped by the agent's ex-ante type. Then the buyer's willingness to pay for the object with homogenous dimensions is higher than for an object with rather different dimensions. For instance, we find that, due to the positive correlation of the two cost parameters, it is cheaper to induce the seller to report that the ex-post type is efficient if he reported his ex-ante type to be efficient, irrespective of his ex-post type.

Which of these cases are economically most relevant? What are reasonable assumptions on the strength of interactions between the items in the bundle the principal consumes? We obtain guidance from the comparative statics properties of the first-best allocation if we are willing to impose that changes in marginal costs of producing one item have more of an impact on the level of that item rather than the other one. If moreover the support of second period information is at least as wide as the support of first period information, then only the case of mild complements and substitutes is relevant. Even for weak substitutes upward distortions are possible. For applications in which these assumptions make sense, we provide a strikingly simple cook-book recipe: optimal contracts can be found by imposing truthtelling constraints *on and off* equilibrium path and the procedure picks up the optimum *even if the truthtelling constraints off path are binding.* In the case of weak complements, the off-path incentive compatibility constraints are slack, while these constraints are binding in the case of weak substitutes.[4]

Last, although waiting enables the agent to deviate in more complicated ways, there is a strictly positive option value of waiting if the exterior and the interior of the house are either strict complements or substitutes for the landowner; that is, the added flexibility is always valuable to the landowner. If, from the buyer's point of view, there is some interdependency between the goods, then the efficient quality level of one item depends on the quality level of the other item, and thereby the efficient quality level of one item depends on the production

[4]This latter result strikes us as pretty surprising, because - as cannot be stressed enough - it has nothing to do with the revelation principle but rather emerges from the solution of the overall maximization problem.

costs of both items. As a consequence, beginning construction works before the constructor has all information about the costs of the interior of the building comes at a loss to the landowner, unless the landowner values each item in the bundle independently of the other item.

3.1.3 Related Literature

Our analysis builds on two branches of the literature: multi-dimensional screening on the one hand and sequential screening on the other hand.

The closest related paper on multi-dimensional screening is Armstrong and Rochet 1999. Armstrong and Rochet analyze a tractable model of two-dimensional screening (for further static multi-dimensional screening problems with two-dimensional information and two-dimensional allocation, see Dana 1993 and Severinov 2008). Armstrong and Rochet assume that the agent knows all his information from the outset, the information is two-dimensional, the allocation problem is two-dimensional, the agent's type is two-dimensional and the mechanism is static. Our paper differs in several aspects from Armstrong and Rochet 1999. First, we assume that, at the beginning of the game, the agent learns one dimension of his type, and, later on, he obtains the other dimension. Second, we consider a sequential screening mechanism; that is, a mechanism with two stages. Third, we allow for substitutability and complementarity between the goods but restrict attention to positive correlation between the informational parameters, while Armstrong and Rochet consider neutral goods but allow for arbitrary correlations. We clearly do not do justice to the multi-dimensional literature in this short account. For an in depth survey, see Rochet and Stole 2003; for general approaches, see Armstrong 1996 and Rochet and Choné 1998.

The seminal paper of the literature on sequential screening is Courty and Li 2000. Sequential screening problems are characterized by a sequential information structure; that is, as in our model, the agent is initially endowed with his privately known, so-called ex-ante type, which is a vague idea about his preferences, and later on refines this idea to his ex-post type, which is his posterior estimate of his preferences. The main difference to Courty and

Li 2000 is that, in their model, the allocation is one-dimensional, while we assume that both, the allocation is two-dimensional and that the ex-ante type determines the cost of producing one allocation and also contains some information about the costs od producing the other item.

To the best of our knowledge, our paper is the first one to consider a setting where the agent's ex-post type is two-dimensional and the allocation is two-dimensional. Our assumptions captures a different set of problems and allows to analyze screening problems where the principal and the agent trade an object that features substitution effects or complementarity effects between its two dimensions.

In the context of sequential screening problems, dynamic lying strategies determine the agent's information rent; where an agent lies a second time, if he lies a first time, arise naturally if the object's two dimensions are strong substitutes or strong complements. Sequentially optimal lies are also analyzed in Eső and Szentes 2007a,b, 2013. The main difference is that they assume that the ex post type is drawn from the full support and one-dimensional. In this setting, an agent who misreported early information will always be able to undo his lie by reporting his ex-post type truthfully (see also Li and Shi 2013). The main difference to our setting is that the ex post type and the allocation are two-dimensional. Therefore the ex-post type does not satisfy the full support assumption. Hence, optimal allocations reflect different trade-offs.

A paper where the optimal mechanism induces a second lie after a first lie is Krähmer and Strausz 2008. The main difference between these two papers is that Krähmer and Strausz analyze a setting where the support of the ex-post type depends on the realization of the ex-ante type. The optimal mechanism induces the agent to lie off the equilibrium path if his ex-ante type is efficient and is sufficiently more important than the ex-post type. In Krähmer and Strausz 2008, the relative importance of ex-ante and ex-post types is an inherent feature of the distribution of these types. The main difference to Krähmer and Strausz 2008 is that we consider a two-dimensional allocation problem with interactions between the two dimensions. The economic intuition why a second lie occurs is very different. If the interactions between the object's two dimensions is weak, then the optimal mechanism does not induce a second lie. However, if the interactions are sufficiently strong, then the second dimension will be designed very closely to the first dimension, irrespective of the realization of the costs of producing the second dimension.

A question related to our timing application is addressed in Krähmer and Strausz 2012, where it is shown that ex post participation constraints eliminate the value of sequential screening in that there is bunching with respect to early information. In that sense, the principal could simply wait for definite information to arrive and not screen until then. Note that this is different in our context where early information is directly payoff relevant; not screening early would expose the principal to a static multi-dimensional screening problem later on; hence, this is suboptimal in our model.

For more recent analyses of sequential screening models, see also Boleslavsky and Said 2012, Krähmer and Strausz 2012, 2013 and Li and Shi 2013; for a combined model of moral hazard and adverse selection, see Garrett and Pavan 2013. Bhaskar 2013 analyzes dynamic deviation strategies in the pure moral hazard model.

Closely related to sequential screening are the papers on dynamic mechanism design.[5] Baron and Besanko 1984 and Battaglini 2005 provide the first general analysis of optimal contracts in this dynamic framework. Battaglini 2005 studies monopolistic selling to customers whose tastes follow a Markov process. Pavan, Segal and Toikka 2014 provide a general model of dynamic mechanism design. In each period, new information arrives and the designer chooses a set of allocation variables as a function of current information and past reports. In each period, the agent's private information is captured by a one-dimensional parameter. This is the key difference to our problem, where there are two payoff relevant parameters that simultaneously affect the agent's payoff. Under this assumption, we obtain a natural taxonomy of cases featuring binding constraints with respect to one-shot deviations or double, dynamic deviations, respectively. The latter case, by definition, fails to satisfy the version of the one-stage-deviation principle by Pavan, Segal and Toikka 2014, which

[5] See Pavan et al. (2014) for a much more extensive survey of the literature on dynamic mechanism design.

applies precisely when the best deviation for the agent is to lie once and then to return to truthful reporting strategies forever after. So, ultimately the qualitative differences of our contracting solutions as compared to those in Pavan, Segal and Toikka 2014 are due to the one-stage-deviation principle applying or failing, respectively. In turn, multi-dimensionality provides a natural reason for the failure of the one-stage-deviation principle.

Complementary to this chapter is contemporaneous work by Battaglini and Lamba 2013 who argue that there are important interactions between the regularity conditions imposed on the screening problem and the length of the time horizon. In the dynamic screening problem separation may not be feasible even though it would be feasible in the static counterpart of the model. In particular, Battaglini and Lamba 2013 provide natural examples where locally optimal contracts fail to satisfy global incentive constraints. Unlike in our model it is the within period incentive constraints that become binding beyond the local ones; in our model, within period incentive compatibility is standard, but the dynamic incentive constraints become binding beyond the local ones. Similar to the present approach, their analysis allows them to explain allocations that could not be rationalized using local constraints only, in particular, dynamic pooling: initial separation followed by pooling in later periods.

Chapter 3 is organized as follows. In section two, we present the model and state the buyer's problem. Section three presents and solves the buyer's problem. Section four discusses the structure of optimal allocations in regular cases where the strength of complementarity/substitutability of goods in the buyer's utility function is limited. Section five gives an example that is outside this regular structure. In section six, we discuss the optimal timing of productive decisions in our model. The final section concludes. Proofs are relegated to Appendix 3.

3.2 The Model

3.2.1 Setup

A buyer contracts with a supplier to obtain two goods in quantities x and y. The buyer's utility is

$$V(x, y) - T,$$

where T is a transfer made to the seller. The seller's payoff is

$$T - \theta x - \eta y,$$

where θ and η are cost shifters.

Contracting is a sequential process. At date 1, the seller knows the realization of θ (but not of η) and the conditional distribution of η given θ, whereas the buyer only knows the joint distribution of types. The cost realizations are binary, so that $\theta \in \{\underline{\theta}, \overline{\theta}\}$ and $\eta \in \{\underline{\eta}, \overline{\eta}\}$, where $\overline{\theta} > \underline{\theta} > 0$ and $\overline{\eta} > \underline{\eta} > 0$. The joint distribution is completely characterized by $\Pr(\theta = \underline{\theta}) = \alpha$ and $\lambda(\theta) \equiv \Pr\{\eta = \underline{\eta} | \theta\}$. At date 2, η becomes known to the seller but not to the buyer. Also, goods are produced and traded in exchange for the transfer T at that date. The game and the information structure is common knowledge.[6]

We place no assumptions on $V(x, y)$ for the time being except that $V(x, y)$ is jointly concave in x and y and that the first unit of consumption is extremely valuable to the buyer, that is $\lim_{x \to 0} V_1(x, y) = \infty$ for all y and $\lim_{y \to 0} V_2(x, y) = \infty$ for all x.[7] Further assumptions will be discussed as we go along.

[6] In our model there are two choice variables, x and y, that interact with two informational variables, θ and η. The essential difference to Courty and Li (2000) is that θx enters the agent's payoff function.

[7] Throughout the paper, $V_i(x, y)$ and $V_{ij}(x, y)$ for $i, j = 1, 2$ denote partial and cross derivatives of the function V with respect to its arguments.

3.2.2 The Buyer's Problem

Invoking the appropriate revelation principle (Myerson 1986), it is without loss of generality to analyze optimal contracting in terms of direct, incentive compatible mechanisms, where the agent announces each piece of information when it arrives. Thus, the contracting game is dynamic and involves two rounds of communication. In the first round at date 1, the seller reports a value $\hat{\theta} \in \{\underline{\theta}, \overline{\theta}\}$; in the second round at date 2, the seller reports a value $\hat{\eta} \in \{\underline{\eta}, \overline{\eta}\}$. The seller is given incentives to announce these values truthfully. This implies in particular, that truthfulness about η is optimal after a truthful report about θ. To rule out all feasible deviations by the seller, we need to analyze also what the seller would announce about η off equilibrium path, that is, had he falsely reported θ in the first round of communication.[8] Since the optimal behavior of the agent in the second round depends on the first round report, $\hat{\theta}$, the first round true type, θ, and the second round true type, η, we need to distinguish between the incremental information that arrives in round two and the agent's private information. That is, in the second period, the agent privately knows which node, identified by the triple $\left(\theta, \hat{\theta}, \eta\right)$, in the game tree has been reached. We let $\hat{\eta}^* \left(\theta, \hat{\theta}, \eta\right)$ for $\hat{\theta} \neq \theta$ denote the optimal report at node $\left(\theta, \hat{\theta}, \eta\right)$ and treat these reports as choice variables (subject to incentive compatibility constraints) of the principal.

It is easy to show that the optimal mechanism is nonstochastic. This is because the principal is risk averse (with respect to lotteries over x and/or y) while the agent only cares about the expected values of such lotteries. Even though the equilibrium concept is a bit different, the proof essentially follows from Myerson 1986.

We can now state the buyer's problem:

$$\max_{x(\cdot,\cdot),y(\cdot,\cdot),T(\cdot,\cdot),\hat{\eta}^*(\cdot,\cdot,\cdot)} \mathbb{E}_\theta \mathbb{E}_{\eta|\theta} \left[V\left(x\left(\theta,\eta\right), y\left(\theta,\eta\right)\right) - T\left(\theta,\eta\right)\right] \tag{3.1}$$

[8]It is important to notice that the revelation principle does not have any implications on reporting off equilibrium path, except for the fact that the agent chooses the optimal report to send as part of his strategy. So, to assess the value of a deviation in the first round of communication, we need to consider the possibility that the optimal thing to do in the second round after a first round lie is to lie again.

s.t.

$$T\left(\overline{\theta},\underline{\eta}\right)-\overline{\theta}x\left(\overline{\theta},\underline{\eta}\right)-\underline{\eta}y\left(\overline{\theta},\underline{\eta}\right)\geq T\left(\overline{\theta},\overline{\eta}\right)-\overline{\theta}x\left(\overline{\theta},\overline{\eta}\right)-\underline{\eta}y\left(\overline{\theta},\overline{\eta}\right), \tag{3.2}$$

$$T\left(\overline{\theta},\overline{\eta}\right)-\overline{\theta}x\left(\overline{\theta},\overline{\eta}\right)-\overline{\eta}y\left(\overline{\theta},\overline{\eta}\right)\geq T\left(\overline{\theta},\underline{\eta}\right)-\overline{\theta}x\left(\overline{\theta},\underline{\eta}\right)-\overline{\eta}y\left(\overline{\theta},\underline{\eta}\right) \tag{3.3}$$

$$T\left(\underline{\theta},\underline{\eta}\right)-\underline{\theta}x\left(\underline{\theta},\underline{\eta}\right)-\underline{\eta}y\left(\underline{\theta},\underline{\eta}\right)\geq T\left(\underline{\theta},\overline{\eta}\right)-\underline{\theta}x\left(\underline{\theta},\overline{\eta}\right)-\underline{\eta}y\left(\underline{\theta},\overline{\eta}\right) \tag{3.4}$$

$$T\left(\underline{\theta},\overline{\eta}\right)-\underline{\theta}x\left(\underline{\theta},\overline{\eta}\right)-\overline{\eta}y\left(\underline{\theta},\overline{\eta}\right)\geq T\left(\underline{\theta},\underline{\eta}\right)-\underline{\theta}x\left(\underline{\theta},\underline{\eta}\right)-\overline{\eta}y\left(\underline{\theta},\underline{\eta}\right) \tag{3.5}$$

$$\begin{aligned}&\mathbb{E}_{\eta|\underline{\theta}}\left[T\left(\underline{\theta},\eta\right)-\underline{\theta}x\left(\underline{\theta},\eta\right)-\eta y\left(\underline{\theta},\eta\right)\right]\\&\geq\mathbb{E}_{\eta|\underline{\theta}}\left[T\left(\overline{\theta},\hat{\eta}^{*}\left(\underline{\theta},\overline{\theta},\eta\right)\right)-\underline{\theta}x\left(\overline{\theta},\hat{\eta}^{*}\left(\underline{\theta},\overline{\theta},\eta\right)\right)-\eta y\left(\overline{\theta},\hat{\eta}^{*}\left(\underline{\theta},\overline{\theta},\eta\right)\right)\right],\end{aligned} \tag{3.6}$$

$$\begin{aligned}&\mathbb{E}_{\eta|\overline{\theta}}\left[T\left(\overline{\theta},\eta\right)-\overline{\theta}x\left(\overline{\theta},\eta\right)-\eta y\left(\overline{\theta},\eta\right)\right]\\&\geq\mathbb{E}_{\eta|\overline{\theta}}\left[T\left(\underline{\theta},\hat{\eta}^{*}\left(\overline{\theta},\underline{\theta},\eta\right)\right)-\overline{\theta}x\left(\underline{\theta},\hat{\eta}^{*}\left(\overline{\theta},\underline{\theta},\eta\right)\right)-\eta y\left(\underline{\theta},\hat{\eta}^{*}\left(\overline{\theta},\underline{\theta},\eta\right)\right)\right],\end{aligned} \tag{3.7}$$

$$\mathbb{E}_{\eta|\underline{\theta}}\left[T\left(\underline{\theta},\eta\right)-\underline{\theta}x\left(\underline{\theta},\eta\right)-\eta y\left(\underline{\theta},\eta\right)\right]\geq 0, \tag{3.8}$$

$$\mathbb{E}_{\eta|\overline{\theta}}\left[T\left(\overline{\theta},\eta\right)-\overline{\theta}x\left(\overline{\theta},\eta\right)-\eta y\left(\overline{\theta},\eta\right)\right]\geq 0, \tag{3.9}$$

and for $\theta\neq\hat{\theta}$

$$\hat{\eta}^{*}\left(\theta,\hat{\theta},\eta\right)\in\arg\max_{\hat{\eta}}T\left(\hat{\theta},\hat{\eta}\right)-\theta x\left(\hat{\theta},\hat{\eta}\right)-\eta y\left(\hat{\theta},\hat{\eta}\right) \tag{3.10}$$

for all $\theta,\hat{\theta}\in\left\{\underline{\theta},\overline{\theta}\right\}$ and $\eta\in\left\{\underline{\eta},\overline{\eta}\right\}$.

Constraints (3.2) through (3.5) are the second period constraints after a truthful report in the first period: after such a truthful report in period one, the seller must find it optimal to be truthful about η as well. We term these constraints "on-path constraints" for the obvious reason. (3.6) and (3.7) are the first period incentive constraints. As of date one, the seller anticipates that after having misreported θ in the first period, he chooses the second period report optimally, as captured by (3.10). Since nodes $\left(\theta,\hat{\theta},\eta\right)$ for $\theta\neq\hat{\theta}$ are off the equilibrium path (on equilibrium path, the first report is truthful), we term constraints ensuring any particular behavior at such off-path nodes "off-path incentive constraints".

(3.8) and (3.9) are the participation constraints.

Before diving into the quite intricate analysis, it is useful to take a bird's eye view of the problem. At the first time of contracting, there are only two possible types - because the seller only knows θ but not yet η. Moreover, the seller decides whether or not he wishes to participate at that date. He anticipates optimal behavior at date 2, so each report gives rise to a continuation value. Due to this structure, our model has much in common with the simple (static) binary model of screening, so much of the logic of that model will carry over. The essential complication relative to the static counterpart is that the continuation values are endogenous and there is no simple shortcut to determine these values.

It is instructive to understand the properties of the first-best allocation.

3.2.3 The First-best

If the buyer and the seller both know θ at the outset and both learn η at date two, then both (3.8) and (3.9) are binding at the optimum and the optimal allocation satisfies

$$V_1\left(x\left(\theta,\eta\right),y\left(\theta,\eta\right)\right)=\theta \tag{3.11}$$

and

$$V_2\left(x\left(\theta,\eta\right),y\left(\theta,\eta\right)\right)=\eta \tag{3.12}$$

for $\theta\in\left\{\underline{\theta},\overline{\theta}\right\}$ and $\eta\in\left\{\underline{\eta},\overline{\eta}\right\}$.

To sharpen our intuition for "relevant" cases in the second-best, it proves useful to ask how the first-best solution depends on the cost parameters. Obviously, x and y move in the same direction in response to changes in the parameters if x and y are complements and move in opposite directions if x and y are substitutes. Beyond that, it is important to understand how strongly these choices respond to information that is learned in period 2.

Lemma 3.2.1 *If x and y are complements ($V_{12}\left(x,y\right)\geq 0$ for all x,y), then the first-best* *allocation, defined by* (3.11) *and* (3.12), *satisfies*

$$y\left(\theta,\underline{\eta}\right)-y\left(\theta,\overline{\eta}\right)\geq(\leq)\,x\left(\theta,\underline{\eta}\right)-x\left(\theta,\overline{\eta}\right)$$

for arbitrary $\overline{\eta}-\underline{\eta}>0$ *if and only if* $V_{12}(x,y)\leq(\geq)-V_{11}(x,y)$ *for all* x,y.
If x *and* y *are substitutes,* ($V_{12}(x,y)\leq 0$ *for all* x,y), *then the first-best allocation satisfies*

$$y\left(\theta,\underline{\eta}\right)-y\left(\theta,\overline{\eta}\right)\geq(\leq)-\left(x\left(\theta,\underline{\eta}\right)-x\left(\theta,\overline{\eta}\right)\right)$$

for arbitrary $\overline{\eta}-\underline{\eta}>0$ *if and only if* $V_{12}(x,y)\geq(\leq)\,V_{11}(x,y)$ *for all* x,y.

Proof. In Appendix 3. ■

If the utility function of the buyer features interactions that are not too strong, then a change in η has a stronger impact on y than on x. We believe this is the natural case, but other cases are possible.[9]

We now address the buyer's problem under asymmetric information.

3.3 Analysis

We assume that the low cost producer in the first period is better to the buyer than the high cost producer in the sense of a weakly positive correlation

Assumption 1: θ and η are weakly positively correlated, that is $\lambda\left(\underline{\theta}\right)\geq\lambda\left(\overline{\theta}\right)$.

First-order stochastic dominance is a regularity condition that is commonly used in the sequential screening literature. Assumption 1 implies the following Lemma:

Lemma 3.3.1 *If* $\lambda\left(\underline{\theta}\right)\geq\lambda\left(\overline{\theta}\right)$, *then* (3.8) *is automatically satisfied if* (3.9) *is.*

[9] In the context of strategic interactions, the natural case would generate stability of a system of best replies; see, e.g., Tirole (1988) for a discussion. Note moreover that joint concavity with respect to x and y requires that $V_{11}V_{22}-V_{12}^2\geq 0$, so a concave function cannot be "irregular" both with respect to changes in η and θ.

Proof. In Appendix 3. ■

The argument is essentially the same as in a static two-type model. We can use the first period incentive constraint (3.6) to show that an allocation that satisfies (3.9) automatically also satisfies (3.8).

Clearly, at least one participation constraint must be binding; otherwise all payments could be lowered and the buyer's payoff could be increased. From Lemma 3.3.1 we can deduce that constraint (3.9) is binding at the optimum. Likewise, at least one of the first period incentive constraints must be binding. Otherwise we could again reduce some payments in a way that keeps incentive compatibility satisfied and increases the buyer's expected payoff. It is easy to see that the critical constraint is (3.6). Which other constraints bind is a relatively complex matter. The reason is that the implications of optimal off-path reporting are quite intricate. We begin with a discussion of the implications of the on-equilibrium path incentive constraints.

Lemma 3.3.2 *$\hat{\eta}^*\left(\underline{\theta},\overline{\theta},\underline{\eta}\right)=\underline{\eta}$ for $x\left(\overline{\theta},\underline{\eta}\right)\geq x\left(\overline{\theta},\overline{\eta}\right)$ and $\hat{\eta}^*\left(\overline{\theta},\underline{\theta},\overline{\eta}\right)=\overline{\eta}$ for $x\left(\underline{\theta},\underline{\eta}\right)\geq x\left(\underline{\theta},\overline{\eta}\right)$. Likewise, $\hat{\eta}^*\left(\underline{\theta},\overline{\theta},\overline{\eta}\right)=\overline{\eta}$ for $x\left(\overline{\theta},\underline{\eta}\right)\leq x\left(\overline{\theta},\overline{\eta}\right)$ and $\hat{\eta}^*\left(\overline{\theta},\underline{\theta},\underline{\eta}\right)=\underline{\eta}$ for $x\left(\underline{\theta},\underline{\eta}\right)\leq x\left(\underline{\theta},\overline{\eta}\right)$.*

Proof. In Appendix 3. ■

The on-path constraints have some, however limited, implications for the optimal reports off path. In particular, it is *never* the case that the agent finds it optimal to lie at *all* off-path nodes in the second period. Depending on the monotonicity properties of the $x-$allocation, there are always some nodes at which truthtelling about the second period incremental information is automatically - by implication of the on-path constraints - induced. The intuition is quite simple. E.g., the on-path constraints of type $\left(\overline{\theta},\underline{\eta}\right)$ make reporting $\hat{\eta}=\underline{\eta}$ optimal for that type. At node $\left(\underline{\theta},\overline{\theta},\underline{\eta}\right)$, the agent has an even stronger incentive to report $\hat{\eta}=\underline{\eta}$ if $x\left(\overline{\theta},\underline{\eta}\right)\geq x\left(\overline{\theta},\overline{\eta}\right)$, because that boosts his extra rent from having exaggerated θ in round one.

The difficulty at this stage is of course that the monotonicity of the $x-$allocation with respect to η is not known and endogenous. Our solution strategy is as follows. Building on the insights from static models of screening, we aim for a reduced problem, where constraints (3.9) and (3.6) hold as equalities, while (3.7) (in addition to (3.8)) is slack. We solve this reduced problem and provide sufficient conditions such that its solution satisfies the neglected constraint (3.7) . In turn, the reduced problem is tackled in a two step procedure, where we determine at step one the cheapest way to implement a given allocation and then determine the optimal allocation in step two. In the first step problem we simultaneously optimize over payments and off-path reports.

3.3.1 The Reduced Problem

If the agent with first period cost $\overline{\theta}$ is indifferent between participating and not, the agent with first period type $\underline{\theta}$ is indifferent between being truthful and lying about θ, and the remaining first period constraints are slack, then the principal faces the standard trade-off between the efficiency of the allocation and the rent that needs to be given to ex ante type $\underline{\theta}$. Denote this rent as Δ. It is useful to split the principal's problem into two steps. In the first step, we take the allocation as given and determine optimal payments that implement the allocation. Implementability of the allocation includes that the incentive constraint of the ex ante type $\overline{\theta}$ needs to be satisfied as well. Formally, letting Ω denote the expected profit ex ante type $\overline{\theta}$ can make by mimicking type $\underline{\theta}$, we require that $\Omega \leq 0$. Once the optimal payments are known, we maximize with respect to the allocation that the principal wishes to implement.

Identifying the minimal payments, while straightforward in the static model, is pretty involved in the present context. The reason is that implementation is much more flexible in the multi-dimensional context and so the pattern of binding constraints is not obvious. The reader who is not interested in the details of this step can skip subsection 3.3.1, consult Lemma 3.3.3 for the solution to the problem, and continue reading from subsection 3.3.1 onwards on a first go. However, to ultimately understand the structure of optimal allocations in sections 3.4 and 3.5 below, the reader needs to go back to subsection 3.3.1 to relate the pattern of binding constraints to the underlying model primitives.

Implementing Given Allocations at Lowest Cost

For a given allocation (x, y)[10], payments to types $\left(\overline{\theta}, \eta\right)$ and optimal off-path reporting at nodes $\left(\underline{\theta}, \overline{\theta}, \eta\right)$ for $\eta \in \left\{\underline{\eta}, \overline{\eta}\right\}$ solve the following problem:

$$\Delta \equiv \min_{\left\{T\left(\overline{\theta},\eta\right), \hat{\eta}^*\left(\underline{\theta},\overline{\theta},\eta\right)\right\}_{\eta\in\left\{\underline{\eta},\overline{\eta}\right\}}} \mathbb{E}_{\eta|\underline{\theta}}\left[T\left(\overline{\theta}, \hat{\eta}^*\left(\underline{\theta}, \overline{\theta}, \eta\right)\right) - \underline{\theta} x\left(\overline{\theta}, \hat{\eta}^*\left(\underline{\theta}, \overline{\theta}, \eta\right)\right) - \eta y\left(\overline{\theta}, \hat{\eta}^*\left(\underline{\theta}, \overline{\theta}, \eta\right)\right)\right] \tag{3.13}$$

$$s.t.$$

$$\hat{\eta}^*\left(\underline{\theta}, \overline{\theta}, \eta\right) \in \arg\max_{\hat{\eta}} T\left(\overline{\theta}, \hat{\eta}\right) - \underline{\theta} x\left(\overline{\theta}, \hat{\eta}\right) - \eta y\left(\overline{\theta}, \hat{\eta}\right) \text{ for } \eta \in \left\{\underline{\eta}, \overline{\eta}\right\}$$

$$\mathbb{E}_{\eta|\overline{\theta}}\left[T\left(\overline{\theta}, \eta\right) - \overline{\theta} x\left(\overline{\theta}, \eta\right) - \eta y\left(\overline{\theta}, \eta\right)\right] = 0,$$

$$(3.2)\,, \text{ and } (3.3)\,.$$

The buyer minimizes the rent that needs to be given to the seller with ex ante type $\underline{\theta}$, taking into account that the optimal reporting strategy of this type in period two can be to misreport his parameter η when he has misreported his parameter θ in the first period. However, if the buyer wishes to implement such a sequential lying strategy - because expected payments can be reduced this way - then he needs to explicitly make sure that the strategy is optimal from the seller's perspective as well.

Once the solution to the first program is found, we can choose payments to types $(\underline{\theta}, \eta)$ and the optimal reporting at nodes $\left(\overline{\theta}, \underline{\theta}, \eta\right)$ for $\eta \in \left\{\underline{\eta}, \overline{\eta}\right\}$ to render constraint (3.7) as slack as can be. Formally, given the payments and reports $\left\{T\left(\overline{\theta}, \eta\right), \hat{\eta}^*\left(\underline{\theta}, \overline{\theta}, \eta\right)\right\}_{\eta\in\left\{\underline{\eta},\overline{\eta}\right\}}$ that solve

[10]Throughout the paper we denote by (x, y) the allocation for all types $(\theta, \eta) \in \left\{\underline{\theta}, \overline{\theta}\right\} \times \left\{\underline{\eta}, \overline{\eta}\right\}$.

program (3.13), payments and reports $\left\{T\left(\underline{\theta},\eta\right),\hat{\eta}^*\left(\overline{\theta},\underline{\theta},\eta\right)\right\}_{\eta\in\left\{\underline{\eta},\overline{\eta}\right\}}$ solve the problem:

$$\Omega\equiv\min_{\left\{T(\underline{\theta},\eta),\hat{\eta}^*\left(\overline{\theta},\underline{\theta},\eta\right)\right\}_{\eta\in\left\{\underline{\eta},\overline{\eta}\right\}}}\mathbb{E}_{\eta|\overline{\theta}}\left[T\left(\underline{\theta},\hat{\eta}^*\left(\overline{\theta},\underline{\theta},\eta\right)\right)-\overline{\theta}x\left(\underline{\theta},\hat{\eta}^*\left(\overline{\theta},\underline{\theta},\eta\right)\right)-\eta y\left(\underline{\theta},\hat{\eta}^*\left(\overline{\theta},\underline{\theta},\eta\right)\right)\right] \tag{3.14}$$

$$s.t.$$

$$\hat{\eta}^*\left(\overline{\theta},\underline{\theta},\eta\right)\in\arg\max_{\hat{\eta}}T\left(\underline{\theta},\hat{\eta}\right)-\overline{\theta}x\left(\underline{\theta},\hat{\eta}\right)-\eta y\left(\underline{\theta},\hat{\eta}\right)\ \text{ for }\eta\in\left\{\underline{\eta},\overline{\eta}\right\}$$

$$\mathbb{E}_{\eta|\underline{\theta}}\left[T\left(\underline{\theta},\eta\right)-\underline{\theta}x\left(\underline{\theta},\eta\right)-\eta y\left(\underline{\theta},\eta\right)\right]=\Delta,$$

$$(3.4)\text{, and }(3.5)\text{,}$$

Notice that we solve problem (3.14) only after having solved problem (3.13). This procedure reflects our solution strategy that is based on reduced problems where constraint (3.7) is slack. As long as (3.7) is slack - formally, as long as $\Omega\leq 0$ - only the solution of problem (3.13) is directly payoff relevant. For this reason, we focus primarily on program (3.13) in the main text and relegate the solution to program (3.14) entirely to Appendix 3. We come back to these results only when we verify that the neglected constraint, (3.7), is indeed satisfied.

The solution to the programs depends on the allocation that the buyer wishes to implement. In particular, define $\Delta x\left(\theta\right)\equiv x\left(\theta,\underline{\eta}\right)-x\left(\theta,\overline{\eta}\right)$, $\Delta y\left(\theta\right)\equiv y\left(\theta,\underline{\eta}\right)-y\left(\theta,\overline{\eta}\right)$, and the following sets

$$\begin{aligned}
\mathbb{X}_i\left(\theta\right)&\equiv\left\{\left\{\left(x\left(\theta,\eta\right),y\left(\theta,\eta\right)\right)\right\}_{\eta\in\left\{\underline{\eta},\overline{\eta}\right\}}\mid\left(\overline{\eta}-\underline{\eta}\right)\Delta y\left(\theta\right)\geq\left(\overline{\theta}-\underline{\theta}\right)\Delta x\left(\theta\right)\geq 0\right\};\\
\mathbb{X}_{ii}\left(\theta\right)&\equiv\left\{\left\{\left(x\left(\theta,\eta\right),y\left(\theta,\eta\right)\right)\right\}_{\eta\in\left\{\underline{\eta},\overline{\eta}\right\}}\mid\left(\overline{\eta}-\underline{\eta}\right)\Delta y\left(\theta\right)\geq-\left(\overline{\theta}-\underline{\theta}\right)\Delta x\left(\theta\right)\geq 0\right\};\\
\mathbb{X}_{iii}\left(\theta\right)&\equiv\left\{\left\{\left(x\left(\theta,\eta\right),y\left(\theta,\eta\right)\right)\right\}_{\eta\in\left\{\underline{\eta},\overline{\eta}\right\}}\mid\left(\overline{\theta}-\underline{\theta}\right)\Delta x\left(\theta\right)\geq\left(\overline{\eta}-\underline{\eta}\right)\Delta y\left(\theta\right)\geq 0\right\};\\
\mathbb{X}_{iv}\left(\theta\right)&\equiv\left\{\left\{\left(x\left(\theta,\eta\right),y\left(\theta,\eta\right)\right)\right\}_{\eta\in\left\{\underline{\eta},\overline{\eta}\right\}}\mid-\left(\overline{\theta}-\underline{\theta}\right)\Delta x\left(\theta\right)\geq\left(\overline{\eta}-\underline{\eta}\right)\Delta y\left(\theta\right)\geq 0\right\}.
\end{aligned}$$

For future reference, also define $\mathbb{X}_j^{int}\left(\theta\right)$ for $j=i,\ldots,iv$ as these same sets when all the defining inequalities are strict, $\mathbb{X}_j\equiv\mathbb{X}_j\left(\underline{\theta}\right)\cup\mathbb{X}_j\left(\overline{\theta}\right)$ for $j=i,\ldots,iv$, and finally $\mathbb{X}\left(\theta\right)\equiv\cup_{j=i}^{iv}\mathbb{X}_j\left(\theta\right)$. These sets are depicted in the following graph:

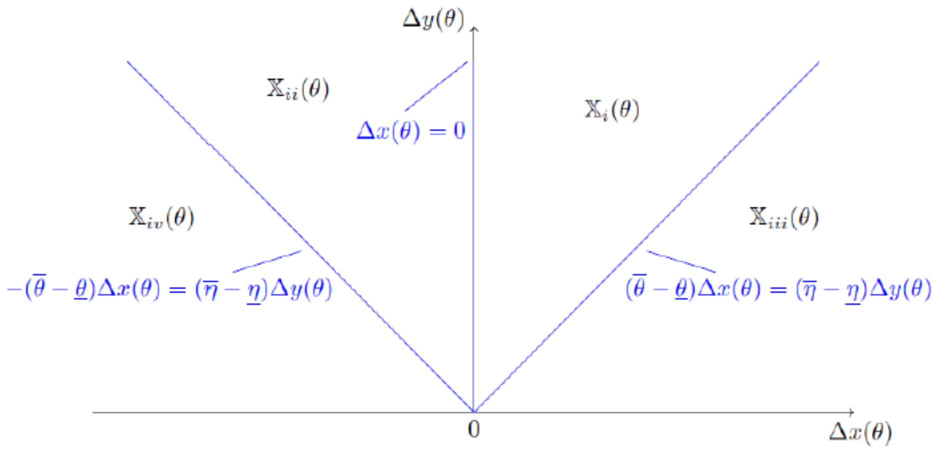


Figure 3.1. The space of implementable allocations is divided into four regions, i through iv, for each type $\theta \in \{\underline{\theta}, \overline{\theta}\}$. The cost minimizing payments that implement allocations within each regime depend on the regime itself.

Only y-allocations that are monotonic in η are incentive compatible. Hence, we only need to consider such allocations. From Lemma 3.3.2 we know that depending on the monotonicity of the x-allocation, truthful reporting is automatic at some nodes off path. Whether it is optimal to induce truthful reporting at the remaining nodes off path depends on which of the sets $\mathbb{X}_j(\theta)$ for $\theta \in \{\underline{\theta}, \overline{\theta}\}$ and $j = i, \ldots, iv$ contain the allocation x, y that is implemented. Note that the sets $\mathbb{X}_j(\theta)$ for $j = i, \ldots, iv$ are defined for both values of θ. The solution to program (3.13) depends on which set $\mathbb{X}_j(\overline{\theta})$ contains the allocation offered to ex ante type $\overline{\theta}$; the solution to (3.14) depends on $\mathbb{X}_j(\underline{\theta})$. Very conveniently, the dividing lines between the sets have isomorphic representations. The complete set of implementable allocations is thus given by $\mathbb{X}_j(\underline{\theta}) \times \mathbb{X}_k(\overline{\theta})$ for $j, k = i, ii, iii, iv$, leaving us with 16 possibilities. However, it turns out that under very natural conditions, the solution of the overall problem has the property that $j = k$, so there are only 4 cases *economically* relevant in our model. Therefore, to economize on space, we just present our result anticipating this result:

Lemma 3.3.3 *For* $(x, y) \in \mathbb{X}_i$

$$\Delta = \Delta_i \equiv \mathbb{E}_{\eta|\underline{\theta}} \left[\left(\overline{\theta} - \underline{\theta} \right) x \left(\overline{\theta}, \eta \right) \right] + \left(\lambda \left(\underline{\theta} \right) - \lambda \left(\overline{\theta} \right) \right) \left(\overline{\eta} - \underline{\eta} \right) y \left(\overline{\theta}, \overline{\eta} \right)$$

incentive compatibility constraints (3.2) *and* (3.5) *are binding and all types report truthfully off path;*
for $(x, y) \in \mathbb{X}_{ii}$

$$\Delta = \Delta_{ii} \equiv \mathbb{E}_{\eta|\overline{\theta}} \left[\left(\overline{\theta} - \underline{\theta} \right) x \left(\overline{\theta}, \eta \right) \right] + \left(\lambda \left(\underline{\theta} \right) - \lambda \left(\overline{\theta} \right) \right) \left(\overline{\eta} - \underline{\eta} \right) y \left(\overline{\theta}, \overline{\eta} \right)$$

all types report truthfully off path and the agent is indifferent between truthfully reporting and lying at nodes $\left(\underline{\theta}, \overline{\theta}, \underline{\eta} \right)$ *and* $\left(\overline{\theta}, \underline{\theta}, \overline{\eta} \right)$*;*
for $(x, y) \in \mathbb{X}_{iii}$

$$\Delta = \Delta_{iii} \equiv \left(\overline{\theta} - \underline{\theta} \right) x \left(\overline{\theta}, \underline{\eta} \right) + \left(1 - \lambda \left(\overline{\theta} \right) \right) \left(\overline{\eta} - \underline{\eta} \right) y \left(\overline{\theta}, \overline{\eta} \right) - \left(1 - \lambda \left(\underline{\theta} \right) \right) \left(\overline{\eta} - \underline{\eta} \right) y \left(\overline{\theta}, \underline{\eta} \right)$$

incentive compatibility constraints (3.2) *and* (3.5) *are binding and* $\hat{\eta}^* \left(\underline{\theta}, \overline{\theta}, \overline{\eta} \right) = \hat{\eta}^* \left(\underline{\theta}, \overline{\theta}, \underline{\eta} \right) = \underline{\eta}$ *and* $\hat{\eta}^* \left(\overline{\theta}, \underline{\theta}, \underline{\eta} \right) = \hat{\eta}^* \left(\overline{\theta}, \underline{\theta}, \overline{\eta} \right) = \overline{\eta}$*;*
for $(x, y) \in \mathbb{X}_{iv}$

$$\Delta = \Delta_{iv} \equiv \left(\overline{\theta} - \underline{\theta} \right) x \left(\overline{\theta}, \overline{\eta} \right) + \lambda \left(\underline{\theta} \right) \left(\overline{\eta} - \underline{\eta} \right) y \left(\overline{\theta}, \overline{\eta} \right) - \lambda \left(\overline{\theta} \right) \left(\overline{\eta} - \underline{\eta} \right) y \left(\overline{\theta}, \underline{\eta} \right)$$

incentive compatibility constraints (3.3) *and* (3.4) *are binding and* $\hat{\eta}^* \left(\underline{\theta}, \overline{\theta}, \underline{\eta} \right) = \hat{\eta}^* \left(\underline{\theta}, \overline{\theta}, \overline{\eta} \right) = \overline{\eta}$ *and* $\hat{\eta}^* \left(\overline{\theta}, \underline{\theta}, \overline{\eta} \right) = \hat{\eta}^* \left(\overline{\theta}, \underline{\theta}, \underline{\eta} \right) = \underline{\eta}$.

Proof. In Appendix 3. ■

Allocations in $\mathbb{X}_i$ induce truthtelling off path automatically in the sense that we can naïvely assume truthtelling off path. Solving program (3.13) under this hypothesis, we find that payments are minimized if constraint (3.2) is binding. In turn, for these payments, it is straightforward to verify that $\hat{\eta}^* \left(\underline{\theta}, \overline{\theta}, \overline{\eta} \right) = \overline{\eta}$. Taken together with Lemma 3, this implies the result. Intuitively, suppose that x is independent of η. In this case, truthtelling about η

is simply a question of monotonicity of y in η. This intuition generalizes to all allocations in $\mathbb{X}_i$ that share the property that x and y move in the same direction and y is more sensitive to changes in η than x is.

For allocations in $\mathbb{X}_{ii}$ conjecturing truthtelling naïvely would prove to be false; the seller would not report truthfully off path if we simply took such behavior as given. While the optimal report off path at nodes $\left(\underline{\theta}, \overline{\theta}, \overline{\eta}\right)$ and $\left(\overline{\theta}, \underline{\theta}, \underline{\eta}\right)$ is indeed to tell the truth, this needs to be ensured explicitly with the appropriate constraints at nodes $\left(\underline{\theta}, \overline{\theta}, \underline{\eta}\right)$ and $\left(\overline{\theta}, \underline{\theta}, \overline{\eta}\right)$. Moreover, these constraints are binding at the optimum. Finally, when the dependency of the $x-$allocation on information η becomes strong, it becomes too costly to insist on truthtelling at all nodes off path. Instead, the cheapest way to implement any given allocation in sets $\mathbb{X}_{iii}$ and $\mathbb{X}_{iv}$ induces some type to lie off path. Intuitively, take again the extreme cases within sets $\mathbb{X}_{iii}$ and $\mathbb{X}_{iv}$ and suppose y is independent of η. Clearly, there is no way to induce truthtelling about η in period 2 in this case. Instead, the seller chooses the report that maximizes his rent from being able to produce x at lower cost, so he chooses the report that maximizes $x\left(\theta, \hat{\eta}\right)$. For example, for $(x, y) \in \mathbb{X}_{iii}$ the seller always reports $\hat{\eta}\left(\underline{\theta}, \overline{\theta}, \eta\right) = \underline{\eta}$, regardless of his true η.

The functional form of the minimal rents that the agent with first period type $\underline{\theta}$ depends on which of the regimes i through iv prevails. The cases are ordered by increasing complexity. Case i (where $(x, y) \in \mathbb{X}_i$) is the standard one, where the agent announces η truthfully in period two regardless of the report about θ. The expected rent of an agent with parameter $\theta = \underline{\theta}$ consists of two parts. First, the agent has a lower cost of producing x than the agent with $\theta = \overline{\theta}$. The expected cost advantage is $\mathbb{E}_{\eta|\underline{\theta}}\left[\left(\overline{\theta} - \underline{\theta}\right) x\left(\overline{\theta}, \eta\right)\right]$, because η (which will be announced truthfully) is not yet known when θ is announced. Secondly, type $\underline{\theta}$ has a higher probability of being relatively more efficient at producing y. Moreover, in period two, an agent with a parameter $\eta = \underline{\eta}$ receives a higher utility than an agent with parameter $\overline{\eta}$, because this agent could always overstate his cost of producing y. At the optimum, the agent of type $\left(\overline{\theta}, \underline{\eta}\right)$ is exactly indifferent between reporting the truth and mimicking type $\left(\overline{\theta}, \overline{\eta}\right)$, so the difference between type $\left(\overline{\theta}, \underline{\eta}\right)' s$ and type $\left(\overline{\theta}, \overline{\eta}\right)' s$ utility is exactly $\left(\overline{\eta} - \underline{\eta}\right) y\left(\overline{\theta}, \overline{\eta}\right)$. The expected additional gain - due to having a low type θ in period one - from this rent arising from having a low rather than a high value of η is exactly equal to $\left(\lambda\left(\underline{\theta}\right)-\lambda\left(\overline{\theta}\right)\right)\left(\overline{\eta}-\underline{\eta}\right)y\left(\overline{\theta},\overline{\eta}\right)$.[11]

Case ii still features truthtelling on and off equilibrium path, but the agent must be kept indifferent between reporting honestly and exaggerating his η parameter at two nodes; in particular at node $\left(\underline{\theta},\overline{\theta},\underline{\eta}\right)$. We can again split the agent's expected rent as of the first period into the expected direct gain from misreporting θ, $\mathbb{E}_{\eta|\underline{\theta}}\left[\left(\overline{\theta}-\underline{\theta}\right)x\left(\overline{\theta},\eta\right)\right]$, and the expected additional gain arising from the combined facts that an agent with a low parameter η obtains a rent in period two and that an agent with parameter $\theta=\underline{\theta}$ forms expectations based on a more favorable distribution of η than an agent with parameter $\theta=\overline{\theta}$ does. Formally, the difference in utilities between types $\left(\overline{\theta},\underline{\eta}\right)$ and $\left(\overline{\theta},\overline{\eta}\right)$ is set so as to keep the agent who has exaggerated θ in the first period from exaggerating η in the second period and so this difference equals $\left(\overline{\theta}-\underline{\theta}\right)\left(x\left(\overline{\theta},\overline{\eta}\right)-x\left(\overline{\theta},\underline{\eta}\right)\right)+\left(\overline{\eta}-\underline{\eta}\right)y\left(\overline{\theta},\overline{\eta}\right)$. Multiplying this term by the difference in distributions, $\left(\lambda\left(\underline{\theta}\right)-\lambda\left(\overline{\theta}\right)\right)$, and simplifying, we obtain Δ_{ii}.

In cases iii and iv, the allocation and the associated cost minimizing payments induce the agent to lie off equilibrium path. To avoid repetition, we focus on case iii only. In this case, the most tempting deviation to the agent with $\theta=\underline{\theta}$ is to exaggerate θ and to underreport η at node $\left(\underline{\theta},\overline{\theta},\overline{\eta}\right)$. Put differently, a double deviation involving both parameters is strictly better to the agent with type $\underline{\theta}$ than a single deviation. As a result, the expected cost advantage due to having a low rather than a high value of θ is simply equal to $\left(\overline{\theta}-\underline{\theta}\right)x\left(\overline{\theta},\underline{\eta}\right)$, because the agent reports $\hat{\eta}\left(\underline{\theta},\overline{\theta},\eta\right)=\underline{\eta}$ for both realizations of η. Moreover, he obtains the utility level that type $\left(\overline{\theta},\underline{\eta}\right)$ obtains, $\left(1-\lambda\left(\overline{\theta}\right)\right)\left(\overline{\eta}-\underline{\eta}\right)y\left(\overline{\theta},\overline{\eta}\right)$, minus the expected loss in case he has a higher η realization in period two than the type he imitates, $\left(1-\lambda\left(\underline{\theta}\right)\right)\left(\overline{\eta}-\underline{\eta}\right)y\left(\overline{\theta},\underline{\eta}\right)$.

The stronger the complementarities, the stronger is the buyer's incentive to design a mechanism that features homogenous goods, even for heterogenous costs of producing the two goods. However, the buyer also has an incentive to minimize the information rent

[11] Pavan et al. (2013) have termed the latter expression an "impulse response function", because the term measures the impact of the agent's current information on future allocation choices.

In our problem, both x and y are determined at date two, so things are slightly different, but the intuition is similar.

and the costs that he pays. The buyer must pay the buyer with the efficient ex-ante type some information rent. Since the buyer prefers the most efficient cost types $\left(\underline{\theta},\underline{\eta}\right)$ and $(\underline{\theta},\overline{\eta})$ to produce efficiently. Therefore she can minimize the seller's information rent only by distorting the allocation of the inefficient ex-ante type $\overline{\theta}$. To minimize the rent of ex-ante type $\underline{\theta}$, the buyer can must distort the allocations of $\left(\overline{\theta},\underline{\eta}\right)$ or of $\left(\overline{\theta},\overline{\eta}\right)$, such that these types' incentive compatibility holds. Relaxing type $\left(\overline{\theta},\eta\right)$'s incentive constraint by downward distorting $y\left(\overline{\theta},\eta\right)$ means tightening the incentive constraint of $\left(\overline{\theta},\eta'\right)$, $\eta\neq\eta'$. The tightened incentive constraint of type $\left(\overline{\theta},\eta'\right)$ can be relaxed by upward distorting $y\left(\overline{\theta},\eta\right)$. Relaxing the incentive constraint of $\left(\overline{\theta},\overline{\eta}\right)$ and tightening the incentive constraint of $\left(\overline{\theta},\underline{\eta}\right)$ is cheaper, although the buyer downward distorts both $x\left(\overline{\theta},\overline{\eta}\right)$ and $x\left(\overline{\theta},\underline{\eta}\right)$. So the buyer should profit from the strong complementarities The reason here is that the positive correlation of θ and η.

The optimal payments and the value of the second minimization problem can be found in Appendix 3. The reason we do not state these things in the main text is that we do not need these results for the discussion of the reduced problem that we now solve.

Optimal Allocations in the Reduced Problem

We can now turn to the design of the optimal allocations in the reduced problem(s). Since we are neglecting constraint (3.7), we allow for any $\{(x(\underline{\theta},\eta),y(\underline{\theta},\eta))\}_{\eta\in\{\underline{\eta},\overline{\eta}\}}\in\mathbb{X}(\underline{\theta})$. Formally, the reduced problem for each constraint set is

$$W_j\equiv\max_{\substack{\left\{\left(x\left(\overline{\theta},\eta\right),y\left(\overline{\theta},\eta\right)\right)\right\}_{\eta\in\{\underline{\eta},\overline{\eta}\}}\in\mathbb{X}_j\left(\overline{\theta}\right)\\ \{(x(\underline{\theta},\eta),y(\underline{\theta},\eta))\}_{\eta\in\{\underline{\eta},\overline{\eta}\}}\in\mathbb{X}(\underline{\theta})}}\mathbb{E}_\theta\mathbb{E}_{\eta|\theta}\left[V\left(x\left(\theta,\eta\right),y\left(\theta,\eta\right)\right)-\theta x\left(\theta,\eta\right)-\eta y\left(\theta,\eta\right)\right]-\alpha\Delta_j, \tag{P$_j$}$$

where Δ_j is defined in Lemma 3.3.3. The principal faces a classical trade-off between efficiency and rent extraction, with the complication that the functional form of the rent expression depends on the qualitative features of the allocation that is being implemented, as explained in Lemma 3.3.3.

The overall optimum for the buyer is

$$W = \max \{W_i, W_{ii}, W_{iii}, W_{iv}\}.$$

The solution has the following simple structure:

Proposition 3.3.1 *Suppose that either* $V_{12}(x,y) \geq 0$ *for all* x,y *or* $V_{12}(x,y) \leq 0$ *for all* x,y. *If in addition* $V_{12}(x,y) \in \left[V_{11}(x,y)\frac{\overline{\eta}-\underline{\eta}}{\overline{\theta}-\underline{\theta}}, -V_{11}(x,y)\frac{\overline{\eta}-\underline{\eta}}{\overline{\theta}-\underline{\theta}}\right]$ *for all* x,y,*then* $W = \max\{W_i, W_{ii}\}$. *Moreover,* $W_i \geq W_{ii}$ *if* $V_{12} \geq 0$ *for all* x,y *and* $W_{ii} > W_i$ *if* $V_{12} < 0$ *for all* x,y.

Proof. In Appendix 3. ■

The intuition is straightforward and easiest to understand with the help of figure 3.2.

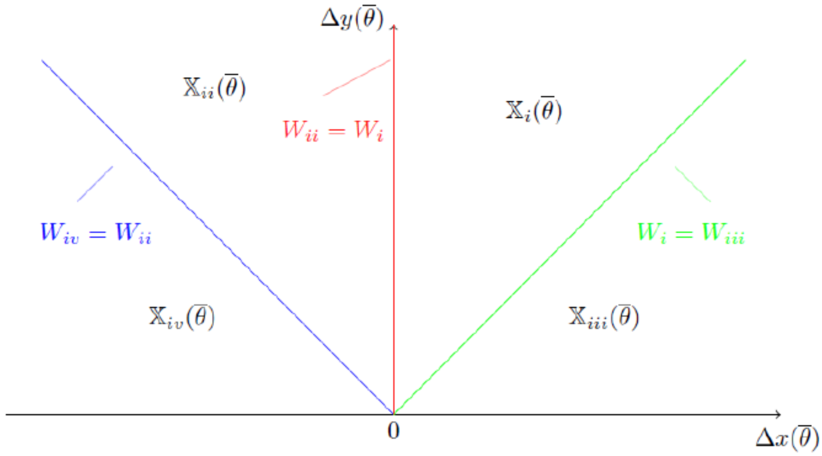


Figure 3.2. Along the dividing line between any two regimes, payoffs from adjacent programs are equal.

The idea to prove the results is as follows. The payoffs in the various regimes have a continuity structure that is displayed in the figure. For allocations that are feasible in two regions, say region i and region ii, the payoffs from programs P_i and P_{ii} are identical for a given allocation. Formally, we have $W_i = W_{ii}$ for a given allocation that satisfies $x(\overline{\theta}, \underline{\eta}) = x(\overline{\theta}, \overline{\eta})$. Moreover, none of the programs P_j is ever so constrained that an allocation in the origin of the diagram is implemented. Hence, we can use simple revealed preference arguments to prove payoff dominance in the cases described in the proposition. For $V_{12} < 0$, the solution to program P_i satisfies $x\left(\overline{\theta}, \underline{\eta}\right) = x\left(\overline{\theta}, \overline{\eta}\right)$, whereas the solution to program P_{ii} does not. Since, the allocation that maximizes program P_i is feasible also under program P_{ii}, but is not chosen, it follows by strict concavity of the problem that the value of the objective under program P_{ii} is strictly higher. Likewise, for $V_{12} \geq 0$, the solution to program P_{ii} satisfies $x\left(\overline{\theta}, \underline{\eta}\right) = x\left(\overline{\theta}, \overline{\eta}\right)$, so the same argument can be made. However, the subtle difference in this case is that the optimal allocation under program P_i might also lie on the feasibility constraint $x\left(\overline{\theta}, \underline{\eta}\right) = x\left(\overline{\theta}, \overline{\eta}\right)$. However, since programs P_i and P_{ii} are identical on the feasibility constraint $x\left(\overline{\theta}, \underline{\eta}\right) = x\left(\overline{\theta}, \overline{\eta}\right)$, we have payoff dominance in the weak sense. Essentially the same arguments can be used to compare payoffs from programs P_i and P_{iii} and between programs P_{ii} and P_{iv}.

Complements versus substitutes are enough to determine whether program P_i or program P_{ii} gives a higher payoff. The reason is as follows. We know from Lemma 3.3.3 that the rent expression Δ_j is identical in cases i and ii except for differences in $\lambda\left(\underline{\theta}\right)$ and $\lambda\left(\overline{\theta}\right)$. Thus, the question is simply which set $\mathbb{X}_j\left(\overline{\theta}\right)$ matches better with the complementarity/substitutability between the goods. If x and y are complements, then both x and y should be allowed to move in the same direction in response to changes in η. If x and y are substitutes, then x and y should be allowed to vary in opposite directions in response to changes in η. Moreover, the proposition not only compares payoffs between programs P_i and P_{ii} but between all programs P_j. Recall the conditions from Lemma 3.2.1 that make y more responsive to changes in η than x. The conditions in Proposition 3.3.1 simply adjust the earlier conditions for differences in the supports of early and late information. In particular, the conditions are *identical* if the supports of the parameters have the same width. A pattern of relatively larger variation in y than in x matches better with the sets $\mathbb{X}_i\left(\overline{\theta}\right)$ and $\mathbb{X}_{ii}\left(\overline{\theta}\right)$ than with the sets $\mathbb{X}_{iii}\left(\overline{\theta}\right)$ and $\mathbb{X}_{iv}\left(\overline{\theta}\right)$. Consequently, under the condition given in the proposition, the maximum of the reduced problem is attained either by problem P_i or P_{ii}.

The reader may verify that the sufficient condition in the proposition captures a relevant parameter restriction with the help of the following example:

Example 3.3.1 $V(x,y) = \beta^2 - \frac{1}{2}(x-\beta)^2 - \frac{1}{2}(y-\beta)^2 + \delta xy.$

In the example, the condition is satisfied for $V_{12}(x,y) = \delta \in \left[-\frac{\overline{\eta}-\underline{\eta}}{\overline{\theta}-\underline{\theta}}, \frac{\overline{\eta}-\underline{\eta}}{\overline{\theta}-\underline{\theta}}\right]$. Note that the utility function is jointly concave in x and y for $\delta \in [-1,1]$. Thus, for $\frac{\overline{\eta}-\underline{\eta}}{\overline{\theta}-\underline{\theta}} \geq 1$, the set of parameter values that *violate* the condition becomes empty. Conversely, there is always a nonempty set of parameter values that generate a concave buyer problem and satisfy the sufficient condition even if $\frac{\overline{\eta}-\underline{\eta}}{\overline{\theta}-\underline{\theta}} < 1$. In this sense - at least in this example - the sufficient condition isolates the important case rather than the pathological one. Therefore, we impose henceforth

Assumption 2: either $V_{12}(x,y) \geq 0$ for all x, y or $V_{12}(x,y) \leq 0$ for all x, y and in addition $V_{12}(x,y) \in \left[V_{11}(x,y)\frac{\overline{\eta}-\underline{\eta}}{\overline{\theta}-\underline{\theta}}, -V_{11}(x,y)\frac{\overline{\eta}-\underline{\eta}}{\overline{\theta}-\underline{\theta}}\right]$ for all x, y.

We solve the full problem under this assumption. However, there are clearly cases that violate Assumption 2. For that reason, we discuss a particular case that violates Assumption 2 in section 5 below.

3.3.2 The Solution to the Full Problem

Obviously, the reduced problem is of interest only if it solves the overall problem; that is, if the solution of the reduced problem satisfies the neglected constraint, (3.7). Checking the neglected constraint requires knowing the set $\mathbb{X}_j(\underline{\theta})$ that contains the allocation offered to the ex-ante type $\underline{\theta}$, $\{x(\underline{\theta},\eta), y(\underline{\theta},\eta)\}_{\eta\in\{\underline{\eta},\overline{\eta}\}}$. Note that this allocation corresponds simply to the first-best allocation for that type. Since intuition and proof for the following result are essentially the same as for Lemma 1, we state without further discussion:

Lemma 3.3.4 *The first-best allocation defined by* (3.11) *and* (3.12) *satisfies* $(x, y) \in$

$$\begin{array}{ll} \mathbb{X}_i & if\ 0 \leq V_{12}(x,y) \leq -\frac{\overline{\eta}-\underline{\eta}}{\overline{\theta}-\underline{\theta}} V_{11}(x,y)\ \ for\ all\ x,y; \\ \mathbb{X}_{ii} & if\ 0 \geq V_{12}(x,y) \geq \frac{\overline{\eta}-\underline{\eta}}{\overline{\theta}-\underline{\theta}} V_{11}(x,y)\ \ for\ all\ x,y; \\ \mathbb{X}_{iii} & if\ V_{12}(x,y) \geq -\frac{\overline{\eta}-\underline{\eta}}{\overline{\theta}-\underline{\theta}} V_{11}(x,y)\ \ for\ all\ x,y; \\ \mathbb{X}_{iv} & if\ V_{12}(x,y) \leq \frac{\overline{\eta}-\underline{\eta}}{\overline{\theta}-\underline{\theta}} V_{11}(x,y)\ \ for\ all\ x,y. \end{array}$$

Moreover, the first-best allocation is in the interior of these sets if the corresponding inequalities are strict.

Proof. In Appendix 3. ■

Combining Proposition 3.3.1 and Lemma 3.3.4, we observe that the solution to the reduced problem satisfies $\{x(\underline{\theta},\eta), y(\underline{\theta},\eta)\}_{\eta\in\{\underline{\eta},\overline{\eta}\}} \in \mathbb{X}_j(\underline{\theta})$ if and only if $\{x(\overline{\theta},\eta), y(\overline{\theta},\eta)\}_{\eta\in\{\underline{\eta},\overline{\eta}\}}$ $\in \mathbb{X}_j(\overline{\theta})$ for $j = i, ii$. Obviously, this was the reason to economize on space in Lemma 3.3.3 in the first place. The reduced problem picks up the overall optimum under natural conditions.

Proposition 3.3.2 *The solution to the reduced problem solves the overall problem under Assumption 2 if in addition either*

I) goods are independent ($V_{12}(x,y) = 0$ for all x, y) or

II)

$$\max_{x,y} \left| \frac{V_{12}}{V_{11}V_{22} - V_{12}^2}(x,y) \right| \leq \frac{\overline{\theta}-\underline{\theta}}{\overline{\eta}-\underline{\eta}} \min_{x,y} \left| \frac{V_{22}}{V_{11}V_{22} - V_{12}^2}(x,y) \right|, \tag{3.15}$$

and either

a) $(x,y) \in \mathbb{X}_i^{int}$ (which holds in particular for α small enough if x, y are strict complements for all x, y), or

b) $(x,y) \in \mathbb{X}_{ii}^{int}$ (which holds in particular for α small enough if x, y are strict substitutes for all x, y) and in addition $\lambda(\underline{\theta}) = \lambda(\overline{\theta})$.

Proof. In Appendix 3. ■

For independent goods, Assumption 2 is enough to guarantee that the reduced problem picks up the overall optimum. For strict complements or substitutes we need to impose additional structure. For strict complements (substitutes) and α sufficiently small, the entire allocation is an element of $\mathbb{X}_i^{int}$ ($\mathbb{X}_{ii}^{int}$). The reason is that in the limit where α tends to zero, the second best allocation converges to the first-best allocation, whose properties we have described in Lemma 3.3.4. Building on this insight, we can go back to Lemma 4 (to be precise to the proof of Lemma 4 in Appendix 3) and check the precise functional form of the neglected constraint, (3.7), and verify whether it is true that $\Omega \leq 0$. Indeed, we have $\Omega \leq 0$ for complements if condition (15) holds; we have $\Omega \leq 0$ for substitutes if condition (15) holds and on top of this the cost parameters are independent, $\lambda(\underline{\theta}) = \lambda(\overline{\theta})$.[12]

Condition (15) restricts V_{12} relative to V_{22}. The new condition is imposed because Ω depends on the allocation offered to both ex ante types, not just the one offered to one particular type. The condition is satisfied in our example for $\delta \in \left[-\frac{\overline{\theta}-\underline{\theta}}{\overline{\eta}-\underline{\eta}}, \frac{\overline{\theta}-\underline{\theta}}{\overline{\eta}-\underline{\eta}}\right]$.The set of parameters that satisfy both Assumption 2 and condition (3.15) is always nonempty. If $\frac{\overline{\theta}-\underline{\theta}}{\overline{\eta}-\underline{\eta}} = 1$, then the conditions are identical; otherwise, one set is a strict subset of the other.

3.4 The Structure of Optimal Allocations

We can now investigate how the optimal allocation depends qualitatively on the interaction between goods in the buyer's utility function. To discuss this question in the simplest possible case, we simply state the result for the case where the optimum for ex ante type $\overline{\theta}$ is an allocation in $\mathbb{X}_i^{int}(\overline{\theta})$ and $\mathbb{X}_{ii}^{int}(\overline{\theta})$. In this case, the optimal allocation for ex ante type $\overline{\theta}$ satisfies

$$V_1\left(x\left(\overline{\theta},\underline{\eta}\right), y\left(\overline{\theta},\underline{\eta}\right)\right) = \overline{\theta} + \frac{\alpha}{(1-\alpha)}\frac{\lambda_j}{\lambda\left(\overline{\theta}\right)}\left(\overline{\theta}-\underline{\theta}\right)$$
$$V_2\left(x\left(\overline{\theta},\underline{\eta}\right), y\left(\overline{\theta},\underline{\eta}\right)\right) = \underline{\eta}$$

[12] Note that the conditions in part II of the proposition are far from necessary. E.g., one can also derive sufficient conditions for the case of substitutes and strictly positive correlation.

and

$$V_1\left(x\left(\overline{\theta},\overline{\eta}\right),y\left(\overline{\theta},\overline{\eta}\right)\right)=\overline{\theta}+\frac{\alpha}{(1-\alpha)}\frac{(1-\lambda_j)}{\left(1-\lambda\left(\overline{\theta}\right)\right)}\left(\overline{\theta}-\underline{\theta}\right)$$
$$V_2\left(x\left(\overline{\theta},\overline{\eta}\right),y\left(\overline{\theta},\overline{\eta}\right)\right)=\overline{\eta}+\frac{\alpha}{(1-\alpha)}\frac{\left(\lambda\left(\underline{\theta}\right)-\lambda\left(\overline{\theta}\right)\right)}{\left(1-\lambda\left(\overline{\theta}\right)\right)}\left(\overline{\eta}-\underline{\eta}\right),$$

where $j=i,ii$ and by convention $\lambda_i=\lambda\left(\underline{\theta}\right)$ and $\lambda_{ii}=\lambda\left(\overline{\theta}\right)$. The optimal allocation for ex ante type $\underline{\theta}$ is given by (3.11) and (3.12).

For the case of complements, the optimal allocation for ex ante type $\overline{\theta}$ displays the standard downward distortions relative to the first-best. For strictly positive complementarities, all allocation variables are strictly below the first-best optimal levels. This is quite different for the case of substitutes, which displays both upward and downward distortions. In particular, $x\left(\overline{\theta},\underline{\eta}\right)$ is distorted downwards and as a result, $y\left(\overline{\theta},\underline{\eta}\right)$ is distorted upwards.

Building on the discussion following Lemma 3.3.3, the intuition for the first-order conditions is straightforward. The first-order conditions for $x\left(\overline{\theta},\eta\right)$ display the trade-off between efficiency and extraction of rents due to lower costs of producing x. $(1-\alpha)\lambda\left(\overline{\theta}\right)$ and $(1-\alpha)\left(1-\lambda\left(\overline{\theta}\right)\right)$, respectively, are the probabilities that the cost realizations equal $\left(\overline{\theta},\underline{\eta}\right)$ and $\left(\overline{\theta},\overline{\eta}\right)$, respectively. These are the weights attached to the efficiency motive. On the other hand, a change in the x allocation affects the agent's expected rent by $\lambda_j\left(\overline{\theta}-\underline{\theta}\right)$ and $(1-\lambda_j)\left(\overline{\theta}-\underline{\theta}\right)$, respectively. A change in the $y\left(\overline{\theta},\underline{\eta}\right)$ allocation does not affect the agent's rent, whereas a change in the $y\left(\overline{\theta},\overline{\eta}\right)$ allocation affects the agent's expected rent by $\left(\lambda\left(\underline{\theta}\right)-\lambda\left(\overline{\theta}\right)\right)\left(\overline{\theta}-\underline{\theta}\right)$. These effects are weighted by α, the probability that $\theta=\underline{\theta}$.

3.5 The Case of Strong Interactions

So far, we have characterized optimal allocations for regular cases, where the strength of interactions between the goods is relatively mild. If the ratio $\frac{\overline{\eta}-\underline{\eta}}{\overline{\theta}-\underline{\theta}}$ is relatively large, then "most" utility functions will display relatively mild interactions between the goods in this sense. This loose statement can be given a very precise meaning in the concrete example of negative quadratic utility. For that case, all concave utility functions satisfy Assumption 2 if the support of second period information is wider than the support of first period information. On the other hand, if the reverse is true, then one can give natural examples, where an allocation outside the sets $\mathbb{X}_i \cup \mathbb{X}_{ii}$ becomes optimal. Specifically, we have the following result:

Proposition 3.5.1 *Suppose that* $\frac{\overline{\eta}-\underline{\eta}}{\overline{\theta}-\underline{\theta}} < 1$ *and consider the quadratic utility function of Example 1 with* $\delta \in \left(\frac{\overline{\eta}-\underline{\eta}}{\overline{\theta}-\underline{\theta}}, 1\right)$. *For that utility function, for* α *sufficiently close to zero, the overall optimal allocation satisfies* $(x, y) \in \mathbb{X}_{iii}$.

Proof. In Appendix 3. ■

For $\delta \in \left(\frac{\overline{\eta}-\underline{\eta}}{\overline{\theta}-\underline{\theta}}, 1\right)$, it follows from Lemma 5 that the solution of the reduced problem is an element of $\mathbb{X}_{iii}^{int}$ for α close to zero. Moreover, it is straightforward to verify that $\Omega \leq 0$ in the example. Hence, we have shown that it can be strictly optimal to induce lying off equilibrium path.

Consider now the structure of the optimal allocation for the case where $(x, y) \in \mathbb{X}_{iii}^{int}$. The first-order conditions for the allocation offered to ex ante type $\overline{\theta}$ are as follows:

$$V_1\left(x\left(\overline{\theta}, \underline{\eta}\right), y\left(\overline{\theta}, \underline{\eta}\right)\right) = \overline{\theta} + \frac{\alpha}{(1-\alpha)} \frac{1}{\lambda\left(\overline{\theta}\right)} \left(\overline{\theta} - \underline{\theta}\right)$$
$$V_2\left(x\left(\overline{\theta}, \underline{\eta}\right), y\left(\overline{\theta}, \underline{\eta}\right)\right) = \underline{\eta} - \frac{\alpha}{(1-\alpha)} \frac{\left(1-\lambda\left(\underline{\theta}\right)\right)}{\lambda\left(\overline{\theta}\right)} \left(\overline{\eta} - \underline{\eta}\right)$$

and

$$V_1\left(x\left(\overline{\theta}, \overline{\eta}\right), y\left(\overline{\theta}, \overline{\eta}\right)\right) = \overline{\theta}$$
$$V_2\left(x\left(\overline{\theta}, \overline{\eta}\right), y\left(\overline{\theta}, \overline{\eta}\right)\right) = \overline{\eta} + \frac{\alpha\left(1-\lambda\left(\overline{\theta}\right)\right)}{(1-\alpha)\left(1-\lambda\left(\overline{\theta}\right)\right)} \left(\overline{\eta} - \underline{\eta}\right).$$

This allocation displays upwards distortions in the quantity $y\left(\overline{\theta}, \underline{\eta}\right)$, for given quantity $x\left(\overline{\theta}, \underline{\eta}\right)$. Since we are considering complements, this upwards distortion does not arise simply as a compensating effect due to a downward distortion in $x\left(\overline{\theta}, \underline{\eta}\right)$, but rather reflects the particular structure of binding incentive constraints for this particular case. Recall from Lemma 3.3.3 that the best deviation of an agent with $\theta = \underline{\theta}$ is to report $\hat{\theta} = \overline{\theta}$ in period one and $\hat{\eta} = \underline{\eta}$ in period two, regardless of the actual realization of η. Hence, the reduction in $x\left(\overline{\theta}, \underline{\eta}\right)$ reflects the fact that the conditional probability of receiving report $\hat{\eta} = \underline{\eta}$ is one; vice versa, there is no rent reduction motive when choosing $x\left(\overline{\theta}, \overline{\eta}\right)$ at all, because the agent with $\theta = \underline{\theta}$ is never going to imitate type $\left(\overline{\theta}, \overline{\eta}\right)$. Recall moreover, that in addition to the rents from producing x more efficiently, the agent with type $\theta = \underline{\theta}$ obtains rents from producing y more efficiently; in particular, the agent would obtain (when deviating to $\hat{\theta} = \overline{\theta}$), the utility level that type $\left(\overline{\theta}, \underline{\eta}\right)$ obtains, $\left(1 - \lambda\left(\overline{\theta}\right)\right)\left(\overline{\eta} - \underline{\eta}\right) y\left(\overline{\theta}, \overline{\eta}\right)$, minus the expected loss in case he has a higher η realization in period two than the type he imitates, $\left(1 - \lambda\left(\underline{\theta}\right)\right)\left(\overline{\eta} - \underline{\eta}\right) y\left(\overline{\theta}, \underline{\eta}\right)$. As a result, all else equal (that is, for a given x-allocation), the principal reduces $y\left(\overline{\theta}, \overline{\eta}\right)$ below the first-best level and increases $y\left(\overline{\theta}, \underline{\eta}\right)$ beyond the first-best level. Whether the overall production levels are above or below first-best depends on the specific utility function.

3.6 Discussion: Sequential Screening and the Value of Waiting

What if x needs to be determined already in period one? We can obtain the optimal mechanism with sequential production from our problem if we add the requirement that

$$x\left(\theta, \underline{\eta}\right) = x\left(\theta, \overline{\eta}\right) \text{ for } \theta \in \left\{\underline{\theta}, \overline{\theta}\right\}. \tag{3.16}$$

Technically, (3.16) is a consistency requirement in the sense that the level of x can only depend on information that is available when the level of x is chosen.

It is straightforward to see that off-path lies are not an issue under this constraint. The reason is that $\theta x\left(\hat{\theta}\right)$ is sunk by the time the report about η needs to be made and moreover enters the seller's profit in an additively separable way. So, seller types who have lied in the past correspond to types with different fixed costs of producing the y good. However, fixed costs do not change the seller's incentive to report about η. So, the on-path incentive constraints automatically ensure that reporting is truthful also off path.

It is also obvious that sequential production cannot do better than delaying production of both goods until all information is there. The reason is that we are simply adding another constraint, (3.16), to the buyer's problem and thereby eliminate some flexibility off equilibrium path (precisely because the on-path constraints automatically imply a particular off-path behavior).

Solving the transfer minimization problems (3.13) and (3.14) for given allocation choices x and y, under the consistency condition (3.16) and its implication of truthfulness of path, we find that at the solutions to these problems constraints (3.2) and (3.9) and (3.6) and (3.5) are binding. Using the optimal payments, the buyer's problem of finding an optimal allocation can be written as

$$\begin{aligned} \max_{x(\theta), y(\theta,\eta)} & \mathbb{E}_\theta \mathbb{E}_{\eta|\theta} \left[V\left(x\left(\theta\right), y\left(\theta,\eta\right)\right) - \theta x\left(\theta\right) - \eta y\left(\theta,\eta\right) \right] \\ & -\alpha \left[\left(\overline{\theta} - \underline{\theta}\right) x\left(\overline{\theta}\right) + \left(\lambda\left(\underline{\theta}\right) - \lambda\left(\overline{\theta}\right)\right) \left(\overline{\eta} - \underline{\eta}\right) y\left(\overline{\theta}, \overline{\eta}\right) \right] \end{aligned}$$

Moreover, the neglected incentive constraint (3.7) is equivalent to

$$\left(\lambda\left(\underline{\theta}\right) - \lambda\left(\overline{\theta}\right)\right) \left(\overline{\eta} - \underline{\eta}\right) \left(y\left(\underline{\theta}, \underline{\eta}\right) - y\left(\overline{\theta}, \overline{\eta}\right)\right) \geq \left(\overline{\theta} - \underline{\theta}\right) \left(x\left(\overline{\theta}\right) - x\left(\underline{\theta}\right)\right).$$

The following proposition is now obvious:

Corollary 3.6.1 *Delayed and early production achieve the same payoff only for independent goods. If* $V_{12}(x,y) > (<) \, 0$ *for all* x, y*, delayed production is strictly better than early production.*

The proof of the statement follows from the discussion in an obvious way and is therefore omitted. The logic is simply that the allocation under sequential production is always feasible under delayed production of both goods but is not chosen at the optimum, except for the case of independent goods.

It is instructive to take a closer look into the losses associated to sequential production.

The allocation offered to ex ante type $\underline{\theta}$ is first-best efficient; that is, there is no distortion at the top. The allocation offered to ex ante type $\overline{\theta}$ satisfies the first-order conditions

$$\mathbb{E}_{\eta|\overline{\theta}}\left[V_1\left(x\left(\overline{\theta}\right), y\left(\overline{\theta}, \eta\right)\right)\right] = \overline{\theta} + \frac{\alpha}{1-\alpha}\left(\overline{\theta} - \underline{\theta}\right),$$

$$V_2\left(x\left(\overline{\theta}\right), y\left(\overline{\theta}, \underline{\eta}\right)\right) = \underline{\eta},$$

and

$$V_2\left(x\left(\overline{\theta}\right), y\left(\overline{\theta}, \overline{\eta}\right)\right) = \overline{\eta} + \frac{\alpha}{1-\alpha}\frac{\lambda\left(\underline{\theta}\right) - \lambda\left(\overline{\theta}\right)}{1-\lambda\left(\overline{\theta}\right)}\left(\overline{\eta} - \underline{\eta}\right).$$

The expected marginal benefit of $x\left(\overline{\theta}\right)$ is equal to $\overline{\theta} + \frac{\alpha}{1-\alpha}\left(\overline{\theta} - \underline{\theta}\right)$. For given allocation $y\left(\overline{\theta}, \eta\right)$, this corresponds to the standard result that $x\left(\overline{\theta}\right)$ is distorted downwards relative to the first-best. Likewise, for given allocation $x\left(\overline{\theta}\right)$, $y\left(\overline{\theta}, \underline{\eta}\right)$ is set efficiently, while $y\left(\overline{\theta}, \overline{\eta}\right)$ is distorted downwards. Whether the entire allocation is higher or lower than first-best depends on the nature of interactions between the goods. For the case of independent goods, the overall allocation relates exactly as stated to the first-best allocation.

For nonzero interactions between the goods, there are two sources of losses for the principal due to choosing x early on. Firstly, it is simply the case that both allocation choices should be adjusted to both cost conditions. Secondly, as we have explained at great lengths, it is sometimes not optimal to insist on truthtelling off path when both x and y are chosen late. Intuitively, it becomes easier to screen the information in the second round of reporting when the principal has more screening instruments available.

Note that in the case of weak substitutes in the sense of Proposition 2, the first-order conditions differ *only* in that the marginal utilities interact with each other; the virtual cost expressions on the right hand side are identical for both timing configurations.[13] It is then straightforward to see how the optimal allocations differ from each other in the more flexible regime with delayed production and in the regime with early production of x. For an allocation in the regime with delayed production in $\mathbb{X}_{ii}^{int}\left(\overline{\theta}\right)$, we have that $y\left(\overline{\theta}, \underline{\eta}\right) > y\left(\overline{\theta}, \overline{\eta}\right)$

[13]In the case of complements, the virtual marginal cost of $x\left(\overline{\theta}, \underline{\eta}\right)$ is increased while the virtual marginal cost of $x\left(\overline{\theta}, \overline{\eta}\right)$ is decreased for given level of y.

and $x\left(\overline{\theta},\overline{\eta}\right) > x\left(\overline{\theta},\underline{\eta}\right)$. If the $x-$allocation is now forced to take the common value $x\left(\overline{\theta}\right)$, then, heuristically, $x\left(\overline{\theta},\overline{\eta}\right)$ is reduced while $x\left(\overline{\theta},\underline{\eta}\right)$ is increased. Since the marginal utility of consuming y still must take on the same value, the $y-$allocation has to respond more to η than it does in the flexible regime. Hence, the variation in the level of y is increased in response to the reduction in the variation in the level of x.

Thus, if the buyer has a choice, then starting production before all information is available is never strictly better than waiting until all information is available. In other words, our model features a nonnegative option value of waiting. The timing of production is irrelevant only in the case where the buyer's utility is additively separable in the utilities from consuming x and y.

3.7 Conclusion

This chapter solves a tractable two-dimensional model of screening where the agent produces two goods, knows the cost parameter of producing one good from the outset, and learns the cost parameter of producing the second good at some later date. We assume positive correlated cost parameters. Depending on whether the goods are complements or substitutes and on how strongly the goods interact, a different pattern of binding constraints arises at the optimum. For weak complements, we obtain a standard solution, where the allocations of the inefficient types are downward distorted, and the principal only needs to worry about single deviations. Note that the solution to the full problem could also be obtained by a naive procedure that imposes truthtelling at all nodes of the game, even at those that are not reached if the agent is truthful early on in the game, but all off-path constraints are slack. Therefore, we can simply ignore the off-path incentive compatibility constraints. For weak substitutes, it is still true that the solution can be obtained by imposing truthtelling on and off equilibrium path. In this case, upward distortions may arise. However, now a truthtelling constraint off the equilibrium path is binding at the optimum. As a result, the solution displays both upward and downward distortions. Finally, in the case of strong interactions between the goods, inducing the agent to lie again after a first may be rent-minimizing for the principal. In this case, upward distortions may arise even in the case of complements.

As a simple by-product of our work we compare our solution to the literature by varying the timing of production in our model. It is always desirable to postpone all decisions until all information is available, if that is feasible. The comparison to the static model of two-dimensional screening is done in companion work. An interesting set of questions that we do not address in this work relates to repeating the interaction between the buyer and the seller.

Clearly, multi-dimensional problems are more complex than one-dimensional ones. However, the timing of the information process imposes quite some structure on our problem. Part of the analysis of sequential screening problems has no grounds whatsoever in the revelation principle per se, but emerges from implementing given allocations at lowest cost (see also Krähmer and Strausz 2008). However, the buyer's preferences impose structure on this problem so that the complicated problem becomes tractable.

We provide a natural setting in which upward distortions may arise as a feature of the optimal mechanism.

Appendix 1

I can formulate this problem for a belief $\gamma = P(i = A)$, $\gamma \in (0,1)$. $((r_A^\gamma, m_A^\gamma), (r_B^\gamma, m_B^\gamma))$ denotes the optimal mechanism provided the seller's belief is γ. For a given belief γ, the monopolist's optimal mechanism $((r_A^\gamma, m_A^\gamma), (r_B^\gamma, m_B^\gamma))$ is the solution to

$$\max_{r_A, r_B, m_A, m_B} \gamma m_A + (1-\gamma) m_B \tag{3.17}$$

subject to

$$\theta_j r_j - m_j \geq 0, \tag{3.18}$$

$$\theta_j r_j - m_j \geq \theta_j r_i - m_i, \tag{3.19}$$

$$r_j \in [0,1] \tag{3.20}$$

for $i \in \{A, B\}$, $j \in \{A, B\}$,$j \neq i$,

where constraint (3.18) is the individual rationality condition of a buyer j, constraint (3.19) is the incentive compatibility condition of buyer j, and constraint (3.20) is the feasibility condition that restricts the quantity to less than 1. Constraint (3.20) must be imposed because the buyer has unit demand in my setting.

This static problem has been investigated in more generality by many authors like Mussa and Rosen 1978 or Maskin and Riley 1984. Applying standard methods I can derive the following well-known solution.

Lemma 3.7.1 *The solution to the static problem* (3.17) *is given by*

$$M^\gamma \equiv \begin{cases} ((1, \theta_A), (0, 0)) & \text{if } \gamma\theta_A \geq \theta_B \\ ((1, \theta_B), (1, \theta_B)) & \text{if } \gamma\theta_A \leq \theta_B \end{cases} . \tag{3.21}$$

The seller's expected revenue is given by

$$\max\left(\gamma\theta_A, \theta_B\right). \tag{3.22}$$

Proof of Lemma 3.7.1. The proof builds on the solution to the optimal nonlinear pricing with two types in Bolton and Dewatripont 2005 pp. 52. I restate their proof in my notation.

Let $((r_A, m_A), (r_B, m_B))$ be the optimal mechanism.

Step 1: At the optimum the individual rationality constraint of type A does not bind.

Note that the following relation holds

$$\theta_A r_A - m_A \geq \theta_A r_B - m_B \geq \theta_B r_B - m_B \geq 0.$$

The first inequality holds by the incentive compatibility constraint of A. The second inequality holds since $\theta_A > \theta_B$. The last inequality holds by the individual rationality constraint of B. Thus, the individual rationality constraint of A is satisfied automatically.

Step 2: The individual rationality constraint of type B must bind.

At the optimum at least one individual rationality constraint must be binding.

Suppose not. Then one can increase both transfers m_A and m_B by the same amount without a violation of any constraint. This is a contradiction to $((r_A, m_A), (r_B, m_B))$ being the optimal.

Step 3: Solve the relaxed problem without the incentive compatibility constraint of type B.

There are only two constraints; the binding individual rationality constraint $\theta_B r_B - m_B = 0$ and the incentive compatibility constraint of A, which is equivalent to

$$\theta_A r_A - (\theta_A - \theta_B) r_B \geq m_A.$$

Since there is no other constraint, this has to be binding.

The relaxed problem is given by

$$\max_{r_A, r_B} \gamma\left(\theta_A r_A - (\theta_A - \theta_B) r_B\right) + (1-\gamma)\theta_B r_B \tag{3.23}$$

One solution candidate is $r_A = 1$ and $r_B = 1$ with transfers $m_A = \theta_A$ and $m_B = 0$. The expected revenue is $\gamma\theta_A$. The other solution candidate is $r_A = 1$ and $r_B = 0$ with transfers $m_A = m_B = \theta_B$. The solution candidate is given by (3.21) providing expected revenue (3.22).

Step 4: Check whether the solution candidate satisfies the neglected constraint.

Substituting for the transfers, the neglected constraint is satisfied if

$$(\theta_A - \theta_B)\, r_A \geq (\theta_A - \theta_B)\, r_B,$$

which is satisfied at both solution candidates.

Hence the solution candidate solves (3.17). ■

Proof of Proposition 1.3.1. If seller 2 buys the purchase history, then she will be able to distinguish the new customer from the old customer. Given that knowledge her expected revenue is the sum of the expected revenue from selling good 2 to the new customer, buyer 2, and the expected revenue from selling good 2 to buyer 1.

Her expected revenue from making buyer 2 an optimal offer

$$\begin{cases} ((1,\theta_A),(0,0)) & \text{if } \beta\theta_A \geq \theta_B \\ ((1,\theta_B),(1,\theta_B)) & \text{if } \beta\theta_A \leq \theta_B \end{cases} \tag{3.24}$$

is equal to $\max(\beta\theta_A, \theta_B)$, since (3.24) is the solution to problem (1.1) is given by Lemma 3.7.1 for prior β and $r = y$ and $m = t$.

Denote seller 2's equilibrium offer to seller 1's customer, buyer 1, who reported h by $((y^*_{hA}, t^*_{hA}), (y^*_{hB}, t^*_{hB}))$. If she does not condition the offer on the buyer's report to seller 1, then seller 2's offer is independent of the report to seller 1, i.e. $((y^*_{AA}, t^*_{AA}), (y^*_{AB}, t^*_{AB})) = ((y^*_{BA}, t^*_{BA}), (y^*_{BB}, t^*_{BB}))$.

If $s_A = 1$ and $s_B = 0$, then the purchase history will be fully revealing. In that case she can perfectly discriminate buyer 1; that is, she offers one unit of her good at θ_A if the history is A and at θ_B if the history is B so that her optimal offer to buyer 1 is

$$\left(\left(y_{hA}^*, t_{hA}^*\right), \left(y_{hB}^*, t_{hB}^*\right)\right) = \begin{cases} \left(\left(1, \theta_A\right), \left(0, 0\right)\right) & \text{if buyer 1 reported } h = A \\ \left(\left(1, \theta_B\right), \left(1, \theta_B\right)\right) & \text{if buyer 1 reported } h = B \end{cases}.$$

Then seller 2's expected revenue, provided she bought a perfectly revealing purchase history, is equal to

$$\begin{cases} \max\left(\beta\theta_A, \theta_B\right) + \theta_A & \text{if } h = A \\ \max\left(\beta\theta_A, \theta_B\right) + \theta_B & \text{if } h = B \end{cases}.$$

■

Proof of Proposition 1.3.2. If seller 2 does not possess the purchase history, then he does not know the customers identity and cannot identify the two buyers. From the main text I know that the probability that a buyer has a type A is equal to $\frac{\lambda+\beta}{2}$. The solution is equal to the solution of the static problem for $\gamma = \frac{\lambda+\beta}{2}$ and $r = y$ and $m = t$. Then by Lemma 3.7.1 her expected revenue from trading with one of her customers is equal to $\max\left(\frac{(\beta+\lambda)}{2}\theta_A, \theta_B\right)$. Her total expected revenue is equal to $\max\left(\left(\beta+\lambda\right)\theta_A, 2\theta_B\right)$. ■

Proof of Proposition 1.3.3. The probability that seller 1's former customer will have demand for good 2 is $\delta_A\alpha + \delta_B\left(1-\alpha\right)$. If seller 1's offer is fully separating so that her customer reports with $s_A = 1$ and $s_B = 0$, then $\mu_A\left(1, 0\right) = 1$ and the probability that buyer 1 reports A given he has positive demand is $\frac{\alpha\delta_A}{\delta_A\alpha+\delta_B(1-\alpha)}$. Therefore $\frac{\alpha\delta_A}{\delta_A\alpha+\delta_B(1-\alpha)}$ is the probability that the type of seller 1's former customer is A if he has positive demand. If the purchase history contains the information that enables seller 2 to fully separate the types according to the reports, then by Proposition 1.3.1 the revenue of seller 2 is equal to

$$\begin{cases} \max\left(\beta\theta_A, \theta_B\right) + \theta_A & \text{if } h = A \\ \max\left(\beta\theta_A, \theta_B\right) + \theta_B & \text{if } h = B \end{cases}.$$

Then her expected revenue under disclosure policy is equal to

$$\begin{array}{c}\left(\delta_A\alpha+\delta_B\left(1-\alpha\right)\right)\left(\begin{array}{c}\max\left(\beta\theta_A,\theta_B\right)+\frac{\alpha\delta_A}{\delta_A\alpha+\delta_B(1-\alpha)}\theta_A\\+\left(1-\frac{\alpha\delta_A}{\delta_A\alpha+\delta_B(1-\alpha)}\right)\theta_B\end{array}\right)\\+\left(1-\left(\delta_A\alpha+\delta_B\left(1-\alpha\right)\right)\right)\max\left(\beta\theta_A,\theta_B\right)\end{array}\tag{3.25}$$

provided she bought the purchase history. If she does not purchase the purchase history, then by Proposition 1.3.2 her expected revenue is $\max\left(\left(\beta+\lambda\right)\theta_A,\theta_B\right)$. The willingness to pay for the purchase history of seller 2 is her expected revenue from buying it that exceeds $\max\left(\left(\beta+\lambda\right)\theta_A,2\theta_B\right)$. Therefore $WTP\left(1,0\right)$ is equal to

$$\left(\delta_A\alpha+\delta_B\left(1-\alpha\right)\right)\left(\begin{array}{c}\max\left(\beta\theta_A,\theta_B\right)+\frac{\alpha\delta_A}{\delta_A\alpha+\delta_B(1-\alpha)}\theta_A\\+\left(1-\frac{\alpha\delta_A}{\delta_A\alpha+\delta_B(1-\alpha)}\right)\theta_B-\max\left(\left(\beta+\lambda\right)\theta_A,2\theta_B\right)\end{array}\right).$$

■

Proof of Proposition 1.3.4. Since the seller faces a myopic problem the solution to her problem is equal to the solution to the static problem provided by Lemma 3.7.1 for belief $\gamma=\alpha$ and $r=x$ and $m=p$. Therefore her expected revenue is equal to $\max\left(\alpha\theta_A,\theta_B\right)$. ■

Proof of Lemma 1.3.1. This proof derives the price of the purchase history in the model provided that seller 1 offers a fully separating mechanism. I substitute $s_A=1$ and $s_B=0$ into (1.11) which gives

$$\begin{array}{ll}\left\{\begin{array}{c}\alpha\delta_A\left(\beta\theta_A+\theta_A-\left(\beta+\lambda\right)\theta_A\right)\\+\left(1-\alpha\right)\delta_B\left(\beta\theta_A+\theta_B-\left(\beta+\lambda\right)\theta_A\right)\end{array}\right\} & \text{if } \left(\beta+\lambda\right)\theta_A\geq 2\theta_B \text{ and } \beta\theta_A\geq\theta_B\\ \left\{\begin{array}{c}\alpha\delta_A\left(\theta_B+\theta_A-\left(\beta+\lambda\right)\theta_A\right)\\+\left(1-\alpha\right)\delta_B\left(\theta_B+\theta_B-\left(\beta+\lambda\right)\theta_A\right)\end{array}\right\} & \text{if } \left(\beta+\lambda\right)\theta_A\geq 2\theta_B \text{ and } \beta\theta_A\leq\theta_B\\ \left\{\begin{array}{c}\alpha\delta_A\left(\beta\theta_A+\theta_A-2\theta_B\right)\\+\left(1-\alpha\right)\delta_B\left(\beta\theta_A+\theta_B-2\theta_B\right)\end{array}\right\} & \text{if } \left(\beta+\lambda\right)\theta_A\leq 2\theta_B \text{ and } \beta\theta_A\geq\theta_B\\ \left\{\begin{array}{c}\alpha\delta_A\left(\theta_B+\theta_A-2\theta_B\right)\\+\left(1-\alpha\right)\delta_B\left(\theta_B+\theta_B-2\theta_B\right)\end{array}\right\} & \text{if } \left(\beta+\lambda\right)\theta_A\leq 2\theta_B \text{ and } \beta\theta_A\leq\theta_B\end{array}.$$

I can simplify this to

$$\begin{array}{ll} \{(1-\alpha)\,\delta_B\theta_B\} & \text{if } (\beta+\lambda)\,\theta_A \geq 2\theta_B \text{ and } \beta\theta_A \geq \theta_B \\ \{\alpha\delta_A\,(\theta_B-\beta\theta_A)+(1-\alpha)\,\delta_B\,(2\theta_B-\beta\theta_A)\} & \text{if } (\beta+\lambda)\,\theta_A \geq 2\theta_B \text{ and } \beta\theta_A \leq \theta_B \\ \left\{\begin{array}{c} \alpha\delta_A\,(\beta\theta_A+\theta_A-2\theta_B) \\ +(1-\alpha)\,\delta_B\,(\beta\theta_A-\theta_B) \end{array}\right\} & \text{if } (\beta+\lambda)\,\theta_A \leq 2\theta_B \text{ and } \beta\theta_A \geq \theta_B \\ \{\alpha\delta_A\,(\theta_A-\theta_B)\} & \text{if } (\beta+\lambda)\,\theta_A \leq 2\theta_B \text{ and } \beta\theta_A \leq \theta_B \end{array}.$$

■

Proof of Theorem 1.3.1. Seller 1's expected payoff is equal to the sum of revenue from selling to the buyer and seller 2's expected willingness to pay. From Lemma 1.3.1 seller 2's expected willingness to pay is known. It remains to plug the expected revenue from offering mechanism $(1,\theta_A-\delta_A\,(\theta_A-\theta_B)\,,(0,0))$ and the price of the purchase history in seller 1's expected revenue and to compare it with the threshold provided in Corollary 1.3.1.

Assume $1>\delta_A$.

Step 1: Show that $(1,\theta_A-\delta_A\,(\theta_A-\theta_B)\,,(0,0))$ satisfies all constraints.

Substitution of $(1,\theta_A-\delta_A\,(\theta_A-\theta_B)\,,(0,0))$ into the constraints gives:

The mechanism satisfies the individual rationality constraint of type B, (1.19), with equality

$$0\theta_B-0=0.$$

The incentive constraint of type B, (1.21) is slack

$$\begin{aligned} 0\theta_B-0 &> 1\theta_B-(\theta_A-\delta_A\,(\theta_A-\theta_B)) \\ &= -\,(1-\delta_A)\,(\theta_A-\theta_B) \end{aligned}$$

if $\delta_A<1$. The individual rationality constraint of type A is slack

$$1\theta_A-(\theta_A-\delta_A\,(\theta_A-\theta_B))>0.$$

The incentive compatibility constraint of type A binds

$$1\theta_A - (\theta_A - \delta_A(\theta_A - \theta_B)) = 0\theta_A - 0 + \delta_A(\theta_A - \theta_B).$$

Since all constraints are satisfied, I can set $s_A = 1$ and $s_A = 0$.

Step 2: If seller 1 offers the screening contract $(1, \theta_A - \delta_A(\theta_A - \theta_B), (0,0))$, then she receives an expected revenue from selling to the buyer of $\alpha(\theta_A - \delta_A(\theta_A - \theta_B))$. Then her expected revenue is equal to

$$\begin{array}{ll}
\{\alpha\theta_A - \alpha\delta_A(\theta_A - \theta_B) + (1-\alpha)\delta_B\theta_B\} & \text{if } (\beta+\lambda)\theta_A \geq 2\theta_B \text{ and } \beta\theta_A \geq \theta_B \\
\left\{\begin{array}{c} \alpha\theta_A - \alpha\delta_A(\theta_A - \theta_B) + \alpha\delta_A(\theta_B - \beta\theta_A) \\ +(1-\alpha)\delta_B(2\theta_B - \beta\theta_A) \end{array}\right\} & \text{if } (\beta+\lambda)\theta_A \geq 2\theta_B \text{ and } \beta\theta_A \leq \theta_B \\
\left\{\begin{array}{c} \alpha\theta_A - \alpha\delta_A(\theta_A - \theta_B) + \alpha\delta_A(\beta\theta_A + \theta_A - 2\theta_B) \\ +(1-\alpha)\delta_B(\beta\theta_A - \theta_B) \end{array}\right\} & \text{if } (\beta+\lambda)\theta_A \leq 2\theta_B \text{ and } \beta\theta_A \geq \theta_B \\
\{\alpha\theta_A - \alpha\delta_A(\theta_A - \theta_B) + \alpha\delta_A(\theta_A - \theta_B)\} & \text{if } (\beta+\lambda)\theta_A \leq 2\theta_B \text{ and } \beta\theta_A \leq \theta_B
\end{array}$$

which simplifies to

$$\begin{array}{ll}
\{\alpha\theta_A + (\alpha\delta_A + (1-\alpha)\delta_B)(\theta_B - \lambda\theta_A)\} & \text{if } (\beta+\lambda)\theta_A \geq 2\theta_B \text{ and } \beta\theta_A \geq \theta_B \\
\left\{\begin{array}{c} \alpha\theta_A \\ +(\alpha\delta_A + (1-\alpha)\delta_B)(2\theta_B - (\beta+\lambda)\theta_A) \end{array}\right\} & \text{if } (\beta+\lambda)\theta_A \geq 2\theta_B \text{ and } \beta\theta_A \leq \theta_B \\
\{\alpha\theta_A + (\alpha\delta_A + (1-\alpha)\delta_B)(\beta\theta_A - \theta_B)\} & \text{if } (\beta+\lambda)\theta_A \leq 2\theta_B \text{ and } \beta\theta_A \geq \theta_B \\
\alpha\theta_A & \text{if } (\beta+\lambda)\theta_A \leq 2\theta_B \text{ and } \beta\theta_A \leq \theta_B
\end{array}.$$

Comparing this expected revenue with $\alpha\theta_A$, I conclude that seller 1 strictly prefers the disclosure policy if either

I) $(\beta+\lambda)\theta_A \geq 2\theta_B$ and $\beta\theta_A \geq \theta_B$ and $\theta_B - \lambda\theta_A > 0$ or

II) $(\beta+\lambda)\theta_A \leq 2\theta_B$ and $\beta\theta_A > \theta_B$. ■

Appendix 2

Appendix 2.A

Lemma 3.7.2 *For $i \in \{1,2\}$ bidder $i's$ maximization problem is given by*

$$\max_{\beta_i(h_i)} \int_{\left\{t_j:\beta_i(h_i)\geq\beta_j^*(h_j)\right\}} \left(at_i + bt_j + \alpha\mathbb{E}\left[Z_i|\, h_i\right] - \beta_j^*\left(h_j\right)\right) f\left(t_j\right) dt_j. \tag{3.26}$$

If $\mathbb{E}\left[Z_1|\, h_1\right] = \mathbb{E}\left[Z_2|\, h_2\right]$, then there is an equilibrium in which bidder 1 and bidder 2 bid, respectively,

$$\begin{aligned} \beta^*\left(h_1\right) &= (a+b)\, t_1 + \alpha\mathbb{E}\left[Z_1|\, h_1\right], \\ \beta^*\left(h_2\right) &= (a+b)\, t_2 + \alpha\mathbb{E}\left[Z_2|\, h_2\right]. \end{aligned}$$

Proof of Lemma 3.7.2. The bidder's equilibrium bidding strategies can be derived by solving a set of first-order conditions in an analogue fashion to Milgrom and Weber 1982a or Maskin 1992. The equilibrium characterization is derived by applying the same techniques and since the bidders are symmetric the equilibrium bidding strategies do not differ from the equilibrium bidding strategies found in Milgrom and Weber 1982a. I provide a derivation for the sake of completeness.

Suppose $\mathbb{E}\left[Z_1|\, h_1\right] = \mathbb{E}\left[Z_2|\, h_2\right]$. The first-order condition for problem (3.26) is given by

$$\left\{ \begin{array}{c} \frac{\partial\beta_j^{*-1}(\beta_i(h_i))}{\partial\beta_i(h_i)} \left(at_i + b\beta_j^{*-1}\left(\beta_i\left(h_i\right)\right) + \alpha\mathbb{E}\left[Z_i|\, h_i\right] - \beta_j^*\left(\beta_j^{*-1}\left(\beta_i\left(h_i\right)\right)\right)\right) \\ \cdot f\left(\beta_j^{*-1}\left(\beta_i\left(h_i\right)\right)\right) \end{array} \right\} = 0.$$

By the symmetry of Z_1 and Z_2 this is only true when the seller conceals Z or if $z_1 = z_2$. In both cases $h_1 = h_2 \iff t_1 = t_2$. Since I am interested in symmetric Bayesian Nash equilibria in continuous strategies which are increasing in t

$$\beta_i(h) = \beta_j(h)$$

for all $t_i \in [\underline{t}, \bar{t}]$. Substitution gives

$$\left\{ \begin{array}{c} \frac{\partial \beta_j^{*-1}(\beta_i(h_i))}{\partial \beta_i(h_i)} \left((a+b)\, t_i + \alpha \mathbb{E}\left[Z_i | \, h_i \right] - \beta(h_i) \right) \\ \cdot f\left(\beta_j^{*-1}(\beta_i(h_i)) \right) \end{array} \right\} = 0,$$

which implies that the equilibrium satisfies

$$\beta^*(h_i) = (a+b)\, t_i + \alpha \mathbb{E}\left[Z_i | \, h_i \right]$$

for $i \in \{1, 2\}$ provided that this constitutes a maximum.

The second-order condition at this point is given by

$$\frac{\partial \beta_j^{*-1}(\beta_i(h_i))}{\partial \beta_i(h_i)} \frac{1}{\beta^{*\prime}(h_i)} (b - (a+b))\, f\left(\beta_j^{*-1}(\beta_i(h_i)) \right) < 0$$

which is true.

Moreover the winning bidder's payoff is positive, since

$$\begin{array}{rcl} (at_i + bt_j + \alpha \mathbb{E}[Z_i | \, h_i] - (a+b)\, t_j + \alpha \mathbb{E}[Z_j | \, h_j]) & > & 0 \\ & \iff & \\ t_i & > & t_j \\ & \iff & \\ \beta(h_i) & > & \beta(h_j) \end{array}$$

which is true whenever bidder i wins. ■

Proof of Proposition 2.3.1. First, I derive the unique inner solution in linear strategies, which are strictly increasing and continuous in the bidders' types. Since I am interested in equilibria in linear strategies, I suppose bidder j's bid is of the linear form

$$\beta_j^{N,*}\left(h_j^N\right) = x_j t_j + y_j.$$

Substitution into the first-order condition of bidder i gives

$$\left\{ \begin{array}{c} \frac{\partial \beta_j^{*-1}\left(\sigma_i^N\left(h_i^N\right)\right)}{\partial \sigma_i^N\left(h_i^N\right)} \left(at_i + b\frac{1}{x_j}\left(\sigma_i^N\left(h_i^N\right) - y_j\right) + \alpha \mathbb{E}\left[Z\right] - \sigma_i^N\left(h_i^N\right)\right) \\ \cdot f\left(\frac{1}{x_j}\left(\sigma_i^N\left(h_i^N\right) - y_j\right)\right) \end{array} \right\} = 0,$$

which is equivalent to

$$\sigma_i^N\left(h_i^N\right) = \frac{x_j}{x_j - b}\left(at_i - b\frac{y_j}{x_j} + \alpha \mathbb{E}\left[Z\right]\right).$$

$\sigma_i^N\left(h_i^N\right)$ is bidder i's best reply to bidder j's linear strategy $\beta_j^{N,*}\left(h_j^N\right)$ and is linear in t_i. I denote bidder i's best reply by $\sigma_i^{BR}\left(\beta_j^{N,*}\left(h_j^N\right)\right)$.

Let $\sigma_j^N\left(h_j^N\right)$ denote bidder j's reply to bidder i's $\sigma_i^{BR}\left(\beta_j^{N,*}\left(h_j^N\right)\right)$. Substitution into bidder j's first-order condition gives

$$\left\{ \begin{array}{c} \frac{\partial \sigma_i^{BR^{-1}}\left(\beta_j^{N,*}\left(\sigma_j^N\left(h_j^N\right)\right)\right)}{\partial \sigma_j^N\left(h_j^N\right)} \\ \cdot\left(at_j + b\frac{1}{a}\left(\frac{x_j - b}{x_j}\sigma_j^N\left(h_j^N\right) + b\frac{y_j}{x_j} - \alpha \mathbb{E}\left[Z\right]\right) + \alpha \mathbb{E}\left[Z\right] - \sigma_j^N\left(h_j^N\right)\right) \\ \cdot f\left(\frac{1}{x_j}\left(\sigma_i^N\left(h_i^N\right) - y_j\right)\right) \end{array} \right\} = 0,$$

which is equivalent to

$$\sigma_j^N\left(h_j^N\right) = \frac{ax_j}{ax_j - bx_j + b^2}\left(at_j + \frac{b^2}{a}\frac{y_j}{x_j} + \frac{a-b}{a}\alpha \mathbb{E}\left[Z\right]\right).$$

This solution is bidder j's best reply to bidder i's best reply to bidder j's linear strategy $\beta_j^{N,*}\left(h_j^N\right)$. The bidders' strategies must be mutually best replies, i.e. I must have

$$\sigma_j^{BR}\left(\sigma_i^{BR}\left(\beta_j^{N,*}\left(h_j^N\right)\right)\right) = \beta_j^{N,*}\left(h_j^N\right).$$

This linear system has a unique solution for x_2 and y_2, namely

$$x_2 = a + b$$

and

$$y_2 = \alpha\mathbb{E}\left[Z\right].$$

Substitution into the best reply functions gives that at stage 2 the unique inner solutions in linear form to the bidders' maximization problems, which are mutually best replies provided the seller conceals her information are given by

$$\begin{aligned} \sigma_j^{BR}\left(\sigma_i^{BR}\left(\beta_j^{N,*}\left(h_j^N\right)\right)\right) &= \beta_j^{N,*}\left(h_j^N\right) = (a+b)\,t_j + \alpha\mathbb{E}\left[Z\right], \\ \sigma_i^{BR}\left(\beta_j^{N,*}\left(h_j^N\right)\right) &= (a+b)\,t_i + \alpha\mathbb{E}\left[Z\right]. \end{aligned} \tag{3.27}$$

It can be shown by substitution that the bidders' expected payoffs are positive (see also Lemma 3.7.2).

There is another equilibrium candidate. There are corner solutions $\beta_i^{N,*}\left(h_i^N\right)$ and $\beta_j^{N,*}\left(h_j^N\right)$ such that bidder i wins with probability 1, i.e.

$$\beta_i^{N,*}\left(h_i^N\right) \geq \beta_j^{N,*}\left(h_j^N\right)$$

for all h_i^N and h_j^N. It is relatively easy to see that bidder i's expected utility must be positive, i.e.

$$\mathbb{E}_{T_j}\left[at_i + bT_j + \alpha\mathbb{E}\left[Z\right] - \beta_j^{N,*}\left(h_j^N\right)\right] \geq 0$$

if bidders play

$$\begin{aligned}
\beta_i^{N,*}\left(h_i^N\right) &= a\bar{t} + bt_i + \alpha\mathbb{E}\left[Z\right], \\
\beta_j^{N,*}\left(h_j^N\right) &= a\underline{t} + bt_i + \alpha\mathbb{E}\left[Z\right].
\end{aligned}$$

However, bidder i's candidate equilibrium strategy $a\underline{t}+bt_i+\alpha\mathbb{E}\left[Z\right]$ is weakly dominated by the strategy (see Definition 2.2.1) to bid his true minimal expected valuation $at_i+b\underline{t}+\alpha\mathbb{E}\left[Z\right]$ since

$$at_i + b\underline{t} + \alpha\mathbb{E}\left[Z\right] > a\underline{t} + bt_i + \alpha\mathbb{E}\left[Z\right],$$

which is true for all t_i. This implies that he wins more often and receives a potentially positive payoff against bidder j by bidding his minimal true valuation.

It remains to be shown that the candidate equilibrium $at_i+b\underline{t}+\alpha\mathbb{E}\left[Z\right]$ and $a\bar{t}+bt_j+\alpha\mathbb{E}\left[Z\right]$, $i \neq j, i, j \in \{1,2\}$, cannot be an equilibrium in increasing and continuous strategies. The reason is that bidder j always wins, but receives negative payoff for some of his types.

Then the seller's expected revenue is given by

$$\begin{aligned}
&\sum_{\substack{i=1\\ j\neq i}}^{2} \int_{\{t_i,t_j:\beta(t_i)\geq\beta(t_j)\}} \left[(a+b)\,t_j + \mathbb{E}\left[Z\right]\right] f\left(t_j\right) dt_j f\left(t_i\right) dt_i \\
=\ & \sum_{\substack{i=1\\ j\neq i}}^{2} \int_{t_i\geq t_j} \left[(a+b)\,t_j + \mathbb{E}\left[Z\right]\right] f\left(t_j\right) dt_j f\left(t_i\right) dt_i \\
=\ & (a+b)\,\mathbb{E}\left[T_{2:2}\right] + \mathbb{E}\left[Z\right].
\end{aligned}$$

■

Proof of Proposition 2.3.2. $a > b \geq 0$ and $(a+b)\,\underline{t} + \alpha z_l - \alpha\frac{b(z_h-z_l)}{a-b}$. In the main text I provide an argument for the uniqueness of this equilibrium in the sense that there is no other equilibrium in linear strategies in which each bidder's strategy is a solution to the first-order condition of the respective bidder's maximization problem. The proof provided here only establishes that there is an equilibrium with strategies

$$\beta_1^*(t_1, z_1, z_2) = (a+b)\,t_1 + \alpha z_1 - \alpha \frac{b(z_2 - z_1)}{a-b} \tag{3.28}$$

and

$$\beta_2^*(t_2, z_2, z_1) = (a+b)\,t_2 + \alpha z_2 - \alpha \frac{b(z_1 - z_2)}{a-b} \tag{3.29}$$

if $a > b$ by proving that these strategies are mutually best responses. I do this in the first two steps of the proof. In the third step I prove that the (3.28) and (3.29) are nonnegative if $(a+b)\,\underline{t} + \alpha z_l - \alpha\frac{b(z_h - z_l)}{a-b}$ holds true and in the last step I show that the equilibrium is efficient if $a > b$. The last step is only given for completeness. Actually I already know that the equilibrium is efficient by Maskin 1992, since the single-crossing condition is satisfied.

Step 1: Assume that 2 plays (3.29) and suppose that bidder 1 has private signal t_1 such that

$$\beta_1^*(t_1, z_1, z_2) \geq \beta_2^*(\underline{t}, z_2, z_1),$$

which is equivalent to

$$t_1 \geq \underline{t} + \alpha \frac{(z_j - z_i)}{(a-b)}.$$

Bidder i's maximization problem is given by maximizes over $\hat{\beta}$

$$\max_{\hat{\beta}} \int_{\left\{t_2 : \hat{\beta} > \beta_2^*(t_2, z_2, z_1)\right\}} \left[a t_1 + b t_2 + \alpha z_1 - \beta_2^*(t_2, z_2, z_1)\right] dF(t_2).$$

I want to show that (3.28) solves this problem.

The first-order condition with respect to $\hat{\beta}$ of bidder 1's problem is given by

$$\frac{\partial \beta_2^{*-1}\left(\hat{\beta}, z_2, z_1\right)}{\partial \hat{\beta}} \left[\begin{array}{c} a t_1 + b \beta_2^{*-1}\left(\hat{\beta}, z_2, z_1\right) \\ +\alpha z_1 - \beta_2^*\left(\beta_2^{*-1}\left(\hat{\beta}, z_2, z_1\right), z\right) \end{array}\right] f\left(\beta_2^{*-1}\left(\hat{\beta}, z\right)\right) = 0.$$

The second derivative is given by

$$\frac{\partial \beta_2^{*-1}\left(\hat{\beta}, z_2, z_1\right)}{\partial \hat{\beta}} \left(\begin{array}{c} \left[b\frac{\partial \beta_2^{*-1}\left(\hat{\beta}, z_2, z_1\right)}{\partial \hat{\beta}} - 1 \right] f\left(\beta_2^{*-1}\left(\hat{\beta}, z_2, z_1\right)\right) \\ + \left[\begin{array}{c} at_1 + b\beta_2^{*-1}\left(\hat{\beta}, z_2, z_1\right) + \alpha z_1 \\ -\beta_2^{*}\left(\beta_2^{*-1}\left(\hat{\beta}, z_2, z_1\right), z_2, z_1\right) \end{array} \right] f'\left(\beta_2^{*-1}\left(\hat{\beta}, z_2, z_1\right)\right) \frac{\partial \beta_2^{*-1}\left(\hat{\beta}, z_2, z_1\right)}{\partial \hat{\beta}} \end{array} \right)$$

I first solve the unconstrained problem and then check the constraint. The solution to the first-order condition is given by

$$\beta_2^{*-1}\left(\hat{\beta}, z_2, z_1\right) = \frac{1}{a+b}\left[\hat{\beta} - \alpha z_2 + \alpha \frac{b}{a-b}\left(z_1 - z_2\right)\right]$$

The first-order condition is given by

$$\frac{1}{a+b}\left[\begin{array}{c} at_1 + \frac{b}{a+b}\left[\hat{\beta} - \alpha z_2 + \alpha\frac{b}{a-b}\left(z_1 - z_2\right)\right] \\ +\alpha z_1 - \hat{\beta} \end{array} \right] f\left(\beta_2^{*-1}\left(\hat{\beta}, z_2, z_1\right)\right) = 0$$

which is equivalent to

$$at_1 + \alpha z_1 + \frac{b}{a+b}\left[-\alpha\frac{a}{a-b}z_2 + \alpha\frac{b}{a-b}z_1\right] = \hat{\beta}\frac{a}{a+b}.$$

Solving this equation for $\hat{\beta}$ gives

$$\begin{aligned} \hat{\beta} &= (a+b)\,t_1 + \alpha z_1 - \alpha\frac{b}{a-b}\left(z_2 - z_1\right) \\ &= \beta_1^{*}\left(t_1, z_1, z_2\right). \end{aligned}$$

The prove for bidder 2 is performed in the same manner and therefore neglected.

I simplify the notation and write $\beta_2^{*-1}\left(\hat{\beta}\right)$ instead of $\beta_2^{*-1}\left(\hat{\beta}, z_2, z_1\right)$. The second-order condition is satisfied if

$$\left.\frac{\partial\beta_2^{*-1}\left(\hat{\beta}\right)}{\partial\hat{\beta}}\left(\begin{array}{c}\left[b\frac{\partial\beta_2^{*-1}(\hat{\beta})}{\partial\hat{\beta}}-1\right]f\left(\beta_2^{*-1}\left(\hat{\beta}\right)\right)\\ +\left[\begin{array}{c}at_1+b\beta_2^{*-1}\left(\hat{\beta}\right)\\ +\alpha z_1-\hat{\beta}\end{array}\right]f'\left(\beta_2^{*-1}\left(\hat{\beta}\right)\right)\frac{\partial\beta_2^{*-1}(\hat{\beta})}{\partial\hat{\beta}}\end{array}\right)\right|_{\hat{\beta}=\beta_1^*(t_1,z_1,z_2)}$$
$$< \quad 0.$$

Substitution gives

$$\left.\frac{1}{a+b}\left(\begin{array}{c}\left[b\frac{1}{a+b}-1\right]f\left(\beta_2^{*-1}\left(\hat{\beta}\right)\right)\\ +\left[\begin{array}{c}at_1+b\left[t_1+\alpha\frac{z_1-z_2}{a-b}\right]+\alpha z_1\\ -\left(\left(a+b\right)t_1+\frac{\alpha z_1 a}{a-b}-\frac{\alpha b z_2}{a-b}\right)\end{array}\right]f'\left(\beta_2^{*-1}\left(\hat{\beta}\right)\right)\frac{\partial\beta_2^{*-1}(\hat{\beta})}{\partial\hat{\beta}}\end{array}\right)\right|_{\hat{\beta}=\beta_1^*(t_1,z_1,z_2)}$$
$$= \quad \frac{1}{a+b}\left(-\frac{a}{a+b}f\left(t_1+\alpha\frac{(z_1-z_2)}{a-b}\right)\right)<0.$$

since $\frac{\partial\beta_2^{*-1}(\hat{\beta})}{\partial\hat{\beta}}=\frac{1}{a+b}$ and $\beta_2^{*-1}\left(\hat{\beta}\right)=\frac{1}{a+b}\left[\left(\left(a+b\right)t_1+\alpha z_1\frac{a}{a-b}-\alpha\frac{b}{a-b}z_2\right)-\alpha z_2+\alpha\frac{b(z_1-z_2)}{a-b}\right]$ $=\left[t_1+\alpha\frac{z_1-z_2}{a-b}\right]$. I conclude that the candidate maximizes bidder i's expected utility given that bidder j bids $\beta_2^*\left(t_2, z_2, z_1\right)$ since $a>0$.

Note that bidder 1 has a positive probability to win since $\beta_1^*\left(t_1, z_1, z_2\right) \geq \beta_1^*\left(\underline{t}, z_1, z_2\right)$.

In the next step I prove that this strategy is also a best reply to $\beta_2^*\left(t_2, z_2, z_1\right)$ for all other types t_1 such that $\beta_1^*\left(t_1, z_1, z_2\right) \leq \beta_1^*\left(\underline{t}, z_1, z_2\right)$, who have a probability to win of 0 playing $\beta_1^*\left(t_1, z_1, z_2\right)$.

Step 2: Assume $z_2 > z_1$. I show that (3.28) is also bidder 1's best response to (3.29) if his type t_1 satisfies $\beta_1^*\left(t_1, z_1, z_2\right) \leq \beta_1^*\left(\underline{t}, z_1, z_2\right) \iff t_1 \leq \underline{t}+\alpha\frac{(z_h-z_l)}{a-b}$.

Clearly, bidder 1 loses with certainty if he uses a bid β, which satisfies $\beta < \beta_2^*\left(\underline{t}, z_2, z_1\right)$. Thus, he is indifferent between any of these bids. Ties occur with probability 0 as F is atomless. Thus, I need to show that any $\beta > \beta_2^*\left(\underline{t}, z_2, z_1\right)$ gives a negative payoff.

If bidder 1 plays (3.28), then his probability of winning is zero. Suppose bidder 1 deviates to a strategy $\beta' = \beta_2^*\left(\underline{t}, z_2, z_1\right)+\epsilon$, $\epsilon > 0$ so that 1 wins with positive probability. Then his expected value of playing that strategy β' is negative for all $t_1 \in \left[\underline{t}, \bar{t}\right]$

$$at_1 + b\underline{t} + \alpha z_l - \left((a+b)\,\underline{t} + \alpha z_h - \alpha \frac{b\,(z_l - z_h)}{a-b} \right) \leq 0$$

if and only if

$$t_1 \leq \underline{t} + \frac{1}{a-b} \alpha\,(z_h - z_l)\,,$$

which is true.

Step 3: Bids are positive if for all t_1, t_2, z_1, z_2 if the following system of inequalities defines a nonempty regime

$$\begin{aligned} (a+b)\,\underline{t} + \alpha z_1 - \alpha \frac{b\,(z_2 - z_1)}{a-b} &\geq 0, \\ (a+b)\,\underline{t} + \alpha z_2 - \alpha \frac{b\,(z_1 - z_2)}{a-b} &\geq 0. \end{aligned}$$

These conditions can be rewritten as

$$\begin{aligned} (a+b)\,\underline{t} + \alpha \frac{az_1 - bz_2}{a-b} &\geq 0, \\ (a+b)\,\underline{t} + \alpha \frac{az_2 - bz_1}{a-b} &\geq 0, \end{aligned}$$

Bids are positive if

$$(a+b)\,\underline{t} + \alpha \min \left\{ \frac{az_2 - bz_1}{a-b}, \frac{az_1 - bz_2}{a-b} \right\} \geq 0$$

for all realizations of z_1 and z_2. $\min\left\{\frac{az_l - bz_h}{a-b}, \frac{az_h - bz_l}{a-b}\right\}$, which is equal to $\frac{az_l - bz_h}{a-b}$ by the single crossing property and

$$\begin{aligned} az_l - bz_h &\leq az_h - bz_l \\ &\Longleftrightarrow \\ (a+b)\,z_l &\leq (a+b)\,z_h \end{aligned}$$

Then bids are positive if

$$(a+b)\underline{t}+\alpha\frac{az_l-bz_h}{a-b}\geq 0.$$

Step 4: The equilibrium is efficient if $\beta_1^*(t_1,z_1,z_2)\geq\beta_2^*(t_2,z_2,z_1)$ if and only if $at_1+bt_2+\alpha z_1\geq at_2+bt_1+\alpha z_2$.

$$\begin{array}{rcl}
\beta_1^*(t_1,z_1,z_2) & \geq & \beta_2^*(t_2,z_2,z_1) \\
 & \Longleftrightarrow & \\
(a+b)\,t_1+\alpha z_1-\alpha\dfrac{b}{a-b}(z_2-z_1) & \geq & (a+b)\,t_2+\alpha z_2-\alpha\dfrac{b}{a-b}(z_1-z_2) \\
 & \Longleftrightarrow & \\
(a+b)\,t_1+\alpha(z_1-z_2)\dfrac{a+b}{a-b} & \geq & (a+b)\,t_2 \\
 & \Longleftrightarrow & \\
at_1+bt_2+\alpha z_1 & \geq & at_2+bt_1+\alpha z_2.
\end{array}$$

■

Proof of Theorem 2.3.1. Assume $(a+b)\underline{t}+\alpha\frac{az_l-bz_h}{a-b}\geq 0$. I split up the proof in three steps. First, I evaluate the seller's gains from publicly disclosing Z at $\alpha=0$ and show that the gains are 0 at $\alpha=0$. Second, I evaluate the derivative of R^D-R^N at $\alpha=0$. Third, I show that the derivative is negative everywhere.

Assume parameters lie in regime A:

By $\bar{t}-\underline{t} > \alpha\frac{1}{a-b}(z_h - z_l)$, the expected revenue is given by (2.10), which can be rewritten as

$$= \left\{ \begin{array}{c} \sum_{\substack{i=1 \\ j\neq i}}^{2} \lambda(1-\lambda) \left[\begin{array}{c} \int_{\underline{t}}^{\bar{t}-\alpha\frac{z_h-z_l}{a-b}} \left(\begin{array}{c} \int\limits_{t_i+\alpha\frac{z_h-z_l}{a-b}}^{\bar{t}} \left(\begin{array}{c} (a+b)\,t_i + \\ \alpha z_h - \alpha\frac{b(z_l-z_h)}{a-b} \end{array} \right) dF(t_j) \\ + \int\limits_{\underline{t}}^{t_i+\alpha\frac{z_h-z_l}{a-b}} \left(\begin{array}{c} (a+b)\,t_j + \\ \alpha z_l - \alpha\frac{b(z_h-z_l)}{a-b} \end{array} \right) dF(t_j) \\ + \int\limits_{\underline{t}}^{\bar{t}} \left(\begin{array}{c} (a+b)\,t_j + \\ \alpha z_l - \alpha\frac{b(z_h-z_l)}{a-b} \end{array} \right) dF(t_j) \\ - \int\limits_{\underline{t}}^{\bar{t}} \left(\begin{array}{c} (a+b)\,t_j + \\ \alpha z_l - \alpha\frac{b(z_h-z_l)}{a-b} \end{array} \right) dF(t_j) \end{array} \right) dF(t_i) \\ + \int_{\bar{t}-\alpha\frac{z_h-z_l}{a-b}}^{\bar{t}} \int\limits_{\underline{t}}^{\bar{t}} \left(\begin{array}{c} (a+b)\,t_j + \\ \alpha z_l - \alpha\frac{b(z_h-z_l)}{a-b} \end{array} \right) dF(t_j)\, dF(t_i) \end{array} \right] \\ + \left(\lambda^2 + (1-\lambda)^2\right) \mathbb{E}\left[T_{2:2}\right] + \lambda^2 \alpha z_h + (1-\lambda)^2 \alpha z_l \end{array} \right\}.$$

Then the impact on the seller's expected revenue is given by

$$\begin{array}{l} R^R - R^N \\ = \left\{ \begin{array}{c} \sum_{\substack{i=1 \\ j\neq i}}^{2} \lambda(1-\lambda)(a+b) \int_{\underline{t}}^{\bar{t}-\alpha\frac{z_h-z_l}{a-b}} \left(\int\limits_{t_i+\alpha\frac{z_h-z_l}{a-b}}^{\bar{t}} \left(\begin{array}{c} (t_i - t_j) \\ +\alpha\frac{z_h-z_l}{a-b} \end{array} \right) dF(t_j) \right) dF(t_i) \\ +2\lambda(1-\lambda)\left((a+b)\,\mathbb{E}[T] + \alpha z_l - \alpha\frac{b(z_h-z_l)}{a-b} \right) \\ + [1-2\lambda(1-\lambda)]\,(a+b)\,\mathbb{E}\left[T_{2:2}\right] + \lambda^2 \alpha z_h + (1-\lambda)^2 \alpha z_l \\ - \left[(1-\lambda)\,\alpha z_l + \lambda \alpha z_h + (a+b)\mathbb{E}[T_{2:2}]\right] \end{array} \right\} \end{array}$$

which can be rewritten as

$$= \lambda(1-\lambda)(a+b)\left\{\begin{array}{c} \sum_{\substack{i=1\\ j\neq i}}^{2} \int_{\underline{t}}^{\bar{t}-\alpha\frac{z_h-z_l}{a-b}} \left(\int\limits_{t_i+\alpha\frac{z_h-z_l}{a-b}}^{\bar{t}} (t_i - t_j)\, dF(t_j) \right) dF(t_i) \\ +2\left(\mathbb{E}[T] - \mathbb{E}[T_{2:2}]\right) \\ +2\alpha\frac{z_h-z_l}{a-b} \left(\int_{\underline{t}}^{\bar{t}-\alpha\frac{z_h-z_l}{a-b}} \int\limits_{t_i+\alpha\frac{z_h-z_l}{a-b}}^{\bar{t}} dF(t_j)\, dF(t_i) - \frac{1}{2} \right) \end{array}\right\}. \tag{3.30}$$

Evaluating $R^D - R^N$ at $\alpha = 0$ gives

$$\begin{aligned} & \left|R^D - R^N\right|_{\alpha=0} \\ = \; & \lambda(1-\lambda)(a+b)\left\{\begin{array}{c} \sum_{\substack{i=1\\ j\neq i}}^{2} \int_{\underline{t}}^{\bar{t}} \int\limits_{t_i}^{\bar{t}} (t_i - t_j)\, dF(t_j)\, dF(t_i) \\ +2\left(\mathbb{E}[T] - \mathbb{E}[T_{2:2}]\right) \end{array}\right\}. \end{aligned}$$

With some tedious calculation one can show that

$$\sum_{\substack{i=1\\ j\neq i}}^{2} \int_{\underline{t}}^{\bar{t}} \int\limits_{t_i}^{\bar{t}} (t_i - t_j)\, dF(t_j)\, dF(t_i) = \left(\mathbb{E}[T_{2:2}] - \mathbb{E}[T_{1:1}]\right).$$

Therefore

$$\begin{aligned} & \left|R^D - R^N\right|_{\alpha=0} \\ = \; & \lambda(1-\lambda)(a+b)\left\{2\mathbb{E}[T] - \mathbb{E}[T_{2:2}] - \mathbb{E}[T_{1:1}]\right\} \end{aligned}$$

which is equal to 0 since $-\mathbb{E}[T_{2:2}] = -2\mathbb{E}[T] + \mathbb{E}[T_{1:1}]$.

Next, I show that the derivative of $R^D - R^N$ at $\alpha = 0$ is 0.

The derivative of $R^D - R^N$ is equal to

$$\begin{aligned}
&\frac{\partial R^D - R^N}{\partial \alpha} \\
= \;& \lambda(1-\lambda)(a+b)\frac{\partial\left\{\begin{array}{c}\sum_{\substack{i=1\\ j\neq i}}^{2}\int_{\underline{t}}^{\bar{t}-\alpha\frac{z_h-z_l}{a-b}}\left(\int\limits_{t_i+\alpha\frac{z_h-z_l}{a-b}}^{\bar{t}}(t_i-t_j)\,dF(t_j)\right)dF(t_i)\\ +2\left(\mathbb{E}[T]-\mathbb{E}[T_{2:2}]\right)\\ +2\alpha\frac{z_h-z_l}{a-b}\left(\int_{\underline{t}}^{\bar{t}-\alpha\frac{z_h-z_l}{a-b}}\int\limits_{t_i+\alpha\frac{z_h-z_l}{a-b}}^{\bar{t}}dF(t_j)\,dF(t_i)-\frac{1}{2}\right)\end{array}\right\}}{\partial\alpha}.
\end{aligned}$$

Since $\lambda(1-\lambda)(a+b) > 0$, by assumption, it suffices to derive $\frac{\frac{\partial R^D - R^N}{\partial\alpha}}{\lambda(1-\lambda)(a+b)}$, which simplifies to

$$\begin{aligned}
&\frac{\frac{\partial R^D - R^N}{\partial\alpha}}{\lambda(1-\lambda)(a+b)} \\
=\;& \left\{\begin{array}{c}\sum_{\substack{i=1\\ j\neq i}}^{2}\left(\begin{array}{c}-\frac{z_h-z_l}{a-b}\int_{\underline{t}}^{\bar{t}-\alpha\frac{z_h-z_l}{a-b}}\left(t_i f\left(t_i+\alpha\frac{z_h-z_l}{a-b}\right)f(t_i)\right)dt_i\\ +\frac{z_h-z_l}{a-b}\int_{\underline{t}}^{\bar{t}-\alpha\frac{z_h-z_l}{a-b}}\left(\begin{array}{c}\left(t_i+\alpha\frac{z_h-z_l}{a-b}\right)\\ \cdot f\left(t_i+\alpha\frac{z_h-z_l}{a-b}\right)f(t_i)\end{array}\right)dt_i\end{array}\right)\\ +2\frac{(z_h-z_l)}{a-b}\left(\begin{array}{c}\alpha\left(\begin{array}{c}-\frac{z_h-z_l}{a-b}F\left(\bar{t}-\alpha\frac{z_h-z_l}{a-b}\right)f\left(\bar{t}\right)\\ +\int_{\underline{t}}^{\bar{t}-\alpha\frac{z_h-z_l}{a-b}}F(t_i)f'\left(t_i+\alpha\frac{z_h-z_l}{a-b}\right)\frac{z_h-z_l}{a-b}dt_i\end{array}\right)\\ +\int_{\underline{t}}^{\bar{t}-\alpha\frac{z_h-z_l}{a-b}}F(t_i)f\left(t_i+\alpha\frac{z_h-z_l}{a-b}\right)dt_i-\frac{1}{2}\end{array}\right)\end{array}\right\},
\end{aligned}$$

which can be simplified to

$$2\frac{z_h-z_l}{a-b}\left(\int_{\underline{t}}^{\bar{t}-\alpha\frac{z_h-z_l}{a-b}}F(t_i)f\left(t_i+\alpha\frac{z_h-z_l}{a-b}\right)dt_i-\frac{1}{2}\right).$$

Clearly, the first derivative is 0 at $\alpha = 0$, since $\left.\int_{\underline{t}}^{\bar{t}-\alpha\frac{z_h-z_l}{a-b}}F(t_i)f\left(t_i+\alpha\frac{z_h-z_l}{a-b}\right)dt_i\right|_{\alpha=0} = \frac{1}{2}\int_{\underline{t}}^{\bar{t}}2F(t_i)f(t_i)\,dt_i = \frac{1}{2}$. This implies that there is an extreme point or a saddle point at

$\alpha = 0$. Moreover if $f\left(\bar{t}\right) > 0$, then

$$\begin{aligned}
&\frac{\partial \int_{\underline{t}}^{\bar{t}-\alpha\frac{z_h-z_l}{a-b}} F\left(t_i\right) f\left(t_i+\alpha\frac{z_h-z_l}{a-b}\right) dt_i}{\partial\alpha} \\
= &\left\{ \begin{array}{c} -\frac{z_h-z_l}{a-b} F\left(\bar{t}-\alpha\frac{z_h-z_l}{a-b}\right) f\left(\bar{t}\right) \\ +\frac{z_h-z_l}{a-b}\int_{\underline{t}}^{\bar{t}-\alpha\frac{z_h-z_l}{a-b}} F\left(t_i\right) f'\left(t_i+\alpha\frac{z_h-z_l}{a-b}\right) dt_i \end{array} \right\} \\
= &-\frac{z_h-z_l}{a-b}\int_{\underline{t}}^{\bar{t}-\alpha\frac{z_h-z_l}{a-b}} f\left(t_i\right) f\left(t_i+\alpha\frac{z_h-z_l}{a-b}\right) dt_i.
\end{aligned}$$

This is negative since $a-b>0$.

The second derivative, $\frac{\partial^2\left(R^D-R^N\right)}{\left(\partial\alpha\right)^2}$, is negative iff $\frac{\frac{\partial^2\left(R^D-R^N\right)}{\left(\partial\alpha\right)^2}}{\lambda(1-\lambda)(a+b)}$ is negative.

$$\begin{aligned}
&\frac{\frac{\partial^2\left(R^D-R^N\right)}{\left(\partial\alpha\right)^2}}{\lambda\left(1-\lambda\right)\left(a+b\right)} \\
= &\ 2\frac{z_h-z_l}{a-b}\left(\frac{\partial \int_{\underline{t}}^{\bar{t}-\alpha\frac{z_h-z_l}{a-b}} F\left(t_i\right) f\left(t_i+\alpha\frac{z_h-z_l}{a-b}\right) dt_i}{\partial\alpha}\right).
\end{aligned}$$

Substitution gives

$$= -2\left(\frac{z_h-z_l}{a-b}\right)^2\int_{\underline{t}}^{\bar{t}-\alpha\frac{z_h-z_l}{a-b}} f\left(t_i\right) f\left(t_i+\alpha\frac{z_h-z_l}{a-b}\right) dt_i,$$

which is negative for all $\alpha \geq 0$.

It follows that the expected revenue is maximized at $\alpha = 0$.

Assume parameters lie in regime B: $(a-b)\frac{\left(\bar{t}-\underline{t}\right)}{\left(z_h-z_l\right)} < \alpha$.

If $(a-b)\frac{\left(\bar{t}-\underline{t}\right)}{\left(z_h-z_l\right)} < \alpha$, then the strong bidder always wins if $z_1 \neq z_2$ and the expected price conditional on $z_1 \neq z_2$ is equal to

$$(a+b)\,\mathbb{E}\left[T\right] + \alpha z_l - \alpha\frac{b\left(z_h-z_l\right)}{a-b}.$$

This implies that the bid of the losing bidder is much lower than $v(t_i, t_i, z_l)$. The expected revenue is equal to

$$\left\{ \begin{array}{c} (a+b)\,\mathbb{E}[T_{2:2}] + \alpha\mathbb{E}[Z] \\ +2\lambda(1-\lambda)\left((a+b)\,\mathbb{E}[T] + \alpha z_l - \alpha\frac{b(z_h-z_l)}{a-b} - ((a+b)\,\mathbb{E}[T_{2:2}] + \alpha\mathbb{E}[Z])\right) \end{array} \right\}.$$

The expected revenue from disclosure is larger than the expected revenue from no disclosure if and only if

$$(a+b)\,\mathbb{E}[T] + \alpha z_l - \alpha\frac{b(z_h-z_l)}{a-b} > (a+b)\,\mathbb{E}[T_{2:2}] + \alpha\mathbb{E}[Z]$$

which can be rewritten as

$$(a+b)\frac{(\mathbb{E}[T] - \mathbb{E}[T_{2:2}])}{\left(\frac{b}{a-b} + \lambda\right)(z_h - z_l)} > \alpha.$$

Since parameters are in regime B, $\alpha > (a-b)\frac{\bar{t}-\underline{t}}{z_h-z_l}$. Together these conditions on α imply

$$(a+b)\frac{(\mathbb{E}[T] - \mathbb{E}[T_{2:2}])}{\left(\frac{b}{a-b} + \lambda\right)(z_h - z_l)} > (a-b)\frac{\bar{t}-\underline{t}}{z_h-z_l}$$

which is equivalent to

$$a\left((\mathbb{E}[T] - \mathbb{E}[T_{2:2}]) - \lambda\left(\bar{t}-\underline{t}\right)\right) > b\left((1-\lambda)\left(\bar{t}-\underline{t}\right) - (\mathbb{E}[T] - \mathbb{E}[T_{2:2}])\right).$$

Hence if $a\left((\mathbb{E}[T] - \mathbb{E}[T_{2:2}]) - \lambda\left(\bar{t}-\underline{t}\right)\right) > b\left((1-\lambda)\left(\bar{t}-\underline{t}\right) - (\mathbb{E}[T] - \mathbb{E}[T_{2:2}])\right)$, then the seller commits to disclosure. ■

Proof of Proposition 2.3.3. Step 1: Equilibrium bidding strategies when the seller conceals her information.

If the seller conceals her information, then the bidders' optimal bidding behaviors at stage 2 is characterized in Proposition 2.3.1.

Step 2: Equilibrium bidding strategies when the seller discloses her information.

Bidder 1's information set is given by $h_1^D = \{t_1, z_1, z_2\}$ and bidder 2's information set is given by $h_2^D = \{t_2, z_2, z_1\}$.

I want to show that bidder i's best reply to $\beta_j^*(t_j, z_j, z_i) = \begin{cases} (a+b)\,t_j + \alpha z_i & \text{if } z_i = z_j \\ a\underline{t} + bt_j + \alpha z_h & \text{if } z_i > z_j \\ a\bar{t} + bt_j + \alpha z_h & \text{if } z_i < z_j \end{cases}$ is $\beta_i(t_i, z_i, z_j) = \begin{cases} (a+b)\,t_i + \alpha z_i & \text{if } z_i = z_j \\ a\underline{t} + bt_i + \alpha z_h & \text{if } z_i < z_j \\ a\bar{t} + bt_i + \alpha z_h & \text{if } z_i > z_j \end{cases}$. I split up the proof in two cases: $z_1 = z_2$ and $z_1 \neq z_2$.

Step 2.1: If $z_1 = z_2 = z$, then I can apply Lemma 3.26 to solve the bidder's maximization problem for $\mathbb{E}[Z_1 | t_1, z, z] = \mathbb{E}[Z_2 | t_2, z, z] = z$. Then the equilibrium bids satisfy

$$\beta_i^{D*}(t_i, z, z) = (a+b)\,t_i + z, i \in \{1, 2\}, z \in \{z_l, z_h\}.$$

If $z_1 = z_2 = z$, then the revenue is equal to

$$(a+b)\,\mathbb{E}[T_{2:2}] + \alpha z.$$

Step 2.2: If $z_1 \neq z_2$, then, without loss of generality, I can restrict attention to $z_1 > z_2$. The proof for $z_1 < z_2$ works analogously and is omitted. I restrict attention to equilibria in which one bidder wins with positive probability.

Consider the candidates

$$\begin{aligned} \beta_1^*(t_1, z_h, z_l) &= a\bar{t} + bt_1 + \alpha z_h, \\ \beta_2^*(t_2, z_l, z_h) &= a\underline{t} + bt_2 + \alpha z_h. \end{aligned}$$

Note that bidder 1 wins with probability 1 if the bidders play these strategies.

Bidder 1's expected payoff is positive which one can check by substitution of the candidates into bidder 1's payoff, that is

$$at_1 + bt_2 + \alpha z_h - (a\underline{t} + bt_2 + \alpha z_h) > 0.$$

Bidder 2's payoff is zero, but would be negative if bidder 2 tried to outbid bidder 1 by bidding $\hat{\beta}_2 > \beta_1^*(t_1, z_h, z_l)$ for some t_1:

$$\max_{\hat{\beta}_2} \int_{\left\{t_1:\hat{\beta}_2>\beta_1^*(t_1,z_h,z_l)\right\}} \left[at_2 - a\bar{t}\right] dF(t_1) < 0$$

for all $\hat{\beta}_2 > \beta_1^*(t_1, z_h, z_l) = a\bar{t} + bt_1 + \alpha z_h$ and $t_2 < \bar{t}$. Since $a\bar{t} + bt_1 + \alpha z_h \geq a\underline{t} + bt_2 + \alpha z_h$, $\beta_2(t_2, z_l, z_h) = a\underline{t}+bt_2+\alpha z_h$ is a best reply to $\beta_1^*(t_1, z_h, z_l)$. Therefore bidder i's best reply to $\beta_j^*(t_j, z_j, z_i) = \begin{cases} (a+b)\,t_j + \alpha z_i & \text{if } z_i = z_j \\ a\underline{t} + bt_j + \alpha z_h & \text{if } z_i > z_j \\ a\bar{t} + bt_j + \alpha z_h & \text{if } z_i < z_j \end{cases}$ is $\beta_i(t_i, z_i, z_j) = \begin{cases} (a+b)\,t_i + \alpha z_i & \text{if } z_i = z_j \\ a\underline{t} + bt_i + \alpha z_h & \text{if } z_i < z_j \\ a\bar{t} + bt_i + \alpha z_h & \text{if } z_i > z_j \end{cases}$.
If $z_1 = z_h$ and $z_2 = z_l$, then the expected revenue condition on $z_1 = z_h$ and $z_2 = z_l$ is given by

$$\int_T (a\underline{t} + bt + \alpha z_h)\, f(t)\, dt = a\underline{t} + b\mathbb{E}[T] + \alpha z_h.$$

This implies together with the first step of the proof that the overall expected revenue after disclosure is given by

$$\begin{aligned} R^D &= \left\{ \begin{array}{c} \lambda^2\left((a+b)\,\mathbb{E}[T_{2:2}] + \alpha z_h\right) \\ +(1-\lambda)^2\left((a+b)\,\mathbb{E}[T_{2:2}] + \alpha z_l\right) \\ +2\lambda(1-\lambda)\left(a\underline{t} + b\mathbb{E}[T] + \alpha z_h\right) \end{array} \right\} \\ &= \left\{ \begin{array}{c} (a+b)\,\mathbb{E}[T_{2:2}] \\ +2\lambda(1-\lambda)\left(a\underline{t} + b\mathbb{E}[T] - (a+b)\,\mathbb{E}[T_{2:2}]\right) + \mathbb{E}[Z_{1:2}] \end{array} \right\}. \end{aligned}$$

■

Proof of Theorem 2.3.2. Substituting R^D from Proposition 2.3.3 and R^N into W gives

$$\left\{ \begin{array}{c} \lambda^2\left((a+b)\,\mathbb{E}[T_{2:2}] + \alpha z_h\right) + (1-\lambda)^2\left((a+b)\,\mathbb{E}[T_{2:2}] + \alpha z_l\right) \\ +2\lambda(1-\lambda)\left(a\underline{t} + b\mathbb{E}[T] + \alpha z_h\right) - \left((a+b)\,\mathbb{E}[T_{2:2}] + \alpha\mathbb{E}[Z]\right) \end{array} \right\},$$

which simplifies to

$$\lambda(1-\lambda)\left(2\left(a\underline{t} + b\mathbb{E}[T] - (a+b)\,\mathbb{E}[T_{2:2}]\right) + \alpha(z_h - z_l)\right).$$

Then $W > 0$ if and only if

$$\alpha > \frac{2\left(\left(a+b\right)\mathbb{E}\left[T_{2:2}\right] - \left(a\underline{t} + b\mathbb{E}\left[T\right]\right)\right)}{\left(z_h - z_l\right)}. \tag{3.31}$$

■

Appendix 2.B: Efficiency

Appendix 2.B is based on Maskin 2003. Consider a second price auction with two bidders. $\beta_i\left(h_i\right)$ denotes bidder i's bid for a given information set h_i.

Definition 3.7.1 *An equilibrium is efficient if the bidder with the highest valuation wins the auction and is allocated the good.*

Superscript * denotes an equilibrium bidding strategy. The following Lemma follows directly from the definition of an efficient equilibrium of the second price auction and is obvious. I state it for completeness.

Lemma 3.7.3 *Suppose that the bidders' preferences satisfy* (2.1)*. In an efficient equilibrium of the second price auction, $\beta_1^*\left(h_1\right), \beta_2^*\left(h_2\right)$, the bidder with the highest private value component wins if and only if*

$$\beta_i^*\left(h_i\right) \geq \beta_j^*\left(h_j\right) \iff \left(a-b\right)t_i + \alpha z_i \geq \left(a-b\right)t_j + \alpha z_j. \tag{3.32}$$

Proof of Lemma 3.7.3. Consider an efficient equilibrium of the English auction with equilibrium bids $\beta_1^*\left(t_1, z_1, z_2\right)$ and $\beta_2^*\left(t_2, z_2, z_1\right)$.

By definition of an efficient equilibrium, we must have

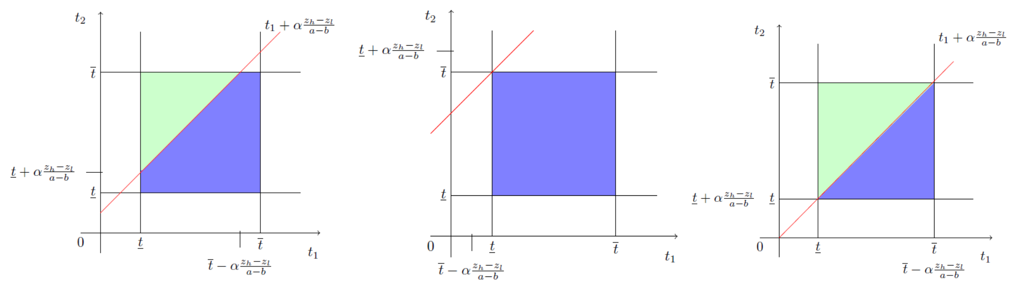
$$\beta_i^*\left(h_i\right) \geq \beta_j^*\left(h_j\right) \iff at_1 + bt_2 + \alpha z_1 \geq at_2 + bt_1 + \alpha z_2.$$

Reformulating this inequality gives

$$\begin{aligned} at_1 + bt_2 + \alpha z_1 &\geq at_2 + bt_1 + \alpha z_2 \iff \\ (a-b)\,t_1 + \alpha z_1 &\geq (a-b)\,t_2 + \alpha z_2. \end{aligned}$$

■

Next, I graphically illustrate the properties of the efficient allocation (see Lemma 3.7.3) as a function of the bidders' valuations. The figures shall illustrate realizations of the bidders' valuations as functions of (t_1, t_2) and for fixed z_1 and z_2. The support of bidder 2's private information t_2 lies in the horizontal dimension and the support of bidder 1's private information t_1 is depicted in the vertical dimension.



For regime A and $(z_1, z_2) = (z_h, z_l)$. | For regime B if $(z_1, z_2) = (z_h, z_l)$. | If (z_1, z_2) such that $z_1 = z_2$.

Figure 3.1: Figure A2.

For given z_1 and z_2, the green color in figures 1 to 3 marks those types (t_1, t_2) for which bidder 2 has the highest valuation and the blue area highlights the set of type profiles for which bidder 1 has the highest valuation. Figure 1 depicts the efficient allocation in regime A for $z_1 = z_h$ and $z_2 = z_l$. In regime A an allocation is efficient if and only if weak bidder 2 wins whenever $t_2 > t_1 + \alpha\frac{z_h - z_l}{a-b}$. Figure 2 depicts the efficient allocation in regime B for $z_1 = z_h$ and $z_2 = z_l$. In regime B the allocation is efficient if the strong bidder always receives the good. Figure 3 illustrates the efficient allocation for the symmetric case, $z_1 = z_2$, in which bidder i has the highest valuation if and only if his private signal exceeds his rival's private signal, i.e. $t_i \geq t_j$.

Appendix 3

Proof of Lemma 3.2.1. The first-best allocation $x(\theta,\eta), y(\theta,\eta)$ satisfies the system of first-order conditions (3.11) and (3.12) for $\theta \in \{\underline{\theta}, \overline{\theta}\}$ and $\eta \in \{\underline{\eta}, \overline{\eta}\}$. Define a new, artificial system of equations by

$$\begin{aligned} V_1(x(\theta,\eta), y(\theta,\eta)) &= \theta \\ V_2(x(\theta,\eta), y(\theta,\eta)) &= \eta \end{aligned} \tag{3.33}$$

for $\theta, \eta \in [\underline{\theta}, \overline{\theta}] \times [\underline{\eta}, \overline{\eta}]$. Note that the domain of the artificial system is obtained by a convexification of the original domain of definition; hence, by construction, the extreme points in the convexified domain are the cost types in the model. However, on the convexified domain, we can use calculus to determine differences between allocation choices. We prove the claims by direct evaluation of the differences. We focus on claim (i); the proof of claim (ii) uses the same methods and is therefore omitted.

Proof of claim (i): Since (3.33) is defined on a convex domain, we can write (by the fundamental theorem of calculus)

$$x(\theta,\overline{\eta}) - x(\theta,\underline{\eta}) = \int_{\underline{\eta}}^{\overline{\eta}} \frac{\partial}{\partial\eta} x(\theta,\eta)\, d\eta.$$

Totally differentiating the system (3.33), we have

$$V_{11}(x(\theta,\eta), y(\theta,\eta))\, dx + V_{12}(x(\theta,\eta), y(\theta,\eta))\, dy = 0$$

$$V_{21}\left(x\left(\theta,\eta\right),y\left(\theta,\eta\right)\right)dx+V_{22}\left(x\left(\theta,\eta\right),y\left(\theta,\eta\right)\right)dy=d\eta$$

By Cramer's rule

$$\frac{dx}{d\eta}=\frac{-V_{12}}{V_{11}V_{22}-V_{12}^{2}}$$

so

$$x\left(\theta,\overline{\eta}\right)-x\left(\theta,\underline{\eta}\right)=\int_{\underline{\eta}}^{\overline{\eta}}\frac{-V_{12}}{V_{11}V_{22}-V_{12}^{2}}\left(\theta,\eta\right)d\eta.$$

So, for any $\theta\in\left[\underline{\theta},\overline{\theta}\right]$, $x\left(\theta,\overline{\eta}\right)\leq x\left(\theta,\underline{\eta}\right)$ for $V_{12}\geq 0$ and $x\left(\theta,\overline{\eta}\right)>x\left(\theta,\underline{\eta}\right)$ for $V_{12}<0$. Thus, these inequalities hold in particular for $\theta\in\left\{\underline{\theta},\overline{\theta}\right\}$.

Again using (3.33), the fundamental theorem, and Cramer's rule, we obtain

$$y\left(\theta,\overline{\eta}\right)-y\left(\theta,\underline{\eta}\right)=\int_{\underline{\eta}}^{\overline{\eta}}\frac{V_{11}}{V_{11}V_{22}-V_{12}^{2}}\left(\theta,\eta\right)d\eta.$$

By concavity, we have $y\left(\theta,\overline{\eta}\right)-y\left(\theta,\underline{\eta}\right)<0.$

Combining these arguments, we have, for any θ,

$$\left(y\left(\theta,\underline{\eta}\right)-y\left(\theta,\overline{\eta}\right)\right)\geq\left(x\left(\theta,\underline{\eta}\right)-x\left(\theta,\overline{\eta}\right)\right)\geq 0$$

iff $V_{12}\geq 0$ and

$$0\geq\int_{\underline{\eta}}^{\overline{\eta}}\frac{V_{11}+V_{12}}{V_{11}V_{22}-V_{12}^{2}}\left(\theta,\eta\right)d\eta,$$

which is satisfied if $V_{12}<-V_{11}$ for all (x,y). Hence, these inequalities hold in particular for $\theta\in\left\{\underline{\theta},\overline{\theta}\right\}$. Likewise, we have

$$\left(x\left(\theta,\underline{\eta}\right)-x\left(\theta,\overline{\eta}\right)\right)\geq\left(y\left(\theta,\underline{\eta}\right)-y\left(\theta,\overline{\eta}\right)\right)\geq 0$$

for $\theta\in\left\{\underline{\theta},\overline{\theta}\right\}$ if $V_{12}>-V_{11}$ for all (x,y).

Proof of claim (ii): Similarly, one shows that for $V_{12} < 0$ we have

$$\left(y\left(\theta,\underline{\eta}\right) - y\left(\theta,\overline{\eta}\right)\right) \geq -\left(x\left(\theta,\underline{\eta}\right) - x\left(\theta,\overline{\eta}\right)\right) \geq 0$$

if $V_{12} \geq V_{11}$ for all (x, y) and

$$-\left(x\left(\theta,\underline{\eta}\right) - x\left(\theta,\overline{\eta}\right)\right) \geq \left(y\left(\theta,\underline{\eta}\right) - y\left(\theta,\overline{\eta}\right)\right)$$

for $V_{12} \leq V_{11}$ for all (x, y). ■

Proof of Lemma 3.3.1. Note first that at least one participation constraint must be binding; otherwise all payments could be reduced by the same amount, resulting in higher buyer surplus. To prove the statement, it suffices to show the standard result that (3.9) together with (3.6) imply (3.8). This is true if $\lambda\left(\underline{\theta}\right) \geq \lambda\left(\overline{\theta}\right)$.

Let $u(\theta, \eta)$ denote equilibrium utility.

From (3.6), we have

$$\begin{aligned}
&\mathbb{E}_{\eta|\underline{\theta}}\left[u\left(\underline{\theta},\eta\right)\right] \\
&= \mathbb{E}_{\eta|\underline{\theta}}\left[T\left(\underline{\theta},\eta\right) - \underline{\theta}x\left(\underline{\theta},\eta\right) - \eta y\left(\underline{\theta},\eta\right)\right] \\
&\geq \mathbb{E}_{\eta|\underline{\theta}}\left[T\left(\overline{\theta},\hat{\eta}^*\left(\underline{\theta},\overline{\theta},\eta\right)\right) - \underline{\theta}x\left(\overline{\theta},\hat{\eta}^*\left(\underline{\theta},\overline{\theta},\eta\right)\right) - \eta y\left(\overline{\theta},\hat{\eta}^*\left(\underline{\theta},\overline{\theta},\eta\right)\right)\right]
\end{aligned}$$

On the other hand

$$\begin{aligned}
&\mathbb{E}_{\eta|\underline{\theta}}\left[T\left(\overline{\theta},\hat{\eta}^*\left(\underline{\theta},\overline{\theta},\eta\right)\right) - \underline{\theta}x\left(\overline{\theta},\hat{\eta}^*\left(\underline{\theta},\overline{\theta},\eta\right)\right) - \eta y\left(\overline{\theta},\hat{\eta}^*\left(\underline{\theta},\overline{\theta},\eta\right)\right)\right] \\
&\geq \mathbb{E}_{\eta|\underline{\theta}}\left[T\left(\overline{\theta},\eta\right) - \underline{\theta}x\left(\overline{\theta},\eta\right) - \eta y\left(\overline{\theta},\eta\right)\right]
\end{aligned}$$

since $\hat{\eta}^*\left(\underline{\theta},\overline{\theta},\underline{\eta}\right)$ and $\hat{\eta}^*\left(\underline{\theta},\overline{\theta},\overline{\eta}\right)$ are chosen optimally. Moreover,

$$\begin{aligned}
&\mathbb{E}_{\eta|\underline{\theta}}\left[T\left(\overline{\theta},\eta\right) - \underline{\theta}x\left(\overline{\theta},\eta\right) - \eta y\left(\overline{\theta},\eta\right)\right] \\
&= \mathbb{E}_{\eta|\underline{\theta}}\left[u\left(\overline{\theta},\eta\right) + \left(\overline{\theta} - \underline{\theta}\right)x\left(\overline{\theta},\eta\right)\right] \\
&\geq \mathbb{E}_{\eta|\underline{\theta}}\left[u\left(\overline{\theta},\eta\right)\right],
\end{aligned}$$

where the last inequality follows since production is non-negative.

Hence, from (3.6) , we have that

$$\mathbb{E}_{\eta|\underline{\theta}}\left[u\left(\underline{\theta},\eta\right)\right] \geq \mathbb{E}_{\eta|\underline{\theta}}\left[u\left(\overline{\theta},\eta\right)\right].$$

Now, from (3.2) it is straightforward to see that

$$u\left(\overline{\theta},\underline{\eta}\right) \geq u\left(\overline{\theta},\overline{\eta}\right) + \left(\overline{\eta}-\underline{\eta}\right) y\left(\overline{\theta},\overline{\eta}\right),$$

and thus $u\left(\overline{\theta},\underline{\eta}\right) \geq u\left(\overline{\theta},\overline{\eta}\right)$. Using $\lambda(\underline{\theta}) \geq \lambda(\overline{\theta})$, we have moreover that

$$\mathbb{E}_{\eta|\underline{\theta}}\left[u\left(\overline{\theta},\eta\right)\right] \geq \mathbb{E}_{\eta|\overline{\theta}}\left[u\left(\overline{\theta},\eta\right)\right].$$

(3.9) written in terms of equilibrium utilities amounts to

$$\mathbb{E}_{\eta|\overline{\theta}}\left[u\left(\overline{\theta},\eta\right)\right] \geq 0,$$

which proves the claim. ■

Proof of Lemma 3.3.2. The proof is by direct inspection. We consider all four off-path types in sequence.

Recall that $u\left(\theta,\eta\right)$ denotes the equilibrium utility of type $\left(\theta,\eta\right)$.

Consider type $\left(\underline{\theta},\overline{\theta},\overline{\eta}\right)$, that is an agent with preference parameters $\underline{\theta},\overline{\eta}$ who has sent a first period report $\hat{\theta}=\overline{\theta}$. By reporting $\hat{\eta}=\overline{\eta}$, he obtains utility

$$T\left(\overline{\theta},\overline{\eta}\right) - \underline{\theta}x\left(\overline{\theta},\overline{\eta}\right) - \overline{\eta}y\left(\overline{\theta},\overline{\eta}\right) = u\left(\overline{\theta},\overline{\eta}\right) + \left(\overline{\theta}-\underline{\theta}\right)x\left(\overline{\theta},\overline{\eta}\right)$$

If he reports $\hat{\eta}=\underline{\eta}$, then he obtains utility

$$T\left(\overline{\theta},\underline{\eta}\right) - \underline{\theta}x\left(\overline{\theta},\underline{\eta}\right) - \overline{\eta}y\left(\overline{\theta},\underline{\eta}\right) = u\left(\overline{\theta},\underline{\eta}\right) + \left(\overline{\theta}-\underline{\theta}\right)x\left(\overline{\theta},\underline{\eta}\right) - \left(\overline{\eta}-\underline{\eta}\right)y\left(\overline{\theta},\underline{\eta}\right).$$

Type $\left(\underline{\theta},\overline{\theta},\overline{\eta}\right)$ prefers to report $\hat{\eta}=\overline{\eta}$ if

$$u\left(\overline{\theta},\overline{\eta}\right)+\left(\overline{\theta}-\underline{\theta}\right)x\left(\overline{\theta},\overline{\eta}\right)\geq u\left(\overline{\theta},\underline{\eta}\right)+\left(\overline{\theta}-\underline{\theta}\right)x\left(\overline{\theta},\underline{\eta}\right)-\left(\overline{\eta}-\underline{\eta}\right)y\left(\overline{\theta},\underline{\eta}\right)$$

From the on equilibrium path constraint 3.3, we know that

$$u\left(\overline{\theta},\overline{\eta}\right)\geq u\left(\overline{\theta},\underline{\eta}\right)-\left(\overline{\eta}-\underline{\eta}\right)y\left(\overline{\theta},\underline{\eta}\right).$$

adding $\left(\overline{\theta}-\underline{\theta}\right)x\left(\overline{\theta},\overline{\eta}\right)$ to both sides we get

$$u\left(\overline{\theta},\overline{\eta}\right)+\left(\overline{\theta}-\underline{\theta}\right)x\left(\overline{\theta},\overline{\eta}\right)\geq u\left(\overline{\theta},\underline{\eta}\right)+\left(\overline{\theta}-\underline{\theta}\right)x\left(\overline{\theta},\overline{\eta}\right)-\left(\overline{\eta}-\underline{\eta}\right)y\left(\overline{\theta},\underline{\eta}\right),$$

which implies that $\hat{\eta}^*\left(\underline{\theta},\overline{\theta},\overline{\eta}\right)=\overline{\eta}$ if

$$x\left(\overline{\theta},\overline{\eta}\right)\geq x\left(\overline{\theta},\underline{\eta}\right).$$

It is easy to demonstrate the other results by the exact same procedure. In particular:

$\hat{\eta}^*\left(\underline{\theta},\overline{\theta},\underline{\eta}\right)=\underline{\eta}$ follows from the on-path constraint (3.2) if $x\left(\overline{\theta},\underline{\eta}\right)\geq x\left(\overline{\theta},\overline{\eta}\right)$;

$\hat{\eta}^*\left(\overline{\theta},\underline{\theta},\underline{\eta}\right)=\underline{\eta}$ follows from the on-path constraint (3.4) if $x\left(\underline{\theta},\overline{\eta}\right)\geq x\left(\underline{\theta},\underline{\eta}\right)$; and

$\hat{\eta}^*\left(\overline{\theta},\underline{\theta},\overline{\eta}\right)=\overline{\eta}$ follows from the on-path constraint (3.5) if $x\left(\underline{\theta},\underline{\eta}\right)\geq x\left(\underline{\theta},\overline{\eta}\right)$. ■

Proof of lemma 3.3.3. We split the proof into two cases, depending on whether $x\left(\theta,\underline{\eta}\right)-x\left(\theta,\overline{\eta}\right)$ is nonnegative or nonpositive. For both cases, we first prove the part concerning the allocations of type $\overline{\theta}$. Afterwards we turn to the allocation for type $\underline{\theta}$.

Preliminaries:

For convenience, note that the on-path constraints (3.2) – (3.5) can be rewritten as follows:

$$T\left(\overline{\theta},\underline{\eta}\right)-T\left(\overline{\theta},\overline{\eta}\right)\geq\overline{\theta}\left(x\left(\overline{\theta},\underline{\eta}\right)-x\left(\overline{\theta},\overline{\eta}\right)\right)+\underline{\eta}\left(y\left(\overline{\theta},\underline{\eta}\right)-y\left(\overline{\theta},\overline{\eta}\right)\right)\tag{3.34}$$

$$T\left(\overline{\theta},\underline{\eta}\right)-T\left(\overline{\theta},\overline{\eta}\right)\leq\overline{\theta}\left(x\left(\overline{\theta},\underline{\eta}\right)-x\left(\overline{\theta},\overline{\eta}\right)\right)+\overline{\eta}\left(y\left(\overline{\theta},\underline{\eta}\right)-y\left(\overline{\theta},\overline{\eta}\right)\right)\tag{3.35}$$

$$T\left(\underline{\theta},\underline{\eta}\right)-T\left(\underline{\theta},\overline{\eta}\right)\geq\underline{\theta}\left(x\left(\underline{\theta},\underline{\eta}\right)-x\left(\underline{\theta},\overline{\eta}\right)\right)+\underline{\eta}\left(y\left(\underline{\theta},\underline{\eta}\right)-y\left(\underline{\theta},\overline{\eta}\right)\right)\tag{3.36}$$

$$T\left(\underline{\theta},\underline{\eta}\right)-T\left(\underline{\theta},\overline{\eta}\right)\leq\underline{\theta}\left(x\left(\underline{\theta},\underline{\eta}\right)-x\left(\underline{\theta},\overline{\eta}\right)\right)+\overline{\eta}\left(y\left(\underline{\theta},\underline{\eta}\right)-y\left(\underline{\theta},\overline{\eta}\right)\right) \tag{3.37}$$

Likewise, the off-path constraints take the following form:

Type $\left(\underline{\theta},\overline{\theta},\overline{\eta}\right)$ prefers to report $\hat{\eta}=\overline{\eta}$ if

$$T\left(\overline{\theta},\underline{\eta}\right)-T\left(\overline{\theta},\overline{\eta}\right)\leq\underline{\theta}\left(x\left(\overline{\theta},\underline{\eta}\right)-x\left(\overline{\theta},\overline{\eta}\right)\right)+\overline{\eta}\left(y\left(\overline{\theta},\underline{\eta}\right)-y\left(\overline{\theta},\overline{\eta}\right)\right). \tag{3.38}$$

and prefers to report $\hat{\eta}=\underline{\eta}$ if

$$T\left(\overline{\theta},\underline{\eta}\right)-T\left(\overline{\theta},\overline{\eta}\right)\geq\underline{\theta}\left(x\left(\overline{\theta},\underline{\eta}\right)-x\left(\overline{\theta},\overline{\eta}\right)\right)+\overline{\eta}\left(y\left(\overline{\theta},\underline{\eta}\right)-y\left(\overline{\theta},\overline{\eta}\right)\right). \tag{3.39}$$

Type $\left(\underline{\theta},\overline{\theta},\underline{\eta}\right)$ prefers to report $\hat{\eta}=\underline{\eta}$ if

$$T\left(\overline{\theta},\underline{\eta}\right)-T\left(\overline{\theta},\overline{\eta}\right)\geq\underline{\theta}\left(x\left(\overline{\theta},\underline{\eta}\right)-x\left(\overline{\theta},\overline{\eta}\right)\right)+\underline{\eta}\left(y\left(\overline{\theta},\underline{\eta}\right)-y\left(\overline{\theta},\overline{\eta}\right)\right) \tag{3.40}$$

and prefers to report $\hat{\eta}=\overline{\eta}$ if

$$T\left(\overline{\theta},\underline{\eta}\right)-T\left(\overline{\theta},\overline{\eta}\right)\leq\underline{\theta}\left(x\left(\overline{\theta},\underline{\eta}\right)-x\left(\overline{\theta},\overline{\eta}\right)\right)+\underline{\eta}\left(y\left(\overline{\theta},\underline{\eta}\right)-y\left(\overline{\theta},\overline{\eta}\right)\right). \tag{3.41}$$

Type $\left(\overline{\theta},\underline{\theta},\underline{\eta}\right)$ prefers to report $\hat{\eta}=\underline{\eta}$ if

$$T\left(\underline{\theta},\underline{\eta}\right)-T\left(\underline{\theta},\overline{\eta}\right)\geq\overline{\theta}\left(x\left(\underline{\theta},\underline{\eta}\right)-x\left(\underline{\theta},\overline{\eta}\right)\right)+\underline{\eta}\left(y\left(\underline{\theta},\underline{\eta}\right)-y\left(\underline{\theta},\overline{\eta}\right)\right) \tag{3.42}$$

and prefers to report $\hat{\eta}=\overline{\eta}$ if

$$T\left(\underline{\theta},\underline{\eta}\right)-T\left(\underline{\theta},\overline{\eta}\right)\leq\overline{\theta}\left(x\left(\underline{\theta},\underline{\eta}\right)-x\left(\underline{\theta},\overline{\eta}\right)\right)+\underline{\eta}\left(y\left(\underline{\theta},\underline{\eta}\right)-y\left(\underline{\theta},\overline{\eta}\right)\right). \tag{3.43}$$

Type $\left(\overline{\theta},\underline{\theta},\overline{\eta}\right)$ prefers to report $\hat{\eta}=\overline{\eta}$ if

$$T\left(\underline{\theta},\underline{\eta}\right)-T\left(\underline{\theta},\overline{\eta}\right)\leq\overline{\theta}\left(x\left(\underline{\theta},\underline{\eta}\right)-x\left(\underline{\theta},\overline{\eta}\right)\right)+\overline{\eta}\left(y\left(\underline{\theta},\underline{\eta}\right)-y\left(\underline{\theta},\overline{\eta}\right)\right) \tag{3.44}$$

and prefers to report $\hat{\eta} = \underline{\eta}$ if

$$T\left(\underline{\theta}, \underline{\eta}\right) - T\left(\underline{\theta}, \overline{\eta}\right) \geq \overline{\theta}\left(x\left(\underline{\theta}, \underline{\eta}\right) - x\left(\underline{\theta}, \overline{\eta}\right)\right) + \overline{\eta}\left(y\left(\underline{\theta}, \underline{\eta}\right) - y\left(\underline{\theta}, \overline{\eta}\right)\right). \tag{3.45}$$

Now we are ready to begin with the proof of the Lemma.

Suppose that $\left(x\left(\overline{\theta}, \underline{\eta}\right) - x\left(\overline{\theta}, \overline{\eta}\right)\right) \geq 0$. By Lemma 3.3.2 this implies that $\hat{\eta}^*\left(\underline{\theta}, \overline{\theta}, \underline{\eta}\right) = \underline{\eta}$. Adding the expected utility of the high type (which is zero by (3.9)) to the objective, we obtain the following problem:

$$\Delta \equiv \min_{\left\{T\left(\overline{\theta},\eta\right)\right\}_{\eta\in\left\{\underline{\eta},\overline{\eta}\right\}}, \hat{\eta}^*\left(\underline{\theta},\overline{\theta},\overline{\eta}\right)} \left\{ \begin{array}{c} \left(\lambda(\underline{\theta})-\lambda\left(\overline{\theta}\right)\right)\left(T\left(\overline{\theta},\underline{\eta}\right)-\overline{\theta}x\left(\overline{\theta},\underline{\eta}\right)-\underline{\eta}y\left(\overline{\theta},\underline{\eta}\right)\right)+\lambda(\underline{\theta})\left(\overline{\theta}-\underline{\theta}\right)x\left(\overline{\theta},\underline{\eta}\right) \\ +(1-\lambda(\underline{\theta}))\left(T\left(\overline{\theta},\hat{\eta}^*\left(\underline{\theta},\overline{\theta},\overline{\eta}\right)\right)-\underline{\theta}x\left(\overline{\theta},\hat{\eta}^*\left(\underline{\theta},\overline{\theta},\overline{\eta}\right)\right)-\overline{\eta}y\left(\overline{\theta},\hat{\eta}^*\left(\underline{\theta},\overline{\theta},\overline{\eta}\right)\right)\right) \\ -\left(1-\lambda\left(\overline{\theta}\right)\right)\left[T\left(\overline{\theta},\overline{\eta}\right)-\overline{\theta}x\left(\overline{\theta},\overline{\eta}\right)-\overline{\eta}y\left(\overline{\theta},\overline{\eta}\right)\right] \end{array} \right\}$$

$$\textit{subject to } (3.34), (3.35), \textit{ and}$$

$$\textit{either } (3.38) \textit{ if } \hat{\eta}^*\left(\underline{\theta}, \overline{\theta}, \overline{\eta}\right) = \overline{\eta}$$

$$\textit{or } (3.39) \textit{ if } \hat{\eta}^*\left(\underline{\theta}, \overline{\theta}, \overline{\eta}\right) = \underline{\eta}.$$

Consider now both possible off-path reports. If $\hat{\eta}^*\left(\underline{\theta}, \overline{\theta}, \overline{\eta}\right) = \overline{\eta}$, then the objective is

$$\begin{aligned} \min_{T\left(\overline{\theta},\underline{\eta}\right)-T\left(\overline{\theta},\overline{\eta}\right)} & \left(\lambda\left(\underline{\theta}\right) - \lambda\left(\overline{\theta}\right)\right)\left[T\left(\overline{\theta}, \underline{\eta}\right) - \overline{\theta}x\left(\overline{\theta}, \underline{\eta}\right) - \underline{\eta}y\left(\overline{\theta}, \underline{\eta}\right)\right] + \lambda\left(\underline{\theta}\right)\left(\overline{\theta} - \underline{\theta}\right)x\left(\overline{\theta}, \underline{\eta}\right) \\ & - \left(\lambda\left(\underline{\theta}\right) - \lambda\left(\overline{\theta}\right)\right)\left[T\left(\overline{\theta}, \overline{\eta}\right) - \overline{\theta}x\left(\overline{\theta}, \overline{\eta}\right) - \overline{\eta}y\left(\overline{\theta}, \overline{\eta}\right)\right] + \left(1 - \lambda\left(\underline{\theta}\right)\right)\left(\overline{\theta} - \underline{\theta}\right)x\left(\overline{\theta}, \overline{\eta}\right) \end{aligned}$$

subject to the constraints $(3.34), (3.35)$, and (3.38). Note that (3.35) is automatically satisfied if (3.38) is. There exists a solution to the problem only if the constraint set is non-empty, that is, if the right-hand side of (3.38) is weakly larger than the right-hand side of (3.34). This is the case for $\left\{x\left(\overline{\theta}, \eta\right), y\left(\overline{\theta}, \eta\right)\right\}_{\eta\in\left\{\underline{\eta},\overline{\eta}\right\}} \in \mathbb{X}_i\left(\overline{\theta}\right)$. In this case (3.34) is binding. Using (3.9) and (3.34) to solve for the optimal payments, we have

$$\begin{pmatrix} T\left(\overline{\theta}, \underline{\eta}\right) \\ T\left(\overline{\theta}, \overline{\eta}\right) \end{pmatrix} = \begin{pmatrix} \overline{\theta}x\left(\overline{\theta}, \underline{\eta}\right) + \underline{\eta}y\left(\overline{\theta}, \underline{\eta}\right) + \left(1 - \lambda\left(\overline{\theta}\right)\right)\left(\overline{\eta} - \underline{\eta}\right)y\left(\overline{\theta}, \overline{\eta}\right) \\ \overline{\theta}x\left(\overline{\theta}, \overline{\eta}\right) + \lambda\left(\overline{\theta}\right)\underline{\eta}y\left(\overline{\theta}, \overline{\eta}\right) + \left(1 - \lambda\left(\overline{\theta}\right)\right)\overline{\eta}y\left(\overline{\theta}, \overline{\eta}\right) \end{pmatrix}. \tag{3.46}$$

Substituting back into the objective we have obtain

$$\Delta_i = \left(\lambda\left(\underline{\theta}\right) - \lambda\left(\overline{\theta}\right)\right)\left(\overline{\eta} - \underline{\eta}\right) y\left(\overline{\theta}, \overline{\eta}\right) + \mathbb{E}_{\eta|\underline{\theta}}\left[\left(\overline{\theta} - \underline{\theta}\right) x\left(\overline{\theta}, \eta\right)\right].$$

On the other hand, if $\hat{\eta}^*\left(\underline{\theta}, \overline{\theta}, \overline{\eta}\right) = \underline{\eta}$, then the problem is

$$\begin{gathered}\min_{T\left(\overline{\theta},\underline{\eta}\right)-T\left(\overline{\theta},\overline{\eta}\right)} \left(1-\lambda\left(\overline{\theta}\right)\right)\left\{T\left(\overline{\theta},\underline{\eta}\right) - T\left(\overline{\theta},\overline{\eta}\right)\right\} \\ + \left(\overline{\theta}-\underline{\theta}\right) x\left(\overline{\theta},\underline{\eta}\right) - \left(1-\lambda\left(\underline{\theta}\right)\right)\left(\overline{\eta}-\underline{\eta}\right) y\left(\overline{\theta},\underline{\eta}\right) \\ -\left(1-\lambda\left(\overline{\theta}\right)\right)\left(\overline{\theta}\left(x\left(\overline{\theta},\underline{\eta}\right) - x\left(\overline{\theta},\overline{\eta}\right)\right) + \underline{\eta} y\left(\overline{\theta},\underline{\eta}\right) - \overline{\eta} y\left(\overline{\theta},\overline{\eta}\right)\right)\end{gathered}$$

subject to the constraints (3.34), (3.35), and (3.39). The right-hand side of (3.34) is weakly larger than the right-hand side of (3.39) for $\left\{x\left(\overline{\theta}, \eta\right), y\left(\overline{\theta}, \eta\right)\right\}_{\eta \in \left\{\underline{\eta}, \overline{\eta}\right\}} \in \mathbb{X}_{iii}\left(\overline{\theta}\right)$ and the reverse is true for $\left\{x\left(\overline{\theta}, \eta\right), y\left(\overline{\theta}, \eta\right)\right\}_{\eta \in \left\{\underline{\eta}, \overline{\eta}\right\}} \in \mathbb{X}_{i}\left(\overline{\theta}\right)$. Clearly, the right-hand side of (3.35) is always larger than the right-hand side of (3.34). Therefore, for $\left\{x\left(\overline{\theta}, \eta\right), y\left(\overline{\theta}, \eta\right)\right\}_{\eta \in \left\{\underline{\eta}, \overline{\eta}\right\}} \in \mathbb{X}_{iii}\left(\overline{\theta}\right)$, at the solution of the problem, constraint (3.34) holds as an equality. It follows that for $\hat{\eta}^*\left(\underline{\theta}, \overline{\theta}, \overline{\eta}\right) = \underline{\eta}$ and $\left\{x\left(\overline{\theta}, \eta\right), y\left(\overline{\theta}, \eta\right)\right\}_{\eta \in \left\{\underline{\eta}, \overline{\eta}\right\}} \in \mathbb{X}_{iii}\left(\overline{\theta}\right)$, the transfers can be taken from (3.46) so that the objective takes value

$$\Delta_{iii} = \left(\overline{\theta} - \underline{\theta}\right) x\left(\overline{\theta}, \underline{\eta}\right) - \left(1 - \lambda\left(\underline{\theta}\right)\right)\left(\overline{\eta} - \underline{\eta}\right) y\left(\overline{\theta}, \underline{\eta}\right) + \left(1 - \lambda\left(\overline{\theta}\right)\right)\left(\overline{\eta} - \underline{\eta}\right) y\left(\overline{\theta}, \overline{\eta}\right).$$

For $\left\{x\left(\overline{\theta}, \eta\right), y\left(\overline{\theta}, \eta\right)\right\}_{\eta \in \left\{\underline{\eta}, \overline{\eta}\right\}} \in \mathbb{X}_{i}\left(\overline{\theta}\right)$ the off-path constraint (3.39) is binding. Substituting for the transfers implies that in this case

$$\hat{\Delta}_i = \left(\lambda\left(\underline{\theta}\right) - \lambda\left(\overline{\theta}\right)\right)\left(\overline{\eta} - \underline{\eta}\right) y\left(\overline{\theta}, \underline{\eta}\right) + \mathbb{E}_{\eta|\overline{\theta}}\left[\left(\overline{\theta} - \underline{\theta}\right) x\left(\overline{\theta}, \eta\right)\right].$$

Since $\hat{\Delta}_i \geq \Delta_i$ if and only if $\left\{x\left(\overline{\theta}, \eta\right), y\left(\overline{\theta}, \eta\right)\right\}_{\eta \in \left\{\underline{\eta}, \overline{\eta}\right\}} \in \mathbb{X}_{i}\left(\overline{\theta}\right)$, for a given allocation $\left\{x\left(\overline{\theta}, \eta\right), y\left(\overline{\theta}, \eta\right)\right\}_{\eta \in \left\{\underline{\eta}, \overline{\eta}\right\}} \in \mathbb{X}_{i}\left(\overline{\theta}\right)$ the optimal payments are given by (3.46), $\hat{\eta}^*\left(\underline{\theta}, \overline{\theta}, \overline{\eta}\right) = \overline{\eta}$ and the information rent by Δ_i.

Next consider the second problem for the case where $x\left(\underline{\theta}, \underline{\eta}\right) \geq x\left(\underline{\theta}, \overline{\eta}\right)$. By lemma 2 this implies that $\hat{\eta}^*\left(\overline{\theta}, \underline{\theta}, \overline{\eta}\right) = \overline{\eta}$. So, the problem can be written as

$$\Omega = \min_{\{T(\underline{\theta},\eta)\}_{\eta\in\{\underline{\eta},\overline{\eta}\}},\hat{\eta}^*\left(\overline{\theta},\underline{\theta},\underline{\eta}\right)} \left\{ \begin{array}{c} \lambda\left(\overline{\theta}\right)\left[T\left(\underline{\theta},\hat{\eta}^*\left(\overline{\theta},\underline{\theta},\eta\right)\right)-\overline{\theta}x\left(\underline{\theta},\hat{\eta}^*\left(\overline{\theta},\underline{\theta},\eta\right)\right)-\underline{\eta}y\left(\underline{\theta},\hat{\eta}^*\left(\overline{\theta},\underline{\theta},\eta\right)\right)\right] \\ +\left(1-\lambda\left(\overline{\theta}\right)\right)\left[T(\underline{\theta},\overline{\eta})-\overline{\theta}x(\underline{\theta},\overline{\eta})-\overline{\eta}y(\underline{\theta},\overline{\eta})\right] \\ -\mathbb{E}_{\eta|\underline{\theta}}[T(\underline{\theta},\eta)-\underline{\theta}x(\underline{\theta},\eta)-\eta y(\underline{\theta},\eta)]+\Delta \end{array} \right\}$$

$$\text{subject to}$$

$$(3.36)\,,(3.37)\,,\ \text{and either}$$

$$(3.42)\ \text{ if } \hat{\eta}^*\left(\overline{\theta}, \underline{\theta}, \underline{\eta}\right) = \underline{\eta},\ \text{ or}$$

$$(3.43)\ \text{ if } \hat{\eta}^*\left(\overline{\theta}, \underline{\theta}, \underline{\eta}\right) = \overline{\eta},$$

where the objective is obtained from substituting the constraint (3.6) as an equality into the objective.

Consider first the case where $\hat{\eta}^*\left(\overline{\theta}, \underline{\theta}, \underline{\eta}\right) = \underline{\eta}$. In this case, the problem is

$$\begin{aligned} &\min_{T\left(\underline{\theta},\underline{\eta}\right)-T(\underline{\theta},\overline{\eta})} -\left(\lambda\left(\underline{\theta}\right)-\lambda\left(\overline{\theta}\right)\right)\left\{T\left(\underline{\theta},\underline{\eta}\right)-T\left(\underline{\theta},\overline{\eta}\right)\right\} \\ &+\left(\lambda\left(\underline{\theta}\right)-\lambda\left(\overline{\theta}\right)\right)\left[\overline{\theta}x\left(\underline{\theta},\underline{\eta}\right)+\underline{\eta}y\left(\underline{\theta},\underline{\eta}\right)-\underline{\theta}x\left(\underline{\theta},\overline{\eta}\right)-\overline{\eta}y\left(\underline{\theta},\overline{\eta}\right)\right] \\ &-\left(1-\lambda\left(\overline{\theta}\right)\right)\left(\overline{\theta}-\underline{\theta}\right)x\left(\underline{\theta},\overline{\eta}\right)-\lambda\left(\underline{\theta}\right)\left(\overline{\theta}-\underline{\theta}\right)x\left(\underline{\theta},\underline{\eta}\right)+\Delta. \end{aligned}$$

subject to the constraints $(3.36)\,,(3.37)$ and $(3.42)\,.$ The right-hand side of (3.36) is weakly smaller than the right-hand side of $(3.42)\,.$ Hence, the constraint set is nonempty if the right hand side of (3.42) is weakly smaller than the right-hand side of $(3.37)\,,$ which is exactly true for $\{x\left(\underline{\theta},\eta\right), y\left(\underline{\theta},\eta\right)\}_{\eta\in\left\{\underline{\eta},\overline{\eta}\right\}} \in \mathbb{X}_i\left(\underline{\theta}\right).$ So, in this case, (3.37) is binding at the solution to the problem. Solving for the transfers from (3.37) and $(3.6)\,,$ we obtain

$$\begin{pmatrix} T\left(\underline{\theta},\underline{\eta}\right) \\ T\left(\underline{\theta},\overline{\eta}\right) \end{pmatrix} = \begin{pmatrix} \Delta+\underline{\theta}x\left(\underline{\theta},\underline{\eta}\right)+\left(\lambda\left(\underline{\theta}\right)\underline{\eta}+\left(1-\lambda\left(\underline{\theta}\right)\right)\overline{\eta}\right)y\left(\underline{\theta},\underline{\eta}\right) \\ \Delta+\underline{\theta}x\left(\underline{\theta},\overline{\eta}\right)+\overline{\eta}y\left(\underline{\theta},\overline{\eta}\right)-\lambda\left(\underline{\theta}\right)\left(\overline{\eta}-\underline{\eta}\right)y\left(\underline{\theta},\underline{\eta}\right) \end{pmatrix}. \tag{3.47}$$

Substituting these transfers back into the objective, we obtain

$$\Omega_i = -\lambda\left(\overline{\theta}\right)\left(\overline{\theta}-\underline{\theta}\right)x\left(\underline{\theta},\underline{\eta}\right)-\left(\lambda\left(\underline{\theta}\right)-\lambda\left(\overline{\theta}\right)\right)\left(\overline{\eta}-\underline{\eta}\right)y\left(\underline{\theta},\underline{\eta}\right)-\left(1-\lambda\left(\overline{\theta}\right)\right)\left(\overline{\theta}-\underline{\theta}\right)x\left(\underline{\theta},\overline{\eta}\right)+\Delta.$$

For future reference, we note that for $(x,y) \in \mathbb{X}_i$, this can be written as

$$\Omega_i = \mathbb{E}_{\eta|\underline{\theta}}\left[\left(\overline{\theta}-\underline{\theta}\right)x\left(\overline{\theta},\eta\right)\right] - \mathbb{E}_{\eta|\overline{\theta}}\left[\left(\overline{\theta}-\underline{\theta}\right)x\left(\underline{\theta},\eta\right)\right] - \left(\lambda\left(\underline{\theta}\right) - \lambda\left(\overline{\theta}\right)\right)\left(\overline{\eta}-\underline{\eta}\right)\left(y\left(\underline{\theta},\underline{\eta}\right) - y\left(\overline{\theta},\overline{\eta}\right)\right).$$

Consider next the case where $\hat{\eta}^*\left(\overline{\theta},\underline{\theta},\underline{\eta}\right) = \overline{\eta}$. In this case, the problem becomes

$$\begin{aligned}
&\min_{T\left(\underline{\theta},\underline{\eta}\right)-T\left(\underline{\theta},\overline{\eta}\right)} -\lambda\left(\underline{\theta}\right)\left\{T\left(\underline{\theta},\underline{\eta}\right) - T\left(\underline{\theta},\overline{\eta}\right)\right\} \\
&-\left(\overline{\theta}-\underline{\theta}\right)x\left(\underline{\theta},\overline{\eta}\right) + \lambda\left(\overline{\theta}\right)\left(\overline{\eta}-\underline{\eta}\right)y\left(\underline{\theta},\overline{\eta}\right) \\
&+\lambda\left(\underline{\theta}\right)\left[\underline{\theta}x\left(\underline{\theta},\underline{\eta}\right) - \underline{\theta}x\left(\underline{\theta},\overline{\eta}\right) + \underline{\eta}y\left(\underline{\theta},\underline{\eta}\right) - \overline{\eta}y\left(\underline{\theta},\overline{\eta}\right)\right] + \Delta.
\end{aligned}$$

subject to the constraints $(3.36), (3.37)$ and (3.43). The right-hand side of (3.43) is larger than the right-hand side of (3.37) for $\{x\left(\underline{\theta},\eta\right), y\left(\underline{\theta},\eta\right)\}_{\eta\in\left\{\underline{\eta},\overline{\eta}\right\}} \in \mathbb{X}_{iii}\left(\underline{\theta}\right)$. In this case, the feasible set is nonempty and at the solution (3.37) is binding; hence the transfers are given by (3.47) and the objective takes value

$$\begin{aligned}
\Omega_{iii} = &-\lambda\left(\underline{\theta}\right)\left(\overline{\eta}-\underline{\eta}\right)y\left(\underline{\theta},\underline{\eta}\right) - \left(\overline{\theta}-\underline{\theta}\right)x\left(\underline{\theta},\overline{\eta}\right) + \lambda\left(\overline{\theta}\right)\left(\overline{\eta}-\underline{\eta}\right)y\left(\underline{\theta},\overline{\eta}\right) \\
&+\lambda\left(\underline{\theta}\right)\underline{\theta}\left(x\left(\underline{\theta},\underline{\eta}\right) - x\left(\underline{\theta},\overline{\eta}\right)\right) + \Delta.
\end{aligned}$$

Again, for future reference, if $(x,y) \in \mathbb{X}_{iii}$, then we can write

$$\begin{aligned}
\Omega_{iii} = &-\lambda\left(\underline{\theta}\right)\left(\overline{\eta}-\underline{\eta}\right)y\left(\underline{\theta},\underline{\eta}\right) + \lambda\left(\overline{\theta}\right)\left(\overline{\eta}-\underline{\eta}\right)y\left(\underline{\theta},\overline{\eta}\right) \\
&+\lambda\left(\underline{\theta}\right)\underline{\theta}\left(x\left(\underline{\theta},\underline{\eta}\right) - x\left(\underline{\theta},\overline{\eta}\right)\right) - \left(1-\lambda\left(\underline{\theta}\right)\right)\left(\overline{\eta}-\underline{\eta}\right)y\left(\overline{\theta},\underline{\eta}\right) + \left(1-\lambda\left(\overline{\theta}\right)\right)\left(\overline{\eta}-\underline{\eta}\right)y\left(\overline{\theta},\overline{\eta}\right).
\end{aligned}$$

For $\{x\left(\underline{\theta},\eta\right), y\left(\underline{\theta},\eta\right)\}_{\eta\in\left\{\underline{\eta},\overline{\eta}\right\}} \in \mathbb{X}_i\left(\underline{\theta}\right)$, the right-hand side of (3.43) is smaller than the right-hand side of (3.37). Moreover, the feasible set is always nonempty and thus at the solution constraint (3.43) is binding. Hence, we can substitute

$$T\left(\underline{\theta},\underline{\eta}\right) - T\left(\underline{\theta},\overline{\eta}\right) = \overline{\theta}\left(x\left(\underline{\theta},\underline{\eta}\right) - x\left(\underline{\theta},\overline{\eta}\right)\right) + \underline{\eta}\left(y\left(\underline{\theta},\underline{\eta}\right) - y\left(\underline{\theta},\overline{\eta}\right)\right)$$

into the objective and obtain

$$\begin{aligned}\hat{\Omega}_i = & -\lambda\left(\underline{\theta}\right)\left\{\left(\overline{\theta}-\underline{\theta}\right)\left(x\left(\underline{\theta},\underline{\eta}\right)-x\left(\underline{\theta},\overline{\eta}\right)\right)+\left(\overline{\eta}-\underline{\eta}\right)y\left(\underline{\theta},\overline{\eta}\right)\right\} \\ & -\left(\overline{\theta}-\underline{\theta}\right)x\left(\underline{\theta},\overline{\eta}\right)+\lambda\left(\overline{\theta}\right)\left(\overline{\eta}-\underline{\eta}\right)y\left(\underline{\theta},\overline{\eta}\right)+\Delta.\end{aligned}$$

We have $\Omega_i \leq \hat{\Omega}_i$ for $\left\{x\left(\underline{\theta},\eta\right),y\left(\underline{\theta},\eta\right)\right\}_{\eta\in\left\{\underline{\eta},\overline{\eta}\right\}} \in \mathbb{X}_i\left(\underline{\theta}\right)$, so $\hat{\eta}^*\left(\overline{\theta},\underline{\theta},\underline{\eta}\right) = \underline{\eta}$ is cheaper to implement in that case.

Next consider the case where $\left(x\left(\overline{\theta},\underline{\eta}\right)-x\left(\overline{\theta},\overline{\eta}\right)\right) \leq 0$. By Lemma 3.3.2, this implies that $\hat{\eta}^*\left(\underline{\theta},\overline{\theta},\overline{\eta}\right) = \overline{\eta}$. Adding and subtracting the expected utility of type $\overline{\theta}$, we can write the objective as

$$\Delta \equiv \min_{\left\{T\left(\overline{\theta},\eta\right)\right\}_{\eta\in\left\{\underline{\eta},\overline{\eta}\right\}},\hat{\eta}^*\left(\underline{\theta},\overline{\theta},\underline{\eta}\right)} \left\{\begin{array}{c} \lambda(\underline{\theta})\left(T\left(\overline{\theta},\hat{\eta}^*\left(\underline{\theta},\overline{\theta},\underline{\eta}\right)\right)-\underline{\theta}x\left(\overline{\theta},\hat{\eta}^*\left(\underline{\theta},\overline{\theta},\underline{\eta}\right)\right)-\underline{\eta}y\left(\overline{\theta},\hat{\eta}^*\left(\underline{\theta},\overline{\theta},\underline{\eta}\right)\right)\right) \\ -\left(\lambda(\underline{\theta})-\lambda\left(\overline{\theta}\right)\right)\left(T\left(\overline{\theta},\overline{\eta}\right)-\overline{\theta}x\left(\overline{\theta},\overline{\eta}\right)-\overline{\eta}y\left(\overline{\theta},\overline{\eta}\right)\right)+(1-\lambda(\underline{\theta}))\left(\overline{\theta}-\underline{\theta}\right)x\left(\overline{\theta},\overline{\eta}\right) \\ -\lambda\left(\overline{\theta}\right)\left[T\left(\overline{\theta},\underline{\eta}\right)-\overline{\theta}x\left(\overline{\theta},\underline{\eta}\right)-\underline{\eta}y\left(\overline{\theta},\underline{\eta}\right)\right] \end{array}\right\}$$

$$\textit{subject to}$$

$$(3.34)\,,(3.35)\ \text{ and either}$$

$$(3.40)\ \text{ if } \hat{\eta}^*\left(\underline{\theta},\overline{\theta},\underline{\eta}\right)=\underline{\eta},\ \text{ or}$$

$$(3.41)\ \text{ if } \hat{\eta}^*\left(\underline{\theta},\overline{\theta},\underline{\eta}\right)=\overline{\eta}.$$

Consider first the case where the off-path report is $\hat{\eta}^*\left(\underline{\theta},\overline{\theta},\underline{\eta}\right) = \underline{\eta}$. In this case, the objective is

$$\Delta \equiv \min_{T\left(\overline{\theta},\underline{\eta}\right)-T\left(\overline{\theta},\overline{\eta}\right)} \left\{\begin{array}{c} \left(\lambda(\underline{\theta})-\lambda\left(\overline{\theta}\right)\right)\left(T\left(\overline{\theta},\underline{\eta}\right)-\overline{\theta}x\left(\overline{\theta},\underline{\eta}\right)-\underline{\eta}y\left(\overline{\theta},\underline{\eta}\right)\right)+\lambda(\underline{\theta})\left(\overline{\theta}-\underline{\theta}\right)x\left(\overline{\theta},\underline{\eta}\right) \\ -\left(\lambda(\underline{\theta})-\lambda\left(\overline{\theta}\right)\right)\left(T\left(\overline{\theta},\overline{\eta}\right)-\overline{\theta}x\left(\overline{\theta},\overline{\eta}\right)-\overline{\eta}y\left(\overline{\theta},\overline{\eta}\right)\right)+(1-\lambda(\underline{\theta}))\left(\overline{\theta}-\underline{\theta}\right)x\left(\overline{\theta},\overline{\eta}\right) \end{array}\right\}$$

subject to the constraints $(3.34)\,,(3.35)\,,$ and $(3.40)\,.$ The right-hand side of (3.40) is always at least as large as the right-hand side of (3.34) (by the fact that $\left(x\left(\overline{\theta},\underline{\eta}\right)-x\left(\overline{\theta},\overline{\eta}\right)\right) \leq 0$). Hence, the constraint set is nonempty if the right-hand side of (3.35) is at least as large as the right-hand side of $(3.40)\,,$ which is precisely the case for $\left\{x\left(\overline{\theta},\eta\right),y\left(\overline{\theta},\eta\right)\right\}_{\eta\in\left\{\underline{\eta},\overline{\eta}\right\}} \in \mathbb{X}_{ii}\left(\overline{\theta}\right)$. Since the objective is increasing in $T\left(\overline{\theta},\underline{\eta}\right)-T\left(\overline{\theta},\overline{\eta}\right)$ and we are minimizing Δ, $T\left(\overline{\theta},\underline{\eta}\right)-T\left(\overline{\theta},\overline{\eta}\right)$ is set as small as possible, implying that (3.40) is binding. We can compute the transfers from (3.40) and (3.9). We obtain

$$\begin{pmatrix} T\left(\overline{\theta},\underline{\eta}\right) \\ T\left(\overline{\theta},\overline{\eta}\right) \end{pmatrix} = \begin{pmatrix} -\lambda\left(\overline{\theta}\right)\left[\left(\overline{\eta}-\underline{\eta}\right)y\left(\overline{\theta},\overline{\eta}\right)-\left(\overline{\theta}-\underline{\theta}\right)\left(x\left(\overline{\theta},\underline{\eta}\right)-x\left(\overline{\theta},\overline{\eta}\right)\right)\right]+\overline{\theta}x\left(\overline{\theta},\overline{\eta}\right)+\overline{\eta}y\left(\overline{\theta},\overline{\eta}\right)+\underline{\theta}\left(x\left(\overline{\theta},\underline{\eta}\right)-x\left(\overline{\theta},\overline{\eta}\right)\right)+\underline{\eta}\left(y\left(\overline{\theta},\underline{\eta}\right)-y\left(\overline{\theta},\overline{\eta}\right)\right) \\ -\lambda\left(\overline{\theta}\right)\left[\left(\overline{\eta}-\underline{\eta}\right)y\left(\overline{\theta},\overline{\eta}\right)-\left(\overline{\theta}-\underline{\theta}\right)\left(x\left(\overline{\theta},\underline{\eta}\right)-x\left(\overline{\theta},\overline{\eta}\right)\right)\right]+\overline{\theta}x\left(\overline{\theta},\overline{\eta}\right)+\overline{\eta}y\left(\overline{\theta},\overline{\eta}\right) \end{pmatrix}. \tag{3.48}$$

Substituting these transfers back into the objective, we obtain

$$\Delta_{ii} \equiv \left(\lambda\left(\underline{\theta}\right)-\lambda\left(\overline{\theta}\right)\right)\left(\left(\overline{\eta}-\underline{\eta}\right)y\left(\overline{\theta},\overline{\eta}\right)\right)+\mathbb{E}_{\eta|\overline{\theta}}\left[\left(\overline{\theta}-\underline{\theta}\right)x\left(\overline{\theta},\eta\right)\right].$$

For $\left\{x\left(\overline{\theta},\eta\right),y\left(\overline{\theta},\eta\right)\right\}_{\eta\in\left\{\underline{\eta},\overline{\eta}\right\}} \in \mathbb{X}_{iv}\left(\overline{\theta}\right)$, no solution with $\hat{\eta}^*\left(\underline{\theta},\overline{\theta},\underline{\eta}\right)=\underline{\eta}$ exists.

Suppose thus that $\hat{\eta}^*\left(\underline{\theta},\overline{\theta},\underline{\eta}\right)=\overline{\eta}$. In this case, the objective is

$$\Delta\equiv\min_{T\left(\overline{\theta},\underline{\eta}\right)-T\left(\overline{\theta},\overline{\eta}\right)}\left\{\lambda\left(\overline{\theta}\right)\left(T\left(\overline{\theta},\overline{\eta}\right)-\overline{\theta}x\left(\overline{\theta},\overline{\eta}\right)-\overline{\eta}y\left(\overline{\theta},\overline{\eta}\right)-\left[T\left(\overline{\theta},\underline{\eta}\right)-\overline{\theta}x\left(\overline{\theta},\underline{\eta}\right)-\underline{\eta}y\left(\overline{\theta},\underline{\eta}\right)\right]\right)+\left(\overline{\theta}-\underline{\theta}\right)x\left(\overline{\theta},\overline{\eta}\right)+\lambda(\underline{\theta})\left(\overline{\eta}-\underline{\eta}\right)y\left(\overline{\theta},\overline{\eta}\right)\right\}$$

subject to (3.34), (3.35), and (3.41). The right-hand side of (3.35) is weakly smaller than the right-hand side of (3.41) for $(x,y)\in\mathbb{X}_{iv}$. Since the objective is decreasing in $T\left(\overline{\theta},\underline{\eta}\right)-T\left(\overline{\theta},\overline{\eta}\right)$ and we seek to minimize the objective function, at the optimum (3.35) must be binding. Thus, we can compute the optimal transfers from (3.35) and (3.9). We obtain

$$\begin{aligned} T\left(\overline{\theta},\underline{\eta}\right) &= \overline{\theta}x\left(\overline{\theta},\underline{\eta}\right)+\mathbb{E}_{\eta|\overline{\theta}}\eta y\left(\overline{\theta},\underline{\eta}\right) \\ T\left(\overline{\theta},\overline{\eta}\right) &= \overline{\theta}x\left(\overline{\theta},\overline{\eta}\right)+\mathbb{E}_{\eta|\overline{\theta}}\eta y\left(\overline{\theta},\underline{\eta}\right)-\overline{\eta}\left(y\left(\overline{\theta},\underline{\eta}\right)-y\left(\overline{\theta},\overline{\eta}\right)\right) \end{aligned} \tag{3.49}$$

Substituting these transfers back into the objective, we obtain

$$\Delta_{iv} \equiv -\lambda\left(\overline{\theta}\right)\left(\overline{\eta}-\underline{\eta}\right)y\left(\overline{\theta},\underline{\eta}\right)+\left(\overline{\theta}-\underline{\theta}\right)x\left(\overline{\theta},\overline{\eta}\right)+\lambda\left(\underline{\theta}\right)\left(\overline{\eta}-\underline{\eta}\right)y\left(\overline{\theta},\overline{\eta}\right).$$

For $\left\{x\left(\overline{\theta},\eta\right),y\left(\overline{\theta},\eta\right)\right\}_{\eta\in\left\{\underline{\eta},\overline{\eta}\right\}} \in \mathbb{X}_{ii}\left(\overline{\theta}\right)$, the right-hand side of (3.41) is weakly smaller than the right-hand side of (3.35). Thus, (3.35) is slack. The right-hand side of (3.34) is smaller than the right-hand side of (3.41), which implies that constraint (3.41) must be binding and we obtain rent Δ_{ii}.

Consider next the second problem in case where $x\left(\underline{\theta},\underline{\eta}\right)\leq x\left(\underline{\theta},\overline{\eta}\right)$. By lemma 3.3.2, this implies that $\hat{\eta}^*\left(\overline{\theta}, \underline{\theta}, \underline{\eta}\right) = \underline{\eta}$. The objective then becomes

$$\Omega = \min_{\{T(\underline{\theta},\eta)\}_{\eta\in\{\underline{\eta},\overline{\eta}\}},\hat{\eta}^*\left(\overline{\theta},\underline{\theta},\overline{\eta}\right)} \begin{pmatrix} -\left(\lambda(\underline{\theta})-\lambda\left(\overline{\theta}\right)\right)\left[T\left(\underline{\theta},\underline{\eta}\right)-\underline{\theta}x\left(\underline{\theta},\underline{\eta}\right)-\underline{\eta}y\left(\underline{\theta},\underline{\eta}\right)\right]-\lambda\left(\overline{\theta}\right)\left(\overline{\theta}-\underline{\theta}\right)x\left(\underline{\theta},\underline{\eta}\right) \\ +\left(1-\lambda\left(\overline{\theta}\right)\right)\left(T\left(\underline{\theta},\hat{\eta}^*\left(\overline{\theta},\underline{\theta},\overline{\eta}\right)\right)-\overline{\theta}x\left(\underline{\theta},\hat{\eta}^*\left(\overline{\theta},\underline{\theta},\overline{\eta}\right)\right)-\overline{\eta}y\left(\underline{\theta},\hat{\eta}^*\left(\overline{\theta},\underline{\theta},\overline{\eta}\right)\right)\right) \\ -(1-\lambda(\underline{\theta}))[T(\underline{\theta},\overline{\eta})-\underline{\theta}x(\underline{\theta},\overline{\eta})-\overline{\eta}y(\underline{\theta},\overline{\eta})]+\Delta \end{pmatrix}$$

subject to

$(3.37), (3.36),$ and either

(3.44) if $\hat{\eta}^*\left(\overline{\theta}, \underline{\theta}, \overline{\eta}\right) = \overline{\eta}$, or

(3.45) if $\hat{\eta}^*\left(\overline{\theta}, \underline{\theta}, \overline{\eta}\right) = \underline{\eta}$.

where we have added the difference between the right- and the left-hand side of (3.6), which is zero by the fact that this constraint binds.

Consider first the possibility that $\hat{\eta}^*\left(\overline{\theta}, \underline{\theta}, \overline{\eta}\right) = \overline{\eta}$. In that case the problem becomes

$$\Omega = \min_{T\left(\underline{\theta},\underline{\eta}\right)-T(\underline{\theta},\overline{\eta})} \begin{pmatrix} -\left(\lambda(\underline{\theta})-\lambda\left(\overline{\theta}\right)\right)\left[T\left(\underline{\theta},\underline{\eta}\right)-\underline{\theta}x\left(\underline{\theta},\underline{\eta}\right)-\underline{\eta}y\left(\underline{\theta},\underline{\eta}\right)\right] \\ +\left(\lambda(\underline{\theta})-\lambda\left(\overline{\theta}\right)\right)\left(T(\underline{\theta},\overline{\eta})-\overline{\theta}x(\underline{\theta},\overline{\eta})-\overline{\eta}y(\underline{\theta},\overline{\eta})\right) \\ -\lambda\left(\overline{\theta}\right)\left(\overline{\theta}-\underline{\theta}\right)x\left(\underline{\theta},\underline{\eta}\right)-(1-\lambda(\underline{\theta}))\left(\overline{\theta}-\underline{\theta}\right)x(\underline{\theta},\overline{\eta})+\Delta \end{pmatrix}$$

subject to $(3.37), (3.36)$ and (3.44).

The right-hand side of (3.44) is always weakly smaller than the right-hand side of (3.37). Hence, (3.37) cannot become binding at the optimum. Moreover, the constraint set is non-empty exactly for $\{x(\underline{\theta},\eta), y(\underline{\theta},\eta)\}_{\eta\in\{\underline{\eta},\overline{\eta}\}} \in \mathbb{X}_{ii}(\underline{\theta})$. Since the objective is decreasing in $T\left(\underline{\theta}, \underline{\eta}\right) - T\left(\underline{\theta}, \overline{\eta}\right)$, at the optimum, (3.44) is binding and we can compute the transfers from (3.44) and (3.6):

$$\begin{pmatrix} T\left(\underline{\theta},\underline{\eta}\right) \\ T(\underline{\theta},\overline{\eta}) \end{pmatrix} = \begin{pmatrix} \Delta+(1-\lambda(\underline{\theta}))\left[\overline{\theta}\left(x\left(\underline{\theta},\underline{\eta}\right)-x(\underline{\theta},\overline{\eta})\right)+\overline{\eta}\left(y\left(\underline{\theta},\underline{\eta}\right)-y(\underline{\theta},\overline{\eta})\right)\right]+\lambda(\underline{\theta})\left(\underline{\theta}x\left(\underline{\theta},\underline{\eta}\right)+\underline{\eta}y\left(\underline{\theta},\underline{\eta}\right)\right)+(1-\lambda(\underline{\theta}))[\underline{\theta}x(\underline{\theta},\overline{\eta})+\overline{\eta}y(\underline{\theta},\overline{\eta})] \\ \Delta-\lambda(\underline{\theta})\left[\overline{\theta}\left(x\left(\underline{\theta},\underline{\eta}\right)-x(\underline{\theta},\overline{\eta})\right)+\overline{\eta}\left(y\left(\underline{\theta},\underline{\eta}\right)-y(\underline{\theta},\overline{\eta})\right)\right]+\lambda(\underline{\theta})\left(\underline{\theta}x\left(\underline{\theta},\underline{\eta}\right)+\underline{\eta}y\left(\underline{\theta},\underline{\eta}\right)\right)+(1-\lambda(\underline{\theta}))[\underline{\theta}x(\underline{\theta},\overline{\eta})+\overline{\eta}y(\underline{\theta},\overline{\eta})] \end{pmatrix}$$

Since (3.44) is binding, we can substitute for

$$T\left(\underline{\theta},\underline{\eta}\right)-T\left(\underline{\theta},\overline{\eta}\right)=\overline{\theta}\left(x\left(\underline{\theta},\underline{\eta}\right)-x\left(\underline{\theta},\overline{\eta}\right)\right)+\overline{\eta}\left(y\left(\underline{\theta},\underline{\eta}\right)-y\left(\underline{\theta},\overline{\eta}\right)\right)$$

into the objective and obtain

$$\Omega_{ii}=-\mathbb{E}_{\eta|\underline{\theta}}\left[\left(\overline{\theta}-\underline{\theta}\right)x\left(\underline{\theta},\eta\right)\right]+\Delta-\left(\lambda\left(\underline{\theta}\right)-\lambda\left(\overline{\theta}\right)\right)\left(\overline{\eta}-\underline{\eta}\right)y\left(\underline{\theta},\underline{\eta}\right)$$

If $(x,y)\in\mathbb{X}_{ii}$, then this can be written as

$$\Omega_{ii}=\mathbb{E}_{\eta|\overline{\theta}}\left[\left(\overline{\theta}-\underline{\theta}\right)x\left(\overline{\theta},\eta\right)\right]-\mathbb{E}_{\eta|\underline{\theta}}\left[\left(\overline{\theta}-\underline{\theta}\right)x\left(\underline{\theta},\eta\right)\right]-\left(\lambda\left(\underline{\theta}\right)-\lambda\left(\overline{\theta}\right)\right)\left(\overline{\eta}-\underline{\eta}\right)\left(y\left(\underline{\theta},\underline{\eta}\right)-y\left(\overline{\theta},\overline{\eta}\right)\right).$$

Consider finally the possibility that $\hat{\eta}^*\left(\overline{\theta},\underline{\theta},\overline{\eta}\right)=\underline{\eta}$. In that case the problem becomes

$$\Omega=\min_{T\left(\underline{\theta},\underline{\eta}\right)-T\left(\underline{\theta},\overline{\eta}\right)}\begin{pmatrix}(1-\lambda(\underline{\theta}))\left\{T\left(\underline{\theta},\underline{\eta}\right)-\underline{\theta}x\left(\underline{\theta},\underline{\eta}\right)-\underline{\eta}y\left(\underline{\theta},\underline{\eta}\right)-\left[T(\underline{\theta},\overline{\eta})-\underline{\theta}x(\underline{\theta},\overline{\eta})-\overline{\eta}y(\underline{\theta},\overline{\eta})\right]\right\}\\-\left(1-\lambda\left(\overline{\theta}\right)\right)\left(\overline{\eta}-\underline{\eta}\right)y\left(\underline{\theta},\underline{\eta}\right)-\left(\overline{\theta}-\underline{\theta}\right)x\left(\underline{\theta},\underline{\eta}\right)+\Delta\end{pmatrix}$$

subject to (3.37), (3.36), and (3.45).

The right-hand side of (3.36) is weakly larger than the right-hand side of (3.45) exactly for $\{x\left(\underline{\theta},\eta\right),y\left(\underline{\theta},\eta\right)\}_{\eta\in\left\{\underline{\eta},\overline{\eta}\right\}}\in\mathbb{X}_{iv}\left(\underline{\theta}\right)$. Moreover, for such allocations, the constraint set is nonempty, and at the solution of the problem $T\left(\underline{\theta},\underline{\eta}\right)-T\left(\underline{\theta},\overline{\eta}\right)$ reaches its lower bound, so (3.36) is binding. The transfers can then be computed from (3.6) and (3.36) :

$$\begin{pmatrix}T\left(\underline{\theta},\underline{\eta}\right)\\T\left(\underline{\theta},\overline{\eta}\right)\end{pmatrix}=\begin{pmatrix}\Delta+\left(1-\lambda\left(\underline{\theta}\right)\right)\left(\overline{\eta}-\underline{\eta}\right)y\left(\underline{\theta},\overline{\eta}\right)+\underline{\theta}x\left(\underline{\theta},\underline{\eta}\right)+\underline{\eta}y\left(\underline{\theta},\underline{\eta}\right)\\\Delta+\underline{\theta}x\left(\underline{\theta},\overline{\eta}\right)+\left(\lambda\left(\underline{\theta}\right)\underline{\eta}+\left(1-\lambda\left(\underline{\theta}\right)\right)\overline{\eta}\right)y\left(\underline{\theta},\overline{\eta}\right)\end{pmatrix}$$

Since (3.36) is binding, we can substitute

$$T\left(\underline{\theta},\underline{\eta}\right)-T\left(\underline{\theta},\overline{\eta}\right)=\underline{\theta}\left(x\left(\underline{\theta},\underline{\eta}\right)-x\left(\underline{\theta},\overline{\eta}\right)\right)+\underline{\eta}\left(y\left(\underline{\theta},\underline{\eta}\right)-y\left(\underline{\theta},\overline{\eta}\right)\right)$$

into the objective and obtain

$$\Omega_{iv}=\left(1-\lambda\left(\underline{\theta}\right)\right)\left(\overline{\eta}-\underline{\eta}\right)y\left(\underline{\theta},\overline{\eta}\right)-\left(1-\lambda\left(\overline{\theta}\right)\right)\left(\overline{\eta}-\underline{\eta}\right)y\left(\underline{\theta},\underline{\eta}\right)-\left(\overline{\theta}-\underline{\theta}\right)x\left(\underline{\theta},\underline{\eta}\right)+\Delta$$

For future reference, if $(x, y) \in \mathbb{X}_{iv}$, then we can write

$$\begin{aligned}\Omega_{iv} = &-\lambda\left(\underline{\theta}\right)\left(\overline{\eta}-\underline{\eta}\right)\left(y\left(\underline{\theta},\overline{\eta}\right)-y\left(\overline{\theta},\overline{\eta}\right)\right)-\left(\overline{\theta}-\underline{\theta}\right)\left(x\left(\underline{\theta},\underline{\eta}\right)-x\left(\overline{\theta},\overline{\eta}\right)\right)\\ &-\lambda\left(\overline{\theta}\right)\left(\overline{\eta}-\underline{\eta}\right)\left(y\left(\overline{\theta},\underline{\eta}\right)-y\left(\underline{\theta},\underline{\eta}\right)\right)-\left(\overline{\eta}-\underline{\eta}\right)\left(y\left(\underline{\theta},\underline{\eta}\right)-y\left(\underline{\theta},\overline{\eta}\right)\right).\end{aligned}$$

For $\{x\left(\underline{\theta},\eta\right), y\left(\underline{\theta},\eta\right)\}_{\eta\in\{\underline{\eta},\overline{\eta}\}} \in \mathbb{X}_{ii}\left(\underline{\theta}\right)$, the right-hand side of (3.45) is weakly larger than the right-hand side of (3.36). Moreover, since the right-hand side of (3.45) is smaller than the right-hand side of (3.37), the constraint set is nonempty. At the solution, (3.45) is binding, so we can substitute for

$$T\left(\underline{\theta},\underline{\eta}\right)-T\left(\underline{\theta},\overline{\eta}\right)=\overline{\theta}\left(x\left(\underline{\theta},\underline{\eta}\right)-x\left(\underline{\theta},\overline{\eta}\right)\right)+\overline{\eta}\left(y\left(\underline{\theta},\underline{\eta}\right)-y\left(\underline{\theta},\overline{\eta}\right)\right)$$

into the objective and obtain

$$\hat{\Omega}_{ii}=\left(1-\lambda\left(\underline{\theta}\right)\right)\left(\overline{\theta}-\underline{\theta}\right)\left(x\left(\underline{\theta},\underline{\eta}\right)-x\left(\underline{\theta},\overline{\eta}\right)\right)-\left(\lambda\left(\underline{\theta}\right)-\lambda\left(\overline{\theta}\right)\right)\left(\overline{\eta}-\underline{\eta}\right)y\left(\underline{\theta},\underline{\eta}\right)-\left(\overline{\theta}-\underline{\theta}\right)x\left(\underline{\theta},\underline{\eta}\right)+\Delta$$

Since $\Omega_{ii} \leq \hat{\Omega}_{ii}$ for any $\{x\left(\underline{\theta},\eta\right), y\left(\underline{\theta},\eta\right)\}_{\eta\in\{\underline{\eta},\overline{\eta}\}} \in \mathbb{X}_{ii}\left(\underline{\theta}\right)$, implementing $\hat{\eta}^*\left(\overline{\theta},\underline{\theta},\overline{\eta}\right)=\underline{\eta}$, the principal cannot gain by implementing this report. ■

Proof of Proposition 3.3.1. The proof of the first statements is given in three parts. Part I establishes properties of the solution of program P_{ii}; part II does likewise for program P_i; finally, part III compares the value of the objectives. The proof of the fact that $W_{iv} \leq W_{ii}$ and $W_{iii} \leq W_i$ is not given here but is available upon request from the authors; it uses essentially the same arguments.

Part I) Consider program P_{ii}. Up to a constant, the Lagrangian of program P_{ii} can be written as

$$\begin{aligned}&(1-\alpha)\,\mathbb{E}_{\eta|\overline{\theta}}\left[V\left(x\left(\overline{\theta},\eta\right),y\left(\overline{\theta},\eta\right)\right)-\overline{\theta}x\left(\overline{\theta},\eta\right)-\eta y\left(\overline{\theta},\eta\right)\right]\\ &-\alpha\left\{\lambda\left(\overline{\theta}\right)\left(\overline{\theta}-\underline{\theta}\right)x\left(\overline{\theta},\underline{\eta}\right)+\left(\lambda\left(\underline{\theta}\right)-\lambda\left(\overline{\theta}\right)\right)\left(\overline{\eta}-\underline{\eta}\right)y\left(\overline{\theta},\overline{\eta}\right)+\left(1-\lambda\left(\overline{\theta}\right)\right)\left(\overline{\theta}-\underline{\theta}\right)x\left(\overline{\theta},\overline{\eta}\right)\right\}\\ &+\phi\left[x\left(\overline{\theta},\overline{\eta}\right)-x\left(\overline{\theta},\underline{\eta}\right)\right]+\mu\left\{\left(\overline{\eta}-\underline{\eta}\right)\left(y\left(\overline{\theta},\underline{\eta}\right)-y\left(\overline{\theta},\overline{\eta}\right)\right)-\left(\overline{\theta}-\underline{\theta}\right)\left(x\left(\overline{\theta},\overline{\eta}\right)-x\left(\overline{\theta},\underline{\eta}\right)\right)\right\}\end{aligned}$$

The conditions of optimality are

$$\left((1-\alpha)\lambda\left(\overline{\theta}\right)\left(V_1\left(x\left(\overline{\theta},\underline{\eta}\right),y\left(\overline{\theta},\underline{\eta}\right)\right)-\overline{\theta}\right)-\alpha\lambda\left(\overline{\theta}\right)\left(\overline{\theta}-\underline{\theta}\right)-\phi+\mu\left(\overline{\theta}-\underline{\theta}\right)\right)=0 \tag{3.50}$$

$$\left((1-\alpha)\left(1-\lambda\left(\overline{\theta}\right)\right)\left(V_1\left(x\left(\overline{\theta},\overline{\eta}\right),y\left(\overline{\theta},\overline{\eta}\right)\right)-\overline{\theta}\right)-\alpha\left(1-\lambda\left(\overline{\theta}\right)\right)\left(\overline{\theta}-\underline{\theta}\right)+\phi-\mu\left(\overline{\theta}-\underline{\theta}\right)\right)=0 \tag{3.51}$$

$$\left((1-\alpha)\lambda\left(\overline{\theta}\right)\left(V_2\left(x\left(\overline{\theta},\underline{\eta}\right),y\left(\overline{\theta},\underline{\eta}\right)\right)-\underline{\eta}\right)+\mu\left(\overline{\eta}-\underline{\eta}\right)\right)=0 \tag{3.52}$$

$$\left((1-\alpha)\left(1-\lambda\left(\overline{\theta}\right)\right)\left(V_2\left(x\left(\overline{\theta},\overline{\eta}\right),y\left(\overline{\theta},\overline{\eta}\right)\right)-\overline{\eta}\right)-\alpha\left(\lambda\left(\underline{\theta}\right)-\lambda\left(\overline{\theta}\right)\right)\left(\overline{\eta}-\underline{\eta}\right)-\mu\left(\overline{\eta}-\underline{\eta}\right)\right)=0. \tag{3.53}$$

We show by contradiction that at most one constraint binds at the optimum of program ii.

Suppose both constraints bind. If $\phi,\mu>0$, then $x\left(\overline{\theta},\underline{\eta}\right)=x\left(\overline{\theta},\overline{\eta}\right)=x\left(\overline{\theta}\right)$ and $y\left(\overline{\theta},\underline{\eta}\right)=y\left(\overline{\theta},\overline{\eta}\right)=y\left(\overline{\theta}\right)$ and the conditions of optimality imply that

$$\left(V_1\left(x\left(\overline{\theta}\right),y\left(\overline{\theta}\right)\right)-\overline{\theta}\right)-\frac{\alpha}{1-\alpha}\left(\overline{\theta}-\underline{\theta}\right)=0 \tag{3.54}$$

and

$$\left(\begin{array}{c} V_2\left(x\left(\overline{\theta}\right),y\left(\overline{\theta}\right)\right) \\ -\lambda\left(\overline{\theta}\right)\underline{\eta}-\left(1-\lambda\left(\overline{\theta}\right)\right)\overline{\eta}-\frac{\alpha}{1-\alpha}\left(\lambda\left(\underline{\theta}\right)-\lambda\left(\overline{\theta}\right)\right)\left(\overline{\eta}-\underline{\eta}\right)\end{array}\right)=0. \tag{3.55}$$

Using (3.52), the Kuhn-Tucker-first-order-optimality-condition for $y\left(\overline{\theta},\underline{\eta}\right)$ and substituting (3.55) we have for $\mu\neq 0$

$$\left((1-\alpha)\lambda\left(\overline{\theta}\right)\left(\lambda\left(\overline{\theta}\right)\underline{\eta}+\left(1-\lambda\left(\overline{\theta}\right)\right)\overline{\eta}+\frac{\alpha}{1-\alpha}\left(\lambda\left(\underline{\theta}\right)-\lambda\left(\overline{\theta}\right)\right)\left(\overline{\eta}-\underline{\eta}\right)-\underline{\eta}\right)+\mu\left(\overline{\eta}-\underline{\eta}\right)\right)=0$$

which simplifies to

$$(1-\alpha)\lambda\left(\overline{\theta}\right)\left(\left(1-\lambda\left(\overline{\theta}\right)\right)+\frac{\alpha}{1-\alpha}\left(\lambda\left(\underline{\theta}\right)-\lambda\left(\overline{\theta}\right)\right)\right)=-\mu$$

This implies $\mu<0$ which contradicts the supposition that both constraints bind at the optimum. It follows that at most one constraint binds at the optimum of program ii.

Further results require a case distinction between $V_{12}<0$ and $V_{12}\geq 0$.

Case I) $V_{12} \geq 0$.

First, we show that if $V_{12} \geq 0$, then either constraint $x\left(\overline{\theta}, \underline{\eta}\right) - x\left(\overline{\theta}, \overline{\eta}\right) \geq 0$ or constraint $\left(\overline{\eta} - \underline{\eta}\right)\left(y\left(\overline{\theta}, \underline{\eta}\right) - y\left(\overline{\theta}, \overline{\eta}\right)\right) - \left(\overline{\theta} - \underline{\theta}\right) x\left(\overline{\theta}, \overline{\eta}\right) - x\left(\overline{\theta}, \underline{\eta}\right) \geq 0$ binds at the optimum of program P_{ii}.

Suppose no constraint binds. Then the first-order conditions with respect to y are given by

$$V_2\left(x\left(\overline{\theta}, \underline{\eta}\right), y\left(\overline{\theta}, \underline{\eta}\right)\right) - \underline{\eta} = 0$$

$$V_2\left(x\left(\overline{\theta}, \overline{\eta}\right), y\left(\overline{\theta}, \overline{\eta}\right)\right) - \overline{\eta} - \frac{\alpha\left(\lambda\left(\underline{\theta}\right) - \lambda\left(\overline{\theta}\right)\right)}{(1-\alpha)\left(1 - \lambda\left(\overline{\theta}\right)\right)}\left(\overline{\eta} - \underline{\eta}\right) = 0.$$

The first-order conditions with respect to x are given by

$$V_1\left(x\left(\overline{\theta}, \underline{\eta}\right), y\left(\overline{\theta}, \underline{\eta}\right)\right) - \overline{\theta} - \frac{\alpha}{1-\alpha}\left(\overline{\theta} - \underline{\theta}\right) = 0$$

$$V_1\left(x\left(\overline{\theta}, \overline{\eta}\right), y\left(\overline{\theta}, \overline{\eta}\right)\right) - \overline{\theta} - \frac{\alpha}{1-\alpha}\left(\overline{\theta} - \underline{\theta}\right) = 0$$

which imply

$$V_1\left(x\left(\overline{\theta}, \underline{\eta}\right), y\left(\overline{\theta}, \underline{\eta}\right)\right) = V_1\left(x\left(\overline{\theta}, \overline{\eta}\right), y\left(\overline{\theta}, \overline{\eta}\right)\right). \tag{3.56}$$

By concavity, $V_{11} < 0$, and $x\left(\overline{\theta}, \overline{\eta}\right) - x\left(\overline{\theta}, \underline{\eta}\right) > 0$, we have

$$V_1\left(x\left(\overline{\theta}, \overline{\eta}\right), y\left(\overline{\theta}, \underline{\eta}\right)\right) < V_1\left(x\left(\overline{\theta}, \underline{\eta}\right), y\left(\overline{\theta}, \underline{\eta}\right)\right) \tag{3.57}$$

Together conditions (3.56) and (3.57) imply

$$V_1\left(x\left(\overline{\theta}, \overline{\eta}\right), y\left(\overline{\theta}, \underline{\eta}\right)\right) < V_1\left(x\left(\overline{\theta}, \overline{\eta}\right), y\left(\overline{\theta}, \overline{\eta}\right)\right). \tag{3.58}$$

By complementarity, $V_{12} \geq 0$, and $y\left(\overline{\theta}, \underline{\eta}\right) - y\left(\overline{\theta}, \overline{\eta}\right) > 0$, we have

$$V_1\left(x\left(\overline{\theta}, \overline{\eta}\right), y\left(\overline{\theta}, \overline{\eta}\right)\right) \leq V_1\left(x\left(\overline{\theta}, \overline{\eta}\right), y\left(\overline{\theta}, \underline{\eta}\right)\right) \tag{3.59}$$

which contradicts (3.58).

It follows that at least one constraint must be binding at the optimum of program P_{ii}.

Next, we show that if $V_{12} \geq 0$, then the optimal allocation satisfies

$$\left(\overline{\eta}-\underline{\eta}\right)\left(y\left(\overline{\theta},\underline{\eta}\right)-y\left(\overline{\theta},\overline{\eta}\right)\right)-\left(\overline{\theta}-\underline{\theta}\right)\left(x\left(\overline{\theta},\overline{\eta}\right)-x\left(\overline{\theta},\underline{\eta}\right)\right)>0.$$

Suppose, contrary to our claim,

$$\left(\overline{\eta}-\underline{\eta}\right)\left(y\left(\overline{\theta},\underline{\eta}\right)-y\left(\overline{\theta},\overline{\eta}\right)\right)-\left(\overline{\theta}-\underline{\theta}\right)\left(x\left(\overline{\theta},\overline{\eta}\right)-x\left(\overline{\theta},\underline{\eta}\right)\right)=0$$

and moreover $\phi = 0$ and $\mu > 0$.

The first-order conditions with respect to x are given by

$$V_1\left(x\left(\overline{\theta},\underline{\eta}\right),y\left(\overline{\theta},\underline{\eta}\right)\right)-\overline{\theta}-\frac{\alpha\lambda\left(\overline{\theta}\right)\left(\overline{\theta}-\underline{\theta}\right)-\mu\left(\overline{\theta}-\underline{\theta}\right)}{\left(1-\alpha\right)\lambda\left(\overline{\theta}\right)}=0$$

$$V_1\left(x\left(\overline{\theta},\overline{\eta}\right),y\left(\overline{\theta},\overline{\eta}\right)\right)-\overline{\theta}-\frac{\alpha\left(1-\lambda\left(\overline{\theta}\right)\right)\left(\overline{\theta}-\underline{\theta}\right)+\mu\left(\overline{\theta}-\underline{\theta}\right)}{\left(1-\alpha\right)\left(1-\lambda\left(\overline{\theta}\right)\right)}=0$$

implying that

$$V_1\left(x\left(\overline{\theta},\underline{\eta}\right),y\left(\overline{\theta},\underline{\eta}\right)\right)<V_1\left(x\left(\overline{\theta},\overline{\eta}\right),y\left(\overline{\theta},\overline{\eta}\right)\right)$$

if and only if

$$\frac{\alpha\lambda\left(\overline{\theta}\right)\left(\overline{\theta}-\underline{\theta}\right)-\mu\left(\overline{\theta}-\underline{\theta}\right)}{\left(1-\alpha\right)\lambda\left(\overline{\theta}\right)}<\frac{\alpha\left(1-\lambda\left(\overline{\theta}\right)\right)\left(\overline{\theta}-\underline{\theta}\right)+\mu\left(\overline{\theta}-\underline{\theta}\right)}{\left(1-\alpha\right)\left(1-\lambda\left(\overline{\theta}\right)\right)}$$

$$\Longleftrightarrow$$

$$0<\mu.$$

However, we must have $V_1\left(x\left(\overline{\theta},\underline{\eta}\right),y\left(\overline{\theta},\underline{\eta}\right)\right)\geq V_1\left(x\left(\overline{\theta},\overline{\eta}\right),y\left(\overline{\theta},\overline{\eta}\right)\right)$.

To see this, note that by $V_{11} < 0$

$$V_1\left(x\left(\overline{\theta},\underline{\eta}\right),y\left(\overline{\theta},\underline{\eta}\right)\right)>V_1\left(x\left(\overline{\theta},\overline{\eta}\right),y\left(\overline{\theta},\underline{\eta}\right)\right).$$

By $V_{12} \geq 0$

$$V_1\left(x\left(\overline{\theta},\overline{\eta}\right),y\left(\overline{\theta},\underline{\eta}\right)\right) \geq V_1\left(x\left(\overline{\theta},\overline{\eta}\right),y\left(\overline{\theta},\overline{\eta}\right)\right)$$

Together these imply that

$$V_1\left(x\left(\overline{\theta},\underline{\eta}\right),y\left(\overline{\theta},\underline{\eta}\right)\right) > V_1\left(x\left(\overline{\theta},\overline{\eta}\right),y\left(\overline{\theta},\overline{\eta}\right)\right),$$

so that the conditions above would imply that $\mu < 0$, a contradiction.

It follows from these arguments that the optimal allocation for $V_{12} \geq 0$ satisfies $x\left(\overline{\theta},\overline{\eta}\right) = x\left(\overline{\theta},\underline{\eta}\right).$

Case II) $V_{12} < 0$.

If $V_{12} < 0$, then the solution to program P_{ii} satisfies $x\left(\overline{\theta},\overline{\eta}\right) - x\left(\overline{\theta},\underline{\eta}\right) > 0$.

Suppose not. We know that $\phi, \mu > 0$ is not possible. So, if $x\left(\overline{\theta},\overline{\eta}\right) = x\left(\overline{\theta},\underline{\eta}\right)$, this would have to imply that $\mu = 0$. So, we would have $x\left(\overline{\theta},\underline{\eta}\right) = x\left(\overline{\theta},\overline{\eta}\right) = x\left(\overline{\theta}\right)$, $y\left(\overline{\theta},\overline{\eta}\right) < y\left(\overline{\theta},\underline{\eta}\right)$ and $\mu = 0$. Adding up of conditions (3.50) and (3.51), the first-order conditions for $x\left(\overline{\theta},\underline{\eta}\right)$ and $x\left(\overline{\theta},\overline{\eta}\right)$, gives

$$\left(\begin{array}{c} \lambda\left(\overline{\theta}\right)\left(V_1\left(x\left(\overline{\theta}\right),y\left(\overline{\theta},\underline{\eta}\right)\right) - \overline{\theta}\right) + \left(1-\lambda\left(\overline{\theta}\right)\right)\left(V_1\left(x\left(\overline{\theta}\right),y\left(\overline{\theta},\overline{\eta}\right)\right) - \overline{\theta}\right) \\ -\frac{\alpha}{1-\alpha}\left(\overline{\theta}-\underline{\theta}\right) \end{array}\right) = 0. \quad (3.60)$$

$V_{12} < 0$ and $y\left(\overline{\theta},\underline{\eta}\right) - y\left(\overline{\theta},\overline{\eta}\right) > 0$ imply that

$$V_1\left(x\left(\overline{\theta}\right),y\left(\overline{\theta},\underline{\eta}\right)\right) < V_1\left(x\left(\overline{\theta}\right),y\left(\overline{\theta},\overline{\eta}\right)\right).$$

Together with (3.60), this implies that

$$V_1\left(x\left(\overline{\theta}\right),y\left(\overline{\theta},\underline{\eta}\right)\right) < \overline{\theta} + \frac{\alpha}{(1-\alpha)}\left(\overline{\theta}-\underline{\theta}\right) < V_1\left(x\left(\overline{\theta}\right),y\left(\overline{\theta},\overline{\eta}\right)\right).$$

Plugging the first of these inequalities into (3.50), we obtain

$$\left((1-\alpha)\lambda\left(\overline{\theta}\right)\frac{\alpha}{(1-\alpha)}\left(\overline{\theta}-\underline{\theta}\right) - \alpha\lambda\left(\overline{\theta}\right)\left(\overline{\theta}-\underline{\theta}\right) - \phi + \mu\left(\overline{\theta}-\underline{\theta}\right)\right) > 0.$$

Plugging the latter of the inequalities into (3.51), we obtain

$$\left((1-\alpha)\left(1-\lambda\left(\overline{\theta}\right)\right)\frac{\alpha}{(1-\alpha)}\left(\overline{\theta}-\underline{\theta}\right)-\alpha\left(1-\lambda\left(\overline{\theta}\right)\right)\left(\overline{\theta}-\underline{\theta}\right)+\phi-\mu\left(\overline{\theta}-\underline{\theta}\right)\right)<0,$$

which simplifies to

$$\mu\left(\overline{\theta}-\underline{\theta}\right)>\phi.$$

For $\mu=0$ this implies $\phi<0$. Hence, $V_{12}<0$ implies that $x\left(\overline{\theta},\overline{\eta}\right)-x\left(\overline{\theta},\underline{\eta}\right)>0$.

Part II) Consider program P_i. Up to a constant, the Lagrangian of program P_i can be written as

$$\begin{array}{l}(1-\alpha)\,\mathbb{E}_{\eta|\overline{\theta}}\left[V\left(x\left(\overline{\theta},\eta\right),y\left(\overline{\theta},\eta\right)\right)-\overline{\theta}x\left(\overline{\theta},\eta\right)-\eta y\left(\overline{\theta},\eta\right)\right]\\ -\alpha\left\{\begin{array}{c}\left(1-\lambda\left(\underline{\theta}\right)\right)\left(\overline{\theta}-\underline{\theta}\right)x\left(\overline{\theta},\overline{\eta}\right)+\lambda\left(\underline{\theta}\right)\left(\overline{\theta}-\underline{\theta}\right)x\left(\overline{\theta},\underline{\eta}\right)\\ +\left(\lambda\left(\underline{\theta}\right)-\lambda\left(\overline{\theta}\right)\right)\left(\overline{\eta}-\underline{\eta}\right)y\left(\overline{\theta},\overline{\eta}\right)\end{array}\right\}\\ +\xi\left[x\left(\overline{\theta},\underline{\eta}\right)-x\left(\overline{\theta},\overline{\eta}\right)\right]+\nu\left[\begin{array}{c}\left(\overline{\eta}-\underline{\eta}\right)\left(y\left(\overline{\theta},\underline{\eta}\right)-y\left(\overline{\theta},\overline{\eta}\right)\right)\\ -\left(\overline{\theta}-\underline{\theta}\right)\left(x\left(\overline{\theta},\underline{\eta}\right)-x\left(\overline{\theta},\overline{\eta}\right)\right)\end{array}\right]\end{array}$$

The conditions of optimality are given by

$$\left(\begin{array}{c}(1-\alpha)\lambda\left(\overline{\theta}\right)\left(V_1\left(x\left(\overline{\theta},\underline{\eta}\right),y\left(\overline{\theta},\underline{\eta}\right)\right)-\overline{\theta}\right)\\ -\alpha\lambda\left(\underline{\theta}\right)\left(\overline{\theta}-\underline{\theta}\right)+\xi-\nu\left(\overline{\theta}-\underline{\theta}\right)\end{array}\right)=0 \tag{3.61}$$

$$(1-\alpha)\lambda\left(\overline{\theta}\right)\left(V_2\left(x\left(\overline{\theta},\underline{\eta}\right),y\left(\overline{\theta},\underline{\eta}\right)\right)-\underline{\eta}\right)+\nu\left(\overline{\eta}-\underline{\eta}\right)=0 \tag{3.62}$$

$$\left(\begin{array}{c}(1-\alpha)\left(1-\lambda\left(\overline{\theta}\right)\right)\left(V_1\left(x\left(\overline{\theta},\overline{\eta}\right),y\left(\overline{\theta},\overline{\eta}\right)\right)-\overline{\theta}\right)\\ -\alpha\left(1-\lambda\left(\underline{\theta}\right)\right)\left(\overline{\theta}-\underline{\theta}\right)-\xi+\nu\left(\overline{\theta}-\underline{\theta}\right)\end{array}\right)=0 \tag{3.63}$$

$$\left(\begin{array}{c}(1-\alpha)\left(1-\lambda\left(\overline{\theta}\right)\right)\left(V_2\left(x\left(\overline{\theta},\overline{\eta}\right),y\left(\overline{\theta},\overline{\eta}\right)\right)-\overline{\eta}\right)\\ -\alpha\left(\lambda\left(\underline{\theta}\right)-\lambda\left(\overline{\theta}\right)\right)\left(\overline{\eta}-\underline{\eta}\right)-\nu\left(\overline{\eta}-\underline{\eta}\right)\end{array}\right)=0 \tag{3.64}$$

$$\xi,\nu\geq 0$$

$$\xi\left[x\left(\overline{\theta},\underline{\eta}\right)-x\left(\overline{\theta},\overline{\eta}\right)\right]=0$$

$$\nu\left[\left(\overline{\eta}-\underline{\eta}\right)\left(y\left(\overline{\theta},\underline{\eta}\right)-y\left(\overline{\theta},\overline{\eta}\right)\right)-\left(\overline{\theta}-\underline{\theta}\right)\left(x\left(\overline{\theta},\underline{\eta}\right)-x\left(\overline{\theta},\overline{\eta}\right)\right)\right]=0$$

First, we show by contradiction that at most one constraint binds at the optimum of program P_i.

So suppose both constraints bind at the optimum, i.e. $\xi, \nu > 0$. If $\xi, \nu > 0$, then $y\left(\overline{\theta}, \underline{\eta}\right) = y\left(\overline{\theta}, \overline{\eta}\right) = y\left(\overline{\theta}\right)$ and $x\left(\overline{\theta}, \underline{\eta}\right) = x\left(\overline{\theta}, \overline{\eta}\right) = x\left(\overline{\theta}\right)$. Then

$$V_1\left(x\left(\overline{\theta}\right), y\left(\overline{\theta}\right)\right) = \overline{\theta} + \frac{\alpha}{(1-\alpha)}\left(\overline{\theta} - \underline{\theta}\right)$$

and

$$V_2\left(x\left(\overline{\theta}\right), y\left(\overline{\theta}\right)\right) = \lambda\left(\overline{\theta}\right)\underline{\eta} + \left(1 - \lambda\left(\overline{\theta}\right)\right)\overline{\eta} + \frac{\alpha\left(\lambda\left(\underline{\theta}\right) - \lambda\left(\overline{\theta}\right)\right)}{(1-\alpha)}\left(\overline{\eta} - \underline{\eta}\right).$$

Using the first-order condition with respect to $y\left(\overline{\theta}, \underline{\eta}\right)$, (3.62), gives

$$V_2\left(x\left(\overline{\theta}\right), y\left(\overline{\theta}\right)\right) = \underline{\eta} - \frac{\nu}{(1-\alpha)\lambda\left(\overline{\theta}\right)}\left(\overline{\eta} - \underline{\eta}\right).$$

Substituting for $V_2\left(x\left(\overline{\theta}\right), y\left(\overline{\theta}\right)\right)$ gives

$$\lambda\left(\overline{\theta}\right)\underline{\eta} + \left(1 - \lambda\left(\overline{\theta}\right)\right)\overline{\eta} + \frac{\alpha\left(\lambda\left(\underline{\theta}\right) - \lambda\left(\overline{\theta}\right)\right)}{(1-\alpha)}\left(\overline{\eta} - \underline{\eta}\right) = \underline{\eta} - \frac{\nu}{(1-\alpha)\lambda\left(\overline{\theta}\right)}\left(\overline{\eta} - \underline{\eta}\right)$$

which simplifies to

$$-\left(1 - \lambda\left(\overline{\theta}\right)\right)(1-\alpha)\lambda\left(\overline{\theta}\right) - \alpha\left(\lambda\left(\underline{\theta}\right) - \lambda\left(\overline{\theta}\right)\right)\lambda\left(\overline{\theta}\right) = \nu,$$

implying that $\nu < 0$. It follows that at the optimum $\xi, \nu > 0$ is not true.

Further results require a case distinction between $V_{12} < 0$ and $V_{12} \geq 0$.

Case I) $V_{12} < 0$.

First, we show that if $V_{12} < 0$, then $x\left(\overline{\theta}, \underline{\eta}\right) - x\left(\overline{\theta}, \overline{\eta}\right) = 0$ at the solution to program P_i.

To show this, we establish first that $V_{12} < 0$ implies that at least one constraint binds. Moreover, we show that $\left(\overline{\eta} - \underline{\eta}\right)\left[y\left(\overline{\theta}, \underline{\eta}\right) - y\left(\overline{\theta}, \overline{\eta}\right)\right] > \left(\overline{\theta} - \underline{\theta}\right)\left[x\left(\overline{\theta}, \underline{\eta}\right) - x\left(\overline{\theta}, \overline{\eta}\right)\right]$ at the optimum of program P_i.

Suppose no constraint binds at the optimum, i.e. $\left(\overline{\eta} - \underline{\eta}\right)\left[y\left(\overline{\theta}, \underline{\eta}\right) - y\left(\overline{\theta}, \overline{\eta}\right)\right] >$

$\left(\overline{\theta}-\underline{\theta}\right)\left[x\left(\overline{\theta},\underline{\eta}\right)-x\left(\overline{\theta},\overline{\eta}\right)\right]>0$. Then $\xi=\nu=0$. The first-order conditions with respect to x are given by

$$V_1\left(x\left(\overline{\theta},\underline{\eta}\right),y\left(\overline{\theta},\underline{\eta}\right)\right)-\overline{\theta}-\frac{\alpha\lambda\left(\underline{\theta}\right)}{(1-\alpha)\lambda\left(\overline{\theta}\right)}\left(\overline{\theta}-\underline{\theta}\right)=0$$

$$V_1\left(x\left(\overline{\theta},\overline{\eta}\right),y\left(\overline{\theta},\overline{\eta}\right)\right)-\overline{\theta}-\frac{\alpha\left(1-\lambda\left(\underline{\theta}\right)\right)}{(1-\alpha)\left(1-\lambda\left(\overline{\theta}\right)\right)}\left(\overline{\theta}-\underline{\theta}\right)=0$$

Since $\frac{\lambda(\underline{\theta})}{\lambda(\overline{\theta})}>\frac{(1-\lambda(\underline{\theta}))}{\left(1-\lambda\left(\overline{\theta}\right)\right)}$, these conditions imply that

$$V_1\left(x\left(\overline{\theta},\underline{\eta}\right),y\left(\overline{\theta},\underline{\eta}\right)\right)>V_1\left(x\left(\overline{\theta},\overline{\eta}\right),y\left(\overline{\theta},\overline{\eta}\right)\right). \tag{3.65}$$

However, by $V_{12}<0$ and $y\left(\overline{\theta},\underline{\eta}\right)-y\left(\overline{\theta},\overline{\eta}\right)>0$

$$V_1\left(x\left(\overline{\theta},\underline{\eta}\right),y\left(\overline{\theta},\underline{\eta}\right)\right)<V_1\left(x\left(\overline{\theta},\underline{\eta}\right),y\left(\overline{\theta},\overline{\eta}\right)\right). \tag{3.66}$$

By $V_{11}<0$ and $x\left(\overline{\theta},\underline{\eta}\right)-x\left(\overline{\theta},\overline{\eta}\right)>0$

$$V_1\left(x\left(\overline{\theta},\underline{\eta}\right),y\left(\overline{\theta},\overline{\eta}\right)\right)<V_1\left(x\left(\overline{\theta},\overline{\eta}\right),y\left(\overline{\theta},\overline{\eta}\right)\right). \tag{3.67}$$

Taken together (3.66) and (3.67) imply that

$$V_1\left(x\left(\overline{\theta},\underline{\eta}\right),y\left(\overline{\theta},\underline{\eta}\right)\right)<V_1\left(x\left(\overline{\theta},\overline{\eta}\right),y\left(\overline{\theta},\overline{\eta}\right)\right).$$

which contradicts (3.65) derived previously from the first-order conditions.

It follows that at least one constraint must bind at the optimum of program P_i if $V_{12}<0$.

Suppose that contrary to our claim, that the solution of program P_i satisfies

$\left(\overline{\eta}-\underline{\eta}\right)\left[y\left(\overline{\theta},\underline{\eta}\right)-y\left(\overline{\theta},\overline{\eta}\right)\right]=\left(\overline{\theta}-\underline{\theta}\right)\left[x\left(\overline{\theta},\underline{\eta}\right)-x\left(\overline{\theta},\overline{\eta}\right)\right]>0$, $\xi=0$ and $\nu>0$. Adding up (3.61) and (3.63) gives

$$\left(\begin{array}{c}(1-\alpha)\lambda\left(\overline{\theta}\right)\left(V_1\left(x\left(\overline{\theta},\underline{\eta}\right),y\left(\overline{\theta},\underline{\eta}\right)\right)-\overline{\theta}\right)-\alpha\lambda\left(\underline{\theta}\right)\left(\overline{\theta}-\underline{\theta}\right)\\+(1-\alpha)\left(1-\lambda\left(\overline{\theta}\right)\right)\left(V_1\left(x\left(\overline{\theta},\overline{\eta}\right),y\left(\overline{\theta},\overline{\eta}\right)\right)-\overline{\theta}\right)-\alpha\left(1-\lambda\left(\underline{\theta}\right)\right)\left(\overline{\theta}-\underline{\theta}\right)\end{array}\right)=0.$$

By $V_{11} < 0$ and $x(\overline{\theta}, \underline{\eta}) - x(\overline{\theta}, \overline{\eta}) > 0$

$$V_1\left(x\left(\overline{\theta}, \underline{\eta}\right), y\left(\overline{\theta}, \underline{\eta}\right)\right) < V_1\left(x\left(\overline{\theta}, \overline{\eta}\right), y\left(\overline{\theta}, \underline{\eta}\right)\right)$$

By $V_{12} < 0$ and $y(\overline{\theta}, \underline{\eta}) - y(\overline{\theta}, \overline{\eta}) > 0$

$$V_1\left(x\left(\overline{\theta}, \overline{\eta}\right), y\left(\overline{\theta}, \underline{\eta}\right)\right) < V_1\left(x\left(\overline{\theta}, \overline{\eta}\right), y\left(\overline{\theta}, \overline{\eta}\right)\right).$$

Taken together, we have

$$V_1\left(x\left(\overline{\theta}, \underline{\eta}\right), y\left(\overline{\theta}, \underline{\eta}\right)\right) < V_1\left(x\left(\overline{\theta}, \overline{\eta}\right), y\left(\overline{\theta}, \overline{\eta}\right)\right).$$

Combining with the implications of the first-order conditions with respect to x we obtain

$$\left(V_1\left(x\left(\overline{\theta}, \underline{\eta}\right), y\left(\overline{\theta}, \underline{\eta}\right)\right) - \overline{\theta}\right) < \frac{\alpha}{(1-\alpha)}\left(\overline{\theta} - \underline{\theta}\right) < \left(V_1\left(x\left(\overline{\theta}, \overline{\eta}\right), y\left(\overline{\theta}, \overline{\eta}\right)\right) - \overline{\theta}\right).$$

Substituting into (3.61), using $\xi = 0$, and simplifying, we have

$$\alpha\left(\left(\lambda\left(\overline{\theta}\right) - \lambda\left(\underline{\theta}\right)\right)\left(\overline{\theta} - \underline{\theta}\right)\right) > \nu\left(\overline{\theta} - \underline{\theta}\right),$$

which would imply that $v < 0$, a contradiction.

It follows that for $V_{12} < 0$, the optimum of program P_i features $x(\overline{\theta}, \underline{\eta}) - x(\overline{\theta}, \overline{\eta}) = 0$.

Case II): $V_{12} \geq 0$.

If $V_{12} \leq -V_{11}\frac{(\overline{\eta}-\underline{\eta})}{(\overline{\theta}-\underline{\theta})}$, then the optimum of program P_i features $(\overline{\eta} - \underline{\eta})\left[y(\overline{\theta}, \underline{\eta}) - y(\overline{\theta}, \overline{\eta})\right] > (\overline{\theta} - \underline{\theta})\left[x(\overline{\theta}, \underline{\eta}) - x(\overline{\theta}, \overline{\eta})\right]$.

We know that both constraints cannot bind simultaneously.

Hence, if $(\overline{\eta} - \underline{\eta})\left[y(\overline{\theta}, \underline{\eta}) - y(\overline{\theta}, \overline{\eta})\right] = (\overline{\theta} - \underline{\theta})\left[x(\overline{\theta}, \underline{\eta}) - x(\overline{\theta}, \overline{\eta})\right]$, then necessarily $x(\overline{\theta}, \underline{\eta}) - x(\overline{\theta}, \overline{\eta}) > 0$. Suppose this is the case, so $\xi = 0$ and $\nu > 0$. Define $Y(\overline{\theta}, \overline{\eta}) = y(\overline{\theta}, \underline{\eta}) - \frac{(\overline{\theta}-\underline{\theta})}{(\overline{\eta}-\underline{\eta})}\left[x(\overline{\theta}, \underline{\eta}) - x(\overline{\theta}, \overline{\eta})\right]$. For $\nu \neq 0$ the first-order conditions with respect to x

are given by

$$\left(\begin{array}{c} (1-\alpha)\lambda\left(\overline{\theta}\right)\left(V_1\left(x\left(\overline{\theta},\underline{\eta}\right),y\left(\overline{\theta},\underline{\eta}\right)\right)-\overline{\theta}\right)-\alpha\lambda\left(\underline{\theta}\right)\left(\overline{\theta}-\underline{\theta}\right) \\ -\frac{\left(\overline{\theta}-\underline{\theta}\right)}{\left(\overline{\eta}-\underline{\eta}\right)}\left((1-\alpha)\left(1-\lambda\left(\overline{\theta}\right)\right)\left(V_2\left(x\left(\overline{\theta},\overline{\eta}\right),Y\left(\overline{\theta},\overline{\eta}\right)\right)-\overline{\eta}\right)-\alpha\left(\lambda\left(\underline{\theta}\right)-\lambda\left(\overline{\theta}\right)\right)\left(\overline{\eta}-\underline{\eta}\right)\right) \end{array}\right)$$
$$=0$$

and

$$\left(\begin{array}{c} (1-\alpha)\left(1-\lambda\left(\overline{\theta}\right)\right)\left(V_1\left(x\left(\overline{\theta},\overline{\eta}\right),y\left(\overline{\theta},\overline{\eta}\right)\right)-\overline{\theta}\right)-\alpha\left(1-\lambda\left(\underline{\theta}\right)\right)\left(\overline{\theta}-\underline{\theta}\right) \\ +\frac{\left(\overline{\theta}-\underline{\theta}\right)}{\left(\overline{\eta}-\underline{\eta}\right)}\left((1-\alpha)\left(1-\lambda\left(\overline{\theta}\right)\right)\left(V_2\left(x\left(\overline{\theta},\overline{\eta}\right),Y\left(\overline{\theta},\overline{\eta}\right)\right)-\overline{\eta}\right)-\alpha\left(\lambda\left(\underline{\theta}\right)-\lambda\left(\overline{\theta}\right)\right)\left(\overline{\eta}-\underline{\eta}\right)\right) \end{array}\right)$$
$$=0.$$

These conditions imply

$$\left(\begin{array}{c} (1-\alpha)\lambda\left(\overline{\theta}\right)\left(V_1\left(x\left(\overline{\theta},\underline{\eta}\right),y\left(\overline{\theta},\underline{\eta}\right)\right)-\overline{\theta}\right)-\alpha\lambda\left(\underline{\theta}\right)\left(\overline{\theta}-\underline{\theta}\right) \\ +(1-\alpha)\left(1-\lambda\left(\overline{\theta}\right)\right)\left(V_1\left(x\left(\overline{\theta},\overline{\eta}\right),y\left(\overline{\theta},\overline{\eta}\right)\right)-\overline{\theta}\right)-\alpha\left(1-\lambda\left(\underline{\theta}\right)\right)\left(\overline{\theta}-\underline{\theta}\right) \end{array}\right)=0. \quad (3.68)$$

Define s such that $s>0$, $s=y\left(\overline{\theta},\underline{\eta}\right)-y\left(\overline{\theta},\overline{\eta}\right)$ and $x\left(\overline{\theta},\underline{\eta}\right)=x\left(\overline{\theta},\overline{\eta}\right)+\frac{\left(\overline{\eta}-\underline{\eta}\right)}{\left(\overline{\theta}-\underline{\theta}\right)}s$. Then by

$$\begin{aligned} & V_1\left(x\left(\overline{\theta},\underline{\eta}\right),y\left(\overline{\theta},\underline{\eta}\right)\right) \\ & =V_1\left(x\left(\overline{\theta},\overline{\eta}\right),y\left(\overline{\theta},\overline{\eta}\right)\right)+\int_0^s\frac{\partial V_1\left(x\left(\overline{\theta},\overline{\eta}\right)+\frac{\left(\overline{\eta}-\underline{\eta}\right)}{\left(\overline{\theta}-\underline{\theta}\right)}k,y\left(\overline{\theta},\overline{\eta}\right)+k\right)}{\partial k}dk \end{aligned}$$

and

$$\frac{\partial V_1\left(x\left(\overline{\theta},\overline{\eta}\right)+\frac{\left(\overline{\eta}-\underline{\eta}\right)}{\left(\overline{\theta}-\underline{\theta}\right)}k,y\left(\overline{\theta},\overline{\eta}\right)+k\right)}{\partial k}=\left(\begin{array}{c} \frac{\left(\overline{\eta}-\underline{\eta}\right)}{\left(\overline{\theta}-\underline{\theta}\right)}V_{11}\left(x\left(\overline{\theta},\overline{\eta}\right)+\frac{\left(\overline{\eta}-\underline{\eta}\right)}{\left(\overline{\theta}-\underline{\theta}\right)}k,y\left(\overline{\theta},\overline{\eta}\right)+k\right) \\ +V_{12}\left(x\left(\overline{\theta},\overline{\eta}\right)+\frac{\left(\overline{\eta}-\underline{\eta}\right)}{\left(\overline{\theta}-\underline{\theta}\right)}k,y\left(\overline{\theta},\overline{\eta}\right)+k\right) \end{array}\right)$$

we have $V_1\left(x\left(\overline{\theta},\underline{\eta}\right),y\left(\overline{\theta},\underline{\eta}\right)\right)\leq V_1\left(x\left(\overline{\theta},\overline{\eta}\right),y\left(\overline{\theta},\overline{\eta}\right)\right)$ since $V_{12}\leq -V_{11}\frac{\left(\overline{\eta}-\underline{\eta}\right)}{\left(\overline{\theta}-\underline{\theta}\right)}$ implies

$$\frac{\partial V_1\left(x\left(\overline{\theta},\overline{\eta}\right)+\frac{\left(\overline{\eta}-\underline{\eta}\right)}{\left(\overline{\theta}-\underline{\theta}\right)}k,y\left(\overline{\theta},\overline{\eta}\right)+k\right)}{\partial k}\leq 0 \text{ for all } k>0$$

and $x\left(\overline{\theta},\underline{\eta}\right)-x\left(\overline{\theta},\overline{\eta}\right)>0$. $V_1\left(x\left(\overline{\theta},\underline{\eta}\right),y\left(\overline{\theta},\underline{\eta}\right)\right)\leq V_1\left(x\left(\overline{\theta},\overline{\eta}\right),y\left(\overline{\theta},\overline{\eta}\right)\right)$ implies by (3.68) that

$$\left(V_1\left(x\left(\overline{\theta},\underline{\eta}\right),y\left(\overline{\theta},\underline{\eta}\right)\right)-\overline{\theta}\right)\leq\frac{\alpha}{(1-\alpha)}\left(\overline{\theta}-\underline{\theta}\right)\tag{3.69}$$

By (3.69) into (3.61)

$$\left(\begin{array}{c}(1-\alpha)\lambda\left(\overline{\theta}\right)\frac{\alpha}{(1-\alpha)}\left(\overline{\theta}-\underline{\theta}\right)\\-\alpha\lambda\left(\underline{\theta}\right)\left(\overline{\theta}-\underline{\theta}\right)+\xi-\nu\left(\overline{\theta}-\underline{\theta}\right)\end{array}\right)\geq 0$$

which is equivalent to

$$-\alpha\left(\lambda\left(\underline{\theta}\right)-\lambda\left(\overline{\theta}\right)\right)\left(\overline{\eta}-\underline{\eta}\right)\geq\nu\left(\overline{\eta}-\underline{\eta}\right)$$

which is true only if $\nu<0$ since $\xi=0$. Hence, we get a contradiction to $\nu>0$ contradicting that $\left(\overline{\eta}-\underline{\eta}\right)\left[y\left(\overline{\theta},\underline{\eta}\right)-y\left(\overline{\theta},\overline{\eta}\right)\right]\geq\left(\overline{\theta}-\underline{\theta}\right)\left[x\left(\overline{\theta},\underline{\eta}\right)-x\left(\overline{\theta},\overline{\eta}\right)\right]$ binds in singularity at the optimum of program P_i.

Part III) Comparison between the programs.

As a preliminary argument, note that if $x\left(\overline{\theta},\overline{\eta}\right)-x\left(\overline{\theta},\underline{\eta}\right)=0$, then the objectives of programs P_i and P_{ii} become identical. To see this, note that the objectives are identical up to the costs of implementation, Δ. Moreover, it is easy to verify from Lemma 3 that $\Delta_{ii}-\Delta_i=0$ for $x\left(\overline{\theta},\overline{\eta}\right)-x\left(\overline{\theta},\underline{\eta}\right)=0$.

For $V_{12}<0$, the maximum of program P_i satisfies $x\left(\overline{\theta},\underline{\eta}\right)-x\left(\overline{\theta},\overline{\eta}\right)=0$, whereas the maximum of program P_{ii} satisfies $x\left(\overline{\theta},\overline{\eta}\right)-x\left(\overline{\theta},\underline{\eta}\right)>0$. Hence, the solution of program P_i is feasible but not chosen. By revealed preference, this implies that the solution to program P_{ii} is preferred.

Likewise, for $V_{12}\geq 0$ the optimum of program P_{ii} satisfies $x\left(\overline{\theta},\overline{\eta}\right)-x\left(\overline{\theta},\underline{\eta}\right)=0$. Hence, the solution is feasible under program P_i. If the solution of program P_i is on the line

$x\left(\overline{\theta},\underline{\eta}\right)-x\left(\overline{\theta},\overline{\eta}\right)=0$, then the objectives are identical and hence the solutions of programs P_i and P_{ii} are identical. If the solution of program P_i is off the line $x\left(\overline{\theta},\underline{\eta}\right)-x\left(\overline{\theta},\overline{\eta}\right)=0$, then, by revealed preference, the solution of program P_i yields a strictly higher expected payoff than the solution of program P_{ii}. Taken together, this implies weak payoff dominance of program P_i. ■

Proof of Lemma 3.3.4. The result is essentially a corollary to Lemma 3.2.1. By the same arguments as given there, we have

$$\left(\overline{\eta}-\underline{\eta}\right)\left(y\left(\theta,\underline{\eta}\right)-y\left(\theta,\overline{\eta}\right)\right)>\left(\overline{\theta}-\underline{\theta}\right)\left(x\left(\theta,\underline{\eta}\right)-x\left(\theta,\overline{\eta}\right)\right)>0$$

iff $V_{12}>0$ and

$$0>\int_{\underline{\eta}}^{\overline{\eta}}\frac{V_{11}+\frac{\left(\overline{\theta}-\underline{\theta}\right)}{\left(\overline{\eta}-\underline{\eta}\right)}V_{12}}{V_{11}V_{22}-V_{12}^2}\left(\theta,\eta\right)d\eta,$$

which is satisfied if $V_{12}<-\frac{\left(\overline{\eta}-\underline{\eta}\right)}{\left(\overline{\theta}-\underline{\theta}\right)}V_{11}$ for all $\left(x,y\right)$.

The proof of the remaining statements uses identical arguments and is therefore omitted. ■

Proof of Proposition 3.3.2. We show that the neglected constraint is satisfied under the assumptions.

Preliminaries:

For convenience, recall that the unconstrained solution (in the sense of unconstrained by the implementation sets $\mathbb{X}_j$ for $j=i$ or $j=ii$, respectively) satisfies

$$V_1\left(x,y\right)\left(\underline{\theta},\eta\right)=\underline{\theta} \tag{3.70}$$

$$V_2\left(x,y\right)\left(\underline{\theta},\eta\right)=\eta$$

for $\eta \in \left\{\underline{\eta}, \overline{\eta}\right\}$ and

$$V_1(x,y)\left(\overline{\theta},\underline{\eta}\right) = \overline{\theta} + \frac{\alpha}{(1-\alpha)}\frac{\lambda_j}{\lambda\left(\overline{\theta}\right)}\left(\overline{\theta}-\underline{\theta}\right) \tag{3.71}$$
$$V_2(x,y)\left(\overline{\theta},\underline{\eta}\right) = \underline{\eta}$$

$$V_1(x,y)\left(\overline{\theta},\overline{\eta}\right) = \overline{\theta} + \frac{\alpha}{(1-\alpha)}\frac{(1-\lambda_j)}{\left(1-\lambda\left(\overline{\theta}\right)\right)}\left(\overline{\theta}-\underline{\theta}\right) \tag{3.72}$$
$$V_2(x,y)\left(\overline{\theta},\overline{\eta}\right) = \overline{\eta} + \frac{\alpha}{(1-\alpha)}\frac{\left(\lambda(\underline{\theta})-\lambda\left(\overline{\theta}\right)\right)}{\left(1-\lambda\left(\overline{\theta}\right)\right)}\left(\overline{\eta}-\underline{\eta}\right),$$

where $j = i, ii$ and by convention $\lambda_i = \lambda(\underline{\theta})$ and $\lambda_{ii} = \lambda\left(\overline{\theta}\right)$. Define the following artificial systems of equations for $\theta, \eta \in \left[\underline{\theta}, \overline{\theta}\right] \times \left[\underline{\eta}, \overline{\eta}\right]$:

$$V_1(x,y)(\theta,\eta) = \theta + \frac{\alpha}{(1-\alpha)}\frac{\lambda_j}{\lambda\left(\overline{\theta}\right)}\left(\theta-\underline{\theta}\right) \tag{3.73}$$
$$V_2(x,y)(\theta,\eta) = \eta$$

and

$$V_1(x,y)(\theta,\eta) = \theta + \frac{\alpha}{(1-\alpha)}\left(\theta-\underline{\theta}\right) \tag{3.74}$$
$$V_2(x,y)(\theta,\eta) = \eta + \frac{\alpha}{(1-\alpha)}\frac{\left(\lambda(\underline{\theta})-\lambda\left(\overline{\theta}\right)\right)}{\left(1-\lambda\left(\overline{\theta}\right)\right)}\left(\eta-\underline{\eta}\right).$$

Note that these systems are defined on convex domains. Moreover, the solution to (3.73) for $\theta = \underline{\theta}$ corresponds to the solution of (3.70), and for $\theta = \overline{\theta}$ and $\eta = \underline{\eta}$, the solution to (3.73) corresponds to the solution of (3.71). Likewise, for $\theta = \underline{\theta}$ and $\eta = \underline{\eta}$, the solution to (3.74) corresponds to the solution to (3.70); for $\theta = \overline{\theta}$ and $\eta = \underline{\eta}$, the solution to (3.74) corresponds to the solution of (3.71) for $\lambda_j = \lambda_{ii} = \lambda\left(\overline{\theta}\right)$; and for $\theta = \overline{\theta}$ and $\eta = \overline{\eta}$, the solution to (3.74) corresponds to the solution to (3.72) for $\lambda_j = \lambda_{ii} = \lambda\left(\overline{\theta}\right)$.

So, systems (3.73) and (3.74) are defined on convex domains. Moreover, the solutions to the systems at extreme points of the domain correspond to the economically meaningful solutions of (3.70), (3.71), and (3.72), respectively. Hence, we can conveniently apply calculus to the artificial system (3.73) and (3.74) to determine differences between allocation choices.

Part I) The case of independent goods: $V_{12} = 0$.

From Proposition 3.3.1 we know that program P_i solves the reduced problem for $V_{12} = 0$. Hence, the neglected constraint takes the form

$$\begin{gathered}\left(\lambda\left(\underline{\theta}\right)-\lambda\left(\overline{\theta}\right)\right)\left(\overline{\eta}-\underline{\eta}\right)\left(y\left(\underline{\theta},\underline{\eta}\right)-y\left(\overline{\theta},\overline{\eta}\right)\right)\\ +\left(\overline{\theta}-\underline{\theta}\right)\left(\mathbb{E}_{\eta|\overline{\theta}}\left[x\left(\underline{\theta},\eta\right)\right]-\mathbb{E}_{\eta|\underline{\theta}}\left[x\left(\overline{\theta},\eta\right)\right]\right)\geq 0.\end{gathered}$$

Sufficient conditions for the neglected constraint to hold are

$$y\left(\underline{\theta},\underline{\eta}\right)-y\left(\overline{\theta},\overline{\eta}\right)\geq 0$$

and

$$\left(\mathbb{E}_{\eta|\overline{\theta}}\left[x\left(\underline{\theta},\eta\right)\right]-\mathbb{E}_{\eta|\underline{\theta}}\left[x\left(\overline{\theta},\eta\right)\right]\right)\geq 0.$$

Moreover, we know again from Proposition 3.3.1 that $x\left(\overline{\theta},\underline{\eta}\right) = x\left(\overline{\theta},\overline{\eta}\right) = x\left(\overline{\theta}\right)$ at the solution. So, the relevant first-order conditions describing the optimum simplify to

$$V_1\left(x\left(\underline{\theta},\eta\right)\right)=\underline{\theta}$$

and

$$V_2\left(y\left(\underline{\theta},\eta\right)\right)=\eta$$

for $\eta \in \left\{\underline{\eta},\overline{\eta}\right\}$,

$$(1-\alpha)\left[V_1\left(x\left(\overline{\theta}\right)\right)-\overline{\theta}\right]=\alpha\left(\overline{\theta}-\underline{\theta}\right),$$

and finally

$$V_2\left(y\left(\overline{\theta},\underline{\eta}\right)\right)=\underline{\eta}$$

and

$$V_2\left(y\left(\overline{\theta},\overline{\eta}\right)\right)=\overline{\eta}+\frac{\alpha}{(1-\alpha)}\frac{\left(\lambda\left(\underline{\theta}\right)-\lambda\left(\overline{\theta}\right)\right)}{\left(1-\lambda\left(\overline{\theta}\right)\right)}\left(\overline{\eta}-\underline{\eta}\right).$$

It is easy to see (by concavity of V), that $x\left(\underline{\theta},\underline{\eta}\right)=x\left(\underline{\theta},\overline{\eta}\right)>x\left(\overline{\theta}\right)$, so $\left(\mathbb{E}_{\eta|\overline{\theta}}\left[x\left(\underline{\theta},\eta\right)\right]-\mathbb{E}_{\eta|\underline{\theta}}\left[x\left(\overline{\theta},\eta\right)\right]\right)\geq 0.$ is satisfied. By the same argument, we also have $y\left(\underline{\theta},\underline{\eta}\right)-y\left(\overline{\theta},\overline{\eta}\right)\geq 0.$

Part II)

1. The case of complements.

For the case of complements with $0\leq V_{12}<-V_{11}\frac{(\overline{\eta}-\underline{\eta})}{(\overline{\theta}-\underline{\theta})}$ for all x,y,, by Lemmas 3 and 4, the neglected constraint (3.7) is equivalent to

$$\begin{aligned}&\left(\lambda\left(\underline{\theta}\right)-\lambda\left(\overline{\theta}\right)\right)\left(\overline{\eta}-\underline{\eta}\right)\left(y\left(\underline{\theta},\underline{\eta}\right)-y\left(\overline{\theta},\overline{\eta}\right)\right)\\&+\left(\overline{\theta}-\underline{\theta}\right)\left(\mathbb{E}_{\eta|\overline{\theta}}\left[x\left(\underline{\theta},\eta\right)\right]-\mathbb{E}_{\eta|\underline{\theta}}\left[x\left(\overline{\theta},\eta\right)\right]\right)\geq 0.\end{aligned}$$

Sufficient conditions for the neglected constraint to hold are

$$y\left(\underline{\theta},\underline{\eta}\right)-y\left(\overline{\theta},\overline{\eta}\right)\geq 0$$

and

$$\left(\mathbb{E}_{\eta|\overline{\theta}}\left[x\left(\underline{\theta},\eta\right)\right]-\mathbb{E}_{\eta|\underline{\theta}}\left[x\left(\overline{\theta},\eta\right)\right]\right)\geq 0.$$

We now provide sufficient conditions such that the unconstrained solution satisfies these monotonicity restrictions.

We can write

$$y\left(\underline{\theta},\underline{\eta}\right)-y\left(\overline{\theta},\overline{\eta}\right)=y\left(\underline{\theta},\underline{\eta}\right)-y\left(\overline{\theta},\underline{\eta}\right)+y\left(\overline{\theta},\underline{\eta}\right)-y\left(\overline{\theta},\overline{\eta}\right).$$

Incentive compatibility with respect to η alone requires that $y\left(\overline{\theta},\underline{\eta}\right)\geq y\left(\overline{\theta},\overline{\eta}\right)$. Hence, a sufficient condition for $y\left(\underline{\theta},\underline{\eta}\right)-y\left(\overline{\theta},\overline{\eta}\right)\geq 0$ is that $\left(y\left(\underline{\theta},\underline{\eta}\right)-y\left(\overline{\theta},\underline{\eta}\right)\right)\geq 0$. In turn, this follows trivially from the fact that $\theta+\frac{\alpha}{(1-\alpha)}\frac{\lambda_j}{\lambda(\overline{\theta})}\left(\theta-\underline{\theta}\right)$ is increasing in θ and thus that an increase in θ reduces x, which by complementarity reduces y.

A sufficient condition for $\left(\mathbb{E}_{\eta|\overline{\theta}}\left[x\left(\underline{\theta},\eta\right)\right]-\mathbb{E}_{\eta|\underline{\theta}}\left[x\left(\overline{\theta},\eta\right)\right]\right)\geq 0$ is that

$$\min_{\eta\in\left\{\underline{\eta},\overline{\eta}\right\}} x\left(\underline{\theta},\eta\right)\geq \max_{\eta\in\left\{\underline{\eta},\overline{\eta}\right\}} x\left(\overline{\theta},\eta\right),$$

which in turn holds if

$$x\left(\underline{\theta},\underline{\eta}\right)\geq x\left(\underline{\theta},\overline{\eta}\right)\geq x\left(\overline{\theta},\underline{\eta}\right)\geq x\left(\overline{\theta},\overline{\eta}\right).$$

It is straightforward to see that $x\left(\underline{\theta},\underline{\eta}\right)\geq x\left(\underline{\theta},\overline{\eta}\right)$, since x and y are complements. Similarly, $x\left(\overline{\theta},\underline{\eta}\right)\geq x\left(\overline{\theta},\overline{\eta}\right)$ follows from the fact that $\frac{\lambda(\underline{\theta})}{\lambda(\overline{\theta})}\geq\frac{(1-\lambda(\underline{\theta}))}{\left(1-\lambda\left(\overline{\theta}\right)\right)}$ and that x and y are complements. So, we need to show that $x\left(\underline{\theta},\overline{\eta}\right)\geq x\left(\overline{\theta},\underline{\eta}\right)$. We can write

$$x\left(\underline{\theta},\overline{\eta}\right)-x\left(\overline{\theta},\underline{\eta}\right)=x\left(\underline{\theta},\overline{\eta}\right)-x\left(\underline{\theta},\underline{\eta}\right)+x\left(\underline{\theta},\underline{\eta}\right)-x\left(\overline{\theta},\underline{\eta}\right).$$

The differences on the right-hand side of this equation can be conveniently computed from (3.73), since we argued above that the types on the right-hand side correspond to extreme points in the domain of definition of (3.73). Differentiating the system of equations (3.73), we obtain

$$x\left(\underline{\theta},\overline{\eta}\right)-x\left(\underline{\theta},\underline{\eta}\right)=\int_{\underline{\eta}}^{\overline{\eta}}\frac{-V_{12}}{V_{11}V_{22}-V_{12}^{2}}\left(\underline{\theta},\eta\right)d\eta=\left(\overline{\eta}-\underline{\eta}\right)\frac{-V_{12}}{V_{11}V_{22}-V_{12}^{2}}\left(\underline{\theta},\hat{\eta}\right).$$

where the first equality follows from setting $\theta=\underline{\theta}$ in (3.73) and applying Cramer's rule and the second equality from the mean value theorem, for some $\hat{\eta}\in\left[\underline{\eta},\overline{\eta}\right]$. Likewise, by setting $\eta=\underline{\eta}$ in (3.73) and $j=i$ so that $\lambda_j=\lambda\left(\underline{\theta}\right)$, and applying Cramer's rule, we have

$$\begin{aligned} x\left(\underline{\theta},\underline{\eta}\right)-x\left(\overline{\theta},\underline{\eta}\right)&=\int_{\overline{\theta}}^{\underline{\theta}}\frac{\partial x\left(\theta,\underline{\eta}\right)}{\partial\theta}d\theta=\left(1+\frac{\alpha}{(1-\alpha)}\frac{\lambda\left(\underline{\theta}\right)}{\lambda\left(\overline{\theta}\right)}\right)\int_{\overline{\theta}}^{\underline{\theta}}\frac{V_{22}}{V_{11}V_{22}-V_{12}^{2}}d\theta\\ &=-\left(\overline{\theta}-\underline{\theta}\right)\left(1+\frac{\alpha}{(1-\alpha)}\frac{\lambda\left(\underline{\theta}\right)}{\lambda\left(\overline{\theta}\right)}\right)\frac{V_{22}}{V_{11}V_{22}-V_{12}^{2}}\left(\hat{\theta},\underline{\eta}\right). \end{aligned}$$

for some $\hat{\theta}\in\left[\underline{\theta},\overline{\theta}\right]$, where the last equality follows again by the mean value theorem.

So, we have $x\left(\underline{\theta},\overline{\eta}\right) \geq x\left(\overline{\theta},\underline{\eta}\right)$ iff

$$\left(\overline{\eta}-\underline{\eta}\right)\frac{-V_{12}}{V_{11}V_{22}-V_{12}^2}\left(\underline{\theta},\hat{\eta}\right)-\left(\overline{\theta}-\underline{\theta}\right)\left(1+\frac{\alpha}{(1-\alpha)}\frac{\lambda\left(\underline{\theta}\right)}{\lambda\left(\overline{\theta}\right)}\right)\frac{V_{22}}{V_{11}V_{22}-V_{12}^2}\left(\hat{\theta},\underline{\eta}\right)\geq 0.$$

In turn, this condition is satisfied if

$$\begin{aligned}&\frac{\left(\overline{\theta}-\underline{\theta}\right)}{\left(\overline{\eta}-\underline{\eta}\right)}\left(1+\frac{\alpha}{(1-\alpha)}\frac{\lambda\left(\underline{\theta}\right)}{\lambda\left(\overline{\theta}\right)}\right)\min_{x,y}\frac{-V_{22}}{V_{11}V_{22}-V_{12}^2}(x,y)\\&\geq \max_{x,y}\frac{V_{12}}{V_{11}V_{22}-V_{12}^2}(x,y).\end{aligned}$$

Since the left-hand side is increasing in α, the condition is hardest to satisfy for $\alpha=0$, which is the condition given in the proposition.

2. The case of substitutes:

For $0>V_{12}>V_{11}\frac{\left(\overline{\eta}-\underline{\eta}\right)}{\left(\overline{\theta}-\underline{\theta}\right)}$ for all x,y, the neglected constraint is equivalent to

$$\left(\begin{array}{c}\left(\lambda\left(\underline{\theta}\right)-\lambda\left(\overline{\theta}\right)\right)\left(\overline{\eta}-\underline{\eta}\right)\left(y\left(\underline{\theta},\underline{\eta}\right)-y\left(\overline{\theta},\overline{\eta}\right)\right)\\+\left(\overline{\theta}-\underline{\theta}\right)x\left(\underline{\theta},\underline{\eta}\right)-\left(\lambda\left(\overline{\theta}\right)\left(\overline{\theta}-\underline{\theta}\right)x\left(\overline{\theta},\underline{\eta}\right)+\left(1-\lambda\left(\overline{\theta}\right)\right)\left(\overline{\theta}-\underline{\theta}\right)x\left(\overline{\theta},\overline{\eta}\right)\right)\end{array}\right)\geq 0.$$

Equivalently, this can be written as

$$\left(\begin{array}{c}\left(\lambda\left(\underline{\theta}\right)-\lambda\left(\overline{\theta}\right)\right)\left(\overline{\eta}-\underline{\eta}\right)\left(y\left(\underline{\theta},\underline{\eta}\right)-y\left(\overline{\theta},\overline{\eta}\right)\right)\\+\left(\overline{\theta}-\underline{\theta}\right)\left(x\left(\underline{\theta},\underline{\eta}\right)-x\left(\overline{\theta},\overline{\eta}\right)\right)-\lambda\left(\overline{\theta}\right)\left(\overline{\theta}-\underline{\theta}\right)\left(x\left(\overline{\theta},\underline{\eta}\right)-x\left(\overline{\theta},\overline{\eta}\right)\right)\end{array}\right)\geq 0.$$

Recall that for $(x,y)\in X_{ii}$, we have

$$\left(\overline{\eta}-\underline{\eta}\right)\left(y\left(\theta,\underline{\eta}\right)-y\left(\theta,\overline{\eta}\right)\right)\geq -\left(\overline{\theta}-\underline{\theta}\right)\left(x\left(\theta,\underline{\eta}\right)-x\left(\theta,\overline{\eta}\right)\right)\geq 0,$$

so the third term on the left-hand side is nonnegative. For the case where $\lambda\left(\underline{\theta}\right)=\lambda\left(\overline{\theta}\right)$, the first term is zero and we only need to show that

$$x\left(\underline{\theta},\underline{\eta}\right)-x\left(\overline{\theta},\overline{\eta}\right)\geq 0.$$

We can write

$$x\left(\overline{\theta},\overline{\eta}\right)-x\left(\underline{\theta},\underline{\eta}\right)=x\left(\overline{\theta},\overline{\eta}\right)-x\left(\overline{\theta},\underline{\eta}\right)+x\left(\overline{\theta},\underline{\eta}\right)-x\left(\underline{\theta},\underline{\eta}\right).$$

The types on the right-hand side correspond to extreme points of the domain of definition of (3.74) . Therefore, we obtain - by the same arguments as used for the complements case -

$$\begin{aligned}x\left(\overline{\theta},\overline{\eta}\right)-x\left(\underline{\theta},\underline{\eta}\right)&=\left(\overline{\eta}-\underline{\eta}\right)\left(1+\frac{\alpha}{(1-\alpha)}\frac{\left(\lambda\left(\underline{\theta}\right)-\lambda\left(\overline{\theta}\right)\right)}{\left(1-\lambda\left(\overline{\theta}\right)\right)}\right)\frac{-V_{12}}{V_{11}V_{22}-V_{12}^2}\left(\overline{\theta},\hat{\eta}\right)\\&+\left(\overline{\theta}-\underline{\theta}\right)\frac{1}{1-\alpha}\frac{V_{22}}{V_{11}V_{22}-V_{12}^2}\left(\hat{\theta},\underline{\eta}\right)\end{aligned}$$

for some values $\hat{\theta}\in\left[\underline{\theta},\overline{\theta}\right]$ and $\hat{\eta}\in\left[\underline{\eta},\overline{\eta}\right]$. Hence, we have $x\left(\underline{\theta},\underline{\eta}\right)-x\left(\overline{\theta},\overline{\eta}\right)\geq 0$, if

$$\begin{aligned}&\frac{\overline{\eta}-\underline{\eta}}{\overline{\theta}-\underline{\theta}}\left(1-\alpha+\alpha\frac{\left(\lambda\left(\underline{\theta}\right)-\lambda\left(\overline{\theta}\right)\right)}{\left(1-\lambda\left(\overline{\theta}\right)\right)}\right)\min_{x,y}\frac{V_{12}}{V_{11}V_{22}-V_{12}^2}\\&\geq\max_{x,y}\frac{V_{22}}{V_{11}V_{22}-V_{12}^2}\end{aligned}$$

Since $\frac{\left(\lambda\left(\underline{\theta}\right)-\lambda\left(\overline{\theta}\right)\right)}{\left(1-\lambda\left(\overline{\theta}\right)\right)}<1$, the expression on the left-hand side of the inequality is smallest for $\alpha=0$, so the condition is satisfied if

$$\frac{\overline{\eta}-\underline{\eta}}{\overline{\theta}-\underline{\theta}}\min_{x,y}\frac{V_{12}}{V_{11}V_{22}-V_{12}^2}\geq\max_{x,y}\frac{V_{22}}{V_{11}V_{22}-V_{12}^2}.$$

Finally, we need to show that the optimal allocations that solve the reduced problems P_i and P_{ii}, respectively, are elements of $\mathbb{X}_i^{int}$ or $\mathbb{X}_{ii}^{int}$, respectively. Recall from Lemma 4 that the first-best allocation is an element of $\mathbb{X}_i^{int}$ or $\mathbb{X}_{ii}^{int}$, respectively, precisely under the conditions that make either program P_i or P_{ii} generate a higher value to the principal. Now consider, for $j=i,ii,iii,iv$, the problems

$$\max_{(x,y)\in\cup_j\mathbb{X}_j}\mathrm{P}_j$$

The solution to each of these problems converges uniformly to the first-best allocation as α goes to zero. It follows that the solution of program P_i is in $\mathbb{X}_i^{int}$ for α close enough to zero if $0 < V_{12} < -\frac{\overline{\eta}-\underline{\eta}}{\overline{\theta}-\underline{\theta}}V_{11}$ and that the solution of program P_{ii} is in $\mathbb{X}_{ii}^{int}$ for α close enough to zero if $\frac{\overline{\eta}-\underline{\eta}}{\overline{\theta}-\underline{\theta}}V_{11} < V_{12} < 0$. ■

Proof of Proposition 3. From Lemma 4, we have conditions such that the first-best allocation is in $\mathbb{X}_i^{int}$. Hence, in the limit as α goes to zero, the allocations that achieve the maxima W_j are in $\mathbb{X}_j^{int}$. So, we need to show that these maximizers satisfy the neglected constraint. We focus on the case of strong complements. Exactly the same argument can be given for strong substitutes.

For the example, for $\delta \in (-1, 1)$ and β sufficiently large to generate interior solutions, the first-best allocation is given by

$$\begin{aligned} x(\theta,\eta) &= \tfrac{1}{1-\delta^2}\left(\beta(1+\delta)-\theta-\delta\eta\right) \\ y(\theta,\eta) &= \tfrac{1}{1-\delta^2}\left(\beta(1+\delta)-\eta-\theta\delta\right) \end{aligned}$$

The neglected constraint for $(x, y) \in \mathbb{X}_{iii}$ takes the form

$$\begin{aligned} 0 \geq {} & \left(\overline{\theta}-\underline{\theta}\right)\left(x\left(\overline{\theta},\underline{\eta}\right)-x\left(\underline{\theta},\overline{\eta}\right)\right)+\left(\overline{\eta}-\underline{\eta}\right)\left(y\left(\overline{\theta},\overline{\eta}\right)-y\left(\overline{\theta},\underline{\eta}\right)\right) \\ & +\lambda\left(\underline{\theta}\right)\left(\overline{\eta}-\underline{\eta}\right)\left(y\left(\overline{\theta},\underline{\eta}\right)-y\left(\underline{\theta},\underline{\eta}\right)\right)+\lambda\left(\overline{\theta}\right)\left(\overline{\eta}-\underline{\eta}\right)\left(y\left(\underline{\theta},\overline{\eta}\right)-y\left(\overline{\theta},\overline{\eta}\right)\right). \end{aligned}$$

The first-best allocation is in $\mathbb{X}_{iii}$ for $\delta > \frac{(\overline{\eta}-\underline{\eta})}{(\overline{\theta}-\underline{\theta})}$. The buyer's problem remains concave for $\delta < 1$. Both conditions are satisfied for a nonempty set of parameters only if $\frac{(\overline{\eta}-\underline{\eta})}{(\overline{\theta}-\underline{\theta})} < 1$. For the example, the neglected constraint is equivalent to

$$\begin{aligned} 0 \geq {} & \left(\overline{\theta}-\underline{\theta}\right)\left(\frac{1}{1-\delta^2}\left(-\left(\overline{\theta}-\underline{\theta}\right)+\delta\left(\overline{\eta}-\underline{\eta}\right)\right)\right)+\left(\overline{\eta}-\underline{\eta}\right)\left(\frac{1}{1-\delta^2}\left(-\left(\overline{\eta}-\underline{\eta}\right)\right)\right) \\ & +\lambda\left(\underline{\theta}\right)\left(\overline{\eta}-\underline{\eta}\right)\left(\frac{1}{1-\delta^2}\left(-\left(\overline{\theta}-\underline{\theta}\right)\delta\right)\right)+\lambda\left(\overline{\theta}\right)\left(\overline{\eta}-\underline{\eta}\right)\left(\frac{1}{1-\delta^2}\left(\left(\overline{\theta}-\underline{\theta}\right)\delta\right)\right), \end{aligned}$$

which is satisfied if $\delta \leq \frac{(\overline{\theta}-\underline{\theta})}{(\overline{\eta}-\underline{\eta})}$. Since $\frac{(\overline{\theta}-\underline{\theta})}{(\overline{\eta}-\underline{\eta})} > 1$, this condition is automatically satisfied. ■

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