The Hotel of Algebraic Surgery

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Contents

Lobby

The classification of manifolds is a celebrity under the problems in topology and attracted a lot of attention during the last century. One important aspect of this problem is to decide whether there is an n-dimensional closed topological manifold in the homotopy type of a given n-dimensional finite Poincaré space X. The “classic surgery theory” developed by Browder, Novikov, Sullivan, Wall, Kirby and Siebenmann provides an algorithm to decide this question in the form of a two-stage obstruction theory, when $n \geq 5$. This two-stage obstruction reflects the crucial questions:

(1) Does X admit a degree one normal map $\widehat{f}$ from a manifold to X?

(2) If the answer is yes, is there a degree one normal map $\widehat{f}$ which is bordant to a homotopy equivalence?

These two question have been dealt with in the following way. A result of Spivak provides us with the Spivak normal fibration $\nu_X : X \to \mathrm{BSG}$, which is a spherical fibration, stably unique in some sense. If X is homotopy equivalent to a closed manifold, then ν_X reduces to a stable topological bundle, say $\overline{\nu}_X : X \to \mathrm{BSTOP}$. This gives a positive answer to question (1). Any reduction $\overline{\nu}_X$ determines a degree one normal map $\widehat{f} : M \to X$ from some n-dimensional closed topological manifold M to X using the Pontrjagin-Thom construction. Such a normal map has a surgery obstruction

$$\theta(\widehat{f}) \in L^w_n(\mathbb{Z}[\pi_1(X)]),$$

which is an element in the L-group* of the group ring $\mathbb{Z}[\pi_1(X)]$. The vanishing of θ is equivalent to a positive answer of question (2). Hence we can reformulate the two-stage obstruction as follows: The Poincaré space X is homotopy equivalent to a manifold if and only if

(1) there are reductions $\overline{\nu}_X : X \to \mathrm{BSTOP}$ of its Spivak normal fibration ν_X and

(2) at least one reduction has an associated degree one normal map $\widehat{f} : M \to X$ such that $\theta(\widehat{f}) = 0$.

The algebraic theory of surgery of Ranicki replaces the two obstructions above with a single obstruction, namely the total surgery obstruction

$$s(X) \in \mathbb{S}_n(X)$$

where $\mathbb{S}_n(X)$ is the n-dimensional structure group of X in the sense of the algebraic theory of surgery, which is a certain abelian group associated to X. It overcomes several drawbacks of the classical approach. First of all, no surgery below the middle dimension in advance is needed anymore before we can apply the whole machinery. Secondly, the algebraic approach provides a uniform definition for the odd and even dimensional cases. Finally, the total surgery obstruction

*Read L^w as Wall's L-group in the sense of the classical theory. Strictly speaking the 'w' should be, additionally to a dedication, a choice of an orientation character $w : \pi_1(X) \to \mathbb{Z}_2$. We use L without the w-superscript for the L-groups in the sense of the algebraic surgery theory.

is defined via an assembly map. In the recent decades there has been a lot of progress in studying these types of maps and especially the related Farrell-Jones and Baum-Connes conjecture [LR05]. One outcome of the algebraic surgery approach is that the Farrell-Jones conjecture implies the Borel conjecture.

A lot of material on algebraic surgery has been developed in the last 40 years and spread out over a few books, some PhD thesis and several papers. Based on the work of Ranicki, Weiss, Quinn, recently Laures and McClure and many others, this thesis gives a complete proof of the total surgery obstruction and the identification of the algebraic surgery exact sequence with its geometric equivalent. In joint work with Macko and Mole there has already been published a first approach [KMM13] to assemble the material concerning the total surgery obstruction in one place in one consistent notation. We wrote it with the intention to create a concise guide to the theory and to fill in the details that we missed when we were learning the theory or that we had heard other mathematicians complain about. However, we still remained sloppy in some points to keep the paper reasonably short. Nevertheless, our attempt to write a self-contained article including all necessary definitions resulted in an unwieldy paper.

Based on our first approach, this thesis is a further development that completely reorganizes and enhances the material in order to make it both a guide for beginners as well as a reliable reference for experts. Unfortunately, this work also fails to achieve completeness in a fully satisfactory way. For the following details we refer to other sources:

- The identification of Wall's surgery obstruction with the quadratic signature $\operatorname{sgn}^{\mathbf{L}_\bullet}_{\mathbb{Z}\pi}$ as carried out by Ranicki in [Ran80a] and [Ran80b]. There is also an upcoming book by Crowley, Lück and Macko which will provide a profound discussion of this identification.

- The construction of the normal signature is a thesis on its own. Here we recap only the absolute case although we need it in full generality as presented in [Wei85a] and [Wei85b]. It is supposed to provide everything we need about normal chain complexes although the connection to our setting is not always obvious.

- The relation between L-groups of algebraic bordism categories and L-spectra, especially the homotopy equivalence between $L_n(\Lambda^{K_+})$ and $L_n(\Lambda_L K)$. Laures and McClure provide a detailed proof in [LM09] based on Ranicki's work ([Ran92, Prop. 13.7], see also [Ran13, Errata for p.140]).

- We do not pay attention to the subtleties of transversality in the topological category as for example the necessarity of using micro and block bundles. We rely on that there is a sufficent notion of transversality in our situation and just use the general term topological bundle. For more details we refer to [KS77] and [FQ90].

- Eventually, we are not able to carry out all algebraic constructions in all detail. You find much more material on these things in Ranicki's extensive book [Ran81] and also in his papers [Ran80a] and [Ran80b].

In order to provide the reader with more insight into the motivation of the unwieldy amount of necessary and abstract definitions, thus, making the proof easier to follow, this thesis is organized in an unconventional way. Its hierarchical structure with different levels of details is based on the ideas of Leron [Ler83]. The details of the structure will be explained in the following chapter, the Reception.

At several points the notation of the original sources appeared not to be outstandingly convincing and I took the liberty of deviating from it. Experts who are used to the original notation find a dictionary in the appendix. We will use the following conventions.

Notation conventions

M	closed manifold of dimension $n \geq 5$
X	Poincaré space of dimension $n \geq 5$
Y	normal space of dimension $n \geq 5$
K	simplicial complex
σ, τ	simplices
ν	spherical fibration
ν_X	Spivak normal fibration of X
$\overline{\nu}, \xi$	bundles
$\mathbb{Z}\pi$	$:= \mathbb{Z}[\pi_1(X)]$, the fundamental group ring of X
R	ring with involution
C, D	chain complexes
φ, ψ, λ	structures on chain complexes
$\mathbb{A}$	additive category
$\mathbb{B}, \mathbb{C}, \mathbb{D}, \mathbb{P}, \mathbb{G}, \mathbb{L}$	categories of chain complexes
$\mathbf{L}, \mathbf{NL}, \mathbf{E}, \mathbf{\Omega}$	spectra
$\widehat{f}$	$:= (\overline{f}, f) \colon M \to X$ a degree one normal map with source a manifold
$\widehat{g}$	$:= (\overline{g}, g) \colon X \to Y$ a degree one normal map with source a Poincaré space
∂	something one dimension lower
δ	something one dimension higher
$\partial\!\!\!\delta$	something of the same dimension (algebraic surgery[†])
$\mathrm{con}^{\varphi/\psi/\gamma}$	chain maps with target the chain complexes where symmetric and quadratic structures respectivly chain bundles live
$\mathrm{sgn}^{\mathbf{L}^\bullet/\mathbf{L}_\bullet/\mathbf{NL}^\bullet}$	creates chain complexes with symmetric/quadratic/normal structures
$\mathrm{sgn}_{\mathbb{Z}\pi}$	creates a structured chain complex in a L-group over $\mathbb{Z}\pi$ (i.e. something Poincaré in the symmetric and quadratic case)

[†]Besides that the symbol $\partial\!\!\!\delta$ looks like a pair of scissors it is justified by the fact that algebraic surgery is a reversible process closely related to the boundary construction denoted by ∂

- $\mathrm{sgn}_{\#}$ creates a structured chain complex not necessarily Poincaré
- $\mathrm{sgn}_{\rightarrow}$ creates a structured chain complex pair
- sgn_X creates something X-mosaicked (i.e. a chain complex in a X-based algebraic bordism category like $\Lambda_L X$ or $\Lambda_G X$)
- Λ algebraic bordism category
- $\Lambda_{L/G/N}$ mosaicked algebraic bordism category with local/global/no Poincaré duality
- $\partial^{S/Q/N}$ boundary construction for symmetric/quadratic/normal chain complexes
- $\partial_{\mathbb{C}}$ boundary construction applied to chain complexes that are $\mathbb{C}$-Poincaré, the result is a chain complex in $\mathbb{C}$

Reception

Welcome to the Hotel of Algebraic Surgery. The hotel serves as the metaphorical frame for the proof of the total surgery obstruction theorem and two corollaries which we discuss at the end of this section.

For your stay here we strongly recommend the use of an electronic device such as a pdf reader which supports hyperlinks.[‡] In that case all you have to know is that, in order to keep this thesis readable, links are not highlighted and appear as usual text but keep in mind that almost everything here is a link. If you do not know or do not remember the meaning of an expression try a click on it. All statements have room numbers. If you do not believe an assertion, a click on the room number will guide you to the next level and you will find a proof. There is also an online version available at `http://surgery-hotel.de`. This thesis is presented there in a self-made pdf viewer especially developed to make this highly linked document easier to read. It provides a foldable tree view for the nested structure of the hotel rooms, definitions are displayed separately at the top and if you keep your mouse over a citation the details of the reference are displayed, if you click on it you get the complete paper.

In case you prefer or are forced to use an old fashioned reading device like paper that does not support clickable links, the margin is extensively used for navigation hints and there has been put some effort in the structure of the proof to keep your reading experience comfortable as well.

The proof is organized in 4 *levels*. Each one provides more details to its preceding level. After each level you find an *in the elevator* paragraph that gives a brief overview of the next level, mentions the main ideas in the proceeding proofs and helps to develop a general picture of what is going on. The levels itself are organized in *rooms*. Each room provides a proof for a statement that was claimed and used in a preceding level. If a new statement is used in a proof it is marked on the margin with its room number and the page number of its proof in the next level is provided.

At the beginning of the rooms you usually find a *porter* paragraph which serves you with more motivational background and references for the material used in the room. It repeats the statement that will be proved and gives an overview of all substatements that will be needed. The notation is explained in Figure 1.

The room itself contains the pure proof without any other distracting material. So if you are basically comfortable with the material, just follow your way through the hotel rooms from level to level as high as you are not willing to believe the statements and hunger for more details. The foundational concepts that are needed all over like the various signatures and boundary constructions, algebraic surgery, Umkehr maps and S-duality, are gathered together in the rooms in the *basement*.

In order to make it easier to find the definitions in the paper version, each room is usually followed up by an *room service* paragraph that gives short and precise definitions of all the terms used in that room. But there is a point where the helpfulness of verbosity bumbs up against the clarity and handiness of compactness so in higher levels we will weaken this paradigm and will

[‡]Especially a back button for hyperlinks is very useful. Usually the keyboard shortcut Cmd+[on MacOS resp. Alt+← on Windows brings you back to the last position.

6 Theorem
The statement we want to prove in this room.

61 Lemma/Proposition
A statement used to prove Theorem 6. Substatements are numbered with an additional digit attached to the number of the room where they are used. So (642) is the second statement used to prove (64) which is part of the proof of (6). Each digit is a link to the corresponding room, i.e. if you click on the 4 in 642 you get to room number 64.

51 (6, 71) Lemma/Proposition
The room numbers in brackets indicate additional rooms where a statement is also needed. So this is a statement that was first stated and used in the proof of room 5 but is also necessary to prove 6 and 71.

[62 → Reference] Exercise/Citation
Statements with white background and the room number in square brackets are not proven in this thesis. They either cite statements of other references or are considered as exercises. So one could say that the black boxes we use are displayed as white boxes in order to remain readable.

⇒ **63 Corollary**
Corollaries are indicated by an implication arrow in front of the box.

Figure 1: Example of a surgery hotel room

not repeat every definition that has already been introduced in a preceding level. An overview of all definitions and notations can be found at the very end in the *help desk* section.

Have a nice stay and enjoy the proof of

Main Theorem (Ranicki). *Let X be a finite Poincaré space of dimension $n \geq 5$. Then X is homotopy equivalent to a closed n-dimensional topological manifold if and only if*

$$0 = s(X) \in \mathbb{S}_n(X).$$

More explicitly: X is a topological space such that $H^{n-*}(X) \cong H_*(X)$ with arbitrary coefficients and has the homotopy type of a finite CW complex. The total surgery obstruction $s(X)$ measures the failure on the chain level of the local homology groups $H_*(X, X\backslash\{x\})$ $(x \in X)$ to be isomorphic to $H^{n-*}(\{x\}) = H_*(\mathbb{R}^n, \mathbb{R}^n\backslash\{0\})$. It vanishes if and only if the cellular $\mathbb{Z}[\pi_1(X)]$-module chain complex $C(\tilde{X})$ of the universal cover $\tilde{X}$ of X is algebraic Poincaré cobordant to the assembly of a local system of $\mathbb{Z}$-module chain complexes over X with Poincaré duality.

We obtain from the Main Theorem and its proof the following two corollaries.

Corollary 1 ([Ran92]Thm. 18.5 [Ran79])**.** *Let M be a manifold of dimension $n \geq 5$. Then there are isomorphisms from the geometric to the algebraic exact surgery sequence for M such that the following diagram commutes.*

$$\begin{array}{ccccccc}
L^w_{n+1}(\mathbb{Z}\pi) & \xrightarrow{action} & \mathscr{S}(M) & \longrightarrow & \mathscr{N}(M) = H^0(M, \mathrm{G/TOP}) & \xrightarrow{\theta} & L^w_n(\mathbb{Z}\pi) \\
\downarrow \cong & & \downarrow \cong & & \downarrow \cong & & \downarrow \cong \\
L_{n+1}(\mathbb{Z}\pi) & \longrightarrow & \mathbb{S}_{n+1}(M) & \longrightarrow & H_n(M; \mathbf{L}_\bullet\langle 1\rangle) & \xrightarrow{A} & L_n(\mathbb{Z}\pi)
\end{array}$$

Note that it is crucial for this that we work in the topological category. In the smooth and PL-category there is no known group structure on $\mathscr{N}(X)$. For more details on that see for example [Lüc02, §5].

Corollary 1 follows from the statement of room 3 and the proof of 22. They provide the commutativity of the diagram and that the vertical maps of the L-groups and the normal invariants are isomorphisms. But then, because of the exactness of both sequences, the map $\mathscr{S}(M) \to \mathbb{S}_{n+1}(M)$ has to be an isomorphism as well.

An important aspect of this identification is that it relates the geometric surgery sequence with the assembly map which leads to the second corollary.

Corollary 2. *The Farrell-Jones-Conjecture implies the Borel conjecture.*

We recall some common terminology to explain these two conjectures.

Definition. A manifold M is rigid if any homotopy equivalence $M \simeq N$ implies that M is actually homeomorphic to N.

Examples of rigid manifolds are surfaces and spheres S^n of arbitrary dimensions according to the (generalized) Poincaré conjecture. Counterexamples that are not rigid in general are Lens spaces and products of spheres.

Definition. A manifold M is aspherical if maps $S^n \to M$ are always null-homotopic for $n > 1$ or, equivalently, if the universal covering is contractible.

Now we have a handy formulation of the Borel conjecture.

Conjecture (Borel)**.** *Aspherical manifolds are rigid.*

More explicitly, the conjecture predicts that two aspherical manifolds M and N are homeomorphic if and only if $\pi_1(M) \cong \pi_1(N)$. In terms of the classical surgery theory this can be reformulated as follows.

Conjecture (Borel)**.** *For an aspherical manifold M the structure set $\mathscr{S}(M)$ consists of a single point.*

When we switch to the algebraic formulation of the surgery exact sequence, this becomes a question of assembly maps. Informally, assembly maps determine K- and L-theory of group rings $R[G]$ by looking at small subgroups of G. Formally, we have the following type of map:

$$A\colon H_n^G(E_{\mathcal{F}}G) \to H_n^G(\mathrm{pt})$$

where G is a discrete group, $\mathcal{F}$ a family of subgroups of G and H^G a G-homology theory. The Farrell-Jones conjecture makes the following statement about assembly maps.

Conjecture (Farrell-Jones)**.** *Let* Vcyc *be the family of virtual cyclic subgroups. Then*

$$A_K\colon H_n^G(E_{\mathrm{Vcyc}}G, \mathbf{K}_\bullet(R)) \to K_n(R[G])$$

and

$$A_L\colon H_n^G(E_{\mathrm{Vcyc}}G, \mathbf{L}_\bullet(R)) \to L_n(R[G])$$

are isomorphisms.

How does this help in our situation? The fundamental group of an aspherical manifold is torsion free, so we have $\mathrm{Vcyc} = \{1, \mathbb{Z}\}$. The Farrell-Jones conjecture for L-theory implies $H_n(M; \mathbf{L}_\bullet(\mathbb{Z})) \xrightarrow{\cong} L_n(\mathbb{Z}[\pi_1(M)])$. Using the identification of the surgery sequences of Corollary 1, this means the structure set vanishes and hence the Borel conjecture holds. There are some subtle details, especially we ignored the decorations which make it necessary to deal also with the K-theoretic assembly map. For more information about assembly maps and the isomorphism conjectures see for example [LR05] and in particular section 1.6 for more details on their geometric implications for manifolds.

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Level 0

In the elevator

As algebraic counterparts of manifolds and Poincaré spaces[§] we consider chain complexes together with a chain equivalence to its dual chain complex. Manifolds and Poincaré spaces differ in their strengths of duality and this is captured basically in the relation of three different types of duality structures on the chain complexes. Namely we have symmetric, quadratic and normal structures. If they provide a chain equivalence to the dual chain complex, they are called Poincaré structures.

The key observation is that the pieces of a (reasonable) subdivided manifold have Poincaré duality, but the pieces of a Poincaré space X are only normal. Normal means that we still have a fundamental class, but it does not provide a Poincaré duality isomorphism. Simply speaking, the Poincaré pieces of a manifold yield symmetric Poincaré structures on the corresponding chain complexes; the normal pieces of a Poincaré complex give normal structures and the difference between symmetric and normal is measured in quadratic terms.[¶] Hence, in the local quadratic structures, we find the information on whether a Poincaré space is homotopy equivalent to a manifold and so that is where the surgery obstruction lives.

The classical surgery obstruction of Wall [Wal99] is an element in the L-groups of quadratic forms (L^w_{2k}) and formations (L^w_{2k+1}). Mishchenko [Mis71] and Ranicki [Ran80a] came up with an analog definition for chain complexes, the quadratic L-group $L_n(R)$ for a ring R with involution as cobordism groups of chain complexes with quadratic Poincaré structures. They are isomorphic to Wall's L-groups but there are some advantages of the chain complex L-groups:

– They are uniformly defined for even and odd dimensions.

– There exist, as mentioned above, the two other notions of structures on chain complexes: the symmetric Poincaré structures and normal structures. Cobordism of chain complexes with such structures defines the symmetric and normal L-groups $L^n(R)$ and $NL^n(R)$, which fit into the important long exact sequence

$$\ldots \longrightarrow L_n(R) \xrightarrow{1+t} L^n(R) \xrightarrow{J} NL^n(R) \xrightarrow{\partial^N} L_{n-1}(R) \longrightarrow \ldots$$

– They can be generalized to the crucial concept of L-groups of algebraic bordism categories. An algebraic bordism category Λ comes with an additive category $\mathbb{A}$ with chain duality. Instead of chain complexes over a ring R we consider chain complexes of objects of $\mathbb{A}$. The explicit definition of the chain duality on $\mathbb{A}$ leads to more sophisticated types of (Poincaré) duality on the chain complexes and thereby to useful subcategories of the category of quadratic, symmetric and normal chain complexes.

[§]Often called geometric Poincaré complexes in the literature. We spare the 'geometric' and call them Poincaré spaces to distinguish them from (algebraic) Poincaré chain complexes although these spaces usually appear in the following as simplicial complexes.

[¶]This is the geometrically motivated view. From the algebraic point of view it seems more convenient to start with the symmetrization map from quadratic to symmetric and to consider normal as the relative term measuring the difference.

These properties enable us to establish the total surgery obstruction. The setting is the following: Let X be a simplicial complex (or at least homotopy equivalent to a simplicial complex) and $\pi = \pi_1(X)$. We consider the algebraic bordism category $\Lambda_L X$ with underlying additive category $\mathbb{A} = \mathbb{Z}_* X$, the additive category of free $\mathbb{Z}$-modules There is also a spectrum version $\mathbf{L}_\bullet(\Lambda_L \operatorname{pt}) =:$ $\mathbf{L}_\bullet$ and we get the L-groups as the generalized homology groups

$$H_n(X; \mathbf{L}_\bullet) \cong L_n(\Lambda_L X).$$

Forgetting the indexing of a $\mathbb{Z}_* X$-chain complex in an equivariant way gives a chain complex over $\mathbb{Z}\pi$ with global Poincaré duality. Global Poincaré means that the assembled quadratic structure is a Poincaré duality isomorphism on the assembled chain complex. This induces the assembly map on L-groups

$$A \colon H_n(X; \mathbf{L}_\bullet) \to L_n(\mathbb{Z}\pi),$$

which encodes the passage from manifolds to Poincaré spaces in terms of Poincaré bordism classes of chain complexes. So the difference between the local manifold world and the global Poincaré complex world is captured in the relative homotopy groups of A, the structure groups $\mathbb{S}_n(X)$, which fit into the algebraic surgery exact sequence

$$\ldots \longrightarrow H_n(X; \mathbf{L}_\bullet) \xrightarrow{A} L_n(\mathbb{Z}\pi) \xrightarrow{\partial^Q_{\mathbb{Z}\pi}} \mathbb{S}_n(X) \xrightarrow{I} H_{n-1}(X; \mathbf{L}_\bullet) \xrightarrow{A} \ldots$$

Andrew Ranicki was able to construct an element $s(X) \in \mathbb{S}_n(X)$ which, as we verify in room 1 and 2, encodes both obstructions of the classical surgery theory.

Main theorem

Porter

The Main Theorem relies heavily on the work of the surgery pioneers Browder [Bro72, Bro62], Novikov [Nov64], Sullivan [Sul96] and Wall [Wal99]. In [Ran80a, Ran80b] Ranicki defined a purely algebraic twin of Wall's surgery obstruction and gave a first account in [Ran79] of how the two-stage obstruction can be unified using this twin brother. But the proof is not independent. Its geometrical implications about manifold homotopy types stem from the classic surgery, summarized below in the cited theorems in room 3 and 4. See [Ran80a] and [Ran80b] or the upcoming book on surgery theory by Crowley, Lück and Macko for more details on the identification of the quadratic signature with Wall's surgery obstruction. Be aware of the corrections for the quadratic construction in [Ran81, p.30].

The complete theory surrounding the total surgery obstruction is developed from scratch in [Ran92], which is the main source of this thesis and to which we refer usually and constantly. Other important contributions are

– transversality in the topological category by Kirby-Siebenmann [KS77] and Freedman-Quinn [FQ90],
– normal transversality by [Qui72],
– the normal L-groups by [Wei85a, Wei85b], and
– the surgery obstruction isomorphism $\pi_n(\mathrm{G/TOP}) \xrightarrow{\cong} L_n(\mathbb{Z})$ for $n \geq 1$ by Siebenmann [KS77, Essay V, Theorem C.1].

We consider only the orientable case. For the adjustments that are necessary for the non-orientable case see [Ran92, Appendix A].

Main theorem [Ran79, Theorem 1]
Let X be a finite Poincaré space of dimension $n \geq 5$. Then X is homotopy equivalent to a closed n-dimensional topological manifold if and only if

$$0 = s(X) \in \mathbb{S}_n(X).$$

1 TSO and bundle reductions
$I(s(X)) = 0$ *if and only if there exists a topological bundle reduction of the Spivak normal fibration* $\nu_X : X \to \mathrm{BSG}$.

2 TSO and quadratic signatures
If $I(s(X)) = 0$ *then we have*

$$\partial^{Q}_{\mathbb{Z}\pi}{}^{-1} s(X) = \{ -\operatorname{sgn}^{\mathbf{L}_\bullet}_{\mathbb{Z}\pi}(\widehat{f}) \in L_n(\mathbb{Z}\pi) \mid \widehat{f} : M \to X \text{ degree one normal map},\ M \text{ manifold} \}.$$

[3 → [Ran80b, Proposition 7 .1]] Quadratic signatures and Wall's surgery obstruction
There is an isomorphism

$$L_n(\mathbb{Z}[\pi_1(X)]) \xrightarrow{\cong} L^w_n(\mathbb{Z}[\pi_1(X)])$$

such that for an n-dimensional normal map $\widehat{f} : M \to X$ $(n \geq 5)$ the quadratic signature $\operatorname{sgn}^{\mathbf{L}_\bullet}_{\mathbb{Z}\pi}(\widehat{f})$ gets mapped to Wall's surgery obstruction $\theta(\widehat{f})$.

[4 → [Wal99]] Wall Surgery Obstruction Theorem
Let $n \geq 5$. An n-dimensional degree one normal map $\widehat{f} : M \to X$ has a surgery obstruction $\theta(\widehat{f}) \in L^w_n(\mathbb{Z}\pi)$ such that $\theta(\widehat{f}) = 0$ if and only if $\widehat{f}$ is bordant to a homotopy equivalence.

Proof Main Theorem

Let X be a finite Poincaré space of dimension $n \geq 5$ and let π denote its fundamental group.

If X is homotopy equivalent to a manifold then its Spivak normal fibration $\nu_X : X \to \mathrm{BSG}$ has a bundle reduction and we get $I(s(X)) = 0$ because of 1. By 2 combined with 3 and 4 the set $\partial^{Q}_{\mathbb{Z}\pi}{}^{-1} s(X)$ contains a vanishing Wall surgery obstruction and hence $s(X) = 0$. 1→p.14 2→p.18

Conversely, ff $s(X) = 0$, 3 then of course its image $I(s(X))$ vanishes and hence by 1 the Spivak normal fibration of X has a topological bundle reduction. Also the preimage $\partial^{Q}_{\mathbb{Z}\pi}{}^{-1} s(X)$ contains 0 and hence by 2 and 3 Wall's surgery obstrucion $\theta(\widehat{f})$ vanishes, meaning that X is homotopy equivalent to a manifold. □

Room service

The maps I and $\partial^{Q}_{\mathbb{Z}\pi}$ and the element $s(X) \in \mathbb{S}_n(X)$ used above are part of the algebraic surgery exact sequence

$$\ldots \longrightarrow H_n(X; \mathbf{L}_\bullet) \xrightarrow{A} L_n(\mathbb{Z}\pi) \xrightarrow{\partial^{Q}_{\mathbb{Z}\pi}} \mathbb{S}_n(X) \xrightarrow{I} H_{n-1}(X; \mathbf{L}_\bullet) \xrightarrow{A} \ldots$$

which we use as a black box for now.

At this stage we only give short definitions of the terms taken from the classic surgery theory. The details for all the other groups, maps and elements used above follow in the proceeding levels. For an introduction and more details about the classic theory see for example [Lüc02] and [Ran02] or the original sources like [Wal99] and [Bro72].

Classic surgery theory

$\boxed{X}$ an n-dimensional Poincaré space, i.e. a finite CW complex together with an orientation homomorphism $w\colon \pi_1(X) \to \{\pm 1\}$ and a fundamental class $[X]$.

- $\boxed{[X]}$ fundamental class for an n-dimensional Poincaré space X is a cycle in the cellular $\mathbb{Z}\pi$-chain complex $C_n(\widetilde{X})$ which represents an n-dimensional homology class in $H_n(X; \mathbb{Z}^w)$ such that $\cdot \cap [X]\colon C^{n-*}(\widetilde{X}) \to C_*(\widetilde{X})$ is a $\mathbb{Z}\pi$-chain homotopy equivalence where $\widetilde{X}$ is the universal covering.

$\boxed{\nu_X : X \to \mathrm{BSG}}$ the Spivak normal fibration of X, i.e. an oriented $(k-1)$-spherical fibration of an n-dimensional Poincaré space X for which a class $\alpha \in \pi_{n+k}(\mathrm{Th}(\nu_X))$ $(k > n+1)$ exists such that $h(\alpha) \cap u = [X]$. Here $u \in H^k(\mathrm{Th}(\nu_X))$ is the Thom class and $h\colon \pi_*(\cdot) \to H_*(\cdot)$ is the Hurewicz map.

- $\boxed{\mathrm{BSG}}$ the classifying space of stable $\mathbb{Z}$-oriented spherical fibrations.
- $\boxed{\mathrm{Th}(\xi)}$ the Thom space of a vector bundle ξ, i.e. the quotient of disk and sphere bundle $D(\xi)/S(\xi)$. This agrees with the mapping cone of the projection map $S(\xi) \to X$ which gives rise to a general definition of the Thom space for a spherical fibration $\nu\colon E \to X$ as $\mathrm{Th}(\nu) \coloneqq \mathcal{C}(\nu)$.

$\boxed{\theta(\widehat{f})}$ Wall's surgery obstruction for a degree one normal map $\widehat{f}\colon M \to X$. It is an element in $L_n^w(\mathbb{Z}[\pi_1(X)])$ and if $n \geq 5$ it vanishes if and only if $\widehat{f}$ is cobordant to a homotopy equivalence $\widehat{f}'\colon M' \to X$.

- $\boxed{L_n^w(R)}$ the Wall surgery groups of quadratic forms for n even resp. of formations for n odd where R is an associative ring with unit and involution.

$\boxed{\widehat{f} \coloneqq (\overline{f}, f)\colon M \to X}$ an n-dimensional degree one normal map, i.e. a commutative square

$$\begin{array}{ccc} \nu_M & \xrightarrow{\overline{f}} & \eta \\ \downarrow & & \downarrow \\ M & \xrightarrow{f} & X \end{array}$$

with $f\colon M \to X$ a map from an n-dimensional manifold M to an n-dimensional Poincaré space X such that $f_*([M]) = [X] \in H_n(X)$, and $\overline{f}\colon \nu_M \to \nu_X$ stable bundle map from the stable normal bundle $\nu_M\colon M \to \mathrm{BSTOP}$ to a stable bundle $\nu_X : X \to \mathrm{BSTOP}$.

- $\boxed{\mathrm{BSTOP}}$ the classifying space of stable $\mathbb{Z}$-oriented topological bundles.

Level 1

In the elevator

In each of the two following rooms 1 and 2 an exact sequence of L-groups plays a key role. They are connected in a braid of exact sequences.

The first one used in room 2 is the surgery exact sequence

$$\ldots \longrightarrow H_n(X;\mathbf{L}_\bullet) \xrightarrow{A} L_n(\mathbb{Z}\pi) \xrightarrow{\partial^Q_{\mathbb{Z}\pi}} \mathbb{S}_n(X) \xrightarrow{I} H_{n-1}(X;\mathbf{L}_\bullet) \xrightarrow{A} \ldots$$

It is a reformulation of the following sequence of quadratic L-groups

$$\ldots \longrightarrow L_n(\mathbb{B},\mathbb{L}) \longrightarrow L_n(\mathbb{B},\mathbb{G}) \longrightarrow L_n(\mathbb{G},\mathbb{L}) \longrightarrow L_{n-1}(\mathbb{B},\mathbb{L}) \longrightarrow \ldots$$

that are cobordism groups of quadratic chain complexes which are

$\mathbb{B}$ounded and $\mathbb{L}$ocally Poincaré in $L_n(\mathbb{B},\mathbb{L})$,

$\mathbb{B}$ounded and $\mathbb{G}$lobally Poincaré in $L_n(\mathbb{B},\mathbb{G})$,

$\mathbb{G}$lobally contractible and $\mathbb{L}$ocally Poincaré in $L_n(\mathbb{G},\mathbb{L})$.

The second exact sequence

$$\ldots \longrightarrow L_n(\mathbb{Z}\pi) \longrightarrow L^n(\mathbb{Z}\pi) \longrightarrow NL^n(\mathbb{Z}\pi) \longrightarrow L_{n-1}(\mathbb{Z}\pi) \longrightarrow \ldots$$

used in room 1 relates the three different duality structures, symmetric, quadratic and normal to each other. There is a variety of constructions which produce symmetric, quadratic and normal structures out of geometric input. For this level, take the existence of the following objects as a black box. They will be defined in the next level. A complete overview and more detailed explanations can be found in the basement. The signatures listed in the table above give elements

	L-group	spectrum	signature
symmetric	$L^n(\mathbb{Z}\pi)$	$\mathbf{L}^\bullet$	$\mathrm{sgn}^{\mathbf{L}^\bullet}_{\mathbb{Z}\pi}$
quadratic	$L_n(\mathbb{Z}\pi)$	$\mathbf{L}_\bullet$	$\mathrm{sgn}^{\mathbf{L}_\bullet}_{\mathbb{Z}\pi}$
normal	$NL^n(\mathbb{Z}\pi)$	$\mathbf{NL}^\bullet$	$\mathrm{sgn}^{\mathbf{NL}^\bullet}_{\mathbb{Z}\pi}$

in the corresponding L-groups over the fundamental group ring $\mathbb{Z}\pi$ for appropriate input.

input	output
X an n-dimensional Poincaré space	$\mathrm{sgn}^{\mathbf{L}^\bullet}_{\mathbb{Z}\pi}(X) \in L^n(\mathbb{Z}\pi)$
$\widehat{f}\colon M \to X$ a degree one normal map	$\mathrm{sgn}^{\mathbf{L}_\bullet}_{\mathbb{Z}\pi}(\widehat{f}) \in L_n(\mathbb{Z}\pi)$
Y an n-dimensional normal space.	$\mathrm{sgn}^{\mathbf{NL}^\bullet}_{\mathbb{Z}\pi}(Y) \in NL^n(\mathbb{Z}\pi)$

where $\pi = \pi_1(X)$ or $\pi_1(Y)$. The quadratic signature $\mathrm{sgn}^{\mathbf{L}_\bullet}_{\mathbb{Z}\pi}(\widehat{f})$ can be identified with Wall's surgery obstruction $\theta(\widehat{f})$.

There is a corresponding homotopy fibration sequence of spectra

$$\mathbf{L}_\bullet \to \mathbf{L}^\bullet \to \mathbf{NL}^\bullet \to \Sigma\mathbf{L}_\bullet$$

which induces the following reformulation of the long exact sequence in terms of homology groups

$$\ldots \to H_n(X;\mathbf{L}_\bullet) \to H_n(X;\mathbf{L}^\bullet) \to H_n(X;\mathbf{NL}^\bullet) \to H_{n-1}(X;\mathbf{L}_\bullet) \to \ldots$$

In order to define the total surgery obstruction $s(X)$ we need an important variant of the signatures above that takes into account the local structure. We call them mosaicked signatures. They take values in the homology groups with the corresponding spectra as coefficients. For the geometric input we need a local structure organized in some way:

input	output
M a triangulated n-dimensional manifold	$\mathrm{sgn}^{\mathbf{L}^\bullet}_M(M) \in H_n(M;\mathbf{L}^\bullet)$
$\widehat{f}\colon M' \to M$ a degree one normal map between n-dimensional triangulated manifolds	$\mathrm{sgn}^{\mathbf{L}_\bullet}_X(\widehat{f}) \in H_n(M;\mathbf{L}_\bullet)$
Y a simplicial normal complex	$\mathrm{sgn}^{\mathbf{NL}^\bullet}_Y(Y) \in H_n(Y;\mathbf{NL}^\bullet)$

The assembly maps which forget the local structure in an equivariant way give back the signatures over the group ring

$$A(\mathrm{sgn}^{\mathbf{L}^\bullet}_X(M)) = \mathrm{sgn}^{\mathbf{L}^\bullet}_{\mathbb{Z}\pi}(M), \qquad A(\mathrm{sgn}^{\mathbf{L}_\bullet}_X(\widehat{f})) = \mathrm{sgn}^{\mathbf{L}_\bullet}_{\mathbb{Z}\pi}(\widehat{f}), \qquad A(\mathrm{sgn}^{\mathbf{NL}^\bullet}_Y(Y)) = \mathrm{sgn}^{\mathbf{NL}^\bullet}_{\mathbb{Z}\pi}(Y).$$

In the special case when Y is a simplicial Poincaré space, i.e. has global Poincaré duality, but is locally only normal, then the mosaicked normal signature gives a refined element called the visible signature

$$\mathrm{sgn}^{VL}_X(Y) \in VL^n(Y).$$

The total surgery obstruction $s(X)$ is obtained as the boundary of the visible signature

$$s(X) = \partial^N_{\mathbb{G}}\,\mathrm{sgn}^{VL}_X \in \mathbb{S}_n(X).$$

1 TSO and bundle reductions

Porter

In the following room, orientations with respect to L-theory spectra play the key role in relating bundle reductions to L-theory. The original source for this is [Ran79]. The statement of room 1 with some comments on the proof can also be found in [Ran92, 16.1 (iii)]. L-spectra were introduced by [Qui70] but in a more geometric way than used here. Further we use the description of E^∞-spectra introduced in [May77].

1 TSO and bundle reductions
$I(s(X)) = 0$ if and only if there exists a topological bundle reduction of the Spivak normal fibration $\nu_X\colon X \to \mathrm{BSG}$.

11 Exact sequence of L-groups [Ran92, Prop. 2.11]
There is a long exact sequence of L-groups

$$\ldots \longrightarrow L_n(R) \xrightarrow{1+t} L^n(R) \xrightarrow{J} NL^n(R) \xrightarrow{\partial^N} L_{n-1}(R) \longrightarrow \ldots.$$

12 Exact sequence of homology L-groups [Ran92, Proposition 15.16]
The diagram

$$\begin{array}{ccccccc} \longrightarrow H_n(X;\mathbf{L}^\bullet) & \longrightarrow & H_n(X;\mathbf{NL}^\bullet) & \xrightarrow{\partial^N_{\mathbb{B}}} & H_{n-1}(X;\mathbf{L}_\bullet) \longrightarrow \\ & \searrow & \uparrow & & \uparrow{\scriptstyle I} \\ & & VL^n(X) & \xrightarrow{\partial^N_{\mathbb{G}}} & \mathbb{S}_n(X) \end{array}$$

is commutative and exact in the top row.

13 The visible mosaicked signature [Ran92, Example 9.13]
For an n-dimensional finite simplicial Poincaré complex X there is a visible signature

$$\operatorname{sgn}_X^{VL}(X) \in NL^n(\Lambda_G X) =: VL^n(X)$$

as a refinement of a normal signature

$$\operatorname{sgn}_X^{\mathbf{NL}^\bullet}(X) \in NL^n(\Lambda_N X).$$

14 Canonical L-orientations [Ran79, p. 284-289][Ran92, 16.1(ii)][KMM13, Prop. 13.3 and 13.4]

(i) For a k-dimensional $\mathbb{Z}$-oriented spherical fibration $\alpha\colon X \to \mathrm{BSG}(k)$ there is a canonical $\mathbf{NL}^\bullet$-orientation

$$u_{\mathbf{NL}^\bullet}(\alpha) \in H^k(Th(\alpha);\mathbf{NL}^\bullet).$$

(ii) For a k-dimensional $\mathbb{Z}$-oriented topological bundle $\beta\colon X \to \mathrm{BSTOP}(k)$ there is a canonical $\mathbf{L}^\bullet$-orientation

$$u_{\mathbf{L}^\bullet}(\beta) \in H^k(Th(\beta);\mathbf{L}^\bullet).$$

(iii) For a k-dimensional $\mathbb{Z}$-oriented topological bundle $\beta\colon X \to \mathrm{BSTOP}(k)$ together with a homotopy $h\colon J(\beta) \simeq \nu_X$ there is a canonical $\mathbf{NL}/\mathbf{L}$-orientation

$$u^{\mathbf{NL}/\mathbf{L}^\bullet}(\beta,h) \in H^k(Th(\nu_X);\mathbf{NL}\langle 1/2\rangle/\mathbf{L}\langle 0\rangle^\bullet).$$

They are related via $J(u^{\mathbf{L}^\bullet}(\beta)) = u^{\mathbf{NL}^\bullet}(J(\beta))$ and $u^{\mathbf{NL}/\mathbf{L}^\bullet}(\beta,h) = (u^{\mathbf{NL}^\bullet}(h), u^{\mathbf{L}^\bullet}(\beta) - u^{\mathbf{L}^\bullet}(\nu_X))$.

15 (16) Orientations and signatures [Ran92, Proposition 16.1]

(i) Let X be an n-dimensional Poincaré space with Spivak normal fibration $\nu_X\colon X \to \mathrm{BSG}$. Then we have $S(u_{\mathbf{NL}}(\nu_X)) = \operatorname{sgn}_X^{\mathbf{NL}^\bullet}(X) \in H_n(X;\mathbf{NL}^\bullet)$.

(ii) Let $\overline{\nu}$ be a topological bundle reduction of the Spivak normal fibration $\nu_X\colon X \to \mathrm{BSG}$ of X and $\widehat{f}\colon M \to X$ its associated degree one normal map. Then we have $S(u_{\mathbf{L}}(\nu_X)) = \operatorname{sgn}_X^{\mathbf{L}^\bullet}(X) \in H_n(X;\mathbf{L}^\bullet)$.

(iii) Let $\widehat{f}\colon M \to M'$ be a degree one normal map of n-dimensional simply-connected topological manifolds with M' triangulated, corresponding to a pair (β,h) with $\beta\colon M' \to \mathrm{BSTOP}$ and $h\colon J(\beta) \simeq \nu_{M'}$. Then we have $S(u^{\mathbf{NL}/\mathbf{L}^\bullet}(\beta,h)) = \operatorname{sgn}_{M'}^{\mathbf{NL}/\mathbf{L}^\bullet}(\widehat{f}) \in H_n(M';\mathbf{NL}\langle 1/2\rangle/\mathbf{L}\langle 0\rangle^\bullet)$.

16 The homotopy pullback square [Ran79, p.291][KMM13, Prop. 13.7]
The following diagram is a homotopy pullback square:

$$\begin{array}{ccc} \mathrm{BSTOP} & \xrightarrow{\operatorname{sgn}_B^{\mathbf{L}^\bullet}} & \mathrm{B}\mathbf{L}^\bullet\mathrm{G} \\ {\scriptstyle J}\downarrow & & \downarrow{\scriptstyle J} \\ \mathrm{BSG} & \xrightarrow{\operatorname{sgn}_B^{\mathbf{NL}^\bullet}} & \mathrm{B}\mathbf{NL}^\bullet\mathrm{G} \end{array}$$

Proof 1

Let X be an n-dimensional Poincaré space and denote its Spivak normal fibration by ν. We assume that X is a simplicial complex. Otherwise we would have to use a reference map $r\colon X \xrightarrow{\simeq} |K|$ to a simplicial complex K and work with mosaicked signatures over K. We denote $t(X) := I(s(X))$. We want to relate the vanishing of $t(X)$ to the existence of bundle reductions. The main idea is to translate the statement about the reduction of the spherical fibration ν into a statement about orientations with respect to L-theory spectra. We split this up into the following four equivalent statements:

(i) $0 = t(X) \in H_{n-1}(X; \mathbf{L}_\bullet)$.

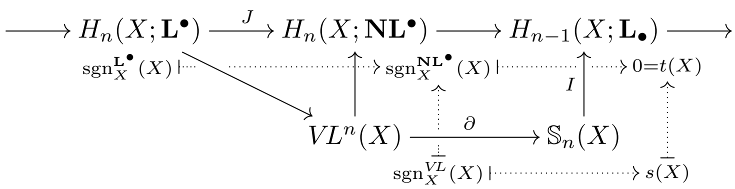
(ii) There exists a preimage $\mathrm{sgn}_X^{\mathbf{L}^\bullet}(X)$ of the normal signature $\mathrm{sgn}_X^{\mathbf{NL}^\bullet}(X)$ under the map $J\colon H_n(X; \mathbf{L}^\bullet) \to H_n(X; \mathbf{NL}^\bullet)$.

(iii) There exists a preimage $u^{\mathbf{L}^\bullet}(\overline{\nu})$ of the normal canonical orientation $u^{\mathbf{NL}^\bullet}(\nu)$ under the map $J\colon H^k(\mathrm{Th}(\nu); \mathbf{L}^\bullet) \to H^k(\mathrm{Th}(\nu); \mathbf{NL}^\bullet)$ such that $S(u^{\mathbf{L}^\bullet}(\overline{\nu})) = \mathrm{sgn}_X^{\mathbf{L}^\bullet}(X)$.

(iv) There exists a lift $\overline{\nu}\colon X \to \mathrm{BSTOP}$ of ν.

We now take a first step into the proof of these equivalences.

11→p.23 *Proof (i)⇔(ii).* The long exact sequence of L-groups induces a long exact sequence of generalized
12→p.28 homology groups which fit into the following commutative diagram.‖



Because X is Poincaré, i.e. locally normal and globally Poincaré, it has a visible X-mosaicked
13→p.35 signature

$$\mathrm{sgn}_X^{VL}(X) \in VL^n(X)$$

as a refinement of the normal X-mosaicked signature

$$\mathrm{sgn}_X^{\mathbf{NL}^\bullet}(X) \in H_n(X; \mathbf{NL}^\bullet).$$

The equivalence follows immediately from the definitions of $s(X)$ and $t(X)$ as images of the appropriate maps. □

Remark. For one direction of the equivalence $(i) \Leftrightarrow (iv)$ we are aiming for this would be already enough. We could conclude $(iv) \Rightarrow (ii)$ from the following geometric background. Suppose ν has a bundle reduction. Then it has an associated degree one normal map $\widehat{f}\colon M \to X$ which can be made transverse to the dual cells of X. For each σ the preimage $(M(\sigma), \partial M(\sigma))$ of the dual cell $(D(\sigma), \partial D(\sigma))$ is an $(n - |\sigma|)$-dimensional submanifold with boundary and we obtain a mosaicked signature over X

$$\mathrm{sgn}_X^{\mathbf{L}^\bullet}(M) \in L^n(\Lambda_L X).$$

The mapping cylinder of the degree one normal map $\widehat{f}$ is a normal cobordism between M and X and produces an algebraic normal cobordism between $J(\mathrm{sgn}_X^{\mathbf{L}^\bullet}(M))$ and $\mathrm{sgn}_X^{\mathbf{NL}^\bullet}(X)$. In other words there exists a lift of $\mathrm{sgn}_X^{\mathbf{NL}^\bullet}(X)$ in the exact sequence above.

For the other direction, orientations with respect to L-theory spectra come into play and we need statement (iii).

‖In the right column we recover a part of the algebraic surgery exact sequence. In fact this is a section of a braid in which the whole surgery sequence is contained.

Proof (ii)⇔(iii). For k large enough we have an embeddng $X \subset S^{n+k}$ and because X is Poincaré we can use the well-known S-duality $\mathrm{Th}(\nu)^* \simeq X_+$ to obtain the isomorphisms

$$S\colon H^k(\mathrm{Th}(\nu); \mathbf{L}^\bullet) \cong H_n(X; \mathbf{L}^\bullet) \quad \text{and} \quad S\colon H^k(\mathrm{Th}(\nu); \mathbf{NL}^\bullet) \cong H_n(X; \mathbf{NL}^\bullet).$$

Let $\overline{\nu}\colon X \to \mathrm{BSTOP}$ be a topological bundle reduction of ν. There are canonical orientations $u^{\mathbf{L}^\bullet}(\overline{\nu})$ and $u^{\mathbf{NL}^\bullet}(\nu)$ and they fit into the following commutative diagram. 14→p.38
15→p.42

$$\begin{array}{ccc}
H^k(\mathrm{Th}(\nu); \mathbf{L}^\bullet) & \xrightarrow[S]{\cong} & H_n(X; \mathbf{L}^\bullet) \\
u^{\mathbf{L}^\bullet}(\overline{\nu}) & \longmapsto & \mathrm{sgn}_X^{\mathbf{L}^\bullet}(X) \\
{\scriptstyle J}\downarrow & & \downarrow{\scriptstyle J} \\
H^k(\mathrm{Th}(\nu); \mathbf{NL}^\bullet) & \xrightarrow[S]{\cong} & H_n(X; \mathbf{NL}^\bullet) \\
u^{\mathbf{NL}^\bullet}(\nu) & \longmapsto & \mathrm{sgn}_X^{\mathbf{NL}^\bullet}(X)
\end{array}$$

□

Proof (iii)⇔(iv). We claim that there is a homotopy pullback square 16→p.47

$$\begin{array}{ccc}
\mathrm{BSTOP} & \xrightarrow{\mathrm{sgn}_B^{\mathbf{L}^\bullet}} & \mathrm{B}\mathbf{L}^\bullet\mathrm{G} \\
{\scriptstyle J}\downarrow & & \downarrow{\scriptstyle J} \\
\mathrm{BSG} & \xrightarrow{\mathrm{sgn}_B^{\mathbf{NL}^\bullet}} & \mathrm{B}\mathbf{NL}^\bullet\mathrm{G}.
\end{array}$$

For a ring spectrum $\mathbf{E}$ according to [May77] there is a one-to-one correspondence between the following maps and pairs of maps

$$(\alpha\colon X \to B\mathbf{E}G) \xleftrightarrow{1-1} (\nu_\alpha\colon X \to \mathrm{BSG}, u^{\mathbf{E}}(\alpha)\colon \mathrm{Th}(\nu_\alpha) \to \mathbf{E}_k)$$

where $u^{\mathbf{E}}$ is an $\mathbf{E}$-orientation.

Now for $\mathbf{E} = \mathbf{NL}$ we get the identification

$$(\mathrm{sgn}_B^{\mathbf{NL}^\bullet} \circ \nu\colon\ X \to \mathrm{B}\mathbf{NL}^\bullet\mathrm{G}) = (\nu, u^{\mathbf{NL}^\bullet}(\nu)).$$

Using the homotopy pullback square from above as indicated in the diagram below it follows that the existence of a lift $\overline{\nu}\colon X \to \mathrm{BSTOP}$ is equivalent to the existence of an orientation $u^{\mathbf{L}^\bullet}(\nu)\colon \mathrm{Th}(\nu) \to \mathbf{L}^\bullet$ such that

$$(\nu, u^{\mathbf{NL}^\bullet}(\nu)) = J \circ (\overline{\nu}, u^{\mathbf{L}^\bullet}(\overline{\nu}))$$

i.e. the symmetric orientation is a lift of the normal orientation $u^{\mathbf{NL}^\bullet}(\nu) = J(u^{\mathbf{L}^\bullet}(\overline{\nu}))$.

$$\begin{array}{ccccc}
X & & \xrightarrow{(\nu, u^{\mathbf{L}^\bullet}(\nu))} & & \\
& \overset{\overline{\nu}}{\searrow} & & & \\
& & \mathrm{BSTOP} & \longrightarrow & \mathrm{B}\mathbf{L}^\bullet\mathrm{G} \\
& {\scriptstyle \nu}\searrow & \downarrow & \lrcorner & \downarrow{\scriptstyle J} \\
& & \mathrm{BSG} & \longrightarrow & \mathrm{B}\mathbf{NL}^\bullet\mathrm{G} \\
& & & \xrightarrow{(\nu, u^{\mathbf{NL}^\bullet}(\nu))} &
\end{array}$$

□

Room service 1

$\boxed{\mathbf{L}^\bullet, \mathbf{L}_\bullet, \mathbf{NL}^\bullet}$ short for $\mathbf{L}^\bullet\langle 0\rangle, \mathbf{L}_\bullet\langle 1\rangle, \mathbf{NL}^\bullet\langle 1/2\rangle$, connective Ω-spectra of Kan Δ-sets; for more details see section 1231.

$\boxed{\Delta\text{-set}}$ a simplicial set without degeneracies.

$\boxed{\text{Kan}}$ is what a Δ-set X is called if every map $\Lambda^n_i \to X$ extends to a map $\Delta^n \to X$; this property is necessary to do homotopy theory on Δ-sets .

$\boxed{\Lambda^n_i := \Delta^n - ((\Delta^n)^{(n)} \cup \partial_i \Delta^n)}$ the subcomplex of Δ^n obtained by removing the interior of Δ^n and a single face of Δ^n.

$\boxed{\text{Th}(\xi)}$ the Thom space of a vector bundle ξ, i.e. the quotient of disk and sphere bundle $D(\xi)/S(\xi)$. This agrees with the mapping cone of the projection map $S(\xi) \to X$ which gives rise to a general definition of the Thom space for a spherical fibration $\nu\colon E \to X$ as $\text{Th}(\nu) := \mathcal{C}(\nu)$.

$\boxed{u^{\mathbf{E}}(\nu)}$ an $\mathbf{E}$-orientation of a $\mathbb{Z}$-oriented spherical fibration $\nu\colon X \to \text{BSG}(k)$ is an element $u^{\mathbf{E}}(\nu) \in H^k(\text{Th}(\nu); \mathbf{E})$ such that $u^{\mathbf{E}}(\nu)$ restricts to a generator of $H^k(\text{Th}(\nu_x); \mathbf{E})$ for each fiber ν_x of ν.

$\boxed{\text{BSG}}$ the classifying space of stable $\mathbb{Z}$-oriented spherical fibrations.

$\boxed{\text{BSTOP}}$ the classifying space of stable $\mathbb{Z}$-oriented topological bundles.

$\boxed{I\colon \mathbb{S}_n(X) \to H_{n-1}(X; \mathbf{L}_\bullet)}$ group homomorphism, see room 12.

$\boxed{S\colon [Y, Z] \xrightarrow{\cong} [S^N, X \wedge Y]}$ the S-duality isomorphism; for an N-dimensional S-duality map $\alpha\colon S^N \to X \wedge Y$ and an arbitrary space Z defined by $S(\gamma) = (\text{id}_Y \wedge \gamma) \circ \alpha$; denotes the induced isomorphism $S\colon H^{N-*}(X; \mathbf{E}) \xrightarrow{\cong} H_*(X, \mathbf{E})$ as well.

2 TSO and quadratic signatures

Porter

By 3 the quadratic signatures establish the connection between the total surgery obstruction and Wall's surgery obstruction. We want to show that the preimage of the total surgery obstruction under the boundary map $\partial^Q_{\mathbb{Z}\pi}\colon L_n(\mathbb{Z}\pi) \to \mathbb{S}_n(X)$ is the subset of $L_n(\mathbb{Z}\pi)$ consisting of quadratic signatures of degree one normal maps $\widehat{f}\colon M \to X$.

By the algebraic π-π-Theorem 1221 the boundary map $\partial^Q_{\mathbb{Z}\pi}$ factors through $\partial^Q_{\mathbb{G}}\colon L_n(\Lambda_G X) \to \mathbb{S}_n(X)$. This enables us to work on both sides with L-groups of algebraic bordism categories. But for this we need a refined version of the quadratic signature $\text{sgn}^{\mathbf{L}_\bullet}_{\mathbb{Z}\pi}$, the mosaicked quadratic signature $\text{sgn}^{\mathbf{L}_\bullet}_X$, which produces a chain complex in $L_n(\Lambda_G X)$ with global Poincaré duality. In fact, we will need even more. In 22 we have to produce something in the source of the assembly map which is the quadratic L-group $L_n(\Lambda_L X) = H_n(X; \mathbf{L}_\bullet\langle 1\rangle)$ where local Poincaré duality is required. It turns out that the difference $\operatorname{sgn}_X^{\mathbf{L}_\bullet}(\widehat{f}) - \operatorname{sgn}^{\mathbf{L}_\bullet}(\widehat{f}_0)$ for two degree one normal maps lives in that group. The prerequisite $I(s(X)) = 0$ of this statement ensures that there exists at least one such degree one normal map $\widehat{f}_0$ that we can use as reference map for this construction.

2 TSO and quadratic signatures
If $I(s(X)) = 0$ then we have

$$\partial^{Q}_{\mathbb{Z}\pi}{}^{-1} s(X) = \{ -\operatorname{sgn}^{\mathbf{L}_\bullet}_{\mathbb{Z}\pi}(\widehat{f}) \in L_n(\mathbb{Z}\pi) \mid \widehat{f}\colon M \to X \text{ degree one normal map},\ M \text{ manifold}\}.$$

21 The algebraic surgery exact sequence [Ran92, Prop. 14.7]
There is a long exact sequence $\ldots \longrightarrow H_n(X;\mathbf{L}_\bullet) \xrightarrow{A} L_n(\mathbb{Z}\pi) \xrightarrow{\partial^Q_{\mathbb{Z}\pi}} \mathbb{S}_n(X) \xrightarrow{I} H_{n-1}(X;\mathbf{L}_\bullet) \xrightarrow{A} \ldots$

22 Coset step
$-\operatorname{sgn}^{\mathbf{L}_\bullet}_{\mathbb{Z}\pi}(\mathscr{N}(X))$ *is a coset of* $\operatorname{im}(A)$ *in* $L_n(\mathbb{Z}\pi)$ *where* $A\colon H_n(X;\mathbf{L}_\bullet) \to L_n(\mathbb{Z}\pi)$ *is the assembly map.*

23 (221) Subset step
$-\operatorname{sgn}^{\mathbf{L}_\bullet}_{\mathbb{Z}\pi}(\mathscr{N}(X)) \subseteq \partial^{Q}_{\mathbb{Z}\pi}{}^{-1}(s(X))$ *where* $\partial^Q_{\mathbb{Z}\pi}\colon L_n(\mathbb{Z}\pi) \to \mathbb{S}_n(X)$ *is the boundary map from the surgery braid.*

Proof 2

We denote the restriction of the quadratic L-group $L_n(\mathbb{Z}\pi)$ to signatures coming from degree one normal maps by

$$-\operatorname{sgn}^{\mathbf{L}_\bullet}_{\mathbb{Z}\pi}(\mathscr{N}(X)) := \{-\operatorname{sgn}^{\mathbf{L}_\bullet}_{\mathbb{Z}\pi}(\widehat{f}) \in L_n(\mathbb{Z}\pi) \mid \widehat{f}\colon M \to X \text{ degree one normal map}, M \text{ manifold}\}.$$

We have to identify $\partial^{Q}_{\mathbb{Z}\pi}{}^{-1}s(X)$ with the set of quadratic signatures $-\operatorname{sgn}^{\mathbf{L}_\bullet}_{\mathbb{Z}\pi}(\mathscr{N}(X))$. The foundation of the proof is that the part

$$H_n(X;\mathbf{L}_\bullet\langle 1\rangle) \xrightarrow{A} L_n(\mathbb{Z}\pi) \xrightarrow{\partial^Q_{\mathbb{Z}\pi}} \mathbb{S}_n(X)$$ 21→p.51

of the surgery exact sequence is exact. Hence $\partial^{Q}_{\mathbb{Z}\pi}{}^{-1}s(X)$ is a coset of $\operatorname{im}(A)$. The right hand side $-\operatorname{sgn}^{\mathbf{L}_\bullet}_{\mathbb{Z}\pi}(\mathscr{N}(X))$ is also coset of $\operatorname{im}(A)$ because of 22 and it is a subset of $\partial^{Q}_{\mathbb{Z}\pi}{}^{-1}s(X)$ by 23 and so both are equal. ☐ 22→p.51 23→p.55

Room service 2

$\boxed{\mathbb{S}_n(X) := L_{n-1}(\mathbb{Z}_*X, \mathbb{G}\langle 1\rangle, \mathbb{L}\langle 1\rangle)}$ the n-dimensional *structure group* of X. An element in $\mathbb{S}_n(X)$ is represented by an 1-connective $(n-1)$-quadratic chain complex in $\mathbb{Z}_*X$ which is globally contractible and locally Poincaré.

$\boxed{\Lambda_G X}$ here short for the algebraic bordism category $\Lambda(\mathbb{Z})_G X = (\mathbb{Z}_*X, \mathbb{B}\langle 1\rangle, \mathbb{G}\langle 1\rangle)$.

$\boxed{\Lambda_L X}$ here short for the algebraic bordism category $\Lambda(\mathbb{Z})_L X = (\mathbb{Z}_*X, \mathbb{B}\langle 1\rangle, \mathbb{L}\langle 1\rangle)$.

$\boxed{\Lambda(\mathbb{Z}) = (\mathbb{A}(\mathbb{Z}), \mathbb{B}(\mathbb{Z}), \mathbb{C}(\mathbb{Z}))}$ denotes the algebraic bordism category with

- $\mathbb{A}(\mathbb{Z})$ the additive category of finitely generated free left $\mathbb{Z}$-modules.
- $\mathbb{B}(\mathbb{Z})$ the category of bounded chain complexes in $\mathbb{A}(\mathbb{Z})$.
- $\mathbb{B}(\mathbb{Z})$ the category of contractible chain complexes in $\mathbb{A}(\mathbb{Z})$.

$\mathbb{Z}_*X$ short for $\mathbb{A}(\mathbb{Z})_*X$, the additive category of X-based free $\mathbb{Z}$-modules with 'non-decreasing' morphisms $\sum_{\tau\geq\sigma} f_{\tau,\sigma}\colon \sum_{\sigma\in X} M_\sigma \to \sum_{\tau\in X} N_\tau$ where $(f_{\tau,\sigma}\colon M_\sigma \to N_\tau)$ are $\mathbb{Z}$-module morphism.

$\mathbb{G}\langle q\rangle\subset \mathbb{G}$ the subcategory of chain complexes of $\mathbb{G}$ which are homotopy equivalent to q-connected chain complexes.

- $\mathbb{G}$ short for $\mathbb{B}(\mathbb{Z})_G X := \{C \in \mathbb{B}(\mathbb{Z}) \mid A(C) \simeq *\}$, the *globally contractible* objects of $\mathbb{B}(\mathbb{Z})$.

$\mathbb{L}\langle q\rangle\subset \mathbb{L}$ the subcategory of chain complexes of $\mathbb{L}$ which are homotopy equivalent to q-connected chain complexes.

Level 2

In the elevator

The diagram on the next page provides an overview of the logical structure of this level. Recall that room 1 realizes the bundle obstruction of the classical theory which means that the Spivak normal fibration of a Poincaré space X has a topological bundle reduction. If we have such a bundle, then the Pontrjagin-Thom construction provides a degree one normal map from a manifold to X. The existence of such a map is used in the proof of 22 and leads to the second stage of the obstruction theory, statement 2, where a certain preimage of the total surgery obstruction is identified as the set of quadratic signatures of all degree one normal maps from manifolds to X. Finally, the isomorphism from 3, which correlates the quadratic signatures to Wall's surgery obstruction, proves the main theorem.

The statement 1 is based on the following six statements:

11 introduces the fundamental objects of the whole proof, the symmetric, quadratic and normal L-groups and shows how they are related via an exact sequence.

12 presents several steps of generalization and adjustments of the concepts introduced in 11 in order to obtain the more sophisticated version of the exact sequence of L-groups we need.

13 constructs the total surgery obstruction which lives in one of the L-groups presented in 12.

14 introduces the L-orientations which have a close connection to bundle theory.

15 realizes the connections between signatures and L-orientations.

16 realizes the connection between L-orientations and bundle reductions.

For statement 2 we use the surgery exact sequence of 21 which is part of the braid we already used to obtain the diagram of 12. This exact sequence together with the two statements 22 and 23 identify in the quadratic L-group $L_n(\mathbb{Z}\pi)$ the subset of all quadratic signatures coming from degree one normal maps $\widehat{f}\colon M \to X$ with the preimage of $s(X) \in \mathbb{S}_n(X)$ under the boundary map $\partial^Q_{\mathbb{Z}\pi}\colon L_n(\mathbb{Z}\pi) \to \mathbb{S}_n(X)$.

The total surgery obstruction

X is homotopy equivalent to a manifold if and only if $0 = s(X) \in \mathbb{S}_n(X) \quad (n \geq 5)$.

3

There is an isomorphism $\Phi \colon L_n^w(\mathbb{Z}\pi) \xrightarrow{\simeq} L_n(\mathbb{Z}\pi)$ such that $\Phi(\theta(\widehat{f})) = \operatorname{sgn}^{\mathbf{L}_\bullet}_{\mathbb{Z}\pi}(\widehat{f})$

4

A degree one normal map $\widehat{f}$ is bordant to a homotopy equivalence if and only if $0 = \theta(\widehat{f}) \in L_n^w(\mathbb{Z}\pi)$.

2

$$\partial^{Q}_{\mathbb{Z}\pi}{}^{-1} s(X) = \underbrace{\left\{\operatorname{sgn}^{\mathbf{L}_\bullet}_{\mathbb{Z}\pi}(\widehat{f}) \mid \widehat{f} \colon M \to X,\ M \text{ manifold}\right\}}_{\operatorname{sgn}^{\mathbf{L}_\bullet}_{\mathbb{Z}\pi}(\mathcal{N}(X))}$$

21

$$\longrightarrow H_n(X; \mathbf{L}_\bullet\langle 1\rangle) \xrightarrow{A} L_n(\mathbb{Z}\pi) \xrightarrow{\partial^Q_{\mathbb{Z}\pi}} \mathbb{S}_n(X) \xrightarrow{I}$$

is exact

22

$\operatorname{sgn}^{\mathbf{L}_\bullet}_{\mathbb{Z}\pi}(\mathcal{N}(X))$ is a coset of $\operatorname{im}(A)$

23

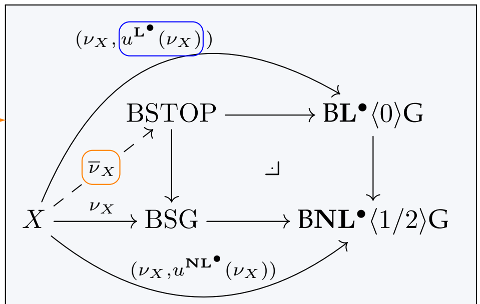
$\operatorname{sgn}^{\mathbf{L}_\bullet}_{\mathbb{Z}\pi}(\mathcal{N}(X))$ is a subset of $\partial^{Q}_{\mathbb{Z}\pi}{}^{-1} s(X)$

There exists a degree one normal map $\widehat{f}_0 \colon M \to X$

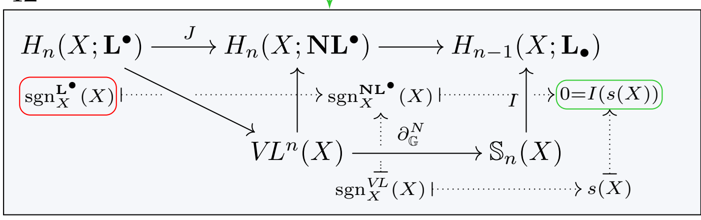
1

$I(s(X)) = 0$ if and only if there exists a bundle reduction $\overline{\nu}_X \colon X \to \mathrm{BSTOP}$ of the Spivak normal fibration $\nu_X \colon X \to \mathrm{BSG}$.

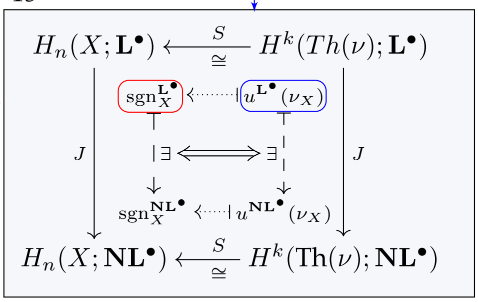
16



12



15



11 $\quad \xrightarrow{1+t} L^n(R) \xrightarrow{J} NL^n(R) \xrightarrow{\partial^N} L_{n-1}(R) \longrightarrow$

13 Construction of $s(X) := \partial^N_{\mathbb{G}} \operatorname{sgn}^{VL}_X(X) \in \mathbb{S}_n(X)$

14 Construction of $u^{\mathbf{L}^\bullet}$ and $u^{\mathbf{NL}^\bullet}$.

11 Exact sequence of L-groups

Porter

The exact sequence as presented in this section is not really used in the proof but it introduces the fundamental objects of the proof in a basic version, i.e. the symmetric, quadratic and normal L-groups over a ring, and establishes the crucial relation between them. In the next section we continue with the more sophisticated version of the exact sequence with generalized homology groups with coefficients in L-spectra. A more detailed account can be found in the basement.

The quadratic L-groups L_n are analogues of the surgery obstruction groups of Wall [Wal99]. They are defined by Ranicki in [Ran80a]. The symmetric L-groups L^n are the algebraic Poincaré cobordism groups of Mishchenko [Mis71]. The normal L-groups are cobordism groups of normal chain bundles introduced by Ranicki [Ran80a, §9] and Weiss [Wei85a]. An overview of these three different types of L-groups can be found in [Ran01]. The constructions that connect these algebraic L-groups to geometry were introduced in [Ran80b]. For the quadratic case take into account the corrections in [Ran81, p.30] and for the normal construction consider also the approach in [Wei85a, Theorem 3.4 and 3.5]. We will give an overview of these constructions in the room service and more details and proofs in the basement.

11 Exact sequence of L-groups [Ran92, Prop. 2.11]
There is a long exact sequence of L-groups

$$\ldots \longrightarrow L_n(R) \xrightarrow{1+t} L^n(R) \xrightarrow{J} NL^n(R) \xrightarrow{\partial^N} L_{n-1}(R) \longrightarrow \ldots .$$

111 Poincaré symmetric and Poincaré normal [Ran92, Proposition 2.6 (ii)]
There is the following natural one-to-one correspondence of homotopy equivalence classes.

$$\begin{array}{ccc} n\text{-normal chain complexes} & & n\text{-symmetric Poincaré chain complexes} \\ (C,(\varphi,\gamma,\chi)) & \xleftrightarrow{1-1} & (C,\varphi) \\ \text{such that } \varphi_0 \text{ is a chain homotopy equivalence} & & \end{array}$$

112 (121, 1411, 164) Quadratic and (normal, Poincaré symmetric) [Ran92, Proposition 2.8 (ii)]
There is the following natural one-to-one correspondence of cobordism classes.

$$\begin{array}{ccc} n\text{-dimensional (normal, symmetric) pairs} & \xleftrightarrow{1-1} & (n-1)\text{-dimensional quadratic chain complexes} \\ (f\colon C\to D,(\delta\varphi,\delta\gamma,\delta\chi),\varphi) & & (C',\psi') \end{array}$$

Additionally, if (C,φ) is Poincaré, then (C',ψ') is Poincaré and vice versa.

Proof 11

Note that for $L^n(R)$ and $L_n(R)$ we consider chain complexes with symmetric and quadratic structures which are Poincaré whereas in $NL^n(R)$ we do not require any part of the normal structures to be a homotopy equivalence. In fact the natural one-to-one correspondence of 111 111→p.60
tells us that a normal structure which is Poincaré is the same as a symmetric Poincaré structure and hence defines a map

$$J\colon L^n(R) \to NL^n(R).$$

Similarly to the long exact sequence of cobordism groups of spaces, we define a relative group $L(J)^n$ as the cobordism group of n-dimensional (normal, symmetric Poincaré) pairs, which fits by definition into a long exact sequence via the maps

$$\begin{array}{ccccc} NL^n(R) & \longrightarrow & L^n(J) & \longrightarrow & L^{n-1}(R) \\ (C,(\varphi,\gamma,\chi)) & \longmapsto & (0 \to C,(\varphi,\gamma,\chi),0) & & \\ & & (C \to D,(\delta\varphi,\delta\gamma,\delta\chi),\varphi) & \longmapsto & (C,\varphi). \end{array}$$

Cobordism of pairs is defined using triads of structured chain complexes. You find a short definition below and more details in [Ran81, §2].

112→p.61 The second natural one-to-one correspondence 112 defines an isomorphism

$$L^n(J) \cong L_{n-1}(R)$$

which gives us the desired exact sequence. □

Remark. As a different approach one can consider instead of J the symmetrization map

$$\begin{aligned} 1+t\colon L_n(R) &\longrightarrow L^n(R), \\ \psi &\longmapsto (1+t)(\psi) \end{aligned}$$

and identify the relative term with the normal L-group by the following one-to-one correspondence proved in [Ran92, Prop. 2.8 (i)]:

$$\begin{array}{c} \text{homotopy equivalence classes} \\ \text{of } n\text{-dimensional (symmetric,} \\ \text{quadratic) Poincaré pairs} \end{array} \xleftrightarrow{1-1} \begin{array}{c} \text{homotopy equivalence classes} \\ \text{of } n\text{-dimensional normal} \\ \text{complexes } (C,(\varphi,\gamma,\chi)). \end{array}$$

Room service 11

$\boxed{R}$ a ring with involution $^-\colon R \to R; r \mapsto \overline{r}$, i.e. it satisfies $\overline{1} = 1, \overline{\overline{r}} = r, \overline{rs} = \overline{s}\,\overline{r}$ and $\overline{r+s} = \overline{r}+\overline{s}$ for $r, s \in R$.

$\boxed{C}$ a chain complex of finitely generated projective left R-modules.

$\boxed{C^{-*}}$ the dual chain complex with $(C^{-*})_k := (C_{-k})^*$ and differential $d_k^{C^{-*}} := (-)^k(d_k^C)^*$.

$\boxed{C \otimes C}$ short for the chain complex of $\mathbb{Z}[\mathbb{Z}_2]$-modules $C^\iota \otimes_R C$ where the $\mathbb{Z}_2$-action swaps the factors.

- $\boxed{C^\iota}$ chain complex of right R-modules obtained from a chain complex C of left R-modules using the involution of R.

$\boxed{\mathscr{C}(f)}$ the algebraic mapping cone with $\mathscr{C}(f)_k := D_k \oplus C_{k-1}$ and differential $d^{\mathscr{C}(f)}(x,y) := (d^D(x) + f(y), -d^C(y))$ for a chain map $f\colon C \to D$.

$\boxed{W}$ the free resolution of the trivial $\mathbb{Z}[\mathbb{Z}_2]$-chain module $\mathbb{Z}$; given by the $\mathbb{Z}[\mathbb{Z}_2]$-chain complex

$$\ldots \xrightarrow{1+t} \mathbb{Z}[\mathbb{Z}_2] \xrightarrow{1-t} \mathbb{Z}[\mathbb{Z}_2] \longrightarrow 0$$

$\boxed{\widehat{W}}$ the complete resolution of the trivial $\mathbb{Z}[\mathbb{Z}_2]$-chain module $\mathbb{Z}$; given by the $\mathbb{Z}[\mathbb{Z}_2]$-chain complex $\dots \xrightarrow{1+t} \mathbb{Z}[\mathbb{Z}_2] \xrightarrow{1-t} \mathbb{Z}[\mathbb{Z}_2] \xrightarrow{1+t} \mathbb{Z}[\mathbb{Z}_2] \xrightarrow{1-t} \dots$

$\boxed{W^{\%}, W_{\%}, \widehat{W}^{\%}}$ denote for a chain complex C the abelian group chain complexes

$$W^{\%}C := \mathrm{Hom}_{\mathbb{Z}[\mathbb{Z}_2]}(W, C \otimes C),$$
$$W_{\%}C := W \otimes_{\mathbb{Z}[\mathbb{Z}_2]} (C \otimes C),$$
$$\widehat{W}^{\%}C := \mathrm{Hom}_{\mathbb{Z}[\mathbb{Z}_2]}(\widehat{W}, C \otimes C).$$

$\boxed{(C, \varphi)}$ an n-symmetric chain complex consisting of a chain complex C and an n-symmetric structure φ;
Poincaré means $\varphi_0 : C^{n-*} \to C$ is a chain equivalence.

- $\boxed{\varphi \in W^{\%}(C)_n}$ a cycle, called an n-symmetric structure

$\boxed{(C, \psi)}$ an n-quadratic chain complex consisting of a chain complex C and an n-quadratic structure ψ;
Poincaré means the symmetrization $(1+t)(\psi)_0 : C^{n-*} \to C$ is a chain equivalence.

- $\boxed{\psi \in W_{\%}(C)_n}$ a cycle, called an n-quadratic structure on C;
  can be represented by a set $\{\varphi_s : C^{n-s-*} \to C_* \mid s \geq 0\}$.

$\boxed{(C, (\varphi, \gamma, \chi))}$ an n-normal chain complex consisting of a chain complex C and an n-normal structure (φ, γ, χ).

- $\boxed{(\varphi, \gamma, \chi)}$ an n-normal structure with
  $\varphi \in {W^{\%}}_n$ an n-symmetric structure,
  $\gamma \in \widehat{W}^{\%}(C^{-*})_0$ a cycle called chain bundle,
  $\chi \in {\widehat{W}^{\%}}_{n+1}$ a chain satisfying $d\chi = J(\varphi) - \widehat{\varphi}^{\%} S^n \gamma$.

$\boxed{L^n(R)}$ the cobordism group of n-symmetric Poincaré chain complexes over R.

$\boxed{L_n(R)}$ the cobordism group of n-quadratic Poincaré chain complexes over R.

$\boxed{NL^n(R)}$ the cobordism group of n-normal chain complexes over R.

- $\boxed{Cobordism}$ of n-dimensional structured chain complexes:
  $(C, \lambda) \sim (C', \lambda') \iff$ there is an $(n+1)$-dimensional structured (Poincaré) pair $(C \oplus C' \to D, \partial\lambda, \lambda \oplus -\lambda')$,
  where Poincaré is only required in the symmetric and quadratic case.

$\boxed{\lambda}$ placeholder for a symmetric φ resp. quadratic ψ or normal structure (φ, γ, χ). We call a pair (C, λ) a *structured chain complex* and a triple $(C \to D, \delta\lambda, \lambda)$ a *structured pair*.

$\boxed{(f\colon C \to D, \delta\varphi, \varphi)}$ an $(n+1)$-symmetric pair with
- $f\colon C \to D$ a chain map
- (C, φ) an n-symmetric chain complex
- $\delta\varphi \in W^{\%}(D)_{n+1}$ such that $d(\delta\varphi) = f^{\%}(\varphi)$ which is equivalent to $(\delta\varphi, \varphi)$ is a cycle in $\mathscr{C}(f^{\%})_{n+1}$.

Poincaré means $(\delta\varphi_0, \varphi_0 f^*)\colon D^{n+1-*} \to \mathscr{C}(f)_*$ is a chain equivalence.

$\boxed{f^{\%}\colon W^{\%}(C) \to W^{\%}(D)}$ the chain map induced by a chain map $f\colon C \to D$; explicitly given by $(f^{\%}(\varphi))_s := f\varphi_s f^*\colon D^{n+s-*} \to D$.

$\boxed{(f\colon C \to D, \delta\psi, \psi)}$ an $(n+1)$-quadratic pair with
- $f\colon C \to D$ a chain map
- (C, ψ) an n-quadratic chain complex
- $\delta\psi \in W_{\%}(D)_{n+1}$ such that $d(\delta\psi) = f_{\%}(\psi)$.

Poincaré means the symmetrization is Poincaré, i.e.
$((1+t)\delta\varphi_0, (1+t)\varphi_0 f^*)\colon D^{n+1-*} \to \mathscr{C}(f)_* = (D_*, C_{*-1})$ is a chain equivalence.

$\boxed{f_{\%}\colon W_{\%}(C) \to W_{\%}(D)}$ the induced chain map

$\boxed{(f\colon C \to D, (\delta\varphi, \delta\gamma, \delta\chi), (\varphi, \gamma, \chi))}$ an $(n+1)$-normal pair with
- $(C, (\varphi, \gamma, \chi))$ an n-normal chain complex
- $(f\colon C \to D, \delta\varphi, \varphi)$ an $(n+1)$-symmetric pair
- $(f, b)\colon (C, \gamma) \to (D, \delta\gamma)$ a map of chain bundles
- $\delta\chi \in \widehat{W}^{\%}(D)_{n+2}$ a chain such that
  $J(\delta\varphi) - \widehat{\delta\varphi_0}^{\%}(\mathcal{S}^{n+1}\delta\gamma) + \widehat{f}^{\%}(\chi - \widehat{\varphi_0}^{\%}(\mathcal{S}^n b)) = d(\delta\chi) \in \widehat{W}^{\%}(D)_{n+1}$.

$\boxed{\widehat{f}^{\%}\colon \widehat{W}^{\%}(C) \to \widehat{W}^{\%}(C)}$ the induced chain map

$\boxed{(f, b)\colon (C, \gamma) \to (C', \gamma')}$ a map of chain bundles where $f\colon C \to C'$ a chain map and $b \in \widehat{W}^{\%}(C^*)_1$ a 1-chain such that $\widehat{f}^{\%}(\gamma) - \gamma' = d(b)$.

$\boxed{L(J)^n}$ the relative L-group of $J\colon L^n(R) \to NL^n(R)$ is the cobordism group of (normal, symmetric Poincaré) pairs $(f\colon C \to D, (\delta\varphi, \delta\gamma, \delta\chi), \varphi)$.

$\boxed{J\colon L^n(R) \to NL^n(R)}$ roughly induced by $j\colon W^{\%}C \to \widehat{W}^{\%}C$; see (111) for more details of how a normal structure (φ, γ, χ) is obtained from a symmetric Poincaré chain complex (C, φ).

$(f\colon C \to D, (\delta\varphi, \delta\gamma, \delta\chi), \varphi)$ an $(n+1)$-dimensional (normal, symmetric Poincaré) pair with

- $(D, \delta\varphi, \delta\gamma, \delta\chi)$ an $(n+1)$-normal chain complex
- $(f\colon C \to D, (\delta\varphi, \varphi))$ an $(n+1)$-symmetric pair
- φ_0 a chain equivalence.

$Cobordism$ of n-dimensional structured pairs:

$(C \xrightarrow{f} D, \delta\lambda, \lambda) \sim (C' \xrightarrow{f'} D', \delta\lambda', \lambda') \iff$ there is an $(n+1)$-dimensional structured (Poincaré) triad

$$\begin{array}{ccc} (C \oplus C', \lambda - \lambda') & \xrightarrow{\begin{pmatrix} f & 0 \\ 0 & f' \end{pmatrix}} & (D \oplus D', \delta\lambda - \delta\lambda') \\ {\scriptstyle \widetilde{f}}\downarrow & \overset{h}{\searrow} & \downarrow{\scriptstyle g} \\ (\widetilde{C}, \delta\widetilde{\lambda}) & \xrightarrow[g']{} & (E, \delta^2\lambda) \end{array}$$

where Poincaré is only required in the symmetric and quadratic case.

$\Gamma = (f, f', g, g'; h, (\varphi, \varphi', \delta\varphi, \delta\varphi'; \delta^2\varphi))$ an $(n+2)$-dimensional symmetric triad, i.e. a commutative square of chain complexes and chain maps

$$\begin{array}{ccc} C & \xrightarrow{f} & D \\ {\scriptstyle f'}\downarrow & \overset{h}{\searrow} & \downarrow{\scriptstyle g} \\ D' & \xrightarrow[g']{} & E \end{array}$$

with

- $h\colon gf \simeq g'f'\colon C \to E_{*+1}$ a chain homotopy
- $(f\colon C \to D, \delta\varphi, \varphi)$ and $(f'\colon C \to D', \delta\varphi', \varphi)$ being $(n+1)$-symmetric pairs
- $\delta^2\varphi \in W^{\%}(E)_{n+2}$ a chain such that
  $d(\delta^2\varphi)_s = g'^{\%}(\delta\varphi')_s - g^{\%}(\delta\varphi)_s + g'f'\varphi_s h^* + h\varphi_s f^* g^* + h^{\%}(\varphi)_s$.

Poincaré means $(f\colon C \to D, \delta\varphi, \varphi)$ and $(f'\colon C \to D', \delta\varphi', \varphi)$ are Poincaré and

$\begin{pmatrix} \delta^2\varphi_0 \\ \delta\varphi_0 g^* \\ f'\varphi_0 h^* + \delta\varphi_0' g'^* \\ \varphi_0 f^* g^* \end{pmatrix} \colon E^{n+2-*} \to \mathscr{C}(\Gamma) := \mathscr{C}\left(\begin{pmatrix} g' & h \\ 0 & f \end{pmatrix} \colon \mathscr{C}(f') \to \mathscr{C}(g)\right)$ is a chain equivalence.

The quadratic case is analog but uses symmetrization for the definition of Poincaré.

$1 + t\colon W_{\%}C \to W^{\%}C$ the symmetrization map defined by

$$(1+t)(\psi)_s = \begin{cases} (1+t)\psi_0 & \text{if } s = 0 \\ 0 & \text{otherwise} \end{cases}$$

induces a map of L-groups $1 + t\colon L_n(R) \to L^n(R)$.

12 Exact sequence of homology L-groups

Porter

In 11 we have introduced the L-groups $L^n(R), L_n(R), NL^n(R)$ for chain complexes of modules over a ring R with (Poincaré) duality structures and proved that they fit in an exact sequence. Now we need a more systematic way to distinguish such duality structures with a different type of Poincaré duality, especially local and global Poincaré duality. For this purpose we generalize the concept of L-groups from chain complexes of R-modules to chain complexes of objects of an additive category. To achieve this we first need a notion of a chain duality for additive categories in order to define λ-structures on the chain complexes. For such an additive category $\mathbb{A}$ with chain duality we can then define an algebraic bordism category Λ that establishes a certain flavour of Poincaré duality by fixing a subcategory $\mathbb{P}$ of the category of chain complexes in $\mathbb{A}$. Then a structured chain complex (C, λ) in Λ is called Poincaré if the algebraic mapping cone of λ_0 is in $\mathbb{P}$.

Take for example the algebraic bordism category $\Lambda(\mathbb{Z})$ with free $\mathbb{Z}$-modules as underlying additive category and choose $\mathbb{P}$ to be the category of contractible chain complexes. Then a Poincaré space X with its Poincaré duality chain equivalence $\cdot \cap [X]$ provides a symmetric Poincaré chain complex in $\Lambda(\mathbb{Z})$.

As in 11 or more precisely 112, we identify certain relative L-groups as absolute L-groups via algebraic surgery. With a special choice of algebraic bordism categories that encode local and global Poincaré duality we obtain a braid of exact sequences of L-groups. Some of these L-groups have a homological description and eventually lead to the desired diagram.

A lot of new terms are used in this section: L-groups of algebraic bordism categories and L-spectra and connective versions of them, X-based chain complexes for a simplicial complex which we will call mosaicked chain complexes and sophisticated versions of chain duality on them, and, finally, generalized homology groups of spectra of Δ-sets. You find short definitions in the room service below. For more motivational background have a look at the elevator to the basement (p. 110). At this stage we only put together the puzzle of exact sequences and identifications. The details of these definitions will become more important on the deeper levels.

The main source for all these constructions is [Ran92, Part I]. For the full details for the transition from L-groups to generalized homology groups consider [LM09].

12 Exact sequence of homology L-groups [Ran92, Proposition 15.16]
The diagram

$$\begin{array}{ccccccc}
\longrightarrow H_n(X;\mathbf{L}^\bullet\langle 0\rangle) & \longrightarrow & H_n(X;\mathbf{NL}^\bullet\langle 1/2\rangle) & \xrightarrow{\partial^N_{\mathbb{B}}} & H_{n-1}(X;\mathbf{L}_\bullet\langle 1\rangle) \longrightarrow \\
 & \searrow & \uparrow & & \uparrow I \\
 & & VL^n(X) & \xrightarrow{\partial^N_{\mathbb{G}}} & \mathbb{S}_n(X)
\end{array}$$

is commutative and exact in the top row.

121 Exact sequences for inclusions of bordism categories [Ran92, Prop. 3.9]
An inclusion functor $(\mathbb{A}, \mathbb{B}, \mathbb{Q}) \to (\mathbb{A}, \mathbb{B}, \mathbb{P})$ *of algebraic bordism categories induces the following long exact sequences in symmetric, quadratic and normal L-groups*

$$\ldots \longrightarrow L^n(\mathbb{B},\mathbb{Q}) \longrightarrow L^n(\mathbb{B},\mathbb{P}) \longrightarrow L^{n-1}(\mathbb{P},\mathbb{Q}) \longrightarrow L^{n-1}(\mathbb{B},\mathbb{Q}) \longrightarrow \ldots,$$

$$\ldots \longrightarrow L_n(\mathbb{B},\mathbb{Q}) \longrightarrow L_n(\mathbb{B},\mathbb{P}) \longrightarrow L_{n-1}(\mathbb{P},\mathbb{Q}) \longrightarrow L_{n-1}(\mathbb{B},\mathbb{Q}) \longrightarrow \ldots,$$

$$\ldots \longrightarrow NL^n(\mathbb{B},\mathbb{Q}) \longrightarrow NL^n(\mathbb{B},\mathbb{P}) \longrightarrow L_{n-1}(\mathbb{P},\mathbb{Q}) \longrightarrow NL^{n-1}(\mathbb{B},\mathbb{Q}) \longrightarrow \ldots.$$

122 Quadratic assembly isomorphism [Ran92, Proposition 15.11]
Let X be a simplicial complex and $\Lambda = \Lambda(\mathbb{Z})$. Then for $n \geq 5$ we have

$$L_n(\Lambda\langle 1\rangle_G X) \cong L_n(\mathbb{Z}\pi).$$

123 (13) L-spectra and homology [LM09, Remark 16.2][Ran92, Proposition 15.9]
Let X be a simplicial complex and $\Lambda = \Lambda(\mathbb{Z})$. Then we have the following equivalences

$$L_n(\Lambda\langle 1\rangle_L X) = H_n(X; \mathbf{L}_\bullet\langle 1\rangle),$$
$$NL^n(\Lambda\langle 0\rangle_L X) = H_n(X; \mathbf{L}^\bullet\langle 0\rangle),$$
$$NL^n(\Lambda\langle 1/2\rangle_N X) = H_n(X; \mathbf{NL}^\bullet\langle 1/2\rangle).$$

[124 → [Ran92, Prop. 5.1]] Local chain duality
The contravariant functor $T_* : \mathbb{A}_*X \to \mathbb{B}(\mathbb{A}_*X)$, $T_*(\sum_{\sigma\in X} M_\sigma)_s(\tau) = (T(\bigoplus_{\tau\leq\tilde{\tau}} M_{\tilde{\tau}})_{s-|\tau|}$ defines a chain duality on $\mathbb{A}_*X$.

Proof 12

We could think of the top row as induced by the fibration sequence of spectra 1411 but the whole diagram is part of a braid of L-groups of certain algebraic bordism categories. This braid is constructed using long exact sequences of triples:

Let $\mathbb{A}$ be an additive category with chain duality and let $\mathbb{Q} \subset \mathbb{P} \subset \mathbb{B}$ be subcategories of the category $\mathbb{B}(\mathbb{A})$ of bounded chain complexes in $\mathbb{A}$. They define algebraic bordism categories $(\mathbb{A}, \mathbb{B}, \mathbb{P})$ and $(\mathbb{A}, \mathbb{B}, \mathbb{Q})$. In the remainder we will usually omit the $\mathbb{A}$ in the notation of an algebraic bordism category. So $L(\mathbb{B}, \mathbb{P})$ will denote the L-group of structured chain complexes in $\mathbb{A}$ that are objects in $\mathbb{B}$ and are $\mathbb{P}$-Poincaré.

By 121, we have for an inclusion functor $(\mathbb{A}, \mathbb{B}, \mathbb{Q}) \to (\mathbb{A}, \mathbb{B}, \mathbb{P})$ the following long exact sequences for symmetric, quadratic and normal L-groups. 121→p.62

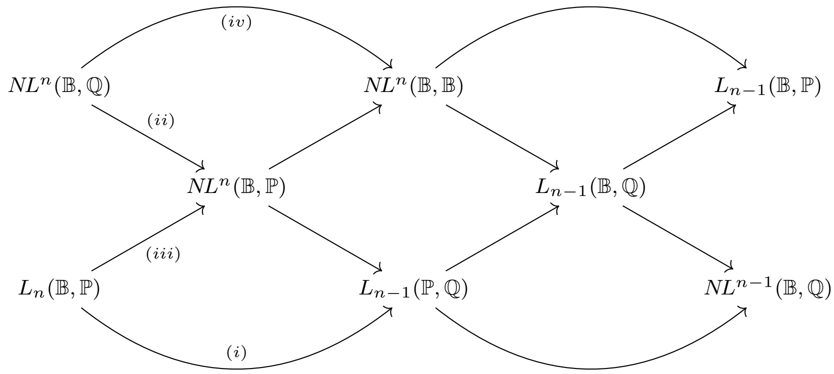
$$\ldots \longrightarrow L^n(\mathbb{B}, \mathbb{Q}) \longrightarrow L^n(\mathbb{B}, \mathbb{P}) \longrightarrow L^{n-1}(\mathbb{P}, \mathbb{Q}) \longrightarrow L^{n-1}(\mathbb{B}, \mathbb{Q}) \longrightarrow \ldots,$$
$$\ldots \longrightarrow L_n(\mathbb{B}, \mathbb{Q}) \longrightarrow L_n(\mathbb{B}, \mathbb{P}) \longrightarrow L_{n-1}(\mathbb{P}, \mathbb{Q}) \longrightarrow L_{n-1}(\mathbb{B}, \mathbb{Q}) \longrightarrow \ldots,$$
$$\ldots \longrightarrow NL^n(\mathbb{B}, \mathbb{Q}) \longrightarrow NL^n(\mathbb{B}, \mathbb{P}) \longrightarrow L_{n-1}(\mathbb{P}, \mathbb{Q}) \longrightarrow NL^{n-1}(\mathbb{B}, \mathbb{Q}) \longrightarrow \ldots.$$

The quadratic term in the sequence of normal L-groups allows us to create a commutative braid as shown below using the quadratic exact sequence for the inclusion functor

(i) $(\mathbb{B}, \mathbb{Q}) \to (\mathbb{B}, \mathbb{P})$

and the normal exact sequence for the inclusions

(ii) $(\mathbb{B}, \mathbb{Q}) \to (\mathbb{B}, \mathbb{P})$ (iii) $(\mathbb{B}, \mathbb{P}) \to (\mathbb{B}, \mathbb{B})$ (iv) $(\mathbb{B}, \mathbb{Q}) \to (\mathbb{B}, \mathbb{B})$.



To plait our algebraic surgery braid we take the following ingredients: In the remainder, the underlying additive category is the additive category of X-based free $\mathbb{Z}$-modules

$$\mathbb{A} = \mathbb{Z}_* X$$

with the chain duality T_* from 124. You find the motivation for this definition in the elevator to the basement. This chain duality on $\mathbb{Z}_*X$ allows us to talk about structured $\mathbb{Z}_*X$-chain complexes (C,λ) where the Poincaré duality map is given by $\lambda_0 : \Sigma^n TC \to C$.

We use the following subcategories of the category of chain complexes in $\mathbb{Z}_*X$.

$\mathbb{B}$ for $\mathbb{B}(\mathbb{Z})_L X = \mathbb{B}(\mathbb{Z}_* X)$, the X-based bounded chain complexes of free $\mathbb{Z}$-modules.

$\mathbb{G}$ for $\mathbb{B}(\mathbb{Z})_G X := \{C \in \mathbb{B} \mid A(C) \simeq *\}$, the *globally contractible* chain complexes of $\mathbb{B}$.

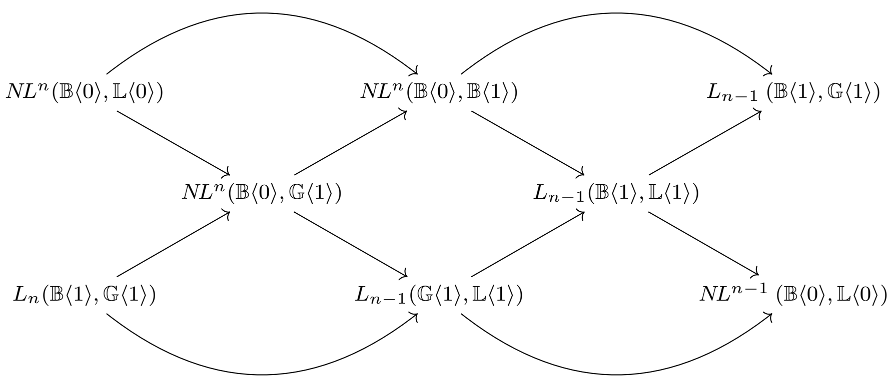
$\mathbb{L}$ for $\mathbb{B}(\mathbb{Z})_L X := \{C \in \mathbb{B} \mid C(\sigma) \simeq *,\ \sigma \in X\}$, the *locally contractible* chain complexes of $\mathbb{B}$.

As a further refinement, we have to take connective versions of these categories. Then, with the inclusions

(i) $(\mathbb{B}\langle 1\rangle, \mathbb{L}\langle 1\rangle) \to (\mathbb{B}\langle 1\rangle, \mathbb{G}\langle 1\rangle)$ (iii) $(\mathbb{B}\langle 0\rangle, \mathbb{G}\langle 1\rangle) \to (\mathbb{B}\langle 0\rangle, \mathbb{B}\langle 1\rangle)$

(ii) $(\mathbb{B}\langle 0\rangle, \mathbb{L}\langle 0\rangle) \to (\mathbb{B}\langle 0\rangle, \mathbb{G}\langle 1\rangle)$ (iv) $(\mathbb{B}\langle 0\rangle, \mathbb{L}\langle 0\rangle) \to (\mathbb{B}\langle 0\rangle, \mathbb{B}\langle 1\rangle)$

and the obvious identification $\mathbb{L}\langle 0\rangle = \mathbb{L}\langle 1\rangle$, we get the braid



Let $\Lambda = \Lambda(\mathbb{Z})$ be the algebraic bordism category of free $\mathbb{Z}$-modules. By 122 and 123 the L-groups of these algebraic bordism categories can be identified as follows. Note that an algebraic bordism category with local or no Poincaré duality (denoted by an L resp. N subscript) is necessary to obtain the corresponding L-groups as a generalized homology group.

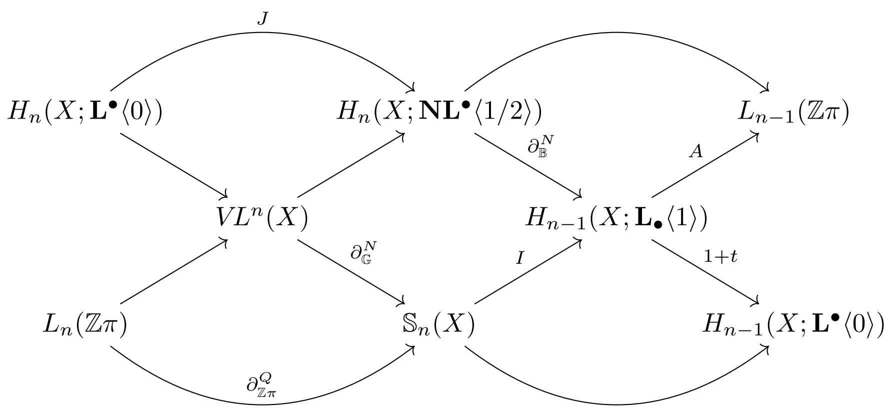
$$L_n(\mathbb{B}\langle 1\rangle, \mathbb{G}\langle 1\rangle) \;:=: L_n(\Lambda\langle 1\rangle_G X) \quad = L_n(\mathbb{Z}\pi),$$ 122→p.65

$$\begin{aligned} L_n(\mathbb{B}\langle 1\rangle, \mathbb{L}\langle 1\rangle) &:=: L_n(\Lambda\langle 1\rangle_L X) &&= H_n(X; \mathbf{L}_\bullet\langle 1\rangle), \\ NL^n(\mathbb{B}\langle 0\rangle, \mathbb{L}\langle 0\rangle) &:=: NL^n(\Lambda\langle 0\rangle_L X) &&= H_n(X; \mathbf{L}^\bullet\langle 0\rangle), \\ NL^n(\mathbb{B}\langle 0\rangle, \mathbb{B}\langle 1\rangle) &:=: NL^n(\Lambda\langle 1/2\rangle_N X) &&= H_n(X; \mathbf{NL}^\bullet\langle 1/2\rangle). \end{aligned}$$ 123→p.65

For the two remaining L-groups we make the following definitions.

$$\begin{aligned} NL^n(\mathbb{B}\langle 0\rangle, \mathbb{G}\langle 1\rangle) &:=: NL^n(\Lambda\langle 1/2\rangle_G X) =: VL^n(X) \\ L_{n-1}(\mathbb{G}\langle 1\rangle, \mathbb{L}\langle 1\rangle) &=: \mathbb{S}_n(X) \end{aligned}$$

We end up with the braid



where the center square is exactly the commutative diagram we were looking for. □

Room service 12

Algebraic bordism categories and L-groups

$L^n(\Lambda)$ the cobordism group of n-symmetric chain complexes (C, φ) in Λ.

$L_n(\Lambda)$ the cobordism group of n-quadratic chain complexes (C, ψ) in Λ.

$NL^n(\Lambda)$ the cobordism group of n-normal chain complexes $(C, (\varphi, \gamma, \chi))$ in Λ.

$\Lambda = (\mathbb{A}, \mathbb{C}, \mathbb{P}, (T, e))$ an algebraic bordism category, usually denoted by Λ or $(\mathbb{A}, \mathbb{C}, \mathbb{P})$, consists of full additive subcategories $\mathbb{P} \subseteq \mathbb{C} \subseteq \mathbb{B}(\mathbb{A})$ where $\mathbb{P}$ is closed under weak equivalences and mapping cones, i.e.

- $\mathscr{C}(f : C \to D) \in \mathbb{P}$ for any chain map f in $\mathbb{P}$,

and additionally any $C \in \mathbb{C}$ satisfies

- $\mathscr{C}(\mathrm{id} : C \to C) \in \mathbb{P}$,
- $\mathscr{C}(e(C) : T^2(C) \xrightarrow{\cong} C) \in \mathbb{P}$.

$\mathbb{A}$ additive category

$\mathbb{B}(\mathbb{A})$ the category of bounded chain complexes in $\mathbb{A}$.

(T, e) chain duality on $\mathbb{A}$ where $T\colon \mathbb{A}^{op} \to \mathbb{B}(\mathbb{A})$ is a functor and $e\colon T^2 \to \mathbb{1}$ a natural transformation with $e_A : T^2 A \xrightarrow{\cong} A$ and $e_{T(A)} \circ T(e_A) = \mathbb{1}$. for $A \in \mathbb{A}$.

(C, λ) in Λ an n-dimensional structured chain complex in $\Lambda = (\mathbb{A}, \mathbb{C}, \mathbb{P})$, i.e. a chain complex $C \in \mathbb{C}$ with an n-dimensional $\mathbb{P}$-Poincaré structure λ.

λ placeholder for a symmetric φ resp. a quadratic ψ or a normal structure (φ, γ, χ).

$\mathbb{P}$-Poincaré is what a structured complex (C, λ) is called if $\partial C := \Sigma^{-1}\mathscr{C}(\lambda_0) \in \mathbb{P}$.

λ_0 stands for φ_0 in the symmetric and normal case and for $(1 + t)(\psi_0)$ in the quadratic case.

$Cobordism$ of n-dimensional structured chain complexes in Λ:

$(C, \lambda) \sim (C', \lambda') \iff$ there is an $(n + 1)$-dimensional structured pair $(C \oplus C' \to D, \delta\lambda, \lambda \oplus -\lambda')$ in Λ.

$f : (C \to D, \delta\lambda, \lambda)$ in Λ a structured pair with $C, D \in \mathbb{B}$, λ is $\mathbb{P}$-Poincaré and $\mathscr{C}(\delta\lambda_0, \lambda_0 f^*) \in \mathbb{P}$.

Mosaicked algebraic bordism categories

$\Lambda(\mathbb{Z}) = (\mathbb{A}(\mathbb{Z}), \mathbb{B}(\mathbb{Z}), \mathbb{C}(\mathbb{Z}))$ denotes the algebraic bordism category with

$\mathbb{A}(\mathbb{Z})$ the additive category of finitely generated free left $\mathbb{Z}$-modules.

$\mathbb{B}(\mathbb{Z})$ the category of bounded chain complexes in $\mathbb{A}(\mathbb{Z})$.

$\mathbb{B}(\mathbb{Z})$ the category of contractible chain complexes in $\mathbb{A}(\mathbb{Z})$.

$\Lambda\langle q\rangle$ for $\Lambda = (\mathbb{A}, \mathbb{C}, \mathbb{P})$ and $q \in \mathbb{Z}$ the q-connective algebraic bordism category $(\mathbb{A}, \mathbb{C}\langle q\rangle, \mathbb{P}\langle q\rangle)$

$\mathbb{C}\langle q\rangle$ the subcategory of chain complexes of $\mathbb{C}$ that are homotopy equivalent to q-connected chain complexes.

$\Lambda\langle 1/2\rangle$ denotes for $\Lambda = (\mathbb{A},\mathbb{C},\mathbb{P})$ the algebraic bordism category $(\mathbb{A},\mathbb{C}\langle 0\rangle,\mathbb{P}\langle 1\rangle)$.

$\Lambda_L X$ for $\Lambda = (\mathbb{A},\mathbb{C},\mathbb{P})$ denotes $(\mathbb{A}_*X,\mathbb{C}_LX,\mathbb{P}_LX,(T_*,e_*))$, the X-mosaicked algebraic bordism category of $\Lambda = (\mathbb{A},\mathbb{C},\mathbb{P})$ with *local Poincaré* duality.

$\Lambda_G X$ for $\Lambda = (\mathbb{A},\mathbb{C},\mathbb{P})$ denotes $(\mathbb{A}_*X,\mathbb{C}_LX,\mathbb{P}_GX,(T_*,e_*))$, the X-mosaicked algebraic bordism category of $\Lambda = (\mathbb{A},\mathbb{C},\mathbb{P})$ with *global Poincaré* duality.

$\Lambda_N X$ for $\Lambda = (\mathbb{A},\mathbb{C},\mathbb{P})$ denotes $(\mathbb{A}_*X,\mathbb{C}_LX,\mathbb{C}_LX,(T_*,e_*))$, the X-mosaicked algebraic bordism category of $\Lambda = (\mathbb{A},\mathbb{C},\mathbb{P})$ with *no Poincaré* duality.

$\mathbb{A}_*X$ additive category of X-based objects in $\mathbb{A}$, i.e.

$$\mathrm{Obj}_{\mathbb{A}_*X} = \Big\{\sum_{\sigma\in X} M_\sigma \mid M_\sigma\in\mathbb{A}, M_\sigma = 0 \text{ except for finitely many } \sigma\Big\},$$

$$\mathrm{Mor}_{\mathbb{A}_*X}\Big(\sum_{\widetilde{\sigma}\in X} M_{\widetilde{\sigma}},\sum_{\widetilde{\tau}\in X} N_{\widetilde{\tau}}\Big) = \Big\{\sum_{\substack{\tau\geq\sigma\\ \sigma,\tau\in X}} f_{\tau,\sigma} \mid (f_{\tau,\sigma}\colon M_\sigma\to N_\tau)\in\mathrm{Mor}_{\mathbb{A}}(M_\sigma,N_\tau)\Big\},$$

where X is a simplicial complex.

$\mathbb{C}_LX := \{C \text{ chain complex in } \mathbb{A}_*X \mid C(\sigma)\in\mathbb{C} \text{ for all } \sigma\in X\}$ for a category $\mathbb{C}$ of chain complexes in $\mathbb{A}$.

$\mathbb{C}_GX := \{C \text{ chain complex in } \mathbb{A}_*X \mid A(C)\in\mathbb{C}\}$ for a category $\mathbb{C}$ of chain complexes in $\mathbb{A}$.

$A\colon\mathbb{A}_*X\to\mathbb{A}(\mathbb{Z}\pi)$ the assembly map defined by $\sum_{\sigma\in X} M_\sigma \mapsto \bigoplus_{\widetilde{\sigma}\in\widetilde{X}} M_{p(\widetilde{\sigma})}$ where $p\colon\widetilde{X}\to X$ is the universal covering.

T_* defined for a chain duality $T\colon\mathbb{A}\to\mathbb{B}(\mathbb{A})$ as the mosaicked chain duality $\mathbb{A}_*X\to\mathbb{B}(\mathbb{A}_*X)$ with $(T_*(\sum_{\sigma\in X} M_\sigma))_r(\tau) = (T(\bigoplus_{\tau\leq\tilde{\tau}} M_{\tilde{\tau}}))_{r-|\tau|}$.

$\mathbb{Z}_*X$ short for $\mathbb{A}(\mathbb{Z})_*X$, the additive category of X-based free $\mathbb{Z}$-modules with 'non-decreasing' morphisms $\sum_{\tau\geq\sigma} f_{\tau,\sigma}\colon \sum_{\sigma\in X} M_\sigma \to \sum_{\tau\in X} N_\tau$ where $(f_{\tau,\sigma}\colon M_\sigma\to N_\tau)$ are $\mathbb{Z}$-module morphism.

$\mathbb{B}$ short for $\mathbb{B}(\mathbb{Z})_LX = \mathbb{B}(\mathbb{Z}_*X)$, the X-based bounded chain complexes of free $\mathbb{Z}$-modules.

$\mathbb{L}$ short for $\mathbb{B}(\mathbb{Z})_LX := \{C\in\mathbb{B}(\mathbb{Z}) \mid C(\sigma)\simeq * \text{ for all } \sigma\in X\}$, the *locally contractible* chain complexes of $\mathbb{B}$.

$\mathbb{G}$ short for $\mathbb{B}(\mathbb{Z})_GX := \{C\in\mathbb{B}(\mathbb{Z}) \mid A(C)\simeq *\}$, the *globally contractible* objects of $\mathbb{B}(\mathbb{Z})$.

L-spectra

$\mathbf{L}^\bullet(\Lambda), \mathbf{L}_\bullet(\Lambda), \mathbf{NL}^\bullet(\Lambda)$ Ω-spectra of pointed Kan Δ-sets defined for an algebraic bordism category Λ by

$$\mathbf{L}^\bullet(\Lambda) = \left\{\mathbf{L}^n(\Lambda) \quad | \ n \in \mathbb{Z}, \mathbf{L}^n(\Lambda)^{(k)} = \{n\text{-dim. } (C,\varphi) \text{ in } \Lambda^L\Delta^k\}\right\},$$

$$\mathbf{L}_\bullet(\Lambda) = \left\{\mathbf{L}_n(\Lambda) \quad | \ n \in \mathbb{Z}, \mathbf{L}_n(\Lambda)^{(k)} = \{n\text{-dim. } (C,\psi) \text{ in } \Lambda^L\Delta^k\}\right\},$$

$$\mathbf{NL}^\bullet(\Lambda) = \left\{\mathbf{NL}^n(\Lambda) \mid n \in \mathbb{Z}, \mathbf{NL}^n(\Lambda)^{(k)} = \{n\text{-dim. } (C,(\varphi,\gamma,\chi)) \text{ in } \Lambda^L\Delta^{k+n}\}\right\};$$

face maps are induced by face inclusions $\partial_i : \Delta^{k-1} \to \Delta^k$, base point is the 0-chain complex.

Δ-set a simplicial set without degeneracies, i.e. a functor $\Delta \to \mathbf{Sets}$, where Δ is the category of finite sets $\{0,\ldots,n\}$, $n \geq 0$ and order-preserving injections and $\mathbf{Sets}$ the category of sets and functions.

Kan is what a Δ-set X is called if every map $\Lambda_i^n \to X$ extends to a map $\Delta^n \to X$; this property is necessary to do homotopy theory on Δ-sets .

$\Lambda_i^n := \Delta^n - ((\Delta^n)^{(n)} \cup \partial_i\Delta^n)$ the subcomplex of Δ^n obtained by removing the interior of Δ^n and a single face of Δ^n.

$K^{(k)}$ the set of k-simplices of a simplicial complex K.

Δ^n the standard n-simplex with ordered simplices $0 < 1 < \ldots < n$.

$\Lambda^L X$ for $\Lambda = (\mathbb{A},\mathbb{C},\mathbb{P})$ denotes $(\mathbb{A}^*X, \mathbb{C}^LX, \mathbb{P}^LX, (T^*, e^*))$, the X-mosaicked algebraic bordism category of $\Lambda = (\mathbb{A},\mathbb{C},\mathbb{P})$ with *local Poincaré* duality.

$\mathbb{Z}^*X$ short for $\mathbb{A}(\mathbb{Z})^*X$, the additive category of X-based free $\mathbb{Z}$-modules with 'non-increasing' morphisms $\sum\limits_{\tau\leq\sigma} f_{\tau,\sigma} : \sum\limits_{\widetilde{\sigma}\in X} M_{\widetilde{\sigma}} \to \sum\limits_{\widetilde{\tau}\in X} N_{\widetilde{\tau}}$ where $(f_{\tau,\sigma} : M_\sigma \to N_\tau)$ are $\mathbb{Z}$-module morphism

$\mathbb{A}^*X$ additive category of X-based objects in $\mathbb{A}$, i.e.

$$\text{Obj}_{\mathbb{A}^*X} = \left\{\sum_{\sigma\in X} M_\sigma \mid M_\sigma \in \mathbb{A}, M_\sigma = 0 \text{ except for finitely many } \sigma\right\},$$

$$\text{Mor}_{\mathbb{A}^*X}\Big(\sum_{\widetilde{\sigma}\in X} M_{\widetilde{\sigma}}, \sum_{\widetilde{\tau}\in X} N_{\widetilde{\tau}}\Big) = \left\{\sum_{\substack{\tau\leq\sigma\\ \sigma,\tau\in X}} f_{\tau,\sigma} \mid (f_{\tau,\sigma} : M_\sigma \to N_\tau) \in \text{Mor}_{\mathbb{A}}(M_\sigma, N_\tau)\right\},$$

where X is a simplicial complex.

$\mathbb{C}^LX := \{C \text{ chain complex in } \mathbb{A}^*X \mid C(\sigma) \in \mathbb{C} \text{ for all } \sigma \in X\}$ for a category $\mathbb{C}$ of chain complexes in $\mathbb{A}$.

T^* defined for a chain duality $T\colon \mathbb{A} \to \mathbb{B}(\mathbb{A})$ as the mosaicked chain duality $\mathbb{A}^*X \to \mathbb{B}(\mathbb{A}^*X)$ with $(T^*(\sum_{\sigma\in X} M_\sigma))_r(\tau) = (T(\bigoplus_{\tau\geq\tilde{\tau}} M_{\tilde{\tau}}))_{r+|\tau|}$.

$\mathbf{L}^\bullet\langle 0\rangle$ short for $\mathbf{L}^\bullet(\Lambda(\mathbb{Z})\langle 0\rangle)$

$\mathbf{L}_\bullet\langle 1\rangle$ short for $\mathbf{L}_\bullet(\Lambda(\mathbb{Z})\langle 1\rangle)$

$\mathbf{NL}^\bullet\langle 1/2\rangle$ short for $\mathbf{NL}^\bullet(\Lambda(\mathbb{Z})\langle 1/2\rangle)$

13 The visible mosaicked signature

Porter

In this section we gather the necessary results to construct the visible signature that produces the total surgery obstruction.

The visible mosaicked signature $\mathrm{sgn}_X^{VL}(X)$ is a refined version of the normal mosaicked signature $\mathrm{sgn}_X^{\mathbf{NL}^\bullet}(X)$ for the case that X is a Poincaré space rather than just a normal space. So we get something which is globally Poincaré, something in $NL^n(\Lambda_G X) =: VL^n(X)$ instead of $NL^n(\Lambda_N X)$. The mosaicked visible L-groups $VL^n(X)$ are related to the visible L-groups of a group ring as defined in [Wei92]. See [Ran92, Remark 9.8] for more details. We will use the following four equivalent descriptions of $\mathrm{sgn}_X^{\mathbf{NL}^\bullet}$ to get from the original definition to the one we will actually use to explicitly construct signatures, but be aware that there is a lot of material involved to get from (i) to (iv).

Let X be an n-dimensional simplicial Poincaré space with Spivak normal fibration $\nu_X\colon X \to \mathrm{BSG}$.

(i) **Structured chain complex description**
In order to get an element to play with in the braid of exact sequences we want our signature

$$\mathrm{sgn}_X^{\mathbf{NL}^\bullet}(X) \in NL^n(\Lambda_N X)$$

to be a normal chain complex mosaicked over X representing a cobordism class in $NL^n(\Lambda_N X)$ which means no Poincaré duality chain equivalence is required.

(ii) **Homological description**
The isomorphisms from 1232 allow us to consider signatures as elements in generalized homology groups with coefficients in L-spectra, i.e.

$$\mathrm{sgn}_X^{\mathbf{NL}^\bullet}(X) \in H_n(X; \mathbf{NL}^\bullet).$$

(iii) **Cohomological NL-cycle description**
In order to work with a specific element and be able to make identifications we use a cohomological description. Using S-duality we obtain such a cohomological description of $\mathrm{sgn}_X^{\mathbf{NL}^\bullet}$. More precisely, for m large enough such that X embeds into $\partial\Delta^{m+1}$, we can represent $\mathrm{sgn}_X^{\mathbf{NL}^\bullet}$ by an assignment

$$[X]^{\mathbf{NL}^\bullet} = \left\{\sigma \mapsto \mathrm{sgn}_X^{\mathbf{NL}^\bullet}(\sigma) \in \mathbf{NL}_{n-m}^{(m-|\sigma|)} \mid \sigma \in X\right\}$$

that has to satisfy certain boundary relations and is called an $\mathbf{NL}^\bullet$-cycle. We see the equivalence to the homological description as follows: A clever construction of an S-dual of X with the homotopy type of $\mathrm{Th}(\nu)$ makes obvious that an $\mathbf{NL}^\bullet$-cycle is an equivalent description of a map $\mathrm{sgn}^{\mathbf{NL}^\bullet}_{\mathrm{Th}(\nu)} \in H^{m-n}(\mathrm{Th}(\nu); \mathbf{NL}^\bullet)$. Then S-duality gives $\mathrm{sgn}^{\mathbf{NL}^\bullet}_X = S(\mathrm{sgn}^{\mathbf{NL}^\bullet}_{\mathrm{Th}(\nu)}) \in H_n(X; \mathbf{NL}^\bullet)$.

(iv) **Geometrical constructive description**
In order to actually construct our signature out of the geometric input of a simplicial Poincaré space X we assign to each simplex a normal space with respect to the simplicial structure of X. This means we assign to each simplex in X a simplex in the spectrum of geometric normal n-ads considered as a spectrum of pointed Δ-sets:

$$[X]^{\mathbf{\Omega}^N_\bullet} = \left\{\sigma \mapsto (X(\sigma), \nu(\sigma), \rho(\sigma)) \in (\mathbf{\Omega}^N_{n-m})^{(m-|\sigma|)} \mid \sigma \in X\right\}.$$

The relative normal construction yields a map $\mathrm{sgn}^{\mathbf{NL}^\bullet}_\Omega : \mathbf{\Omega}^N_\bullet \to \mathbf{NL}^\bullet$ and eventually we define the normal signature to be

$$\mathrm{sgn}^{\mathbf{NL}^\bullet}_X(X) \coloneqq \mathrm{sgn}^{\mathbf{NL}^\bullet}_\Omega([X]^{\mathbf{\Omega}^N_\bullet}).$$

The visible L-group $VL^n(X) = NL^n(\Lambda_G X)$ cannot be described as a homology group so we cannot go through the steps above to get a visible signature sgn^{VL}_X. Instead we have to check that the construction of $\mathrm{sgn}^{\mathbf{NL}^\bullet}_X$ produces an element in $VL^n(X)$ if X is Poincaré. In fact, we define $\mathrm{sgn}^{\mathbf{NL}^\bullet}_X$ only for the case that X is Poincaré. For a more general treatment, where X is allowed to be a normal space see [Ran13, Errata for page 103]. All the material used in this section is based on the last chapters of part one of [Ran92].

13 The visible mosaicked signature [Ran92, Example 9.13]
For an n-dimensional finite simplicial Poincaré complex X there is a visible signature

$$\mathrm{sgn}^{VL}_X(X) \in NL^n(\Lambda_G X) =: VL^n(X)$$

as a refinement of a normal signature

$$\mathrm{sgn}^{\mathbf{NL}^\bullet}_X(X) \in NL^n(\Lambda_N X).$$

131 (15) $E_\bullet$-cycles [Ran92, Prop. 12.8]
Let X be a finite simplicial complex and $m \in \mathbb{N}$ large enough such that there is an embedding of X into $\partial\Delta^{m+1}$. Then an n-dimensional $\mathbf{E}$-cycle $[K]^{\mathbf{E}}$ of X in $\partial\Delta^{m+1}$ defines an element in $H_n(X; \mathbf{E})$.

132 (15) Normal cycles [KMM13, Construction 11.1, 11.2 and 11.3]
Let X be a finite simplicial Poincaré space of dimension n embedded into $\partial\Delta^{m+1}$ for an $m \gg n$ large enough. There is an n-dimensional $\mathbf{\Omega}_N$-cycle, i.e. a collection of assignments

$$[X]^{\mathbf{\Omega}^N_\bullet} = \Big\{\sigma \mapsto x(\sigma) = (X(\sigma), \nu(\sigma), \rho(\sigma)) \in (\mathbf{\Omega}^N_{n-m})^{(m-|\sigma|)} \;\Big|\; \sigma \in X,\ \partial_i x(\sigma) = x(\delta_i\sigma) \text{ for all } \sigma, \delta_i\sigma \in X, 0 \le i \le m-|\sigma|\Big\}.$$

123 (13) L-spectra and homology [LM09, Remark 16.2][Ran92, Proposition 15.9]

$$H_n(X; \mathbf{NL}^\bullet) \cong NL^n(\Lambda\langle 1/2\rangle_N X)$$

141 (13, 15) Signature spectra maps [Ran79]
There is a map of spectra $\mathrm{sgn}^{\mathbf{NL}^\bullet}_\Omega : \mathbf{\Omega}^N_\bullet \to \mathbf{NL}^\bullet$.

Proof 13

We start with developing the definition of the normal mosaicked signature $\mathrm{sgn}_X^{\mathbf{NL}^\bullet}$. Let X be a finite simplicial Poincaré space of dimension n embedded into $\partial\Delta^{m+1}$ for an $m \gg n$ large enough. By 132, there is an n-dimensional $\mathbf{\Omega}_N$-cycle, i.e. a collection of assignments 132→p.75

$$[X]^{\mathbf{\Omega}^N_\bullet} = \Big\{\sigma \mapsto x(\sigma) = (X(\sigma), \nu(\sigma), \rho(\sigma)) \in (\mathbf{\Omega}^N_{n-m})^{(m-|\sigma|)} \;\Big|\; \sigma \in X,\ \partial_i x(\sigma) = x(\delta_i\sigma) \text{ for all } \sigma, \delta_i\sigma \in X, 0 \le i \le m - |\sigma|\Big\}.$$

As proven in 131 this is via S-duality essentially the same as a representative of a homology class in $H_n(X; \mathbf{\Omega}^N_\bullet)$. Applying the normal signature map $\mathrm{sgn}_\Omega^{\mathbf{NL}^\bullet} : \mathbf{\Omega}^N_\bullet \to \mathbf{NL}^\bullet$ produces an $\mathbf{NL}^\bullet$-cycle $[X]^{\mathbf{NL}^\bullet} \in H_n(X; \mathbf{NL}^\bullet)$. Finally, we use the isomorphism $H_n(X; \mathbf{NL}^\bullet) \cong NL^n(\Lambda_N X)$ we have already encountered in 12 and which is proved in 123 to obtain our normal mosaicked chain complex $\mathrm{sgn}_X^{\mathbf{NL}^\bullet}$. 131→p.72 141→p.78 123→p.65

The assembly of $\mathrm{sgn}_X^{\mathbf{NL}^\bullet}$ is given by $A(\mathrm{sgn}_X^{\mathbf{NL}^\bullet}(X)) = \mathrm{sgn}_{\mathbb{Z}\pi}^{\mathbf{NL}^\bullet}(X) =: (C(\widetilde{X}), (\varphi, \gamma, \chi)) \in NL^n(\mathbb{Z}\pi)$. But X was Poincaré and hence $\varphi = \mathrm{con}_X^\varphi([X])$ is a chain equivalence. So, actually, we have a well-defined element

$$\mathrm{sgn}_X^{VL}(X) := \mathrm{sgn}_X^{\mathbf{NL}^\bullet}(X) \in VL^n(X) := NL^n(\Lambda_G X)$$

if X is Poincaré. □

Room service 13

For the definition of algebraic bordism categories, L-groups and L-spectra see the room service of 12 on page 31.

(Y, ν, ρ) an n-dimensional normal space consisting of a topological space Y together with an oriented k-dimensional spherical fibration $\nu : Y \to \mathrm{BSG}(k)$ and a map $\rho : S^{n+k} \to \mathrm{Th}(\nu)$.

$[K]^{\mathbf{E}} \in H_n(K; \mathbf{E})$ an n-dimensional $\mathbf{E}$-cycle of a simplicial complex $K \subset \partial\Delta^{m+1}$ defined by a collection $\Big\{[K]^{\mathbf{E}}(\sigma) \in \mathbf{E}_{n-m}^{(m-|\sigma|)} \mid \sigma \in K\Big\}$ such that $\partial_i [K]^{\mathbf{E}}(\sigma) = [K]^{\mathbf{E}}(\delta_i\sigma)$ if $\delta_i\sigma \in K$ and $\emptyset$ otherwise.

$\mathbf{NL}^\bullet$ short for $\mathbf{NL}^\bullet(\Lambda(\mathbb{Z})\langle 1/2\rangle)$.

- $\mathbf{NL}^\bullet(\Lambda) = \big\{\mathbf{NL}^n(\Lambda) \mid n \in \mathbb{Z}, \mathbf{NL}^n(\Lambda)^{(k)} = \{n\text{-dim. } (C, (\varphi, \gamma, \chi)) \text{ in } \Lambda^L\Delta^{k+n}\}\big\}$ an Ω-spectrum of pointed Kan Δ-sets .
- $\Lambda(\mathbb{Z}) = (\mathbb{A}(\mathbb{Z}), \mathbb{B}(\mathbb{Z}), \mathbb{C}(\mathbb{Z}))$ denotes the algebraic bordism category with
  - $\mathbb{A}(\mathbb{Z})$ the category of finitely generated free left $\mathbb{Z}$-modules,
  - $\mathbb{B}(\mathbb{Z})$ the bounded chain complexes in $\mathbb{A}(\mathbb{Z})$,
  - $\mathbb{C}(\mathbb{Z})$ the contractible chain complexes of $\mathbb{B}(\mathbb{Z})$.

$A : \mathbb{A}_* X \to \mathbb{A}(\mathbb{Z}\pi)$ the assembly map defined by $\sum_{\sigma \in X} M_\sigma \mapsto \bigoplus_{\widetilde{\sigma} \in \widetilde{X}} M_{p(\widetilde{\sigma})}$ where $p : \widetilde{X} \to X$ is the universal covering.

$\boxed{\mathbf{\Omega}^N_\bullet}$ the Ω-spectrum of Kan Δ-sets defined by

$$(\mathbf{\Omega}^N_n)^{(k)} = \{(X_{\Delta^k}, \nu, \rho) \mid (n+k) - \text{dimensional normal space } (k+2)\text{-ad, i.e. } X_{\Delta^k} = (X, \partial_0 X, \ldots, \partial_k X) \text{ s.t. } \partial_0 X \cap \ldots \cap \partial_k X = \emptyset, \; \nu\colon X \to \mathrm{BSG}(r) \text{ an } (r-1)\text{-spherical fibration}, \; \rho\colon \Delta^{n+k+r} \to \mathrm{Th}(\nu) \text{ s.t. } \rho(\partial_i \Delta^{n+k+r}) \subset \mathrm{Th}(\nu|_{\partial_i X}) \}$$

The face maps $\partial_i\colon (\mathbf{\Omega}^N_n)^{(k)} \to (\mathbf{\Omega}^N_n)^{(k-1)}$ are given by

$\partial_i(X) = (\partial_i X,\ \partial_i X \cap \partial_0 X, \ldots, \partial_i X \cap \partial_{i-1} X,\ \partial_i X \cap \partial_{i+1} X, \ldots, \partial_i X \cap \partial_k X)$.

$\boxed{\mathrm{sgn}^{\mathbf{NL}^\bullet}_{\Omega}\colon \mathbf{\Omega}^N_\bullet \to \mathbf{NL}^\bullet}$ the normal signature map; based on the normal signature $\mathrm{sgn}^{\mathbf{NL}^\bullet}_{\mathbb{Z}\pi}$.

14 Canonical L-orientations

Porter

In this room we construct the L-orientations which will be related to the corresponding L-signatures in the next room.

We obtain these orientations with respect to the L-spectra $\mathbf{L}^\bullet$ and $\mathbf{NL}^\bullet$ from orientations with respect to the Thom spectra $\mathbf{MSG}$ and $\mathbf{MSTOP}$. Since any spherical fibration α and topological bundle β is a pullback of the universal one, we obtain canonical orientations $u^G(\alpha)\colon \mathrm{Th}(\alpha) \to \mathbf{MSG}(k)$, $u^T(\beta)\colon \mathrm{Th}(\beta) \to \mathbf{MSTOP}(k)$ and $u^{G/T}(\beta, h)\colon \mathrm{Th}(\nu_X) \to \mathbf{MS}(\mathrm{G/TOP})(k)$.

Composed with the Pontrjagin-Thom equivalences and the signature maps these Thom spectra orientations yield the desired L-spectra orientations. The Pontrjagin-Thom equivalence $\mathbf{\Omega}^{STOP}_\bullet \simeq \mathbf{MSTOP}$ is obtained from topological transversality of Kirby-Siebenmann [KS77] and Freedman-Quinn [FQ90]. The normal transversality used for the equivalence $\mathbf{\Omega}^N_\bullet \simeq \mathbf{MSG}$ is defined explicitly in [Ran13] by $(Y, \nu, \rho) \mapsto u^G(\nu) \circ \rho$. The inverse is given by mapping ρ to $(\mathrm{BSG}, \gamma_{SG}, \rho)$.

14 Canonical L-orientations [Ran79, p. 284-289][Ran92, 16.1(ii)][KMM13, Prop. 13.3 and 13.4]

(i) For a k-dimensional $\mathbb{Z}$-oriented spherical fibration $\alpha\colon X \to \mathrm{BSG}(k)$ there is a canonical $\mathbf{NL}^\bullet$-orientation

$$u_{\mathbf{NL}^\bullet}(\alpha) \in H^k(Th(\alpha); \mathbf{NL}^\bullet).$$

(ii) For a k-dimensional $\mathbb{Z}$-oriented topological bundle $\beta\colon X \to \mathrm{BSTOP}(k)$ there is a canonical $\mathbf{L}^\bullet$-orientation

$$u_{\mathbf{L}^\bullet}(\beta) \in H^k(Th(\beta); \mathbf{L}^\bullet).$$

(iii) For a k-dimensional $\mathbb{Z}$-oriented topological bundle $\beta\colon X \to \mathrm{BSTOP}(k)$ together with a homotopy $h\colon J(\beta) \simeq \nu_X$ there is a canonical $\mathbf{NL}/\mathbf{L}$-orientation

$$u^{\mathbf{NL}/\mathbf{L}^\bullet}(\beta, h) \in H^k(Th(\nu_X); \mathbf{NL}/\mathbf{L}^\bullet).$$

They are related via $J(u^{\mathbf{L}^\bullet}(\beta)) = u^{\mathbf{NL}^\bullet}(J(\beta))$ and $u^{\mathbf{NL}/\mathbf{L}^\bullet}(\beta, h) = (u^{\mathbf{NL}^\bullet}(h), u^{\mathbf{L}^\bullet}(\beta) - u^{\mathbf{L}^\bullet}(\nu_X))$.

141 (13, 15) Signature spectra maps [Ran79]
The relative symmetric and relative normal construction induce maps of spectra

$$\begin{aligned}&\mathrm{sgn}_\Omega^{\mathbf{L}^\bullet}\colon \mathbf{\Omega}_\bullet^{STOP} \to \mathbf{L}^\bullet, \quad \mathrm{sgn}_\Omega^{\mathbf{NL}^\bullet}\colon \mathbf{\Omega}_\bullet^{N} \to \mathbf{NL}^\bullet,\\ &\mathrm{sgn}_\Omega^{\mathbf{NL}/\mathbf{L}^\bullet}\colon \Sigma^{-1}\mathbf{\Omega}_\bullet^{\mathrm{N,STOP}} \to \mathbf{NL}/\mathbf{L}^\bullet, \quad \text{and}\\ &\mathrm{sgn}_\Omega^{\mathbf{L}_\bullet}\colon \Sigma^{-1}\mathbf{\Omega}_\bullet^{\mathrm{N,STOP}} \to \mathbf{L}_\bullet.\end{aligned}$$

[142 → [KS77, Essay III][FQ90, chapter 9][Ran92, Errata]] Transversality
There is topological and normal transversality which induce homotopy equivalences

$$\widetilde{c}\colon \mathbf{\Omega}_\bullet^{STOP} \simeq \mathbf{MSTOP}, \quad c\colon \mathbf{\Omega}_\bullet^{N} \simeq \mathbf{MSG} \quad \text{and} \quad c/\widetilde{c}\colon \Sigma^{-1}\mathbf{\Omega}_\bullet^{\mathrm{N,STOP}} \simeq \mathbf{MS}(\mathrm{G/TOP}).$$

Proof 14

First, we need orientations with respect to the Thom spectra $\mathbf{MSG}, \mathbf{MSTOP}$ and $\mathbf{MS}(\mathrm{G/TOP})$. Any spherical fibration $\alpha\colon X \to \mathrm{BSG}(k)$ is a pullback of the universal fibration γ_{SG} via the classifying map α. The induced map on the Thom spaces $\mathrm{Th}(\alpha) \to \mathrm{Th}(\gamma_{\mathrm{SG}}) = \mathbf{MSG}(k)$ defines a canonical $\mathbf{MSG}$-orientation $u^G(\alpha) \in H^k(\mathrm{Th}(\alpha); \mathbf{MSG})$. Likewise, we get a canonical $\mathbf{MSTOP}$-orientation $u^T(\beta) \in H^k(\mathrm{Th}(\beta); \mathbf{MSTOP})$ for a topological bundle $\beta\colon X \to \mathrm{BSTOP}$. The homotopy h gives us a spherical fibration over $X \times I$ with the canonical orientation $u^G(h)$ which we view as a homotopy between the orientation $J(u^T(\beta))$ and $J(u^T(\nu_X))$. We obtain a $\mathbf{MS}(\mathrm{G/TOP})$-orientation $u^{G/T}(\beta, h) := (u^G(h), u^T(\beta) - u^T(\nu_X)) \in H^k(\mathrm{Th}(\nu_X); \mathbf{MS}(\mathrm{G/TOP}))$. The homotopy inverses of the transversality homotopy equivalences $c\colon \mathbf{\Omega}_\bullet^N \xrightarrow{\simeq} \mathbf{MSG}$, $\widetilde{c}\colon \mathbf{\Omega}_\bullet^{STOP} \xrightarrow{\simeq} \mathbf{MSTOP}$ and $c/\widetilde{c}\colon \Sigma^{-1}\mathbf{\Omega}_\bullet^{\mathrm{N,STOP}} \to \mathbf{MS}(\mathrm{G/TOP})$ of 142 composed with the signature maps $\mathrm{sgn}_\Omega^{\mathbf{L}^\bullet}$, $\mathrm{sgn}_\Omega^{\mathbf{NL}^\bullet}$ and $\mathrm{sgn}_\Omega^{\mathbf{NL}/\mathbf{L}^\bullet}$ give the canonical orientations $u^{\mathbf{NL}^\bullet}, u^{\mathbf{L}^\bullet}$ and $u^{\mathbf{NL}/\mathbf{L}^\bullet}$. In the symmetric case there 141→p.78
is an explicit construction of $\widetilde{c}$ and we have

$$\begin{array}{ccccccc} u^{\mathbf{L}^\bullet}\colon \mathrm{Sing}(\mathrm{Th}(\beta)) & \xrightarrow{u^T(\beta)\circ -} & \mathrm{Sing}(\mathbf{MSTOP}(k)) & \xrightarrow{\widetilde{c}^{-1}} & \mathbf{\Omega}_{-k}^{STOP} & \xrightarrow{\mathrm{sgn}_\Omega^{\mathbf{L}^\bullet}} & \mathbf{L}^{-k} \\ \rho\colon \Delta^k \to \mathrm{Th}(\beta) & \longmapsto & u^T(\beta)\circ\rho & \longmapsto & M_{\Delta^k} & \longmapsto & (C_{\Delta^k}, \varphi_{\Delta^k}) \end{array}$$

where $M_{\Delta^k} := (M, \partial_0 M, \ldots, \partial_k M)$ is a manifold k-ad obtained by making $\widehat{\rho} := u^T(\beta)\circ\rho$ transverse to $\mathrm{BSTOP}(k) \subset \mathbf{MSTOP}(k)$ and by taking preimages, i.e. set $M := \widehat{\rho}^{-1}(\mathrm{BSTOP}(k))$ and $\partial_i M := (\widehat{\rho}|_{\partial_i\Delta^n})^{-1}(\mathrm{BSTOP}(k))$. Use the symmetric construction in the shape of the symmetric signature map from 141 to obtain the symmetric algebraic chain complex $(C_{\Delta^k}, \varphi_{\Delta^k})$ in $\mathbb{Z}^* X^L \Delta^n$.

In the other cases we have to be content with the existence of inverses c^{-1} and $c/\widetilde{c}^{-1}$ but this will be enough for our purposes. The $u^{\mathbf{L}^\bullet}$ and $u^{\mathbf{NL}^\bullet}$ orientation fit into the following commutative diagram.

$$\begin{array}{ccccccc} u^{\mathbf{L}^\bullet}\colon \mathrm{Sing}(\mathrm{Th}(\beta)) & \xrightarrow{u^T(\beta)\circ -} & \mathrm{Sing}(\mathbf{MSTOP}(k)) & \xrightarrow{\widetilde{c}^{-1}} & \mathbf{\Omega}_{-k}^{STOP} & \xrightarrow{\mathrm{sgn}_\Omega^{\mathbf{L}^\bullet}} & \mathbf{L}^{-k} \\ & & \downarrow J & & \downarrow J & & \downarrow J \\ u^{\mathbf{NL}^\bullet}\colon \mathrm{Sing}(\mathrm{Th}(J(\beta))) & \xrightarrow{u^G} & \mathrm{Sing}(\mathbf{MSG}(k)) & \xrightarrow{c^{-1}} & \mathbf{\Omega}_{-k}^{N} & \xrightarrow{\mathrm{sgn}_\Omega^{\mathbf{NL}^\bullet}} & \mathbf{NL}^{-k} \end{array}$$

□

Room service 14

$\boxed{\nu_X : X \to \mathrm{BSG}}$ the Spivak normal fibration of X, i.e. an oriented $(k-1)$-spherical fibration of an n-dimensional Poincaré space X for which a class $\alpha \in \pi_{n+k}(\mathrm{Th}(\nu_X))$ $(k > n+1)$ exists such that $h(\alpha) \cap u = [X]$. Here $u \in H^k(\mathrm{Th}(\nu_X))$ is the Thom class and $h : \pi_*(\cdot) \to H_*(\cdot)$ is the Hurewicz map.

$\boxed{\mathrm{BSG}}$ the classifying space of stable $\mathbb{Z}$-oriented spherical fibrations.

$\boxed{\mathrm{BSTOP}}$ the classifying space of stable $\mathbb{Z}$-oriented topological bundles.

$\boxed{\mathbf{MSTOP}}$ the Thom spectrum of the universal stable $\mathbb{Z}$-oriented topological bundles over the classifying space BSTOP with the k-th space the Thom space $\mathbf{MSTOP}(k) = \mathrm{Th}(\gamma_{\mathrm{STOP}}(k))$ of the universal k-dimensional bundle $\gamma_{\mathrm{STOP}}(k)$ over BSTOP.

$\boxed{\mathbf{MSG}}$ the Thom spectrum of the universal stable $\mathbb{Z}$-oriented spherical fibrations over BSG with the k-th space the Thom space $\mathbf{MSG}(k) = \mathrm{Th}(\gamma_{\mathrm{SG}}(k))$ of the universal k-dimensional spherical fibration $\gamma_{\mathrm{SG}}(k)$ over $\mathrm{BSG}(k)$.

$\boxed{\mathbf{MS}(\mathrm{G/TOP})}$ the fiber of $J : \mathbf{MSTOP} \to \mathbf{MSG}$.

$\boxed{\mathbf{\Omega}^N_\bullet}$ the Ω-spectrum of Kan Δ-sets defined by

$$\begin{aligned}(\mathbf{\Omega}^N_n)^{(k)} = \{(X_{\Delta^k}, \nu, \rho) \mid (n+k) - \text{dimensional normal space } (k+2)\text{-ad, i.e.}\\ X_{\Delta^k} = (X, \partial_0 X, \ldots, \partial_k X) \text{ s.t. } \partial_0 X \cap \ldots \cap \partial_k X = \emptyset,\\ \nu : X \to \mathrm{BSG}(r) \text{ an } (r-1)\text{-spherical fibration,}\\ \rho : \Delta^{n+k+r} \to \mathrm{Th}(\nu) \text{ s.t. } \rho(\partial_i \Delta^{n+k+r}) \subset \mathrm{Th}(\nu|_{\partial_i X}) \quad \}\end{aligned}$$

The face maps $\partial_i : (\mathbf{\Omega}^N_n)^{(k)} \to (\mathbf{\Omega}^N_n)^{(k-1)}$are given by

$\partial_i(X) = (\partial_i X,\ \partial_i X \cap \partial_0 X, \ldots, \partial_i X \cap \partial_{i-1} X,\ \partial_i X \cap \partial_{i+1} X, \ldots, \partial_i X \cap \partial_k X).$

$\boxed{\mathbf{\Omega}^{STOP}_\bullet}$ the Ω-spectrum of Kan Δ-sets defined by

$$\begin{aligned}(\mathbf{\Omega}^{STOP}_n)^{(k)} = \{(M, \partial_0 M, \ldots, \partial_k M) \mid (n+k) - \text{dimensional manifold}\\ (k+2)\text{-ad such that } \partial_0 M \cap \ldots \cap \partial_k M = \emptyset\}.\end{aligned}$$

The face maps $\partial_i : (\mathbf{\Omega}^{STOP}_n)^{(k)} \to (\mathbf{\Omega}^{STOP}_n)^{(k-1)}$ are given by
$\partial_i(M) = (\partial_i M,\ \partial_i M \cap \partial_0 M, \ldots, \partial_i M \cap \partial_{i-1} M,\ \partial_i M \cap \partial_{i+1} M, \ldots, \partial_i M \cap \partial_k M).$

$\boxed{\Sigma^{-1}\mathbf{\Omega}^{\mathrm{N,STOP}}_\bullet}$ the Ω-spectrum of Δ-sets obtained as the fiber of canonical the map of spectra $\mathbf{\Omega}^{STOP}_\bullet \to \mathbf{\Omega}^N_\bullet$.

$\boxed{u^{\mathbf{E}}(\nu)}$ an $\mathbf{E}$-orientation of a $\mathbb{Z}$-oriented spherical fibration $\nu : X \to \mathrm{BSG}(k)$ is an element $u^{\mathbf{E}}(\nu) \in H^k(\mathrm{Th}(\nu); \mathbf{E})$ such that $u^{\mathbf{E}}(\nu)$ restricts to a generator of $H^k(\mathrm{Th}(\nu_x); \mathbf{E})$ for each fiber ν_x of ν.

$u^G(\beta) \in H^k(\mathrm{Th}(\beta); \mathbf{MSG})$ the canonical **MSG**-orientation of β which is a map on the Thom spaces $\mathrm{Th}(\beta) \to \mathrm{Th}(\gamma_{SG})$ induced by the classifying map of a k-dimensional $\mathbb{Z}$-oriented spherical fibration $\beta\colon X \to \mathrm{BSG}(k)$.

$u^T(\alpha) \in H^k(\mathrm{Th}(\alpha); \mathbf{MSTOP})$ the canonical **MSTOP**-orientation of α which is a map on the Thom spaces $\mathrm{Th}(\alpha) \to \mathrm{Th}(\gamma_{STOP})$ induced by the classifying map of a k-dimensional $\mathbb{Z}$-oriented topological bundle $\alpha\colon X \to \mathrm{BSG}(k)$.

$u^{G/T}(\nu, h) \in H^k(\mathrm{Th}(\nu_X); \mathbf{MS}(\mathrm{G/TOP}))$ the preferred lift of $u^T(\nu)$ for a bundle reduction ν of the Spivak normal fibration ν_X, determined by the homotopy $h\colon \mathrm{Th}(\nu_X) \times [0,1] \to \mathbf{MSG}$ between $J(\nu)$ and $J(\nu_X)$.

$\mathbf{L}^\bullet$ short for $\mathbf{L}^\bullet(\Lambda(\mathbb{Z})\langle 0\rangle)$.

$\mathbf{NL}^\bullet$ short for $\mathbf{NL}^\bullet(\Lambda(\mathbb{Z})\langle 1/2\rangle)$.

$\mathbf{NL}/\mathbf{L}^\bullet := \mathrm{Fiber}(J\colon \mathbf{L}^\bullet \to \mathbf{NL}^\bullet)$.

$\mathbf{L}^\bullet(\Lambda), \mathbf{NL}^\bullet(\Lambda)$ Ω-spectra of pointed Kan Δ-sets defined for an algebraic bordism category Λ by

$$\mathbf{L}^\bullet(\Lambda) = \{\mathbf{L}^n(\Lambda) \mid n \in \mathbb{Z}, \mathbf{L}^n(\Lambda)^{(k)} = n\text{-dim. symmetric complexes in } \Lambda^L\Delta^k\},$$

$$\mathbf{NL}^\bullet(\Lambda) = \{\mathbf{NL}^n(\Lambda) \mid n \in \mathbb{Z}, \mathbf{NL}^n(\Lambda)^{(k)} = n\text{-dim. normal complexes in } \Lambda^L\Delta^{k+n}\};$$

face maps are induced by face inclusions $\partial_i : \Delta^{k-1} \to \Delta^k$, the base point is the 0-chain complex.

$\Lambda^L X$ for $\Lambda = (\mathbb{A}, \mathbb{C}, \mathbb{P})$ denotes $(\mathbb{A}^*X, \mathbb{C}^L X, \mathbb{P}^L X, (T^*, e^*))$, the X-mosaicked algebraic bordism category of $\Lambda = (\mathbb{A}, \mathbb{C}, \mathbb{P})$ with *local Poincaré* duality.

$\mathbb{Z}^*X$ short for $\mathbb{A}(\mathbb{Z})^*X$, the additive category of X-based free $\mathbb{Z}$-modules with 'non-increasing' morphisms $\sum\limits_{\tau\leq\sigma} f_{\tau,\sigma}\colon \sum\limits_{\tilde{\sigma}\in X} M_{\tilde{\sigma}} \to \sum\limits_{\tilde{\tau}\in X} N_{\tilde{\tau}}$ where $(f_{\tau,\sigma}\colon M_\sigma \to N_\tau)$ are $\mathbb{Z}$-module morphism

$\mathbb{C}^L X := \{C \text{ chain complex in } \mathbb{A}^*X \mid C(\sigma) \in \mathbb{C} \text{ for all } \sigma \in X\}$ for a category $\mathbb{C}$ of chain complexes in $\mathbb{A}$.

$\Lambda(\mathbb{Z}) = (\mathbb{A}(\mathbb{Z}), \mathbb{B}(\mathbb{Z}), \mathbb{C}(\mathbb{Z}))$ denotes the algebraic bordism category with

- $\mathbb{A}(\mathbb{Z})$ the category of finitely generated free left $\mathbb{Z}$-modules,
- $\mathbb{B}(\mathbb{Z})$ the bounded chain complexes in $\mathbb{A}(\mathbb{Z})$,
- $\mathbb{C}(\mathbb{Z})$ the contractible chain complexes of $\mathbb{B}(\mathbb{Z})$.

$\Lambda = (\mathbb{A}, \mathbb{C}, \mathbb{P}, (T, e))$ an algebraic bordism category, usually denoted by Λ or $(\mathbb{A}, \mathbb{C}, \mathbb{P})$, consists of full additive subcategories $\mathbb{P} \subseteq \mathbb{C} \subseteq \mathbb{B}(\mathbb{A})$ where $\mathbb{P}$ is closed under weak equivalences and mapping cones, i.e.

- $\mathscr{C}(f : C \to D) \in \mathbb{P}$ for any chain map f in $\mathbb{P}$,

and additionally any $C \in \mathbb{C}$ satisfies

- $\mathscr{C}(\mathrm{id} : C \to C) \in \mathbb{P}$,
- $\mathscr{C}(e(C) : T^2(C) \xrightarrow{\simeq} C) \in \mathbb{P}$.

$\Lambda\langle q \rangle$ for $\Lambda = (\mathbb{A}, \mathbb{C}, \mathbb{P})$ and $q \in \mathbb{Z}$ the q-connective algebraic bordism category $(\mathbb{A}, \mathbb{C}\langle q \rangle, \mathbb{P}\langle q \rangle)$

$\mathbb{C}\langle q \rangle$ the subcategory of chain complexes of $\mathbb{C}$ that are homotopy equivalent to q-connected chain complexes.

$\Lambda\langle 1/2 \rangle$ denotes for $\Lambda = (\mathbb{A}, \mathbb{C}, \mathbb{P})$ the algebraic bordism category $(\mathbb{A}, \mathbb{C}\langle 0 \rangle, \mathbb{P}\langle 1 \rangle)$.

Sing singular simplicial complex

15 Orientations and signatures

Porter

In this room we identify the L-spectra orientations with the corresponding signatures. For a Poincaré space X the orientations $u^{\mathbf{L}^\bullet}(\nu_X) \in H^k(\mathrm{Th}(\alpha); \mathbf{L}^\bullet\langle 0 \rangle)$ and $u^{\mathbf{NL}^\bullet}(\nu_X) \in H^k(\mathrm{Th}(\beta); \mathbf{NL}^\bullet\langle 1/2 \rangle)$ are defined combining the Thom spectra orientations u^G and u^T with transversality and the signature maps $\mathrm{sgn}_\Omega^{\mathbf{NL}^\bullet}$ and $\mathrm{sgn}_\Omega^{\mathbf{L}^\bullet}$. The signatures $\mathrm{sgn}_X^{\mathbf{NL}^\bullet}(X) \in H_n(X; \mathbf{NL}^\bullet\langle 1/2 \rangle)$ and $\mathrm{sgn}_X^{\mathbf{L}^\bullet}(X) \in H_n(X; \mathbf{L}^\bullet\langle 0 \rangle)$ are defined by applying the same signature maps $\mathrm{sgn}_\Omega^{\mathbf{NL}^\bullet}$ and $\mathrm{sgn}_\Omega^{\mathbf{L}^\bullet}$ to $\mathbf{\Omega}_\bullet^N$- and $\mathbf{\Omega}_\bullet^{STOP}$-cycles $[X]^{\mathbf{\Omega}_\bullet^N}$ and $[X]^{\mathbf{\Omega}_\bullet^{STOP}}$. But these cycles actually define elements in the cohomology groups of an S-dual of X, more precisely simplicial maps $\Sigma^m/\overline{X} \to \mathbf{\Omega}_{-k}^N$ resp. $\Sigma^m/\overline{X} \to \mathbf{\Omega}_{-k}^{STOP}$. We used S-duality to consider these maps as elements in the homology groups of X. So at the end we only have to prove that transversality mediates between Thom orientations u^G, u^T and bordism cycles $[X]^{\mathbf{\Omega}_\bullet^N}$ and $[X]^{\mathbf{\Omega}_\bullet^{STOP}}$.

There are no new substatements in this room. It is more or less only a matter of putting all the definitions and the statements used for these definitions together.

15 (16) Orientations and signatures [Ran92, Proposition 16.1]

(i) Let X be an n-dimensional Poincaré space with Spivak normal fibration $\nu_X : X \to \mathrm{BSG}$. Then we have $S(u_{\mathbf{NL}}(\nu_X)) = \mathrm{sgn}_X^{\mathbf{NL}^\bullet}(X) \in H_n(X; \mathbf{NL}^\bullet\langle 1/2 \rangle)$.

(ii) Let $\overline{\nu}$ be a topological bundle reduction of the Spivak normal fibration $\nu_X : X \to \mathrm{BSG}$ of X and $\widehat{f} : M \to X$ its associated degree one normal map. Then we have $S(u_{\mathbf{L}}(\nu_X)) = \mathrm{sgn}_X^{\mathbf{L}^\bullet}(X) \in H_n(X; \mathbf{L}^\bullet\langle 0 \rangle)$.

(iii) Let $\widehat{f} : M \to M'$ be a degree one normal map of n-dimensional simply-connected topological manifolds with M' triangulated, corresponding to a pair (β, h) with $\beta : M' \to \mathrm{BSTOP}$ and $h : J(\beta) \simeq \nu_{M'}$. Then we have $S(u^{\mathbf{NL}/\mathbf{L}^\bullet}(\beta, h)) = \mathrm{sgn}_{M'}^{\mathbf{NL}/\mathbf{L}^\bullet}(\widehat{f}) \in H_n(M'; \mathbf{NL}\langle 1/2 \rangle / \mathbf{L}\langle 0 \rangle^\bullet)$.

131 (15) $E_\bullet$-cycles [Ran92, Prop. 12.8]
Let X be a finite simplicial complex and $m \in \mathbb{N}$ large enough such that there is an embedding of X into $\partial\Delta^{m+1}$. Then an n-dimensional $\mathbf{E}$-cycle $[K]^{\mathbf{E}}$ of X in $\partial\Delta^{m+1}$ defines an element in $H_n(X;\mathbf{E})$.

1311 (132, 15) Simplicial dual complex [Ran92, §12]
There is an isomorphism of simplicial complexes $\Phi\colon (\partial\Delta^{m+1})' \xrightarrow{\cong} (\Sigma^m)'$ such that for each $\sigma^* \in \Sigma^m$ we have

$$\Phi(D(\sigma, \partial\Delta^{m+1})) = \sigma^*.$$

132 (15) Normal cycles [KMM13, Construction 11.1, 11.2 and 11.3]
Let X be a finite simplicial Poincaré space of dimension n embedded into $\partial\Delta^{m+1}$ for an $m \gg n$ large enough. There is an n-dimensional $\mathbf{\Omega}_N$-cycle, i.e. a collection of assignments

$$[X]^{\mathbf{\Omega}^N_\bullet} = \Big\{\sigma \mapsto x(\sigma) = (X(\sigma), \nu(\sigma), \rho(\sigma)) \in (\mathbf{\Omega}^N_{n-m})^{(m-|\sigma|)} \;\Big|\; \sigma \in X,\ \partial_i x(\sigma) = x(\delta_i\sigma) \text{ for all } \sigma, \delta_i\sigma \in X, 0 \le i \le m - |\sigma|\Big\}.$$

2221 (15) (normal, manifold)-cycles [KMM13, Construction 11.9]
Let $\widehat{f}\colon M \to M'$ be a degree one map of n-dimensional topological manifolds such that M' is triangulated. Then there is a $\Sigma^{-1}\mathbf{\Omega}^{\mathrm{N,STOP}}_\bullet$-cycle

$$[\widehat{f}]^{\Sigma^{-1}\mathbf{\Omega}^{\mathrm{N,STOP}}_\bullet} \in H_n(M';\Sigma^{-1}\mathbf{\Omega}^{\mathrm{N,STOP}}_\bullet)$$

such that $\mathrm{sgn}^{\mathbf{L}_\bullet}_X(X) = \mathrm{sgn}^{\mathbf{L}_\bullet}_\Omega([\widehat{f}]^{\Sigma^{-1}\mathbf{\Omega}^{\mathrm{N,STOP}}_\bullet}) \in H_n(X;\mathbf{L}_\bullet\langle 1\rangle)$.

141 (13, 15) Signature spectra maps [Ran79]
The relative symmetric and relative normal construction induce maps of spectra

$$\mathrm{sgn}^{\mathbf{L}^\bullet}_\Omega\colon \mathbf{\Omega}^{STOP}_\bullet \to \mathbf{L}^\bullet\langle 0\rangle, \quad \mathrm{sgn}^{\mathbf{NL}^\bullet}_\Omega\colon \mathbf{\Omega}^N_\bullet \to \mathbf{NL}^\bullet\langle 1/2\rangle,$$
$$\mathrm{sgn}^{\mathbf{NL}/\mathbf{L}^\bullet}_\Omega\colon \Sigma^{-1}\mathbf{\Omega}^{\mathrm{N,STOP}}_\bullet \to \mathbf{NL}\langle 1/2\rangle/\mathbf{L}\langle 0\rangle^\bullet, \quad \textit{and}$$
$$\mathrm{sgn}^{\mathbf{L}_\bullet}_\Omega\colon \Sigma^{-1}\mathbf{\Omega}^{\mathrm{N,STOP}}_\bullet \to \mathbf{L}_\bullet\langle 1\rangle.$$

Proof 15

For simplicity we assume that we have a simplicial structure on X itself rather than using a homotopy equivalence $r\colon X \to K$ and preimages of dual cells.

Let $m \ge 0$ be large enough such that we have an embedding $X \subset \partial\Delta^{m+1}$. Recall that for a Poincaré space the Thom space is an S-dual. Hence the S-dual $X^* := \Sigma^m/\overline{X}$ as constructed in 1312 1312→p.74 gives a simplicial description of the Thom space $\mathrm{Th}(\nu_X)$. The equivalence $\Sigma^m/\overline{X} \simeq \mathrm{Sing}(\mathrm{Th}(\nu_X))$ is given by $\sigma^* \mapsto \big(\rho(\sigma)\colon \Delta^r \to |\sigma^*|/|\overline{X}| \simeq D(\nu_X(\sigma)) \cup * \simeq \mathrm{Th}(\nu_X(\sigma))\big)$ for $\sigma^* \in \Sigma^m$. See 132 for 132→p.75 more details.

The definition of the signatures $\mathrm{sgn}^{\mathbf{NL}^\bullet}_X$ and $\mathrm{sgn}^{\mathbf{L}^\bullet}_X$ via the cycles $[X]^{\mathbf{\Omega}^N_\bullet}$ and $[X]^{\mathbf{\Omega}^{STOP}_\bullet}$ implicitly used S-duality. In fact, for $\mathbf{\Omega}_\bullet = \mathbf{\Omega}^N_\bullet, \Sigma^{-1}\mathbf{\Omega}^{\mathrm{N,STOP}}_\bullet$ these cycles define simplicial maps

$$[X^*]^{\mathbf{\Omega}_\bullet} := S^{-1}([X]^{\mathbf{\Omega}_\bullet}) \in H^{m-n}(\Sigma^m, \overline{X}; \mathbf{\Omega}_\bullet)$$

given by $\sigma^* \mapsto [X]^{\mathbf{\Omega}_\bullet}(\sigma)$.

Let $\sigma^* \in \Sigma^m$ be an r-dimensional simplex. The diagram below proves part (i). The diagram commutes because the homotopy equivalence c between $\mathbf{\Omega}^N_\bullet$ and $\mathbf{MSG}$ is based on the same classifying map for ν_X which defines the u^G-orientation. The only difference is that for a $\sigma^* \in \Sigma^m$ the orientation uses the classifying map for ν_X while c restricts the classifying map to $\nu_{X(\sigma)}$. But

$\rho(\sigma)$ lives over $X(\sigma)$ anyway.

$$\begin{array}{ccccc}
X^* & \xrightarrow{[X^*]^{\mathbf{\Omega}^N_\bullet}} & \mathbf{\Omega}^N_{n-m} & \xrightarrow{\mathrm{sgn}^{\mathbf{NL}^\bullet}_\Omega} & \mathbf{NL}^{n-m}\langle 1/2\rangle \\
\simeq\big\downarrow & & c\big\downarrow\simeq & & \\
\mathrm{Sing\,Th}(\nu_X) & \xrightarrow{-\circ u^G(\nu_X)} & \mathrm{Sing}\,\mathbf{MSG}(n-m) & &
\end{array}$$

$$\sigma^* \mapsto (X(\sigma),\nu_X(\sigma),\rho(\sigma)), \qquad \rho(\sigma) \mapsto u^G(\nu_X)\circ\rho(\sigma) = u^G(\nu_X(\sigma))\circ\rho(\sigma)$$

In the second case we have a simplicial map $[X^*]^{\mathbf{\Omega}^{STOP}_\bullet} \in H^{m-n}(\Sigma^m, \overline{X}; \mathbf{\Omega}^{STOP}_\bullet)$. But this time a map $\rho(\sigma)\colon \Delta^r \to \mathrm{Th}(\nu_X)$ is not explicitly part of the data. The simplicial map $[X^*]^{\mathbf{\Omega}^{STOP}_\bullet}$ is defined by $\sigma^* \mapsto M(\sigma)_{\Delta^r}$ where $M(\sigma)_{\Delta^r}$ is the $(m-n)$-dimensional manifold r-ad obtained as preimage of $D(\sigma, X) \subset \mathrm{Th}(\overline{\nu}_X)$ under the projection $pr\colon \Sigma^m \to \Sigma^m/\overline{X}$ after making pr transverse to the dual cells of X.

Let pr_σ be the restriction of pr to $M(\sigma)_{\Delta^r}$. It is covered by a map of stable (micro) bundles $\overline{\nu}_{M(\sigma)_{\Delta^r}} \to \overline{\nu}_X$. The following diagram proves the statement.

$$\begin{array}{ccccc}
X^* & \xrightarrow{[X^*]^{\mathbf{\Omega}^{STOP}_\bullet}} & \mathbf{\Omega}^{STOP}_{n-m} & \xrightarrow{\mathrm{sgn}^{\mathbf{NL}^\bullet}_\Omega} & \mathbf{L}^{n-m}\langle 0\rangle \\
\simeq\big\downarrow & & \widetilde{c}\big\downarrow\simeq & & \\
\mathrm{Sing\,Th}(\overline{\nu}_X) & \xrightarrow{-\circ u^T(\overline{\nu}_X)} & \mathrm{Sing}\,\mathbf{MSTOP}(n-m) & &
\end{array}$$

In order to see that it commutes we have a look at a single simplex in the diagram below. Recall
1311→p.73 that under the isomorphism Φ of 1311 the dual cell $D(\sigma, X)$ is mapped to $|\sigma^*|$. The **MSTOP**-orientations are given by the classifying maps and $\overline{\nu}_{M(\sigma)_{\Delta^r}}$ was obtained as a pullback of $u^T(\overline{\nu}_X)$ along pr_σ. Hence we have $pr_\sigma \circ u^T(\overline{\nu}_X) = u^T(\overline{\nu}_{M(\sigma)_{\Delta^r}})$.

$$\begin{array}{ccccc}
\sigma^* & \longmapsto & M(\sigma)_{\Delta^r} & & \\
\big\downarrow & & \big\downarrow & & \\
 & & (\Delta^r \longrightarrow |M(\sigma)_\Delta| & \xrightarrow{u^T(\overline{\nu}_{M(\sigma)_{\Delta^r}})} & \mathbf{MSTOP}(n-m)) \\
\big\downarrow & & \big\downarrow pr_\sigma & & \\
(\Delta^r \to |\sigma^*|) & \longmapsto & (\Delta^r \longrightarrow |\sigma^*| & \xrightarrow{u^T(\overline{\nu}_X)} & \mathbf{MSTOP}(n-m))
\end{array}$$

When we consider the fibers $\Sigma^{-1}\mathbf{\Omega}^{\mathrm{N,STOP}}_\bullet$ and $\mathbf{NL}\langle 1/2\rangle/\mathbf{L}\langle 0\rangle^\bullet$ and the corresponding orientations and signatures, the first two cases yield the commutative diagram

$$\begin{array}{ccccc}
X^* & \xrightarrow{[\widehat{f}]^{\Sigma^{-1}\mathbf{\Omega}^{\mathrm{N,STOP}}_\bullet}} & \Sigma^{-1}\mathbf{\Omega}^{\mathrm{N,STOP}}_{n-m} & \xrightarrow{\mathrm{sgn}^{\mathbf{NL}/\mathbf{L}^\bullet}_\Omega} & \mathbf{NL}\langle 1/2\rangle/\mathbf{L}\langle 0\rangle^{n-m} \\
\simeq\big\downarrow & & c/\widetilde{c}\big\downarrow\simeq & & \\
\mathrm{Sing\,Th}(\nu_X) & \xrightarrow{-\circ u^{G/T}(\nu_X)} & \mathrm{Sing}\,\mathbf{MS}(\mathrm{G/TOP})(n-m) & &
\end{array}$$

which proves case (iii). □

Room service 15

For the definitions of the involved spectra see the room service of the preceding section.

Cycles

M_{Δ^k} manifold k-ad consisting of a manifold M and submanifolds $\partial_0 M, \ldots, \partial_k M$ such that $\partial_0 M \cap \ldots \cap \partial_k M = \emptyset$.

$[K]^{\mathbf{E}} \in H_n(K;\mathbf{E})$ an n-dimensional $\mathbf{E}$-cycle of a simplicial complex $K \subset \partial\Delta^{m+1}$ defined by a collection $\left\{[K]^{\mathbf{E}}(\sigma) \in \mathbf{E}_{n-m}^{(m-|\sigma|)} \mid \sigma \in K\right\}$ such that $\partial_i[K]^{\mathbf{E}}(\sigma) = [K]^{\mathbf{E}}(\delta_i\sigma)$ if $\delta_i\sigma \in K$ and $\emptyset$ otherwise.

$\Rightarrow$ $[X]^{\mathbf{\Omega}_\bullet^{STOP}} \in H_n(X;\mathbf{\Omega}_\bullet^{STOP})$ an n-dimensional $\mathbf{\Omega}_\bullet^{STOP}$-cycle which assigns to each $\sigma \in X \subset \partial\Delta^{m+1}$ the $(m-n)$-dimensional manifold $(m-|\sigma|+2)$-ad $M[\sigma] := [X]^{\mathbf{\Omega}_\bullet^{STOP}}(\sigma) := pr^{-1}(D(\sigma, X)) \in (\mathbf{\Omega}_{m-n}^{STOP})^{(m-|\sigma|)}$ using a simplicial Pontrjagin-Thom construction for $pr\colon \Sigma^m \to \Sigma^m/\Phi(\overline{X}) \simeq \mathrm{Sing}(\mathrm{Th}(\nu_X))$.

$\Rightarrow$ $[X]^{\mathbf{\Omega}_\bullet^N} \in H_n(X;\mathbf{\Omega}_\bullet^N)$ an n-dimensional $\mathbf{\Omega}_\bullet^N$-cycle which assigns to each $\sigma \in X$ a $(m-n)$-dimensional normal $(m-|\sigma|+2)$-ad $(X[\sigma], \nu(\sigma), \rho(\sigma))$ as constructed in 132.

$\Rightarrow$ $[\widehat{f}]^{\Sigma^{-1}\mathbf{\Omega}_\bullet^{\mathrm{N,STOP}}} \in H_n(X;\Sigma^{-1}\mathbf{\Omega}_\bullet^{\mathrm{N,STOP}})$ a $\mathbf{\Omega}_\bullet^N$-cobordism class of $\mathbf{\Omega}_\bullet^{STOP}$-cycle for a degree one normal map $\widehat{f}\colon M \to M'$ which assigns an $(m-|\sigma|)$-ad $(W(\sigma), \nu_{\overline{f}(\sigma)}, \rho(\overline{f}(\sigma)), M(\sigma) \amalg M(\sigma))$ to each $\sigma \in M'$ (see 2221).

- $\widehat{f}_\Delta := \bigcup_{\sigma\in X} \widehat{f}[\sigma]\colon M[\sigma] \to X[\sigma]$ the decomposition of a degree one normal map $\widehat{f}\colon M \to X$ into degree one normal maps $\widehat{f}[\sigma] = \widehat{f}|_{\widehat{f}^{-1}(X[\sigma])}$ of $(n-|\sigma|)$-dimensional manifold $(m-|\sigma|)$-ads.

Signatures

$\mathrm{sgn}_X^{\mathbf{NL}^\bullet}(X) := \mathrm{sgn}_\Omega^{\mathbf{NL}^\bullet}([X]^{\mathbf{\Omega}_\bullet^N}) \in H_n(X;\mathbf{NL}^\bullet) \cong NL^n(\Lambda_N X)$ the X-mosaicked normal signature defined here only for an n-dimensional Poincaré space X.

$\mathrm{sgn}_X^{\mathbf{L}^\bullet}(X) := \mathrm{sgn}_\Omega^{\mathbf{L}^\bullet}([X]_f^{\mathbf{\Omega}_\bullet^{STOP}}) \in H_n(X;\mathbf{L}^\bullet) \cong L^n(\Lambda_L X)$ the mosaicked symmetric signature for a Poincaré space X with a degree one normal map $\widehat{f}\colon M \to X$.

$\mathrm{sgn}_X^{\mathbf{NL}/\mathbf{L}^\bullet}(\widehat{f}) := \mathrm{sgn}_\Omega^{\mathbf{NL}/\mathbf{L}^\bullet}([\widehat{f}]^{\Sigma^{-1}\mathbf{\Omega}_\bullet^{\mathrm{N,STOP}}}) \in H_n(M';\mathbf{NL}/\mathbf{L}^\bullet)$ the mosaicked normal/symmetric signature over X defined for a degree one normal map $\widehat{f}\colon M \to M'$ between manifolds (see 2221).

Orientations

$u^{\mathbf{E}}(\nu)$ an $\mathbf{E}$-orientation of a $\mathbb{Z}$-oriented spherical fibration $\nu\colon X \to \mathrm{BSG}(k)$ is an element $u^{\mathbf{E}}(\nu) \in H^k(\mathrm{Th}(\nu);\mathbf{E})$ such that $u^{\mathbf{E}}(\nu)$ restricts to a generator of $H^k(\mathrm{Th}(\nu_x);\mathbf{E})$ for each fiber ν_x of ν.

⇒ $u^{\mathbf{NL}^\bullet}(\beta) \in H^k(\mathrm{Th}(\beta); \mathbf{NL}^\bullet)$ the canonical $\mathbf{NL}^\bullet$-orientation for a k-dimensional $\mathbb{Z}$-oriented spherical fibration $\beta\colon X \to \mathrm{BSG}(k)$ given by the composition

$$\mathrm{Th}(\beta) \xrightarrow{u^G(\beta)} \mathbf{MSG}(k) \xrightarrow{c^{-1}} \mathbf{\Omega}^N_{-k} \xrightarrow{\mathrm{sgn}^{\mathbf{NL}^\bullet}_\Omega} \mathbf{NL}^{-k}\langle 1/2\rangle.$$

$u^G(\beta) \in H^k(\mathrm{Th}(\beta); \mathbf{MSG})$ the canonical $\mathbf{MSG}$-orientation of β which is a map on the Thom spaces $\mathrm{Th}(\beta) \to \mathrm{Th}(\gamma_{SG})$ induced by the classifying map of a k-dimensional $\mathbb{Z}$-oriented spherical fibration $\beta\colon X \to \mathrm{BSG}(k)$.

⇒ $u^{\mathbf{L}^\bullet}(\alpha) \in H^k(\mathrm{Th}(\alpha); \mathbf{L}^\bullet)$ the canonical $\mathbf{L}^\bullet$-orientation for a k-dimensional $\mathbb{Z}$-oriented topological bundle $\alpha\colon X \to \mathrm{BSTOP}(k)$ is given by the composition

$$\mathrm{Th}(\alpha) \xrightarrow{u^T(\alpha)} \mathbf{MSTOP}(k) \xrightarrow{\tilde{c}^{-1}} \mathbf{\Omega}^{STOP}_{-k} \xrightarrow{\mathrm{sgn}^{\mathbf{L}^\bullet}_\Omega} \mathbf{L}^{-k}\langle 0\rangle.$$

$u^T(\alpha) \in H^k(\mathrm{Th}(\alpha); \mathbf{MSTOP})$ the canonical $\mathbf{MSTOP}$-orientation of α which is a map on the Thom spaces $\mathrm{Th}(\alpha) \to \mathrm{Th}(\gamma_{STOP})$ induced by the classifying map of a k-dimensional $\mathbb{Z}$-oriented topological bundle $\alpha\colon X \to \mathrm{BSG}(k)$.

⇒ $u^{\mathbf{NL}/\mathbf{L}^\bullet}(\alpha, h) = (u^{\mathbf{NL}^\bullet}(h), u^{\mathbf{L}^\bullet}(\alpha) - u^{\mathbf{L}^\bullet}(\overline{\nu}_X)) \in H^k(\mathrm{Th}(\alpha); \mathbf{NL}\langle 1/2\rangle/\mathbf{L}\langle 0\rangle^\bullet)$ the canonical $\mathbf{NL}\langle 1/2\rangle/\mathbf{L}\langle 0\rangle^\bullet$-orientation given by the composition

$$\mathrm{Th}(\alpha) \xrightarrow{u^{G/T}(\alpha)} \mathbf{MS}(\mathrm{G/TOP})(k) \to \Sigma^{-1}\mathbf{\Omega}^{\mathrm{N,STOP}}_{-k} \xrightarrow{\mathrm{sgn}^{\mathbf{NL}/\mathbf{L}^\bullet}_\Omega} \mathbf{NL}\langle 1/2\rangle/\mathbf{L}\langle 0\rangle^{\cdot}$$

for a k-dimensional $\mathbb{Z}$-oriented topological bundle $\alpha\colon X \to \mathrm{BSTOP}(k)$ together with a homotopy $h\colon J(\alpha) \simeq J(\overline{\nu}_X)$

$u^{G/T}(\nu, h) \in H^k(\mathrm{Th}(\nu_X); \mathbf{MS}(\mathrm{G/TOP}))$ the preferred lift of $u^T(\nu)$ for a bundle reduction ν of the Spivak normal fibration ν_X, determined by the homotopy $h\colon \mathrm{Th}(\nu_X) \times [0,1] \to \mathbf{MSG}$ between $J(\nu)$ and $J(\nu_X)$.

$X^* := (\Sigma^m/\Phi(\overline{X}))'$ a simplicial S-dual of X (see 1312).

MSG the Thom spectrum of the universal stable $\mathbb{Z}$-oriented spherical fibrations over BSG with the k-th space the Thom space $\mathbf{MSG}(k) = \mathrm{Th}(\gamma_{\mathrm{SG}}(k))$ of the universal k-dimensional spherical fibration $\gamma_{\mathrm{SG}}(k)$ over $\mathrm{BSG}(k)$.

MSTOP the Thom spectrum of the universal stable $\mathbb{Z}$-oriented topological bundles over the classifying space BSTOP with the k-th space the Thom space $\mathbf{MSTOP}(k) = \mathrm{Th}(\gamma_{\mathrm{STOP}}(k))$ of the universal k-dimensional bundle $\gamma_{\mathrm{STOP}}(k)$ over BSTOP.

BSG the classifying space of stable $\mathbb{Z}$-oriented spherical fibrations.

BSTOP the classifying space of stable $\mathbb{Z}$-oriented topological bundles.

16 The homotopy pullback square

Porter

In this section we finally establish the link to the bundle reductions. We have to show that a certain diagram is a homotopy pullback. This is done by proving that the induced map on the fibers of the vertical maps induces an isomorphism on homotopy groups, namely the surgery obstruction isomorphism as briefly mentioned in [Ran79, p.291].

16 The homotopy pullback square [Ran79, p.291][KMM13, Prop. 13.7]
The following diagram is a homotopy pullback square:

$$\begin{array}{ccc} \mathrm{BSTOP} & \xrightarrow{\mathrm{sgn}_B^{\mathbf{L}^\bullet}} & \mathrm{B}\mathbf{L}^\bullet\mathrm{G} \\ {\scriptstyle J}\downarrow & & \downarrow{\scriptstyle J} \\ \mathrm{BSG} & \xrightarrow{\mathrm{sgn}_B^{\mathbf{NL}^\bullet}} & \mathrm{B}\mathbf{NL}^\bullet\mathrm{G} \end{array}$$

161 Fibration sequence of classifying spaces [Ran79, p.290][KMM13, Prop. 13.6]
There are the following homotopy fibration sequences of spaces:

$$\mathbf{L}_0\langle 1\rangle \simeq \mathbf{NL}/\mathbf{L}^0 \to \mathrm{B}\mathbf{L}^\bullet\mathrm{G} \to \mathrm{B}\mathbf{NL}^\bullet\mathrm{G}$$

and

$$\mathbf{L}_0\langle 1\rangle \simeq \mathbf{NL}/\mathbf{L}^0 \to \mathbf{L}^\otimes \to \mathbf{NL}^\otimes.$$

15 (16) Orientations and signatures [Ran92, Proposition 16.1]

(i) Let X be an n-dimensional Poincaré space with Spivak normal fibration $\nu_X : X \to \mathrm{BSG}$. Then we have $S(u_{\mathbf{NL}}(\nu_X)) = \mathrm{sgn}_X^{\mathbf{NL}^\bullet}(X) \in H_n(X; \mathbf{NL}^\bullet)$.

(ii) Let $\overline{\nu}$ be a topological bundle reduction of the Spivak normal fibration $\nu_X : X \to \mathrm{BSG}$ of X and $\widehat{f} : M \to X$ its associated degree one normal map. Then we have $S(u_{\mathbf{L}}(\nu_X)) = \mathrm{sgn}_X^{\mathbf{L}^\bullet}(X) \in H_n(X; \mathbf{L}^\bullet)$.

(iii) Let $\widehat{f} : M \to M'$ be a degree one normal map of n-dimensional simply-connected topological manifolds with M' triangulated, corresponding to a pair (β, h) with $\beta : M' \to \mathrm{BSTOP}$ and $h : J(\beta) \simeq \nu_{M'}$. Then we have $S(u^{\mathbf{NL}/\mathbf{L}^\bullet}(\beta, h)) = \mathrm{sgn}_{M'}^{\mathbf{NL}/\mathbf{L}^\bullet}(\widehat{f}) \in H_n(M'; \mathbf{NL}/\mathbf{L}^\bullet)$.

A29 (221, 23) Mosaicked quadratic signature [Ran92, Example 9.14]
Let $\widehat{f} : M \to X$ be a degree one normal map from a closed topological manifold to a Poincaré space both of dimension n. Let $r : X \to K$ be a map to a simplicial complex K. There is a mosaicked quadratic signature

$$\mathrm{sgn}_K^{\mathbf{L}_\bullet}(\widehat{f}) \in L_n(\Lambda_G K)$$

with $A(\mathrm{sgn}_X^{\mathbf{L}_\bullet}(\widehat{f})) = \mathrm{sgn}_{\mathbb{Z}\pi}^{\mathbf{L}_\bullet}(\widehat{f})$. If X is a manifold, there is a refined version

$$\mathrm{sgn}_K^{\mathbf{L}_\bullet}(\widehat{f}) \in L_n(\Lambda_L K).$$

162 Mosaicked normal/symmetric signature
Let $\widehat{f} : M \to X$ be a degree one normal map and X triangulated. There is a normal/symmetric signature $\mathrm{sgn}_X^{\mathbf{NL}/\mathbf{L}^\bullet}(\widehat{f})$ such that $\partial(\mathrm{sgn}_X^{\mathbf{NL}/\mathbf{L}^\bullet}(\widehat{f})) = \mathrm{sgn}_X^{\mathbf{L}_\bullet}(\widehat{f})$.

163 Assembly isomorphims for simply-connected manifolds
Let M be a simply-connected n-dimensional manifold. Then the assembly map $A : H_n(M, \mathbf{L}_\bullet) \to L_n(\mathbb{Z})$ is an isomorphism.

Corollary

164 (22) Quadratic signature isomorphism
The quadratic signature defines an isomorphism $\mathrm{sgn}_{\mathrm{G/TOP}}^{\mathbf{L}_\bullet} : [X; \mathrm{G/TOP}] \xrightarrow{\cong} H^0(X; \mathbf{L}_\bullet)$.

Proof 16

There will be no change of the connectivity conditions made on the involved spectra so we omit them in the notation. The symmetric spectrum is always 0-connected, the normal spectrum 1/2-connected and the quadratic spectrum is supposed to be 1-connected.

In the commutative diagram that we want to prove to be a homotopy pullback, we have, on 161→p.81 the left hand side, the fiber $\mathrm{G/TOP}$ and on the right hand side 161 identifies the fiber as $\mathbf{L}_0\langle 1\rangle$. We show that the induced map $\Phi\colon \mathrm{G/TOP} \to \mathbf{L}_0\langle 1\rangle$ induces isomorphisms $\Phi_n\colon [S^n, \mathrm{G/TOP}] \xrightarrow{\cong} [S^n, \mathbf{L}_0\langle 1\rangle]$ on the homotopy groups for all $n \geq 0$ and hence is a homotopy equivalence.

As a first step, we point out that the surgery obstruction map realizes an isomorphism between these two groups. Then it remains to show that the induced map Φ_n is actually the same map.

An element in $[S^n, \mathrm{G/TOP}]$ is given by a topological bundle $\alpha\colon S^n \to \mathrm{BSTOP}$ together with a null-homotopy $h\colon S^n \times I \to \mathrm{BSG}$ of $J(\alpha)$. It is equivalently given by a degree one normal map $\widehat{f}\colon M \to S^n$ using the well-known isomorphism $\mathscr{N}(S^n) \cong [S^n, \mathrm{G/TOP}]$. It is also known that the surgery obstruction map $\theta = \mathrm{sgn}^{\mathbf{L}_\bullet}_{\mathbb{Z}\pi}\colon \mathscr{N}(S^n) \to L_n(\mathbb{Z})$ is an isomorphism for $n \geq 1$. On the other side we have the chain of isomorphisms

$$[S^n, \mathbf{L}_0\langle 1\rangle] \xrightarrow{i} H^0(S^n; \mathbf{L}_\bullet) \xrightarrow{T} H^k(\mathrm{Th}(J(\alpha)); \mathbf{L}_\bullet) \xrightarrow{S} H_n(S^n; \mathbf{L}_\bullet) \xrightarrow{A} L_n(\mathbb{Z}).$$

The maps T and S are the Thom and S-duality isomorphisms. That the assembly map A is an 22→p.51 isomorphism follows from the proof of 22 for the special case that X is the manifold S^n. We can choose f_0 to be the identiy so that the commutative diagram, that we use there simplifies to

$$\begin{array}{ccc} \mathscr{N}(X) & & \\ {\scriptstyle \mathrm{sgn}^{\mathbf{L}_\bullet}_X}\downarrow & \searrow {\scriptstyle \mathrm{sgn}^{\mathbf{L}_\bullet}_{\mathbb{Z}\pi}} & \\ H_n(X; \mathbf{L}_\bullet) & \xrightarrow{A} & L_n(\mathbb{Z}) \end{array}$$

We prove in 22 that $\mathrm{sgn}^{\mathbf{L}_\bullet}_X$ is an isomorphism. In this simply-connected case $\mathrm{sgn}^{\mathbf{L}_\bullet}_{\mathbb{Z}\pi}$ isomorphism as well because of 1631 and therefore A is an isomorphism.

Remark. There is the subtlety that we use in 22 the homotopy equivalence between $\mathrm{G/TOP}$ and $\mathbf{L}_0\langle 1\rangle$ which might look like a circular argument. But we do not have to use the map Φ induced by the homotopy pullback with which we deal here. Instead we use the homotopy equivalence $\mathrm{Sing}\,\mathrm{G/TOP} \simeq \mathbf{L}_0\langle 1\rangle$ explicitly given as follows: A k-simplex $\Delta^k \to \mathrm{G/TOP}$ can also be described as a map $f\colon M \to \Delta^k$ and we assign to it the quadratic chain complex $\mathrm{sgn}^{\mathbf{L}_\bullet}_{\Delta^k}(f)$. This induces on homotopy groups the surgery obstruction isomorphism $\pi_n(\mathrm{G/TOP}) \to L_n(\mathbb{Z})$ of 1631.

22→p.51 In the proof of 22 we identify the assembly with the surgery obstruction map because of 1221. A simplex Hence we can take for f_0 the identity.

It remains to show that this composition commutes with Φ, more precisely that $A \circ S \circ T \circ i \circ \Phi(\alpha, h) = \mathrm{sgn}^{\mathbf{L}_\bullet}_{\mathbb{Z}\pi}(\widehat{f})$ but we will work in the actual fiber $\mathbf{NL}/\mathbf{L}^\bullet$ of $\mathbf{L}^\bullet \to \mathbf{NL}^\bullet$ and use the identification of the fiber with the quadratic spectrum $\mathbf{L}_\bullet$ at the very end. The situation is summarized in the following diagram.

$$\begin{array}{ccc}
[S^n, \mathrm{G/TOP}] & \xrightarrow{\Phi} & [S^n, \mathbf{NL}/\mathbf{L}^0] \\
\downarrow \cong & & \downarrow S\circ T\circ i \\
\mathscr{N}(S^n) & & H_n(S^n; \mathbf{NL}/\mathbf{L}^\bullet) \\
\theta(\widehat{f})=\mathrm{sgn}^{\mathbf{L}^\bullet}_{\mathbb{Z}}(\widehat{f}) \downarrow \cong & & \downarrow A \cong \\
L_n(\mathbb{Z}) & \xleftarrow[\text{surgery}]{\text{algebraic}} & L(J)^n
\end{array}$$

We need to have a closer look at how the element $\Phi(\alpha, h) \in [S^n; \mathbf{NL}/\mathbf{L}^0]$ is obtained from $(\alpha, h) \in [S^n; \mathrm{G/TOP}]$. The map $\mathrm{sgn}^{\mathbf{L}^\bullet}_B(\alpha)\colon S^n \to \mathrm{B}\mathbf{L}^\bullet\mathrm{G}$ is given by the pair $(J(\alpha), u^{\mathbf{L}^\bullet}(\alpha))$ and similarly the map $\mathrm{sgn}^{\mathbf{NL}^\bullet}_B(h)\colon S^n \times I \to \mathrm{B}\mathbf{NL}^\bullet\mathrm{G}$ is given by the pair $(h, u^{\mathbf{NL}^\bullet}(h))$. By the fibration sequence $\mathbf{L}^\otimes \to \mathrm{B}\mathbf{L}^\bullet\mathrm{G} \to \mathrm{BSG}$ we obtain from α a map $\alpha^\otimes\colon S^n \to \mathbf{L}^\otimes$ because of the null-homotopy $h\colon J(\alpha) \simeq \mathrm{pt}$ in BSG. Similarly, with the fibration sequence $\mathbf{NL}^\otimes \to \mathrm{B}\mathbf{NL}^\bullet\mathrm{G} \to \mathrm{BSG}$ we obtain from h a map $h^\otimes\colon S^n \otimes I \to \mathbf{NL}^\otimes$ which is a null-homotopy of $J(\alpha^\otimes)$. Together, they yield a map $(\alpha^\otimes, h^\otimes)\colon S^n \to \mathbf{NL}/\mathbf{L}^0$ using the second fibration sequence from 161.

Now we extend the diagram from above. Because $\mathbf{L}^\bullet$ and $\mathbf{NL}^\bullet$ are ring spectra, the orientations $u^{\mathbf{NL}^\bullet}(\nu_X)$ and $u^{\mathbf{L}^\bullet}(\overline{\nu}_X)$ induce Thom isomorphisms. The fiber $\mathbf{NL}/\mathbf{L}^\bullet$ is not a ring spectrum but a module spectrum over $\mathbf{L}^\bullet$. Hence $u^{\mathbf{NL}^\bullet}$ and $u^{\mathbf{L}^\bullet}$ also induce a compatible Thom isomorphism in $\mathbf{NL}/\mathbf{L}^\bullet$. Together with S-duality and assembly maps we obtain

$$\begin{array}{ccccc}
[S^n, \mathbf{NL}/\mathbf{L}^0] & & & & \\
\downarrow \cong & & & & \\
H^0(S^n; \mathbf{NL}/\mathbf{L}^\bullet) & \longrightarrow & H^0(S^n; \mathbf{L}^\bullet) & \longrightarrow & H^0(S^n; \mathbf{NL}^\bullet) \\
\cdot\cup u^{\mathbf{NL}/\mathbf{L}^\bullet} \downarrow \cong & & \cdot\cup u^{\mathbf{L}^\bullet} \downarrow \cong & & \cdot\cup u^{\mathbf{NL}^\bullet} \downarrow \cong \\
H^k(\mathrm{Th}(\alpha); \mathbf{NL}/\mathbf{L}^\bullet) & \longrightarrow & H^k(\mathrm{Th}(\alpha); \mathbf{L}^\bullet) & \longrightarrow & H^k(\mathrm{Th}(\alpha); \mathbf{NL}^\bullet) \\
S \downarrow \cong & & S \downarrow \cong & & S \downarrow \cong \\
H_n(S^n; \mathbf{NL}/\mathbf{L}^\bullet) & \longrightarrow & H_n(S^n; \mathbf{L}^\bullet) & \longrightarrow & H_n(S^n; \mathbf{NL}^\bullet) \\
A \downarrow \cong & & A \downarrow \cong & & A \downarrow \cong \\
L(J)^n \cong L_n(\mathbb{Z}) & \longrightarrow & L^n(\mathbb{Z}) & \xrightarrow{J} & NL^n(\mathbb{Z})
\end{array}$$

where all homomorphisms are induced by maps of spaces.

We want to identify the element we obtain from (α, h) in $L_n(\mathbb{Z})$. Via the Thom isomorphism we obtain the orientation $u^{\mathbf{NL}/\mathbf{L}^\bullet}(\alpha, h)$ in $H^k(\mathrm{Th}(\alpha); \mathbf{NL}^\bullet)$. Applying S-duality yields by 15 (iii) the normal/symmetric signature $\mathrm{sgn}^{\mathbf{NL}/\mathbf{L}^\bullet}_{S^n}(\widehat{f})$ of the corresponding degree one normal map $\widehat{f}$. By 162 this is, via algebraic surgery, the same as the mosaicked quadratic signature $\mathrm{sgn}^{\mathbf{L}_\bullet}_{S^n}(\widehat{f})$ 162→p.82
which has the property that assembly leads to $\mathrm{sgn}^{\mathbf{L}_\bullet}_{\mathbb{Z}\pi}(\widehat{f})$. Hence the effect of algebraic surgery on the (normal, symmetric Poincaré) pair $A(\mathrm{sgn}^{\mathbf{NL}/\mathbf{L}^\bullet}_{S^n}(\widehat{f}))$ in $L(J)^n$ is $\mathrm{sgn}^{\mathbf{L}_\bullet}_{\mathbb{Z}\pi}(\widehat{f})$. More explicitly, α leads to the element $\mathrm{sgn}^{\mathbf{L}^\bullet}_{\mathbb{Z}\pi}(S^n)$ in $NL^n(\mathbb{Z})$ and the relative term obtained from (α, h) is given by $(\mathrm{sgn}^{\mathbf{NL}^\bullet}_{\mathbb{Z}\pi}(\mathcal{M}(f)), \mathrm{sgn}^{\mathbf{L}^\bullet}_{\mathbb{Z}\pi}(M) - \mathrm{sgn}^{\mathbf{L}^\bullet}_{\mathbb{Z}\pi}(S^n))$. The identification with $\mathrm{sgn}^{\mathbf{L}_\bullet}_{\mathbb{Z}\pi}(\widehat{f})$ is carried out in 1621. 1621→p.84

□

Room service 16

$\mathbf{L}^\bullet, \mathbf{L}_\bullet, \mathbf{NL}^\bullet$ short for $\mathbf{L}^\bullet\langle 0\rangle$,$\mathbf{L}_\bullet\langle 1\rangle$, and $\mathbf{NL}^\bullet\langle 1/2\rangle$ respectively.

$\mathbf{L}^\bullet\langle q\rangle, \mathbf{L}_\bullet\langle q\rangle, \mathbf{NL}^\bullet\langle q\rangle$ short for $\mathbf{L}^\bullet(\Lambda(\mathbb{Z})\langle q\rangle), \mathbf{L}_\bullet(\Lambda(\mathbb{Z})\langle q\rangle), \mathbf{NL}^\bullet(\Lambda(\mathbb{Z})\langle q\rangle)$.

$\Lambda(\mathbb{Z}) = (\mathbb{A}(\mathbb{Z}), \mathbb{B}(\mathbb{Z}), \mathbb{C}(\mathbb{Z}))$ denotes the algebraic bordism category with

- $\mathbb{A}(\mathbb{Z})$ the category of finitely generated free left $\mathbb{Z}$-modules,
- $\mathbb{B}(\mathbb{Z})$ the bounded chain complexes in $\mathbb{A}(\mathbb{Z})$,
- $\mathbb{C}(\mathbb{Z})$ the contractible chain complexes of $\mathbb{B}(\mathbb{Z})$.

$\Lambda\langle q\rangle$ for $\Lambda = (\mathbb{A}, \mathbb{C}, \mathbb{P})$ and $q \in \mathbb{Z}$ the q-connective algebraic bordism category $(\mathbb{A}, \mathbb{C}\langle q\rangle, \mathbb{P}\langle q\rangle)$

$\mathbb{C}\langle q\rangle$ the subcategory of chain complexes of $\mathbb{C}$ that are homotopy equivalent to q-connected chain complexes.

$\Lambda\langle 1/2\rangle$ denotes for $\Lambda = (\mathbb{A}, \mathbb{C}, \mathbb{P})$ the algebraic bordism category $(\mathbb{A}, \mathbb{C}\langle 0\rangle, \mathbb{P}\langle 1\rangle)$.

$\mathbf{NL}/\mathbf{L}^\bullet := \mathrm{Fiber}(J\colon \mathbf{L}^\bullet \to \mathbf{NL}^\bullet)$.

$J\colon L^n(R) \to NL^n(R)$ roughly induced by $j\colon W^{\%}C \to \widehat{W}^{\%}C$; see (111) for more details of how a normal structure (φ, γ, χ) is obtained from a symmetric Poincaré chain complex (C, φ).

$L(J)^n$ the relative L-group of $J\colon L^n(R) \to NL^n(R)$ is the cobordism group of (normal, symmetric Poincaré) pairs $(f\colon C \to D, (\delta\varphi, \delta\gamma, \delta\chi), \varphi)$.

L^n, L_n, NL^n are the cobordism groups of symmetric Poincaré, quadratic Poincaré, normal chain complexes.

$\mathbf{L}^\otimes$ the component of $1 \in \mathbb{Z}$ in $\mathbf{L}^\bullet$.

$\mathbf{NL}^\otimes$ the component of $1 \in \mathbb{Z}$ in $\mathbf{NL}^\bullet$.

$\theta(\widehat{f})$ Wall's surgery obstruction for a degree one normal map $\widehat{f}\colon M \to X$. It is an element in $L_n^w(\mathbb{Z}[\pi_1(X)])$ and if $n \geq 5$ it vanishes if and only if $\widehat{f}$ is cobordant to a homotopy equivalence $\widehat{f}'\colon M' \to X$.

$\mathscr{N}(X)$ the normal invariants of a geometric Poincaré complex X. An element of $\mathscr{N}(X)$ can be represented in two different ways which are identified via the Pontrjagin-Thom construction:

- by a degree one normal map $(f, b)\colon M \to X$ from a manifold M to X or
- by a pair (ν, h) where $\nu\colon X \to \mathrm{BSTOP}$ is a stable topological bundle on X and $h\colon J(\nu) \simeq \nu_X$ is a homotopy from the underlying spherical fibration of ν to the Spivak normal fibration $\nu_X$of X.

$u^{\mathbf{NL}/\mathbf{L}^\bullet}(\alpha, h) = (u^{\mathbf{NL}^\bullet}(h), u^{\mathbf{L}^\bullet}(\alpha) - u^{\mathbf{L}^\bullet}(\overline{\nu}_X)) \in H^k(\mathrm{Th}(\alpha); \mathbf{NL}/\mathbf{L}^\bullet)$

the canonical $\mathbf{NL}/\mathbf{L}^\bullet$-orientation given by the composition

$$\mathrm{Th}(\alpha) \xrightarrow{u^{G/T}(\alpha)} \mathbf{MS}(\mathrm{G/TOP})(k) \to \Sigma^{-1}\mathbf{\Omega}^{\mathrm{N,STOP}}_{-k} \xrightarrow{\mathrm{sgn}^{\mathbf{NL}/\mathbf{L}^\bullet}_{\Omega}} \mathbf{NL}/\mathbf{L}^{-k}.$$

for a k-dimensional $\mathbb{Z}$-oriented topological bundle $\alpha\colon X \to \mathrm{BSTOP}(k)$ together with a homotopy $h\colon J(\alpha) \simeq J(\overline{\nu}_X)$

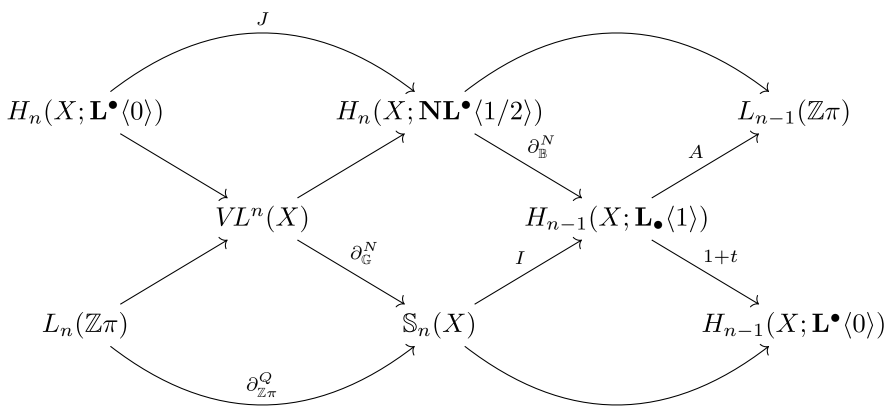
21 The algebraic surgery exact sequence

21 The algebraic surgery exact sequence [Ran92, Prop. 14.7]
There is a long exact sequence

$$\ldots \longrightarrow H_n(X; \mathbf{L}_\bullet) \xrightarrow{A} L_n(\mathbb{Z}\pi) \xrightarrow{\partial^Q_{\mathbb{Z}\pi}} \mathbb{S}_n(X) \xrightarrow{I} H_{n-1}(X; \mathbf{L}_\bullet) \xrightarrow{A} \ldots$$

Proof 21

The existence of this long exact sequence is a direct consequence of the proof of 12. The following exact braid was constructed there and the sequence we are looking for is a part of it. 12→p.28



□

22 Coset step

Porter

For a degree one normal map $\widehat{e}\colon M \to M'$ between manifolds M and M' the quadratic signature $\mathrm{sgn}^{\mathbf{L}_\bullet}_{\mathbb{Z}\pi}(\widehat{e})$ considered as a map $\mathscr{N}(M') \to L_n(\mathbb{Z}\pi)$ immediately factors through the assembly map $A\colon H_n(M'; \mathbf{L}_\bullet\langle 1\rangle) \to L_n(\mathbb{Z}\pi)$ using the mosaicked quadratic signature $\mathrm{sgn}^{\mathbf{L}_\bullet}_{M'}(\widehat{e}) \in L_n(\Lambda_L M') = H_n(M'; \mathbf{L}_\bullet\langle 1\rangle)$. But we need the same statement for a degree one normal map $\widehat{f}\colon M \to X$ with a Poincaré space as target. Thus, the output of the quadratic signature $\mathrm{sgn}^{\mathbf{L}_\bullet}_X(\widehat{f})$ is only globally Poincaré and hence lives in $L_n(\Lambda_G X)$. The solution for obtaining a quadratic chain complex in

$L_n(\Lambda_L X)$ is to fix a degree one normal map $\widehat{f}_0 \colon M \to X$ and to work relatively to $\widehat{f}_0$. This is the core of the following proof.

22 Coset step
$-\operatorname{sgn}^{\mathbf{L}_\bullet}_{\mathbb{Z}\pi}(\mathscr{N}(X))$ *is a coset of* $\operatorname{im}(A)$ *in* $L_n(\mathbb{Z}\pi)$ *where* $A\colon H_n(X;\mathbf{L}_\bullet) \to L_n(\mathbb{Z}\pi)$ *is the assembly map.*

221 Difference of quadratic signatures [KMM13, section 14.5]
Let $\widehat{f}_i \colon M_i \to X$ *with* $i = 0,1$ *be two degree one normal maps. Then the difference of their mosaicked quadratic signature defines an element*

$$\operatorname{sgn}^{\mathbf{L}_\bullet}_X(\widehat{f}_1) - \operatorname{sgn}^{\mathbf{L}_\bullet}_X(\widehat{f}_0) \in L_n(\Lambda_L X) = H_n(X;\mathbf{L}_\bullet\langle 1\rangle)$$

such that $A(\operatorname{sgn}^{\mathbf{L}_\bullet}_X(\widehat{f}_1) - \operatorname{sgn}^{\mathbf{L}_\bullet}_X(\widehat{f}_0)) = \operatorname{sgn}^{\mathbf{L}_\bullet}_{\mathbb{Z}\pi}(\widehat{f}_1) - \operatorname{sgn}^{\mathbf{L}_\bullet}_{\mathbb{Z}\pi}(\widehat{f}_0) \in L_n(\mathbb{Z}\pi)$.

222 (normal, manifold)-cycles for Poincaré spaces [KMM13, Lemma 14.16]
Let $\widehat{f}_i := (\overline{f}_i, f_i)\colon M_i \to X$ *with* $i = 0,1$ *be two n-dimensional degree one normal maps from topological manifolds to Poincaré spaces. Then there exists a* $\Sigma^{-1}\mathbf{\Omega}^{\mathrm{N,STOP}}_\bullet$*-cycle*

$$[\widehat{f}_1, \widehat{f}_0]^{\Sigma^{-1}\mathbf{\Omega}^{\mathrm{N,STOP}}_\bullet} \in H_n(X;\Sigma^{-1}\mathbf{\Omega}^{\mathrm{N,STOP}}_\bullet)$$

such that $\operatorname{sgn}^{\mathbf{L}_\bullet}_X(\widehat{f}_1) - \operatorname{sgn}^{\mathbf{L}_\bullet}_X(\widehat{f}_0) = \operatorname{sgn}^{\mathbf{L}_\bullet}_\Omega([\widehat{f}_1, \widehat{f}_0]^{\Sigma^{-1}\mathbf{\Omega}^{\mathrm{N,STOP}}_\bullet}) \in H_n(X;\mathbf{L}_\bullet\langle 1\rangle)$.

223 (normal,manifold)-cycles and MSTOP-orientations
Given a degree one normal map $\widehat{f}_0 \colon M \to X$*, there is a commutative diagram*

$$\begin{array}{ccccc}
\mathscr{N}(X) & & \xrightarrow{[-,\widehat{f}_0]^{\Sigma^{-1}\mathbf{\Omega}^{\mathrm{N,STOP}}_\bullet}} & & H_n(X;\Sigma^{-1}\mathbf{\Omega}^{\mathrm{N,STOP}}_\bullet) \\
\downarrow{\scriptstyle t(-,\widehat{f}_0)} & & & & \uparrow{\scriptstyle S} \\
[X;\mathrm{G/TOP}] & \xrightarrow{\widetilde{\Gamma}} & H^0(X;\Sigma^{-1}\mathbf{\Omega}^{\mathrm{N,STOP}}_\bullet) & \xrightarrow{u^T(\nu_0)} & H^k(Th(\nu_X);\Sigma^{-1}\mathbf{\Omega}^{\mathrm{N,STOP}}_\bullet)
\end{array}$$

i.e. $[-,\widehat{f}_0]^{\Sigma^{-1}\mathbf{\Omega}^{\mathrm{N,STOP}}_\bullet} = S(\widetilde{\Gamma}(t(-,\widehat{f}_0)) \cup u^T(\nu_0))$.

224 Classification of normal invariants [Wal99, chapter 10]
Let X be a Poincaré space and $\widehat{f}_0 \in \mathscr{N}(X)$ *a degree one normal map. Then there is a bijection*

$$t(-,\widehat{f}_0)\colon \mathscr{N}(X) \xrightarrow{\cong} [X;\mathrm{G/TOP}].$$

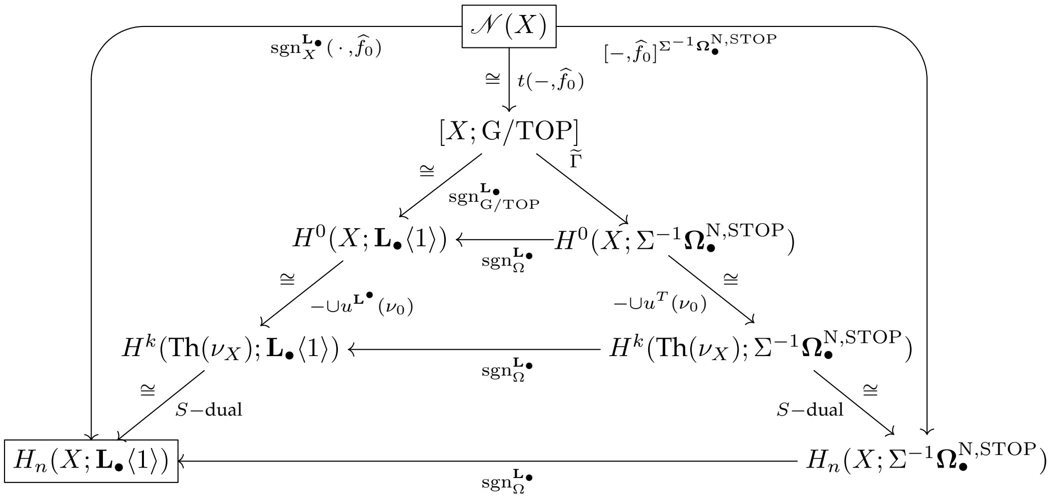
164 (22) Quadratic signature isomorphism
The quadratic signature defines an isomorphism $\operatorname{sgn}^{\mathbf{L}_\bullet}_{\mathrm{G/TOP}}\colon [X;\mathrm{G/TOP}] \xrightarrow{\cong} H^0(X;\mathbf{L}_\bullet\langle 1\rangle)$.

Proof 22

221→p.86 Fix a degree one normal map $\widehat{f}_0 \colon M_0 \to X$ for the rest of the proof. By 221, for any degree one normal map $\widehat{f}\colon M \to X$ the difference $\operatorname{sgn}^{\mathbf{L}_\bullet}_X(\widehat{f}) - \operatorname{sgn}^{\mathbf{L}_\bullet}_X(\widehat{f}_0)$ defines an element in $H_n(X;\mathbf{L}_\bullet\langle 1\rangle)$. We write $\operatorname{sgn}^{\mathbf{L}_\bullet}_X(\,\cdot\,,\widehat{f}_0)$ for short of $\operatorname{sgn}^{\mathbf{L}_\bullet}_X(\,\cdot\,) - \operatorname{sgn}^{\mathbf{L}_\bullet}_X(\widehat{f}_0)$. The assembly map applied to $\operatorname{sgn}^{\mathbf{L}_\bullet}_X(\,\cdot\,,\widehat{f}_0)$ returns the quadratic signature $\operatorname{sgn}^{\mathbf{L}_\bullet}_{\mathbb{Z}\pi}(\widehat{f}) - \operatorname{sgn}^{\mathbf{L}_\bullet}_{\mathbb{Z}\pi}(\widehat{f}_0)$. Hence, we have to show that in the commutative diagram below the vertical map $\operatorname{sgn}^{\mathbf{L}_\bullet}_X(\,\cdot\,,\widehat{f}_0)$ is a bijection.

$$\begin{array}{ccc}
\mathscr{N}(X) & & \\
\downarrow{\scriptstyle \operatorname{sgn}^{\mathbf{L}_\bullet}_X(\,\cdot\,,\widehat{f}_0)} & \searrow{\scriptstyle \operatorname{sgn}^{\mathbf{L}_\bullet}_{\mathbb{Z}\pi}(\,\cdot\,)-\operatorname{sgn}^{\mathbf{L}_\bullet}_{\mathbb{Z}\pi}(\widehat{f}_0)} & \\
H_n(X;\mathbf{L}_\bullet\langle 1\rangle) & \xrightarrow{A} & L_n(\mathbb{Z}[\pi_1 X])
\end{array}$$

This is done by decomposing $\mathrm{sgn}_X^{\mathbf{L}_\bullet}(\,\cdot\,,\widehat{f}_0)$ into four maps which fit into a diagram as follows.



We verify that the top left corner of the diagram commutes by proving that all other paths in the diagram commute. Then it suffices that the four maps on the left hand side of the pyramid are isomorphisms.

Commutativity. The map

$$\widetilde{\Gamma}\colon \mathrm{G/TOP} \to \Sigma^{-1}\mathbf{\Omega}_0^{\mathrm{N,STOP}}$$

in the diagram is defined as follows. It associates to an l-simplex $\widehat{f}\colon M \to \Delta^l$ in $\mathrm{G/TOP}$ an l-simplex of $\Sigma^{-1}\mathbf{\Omega}_\bullet^{\mathrm{N,STOP}}$ which is an $(l+1)$-dimensional l-ad of (normal, topological manifold) pairs $(\mathscr{M}(f), M \amalg -\Delta^l)$ where the normal structure comes from the bundle map $\overline{f}$.

The proof of 16 shows that the quadratic signature map $\mathrm{sgn}^{\mathbf{L}_\bullet}_{\mathrm{G/TOP}}\colon \mathrm{G/TOP} \to \mathbf{L}_0\langle 1\rangle$ can be thought of as a composition of the two maps 16→p.47

$$\widetilde{\Gamma}\colon \mathrm{G/TOP} \to \Sigma^{-1}\mathbf{\Omega}_0^{\mathrm{N,STOP}} \qquad \text{and} \qquad \mathrm{sgn}_\Omega^{\mathbf{L}_\bullet}\colon \Sigma^{-1}\mathbf{\Omega}_0^{\mathrm{N,STOP}} \to \mathbf{L}_0\langle 1\rangle.$$

Hence, the triangle at the peak of the pyramid in the diagram above commutes. The square beneath the peak commutes because of the naturality of the cup product with respect to the coefficient spectra and because the canonical $\mathbf{L}^\bullet$-orientation of a stable topological bundle is the image of the canonical $\mathbf{MSTOP}$-orientation. 14→p.38

The commutativity of the bottom square follows from the naturality of the S-duality with respect to the coefficient spectra.

The proof that the outer rounded square commutes, i.e.

$$\mathrm{sgn}_X^{\mathbf{L}_\bullet}(\,\cdot\,,\widehat{f}_0) = \mathrm{sgn}_\Omega^{\mathbf{L}_\bullet}([-,\widehat{f}_0]^{\Sigma^{-1}\mathbf{\Omega}_\bullet^{\mathrm{N,STOP}}}),$$ 222→p.88

and that the upper right hand corner commutes, i.e. 223→p.91

$$[-,\widehat{f}_0]^{\Sigma^{-1}\mathbf{\Omega}_\bullet^{\mathrm{N,STOP}}} = S(\widetilde{\Gamma}(\widetilde{x}) \cup u^T(\nu_0)),$$

will be given in the next level in the sections 222 and 223.

Bijectivity. The bijectivity of the first map $t(-,\widehat{f}_0)\colon \mathscr{N}(X) \xrightarrow{\cong} [X;\mathrm{G/TOP}]$ is a well-known
224→p.94 fact. We provide some comments on that in 224. As for the second map $\mathrm{sgn}^{\mathbf{L}_\bullet}_{\mathbb{Z}\pi}\colon [X;\mathrm{G/TOP}] \xrightarrow{\cong}$
164→p.86 $H^0(X;\mathbf{L}_\bullet\langle 1\rangle)$, the isomorphism is a consequence of the proof of 16 as stated in 164.

The third map is given by the cup product with an $\mathbf{L}^\bullet\langle 0\rangle$-orientation. Recall that $\mathbf{L}_\bullet\langle 1\rangle$ is a module spectrum over the ring spectrum $\mathbf{L}^\bullet\langle 0\rangle$. The cup product

$$- \cup - \colon H^p(X;\mathbf{L}_\bullet\langle 1\rangle) \otimes H^q(\mathrm{Th}(\xi);\mathbf{L}^\bullet\langle 0\rangle) \longrightarrow H^{p+q}(\mathrm{Th}(\xi);\mathbf{L}_\bullet\langle 1\rangle),$$
$$x \otimes y \longmapsto x \cup y$$

is given by the composition

$$x \cup y\colon \mathrm{Th}(\xi) \xrightarrow{\widetilde{\Delta}} X_+ \wedge \mathrm{Th}(\xi) \xrightarrow{x\wedge y} \mathbf{L}_p\langle 1\rangle \wedge \mathbf{L}^q\langle 0\rangle \to \mathbf{L}_{p+q}\langle 1\rangle.$$

In this special case where y is an $\mathbf{L}^\bullet\langle 0\rangle$-orientation the cup product gives the Thom isomorphism $H^p(X;\mathbf{L}_\bullet\langle 1\rangle) \to H^{p+q}(\mathrm{Th}(\xi);\mathbf{L}_\bullet\langle 1\rangle)$.

The last map of the composition is the S-duality isomorphism. □

Room service 22

$\boxed{\mathscr{N}(X)}$ the normal invariants of a geometric Poincaré complex X. An element of $\mathscr{N}(X)$ can be represented in two different ways which are identified via the Pontrjagin-Thom construction:

- by a degree one normal map $(f,b)\colon M \to X$ from a manifold M to X or
- by a pair (ν,h) where $\nu\colon X \to \mathrm{BSTOP}$ is a stable topological bundle on X and $h\colon J(\nu) \simeq \nu_X$ is a homotopy from the underlying spherical fibration of ν to the Spivak normal fibration ν_X of X.

$\boxed{u^{\mathbf{L}^\bullet}(\alpha) \in H^k(\mathrm{Th}(\alpha);\mathbf{L}^\bullet)}$ the canonical $\mathbf{L}^\bullet$-orientation for a k-dimensional $\mathbb{Z}$-oriented topological bundle $\alpha\colon X \to \mathrm{BSTOP}(k)$ is given by the composition

$$\mathrm{Th}(\alpha) \xrightarrow{u^T(\alpha)} \mathbf{MSTOP}(k) \xrightarrow{\widetilde{c}^{-1}} \mathbf{\Omega}^{STOP}_{-k} \xrightarrow{\mathrm{sgn}^{\mathbf{L}^\bullet}_{\Omega}} \mathbf{L}^{-k}\langle 0\rangle.$$

$\boxed{u^T(\alpha) \in H^k(\mathrm{Th}(\alpha);\mathbf{MSTOP})}$ the canonical $\mathbf{MSTOP}$-orientation of α which is a map on the Thom spaces $\mathrm{Th}(\alpha) \to \mathrm{Th}(\gamma_{STOP})$ induced by the classifying map of a k-dimensional $\mathbb{Z}$-oriented topological bundle $\alpha\colon X \to \mathrm{BSG}(k)$.

$\boxed{\widetilde{\Delta}\colon \mathrm{Th}(\nu) \simeq \frac{V}{\partial V} \xrightarrow{\Delta} \frac{V\times V}{V\times\partial V} \simeq \mathrm{Th}(\nu)\wedge X_+}$ the generalized diagonal map where V is the mapping cylinder of the projection map of ν and ∂V the total space of ν.

$\boxed{\widehat{f} := (\overline{f},f)\colon M \to X}$ an n-dimensional degree one normal map, i.e. a commutative square

$$\begin{array}{ccc} \nu_M & \xrightarrow{\overline{f}} & \eta \\ \downarrow & & \downarrow \\ M & \xrightarrow{f} & X \end{array}$$

with $f\colon M \to X$ a map from an n-dimensional manifold M to an n-dimensional Poincaré space X such that $f_*([M]) = [X] \in H_n(X)$, and $\overline{f}\colon \nu_M \to \nu_X$ stable bundle map from the stable normal bundle $\nu_M\colon M \to \mathrm{BSTOP}$ to a stable bundle $\nu_X\colon X \to \mathrm{BSTOP}$.

$\boxed{\mathrm{sgn}^{\mathbf{L}_\bullet}_X(\widehat{f}) \in L_n(\Lambda_G X)}$ the mosaicked quadratic signature over X constructed in the following way: make f transverse to the dual cells $D(\sigma,K)$, then each σ-component is defined as the quadratic pair signature $\mathrm{sgn}^{\mathbf{L}_\bullet}_{\to}(\widehat{f}[\sigma],\partial\widehat{f}[\sigma])$.

$\boxed{\mathrm{sgn}^{\mathbf{L}_\bullet}_{\mathrm{G/TOP}} \colon \mathrm{G/TOP} \to \mathbf{L}_\bullet\langle 1\rangle}$ the quadratic signature induced by the pullback square of 16.

$\boxed{\mathrm{sgn}^{\mathbf{L}^\bullet}_{\Omega} \colon \mathbf{\Omega}^{STOP}_\bullet \to \mathbf{L}^\bullet\langle 0\rangle}$ the symmetric signature map defined for a k-simplex by $X_{\Delta^k} \mapsto \mathrm{sgn}^{\mathbf{L}^\bullet}_{\Delta^k}(X_{\Delta^k})$.

$\boxed{\mathbf{\Omega}^{STOP}_\bullet}$ the Ω-spectrum of Kan Δ-sets defined by

$$(\mathbf{\Omega}^{STOP}_n)^{(k)} = \{(M, \partial_0 M, \dots, \partial_k M) \mid (n+k) - \text{dimensional manifold } (k+2)\text{-ad such that } \partial_0 M \cap \dots \cap \partial_k M = \emptyset\}.$$

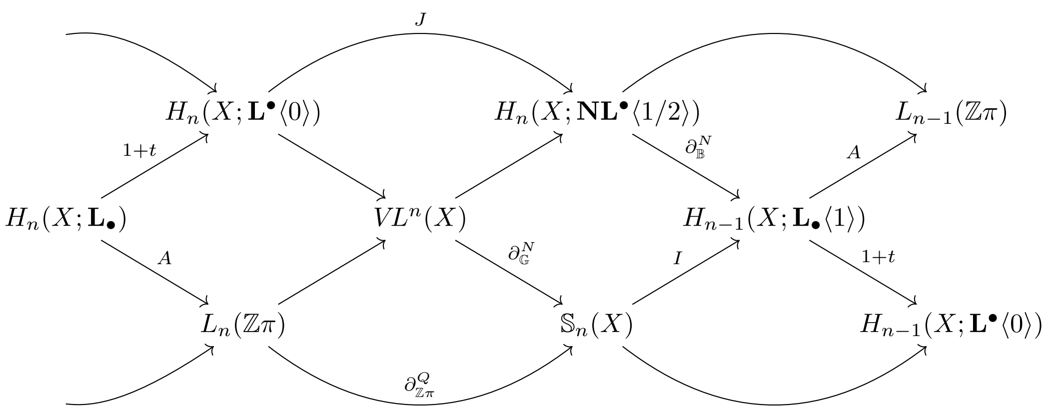
The face maps $\partial_i \colon (\mathbf{\Omega}^{STOP}_n)^{(k)} \to (\mathbf{\Omega}^{STOP}_n)^{(k-1)}$ are given by
$\partial_i(M) = (\partial_i M,\ \partial_i M \cap \partial_0 M, \dots, \partial_i M \cap \partial_{i-1} M,\ \partial_i M \cap \partial_{i+1} M, \dots, \partial_i M \cap \partial_k M)$.

$\boxed{\Sigma^{-1}\mathbf{\Omega}^{\mathrm{N,STOP}}_\bullet}$ the Ω-spectrum of Δ-sets obtained as the fiber of canonical the map of spectra $\mathbf{\Omega}^{STOP}_\bullet \to \mathbf{\Omega}^{N}_\bullet$.

23 Subset step

Porter

Recall the algebraic surgery braid from 12:



The aim here is to show that the subset of $L_n(\mathbb{Z}\pi)$, consisting of all quadratic signatures $\mathrm{sgn}^{\mathbf{L}_\bullet}_{\mathbb{Z}\pi}(\widehat{f})$ produced by degree one normal maps $\widehat{f} \colon M \to X$ between manifolds and X, is a subset of the preimage of the total surgery obstruction $s(X) \in \mathbb{S}_n(X)$ under the boundary map $\partial^Q_{\mathbb{Z}\pi} \colon L_n(\mathbb{Z}\pi) \to \mathbb{S}_n(X) = L_{n-1}(\mathbb{G}\langle 1\rangle, \mathbb{L}\langle 1\rangle)$. The total surgery obstruction was defined as the image of the visible signature $\mathrm{sgn}^{VL}_X(X)$ under the boundary map $\partial^N_{\mathbb{G}} \colon VL^n(X) \to \mathbb{S}_n(X)$. The visible signature was defined as the normal signature $\mathrm{sgn}^{\mathbf{NL}^\bullet}_X$ applied to the special case that X is a Poincaré space. Hence, the mosaicked normal structure produced by $\mathrm{sgn}^{\mathbf{NL}^\bullet}_X$ is globally Poincaré (i.e. Poincaré after assembly) and lives in $NL^n(\Lambda_G X) =: VL^n(X)$ instead of $NL^n(\Lambda_N X) = H_n(X;\mathbf{NL}^\bullet)$.

In the final version of the braid as displayed above, $L_n(\Lambda_G X)$ was replaced by $L_n(\mathbb{Z}\pi)$ using the algebraic π-π-Theorem 1221. For the proof we need to go back to $L_n(\Lambda_G X)$ and work over algebraic bordism categories and consider instead the boundary map

$$\partial^Q_{\mathbb{G}} : L_n(\mathbb{B}\langle 1\rangle, \mathbb{G}\langle 1\rangle) \to L_{n-1}(\mathbb{G}\langle 1\rangle, \mathbb{L}\langle 1\rangle).$$

Therefore we need a mosaicked quadratic signature $\operatorname{sgn}^{\mathbf{L}_\bullet}_X(\widehat{f}) \in L_n(\mathbb{B}\langle 1\rangle, \mathbb{G}\langle 1\rangle)$ which is mapped to $\operatorname{sgn}^{\mathbf{L}_\bullet}_{\mathbb{Z}\pi}(\widehat{f}) \in L_n(\mathbb{Z}\pi)$ under the algebraic π-π-isomorphism. This is obtained by the spectral quadratic construction.

Now the key to the proof of 23 is that the quadratic boundary of the normal signature $\operatorname{sgn}^{\mathbf{NL}^\bullet}_X(X)$ can also be obtained by using the spectral quadratic construction. We will prove this explicitly for the absolute case. The fact that this holds also for the relative and the mosaicked case is a consequence of the long and technical proof of Theorem 7.1 in [Wei85b]. We will use this as a black box.

The original source for this room is [Ran81, section 7.3 and 7.4] with additional contributions from [Ran92, p.192] and [Wei85a, Wei85b]. Most of the details we give here can also be found in [KMM13, section 14.1 to 14.4].

23 (221) Subset step
$-\operatorname{sgn}^{\mathbf{L}_\bullet}_{\mathbb{Z}\pi}(\mathscr{N}(X)) \subseteq \partial^{Q\ -1}_{\mathbb{Z}\pi}(s(X))$ *where* $\partial^Q_{\mathbb{Z}\pi} : L_n(\mathbb{Z}\pi) \to \mathbb{S}_n(X)$ *is the boundary map from the surgery braid.*

A29 (221, 23) Mosaicked quadratic signature [Ran92, Example 9.14]
Let $\widehat{f} : M \to X$ *be a degree one normal map from a closed topological manifold to a Poincaré space both of dimension* n. *Let* $r : X \to K$ *be a map to a simplicial complex* K. *There is a mosaicked quadratic signature*

$$\operatorname{sgn}^{\mathbf{L}_\bullet}_K(\widehat{f}) \in L_n(\Lambda_G K)$$

with $A(\operatorname{sgn}^{\mathbf{L}_\bullet}_X(\widehat{f})) = \operatorname{sgn}^{\mathbf{L}_\bullet}_{\mathbb{Z}\pi}(\widehat{f})$. *If* X *is a manifold, there is a refined version*

$$\operatorname{sgn}^{\mathbf{L}_\bullet}_K(\widehat{f}) \in L_n(\Lambda_L K).$$

[1221 (23) → [Ran92, Thm. 10.6]] Algebraic π-π-Theorem
The assembly map $A : L_n(\Lambda_G X) \to L_n(\mathbb{Z}\pi)$ *defined by* $M \mapsto \bigoplus_{\widetilde{\sigma}\in\widetilde{X}} M(p(\widetilde{\sigma}))$ *is an isomorphism for* $n \in \mathbb{Z}$.

231 The absolute case
Let $\widehat{g} : N \to Y$ *be a degree one normal map from a Poincaré space* N *to a normal space* Y *both of dimension* n. *There is a homotopy equivalence of quadratic chain complexes*

$$h : \partial^Q \operatorname{sgn}^{\mathbf{L}_\bullet}_{\#}(\widehat{g}) \xrightarrow{\simeq} -\partial^N \operatorname{sgn}^{\mathbf{NL}^\bullet}_{\mathbb{Z}\pi}(Y) \in L_{n-1}(\mathbb{Z}\pi).$$

232 The relative case
Let $(\delta\widehat{g}, \widehat{g}) : (N, A) \to (Y, B)$ *be a degree one normal map from a Poincaré pair* (N, A) *to a normal pair* (Y, B) *both of dimension* $(n+1)$. *Then there is a homotopy equivalence of quadratic pairs*

$$h : \partial^Q_{\to} \operatorname{sgn}^{\mathbf{L}_\bullet}_{\to}(\delta\widehat{g}, \widehat{g}) \simeq -\partial^N_{\to} \operatorname{sgn}^{\mathbf{NL}^\bullet}_{\to}(Y, B).$$

Proof 23

A29→p.136 Note that there is a mosaicked quadratic signature $\operatorname{sgn}^{\mathbf{L}_\bullet}_X(\widehat{f}) \in L_n(\Lambda_G X)$ such that $A(\operatorname{sgn}^{\mathbf{L}_\bullet}_X(\widehat{f})) =$
1221→[Ran92] $\operatorname{sgn}^{\mathbf{L}_\bullet}_{\mathbb{Z}\pi}(\widehat{f}) \in L_n(\mathbb{Z}\pi)$ and recall that in this case the assembly map $A : L_n(\Lambda_G X) \to L_n(\mathbb{Z}\pi)$ is an

isomorphism. So it suffices to show

$$\partial^Q_{\mathbb{G}} \operatorname{sgn}^{\mathbf{L}_\bullet}_X(\widehat{f}) \simeq -\partial^N_{\mathbb{G}} \operatorname{sgn}^{VL}_X(X) =: s(X). \tag{23.1}$$

To demonstrate the general scheme of the proof in detail we deal in 231 with the absolute case 231→p.94

$$\partial^Q \operatorname{sgn}^{\mathbf{L}_\bullet}_{\mathbb{Z}\pi}(\widehat{f}) \simeq -\partial^N \operatorname{sgn}^{\mathbf{NL}^\bullet}_{\mathbb{Z}\pi}(Y) \in L_n(\mathbb{Z}\pi)$$

where Y is a normal space. In the mosaicked version we only consider Poincaré spaces but in the subdivision each piece $(X[\sigma], \partial X[\sigma])$ is only a normal pair in general. So we need the relative statement 232 232→p.101

$$\partial^Q_{\to} \operatorname{sgn}^{\mathbf{L}_\bullet}_{\to}(\delta\widehat{g}, \widehat{g}) \simeq -\partial^N_{\to} \operatorname{sgn}^{\mathbf{NL}^\bullet}_{\to}(Y, B)$$

for (Y, B) a normal pair and $(\delta\widehat{g}, \widehat{g})\colon (N, A) \to (Y, B)$ a degree one normal map from a manifold with boundary to a normal pair. This can be generalized for k-ads to obtain the mosaicked version (23.1). We obtain a collection of $(n - |\sigma|)$-dimensional quadratic $(m - |\sigma|)$-ads indexed by simplices of Y and produced by the relative spectral quadratic construction. Now for each simplex we apply the relative statement 232. We start with the top dimensional simplices. Inductively, we can make all the homotopy equivalences fit together. □

Room service 23

$\boxed{\operatorname{sgn}^{\mathbf{L}_\bullet}_X(\widehat{f}) \in L_n(\Lambda_G X)}$ the mosaicked quadratic signature over X constructed in the following way: make f transverse to the dual cells $D(\sigma, K)$, then each σ-component is defined as the quadratic pair signature $\operatorname{sgn}^{\mathbf{L}_\bullet}_{\to}(\widehat{f}[\sigma], \partial\widehat{f}[\sigma])$.

$\boxed{\widehat{f} := (\overline{f}, f)\colon M \to X}$ an n-dimensional degree one normal map, i.e. a commutative square

$$\begin{array}{ccc} \nu_M & \xrightarrow{\overline{f}} & \eta \\ \downarrow & & \downarrow \\ M & \xrightarrow{f} & X \end{array}$$

with $f\colon M \to X$ a map from an n-dimensional manifold M to an n-dimensional Poincaré space X such that $f_*([M]) = [X] \in H_n(X)$, and $\overline{f}\colon \nu_M \to \nu_X$ stable bundle map from the stable normal bundle $\nu_M\colon M \to$ BSTOP to a stable bundle $\nu_X\colon X \to$ BSTOP.

$\boxed{\operatorname{sgn}^{\mathbf{L}_\bullet}_{\to}(\delta\widehat{g}, \widehat{g}) = (f\colon C \to D, \delta\psi, \psi)}$ the quadratic pair signature for a degree one normal map $(\delta\widehat{g}, \widehat{g})\colon (N, A) \to (Y, B)$ from a Poincaré pair (N, A) to a normal pair (Y, B) defined as follows.

The maps
$$\delta\Gamma := \Gamma_N \circ \mathrm{Th}(\overline{g})/\mathrm{Th}(\partial\overline{g})^*\colon (\mathrm{Th}(\nu_Y)/\mathrm{Th}(\nu_B))^* \to \Sigma^p X_+$$
$$\Gamma := \Gamma_A \circ \mathrm{Th}(\partial\overline{g})^* \quad\colon \Sigma^{-1}\mathrm{Th}(\nu_B)^* \to \Sigma^p A_+$$

induce chain maps
$$g^!_i\colon C^{n+1-*}(Y, B) \to C(N)$$
$$g^!\colon C^{n-*}(B) \to C(A).$$

The chain complexes $C := \mathscr{C}(\partial g^!)$, $D := \mathscr{C}(g^!)$ are defined as the mapping cones on the induced maps and the pair structure $(\delta\psi, \psi) = \operatorname{con}^{\delta\psi^!, \psi^!}_{\delta\Gamma, \Gamma}(u^*_{\nu_Y})$ is obtained from the relative spectral quadratic construction applied to the S-dual of a choice of the Thom class $u(\nu_Y)$.

$\boxed{\partial\operatorname{gn}^{\mathbf{L}_\bullet}_{\mathbb{Z}\pi}(Y)}$ the quadratic boundary signature for an n-dimensional normal space (Y, ν, ρ) is an $(n-1)$-quadratic chain complex $(\partial C', \partial\psi')$ in $L_{n-1}(\mathbb{Z}\pi)$ (see B27). B27→p.155

$\boxed{\Gamma_Y := S^{-1}(\widetilde{\Delta} \circ \rho) \colon \mathrm{Th}(\nu)^* \to \Sigma^p Y_+}$ the semi-stable map obtained for an n-dimensional normal space (Y, ν, ρ) with an N-dimensional S-dual $\mathrm{Th}(\nu)^*$ of its Thom space and $p = N - (n+k)$.

$\boxed{\partial \mathrm{gn}^{\mathbf{L}_\bullet}_{\to}(Y, B)}$ the quadratic boundary pair signature for an n-dimensional pair of normal spaces (Y, B) is the $(n-1)$-quadratic Poincaré pair $(\partial C(B) \to \partial_+ C(Y), (\delta\psi, \psi))$ obtained by using the boundary construction and the spectral quadratic construction (see B28).

$\boxed{(\widehat{f}[\sigma], \partial\widehat{f}[\sigma]) = ((\overline{f}[\sigma], f[\sigma]), (\partial\overline{f}[\sigma], \partial f[\sigma]))}$ an n-dimensional degree one normal map

$$\begin{array}{ccc} (\nu_M|_{M[\sigma]}, \nu_M|_{\partial M[\sigma]}) & \xrightarrow{(\overline{f}, \partial\overline{f})} & (\nu_X|_{X[\sigma]}, \nu_X|_{\partial X[\sigma]}) \\ \downarrow & & \downarrow \\ (M[\sigma], \partial M[\sigma]) := (f^{-1}, \partial f^{-1})(X[\sigma], \partial X[\sigma]) & \xrightarrow{(f, \partial f)} & (X[\sigma], \partial X[\sigma]), \end{array}$$

denoted $(f[\sigma], \partial f[\sigma]) \colon (M[\sigma], \partial M[\sigma]) \to (X[\sigma], \partial X[\sigma])$ for short, from an $(n-|\sigma|)$-dimensional manifold with boundary to an $(n-|\sigma|)$-dimensional normal pair obtained from a degree one normal map $\widehat{f}$ after making f transverse to a K-dissection $\bigcup_{\sigma \in K} X[\sigma]$ of X .

$\boxed{X[\sigma]}$ is defined for a map $r \colon X \to K$ to a simplicial complex as the preimage of the dual cell $D(\sigma, K)$ after making r transverse. If X is a simplicial complex itself, choose r to be the identity. The subdivision $X = \bigcup_{\sigma \in K} X[\sigma]$ is called a K-dissection of X.

$\boxed{\mathrm{sgn}^{VL}_X(X) \in VL^n(X)}$ defined for a Poincaré space X as the normal signature $\mathrm{sgn}^{\mathbf{NL}^\bullet}_X(X)$.

$\boxed{\mathrm{sgn}^{\mathbf{NL}^\bullet}_X(X) := \mathrm{sgn}^{\mathbf{NL}^\bullet}_\Omega([X]^{\mathbf{\Omega}^N_\bullet}) \in H_n(X; \mathbf{NL}^\bullet) \cong NL^n(\Lambda_N X)}$ the X-mosaicked normal signature defined here only for an n-dimensional Poincaré space X.

$\boxed{\mathrm{sgn}^{\mathbf{NL}^\bullet}_\Omega \colon \mathbf{\Omega}^N_\bullet \to \mathbf{NL}^\bullet\langle 1/2 \rangle}$ the normal signature map; based on the normal signature $\mathrm{sgn}^{\mathbf{NL}^\bullet}_{\mathbb{Z}\pi}$.

$\boxed{[X]^{\mathbf{\Omega}^N_\bullet} \in H_n(X; \mathbf{\Omega}^N_\bullet)}$ an n-dimensional $\mathbf{\Omega}^N_\bullet$-cycle which assigns to each $\sigma \in X$ a $(m-n)$-dimensional normal $(m - |\sigma| + 2)$-ad $(X[\sigma], \nu(\sigma), \rho(\sigma))$ as constructed in 132.

$\boxed{f^! \colon C(\widetilde{X}) \to C(\widetilde{M})}$ the Umkehr map of a degree one normal map $\widehat{f} \colon M \to X$ of Poincaré spaces M and X. We obtain a stable equivariant map $F \colon \Sigma^k \widetilde{X}_+ \to \Sigma^k \widetilde{M}_+$ for some $k \in \mathbb{N}$ and define $f^!$ as the composition $C(\widetilde{X}) \xrightarrow{\Sigma_X} \Sigma^{-k} C(\Sigma^k \widetilde{X}_+) \xrightarrow{F} \Sigma^{-k} C(\Sigma^k \widetilde{M}_+) \xrightarrow{\Sigma_X^{-1}} C(\widetilde{M})$.

$\boxed{\mathrm{con}^{(\delta\psi^!, \psi^!)}_{G,F} \colon \Sigma^{-p} \widetilde{C}(X, A) \to \mathscr{C}(j, i)_\%}$ a chain map called relative spectral quadratic construc-
A25→p.133 tion; defined for a semi-stable map of pairs $(G, F) \colon \to (X, A) \to \Sigma^p(Y, B)$ (see A25).

$\boxed{\Lambda_G X}$ here short for the algebraic bordism category $(\mathbb{Z}_* X, \mathbb{B}\langle 1 \rangle, \mathbb{G}\langle 1 \rangle)$.

$\boxed{\Lambda_N X}$ here short for the algebraic bordism category $(\mathbb{Z}_* X, \mathbb{B}\langle 1 \rangle, \mathbb{B}\langle 0 \rangle)$.

- $\boxed{\mathbb{B}}$ short for $\mathbb{B}(\mathbb{Z})_L X = \mathbb{B}(\mathbb{Z}_* X)$, the X-based bounded chain complexes of free $\mathbb{Z}$-modules.
- $\boxed{\mathbb{G}}$ short for $\mathbb{B}(\mathbb{Z})_G X \coloneqq \{C \in \mathbb{B}(\mathbb{Z}) \mid A(C) \simeq *\}$, the *globally contractible* objects of $\mathbb{B}(\mathbb{Z})$.
- $\boxed{\mathbb{L}}$ short for $\mathbb{B}(\mathbb{Z})_L X \coloneqq \{C \in \mathbb{B}(\mathbb{Z}) \mid C(\sigma) \simeq * \text{ for all } \sigma \in X\}$, the *locally contractible* chain complexes of $\mathbb{B}$.
- $\boxed{\mathbb{C}\langle q\rangle \subset \mathbb{C}}$ the subcategory of a category of chain complexes $\mathbb{C}$ restricted to chain complexes which are homotopy equivalent to q-connected chain complexes.

Level 3 and 4

In the elevator

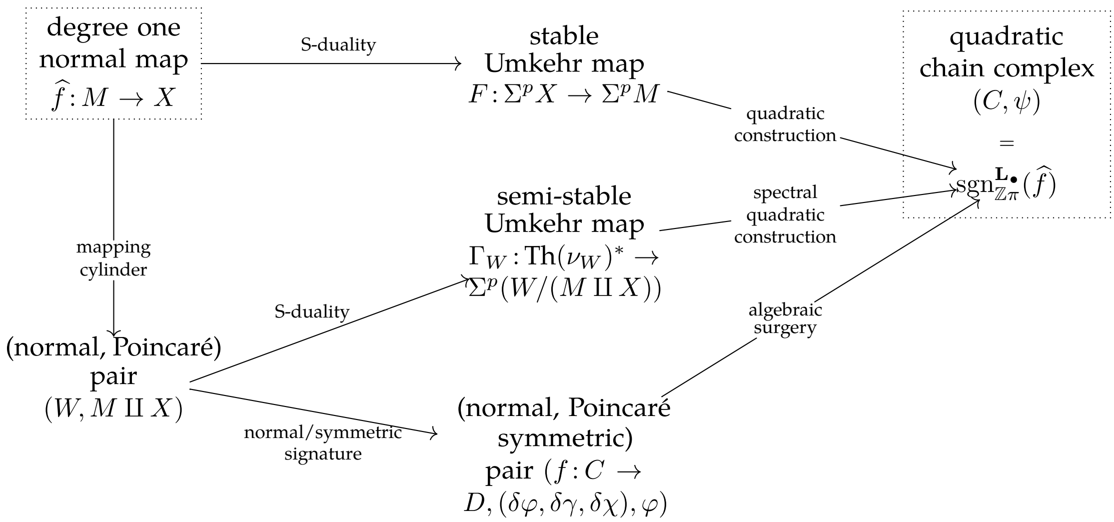
At this level of detail we stop ordering the rooms by level and group the rooms together with their subrooms of the next level. We now come to theheart of the proof that will involve a lot of technical details. For non-experts we recommend to have a look at the basics explained in the basement. Based on the construction introduced there the following concepts are developed and used in this level:

– the relation between structured pairs and structured chain complexes using algebraic surgery in order to establish certain fibration sequences and exact sequences (111, 112)

– the construction of L-spectra (123, 1231,1232)

– an explicit simplicial description of elements in homology groups with coefficients in L-spectra (131,1311,1312)

Further on we use these constructions in order to deal with signatures and orientations. Eventually, a crucial detail in the proof is that we play with three different ways of how we obtain a quadratic signature. They enable us to switch from the purely algebraical defined object to a more geometrical approach using cobordism of mapping cylinders.



111 Poincaré symmetric and Poincaré normal

Porter

In contrast to the symmetric and quadratic L-groups, for the normal L-groups we do not require the normal complexes to be Poincaré, i.e. that φ_0 is a homotopy equivalence. The reason is the following statement that a Poincaré normal complex would carry the same information as a symmetric one. This establishes a map from symmetric to normal L-groups.

111 Poincaré symmetric and Poincaré normal [Ran92, Proposition 2.6 (ii)]
There is the following natural one-to-one correspondence of homotopy equivalence classes.

n-normal chain complexes $(C,(\varphi,\gamma,\chi))$ *such that* φ_0 *is a chain homotopy equivalence*	$\overset{1-1}{\longleftrightarrow}$	*n-symmetric Poincaré chain complexes* (C,φ)

Proof 111

In order to obtain the symmetric Poincaré chain complex just pick (C,φ) from the normal chain complex $(C,(\varphi,\gamma,\chi))$. For the other direction we construct a normal structure for a symmetric Poincaré chain complex as follows. By definition, being Poincaré means φ_0 is a chain equivalence. Using a chain homotopy inverse $\widehat{\varphi_0}^{\%-1}$ of $\widehat{\varphi_0}^{\%}$ we obtain a chain bundle $\gamma \in \widehat{W}^{\%}(C^{n-*})_0$ as the image of $\varphi \in W^{\%}(C)_n$ under the composition

$$W^{\%}C_n \xrightarrow{j} \widehat{W}^{\%}C_n \xrightarrow{\widehat{\varphi_0}^{\%-1}} \widehat{W}^{\%}(\Sigma^n C^{n-*})_n \xrightarrow{S^{-n}} \widehat{W}^{\%}(C^{-*})_0.$$

The chain homotopy $\widehat{\varphi_0}^{\%} \circ (\widehat{\varphi_0}^{\%})^{-1} \simeq 1$ yields the chain $\chi \in \widehat{W}^{\%}C_{n+1}$, the last missing piece for a normal structure on C. □

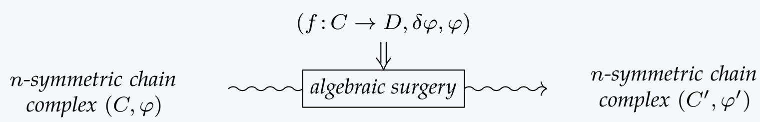
112 Quadratic and (normal, Poincaré symmetric)

112 (121, 1411, 164) Quadratic and (normal, Poincaré symmetric) [Ran92, Proposition 2.8 (ii)]
There is the following natural one-to-one correspondence of cobordism classes.

n-dimensional (normal, symmetric) pairs $(f : C \to D, (\delta\varphi, \delta\gamma, \delta\chi), \varphi)$	$\overset{1-1}{\longleftrightarrow}$	*$(n-1)$-dimensional quadratic chain complexes* (C', ψ')

Additionally, if (C,φ) is Poincaré, then (C',ψ') is Poincaré and vice versa.

B1 (112, 11, 121) Algebraic surgery [Ran92, Def. 1.12]
Let (C,φ) be an n-symmetric chain complex. The effect of algebraic surgery of an $(n+1)$-symmetric pair $(f : C \to D, \delta\varphi, \varphi)$ on (C,φ) is an n-symmetric chain complex (C',φ'). It is Poincaré if and only if (C,φ) is Poincaré. Moreover, we have $\partial C \simeq \partial C'$.



The same construction works for quadratic chain complexes as well.

B24 (112, 231) Quadratic boundary for normal
An n-normal chain complex $(C,(\varphi,\gamma,\chi)) \in NL^n(\mathbb{Z}\pi)$ has an $(n-1)$-quadratic Poincaré boundary

$$\partial^N(C,(\varphi,\gamma,\chi)) =: (\partial C, \partial\psi)$$

which defines a map $\partial^N : NL^n(\mathbb{Z}\pi) \longrightarrow L_{n-1}(\mathbb{Z}\pi); \quad (C,(\varphi,\gamma,\chi)) \mapsto (\partial C, \partial\psi)$.

Proof 112

Let $(f\colon C \to D, (\delta\varphi, \delta\gamma, \delta\chi), \varphi)$ be a (normal, symmetric) pair. Perform algebraic surgery with
B1→p.143 the symmetric pair $(f\colon C \to D, \delta\varphi, \varphi)$ on (C,φ) to obtain a symmetric chain complex (C',φ'). A
B24→p.152 diagram similar to that used in the proof of B24 yields the quadratic Poincaré structure (replace $\widehat{\varphi_0}^{\%}$ by $\widehat{\varphi_{f^*}}^{\%}$).

Conversely, we start with an $(n-1)$-quadratic chain complex (C',ψ'). The corresponding n-dimensional (normal, Poincaré symmetric) pair is then given by $(C' \to 0, 0, (1+t)\psi)$. □

Room service 112

$(f\colon C \to D, (\delta\varphi, \delta\gamma, \delta\chi), (\varphi,\gamma,\chi))$ an $(n+1)$-normal pair with

- $(C,(\varphi,\gamma,\chi))$ an n-normal chain complex
- $(f\colon C \to D, \delta\varphi, \varphi)$ an $(n+1)$-symmetric pair
- $(f,b)\colon (C,\gamma) \to (D,\delta\gamma)$ a map of chain bundles
- $\delta\chi \in \widehat{W}^{\%}(D)_{n+2}$ a chain such that

$$J(\delta\varphi) - \widehat{\delta\varphi_0}^{\%}(\mathcal{S}^{n+1}\delta\gamma) + \widehat{f}^{\%}(\chi - \widehat{\varphi_0}^{\%}(\mathcal{S}^n b)) = d(\delta\chi) \in \widehat{W}^{\%}(D)_{n+1}.$$

$\varphi_{f^*} = \mathrm{ev}_r(\delta\varphi,\varphi) \simeq \begin{pmatrix} \delta\varphi_0 \\ \varphi_0 f^* \end{pmatrix} : D^{n-*} \to \mathscr{C}(f)$ a chain map defined for an n-symmetric pair $(f\colon C \to D, \delta\varphi, \varphi)$.

121 Exact sequences for inclusions of bordism categories

121 Exact sequences for inclusions of bordism categories [Ran92, Prop. 3.9]
An inclusion functor $(\mathbb{A},\mathbb{B},\mathbb{Q}) \to (\mathbb{A},\mathbb{B},\mathbb{P})$ of algebraic bordism categories induces the following long exact sequences in symmetric, quadratic and normal L-groups

$$\ldots \longrightarrow L^n(\mathbb{B},\mathbb{Q}) \longrightarrow L^n(\mathbb{B},\mathbb{P}) \longrightarrow L^{n-1}(\mathbb{P},\mathbb{Q}) \longrightarrow L^{n-1}(\mathbb{B},\mathbb{Q}) \longrightarrow \ldots,$$

$$\ldots \longrightarrow L_n(\mathbb{B},\mathbb{Q}) \longrightarrow L_n(\mathbb{B},\mathbb{P}) \longrightarrow L_{n-1}(\mathbb{P},\mathbb{Q}) \longrightarrow L_{n-1}(\mathbb{B},\mathbb{Q}) \longrightarrow \ldots,$$

$$\ldots \longrightarrow NL^n(\mathbb{B},\mathbb{Q}) \longrightarrow NL^n(\mathbb{B},\mathbb{P}) \longrightarrow L_{n-1}(\mathbb{P},\mathbb{Q}) \longrightarrow NL^{n-1}(\mathbb{B},\mathbb{Q}) \longrightarrow \ldots.$$

[1211 → [Ran92, Prop. 3.8]]
A functor $F\colon \Lambda \to \Lambda'$ of algebraic bordism categoriesinduces a map of L-groups.

112 (121, 1411, 164) Quadratic and (normal, Poincaré symmetric) [Ran92, Proposition 2.8 (ii)]
There is the following natural one-to-one correspondence of cobordism classes.

$$\begin{array}{ccc} n\text{-dimensional} & & (n-1)\text{-dimensional} \\ \text{(normal, symmetric) pairs} & \xleftrightarrow{1-1} & \text{quadratic chain complexes} \\ (f\colon C \to D, (\delta\varphi, \delta\gamma, \delta\chi), \varphi) & & (C', \psi') \end{array}$$

Additionally, if (C, φ) is Poincaré, then (C', ψ') is Poincaré and vice versa.

B1 (112, 11, 121) Algebraic surgery [Ran92, Def. 1.12]
Let (C, φ) be an n-symmetric chain complex. The effect of algebraic surgery of an $(n+1)$-symmetric pair $(f\colon C \to D, \delta\varphi, \varphi)$ on (C, φ) is an n-symmetric chain complex (C', φ'). It is Poincaré if and only if (C, φ) is Poincaré. Moreover, we have $\partial C \simeq \partial C'$.

$$\begin{array}{ccccc} & & (f\colon C \to D, \delta\varphi, \varphi) & & \\ & & \Downarrow & & \\ n\text{-symmetric chain complex } (C, \varphi) & \rightsquigarrow & \boxed{\text{algebraic surgery}} & \rightsquigarrow & n\text{-symmetric chain complex } (C', \varphi') \end{array}$$

The same construction works for quadratic chain complexes as well.

Proof 121

A functor $F\colon \Lambda \to \Lambda'$ of algebraic bordism categoriesinduces a map of L-groups. For an arbitrary 1211→[Ran92]
functor F there are relative L-groups $L^n(F)$, $L_n(F)$ and $NL^n(F)$ which fit into the long exact sequences

$$\ldots \longrightarrow L_n(\Lambda) \longrightarrow L_n(\Lambda') \longrightarrow L_n(F) \longrightarrow L_{n-1}(\Lambda) \longrightarrow \ldots,$$

$$\ldots \longrightarrow L^n(\Lambda) \longrightarrow L^n(\Lambda') \longrightarrow L^n(F) \longrightarrow L^{n-1}(\Lambda) \longrightarrow \ldots,$$

$$\ldots \longrightarrow NL^n(\Lambda) \longrightarrow NL^n(\Lambda') \longrightarrow NL^n(F) \longrightarrow NL^{n-1}(\Lambda) \longrightarrow \ldots.$$

Similar to the proof of the long exact sequence of L-groups in 11 we get a localized sequence by 11→p.23
replacing the relative term. For the special case that $F\colon (\mathbb{A}, \mathbb{B}, \mathbb{Q}) \to (\mathbb{A}, \mathbb{B}, \mathbb{P})$ is an inclusion of algebraic bordism categories, i.e. $\mathbb{Q} \subset \mathbb{P} \subset \mathbb{B}$, we prove that the relative L-groups are given as follows.

(i) $L^n(F) \cong L^{n-1}(\mathbb{A}, \mathbb{P}, \mathbb{Q})$,

(ii) $L_n(F) \cong L_{n-1}(\mathbb{A}, \mathbb{P}, \mathbb{Q})$,

(iii) $NL^n(F) \cong L_{n-1}(\mathbb{A}, \mathbb{P}, \mathbb{Q})$.

In the symmetric case (i) an element in $L^n(F)$ is an n-symmetric pair $(f\colon C \to D, \delta\varphi, \varphi)$ in $(\mathbb{A}, \mathbb{B}, \mathbb{P})$ such that (C, φ) is an $(n-1)$-symmetric chain complex in $(\mathbb{A}, \mathbb{B}, \mathbb{Q})$. We define the map

$$L^n(F) \to L_{n-1}(\mathbb{A}, \mathbb{P}, \mathbb{Q}) \qquad \text{by} \qquad (f\colon C \to D, \delta\varphi, \varphi) \mapsto (C', \varphi')$$

where (C', φ') is the effect of algebraic surgery on $(f\colon C \to D, \delta\varphi, \varphi)$. By B1, algebraic surgery B1→p.143
preserves the homotopy type of the boundary of C. Hence (C', φ') is $\mathbb{Q}$-Poincaré and defines an element in $L^n(\mathbb{A}, \mathbb{P}, \mathbb{Q})$.

The quadratic case is analog. In the normal case we additionally use 112 to obtain a quadratic 112→p.61
structure. □

Room service 121

$\Lambda = (\mathbb{A}, \mathbb{C}, \mathbb{P}, (T, e))$ an algebraic bordism category, usually denoted by Λ or $(\mathbb{A}, \mathbb{C}, \mathbb{P})$, consists of full additive subcategories $\mathbb{P} \subseteq \mathbb{C} \subseteq \mathbb{B}(\mathbb{A})$ where $\mathbb{P}$ is closed under weak equivalences and mapping cones, i.e.
- $\mathscr{C}(f : C \to D) \in \mathbb{P}$ for any chain map f in $\mathbb{P}$,

and additionally any $C \in \mathbb{C}$ satisfies
- $\mathscr{C}(\mathrm{id} : C \to C) \in \mathbb{P}$,
- $\mathscr{C}(e(C) : T^2(C) \xrightarrow{\simeq} C) \in \mathbb{P}$.

$F : \Lambda \to \Lambda'$ a functor of algebraic bordism categories is a covariant functor of additive categories, such that
- $F(B) \in \mathbb{B}'$ for every $B \in \mathbb{B}$
- $F(C) \in \mathbb{P}'$ for every $C \in \mathbb{P}$
- for every $A \in \mathbb{A}$ there is a natural chain map $G(A) : T'F(A) \to FT(A)$ such that

$$\begin{array}{ccc} T'FT(A) & \xrightarrow{GT(A)} & FT^2(A) \\ \downarrow{\scriptstyle T'G(A)} & & \downarrow{\scriptstyle Fe(A)} \\ T'^2F(A) & \xrightarrow{e'F(A)} & F(A) \end{array}$$

  commutes and $\mathscr{C}(G(A)) \in \mathbb{P}'$.

(C, λ) in Λ an n-dimensional structured chain complex in $\Lambda = (\mathbb{A}, \mathbb{C}, \mathbb{P})$, i.e. a chain complex $C \in \mathbb{C}$ with an n-dimensional $\mathbb{P}$-Poincaré structure λ.

$f : (C \to D, \delta\lambda, \lambda)$ in Λ a structured pair with $C, D \in \mathbb{B}$, λ is $\mathbb{P}$-Poincaré and $\mathscr{C}(\delta\lambda_0, \lambda_0 f^*) \in \mathbb{P}$.

λ placeholder for a symmetric φ resp. quadratic ψ or normal structure (φ, γ, χ). We call a pair (C, λ) a *structured chain complex* and a triple $(C \to D, \delta\lambda, \lambda)$ a *structured pair*.

λ_0 stands for φ_0 in the symmetric and normal case and for $(1 + t)(\psi_0)$ in the quadratic case.

$\mathbb{P}$-Poincaré is what a structured complex (C, λ) is called if $\partial C := \Sigma^{-1}\mathscr{C}(\lambda_0) \in \mathbb{P}$.

$L^n(\Lambda), L_n(\Lambda), NL^n(\Lambda)$ the cobordism groups of n-dimensional symmetric, quadratic, and normal chain complexes in Λ respectively.

$Cobordism$ of n-dimensional structured chain complexes in Λ:

$(C, \lambda) \sim (C', \lambda') \iff$ there is an $(n + 1)$-dimensional structured pair $(C \oplus C' \to D, \delta\lambda, \lambda \oplus -\lambda')$ in Λ.

$L^n(F), L_n(F), NL^n(F)$ n-dimensional relative L-groups consisting, up to cobordism, of pairs $((C, \lambda), (F(C) \to D, \delta\lambda, \lambda))$ where (C, λ) is an $(n - 1)$-dimensional structured chain complex in Λ and $(F(C) \to D, \delta\lambda, \lambda)$ an n-dimensional structured pair in Λ'.

122 Quadratic assembly isomorphism

We only provide a comment on the connectivity condition here. The main work is done in the algebraic π-π-Theorem due to Ranicki.

122 Quadratic assembly isomorphism [Ran92, Proposition 15.11]
Let X be a simplicial complex and $\Lambda = \Lambda(\mathbb{Z})$. Then for $n \geq 5$ we have

$$L_n(\Lambda\langle 1\rangle_G X) \cong L_n(\mathbb{Z}\pi).$$

[1221 (23) → [Ran92, Thm. 10.6]] Algebraic π-π-Theorem
The assembly map $A\colon L_n(\Lambda_G X) \to L_n(\mathbb{Z}\pi)$ defined by $M \mapsto \bigoplus_{\widetilde{\sigma}\in\widetilde{X}} M(p(\widetilde{\sigma}))$ is an isomorphism for $n \in \mathbb{Z}$.

Proof 122

In fact, the more general result holds that $A\colon L_n(\Lambda\langle q\rangle_G X) \to L_n(\mathbb{Z}\pi)$ is an isomorphism for $n \geq 2q$. In this situation the forgetful map $L_n(\Lambda\langle q\rangle_G X) \to L_n(\Lambda_G X)$ has an inverse defined by sending an n-quadratic chain complex $(C, \psi) \in L_n(\Lambda_G X)$ to the quadratic chain complex obtained by algebraic surgery below the middle dimension using the quadratic pair $(C \to D, 0, \psi)$ with

$$D_r = \begin{cases} C_r & \text{if } 2r > n+1 \\ 0 & \text{otherwise} \end{cases}$$

Composed with the isomorphism from 1221 we obtain the desired isomorphism. □

Room service 122

$\boxed{\Lambda\langle q\rangle_G X}$ here short for $\Lambda(\mathbb{Z})\langle q\rangle_G X = (\mathbb{Z}_* X, \mathbb{B}\langle q\rangle, \mathbb{G}\langle q\rangle, (T_*, e_*))$, the q-connected algebraic bordism category of X-based free $\mathbb{Z}$-modules with *global Poincaré* duality where $q \geq 0$. For an arbitrary algebraic bordism category $\Lambda = (\mathbb{A}, \mathbb{C}, \mathbb{P}, (T, e))$ defined as $(\mathbb{A}_* X, \mathbb{C}\langle q\rangle_L X, \mathbb{P}\langle q\rangle_G X, (T_*, e_*))$.

123 L-spectra and homology

We prove in this room that certain L-groups are obtained as homology groups. Note that the second line in the display below is not a typo. It is a consequence of 111 which says that normal Poincaré is the same as symmetric Poincaré.

123 (13) L-spectra and homology [LM09, Remark 16.2][Ran92, Proposition 15.9]
Let X be a simplicial complex and $\Lambda = \Lambda(\mathbb{Z})$. Then we have the following equivalences

$$L_n(\Lambda\langle 1\rangle_L X) = H_n(X; \mathbf{L}_\bullet\langle 1\rangle),$$
$$NL^n(\Lambda\langle 0\rangle_L X) = H_n(X; \mathbf{L}^\bullet\langle 0\rangle),$$
$$NL^n(\Lambda\langle 1/2\rangle_N X) = H_n(X; \mathbf{NL}^\bullet\langle 1/2\rangle).$$

1231 L-spectra [Ran92, Prop. 13.4]
There are Ω-spectra of pointed Kan Δ-sets

$$\mathbf{L}^\bullet(\Lambda) := \{\mathbf{L}^n(\Lambda) \mid n \in \mathbb{Z}\}, \quad \mathbf{L}_\bullet(\Lambda) := \{\mathbf{L}_n(\Lambda) \mid n \in \mathbb{Z}\}, \quad \mathbf{NL}^\bullet(\Lambda) := \{\mathbf{NL}^n(\Lambda) \mid n \in \mathbb{Z}\}$$

with homotopy groups

$$\pi_n(\mathbf{L}^\bullet(\Lambda)) \cong L^n(\Lambda), \quad \pi_n(\mathbf{L}_\bullet(\Lambda)) \cong L_n(\Lambda), \quad \pi_n(\mathbf{NL}^\bullet(\Lambda)) \cong NL^n(\Lambda).$$

1232 L-spectra and smash products [LM09, Remark 16.2][Ran92, Prop. 13.7]
Let K be finite simplicial complex and Λ an algebraic bordism category. Then

$$K_+ \wedge \mathbf{L}_\bullet(\Lambda) \simeq \mathbf{L}_\bullet(\Lambda_L K)$$
$$K_+ \wedge \mathbf{L}^\bullet(\Lambda) \simeq \mathbf{L}^\bullet(\Lambda_L K)$$
$$K_+ \wedge \mathbf{NL}^\bullet(\Lambda) \simeq \mathbf{NL}^\bullet(\Lambda_N K)$$

Proof 123

1232→p.70
1231→p.67
We obtain our result immediately from 1232 and 1231. In the quadratic case we have

$$L_n(\Lambda_L X) \overset{1231}{\cong} \pi_n(\mathbf{L}_\bullet(\Lambda)) \overset{1232}{\cong} \pi_n(K_+ \wedge \mathbf{L}_\bullet(\Lambda)) =: H_n(X, \mathbf{L}_\bullet(\Lambda)).$$

111→p.60
The third case is analog. For the second one use in addition the fact that by 111 there is a one-to-one correspondence between chain complexes in $NL^n(\Lambda_L X)$ and $L^n(\Lambda_L X)$. □

Room service 123

$\Lambda\langle q\rangle_L X$ here short for $\Lambda(\mathbb{Z})\langle q\rangle_L X = (\mathbb{Z}_* X, \mathbb{B}\langle q\rangle, \mathbb{L}\langle q\rangle, (T_*, e_*))$, the q-connected algebraic bordism category of X-based free $\mathbb{Z}$-modules with *local Poincaré* duality where $q \geq 0$. For an arbitrary algebraic bordism category $\Lambda = (\mathbb{A}, \mathbb{C}, \mathbb{P}, (T, e))$ defined as $(\mathbb{A}_* X, \mathbb{C}\langle q\rangle_L X, \mathbb{P}\langle q\rangle_L X, (T_*, e_*))$.

$\Lambda\langle 1/2\rangle_N X$ here short for $(\mathbb{Z}_* X, \mathbb{B}\langle 0\rangle, \mathbb{B}\langle 1\rangle, (T_*, e_*))$, the $1/2$-connected algebraic bordism category of X-based free $\mathbb{Z}$-modules with *no Poincaré* duality. For an arbitrary algebraic bordism category $\Lambda = (\mathbb{A}, \mathbb{C}, \mathbb{P}, (T, e))$ defined as $(\mathbb{A}_* X, \mathbb{C}\langle 0\rangle_L X, \mathbb{C}\langle 1\rangle_L X, (T_*, e_*))$.

$\mathbb{B}$ short for $\mathbb{B}(\mathbb{Z})_L X = \mathbb{B}(\mathbb{Z}_* X)$, the X-based bounded chain complexes of free $\mathbb{Z}$-modules.

$\mathbb{L}$ short for $\mathbb{B}(\mathbb{Z})_L X := \{C \in \mathbb{B}(\mathbb{Z}) \mid C(\sigma) \simeq * \text{ for all } \sigma \in X\}$, the *locally contractible* chain complexes of $\mathbb{B}$.

$\mathbb{C}\langle q\rangle$ the subcategory of chain complexes of $\mathbb{C}$ that are homotopy equivalent to q-connected chain complexes.

$\mathbb{C}_L X := \{C \text{ chain complex in } \mathbb{A}_* X \mid C(\sigma) \in \mathbb{C} \text{ for all } \sigma \in X\}$ for a category $\mathbb{C}$ of chain complexes in $\mathbb{A}$.

$\boxed{\mathbb{Z}_*X}$ short for $\mathbb{A}(\mathbb{Z})_*X$, the additive category of X-based free $\mathbb{Z}$-modules with 'non-decreasing' morphisms $\sum\limits_{\tau\geq\sigma} f_{\tau,\sigma}\colon \sum\limits_{\sigma\in X} M_\sigma \to \sum\limits_{\tau\in X} N_\tau$ where $(f_{\tau,\sigma}\colon M_\sigma \to N_\tau)$ are $\mathbb{Z}$-module morphism.

$\boxed{\mathbf{L}^\bullet\langle q\rangle, \mathbf{L}_\bullet\langle q\rangle, \mathbf{NL}^\bullet\langle q\rangle}$ short for $\mathbf{L}^\bullet(\Lambda(\mathbb{Z})\langle q\rangle), \mathbf{L}_\bullet(\Lambda(\mathbb{Z})\langle q\rangle), \mathbf{NL}^\bullet(\Lambda(\mathbb{Z})\langle q\rangle)$.

1231 L-spectra

Porter

We want to construct L-theory spectra whose homotopy groups are the L-groups in order to obtain signatures as elements in generalized homology theories. The technology used for the construction are Δ-sets, i.e. simplicial sets without degeneracies.

We recall the basic definitions. Let Δ be the prototype of a simplicial complex, i.e. the category with

$$\begin{aligned} \mathrm{obj}_\Delta\colon &\ \text{sets } [n] \coloneqq \{0,\ldots,n\} \text{ for } n \geq 0,\\ \mathrm{mor}_\Delta([n],[m])\colon &\ \text{strictly order preserving functions } [n] \to [m]. \end{aligned}$$

A Δ-set X is a contravariant functor from Δ to the category of sets. We denote by $X^{(k)} = X([k])$ the k-skeleton and call the elements k-simplices. A map of Δ-sets is a natural transformation.

Remark.

- An alternative description for a Δ-set X is a sequence of sets $X^{(k)}, k \geq 0$ with maps $\partial_i\colon X^{(k)} \to X^{(k-1)} \quad (0 \leq i \leq k)$ such that $\partial_i\partial_j = \partial_{j-1}\partial_i$ when ever $j < i$.
- A simplicial complex with ordered vertices defines a Δ-set, but not the other way round. Consider the circle as standard one simplex where the vertices are identified. It is no longer a simplicial complex because the faces of the one simplex are not unique but it is a Δ-set. However, one can take a subdivision to obtain a simplicial complex.
- Simplicial maps in general do not yield maps of Δ-sets, e.g. the map which collapses the standard one simplex to a vertex.

In order to do homotopy theory we have to introduce fibrant Δ-sets, which are called Kan Δ-sets after [Kan55]. Let Λ^n_i be a simplicial complex obtained from the standard n-simplex Δ^n by removing the interior and the i-th face. A Δ-set satisfies the Kan condition if each map $f\colon \Lambda^n_i \to X$ extends to a map $\widetilde{f}\colon \Delta^n \to X$. With Kan Δ-sets we can define homotopy groups and loop spaces and construct spectra. More details on the definitions are in the room service section below. The original source is [RS71]. Summaries of the topic related to our setting can be found in [RW07] and [Ran92, §11].

1231 L-spectra [Ran92, Prop. 13.4]
There are Ω-spectra of pointed Kan Δ-sets

$$\mathbf{L}^\bullet(\Lambda) \coloneqq \{\mathbf{L}^n(\Lambda) \mid n \in \mathbb{Z}\}, \quad \mathbf{L}_\bullet(\Lambda) \coloneqq \{\mathbf{L}_n(\Lambda) \mid n \in \mathbb{Z}\}, \quad \mathbf{NL}^\bullet(\Lambda) \coloneqq \{\mathbf{NL}^n(\Lambda) \mid n \in \mathbb{Z}\}$$

with homotopy groups

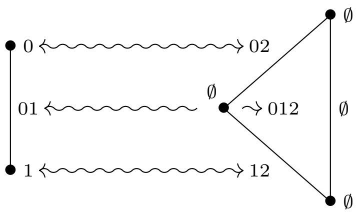
$$\pi_n(\mathbf{L}^\bullet(\Lambda)) \cong L^n(\Lambda), \quad \pi_n(\mathbf{L}_\bullet(\Lambda)) \cong L_n(\Lambda), \quad \pi_n(\mathbf{NL}^\bullet(\Lambda)) \cong NL^n(\Lambda).$$

Proof 1231

We only consider the symmetric case. For the proof of the Kan extension see [Ran92, Prop. 13.4]. Below we repeat the proofs for the Ω-spectrum property and the identification of the homotopy groups.

For the Ω-spectrum property we will see that in fact $(\mathbf{L}^{n+1}(\Lambda))^{(k)}$ and $(\Omega\mathbf{L}^n(\Lambda))^{(k)}$ are different descriptions of the same Δ-set . Be aware that the indexing is reversed compared to the usual way, namely, if $\mathbf{E}$ is any of the spectra above we have $\mathbf{E}_{n+1} \simeq \Omega\mathbf{E}_n$.

Let Λ be an algebraic bordism category and K a finite pointed Δ-set. There is a one-to-one correspondence between m-dimensional simplices in Δ^k and $(m+1)$-dimensional simplices in $\Omega\Delta^k$ realized by mapping a simplex $\sigma := \{i_0, \ldots, i_m\}$ in Δ^k to $\widetilde{\sigma} := \{i_0, \ldots, i_m, m+1\}$ in $\Omega\Delta^k$. Here is an example for $k = 1$:



This leads to an isomorphism between $\mathbf{L}^{n+1}(\Lambda)$ and $\Omega\mathbf{L}^n(\Lambda)$ as follows. Let (C, φ) be an $(n+1)$-symmetric chain complex in $\Lambda^L\Delta^k$ which is a k-simplex in $\mathbf{L}^{n+1}(\Lambda)$. Then (C', φ'), defined by

$$(C'(\widetilde{\sigma}), \varphi'(\widetilde{\sigma}))) := \begin{cases} (C(\sigma), \varphi(\sigma)) & \text{if } \sigma \in \Delta^k \\ (0,0) & \text{if } \sigma \in \partial_0\Delta^{k+1} \cup \{k+1\}\,, \end{cases}$$

is a symmetric chain complex in $\Lambda^L(\Delta^{k+1}, \partial_0\Delta^{k+1} \cup \{k+1\}) = \Lambda^L(\Omega\Delta^k)$, which is a k-simplex in $\Omega\mathbf{L}^n(\Lambda)$. Because of the dimension shift of the underlying simplices, the symmetric structure $\varphi'_0(\widetilde{\sigma})\colon C'^{n+|\widetilde{\sigma}|}(\widetilde{\sigma}) = C^{n+1+|\sigma|}(\sigma) \xrightarrow{\varphi_0(\sigma)} C(\sigma) = C'(\widetilde{\sigma})$ is now n-dimensional on C'. To make this clear: φ was $(n+1)$-dimensional, meaning $(n+1)$-dimensional on 0-simplices. Then φ' is trivial on all 0-simplices and $(n+1)$-dimensional on 1-simplices and hence considered as an n-dimensional structure on the whole chain complex C'.

The proof of $\pi_n(\mathbf{L}^\bullet(\Lambda)) = \mathbf{L}^n(\Lambda)$ is essentially the same story. A k-simplex in $\pi_n(\mathbf{L}^m(\Lambda))$ is an m-dimensional symmetric chain complex (C, φ) in $\Lambda^L K$ with $C(\sigma) = 0$ for all $\sigma \in K$ with $|\sigma| < n$. There is only one non-trivial column chain complex in C, namely $C(\tau)$ for the top-dimensional simplex τ with $|\tau| = n$. This means that $(C(\tau), \varphi(\tau))$ is an $(n+m)$-dimensional symmetric chain complex in Λ. The homotopy relation corresponds to the cobordism relation; hence $\pi_n(\mathbf{L}^\bullet(\Lambda)) = \pi_{n+m}(\mathbf{L}^{-m}(\Lambda)) = \mathbf{L}^n(\Lambda)$. □

Room service 1231

$\boxed{\Lambda^L X}$ for $\Lambda = (\mathbb{A}, \mathbb{C}, \mathbb{P})$ denotes $(\mathbb{A}^*X, \mathbb{C}^LX, \mathbb{P}^LX, (T^*, e^*))$, the X-mosaicked algebraic bordism category of $\Lambda = (\mathbb{A}, \mathbb{C}, \mathbb{P})$ with *local Poincaré* duality.

$\boxed{\mathbb{A}^*X}$ additive category of X-based objects in $\mathbb{A}$, i.e.

$$\mathrm{Obj}_{\mathbb{A}^*X} = \Big\{ \sum_{\sigma \in X} M_\sigma \mid M_\sigma \in \mathbb{A}, M_\sigma = 0 \text{ except for finitely many } \sigma \Big\},$$

$$\mathrm{Mor}_{\mathbb{A}^*X}\big(\sum_{\tilde\sigma \in X} M_{\tilde\sigma}, \sum_{\tilde\tau \in X} N_{\tilde\tau}\big) = \Big\{ \sum_{\substack{\tau \le \sigma \\ \sigma,\tau \in X}} f_{\tau,\sigma} \mid (f_{\tau,\sigma} \colon M_\sigma \to N_\tau) \in \mathrm{Mor}_{\mathbb{A}}(M_\sigma, N_\tau) \Big\},$$

where X is a simplicial complex.

$\boxed{\mathbb{C}^L X := \{C \text{ chain complex in } \mathbb{A}^*X \mid C(\sigma) \in \mathbb{C} \text{ for all } \sigma \in X\}}$ for a category $\mathbb{C}$ of chain complexes in $\mathbb{A}$.

$\boxed{T^*}$ defined for a chain duality $T \colon \mathbb{A} \to \mathbb{B}(\mathbb{A})$ as the mosaicked chain duality $\mathbb{A}^*X \to \mathbb{B}(\mathbb{A}^*X)$ with $(T^*(\sum_{\sigma \in X} M_\sigma))_r(\tau) = (T(\bigoplus_{\tau \ge \tilde\tau} M_{\tilde\tau}))_{r+|\tau|}$.

$\boxed{\mathbf{L}^\bullet(\Lambda), \mathbf{L}_\bullet(\Lambda), \mathbf{NL}^\bullet(\Lambda)}$ Ω-spectra of pointed Kan Δ-sets defined for an algebraic bordism category Λ by

$$\mathbf{L}^\bullet(\Lambda) = \Big\{\mathbf{L}^n(\Lambda) \quad \mid n \in \mathbb{Z}, \mathbf{L}^n(\Lambda)^{(k)} = \{n\text{-dim. } (C,\varphi) \text{ in } \Lambda^L\Delta^k\}\Big\},$$

$$\mathbf{L}_\bullet(\Lambda) = \Big\{\mathbf{L}_n(\Lambda) \quad \mid n \in \mathbb{Z}, \mathbf{L}_n(\Lambda)^{(k)} = \{n\text{-dim. } (C,\psi) \text{ in } \Lambda^L\Delta^k\}\Big\},$$

$$\mathbf{NL}^\bullet(\Lambda) = \Big\{\mathbf{NL}^n(\Lambda) \mid n \in \mathbb{Z}, \mathbf{NL}^n(\Lambda)^{(k)} = \{n\text{-dim. } (C,(\varphi,\gamma,\chi)) \text{ in } \Lambda^L\Delta^{k+n}\}\Big\};$$

face maps are induced by face inclusions $\partial_i \colon \Delta^{k-1} \to \Delta^k$, base point is the 0-chain complex.

$\boxed{\Delta\text{-set}}$ a simplicial set without degeneracies, i.e. a functor $\Delta \to \mathbf{Sets}$, where Δ is the category of finite sets $\{0,\ldots,n\}, n \ge 0$ and order-preserving injections and **Sets** the category of sets and functions.

$\boxed{\emptyset}$ the base point simplex in a pointed Δ-set given in every dimension with $\partial_i\emptyset = \emptyset$.

$\boxed{\Delta\text{-map } f \colon K \to L}$ a natural transformation, i.e. a collection $\{f_n \colon K^{(n)} \to L^{(n)} \mid n \ge 0\}$ such that $\partial_i \circ f_n = \partial_i \circ f_{n+1}$.

$\boxed{\text{Kan}}$ is what a Δ-set X is called if every map $\Lambda^n_i \to X$ extends to a map $\Delta^n \to X$; this property is necessary to do homotopy theory on Δ-sets .

$\boxed{\Lambda^n_i := \Delta^n - ((\Delta^n)^{(n)} \cup \partial_i\Delta^n)}$ the subcomplex of Δ^n obtained by removing the interior of Δ^n and a single face of Δ^n.

$\boxed{\mathbf{E}}$ an Ω-spectrum of pointed Kan Δ-sets, like $\mathbf{L}^\bullet(\Lambda), \mathbf{L}_\bullet(\Lambda), \mathbf{NL}^\bullet(\Lambda)$, is a sequence $\mathbf{E}_n$ of pointed Kan Δ-sets together with homotopy equivalences $\mathbf{E}_n \xrightarrow{\simeq} \Omega\mathbf{E}_{n+1}$. Note that the indexing is reversed compared to the usual convention.

$\boxed{\Omega K := K^{S^1}}$ the loop Δ-set; can be expressed as the Δ-set with n-simplices $\{\sigma \in K^{(n+1)} \mid \partial_0\partial_1 \ldots \partial_n\sigma = \emptyset \in K^{(0)}, \partial_{n+1}\sigma = \emptyset \in K^{(n)}\}$.

S^n the pointed Δ-set with base simplices in all dimensions and only one additional simplex in dimension n.

K^L the function Δ-set with n-simplices the Δ-maps $K \otimes \Delta^n \to L$; face maps are induced by $\partial_i : \Delta^n \to \Delta^{n-1}$.

$K \otimes L$ the geometric product Δ-set with one p-simplex for each equivalence class of triples

$$(m\text{-simplex } \sigma \in K, n\text{-simplex } \tau \in L, p\text{-simplex } \rho \in \Delta^m \times \Delta^n),$$

subject to the equivalence relation generated by $(\sigma, \tau, \rho) \sim (\sigma', \tau', \rho')$ if there exist Δ-maps $f : \Delta^m \to \Delta^{m'}, g : \Delta^n \to \Delta^{n'}$ such that $\sigma = f^*\sigma', \tau = g^*\tau', (f \times g)_*(\rho) = \rho'$.

$K \times L$ the product for ordered simplicial complexes with 0-simplices $K^{(0)} \times L^{(0)}$ and vertices $(a_0, b_0), \ldots, (a_n, b_n)$ span an n-simplex if and only if $a_0 \leq \ldots \leq a_n, b_0 \leq \ldots \leq b_n$ and $(a_r, b_r) \neq (a_{r+1}, b_{r+1})$.

$\pi_n(\mathbf{E}) := \pi_{n+k}(\mathbf{E}_{-k})$ for $n, k \in \mathbb{Z}, n + k \geq 0$.

$\pi_n(K) = [\partial\Delta^{n+1}, K]$ the pointed homotopy groups; can be expressed as the set of equivalence classes $\{\sigma \in K^{(n)} \mid \partial_i\sigma = \emptyset, 0 \leq i \leq n\}$ with $\sigma \sim \tau$ if there exists $\rho \in K^{(n+1)}$ with faces $\partial_0\rho = \sigma$, $\partial_1\rho = \tau$ and $\emptyset$ otherwise. The composition is defined by $\sigma_0 \cdot \sigma_1 = \tau$ if there is a $\rho \in K^{(n+1)}$ with faces σ_0, σ_1, τ and $\emptyset$ otherwise.

homotopy of Δ-maps a homotopy between two Δ-maps $f_0, f_1 : K \to L$ is a Δ-map $h : K \otimes \Delta^1 \to L$ with $h(\sigma \otimes i) = f_i(\sigma) \in L^{(n)}$ $(\sigma \in K^{(n)}, i = 0, 1)$.

$K \wedge L = K \otimes L/(K \otimes \emptyset_L \cup \emptyset_K \otimes L)$ the smash product of pointed Δ-sets.

$|K| := (\coprod_{n \geq 0} \Delta^n \times K^{(n)}/_\sim)$ the geometric realization of a simplicial complex K with $\sim$ the equivalence relation generated by $(a, \partial_i\sigma) \sim (\delta_i a, \sigma)$ $\quad (a \in \Delta^{n-1}, \sigma \in K^{(n)})$ with $\delta_i : \Delta^{n-1} \to \Delta^n$ $\quad (0 \leq i \leq n)$ the inclusion of the i-th face.

1232 L-spectra and smash products

Porter

We only give the foundational idea as suggested in [Ran92, p.140]. For a complete account we refer to [LM09]. There is also a different approach by Weiss [Wei92] which proves that the homology groups $H_n(X; \mathbf{L}_\bullet(\mathbb{Z})), H_n(X; \mathbf{L}^\bullet(\mathbb{Z})), H_n(X; \mathbf{NL}^\bullet(\mathbb{Z}))$ are induced by homotopy invariant and excisive functors from X to $\mathbf{L}_\bullet(\Lambda_L X)$, $\mathbf{L}^\bullet(\Lambda_L X)$ and $\mathbf{NL}^\bullet(\Lambda_N X)$ and hence they are natural in X.

1232 L-spectra and smash products [LM09, Remark 16.2][Ran92, Prop. 13.7]
Let K be finite simplicial complex and Λ an algebraic bordism category. Then

$$K_+ \wedge \mathbf{L}_\bullet(\Lambda) \simeq \mathbf{L}_\bullet(\Lambda_L K)$$
$$K_+ \wedge \mathbf{L}^\bullet(\Lambda) \simeq \mathbf{L}^\bullet(\Lambda_L K)$$
$$K_+ \wedge \mathbf{NL}^\bullet(\Lambda) \simeq \mathbf{NL}^\bullet(\Lambda_N K)$$

Proof 1232

We only deal with the quadratic case. The symmetric and normal cases are analog. We use an embedding $i\colon K \to \partial\Delta^{m+1}$ $(m \in \mathbb{N}$ large enough), the supplement construction $\overline{K}$ and the simplicial complex Σ^m in order to decompose the equivalence we want to prove into the following three equivalences

$$\mathbf{L}_\bullet(\Lambda_L K) \overset{(a)}{\cong} \mathbf{L}_\bullet(\Lambda^L(\Sigma^m, \Phi(\overline{K}))) \overset{(b)}{\simeq} \mathbf{L}_\bullet(\Lambda)^{(\Sigma^m, \Phi(\overline{K}))} \overset{(c)}{\simeq} K_+ \wedge \mathbf{L}_\bullet(\Lambda).$$

(a) $\Lambda_L K \cong \Lambda^L(\Sigma^m, \Phi(\overline{K}))$
There is a one-to-one correspondence between k-simplices σ in $\partial\Delta^{m+1}$ and $(m-k)$-simplices in Σ^m by definition. It can be refined to a one-to-one correspondence between k-simplices of K and $(m-k)$-simplices of $\Sigma^m \setminus \Phi(\overline{K})$. The equivalence follows from the property that $\sigma \leq \tau$ in K if and only if $\sigma^* \geq \tau^*$ in $\Sigma^m \setminus \Phi(\overline{K})$ and the analogous opposite between the chain dualities T_* and T^*.

(b) $\mathbf{L}_\bullet(\Lambda^L K) \simeq \mathbf{L}_\bullet(\Lambda)^{K_+}$
The basic idea is that in the category $\Lambda^L K$ every morphism $M(\sigma) \to N(\tau)$ has the property that the target simplex τ is contained in the source simplex σ. Hence we can split an n-quadratic chain complex (C, ψ) in $\Lambda^L K$ into a collection of n-quadratic chain complexes $\{(C_\sigma, \psi_\sigma) \in \Lambda^L \Delta^{|\sigma|} \mid \sigma \in K\}$ over standard simplices such that the (C_σ, ψ_σ) are related to each other in the same way the corresponding simplices are related to each other in K, i.e. $C_\sigma(\partial_i\sigma) = C_{\partial_i\sigma}(\partial_i\sigma)$ for all $\sigma \in K$. The complex (C_σ, ψ_σ) is a $|\sigma|$-simplex in $\mathbf{L}_n(\Lambda)$ and the compatibility conditions are contained in the notion of Δ-maps. Hence (C, ψ) is the same as a Δ-map $f\colon K_+ \to \mathbf{L}_n(\Lambda)$ with $f(\sigma) = (C_\sigma, \psi_\sigma)$.

In [LM09] it is shown that this leads to a homotopy equivalence $\mathbf{L}_\bullet(\Lambda^L K) \simeq \mathbf{L}_\bullet(\Lambda)^{K_+}$ by identifying a k-simplex in $\mathbf{L}_n(\Lambda^L K)$ which is an n-quadratic chain complex in $\Lambda^L K^L \Delta^n$ with an n-quadratic chain complex in $\Lambda^L(K \otimes \Delta^n)$ which is by the argument above a Δ-map $K \otimes \Delta^n \to \mathbf{L}_n$ and hence a k-simplex in $\mathbf{L}_n(\Lambda)^{K_+}$.

(c) $\mathbf{L}_\bullet(\Lambda)^{(\Sigma^m, \Phi(\overline{K}))} \simeq K_+ \wedge \mathbf{L}_n(\Lambda)$
This equivalence is a spectrum version of the isomorphism of 1312. □ 1312→p.74

Room service 1232

For details on Δ-sets and spectra see the room service of the previous section.

$\boxed{\Sigma^m}$ a simplicial complex dual to $\partial\Delta^{m+1}$ with one k-simplex σ^* for every $(m-k)$-simplex σ in $\partial\Delta^{m+1}$ and face maps

$$\partial_i \colon (\Sigma^m)^{(k)} := \left\{\sigma^* \mid \sigma \in (\partial\Delta^{m+1})^{(m-k)}\right\} \to (\Sigma^m)^{(k-1)}; \quad \sigma^* \mapsto (\delta_i\sigma)^* \qquad (0 \leq i \leq k)$$

i.e. if the $(m-k)$-simplex σ is spanned by the vertices $\{0, 1, \ldots, m+1\} \setminus \{j_0, \ldots, j_k\}$ than σ^* is spanned by the vertices $\{j_0, \ldots, j_k\}$ and $\partial_i(\sigma^*) = (\delta_i\sigma)^* = (\sigma \cup \{j_i\})^* = \sigma^* \setminus \{j_i\}$.

$\boxed{\sigma^* \in \Sigma^m}$ the dual k-simplex with $\partial_i\sigma^* = (\delta_i\sigma)^*$ for a $(m-k)$-simplex $\sigma \in \partial\Delta^{m+1}$.

$\boxed{\overline{K} := K \div \partial\Delta^{m+1}}$ the supplement of K embedded into $\partial\Delta^{m+1}$ for $m \in \mathbb{N}$ large enough.

$\boxed{L \div K := \{\sigma' \in L' \mid \text{ no vertex of } \sigma' \text{ lies in } K'\} = \bigcup_{\sigma \in L, \sigma \notin K} D(\sigma, L) \subset L'}$ the supplement of a subcomplex K of a simplicial complex L, i.e. the subcomplex of L' spanned by all of the vertices of $L' - K'$.

$\boxed{\Phi \colon (\partial\Delta^{m+1})' \xrightarrow{\cong} (\Sigma^m)'}$ an isomorphism of simplicial complexes that maps dual cells in $\partial\Delta^{m+1}$ to simplices in Σ^m. For more details see 1311.

$\boxed{T_*}$ defined for a chain duality $T \colon \mathbb{A} \to \mathbb{B}(\mathbb{A})$ as the mosaicked chain duality $\mathbb{A}_*K \to \mathbb{B}(\mathbb{A}_*K)$ with $(T_*(\sum_{\sigma \in K} M_\sigma))_r(\tau) = (T(\bigoplus_{\tau \leq \tilde{\tau}} M_{\tilde{\tau}}))_{r-|\tau|}$.

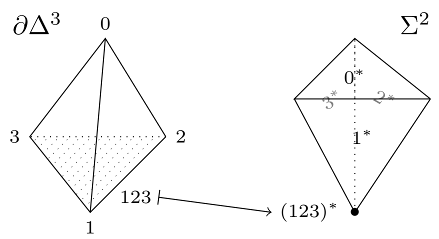
$\boxed{T^*}$ defined for a chain duality $T \colon \mathbb{A} \to \mathbb{B}(\mathbb{A})$ as the mosaicked chain duality $\mathbb{A}^*K \to \mathbb{B}(\mathbb{A}^*K)$ with $(T^*(\sum_{\sigma \in K} M_\sigma))_r(\tau) = (T(\bigoplus_{\tau \geq \tilde{\tau}} M_{\tilde{\tau}}))_{r+|\tau|}$.

131 $E_\bullet$-cycles

Porter

The outcome of this statement is an explicit and handy description of elements in $H_n(X; \mathbf{E})$ called $\mathbf{E}$-cycles. We think of homology as cohomology of the S-dual in order to get Δ-maps as elements. Then a subtle combinatorial construction of the S-dual enables us to define the maps actually on the simplices of X instead of the S-dual of X. More precisely, in order to determine an element in $H_n(X; \mathbf{E})$ we have to assign for each $\sigma \in X$ an $(m - |\sigma|)$-dimensional simplex in $\mathbf{E}_{n-m}$ such that a reversed boundary relation is satisfied.

The crucial ingredient for this construction is the manner in which the two simplicial complexes $\partial\Delta^{m+1}$ and Σ^m are related to each other. First, there is the duality relation coming immediately from the definition of Σ^m; that is Σ^m has one k-simplex σ^* for each $(m-k)$-simplex σ of $\partial\Delta^{m+1}$ and σ^* is a face of τ^* if and only if τ is a face of σ.



But when we pass on to the barycentric subdivision of $\partial\Delta^{m+1}$ and Σ^m, we get another relation: An isomorphism Φ of simplicial complexes which identifies dual cells in $\partial\Delta^{m+1}$ with simplices in the non-subdivided complex Σ^m. The reference for this is [Ran92, §12].

131 (15) $E_\bullet$-cycles [Ran92, Prop. 12.8]
Let X be a finite simplicial complex and $m \in \mathbb{N}$ large enough such that there is an embedding of X into $\partial\Delta^{m+1}$. Then an n-dimensional $\mathbf{E}$-cycle $[K]^{\mathbf{E}}$ of X in $\partial\Delta^{m+1}$ defines an element in $H_n(X;\mathbf{E})$.

1311 (132, 15) Simplicial dual complex [Ran92, §12]
There is an isomorphism of simplicial complexes $\Phi\colon (\partial\Delta^{m+1})' \xrightarrow{\cong} (\Sigma^m)'$ such that for each $\sigma^* \in \Sigma^m$ we have

$$\Phi(D(\sigma, \partial\Delta^{m+1})) = \sigma^*.$$

1312 (15) Simplicial description of homology [Ran92, Proposition 12.4]
Let $\mathbf{E}$ be an Ω-spectrum of Kan Δ-sets and K a finite simplicial complex. Then for $m \in \mathbb{N}$ large enough such that there is an embedding $K \subset \partial\Delta^{m+1}$ we have

$$S\colon H_n(K;\mathbf{E}) \cong H^{m-n}(\Sigma^m, \Phi(\overline{K}); \mathbf{E}).$$

Proof 131

We use the dual simplicial complex of 1311 to define, with the data of an n-dimensional $\mathbf{E}$-cycle $[K]^{\mathbf{E}}$, a Δ-map 1311→p.73

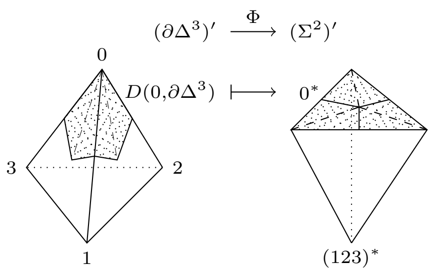
$$(\Sigma^m, \Phi(\overline{X})) \longrightarrow \mathbf{E}_{n-m}; \quad \sigma^* \mapsto \begin{cases} [X]^{\mathbf{E}}(\sigma) & \sigma \in K \\ \emptyset & \sigma \notin K \end{cases}$$

and hence an element in $H^{m-n}(\Sigma^m, \Phi(\overline{X}); \mathbf{E})$. By 1312, this is via S-duality isomorphic to $H_n(X;\mathbf{E})$. □ 1312→p.74

Remark. We can define two $\mathbf{E}$-cycles x_0 and x_1 to be cobordant if there is a Δ-map $y\colon (\Sigma^m, \Phi(\overline{X})) \otimes \Delta^1 \to (\mathbf{E}_{n-m}, \emptyset)$ such that $y(\sigma \times i) = x_i(\sigma) \in \mathbf{E}_{n-m}^{m-|\sigma|}$ for $\sigma \in X$ and $i = 0, 1$. This corresponds to the homotopy relation of Δ-maps and thus there is a one-to-one relation between cobordism classes of n-dimensional $\mathbf{E}$-cycles in X and elements of $H_n(X;\mathbf{E})$.

Proof 1311 (Simplicial dual complex)

Define $\Phi\colon (\partial\Delta^{m+1})' \to (\Sigma^m)'$ for an n-simplex $\sigma = \{\hat\sigma_0, \dots, \hat\sigma_n\}$ to be the n-simplex $\Phi(\sigma) = \{\hat\sigma_0{}^*, \dots, \hat\sigma_n{}^*\}$ where $\hat\sigma_0 < \dots < \hat\sigma_n$ and hence $\hat\sigma_0{}^* > \dots > \hat\sigma_n{}^*$. Therefore, a dual cell $D(\sigma, K)$, consisting of all simplices $\{\hat\sigma_0, \dots, \hat\sigma_p\}$ such that $\sigma \le \sigma_0 < \sigma_1 < \dots < \sigma_p$ gets mapped to the subcomplex $(\sigma^*)'$ consisting of the simplices $\{\hat\sigma_0{}^*, \dots, \hat\sigma_p{}^*\}$ with $\sigma \ge \sigma_0{}^* > \sigma_1{}^* > \dots > \sigma_p{}^*$. See the picture below for an example for $m = 2$. □



Remark. The isomorphism Φ induces a homeomorphism of geometric realizations with the property $|\Phi(D(\sigma, K))| \cong |\Delta^{m-|\sigma|}|$.

Proof 1312 (Simplicial description of homology)

We think of K embedded in Σ^m via the isomorphism Φ. The complex $\Sigma^m/\Phi(\overline{K})$ is the quotient of Σ^m by the complement of a neighborhood of K. This is a well-known construction of an m-dimensional S-dual of K, which is proved in detail for example in [Whi62, p. 265]. The construction there provides an explicit simplicial construction of the reduced diagonal map $\Delta' \colon \Sigma^m \to K_+ \wedge (\Sigma^m/\Phi(\overline{K}))$. On spectrum level, after reindexing the spectrum $\mathbf{F}_n := (\mathbf{E}_{n-m}, \emptyset)^{(\Sigma^m, \Phi(\overline{K}))}$ it gives a map of Δ-sets

$$\mathbf{F}_n \to (K_+ \wedge \mathbf{E}_{n-m})^{\Sigma^m} \simeq \Omega_m(K_+ \wedge \mathbf{E}_{n-m}) \simeq (K_+ \wedge \mathbf{E})_n$$

that maps a p-simplex $\sigma \colon \Sigma^m \wedge \Delta^p{}_+ \to \mathbf{E}_{n-m}$ to the composition

$$\Sigma^m \wedge \Delta^p{}_+ \xrightarrow{\Delta' \wedge \mathrm{id}} K_+ \wedge (\Sigma^m/\Phi(\overline{K})) \wedge \Delta^p{}_+ \xrightarrow{\mathrm{id} \wedge \sigma} K_+ \wedge \mathbf{E}_{n-m}$$

and induces the Alexander S-duality isomorphism on the generalized homology groups.

$$H^{m-n}(\Sigma^m, \Phi(\overline{K}); \mathbf{E}) = \pi_n(\mathbf{F}) \cong \pi_n(K_+ \wedge \mathbf{E}) = H_n(K; \mathbf{E})$$

See [Ran92, §12, especially Remark 12.5] for more details. □

Room service 131, 1311 and 1312

$\mathbf{E}$ an Ω ring spectrum of Kan Δ-sets, e.g. $\mathbf{NL}^\bullet$ or $\mathbf{L}^\bullet$.

$\Omega K := K^{S^1}$ the loop Δ-set; can be expressed as the Δ-set with n-simplices $\{\sigma \in K^{(n+1)} \mid \partial_0\partial_1 \dots \partial_n\sigma = \emptyset \in K^{(0)}, \partial_{n+1}\sigma = \emptyset \in K^{(n)}\}$.

S^n the pointed Δ-set with base simplices in all dimensions and only one additional simplex in dimension n.

$[K]^{\mathbf{E}} \in H_n(K; \mathbf{E})$ an n-dimensional $\mathbf{E}$-cycle of a simplicial complex $K \subset \partial\Delta^{m+1}$ defined by a collection $\left\{[K]^{\mathbf{E}}(\sigma) \in \mathbf{E}_{n-m}^{(m-|\sigma|)} \mid \sigma \in K\right\}$ such that $\partial_i[K]^{\mathbf{E}}(\sigma) = [K]^{\mathbf{E}}(\delta_i\sigma)$ if $\delta_i\sigma \in K$ and $\emptyset$ otherwise.

$H_n(K, \mathbf{E}) := \pi_n(K_+ \wedge \mathbf{E}) = \lim_k \pi_{n+k}(K_+ \wedge \mathbf{E}_{-k})$ the $\mathbf{E}$-homology groups for a locally finite Δ-set K.

$H^n(K, \mathbf{E}) := \pi_{-n}(\mathbf{E}^{K_+}) = [K_+, \mathbf{E}_{-n}]$ the $\mathbf{E}$-cohomology groups for a locally finite Δ-set K.

K_+ for a Δ-set K the pointed Δ-set with n-simplices $K^{(n)} \cup \{\emptyset\} \quad (n \geq 0)$.

$\emptyset$ the base point simplex in a pointed Δ-set given in every dimension with $\partial_i\emptyset = \emptyset$.

$K \wedge L = K \otimes L/(K \otimes \emptyset_L \cup \emptyset_K \otimes L)$ the smash product of pointed Δ-sets.

$\overline{K} := K \div \partial\Delta^{m+1}$ the supplement of K embedded into $\partial\Delta^{m+1}$ for $m \in \mathbb{N}$ large enough.

$L \div K := \{\sigma' \in L' \mid \text{ no vertex of } \sigma' \text{ lies in } K'\} = \bigcup_{\sigma \in L, \sigma \notin K} D(\sigma, L) \subset L'$ the supplement of a subcomplex K of a simplicial complex L, i.e. the subcomplex of L' spanned by all of the vertices of $L' - K'$.

$\Phi\colon (\partial\Delta^{m+1})' \xrightarrow{\cong} (\Sigma^m)'$ an isomorphism of simplicial complexes that maps dual cells in $\partial\Delta^{m+1}$ to simplices in Σ^m. For more details see 1311.

$\partial\Delta^{m+1}$ the boundary of the standard simplex.

Σ^m a simplicial complex dual to $\partial\Delta^{m+1}$ with one k-simplex σ^* for every $(m-k)$-simplex σ in $\partial\Delta^{m+1}$ and face maps

$$\partial_i\colon (\Sigma^m)^{(k)} := \left\{\sigma^* \mid \sigma \in (\partial\Delta^{m+1})^{(m-k)}\right\} \to (\Sigma^m)^{(k-1)}; \quad \sigma^* \mapsto (\delta_i\sigma)^* \qquad (0 \leq i \leq k)$$

i.e. if the $(m-k)$-simplex σ is spanned by the vertices $\{0, 1, \ldots, m+1\} \setminus \{j_0, \ldots, j_k\}$ than σ^* is spanned by the vertices $\{j_0, \ldots, j_k\}$ and $\partial_i(\sigma^*) = (\delta_i\sigma)^* = (\sigma \cup \{j_i\})^* = \sigma^* \setminus \{j_i\}$.

$\sigma^* \in \Sigma^m$ the dual k-simplex with $\partial_i\sigma^* = (\delta_i\sigma)^*$ for a $(m-k)$-simplex $\sigma \in \partial\Delta^{m+1}$.

$\delta_i\colon (\partial\Delta^{m+1})^{(m-k)} \to (\partial\Delta^{m+1})^{(m-k+1)}$ given by

$$\sigma = \{0, \ldots, m+1\} \setminus \{j_0, \ldots, j_k\} \mapsto \sigma \cup \{j_i\} \qquad (j_0 < j_1 < \cdots < j_k)$$

where $\{0, 1, \ldots, m+1\}$ are the vertices of $\partial\Delta^{m+1}$.

132 Normal cycles

Porter

In this room we construct an important geometric ingredient for the total surgery obstruction. We prove that we can subdivide a Poincaré space X in pieces of normal spaces in such a way that we can apply the normal signature in order to obtain a mosaicked normal signature $\mathrm{sgn}_X^{\mathbf{NL}^\bullet}$ producing something in the algebraic bordism category $\Lambda_G X$ that will be used for the definition of the total surgery obstruction. The existence of such a signature was claimed in [Ran92, Example 9.12] and a few more details are given in [Ran13, Errata for p.103]. The details of the geometric construction below have been already published in [KMM13, section 11].

132 (15) Normal cycles [KMM13, Construction 11.1, 11.2 and 11.3]
Let X be a finite simplicial Poincaré space of dimension n embedded into $\partial\Delta^{m+1}$ for an $m \gg n$ large enough. There is an n-dimensional $\mathbf{\Omega}_N$-cycle, i.e. a collection of assignments

$$[X]^{\mathbf{\Omega}_\bullet^N} = \Big\{\sigma \mapsto x(\sigma) = (X(\sigma), \nu(\sigma), \rho(\sigma)) \in (\mathbf{\Omega}_{n-m}^N)^{(m-|\sigma|)} \;\Big|\; \sigma \in X,\ \partial_i x(\sigma) = x(\delta_i\sigma) \text{ for all } \sigma, \delta_i\sigma \in X, 0 \leq i \leq m - |\sigma|\Big\}.$$

1311 (132, 15) Simplicial dual complex [Ran92, §12]
There is an isomorphism of simplicial complexes $\Phi\colon (\partial\Delta^{m+1})' \xrightarrow{\cong} (\Sigma^m)'$ *such that for each* $\sigma^* \in \Sigma^m$ *we have*

$$\Phi(D(\sigma, \partial\Delta^{m+1})) = \sigma^*.$$

Proof 132

Let (X, ν_X, ρ) be the associated normal space of the Poincaré space X where ν_X is the Spivak normal fibration of X with fiber S^{m-n} and $\rho\colon S^m \to \mathrm{Th}(\nu_X)$ the Pontrjagin-Thom collapse map. We have to construct for $\sigma \in X$ an $(n-|\sigma|)$-dimensional normal space $x(\sigma) = (X(\sigma), \nu_X(\sigma), \rho(\sigma))$. The first two entries of the triple $x(\sigma)$ are defined as follows

$$\begin{aligned} X(\sigma) &= |D(\sigma, X)|, \\ \nu(\sigma) &= \nu_X \circ \mathrm{incl}\colon X(\sigma) \hookrightarrow X \to \mathrm{BSG}(m-n-1). \end{aligned}$$

For the definition of the remaining map $\rho(\sigma)\colon \Delta^{m-|\sigma|} \to \mathrm{Th}(\nu(\sigma))$ we need some ideas from [BM52, §2] about supplements of simplicial complexes together with the crucial insight from [Ran92, p.123] into how dual cells in $\partial\Delta^{m+1}$ can be identified with simplices in an appropriate dual complex Σ^m. We do not need these constructions in all generality but only the special case where X is embedded into $\partial\Delta^{m+1}$ for some $m > 0$. For simplicity we abbreviate the supplement $\partial\Delta^{m+1} \div X$ by $\overline{X}$.

Every simplex of $\partial\Delta^{m+1}$ which is neither in X' nor in $\overline{X}$ is the join of a simplex of X' and a simplex of $\overline{X}$. Thus there is an embedding $|\partial\Delta^{m+1}| \hookrightarrow |X| * |\overline{X}|$. Using this embedding we get a description of $|\partial\Delta^{m+1}|$ in coordinates (x, t, y) where $x \in |X|$, $y \in |\overline{X}|$ and $0 \leq t \leq 1$. If $t = 0$, then $(x, t, y) \in |X|$, while if $t = 1$, then $(x, t, y) \in |\overline{X}|$. Let

$$\begin{aligned} N \coloneqq N(X') &\coloneqq \left\{ (x, t, y) \in |\partial\Delta^{m+1}| \;\middle|\; t \leq \frac{1}{2} \right\}, \\ \overline{N} \coloneqq N(\overline{X}) &\coloneqq \left\{ (x, t, y) \in |\partial\Delta^{m+1}| \;\middle|\; t \geq \frac{1}{2} \right\}. \end{aligned}$$

Then N and $\overline{N}$ are closed neighborhoods of $|X|$ and $|\overline{X}|$ and $|\partial\Delta^{m+1}| = N \cup \overline{N}$. There are obvious deformation retractions $r\colon N \to |X|$ and $\overline{r}\colon \overline{N} \to |\overline{X}|$.

For m large enough the homotopy fiber of the projection map $\partial r\colon \partial N = N \cap \overline{N} \to X$ is homotopy equivalent to S^{m-n-1} and the associated spherical fibration is the Spivak normal fibration ν_X. In more detail, there is a (D^{m-n}, S^{m-n-1})-fibration $p\colon (D(\nu_X), S(\nu_X)) \to X$ and a homotopy equivalence of pairs $h\colon (N, \partial N) \to (D(\nu_X), S(\nu_X))$ such that the following diagram commutes

$$\begin{array}{ccc} (N, \partial N) & \xrightarrow{\quad h \quad} & (D(\nu_X), S(\nu_X)) \\ & {\scriptstyle r}\searrow \quad \swarrow{\scriptstyle p} & \\ & X & \end{array}$$

The map p is now an honest fibration.

Let $c\colon \overline{N} \to *$ be the collapse map. We recover ρ_X by

$$\rho_X\colon S^m \cong |\partial\Delta^{m+1}| \cong N \cup \overline{N} \xrightarrow{h \cup c} D(\nu_X) \cup \{*\} \cong \mathrm{Th}(\nu_X).$$

Now, we use the data above to get a dissection of ρ_X in accordance with the dissection $\bigcup_{\sigma\in X} X(\sigma)$ of X. For $\sigma \in X$ let

$$N(\sigma) := N \cap (X(\sigma) * |\overline{X}|),$$
$$\overline{N}(\sigma) := \overline{N} \cap (X(\sigma) * |\overline{X}|).$$

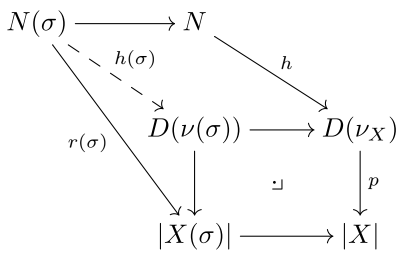
Then we have the dissection $N = \bigcup_{\sigma\in X} N(\sigma)$ and the deformation retraction r restricts for each $N(\sigma)$ to a deformation retraction

$$r(\sigma) = r|_{N(\sigma)} \colon N(\sigma) \to X(\sigma).$$

and $N(\sigma) \cup \overline{N}(\sigma) = |D(\sigma, \partial\Delta^{m+1})|$.

The isomorphism Φ from 1311 gives us already the right source $D(\sigma, \partial\Delta^{m+1}) \cong \Delta^{m-|\sigma|}$ for the map $\rho(\sigma)$. 1311→p.73

For the target $\mathrm{Th}(\nu(\sigma))$ we have to solve the problem that, in general the projections $\partial r(\sigma) \colon \partial N(\sigma) = N(\sigma) \cap \overline{N}(\sigma) \to X(\sigma)$ do not give a spherical fibration and especially not $\nu(\sigma)$. We defined $\nu(\sigma)$ as pullback of ν_X along the inclusion $X(\sigma) \hookrightarrow X$. The associated disc fibration $D(\nu(\sigma))$ is a pullback as indicated by the following diagram.



Since the two compositions $N(\sigma) \to |X(\sigma)| \to |X|$ and $N(\sigma) \to N \to D(\nu_X) \to |X|$ commute we obtain the dashed map $\rho(N(\sigma)) \colon N(\sigma) \to D(\nu(\sigma))$.

Let $c(\sigma) \colon \overline{N}(\sigma) \to \{*\}$ be the collapse map. Finally, we define

$$\rho(\sigma) \colon \Delta^{m-|\sigma|} \cong N(\sigma) \cup \overline{N}(\sigma) \xrightarrow{h(\sigma)\cup c(\sigma)} D(\nu(\sigma)) \cup \{*\} \cong \mathrm{Th}(\nu(\sigma)). \qquad \square$$

Room service 132

(Y, ν, ρ) an n-dimensional normal space consisting of a topological space Y together with an oriented k-dimensional spherical fibration $\nu \colon Y \to \mathrm{BSG}(k)$ and a map $\rho \colon S^{n+k} \to \mathrm{Th}(\nu)$.

$D(\sigma, K)$ dual cell of a simplex σ in a simplicial complex K is the subcomplex of the barycentric subdivision K' defined by

$$D(\sigma, K) = \{\widehat{\sigma}_0\widehat{\sigma}_1 \ldots \widehat{\sigma}_r \mid \sigma \le \sigma_0 < \sigma_1 < \ldots < \sigma_r\}.$$

$L \div K := \{\sigma' \in L' \mid \text{ no vertex of } \sigma' \text{ lies in } K'\} = \bigcup_{\sigma \in L, \sigma \notin K} D(\sigma, L) \subset L'$ the supplement of a subcomplex K of a simplicial complex L, i.e. the subcomplex of L' spanned by all of the vertices of $L' - K'$.

$\overline{K} := K \div \partial\Delta^{m+1}$ the supplement of K embedded into $\partial\Delta^{m+1}$ for $m \in \mathbb{N}$ large enough.

$\Phi\colon (\partial\Delta^{m+1})' \xrightarrow{\cong} (\Sigma^m)'$ an isomorphism of simplicial complexes that maps dual cells in $\partial\Delta^{m+1}$ to simplices in Σ^m. For more details see 1311.

K' the first barycentric subdivision of a simplicial complex K.

$\hat{\sigma}$ the vertex in K' given by the barycenter of the simplex $\sigma \in K$.

$|K| * |L|$ the join of two topological spaces X and Y obtained from $X \times I \times Y$ by identifying $x \times 0 \times Y$ with x for all $x \in X$ and $X \times 1 \times y$ with y for all $y \in Y$ [Whi50, p. 202, III]. Thus each point of $X * Y$ lies on a unique line segment joining a point of X to a point Y.

$[K]^{\mathbf{E}} \in H_n(K; \mathbf{E})$ an n-dimensional $\mathbf{E}$-cycle of a simplicial complex $K \subset \partial\Delta^{m+1}$ defined by a collection $\left\{[K]^{\mathbf{E}}(\sigma) \in \mathbf{E}_{n-m}^{(m-|\sigma|)} \mid \sigma \in K\right\}$ such that $\partial_i [K]^{\mathbf{E}}(\sigma) = [K]^{\mathbf{E}}(\delta_i \sigma)$ if $\delta_i \sigma \in K$ and $\emptyset$ otherwise.

$\mathbf{\Omega}^N_\bullet$ the Ω-spectrum of Kan Δ-sets defined by

$$\begin{aligned}(\mathbf{\Omega}^N_n)^{(k)} = \{(X_{\Delta^k}, \nu, \rho) \mid\ & (n+k)-\text{dimensional normal space } (k+2)\text{-ad, i.e.}\\ & X_{\Delta^k} = (X, \partial_0 X, \ldots, \partial_k X) \text{ s.t. } \partial_0 X \cap \ldots \cap \partial_k X = \emptyset,\\ & \nu\colon X \to \mathrm{BSG}(r) \text{ an } (r-1)\text{-spherical fibration},\\ & \rho\colon \Delta^{n+k+r} \to \mathrm{Th}(\nu) \text{ s.t. } \rho(\partial_i \Delta^{n+k+r}) \subset \mathrm{Th}(\nu|_{\partial_i X}) \quad \}\end{aligned}$$

The face maps $\partial_i\colon (\mathbf{\Omega}^N_n)^{(k)} \to (\mathbf{\Omega}^N_n)^{(k-1)}$ are given by
$\partial_i(X) = (\partial_i X,\ \partial_i X \cap \partial_0 X, \ldots, \partial_i X \cap \partial_{i-1} X,\ \partial_i X \cap \partial_{i+1} X, \ldots, \partial_i X \cap \partial_k X)$.

141 Signature spectra maps

Porter

The L-spectra are constructed using chain complexes over $\mathbb{Z}^*\Delta^k$. So for the signature maps we need mosaicked signatures slightly different from $\mathrm{sgn}_X^{\mathbf{L}^\bullet}$ and $\mathrm{sgn}_X^{\mathbf{NL}/\mathbf{L}^\bullet}$ as constructed for example in A17 or A29. Instead of the dual cells we have to use the simplices itself in the construction. We provide some details for the symmetric case. The normal case is much more complicated and we refer for it to [Wei85b].

141 (13, 15) Signature spectra maps [Ran79]
The relative symmetric and relative normal construction induce maps of spectra

$$\begin{aligned}&\mathrm{sgn}_\Omega^{\mathbf{L}^\bullet}\colon \mathbf{\Omega}_\bullet^{STOP} \to \mathbf{L}^\bullet\langle 0\rangle, \quad \mathrm{sgn}_\Omega^{\mathbf{NL}^\bullet}\colon \mathbf{\Omega}_\bullet^{N} \to \mathbf{NL}^\bullet\langle 1/2\rangle,\\ &\mathrm{sgn}_\Omega^{\mathbf{NL}/\mathbf{L}^\bullet}\colon \Sigma^{-1}\mathbf{\Omega}_\bullet^{\mathrm{N,STOP}} \to \mathbf{NL}\langle 1/2\rangle/\mathbf{L}\langle 0\rangle^\bullet, \quad \textit{and}\\ &\mathrm{sgn}_\Omega^{\mathbf{L}_\bullet}\colon \Sigma^{-1}\mathbf{\Omega}_\bullet^{\mathrm{N,STOP}} \to \mathbf{L}_\bullet\langle 1\rangle.\end{aligned}$$

1411 (161) L-spectra fibration sequences [Ran92, Prop. 15.16]
There are homotopy fibration sequences

$$\mathbf{L}_\bullet \simeq \mathbf{NL}/\mathbf{L}^\bullet \xrightarrow{1+t} \mathbf{L}^\bullet \xrightarrow{J} \mathbf{NL}^\bullet \quad \textit{and} \quad \mathbf{L}_\bullet\langle 1\rangle \simeq \mathbf{NL}\langle 1/2\rangle/\mathbf{L}\langle 0\rangle^\bullet \xrightarrow{1+t} \mathbf{L}^\bullet\langle 0\rangle \xrightarrow{J} \mathbf{NL}^\bullet\langle 1/2\rangle.$$

112 (121, 1411, 164) Quadratic and (normal, Poincaré symmetric) [Ran92, Proposition 2.8 (ii)]
There is the following natural one-to-one correspondence of cobordism classes.

n-dimensional (normal, symmetric) pairs $(f\colon C \to D, (\delta\varphi, \delta\gamma, \delta\chi), \varphi)$	$\xleftrightarrow{1-1}$	$(n-1)$-dimensional quadratic chain complexes (C', ψ')

Additionally, if (C, φ) is Poincaré, then (C', ψ') is Poincaré and vice versa.

Proof 141

(a) $\mathrm{sgn}_\Omega^{\mathbf{L}^\bullet}\colon \mathbf{\Omega}_\bullet^{STOP} \to \mathbf{L}^\bullet\langle 0\rangle$
Let $M_{\Delta^k} = (M, \partial_0 M, \ldots, \partial_k M)$ be an $(n+k)$-dimensional manifold k-ad and denote by M_σ for $\sigma \in \Delta^k$ the boundary component of M_{Δ^k} which corresponds to σ. Let C_{Δ^k} be the chain subcomplex of the singular chain complex $C(M)$ consisting of simplices that respect the k-ad structure such that each singular simplex is contained in some M_σ. It is still chain homotopy equivalent to $C(M)$ but can now be considered to be a chain complex in $\mathbb{Z}^*\Delta^k$ by $C_{\Delta^k}(\sigma) = C(M_\sigma, \partial M_\sigma)$. The dual chain complex $T^*C_{\Delta^k}$ is given by $T^*C_{\Delta^k}(\sigma) = C^{|\sigma|-*}(M_\sigma)$. A generalization of the relative symmetric construction A14 gives a chain map

$$\varphi_{\Delta^k}\colon \Sigma^{-k}C(M, \partial M) \to W^{\%}(C(M)) \text{ over } \mathbb{Z}^*\Delta^k$$

which, applied to the fundamental class $[M] \in C_{n+k}(M, \partial M)$, produces an n-symmetric chain complex $(C_{\Delta^k}, \varphi_{\Delta^k})$ in $\Lambda^L\Delta^k$ because the maps

$$\varphi_{\Delta^k}(\sigma)_0\colon C^{n-k+|\sigma|-*}(M_\sigma) \to C(M_\sigma, \partial M_\sigma)$$

are given by taking the cap products with the fundamental classes $[M_\sigma] \in C_{n-k+|\sigma|}(M_\sigma, \partial M_\sigma)$ and hence are chain homotopy equivalences with contractible mapping cones.

Since the geometric input yield only chain complexes concentrated in non-negative degrees, the connectivity requirement is fulfilled.

(b) $\mathrm{sgn}_\Omega^{\mathbf{NL}^\bullet}\colon \mathbf{\Omega}_\bullet^{N} \to \mathbf{NL}^\bullet\langle 1/2\rangle$
For the second case use the relative normal construction. The full details can be found in [Wei85b, section 7]. Why we can expect to obtain a $1/2$-connective normal chain complex is explained in [Ran92, p. 178].

(c) $\mathrm{sgn}_\Omega^{\mathbf{NL}/\mathbf{L}^\bullet}\colon \Sigma^{-1}\mathbf{\Omega}_\bullet^{\mathrm{N,STOP}} \to \mathbf{NL}\langle 1/2\rangle/\mathbf{L}\langle 0\rangle^\bullet$ and $\mathrm{sgn}_\Omega^{\mathbf{L}_\bullet}\colon \Sigma^{-1}\mathbf{\Omega}_\bullet^{\mathrm{N,STOP}} \to \mathbf{L}_\bullet\langle 1\rangle$
The existence of these maps follows from (a) and (b) and the fibration sequence of 1411. □ 1411→p.79

Proof 1411 (L-spectra fibration sequences)

112→p.61 By 112, we can identify the fiber of the map $\mathbf{L}^\bullet \to \mathbf{NL}^\bullet$ with $\mathbf{L}_\bullet$ using algebraic surgery. In the connective case the 1-connective Poincaré structure in $\mathbf{NL}^\bullet\langle 1/2\rangle$ ensures that we obtain something in $\mathbf{L}_\bullet\langle 1\rangle$. □

Room service 141 and 1411

M_{Δ^k} manifold k-ad consisting of a manifold M and submanifolds $\partial_0 M, \ldots, \partial_k M$ such that $\partial_0 M \cap \ldots \cap \partial_k M = \emptyset$.

$\Lambda^L X$ for $\Lambda = (\mathbb{A}, \mathbb{C}, \mathbb{P})$ denotes $(\mathbb{A}^* X, \mathbb{C}^L X, \mathbb{P}^L X, (T^*, e^*))$, the X-mosaicked algebraic bordism category of $\Lambda = (\mathbb{A}, \mathbb{C}, \mathbb{P})$ with *local Poincaré* duality.

$\mathbf{L}^\bullet\langle q\rangle, \mathbf{L}_\bullet\langle q\rangle, \mathbf{NL}^\bullet\langle q\rangle$ short for $\mathbf{L}^\bullet(\Lambda(\mathbb{Z})\langle q\rangle), \mathbf{L}_\bullet(\Lambda(\mathbb{Z})\langle q\rangle), \mathbf{NL}^\bullet(\Lambda(\mathbb{Z})\langle q\rangle)$.

$\Lambda(\mathbb{Z}) = (\mathbb{A}(\mathbb{Z}), \mathbb{B}(\mathbb{Z}), \mathbb{C}(\mathbb{Z}))$ denotes the algebraic bordism category with
- $\mathbb{A}(\mathbb{Z})$ the category of finitely generated free left $\mathbb{Z}$-modules,
- $\mathbb{B}(\mathbb{Z})$ the bounded chain complexes in $\mathbb{A}(\mathbb{Z})$,
- $\mathbb{C}(\mathbb{Z})$ the contractible chain complexes of $\mathbb{B}(\mathbb{Z})$.

$\Lambda\langle q\rangle$ for $\Lambda = (\mathbb{A}, \mathbb{C}, \mathbb{P})$ and $q \in \mathbb{Z}$ the q-connective algebraic bordism category $(\mathbb{A}, \mathbb{C}\langle q\rangle, \mathbb{P}\langle q\rangle)$

$\Lambda\langle 1/2\rangle$ denotes for $\Lambda = (\mathbb{A}, \mathbb{C}, \mathbb{P})$ the algebraic bordism category $(\mathbb{A}, \mathbb{C}\langle 0\rangle, \mathbb{P}\langle 1\rangle)$.

$\mathbb{C}\langle q\rangle$ the subcategory of chain complexes of $\mathbb{C}$ that are homotopy equivalent to q-connected chain complexes.

$\mathbf{NL}\langle 1/2\rangle/\mathbf{L}\langle 0\rangle^\bullet := \mathrm{Fiber}(J : \mathbf{L}^\bullet\langle 0\rangle \to \mathbf{NL}^\bullet\langle 1/2\rangle)$

$J : L^n(R) \to NL^n(R)$ roughly induced by $j : W^{\%}C \to \widehat{W}^{\%}C$; see (111) for more details of how a normal structure (φ, γ, χ) is obtained from a symmetric Poincaré chain complex (C, φ).

$\mathbf{\Omega}^{STOP}_\bullet$ the Ω-spectrum of Kan Δ-sets defined by

$$(\mathbf{\Omega}^{STOP}_n)^{(k)} = \{(M, \partial_0 M, \ldots, \partial_k M) \mid (n+k)-\text{dimensional manifold } (k+2)\text{-ad such that } \partial_0 M \cap \ldots \cap \partial_k M = \emptyset\}.$$

$\mathbf{\Omega}^N_\bullet$ the Ω-spectrum of Kan Δ-sets defined by

$$\begin{aligned}(\mathbf{\Omega}^N_n)^{(k)} = \{(X_{\Delta^k}, \nu, \rho) \mid\ & (n+k)-\text{dimensional normal space } (k+2)\text{-ad, i.e.}\\ & X_{\Delta^k} = (X, \partial_0 X, \ldots, \partial_k X) \text{ s.t. } \partial_0 X \cap \ldots \cap \partial_k X = \emptyset,\\ & \nu : X \to \mathrm{BSG}(r) \text{ an } (r-1)\text{-spherical fibration},\\ & \rho : \Delta^{n+k+r} \to \mathrm{Th}(\nu) \text{ s.t. } \rho(\partial_i \Delta^{n+k+r}) \subset \mathrm{Th}(\nu|_{\partial_i X}) \quad \}\end{aligned}$$

$\boxed{\Sigma^{-1}\mathbf{\Omega}_\bullet^{\mathrm{N,STOP}}}$ the Ω-spectrum of Δ-sets obtained as the fiber of canonical the map of spectra $\mathbf{\Omega}_\bullet^{STOP} \to \mathbf{\Omega}_\bullet^{N}$.

161 Fibration sequence of classifying spaces

161 Fibration sequence of classifying spaces [Ran79, p.290][KMM13, Prop. 13.6]
There are the following homotopy fibration sequences of spaces:

$$\begin{aligned} & \mathbf{L}_0\langle 1\rangle \simeq \mathbf{NL}\langle 1/2\rangle/\mathbf{L}\langle 0\rangle^0 \to \mathrm{B}\mathbf{L}^\bullet\langle 0\rangle\mathrm{G} \to \mathrm{B}\mathbf{NL}^\bullet\langle 1/2\rangle\mathrm{G} \\ \textit{and} \quad & \mathbf{L}_0\langle 1\rangle \simeq \mathbf{NL}\langle 1/2\rangle/\mathbf{L}\langle 0\rangle^0 \to \mathbf{L}^\otimes\langle 0\rangle \to \mathbf{NL}^\otimes\langle 1/2\rangle. \end{aligned}$$

[1611 → [May77, section III.2]]
For an Ω ring spectrum $\mathbf{E}$ *with* $\pi_0(\mathbf{E}) = \mathbb{Z}$ *there is the following homotopy fibration sequence*

$$\mathbf{E}_\otimes \xrightarrow{i} \mathrm{B}\mathbf{E}\mathrm{G} \to \mathrm{BSG}.$$

1411 (161) L-spectra fibration sequences [Ran92, Prop. 15.16]
There are homotopy fibration sequences

$$\mathbf{L}_\bullet \simeq \mathbf{NL}/\mathbf{L}^\bullet \xrightarrow{1+t} \mathbf{L}^\bullet \xrightarrow{J} \mathbf{NL}^\bullet \quad \textit{and} \quad \mathbf{L}_\bullet\langle 1\rangle \simeq \mathbf{NL}\langle 1/2\rangle/\mathbf{L}\langle 0\rangle^\bullet \xrightarrow{1+t} \mathbf{L}^\bullet\langle 0\rangle \xrightarrow{J} \mathbf{NL}^\bullet\langle 1/2\rangle.$$

Proof 161

Both spectra $\mathbf{L}^\bullet\langle 0\rangle$ and $\mathbf{NL}^\bullet\langle 1/2\rangle$ are ring spectra with $\pi_0 \cong \mathbb{Z}$. Consider the sequence of 1611 for these spectra together with the map J between them. We obtain a diagram as follows with an induced map on the fibers.

$$\begin{array}{ccc} \mathbf{L}^\otimes\langle 0\rangle & \longrightarrow & \mathbf{NL}^\otimes\langle 1/2\rangle \\ \downarrow & & \downarrow \\ \mathrm{B}\mathbf{L}^\bullet\langle 0\rangle\mathrm{G} & \xrightarrow{J} & \mathrm{B}\mathbf{NL}^\bullet\langle 1/2\rangle\mathrm{G} \\ \downarrow & & \downarrow \\ \mathrm{BSG} & = & \mathrm{BSG} \end{array}$$

From 1411 we have the fibration sequence $\mathbf{L}_0\langle 1\rangle \xrightarrow{1+t} \mathbf{L}^0\langle 0\rangle \xrightarrow{J} \mathbf{NL}^0\langle 1/2\rangle$. Replace the symmetrization map $(1+t)$ by the map given on the l-simplices by 1411→p.79

$$\begin{aligned} (1+t)^\otimes : \mathbf{L}_0\langle 1\rangle &\longrightarrow \mathbf{L}^\otimes\langle 0\rangle, \\ (C,\psi) &\longmapsto (1+t)(C,\psi) + (C(\Delta^l), \mathrm{con}^\varphi_{\Delta^l}([\Delta^l])). \end{aligned}$$

It maps the component of 0 in $\mathbf{L}_0\langle 1\rangle$ to the component of 1 in $\mathbf{L}^0\langle 0\rangle$. We obtain the fibration sequence

$$\mathbf{L}_0\langle 1\rangle \to \mathbf{L}^\otimes\langle 0\rangle \to \mathbf{NL}^\otimes\langle 1/2\rangle. \qquad \square$$

Room service 161

$i\colon \mathbf{E}_\otimes \to \mathrm{B}\mathbf{E}\mathrm{G}$ If $c\colon X \to \mathbf{E}_\otimes$ is a map then $i(c)\colon X \to \mathrm{B}\mathbf{E}\mathrm{G}$ is given by the pair $(\varepsilon, u_{\mathbf{E}}(i(c)))$ with $u_{\mathbf{E}}(i(c))\colon \mathrm{Th}(\varepsilon) \xrightarrow{\widetilde{\Delta}} X_+ \wedge \mathrm{Th}(\varepsilon) \xrightarrow{c\wedge\Sigma^k(1)} \mathbf{E}_\otimes \wedge \mathbf{E}_k \to \mathbf{E}_k$.

$\mathbf{E}_\otimes$ the component of $1 \in \mathbb{Z}$ for an Ω ring spectrum $\mathbf{E}$ with $\pi_0(\mathbf{E}) = \mathbb{Z}$, e.g. $\mathbf{L}^\otimes\langle 0\rangle$ or $\mathbf{NL}^\otimes\langle 1/2\rangle$.

$\mathrm{B}\mathbf{E}\mathrm{G}$ the classifying space of spherical fibrations with $\mathbf{E}$-orientation; a map $X \to \mathrm{B}\mathbf{E}\mathrm{G}$ is given by a pair $(\nu, u^{\mathbf{E}})$ with ν a spherical fibration and $u^{\mathbf{E}}$ an $\mathbf{E}$-orientation.

$u^{\mathbf{E}}(\nu)$ an $\mathbf{E}$-*orientation* of a $\mathbb{Z}$-oriented spherical fibration $\nu\colon X \to \mathrm{BSG}(k)$ for a ring spectrum $\mathbf{E}$, is a homotopy class of maps $u^{\mathbf{E}}(\nu)\colon \mathbf{T}(\nu) \to \mathbf{E}$ such that for each $x \in X$ the restriction $u^{\mathbf{E}}(\nu)_x\colon \mathbf{T}(\nu_x) \to \mathbf{E}$ to the fiber ν_x of ν over x represents a generator of $\mathbf{E}^*(\mathbf{T}(\nu_x)) \cong \mathbf{E}^*(S^k)$ which under the Hurewicz homomorphism $\mathbf{E}^*(\mathbf{T}(\nu_x)) \to H^*(\mathbf{T}(\nu_X);\mathbb{Z})$ maps to the chosen $\mathbb{Z}$-orientation.

$J\colon L^n(R) \to NL^n(R)$ roughly induced by $j\colon W^{\%}C \to \widehat{W}^{\%}C$; see (111) for more details of how a normal structure (φ, γ, χ) is obtained from a symmetric Poincaré chain complex (C, φ).

162 Mosaicked normal/symmetric signature

162 Mosaicked normal/symmetric signature
Let $\widehat{f}\colon M \to X$ be a degree one normal map and X triangulated. There is a normal/symmetric signature $\mathrm{sgn}^{\mathbf{NL}/\mathbf{L}^\bullet}_X(\widehat{f})$ such that $\partial\!\!\!\!\partial(\mathrm{sgn}^{\mathbf{NL}/\mathbf{L}^\bullet}_X(\widehat{f})) = \mathrm{sgn}^{\mathbf{L}_\bullet}_X(\widehat{f})$.

1621 Alternative quadratic signatures [KMM13, Example 3.26][Ran81, Prop. 7.4.1][Ran92, Remark 2.16][Wei85b, Theorem 7.1]
Let $\widehat{f}\colon M \to X$ be a degree one normal map of n-dimensional Poincaré spaces with Spivak normal fibrations ν_M and ν_X. Let $W = \mathscr{M}(f)$ be the mapping cylinder. The quadratic boundary pair signature $\partial\mathrm{gn}^{\mathbf{L}_\bullet}_{\to}$ applied to the $(n+1)$-dimensional (normal, Poincaré) pair of spaces

$$Z := ((W, M \amalg X), (\nu_W, \nu_{M\amalg X}), (\rho_W, \rho_{M\amalg X}))$$

gives a quadratic pair $\partial\mathrm{gn}^{\mathbf{L}_\bullet}_{\to}(Z) = (i\colon \partial C' \to \partial D', \partial\delta\psi', \partial\psi')$ such that the following equivalences hold:

$$\partial\!\!\!\!\partial\mathrm{sgn}^{\mathbf{NL}/\mathbf{L}^\bullet}_{\to}(\widehat{f}) \overset{(1)}{\simeq} (\partial D', \delta\psi') \overset{(2)}{\simeq} \mathrm{sgn}^{\mathbf{L}_\bullet}_{\mathbb{Z}\pi}(\widehat{f}).$$

If $\widehat{f}$ is replaced by a degree one normal $\widehat{g}\colon N \to Y$ from a Poincaré space to a normal space the same construction yields a quadratic pair such that

$$\partial\!\!\!\!\partial\mathrm{sgn}^{\mathbf{NL}^\bullet}_{\to}(\widehat{g}) = (\partial D', \delta\psi') = \mathrm{sgn}^{\mathbf{L}_\bullet}_{\#}(\widehat{g}).$$

Proof 162

For each $\sigma \in X$ we have a map $\widehat{f}[\sigma]$ which yields an $(n+1-|\sigma|)$-dimensional pair of normal $(m-|\sigma|)$-ads

$$(W[\sigma]), M[\sigma] \amalg X[\sigma], \nu_{\overline{f}(\sigma)}, \rho(\overline{f}(\sigma))).$$

1621→p.84 The statement of 1621 generalizes to the relative case. Use the quadratic pair signature $\mathrm{sgn}^{\mathbf{L}_\bullet}_{\to}$

on one side and on the other take the complete boundary pair of $\mathrm{sgn}_{\to}^{\mathbf{NL}/\mathbf{L}^\bullet}$ instead of only the effect of algebraic surgery. We obtain equivalences

$$\partial^N_{\to}\,\mathrm{sgn}_{\to}^{\mathbf{NL}^\bullet}(\widehat{f}[\sigma]) = \partial\mathrm{gn}_{\to}^{\mathbf{L}_\bullet}(W[\sigma], M[\sigma] \amalg X[\sigma]) = \mathrm{sgn}_{\to}^{\mathbf{L}_\bullet}(\widehat{f}[\sigma]).$$

Hence, the boundary of the pair

$$\mathrm{sgn}_{\to}^{\mathbf{NL}^\bullet}(W[\sigma], M[\sigma] \amalg X[\sigma]) = (\mathrm{sgn}_{\mathbb{Z}\pi}^{\mathbf{NL}^\bullet}(W[\sigma]), \mathrm{sgn}_{\mathbb{Z}\pi}^{\mathbf{NL}^\bullet}(M[\sigma]) - \mathrm{sgn}_{\mathbb{Z}\pi}^{\mathbf{NL}^\bullet}(X[\sigma]))$$

gives a quadratic chain complex that coincides with the σ-component of $\mathrm{sgn}_X^{\mathbf{L}_\bullet}(\widehat{f})$. □

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$\mathrm{sgn}_X^{\mathbf{NL}/\mathbf{L}^\bullet}(\widehat{f}) := \mathrm{sgn}_{\Omega}^{\mathbf{NL}/\mathbf{L}^\bullet}([\widehat{f}]^{\Sigma^{-1}\mathbf{\Omega}_\bullet^{\mathrm{N,STOP}}}) \in H_n(M'; \mathbf{NL}/\mathbf{L}^\bullet)$ the mosaicked normal/symmetric signature over X defined for a degree one normal map $\widehat{f}: M \to M'$ between manifolds (see 2221).

$[\widehat{f}]^{\Sigma^{-1}\mathbf{\Omega}_\bullet^{\mathrm{N,STOP}}} \in H_n(X; \Sigma^{-1}\mathbf{\Omega}_\bullet^{\mathrm{N,STOP}})$ a $\mathbf{\Omega}_\bullet^N$-cobordism class of $\mathbf{\Omega}_\bullet^{STOP}$-cycle for a degree one normal map $\widehat{f}: M \to M'$ which assigns an $(m-|\sigma|)$-ad $(W(\sigma), \nu_{\overline{f}(\sigma)}, \rho(\overline{f}(\sigma)), M(\sigma) \amalg M(\sigma))$ to each $\sigma \in M'$ (see 2221).

$\mathrm{sgn}_{\Omega}^{\mathbf{NL}/\mathbf{L}^\bullet}: \Sigma^{-1}\mathbf{\Omega}_\bullet^{\mathrm{N,STOP}} \to \mathbf{NL}\langle 1/2\rangle/\mathbf{L}\langle 0\rangle^\bullet$ the normal/symmetric signature map induced by the maps $\mathrm{sgn}_{\Omega}^{\mathbf{NL}^\bullet}$ and $\mathrm{sgn}_{\Omega}^{\mathbf{L}^\bullet}$ and the fibration sequence of 161.

$\mathrm{sgn}_X^{\mathbf{L}_\bullet}(\widehat{f}) \in H_n(X; \mathbf{L}_\bullet(\mathbb{Z}))$ the mosaicked quadratic signature over X defined as follows. Make f transverse to the dual cells $D(\sigma, K)$. Then each σ-component is the quadratic pair signature $\mathrm{sgn}_{\to}^{\mathbf{L}_\bullet}(\widehat{f}[\sigma], \partial\widehat{f}[\sigma])$.

$\mathrm{sgn}_{\to}^{\mathbf{NL}/\mathbf{L}^\bullet}(Y, X) = (f: C \to D, (\delta\varphi, \delta\gamma, \delta\chi), \varphi) \in L(J)^n$ the normal/symmetric pair signature, defined for a pair of (normal, Poincaré)-spaces (Y, X) by $(D, (\delta\varphi, \delta\gamma, \delta\chi)) = \mathrm{sgn}_{\mathbb{Z}\pi}^{\mathbf{NL}^\bullet}(Y)$ and $(C, \varphi) = \mathrm{sgn}_{\mathbb{Z}\pi}^{\mathbf{L}^\bullet}(X)$.

$\partial\mathrm{gn}_{\to}^{\mathbf{L}_\bullet}(Y, B)$ the quadratic boundary pair signature for an n-dimensional pair of normal spaces (Y, B) is the $(n-1)$-quadratic Poincaré pair $(\partial C(B) \to \partial_+ C(Y), (\delta\psi, \psi))$ obtained by using the boundary construction and the spectral quadratic construction (see B28).

$\partial^N_{\to}$ the relative quadratic boundary construction, produces for an n-normal pair an $(n-1)$-quadratic Poincaré pair usually denoted $(\partial f: \partial C \to \partial_f D, \partial(\delta\psi, \psi))$.

$\partial\mathfrak{S}: \mathbf{NL}/\mathbf{L}^\bullet \to \mathbf{L}_\bullet$ here the map defined by algebraic surgery (see 112).

$\widehat{f}_\Delta := \bigcup_{\sigma \in X} \widehat{f}[\sigma]: M[\sigma] \to X[\sigma]$ the decomposition of a degree one normal map $\widehat{f}: M \to X$ into degree one normal maps $\widehat{f}[\sigma] = \widehat{f}|_{\widehat{f}^{-1}(X[\sigma])}$ of $(n-|\sigma|)$-dimensional manifold $(m-|\sigma|)$-ads.

$X[\sigma]$ is defined for a map $r: X \to K$ to a simplicial complex as the preimage of the dual cell $D(\sigma, K)$ after making r transverse. If X is a simplicial complex itself, choose r to be the identity. The subdivision $X = \bigcup_{\sigma \in K} X[\sigma]$ is called a K-dissection of X.

1621 Alternative quadratic signatures

Porter

The proof is taken from [KMM13, Example 3.26], based on [Ran81, Prop. 7.4.1] and [Ran92, Remark 2.16] and also uses [Wei85b, Theorem 7.1].

1621 Alternative quadratic signatures [KMM13, Example 3.26][Ran81, Prop. 7.4.1][Ran92, Remark 2.16][Wei85b, Theorem 7.1]
Let $\widehat{f}: M \to X$ be a degree one normal map of n-dimensional Poincaré spaces with Spivak normal fibrations ν_M and ν_X. Let $W = \mathscr{M}(f)$ be the mapping cylinder. The quadratic boundary pair signature $\partial gn^{\mathbf{L}_\bullet}_{\to}$ applied to the $(n+1)$-dimensional (normal, Poincaré) pair of spaces

$$Z := ((W, M \amalg X), (\nu_W, \nu_{M \amalg X}), (\rho_W, \rho_{M \amalg X}))$$

gives a quadratic pair $\partial gn^{\mathbf{L}_\bullet}_{\to}(Z) = (i: \partial C' \to \partial D', \partial\delta\psi', \partial\psi')$ such that the following equivalences hold:

$$\partial\!\!\!\!\mathcal{B}\mathrm{sgn}^{\mathbf{NL}/\mathbf{L}^\bullet}_{\to}(\widehat{f}) \overset{(1)}{\simeq} (\partial D', \delta\psi') \overset{(2)}{\simeq} \mathrm{sgn}^{\mathbf{L}_\bullet}_{\mathbb{Z}\pi}(\widehat{f}).$$

If $\widehat{f}$ is replaced by a degree one normal $\widehat{g}: N \to Y$ from a Poincaré space to a normal space the same construction yields a quadratic pair such that

$$\partial\!\!\!\!\mathcal{B}\mathrm{sgn}^{\mathbf{NL}^\bullet}_{\to}(\widehat{g}) = (\partial D', \delta\psi') = \mathrm{sgn}^{\mathbf{L}_\bullet}_{\#}(\widehat{g}).$$

B28 (1621, 232) Quadratic boundary pair signature
Let $((Y, B), \nu, (\rho_Y, \rho_B))$ be an n-dimensional pair of normal spaces. There is a quadratic boundary pair signature

$$\partial gn^{\mathbf{L}_\bullet}_{\to}(Y, B) = (\partial j: \partial C' \to \partial D', (\partial\delta\psi, \partial\psi))$$

producing an $(n-1)$-quadratic Poincaré pair.

[16211 → [Ran81, Prop. 7.3.1 (iv)]] Property of the spectral quadratic construction for stable maps
Let $\Gamma_Y: X \to \Sigma^p Y$ be a semi-stable map. If there is a space X_0 such that $\Gamma_Y: \Sigma^p X_0 = X \to \Sigma^p Y$ is in fact a stable map, then

$$\mathrm{con}^{\psi^!}_{\Gamma_Y} = e_\% \circ \mathrm{con}^{\psi}_{\Gamma_Y}$$

where $e: C(Y) \to \mathscr{C}(\gamma_Y)$ is the inclusion.

Proof 1621

B28→p.157 The equivalence (1) stated in 1621 is immediately from B28. Recall that the effect of algebraic surgery is the same as the target part in the boundary pair.

We now prove the second equivalence. The bundle map $\overline{f}$ induces the k-dimensional spherical fibration ν_W over W, and ρ_M and ρ_X induce the map

$$\rho_{M \amalg X}: (D^{n+1+k}, S^{n+k}) \to (\mathrm{Th}(\nu_W), \mathrm{Th}(\nu_{M \amalg X})).$$

Denote by $j_M: M \to W$ the inclusion and by $\mathrm{pr}: W \to X$ the projection which is a homotopy equivalence. From the first equivalence we already know that $\Sigma^{-1}\mathscr{C}(\varphi_{i^*})$ is equivalent to the chain complex of $\partial\mathrm{gn}^{\mathbf{L}\bullet}_{\mathbb{Z}\pi}(Z)$. The following commutative diagram identifies $\Sigma^{-1}(\mathscr{C}\varphi_{i^*})$ with the underlying chain complex and $\Sigma^{-1}\mathscr{C}(^!)$ of $\mathrm{sgn}^{\mathbf{L}\bullet}_{\mathbb{Z}\pi}(\widehat{f})$.

$$\begin{array}{ccc} C(W)^{n+1-*} & \xrightarrow{\varphi_{i^*}} & C(W, M \amalg X) \\ \big\downarrow j_M^* & & \big\downarrow \simeq \\ C(M)^{n+1-*} & \xrightarrow[\simeq]{\varphi_0|_M} & C(\Sigma M) \\ \big\uparrow f^* & & \big\uparrow ! \\ C(X)^{n+1-*} & \xrightarrow[\simeq]{\varphi_0|_X} & C(\Sigma X) \end{array} \qquad \mathrm{pr}_X^* \colon C(X)^{n+1-*} \xrightarrow{\simeq} C(W)^{n+1-*}$$

Next we identify the quadratic structures. Using the homotopy equivalence j_X and S-duality we recognize the diagram above induced by the following diagram of spaces.

$$\begin{array}{ccc} \mathrm{Th}(\nu_W)^* & \xrightarrow{\Gamma_W} & \Sigma^p(W/(M \amalg X)) \\ \big\downarrow \mathrm{Th}(j_M)^* & & \big\downarrow \simeq \\ \mathrm{Th}(\nu_M)^* & \xrightarrow[\simeq]{\gamma_M} & \Sigma^{p+1}M_+ \\ \big\uparrow T(\overline{f})^* & & \big\uparrow F \\ \mathrm{Th}(\nu_X)^* & \xrightarrow[\simeq]{\gamma_X} & \Sigma^{p+1}X \end{array} \qquad \mathrm{Th}(\mathrm{pr}_X)^* \colon \mathrm{Th}(\nu_X)^* \xrightarrow{\simeq} \mathrm{Th}(\nu_W)^*$$

It identifies the semi-stable map Γ_W with the stable map F. Hence we can use the property 16211 of the spectral quadratic construction to obtain

$$\mathrm{con}^{\psi !}_{\Gamma_W} = e_{\%} \circ \mathrm{con}^{\psi}_{F}\,.$$

The Thom class $u(\nu(W))$ restricts to $u(\nu_X)$ and hence the duals $u(\nu_W)^*$and $u(\nu_X) = \Sigma^X$ are also identified. The uniqueness of desuspensions gives the identification of the equivalence classes of the quadratic structures $e_{\%}\,\mathrm{con}^{\psi}_{F}([X]) \sim \partial\psi$.

The proof for $\widehat{g}$ where the target of the map is only a normal space is almost the same with the only difference that the map $\varphi_0|_Y$, which replaces $\varphi_0|_X$, is no longer an equivalence. □

Room service 1621

$\boxed{\mathrm{sgn}^{\mathbf{L}\bullet}_{\mathbb{Z}\pi}(\widehat{f}) = (C, \psi) = (\mathscr{C}(^!), e_{\%}\,\mathrm{con}^{\psi}_{F}([X])) \in L_n(\mathbb{Z}\pi)}$ an n-quadratic chain complex, called quadratic signature, where $F\colon \Sigma^p\widetilde{X}_+ \to \Sigma^p\widetilde{M}$ is the stable map obtained from the degree one normal map $\widehat{f}$ by equivariant S-duality (see A26).

$\boxed{\partial C := \Sigma^{-1}\mathscr{C}(\varphi_0)}$ the boundary chain complex.

$\boxed{\varphi_{f^*} = \mathrm{ev}_r(\delta\varphi, \varphi) \simeq \begin{pmatrix} \delta\varphi_0 \\ \varphi_0 f^* \end{pmatrix} : D^{n-*} \to \mathscr{C}(f)}$ a chain map defined for an n-symmetric pair $(f\colon C \to D, \delta\varphi, \varphi)$.

$\boxed{\varphi_f = \mathrm{ev}_l(\delta\varphi, \varphi) \simeq (\delta\varphi_0, f\varphi_0) : \mathscr{C}(f)^{n-*} \to D}$ a chain map defined for an n-symmetric pair $(f : C \to D, \delta\varphi, \varphi)$.

$\boxed{\partial\psi := \left(\begin{smallmatrix} 1+t \\ \mathcal{S} \end{smallmatrix}\right)^{-1} (\mathcal{S}^{-1}e^{\%}(\varphi), e_{\%}(\psi))}$ the quadratic boundary structure.

$\boxed{\Gamma_Y := S^{-1}(\widetilde{\Delta} \circ \rho) : \mathrm{Th}(\nu)^* \to \Sigma^p Y_+}$ the semi-stable map obtained for an n-dimensional normal space (Y, ν, ρ) with an N-dimensional S-dual $\mathrm{Th}(\nu)^*$ of its Thom space and $p = N - (n + k)$.

$\boxed{\partial\mathrm{gn}^{\mathbf{L}_\bullet}_{\to}(Y, B)}$ the quadratic boundary pair signature for an n-dimensional pair of normal spaces (Y, B) is the $(n-1)$-quadratic Poincaré pair $(\partial C(B) \to \partial_+ C(Y), (\delta\psi, \psi))$ obtained by using the boundary construction and the spectral quadratic construction (see B28).

164 Quadratic signature isomorphism

164 (22) Quadratic signature isomorphism
The quadratic signature defines an isomorphism $\mathrm{sgn}^{\mathbf{L}_\bullet}_{\mathrm{G/TOP}} : [X; \mathrm{G/TOP}] \xrightarrow{\cong} H^0(X; \mathbf{L}_\bullet\langle 1 \rangle)$.

112 (121, 1411, 164) Quadratic and (normal, Poincaré symmetric) [Ran92, Proposition 2.8 (ii)]
There is the following natural one-to-one correspondence of cobordism classes.

$$\begin{array}{ccc} n\text{-dimensional} & & (n-1)\text{-dimensional} \\ \text{(normal, symmetric) pairs} & \xleftrightarrow{1-1} & \text{quadratic chain complexes} \\ (f : C \to D, (\delta\varphi, \delta\gamma, \delta\chi), \varphi) & & (C', \psi') \end{array}$$

Additionally, if (C, φ) is Poincaré, then (C', ψ') is Poincaré and vice versa.

Proof 164

16→p.47 This statement is part of the proof of 16. We only have to replace $\mathbf{NL}/\mathbf{L}^\bullet$ by $\mathbf{L}_\bullet$ using 112. Then
112→p.61 both sides are the fibers of the homotopy pullback there and we proved that the induced map on these fibers is an isomorphism and agrees with $\mathrm{sgn}^{\mathbf{L}_\bullet}_{\mathbb{Z}\pi}$. □

221 Difference of quadratic signatures

221 Difference of quadratic signatures [KMM13, section 14.5]
Let $\widehat{f}_i : M_i \to X$ with $i = 0, 1$ be two degree one normal maps. Then the difference of their mosaicked quadratic signature defines an element

$$\mathrm{sgn}^{\mathbf{L}_\bullet}_X(\widehat{f}_1) - \mathrm{sgn}^{\mathbf{L}_\bullet}_X(\widehat{f}_0) \in L_n(\Lambda_L X) = H_n(X; \mathbf{L}_\bullet\langle 1 \rangle)$$

such that $A(\mathrm{sgn}^{\mathbf{L}_\bullet}_X(\widehat{f}_1) - \mathrm{sgn}^{\mathbf{L}_\bullet}_X(\widehat{f}_0)) = \mathrm{sgn}^{\mathbf{L}_\bullet}_{\mathbb{Z}\pi}(\widehat{f}_1) - \mathrm{sgn}^{\mathbf{L}_\bullet}_{\mathbb{Z}\pi}(\widehat{f}_0) \in L_n(\mathbb{Z}\pi)$.

A29 (221, 23) Mosaicked quadratic signature [Ran92, Example 9.14]
Let $\widehat{f}\colon M \to X$ *be a degree one normal map from a closed topological manifold to a Poincaré space both of dimension* n. *Let* $r\colon X \to K$ *be a map to a simplicial complex* K. *There is a mosaicked quadratic signature*

$$\mathrm{sgn}_K^{\mathbf{L}_\bullet}(\widehat{f}) \in L_n(\Lambda_G K)$$

with $A(\mathrm{sgn}_X^{\mathbf{L}_\bullet}(\widehat{f})) = \mathrm{sgn}_{\mathbb{Z}\pi}^{\mathbf{L}_\bullet}(\widehat{f})$. *If* X *is a manifold, there is a refined version*

$$\mathrm{sgn}_K^{\mathbf{L}_\bullet}(\widehat{f}) \in L_n(\Lambda_L K).$$

[2211 → [RW90, Prop. 2.9]]
A chain complex C *over* $\mathbb{Z}_*X$ *is contractible if and only if* $C(\sigma)$ *over* $\mathbb{Z}$ *is contractible for all* $\sigma \in X$.

23 (221) Subset step
$-\,\mathrm{sgn}_{\mathbb{Z}\pi}^{\mathbf{L}_\bullet}(\mathscr{N}(X)) \subseteq \partial_{\mathbb{Z}\pi}^{Q}{}^{-1}(s(X))$ *where* $\partial_{\mathbb{Z}\pi}^{Q}\colon L_n(\mathbb{Z}\pi) \to \mathbb{S}_n(X)$ *is the boundary map from the surgery braid.*

Proof 221

The mosaicked quadratic signature $\mathrm{sgn}_X^{\mathbf{L}_\bullet}(\widehat{f})$ produces an n-quadratic chain complex in $\Lambda_G X$. In order to obtain an element in $\Lambda_L X$ it has to be locally Poincaré. Using 2211 this is equivalent 2211→[RW90] to the boundary being contractible. So it is enough to prove that both signatures $\mathrm{sgn}_X^{\mathbf{L}_\bullet}(\widehat{f}_0)$ and $\mathrm{sgn}_X^{\mathbf{L}_\bullet}(\widehat{f}_1)$ are represented by quadratic chain complexes with homotopic boundaries. But this follows immediately from the equivalence 23.1 in the proof of 23 where it was shown that the 23→p.55 boundary of the mosaicked quadratic signature for a degree one normal map is the total surgery obstruction, which depends only on the homotopy type of X. □

Room service 221

$\boxed{\Lambda_L X}$ for $\Lambda = (\mathbb{A}, \mathbb{C}, \mathbb{P})$ denotes $(\mathbb{A}_*X, \mathbb{C}_L X, \mathbb{P}_L X, (T_*, e_*))$, the X-mosaicked algebraic bordism category of $\Lambda = (\mathbb{A}, \mathbb{C}, \mathbb{P})$ with *local Poincaré* duality.

$\boxed{\Lambda_G X}$ for $\Lambda = (\mathbb{A}, \mathbb{C}, \mathbb{P})$ denotes $(\mathbb{A}_*X, \mathbb{C}_L X, \mathbb{P}_G X, (T_*, e_*))$, the X-mosaicked algebraic bordism category of $\Lambda = (\mathbb{A}, \mathbb{C}, \mathbb{P})$ with *global Poincaré* duality.

$\boxed{\partial_{\mathbb{G}}^{Q}\colon L_n(\mathbb{B}\langle 1\rangle, \mathbb{G}\langle 1\rangle) \to L_{n-1}(\mathbb{G}\langle 1\rangle, \mathbb{L}\langle 1\rangle) = \mathbb{S}_n(X)}$ the boundary map induced by the boundary construction ∂^Q.

$\boxed{\partial_{\mathbb{G}}^{N}\colon NL^n(\mathbb{B}\langle 0\rangle, \mathbb{G}\langle 1\rangle) = VL^n(X) \to L_{n-1}(\mathbb{G}\langle 1\rangle, \mathbb{L}\langle 1\rangle) = \mathbb{S}_n(X)}$ the boundary map induced by the boundary construction ∂^N.

$\boxed{\mathrm{sgn}_X^{VL}(X) \in VL^n(X)}$ defined for a Poincaré space X as the normal signature $\mathrm{sgn}_X^{\mathbf{NL}^\bullet}(X)$.

$\boxed{\mathrm{sgn}_X^{\mathbf{NL}^\bullet}(X) := \mathrm{sgn}_\Omega^{\mathbf{NL}^\bullet}([X]^{\mathbf{\Omega}_\bullet^N}) \in H_n(X; \mathbf{NL}^\bullet) \cong NL^n(\Lambda_N X)}$ the X-mosaicked normal signature defined here only for an n-dimensional Poincaré space X.

$\boxed{\mathrm{sgn}^{\mathbf{L}_\bullet}_X(\widehat{f}) \in L_n(\Lambda_G X)}$ the mosaicked quadratic signature over X constructed in the following way: make f transverse to the dual cells $D(\sigma, K)$, then each σ-component is defined as the quadratic pair signature $\mathrm{sgn}^{\mathbf{L}_\bullet}_{\to}(\widehat{f}[\sigma], \partial\widehat{f}[\sigma])$.

222 (normal, manifold)-cycles for Poincaré spaces

Porter

We generalize the construction of $\Sigma^{-1}\mathbf{\Omega}^{\mathrm{N,STOP}}_\bullet$-cycles from 2221 for degree normal maps with target Poincaré spaces. This is achieved by using two maps and gluing their mapping cylinder together so that we get rid of the Poincaré spaces and the remaining boundary components are manifolds.

222 (normal, manifold)-cycles for Poincaré spaces [KMM13, Lemma 14.16]
Let $\widehat{f}_i := (\overline{f}_i, f_i)\colon M_i \to X$ with $i = 0, 1$ be two n-dimensional degree one normal maps from topological manifolds to Poincaré spaces. Then there exists a $\Sigma^{-1}\mathbf{\Omega}^{\mathrm{N,STOP}}_\bullet$-cycle

$$[\widehat{f}_1, \widehat{f}_0]^{\Sigma^{-1}\mathbf{\Omega}^{\mathrm{N,STOP}}_\bullet} \in H_n(X; \Sigma^{-1}\mathbf{\Omega}^{\mathrm{N,STOP}}_\bullet)$$

such that $\mathrm{sgn}^{\mathbf{L}_\bullet}_X(\widehat{f}_1) - \mathrm{sgn}^{\mathbf{L}_\bullet}_X(\widehat{f}_0) = \mathrm{sgn}^{\mathbf{L}_\bullet}_\Omega([\widehat{f}_1, \widehat{f}_0]^{\Sigma^{-1}\mathbf{\Omega}^{\mathrm{N,STOP}}_\bullet}) \in H_n(X; \mathbf{L}_\bullet\langle 1\rangle)$.

2221 (15) (normal, manifold)-cycles [KMM13, Construction 11.9]
Let $\widehat{f}\colon M \to M'$ be a degree one map of n-dimensional topological manifolds such that M' is triangulated. Then there is a $\Sigma^{-1}\mathbf{\Omega}^{\mathrm{N,STOP}}_\bullet$-cycle

$$[\widehat{f}]^{\Sigma^{-1}\mathbf{\Omega}^{\mathrm{N,STOP}}_\bullet} \in H_n(M'; \Sigma^{-1}\mathbf{\Omega}^{\mathrm{N,STOP}}_\bullet)$$

such that $\mathrm{sgn}^{\mathbf{L}_\bullet}_X(X) = \mathrm{sgn}^{\mathbf{L}_\bullet}_\Omega([\widehat{f}]^{\Sigma^{-1}\mathbf{\Omega}^{\mathrm{N,STOP}}_\bullet}) \in H_n(X; \mathbf{L}_\bullet\langle 1\rangle)$.

1621 Alternative quadratic signatures[KMM13, Example 3.26][Ran81, Prop. 7.4.1][Ran92, Remark 2.16][Wei85b, Theorem 7.1]
Let $\widehat{f}\colon M \to X$ be a degree one normal map of n-dimensional Poincaré spaces with Spivak normal fibrations ν_M and ν_X. Let $W = \mathscr{M}(f)$ be the mapping cylinder. The quadratic boundary pair signature $\partial\mathrm{gn}^{\mathbf{L}_\bullet}_{\to}$ applied to the $(n+1)$-dimensional (normal, Poincaré) pair of spaces

$$Z := ((W, M \amalg X), (\nu_W, \nu_{M\amalg X}), (\rho_W, \rho_{M\amalg X}))$$

gives a quadratic pair $\partial\mathrm{gn}^{\mathbf{L}_\bullet}_{\to}(Z) = (i\colon \partial C' \to \partial D', \partial\delta\psi', \partial\psi')$ such that the following equivalences hold:

$$\partial\!\!\!\mathrm{6}\mathrm{sgn}^{\mathbf{NL}/\mathbf{L}^\bullet}_{\to}(\widehat{f}) \overset{(1)}{\simeq} (\partial D', \delta\psi') \overset{(2)}{\simeq} \mathrm{sgn}^{\mathbf{L}_\bullet}_{\mathbb{Z}\pi}(\widehat{f}).$$

If $\widehat{f}$ is replaced by a degree one normal $\widehat{g}\colon N \to Y$ from a Poincaré space to a normal space the same construction yields a quadratic pair such that

$$\partial\!\!\!\mathrm{6}\mathrm{sgn}^{\mathbf{NL}^\bullet}_{\to}(\widehat{g}) = (\partial D', \delta\psi') = \mathrm{sgn}^{\mathbf{L}_\bullet}_{\#}(\widehat{g}).$$

Proof 222

Let $W_i = \mathscr{M}(\widehat{f}_i)$ be the mapping cylinders for $i = 0, 1$.

2221→p.90 We start with defining $[\widehat{f}_1, \widehat{f}_0]^{\Sigma^{-1}\mathbf{\Omega}^{\mathrm{N,STOP}}_\bullet}$. It is based on the definition 2221 of (normal, manifold)-cycles $[\widehat{f}]^{\Sigma^{-1}\mathbf{\Omega}^{\mathrm{N,STOP}}_\bullet}$ for degree one normal maps between manifolds. Now the target is only a

Poincaré space. Hence the dissections

$$\widehat{f}_i = \bigcup_{\sigma \in X} (\overline{f}_i(\sigma), f_i(\sigma)) \colon M_i(\sigma) \to X(\sigma)$$

for $i = 0, 1$ give rise to cobordisms of $\mathbf{\Omega}^N_\bullet$-cycles

$$[\widehat{f}_i]^{\mathbf{\Omega}^N_\bullet} = \{\sigma \mapsto (W_i(\sigma), \nu(b_i(\sigma)), \rho(b_i(\sigma))) \mid \sigma \in X\}$$

but not of $\Sigma^{-1}\mathbf{\Omega}^{\mathrm{N,STOP}}_\bullet$-cycles since the ends of the cobordisms given by X are not topological manifolds. But these ends are equal for $i = 0, 1$ and so we can glue the two cobordisms together along these ends and we obtain for each $\sigma \in X$ the $(n + 1 - |\sigma|)$-dimensional pair of normal $(m - |\sigma|)$-ads

$$\big(W_1(\sigma) \cup_{X(\sigma)} W_0(\sigma) f, M_1(\sigma) \amalg -M_0(\sigma),\ \nu(b_1(\sigma)) \cup_{\nu_{X(\sigma)}} \nu(b_0(\sigma)),\ \rho(b_1(\sigma)) \cup_{\rho(\sigma)} \rho(b_0(\sigma))\big)$$

which now fit together to produce a $\Sigma^{-1}\mathbf{\Omega}^{N,\mathrm{STOP}}_\bullet$-cycle. This gives the desired signature

$$[\widehat{f}_1, \widehat{f}_2]^{\Sigma^{-1}\mathbf{\Omega}^{\mathrm{N,STOP}}_\bullet} \in H_n(X; \Sigma^{-1}\mathbf{\Omega}^{\mathrm{N,STOP}}_\bullet).$$

Now consider the left hand side of the equation we want to proof. By definition, each summand $\mathrm{sgn}^{\mathbf{L}_\bullet}_X(\widehat{f}_i)$ on a simplex σ is given by the quadratic pair signature $\mathrm{sgn}^{\mathbf{L}_\bullet}_\to(\widehat{f}(\sigma), \partial\widehat{f}(\sigma))$. But 1621 1621→p.84 gives an alternative description, namely we obtain $\mathrm{sgn}^{\mathbf{L}_\bullet}_X(\widehat{f}_i)(\sigma)$ via algebraic surgery on the algebraic normal pair extracted from the geometric normal pair $(W_i(\sigma), M_i(\sigma) \amalg X(\sigma))$. Subtracting the signatures $\mathrm{sgn}^{\mathbf{L}_\bullet}_X(\widehat{f}_i)$ corresponds to taking disjoint union of these normal pairs and reversing the orientation on the one labeled with $i = 0$. Denote the σ-component by

$$Z(\sigma) := (W_1(\sigma) \amalg -W_0(\sigma), M_1(\sigma) \amalg X(\sigma) \amalg -M_0(\sigma) \amalg -X(\sigma)).$$

On the other side of the equation the underlying normal pair for each simplex σ of the just defined $\Sigma^{-1}\mathbf{\Omega}^{\mathrm{N,STOP}}_\bullet$-signature is given by

$$(W_1(\sigma) \cup_{X(\sigma)} W_0(\sigma), M_1(\sigma) \amalg -M_0(\sigma)).$$

But there is a geometric cobordism between these two pairs that induces an algebraic cobordism between the algebraic normal pairs $[\widehat{f}_1, \widehat{f}_2]^{\Sigma^{-1}\mathbf{\Omega}^{\mathrm{N,STOP}}_\bullet}(\sigma)$ and $\mathrm{sgn}^{\mathbf{NL}^\bullet}_\to(Z(\sigma))$. The image of $[\widehat{f}_1, \widehat{f}_2]^{\Sigma^{-1}\mathbf{\Omega}^{\mathrm{N,STOP}}_\bullet}$ under the map $\mathrm{sgn}^{\mathbf{L}_\bullet}_\Omega \colon \Sigma^{-1}\mathbf{\Omega}^{\mathrm{N,STOP}}_\bullet \to \mathbf{L}_\bullet\langle 1\rangle$ as well as $\mathrm{sgn}^{\mathbf{L}_\bullet}_X(\widehat{f}_1) - \mathrm{sgn}^{\mathbf{L}_\bullet}_X(\widehat{f}_0)$ are both obtained from these pairs via algebraic surgery. Thus they are cobordant. □

Room service 222

$\mathrm{sgn}^{\mathbf{L}_\bullet}_X(\widehat{f}) \in L_n(\Lambda_G X)$ the mosaicked quadratic signature over X constructed in the following way: make f transverse to the dual cells $D(\sigma, K)$, then each σ-component is defined as the quadratic pair signature $\mathrm{sgn}^{\mathbf{L}_\bullet}_\to(\widehat{f}[\sigma], \partial\widehat{f}[\sigma])$.

$(\widehat{f}[\sigma], \partial\widehat{f}[\sigma]) = ((\overline{f}[\sigma], f[\sigma]), (\partial\overline{f}[\sigma], \partial f[\sigma]))$ an n-dimensional degree one normal map

$$\begin{array}{ccc} (\nu_M|_{M[\sigma]}, \nu_M|_{\partial M[\sigma]}) & \xrightarrow{(\overline{f}, \partial\overline{f})} & (\nu_X|_{X[\sigma]}, \nu_X|_{\partial X[\sigma]}) \\ \downarrow & & \downarrow \\ (M[\sigma], \partial M[\sigma]) := (f^{-1}, \partial f^{-1})(X[\sigma], \partial X[\sigma]) & \xrightarrow{(f, \partial f)} & (X[\sigma], \partial X[\sigma]), \end{array}$$

denoted $(f[\sigma], \partial f[\sigma]) : (M[\sigma], \partial M[\sigma]) \to (X[\sigma], \partial X[\sigma])$ for short, from an $(n - |\sigma|)$-dimensional manifold with boundary to an $(n - |\sigma|)$-dimensional normal pair obtained from a degree one normal map $\widehat{f}$ after making f transverse to a K-dissection $\bigcup_{\sigma \in K} X[\sigma]$ of X .

$[K]^{\mathbf{E}} \in H_n(K; \mathbf{E})$ an n-dimensional $\mathbf{E}$-cycle of a simplicial complex $K \subset \partial\Delta^{m+1}$ defined by a collection $\left\{[K]^{\mathbf{E}}(\sigma) \in \mathbf{E}_{n-m}^{(m-|\sigma|)} \mid \sigma \in K\right\}$ such that $\partial_i [K]^{\mathbf{E}}(\sigma) = [K]^{\mathbf{E}}(\delta_i \sigma)$ if $\delta_i \sigma \in K$ and $\emptyset$ otherwise.

$\Sigma^{-1}\mathbf{\Omega}_\bullet^{\mathrm{N,STOP}}$ the Ω-spectrum of Δ-sets obtained as the fiber of canonical the map of spectra $\mathbf{\Omega}_\bullet^{STOP} \to \mathbf{\Omega}_\bullet^{N}$.

$\mathrm{sgn}_\Omega^{\mathbf{L}_\bullet} : \Sigma^{-1}\mathbf{\Omega}_\bullet^{\mathrm{N,STOP}} \to \mathbf{L}_\bullet\langle 1\rangle$ the quadratic signature map given by the normal/symmetric signature map and the identification $\mathbf{NL}\langle 1/2\rangle/\mathbf{L}\langle 0\rangle^\bullet \simeq \mathbf{L}_\bullet\langle 1\rangle$.

$\mathrm{sgn}_\to^{\mathbf{L}_\bullet}(\delta\widehat{g}, \widehat{g}) = (G^! : C^! \to D^!, \delta\psi^!, \psi^!)$ the quadratic pair signature for a degree one normal map $(\delta\widehat{g}, \widehat{g}) : (N, A) \to (Y, B)$ from a Poincaré pair (N, A) to a normal pair (Y, B) (see A28).

$\Gamma_Y := S^{-1}(\widetilde{\Delta} \circ \rho) : \mathrm{Th}(\nu)^* \to \Sigma^p Y_+$ the semi-stable map obtained for an n-dimensional normal space (Y, ν, ρ) with an N-dimensional S-dual $\mathrm{Th}(\nu)^*$ of its Thom space and $p = N - (n + k)$.

2221 (normal, manifold)-cycles

Porter

From a degree one normal map $\widehat{f} : M \to M'$ between topological manifolds we obtain a (normal, topological manifold) pair $(\mathcal{M}(f), M \amalg M')$ which we can consider as a simplex in $\Sigma^{-1}\mathbf{\Omega}_\bullet^{\mathrm{N,STOP}}$ and which leads to a (normal, symmetric Poincaré) pair, i.e. a simplex in $\mathbf{NL}\langle 1/2\rangle/\mathbf{L}\langle 0\rangle^\bullet$.

2221 (15) (normal, manifold)-cycles [KMM13, Construction 11.9]
Let $\widehat{f} : M \to M'$ be a degree one map of n-dimensional topological manifolds such that M' is triangulated. Then there is a $\Sigma^{-1}\mathbf{\Omega}_\bullet^{\mathrm{N,STOP}}$-cycle

$$[\widehat{f}]^{\Sigma^{-1}\mathbf{\Omega}_\bullet^{\mathrm{N,STOP}}} \in H_n(M'; \Sigma^{-1}\mathbf{\Omega}_\bullet^{\mathrm{N,STOP}})$$

such that $\mathrm{sgn}_X^{\mathbf{L}_\bullet}(X) = \mathrm{sgn}_\Omega^{\mathbf{L}_\bullet}([\widehat{f}]^{\Sigma^{-1}\mathbf{\Omega}_\bullet^{\mathrm{N,STOP}}}) \in H_n(X; \mathbf{L}_\bullet\langle 1\rangle)$.

Proof 2221

We can assume that f is transverse to the dual cell decomposition of M'. Consider the dissection

$$M' = \bigcup_{\sigma\in M'} M'(\sigma) \qquad \widehat{f} = \bigcup_{\sigma\in M'} (\overline{f}(\sigma), f(\sigma)) \colon M(\sigma) \to M'(\sigma)$$

where each $(f(\sigma), \overline{f}(\sigma))$ is a degree one normal map of $(n-|\sigma|)$-dimensional manifolds $(m-|\sigma|)$-ads. We obtain an assignment which associates to each $\sigma \in M'$ an $(n+1-|\sigma|)$-dimensional pair of normal $(m-|\sigma|)$-ads

$$\sigma \mapsto (W(\sigma), \nu_{\overline{f}(\sigma)}, \rho(\overline{f}(\sigma)), M(\sigma) \amalg M'(\sigma)).$$

These fit together to produce an $\mathbf{\Omega}^N_\bullet$-cobordism of $\mathbf{\Omega}^{STOP}_\bullet$-cycles or equivalently a $\Sigma^{-1}\mathbf{\Omega}^{\mathrm{N,STOP}}_\bullet$-cycle providing us with an element

$$[\widehat{f}]^{\Sigma^{-1}\mathbf{\Omega}^{\mathrm{N,STOP}}_\bullet} \in H_n(X; \Sigma^{-1}\mathbf{\Omega}^{\mathrm{N,STOP}}_\bullet).$$

Composing with the normal signature map $\mathrm{sgn}^{\mathbf{NL}^\bullet}_\Omega \colon \mathbf{\Omega}^N_\bullet \to \mathbf{NL}^\bullet\langle 1/2\rangle$ produces a $\mathbf{NL}^\bullet\langle 1/2\rangle$-cobordism, which can be seen as an $(n+1)$-dimensional (normal, symmetric Poincaré) pair over $\mathbb{Z}_*X$

$$\mathrm{sgn}^{\mathbf{NL}/\mathbf{L}^\bullet}_X(\widehat{f}) = \mathrm{sgn}^{\mathbf{NL}/\mathbf{L}^\bullet}_\Omega([\widehat{f}]^{\Sigma^{-1}\mathbf{\Omega}^{\mathrm{N,STOP}}_\bullet}) \in H_n(X; \mathbf{NL}\langle 1/2\rangle/\mathbf{L}\langle 0\rangle^\bullet).$$

Adapting the proof of 1621 shows that the quadratic chain complex over $\mathbb{Z}_*X$ obtained this way coincides with the quadratic signature $\mathrm{sgn}^{\mathbf{L}_\bullet}_X(\widehat{f})$. □ 1621→p.84

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$\boxed{\mathrm{sgn}^{\mathbf{NL}^\bullet}_\Omega \colon \mathbf{\Omega}^N_\bullet \to \mathbf{NL}^\bullet\langle 1/2\rangle}$ the normal signature map; based on the normal signature $\mathrm{sgn}^{\mathbf{NL}^\bullet}_{\mathbb{Z}\pi}$.

$\boxed{\mathrm{sgn}^{\mathbf{NL}/\mathbf{L}^\bullet}_X(\widehat{f}) := \mathrm{sgn}^{\mathbf{NL}/\mathbf{L}^\bullet}_\Omega([\widehat{f}]^{\Sigma^{-1}\mathbf{\Omega}^{\mathrm{N,STOP}}_\bullet}) \in H_n(M'; \mathbf{NL}/\mathbf{L}^\bullet)}$ the mosaicked normal/symmetric signature over X defined for a degree one normal map $\widehat{f} \colon M \to M'$ between manifolds (see 2221).

$\boxed{\mathbf{NL}\langle 1/2\rangle/\mathbf{L}\langle 0\rangle^\bullet := \mathrm{Fiber}(J \colon \mathbf{L}^\bullet\langle 0\rangle \to \mathbf{NL}^\bullet\langle 1/2\rangle)}$

$\boxed{[K]^{\mathbf{E}} \in H_n(K; \mathbf{E})}$ an n-dimensional $\mathbf{E}$-cycle of a simplicial complex $K \subset \partial\Delta^{m+1}$ defined by a collection $\left\{[K]^{\mathbf{E}}(\sigma) \in \mathbf{E}^{(m-|\sigma|)}_{n-m} \mid \sigma \in K\right\}$ such that $\partial_i[K]^{\mathbf{E}}(\sigma) = [K]^{\mathbf{E}}(\delta_i\sigma)$ if $\delta_i\sigma \in K$ and $\emptyset$ otherwise.

223 (normal,manifold)-cycles and MSTOP-orientations

Porter

To show that the diagram below commutes we use once again canonical orientations. This time with respect to the $\mathbf{MS}(\mathrm{G/TOP})$ spectrum, the fiber of $J \colon \mathbf{MSTOP} \to \mathbf{MSG}$. It comes into play using the Pontrjagin-Thom map. We will relate both paths in the diagram to the same orientation $u^{G/T}$. But for a start we will work with the push-forward living in $\mathbf{MSTOP}$ and then refine the statement to $\mathbf{MS}(\mathrm{G/TOP})$ by using the homotopies of the alternative description for elements in $\mathscr{N}(X)$ and $[X;\mathrm{G/TOP}]$ (see 224).

In order to make the formulas less space-consuming we use $\mathbf{T}$ for $\mathbf{MSTOP}$, $\mathbf{G}$ for $\mathbf{MSG}$ and $\mathbf{G/T}$ for $\mathbf{MS}(\mathrm{G/TOP})$.

223 (normal,manifold)-cycles and MSTOP-orientations

Given a degree one normal map $\widehat{f}_0\colon M \to X$, there is a commutative diagram

$$\begin{array}{ccc} \mathscr{N}(X) & \xrightarrow{[-,\widehat{f}_0]^{\Sigma^{-1}\mathbf{\Omega}^{\mathrm{N,STOP}}_\bullet}} & H_n(X;\Sigma^{-1}\mathbf{\Omega}^{\mathrm{N,STOP}}_\bullet) \\ \downarrow{\scriptstyle t(-,\widehat{f}_0)} & & \uparrow{\scriptstyle S} \\ [X;\mathrm{G/TOP}] \xrightarrow{\widetilde{\Gamma}} H^0(X;\Sigma^{-1}\mathbf{\Omega}^{\mathrm{N,STOP}}_\bullet) & \xrightarrow{u^T(\nu_0)} & H^k(Th(\nu_X);\Sigma^{-1}\mathbf{\Omega}^{\mathrm{N,STOP}}_\bullet) \end{array}$$

i.e. $[-,\widehat{f}_0]^{\Sigma^{-1}\mathbf{\Omega}^{\mathrm{N,STOP}}_\bullet} = S(\widetilde{\Gamma}(t(-,\widehat{f}_0)) \cup u^T(\nu_0))$.

2231 [KMM13, Prop. 14.19]

Let X be an n-dimensional Poincaré space, $\widehat{f}_0\colon M_0 \to X$, $\widehat{f}\colon M \to X$ degree one normal maps and $\nu, \nu_0\colon X \to \mathrm{BSTOP}$ topological bundle reductions of the Spivak normal fibration ν_X. Then we have

$$S(u^{G/T}((\nu,h),(\nu_0,h_0)) = [\widehat{f},\widehat{f}_0]^{\Sigma^{-1}\mathbf{\Omega}^{\mathrm{N,STOP}}_\bullet} \in H_n(X;\mathbf{G/T}).$$

15 (16) Orientations and signatures [Ran92, Proposition 16.1]

(iii) Let $\widehat{f}\colon M \to M'$ be a degree one normal map of n-dimensional simply-connected topological manifolds with M' triangulated, corresponding to a pair (β,h) with $\beta\colon M' \to \mathrm{BSTOP}$ and $h\colon J(\beta)\simeq \nu_X$. Then we have

$$S(u^{\mathbf{NL}/\mathbf{L}^\bullet}(\beta,h)) = \mathrm{sgn}^{\mathbf{NL}/\mathbf{L}^\bullet}_{M'}(\widehat{f}) \in H_n(M';\mathbf{NL}\langle 1/2\rangle/\mathbf{L}\langle 0\rangle^\bullet).$$

Proof 223

Use the Pontrjagin-Thom map $\Sigma^{-1}\mathbf{\Omega}^{\mathrm{N,STOP}}_\bullet \simeq \mathbf{G/T}$ to consider $\widetilde{\Gamma}$ as a map

$$[X;\mathrm{G/TOP}] \to H^0(X;\mathbf{G/T}).$$

Now let $(\overline{\nu},\overline{h})$ represent an element of $[X;\mathrm{G/TOP}]$ and set $(\nu,h) = (\overline{\nu}\oplus\nu_0,\overline{h}\oplus h)$ so that we
2231→p.93 have $t((\nu,h),(\nu_0,h_0)) = (\overline{\nu},\overline{h})$. Apply 2231 to rewrite the claimed formula as

$$S^{-1}([\widehat{f},\widehat{f}_0]^{\Sigma^{-1}\mathbf{\Omega}^{\mathrm{N,STOP}}_\bullet}) = u^{G/T}((\nu,h),(\nu_0,h_0)) = \widetilde{\Gamma}(\overline{\nu},\overline{h}) \cup u^T(\nu_0)$$

where $\widehat{f},\widehat{f}_0 \in \mathscr{N}(X)$ are the degree one normal maps represented by (ν,h) and (ν_0,h_0). Recall that $\overline{h}\colon J(\overline{\nu}) \simeq \varepsilon$ and that there are canonical orientations $u^T(\nu)$ and $u^T(\varepsilon)$ and the homotopy $u^G(\overline{h})\colon u^G(J(\nu)) \simeq u^G(\varepsilon)$. Then $\widetilde{\Gamma}$ maps to $(\overline{\nu},\overline{h})$ to the unique lift of $u^T(\nu) - u^T(\varepsilon)$ determined by the homotopy $\overline{h}$ and given by the pair $(u^T(\nu)-u^T(\varepsilon), u^G(\overline{h}))$. Define Γ to be the composition

$$[X;\mathrm{G/TOP}] \xrightarrow{\widetilde{\Gamma}} H^0(X;\mathbf{G/T}) \xrightarrow{\mathrm{incl}} H^0(X;\mathbf{T})$$

which is given by forgetting the homotopy $u^G(\overline{h})$, i.e.

$$\Gamma(\overline{\nu},\overline{h}) = u^T(\overline{\nu}) - u^T(\varepsilon)\colon \mathrm{Th}(\overline{\nu}) \simeq \Sigma^k\Delta^l{}_+ \simeq \mathrm{Th}(\varepsilon) \to \mathbf{T}.$$

The Thom isomorphism

$$-\cup u^T(\nu_0)\colon H^0(X;\mathbf{T}) \to H^k(\mathrm{Th}(\nu_X);\mathbf{T})$$

applied to $u^T(\overline{\nu})$ is given by the composition

$$\mathrm{Th}(\nu_0) \xrightarrow{\widetilde{\Delta}} \Sigma^l X_+ \wedge \mathrm{Th}(\nu_0) \xrightarrow{u^T(\overline{\nu})\wedge u^T(\nu_0)} \mathbf{T}\wedge\mathbf{T} \xrightarrow{\oplus} \mathbf{T}.$$

From the relationship between Whitney sum and the cross product and the diagonal map we obtain that

$$u^T(\nu) = u^T(\overline{\nu}) \cup u^T(\nu_0) \qquad \text{and} \qquad u^T(\nu_0) = u^T(\varepsilon) \cup u^T(\nu_0).$$

Subtracting these equations we obtain

$$u^T(\nu) - u^T(\nu_0) = \Gamma(\overline{\nu}, \overline{h}) \cup u^T(\nu_0).$$

Now the left hand side lifts to $u^{G/T}((\nu,h),(\nu_0,h_0))$ using the null-homotopy $J(u^T(\nu) - u^T(\nu_0))$ coming from $h\cup h_0\colon J(\nu)\simeq J(\nu_0)$. On the right hand side we have the null-homotopy $\overline{h}\colon J(\overline{\nu})\simeq\varepsilon$. Applying the cup product with $u^T(\nu_0)$ to this null-homotopy corresponds to taking the Whitney sum with ν_0 and produces the homotopy $\overline{h}\oplus\mathrm{id}_{\nu_0}\colon J(\overline{\nu})\simeq J(\nu_0)$. Now the claim that

$$u^{G/T}((\nu,h),(\nu_0,h_0)) = \widetilde{\Gamma}(\overline{\nu},\overline{h}) \cup u^T(\nu_0)$$

follows from the Spivak normal fibration's property that fiberwise homotopy equivalences of stable topological bundle reductions are stably fiberwise homotopic. □

Proof 2231

This follows from 15 (iii). We only have to omit the signature map $\mathrm{sgn}_\Omega^{\mathbf{NL}/\mathbf{L}^\bullet}\colon \Sigma^{-1}\mathbf{\Omega}_\bullet^{\mathrm{N,STOP}} \to$ 15→p.42
$\mathbf{NL}\langle 1/2\rangle/\mathbf{L}\langle 0\rangle^\bullet$which we used to push the equivalence forward from $\Sigma^{-1}\mathbf{\Omega}_\bullet^{\mathrm{N,STOP}} \simeq \mathbf{G}/\mathbf{T}$ to $\mathbf{NL}\langle 1/2\rangle/\mathbf{L}\langle 0\rangle^\bullet$ and we have to replace the cycle $[\widehat{f}]^{\Sigma^{-1}\mathbf{\Omega}_\bullet^{\mathrm{N,STOP}}}$ and the orientation $u^{G/T}(\nu,h)$ by the generalized versions $[\widehat{f},\widehat{f}_0]^{\Sigma^{-1}\mathbf{\Omega}_\bullet^{\mathrm{N,STOP}}}$ and $u^{G/T}((\nu,h),(\nu_0,h_0))$ for pairs of degree one normal maps with targets Poincaré spaces. □

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$u^T(\alpha)\in H^k(\mathrm{Th}(\alpha);\mathbf{T})$ the canonical $\mathbf{T}$-orientation of α which is a map on the Thom spaces $\mathrm{Th}(\alpha)\to\mathrm{Th}(\gamma_{STOP})$ induced by the classifying map of a k-dimensional $\mathbb{Z}$-oriented topological bundle $\alpha\colon X\to\mathrm{BSG}(k)$.

$u^{G/T}(\nu,h)\in H^k(\mathrm{Th}(\nu_X);\mathbf{G}/\mathbf{T})$ the preferred lift of $u^T(\nu)$ for a bundle reduction ν of the Spivak normal fibration ν_X, determined by the homotopy $h\colon\mathrm{Th}(\nu_X)\times[0,1]\to\mathbf{G}$ between $J(\nu)$ and $J(\nu_X)$.

$u^{G/T}((\nu,h),(\nu_0,h_0))\in H^k(\mathrm{Th}(\nu_X);\mathbf{G}/\mathbf{T})$ the preferred lift of $u^T(\nu)-u^T(\nu_0)$ for two bundle reductions ν,ν_0 of the Spivak normal fibration ν_X. The lift is obtained from the homotopy $h_0\cup h\colon\mathrm{Th}(\nu_X)\times[0,1]\to\mathbf{G}$ between $J(\nu)$ and $J(\nu_0)$. If X is a manifold with a preferred topological bundle $\overline{\nu}_X$, define $u^{G/T}(\nu) = u^{G/T}(\nu,\overline{\nu}_X)$.

$\boxed{S\colon [Y,Z] \xrightarrow{\cong} [S^N, X \wedge Y]}$ the S-duality isomorphism; for an N-dimensional S-duality map $\alpha\colon S^N \to X \wedge Y$ and an arbitrary space Z defined by $S(\gamma) = (\mathrm{id}_Y \wedge \gamma) \circ \alpha$; denotes the induced isomorphism $S\colon H^{N-*}(X;\mathbf{E}) \xrightarrow{\cong} H_*(X, \mathbf{E})$ as well.

$\boxed{\widetilde{\Delta}\colon \mathrm{Th}(\nu) \simeq \frac{V}{\partial V} \xrightarrow{\Delta} \frac{V \times V}{V \times \partial V} \simeq \mathrm{Th}(\nu) \wedge X_+}$ the generalized diagonal map where V is the mapping cylinder of the projection map of ν and ∂V the total space of ν.

224 Classification of normal invariants

We give a brief construction of this equivalence. The details can be found in [Wal99, chapter 10].

224 Classification of normal invariants [Wal99, chapter 10]
Let X be a Poincaré space and $\widehat{f}_0 \in \mathscr{N}(X)$ a degree one normal map. Then there is a bijection

$$t(-, \widehat{f}_0)\colon \mathscr{N}(X) \xrightarrow{\cong} [X; \mathrm{G/TOP}].$$

Proof 224

An element in the set $[X;\mathrm{G/TOP}]$ of homotopy classes of maps from X to $\mathrm{G/TOP}$ can be represented by a pair $(\overline{\nu}, \overline{h})$, where $\overline{\nu}\colon X \to \mathrm{BSTOP}$ is a stable topological bundle on X and $\overline{h}\colon J(\overline{\nu}) \simeq *$ is a homotopy from the underlying spherical fibration to the trivial spherical fibration represented by the constant map.

An element $\widehat{f} \in \mathscr{N}(X)$ which is a degree one normal map $\widehat{f}\colon M \to X$ can also be represented by a pair (ν, h) where $\nu\colon X \to \mathrm{BSTOP}$ is again a stable topological bundle but this time $h\colon J(\nu) \simeq \nu_X$ is a homotopy to the Spivak normal fibration ν_X. These two descriptions of $\mathscr{N}(X)$ are identified by using the Pontrjagin-Thom construction.

The set $[X;\mathrm{G/TOP}]$ is a group under the Whitney sum operation and by using the pair description we can define an action on $\mathscr{N}(X)$ by

$$\begin{aligned}[X;\mathrm{G/TOP}] \times \mathscr{N}(X) &\longrightarrow \mathscr{N}(X),\\ ((\overline{\nu},\overline{h}),(\nu,h)) &\longmapsto (\overline{\nu}\oplus\nu, \overline{h}\oplus h).\end{aligned}$$

This action is free and transitive and hence any choice of an element $\widehat{f}_0 \in \mathscr{N}(X)$ gives a bijection $[X;\mathrm{G/TOP}] \cong \mathscr{N}(X)$. We denote the inverse by

$$t(-, \widehat{f}_0)\colon \mathscr{N}(X) \to [X;\mathrm{G/TOP}].$$ □

231 The absolute case

231 The absolute case
Let $\widehat{g}\colon N \to Y$ be a degree one normal map from a Poincaré space N to a normal space Y both of dimension n. There is a homotopy equivalence of quadratic chain complexes

$$h\colon \partial^Q \operatorname{sgn}^{\mathbf{L}_\bullet}_{\#}(\widehat{g}) \xrightarrow{\simeq} -\partial^N \operatorname{sgn}^{\mathbf{NL}^\bullet}_{\mathbb{Z}\pi}(Y) \in L_{n-1}(\mathbb{Z}\pi).$$

For defining the left side

A27 (231) Quadratic signature for normal target spaces [Ran81, Prop. 7.3.4 (iv)]
Let $\widehat{g}\colon N \to Y$ be a degree one normal map from an Poincaré space to a normal space both of dimension n. There is a quadratic signature

$$\mathrm{sgn}^{\mathbf{L}_\bullet}_{\#}(\widehat{g}) = (C^!, \psi^!) := (\mathscr{C}(g^!), \mathrm{con}^{\psi^!}_{\Gamma_Y}(u^*_{\nu_Y}))$$

producing an n-quadratic chain complex (not necessarily Poincaré) such that $(1+t)(\psi^!) = e^{\%}_{g^!}(\varphi_N)$.

B23 (231) Quadratic boundary
An n-quadratic chain complex $(C, \psi) \in L_n(\mathbb{Z}\pi)$ has an $(n-1)$-quadratic boundary

$$\partial^Q(C, \psi) := (\partial C, \partial\psi) := (\Sigma^{-1}\mathscr{C}(\varphi_0), \left(\begin{smallmatrix}1+t\\ \mathcal{S}\end{smallmatrix}\right)^{-1}(\mathcal{S}^{-1}e^{\%}(\varphi), e_{\%}(\psi))$$

where $\varphi = (1+t)\psi$.

For defining the right side

A33 (231) Normal signature [Ran80b, §9][Wei85a, Theorem 3.4]
Let (Y, ν, ρ) be an n-dimensional normal space. There is a normal signature

$$\mathrm{sgn}^{\mathbf{NL}^\bullet}_{\mathbb{Z}\pi}(Y) \in NL^n(\mathbb{Z}\pi)$$

producing an n-normal chain complex $(C, (\varphi, \gamma, \chi))$.

B24 (112, 231) Quadratic boundary for normal
An n-normal chain complex $(C, (\varphi, \gamma, \chi)) \in NL^n(\mathbb{Z}\pi)$ has an $(n-1)$-quadratic Poincaré boundary

$$\partial^N(C, (\varphi, \gamma, \chi)) =: (\partial C, \partial\psi)$$

which defines a map $\partial^N : NL^n(\mathbb{Z}\pi) \longrightarrow L_{n-1}(\mathbb{Z}\pi); \quad (C, (\varphi, \gamma, \chi)) \mapsto (\partial C, \partial\psi)$.

For identifying both sides

B27 (231) Quadratic boundary signature [KMM13, Constr. 3.25][Ran81, Prop. 7.4.1][Wei85b, Theorem 7.1]
Let (Y, ν, ρ) be an n-dimensional normal space. The quadratic boundary signature

$$\partial gn^{\mathbf{L}_\bullet}_{\mathbb{Z}\pi}(Y) =: (\partial C', \partial\psi') \in L_{n-1}(\mathbb{Z}\pi)$$

produces an $(n-1)$-quadratic Poincaré chain complex such that $(\partial C', \partial\psi') = \partial^N(\mathrm{sgn}^{\mathbf{NL}^\bullet}_{\mathbb{Z}\pi}(Y)) \in L_{n-1}(\mathbb{Z}\pi)$.

2311 Property of spectral quadratic construction [Ran81, Proposition 7.3.1 (v)]
Let $F\colon X \to \Sigma^p Y$ and $F'\colon X' \to \Sigma^p Y'$ be semi-stable maps fitting into the following commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{F} & \Sigma^p Y \\ \downarrow{\scriptstyle G_X} & & \downarrow{\scriptstyle G_Y} \\ X' & \xrightarrow{F'} & \Sigma^p Y' \end{array}$$

inducing the commutative diagram of chain complexes

$$\begin{array}{ccccc} \Sigma^{-p}\widetilde{C}(X) & \xrightarrow{f} & \widetilde{C}(Y) & \xrightarrow{e} & \mathscr{C}(f) \\ \downarrow{\scriptstyle g_X} & & \downarrow{\scriptstyle g_Y} & & \downarrow{\scriptstyle \left(\begin{smallmatrix} g_Y & 0 \\ 0 & g_X \end{smallmatrix}\right)} \\ \Sigma^{-p}\widetilde{C}(X') & \xrightarrow{f'} & \widetilde{C}(Y') & \xrightarrow{e'} & \mathscr{C}(f'). \end{array}$$

Then the spectral quadratic constructions of F and F' are related by

$$\mathrm{con}^{\psi^!}_{F'} \circ g_X = \left(\begin{smallmatrix} g_Y & 0 \\ 0 & g_X \end{smallmatrix}\right)_{\%} \circ \mathrm{con}^{\psi^!}_{F} + (e')_{\%} \circ \mathrm{con}^{\psi}_{G_Y} \circ f.$$

Firstly, denoting $\partial^Q \operatorname{sgn}^{\mathbf{L}_\bullet}_{\#}(\widehat{g})$ by $(\partial C^!, \delta\psi^!)$ and $\partial^N \operatorname{sgn}^{\mathbf{NL}^\bullet}_{\mathbb{Z}\pi}(Y)$ by $(\partial C, \psi)$, we identify the chain complexes ∂C and $\partial C^!$. The next step is the identification of the quadratic structures $\delta\psi$ and $\psi^!$ on these chain complexes.

Proof 231 - The boundary chain complexes

Recall that φ in $(C, (\varphi, \gamma, \chi)) \coloneqq \operatorname{sgn}^{\mathbf{NL}^\bullet}_{\mathbb{Z}\pi}(Y)$ is the symmetric structure produced by applying the symmetric construction $\operatorname{con}^{\varphi}_{Y}$ to a fundamental cycle $[Y]$.

Consider the following commutative diagram where all sequences are cofibrations and the map μ is the induced map.

$$\begin{array}{ccccc}
0 & \longrightarrow & C(Y)^{n-*} & \xrightarrow{\text{id}} & C(Y)^{n-*} \\
\downarrow & & \downarrow{\scriptstyle g^!} & & \downarrow{\scriptstyle \varphi_0} \\
\Sigma^{-1}\mathscr{C}(g) & \xrightarrow{p_g} & C(N) & \xrightarrow{g} & C(Y) \\
\downarrow{\scriptstyle \text{id}} & & \downarrow{\scriptstyle e_{g^!}} & & \\
\Sigma^{-1}\mathscr{C}(g) & \xrightarrow{\mu} & \mathscr{C}(g^!) & &
\end{array}$$

The diagram induces a homotopy equivalence $\mathscr{C}(\mu) \xrightarrow{\simeq} \mathscr{C}(\varphi_0)$ in the lower right corner. Using the Poincaré duality of N we obtain in the lower left corner of the diagram the homotopy equivalence

$$\begin{pmatrix} 0 & \text{id} \\ (\varphi_N)_0 & 0 \end{pmatrix} : \mathscr{C}(g^!)^{n-*} = C(N)^{n-*} \oplus C(Y)_{*-1} \xrightarrow{\simeq} C(N) \oplus C(Y)_{*-1} = \Sigma^{-1}\mathscr{C}(g).$$

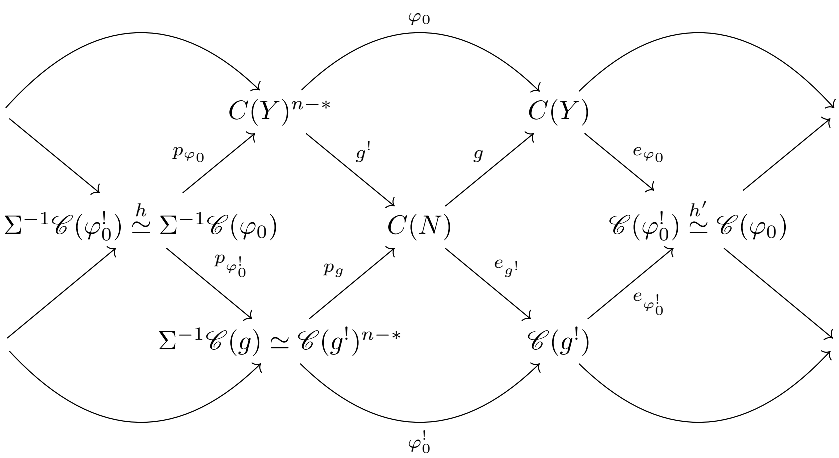
Composed with μ we recover the map $\varphi^!_0 : \mathscr{C}(g^!)^{n-*} \to \mathscr{C}(g^!)$. We end up with a homotopy equivalence

$$h' : \Sigma\partial C^! = \mathscr{C}(\varphi^!_0) \xrightarrow{\simeq} \mathscr{C}(\mu) \xrightarrow{\simeq} \mathscr{C}(\varphi_0) = \Sigma\partial C$$

which satisfies the equation

$$e_{g^!} \circ e_{\varphi^!_0} \circ h' = g \circ e_{\varphi_0} \tag{231.1}$$

as we see in the right square of the following braid. We will need this equation later for identifying the boundary structures.



On the left hand side of the braid we obtain the homotopy equivalence we are looking for,

$$h\colon \partial C^! = \Sigma^{-1}\mathscr{C}(\varphi_0^!) \simeq \Sigma^{-1}\mathscr{C}(\varphi_0) = \partial C,$$

which is related to h' by $-\Sigma(h) = h'$ due to the sign conventions used for suspensions. □

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$\partial C = \Sigma^{-1}\mathscr{C}(\varphi_0)$ the boundary chain complex produced by $\partial^N \operatorname{sgn}^{\mathbf{NL}^\bullet}_{\mathbb{Z}\pi}(Y)$.

- $\varphi = \operatorname{con}^{\varphi}_Y([Y])$ the symmetric structure produced by the symmetric construction for Y and part of the normal signature $\operatorname{sgn}^{\mathbf{NL}^\bullet}_{\mathbb{Z}\pi}(Y)$.
- ∂^N the quadratic boundary construction, produces for an n-normal chain complex $(C,(\varphi,\gamma,\chi))$ an $(n-1)$-quadratic Poincaré chain complex denoted $(\partial C, \partial\psi)$; induces a map $NL^n(R) \to L_{n-1}(R)$ (see B24).
- $\operatorname{sgn}^{\mathbf{NL}^\bullet}_{\mathbb{Z}\pi}(Y) = (C,(\varphi,\gamma,\chi)) \in NL^n(\mathbb{Z}\pi)$ an n-normal chain complex called the normal signature of Y (see A33).
- $Y = (Y,\nu,\rho)$ an n-dimensional normal space.

$\partial C^! = \Sigma^{-1}\mathscr{C}(\varphi_0^!)$ the boundary chain complex produced by $\partial^Q \operatorname{sgn}^{\mathbf{L}_\bullet}_{\#}\widehat{g}$.

- $\varphi^! = (1+t)\psi^! \in W^{\%}(\mathscr{C}(g^!))_n$ the symmetric structure given by the symmetrization of the quadratic structure $\psi^!$ of $\operatorname{sgn}^{\mathbf{L}_\bullet}_{\#}(\widehat{g})$.
- ∂^Q the quadratic boundary construction, produces for an n-quadratic chain complex $(C^!,\psi^!)$ an $(n-1)$-quadratic Poincaré chain complex denoted by $(\partial C^!, \partial\psi^!)$ (see B23).
- $\operatorname{sgn}^{\mathbf{L}_\bullet}_{\#}(\widehat{g}) =: (C^!,\psi^!)$ an n-quadratic chain complex, called the quadratic signature of $\widehat{g}$ (see A27).
  - $C^! = \mathscr{C}(g^!)$ the chain complex produced by the quadratic signature $\operatorname{sgn}^{\mathbf{L}_\bullet}_{\#}(\widehat{g})$.
    - $g^!$ the chain map given by the composition $C(Y)^{n-*} \xrightarrow{g^*} C(N)^{n-*} \xrightarrow{(\varphi_N)_0} C(N)$.
      - $\varphi_N = \operatorname{con}^{\varphi}_N([N]) \in W^{\%}(C(N))_n$
  - $\psi^! = \operatorname{con}^{\psi^!}_{\Gamma^!}(u^*_{\nu_Y}) \in W_{\%}(\mathscr{C}(g^!))_n$ the n-quadratic structure produced by the spectral quadratic construction, part of the quadratic signature $\operatorname{sgn}^{\mathbf{L}_\bullet}_{\#}\widehat{g}$.
- $(\overline{g}, g)$ a degree one normal map $g\colon N \to Y$ from a Poincaré space N to a normal space Y both of dimension n.

$\boxed{e\colon C \to \mathscr{C}(\varphi_0)}$ the inclusion; with a map $\alpha\colon C \to D$ as subscript e_α denotes the inclusion $D \to \mathscr{C}(\alpha)$.

$\boxed{p\colon \mathscr{C}(\varphi_0) \to \Sigma C^{n-*}}$ the projection; with a map $\alpha\colon C \to D$ as subscript e_α denotes the projection $\mathscr{C}(\alpha) \to \Sigma C$.

Proof 231 - The boundary structures

B27→p.155 Instead of the quadratic structure $\partial\psi$ of the quadratic boundary of the normal signature of $\operatorname{sgn}^{\mathbf{NL}^\bullet}_{\mathbb{Z}\pi}(Y)$ we use the equivalent quadratic structure $\partial\psi'$ of the quadratic boundary signature $\partial\operatorname{sgn}^{\mathbf{L}_\bullet}_{\mathbb{Z}\pi}(Y)$. It is obtained in a similar way as $\partial\psi^!$, namely by using the following cofibration sequence of chain complexes

$$\begin{array}{ccccc}
\Sigma W_\%(\partial C) & \xrightarrow{\binom{1+t}{\mathcal{S}}} & \Sigma W^\%(\partial C) \oplus W_\%(\Sigma\partial C) & \xrightarrow{\mathcal{S}-(1+t)} & W^\%(\Sigma\partial C) \\
\partial\psi^! & \dashmapsto & (\partial\varphi^!, S\partial\psi^!) & \longmapsto & (\mathcal{S}(\partial\varphi^!) - (1+t)S\partial\psi^!) = 0 \\
\partial\psi' & \dashmapsto & (\partial\varphi, S\partial\psi) & \longmapsto & (\mathcal{S}(\partial\varphi) - (1+t)S\partial\psi) = 0
\end{array}$$

where $S\partial\psi^!$ and $S\partial\psi$ are defined as follows

$$\begin{aligned}
S\partial\psi^! &:= (e_{\varphi_0^!})_\%(\psi^!) \quad \in W_\%(\Sigma\partial C^!)_n \simeq W_\%(\Sigma\partial C)_n \\
S\partial\psi &:= \operatorname{con}^{\psi^!}_{\Gamma_Y}(u(\nu)^*) \in W_\%(\Sigma\partial C)_n
\end{aligned}$$

They satisfy the equations

$$\begin{aligned}
(1+t)S\partial\psi^! &= (1+t)(e_{\psi_0^!})_\%(\psi^!) = e^\%_{\varphi_0^!}((1+t)\psi^!) = e^\%_{\varphi_0^!}\varphi^! = \mathcal{S}(\partial\varphi^!) \\
(1+t)S\partial\psi &= (e_{\varphi_0})_\%(\varphi) = \mathcal{S}(\partial\varphi)
\end{aligned}$$

and hence define the preimages $\partial\psi'$ and $\partial\psi^!$ as indicated above.

Thus, in order to identify the quadratic boundary structures $\partial\psi' = -\partial\psi^!$ it is enough to identify the symmetric boundary structures $\partial\varphi = -\partial\varphi^!$ and the suspended quadratic boundary structures $S\partial\psi = -S\partial\psi^!$.

We begin with the symmetric components $\partial\varphi^!$ and $\partial\varphi$. By the properties of the quadratic signature (coming from the basic properties 2311 of the spectral quadratic construction) we have the following description of the symmetric structure

$$\varphi^! = (1+t)\psi^! = e^\%_{g^!}(\varphi_N)$$

and by the naturality of the symmetric construction we have

$$\varphi = g^\%(\varphi(N)).$$

So the symmetric boundaries are both given by φ_N in the following way:

$$\partial\varphi^! = \mathcal{S}^{-1}e^\%_{\varphi_0^!}(\varphi^!) = \mathcal{S}^{-1}e^\%_{\varphi_0^!} \circ e^\%_{g^!}(\varphi_N) \in W^\%(\Sigma^{-1}\mathscr{C}(\varphi^!))_{n-1}$$

and

$$\partial\varphi = \mathcal{S}^{-1}e^\%_{\varphi_0}(\varphi) = \mathcal{S}^{-1}e^\%_{\varphi_0} \circ g^\%(\varphi_N) \in W^\%(\Sigma^{-1}\mathscr{C}(\varphi))_{n-1}.$$

From the right square of the commutative braid we get the following equation for the suspended boundaries.

$$(h')^{\%}\mathcal{S}(\partial\varphi^!) = (h')^{\%} \circ e^{\%}_{\varphi_0^!} \circ e^{\%}_{g^!}(\varphi_N) = e^{\%}_{\varphi_0} \circ g^{\%}(\varphi_N) = \mathcal{S}(\partial\varphi)$$

Because of the injectivity of the suspension map and the sign conventions, we have $h^{\%}(\partial\varphi^!) = -\partial\varphi$.

Now it remains to identify the quadratic structures $S\partial\psi^!$ and $S\partial\psi$. The quadratic signature $\psi^!$ was obtained by applying the spectral quadratic construction $\mathrm{con}^{\psi^!}_{\Gamma^!}$ of the semi-stable map

$$\Gamma^! : \mathrm{Th}(\nu_Y)^* \xrightarrow{\mathrm{Th}(\overline{g})} \mathrm{Th}(\nu_N)^* \xrightarrow{\Gamma_N} \Sigma^p N_+$$

to the S-dual $u(\nu_Y)^*$ of the Thom class of ν_Y. Similarly, we get the quadratic signature ψ by applying the spectral quadratic construction $\mathrm{con}^{\psi^!}_{\Gamma_Y}$ of the semi-stable map

$$\Gamma_Y : \mathrm{Th}(\nu_Y)^* \longrightarrow \Sigma^p Y_+$$

to the same Thom class $u(\nu_Y)^*$. Now we use 2311 for the following diagram.

$$\begin{array}{ccc} \mathrm{Th}(\nu_Y)^* & \xrightarrow{\Gamma^!} & \Sigma^p N_+ \\ \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle \Sigma^p g} \\ \mathrm{Th}(\nu_Y)^* & \xrightarrow{\Gamma_Y} & \Sigma^p Y_+ \end{array}$$

On the chain level we get the induced diagram

$$\begin{array}{ccccc} \Sigma^{-p}C(\mathrm{Th}(\nu_Y)^*) & \xrightarrow{\gamma^!} & C(N) & \longrightarrow & \mathscr{C}(\gamma^!) \\ \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle g_*} & & \downarrow{\scriptstyle \left(\begin{smallmatrix} g_* & 0 \\ 0 & 1 \end{smallmatrix}\right)} \\ \Sigma^{-p}C(\mathrm{Th}(\nu_Y)^*) & \xrightarrow{\gamma_Y} & C(Y) & \xrightarrow{e_{\gamma_Y}} & \mathscr{C}(\gamma_Y) \end{array}$$

Via S-duality and Thom isomorphism (see the diagram B27.1 in B27) this is equivalent to

$$\begin{array}{ccccc} C(Y)^{n-*} & \xrightarrow{g^!} & C(N) & \longrightarrow & \mathscr{C}(g^!) \\ \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle g_*} & & \downarrow{\scriptstyle \left(\begin{smallmatrix} g_* & 0 \\ 0 & 1 \end{smallmatrix}\right)} \\ C(Y)^{n-*} & \xrightarrow{\varphi_0} & C(Y) & \xrightarrow{e_{\varphi_0}} & \mathscr{C}(\varphi_0) \end{array}$$

We end up with the relation $\mathrm{con}^{\psi^!}_{\Gamma_Y} = \left(\begin{smallmatrix} g_* & 0 \\ 0 & 1 \end{smallmatrix}\right)_{\%} \circ \mathrm{con}^{\psi^!}_{\Gamma^!} + (e_{\lambda_0})_{\%} \circ \mathrm{con}^{\psi}_{\Sigma^p g} \circ g^!$. However, the stable map $\Sigma^p g$ comes from a map of spaces $g\colon N \to Y$ and hence $\mathrm{con}^{\psi}_{\Sigma^p g} = 0$. The following commutative diagram remains.

$$\begin{array}{ccc} C(Y)^{n-*} & \xrightarrow{\mathrm{con}^{\psi^!}_{\Gamma^!}} & W_{\%}(\mathscr{C}(g^!))_n \\ & \searrow{\scriptstyle \mathrm{con}^{\psi^!}_{\Gamma_Y}} & \downarrow{\scriptstyle \left(\begin{smallmatrix} g_* & 0 \\ 0 & 1 \end{smallmatrix}\right)_{\%}} \\ & & W_{\%}(\mathscr{C}(\varphi_0))_n \end{array}$$

From the braid we get the identification $\begin{pmatrix} g_* & 0 \\ 0 & 1 \end{pmatrix}_{\%} = (h' \circ e_{\varphi^!_0})_{\%}$. Hence we obtain that

$$\begin{aligned} h'_{\%}(S\partial\psi^!) &= h'_{\%} \circ (e_{\varphi^!_0})_{\%}(\psi^!) \\ &= (h' \circ e_{\varphi^!_0})_{\%} \operatorname{con}^{\psi^!}_{\Gamma^!}(u(\nu_Y)^*) \\ &= \operatorname{con}^{\psi^!}_{\Gamma_Y}(u(\nu_Y)^*) \\ &= S\partial\psi. \end{aligned}$$ □

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$\boxed{\partial\psi^! = \left(\tfrac{1+t}{\mathcal{S}}\right)^{-1}(\partial\varphi^!, S\partial\psi^!) \in W_{\%}(\partial C^!)}$ the $(n-1)$-quadratic Poincaré structure of $\partial^Q \operatorname{sgn}^{\mathbf{L}^\bullet}_{\#}(\widehat{g})$.

- $\boxed{\partial\varphi^! = \mathcal{S}^{-1} e^{\%}_{\varphi^!_0}(\varphi^!) \in W^{\%}(\partial C^!)_{n-1}}$ the $(n-1)$-symmetric boundary structure of $(C^!, \varphi^!)$ (see B21).
- $\boxed{S\partial\psi^! = (e_{\varphi^!_0})_{\%}(\psi^!) \in W_{\%}(\Sigma\partial C^!)_n}$

$\boxed{\partial\psi}$ the $(n-1)$-quadratic Poincaré structure of $\partial^N \operatorname{sgn}^{\mathbf{NL}^\bullet}_{\mathbb{Z}\pi}(Y)$, equivalent to $\partial\psi'$. (see symmetric-B24 and B27).

$\boxed{\partial\psi' = \left(\tfrac{1+t}{\mathcal{S}}\right)^{-1}(\partial\varphi, \operatorname{con}^{\psi^!}_{\Gamma_Y}(u^*_{\nu_Y}))}$ the $(n-1)$-quadratic structure produced by the quadratic boundary signature $\partial\mathrm{gn}^{\mathbf{L}^\bullet}_{\mathbb{Z}\pi}$.

- $\boxed{\partial\varphi^! = \mathcal{S}^{-1} e^{\%}_{\varphi_0}(\varphi) \in W^{\%}(\partial C)_{n-1}}$ the $(n-1)$-symmetric boundary structure of (C, ψ).
- $\boxed{S\partial\psi = \operatorname{con}^{\psi^!}_{\Gamma_Y}(u(\nu)^*) \in W_{\%}(\Sigma\partial C)_n}$

$\boxed{\mathcal{S}\colon W^{\%}C \to \Sigma^{-1}W^{\%}(\Sigma C))}$ the suspension map; defined by $(\mathcal{S}(\varphi))_k := \varphi_{k-1}$.

$\boxed{1+t\colon W_{\%}C \to W^{\%}C}$ the symmetrization map defined by

$$(1+t)(\psi)_s = \begin{cases} (1+t)\psi_0 & \text{if } s = 0 \\ 0 & \text{otherwise} \end{cases}$$

induces a map of L-groups $1+t\colon L_n(R) \to L^n(R)$.

$\boxed{(\partial C, \partial\varphi)}$ the symmetric boundary of an n-symmetric chain complex obtained from algebraic surgery on the pair $(0 \to C, \varphi, 0)$, i.e. $\partial C = \Sigma^{-1}\mathscr{C}(\varphi_0)$, $\partial\varphi = \mathcal{S}^{-1}e^{\%}(\varphi)$ where $e\colon C \to \mathscr{C}(\varphi_0)$ is the inclusion (see B21 for more details).

$\boxed{\operatorname{con}^{\psi^!}_F\colon \widetilde{C}(X)_{p+*} \to W_{\%}(\mathscr{C}(f))}$ a chain map called the spectral quadratic construction; defined for a semi-stable map $F\colon X \to \Sigma^p Y$ where $f\colon \widetilde{C}(X)_{p+*} \to \widetilde{C}(\Sigma^p Y)_{p+*} \simeq \widetilde{C}(Y)$ is the chain map induced by F.

$\boxed{\mathrm{con}^{\psi}_{F} : C(X) \to W_{\%}(C(Y))}$ a chain map called quadratic construction; defined for a stable map $F : \Sigma^p X \to \Sigma^p Y$ of pointed topological spaces X, Y.

232 The relative case

The relative case is, in principle, analog to the absolute case 231. We do not give an explicit definition of the normal pair signature $\mathrm{sgn}^{\mathbf{NL}^\bullet}_{\to}$ and the quadratic boundary $\partial^N_{\to}$ of a normal pair. Instead we use the boundary pair signature and refer to the proof of [Wei85b, Theorem 7.1] for the validation that this is a homotopy equivalent replacement.

232 The relative case
Let $(\delta\widehat{g}, \widehat{g}) : (N, A) \to (Y, B)$ be a degree one normal map from a Poincaré pair (N, A) to a normal pair (Y, B) both of dimension $(n+1)$. Then there is a homotopy equivalence of quadratic pairs

$$h : \partial^Q_{\to} \, \mathrm{sgn}^{\mathbf{L}_\bullet}_{\to}(\delta\widehat{g}, \widehat{g}) \simeq -\partial^N_{\to} \, \mathrm{sgn}^{\mathbf{NL}^\bullet}_{\to}(Y, B).$$

A28 (232) Relative quadratic signature for normal target spaces [Ran80b, Prop. 6.4]
Let $(\delta\widehat{g}, \widehat{g}) : (N, A) \to (Y, B)$ be a degree one normal map from a Poincaré pair (N, A) to a normal pair (Y, B) both of dimension n. There is an n-quadratic pair called the relative quadratic signature

$$\mathrm{sgn}^{\mathbf{L}_\bullet}_{\to}(\delta\widehat{g}, \widehat{g}) = (G^! : C^! \to D^!, (\delta\psi^!, \psi^!)).$$

such that

$$(1+t)(\delta\psi^!, \psi^!) = {e_{g^!_i, g^!}}^{\%} \circ \mathrm{con}^{\delta\varphi,\varphi}_{N,A}([N, A]).$$

B25 (232) Quadratic boundary pair
Let $(G^! : C^! \to D^!, (\delta\psi^!, \psi^!))$ be an n-quadratic pair and $(\delta\varphi^!, \varphi^!) = (1+t)(\delta\psi^!, \psi^!)$ and $\varphi^!_{G^!} = (\delta\varphi^!_0, G^!\varphi^!_0) : \mathscr{C}(G^!)^{n-*} \to D^!$. Then

$$\begin{pmatrix} \partial G^! : \partial C^! \to \partial_{G^!} D^!, \\ \partial\delta\psi^!, \\ \partial\psi^! \end{pmatrix} := \begin{pmatrix} \partial G^! : \Sigma^{-1}\mathscr{C}(\varphi^!_0) \to \Sigma^{-1}\mathscr{C}(\varphi^!_{G^!}), \\ \binom{1+t}{\mathcal{S}}^{-1} (\mathcal{S}^{-1}(e^{\%}_{\varphi^!_{G^!}} \delta\varphi^!), (e_{\varphi^!_{G^!}})_{\%}\delta\psi^!), \\ \binom{1+t}{\mathcal{S}}^{-1} (\mathcal{S}^{-1}(e^{\%}_{\varphi^!} \varphi^!), (e_{\varphi^!})_{\%}\psi^!) \end{pmatrix}$$

defines an $(n-1)$-quadratic Poincaré pair called the boundary and denoted $\partial^Q_{\to}$.

[2321 → [Wei85b, Theorem 7.1]]
For a normal pair $((Y, B), \nu, (\rho_Y, \rho_B))$ there is a normal pair $\mathrm{sgn}^{\mathbf{NL}^\bullet}_{\to}$ such that the following homotopy equivalence holds.

$$\partial^N_{\to} \, \mathrm{sgn}^{\mathbf{NL}^\bullet}_{\to}(Y, B) \simeq \partial gn^{\mathbf{L}_\bullet}_{\to}(Y, B)$$

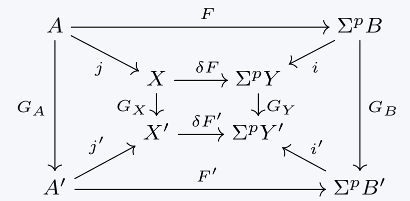
where $\partial gn^{\mathbf{L}_\bullet}_{\to}(Y, B)$ is the boundary pair signature as defined in B28.

B28 (1621, 232) Quadratic boundary pair signature
Let $((Y, B), \nu, (\rho_Y, \rho_B))$ be an n-dimensional pair of normal spaces. There is a quadratic boundary pair signature

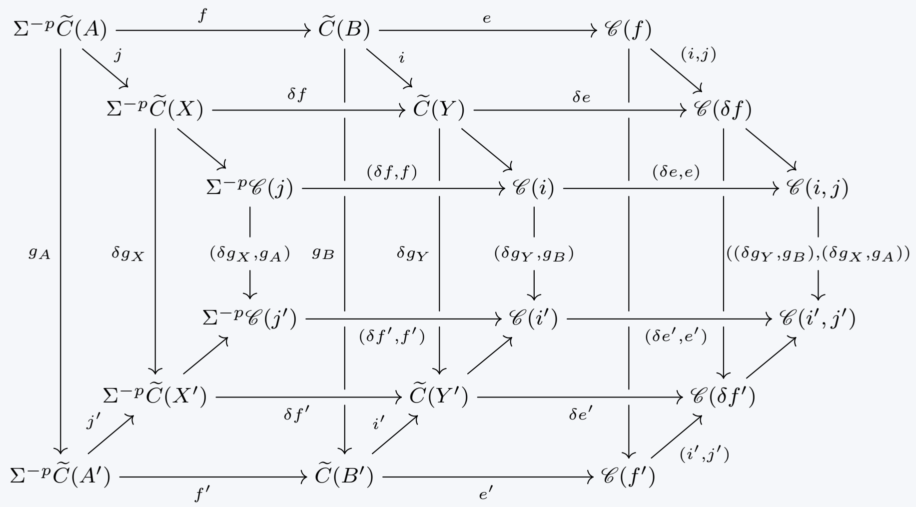
$$\partial gn^{\mathbf{L}_\bullet}_{\to}(Y, B) = (\partial j : \partial C' \to \partial D', (\partial\delta\psi, \partial\psi))$$

producing an $(n-1)$-quadratic Poincaré pair.

2322 Property of the relative spectral quadratic construction
Let $(\delta F, F)\colon (X, A) \to \Sigma^p(Y, B)$ and $(\delta F', F')\colon (X', A') \to \Sigma^p(Y', A')$ be semi-stable maps of pairs of spaces fitting into a commutative diagram as follows



and inducing the commutative diagram of chain complexes



Then the relative spectral quadratic constructions of $(\delta F, F)$ and $(\delta F', F')$ are related by

$$\mathrm{con}^{\delta\psi^!,\psi^!}_{\delta F',F'} \circ (\delta g_X, g_A) = \begin{pmatrix} \begin{smallmatrix} \delta g_Y & 0 \\ 0 & g_B \end{smallmatrix} & \\ & \begin{smallmatrix} \delta g_X & 0 \\ 0 & g_A \end{smallmatrix} \end{pmatrix} \circ \mathrm{con}^{\delta\psi^!,\psi^!}_{\delta F,F} + (\delta e', e')_{\%} \circ \mathrm{con}^{\delta\psi,\psi}_{G_Y,G_B} \circ (\delta f, f).$$

Proof 232

We start with recalling the definitions of the involved structured chain complexes. Then we proceed with identifying the underlying chain complexes of $\partial^{Q}_{\to} \operatorname{sgn}^{\mathbf{L}_\bullet}_{\to}(\delta\widehat{g}, \widehat{g})$ and $\partial\operatorname{gn}^{\mathbf{L}_\bullet}_{\to}(Y, B)$ before we finally prove that the quadratic structures are homotopic.

We will use the following notations.

The foundational maps of spaces are $\begin{array}{ccc} A & \xrightarrow{i} & N \\ \downarrow g & & \downarrow \delta g \\ B & \xrightarrow{j} & Y \end{array}$ and we use the same notation for the induced

maps of chain complexes $\begin{array}{ccc} C(A) & \xrightarrow{i} & C(N) \\ \downarrow g & & \downarrow \delta g \\ C(B) & \xrightarrow{j} & C(Y). \end{array}$. Denote the symmetric structures obtained

from the relative symmetric construction for the fundamental classes of (Y, B) and (N, A) by

$(\delta\varphi,\varphi) = \mathrm{con}^{\delta\varphi,\varphi}_{Y,B}([Y])$ and $(\delta\varphi_N,\varphi_A) = \mathrm{con}^{\delta\varphi,\varphi}_{N,A}([N])$. By C6 we obtain maps C6→p.165

$$\begin{aligned}\Gamma_Y &: (\mathrm{Th}(\nu_Y)/\mathrm{Th}(\nu_B))^* \to \Sigma^p Y_+\\ \Gamma_B &: \Sigma^{-1}\mathrm{Th}(\nu_B)^* \to \Sigma^p B_+,\end{aligned}$$

with induced chain maps

$$\begin{aligned}(\gamma_Y &: \widetilde{C}((\mathrm{Th}(\nu_Y)/\mathrm{Th}(\nu_B))^*) \to \widetilde{C}(Y)) \simeq (\varphi_j : \mathscr{C}(j)^{n+1-*} \to C(Y)),\\ (\gamma_B &: \widetilde{C}(\Sigma^{-1}\mathrm{Th}(\nu_B)^*)_{p+*} \to \widetilde{C}(B)) \simeq (\varphi_0 : C(B)^{n-*} \to C(B)).\end{aligned}$$

Using the degree one normal map $(\delta\widehat{g},\widehat{g}) : (N,A) \to (Y,B)$ and C5 we get a similar pair of maps denoted C5→p.165

$$\begin{aligned}\Gamma^!_Y &= \Gamma_N \circ (\mathrm{Th}(\delta\overline{g})/\mathrm{Th}(\overline{g}))^* : (\mathrm{Th}(\nu_Y)/\mathrm{Th}(\nu_B))^* \to \Sigma^p N_+,\\ \Gamma^!_B &= \Gamma_A \circ \mathrm{Th}(\overline{g})^* : \Sigma^{-1}\mathrm{Th}(\nu_B)^* \to \Sigma^p A_+\end{aligned}$$

with the induced chain maps ${\gamma_Y}^!$ and $\gamma^!_B$ and the chain homotopic versions

$$\begin{aligned}g^!_i &: \mathscr{C}(j)^{n-*} \xrightarrow{(\delta g^*,g^*)} \mathscr{C}^{n-*}(i) \xrightarrow{((\delta\varphi_N)_0,(\varphi_A)_0 i)} C(N),\\ g^! &: C^{n-1-*}(B) \xrightarrow{g^*} C^{n-1-*}(A) \xrightarrow{(\varphi_A)_0} C(A).\end{aligned}$$

We denote the n-quadratic Poincaré pair of the right hand side of the homotopy equivalence that we are aiming to prove by B28→p.157

$$\partial\mathrm{gn}^{\mathbf{L}\bullet}_{\to}(Y,B) =: \big(\ \partial j : \Sigma^{-1}\partial C := \Sigma^{-1}\mathscr{C}(\varphi_0) \to \partial_j D := \mathscr{C}(\varphi_j),\ (\partial\delta\psi,\partial\psi) := \mathrm{con}^{\delta\psi^!,\psi^!}_{\Gamma_Y,\Gamma_B}(u_{\nu_Y})^* \in \mathscr{C}(\partial j_{\%})_n\ \big).$$

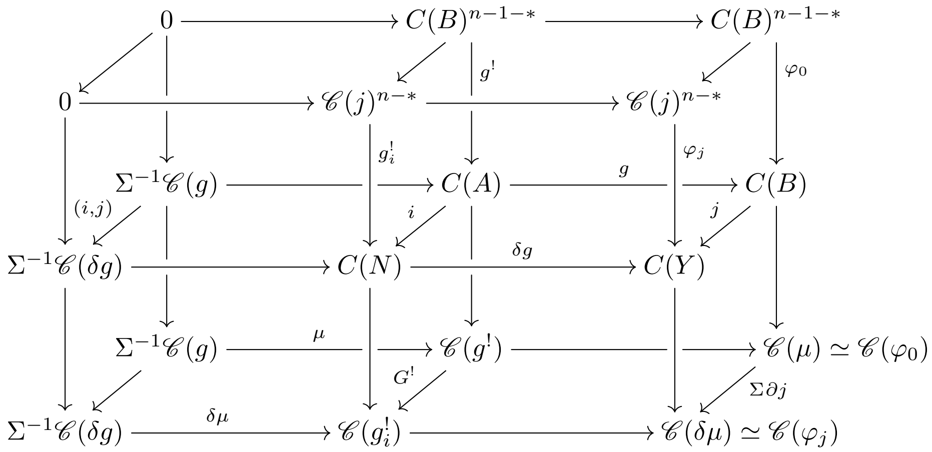
On the left hand side we use the $(n+1)$-quadratic pair A28→p.134

$$\mathrm{sgn}^{\mathbf{L}\bullet}_{\to}(\delta\widehat{g},\widehat{g}) =: (\ G^! : C^! := \mathscr{C}(g^!) \to D^! := \mathscr{C}(g^!_i),\ (\delta\psi^!,\psi^!) := \mathrm{con}^{\delta\psi^!,\psi^!}_{{\Gamma_Y}^!,{\Gamma_A}^!}({u_{\nu_y}}^*) \in \mathscr{C}({G^!}_{\%})_{n+1}\ )$$

and the symmetrization $(\delta\varphi^!,\varphi^!) := (1+t)(\delta\psi^!,\psi^!) \in \mathscr{C}({G^!}^{\%})_{n+1}$ of its quadratic structure to define the n-quadratic Poincaré pair

$$\partial^Q_{\to}\,\mathrm{sgn}^{\mathbf{L}\bullet}_{\to}(\delta\widehat{g},\widehat{g}) =: (\ \partial G^! : \partial C^! := \Sigma^{-1}\mathscr{C}(\varphi^!_0) \to \partial_{G^!} D^! := \Sigma^{-1}\mathscr{C}(\varphi^!_{G^!}),\ (\partial\delta\psi^!,\partial\psi^!) \in \mathscr{C}(\partial {G^!}_{\%})_n\ ).$$

Identification of the chain complexes. We use the following diagram of cofibrations analog to the absolute case



(232.1)

At the bottom we recover in the induced maps μ and $\delta\mu$ the duality maps

$$\varphi_0^! : \mathscr{C}(g^!)^{n-1-*} \to \mathscr{C}(g^!)$$
$$\varphi_{G^!}^! : \mathscr{C}(G^!)^{n-*} \to \mathscr{C}(g_i^!)$$

that were used to produce the boundary chain complexes $\partial C^!$ and $\partial_{G^!} D^!$. Note that the back side
231→p.94 of the cube is exactly the diagram from the absolute case 231. So again, with some inspection and using Poincaré duality of A we can identify $\Sigma^{-1}\mathscr{C}(g) \simeq \mathscr{C}(g^!)^{n-*+1}$ and $\mu \simeq \varphi_0^!$ and hence we obtain a homotopy equivalence

$$h' : \Sigma\partial C^! = \mathscr{C}(\varphi_0^!) \simeq \mathscr{C}(\varphi_0) = \Sigma\partial C.$$

Analogously, at the front of the cube we get a homotopy equivalence which identifies $\delta\mu$ and $\varphi_{G^!}^!$ and so we obtain from

$$\begin{aligned}\varphi_{G^!}^! : \mathscr{C}(G^!)^{n-*} &= C(N)^{n-*} \oplus C(Y)_{*+1} \oplus C(B)_* \oplus C(A)^{n-*-1} \oplus C(B)_{*+1} \\ &\simeq C(Y)_{*+1} \oplus C(N) \\ &= \Sigma^{-1}\mathscr{C}(\delta g) \xrightarrow{\delta\mu} \mathscr{C}(g_i^!)\end{aligned}$$

in the bottom right corner at the front of the cube the homotopy equivalence

$$\delta h' : \Sigma\partial_{G^!} D^! = \mathscr{C}(\varphi_{G^!}^!) \xrightarrow{\simeq} \mathscr{C}(\varphi_j) = \Sigma\partial_j D.$$

The sign conventions for suspensions lead to homotopy equivalences

$$(\delta h, h) : (\partial_{G^!} D^!, \partial C^!) \to (\partial_j D, \partial C)$$

with $\Sigma(\delta h, h) = -(\delta h', h')$.

Identification of the quadratic structures. The pair $(\delta\varphi, \varphi)$ denotes the symmetric pair structure obtained by applying the relative symmetric signature $\mathrm{sgn}_{\to}^{\mathbf{L}^\bullet}$ to (Y, B). The Poincaré quadratic
B28→p.157 structure of the quadratic pair $\partial\mathrm{gn}_{\to}^{\mathbf{L}_\bullet}(Y, B)$ is given by $\left(\begin{smallmatrix} \mathcal{S} \\ 1+t \end{smallmatrix}\right)^{-1}(\Psi, \Phi)$ with

$$\begin{aligned}\Psi &:= (S\partial\delta\psi, S\partial\psi) := S\partial_{\to}^{Q}(\delta\psi, \psi) = \mathrm{con}_{\Gamma_Y, \Gamma_B}^{\delta\psi^!, \psi^!}(u_{\nu_Y})^* \\ \Phi &:= (\partial\delta\varphi, \partial\varphi) \quad := \partial_{\to}^{S}(\delta\varphi, \varphi) \quad = \mathcal{S}^{-1}(1+t)(\Psi)\end{aligned}$$

We want to identify this pair with the quadratic structure of the boundary of $\operatorname{sgn}^{\mathbf{L}_\bullet}_{\to}(\delta\widehat{g}, \widehat{g})$. The quadratic structure $(\delta\psi^!, \psi^!)$ of the pair $\operatorname{sgn}^{\mathbf{L}_\bullet}_{\to}(\delta\widehat{g}, \widehat{g})$ is produced by the relative spectral quadratic construction $\operatorname{con}^{\delta\psi^!,\psi^!}_{\Gamma^!_Y,\Gamma^!_B}$. In order to obtain a Poincaré quadratic structure on the boundary we A25→p.133
have to apply the map $(e^{\%}_{\varphi^!_j}, e^{\%}_{\varphi^!_0}) : \mathscr{C}(j_{\%}) \to \mathscr{C}(\Sigma\partial G^!{}_{\%})$ induced by the following diagram. B25→p.154

$$\begin{array}{ccccc}
\mathscr{C}(g^!) & \xrightarrow{G^!} & \mathscr{C}(g^!_i) & \xrightarrow{e_{G^!}} & \mathscr{C}(G^!) \\
\downarrow{\scriptstyle e_{\varphi^!_0}} & & \downarrow{\scriptstyle e_{\varphi^!_{G^!}}} & & \downarrow{\scriptstyle (e_{\varphi^!_j}, e_{\varphi^!_0})} \\
\mathscr{C}(\varphi^!_0) & \xrightarrow{\Sigma\partial G^!} & \mathscr{C}(\varphi^!_{G^!}) & \xrightarrow{e_{\Sigma\partial G^!}} & \mathscr{C}(\Sigma\partial G^!)
\end{array}$$

We denote the Poincaré quadratic structure of $\partial^Q_{\to} \operatorname{sgn}^{\mathbf{L}_\bullet}_{\to}(\delta\widehat{g}, \widehat{g})$ by

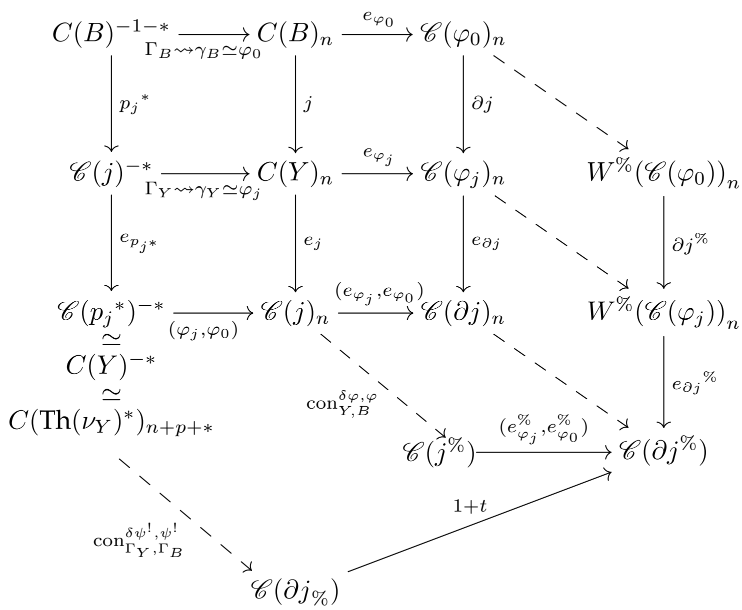
$$(\partial\delta\psi^!, \partial\psi^!) = \left(\begin{smallmatrix} \mathcal{S} \\ 1+t \end{smallmatrix}\right)^{-1} (\Phi^!, \Psi^!)$$

with

$$\Psi^! := S\partial^Q_{\to}(\delta\psi^!, \psi^!) := (S\partial\delta\psi^!, S\partial\psi^!) = (e^{\%}_{\varphi^!_{G^!}}, e^{\%}_{\varphi^!_0}) \circ \operatorname{con}^{\delta\psi^!,\psi^!}_{\Gamma^!_Y,\Gamma^!_B}(u_{\nu_Y})^*$$
$$\Phi^! := \partial^S_{\to}(\delta\varphi^!, \varphi^!) \quad := (\partial\delta\varphi^!, \partial\varphi^!) \qquad = \mathcal{S}^{-1} \circ (1+t)(\Psi^!).$$

So, using this notation we have to proof $\Psi \simeq \Psi^!$ and $\Phi \simeq \Phi^!$.

We start with the symmetric structures Φ and $\Phi^!$. According to its definition the relative spectral quadratic construction for the semi-stable map (Γ_Y, Γ_B) fits into a commutative diagram A25→p.133
as follows.



We have
$$\mathcal{S}\Phi = (1+t)\operatorname{con}^{\delta\psi^!,\psi^!}_{\Gamma_Y,\Gamma_B}(u_{\nu_Y}{}^*) = (e^{\%}_{\varphi_j}, e^{\%}_{\varphi_0}) \circ \operatorname{con}^{\delta\varphi,\varphi}_{Y,B} \circ (\varphi_j, \varphi_0)(u_{\nu_Y}{}^*)$$
$$= (e^{\%}_{\varphi_j}, e^{\%}_{\varphi_0})(\delta\varphi, \varphi).$$
$$= (e^{\%}_{\varphi_j}, e^{\%}_{\varphi_0}) \circ (\delta g^{\%}, g^{\%})(\delta\varphi_N, \varphi_A)$$

where the last step uses the naturality of the symmetric construction, i.e. that $(\delta g^{\%}, g^{\%}) \circ \mathrm{con}^{\delta\varphi,\varphi}_{N,A}([N]) = \mathrm{con}^{\delta\varphi,\varphi}_{Y,B}([Y])$.

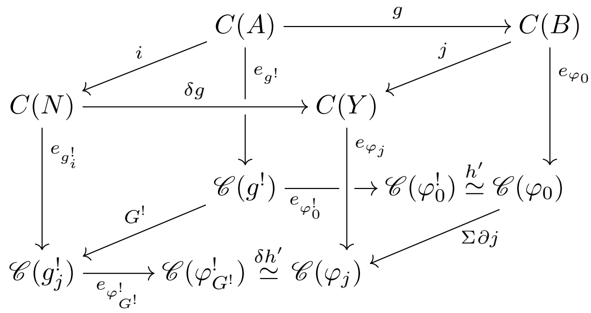
For $\Phi^!$ the basic property of the relative spectral quadratic construction leads to

$$\begin{aligned} \mathcal{S}\Phi^! &= ({e_{\varphi^!_{G^!}}}^{\%}, {e_{\varphi^!_0}}^{\%}) \circ (1+t)(\delta\psi^!, \psi^!) \\ &\qquad ({e_{\varphi^!_{G^!}}}^{\%}, {e_{\varphi^!_0}}^{\%})(1+t)(\delta\psi^!, \psi^!) = ({e_{\varphi^!_{G^!}}}^{\%}, {e_{\varphi^!_0}}^{\%})(e^{\%}_{g^!_i}, e^{\%}_{g^!}) \circ \mathrm{con}^{\delta\varphi,\varphi}_{N,A} \circ (g^!_i, g^!)({u_{\nu_Y}}^{*}). \end{aligned}$$

We end up with the following description of both symmetric structures in terms of $(\delta\varphi_N, \varphi_A)$.

$$\begin{aligned} \mathcal{S}(\Phi^!) &= (e^{\%}_{\varphi^!_{G^!}} \circ e^{\%}_{g^!_i}(\delta\varphi_N), e^{\%}_{\varphi^!_0} \circ e^{\%}_{g^!}(\varphi_A)) \\ \mathcal{S}(\Phi) &= (e^{\%}_{\varphi_j} \circ \delta g^{\%}(\delta\varphi_N), e^{\%}_{\varphi_0} \circ g^{\%}(\varphi_A)) \end{aligned}$$

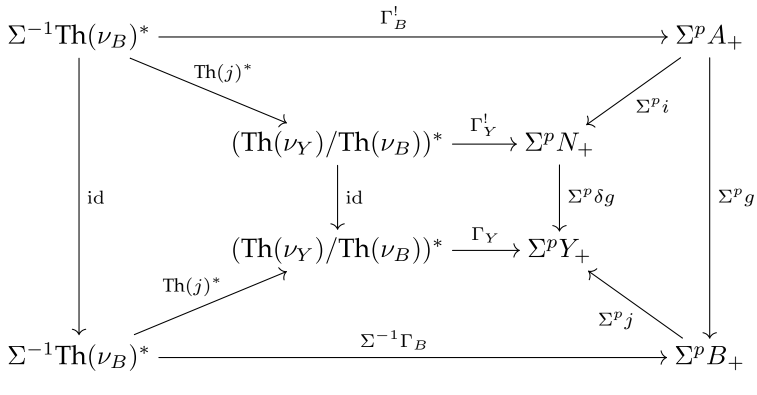
The identification of the second components is literally the same as in the absolute case. The cube below adapted from the lower right cube in (232.1) identifies both components via the homotopy equivalences $\delta h'$ and h'.



(232.2)

By the injectivity of the suspension map $\mathcal{S}$ and the sign conventions for suspensions we obtain $\Phi \simeq -\Phi^!$.

Now it remains to identify the quadratic components Ψ and $\Psi^!$. We apply 2322 to the following diagram.



The inner square of the induced diagram of chain complexes looks as follows

$$\begin{array}{ccccc}
\Sigma^{-p}\mathscr{C}(\mathrm{Th}(j)^*) & \xrightarrow{(\gamma^!_Y,\gamma^!_B)} & \mathscr{C}(i) & \xrightarrow{e_{\gamma^!_Y,\gamma^!_B}} & \mathscr{C}(\gamma^!_Y,\gamma^!_B) \simeq \mathscr{C}(i,\mathrm{Th}(j)^*) \\
\downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle \begin{pmatrix} \delta g_* & 0 \\ 0 & g_* \end{pmatrix}} & & \downarrow{\scriptstyle \begin{pmatrix} \delta g_* & 0 & & \\ 0 & g_* & & \\ & & 1 & 0 \\ & & 0 & 1 \end{pmatrix}} \\
\Sigma^{-p}\mathscr{C}(\mathrm{Th}(j)^*) & \xrightarrow{(\gamma_Y,\gamma_B)} & \mathscr{C}(j) & \xrightarrow{e_{\gamma_Y,\gamma_B}} & \mathscr{C}(\gamma_Y,\gamma_B) \simeq \mathscr{C}(j,\mathrm{Th}(j)^*)
\end{array}$$

and we obtain

$$\mathrm{con}^{\delta\psi^!,\psi^!}_{\Gamma_Y,\Gamma_B} \circ (\mathrm{id},\mathrm{id}) = \begin{pmatrix} \delta g_* & 0 & & \\ 0 & g_* & & \\ & & 1 & 0 \\ & & 0 & 1 \end{pmatrix}_{\%} \circ \mathrm{con}^{\delta\psi^!,\psi^!}_{\Gamma^!_Y,\Gamma^!_B} + (e_{\gamma_Y,\gamma_B})_{\%} \circ \mathrm{con}^{\delta\psi,\psi}_{\Sigma^p\delta g,\Sigma^p g} \circ (\gamma^!_Y,\gamma^!_B).$$

The relative quadratic construction for a stable map $\Sigma^p(\delta g, g)$ that comes from a map of spaces $(\delta g, g)$ vanishes and the following diagram remains:

$$\begin{array}{ccc}
C(\mathrm{Th}(\nu_Y))_{p+*} & \xrightarrow{\mathrm{con}^{\delta\psi^!,\psi^!}_{\Gamma^!_Y,\Gamma^!_B}} & \mathscr{C}(G^!{}_{\%}) \\
& {\scriptstyle \mathrm{con}^{\delta\psi^!,\psi^!}_{\Gamma_Y,\Gamma_B}} \searrow & \downarrow{\scriptstyle \begin{pmatrix} \delta g_* & 0 & & \\ 0 & g_* & & \\ & & 1 & 0 \\ & & 0 & 1 \end{pmatrix}_{\%}} \\
& & \mathscr{C}(\partial i_{\%})
\end{array}$$

From 232.2 we get the identification $\begin{pmatrix} \delta g_* & 0 & & \\ 0 & g_* & & \\ & & 1 & 0 \\ & & 0 & 1 \end{pmatrix}_{\%} = (\delta h', h)_{\%} \circ ((e_{\varphi^!_j})_{\%}, (e_{\varphi^!_0})_{\%})$. Hence we obtain

$$\begin{aligned}
(\delta h', h')_{\%}(\Psi^!) &= (\delta h', h')_{\%} \circ ((e_{\varphi^!_j})_{\%}, (e_{\varphi^!_0})_{\%}) \circ \mathrm{con}^{\delta\psi^!,\psi^!}_{\Gamma^!_Y,\Gamma^!_B}(u(\nu_Y)^*) \\
&= \mathrm{con}^{\delta\psi^!,\psi^!}_{\Gamma_Y,\Gamma_B}(u(\nu_Y)^*) \qquad = \Psi. \quad \square
\end{aligned}$$

Room service 232

$(f\colon C \to D, \delta\varphi, \varphi)$ an $(n+1)$-symmetric pair with

- $f\colon C \to D$ a chain map
- (C, φ) an n-symmetric chain complex
- $\delta\varphi \in W^{\%}(D)_{n+1}$ such that $d(\delta\varphi) = f^{\%}(\varphi)$ which is equivalent to $(\delta\varphi, \varphi)$ is a cycle in $\mathscr{C}(f^{\%})_{n+1}$.

Poincaré means $(\delta\varphi_0, \varphi_0 f^*)\colon D^{n+1-*} \to \mathscr{C}(f)_*$ is a chain equivalence.

$(f\colon C \to D, \delta\psi, \psi)$ an $(n+1)$-quadratic pair with

- $f\colon C \to D$ a chain map
- (C, ψ) an n-quadratic chain complex
- $\delta\psi \in W_{\%}(D)_{n+1}$ such that $d(\delta\psi) = f_{\%}(\psi)$.

Poincaré means the symmetrization is Poincaré, i.e.
$((1+t)\delta\varphi_0, (1+t)\varphi_0 f^*)\colon D^{n+1-*} \to \mathscr{C}(f)_* = (D_*, C_{*-1})$ is a chain equivalence.

$\boxed{\mathrm{sgn}^{\mathbf{L}_\bullet}_{\to}(\delta\widehat{g},\widehat{g}) = (G^! : C^! \to D^!, \delta\psi^!, \psi^!)}$ the quadratic pair signature for a degree one normal map $(\delta\widehat{g},\widehat{g}) : (N,A) \to (Y,B)$ from a Poincaré pair (N,A) to a normal pair (Y,B) (see A28).

$\boxed{\partial\mathrm{gn}^{\mathbf{L}_\bullet}_{\to}(Y,B)}$ the quadratic boundary pair signature for an n-dimensional pair of normal spaces (Y,B) is the $(n-1)$-quadratic Poincaré pair $(\partial C(B) \to \partial_+ C(Y), (\delta\psi,\psi))$ obtained by using the boundary construction and the spectral quadratic construction (see B28).

$\boxed{\partial^Q_{\to}}$ the relative quadratic boundary construction, produces for an n-quadratic pair an $(n-1)$-quadratic Poincaré pair usually denoted $(\partial f : \partial C \to \partial_f D, \partial(\delta\psi,\psi))$.

$\boxed{(\partial f : \partial C \to \partial_{f^*} D, \partial_{f^*}\delta\varphi, \partial\varphi)}$ the symmetric boundary of an $(n+1)$-symmetric pair $(f : C \to D, \delta\varphi, \varphi)$ not necessarily Poincaré. It is an n-symmetric Poincaré pair with

- $(\partial C, \partial\varphi)$ the symmetric boundary of (C,φ)
- $\partial_{f^*} D = \mathscr{C}(\begin{pmatrix} \delta\varphi_0 \\ \varphi_0 f^* \end{pmatrix} : D^{n+1-*} \to \mathscr{C}(f))$
- $\partial f = \begin{pmatrix} f & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix} : \partial C = C_{r+1} \oplus C^{n-r-1} \to D_{r+1} \oplus D^{n-r} \oplus C^{n-r-1} = \partial_{f^*} D.$

$\boxed{(\partial C, \partial\varphi)}$ the symmetric boundary of an n-symmetric chain complex obtained from algebraic surgery on the pair $(0 \to C, \varphi, 0)$, i.e. $\partial C = \Sigma^{-1}\mathscr{C}(\varphi_0)$, $\partial\varphi = \mathcal{S}^{-1}e^{\%}(\varphi)$ where $e : C \to \mathscr{C}(\varphi_0)$ is the inclusion (see B21 for more details).

$\boxed{W[r,s]}$ the $\mathbb{Z}[\mathbb{Z}_2]$-module chain complex

$$\cdots 0 \longrightarrow \mathbb{Z}[\mathbb{Z}_2] \xrightarrow{1+(-)^s t} \mathbb{Z}[\mathbb{Z}_2] \xrightarrow{1+(-)^{(s-1)} t} \ldots \xrightarrow{1+(-)^{(r+1)} t} \mathbb{Z}[\mathbb{Z}_2] \longrightarrow 0 \cdots$$

with $W[r,s]_n = 0$ for $n > s$ and $n < r$.

$\boxed{G^! : C^! \to D^!}$ the map obtained from the relative quadratic signature.

$\boxed{g^! : C(B)^{n-*+1} \to C(A)}$ the chain map given by the composition $(\varphi_B)_0 \circ g$.

$\boxed{g^!_i : C(Y)^{n-*} \to \mathscr{C}(i)}$ the chain map given by the composition $\varphi_i \circ \delta g^*$ for a map $i : B \to Y$ the inclusion.

$\boxed{g^!_{i^*} : C(Y,B)^{n-*} \to C(N)}$ the chain map given by the composition $\varphi_{i^*} \circ (\delta g, g)^*$ with $i : A \to N$ the inclusion.

Basement

Contents

In the elevator

Forms on chain complexes

The rooms in the Basenment introduce the foundational algebraic concepts and contstructions we use all over in the hotel, in particular all those kinds of different signatures that we use to produce elements in different types of L-groups.

We motivate here the setting in which these signatures live, namely the L-groups for chain complexes with their three different flavours - symmetric, quadratic and normal - and the following steps of generalization of this concept:

1. L-groups for additive categories with chain duality instead of R-modules.
2. Structured complexes which are Poincaré with respect to an algebraic bordism category instead of just requiring a chain equivalence to the dual chain complex.
3. A kind of controlled algebraic bordism categories which realize local Poincaré duality.

These steps are necessary to define the mosaicked signatures that will finally realize the total surgery obstruction.

We are interested in Poincaré spaces and manifolds and their differences. It turns out that we already find in chain complexes an algebraic tool which is almost powerful enough for our purposes. But we need some additional structures on the chain complexes.

The extrinsic motivation for the three different structures symmetric, quadratic and normal is the existence of the signatures in the picture below. Basically, a chain complex with an n-symmetric (Poincaré) structure φ represents a space X with a (fundamental) cycle $[X] \in C(X)_n$ and a chain complex with a normal structure (φ, γ, χ) represents a normal space (Y, ν, ρ). There is no direct geometric interpretation for quadratic structures but we are able to construct quadratic structures out of degree one normal maps.

Poincaré space X	$\rightsquigarrow$	symmetric Poincaré structure $\varphi \in W^{\%}C(X)_n$
degree one normal map $\widehat{f}\colon M \to X$	$\rightsquigarrow$	quadratic Poincaré structure $\psi \in W_{\%}\mathscr{C}(f^!)_n$
normal space (Y, ν, ρ)	$\rightsquigarrow$	normal structure $(\varphi, \gamma, \chi) \ \in \ W^{\%}C(X)_n \oplus$ $\widehat{W}^{\%}(C(X)^{-*})_0 \oplus \widehat{W}^{\%}C(X)_{n+1}$

where $\pi = \pi_1(X)$ in the first case, $\pi = \pi_1(M)$ in the second and $\pi = \pi_1(Y)$ in the last one and all spaces are of dimension n.

The conceptional idea behind these structure on chain complexes is to develop L-theory for chain complexes instead of modules motivated by Quillen's approach to algebraic K-theory. The original (quadratic) L-groups were defined by Wall using quadratic forms. We are now going to retrace the quadratic and symmetric forms in the definition of the chain complex version of the L-groups.

Let M be a free R-module. A bilinear form over R is an element in $\mathrm{Hom}_R(M, M^*)$ or, using the adjoint description, in $(M \otimes M)^*$. The involution $T\colon (M \otimes M)^* \to (M \otimes M)^*, \quad T(\lambda)(x, y) = \overline{\lambda(y, x)}$

defines a $\mathbb{Z}_2$-action on bilinear forms. A symmetric form can then be considered as a $\mathbb{Z}_2$-fix point which is an element in $\mathrm{Hom}_{\mathbb{Z}[\mathbb{Z}_2]}(\mathbb{Z}, \mathrm{Hom}_R(M, M^*))$ and a quadratic form is a $\mathbb{Z}_2$-orbit which is an element in $\mathbb{Z} \otimes_{\mathbb{Z}[\mathbb{Z}_2]} \mathrm{Hom}_R(M, M^*)$.

This translates almost immediately to chain complexes but we would like to obtain cup products as symmetric forms. This makes two adjustments necessary: Firstly, we have to use a dual description in order to get the right source and target. So we consider an element in $(C \otimes_R C) \cong \mathrm{Hom}_R(C^{-*}, C)$ as a form on a chain complex C. The involution is given by

$$T\colon C \otimes C \longrightarrow C \otimes C; \quad x \otimes y \mapsto (-1)^{|x|\cdot|y|} y \otimes x.$$

Secondly, we need a homotopy invariant notion. Hence instead of fixed points and orbits we consider homotopy fixed points and homotopy orbits. So a symmetric form on a chain complex C, which we call a symmetric structure on C, is defined to be a cycle in $\mathrm{Hom}_{\mathbb{Z}[\mathbb{Z}_2]}(W, C \otimes_R C)_*$. Accordingly, a quadratic structure is a cycle in $(W \otimes_{\mathbb{Z}[\mathbb{Z}_2]} C \otimes_R C)_*$. In the next subsections we provide more details about structures on chain complexes.

Symmetric structures

An n-symmetric structure $\varphi \in W_{\%}(C(X))_n = \mathrm{Hom}_{\mathbb{Z}[\mathbb{Z}_2]}(W, C(X) \otimes C(X))$ for an n-dimensional Poincaré space X is meant to encode the Poincaré duality of X. On the chain level Poincaré duality means that there is a fundamental cycle $[X] \in C(X)_n$ such that the cap product

$$\cdot \cap [X] \colon C(X)^{n-*} \to C(X)_*$$

is a chain equivalence. In general we can define for each $\sigma \in C(X)_n$ a chain map

$$\cdot \cap \sigma \colon C(X)^{n-*} \to C(X)_*.$$

Using the slant isomorphism, $\cdot \cap \sigma$ corresponds to an element φ_σ in $C(X) \otimes C(X)$ and as a first step to the construction of symmetric structures we can make the ad hoc definition of a chain map

$$\mathrm{con}_X^{\varphi_0} \colon C(X) \longrightarrow C(X) \otimes C(X); \quad \sigma \mapsto \varphi_\sigma.$$

We have a closeer look at this. In homology we almost immediately recover $\mathrm{con}_X^{\varphi_0}$ induced by the diagonal map $d\colon X \to X \times X$. For that, recall the definition of the cap product, for example as given in [Mas91, XIII.3]:

$$\begin{aligned} \cdot \cap \cdot \colon H^p(X) \otimes H_q(X) &\longrightarrow H_{q-p}(X), \\ u \otimes v &\longmapsto u \backslash d_*(v) \end{aligned}$$

which involves the slant product

$$\backslash \colon H^p(X) \otimes H_q(X \times X) \longrightarrow H_{q-p}(X),$$

which is induced by the chain map

$$\backslash \colon C^p(X) \otimes C_q(X \times X) \longrightarrow C_{q-p}(X), \qquad u \otimes v \longmapsto u \backslash\backslash \zeta(v)$$

where $\zeta \colon C(X \times X) \longrightarrow C(X) \otimes C(X)$ is the Eilenberg-Zilber chain map and the chain map $\backslash\backslash \colon C^p(Y) \otimes [C(X) \otimes C(Y)]_q \to C(X)_{q-p}$ is defined by $f \otimes a \otimes b \mapsto (-1)^{|a||f|} a \otimes f(b)$ with $f(b) = 0$ for $|b| \neq |f|$.

Using the general adjoint isomorphism $\mathrm{Hom}(B \otimes A, C) \cong \mathrm{Hom}(A, \mathrm{Hom}(B, C))$ we can rewrite the slant product as the homomorphism

$$\backslash : H_q(X \times X) \to \mathrm{Hom}(H^p(X), H_{q-p}(X)).$$

Precomposed with the diagonal map we get the cap product

$$\begin{array}{ccccc} H_q(X) & \xrightarrow{d_*} & H_q(X \times X) & \xrightarrow{(\backslash\backslash \circ \zeta)_*} & \mathrm{Hom}(H^p(X), H_{q-p}(X)) \\ u & \longmapsto & & & \cdot \cap u \end{array}$$

which looks as follows on the chain level:

$$C(X)_q \xrightarrow{d_*} C_q(X \times X) \xrightarrow{\zeta} (C(X) \otimes C(X))_q \xrightarrow{\backslash\backslash} \mathrm{Hom}(C^p(X), C_{q-p}(X)).$$

We recover $\mathrm{con}_X^{\varphi_0}$ as the composition $\zeta \circ d_*$ but it has a defect compared to its geometric origin, the diagonal map d. We lost the symmetry in the target: Let $t \colon X \times X \to X \times X$ be the switch map $t(x, y) = (y, x)$. Then d is fixed under the $\mathbb{Z}_2$-action by t, but for d_* we have to take a cellular approximation of the diagonal map d which is not symmetric. Moreover, consider the following diagram where $T \colon C_p \otimes C_q \to C_p \otimes C_q$ is given by $x \otimes y \mapsto (-1)^{|x||y|} y \otimes x$.

$$\begin{array}{ccccc} \mathrm{con}_X^{\varphi_0} \colon C(X) & \xrightarrow{d_*} & C(X \times X) & \xrightarrow{\zeta} & C(X) \otimes C(X) \\ = & & \downarrow t_* & & \downarrow T \\ \mathrm{con}_X^{\varphi_0} \colon C(X) & \xrightarrow{d_*} & C(X \times X) & \xrightarrow{\zeta} & C(X) \otimes C(X) \end{array}$$

The diagram does not commute on the nose but the Eilenberg-Zilber map is constructed using the acyclic model theorem [EM53] and the uniqueness part of the theorem provides at least a natural chain homotopy

$$\mathrm{con}_X^{\varphi_1} \colon \zeta \circ d_* \simeq \zeta \circ t_* \circ d_* \simeq T \circ \zeta \circ d_* \colon C(X) \to C(X) \otimes C(X).$$

Returning to the isomorphic description of $C \otimes C$ as $\mathrm{Hom}(C^*, C)$ the definition of T changes to

$$T \colon \; \mathrm{Hom}(C^*, C) \longrightarrow \mathrm{Hom}(C^*, C); \quad \varphi \mapsto \varphi^*.$$

We consider for a cycle $\sigma \in C(X)$ the chain map $\backslash\backslash \circ \mathrm{con}_X^{\varphi_0}(\sigma) = \backslash\backslash \circ \zeta \circ d_*(\sigma) \colon C(X)^{n-*} \to C(X)_*$ and denote it by φ_0, which will serve as the seed of a symmetric duality structure on $C(X)$ for σ. It grows to a countable collection of higher chain homotopies. First we include in the data of a symmetric structure for σ the chain homotopy $\varphi_1 \coloneqq \backslash\backslash \circ \mathrm{con}_X^{\varphi_1}(\sigma) \colon \varphi_0 \simeq T\varphi_0$. Now φ_1 itself is again chain homotopic to $T\varphi_1$ and there is a chain homotopy $\varphi_2 \colon \varphi_1 \simeq T\varphi_1$ and so on. So we define a symmetric structure as a collection of chain homotopies with increasing degrees

$$\varphi \coloneqq \left\{ \varphi_s \colon C^{n+s-*} \to C_* \mid s \geq 0 \right\}. \tag{232.3}$$

The dimension of a symmetric chain complex (C, φ) is the degree of the original seed, the possibly Poincaré duality map φ_0. We call (C, φ) a symmetric Poincaré complex if φ_0 is a chain equivalence.

To justify the condensed algebraic definition of φ as an element in $\mathrm{Hom}_{\mathbb{Z}[\mathbb{Z}_2]}(W, C \otimes C)$ we have a look at the geometric background. Let $\mathbb{Z}_2$ act on $X \times X$ via t. A $\mathbb{Z}_2$-equivariant map $w \colon S^\infty = \bigcup_{n=0}^\infty S^n \to X \times X$ is the same as

– a point $\omega_0 \in X$,
– a path $\omega_1 : [0,1] \to X$ from ω_0 to $t\omega_0$,
– a homotopy $\omega_2 : \omega_1 \simeq t\omega_1$ relative $\{0,1\}$,
– a homotopy $\omega_3 : \omega_2 \simeq t\omega_2$ and so on ...

Observe that $W = C(S^\infty)$ is the cellular $\mathbb{Z}[\mathbb{Z}_2]$-module chain complex of $S^\infty = (e^0 \cup te^0) \cup (e^1 \cup te^1) \cup \ldots$ with $\partial e^i = e^{i-1} - te^{i-1}$. A cycle $\varphi(\cdot) \in \mathrm{Hom}_{\mathbb{Z}[\mathbb{Z}_2]}(W, C \otimes C)$ is essentially the same as the collection φ of higher homotopies. For that consider the following diagram.

$$\begin{array}{ccccccccc}
\ldots \longrightarrow & \mathbb{Z}[\mathbb{Z}_2] & \xrightarrow{1+t} & \mathbb{Z}[\mathbb{Z}_2] & \xrightarrow{1-t} & \mathbb{Z}[\mathbb{Z}_2] & \longrightarrow & 0 \\
& \varphi\downarrow \ \ 1 \mapsto & & \varphi\downarrow \ \ 1 \mapsto & & \varphi\downarrow \ \ 1 \mapsto & & \downarrow \\
\ldots \longrightarrow & (C \otimes C)_{n+2} & \longrightarrow & (C \otimes C)_{n+1} & \longrightarrow & (C \otimes C)_n & \longrightarrow & (C \otimes C)_{n-1} \\
& & & & & \varphi_0 & & \\
& & & \varphi_1 & \mapsto & \varphi_0 - T\varphi_0 & & \\
& \varphi_2 & \mapsto & \varphi_1 + T\varphi_1 & & & &
\end{array}$$

Evaluating $\varphi(\cdot)$ on the generator of $\mathbb{Z}[\mathbb{Z}_2]$ together with the cycle condition we get the adjoint description of the symmetric structure as defined in 232.3:

$$\varphi(1) = \left\{ \varphi_s \in (C \otimes C)_s \mid d^{C \otimes C}(\varphi_s) = \varphi_{s-1} \pm t\varphi_{s-1}, s \geq 0 \right\}.$$

Note that these higher homotopies become important to define a gluing operation for chain complexes which is needed to prove that cobordism is an equivalence relation on chain complex level.

Quadratic structures

An n-quadratic structure $\psi \in W_{\%}C(X)_n$ is a refinement of a symmetric structure. Similarly to the symmetric structure a quadratic structure can be described as a collection of chain maps

$$\psi = \left\{ \psi_s : C^{n-s-*} \to C_* \to \ \mid s \leq 0 \right\}$$

but this time with a decreasing instead of an increasing degree. It becomes a symmetric structure via the symmetrization map $(1+t)$. Non-trivial quadratic structures occur when we start with a stable map

$$F : \Sigma^k X \to \Sigma^k Y$$

between pointed spaces X and Y. We get such a map for example for a degree one normal map $\widehat{f} : M \to X$ by using S-duality.

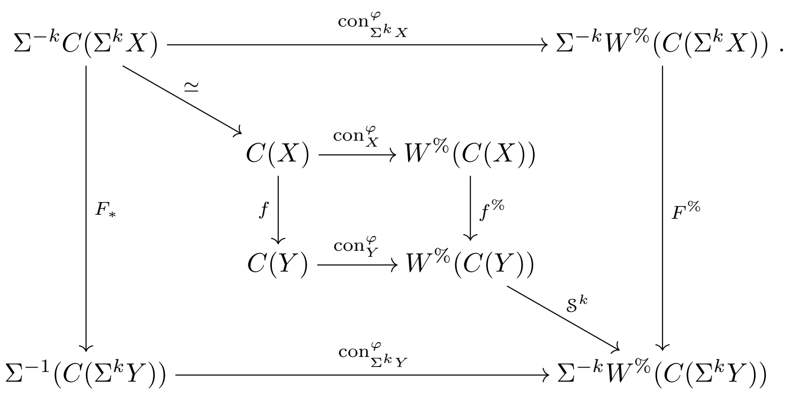
On chain complexes a stable map F gives us a chain map

$$f : C(X) \xrightarrow{\Sigma_X} \Sigma^{-k}(C(\Sigma^k X)) \xrightarrow{F_*} \Sigma^{-k}(C(\Sigma^k Y)) \xrightarrow{\Sigma_X^{-1}} C(Y).$$

But since f is not induced by a map of spaces it might not commute with the symmetric construction but the difference $f^{\%} \operatorname{con}_X^{\varphi} - \operatorname{con}_Y^{\varphi} f$ will become null-homotopic after k-fold suspension because we recover the map F:

$$\mathcal{S}^k(f^{\%} \operatorname{con}_X^{\varphi} - \operatorname{con}_Y^{\varphi} f) \simeq (F_*^{\%} \operatorname{con}_{\Sigma^k X}^{\varphi} - \operatorname{con}_{\Sigma^k Y}^{\varphi} F_*) : C(X) \to W^{\%}(C(Y)).$$

The algebraic situation is summarized in the diagram below with a commuting outer square but the inner square does not necessarily commute.



Now we use the fact that the short exact sequence of chain complexes

$$0 \longrightarrow W^{\%}C \longrightarrow \widehat{W}^{\%}C \longrightarrow \Sigma W_{\%}C \longrightarrow 0$$

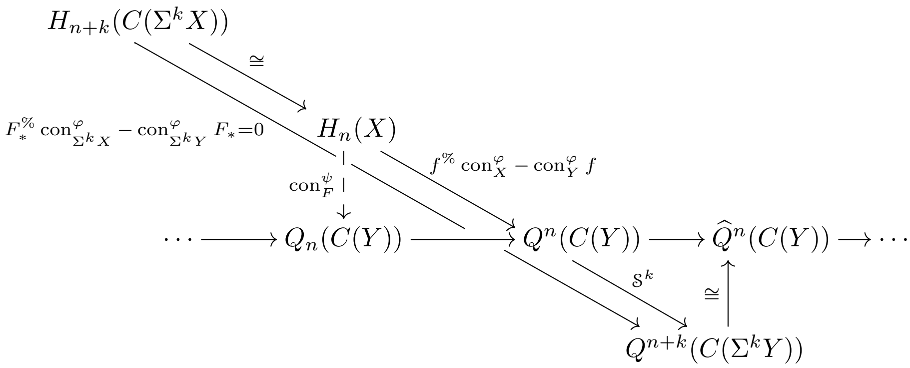
induces in homology the long exact sequence of Q-groups

$$\cdots \longrightarrow Q_{n-1}(C) \longrightarrow Q^n(C) \longrightarrow \widehat{Q}^n(C) \longrightarrow Q_n(C) \longrightarrow \cdots$$

and that the hyperquadratic Q-groups are the stabilization of the symmetric Q-groups via the suspension map $\mathcal{S}\colon Q^n(C) \to Q^{n+1}(\Sigma C)$:

$$\widehat{Q}^n(C) \cong \operatorname*{colim}_{k\to\infty} Q^{n+k}(\Sigma^k C).$$

We obtain the following diagram where the diagonal maps come from the diagram above and the horizontal line is the long exact sequence of Q-groups.



The image of $f^{\%}\,\mathrm{con}^{\varphi}_X - \mathrm{con}^{\varphi}_Y f$ vanishes in $\widehat{Q}^n(C(Y))$ and because X is Poincaré we have $H_n(X) \cong \mathbb{Z}$. Hence the exact sequence gives us a lift to $Q_n(C(Y))$. Additionally, on the chain level there is a preferred choice of the null-homotopy $\mathcal{S}^k(f^{\%}\,\mathrm{con}^{\varphi}_X - \mathrm{con}^{\varphi}_Y f) \simeq 0$ which defines a map

$$\mathrm{con}^{\psi}_F\colon C(X) \to W_{\%}(C(Y)),$$

the quadratic construction.

Example. We give a geometric example where we need suspension to obtain a map that is compatible with the symmetric construction. Denote by T the torus $S^1 \times S^1$ and by M the 2-sphere 'with ears' $S^1 \vee S^2 \vee S^1$. They both can be equipped with a CW-structure with one 0-cell, two 1-cells a and b and one 2-cell x. On the chain level we obtain a chain map $f\colon C(T) \to C(M)$ by sending generators to generators. This does not commute with the symmetric construction because on $C(M)$ the cup-product of the generators a and b vanishes whereas on $C(T)$ we obtain the generator x. But after suspension there is a homotopy equivalence $F\colon \Sigma T \to \Sigma M$ and hence the induced map on the suspended chain complexes is compatible with the symmetric construction.

Normal structures

Normal structures are related to normal spaces. A normal space is a generalization of a Poincaré space. It becomes important when we compare the local structure of a manifold and a Poincaré space. Locally, a manifold has still Poincaré duality whereas a Poincaré space is locally only a normal space.

		$\xrightarrow{\text{weaker}}$	
global geometric object	manifold	Poincaré	normal
local algebraic structure	Poincaré	normal	normal

A normal space is a triple (Y, ν, ρ) consisting of a space Y with a k-dimensional oriented spherical fibration ν and a map $\rho\colon S^{n+k} \to \text{Th}(\nu)$ to the Thom space of ν. This triple together with the Hurewicz homomorphism h and a choice $u(\nu) \in \tilde{C}^k(\text{Th}(\nu))$ of the Thom class of ν determines a fundamental class

$$[Y] := u(\nu) \cap h(\rho).$$

The cap product with the fundamental class does not need to be a chain equivalence in contrast to when Y is a Poincaré space. The dimension of a normal space is the dimension of the source sphere of ρ minus the dimension of the spherical fibration and agrees with the dimension of the fundamental class.

A normal structure is a 3-tuple (φ, γ, χ) where $\varphi \in W^{\%}C_n$ is a symmetric structure, $\gamma \in \widehat{W}^{\%}(C^*)_0$ a chain bundle and $\chi \in \widehat{W}^{\%}(C)_{n+1}$ a chain which relates γ and φ. They represent the following geometric data.

(chain complex)	$C \longleftrightarrow Y$	(space)
(symmetric structure)	$\varphi \longleftrightarrow [Y]$	(fundamental class)
(chain bundle)	$\gamma \longleftrightarrow \nu$	(spherical fibration)
(link between φ and γ)	$\chi \longleftrightarrow \rho$	(link between $[Y]$ and ν)
$J(\varphi) = d\chi + \widehat{\varphi_0}^{\%}(S^n\gamma)$		$[Y] = u(\nu) \cap h(\rho)$

For a normal space (Y, ν, ρ) with fundamental class $[Y]$ and a choice of the Thom class $u(\nu) \in \widetilde{C}(\text{Th}(\nu))$ we obtain a normal structure in the following way.

$C = C(\widetilde{Y})$, the (cellular) chain complex of the universal cover.

$\varphi = \text{con}_Y^{\varphi}([Y])$, the symmetric construction applied to $[Y]$.

$\gamma = \text{con}_\nu^{\gamma}(u(\nu))$, using the chain bundle construction as defined in A3.

Additive categories with chain duality

Let $\mathbb{A}$ be an additive category and $\mathbb{B}(\mathbb{A})$ the bounded chain complexes in $\mathbb{A}$. Our duality structures live in $W^{\%}C = \operatorname{Hom}_{\mathbb{Z}[\mathbb{Z}_2]}(W, C \otimes_{\mathbb{A}} C)$ resp. in $W_{\%}C$ and $\widehat{W}^{\%}C$. So for a chain complex $C \in \mathbb{B}(\mathbb{A})$ we have to make sense of $C \otimes_{\mathbb{A}} C$ and give it a $\mathbb{Z}_2$-action.

For $\mathbb{A}$ the additive category of free R-Modules we have the chain dual $T(C) \coloneqq C^{-*}$ which is related to the tensor product via the slant isomorphism

$$\begin{aligned} -\backslash -\colon C \otimes_R D &\xrightarrow{\cong} \operatorname{Hom}_R(TC, D), \\ x \otimes y &\longmapsto (f \mapsto \overline{f(x)} \cdot y) \end{aligned}$$

and satisfies $T^2(C) \cong C$. So we need a functor $T\colon \mathbb{B}(\mathbb{A})^{\mathrm{op}} \to \mathbb{B}(\mathbb{A})$ together with a natural transformation $e\colon T^2 \to \mathbb{1}$ such that

$$\begin{aligned} e_C\colon T^2C &\xrightarrow{\cong} C \\ e_{T(C)} \circ T(e_C) &= \mathbb{1}. \end{aligned}$$

Then we define

$$C \otimes_{\mathbb{A}} D \coloneqq \operatorname{Hom}_{\mathbb{A}}(TC, D).$$

The missing part is now a $\mathbb{Z}_2$-action on $C \otimes_{\mathbb{A}} C$ but T gives a natural map

$$T\colon \operatorname{Hom}_{\mathbb{A}}(C, D) \to \operatorname{Hom}_{\mathbb{A}}(T(D), T(C))$$

of chain complexes. This induces the maps

$$T_{C,D}\colon C \otimes_{\mathbb{A}} D = \operatorname{Hom}_{\mathbb{A}}(TC, D) \xrightarrow{T} \operatorname{Hom}_{\mathbb{A}}(TD, T^2C) \xrightarrow{(e_C)_*} \operatorname{Hom}_{\mathbb{A}}(TD, C) = D \otimes_{\mathbb{A}} C$$

and

$$T_{D,C}\colon D \otimes_{\mathbb{A}} C = \operatorname{Hom}_{\mathbb{A}}(TD, C) \xrightarrow{T} \operatorname{Hom}_{\mathbb{A}}(TC, T^2D) \xrightarrow{(e_C)_*} \operatorname{Hom}_{\mathbb{A}}(TC, D) = C \otimes_{\mathbb{A}} D$$

with $T_{C,D} \circ T_{D,C} = \mathrm{id} = T_{D,C} \circ T_{C,D}$.

Hence $T_{C,C}\colon C \otimes_{\mathbb{A}} C \to C \otimes_{\mathbb{A}} C$ is an involution which makes $C \otimes_{\mathbb{A}} C$ a $\mathbb{Z}[\mathbb{Z}_2]$-module chain complex.

We will encounter chain dualities where the dual of a single chain module is allowed to be a chain complex. So we actually define a chain duality as a functor

$$\widetilde{T}\colon \mathbb{A}^{\mathrm{op}} \to \mathbb{B}(\mathbb{A})$$

which induces a functor

$$T\colon \mathbb{B}(\mathbb{A}) \to \mathbb{B}(\mathbb{A})$$

via the total complex of the double complex $\widetilde{T}(C_i)_j$ for $C \in \mathbb{B}(\mathbb{A})$.

Algebraic bordism categories

Algebraic bordism categories are designed to capture the zoo of different (Poincaré duality) structures and especially the different local (Poincaré duality) structures in one homogeneous concept. Varying these algebraic bordism categories we can organize these structures in a braid of exact sequences as shown in 12.

We defined a structured chain complex (C, λ) to be Poincaré if the algebraic mapping cone $\mathscr{C}(\lambda_0)$ is contractible. We would like to have a more flexible concept of being Poincaré. Instead of requiring $\mathscr{C}(\lambda_0)$ to be contractible we rather choose a distinguished subcategory $\mathbb{P} \subseteq \mathbb{B}(\mathbb{A})$ of chain complexes and call (C, λ) to be $\mathbb{P}$-Poincaré if $\mathscr{C}(\lambda_0) \in \mathbb{P}$. We formalize this in the definition of an algebraic bordism category $\Lambda = (\mathbb{A}, \mathbb{C}, \mathbb{P}, (T, e))$, usually denoted by Λ or $(\mathbb{A}, \mathbb{C}, \mathbb{P})$. It consists of full additive subcategories $\mathbb{P} \subseteq \mathbb{C} \subseteq \mathbb{B}(\mathbb{A})$ where $\mathbb{P}$ is closed under weak equivalences and mapping cones, i.e.

1. $\mathscr{C}(f\colon C \to D) \in \mathbb{P}$ for any chain map f in $\mathbb{P}$,

and additionally any $C \in \mathbb{C}$ satisfies

2. $\mathscr{C}(\mathrm{id}\colon C \to C) \in \mathbb{P}$ and

3. $\mathscr{C}(e(C)\colon T^2(C) \xrightarrow{\simeq} C) \in \mathbb{P}$.

Local chain duality

For K a triangulated manifold we get $(n - |\sigma|)$-dimensional manifolds $D(\sigma, K)$ with boundary $\partial D(\sigma, K) = \cup_{\sigma \leq \tau} D(\tau, K)$ for all simplices $\sigma \in K$. We would like to encode the Poincaré duality of all these manifolds $D(\sigma, K)$ in one single chain complex. Therefore we define $\mathbb{A}_* X$ to be the additive category with objects and morphisms given by

$$\mathrm{Obj}_{\mathbb{A}_* X} = \Big\{ \sum_{\sigma \in X} M_\sigma \mid M_\sigma \in \mathbb{A}, M_\sigma = 0 \text{ except for finitely many } \sigma \Big\},$$

$$\mathrm{Mor}_{\mathbb{A}_* X}\Big(\sum_{\widetilde{\sigma} \in X} M_{\widetilde{\sigma}}, \sum_{\widetilde{\tau} \in X} N_{\widetilde{\tau}}\Big) = \Big\{ \sum_{\substack{\tau \geq \sigma \\ \sigma, \tau \in X}} f_{\tau,\sigma} \mid (f_{\tau,\sigma}\colon M_\sigma \to N_\tau) \in \mathrm{Mor}_{\mathbb{A}}(M_\sigma, N_\tau) \Big\},$$

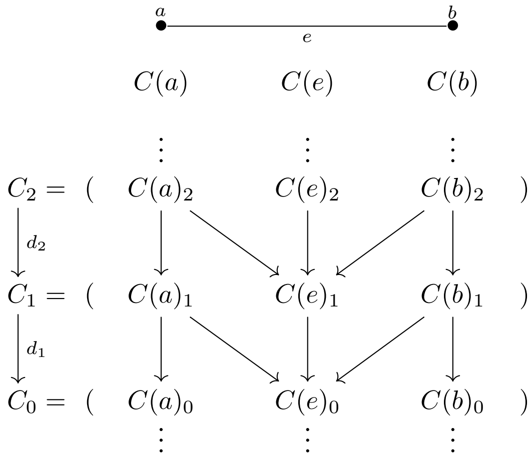
where X is a simplicial complex.

Analogously, there is a category $\mathbb{A}^* X$ with the same objects but the components $f_{\sigma,\tau}$ of a morphism are restricted to the case $\tau \leq \sigma$. They are used for the construction of L-spectra but now we concentrate on the first lower star case**.

A chain complex in the category $\mathbb{A}_* X$ consists of chain complexes $C(\sigma)$ for each $\sigma \in X$ with additional maps $C(\sigma)_n \to C(\tau)_{n-1}$ in each degree n and for all simplices $\sigma < \tau$.

**To justify the position of the star note that the morphisms in $\mathbb{A}_* X$ behave covariantly compared to the morphism in Δ where as the morphisms in $\mathbb{A}^* X$ are contravariantly compatible.

Example. Let $X = \Delta^1$ be the standard one simplex. A chain complex C in $\mathbb{A}_*X$ looks as follows.



The property of the differentials that $d_k \circ d_{k-1} = 0$ is passed on to the columns so that each column chain complex separately is a chain complex.

For a given chain duality T on $\mathbb{A}$ the induced chain duality T_* in $\mathbb{A}_*X$ is defined as follows. Note that $\sum$ indicates an object in $\mathbb{A}_*X$ and $\bigoplus$ defines an object in $\mathbb{A}$. We use the notation M_σ for a single $\mathbb{A}$-object in an $\mathbb{A}_*X$-object M and $C(\sigma)$ for the $\mathbb{A}$-chain complex obtained from an $\mathbb{A}_*X$-chain complex C by restricting in every degree the chain modules C_k to $(C_k)_\sigma$.

$$T_*\colon \mathbb{A}_*X \to \mathbb{B}(\mathbb{A}_*X), \qquad T_*(\sum_{\sigma \in X} M_\sigma)(\tau)_s := T(\bigoplus_{\tau \leq \tilde{\tau}} M_{\tilde{\tau}})_{s-|\tau|}.$$

To explain the purpose of this definition we decompose $T_*\colon \mathbb{A}_*X \to \mathbb{B}(\mathbb{A}_*X)$ into the following three functors

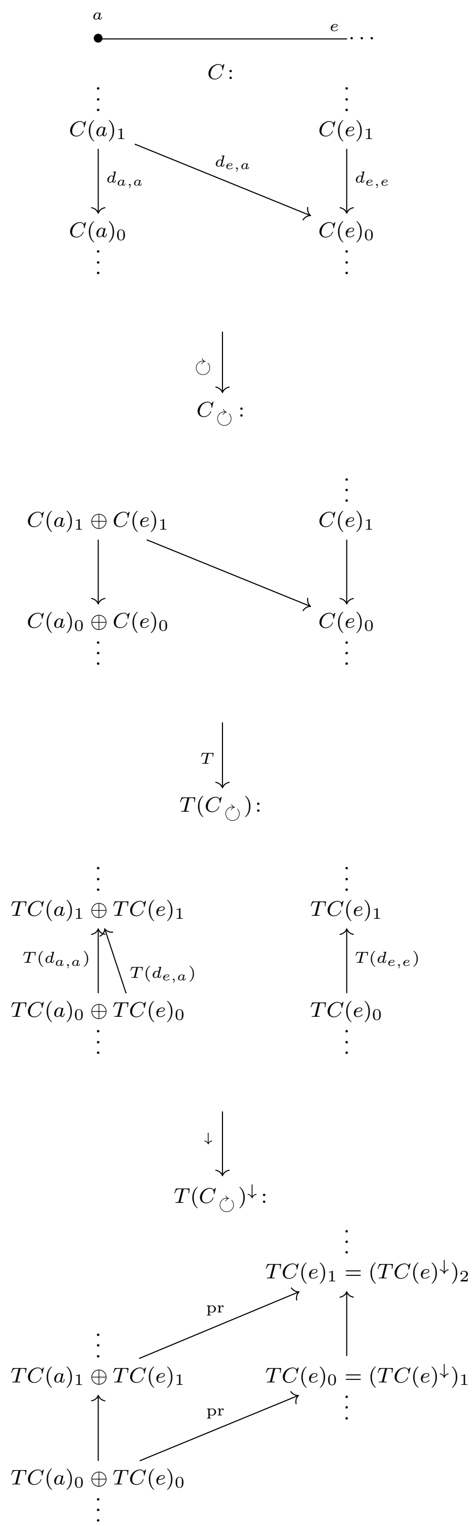
$$T_*\colon \mathbb{A}_*X \xrightarrow{\circlearrowright} \mathbb{A}^*X \xrightarrow{T^*} \mathbb{A}^*X \xrightarrow{\downarrow} \mathbb{B}(\mathbb{A}_*X)$$

where T^* is the chain duality induced by T on $\mathbb{A}^*X$, the subcategory of $\mathbb{A}_*X$ with non-trivial morphisms only between the same simplices. The other two functors $\circlearrowright$ and $\downarrow$ will be defined and explained in the following example.

Remark. Now we obtain with T_* a chain duality where the dual of a chain module is a chain complex. We use the total chain complex $T_*(C)_n = \bigoplus_{n=r-s} T_*(C_r)_s$ to define the dual of a chain complex C.

Example. Let K be an n-dimensional triangulated manifold and C the chain complex in $\mathbb{Z}_*X$ with $C(\sigma) = C(D(\sigma), \partial D(\sigma))$ the simplicial chain complex of the dual cell $D(\sigma, K)$ relative boundary. Let (T, e) be the standard duality of $\mathbb{Z}$-modules given by $T(M) := \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Z})$. The dual chain complex $T_*(C)$ is designed in such a way that an n-symmetric Poincaré structure $\varphi_0\colon \Sigma^n T_*(C) \to C$ restricted to $\varphi_0(\sigma)\colon \Sigma^n T_*(C)(\sigma) \to C(\sigma)$ for a simplex $\sigma \in X$ can encode the Poincaré duality map of the dual cell $D(\sigma, X)$ itself.

To achieve this, the definition of T_* has to deal with three difficulties which we will explain now with a simultaneous look at a 1- and a 0-simplex of K.



(1) The dual cell $D(\sigma, K)$ is a manifold with boundary, hence, Poincaré duality of C restricted to $C(\sigma)$ means in fact Poincaré-Lefschetz duality.

Solution:
$C(\sigma)$ is the cellular chain complex of $D(\sigma, K)$ relative boundary. So for its Poincaré-Lefschetz dual we collect the boundary components. The boundary consists of the dual cells of the simplices that σ is a face of. So we define $C(\overset{\circlearrowright}{\sigma}) := C_{\circlearrowright}(\sigma) := \bigoplus_{\tau \geq \sigma} C(\tau)$.

(2) After applying the chain duality of $\mathbb{A}$ to $C_{\circlearrowright}$ the reversed maps would be no longer morphisms in $\mathbb{A}_*X$.

Solution:
The target $C(\overset{\circlearrowright}{e})$ of a differential $d_{e,a}$ is now also part of the source $C(\overset{\circlearrowright}{a})$. So we can assemble the differentials between different columns in the source columns by defining $d^{\circlearrowright}_{\sigma,\sigma} = \bigoplus_{\sigma \leq \tau} d_{\tau,\sigma}$ and $d^{\circlearrowright}_{\tau,\sigma} = 0$ for $\tau \neq \sigma$.

(3) The dual cell $D(\sigma, K)$ is an $(n - |\sigma|)$-dimensional manifold. So n-dimensional duality on C should be an $(n - |\sigma|)$-dimensional duality on $C(\sigma)$.

Solution:
We shift each column chain complex $C(\sigma)$ according to the dimension of σ into the right position such that a chain map $\varphi\colon T_*(C) \to C$ of degree n is in fact a chain map of degree $n - |\sigma|$ on $T_*(C)(\sigma) \to C(\sigma)$. So we define $C^{\downarrow}(\sigma)_n := C(\sigma)_{n-|\sigma|}$ and introduce projections as the new maps between the columns in $T_*(C)$.

Local chain duality

For $\mathbb{A}^*X$ the motivating picture is that the Poincaré duality restricted to a simplex is meant to encode the Poincaré duality of the simplex itself. So everything is reversed. The boundary is coming from the faces of σ so $\circlearrowright$ collects all chain complexes $C(\tau)$ with $\tau \leq \sigma$ and Poincaré duality of σ is σ-dimensional so $^{\uparrow}$ shifts the chain complex accordingly. We use the notations $C\circlearrowright_{\sigma\, k} = \bigoplus_{\sigma \leq \tau} C(\tau)_k$ and $C^{\uparrow}(\sigma)_k = C(\sigma)_{k+|\sigma|}$ to define

$$(T^*C)(\sigma)_k = (^{\uparrow} \circ T \circ \circlearrowright)(C(\sigma))_k = T(\bigoplus_{\sigma \geq \tau} C(\tau))_{k+|\sigma|}.$$

Room service

$\boxed{\mathrm{con}^{\varphi}_X \colon C(X) \to W^{\%}(C(X))}$ a chain map called symmetric construction; defined for a topological space X.

$\boxed{\mathrm{con}^{\psi}_F \colon C(X) \to W_{\%}(C(Y))}$ a chain map called quadratic construction; defined for a stable map $F \colon \Sigma^p X \to \Sigma^p Y$ of pointed topological spaces X, Y.

$\boxed{\mathrm{con}^{\gamma}_{\nu} \colon \tilde{C}^k(\mathrm{Th}(\nu)) \to \widehat{W}^{\%}(C(X)^{-*})_0}$ a chain map called chain bundle construction; defined for a k-dimensional spherical fibration ν.

$\boxed{C}$ a chain complex; either an element in $\mathbb{B}(R)$ or, more generally, in $\mathbb{B}(\mathbb{A})$.

- $\boxed{\mathbb{B}(R)}$ the category of bounded chain complexes of finitely generated projective left R-modules.
- $\boxed{\mathbb{B}(\mathbb{A})}$ the category of bounded chain complexes in $\mathbb{A}$.

$\boxed{C \otimes C}$ short for the chain complex of $\mathbb{Z}[\mathbb{Z}_2]$-modules $C^{\iota} \otimes_R C$, or, more generally, $C \otimes_{\mathbb{A}} C := \mathrm{Hom}(T(C), C)$.

- $\boxed{C^{\iota}}$ chain complex of right R-modules obtained from a chain complex C of left R-modules using the involution of R.

$\boxed{\mathscr{C}(f)}$ the algebraic mapping cone with $\mathscr{C}(f)_k := D_k \oplus C_{k-1}$ and differential $d^{\mathscr{C}(f)}(x, y) := (d^D(x) + f(y), -d^C(y))$ for a chain map $f \colon C \to D$.

$\boxed{t}$ the generator of $\mathbb{Z}_2$; acts on $C \otimes_R C$ via $t(x \otimes y) = (-)^{|x||y|} y \otimes x$ and on $C \otimes_{\mathbb{A}} C$ via $t(x \otimes y) = T_{C,C}(x \otimes y)$.

$\boxed{T_{C,D} \colon C \otimes_{\mathbb{A}} \to D \otimes_{\mathbb{A}}}$ a map defined as the composition $\mathrm{Hom}_{\mathbb{A}}(TC, D) \xrightarrow{T} \mathrm{Hom}_{\mathbb{A}}(TD, T^2C) \xrightarrow{(e_C)_*} \mathrm{Hom}_{\mathbb{A}}(TD, C)$.

A Constructions and signatures

Porter

In the following rooms we introduce the plant of constructions which produce algebraic structures out of geometric data. The basic machines are the following three constructions which produce symmetric and quadratic structures and chain bundles††. For certain geometric input these constructions are used to construct symmetric, quadratic and normal signatures as elements in appropriate L-groups.

$X \rightsquigarrow$ [symmetric construction con^{φ}_X] $\rightsquigarrow \varphi \in W^{\%}(C(X))_n$ A1→p.125

$F\colon \Sigma^p X \to \Sigma^p Y \rightsquigarrow$ [quadratic construction con^{ψ}_F] $\rightsquigarrow \psi \in W_{\%}(C(Y))_n$ A2→p.131

$\nu\colon Z \to \mathrm{BSG}(k) \rightsquigarrow$ [chain bundle construction $\mathrm{con}^{\gamma}_{\nu}$] $\rightsquigarrow \gamma \in \widehat{W}^{\%}(C(Z)^{-*})_0$ A3→p.138

where X and Y are topological spaces, F a map of suspended pointed topological spaces and $\nu\colon Z \to \mathrm{BSG}(k)$ a spherical fibration with Z a finite CW-complex. There is a variant of the quadratic construction for a semi-stable map

$F\colon X \to \Sigma^p Y \rightsquigarrow$ [spectral quadratic construction $\mathrm{con}^{\psi^!}_F$] $\rightsquigarrow \partial\psi' \in W_{\%}(\mathscr{C}(f))_n$ A24→p.132

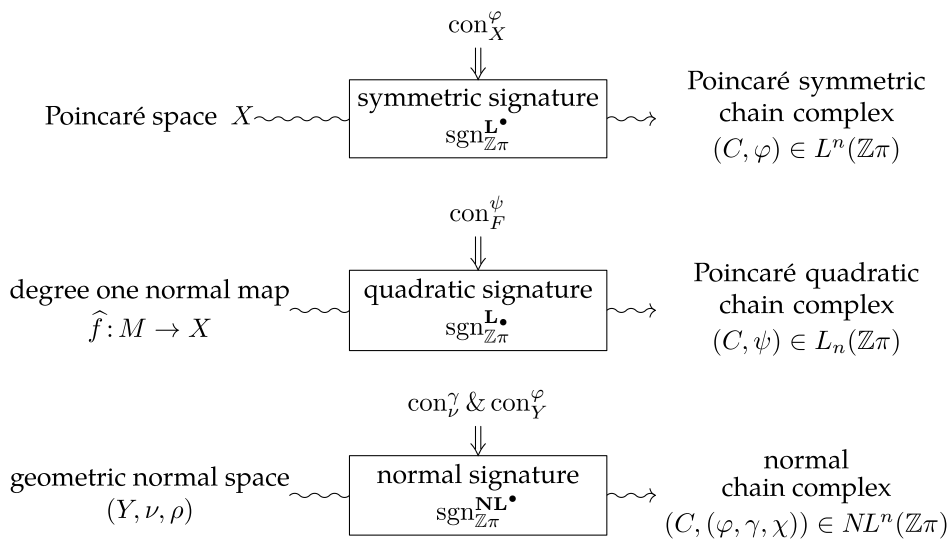
where $f\colon \widetilde{C}(X)_{p+*} \to \widetilde{C}(Y)_*$ is the chain map induced by F.

The symmetric construction con^{φ}_X is a consequence of the Alexander-Whitney diagonal approximation $C(X) \to C(X) \otimes C(X)$. It commutes with maps induced by maps of topological spaces and with suspensions of chain complexes but not with chain maps in general. This leads to the definition of the quadratic construction con^{φ}_X which measures the failure of the symmetric construction to commute with a chain map $f\colon C(X) \to C(Y)$ derived from a stable map $F\colon \Sigma^p X \to \Sigma^p Y$. The chain bundle construction $\mathrm{con}^{\gamma}_{\nu}$ for a spherical fibration ν combines the symmetric construction for the Thom space of ν with Thom- and S-duality equivalences.

For some special geometric input these constructions give rise to elements in L-groups in all

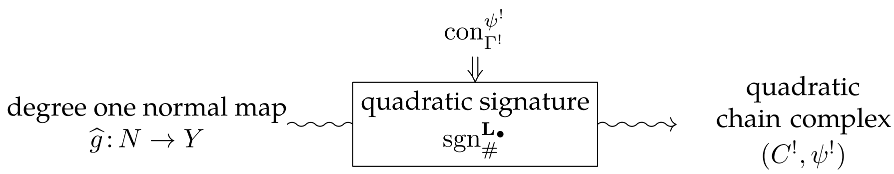
††The chain bundle construction appears in the original literature as hyperquadratic construction.

flavours. Here is an overview.



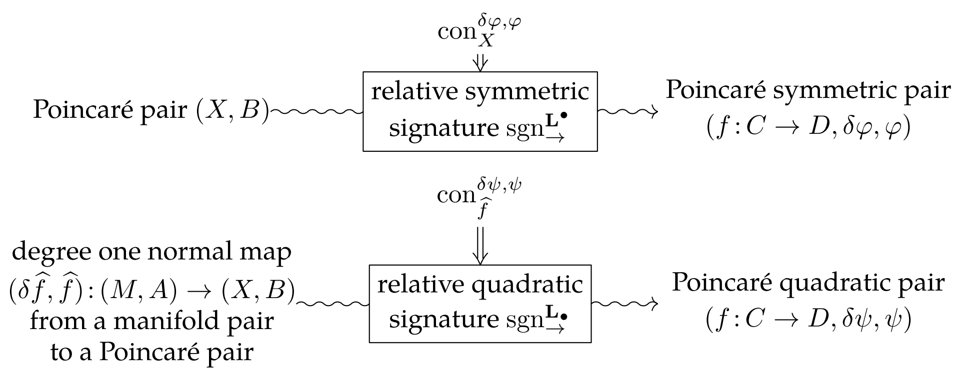
A15→p.126
A26→p.133
A33→p.139

For the quadratic signature there is a generalization for degree one normal maps with target normal spaces using the spectral quadratic construction. The outcome is a non-Poincaré quadratic complex and thereby not an element in an L-group but it will be used to tessellate quadratic mosaicked signatures.



A27→p.134

There are also relative versions of the constructions which give rise to relative signatures.



A15→p.126
A26→p.133

$\mathrm{con}_{\nu}^{\delta\gamma,\gamma}$ & $\mathrm{con}_{Y}^{\delta\varphi,\varphi}$ ⇓

normal pair $((Y,B),\nu,(\rho_Y,\rho_B))$ ⇝ relative normal signature $\mathrm{sgn}_{\to}^{\mathbf{NL}^\bullet}$ ⇝ normal pair $(f\colon C\to D, ((\delta\varphi,\delta\gamma,\delta\chi),(\varphi,\gamma,\chi))$ A33→p.139

$\mathrm{con}_{\Gamma^!}^{\delta\psi^!,\psi^!}$ ⇓

degree one normal map $(\delta\widehat{g},\widehat{g})\colon(N,A)\to(Y,B)$ from a Poincaré pair to a normal pair ⇝ relative quadratic signature $\mathrm{sgn}_{\to}^{\mathbf{L}_\bullet}$ ⇝ quadratic pair $(G^!\colon C^!\to D^!,\delta\psi^!,\psi^!)$ A27→p.134

The relative versions for pairs can be generalized for ads. So for subdivided versions of the geometric input from above we have the mosaicked signatures. Let M, X and Y be as above but equipped with a map r to a simplicial complex K.

manifold M ⇝ mosaicked symmetric signature $\mathrm{sgn}_K^{\mathbf{L}^\bullet}$ ⇝ locally Poincaré symmetric chain complex in $L^n(\Lambda_L K)$

Poincaré space X ⇝ mosaicked symmetric signature $\mathrm{sgn}_K^{\mathbf{L}^\bullet}$ ⇝ globally Poincaré symmetric chain complex in $L^n(\Lambda_G K)$ A17→p.126

degree one normal map $\widehat{f}\colon M'\to M$ ⇝ mosaicked quadratic signature $\mathrm{sgn}_K^{\mathbf{L}_\bullet}$ ⇝ locally Poincaré quadratic chain complex in $L_n(\Lambda_L K)$

degree one normal map $\widehat{f}\colon M'\to X$ ⇝ mosaicked quadratic signature $\mathrm{sgn}_K^{\mathbf{L}_\bullet}$ ⇝ globally Poincaré quadratic chain complex in $L_n(\Lambda_G K)$ A29→p.136

normal space Y ⇝ mosaicked normal signature $\mathrm{sgn}_K^{\mathbf{NL}^\bullet}$ ⇝ (non-Poincaré) normal chain complex in $NL^n(\Lambda_N K)$

Poincaré space X ⇝ mosaicked normal signature $\mathrm{sgn}_K^{\mathbf{NL}^\bullet}$ ⇝ globally Poincaré normal chain complex in $NL^n(\Lambda_G K)$ A38→p.138

A1 Symmetric

The symmetric construction con_X^{φ} is a direct consequence of the Alexander-Whitney diagonal approximation $C(X) \to C(X) \otimes C(X)$ which gives a chain map

$$\mathrm{con}_X^{\varphi} \colon C(X) \to W^{\%}(C(X)).$$

Evaluated on a cycle $[X] \in C(X)_n$ we get an n-symmetric structure $\varphi_n = \mathrm{con}_X^{\varphi}([X])$. The symmetric construction commutes with maps induced by maps of topological spaces and with suspensions of chain complexes but not with chain maps in general. This leads to the definition of the quadratic construction in the next section A2.

The original source for the symmetric construction is [Ran80a, §1]. There you find the details for the equivariant version which we omit here and for the properties listed below which we will not verify here. The symmetric construction is based on the Eilenberg-Zilber map and acyclic models. You find more details on that in [EM53] or in standard textbooks like [DK01, §3].

A1 Symmetric construction [Ran80b, Prop. 1.2]
Let X be a topological space with the singular chain complex $C(X)$. There is a natural chain map

$$\mathrm{con}_X^{\varphi} \colon C(X) \to W^{\%}(C(X)).$$

Properties

⇒ **[A12 → [Ran80b, Prop. 1.1]]**
The symmetric construction is functorial with respect to maps of spaces, i.e. let $f\colon X \to Y$ be a map of topological spaces. Then the following diagram commutes.

$$\begin{array}{ccc} C(X) & \xrightarrow{\mathrm{con}_X^{\varphi}} & W^{\%}(C(X)) \\ \downarrow f & & \downarrow f^{\%} \\ C(Y) & \xrightarrow{\mathrm{con}_Y^{\varphi}} & W^{\%}(C(Y)) \end{array}$$

⇒ **[A13 → [Ran80b, Prop. 1.4]]**
The symmetric construction commutes with algebraic and geometric suspension up to a chain homotopy Γ_X.

$$\begin{array}{ccccc} C_*(X) & \xrightarrow{\mathrm{con}_X^{\varphi}} & W^{\%}(C(X)) & \xrightarrow{S} & \Sigma^{-1}W^{\%}(\Sigma C(X)) \\ \downarrow \Sigma & & \overset{\Gamma_X}{\searrow} & & \downarrow \Sigma^{\%} \\ C_{1+*}(\Sigma X) & & \xrightarrow{\mathrm{con}_{\Sigma X}^{\varphi}} & & \Sigma^{-1}W^{\%}(C(\Sigma X)) \end{array}$$

Variations

⇒ **A14 Relative symmetric construction [Ran80b, Prop. 6.1]**
Let (X, A) be a pair of topological spaces and $i\colon C(A) \to C(X)$ the induced chain map. There is a chain map

$$\mathrm{con}_{X,A}^{\delta\varphi,\varphi} \colon \mathscr{C}(i) \to \mathscr{C}(i^{\%}).$$

compatible with con_X^{φ}.

Signatures

A15 Symmetric signature
Let X be an n-dimensional Poincaré space. There is a symmetric signature

$$\mathrm{sgn}^{\mathbf{L}^\bullet}_{\mathbb{Z}\pi}(X) \in L^n(\mathbb{Z}\pi)$$

producing an n-symmetric Poincaré chain complex (C, φ).

A16 Relative symmetric signature
Let (X, A) be an $(n+1)$-dimensional Poincaré pair. There is a relative symmetric signature

$$\mathrm{sgn}^{\mathbf{L}^\bullet}_{\to}(X, A)$$

producing an $(n+1)$-symmetric Poincaré pair $(f : C \to D, \delta\varphi, \varphi)$.

A17 (15, 16) Mosaicked symmetric signature [Ran92, Example 6.2 and 9.13]
Let X be an n-dimensional Poincaré space and $r : X \simeq |K|$ a map to a simplicial complex K. There is a mosaicked symmetric signature

$$\mathrm{sgn}^{\mathbf{L}^\bullet}_{K}(X) \in L^n(\Lambda_G K)$$

such that $A(\mathrm{sgn}^{\mathbf{L}^\bullet}_{K}) = \mathrm{sgn}^{\mathbf{L}^\bullet}_{\mathbb{Z}\pi}(X) \in L^n(\mathbb{Z}\pi)$. If X is an n-dimensional manifold there is a refined version

$$\mathrm{sgn}^{\mathbf{L}^\bullet}_{K}(X) \in L^n(\Lambda_L K).$$

Proof A1 (Symmetric construction)

[A11 → [DK01, §3]]
There are two functors

$$F : (X, Y) \mapsto C(X \times Y) \qquad \text{and} \qquad F' : (X, Y) \mapsto C(X) \otimes C(Y)$$

from the category of pairs of spaces to the category of chain complexes which are naturally equivalent, i.e. there exist natural transformations $\zeta : F \to F'$ and $\zeta^{-1} : F' \to F$ so that for any pair (X, Y) the composites

$$C(X \times Y) \xrightarrow{\zeta} C(X) \otimes C(Y) \xrightarrow{\zeta^{-1}} C(X \times Y)$$

and

$$C(X) \otimes C(Y) \xrightarrow{\zeta^{-1}} C(X \times Y) \xrightarrow{\zeta} C(X) \otimes C(Y)$$

are chain homotopic to the identity. Moreover, any two choices of ζ resp. ζ^{-1} are naturally chain homotopic.

Choose an Eilenberg-Zilber map (A11)

$$\zeta_0 : C(X \times X) \to C(X) \otimes C(X).$$

It is natural in both factors and $\mathbb{Z}_2$-equivariant up to homotopy with respect to the switch action on the factors. This means the following diagram commutes up to chain homotopy.

$$\begin{array}{ccc} C(X \times X) & \xrightarrow{\zeta_0} & C(X) \otimes C(X) \\ \big\downarrow \text{switch} & \overset{\zeta_1}{\leadsto} & \big\downarrow t \\ C(X \times X) & \xrightarrow{\zeta_0} & C(X) \otimes C(X) \end{array}$$

The chain homotopy $\zeta_1 : C(X \times X) \to (C(X) \otimes C(X))_{*+1}$ is again $\mathbb{Z}_2$-equivariant up to a chain homotopy ζ_2. This collection $\{\zeta_n \mid n \geq 0\}$ of chain homotopies defines in fact a map

$$\zeta : C(X \times X) \to \mathrm{Hom}_{\mathbb{Z}[\mathbb{Z}_2]}(W, C \otimes C).$$

Composed with the geometrically induced diagonal chain map d_*, the Eilenberg-Zilber map defines an Alexander-Whitney diagonal approximation, our symmetric construction

$$\mathrm{con}^{\varphi}_X : C(X) \xrightarrow{d_*} C(X \times X) \xrightarrow{\zeta} W^{\%}(C(X)). \qquad \square$$

Proof A14 (Relative symmetric construction)

We obtain the chain map $\mathrm{con}^{\delta\varphi,\varphi}_{X,A} : \mathscr{C}(i) \to \mathscr{C}(i^{\%})$ immediately by the naturality of the symmetric
A1→p.125 construction A1:

$$\begin{array}{ccccc} C(A) & \xrightarrow{i} & C(X) & \longrightarrow & \mathscr{C}(i) \\ \downarrow \mathrm{con}^{\varphi}_A & & \downarrow \mathrm{con}^{\varphi}_X & & \downarrow \mathrm{con}^{\delta\varphi,\varphi}_{X,A} \\ W^{\%}(C(A)) & \xrightarrow{i^{\%}} & W^{\%}(C(X)) & \longrightarrow & \mathscr{C}(i^{\%}) \end{array}$$

$\square$

Proof A15 (Symmetric signature)

Let $\widetilde{X}$ be the universal covering of an n-dimensional Poincaré space X and $\pi = \pi_1(X)$. The symmetric construction $\mathrm{con}^{\varphi}_{\widetilde{X}}$ yields a chain map of $\mathbb{Z}\pi$-modules. Applying $\mathbb{Z} \otimes_{\mathbb{Z}\pi}$ _ we obtain the equivariant symmetric construction, a chain map of chain complexes of abelian groups

$$\mathrm{con}^{\varphi}_{\widetilde{X}} : C(X) \to W^{\%}(C(\widetilde{X})) = \mathrm{Hom}_{\mathbb{Z}[\mathbb{Z}_2]}(W, C(\widetilde{X}) \otimes_{\mathbb{Z}\pi} C(\widetilde{X})).$$

Evaluate the equivariant symmetric construction on a fundamental cycle $[X]$ to define $\varphi = \mathrm{con}^{\varphi}_X([\widetilde{X}])$. Inspection of the symmetric construction shows that the map of $\mathbb{Z}\pi$-module chain complexes $\varphi_0 : C(\widetilde{X})^{n-*} \to C(\widetilde{X})_*$ is given by taking cap product with $[\widetilde{X}]$ and thus a chain homotopy equivalence. We define the symmetric signature of X to be

$$\mathrm{sgn}^{\mathbf{L}^\bullet}_{\mathbb{Z}\pi}(X) = [(C(\widetilde{X}), \varphi)] \in L^n(\mathbb{Z}\pi).$$

The cobordism class in $L^n(\mathbb{Z}\pi)$ does not depend on the choice of $[X]$. $\square$

Proof A16 (Relative symmetric signature)

Use the relative symmetric construction of A14 to generalize the symmetric signature $\mathrm{sgn}^{\mathbf{L}^\bullet}_{\mathbb{Z}\pi}$ for pairs to obtain the relative symmetric signature $\mathrm{sgn}^{\mathbf{L}^\bullet}_{\to}$. $\square$

Proof A17 (Mosaicked symmetric signature)

By topological transversality we can make r transverse to the dual cells $D(\sigma, K)$. We obtain a K-dissection $X = \bigcup_{\sigma \in K} X[\sigma]$ where $X[\sigma] = r^{-1}(D(\sigma, K))$.

Let $C(X)$ be the subcomplex of the singular chain complex of X such that the image of each singular chain is contained in some $X[\sigma]$. This chain complex is chain homotopy equivalent to the full singular chain complex. We consider $C(X)$ as a K-mosaicked chain complex in $\mathbb{B}(\mathbb{Z}_*K)$ by defining

$$C(X)(\sigma) = C(X[\sigma], \partial X[\sigma])$$

for $\sigma \in K$.

The relative symmetric construction A14 can be generalized to a chain map

$$\mathrm{con}_K^{\varphi_K} : C(X) \to W^{\%}(C(X))$$

over $\mathbb{Z}_* X$. Evaluated on a cycle $[X] \in C(X)_n$ it produces an n-dimensional K-mosaicked symmetric structure $\varphi_K = \mathrm{con}_K^{\varphi_K}([X])$ and hence an n-dimensional K-mosaicked symmetric chain complex $(C(X), \varphi_K)$ in $\Lambda_G K$ whose σ-component

$$\varphi_K(\sigma)_0 : C^{n-|\sigma|-*}(X[\sigma]) \to C(X[\sigma], \partial X[\sigma])$$

is the cap product with the class $[X[\sigma]] \coloneqq \partial_\sigma([X])$. The cobordism class of $(C(X), \varphi_K)$ does not depend on the choice of the fundamental class and hence defines an element $\mathrm{sgn}_K^{\mathbf{L}^\bullet}(X)$ in $L^n(\Lambda_G K)$.

If X is a manifold, then each $X[\sigma]$ is an $(n - |\sigma|)$-dimensional manifold with boundary and hence satisfies Poincaré duality. If we use the fundamental class $[X]$ to obtain φ_K, then $\varphi_K(\sigma)_0$ is the cap product with the fundamental class $[X[\sigma]]$ and hence a homotopy equivalence. The symmetric signature refines to an element

$$\mathrm{sgn}_K^{\mathbf{L}^\bullet}(K) \in L^n(\Lambda_L K). \qquad \square$$

Room service A1

$\boxed{C(X)}$ the singular chain complex for a space X.

$\boxed{t}$ the generator of $\mathbb{Z}_2$; acts on $C \otimes_R C$ via $t(x \otimes y) = (-)^{|x||y|} y \otimes x$ and on $C \otimes_{\mathbb{A}} C$ via $t(x \otimes y) = T_{C,C}(x \otimes y)$.

$\boxed{W^{\%}}$ a functor defined for a chain complex C by $W^{\%}(C) \coloneqq \mathrm{Hom}_{\mathbb{Z}[\mathbb{Z}_2]}(W, C \otimes C)$.

- $\boxed{W}$ the free resolution of the trivial $\mathbb{Z}[\mathbb{Z}_2]$-chain module $\mathbb{Z}$; given by the $\mathbb{Z}[\mathbb{Z}_2]$-chain complex $\ldots \xrightarrow{1+t} \mathbb{Z}[\mathbb{Z}_2] \xrightarrow{1-t} \mathbb{Z}[\mathbb{Z}_2] \longrightarrow 0$

$\boxed{L^n(R)}$ the cobordism group of n-symmetric Poincaré chain complexes over R.

- $\boxed{t}$ the generator of $\mathbb{Z}_2$; acts on $C \otimes_R C$ via $t(x \otimes y) = (-)^{|x||y|} y \otimes x$ and on $C \otimes_{\mathbb{A}} C$ via $t(x \otimes y) = T_{C,C}(x \otimes y)$.
- $\boxed{(C, \varphi)}$ an n-symmetric chain complex consisting of a chain complex C and an n-symmetric structure φ;
  Poincaré means $\varphi_0 : C^{n-*} \to C$ is a chain equivalence.

$\boxed{X}$ an n-dimensional Poincaré space, i.e. a finite CW complex together with an orientation homomorphism $w : \pi_1(X) \to \{\pm 1\}$ and a fundamental class $[X]$.

$[X]$ fundamental class for an n-dimensional Poincaré space X is a cycle in the cellular $\mathbb{Z}\pi$-chain complex $C_n(\widetilde{X})$ which represents an n-dimensional homology class in $H_n(X;\mathbb{Z}^w)$ such that $\cdot \cap [X]\colon C^{n-*}(\widetilde{X}) \to C_*(\widetilde{X})$ is a $\mathbb{Z}\pi$-chain homotopy equivalence where $\widetilde{X}$ is the universal covering.

$X[\sigma]$ is defined for a map $r\colon X \to K$ to a simplicial complex as the preimage of the dual cell $D(\sigma, K)$ after making r transverse. If X is a simplicial complex itself, choose r to be the identity. The subdivision $X = \bigcup_{\sigma\in K} X[\sigma]$ is called a K-dissection of X.

$\partial_\sigma\colon C(K) \to \Sigma^{|\sigma|}C(\sigma)$ chain map defined for each simplex $\sigma = \langle v_0, v_1, \dots, v_{|\sigma|}\rangle$ in K by the composition

$$C(K) = \sum_{\tau\in K} C(\tau)_n \xrightarrow{\text{proj.}} C(\sigma_0)_n \xrightarrow{d_1} C(\sigma_1)_{n_1} \xrightarrow{d_2} \dots \xrightarrow{d_{|\sigma|}} C(\sigma)_{n-\sigma}$$

with $\sigma_j = \langle v_0, \dots, v_j\rangle$ and $d_j = d^{\sigma_j,\sigma_{j+1}}_{n-j+1}$ the relevant component of $d^{C(K)}_{n-j+1}\colon C(K)_{n-j+1} \to C(K)_{n-j}$ (see [Ran92, Def. 8.2]).

$\langle v_0, v_1, \dots, v_j\rangle$ defines a simplex spanned by the vertices $v_0, \dots, v_j$.

$\Lambda_G X$ for $\Lambda = (\mathbb{A}, \mathbb{C}, \mathbb{P})$ denotes $(\mathbb{A}_*X, \mathbb{C}_L X, \mathbb{P}_G X, (T_*, e_*))$, the X-mosaicked algebraic bordism category of $\Lambda = (\mathbb{A}, \mathbb{C}, \mathbb{P})$ with *global Poincaré* duality.

$\Lambda_L X$ for $\Lambda = (\mathbb{A}, \mathbb{C}, \mathbb{P})$ denotes $(\mathbb{A}_*X, \mathbb{C}_L X, \mathbb{P}_L X, (T_*, e_*))$, the X-mosaicked algebraic bordism category of $\Lambda = (\mathbb{A}, \mathbb{C}, \mathbb{P})$ with *local Poincaré* duality.

A2 Quadratic

We give a short introduction on how quadratic structures are obtained, followed up by an overview of different variants of quadratic constructions, their properties and the signatures derived from them.

A stable map $F\colon \Sigma^p X \to \Sigma^p Y$ defines a chain map

$$f\colon C(X) \xrightarrow{\Sigma^p_X} \Sigma^{-p}C(\Sigma^p X) \xrightarrow{F_*} \Sigma^{-p}C(\Sigma^p Y) \xrightarrow{\Sigma^{-p}_Y} C(Y)$$

which is not (necessarily) induced by a geometric map $X \to Y$. Hence we get a non-trivial chain map

$$f^{\%} \circ \mathrm{con}^{\varphi}_X(X) - \mathrm{con}^{\varphi}_X(Y) \circ f_* \colon C(X) \to W^{\%}(C(Y))$$

in the symmetric world measuring the square's failure

$$\begin{array}{ccc} C(X) & \xrightarrow{\mathrm{con}^{\varphi}_X(X)} & W^{\%}C(X) \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f^{\%}} \\ C(Y) & \xrightarrow{\mathrm{con}^{\varphi}_X(Y)} & W^{\%}C(Y) \end{array}$$

to commute. The stabilized term $\mathcal{S}(f^{\%}\,\mathrm{con}^{\varphi}_X - \mathrm{con}^{\varphi}_Y\, f) \simeq F^{\%}_*\,\mathrm{con}^{\varphi}_{\Sigma^k X} - \mathrm{con}^{\varphi}_{\Sigma^k Y}\, F_*$ vanishes because we recover the geometrically induced chain map F_*. The suspension homomorphism is an isomorphism for structures in $\widehat{W}^{\%}$. The short exact sequence

$$0 \to W_{\%}C(Y) \to W^{\%}C(Y) \to \widehat{W}^{\%}C(Y) \to 0$$

together with the fact that the suspension homomorphism $\mathcal{S}$ is an isomorphism on $\widehat{W}^{\%}$ leads to a preferred lift in $W_{\%}$, the quadratic construction

$$\mathrm{con}^{\psi}_F(X)\colon C(X) \to W_{\%}(C(Y))$$

with the property that $(1+t)\cdot \mathrm{con}^{\psi}_F(X) = f^{\%} \circ \mathrm{con}^{\varphi}_X(X) - \mathrm{con}^{\varphi}_X(Y) \circ f \colon C(X) \to W^{\%}(C(Y))$.

This can be applied in the following geometric situations. For a degree one normal map $\widehat{f}\colon M \to X$ between manifolds or Poincaré spaces we obtain a stable Umkehr map $F\colon \Sigma^p X_+ \to \Sigma^p M_+$ so that we can use the quadratic construction directly to define a quadratic signature for $\widehat{f}$. If the target space is normal, we obtain only a semi-stable Umkehr map. For this case we need a variant of the quadratic construction, the spectral quadratic construction A24 in order to obtain the signature as defined in A27.

The quadratic construction was introduced by Ranicki in [Ran80b] and with corrected details defined in [Ran81].

A2 Quadratic construction [Ran80b, Prop. 1.5][Ran81, p.30]
Let $F\colon \Sigma^p X \to \Sigma^p Y$ be a map of pointed topological spaces. There is a chain map

$$\mathrm{con}^{\psi}_F\colon C(X) \to W_{\%}(C(Y))$$

such that $(1+t)\circ \mathrm{con}^{\psi}_F = f^{\%}\,\mathrm{con}^{\varphi}_X - \mathrm{con}^{\varphi}_Y\, f$.

Properties

⇒ **[A22 → [Ran80b, Prop. 1.5 (iii)]]**
For another map $G\colon \Sigma^p Y \to \Sigma^p Z$ of pointed topological spaces we have $\mathrm{con}^{\psi}_{G\circ F} = g_{\%}\,\mathrm{con}^{\psi}_{F} + \mathrm{con}^{\psi}_{G}\, f$.

Variations

⇒ **A23 Relative quadratic construction**
Let $(j\colon A \to X)$ and $(i\colon Y \to B)$ be pairs of of pointed topological spaces and $(\delta F, F)\colon (\Sigma^p X, \Sigma^p A) \to (\Sigma^p Y, \Sigma^p B)$ a map. Then there is a chain map

$$\mathrm{con}^{\delta\psi,\psi}_{\delta F,F}\colon \mathscr{C}(j) \to \mathscr{C}(i_{\%})$$

compatible with con^{ψ}_{F}.

⇒ **A24 (A27) Spectral quadratic construction [Ran81, Proposition 7.3.1]**
Let $F\colon X \to \Sigma^p Y$ be a semi-stable map between pointed topological spaces and $f\colon \widetilde{C}(X)_{p+*} \to \widetilde{C}(Y)_*$ the induced chain map. There is a natural chain map

$$\mathrm{con}^{\psi^!}_{F}\colon \widetilde{C}(X)_{p+*} \to W_{\%}(\mathscr{C}(f))$$

such that

$$(1+t)\circ \mathrm{con}^{\psi^!}_{F} = e^{\%} \circ \mathrm{con}^{\varphi}_{Y} \circ f$$

where $e\colon \widetilde{C}(Y_*) \to \mathscr{C}(f)$ is the inclusion.

⇒ **A25 (A28, B28, 232) Relative spectral quadratic construction [Ran81, Proposition 7.3.1]**
Let $(\delta F, F)\colon (N, A) \to \Sigma^p(Y, B)$ be a semi-stable map between pairs of pointed topological spaces inducing the following commutative diagram of chain maps.

$$\begin{array}{ccccc}
\widetilde{C}(A)_{p+*} & \xrightarrow{f} & \widetilde{C}(B) & \longrightarrow & \mathscr{C}(f) \\
\downarrow{\scriptstyle j} & & \downarrow{\scriptstyle i} & & \downarrow{\scriptstyle (i,j)} \\
\widetilde{C}(N)_{p+*} & \xrightarrow{\delta f} & \widetilde{C}(Y) & \longrightarrow & \mathscr{C}(g) \\
\downarrow & & \downarrow & & \downarrow \\
\widetilde{C}(N,A)_{p+*} & \xrightarrow{(\delta f, f)} & \widetilde{C}(Y,B) & \xrightarrow{e} & \mathscr{C}(j,i)
\end{array}$$

There is a chain map $\quad \mathrm{con}^{\delta\psi^!,\psi^!}_{\delta F,F}\colon \widetilde{C}(N,A) \to \mathscr{C}((i,j)_{\%})$

such that $\quad (1+t)\circ \mathrm{con}^{\delta\psi^!,\psi^!}_{\delta F,F} = e^{\%} \circ \mathrm{con}^{\delta\varphi,\varphi}_{Y,B} \circ (\delta f, f)$.

Signatures

⇒ **A26 Quadratic signature [Ran80b, p.229]**
Let $\widehat{f}\colon M \to X$ be a degree one normal map between n-dimensional Poincaré spaces. There is a quadratic signature

$$\mathrm{sgn}^{\mathbf{L}_\bullet}_{\mathbb{Z}\pi}(\widehat{f}) \in L_n(\mathbb{Z}\pi)$$

producing an n-quadratic Poincaré chain complex (C, ψ).

⇒ **A27 (231) Quadratic signature for normal target spaces [Ran81, Prop. 7.3.4 (iv)]**
Let $\widehat{g}\colon N \to Y$ be a degree one normal map from an Poincaré space to a normal space both of dimension n. There is a quadratic signature

$$\mathrm{sgn}^{\mathbf{L}_\bullet}_{\#}(\widehat{g}) = (C^!, \psi^!) := (\mathscr{C}(g^!), \mathrm{con}^{\psi^!}_{\Gamma_Y}(u^*_{\nu_Y}))$$

producing an n-quadratic chain complex (not necessarily Poincaré) such that $(1+t)(\psi^!) = e^{\%}_{g^!}(\varphi_N)$.

A28 (232) Relative quadratic signature for normal target spaces [Ran80b, Prop. 6.4]
Let $(\delta\widehat{g}, \widehat{g})\colon (N, A) \to (Y, B)$ be a degree one normal map from a Poincaré pair (N, A) to a normal pair (Y, B) both of dimension n. There is an n-quadratic pair called the relative quadratic signature

$$\mathrm{sgn}^{\mathbf{L}_\bullet}_{\to}(\delta\widehat{g}, \widehat{g}) = (G^! \colon C^! \to D^!, (\delta\psi^!, \psi^!)).$$

such that

$$(1+t)(\delta\psi^!, \psi^!) = e_{g_i^!, g^!}{}^{\%} \circ \mathrm{con}^{\delta\varphi;\varphi}_{N,A}([N, A]).$$

A29 (221, 23) Mosaicked quadratic signature [Ran92, Example 9.14]
Let $\widehat{f}\colon M \to X$ be a degree one normal map from a closed topological manifold to a Poincaré space both of dimension n. Let $r\colon X \to K$ be a map to a simplicial complex K. There is a mosaicked quadratic signature

$$\mathrm{sgn}^{\mathbf{L}_\bullet}_{K}(\widehat{f}) \in L_n(\Lambda_G K)$$

with $A(\mathrm{sgn}^{\mathbf{L}_\bullet}_{X}(\widehat{f})) = \mathrm{sgn}^{\mathbf{L}_\bullet}_{\mathbb{Z}\pi}(\widehat{f})$. If X is a manifold, there is a refined version

$$\mathrm{sgn}^{\mathbf{L}_\bullet}_{K}(\widehat{f}) \in L_n(\Lambda_L K).$$

Proof A2 (Quadratic construction)

[A13 → [Ran80b, Prop. 1.4]]
The symmetric construction commutes with algebraic and geometric suspension up to a chain homotopy Γ_X.

$$\begin{array}{ccccc} C_*(X) & \xrightarrow{\mathrm{con}^{\varphi}_X} & W^{\%}(C(X)) & \xrightarrow{\mathcal{S}} & \Sigma^{-1}W^{\%}(\Sigma C(X)) \\ \downarrow{\scriptstyle \Sigma} & & \overset{\Gamma_X}{\rightsquigarrow} & & \downarrow{\scriptstyle \Sigma^{\%}} \\ C_{1+*}(\Sigma X) & & \xrightarrow{\mathrm{con}^{\varphi}_{\Sigma X}} & & \Sigma^{-1}W^{\%}(C(\Sigma X)) \end{array}$$

A21 Stable symmetric is hyperquadratic [Ran80a, p.106]
We have a chain homotopy equivalence

$$\operatorname*{colim}_{p\to\infty} \mathcal{S}^p(W^{\%}C) \simeq \widehat{W}^{\%}C.$$

Suppose that p in $F\colon \Sigma^p X \to \Sigma^p Y$ is large enough such that $\mathcal{S}^p(f^{\%}\,\mathrm{con}^{\varphi}_X - \mathrm{con}^{\varphi}_Y\, f)$ is stable, if necessary suspend F. The following pullback gives us a chain complex $\overline{C}(X)$ homotopy equivalent to $C(X)$.

$$\begin{array}{ccc} \overline{C}(X) & \longrightarrow & C(X) \\ \downarrow{\scriptstyle \overline{f}} & & \downarrow{\scriptstyle \Sigma^{-p}F_*\circ\Sigma_X} \\ C(Y) & \xrightarrow{\simeq} & \Sigma^{-p}C(\Sigma^p Y) \end{array}$$

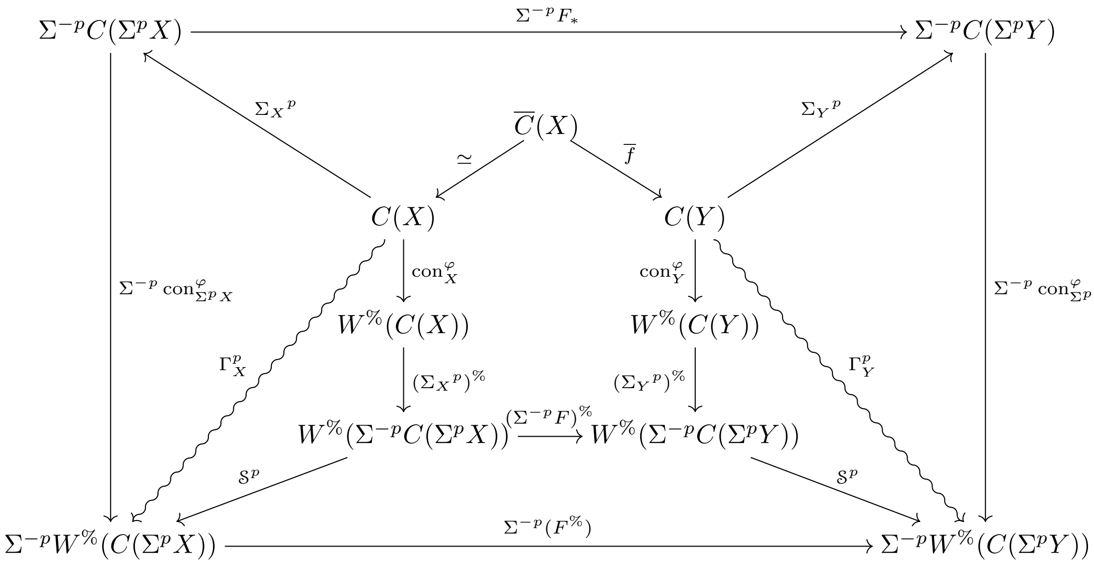
A model for $\overline{C}(X)$ is the desuspended mapping cone $\Sigma^{-1}\mathscr{C}(C(X)\oplus C(Y) \to \Sigma^{-p}C(\Sigma^p Y))$. From the properties of the symmetric construction A13 we have a natural chain homotopy

$$\Gamma_X \colon \Sigma C(X) \to W^{\%}(C(\Sigma X))_{n+1}$$

up to which the following diagram commutes.

$$\begin{array}{ccc} \Sigma C(X) & \xrightarrow{\mathcal{S}\circ\Sigma(\mathrm{con}^{\varphi}_X)} & W^{\%}(\Sigma C(X)) \\ \downarrow{\scriptstyle \Sigma_X} & & \downarrow{\scriptstyle \Sigma_X{}^{\%}} \\ C(\Sigma X) & \xrightarrow{\mathrm{con}^{\varphi}_{\Sigma X}} & W^{\%}(C(\Sigma X)) \end{array}$$

Now we consider the following diagram.



The lower trapezoid commutes strictly and on the left and right hand side we recover the diagram from above. Start a diagram chase with an element (x, y, z) in the pullback $\overline{C}(X)_n$ where $z \in \Sigma^{-p}C(\Sigma^p Y)$ is a $(n+1)$-chain connecting $x \in C(X)_n$ and $y \in C(Y)_n$. It leads to an element in
A21→p.132 the fiber of $\mathcal{S}^p \colon W^{\%}C(Y) \to \Sigma^{-p}W^{\%}(\Sigma^p C(Y))$. Using the equivalence of A21 and the cofibration sequence $W_{\%}(C) \to W^{\%}(C) \to \widehat{W}^{\%}(C)$ we identify the fiber with $W_{\%}C(Y)$ and hence obtain a quadratic structure. □

Proof A21 (Stable symmetric is hyperquadratic)

For $\widehat{W}^{\%}C$ the suspension map $\mathcal{S} \colon \widehat{W}^{\%}C \to \Sigma^{-1}\widehat{W}^{\%}(\Sigma C))$ is a chain equivalence and we have $\mathrm{hocolim}_k\, \Sigma^{-k}W_{\%}\Sigma^k C \simeq 0$. Hence we obtain from

$$\begin{array}{ccccccccc}
0 & \longrightarrow & W^{\%}C & \longrightarrow & \widehat{W}^{\%}C & \longrightarrow & \Sigma W_{\%}C & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \mathcal{S}^k} & & \downarrow{\scriptstyle \mathcal{S}^k} & & \downarrow{\scriptstyle \mathcal{S}^k} & & \\
0 & \longrightarrow & \Sigma^{-k}W^{\%}(\Sigma^k C) & \longrightarrow & \Sigma^{-k}\widehat{W}^{\%}(\Sigma^k C) & \longrightarrow & \Sigma^{-k+1}W_{\%}(\Sigma^k C) & \longrightarrow & 0
\end{array}$$

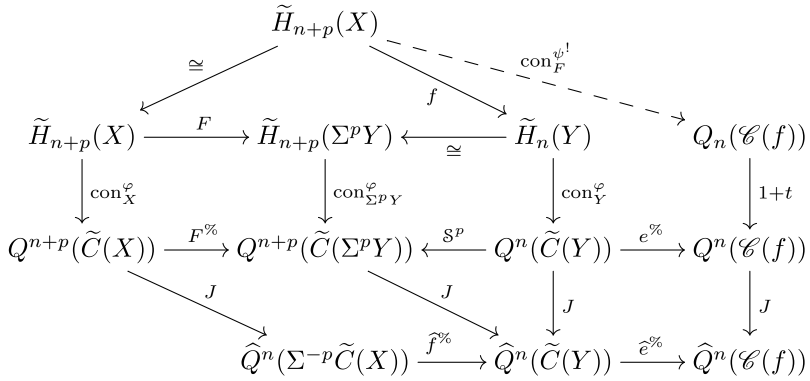
that $\widehat{W}^{\%}C \simeq \Sigma^{-k}W^{\%}\Sigma^k C$ which proves the statement. □

Proof A23 (Relative quadratic construction)

The construction of A2 can be generalized to the relative case. □

Proof A24 (Spectral quadratic construction)

The existence of the spectral quadratic construction $\mathrm{con}_F^{\psi!}$ can be read off the following commutative diagram.



The functor $C \mapsto \widehat{Q}^n(C)$ is a generalized cohomology theory on the category of chain complexes [Wei85a, Theorem 1.1]. So the last row is exact and the right column is also exact due to the long exact sequence of Q-groups. □

Proof A25 (Relative spectral quadratic construction)

The existence of the relative spectral quadratic construction $\mathrm{con}_{\delta F, F}^{\delta\psi^!, \psi^!}$ follows immediately from the naturality of the diagram used to construct the spectral quadratic construction A24. □

Proof A26 (Quadratic signature)

C (A26) Umkehr maps [Ran80b, Prop. 4.2]
Let $\widehat{f}\colon N \to X$ be a degree one normal map between Poincaré spaces both of dimension n. There is a stable geometric Umkehr map

$$F\colon \Sigma^p X_+ \to \Sigma^p N_+$$

such that $\Sigma^p f_+ \circ F \simeq \mathrm{id}\colon \Sigma^p X_+ \to \Sigma^p X_+$ and such that the induced chain map $F_*\colon \widetilde{C}(\Sigma^p X_+) \to \widetilde{C}(\Sigma^p N_+)$ is chain homotopic to the composition

$$f^!\colon C(X) \xrightarrow{(\varphi_X)_0^{-1}} C(X)^{n-*} \xrightarrow{f^*} C(N)^{n-*} \xrightarrow{(\varphi_N)_0} C(N).$$

From the degree one normal map $\widehat{f}\colon N \to X$ of n-dimensional Poincaré spaces we obtain a stable equivariant Umkehr map $F\colon \Sigma^p \widetilde{X}_+ \to \Sigma^+ \widetilde{N}_+$ of the universal coverings $\widetilde{X}$ and $\widetilde{N}$ by using C→p.161 equivariant S-duality as described in [Ran80b, §3]. Similarly to the symmetric case there is a equivariant version of the quadratic construction for F

$$\mathrm{con}_F^{\psi}\colon C(X) \to W_{\%}(C(\widetilde{N})).$$

Evaluated on a fundamental cycle $[N]$ it gives rise to an n-quadratic structure $\psi = \mathrm{con}_F^{\psi}([X]) \in W_{\%}(\widetilde{N})$. The stable map F induces the Umkehr chain map

$$f^!\colon C(\widetilde{X}) \xrightarrow{\Sigma_X} \Sigma^{-p} C(\Sigma^p \widetilde{X}_+) \xrightarrow{F} \Sigma^{-p} C(\Sigma^p \widetilde{N}_+) \xrightarrow{\Sigma_X^{-1}} C(\widetilde{N}).$$

With the inclusion map $e\colon C(\widetilde{N}) \to \mathscr{C}(f^!)$ we push ψ forward to a quadratic structure on the cone of $f^!$ and eventually define the quadratic signature $\mathrm{sgn}_{\mathbb{Z}\pi}^{\mathbf{L}_\bullet}(\widehat{f})$ as the n-quadratic Poincaré chain complex cobordism class

$$[(\mathscr{C}(f^!), e_{\%}\psi)] \in L_n(\mathbb{Z}[\pi_1(X)]).$$

The cobordism class is independent from the choice of the fundamental cycle. □

Remark. If M is an n-dimensional oriented manifold then the quadratic signature only depends on the normal cobordism class of $\widehat{f}$ in the set of the normal invariants $\mathscr{N}(X)$ and provides us with a map

$$\operatorname{sgn}^{\mathbf{L}_\bullet}_{\mathbb{Z}\pi} \colon \mathscr{N}(X) \to L_n(\mathbb{Z}\pi).$$

Proof A27 (Quadratic signature for normal target spaces)

A24 (A27) Spectral quadratic construction [Ran81, Proposition 7.3.1]
Let $F\colon X \to \Sigma^p Y$ be a semi-stable map between pointed topological spaces and $f\colon \widetilde{C}(X)_{p+*} \to \widetilde{C}(Y)_*$ the induced chain map. There is a natural chain map

$$\operatorname{con}^{\psi^!}_F \colon \widetilde{C}(X)_{p+*} \to W_{\%}(\mathscr{C}(f))$$

such that

$$(1+t) \circ \operatorname{con}^{\psi^!}_F = e^{\%} \circ \operatorname{con}^{\varphi}_Y \circ f$$

where $e\colon \widetilde{C}(Y_*) \to \mathscr{C}(f)$ is the inclusion.

C3 (A27) Umkehr map for normal targets
Let $\widehat{g}\colon N \to Y$ be a degree one normal map from a Poincaré space N to a normal space (Y, ν, ρ) both of dimension n. There is a semi-stable geometric Umkehr map

$$\Gamma^! \colon Th(\nu_Y)^* \to \Sigma^p N_+$$

such that the induced chain map $\gamma^!\colon \widetilde{C}(Th(\nu_Y)^*) \to \widetilde{C}(\Sigma^p N_+)$ is chain homotopic to the composition

$$g^! \colon C(Y)^{n-*} \xrightarrow{g^*} C(N)^{n-*} \xrightarrow{(\varphi_N)_0} C(N)$$

where $\varphi_N = \operatorname{con}^{\varphi}_N([N])$.

A24→p.132
C3→p.164

Basically we obtain $(C^!, \psi^!)$ by applying A24 to C3. Use the chain map $g^!$ from C3 to define $C^! := \mathscr{C}(g^!)$. The spectral quadratic construction $\operatorname{con}^{\psi^!}_{\Gamma^!}$ from A24 evaluated on the S-dual of a Thom class u_ν produces the quadratic structure $\psi^! \in W_{\%}\mathscr{C}(\gamma^!)$. By C3 we have $\gamma^! \simeq g^!$ and hence we can consider $\psi^!$ as a structure on $C^!$. The property $(1+t)\operatorname{con}^{\psi^!}_{\Gamma^!} = e^{\%}_{\gamma^!} \circ \operatorname{con}^{\varphi}_N \circ \gamma^!$ from A24 together with the chain homotopy $g^! \simeq \gamma^!$ and the fact that g is of degree one yields that

$$(1+t)\psi^! := (1+t)\operatorname{con}^{\psi^!}_{\Gamma^!}(u_\nu{}^*) = e^{\%}_{g^!}\operatorname{con}^{\varphi}_N([N]) =: e^{\%}_{g^!}\varphi_N.$$ □

Proof A28 (Relative quadratic signature for normal target spaces)

C5 (A28) Relative Umkehr maps for normal targets
Let $(\delta\widehat{g}, \widehat{g})\colon (N, A) \to (Y, B)$ be a degree one normal map from a Poincaré pair to a normal pair both of dimension $(n+1)$ with $j\colon B \to Y$ and $i\colon A \to N$ the inclusion maps. There are semi-stable geometric Umkehr maps $\Gamma^!_Y, \Gamma^!_B, \Gamma^!_{Y,B}$ which fit into the following commutative diagram

$$\begin{array}{ccccccc}
\Sigma^{-1}Th(\nu_B)^* & \xrightarrow{i} & (Th(\nu_Y)/Th(\nu_B))^* & \longrightarrow & Th(\nu_Y)^* & \longrightarrow & Th(\nu_A)^* \\
\downarrow{\scriptstyle \Gamma^!_B} & & \downarrow{\scriptstyle \Gamma^!_Y} & & \downarrow{\scriptstyle \Gamma^!_{Y,B}} & & \downarrow{\scriptstyle \Sigma\Gamma^!_B} \\
\Sigma^p A_+ & \xrightarrow{j} & \Sigma^p N_+ & \longrightarrow & \Sigma^p N/A & \longrightarrow & \Sigma^{p+1} A_+
\end{array}$$

C5 (A28) (cont.)
and the induced chain maps

$$\begin{aligned}\gamma^!_B &: \widetilde{C}(Th(\nu_B)^*) \to \widetilde{C}(\Sigma^{p+1}A_+)\\ \gamma^!_Y &: \widetilde{C}((Th(\nu_Y)/Th(\nu_B))^*) \to \widetilde{C}(\Sigma^p N_+)\\ \gamma^!_{Y,B} &: \widetilde{C}(Th(\nu_Y)^*) \to \widetilde{C}(\Sigma^p N/A)\end{aligned}$$

are chain homotopic to

$$\begin{aligned}g^! &: C(B)^{n+1-*} \xrightarrow{g^*} C(A)^{n+1-*} \xrightarrow{(\varphi_B)_0} C(A)\\ g^!_i &: \mathscr{C}(j)^{n-*} \xrightarrow{(\delta g, g)^*} \mathscr{C}(i)^{n-*} \xrightarrow{\varphi_i} C(N)\\ g^!_{i^*} &: C(Y)^{n-*} \xrightarrow{\delta g^*} C(N)^{n-*} \xrightarrow{\varphi_{i^*}} \mathscr{C}(i).\end{aligned}$$

A25 (A28, B28, 232) Relative spectral quadratic construction [Ran81, Proposition 7.3.1]
Let $(\delta F, F)\colon (N, A) \to \Sigma^p(Y, B)$ *be a semi-stable map between pairs of pointed topological spaces inducing the following commutative diagram of chain maps.*

$$\begin{array}{ccccc}\widetilde{C}(A)_{p+*} & \xrightarrow{f} & \widetilde{C}(B) & \longrightarrow & \mathscr{C}(f)\\ \downarrow{\scriptstyle j} & & \downarrow{\scriptstyle i} & & \downarrow{\scriptstyle (i,j)}\\ \widetilde{C}(N)_{p+*} & \xrightarrow{\delta f} & \widetilde{C}(Y) & \longrightarrow & \mathscr{C}(g)\\ \downarrow & & \downarrow & & \downarrow\\ \widetilde{C}(N,A)_{p+*} & \xrightarrow{(\delta f, f)} & \widetilde{C}(Y,B) & \xrightarrow{e} & \mathscr{C}(j,i)\end{array}$$

There is a chain map $\quad \mathrm{con}^{\delta\psi^!,\psi^!}_{\delta F,F}\colon \widetilde{C}(N,A) \to \mathscr{C}((i,j)_\%)$

such that $\quad (1+t)\circ \mathrm{con}^{\delta\psi^!,\psi^!}_{\delta F,F} = e^\% \circ \mathrm{con}^{\delta\varphi,\varphi}_{Y,B} \circ (\delta f, f).$

Let be $(\delta\varphi(N), \varphi(A)) = \mathrm{con}^{\delta\varphi,\varphi}_{N,A}([N,A])$ the symmetric structure obtained from a fundamental cycle $[N, A]$ and denote

$$\begin{aligned}g^! &: C^{n-1-*}(B) \xrightarrow{g^*} C^{n-1-*}(A) \xrightarrow{\varphi(A)_0} C(A)\\ g^!_i &: \mathscr{C}(i)^{n-*} \xrightarrow{(\delta g^*, g^*)} \mathscr{C}^{n-*}(j) \xrightarrow{(\delta\varphi(N)_0, \varphi(A)_0 j)} C(N).\end{aligned}$$

We define the chain complexes $C^!$ and $D^!$ of $\mathrm{sgn}^{\mathbf{L}_\bullet}_{\to}$ to be the mapping cones $\mathscr{C}(g^!)$ and $\mathscr{C}(g^!_i)$ with the induced map $G^!$.

$$\begin{array}{ccccc}C^{n-1-*}(B) & \xrightarrow{g^!} & C(A) & \longrightarrow & \mathscr{C}(g^!)\\ \downarrow{\scriptstyle \partial^*} & & \downarrow{\scriptstyle j} & & \downarrow{\scriptstyle G^!}\\ \mathscr{C}(i)^{n-*} & \xrightarrow{g^!_i} & C(N) & \longrightarrow & \mathscr{C}(g^!_i)\end{array}$$

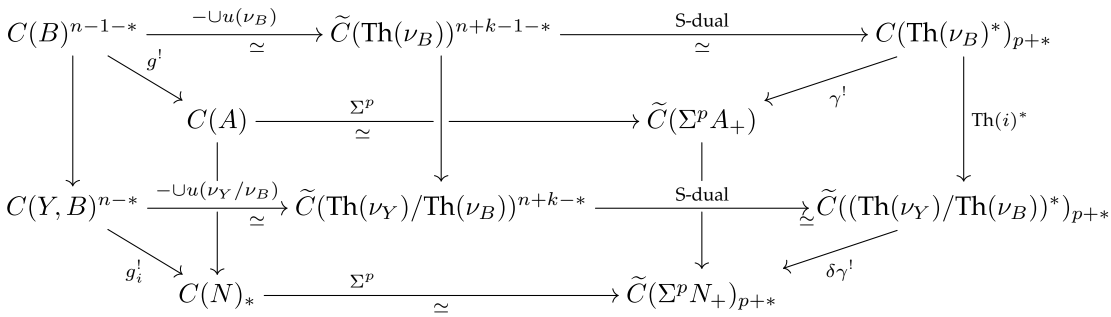
We want to use the relative spectral quadratic construction to obtain the quadratic structure. By C6 we obtain the necessary semi-stable maps C6→p.165

$$\begin{aligned}\Gamma^!_Y &: (\mathrm{Th}(\nu_Y)/\mathrm{Th}(\nu_B))^* \xrightarrow{(\mathrm{Th}(\delta\overline{g})/\mathrm{Th}(\overline{g}))^*} (\mathrm{Th}(\nu_N)/\mathrm{Th}(\nu_A))^* \xrightarrow{\Gamma_N} \Sigma^p N_+\\ \Gamma^!_B &: \Sigma^{-1}\mathrm{Th}(\nu_B)^* \xrightarrow{\mathrm{Th}(\overline{g})^*} \Sigma^{-1}\mathrm{Th}(\nu_A)^* \xrightarrow{\Gamma_A} \Sigma^p A_+\end{aligned}$$

with induced chain maps

$$\gamma^!_Y : \widetilde{C}((\mathrm{Th}(\nu_Y)/\mathrm{Th}(\nu_B))^*) \to \widetilde{C}(N)$$
$$\gamma^!_B : \widetilde{C}(\mathrm{Th}(\nu_B)^*)_{p+*} \to \widetilde{C}(A).$$

The construction is now analog to the absolute case A27 and summarized in the following diagram



A25→p.133 From the relative spectral quadratic construction $\mathrm{con}^{\delta\psi^!,\psi^!}_{\Gamma^!_Y,\Gamma^!_B}$ evaluated on an S-dual Thom class $u(\nu_Y)^*$ we get a quadratic structure $(\delta\psi^!,\psi^!) \in \mathscr{C}((\gamma^!_Y,\gamma^!_B)_{\%})$. By C6 we have $(\gamma^!_Y,\gamma^!_B) \simeq (g^!_i, g^!)$ and hence $(G^! \colon C^! \to D^!, (\delta\psi^!,\psi^!))$ is a well-defined quadratic pair. With the property $(1+t)(\delta\psi^!,\psi^!) = e_{(\gamma^!_Y,\gamma^!_B)} \circ \mathrm{con}^{\delta\psi^!,\psi^!}_{\Gamma^!_Y,\Gamma^!_B} \circ (\gamma^!_Y,\gamma^!_B)$ of A25 we obtain the relation $(1+t)(\delta\psi^!,\psi^!) = e_{g^!_i,g^!}{}^{\%} \circ \mathrm{con}^{\delta\varphi,\varphi}_{N,A}([N,A])$.

Note that the outcome is usually not Poincaré so $\mathrm{sgn}^{\mathbf{L}_\bullet}_{\to}$ does not produce an element in an L-group but it is used to define the mosaicked quadratic signature $\mathrm{sgn}^{\mathbf{L}_\bullet}_X$. ☐

Proof A29 (Mosaicked quadratic signature)

> **A29 (221, 23) Mosaicked quadratic signature [Ran92, Example 9.14]**
> *Let $\widehat{f} \colon M \to X$ be a degree one normal map from a closed topological manifold to a Poincaré space both of dimension n. Let $r \colon X \to K$ be a map to a simplicial complex K. There is a mosaicked quadratic signature*
> $$\mathrm{sgn}^{\mathbf{L}_\bullet}_K(\widehat{f}) \in L_n(\Lambda_G K)$$
> *with $A(\mathrm{sgn}^{\mathbf{L}_\bullet}_X(\widehat{f})) = \mathrm{sgn}^{\mathbf{L}_\bullet}_{\mathbb{Z}\pi}(\widehat{f})$. If X is a manifold, there is a refined version*
> $$\mathrm{sgn}^{\mathbf{L}_\bullet}_K(\widehat{f}) \in L_n(\Lambda_L K).$$

We start with the case that X is a manifold and then provide a few comments about the Poincaré case.

Manifold case Make f transverse to the K-dissection of X. Then we can consider f as a collection of degree one normal maps $(\widehat{f}[\sigma], \partial\widehat{f}[\sigma]) \colon (M[\sigma], \partial M[\sigma]) \to (X[\sigma], \partial X[\sigma])$ between manifolds with boundary ($\sigma \in K$). The relative quadratic construction $\mathrm{con}^{\delta\psi,\psi}_{\delta F,F}$ of A23 can be generalized for the stable Umkehr map $F \colon \Sigma^p X_+ \to Y_+$ of f to a chain map

$$\mathrm{con}^{\psi_K}_F \colon C(M) \to W_{\%}(C(X))$$

over $\mathbb{Z}_*K$. Evaluated on a fundamental class $[X] \in C(X)_n$ it gives an n-dimensional K-mosaicked quadratic structure $\psi_K = \mathrm{con}_F^{\psi_K}(\widehat{f})$ whose components for each $\sigma \in K$ are quadratic pairs $(C[\sigma] \to D[\sigma], (\delta\psi, \psi))$ obtained from the relative quadratic construction $\mathrm{con}_{\delta F, F}^{\delta\psi,\psi}(f[\sigma], f[\partial\sigma])$ evaluated on the fundamental classes $[X[\sigma], \partial X[\sigma]]$. We obtain an n-quadratic chain complex $(C(X), \psi_K)$ in $\Lambda_L X$ where

$$(1+t)(\psi_K(\sigma))_0 \colon \Sigma^n TC(X)(\sigma) = C(X[\sigma])^{n-*} \to C(X)(\sigma) = C(X[\sigma], \partial X[\sigma])$$

equals the cap product with $[X[\sigma], \partial X[\sigma]]$ which realizes the Poincaré-Lefschetz chain homotopy equivalence.

The cobordism class of $(C(X), \psi_K)$ does not depend on the choice of the fundamental class and hence this construction defines a quadratic signature

$$\mathrm{sgn}_X^{\mathbf{L}_\bullet}(\widehat{f}) \in L_n(\Lambda_L X).$$

Poincaré space case If X is only a Poincaré space, we use in principal the same construction as in the manifold case. An important difference is that $(X[\sigma], \partial X[\sigma])$ is now only a normal pair and there is no stable Umkehr map but by C5 we obtain at least a semi-stable Umkehr map. So we have to generalize the relative spectral quadratic construction to a chain map over $\mathbb{Z}_*X$ to obtain a K-mosaicked quadratic structure ψ_K. The σ-components $(C(X)(\sigma), \psi(X)(\sigma))$ are given by the the relative quadratic signature for normal spaces $\mathrm{sgn}_{\to}^{\mathbf{L}_\bullet}(f[\sigma], \partial f[\sigma])$. Assembly produces the quadratic Poincaré structure $A(\psi) = \mathrm{con}_F^{\psi}([X])$ on X. Thus $\mathrm{sgn}_X^{\mathbf{L}_\bullet}(\widehat{f})$ is globally Poincaré and hence a chain complex in $\Lambda_G X$. □

C5→p.165
A25→p.133
A28→p.134

Room service A2

$(\widehat{f}[\sigma], \partial\widehat{f}[\sigma]) = ((\overline{f}[\sigma], f[\sigma]), (\partial\overline{f}[\sigma], \partial f[\sigma]))$ an n-dimensional degree one normal map

$$\begin{array}{ccc} (\nu_M|_{M[\sigma]}, \nu_M|_{\partial M[\sigma]}) & \xrightarrow{(\overline{f}, \partial\overline{f})} & (\nu_X|_{X[\sigma]}, \nu_X|_{\partial X[\sigma]}) \\ \downarrow & & \downarrow \\ (M[\sigma], \partial M[\sigma]) := (f^{-1}, \partial f^{-1})(X[\sigma], \partial X[\sigma]) & \xrightarrow{(f, \partial f)} & (X[\sigma], \partial X[\sigma]), \end{array}$$

denoted $(f[\sigma], \partial f[\sigma]) \colon (M[\sigma], \partial M[\sigma]) \to (X[\sigma], \partial X[\sigma])$ for short, from an $(n - |\sigma|)$-dimensional manifold with boundary to an $(n - |\sigma|)$-dimensional normal pair obtained from a degree one normal map $\widehat{f}$ after making f transverse to a K-dissection $\bigcup_{\sigma \in K} X[\sigma]$ of X .

$X[\sigma]$ is defined for a map $r \colon X \to K$ to a simplicial complex as the preimage of the dual cell $D(\sigma, K)$ after making r transverse. If X is a simplicial complex itself, choose r to be the identity. The subdivision $X = \bigcup_{\sigma \in K} X[\sigma]$ is called a K-dissection of X.

$[Y]$ fundamental class for a normal space (Y, ν, ρ) is a cycle in $C_n(\widetilde{Y})$ which represents an n-dimensional homology class in $H_n(Y; \mathbb{Z}^w)$ given by $[Y] = u(\nu) \cap h(\rho)$ where h is the Hurewicz homomorphism. If Y is a Poincaré space, then the term fundamental class implies that ν and ρ have been chosen in such a way that $\cdot \cap [Y] \colon H^{n-*}(Y) \to H_*(Y)$ is an isomorphism.

$\boxed{C(\sigma)}$ denotes for a chain complex

$$C\colon \ldots \longrightarrow \sum_{\sigma\in X}(C_n)_\sigma \xrightarrow{\sum(f_n)_{\tau,\sigma}} \sum_{\sigma\in X}(C_{n-1})_\sigma \xrightarrow{\sum(f_{n-1})_{\tau,\sigma}} \sum_{\sigma\in X}(C_{n-2})_\sigma \xrightarrow{\sum(f_{n-2})_{\tau,\sigma}} \ldots$$

in $\mathbb{A}_*X$ or $\mathbb{A}^*X$ the chain complex in $\mathbb{A}$ given by

$$C(\sigma)\colon \ldots \longrightarrow (C_n)_\sigma \xrightarrow{(f_n)_{\sigma,\sigma}} (C_{n-1})_\sigma \xrightarrow{(f_{n-1})_{\sigma,\sigma}} (C_{n-2})_\sigma \xrightarrow{(f_{n-2})_{\sigma,\sigma}} \ldots$$

$\boxed{(C,\psi) \text{ in } \Lambda}$ a quadratic chain complex which is $\mathbb{P}$-Poincaré and $C \in \mathbb{C}$.

A3 Normal

This section gives a basic introduction into the way of how normal chain complexes are obtained from geometric input. The first step is what we call here the chain bundle construction. It was introduced by Ranicki in [Ran80b, §9] as hyperquadratic construction. We restrict ourselves to the non-equivariant case here. To complete a chain bundle to a normal structure we need a symmetric structure as well and relate it to the chain bundle by a certain chain. We do this in a sketchy way for the absolute case. The construction of the normal signature in full generality as stated in A34 and A38 is extensive and complicated. We refer to [Wei85a, Wei85b] for the full details.

A3 Chain bundle construction [Ran80b, Prop. 9.1]
Let $\nu\colon X \to \mathrm{BSG}$ a k-dimensional oriented spherical fibration. There is a chain map

$$\mathrm{con}^\gamma_\nu\colon \widetilde{C}^k(Th(\nu))^* \to \widehat{W}^{\%}(C(X)^{-*})_0.$$

Properties

[A31 → [Ran80b, Prop. 9.1 (ii)]]
The chain bundle construction is functorial with respect to maps of spaces, i.e. let $f: M \to X$ be a map of finite CW-complexes. Then the following diagram commutes

$$\begin{array}{ccc} C(X) & \xrightarrow{\mathrm{con}^\gamma_X} & \widehat{W}^{\%}(C(X)^{-*}) \\ \big\downarrow f & & \big\downarrow f^{\%} \\ C(Y) & \xrightarrow{\mathrm{con}^\gamma_Y} & \widehat{W}^{\%}(C(Y)^{-*}) \end{array}$$

[A32 → [Ran80b, Prop. 9.1 (iii)]]
The symmetric construction commutes with algebraic and geometric suspension up to a chain homotopy Γ_X.

$$\begin{array}{ccccc} C(X) & \xrightarrow{\mathrm{con}^\gamma_X} & \widehat{W}^{\%}(C(X)^{-*}) & \xrightarrow{S} & \Sigma^{-1}\widehat{W}^{\%}(\Sigma C(X)^{-*}) \\ \big\downarrow \Sigma & & \overset{\Gamma_X}{\rightsquigarrow} & & \big\downarrow \Sigma^{\%} \\ C_{n+*}(\Sigma X) & & \xrightarrow{\mathrm{con}^\gamma_{\Sigma X}} & & \Sigma^{-1}W^{\%}(C(\Sigma X)^{-*}) \end{array}$$

Signatures

⇒ **A33 (231) Normal signature [Ran80b, §9][Wei85a, Theorem 3.4]**
Let (Y, ν, ρ) be an n-dimensional normal space. There is a normal signature

$$\operatorname{sgn}^{\mathbf{NL}^\bullet}_{\mathbb{Z}\pi}(Y) \in NL^n(\mathbb{Z}\pi)$$

producing an n-normal chain complex $(C, (\varphi, \gamma, \chi))$.

⇒ **[A34 (232) → [Wei85a, Theorem 3.5 and 7.1]] Relative normal signature**
Let $((Y, B), \nu, (\rho_Y, \rho_B))$ be an n-dimensional pair of normal spaces. There is a normal pair signature

$$\operatorname{sgn}^{\mathbf{NL}^\bullet}_{\to}(Y, A)$$

producing an n-dimensional normal pair $(f : C \to D, (\delta\varphi, \delta\gamma, \delta\chi), (\varphi, \gamma, \chi))$.

⇒ **[A38 → [Ran92, Example 9.15][Wei85a, Theorem 7.1]] Mosaicked normal signature**
Let $Y := (Y; Y_{0,\dots} Y_n, \nu, \rho)$ be a normal $(n+2)$ -ad. There is a mosaicked normal signature $\operatorname{sgn}^{\mathbf{NL}^\bullet}_{Y}(Y) \in NL^n(\Lambda_G X)$.

Proof A3 (Chain bundle construction)

The definition of $\operatorname{con}^\gamma_\nu$ involves the following maps.

- The suspension homomorphism $\mathcal{S} : W^\% C \to \Sigma^{-1} W^\%(\Sigma C)$.
- The S-duality equivalence $C(Y) \xrightarrow{\simeq} C(Y^*)^{N-*}$ for a choice of an N-dimensional S-dual Y^* of Y for some N. There is a π-equivariant version of S-duality as well but for simplicity we restrict ourselves here to the non-equivariant case.
- The Thom chain equivalence $T : \widetilde{C}(\mathrm{Th}(\nu))^{-*} \xrightarrow{\simeq} C(Y)^{-k-*}$.
- The symmetric construction $\operatorname{con}^\varphi_X : C(X) \to W^\%(C(X))$ of A1.
- The chain map $j : W^\% C \to \widehat{W}^\% C$ induced by the inclusion $W^\% \to \widehat{W}^\%$.

The composition of the S-duality and the Thom equivalence defines a chain homotopy equivalence

$$ST : \widetilde{C}(\mathrm{Th}(\nu)^*) \to C(\mathrm{Th}(\nu))^{N-*} \to C(Y)^{N-k-*}.$$

We use the induced map $ST^\% : W^\%(\widetilde{C}(\mathrm{Th}(\nu)^*))_{N-k} \to W^\%(C(Y)^{N-k-*})_{N-k}$ together with the composition

$$\begin{aligned} W^\%(C(Y)^{N-k-*})_{N-k} &\xrightarrow{\quad j \quad} \widehat{W}^\%(C(Y)^{N-k-*})_{N-k} \\ &\xrightarrow{\mathcal{S}^{-(N-k)}} \widehat{W}^\%(C(Y)^{-*})_0 \end{aligned}$$

to define $\operatorname{con}^\gamma_\nu := \mathcal{S}^{-(N-k)} \circ J \circ ST^\%$. □

Proof A33 (Normal signature)

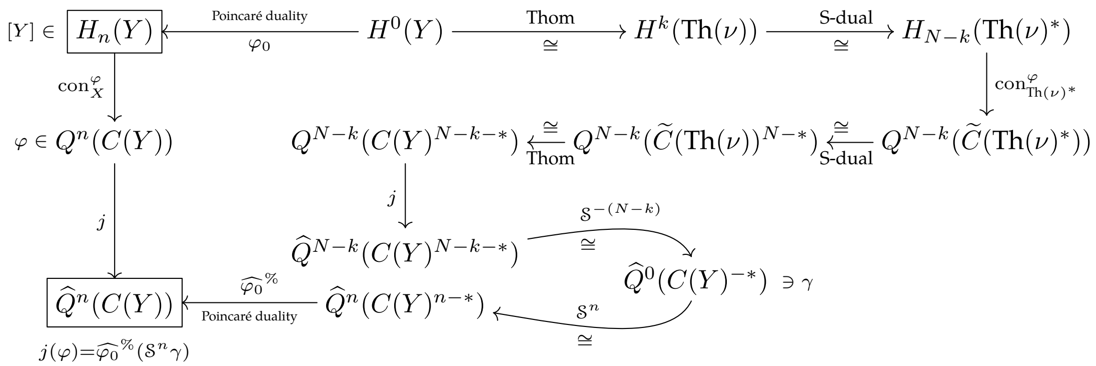
Let Y be a finite CW-complex and (Y, ν, ρ) an n-dimensional geometric normal space with a choice of the Thom class $u(\nu_X) \in \widetilde{C}(\mathrm{Th}(\nu_X))$ whose associated fundamental class is denoted $[Y]$.

We would like to construct an n-normal chain complex $(C, (\varphi, \gamma, \chi))$.

We make two simplifications: We construct only a chain bundle homology class instead of a cycle in the corresponding chain complex and we work in a non-equivariant setting, thus construct only an element in $NL^n(\mathbb{Z})$.

We define the underlying chain complex as $C = C(Y)$ and use the symmetric construction A1→p.125 from A1 for the definition of $\varphi = \mathrm{con}^{\varphi}_X([Y])$. The Thom class has an N-dimensional S-dual A3→p.138 $u(\nu)^* \in \widetilde{C}(\mathrm{Th}(\nu)^*)_{N-k}$ for some N. We apply the the chain bundle construction from A3 to obtain $\gamma = \mathrm{con}^{\gamma}_{\nu}(u(\nu)^*)$.

It remains to relate $J(\gamma)$ and $\widehat{\varphi_0}^{\%}(\mathcal{S}^n\gamma)$ via a chain χ. The commutativity of the following diagram shows that they agree in homology which proves the existence of a chain $\chi \in \widehat{W}^{\%}(C)_{n+1}$ such that $d(\chi) = J(\varphi) - \widehat{\varphi_0}^{\%}(\mathcal{S}^n\gamma)$. Recall that we applied the symmetric construction to the fundamental class $[Y]$ to obtain φ and so the chain map φ_0 is the cap product with $[Y]$. For a more explicit construction and a proof that χ can actually be chosen in a canonical way (see [Wei85b, §7]).



□

Room service A3 and A33

$\boxed{\mathrm{con}^{\varphi}_X : C(X) \to W^{\%}(C(X))}$ a chain map called symmetric construction; defined for a topological space X.

$\boxed{\mathrm{con}^{\gamma}_{\nu} : \tilde{C}^k(\mathrm{Th}(\nu)) \to \widehat{W}^{\%}(C(X)^{-*})_0}$ a chain map called chain bundle construction; defined for a k-dimensional spherical fibration ν.

$\boxed{J : L^n(R) \to NL^n(R)}$ roughly induced by $j : W^{\%}C \to \widehat{W}^{\%}C$; see (111) for more details of how a normal structure (φ, γ, χ) is obtained from a symmetric Poincaré chain complex (C, φ).

$\boxed{Q^n, Q_n, \widehat{Q}^n}$ the n-dimensional Q-groups defined for a chain complex C by

$$Q(C)^n := H_n(W^{\%}C),$$
$$Q(C)_n := H_n(W_{\%}C),$$
$$\widehat{Q}(C)^n := H_n(\widehat{W}^{\%}C).$$

$W^{\%}, W_{\%}, \widehat{W}^{\%}$ denote for a chain complex C the abelian group chain complexes

$$W^{\%}C \coloneqq \mathrm{Hom}_{\mathbb{Z}[\mathbb{Z}_2]}(W, C \otimes C),$$
$$W_{\%}C \coloneqq W \otimes_{\mathbb{Z}[\mathbb{Z}_2]} (C \otimes C),$$
$$\widehat{W}^{\%}C \coloneqq \mathrm{Hom}_{\mathbb{Z}[\mathbb{Z}_2]}(\widehat{W}, C \otimes C).$$

W the free resolution of the trivial $\mathbb{Z}[\mathbb{Z}_2]$-chain module $\mathbb{Z}$; given by the $\mathbb{Z}[\mathbb{Z}_2]$-chain complex $\ldots \xrightarrow{1+t} \mathbb{Z}[\mathbb{Z}_2] \xrightarrow{1-t} \mathbb{Z}[\mathbb{Z}_2] \longrightarrow 0$

$\widehat{W}$ the complete resolution of the trivial $\mathbb{Z}[\mathbb{Z}_2]$-chain module $\mathbb{Z}$; given by the $\mathbb{Z}[\mathbb{Z}_2]$-chain complex $\ldots \xrightarrow{1+t} \mathbb{Z}[\mathbb{Z}_2] \xrightarrow{1-t} \mathbb{Z}[\mathbb{Z}_2] \xrightarrow{1+t} \mathbb{Z}[\mathbb{Z}_2] \xrightarrow{1-t} \ldots$

$f^{\%} \colon W^{\%}(C) \to W^{\%}(D)$ the chain map induced by a chain map $f \colon C \to D$; explicitly given by $(f^{\%}(\varphi))_s \coloneqq f\varphi_s f^* \colon D^{n+s-*} \to D$.

$\mathcal{S} \colon W^{\%}C \to \Sigma^{-1}W^{\%}(\Sigma C)$ the suspension map; defined by $(\mathcal{S}(\varphi))_k \coloneqq \varphi_{k-1}$ if $k \geq 1$ and zero otherwise; induces a map $Q^n(C) \to Q^{n+1}(\Sigma C)$ and an isomorphism $\widehat{Q}^n(C) \xrightarrow{\cong} \widehat{Q}^{n+1}(\Sigma C)$.

B Algebraic surgery and algebraic boundaries

Algebraic surgery is the algebraic analogue of geometric surgery. The relation between the algebraic and the geometric approach can be seen as follows. Let M' be the manifold obtained by surgery on a manifold M and let W be the cobordism between M and M' given by the trace of the surgery. We get the diagram displayed on the left hand side below. The algebraic version can be described with the diagram on the right hand side.

$$
\begin{array}{ccccc}
C(M) \xrightarrow{i} C(W,M') & & & & \Sigma C(M') \\
& \searrow^{e} & & \nearrow^{e'} & \\
& & C(W, M \cup M') & & \\
& \nearrow & & & \\
C(M') \xrightarrow{i'} C(W,M) & & & &
\end{array}
\qquad
\begin{array}{ccccc}
C \xrightarrow{f} D & & & & \Sigma C' \\
& \searrow^{e} & & \nearrow^{e'} & \\
& & \mathscr{C}(f) & & \\
& \nearrow & & & \\
C' \xrightarrow{f'} D^{n+1-*} & \begin{pmatrix} \delta\varphi_0 \\ \varphi_0 f^* \end{pmatrix} & & &
\end{array}
$$

First of all, algebraic surgery is used in [Ran80b] to concentrate finite chain complexes in one degree which leads to the equivalence between the algebraic L-groups and Wall's L-groups. But there are other important aspects. A special application of algebraic surgery gives rise to the boundary construction that relates relative terms to absolute terms, e.g. (normal, symmetric Poincaré) pairs to quadratic chain complexes, and is used to establish the long exact sequences of L-groups.

B1 (112, 11, 121) Algebraic surgery [Ran92, Def. 1.12]
Let (C, φ) be an n-symmetric chain complex. The effect of algebraic surgery of an $(n+1)$-symmetric pair $(f\colon C \to D, \delta\varphi, \varphi)$ on (C, φ) is an n-symmetric chain complex (C', φ'). It is Poincaré if and only if (C, φ) is Poincaré. Moreover, we have $\partial C \simeq \partial C'$.

$(f\colon C \to D, \delta\varphi, \varphi)$
$\Downarrow$
n-symmetric chain complex (C, φ) ⇝ *algebraic surgery* ⇝ *n-symmetric chain complex (C', φ')*

The same construction works for quadratic chain complexes as well.

B2 Boundary (Definition)
The boundary $(\partial C, \partial\lambda)$ of an n-dimensional structured chain complex (C, λ) defined as the effect of algebraic surgery of the pair $(0 \to C, (\lambda, 0))$ on $(0, 0)$ is a chain complex ∂C with an $(n-1)$-dimensional Poincaré structure which is symmetric if λ is symmetric and quadratic if λ is quadratic or normal.

B3 Algebraic Thom construction [Ran80a, 3.4][Ran92, Prop. 1.15]
There is the following one-to-one correspondence of homotopy classes:

n-dimensional symmetric Poincaré pairs $(f\colon C \to D, \delta\lambda, \varphi)$ $\xleftrightarrow{1-1}$ *n-dimensional symmetric complexes (C', φ')*

Proof B1 (Algebraic surgery)

> **B11**
> *Let $C \xrightarrow{f} D \xrightarrow{e} E$ be cofibration sequence of chain complexes. Then there is a cofibration sequence*
>
> $$W^{\%}C \to W^{\%}D \to W^{\%}E \times_{E\otimes E} (D \otimes E) \to \Sigma W^{\%}C \to \Sigma W^{\%}D$$
>
> *A triple (x, z, y) with $x \in W^{\%}(E)_n, z \in (E \otimes E)_{n+1}, y \in (D \otimes E)_n$ with $z\colon ev(x) \simeq (e \otimes \mathrm{id})(y)$ defines an n-symmetric pair $(f\colon C \to D, \delta\varphi, \varphi)$ and the up to homotopy commutative diagram*
>
> $$\begin{array}{ccccccc} C^{n-1-*} & \longrightarrow & E^{n-*} & \xrightarrow{e^*} & D^{n-*} & \xrightarrow{f^*} & C^{n-*} \\ \downarrow{\scriptstyle ev(\varphi)\simeq\varphi_0} & & \downarrow{\scriptstyle ev_l(\delta\varphi,\varphi)\simeq y^*} & & \downarrow{\scriptstyle ev_r(\delta\varphi,\varphi)\simeq y} & & \downarrow{\scriptstyle ev(\varphi)\simeq\varphi_0} \\ C & \xrightarrow{f} & D & \xrightarrow{e} & E & \longrightarrow & C_{n-1} \end{array}$$
>
> *An analog statement holds for the quadratic case.*

We define the effect of algebraic surgery of a pair $(f\colon C \to D, \delta\varphi, \varphi)$ on an n-symmetric chain complex (C, φ). The resultant n-symmetric chain complex will be denoted by (C', φ'). For the pair $(\delta\varphi, \varphi)$ we have the chain map $\varphi_{f^*} = \mathrm{ev}_r(\delta\varphi, \varphi)\colon D^{n+1-*} \to \mathscr{C}(f)$. The chain complex C' is given by $\Sigma^{-1}\mathscr{C}(\varphi_{f^*})$ and yields a map $f'\colon C' \to D^{n+1-*}$. We obtain the n-symmetric structure φ' on C' as follows. Apply B11 to the cofibration sequence B11→p.144

$$C' \xrightarrow{f'} D^{n+1-*} \xrightarrow{\varphi_{f^*}} \mathscr{C}(f) \xrightarrow{e'} \Sigma C'$$

to obtain the new cofibration sequence

$$W^{\%}C' \to W^{\%}D^{n+1-*} \to P \to \Sigma W^{\%}C'$$

with $P \coloneqq W^{\%}\mathscr{C}(f) \times_{\mathscr{C}(f)\otimes\mathscr{C}(f)} (D^{n+1-*} \otimes \mathscr{C}(f))$.

We show that the triple $(\delta\varphi/\varphi, (\delta\varphi/\varphi)_1, e)$ defines an element in the pullback P.

$$\begin{array}{ccc} P & \longrightarrow & D^* \otimes \mathscr{C}(f) \\ \downarrow & & \downarrow{\scriptstyle \varphi_{f^*}\otimes\mathrm{id}} \\ W^{\%}\mathscr{C}(f) & \xrightarrow{\mathrm{ev}} & \mathscr{C}(f) \otimes \mathscr{C}(f) \end{array}$$

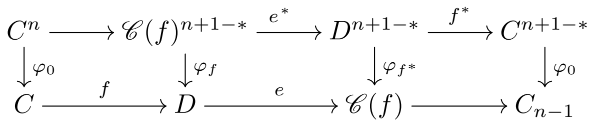
The first entry $\delta\varphi/\varphi$ is the Thom structure as constructed in B3 and the chain map $e\colon D \to \mathscr{C}(f)$ is considered as an element in $D^* \otimes \mathscr{C}(f)$. With $\varphi_{f^*} \coloneqq \mathrm{ev}_l(\delta\varphi, \varphi), \varphi_f \coloneqq \mathrm{ev}_r(\delta\varphi, \varphi)$ we recover $\delta\varphi/\varphi$ and e in $\mathscr{C}(f) \otimes \mathscr{C}(f)$ as the compositions $\varphi_{f^*} \circ e$ and $e^* \circ \varphi_f$ which are homotopic via $\delta\varphi/\varphi_1$. B3→p.159

$$\begin{array}{ccc} \mathscr{C}(f)^{n+1-*} & \xrightarrow{\varphi_f} & D \\ \downarrow{\scriptstyle e^*} & \overset{(\delta\varphi/\varphi)_1}{\leadsto} & \downarrow{\scriptstyle e} \\ D^{n+1-*} & \xrightarrow{\varphi_{f^*}} & \mathscr{C}(f) \end{array}$$

Push it forward in the cofibration sequence above to obtain the symmetric structure $\varphi' \in \Sigma W^{\%}C'$.

It remains to show that φ is Poincaré if and only if φ' is Poincaré. For the pair $(f\colon C \to D, \delta\varphi, \varphi)$

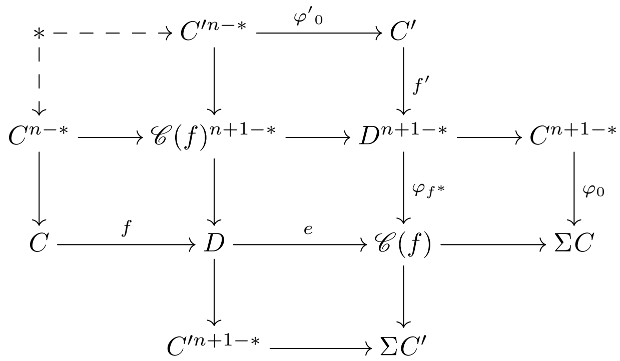
B11→p.144 we obtain the following diagram from B11.



For the cofibration

$$C' \xrightarrow{f'} D^{n+1-*} \xrightarrow{\varphi_{f^*}} \mathscr{C}(f) \xrightarrow{e'} \mathscr{C}(\varphi_{f^*}) = \Sigma C'$$

B11→p.144 we obtain from B11 a similar diagram. They can be combined to a diagram as follows.



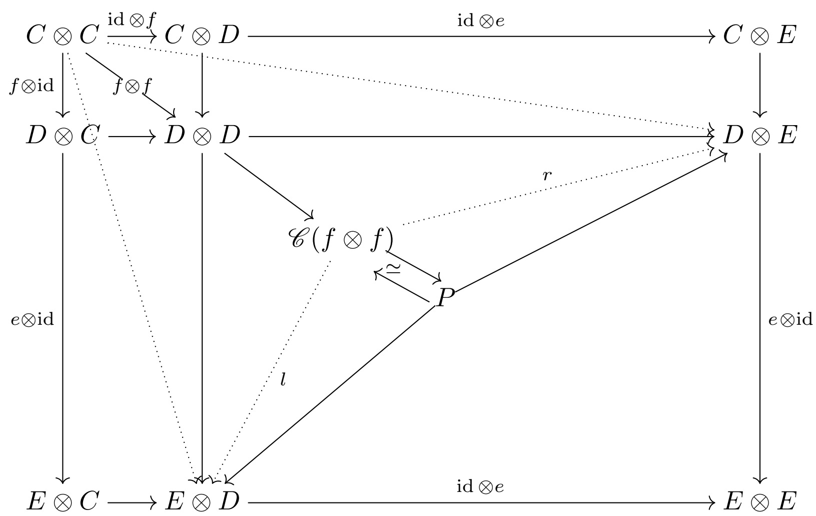
We see that φ_0 is a chain equivalence if and only if ${\varphi'}_0$ is one and also that $(\partial C, \partial\varphi) := \Sigma^{-1}\mathscr{C}(\varphi_0) \simeq \Sigma^{-1}\mathscr{C}\varphi'_0 =: (\partial C, \partial\varphi)'$ holds. □

Proof B11

We want to replace $\mathscr{C}(f^{\%})$ in the cofibration sequence

$$W^{\%}C \xrightarrow{f^{\%}} W^{\%}D \xrightarrow{e_{f^{\%}}} \mathscr{C}(f^{\%}) \longrightarrow \Sigma W^{\%}C \longrightarrow \Sigma W^{\%}D$$

by a homotopy pullback. Consider the following diagram where all vertical and horizontal sequences are cofibrations.



The total cofiber $\mathscr{C}(f\otimes f)$ of the upper left square of the map $f\otimes f$ is homotopy equivalent to the pullback P of the lower right square. Notice that $W^{\%}(-)$ does not respect cofibrations but $\mathrm{Hom}(W,-)$ does. Hence, by applying $\mathrm{Hom}(W,-)$ to the cofibration in the diagonal of the diagram above we obtain the cofibration sequence $W^{\%}C \xrightarrow{f^{\%}} W^{\%}D \to \mathrm{Hom}(W,\mathscr{C}(f\otimes f))$ and the equivalence $\mathscr{C}(f^{\%}) = \mathrm{Hom}(W,\mathscr{C}(f\otimes f))$. Now rewrite the homotopy pullback

$$\begin{array}{ccc} \mathscr{C}(f\otimes f) & \xrightarrow{r} & D\otimes E \\ {\scriptstyle l}\downarrow & & \downarrow{\scriptstyle \mathrm{id}\otimes e} \\ E\otimes D & \xrightarrow{e\otimes \mathrm{id}} & E\otimes E \end{array} \quad \text{as} \quad \begin{array}{ccc} \mathscr{C}(f\otimes f) & \longrightarrow & (E\otimes D)\oplus(D\otimes E) \\ \downarrow & \lrcorner & \downarrow \\ E\otimes E & \longrightarrow & (E\otimes E)\oplus(E\otimes E) \end{array}$$

and apply again $\mathrm{Hom}(W,-)$. Use Shapiro's Lemma for the entries in the right column to obtain

$$\begin{array}{ccc} \mathscr{C}(f^{\%}) & \longrightarrow & D\otimes E \\ \downarrow & & \downarrow{\scriptstyle e\otimes \mathrm{id}} \\ W^{\%}(E) & \xrightarrow{\mathrm{ev}} & E\otimes E. \end{array}$$

So we can replace $\mathscr{C}(f^{\%})$ by the homotopy pullback $P := W^{\%}E \times_{E\otimes E} D\otimes E$ to obtain the cofibration sequence

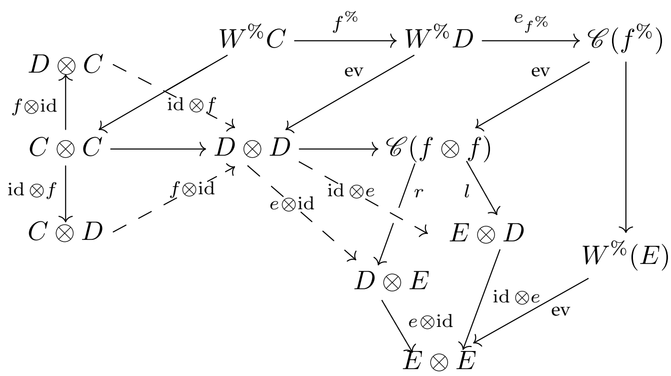
$$W^{\%}C \longrightarrow W^{\%}D \longrightarrow W^{\%}E \times_{E\otimes E} D\otimes E \xrightarrow{q^{\%}} \Sigma W^{\%}C \xrightarrow{\Sigma f^{\%}} \Sigma W^{\%}D.$$

Now we want to construct a symmetric pair out of a triple $(x,y,z)\in P_n$. From the cofibration above we obtain an $(n-1)$-symmetric structure $\varphi := q^{\%}(x,y,z)\in \Sigma W^{\%}C_n$ and a null-homotopy $\delta\varphi\colon \Sigma f^{\%}(\varphi)\simeq 0$ which defines an n-symmetric pair $(f\colon C\to D, \delta\varphi,\varphi)$.

For a chain complex C resp. a chain map $f\colon C\to D$ we have the following evaluation maps

$$\begin{aligned} &\mathrm{ev}\colon W^{\%}C \to C\otimes C && = \mathrm{Hom}(C^*,C); && \varphi\mapsto\varphi_0 \\ &\mathrm{ev}\colon \mathscr{C}(f^{\%}) \to \mathscr{C}(f\otimes f) && = \mathrm{Hom}(D^*,D)\oplus\mathrm{Hom}(C^*,C)_{*-1}; && (\delta\varphi,\varphi)\mapsto(\delta\varphi_0,\varphi_0) \end{aligned}$$

Consider the following diagram.



The null-homotopic compositions $(\mathrm{id}\otimes e)\circ(\mathrm{id}\otimes f)$ and $(e\otimes\mathrm{id})\circ(f\otimes\mathrm{id})$ (the dashed arrows in the diagram) induce maps r and l. We denote by ev_r and ev_l the compositions $r\circ\mathrm{ev}$ and $l\circ\mathrm{ev}$. We have $y\simeq \mathrm{ev}_r(\delta\varphi,\varphi)\in(D\otimes E)_n$ and $y^*\simeq\mathrm{ev}_l(\delta\varphi,\varphi)\in(E\otimes D)_n$. Considered as maps $D^{n-*}\to E$ and $E^{n-*}\to D$ they fit in the desired diagram.

Replacing $\mathrm{Hom}(W,-)$ by $(W\otimes -)$ proves the quadratic case. □

Room service B1 and B11

$(f: C \to D, \delta\varphi, \varphi)$ an $(n+1)$-symmetric pair with
- $f: C \to D$ a chain map
- (C, φ) an n-symmetric chain complex
- $\delta\varphi \in W^{\%}(D)_{n+1}$ such that $d(\delta\varphi) = f^{\%}(\varphi)$ which is equivalent to $(\delta\varphi, \varphi)$ is a cycle in $\mathscr{C}(f^{\%})_{n+1}$.

Poincaré means $(\delta\varphi_0, \varphi_0 f^*): D^{n+1-*} \to \mathscr{C}(f)_*$ is a chain equivalence.

$(f: C \to D, \delta\psi, \psi)$ an $(n+1)$-quadratic pair with
- $f: C \to D$ a chain map
- (C, ψ) an n-quadratic chain complex
- $\delta\psi \in W_{\%}(D)_{n+1}$ such that $d(\delta\psi) = f_{\%}(\psi)$.

Poincaré means the symmetrization is Poincaré, i.e. $((1+t)\delta\varphi_0, (1+t)\varphi_0 f^*): D^{n+1-*} \to \mathscr{C}(f)_* = (D_*, C_{*-1})$ is a chain equivalence.

$e: C \to \mathscr{C}(\varphi_0)$ the inclusion; with a map $\alpha: C \to D$ as subscript e_α denotes the inclusion $D \to \mathscr{C}(\alpha)$.

$\mathrm{id} \otimes e: \mathscr{C}(f) \otimes D \to \mathscr{C}(f) \otimes \mathscr{C}(f)$

$e \otimes \mathrm{id}: D \otimes \mathscr{C}(f) \to \mathscr{C}(f) \otimes \mathscr{C}(f)$

$\mathrm{ev}: W^{\%}C \to C \otimes C$ the evaluation map given by $\varphi \mapsto \varphi_0$.

$\mathrm{ev}_l: \mathscr{C}(f^{\%}) \to \mathscr{C}(f) \otimes D$ the left evaluation map for a chain map $f: C \to D$ given by $(\delta\varphi, \varphi) \mapsto \varphi_f$.

$\mathrm{ev}_r: \mathscr{C}(f^{\%}) \to D \otimes \mathscr{C}(f)$ the right evaluation map for a chain map $f: C \to D$ given by $(\delta\varphi, \varphi) \mapsto \varphi_{f^*}$.

$\varphi_{f^*} = \mathrm{ev}_r(\delta\varphi, \varphi) \simeq \begin{pmatrix} \delta\varphi_0 \\ \varphi_0 f^* \end{pmatrix}: D^{n-*} \to \mathscr{C}(f)$ a chain map defined for an n-symmetric pair $(f: C \to D, \delta\varphi, \varphi)$.

$\varphi_f = \mathrm{ev}_l(\delta\varphi, \varphi) \simeq (\delta\varphi_0, f\varphi_0): \mathscr{C}(f)^{n-*} \to D$ a chain map defined for an n-symmetric pair $(f: C \to D, \delta\varphi, \varphi)$.

$\delta\varphi/\varphi \in W^{\%}\mathscr{C}(f)$ the image of a symmetric pair structure as constructed in B3.

B2 Boundaries

Porter

There are various types of boundary constructions that are special cases of algebraic surgery. They produce out of n-dimensional non-Poincaré structured chain complexes new Poincaré chain complexes of one dimension less. Note that in the normal case the boundary becomes quadratic.

n-dimensional		**(n − 1)-dimensional**	
symmetric (C, φ)	boundary construction ∂^S	Poincaré symmetric $(\partial C, \partial\varphi)$	B21→p.149
quadratic (C, ψ)	boundary construction ∂^Q	Poincaré quadratic $(\partial C, \partial\psi)$	B23→p.151
normal $(C, (\varphi, \gamma, \chi))$	boundary construction ∂^N	Poincaré *quadratic* $(\partial C, \partial\psi)$	B24→p.152

There also boundary constructions for pairs.

n-dimensional		**(n − 1)-dimensional**	
symmetric $(f: C \to D, \delta\varphi, \varphi)$	boundary pair construction $\partial^S_\to$	Poincaré symmetric pair $(\partial f: \partial C \to \partial_{f^*} D, \partial_{f^*}\delta\varphi, \partial\varphi)$	B22→p.150
quadratic $(f: C \to D, \delta\psi, \psi)$	boundary pair construction $\partial^Q_\to$	Poincaré quadratic pair $(\partial f: \partial C \to \partial_{f^*} D, \partial_{f^*}\delta\psi, \partial\psi)$	B25→p.154
normal $(f: C \to D, (\delta\varphi, \delta\gamma, \delta\chi), (\varphi, \gamma, \chi))$	boundary pair construction $\partial^N_\to$	Poincaré quadratic pair $(\partial f: \partial C \to \partial_{f^*} D, \partial_{f^*}\delta\psi, \partial\psi)$	

These boundary constructions are used to define special variants of signatures that we use in some proofs. The boundary constructions are purely algebraically defined whereas the following boundary signatures produce structured chain complexes out of geometrical input.

n-dimensional		**(n − 1)-dimensional**	
normal space (Y, ν, ρ)	boundary signature $\partial\mathrm{gn}^{\mathbf{L}^\bullet}_{\mathbb{Z}\pi}$	Poincaré quadratic $(\partial C', \partial\psi')$	B27→p.155
normal pair (Y, B)	relative boundary signature $\partial\mathrm{gn}^{\mathbf{L}^\bullet}_\to$	quadratic $(f: C \to D, \delta\psi, \psi)$	
(normal, Poincaré) pair (Y, B)	relative boundary signature $\partial\mathrm{gn}^{\mathbf{L}^\bullet}_\to$	Poincaré quadratic $(f: C \to D, \delta\psi, \psi)$	B28→p.157

We apply these signatures in 1621 and 2321 to obtain alternative descriptions of the quadratic signatures for degree one normal maps.

B2 Boundary (Definition)
The boundary $(\partial C, \partial\lambda)$ of an n-dimensional structured chain complex (C, λ) defined as the effect of algebraic surgery of the pair $(0 \to C, (\lambda, 0))$ on $(0, 0)$ is a chain complex ∂C with an $(n-1)$-dimensional Poincaré structure which is symmetric if λ is symmetric and quadratic if λ is quadratic or normal.

Symmetric boundary constructions

B21 Symmetric boundary
An n-symmetric chain complex $(C, \varphi) \in L^n(\mathbb{Z}\pi)$ has an $(n-1)$-symmetric Poincaré boundary

$$\partial^S(C, \varphi) := (\partial C, \partial\varphi)$$

such that $e^{\%}_{\varphi_0}(\varphi) = \mathcal{S}(\partial\varphi)$ where $e: C \to \mathscr{C}(\varphi_0) = \Sigma^{-1}\partial C$ is the inclusion.

B22 Symmetric boundary pair [MR90, Proof of Prop. 3.8]
Let $(f: C \to D, \delta\varphi, \varphi)$ be an n-symmetric pair. There is an $(n-1)$-symmetric Poincaré pair $(\partial f: \partial C \to \partial_{f^*} D, (\partial\delta\varphi, \partial\varphi))$ called the symmetric boundary pair.

Quadratic boundary constructions

B23 (231) Quadratic boundary
An n-quadratic chain complex $(C, \psi) \in L_n(\mathbb{Z}\pi)$ has an $(n-1)$-quadratic boundary

$$\partial^Q(C, \psi) := (\partial C, \partial\psi) := (\Sigma^{-1}\mathscr{C}(\varphi_0), \left(\begin{smallmatrix}1+t \\ \mathcal{S}\end{smallmatrix}\right)^{-1}(\mathcal{S}^{-1}e^{\%}(\varphi), e_{\%}(\psi))$$

where $\varphi = (1+t)\psi$.

B24 (112, 231) Quadratic boundary for normal
An n-normal chain complex $(C, (\varphi, \gamma, \chi)) \in NL^n(\mathbb{Z}\pi)$ has an $(n-1)$-quadratic Poincaré boundary

$$\partial^N(C, (\varphi, \gamma, \chi)) =: (\partial C, \partial\psi)$$

which defines a map $\partial^N: NL^n(\mathbb{Z}\pi) \longrightarrow L_{n-1}(\mathbb{Z}\pi); \quad (C, (\varphi, \gamma, \chi)) \mapsto (\partial C, \partial\psi)$.

B25 (232) Quadratic boundary pair
Let $(G^! : C^! \to D^!, (\delta\psi^!, \psi^!))$ be an n-quadratic pair and $(\delta\varphi^!, \varphi^!) = (1+t)(\delta\psi^!, \psi^!)$ and $\varphi^!_{G^!} = (\delta\varphi^!_0, G^!\varphi^!_0): \mathscr{C}(G^!)^{n-*} \to D^!$. Then

$$\begin{pmatrix} \partial G^! : \partial C^! \to \partial_{G^!} D^!, \\ \partial\delta\psi^!, \\ \partial\psi^! \end{pmatrix} := \begin{pmatrix} \partial G^! : \Sigma^{-1}\mathscr{C}(\varphi^!_0) \to \Sigma^{-1}\mathscr{C}(\varphi^!_{G^!}), \\ \left(\begin{smallmatrix}1+t \\ \mathcal{S}\end{smallmatrix}\right)^{-1}(\mathcal{S}^{-1}(e^{\%}_{\varphi^!_{G^!}}\delta\varphi^!), (e_{\varphi^!_{G^!}})_{\%}\delta\psi^!), \\ \left(\begin{smallmatrix}1+t \\ \mathcal{S}\end{smallmatrix}\right)^{-1}(\mathcal{S}^{-1}(e^{\%}_{\varphi^!}\varphi^!), (e_{\varphi^!})_{\%}\psi^!) \end{pmatrix}$$

defines an $(n-1)$-quadratic Poincaré pair called the boundary and denoted $\partial^Q_{\to}$.

Boundary signatures

B27 (231) Quadratic boundary signature [KMM13, Constr. 3.25][Ran81, Prop. 7.4.1][Wei85b, Theorem 7.1]
Let (Y, ν, ρ) be an n-dimensional normal space. The quadratic boundary signature

$$\partial gn^{\mathbf{L}_\bullet}_{\mathbb{Z}\pi}(Y) =: (\partial C', \partial\psi') \in L_{n-1}(\mathbb{Z}\pi)$$

produces an $(n-1)$-quadratic Poincaré chain complex such that $(\partial C', \partial\psi') = \partial^N(\mathrm{sgn}^{\mathbf{NL}^\bullet}_{\mathbb{Z}\pi}(Y)) \in L_{n-1}(\mathbb{Z}\pi)$.

B28 (1621, 232) Quadratic boundary pair signature
Let $((Y, B), \nu, (\rho_Y, \rho_B))$ be an n-dimensional pair of normal spaces. There is a quadratic boundary pair signature

$$\partial gn^{\mathbf{L}_\bullet}_{\to}(Y, B) = (\partial j: \partial C' \to \partial D', (\partial\delta\psi, \partial\psi))$$

producing an $(n-1)$-quadratic Poincaré pair.

Proof B21 (Symmetric boundary)

> **B11**
> *Let $C \xrightarrow{f} D \xrightarrow{e} E$ be cofibration sequence of chain complexes. Then there is a cofibration sequence*
>
> $$W^{\%}C \to W^{\%}D \to W^{\%}E \times_{E\otimes E} (D \otimes E) \to \Sigma W^{\%}C \to \Sigma W^{\%}D$$
>
> *A triple (x, z, y) with $x \in W^{\%}(E)_n, z \in (E \otimes E)_{n+1}, y \in (D \otimes E)_n$ with $z\colon ev(x) \simeq (e \otimes \mathrm{id})(y)$ defines an n-symmetric pair $(f\colon C \to D, \delta\varphi, \varphi)$ and the up to homotopy commutative diagram*
>
> $$\begin{array}{ccccccc} C^{n-1-*} & \longrightarrow & E^{n-*} & \xrightarrow{e^*} & D^{n-*} & \xrightarrow{f^*} & C^{n-*} \\ \downarrow{\scriptstyle ev(\varphi)\simeq\varphi_0} & & \downarrow{\scriptstyle ev_l(\delta\varphi,\varphi)\simeq y^*} & & \downarrow{\scriptstyle ev_r(\delta\varphi,\varphi)\simeq y} & & \downarrow{\scriptstyle ev(\varphi)\simeq\varphi_0} \\ C & \xrightarrow{f} & D & \xrightarrow{e} & E & \longrightarrow & C_{n-1} \end{array}$$
>
> *An analog statement holds for the quadratic case.*

We give the explicit construction below but note that the boundary is obtained as the effect of algebraic surgery on $(0, 0)$ by the pair $(0 \to C, (\varphi, 0))$.

Let (C, φ) be an n-symmetric chain complex. We define the chain complex of its symmetric boundary as $\partial C = \Sigma^{-1}\mathscr{C}(\varphi_0)$, the homotopy fiber of the possibly Poincaré duality map φ_0. We obtain the $(n-1)$-symmetric structure $\partial\varphi$ as follows. Use B11 for the cofibration sequence B11→p.144

$$\partial C \xrightarrow{i} C^{n-*} \xrightarrow{\varphi_0} C \xrightarrow{e} \Sigma\partial C$$

to obtain the cofibration sequence

$$W^{\%}\partial C^{n-*} \to \underbrace{W^{\%}C \times_{C\otimes C} (C^{n-*} \otimes C)}_{P} \to \Sigma W^{\%}\partial C \to \Sigma W^{\%}C^{n-*}$$

where P is the homotopy pullback

$$\begin{array}{ccc} P & \longrightarrow & C^{n-*} \otimes C = \mathrm{Hom}(C, C) \\ \downarrow & & \downarrow{\scriptstyle {\varphi_0}^* \otimes \mathrm{id}} \\ W^{\%}C & \xrightarrow{\mathrm{ev}} & C \otimes C = \mathrm{Hom}(C^{n-*}, C). \end{array}$$

The triple $(\varphi, \varphi_1, \mathrm{id})$ defines an element in P and the pushforward in the cofibration sequence defines an element $\partial\varphi \in \Sigma W^{\%}\partial C$. In this situation the commutative diagram from B11 looks as follows.

$$\begin{array}{ccccccccc} \longrightarrow & \partial C^n & \longrightarrow & C^{n+1-*} & \longrightarrow & C & \longrightarrow & \Sigma\partial C^n & \longrightarrow \\ & \downarrow{\scriptstyle \partial\varphi_0} & & \downarrow{\scriptstyle \mathrm{id}^*} & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle \partial\varphi_0} & \\ \longrightarrow & \partial C & \xrightarrow{i} & C^{n+1-*} & \longrightarrow & C & \longrightarrow & \Sigma\partial C & \longrightarrow \end{array}$$

Hence $\partial\varphi_0$ is a chain equivalence. The equivalence $e^{\%}_{\varphi_0}(\varphi) = \mathcal{S}(\partial\varphi)$ is obtained from the following commutative diagram.

$$\begin{array}{ccc}
P & \longrightarrow & \Sigma W^{\%}\partial C \\
\big\downarrow{\scriptstyle (\varphi,\varphi_1,\mathrm{id})} & \cdots\cdots\cdots\rightarrow \partial\varphi & \big\downarrow{\scriptstyle S} \\
W^{\%}C & \xrightarrow{e^{\%}_{\varphi_0}} & W^{\%}\Sigma\partial C \\
\vdots\downarrow & & \\
\varphi & &
\end{array}$$

□

Proof B22 (Symmetric boundary pair)

Let $(\partial C, \partial\varphi)$ the symmetric boundary of (C, φ) as defined in B21. Recall the commutative diagram that we used to define algebraic surgery.

$$\begin{array}{ccccccc}
\partial C & \dashrightarrow & C'^{n-*} & \xrightarrow{\varphi'_0} & C' & & \\
\downarrow & & \downarrow & & \downarrow & & \\
C^{n-*} & \longrightarrow & \mathscr{C}(f)^{n+1-*} & \xrightarrow{e^*} & D^{n+1-*} & \longrightarrow & C^{n+1-*} \\
\downarrow{\scriptstyle \varphi_0} & & \downarrow{\scriptstyle \varphi_f} & & \downarrow{\scriptstyle \varphi_{f^*}} & & \downarrow{\scriptstyle \varphi_0} \\
C & \xrightarrow{f} & D & \xrightarrow{e} & \mathscr{C}(f) & \longrightarrow & \Sigma C \\
& & \downarrow & & \downarrow & & \\
& & C'^{n+1-*} & \longrightarrow & \Sigma C' & &
\end{array}$$

We define $\partial_{f^*}D = C'^{n-*}$ where (C', φ') is the effect of algebraic surgery of the pair $(f\colon C \to D, \delta\varphi, \varphi)$ on (C, φ). From the diagram above we obtain the cofibration sequence

$$\partial C \xrightarrow{\partial f} \partial_{f^*}D \xrightarrow{\varphi'_0} \partial_{f^*}D^{n-*}$$

B11→p.144 Analogously to the absolute case B21, using B11 we obtain from the triple $(\varphi', \varphi'_1, \mathrm{id}) \in$

$$W^{\%}\partial_{f^*}D^{n-*} \times_{\partial_{f^*}D^{n-*}\otimes\partial_{f^*}D^{n-*}} (\partial_{f^*}D \otimes \partial_{f^*}D^{n-*}) \to \Sigma W^{\%}\partial C \to \Sigma W^{\%}\partial_{f^*}D$$

the symmetric structure $\partial\varphi' \simeq \partial\varphi \in \Sigma W^{\%}\partial C$. The null-homotopy $\delta\partial\varphi'\colon \partial f^{\%}(\partial\varphi') \simeq 0$ we obtain in $\Sigma W^{\%}\partial_{f^*}D$ completes the symmetric pair structure $(\delta\partial\varphi', \partial\varphi')$. The commutative diagram from B11 proves that the pair is Poincaré:

$$\begin{array}{ccccccc}
\partial C^{n-1-*} & \longrightarrow & (\partial_{f^*}D^{n-*})^{n-*} & \xrightarrow{e^*} & \partial_{f^*}D^{n-*} & \xrightarrow{(\partial f)^*} & \partial C^{n-*} \\
\downarrow{\scriptstyle \mathrm{ev}(\partial\varphi')\simeq\partial\varphi'_0} & & \downarrow{\scriptstyle \mathrm{ev}_l(\delta\partial\varphi',\partial\varphi')\simeq\mathrm{id}} & & \downarrow{\scriptstyle \mathrm{ev}_r(\delta\partial\varphi',\partial\varphi')\simeq\mathrm{id}^*} & & \downarrow{\scriptstyle \mathrm{ev}(\partial\varphi')\simeq\partial\varphi'_0} \\
\partial C & \xrightarrow{\partial f} & \partial_{f^*}D & \xrightarrow{\varphi_0} & \partial_{f^*}D^{n-*} & \longrightarrow & \partial C_{n-1}
\end{array}$$

□

Proof B23 (Quadratic boundary)

B21 Symmetric boundary
An n-symmetric chain complex $(C, \varphi) \in L^n(\mathbb{Z}\pi)$ has an $(n-1)$-symmetric Poincaré boundary

$$\partial^S(C, \varphi) \coloneqq (\partial C, \partial\varphi)$$

such that $e^{\%}_{\varphi_0}(\varphi) = \mathcal{S}(\partial\varphi)$ where $e\colon C \to \mathscr{C}(\varphi_0) = \Sigma^{-1}\partial C$ is the inclusion.

Let (C, ψ) be an n-quadratic chain complex. From the symmetric chain complex $(C, \varphi = (1+t)\psi)$ we obtain the symmetric boundary $(\partial C, \partial\varphi)$ with

$$\partial\varphi \coloneqq \mathcal{S}^{-1}e^{\%}(\varphi) \in W^{\%}(\partial C)$$ B21→p.149

and there is also the quadratic structure $S\partial\psi \coloneqq e_{\%}(\psi) \in W_{\%}(\Sigma\partial C)$ on the suspended boundary such that $(1+t)S\partial\psi = \mathcal{S}(\partial\varphi)$. From the following cofibration sequence of chain complexes

$$\begin{array}{ccccc} \Sigma W_{\%}(\partial C) & \xrightarrow{\binom{1+t}{\mathcal{S}}} & \Sigma W^{\%}(\partial C) \oplus W_{\%}(\Sigma\partial C) & \xrightarrow{\mathcal{S}-(1+t)} & W^{\%}(\Sigma\partial C) \\ \partial\psi & \dashmapsto & (\partial\varphi, S\partial\psi) & \longmapsto & 0 \end{array}$$

we get a quadratic structure $\partial\psi \in W_{\%}(\partial C)_{n-1}$ unique up to equivalence. Define $(\partial C, \partial\psi)$ as the quadratic boundary of (C, ψ). □

Room service B21, B22 and B23

$\boxed{\Sigma C}$ the suspended chain complex C shifted one to the left, i.e. $\Sigma C_n = C_{n-1}, d^{\Sigma C} = -d^C$.

$\boxed{f^{\%}\colon W^{\%}(C) \to W^{\%}(D)}$ the chain map induced by a chain map $f\colon C \to D$; explicitly given by $(f^{\%}(\varphi))_s \coloneqq f\varphi_s f^*\colon D^{n+s-*} \to D$.

$\boxed{(f\colon C \to D, \delta\varphi, \varphi)}$ an $(n+1)$-symmetric pair with
- $f\colon C \to D$ a chain map
- (C, φ) an n-symmetric chain complex
- $\delta\varphi \in W^{\%}(D)_{n+1}$ such that $d(\delta\varphi) = f^{\%}(\varphi)$ which is equivalent to $(\delta\varphi, \varphi)$ is a cycle in $\mathscr{C}(f^{\%})_{n+1}$.

Poincaré means $(\delta\varphi_0, \varphi_0 f^*)\colon D^{n+1-*} \to \mathscr{C}(f)_*$ is a chain equivalence.

$\boxed{(\partial C, \partial\varphi)}$ the symmetric boundary of an n-symmetric chain complex obtained from algebraic surgery on the pair $(0 \to C, \varphi, 0)$, i.e. $\partial C = \Sigma^{-1}\mathscr{C}(\varphi_0)$, $\partial\varphi = \mathcal{S}^{-1}e^{\%}(\varphi)$ where $e\colon C \to \mathscr{C}(\varphi_0)$ is the inclusion (see B21 for more details).

$\boxed{e\colon C \to \mathscr{C}(\varphi_0)}$ the inclusion; with a map $\alpha\colon C \to D$ as subscript e_α denotes the inclusion $D \to \mathscr{C}(\alpha)$.

$\boxed{\mathcal{S}\colon W^{\%}C \to \Sigma^{-1}W^{\%}(\Sigma C))}$ the suspension map; defined by $(\mathcal{S}(\varphi))_k \coloneqq \varphi_{k-1}$.

B24 Quadratic boundary for normal

Porter

112→p.61

This is a special case of the one-to-one correspondence between (normal, symmetric) pairs and quadratic chain complexes one dimension lower proven in 112. Just consider $(C,(\varphi,\gamma,\chi))$ as a pair $(0 \to C,(\varphi,\gamma,\chi),0)$. In fact, the following relation holds. An n-symmetric chain complex (C,φ) has an $(n-1)$-quadratic boundary $(\partial C,\partial\psi)$ if and only if the symmetric structure can be extended to a normal structure (φ,γ,χ). But we need only the transition from normal to quadratic. We will deal only with elements in Q-groups. For the construction of explicit cycles see [Wei85b] especially sections §4 and §5.

B24 (112, 231) Quadratic boundary for normal
An n-normal chain complex $(C,(\varphi,\gamma,\chi)) \in NL^n(\mathbb{Z}\pi)$ has an $(n-1)$-quadratic Poincaré boundary

$$\partial^N(C,(\varphi,\gamma,\chi)) =: (\partial C,\partial\psi)$$

which defines a map $\partial^N : NL^n(\mathbb{Z}\pi) \longrightarrow L_{n-1}(\mathbb{Z}\pi); \quad (C,(\varphi,\gamma,\chi)) \mapsto (\partial C,\partial\psi)$.

B241 Exact sequence of Q-groups
For a chain complex C there is an exact sequence of Q-groups

$$\ldots \longrightarrow Q_n(C) \xrightarrow{1+t} Q^n(C) \xrightarrow{J} \widehat{Q}^n(C) \xrightarrow{\partial} Q_{n-1}(C) \xrightarrow{1+t} \ldots$$

B21 Symmetric boundary
An n-symmetric chain complex $(C,\varphi) \in L^n(\mathbb{Z}\pi)$ has an $(n-1)$-symmetric Poincaré boundary

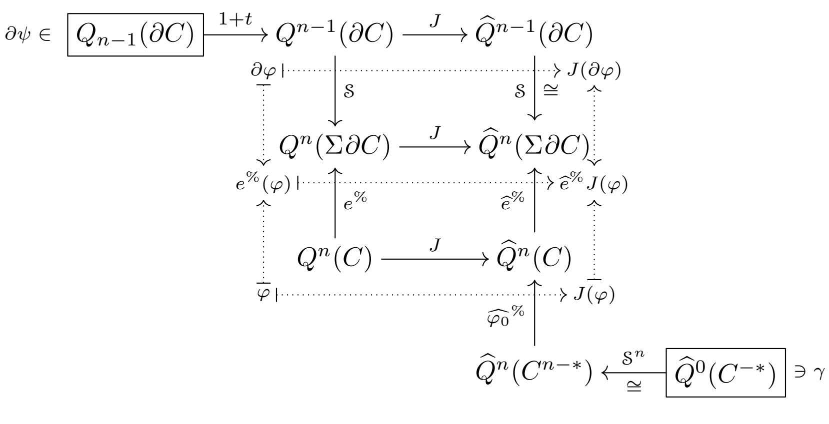
$$\partial^S(C,\varphi) := (\partial C,\partial\varphi)$$

such that $e^{\%}_{\varphi_0}(\varphi) = \mathcal{S}(\partial\varphi)$ where $e : C \to \mathscr{C}(\varphi_0) = \Sigma^{-1}\partial C$ is the inclusion.

Proof B24

B241→p.153

Let $(C,(\varphi,\gamma,\chi))$ be an n-normal chain complex. We take the symmetric boundary $(\partial C,\partial\varphi)$ of its symmetric structure. Consider the following commutative diagram. We want to relate $\partial\psi$ in the upper left corner with γ in the lower right corner. The right column in the diagram is exact in $\widehat{Q}^n(C)$ and the top row is a part of the long exact sequence of Q-groups .



From the symmetric structure φ we obtain the symmetric boundary structure $\partial^S\varphi = \mathcal{S}^{-1}(e^{\%}(\varphi)) \in Q^{n-1}(\partial^S C)$. A diagram chase gives the equation B21→p.149

$$\widehat{e}^{\%}(J(\varphi)) = \mathcal{S}(J(\partial^S\varphi)) \in \widehat{Q}^n(\mathscr{C}(\varphi_0)).$$

It follows that $\partial^S\varphi$ has a preimage in $Q_{n-1}(\partial^S C)$ if and only if $J(\varphi)$ has a preimage in $\widehat{Q}^n(C^{n-*}) \cong \widehat{Q}^0(C^{-*})$. The first preimage is equivalent to the existence of a quadratic refinement $\partial\psi$ of $\partial\varphi$ with $(1+t)\partial\psi = \partial\varphi$. The latter preimage is equivalent to the existence of a normal refinement (φ, γ, χ) of φ with $d\chi = J(\varphi) - \widehat{\varphi_0}^{\%}\mathcal{S}^n(\gamma)$. □

Proof B241 (Exact sequence of Q-groups)

The Q-groups sequence is the long exact homology sequence of the the short exact sequence

$$0 \to W^{\%}C \to \widehat{W}^{\%}C \to \Sigma W_{\%}C \to 0$$

induced by the short exact sequence

$$0 \to \Sigma^{-1}W^{-*} \to \widehat{W} \to W \to 0.$$ □

Room service B24

$(\partial C, \partial\varphi)$ the symmetric boundary of an n-symmetric chain complex obtained from algebraic surgery on the pair $(0 \to C, \varphi, 0)$, i.e. $\partial C = \Sigma^{-1}\mathscr{C}(\varphi_0)$, $\partial\varphi = \mathcal{S}^{-1}e^{\%}(\varphi)$ where $e\colon C \to \mathscr{C}(\varphi_0)$ is the inclusion (see B21 for more details).

- **$e\colon C \to \mathscr{C}(\varphi_0)$** the inclusion; with a map $\alpha\colon C \to D$ as subscript e_α denotes the inclusion $D \to \mathscr{C}(\alpha)$.

$Q^n, Q_n, \widehat{Q}^n$ the n-dimensional Q-groups defined for a chain complex C by

$$Q(C)^n := H_n(W^{\%}C),$$
$$Q(C)_n := H_n(W_{\%}C),$$
$$\widehat{Q}(C)^n := H_n(\widehat{W}^{\%}C).$$

- **$W^{\%}, W_{\%}, \widehat{W}^{\%}$** denote for a chain complex C the abelian group chain complexes

$$W^{\%}C := \mathrm{Hom}_{\mathbb{Z}[\mathbb{Z}_2]}(W, C \otimes C),$$
$$W_{\%}C := W \otimes_{\mathbb{Z}[\mathbb{Z}_2]} (C \otimes C),$$
$$\widehat{W}^{\%}C := \mathrm{Hom}_{\mathbb{Z}[\mathbb{Z}_2]}(\widehat{W}, C \otimes C).$$

  - **$\widehat{W}$** the complete resolution of the trivial $\mathbb{Z}[\mathbb{Z}_2]$-chain module $\mathbb{Z}$; given by the $\mathbb{Z}[\mathbb{Z}_2]$-chain complex $\ldots \xrightarrow{1+t} \mathbb{Z}[\mathbb{Z}_2] \xrightarrow{1-t} \mathbb{Z}[\mathbb{Z}_2] \xrightarrow{1+t} \mathbb{Z}[\mathbb{Z}_2] \xrightarrow{1-t} \ldots$

  - **W** the free resolution of the trivial $\mathbb{Z}[\mathbb{Z}_2]$-chain module $\mathbb{Z}$; given by the $\mathbb{Z}[\mathbb{Z}_2]$-chain complex $\ldots \xrightarrow{1+t} \mathbb{Z}[\mathbb{Z}_2] \xrightarrow{1-t} \mathbb{Z}[\mathbb{Z}_2] \longrightarrow 0$

$(C,(\varphi,\gamma,\chi))$ an n-normal chain complex consisting of a chain complex C and an n-normal structure (φ,γ,χ).

- (φ,γ,χ) an n-normal structure with
  $\varphi \in W^{\%}{}_n$ an n-symmetric structure,
  $\gamma \in \widehat{W}^{\%}(C^{-*})_0$ a cycle called chain bundle,
  $\chi \in \widehat{W}^{\%}{}_{n+1}$ a chain satisfying $d\chi = J(\varphi) - \widehat{\varphi}^{\%}\mathcal{S}^n\gamma$.

- $\chi \in \widehat{W}^{\%}(C)_{n+1}$ a chain satisfying $d\chi = J(\varphi) - \widehat{\varphi}^{\%}(\mathcal{S}^n\gamma)$.

$j\colon W^{\%}C \to \widehat{W}^{\%}C$ induced by the projection $\widehat{W} \to W$, induces a map of Q-groups $j\colon Q^n(R) \to \widehat{Q}^n(R)$.

$1+t\colon W_{\%}C \to W^{\%}C$ the symmetrization map defined by

$$(1+t)(\psi)_s = \begin{cases} (1+t)\psi_0 & \text{if } s=0 \\ 0 & \text{otherwise} \end{cases}$$

induces a map of L-groups $1+t\colon L_n(R) \to L^n(R)$.

B25 Quadratic boundary pair

B25 (232) Quadratic boundary pair
Let $(G^!\colon C^! \to D^!, (\delta\psi^!, \psi^!))$ be an n-quadratic pair and $(\delta\varphi^!, \varphi^!) = (1+t)(\delta\psi^!, \psi^!)$ and $\varphi^!_{G^!} = (\delta\varphi^!_0, G^!\varphi^!_0)\colon \mathscr{C}(G^!)^{n-*} \to D^!$. Then

$$\begin{pmatrix} \partial G^!\colon \partial C^! \to \partial_{G^!} D^!, \\ \partial\delta\psi^!, \\ \partial\psi^! \end{pmatrix} := \begin{pmatrix} \partial G^!\colon \Sigma^{-1}\mathscr{C}(\varphi^!_0) \to \Sigma^{-1}\mathscr{C}(\varphi^!_{G^!}), \\ \left(\tfrac{1+t}{\mathcal{S}}\right)^{-1}(\mathcal{S}^{-1}(e^{\%}_{\varphi^!_{G^!}}\delta\varphi^!), (e_{\varphi^!_{G^!}})_{\%}\delta\psi^!), \\ \left(\tfrac{1+t}{\mathcal{S}}\right)^{-1}(\mathcal{S}^{-1}(e^{\%}_{\varphi^!}\varphi^!), (e_{\varphi^!})_{\%}\psi^!) \end{pmatrix}$$

defines an $(n-1)$-quadratic Poincaré pair called the boundary and denoted $\partial^{Q}_{\rightarrow}$.

Proof B25

Define $\partial C^! = \mathscr{C}(\varphi_0)$ and $\partial_{G^!} D^! = \mathscr{C}(\varphi_{G^!})$ and get the map $\partial G^!$ by the following diagram of cofibrations.

$$\begin{array}{ccccc}
C^{!n-*-1} & \longrightarrow & \mathscr{C}(G^!)^{n-*} & \longrightarrow & D^{!n-*} \\
\downarrow{\scriptstyle \varphi_0} & & \downarrow{\scriptstyle \varphi_{G^!}} & & \downarrow{\scriptstyle (\varphi_{G^!},\varphi_0)} \\
C^! & \xrightarrow{G^!} & D^! & \longrightarrow & \mathscr{C}(G^!) \\
\downarrow{\scriptstyle e_{\varphi_0}} & & \downarrow{\scriptstyle e_{\varphi_{G^!}}} & & \downarrow \\
\partial C^! & \xrightarrow{\partial G^!} & \partial_- D^! & \longrightarrow & \mathscr{C}(\partial G^!)
\end{array}$$

We obtain a quadratic structure $S\partial(\delta\psi,\psi) = ((e_{\varphi_f})_\%, (e_{\varphi_0})_\%)(\delta\psi,\psi) \in \mathscr{C}(\partial G^!{}_\%)$. The zero component $(1+t)S\partial(\delta\psi,\psi))$ is null-homotopic and hence we can desuspend it to $\partial(\delta\varphi,\varphi) = \mathcal{S}^{-1} \circ (1+t)S\partial(\delta\psi,\psi)$. From the short exact sequence we obtain

$$\partial(\delta\psi,\psi) = \begin{pmatrix} 1+t \\ \mathcal{S} \end{pmatrix}^{-1} (\partial(\delta\varphi,\varphi), S\partial(\delta\psi,\psi)). \qquad \square$$

B27 Quadratic boundary signature

B27 (231) Quadratic boundary signature[KMM13, Constr. 3.25][Ran81, Prop. 7.4.1][Wei85b, Theorem 7.1]
Let (Y,ν,ρ) be an n-dimensional normal space. The quadratic boundary signature

$$\partial gn^{\mathbf{L}_\bullet}_{\mathbb{Z}\pi}(Y) =: (\partial C', \partial\psi') \in L_{n-1}(\mathbb{Z}\pi)$$

produces an $(n-1)$-quadratic Poincaré chain complex such that $(\partial C', \partial\psi') = \partial^N(\mathrm{sgn}^{\mathbf{NL}^\bullet}_{\mathbb{Z}\pi}(Y)) \in L_{n-1}(\mathbb{Z}\pi)$.

C4 S-duality (Umkehr map of the identity)
Let (Y,ν,ρ) be an n-dimensional normal space. There is a semi-stable map

$$\Gamma_Y : Th(\nu_Y)^* \to \Sigma^p Y_+$$

such that the induced chain map $\gamma_Y : \widetilde{C}(Th(\nu_Y)^*) \to \widetilde{C}(\Sigma^p Y_+)$ is chain homotopic to

$$(\varphi_Y)_0 = \mathrm{con}^\varphi_Y([Y])_0 : C(Y)^{n-*} \to C(Y).$$

A24 (A27) Spectral quadratic construction [Ran81, Proposition 7.3.1]
Let $F : X \to \Sigma^p Y$ be a semi-stable map between pointed topological spaces and $f : \widetilde{C}(X)_{p+*} \to \widetilde{C}(Y)_*$ the induced chain map. There is a natural chain map

$$\mathrm{con}^{\psi^!}_F : \widetilde{C}(X)_{p+*} \to W_\%(\mathscr{C}(f))$$

such that

$$(1+t) \circ \mathrm{con}^{\psi^!}_F = e^\% \circ \mathrm{con}^\varphi_Y \circ f$$

where $e : \widetilde{C}(Y_*) \to \mathscr{C}(f)$ is the inclusion.

Proof B27

Let $(C,(\varphi,\gamma,\chi))$ denote the n-normal chain complex $\mathrm{sgn}^{\mathbf{NL}^\bullet}_{\mathbb{Z}\pi}(Y)$ with $\varphi = \mathrm{con}^\varphi_Y([Y])$ and denote its quadratic boundary $\partial^N \mathrm{sgn}^{\mathbf{NL}^\bullet}_{\mathbb{Z}\pi}(Y)$ as constructed in B24 by $(\partial C = \Sigma^{-1}\mathscr{C}(\varphi_0), \partial\psi)$. Define the B24→p.152

A24→p.132 chain complex of $\partial \mathrm{gn}^{\mathbf{L}\bullet}_{\mathbb{Z}\pi}(Y)$ to be $\partial C' := \Sigma^{-1}\mathscr{C}(\gamma_Y)$. By A24 we have that $(1+t)\,\mathrm{con}^{\psi^!}_{\Gamma_Y}(u(\nu)^*) = e^{\%}(\varphi) \in \Sigma W^{\%}\mathscr{C}(\gamma_Y)_n$. Hence we can use the exact sequence

$$\Sigma W_{\%}(\partial C') \xrightarrow{\binom{1+t}{\mathcal{S}}} \Sigma W^{\%}(\partial C') \oplus W_{\%}(\Sigma \partial C') \xrightarrow{\mathcal{S}-(1+t)} W^{\%}(\Sigma \partial C')$$

to define the quadratic structure of $\partial \mathrm{gn}^{\mathbf{L}\bullet}_{\mathbb{Z}\pi}(Y)$ to be $\partial\psi' = \binom{1+t}{\mathcal{S}}^{-1}(e^{\%}(\varphi), \mathrm{con}^{\psi^!}_{\Gamma_Y}(u(\nu)^*))$

Now let N be large enough such that there is an N-dimensional S-dual $\mathrm{Th}(\nu)^*$ of $\mathrm{Th}(\nu)$ and set
C4→p.165 $p = N-(n+k)$. From the proof of C4 we obtain the following homotopy commutative diagram.

$$\begin{array}{ccccc} C(Y)^{n-*} & \xrightarrow[\simeq]{-\cup u(\nu)} & \widetilde{C}(\mathrm{Th}(\nu))^{n+k-*} & \xrightarrow[\simeq]{\text{S-dual}} & C(\mathrm{Th}(\nu)^*)_{p+*} \\ \downarrow{\scriptstyle \varphi_0} & & & & \downarrow{\scriptstyle \gamma_Y} \\ C(Y) = \widetilde{C}(Y_+) & & \xrightarrow[\simeq]{\mathcal{S}^p} & & \widetilde{C}(\Sigma^p Y_+)_{p+*} \end{array} \qquad \text{(B27.1)}$$

It identifies the (suspended) chain complexes $\mathscr{C}(\varphi_0) \simeq \mathscr{C}(\gamma_Y)$ of the two boundaries $\partial \mathrm{gn}^{\mathbf{L}\bullet}_{\mathbb{Z}\pi}(Y)$ and $\partial^N \mathrm{sgn}^{\mathbf{NL}^\bullet}_{\mathbb{Z}\pi}(Y)$. Inspection shows that the combination of the diagrams used for the normal
A33→p.139 signature A33 and its quadratic boundary B24 describes the same reason for the existence of $\partial\psi'$
B24→p.152 as the diagram for the spectral quadratic construction A24 together with the cofibration sequence
A24→p.132 used to obtain $\partial\psi$. □

Room service B27

$\boxed{\partial \mathrm{gn}^{\mathbf{L}\bullet}_{\mathbb{Z}\pi}(Y)}$ the quadratic boundary signature for an n-dimensional normal space (Y, ν, ρ) is
B27→p.155 an $(n-1)$-quadratic chain complex $(\partial C', \partial\psi')$ in $L_{n-1}(\mathbb{Z}\pi)$ (see B27).

$\boxed{\partial C' = \Sigma^{-1}\mathscr{C}(\gamma_Y)}$ the chain complex of the quadratic boundary signature $\partial \mathrm{gn}^{\mathbf{L}\bullet}_{\mathbb{Z}\pi}$.

$\boxed{\partial\psi' = \binom{1+t}{\mathcal{S}}^{-1}(\partial\varphi, \mathrm{con}^{\psi^!}_{\Gamma_Y}(u^*_{\nu_Y}))}$ the $(n-1)$-quadratic structure produced by the quadratic boundary signature $\partial \mathrm{gn}^{\mathbf{L}\bullet}_{\mathbb{Z}\pi}$.

$\boxed{\mathrm{con}^{\psi^!}_F : \widetilde{C}(X)_{p+*} \to W_{\%}(\mathscr{C}(f))}$ a chain map called the spectral quadratic construction; defined for a semi-stable map $F\colon X \to \Sigma^p Y$ where $f\colon \widetilde{C}(X)_{p+*} \to \widetilde{C}(\Sigma^p Y)_{p+*} \simeq \widetilde{C}(Y)$ is the chain map induced by F.

$\boxed{\gamma_Y\colon C(\mathrm{Th}(\nu)^*)_{n+p} \to C(Y)}$ the chain map induced by the semi-stable map Γ_Y of an n-dimensional normal space (Y, ν, ρ) with $\mathrm{Th}(\nu)^*$ the N-dimensional S-dual of $\mathrm{Th}(\nu)$ and $p = N-(n+k)$.

$\boxed{\Gamma_Y := S^{-1}(\widetilde{\Delta} \circ \rho)\colon \mathrm{Th}(\nu)^* \to \Sigma^p Y_+}$ the semi-stable map obtained for an n-dimensional normal space (Y, ν, ρ) with an N-dimensional S-dual $\mathrm{Th}(\nu)^*$ of its Thom space and $p = N-(n+k)$.

$\boxed{\widetilde{\Delta}\colon \mathrm{Th}(\nu) \simeq \frac{V}{\partial V} \xrightarrow{\Delta} \frac{V\times V}{V\times \partial V} \simeq \mathrm{Th}(\nu)\wedge X_+}$ the generalized diagonal map where V is the mapping cylinder of the projection map of ν and ∂V the total space of ν.

$S\colon [Y,Z] \xrightarrow{\cong} [S^N, X\wedge Y]$ the S-duality isomorphism; for an N-dimensional S-duality map $\alpha\colon S^N \to X\wedge Y$ and an arbitrary space Z defined by $S(\gamma) = (\mathrm{id}_Y \wedge \gamma)\circ \alpha$; denotes the induced isomorphism $S\colon H^{N-*}(X;\mathbf{E}) \xrightarrow{\cong} H_*(X,\mathbf{E})$ as well.

$\alpha\colon S^N \to X\wedge Y$ an N-dimensional S-duality map, i.e. the slant product maps

$$\alpha_*([S^N])\backslash\cdot\colon \tilde{C}(X)^{N-*} \to \tilde{C}(Y) \quad \text{and} \quad \alpha_*([S^N])\backslash\cdot\colon \tilde{C}(Y)^{N-*} \to \tilde{C}(X)$$

are chain equivalences.

$\mathrm{con}^{\varphi}_X\colon C(X) \to W^{\%}(C(X))$ a chain map called symmetric construction; defined for a topological space X.

$\mathcal{S}\colon W^{\%}C \to \Sigma^{-1}W^{\%}(\Sigma C))$ the suspension map; defined by $(\mathcal{S}(\varphi))_k \coloneqq \varphi_{k-1}$.

B28 Quadratic boundary pair signature

B28 (1621, 232) Quadratic boundary pair signature
Let $((Y,B),\nu,(\rho_Y,\rho_B))$ be an n-dimensional pair of normal spaces. There is a quadratic boundary pair signature

$$\partial gn^{\mathbf{L}\bullet}_{\to}(Y,B) = (\partial j\colon \partial C' \to \partial D', (\partial\delta\psi, \partial\psi))$$

producing an $(n-1)$-quadratic Poincaré pair.

C6 Relative S-duality
Let $((Y,B),\nu,(\rho_Y,\rho_B))$ be an $(n+1)$-dimensional pair of normal spaces with $j\colon B\to Y$ the inclusion map. There are semi-stable geometric Umkehr maps $\Gamma_Y,\Gamma_B,\Gamma_{Y,B}$ which fit into the following commutative diagram

$$\begin{array}{ccccccc}
\Sigma^{-1}Th(\nu_B)^* & \xrightarrow{i} & (Th(\nu_Y)/Th(\nu_B))^* & \longrightarrow & Th(\nu_Y)^* & \longrightarrow & Th(\nu_B)^* \\
\downarrow{\Gamma_B} & & \downarrow{\Gamma_Y} & & \downarrow{\Gamma_{Y,B}} & & \downarrow{\Sigma\Gamma_B} \\
\Sigma^p B_+ & \xrightarrow{j} & \Sigma^p Y_+ & \longrightarrow & \Sigma^p Y/B & \longrightarrow & \Sigma^{p+1}B_+
\end{array}$$

and the induced chain maps

$$\begin{aligned}
\gamma_B &\colon \widetilde{C}(Th(\nu_B)^*) \to \widetilde{C}(\Sigma^{p+1}B_+)\\
\gamma_Y &\colon \widetilde{C}((Th(\nu_Y)/Th(\nu_B))^*) \to \widetilde{C}(\Sigma^p Y_+)\\
\gamma_{Y,B} &\colon \widetilde{C}(Th(\nu_Y)^*) \to \widetilde{C}(\Sigma^p Y/B)
\end{aligned}$$

are chain homotopic to

$$\begin{aligned}
(\varphi_B)_0 &\colon C(B)^{n+1-*} \to C(B)\\
\varphi_j &\colon \mathscr{C}(j)^{n-*} \to C(Y)\\
\varphi_{j^*} &\colon C(Y)^{n-*} \to \mathscr{C}(j).
\end{aligned}$$

A25 (A28, B28, 232) Relative spectral quadratic construction [Ran81, Proposition 7.3.1]
Let $(\delta F, F)\colon (N, A) \to \Sigma^p(Y, B)$ be a semi-stable map between pairs of pointed topological spaces inducing the following commutative diagram of chain maps.

$$\begin{array}{ccccc}
\widetilde{C}(A)_{p+*} & \xrightarrow{f} & \widetilde{C}(B) & \longrightarrow & \mathscr{C}(f) \\
\downarrow{\scriptstyle j} & & \downarrow{\scriptstyle i} & & \downarrow{\scriptstyle (i,j)} \\
\widetilde{C}(N)_{p+*} & \xrightarrow{\delta f} & \widetilde{C}(Y) & \longrightarrow & \mathscr{C}(g) \\
\downarrow & & \downarrow & & \downarrow \\
\widetilde{C}(N, A)_{p+*} & \xrightarrow{(\delta f, f)} & \widetilde{C}(Y, B) & \xrightarrow{e} & \mathscr{C}(j, i)
\end{array}$$

There is a chain map $\quad \mathrm{con}^{\delta\psi^!, \psi^!}_{\delta F, F} \colon \widetilde{C}(N, A) \to \mathscr{C}((i, j)_{\%})$

such that $\quad (1 + t) \circ \mathrm{con}^{\delta\psi^!, \psi^!}_{\delta F, F} = e^{\%} \circ \mathrm{con}^{\delta\varphi, \varphi}_{Y, B} \circ (\delta f, f).$

Proof B28

Let $j\colon C(B) \to C(Y)$ be the inclusion and denote by $(\delta\varphi, \varphi) \in \mathscr{C}(j^{\%})_n$ the symmetric structure obtained by applying relative symmetric construction $\mathrm{con}^{\delta\varphi, \varphi}_{Y, B}$ to the fundamental class $([Y, B])$. Define the underlying chain map of $\partial\mathrm{gn}^{\mathbf{L}^\bullet}_{\to}(Y, B)$ to be the map ∂j induced by the following diagram.

$$\begin{array}{ccccccc}
\partial C = \Sigma^{-1}\mathscr{C}(\varphi_0) & \longrightarrow & C(B)^{n-1*} & \xrightarrow{\varphi_0} & C(B) & \longrightarrow & \mathscr{C}(\varphi_0) \\
\downarrow{\scriptstyle \partial j} & & \downarrow{\scriptstyle q_i} & & \downarrow{\scriptstyle i} & & \downarrow{\scriptstyle \Sigma\partial j} \\
\partial_{j^*} D = \Sigma^{-1}\mathscr{C}(\varphi_j) & \longrightarrow & \mathscr{C}(i)^{n-*} & \xrightarrow{\varphi_j} & C(Y) & \longrightarrow & \mathscr{C}(\varphi_j)
\end{array}$$

Now we are looking for an n-quadratic structure

$$(\partial\delta\psi, \partial\psi) \in \mathscr{C}(\partial j_{\%})_n = W_{\%}(\partial_{j^*} D)_n \oplus W_{\%}(\partial C)_{n-1}.$$

C6→p.165 From C6 we obtain the maps

$$\Gamma_Y \colon (\mathrm{Th}(\nu_Y)/\mathrm{Th}(\nu_B))^* \to \Sigma^p Y_+$$
$$\Gamma_B \colon \Sigma^{-1}\mathrm{Th}(\nu_B)^* \to \Sigma^p B_+,$$

with the induced chain maps γ_Y and γ_B that are chain homotopic to φ_j and φ_0. So the relative
A25→p.133 spectral quadratic construction A25 produces a quadratic pair structure

$$(S\delta\psi', S\psi') = \mathrm{con}^{\delta\psi^!, \psi^!}_{\Gamma_Y, \Gamma_B}(u_{\nu_Y})^*$$

in $\mathscr{C}((\gamma_Y, \gamma_B)_{\%})_n \simeq \mathscr{C}((\varphi_j, \varphi_0)_{\%})_n \simeq \mathscr{C}((\varphi_{j^*})_{\%})_n \simeq \mathscr{C}(\partial j_{\%})_n = W_{\%}(\mathscr{C}(\varphi_j))_n \oplus W_{\%}(\mathscr{C}(\varphi_0))_{n-1}$, which can be read off the following diagram.

$$\begin{array}{ccccccc}
C(B)^{n-1*} & \xrightarrow{q_j^*} & \mathscr{C}(j)^{n-*} & \xrightarrow{e_j^*} & \mathscr{C}(q_j^*) & \simeq & C(Y)^{n-*} \\
\downarrow{\scriptstyle \varphi_0} & & \downarrow{\scriptstyle \varphi_j} & & \downarrow{\scriptstyle (\varphi_j, \varphi_0)} & & \downarrow{\scriptstyle \varphi_{j^*} = \left(\begin{smallmatrix} \delta\varphi_0 \\ \varphi_0 j^* \end{smallmatrix}\right)} \\
C(B) & \xrightarrow{j} & C(Y) & \xrightarrow{e_j} & \mathscr{C}(j) & = & \mathscr{C}(j) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\mathscr{C}(\varphi_0) & \xrightarrow{\Sigma\partial j} & \mathscr{C}(\varphi_j) & \dashrightarrow & \mathscr{C}(\Sigma\partial j) & \simeq & \mathscr{C}(\varphi_{j^*})
\end{array}$$

Consider both components separately. The same argument as in the absolute case B27 yields desuspended structures $\partial\delta\psi' = \left(\begin{smallmatrix} 1+t \\ \mathcal{S} \end{smallmatrix}\right)^{-1} (\partial\delta\varphi, S\delta\psi')$ and $\partial\psi' = \left(\begin{smallmatrix} 1+t \\ \mathcal{S} \end{smallmatrix}\right)^{-1} (\partial\varphi, S\psi')$ where $(\partial\delta\varphi, \partial\varphi)$ is the symmetric boundary pair from B22. ☐ B27→p.155 B22→p.150

Room service B28

$e : C \to \mathscr{C}(\varphi_0)$ the inclusion; with a map $\alpha : C \to D$ as subscript e_α denotes the inclusion $D \to \mathscr{C}(\alpha)$.

B3 Algebraic Thom construction

Porter

In geometry we can pass from (X, A) to X/A and obtain equivalent chain complexes. The algebraic analogue for algebraic complexes is to pass from $\mathscr{C}(f^{\%})$ to $W^{\%}\mathscr{C}(f)$. We would like to understand this passage.

B3 Algebraic Thom construction [Ran80a, 3.4][Ran92, Prop. 1.15]
There is the following one-to-one correspondence of homotopy classes:

$$\begin{array}{ccc} n\text{-dimensional} & & n\text{-dimensional} \\ \text{symmetric Poincaré pairs} & \overset{1-1}{\longleftrightarrow} & \text{symmetric complexes} \\ (f : C \to D, \delta\lambda, \varphi) & & (C', \varphi') \end{array}$$

B21 Symmetric boundary
An n-symmetric chain complex $(C, \varphi) \in L^n(\mathbb{Z}\pi)$ has an $(n-1)$-symmetric Poincaré boundary

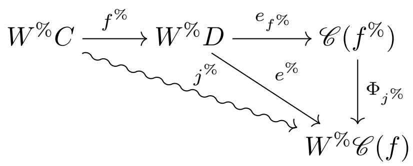
$$\partial^S(C, \varphi) := (\partial C, \partial\varphi)$$

such that $e^{\%}_{\varphi_0}(\varphi) = \mathcal{S}(\partial\varphi)$ where $e : C \to \mathscr{C}(\varphi_0) = \Sigma^{-1}\partial C$ is the inclusion.

Proof B3

Given an n-symmetric Poincaré pair $(f : C \to D, \delta\varphi, \varphi)$ we want to construct an n-symmetric chain complex (C', φ'). Define $C' = \mathscr{C}(f)$. The symmetric structure $\varphi' \in W^{\%}\mathscr{C}(f)$ on C' is obtained as follows.

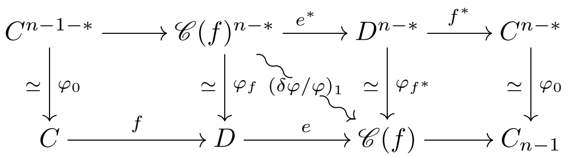
Note that $W^{\%}(-)$ is compatible with chain homotopies but not with cofibrations. Hence we obtain from the cofibration $C \xrightarrow{f} D \xrightarrow{e} \mathscr{C}(f)$ a null homotopy $j^{\%} : e^{\%} f^{\%} = (ef)^{\%} \simeq 0$ but the cofiber $\mathscr{C}(f^{\%})$ is not chain equivalent to $W^{\%}(\mathscr{C}(f))$. But at least we obtain a map $\Phi_{j^{\%}}\colon \mathscr{C}(f^{\%}) \to W^{\%}\mathscr{C}(f)$ induced by the null-homotopy $j^{\%}$:



Denote by $\delta\varphi/\varphi$ the image of $(\delta\varphi, \varphi) \in \mathscr{C}(f^{\%})_n$ under $\Phi_{j^{\%}}$. There is a canonical choice of $\Phi_{j^{\%}}$ that gives the explicit description $\delta\varphi/\varphi = e^{\%}(\delta\varphi) + j^{\%}(\varphi)$. Define $\varphi' = \delta\varphi/\varphi$.

Now conversely, given an n-symmetric chain complex (C, φ). We use the boundary construction of B21 to obtain an $(n-1)$-symmetric Poincaré chain complex $(\partial C, \partial\varphi)$. Then $g\colon (\partial C \to C^{n-*}, (0, \partial\varphi))$ defines an n-symmetric Poincaré pair. B21→p.149

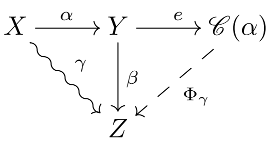
We have $(\Sigma^{-1}\mathscr{C}(\partial C \to C^{n-*}), 0/\partial\varphi) \simeq (C, \varphi)$. For the other equivalence $(f\colon C \to D, \delta\varphi, \varphi) \simeq (g\colon \partial\mathscr{C}(f) = \Sigma^{-1}\mathscr{C}(\mathscr{C}(f)^{n-*} \xrightarrow{(\delta\varphi/\varphi)_0} \mathscr{C}(f)) \to \mathscr{C}(f)^{n-*}, (0, \delta\varphi/\varphi))$ use that we started with a Poincaré pair and the following up to homotopy commutative diagram.



The homotopy equivalence of pairs involves the definition of triad which is given below. $\square$

Room service B3

$\boxed{\Phi_\gamma}$ is the chain map induced by a null-homotopy $\gamma\colon \beta \circ \alpha \simeq 0$, fitting in the following diagram with



and given by $\mathscr{C}(\alpha) \longrightarrow Z; \quad (y, x) \mapsto \beta(y) + \gamma(x)$.

$\boxed{(f\colon C \to D, \delta\varphi, \varphi) \simeq (f'\colon C' \to D', \delta\varphi', \varphi')}$ two symmetric pairs are homotopy equivalent if there is a triad $\Gamma = (f, f', g, g'; h)$ such that g and h are homotopy equivalences and $(g, g'; h)^{\%}(\delta\varphi, \varphi) = (\delta\varphi', \varphi')$

$\Gamma = (f, f', g, g'; h, (\varphi, \varphi', \delta\varphi, \delta\varphi'; \delta^2\varphi))$ an $(n+2)$-dimensional symmetric triad, i.e. a commutative square of chain complexes and chain maps

$$\begin{array}{ccc} C & \xrightarrow{f} & D \\ {\scriptstyle f'}\downarrow & \searrow^{h} & \downarrow{\scriptstyle g} \\ D' & \xrightarrow[g']{} & E \end{array}$$

with

- $h : gf \simeq g'f' : C \to E_{*+1}$ a chain homotopy
- $(f : C \to D, \delta\varphi, \varphi)$ and $(f' : C \to D', \delta\varphi', \varphi)$ being $(n+1)$-symmetric pairs
- $\delta^2\varphi \in W^{\%}(E)_{n+2}$ a chain such that
  $d(\delta^2\varphi)_s = g'^{\%}(\delta\varphi')_s - g^{\%}(\delta\varphi)_s + g'f'\varphi_s h^* + h\varphi_s f^* g^* + h^{\%}(\varphi)_s$.

Poincaré means $(f : C \to D, \delta\varphi, \varphi)$ and $(f' : C \to D', \delta\varphi', \varphi)$ are Poincaré and
$\begin{pmatrix} \delta^2\varphi_0 \\ \delta\varphi_0 g^* \\ f'\varphi_0 h^* + \delta\varphi_0' g'^* \\ \varphi_0 f^* g^* \end{pmatrix} : E^{n+2-*} \to \mathscr{C}(\Gamma) := \mathscr{C}\left(\begin{pmatrix} g' & h \\ 0 & f \end{pmatrix} : \mathscr{C}(f') \to \mathscr{C}(g)\right)$ is a chain equivalence.
The quadratic case is analog but uses symmetrization for the definition of Poincaré.

$(g, g'; h)^{\%} : W^{\%}(\mathrm{Hom}_A(D^*, D)_* \oplus \mathrm{Hom}_A(C^*, C)_{*-1}) \to W^{\%}(\mathrm{Hom}_A(D'^*, D')_* \oplus \mathrm{Hom}_A(C'^*, C')_{*-1})$
given by $(\delta\varphi_s, \varphi_s) \mapsto (g^{\%}(\delta\varphi_s) \pm h\varphi_s f^* g^* \pm f'g'\varphi_s h^* \pm h^{\%}(T\varphi_{s-1}), f'^{\%}(\varphi_s))$ (see [Ran81, p.42])

$\varphi_{f^*} = \mathrm{ev}_r(\delta\varphi, \varphi) \simeq \begin{pmatrix} \delta\varphi_0 \\ \varphi_0 f^* \end{pmatrix} : D^{n-*} \to \mathscr{C}(f)$ a chain map defined for an n-symmetric pair $(f : C \to D, \delta\varphi, \varphi)$.

$\varphi_f = \mathrm{ev}_l(\delta\varphi, \varphi) \simeq (\delta\varphi_0, f\varphi_0) : \mathscr{C}(f)^{n-*} \to D$ a chain map defined for an n-symmetric pair $(f : C \to D, \delta\varphi, \varphi)$.

C Umkehr maps and S-duality

Porter

The important insight we gain from the following results is that Poincaré duality can be seen as S-duality together with Thom equivalence. The proof is taken from [Ran80b, Prop. 4.2] but we restrict ourselves here to the non-equivariant case in order to neaten up the notation. We refer to [Ada74] for more details on S-duality and to [Ran80b, §3] for a full treatment of equivariant πS-duality.

C (A26) Umkehr maps [Ran80b, Prop. 4.2]
Let $\widehat{f}\colon N \to X$ be a degree one normal map between Poincaré spaces both of dimension n. There is a stable geometric Umkehr map

$$F\colon \Sigma^p X_+ \to \Sigma^p N_+$$

such that $\Sigma^p f_+ \circ F \simeq \mathrm{id}\colon \Sigma^p X_+ \to \Sigma^p X_+$ and such that the induced chain map $F_*\colon \widetilde{C}(\Sigma^p X_+) \to \widetilde{C}(\Sigma^p N_+)$ is chain homotopic to the composition

$$f^!\colon C(X) \xrightarrow{(\varphi_X)_0^{-1}} C(X)^{n-*} \xrightarrow{f^*} C(N)^{n-*} \xrightarrow{(\varphi_N)_0} C(N).$$

[C2 → [Ada74]] S-duality properties
For N large enough the following holds.

(i) For every finite CW-complex X there is an N-dimensional S-dual, which we denote X^*.

(ii) The suspended S-dual ΣX^* is an $(N+1)$-dimensional S-dual of X.

(iii) For any space Z and a S-duality map $\alpha\colon S^N \to X \wedge X^*$ we have isomorphisms

$$S\colon [X, Z] \cong [S^N, Z \wedge X^*] \qquad \gamma \mapsto \alpha\backslash\gamma = (\gamma \wedge \mathrm{id}_{X^*}) \circ \alpha$$
$$S\colon [X^*, Z] \cong [S^N, X \wedge Z] \qquad \gamma \mapsto \alpha\backslash\gamma = (\mathrm{id}_X \wedge \gamma) \circ \alpha$$

(iv) A map $f\colon X \to Y$ induces a map $f^*\colon Y^* \to X^*$ via the isomorphism

$$[X, Y] \cong [S^N, Y \wedge X] \cong [Y^*, X^*].$$

(v) If $X \to Y \to Z$ is a cofibration sequence, then $Z^* \to Y^* \to X^*$ is a cofibration sequence.

Variations

⇒ **C3 (A27) Umkehr map for normal targets**
Let $\widehat{g}\colon N \to Y$ be a degree one normal map from a Poincaré space N to a normal space (Y, ν, ρ) both of dimension n. There is a semi-stable geometric Umkehr map

$$\Gamma^!\colon Th(\nu_Y)^* \to \Sigma^p N_+$$

such that the induced chain map $\gamma^!\colon \widetilde{C}(Th(\nu_Y)^*) \to \widetilde{C}(\Sigma^p N_+)$ is chain homotopic to the composition

$$g^!\colon C(Y)^{n-*} \xrightarrow{g^*} C(N)^{n-*} \xrightarrow{(\varphi_N)_0} C(N)$$

where $\varphi_N = \mathrm{con}_N^{\varphi}([N])$.

⇒ **C4 S-duality (Umkehr map of the identity)**
Let (Y, ν, ρ) be an n-dimensional normal space. There is a semi-stable map

$$\Gamma_Y\colon Th(\nu_Y)^* \to \Sigma^p Y_+$$

such that the induced chain map $\gamma_Y\colon \widetilde{C}(Th(\nu_Y)^*) \to \widetilde{C}(\Sigma^p Y_+)$ is chain homotopic to

$$(\varphi_Y)_0 = \mathrm{con}_Y^{\varphi}([Y])_0\colon C(Y)^{n-*} \to C(Y).$$

$\Rightarrow$

C5 (A28) Relative Umkehr maps for normal targets

Let $(\delta\widehat{g}, \widehat{g})\colon (N, A) \to (Y, B)$ be a degree one normal map from a Poincaré pair to a normal pair both of dimension $(n+1)$ with $j\colon B \to Y$ and $i\colon A \to N$ the inclusion maps. There are semi-stable geometric Umkehr maps $\Gamma^!_Y, \Gamma^!_B, \Gamma^!_{Y,B}$ which fit into the following commutative diagram

$$\begin{array}{ccccccc}
\Sigma^{-1}Th(\nu_B)^* & \xrightarrow{i} & (Th(\nu_Y)/Th(\nu_B))^* & \longrightarrow & Th(\nu_Y)^* & \longrightarrow & Th(\nu_A)^* \\
\downarrow{\scriptstyle \Gamma^!_B} & & \downarrow{\scriptstyle \Gamma^!_Y} & & \downarrow{\scriptstyle \Gamma^!_{Y,B}} & & \downarrow{\scriptstyle \Sigma\Gamma^!_B} \\
\Sigma^p A_+ & \xrightarrow{j} & \Sigma^p N_+ & \longrightarrow & \Sigma^p N/A & \longrightarrow & \Sigma^{p+1} A_+
\end{array}$$

and the induced chain maps

$$\begin{aligned}
\gamma^!_B &\colon \widetilde{C}(Th(\nu_B)^*) \to \widetilde{C}(\Sigma^{p+1}A_+) \\
\gamma^!_Y &\colon \widetilde{C}((Th(\nu_Y)/Th(\nu_B))^*) \to \widetilde{C}(\Sigma^p N_+) \\
\gamma^!_{Y,B} &\colon \widetilde{C}(Th(\nu_Y)^*) \to \widetilde{C}(\Sigma^p N/A)
\end{aligned}$$

are chain homotopic to

$$\begin{aligned}
g^! &\colon C(B)^{n+1-*} \xrightarrow{g^*} C(A)^{n+1-*} \xrightarrow{(\varphi_B)_0} C(A) \\
g^!_i &\colon \mathscr{C}(j)^{n-*} \xrightarrow{(\delta g, g)^*} \mathscr{C}(i)^{n-*} \xrightarrow{\varphi_i} C(N) \\
g^!_{i^*} &\colon C(Y)^{n-*} \xrightarrow{\delta g^*} C(N)^{n-*} \xrightarrow{\varphi_{i^*}} \mathscr{C}(i).
\end{aligned}$$

$\Rightarrow$

C6 Relative S-duality

Let $((Y, B), \nu, (\rho_Y, \rho_B))$ be an $(n+1)$-dimensional pair of normal spaces with $j\colon B \to Y$ the inclusion map. There are semi-stable geometric Umkehr maps $\Gamma_Y, \Gamma_B, \Gamma_{Y,B}$ which fit into the following commutative diagram

$$\begin{array}{ccccccc}
\Sigma^{-1}Th(\nu_B)^* & \xrightarrow{i} & (Th(\nu_Y)/Th(\nu_B))^* & \longrightarrow & Th(\nu_Y)^* & \longrightarrow & Th(\nu_B)^* \\
\downarrow{\scriptstyle \Gamma_B} & & \downarrow{\scriptstyle \Gamma_Y} & & \downarrow{\scriptstyle \Gamma_{Y,B}} & & \downarrow{\scriptstyle \Sigma\Gamma_B} \\
\Sigma^p B_+ & \xrightarrow{j} & \Sigma^p Y_+ & \longrightarrow & \Sigma^p Y/B & \longrightarrow & \Sigma^{p+1} B_+
\end{array}$$

and the induced chain maps

$$\begin{aligned}
\gamma_B &\colon \widetilde{C}(Th(\nu_B)^*) \to \widetilde{C}(\Sigma^{p+1}B_+) \\
\gamma_Y &\colon \widetilde{C}((Th(\nu_Y)/Th(\nu_B))^*) \to \widetilde{C}(\Sigma^p Y_+) \\
\gamma_{Y,B} &\colon \widetilde{C}(Th(\nu_Y)^*) \to \widetilde{C}(\Sigma^p Y/B)
\end{aligned}$$

are chain homotopic to

$$\begin{aligned}
(\varphi_B)_0 &\colon C(B)^{n+1-*} \to C(B) \\
\varphi_j &\colon \mathscr{C}(j)^{n-*} \to C(Y) \\
\varphi_{j^*} &\colon C(Y)^{n-*} \to \mathscr{C}(j).
\end{aligned}$$

Proof C

We denote the stable homotopy classes of pointed maps from N_+ to X_+ by $[N, X]$. Since N and X are both Poincaré we have for k large enough normal structures

$$(\nu_X\colon X \to \mathrm{BSG}(k), \rho_X\colon S^{n+k} \to \mathrm{Th}(\nu_X))$$

and

$$(\nu_N\colon N \to \mathrm{BSG}(k), \rho_N\colon S^{n+k} \to \mathrm{Th}(\nu_N))$$

and S-duality maps

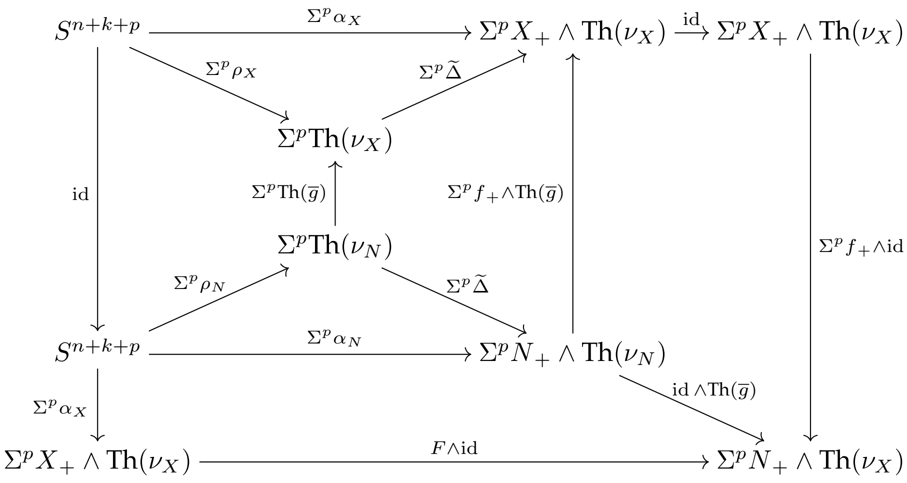
$$\alpha_N : S^{n+k} \to N_+ \wedge \mathrm{Th}(\nu_N) \qquad \text{and} \qquad \alpha_X : S^{n+k} \to X_+ \wedge \mathrm{Th}(\nu_X).$$

The composition

$$[\mathrm{Th}(\nu_N), \mathrm{Th}(\nu_X)] \xrightarrow{(\alpha_N \backslash -)} [S^{n+k}, \mathrm{Th}(\nu_X) \wedge N_+] \xrightarrow{(\alpha_X \backslash -)^{-1}} [X_+, N_+]$$

of the S-duality isomorphism from C2 (iii) evaluated on $\mathrm{Th}(\overline{g}) : \mathrm{Th}(\nu_N) \to \mathrm{Th}(\nu_X)$ yields the stable map $F : \Sigma^p X_+ \to \Sigma^p N_+$.

Working round the stable homotopy commutative diagram



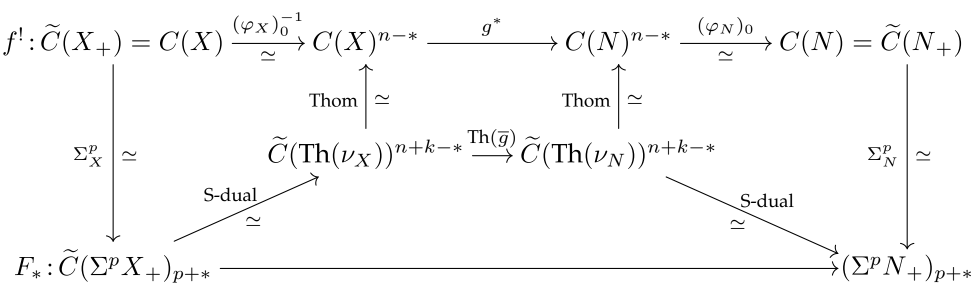
we obtain that

$$((\Sigma^p f_+ \circ F) \wedge \mathrm{id}) \circ (\Sigma^p \alpha_X) \simeq (\Sigma^p \alpha_X) : S^{n+k+p} \to \Sigma^p X_+ \wedge \mathrm{Th}(\nu_X).$$

By C2 (ii) the map $\Sigma^p \alpha_X$ is also an S-duality map and it follows that

$$\Sigma^p f_+ \circ F \simeq \mathrm{id} : \Sigma^p X_+ \to \Sigma^p X_+$$

for p large enough. From the homotopy commutative diagram



we gain the induced Umkehr chain map $f^! : C(X) \to C(N)$. $\square$

Proof C3

Now, because Y is only normal, the S-dual of its Thom space is not Y itself. So we have to use an unspecific S-dual $\mathrm{Th}(\nu_Y)^*$ for one of the S-duality maps

$$\alpha_N\colon S^{n+k+p} \to N_+ \wedge \mathrm{Th}(\nu_N) \qquad \text{and} \qquad \alpha_Y\colon S^{n+k+p} \to \mathrm{Th}(\nu_Y)^* \wedge \mathrm{Th}(\nu_Y)$$

with p chosen large enough. Analogously to C, the composition

$$[\mathrm{Th}(\nu_N), \mathrm{Th}(\nu_Y)] \xrightarrow{(\alpha_N \backslash -)} [S^{n+k}, \mathrm{Th}(\nu_Y) \wedge \Sigma^p N_+] \xrightarrow{(\alpha_Y \backslash -)^{-1}} [\mathrm{Th}(\nu_Y)^*, \Sigma^p N_+]$$

applied to $[\mathrm{Th}(\overline{g})]$ produces a stable map $\Gamma^!\colon \mathrm{Th}(\nu_Y)^* \to \Sigma^p N_+$.

We loose the upper left corner in the homotopy commutative diagram above but S-duality and Thom equivalence still give rise to the induced Umkehr chain map $g^!\colon C(Y)^{n-*} \to C(N)$.

$$\begin{array}{ccccc}
 & g^!\colon C(Y)^{n-*} & \xrightarrow{g^*} & C(N)^{n-*} \xrightarrow[\simeq]{(\varphi_N)_0} & C(N) = \widetilde{C}(N_+) \\
 & \uparrow \text{Thom}\ \simeq & & \uparrow \text{Thom}\ \simeq \quad \nearrow \text{S-dual}\ \simeq & \\
 & \widetilde{C}(\mathrm{Th}(\nu_Y))^{n+k-*} & \xrightarrow{\mathrm{Th}(\overline{g})} & \widetilde{C}(\mathrm{Th}(\nu_N))^{n+k-*} & \downarrow \Sigma^p_N\ \simeq \\
 \nearrow \text{S-dual}\ \simeq & & & & \\
 \gamma^!\colon \widetilde{C}(\mathrm{Th}(\nu_Y)^*)_{p+*} & & \longrightarrow & & \widetilde{C}(\Sigma^p N_+)_{p+*}
\end{array}$$

□

Proof C4

In this case we use the S-duality map $S^{n+k+p} \to \mathrm{Th}(\nu) \wedge \mathrm{Th}(\nu)^*$. From C2 (iii) we have an isomorphism

$$S^{-1}\colon [S^{n+k}, \mathrm{Th}(\nu) \wedge Y_+] \cong [\mathrm{Th}(\nu)^*, \Sigma^p Y_+]$$

that yields a map $\Gamma_Y = S^{-1}(\widetilde{\Delta} \circ \rho)\colon \mathrm{Th}(\nu)^* \to \Sigma^p Y_+$. From the homotopy commutative diagram in C remains

$$\begin{array}{cccc}
 & (\varphi_Y)_0\colon C(Y)^{n-*} & \longrightarrow & C(Y) = \widetilde{C}(Y_+) \\
 & \uparrow \text{Thom}\ \simeq & & \\
 & \widetilde{C}(\mathrm{Th}(\nu_Y)^*)^{n+k-*} & & \simeq\ \downarrow \Sigma^p_Y \\
 \nearrow \text{S-dual}\ \simeq & & & \\
 \gamma_Y\colon \widetilde{C}(\mathrm{Th}(\nu_Y)^*)_{p+*} & & \longrightarrow & \widetilde{C}(\Sigma^p Y_+)_{p+*}
\end{array}$$

which identifies the chain maps γ_Y and $(\varphi_Y)_0$. If Y is Poincaré, then p can be chosen to be 0 and γ_Y and Σ^p_Y to be the identity. We recover the Poincaré duality map $(\varphi_Y)_0$ as the composition of S-duality and Thom equivalence. □

Proof C5 and C6

Choose p such that there is an S-duality map $S^{n+k+p}\colon \mathrm{Th}(\nu_Y) \wedge \mathrm{Th}(\nu_Y)^*$. In the absolute case we used the generalized diagonal map $\widetilde{\Delta}\colon \mathrm{Th}(\nu) \to \mathrm{Th}(\nu) \wedge X_+$ and the one-to-one correspondence $S^{-1}\colon [S^{n+k}, \mathrm{Th}(\nu) \wedge X_+] \cong [\mathrm{Th}(\nu)^*, \Sigma^p X_+]$ to obtain a map $\Gamma := S^{-1}(\widetilde{\Delta} \circ \rho)\colon \mathrm{Th}(\nu)^* \to \Sigma^p X_+$. Now we have three diagonal maps

$$\begin{aligned}
&\Delta_1\colon \mathrm{Th}(\nu_B) \to \mathrm{Th}(\nu_B) \wedge B_+,\\
&\Delta_2\colon \mathrm{Th}(\nu_Y)/\mathrm{Th}(\nu_B) \to \mathrm{Th}(\nu_Y)/\mathrm{Th}(\nu_B) \wedge Y_+,\\
&\Delta_3\colon \mathrm{Th}(\nu_Y) \to \mathrm{Th}(\nu_Y) \wedge Y/B
\end{aligned}$$

and the one-to-one-correspondences from C2 (iii)

$$\begin{aligned}
[S^{n+k}, \mathrm{Th}(\nu_B) \wedge B_+] &\xrightarrow{\cong} [\mathrm{Th}(\nu_B)^*, \Sigma^{p+1} B_+],\\
[S^{n+k+1}, \mathrm{Th}(\nu_Y)/\mathrm{Th}(\nu_B) \wedge Y_+] &\xrightarrow{\cong} [(\mathrm{Th}(\nu_Y)/\mathrm{Th}(\nu_B))^*, \Sigma^p Y_+],\\
[S^{n+k+1}, \mathrm{Th}(\nu_Y)/\mathrm{Th}(\nu_B) \wedge Y/B] &\xrightarrow{\cong} [(\mathrm{Th}(\nu_Y)^*, \Sigma^p Y/B],
\end{aligned}$$

which induce the duality maps

$$\begin{aligned}
\Sigma\Gamma_B &= S^{-1}(\Delta_1 \circ \rho_B)\colon \mathrm{Th}(\nu_B) \to \Sigma^{p+1} B,\\
\Gamma_Y &= S^{-1}(\Delta_2 \circ \rho_Y/\rho_B)\colon (\mathrm{Th}(\nu_Y)/\mathrm{Th}(\nu_B))^* \to \Sigma^p Y,\\
\Gamma_{Y,B} &= S^{-1}(\Delta_3 \circ \rho_Y/\rho_B)\colon \mathrm{Th}(\nu_Y) \to \Sigma^p (Y/B).
\end{aligned}$$

From C2 (v) we obtain the desired commutative diagrams. The chain homotopy equivalences for the corresponding chain maps are obtained by adapting the diagrams of C4 and C3 to the relative case. □

Room service C

$\boxed{K_+}$ for a Δ-set K the pointed Δ-set with n-simplices $K^{(n)} \cup \{\emptyset\}$ $(n \geq 0)$.

$\boxed{\mathrm{Th}(\xi)}$ the Thom space of a vector bundle ξ, i.e. the quotient of disk and sphere bundle $D(\xi)/S(\xi)$. This agrees with the mapping cone of the projection map $S(\xi) \to X$ which gives rise to a general definition of the Thom space for a spherical fibration $\nu\colon E \to X$ as $\mathrm{Th}(\nu) := \mathcal{C}(\nu)$.

$\boxed{(Y, \nu, \rho)}$ an n-dimensional normal space consisting of a topological space Y together with an oriented k-dimensional spherical fibration $\nu\colon Y \to \mathrm{BSG}(k)$ and a map $\rho\colon S^{n+k} \to \mathrm{Th}(\nu)$.

$\boxed{((Y, B), \nu, (\rho_Y, \rho_B))}$ an $(n+1)$-dimensional geometric normal pair consisting of

- a finite CW pair (Y, B) with
- an oriented $(k-1)$-spherical fibration $\nu\colon Y \to \mathrm{BSG}(k)$
- a map $(\rho_Y, \rho_B)\colon (D^{n+k}, S^{n+k+1}) \to (\mathrm{Th}(\nu), \mathrm{Th}(\nu|_B))$
- and $(B, \nu|_B, \rho_B)$ an $(n-1)$-dimensional normal space.

$\boxed{\widehat{f} := (\overline{f}, f)\colon M \to X}$ an n-dimensional degree one normal map, i.e. a commutative square

$$\begin{array}{ccc} \nu_M & \xrightarrow{\overline{f}} & \eta \\ \downarrow & & \downarrow \\ M & \xrightarrow{f} & X \end{array}$$

with $f\colon M \to X$ a map from an n-dimensional manifold M to an n-dimensional Poincaré space X such that $f_*([M]) = [X] \in H_n(X)$, and $\overline{f}\colon \nu_M \to \nu_X$ stable bundle map from the stable normal bundle $\nu_M\colon M \to \mathrm{BSTOP}$ to a stable bundle $\nu_X\colon X \to \mathrm{BSTOP}$.

$\boxed{\widehat{g}}$ a degree one normal map $(\overline{g}, g)\colon X \to Y$ between an n-dimensional Poincaré space X and a normal space Y.

$\boxed{\widetilde{\Delta}\colon \mathrm{Th}(\nu) \simeq \frac{V}{\partial V} \xrightarrow{\Delta} \frac{V \times V}{V \times \partial V} \simeq \mathrm{Th}(\nu) \wedge X_+}$ the generalized diagonal map where V is the mapping cylinder of the projection map of ν and ∂V the total space of ν.

$\boxed{\Sigma_X\colon C(X) \to \Sigma^{-1} C(\Sigma X)}$ the natural suspension chain equivalence obtained from acyclic models.

$\boxed{\alpha\backslash -}$ denotes for a map $\alpha\colon S^N \to X \wedge Y$ the two geometric slant products $[X, Z] \to [S^N, Z \wedge Y]; \gamma \mapsto (\gamma \wedge \mathrm{id}) \circ \alpha)$ and $[Y, Z] \to [S^N, X \wedge Z]; \gamma \mapsto (\mathrm{id} \wedge \gamma) \circ \alpha)$ (see [Ran80b, §3]).

$\boxed{\varphi_f = \mathrm{ev}_l(\delta\varphi, \varphi) \simeq (\delta\varphi_0, f\varphi_0)\colon \mathscr{C}(f)^{n-*} \to D}$ a chain map defined for an n-symmetric pair $(f\colon C \to D, \delta\varphi, \varphi)$.

$\boxed{\varphi_{f^*} = \mathrm{ev}_r(\delta\varphi, \varphi) \simeq \begin{pmatrix} \delta\varphi_0 \\ \varphi_0 f^* \end{pmatrix} \colon D^{n-*} \to \mathscr{C}(f)}$ a chain map defined for an n-symmetric pair $(f\colon C \to D, \delta\varphi, \varphi)$.

Help desk

Contents

Tree of definitions

Geometric foundations

$\boxed{\mathcal{M}(f)}$ the mapping cylinder $(X \times [0,1] \amalg Y)/_{(x,1)\sim f(x)}$ for a map $f\colon X \to Y$.

$\boxed{\mathcal{C}(f)}$ the mapping cone $(X \times [0,1] \amalg Y)/_{(x,0)\sim(x',0),(x,1)\sim f(x)}$ for a map $f\colon X \to Y$.

$\boxed{S\colon [Y,Z] \xrightarrow{\cong} [S^N, X \wedge Y]}$ the S-duality isomorphism; for an N-dimensional S-duality map $\alpha\colon S^N \to X \wedge Y$ and an arbitrary space Z defined by $S(\gamma) = (\mathrm{id}_Y \wedge \gamma) \circ \alpha$; denotes the induced isomorphism $S\colon H^{N-*}(X;\mathbf{E}) \xrightarrow{\cong} H_*(X,\mathbf{E})$ as well.

$\boxed{\alpha\colon S^N \to X \wedge Y}$ an N-dimensional S-duality map, i.e. the slant product maps

$$\alpha_*([S^N])\backslash\cdot\colon \tilde{C}(X)^{N-*} \to \tilde{C}(Y) \quad \text{and} \quad \alpha_*([S^N])\backslash\cdot\colon \tilde{C}(Y)^{N-*} \to \tilde{C}(X)$$

are chain equivalences.

$\mathrm{Th}(\xi)$ the Thom space of a vector bundle ξ, i.e. the quotient of disk and sphere bundle $D(\xi)/S(\xi)$. This agrees with the mapping cone of the projection map $S(\xi) \to X$ which gives rise to a general definition of the Thom space for a spherical fibration $\nu\colon E \to X$ as $\mathrm{Th}(\nu) := \mathcal{C}(\nu)$.

⇒ **$u(\nu) \in H^k(\mathrm{Th}(\nu))$** the Thom class of an oriented $(k-1)$-dimensional spherical fibration $\nu\colon X \to \mathrm{BSG}(k)$ resp. an oriented k-plane bundle $\nu\colon X \to \mathrm{BSO}(k)$ that defines isomorphisms via the cup and cap products $u(\nu)\cap\cdot\colon H_*(\mathrm{Th}(\nu)) \to H_{*-k}(X)$ and $u(\nu)\cup\cdot\colon H^* \to H^{*+k}(\mathrm{Th}(\nu))$.

⇒ **$u(\nu)^* \in H^{N-k}(\mathrm{Th}(\nu)^*)$** the N-dimensional S-dual of a Thom class.

X an n-dimensional Poincaré space, i.e. a finite CW complex together with an orientation homomorphism $w\colon \pi_1(X) \to \{\pm 1\}$ and a fundamental class $[X]$.

$[X]$ fundamental class for an n-dimensional Poincaré space X is a cycle in the cellular $\mathbb{Z}\pi$-chain complex $C_n(\widetilde{X})$ which represents an n-dimensional homology class in $H_n(X;\mathbb{Z}^w)$ such that $\cdot \cap [X]\colon C^{n-*}(\widetilde{X}) \to C_*(\widetilde{X})$ is a $\mathbb{Z}\pi$-chain homotopy equivalence where $\widetilde{X}$ is the universal covering.

$\mathbb{Z}^w$ denotes the $\mathbb{Z}\pi$-module whose underlying abelian group is $\mathbb{Z}$ and $g \in \pi$ acts on it by $w(g)\cdot \mathrm{id}$.

(Y,ν,ρ) an n-dimensional normal space consisting of a topological space Y together with an oriented k-dimensional spherical fibration $\nu\colon Y \to \mathrm{BSG}(k)$ and a map $\rho\colon S^{n+k} \to \mathrm{Th}(\nu)$.

$[Y]$ fundamental class for a normal space (Y,ν,ρ) is a cycle in $C_n(\widetilde{Y})$ which represents an n-dimensional homology class in $H_n(Y;\mathbb{Z}^w)$ given by $[Y] = u(\nu) \cap h(\rho)$ where h is the Hurewicz homomorphism. If Y is a Poincaré space, then the term fundamental class implies that ν and ρ have been chosen in such a way that $\cdot\cap[Y]\colon H^{n-*}(Y) \to H_*(Y)$ is an isomorphism.

$(X,\partial X)$ an n-dimensional geometric Poincaré pair $(X,\partial X)$ is a finite CW pair together with an orientation homomorphism $w\colon \pi_1(X) \to \{\pm 1\}$ and a fundamental class $[X] \in H_n(X,\partial X;\mathbb{Z}^w)$ such that the cap products $-\cap[X]\colon H^*(\widetilde{X}) \to H_{n-*}(\widetilde{X},\partial\widetilde{X})$ are isomorphisms and ∂X is an $(n-1)$-dimensional Poincaré space.

$((Y,B),\nu,(\rho_Y,\rho_B))$ an $(n+1)$-dimensional geometric normal pair consisting of

- a finite CW pair (Y,B) with
- an oriented $(k-1)$-spherical fibration $\nu\colon Y \to \mathrm{BSG}(k)$
- a map $(\rho_Y,\rho_B)\colon (D^{n+k}, S^{n+k+1}) \to (\mathrm{Th}(\nu), \mathrm{Th}(\nu|_B))$
- and $(B,\nu|_B,\rho_B)$ an $(n-1)$-dimensional normal space.

$\boxed{Cobordism}$ of normal spaces:

$(Y, \nu, \rho) \sim (Y', \nu', \rho') \iff$ there is a normal pair $(W, Y \amalg Y', \nu_W, \rho_W)$ such that $\nu_W|Y = \nu$, $\nu_W|Y' = \nu'$ and the following compatibility with the collapse maps ρ and ρ' is satisfied:

Let Y^c be an open collar of the boundary Y in W and $c\colon \mathrm{Th}(\nu_W)/\mathrm{Th}(\nu) \to \mathrm{Th}(\nu_W)/(\mathrm{Th}(\nu) \cup (\mathrm{Th}(\nu_W) - Y^c) \simeq S^1 \wedge \mathrm{Th}(\nu)$ the projection which collapses everything outside of Y^c. Then we require $c \circ \rho_W = \mathrm{id} \wedge \rho\colon S^1 \wedge S^k \xrightarrow{\simeq} S^{k+1} \to \mathrm{Th}(\nu_W)/\mathrm{Th}(\nu) \to S^1 \wedge \mathrm{Th}(\nu)$ and analogously for Y' and ρ'.

$\boxed{X^*}$ the S-dual of X.

$\boxed{\widetilde{X}}$ the universal covering of X.

$\boxed{\widehat{f} := (\overline{f}, f)\colon M \to X}$ an n-dimensional degree one normal map, i.e. a commutative square

$$\begin{array}{ccc} \nu_M & \xrightarrow{\overline{f}} & \eta \\ \downarrow & & \downarrow \\ M & \xrightarrow{f} & X \end{array}$$

with $f\colon M \to X$ a map from an n-dimensional manifold M to an n-dimensional Poincaré space X such that $f_*([M]) = [X] \in H_n(X)$, and $\overline{f}\colon \nu_M \to \nu_X$ stable bundle map from the stable normal bundle $\nu_M\colon M \to \mathrm{BSTOP}$ to a stable bundle $\nu_X\colon X \to \mathrm{BSTOP}$.

$\boxed{\deg(f)}$ the *degree* of a map $f\colon M \to X$ of connected n-dimensional geometric Poincaré complexes; defined as the integer satisfying $f_*([M]) = \deg(f)[X] \in H_n(X; \mathbb{Z})$

$\Rightarrow$ $\boxed{(\delta\widehat{f}, \widehat{f})\colon (M, A) \to (X, B)}$ a degree one normal map from a manifold with boundary to a Poincaré pair (X, B) both of dimension $(n+1)$.

$\boxed{\widehat{g}}$ a degree one normal map $(\overline{g}, g)\colon X \to Y$ between an n-dimensional Poincaré space X and a normal space Y.

$\Rightarrow$ $\boxed{(\delta\widehat{g}, \widehat{g})\colon (X, A) \to (Y, B)}$ a degree one normal map from a Poincaré pair (X, A) to a normal pair (Y, B) both of dimension $(n+1)$.

$\boxed{\nu_X\colon X \to \mathrm{BSG}}$ the Spivak normal fibration of X, i.e. an oriented $(k-1)$-spherical fibration of an n-dimensional Poincaré space X for which a class $\alpha \in \pi_{n+k}(\mathrm{Th}(\nu_X))$ $(k > n+1)$ exists such that $h(\alpha) \cap u = [X]$. Here $u \in H^k(\mathrm{Th}(\nu_X))$ is the Thom class and $h\colon \pi_*(\cdot) \to H_*(\cdot)$ is the Hurewicz map.

$\boxed{\varepsilon}$ the trivial spherical fibration over X.

$\boxed{\overline{\nu}_X}$ the topological bundle lift of the Spivak normal fibration ν_X. If X is a manifold $\overline{\nu}_X$ is the stable normal bundle of X.

$\boxed{\widetilde{\Gamma}\colon \mathrm{G/TOP} \to \Sigma^{-1}\mathbf{\Omega}_0^{\mathrm{N,STOP}}}$ associates to an l-simplex $\widehat{f}\colon M \to \Delta^l$ in $\mathrm{G/TOP}$ an l-simplex of $\Sigma^{-1}\mathbf{\Omega}_\bullet^{\mathrm{N,STOP}}$ which is an $(l+1)$-dimensional l-ad of (normal, topological manifold) pairs $(\mathscr{M}(f), M \amalg -\Delta^l)$ where the normal structure comes from the bundle map $\overline{f}$.

$\widetilde{\Delta}\colon \mathrm{Th}(\nu) \simeq \frac{V}{\partial V} \xrightarrow{\Delta} \frac{V\times V}{V\times \partial V} \simeq \mathrm{Th}(\nu)\wedge X_+$ the generalized diagonal map where V is the mapping cylinder of the projection map of ν and ∂V the total space of ν.

Constructions from the classical surgery theory

$\mathscr{N}(X)$ the normal invariants of a geometric Poincaré complex X. An element of $\mathscr{N}(X)$ can be represented in two different ways which are identified via the Pontrjagin-Thom construction:

- by a degree one normal map $(f,b)\colon M \to X$ from a manifold M to X or
- by a pair (ν, h) where $\nu\colon X \to \mathrm{BSTOP}$ is a stable topological bundle on X and $h\colon J(\nu) \simeq \nu_X$ is a homotopy from the underlying spherical fibration of ν to the Spivak normal fibration ν_X of X.

$\mathscr{S}(X)$ the structure set, i.e. the set of equivalence classes of homotopy equivalences $f\colon M \to X$ from closed manifolds to X where two maps $f_0\colon M_0 \to X$, $f_1\colon M_1 \to X$ are equivalent if there exists a cobordism (W, M_0, M_1) together with a homotopy equivalences $F\colon W \to X\times[0,1]$ such that $F|_{M_0} = f_0\colon M_0 \to X\times\{0\}$ and $F|_{M_1} = f_1\colon M_1 \to X\times\{1\}$.

$\theta(\widehat{f})$ Wall's surgery obstruction for a degree one normal map $\widehat{f}\colon M \to X$. It is an element in $L^w_n(\mathbb{Z}[\pi_1(X)])$ and if $n \geq 5$ it vanishes if and only if $\widehat{f}$ is cobordant to a homotopy equivalence $\widehat{f}'\colon M' \to X$.

$L^w_n(R)$ the Wall surgery groups of quadratic forms for n even resp. of formations for n odd where R is an associative ring with unit and involution.

$L^w_{2k}(R)$ the 2k-dimensional Wall surgery group of quadratic forms is defined as the abelian group of equivalence classes $[(F,\varphi)]$ of non-degenerate $(-1)^k$-quadratic forms (F,φ) such that F is a finitely generated free R-module. Two such forms (F,φ) and (F',φ') are equivalent if they are isomorphic up to stabilization with hyperbolic forms, i.e. there are integer $u, u' \geq 0$ such that

$$(F,\varphi)\oplus H_\varepsilon(R)^u \cong (F',u')\oplus H_\varepsilon(R)^{u'}.$$

Addition is given by the addition of quadratic forms.

$L^w_{2k+1}(R)$ the (2k+1)-dimensional Wall surgery group of quadratic formations is defined as the abelian group of equivalence classes $[(P,\varphi;F,G)]$ of $(-1)^k$-quadratic formations $(P,\varphi;F,G)$ such that P, F, and G are finitely generated free R-modules. Two such formations $(P,\varphi;F,G)$ and $(P',\varphi';F',G')$ are equivalent if there exists $(-1)^{k+1}$-quadratic forms (Q,u) and (Q',u') with finitely generated free R-modules Q and Q' and finitely generated free R-modules S and S' such that

$$(P,\varphi;F,G)\oplus\partial(Q,u)\oplus(H_\varepsilon(S);S,S^*) \cong (P',\varphi';F',G')\oplus\partial(Q',u')\oplus(H_\varepsilon(S');S',(S')^*).$$

Addition is given by the sum of quadratic formations.

$H_\varepsilon(P)$ the standard hyperbolic ε-quadratic form for a finitely generated projective R-module P is given by the R-module $P \oplus P^*$ and the R-homomorphism

$$\phi\colon (P \oplus P^*) \xrightarrow{\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}} P^* \oplus P \xrightarrow{\mathrm{id} \oplus e(P)} P^*(P^*)^* = (P \oplus P^*)^*.$$

$t(-, \widehat{f}_0)\colon \mathscr{N}(X) \to [X; \mathrm{G/TOP}]$ the inverse of the bijection $[X; \mathrm{G/TOP}] \cong \mathscr{N}(X)$ induced by
224→p.94 the action of $[X; \mathrm{G/TOP}]$ on $\mathscr{N}(X)$ (see 224)

$[X; \mathrm{G/TOP}]$ homotopy class of maps from X to $\mathrm{G/TOP}$.

Algebraic foundations

C a chain complex; either an element in $\mathbb{B}(R)$ or, more generally, in $\mathbb{B}(\mathbb{A})$.

$\mathbb{B}(R)$ the category of bounded chain complexes of finitely generated projective left R-modules.

R a ring with involution $^-\colon R \to R; r \mapsto \overline{r}$, i.e. it satisfies $\overline{1} = 1, \overline{\overline{r}} = r, \overline{rs} = \overline{s}\,\overline{r}$ and $\overline{r+s} = \overline{r} + \overline{s}$ for $r, s \in R$.

$\mathbb{B}(\mathbb{A})$ the category of bounded chain complexes in $\mathbb{A}$.

$\mathbb{A}$ additive category

$\tilde{C}$ the reduced chain complex of a nonnegative chain complex C is defined by $\tilde{C}_q = C_q$ for $q \neq 0$ and $\tilde{C}_0 = \ker(\varepsilon)$. For a singular chain complex $\Delta(X)$ the augmentation $\varepsilon\colon \Delta_0(X) \to \mathbb{Z}$ is defined by $\varepsilon(\sigma) = 1$ for every singular 0-simplex σ; resp. for a simplicial chain complex $C_0(K)$ by $\varepsilon(v) = 1$ for every vertex v of K.

C^{-*} the dual chain complex with $(C^{-*})_k := (C_{-k})^*$ and differential $d_k^{C^{-*}} := (-)^k(d_k^C)^*$.

ΣC the suspended chain complex C shifted one to the left, i.e. $\Sigma C_n = C_{n-1}, d^{\Sigma C} = -d^C$.

$\Sigma^{-1}C$ the desuspended chain complex C shifted one to the right, i.e. $\Sigma^{-1}C_n = C_{n+1}$.

$\mathscr{M}(f)$ the algebraic mapping cylinder with $\mathscr{M}(f)_k := D_k \oplus C_k \oplus C_{k-1}$ and differential $d^{\mathscr{M}(f)}(x, y, z) := (d^D(x) + f(y), d^C(y) - z, -d^C(z))$ for a chain map $f\colon C \to D$.

$\mathscr{C}(f)$ the algebraic mapping cone with $\mathscr{C}(f)_k := D_k \oplus C_{k-1}$ and differential $d^{\mathscr{C}(f)}(x, y) := (d^D(x) + f(y), -d^C(y))$ for a chain map $f\colon C \to D$.

$\boxed{(T,e)}$ chain duality on $\mathbb{A}$ where $T\colon \mathbb{A}^{op} \to \mathbb{B}(\mathbb{A})$ is a functor and $e\colon T^2 \to \mathbb{1}$ a natural transformation with $e_A : T^2A \xrightarrow{\cong} A$ and $e_{T(A)} \circ T(e_A) = \mathbb{1}$. for $A \in \mathbb{A}$.

$\boxed{W^{\%}, W_{\%}, \widehat{W}^{\%}}$ denote for a chain complex C the abelian group chain complexes

$$W^{\%}C \coloneqq \mathrm{Hom}_{\mathbb{Z}[\mathbb{Z}_2]}(W, C \otimes C),$$
$$W_{\%}C \coloneqq W \otimes_{\mathbb{Z}[\mathbb{Z}_2]} (C \otimes C),$$
$$\widehat{W}^{\%}C \coloneqq \mathrm{Hom}_{\mathbb{Z}[\mathbb{Z}_2]}(\widehat{W}, C \otimes C).$$

$\boxed{W}$ the free resolution of the trivial $\mathbb{Z}[\mathbb{Z}_2]$-chain module $\mathbb{Z}$; given by the $\mathbb{Z}[\mathbb{Z}_2]$-chain complex $\ldots \xrightarrow{1+t} \mathbb{Z}[\mathbb{Z}_2] \xrightarrow{1-t} \mathbb{Z}[\mathbb{Z}_2] \longrightarrow 0$

$\boxed{\widehat{W}}$ the complete resolution of the trivial $\mathbb{Z}[\mathbb{Z}_2]$-chain module $\mathbb{Z}$; given by the $\mathbb{Z}[\mathbb{Z}_2]$-chain complex $\ldots \xrightarrow{1+t} \mathbb{Z}[\mathbb{Z}_2] \xrightarrow{1-t} \mathbb{Z}[\mathbb{Z}_2] \xrightarrow{1+t} \mathbb{Z}[\mathbb{Z}_2] \xrightarrow{1-t} \ldots$

$\boxed{C \otimes C}$ short for the chain complex of $\mathbb{Z}[\mathbb{Z}_2]$-modules $C^t \otimes_R C$, or, more generally, $C \otimes_{\mathbb{A}} C \coloneqq \mathrm{Hom}(T(C), C)$.

$\boxed{C^t}$ chain complex of right R-modules obtained from a chain complex C of left R-modules using the involution of R.

$\boxed{\mathrm{Hom}(C,D)}$ the Hom-complex for chain complexes C, D; defined by $\mathrm{Hom}(C,D)_n = \bigoplus_{q-p=n} \mathrm{Hom}(C_p, D_q)$ and $d(f) = d_D f - (-1)^n f d_C$.

$\Rightarrow \boxed{f^{\%} : W^{\%}(C) \to W^{\%}(D)}$ the chain map induced by a chain map $f : C \to D$; explicitly given by $(f^{\%}(\varphi))_s \coloneqq f \varphi_s f^* : D^{n+s-*} \to D$.

$\Rightarrow \boxed{f_{\%} : W_{\%}(C) \to W_{\%}(D)}$ the chain map induced by a chain map $f : C \to D$

$\Rightarrow \boxed{\widehat{f}^{\%} : \widehat{W}^{\%}(C) \to \widehat{W}^{\%}(C)}$ the chain map induced by a chain map $f : C \to D$

$\Rightarrow \boxed{Q^n, Q_n, \widehat{Q}^n}$ the n-dimensional Q-groups defined for a chain complex C by

$$Q(C)^n \coloneqq H_n(W^{\%}C),$$
$$Q(C)_n \coloneqq H_n(W_{\%}C),$$
$$\widehat{Q}(C)^n \coloneqq H_n(\widehat{W}^{\%}C).$$

$\boxed{W[r,s]}$ the $\mathbb{Z}[\mathbb{Z}_2]$-module chain complex

$$\cdots 0 \longrightarrow \mathbb{Z}[\mathbb{Z}_2] \xrightarrow{1+(-)^s t} \mathbb{Z}[\mathbb{Z}_2] \xrightarrow{1+(-)^{(s-1)}t} \ldots \xrightarrow{1+(-)^{(r+1)}t} \mathbb{Z}[\mathbb{Z}_2] \longrightarrow 0 \cdots$$

with $W[r,s]_n = 0$ for $n > s$ and $n < r$.

$\boxed{C(X)}$ the singular chain complex for a space X.

$\boxed{I}$ chain complex with

$$I_n = \begin{cases} \mathbb{Z} = \langle e \rangle & \text{for } n = 1, \\ \mathbb{Z} \times \mathbb{Z} = \langle e_1, e_2 \rangle & \text{for } n = 0 \\ 0 & \text{otherwise} \end{cases} ; \quad d_1 : \mathbb{Z} \to \mathbb{Z} \times \mathbb{Z}; e \mapsto e_1 - e_2$$

Chain complex maps

$\boxed{1 + t \colon W_\% C \to W^\% C}$ the symmetrization map defined by

$$(1 + t)(\psi)_s = \begin{cases} (1 + t)\psi_0 & \text{if } s = 0 \\ 0 & \text{otherwise} \end{cases}$$

induces a map of L-groups $1 + t \colon L_n(R) \to L^n(R)$.

$\boxed{t}$ the generator of $\mathbb{Z}_2$; acts on $C \otimes_R C$ via $t(x \otimes y) = (-)^{|x||y|} y \otimes x$ and on $C \otimes_{\mathbb{A}} C$ via $t(x \otimes y) = T_{C,C}(x \otimes y)$.

$\boxed{\mathcal{S} \colon W^\% C \to \Sigma^{-1} W^\% (\Sigma C)}$ the suspension map; defined by $(\mathcal{S}(\varphi))_k := \varphi_{k-1}$ if $k \geq 1$ and zero otherwise; induces a map $Q^n(C) \to Q^{n+1}(\Sigma C)$ and an isomorphism $\widehat{Q}^n(C) \xrightarrow{\cong} \widehat{Q}^{n+1}(\Sigma C)$.

$\boxed{\backslash \colon C \otimes D \to \mathrm{Hom}(C^*, D)}$ the slant chain map defined by $f \otimes (x \otimes y) \mapsto \overline{f(x).y}$.

$\boxed{e \colon C \to \mathscr{C}(\varphi_0)}$ the inclusion; with a map $\alpha \colon C \to D$ as subscript e_α denotes the inclusion $D \to \mathscr{C}(\alpha)$.

$\boxed{p \colon \mathscr{C}(\varphi_0) \to \Sigma C^{n-*}}$ the projection; with a map $\alpha \colon C \to D$ as subscript e_α denotes the projection $\mathscr{C}(\alpha) \to \Sigma C$.

$\boxed{\gamma_Y \colon C(\mathrm{Th}(\nu)^*)_{n+p} \to C(Y)}$ the chain map induced by the semi-stable map Γ_Y of an n-dimensional normal space (Y, ν, ρ) with $\mathrm{Th}(\nu)^*$ the N-dimensional S-dual of $\mathrm{Th}(\nu)$ and $p = N - (n + k)$.

$\boxed{f^! \colon C(\widetilde{X}) \to C(\widetilde{M})}$ the Umkehr map of a degree one normal map $\widehat{f} \colon M \to X$ of Poincaré spaces M and X. We obtain a stable equivariant map $F \colon \Sigma^k \widetilde{X}_+ \to \Sigma^k \widetilde{M}_+$ for some $k \in \mathbb{N}$ and define $f^!$ as the composition $C(\widetilde{X}) \xrightarrow{\Sigma_X} \Sigma^{-k} C(\Sigma^k \widetilde{X}_+) \xrightarrow{F} \Sigma^{-k} C(\Sigma^k \widetilde{M}_+) \xrightarrow{\Sigma_X^{-1}} C(\widetilde{M})$.

$\boxed{\zeta \colon C(X \times X) \to C(X) \otimes C(X)}$ Eilenberg-Zilber map

$\boxed{\mathrm{ev} \colon W^\% C \to C \otimes C}$ the evaluation map given by $\varphi \mapsto \varphi_0$.

$\boxed{\mathrm{ev}_l \colon \mathscr{C}(f^\%) \to \mathscr{C}(f) \otimes D}$ the left evaluation map for a chain map $f \colon C \to D$ given by $(\delta\varphi, \varphi) \mapsto \varphi_f$.

$\varphi_f = \mathrm{ev}_l(\delta\varphi, \varphi) \simeq (\delta\varphi_0, f\varphi_0) : \mathscr{C}(f)^{n-*} \to D$ a chain map defined for an n-symmetric pair $(f : C \to D, \delta\varphi, \varphi)$.

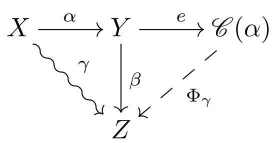
$l : \mathscr{C}(f \otimes f) \to \mathscr{C}(f) \otimes D$

$\mathrm{ev}_r : \mathscr{C}(f^{\%}) \to D \otimes \mathscr{C}(f)$ the right evaluation map for a chain map $f : C \to D$ given by $(\delta\varphi, \varphi) \mapsto \varphi_{f^*}$.

$\varphi_{f^*} = \mathrm{ev}_r(\delta\varphi, \varphi) \simeq \begin{pmatrix} \delta\varphi_0 \\ \varphi_0 f^* \end{pmatrix} : D^{n-*} \to \mathscr{C}(f)$ a chain map defined for an n-symmetric pair $(f : C \to D, \delta\varphi, \varphi)$.

$r : \mathscr{C}(f \otimes f) \to D \otimes \mathscr{C}(f)$

Φ_γ is the chain map induced by a null-homotopy $\gamma : \beta \circ \alpha \simeq 0$, fitting in the following diagram with



and given by $\mathscr{C}(\alpha) \longrightarrow Z; \quad (y, x) \mapsto \beta(y) + \gamma(x)$.

$\partial_\sigma : C(K) \to \Sigma^{|\sigma|} C(\sigma)$ chain map defined for each simplex $\sigma = \langle v_0, v_1, \ldots, v_{|\sigma|} \rangle$ in K by the composition

$$C(K) = \sum_{\tau \in K} C(\tau)_n \xrightarrow{\text{proj.}} C(\sigma_0)_n \xrightarrow{d_1} C(\sigma_1)_{n_1} \xrightarrow{d_2} \ldots \xrightarrow{d_{|\sigma|}} C(\sigma)_{n-\sigma}$$

with $\sigma_j = \langle v_0, \ldots, v_j \rangle$ and $d_j = d^{\sigma_j, \sigma_{j+1}}_{n-j+1}$ the relevant component of $d^{C(K)}_{n-j+1} : C(K)_{n-j+1} \to C(K)_{n-j}$ (see [Ran92, Def. 8.2]).

$\langle v_0, v_1, \ldots, v_j \rangle$ defines a simplex spanned by the vertices $v_0, \ldots, v_j$.

$\mathrm{th} : \mathscr{C}(f^{\%}) \to W^{\%}(\mathscr{C}(f))$ the map from the algebraic Thom construction.

$\Sigma_X : C(X) \to \Sigma^{-1} C(\Sigma X)$ the natural suspension chain equivalence obtained from acyclic models.

$\Sigma_X^{-1} : \Sigma^{-1} C(\Sigma X) \to C(X)$ the desuspension chain equivalence; note that it is not natural in X.

Structured chain complexes

(C,φ) an n-symmetric chain complex consisting of a chain complex C and an n-symmetric structure φ;
Poincaré means $\varphi_0 : C^{n-*} \to C$ is a chain equivalence.

- $\varphi \in W^{\%}(C)_n$ a cycle, called an n-symmetric structure
- φ_0 the 0-component of an n-symmetric structure $\varphi = \{\varphi_s : C^{n+s-*} \to C_* \mid s \geq 0\} \in W^{\%}C$. If φ was obtained by applying the symmetric construction con_X^{φ} to a class $[X]$ then $\varphi_0 = -\cap[X]$.

(C,ψ) an n-quadratic chain complex consisting of a chain complex C and an n-quadratic structure ψ;
Poincaré means the symmetrization $(1+t)(\psi)_0 : C^{n-*} \to C$ is a chain equivalence.

- $\psi \in W_{\%}(C)_n$ a cycle, called an n-quadratic structure on C; can be represented by a set $\{\varphi_s : C^{n-s-*} \to C_* \mid s \geq 0\}$.

$(C,(\varphi,\gamma,\chi))$ an n-normal chain complex consisting of a chain complex C and an n-normal structure (φ,γ,χ).

- (φ,γ,χ) an n-normal structure
  - $\varphi \in W^{\%}(C)_n$ a cycle, called an n-symmetric structure.
  - $\gamma \in \widehat{W}^{\%}(C^{-*})_0$ a cycle called chain bundle
  - $\chi \in \widehat{W}^{\%}(C)_{n+1}$ a chain satisfying $d\chi = J(\varphi) - \widehat{\varphi}^{\%}(\mathcal{S}^n\gamma)$.

$L^n(R)$ the cobordism group of n-symmetric Poincaré chain complexes over R.

$L_n(R)$ the cobordism group of n-quadratic Poincaré chain complexes over R.

$NL^n(R)$ the cobordism group of n-normal chain complexes over R.

- $Cobordism$ of n-dimensional structured chain complexes:

  $$(C,\lambda) \sim (C',\lambda') \iff \text{there is an } (n+1)\text{-dimensional structured (Poincaré) pair } (C \oplus C' \to D, \partial\lambda, \lambda \oplus -\lambda'),$$

  where Poincaré is only required in the symmetric and quadratic case.

  - λ placeholder for a symmetric φ resp. quadratic ψ or normal structure (φ,γ,χ). We call a pair (C,λ) a *structured chain complex* and a triple $(C \to D, \delta\lambda, \lambda)$ a *structured pair*.

$(f\colon C\to D, \delta\varphi, \varphi)$ an $(n+1)$-symmetric pair with
- $f\colon C\to D$ a chain map
- (C,φ) an n-symmetric chain complex
- $\delta\varphi \in W^{\%}(D)_{n+1}$ such that $d(\delta\varphi) = f^{\%}(\varphi)$ which is equivalent to $(\delta\varphi,\varphi)$ is a cycle in $\mathscr{C}(f^{\%})_{n+1}$.

Poincaré means $(\delta\varphi_0, \varphi_0 f^*)\colon D^{n+1-*} \to \mathscr{C}(f)_*$ is a chain equivalence.

$(f\colon C\to D, \delta\psi, \psi)$ an $(n+1)$-quadratic pair with
- $f\colon C\to D$ a chain map
- (C,ψ) an n-quadratic chain complex
- $\delta\psi \in W_{\%}(D)_{n+1}$ such that $d(\delta\psi) = f_{\%}(\psi)$.

Poincaré means the symmetrization is Poincaré, i.e.
$((1+t)\delta\varphi_0, (1+t)\varphi_0 f^*)\colon D^{n+1-*} \to \mathscr{C}(f)_* = (D_*, C_{*-1})$ is a chain equivalence.

$(f\colon C\to D, (\delta\varphi,\delta\gamma,\delta\chi), (\varphi,\gamma,\chi))$ an $(n+1)$-normal pair with
- $(C,(\varphi,\gamma,\chi))$ an n-normal chain complex
- $(f\colon C\to D, \delta\varphi, \varphi)$ an $(n+1)$-symmetric pair
- $(f,b)\colon (C,\gamma)\to(D,\delta\gamma)$ a map of chain bundles
- $\delta\chi \in \widehat{W}^{\%}(D)_{n+2}$ a chain such that

  $J(\delta\varphi) - \widehat{\delta\varphi_0}^{\%}(\mathcal{S}^{n+1}\delta\gamma) + \widehat{f}^{\%}(\chi - \widehat{\varphi_0}^{\%}(\mathcal{S}^n b)) = d(\delta\chi) \in \widehat{W}^{\%}(D)_{n+1}$.

$(f,b)\colon (C,\gamma)\to(C',\gamma')$ a map of chain bundles where $f\colon C\to C'$ a chain map and $b\in \widehat{W}^{\%}(C^*)_1$ a 1-chain such that $\widehat{f}^{\%}(\gamma) - \gamma' = d(b)$.

$L(J)^n$ the relative L-group of $J\colon L^n(R) \to NL^n(R)$ is the cobordism group of (normal, symmetric Poincaré) pairs $(f\colon C\to D, (\delta\varphi,\delta\gamma,\delta\chi),\varphi)$.

$J\colon L^n(R)\to NL^n(R)$ roughly induced by $j\colon W^{\%}C \to \widehat{W}^{\%}C$; see (111) for more details of how a normal structure (φ,γ,χ) is obtained from a symmetric Poincaré chain complex (C,φ).

$j\colon W^{\%}C\to\widehat{W}^{\%}C$ induced by the projection $\widehat{W} \to W$, induces a map of Q-groups $j\colon Q^n(R)\to\widehat{Q}^n(R)$.

$(f\colon C\to D, (\delta\varphi,\delta\gamma,\delta\chi),\varphi)$ an $(n+1)$-dimensional (normal, symmetric Poincaré) pair with
- $(D,\delta\varphi,\delta\gamma,\delta\chi)$ an $(n+1)$-normal chain complex
- $(f\colon C\to D, (\delta\varphi,\varphi))$ an $(n+1)$-symmetric pair
- φ_0 a chain equivalence.

└ $\boxed{Cobordism}$ of n-dimensional structured pairs:

$(C \xrightarrow{f} D, \delta\lambda, \lambda) \sim (C' \xrightarrow{f'} D', \delta\lambda', \lambda') \iff$ there is an $(n+1)$-dimensional structured (Poincaré) triad

$$\begin{array}{ccc} (C \oplus C', \lambda - \lambda') & \xrightarrow{\begin{pmatrix} f & 0 \\ 0 & f' \end{pmatrix}} & (D \oplus D', \delta\lambda - \delta\lambda') \\ \downarrow \widetilde{f} & \overset{h}{\rightsquigarrow} & \downarrow g \\ (\widetilde{C}, \delta\widetilde{\lambda}) & \xrightarrow[g']{} & (E, \delta^2\lambda) \end{array}$$

where Poincaré is only required in the symmetric and quadratic case.

└ $\boxed{\Gamma = (f, f', g, g'; h, (\varphi, \varphi', \delta\varphi, \delta\varphi'; \delta^2\varphi))}$ an $(n+2)$-dimensional symmetric triad, i.e. a commutative square of chain complexes and chain maps

$$\begin{array}{ccc} C & \xrightarrow{f} & D \\ \downarrow f' & \overset{h}{\rightsquigarrow} & \downarrow g \\ D' & \xrightarrow[g']{} & E \end{array}$$

with

- $h\colon gf \simeq g'f'\colon C \to E_{*+1}$ a chain homotopy
- $(f\colon C \to D, \delta\varphi, \varphi)$ and $(f'\colon C \to D', \delta\varphi', \varphi)$ being $(n+1)$-symmetric pairs
- $\delta^2\varphi \in W^{\%}(E)_{n+2}$ a chain such that
  $d(\delta^2\varphi)_s = g'^{\%}(\delta\varphi')_s - g^{\%}(\delta\varphi)_s + g'f'\varphi_s h^* + h\varphi_s f^* g^* + h^{\%}(\varphi)_s$.

Poincaré means $(f\colon C \to D, \delta\varphi, \varphi)$ and $(f'\colon C \to D', \delta\varphi', \varphi)$ are Poincaré and
$\begin{pmatrix} \delta^2\varphi_0 \\ \delta\varphi_0 g^* \\ f'\varphi_0 h^* + \delta\varphi_0' g'^* \\ \varphi_0 f^* g^* \end{pmatrix} : E^{n+2-*} \to \mathscr{C}(\Gamma) := \mathscr{C}\left(\begin{pmatrix} g' & h \\ 0 & f \end{pmatrix} : \mathscr{C}(f') \to \mathscr{C}(g)\right)$ is a chain equivalence.
The quadratic case is analog but uses symmetrization for the definition of Poincaré.

$\boxed{(f, \chi)\colon (C, \varphi) \to (C', \varphi')}$ a morphism of n-symmetric chain complexes consisting of a chain map $f\colon C \to C'$ together with a chain $\chi \in (W^{\%}(C'))_{n+1}$ such that $\varphi' - f(\varphi) = d\chi$.

$\boxed{(f\colon C \to D, \delta\varphi, \varphi) \simeq (f'\colon C' \to D', \delta\varphi', \varphi')}$ two symmetric pairs are homotopy equivalent if there is a triad $\Gamma = (f, f', g, g'; h)$ such that g and h are homotopy equivalences and $(g, g'; h)^{\%}(\delta\varphi, \varphi) = (\delta\varphi', \varphi')$

$\boxed{(g, g'; h)^{\%}\colon W^{\%}(\mathrm{Hom}_A(D^*, D)_* \oplus \mathrm{Hom}_A(C^*, C)_{*-1}) \to W^{\%}(\mathrm{Hom}_A(D'^*, D')_* \oplus \mathrm{Hom}_A(C'^*, C')_{*-1})}$ given by $(\delta\varphi_s, \varphi_s) \mapsto (g^{\%}(\delta\varphi_s) \pm h\varphi_s f^* g^* \pm f'g'\varphi_s h^* \pm h^{\%}(T\varphi_{s-1}), f'^{\%}(\varphi_s))$ (see [Ran81, p.42])

$\boxed{\partial\!\!\!\!\mathscr{C}(C \xrightarrow{f} D, \theta, \varphi) = (\Sigma^{-1}\mathscr{C}(\varphi_{f^*}), \varphi')}$ an n-symmetric chain complex, the effect of algebraic surgery on an $(n+1)$-symmetric pair $(f\colon C \to D, \delta\varphi, \varphi)$ (see B1).

Boundary constructions

(C', φ') symmetric chain complex obtained from algebraic surgery (see B1

$\varphi_{f^*} = \mathrm{ev}_r(\delta\varphi, \varphi) \simeq \begin{pmatrix} \delta\varphi_0 \\ \varphi_0 f^* \end{pmatrix} : D^{n-*} \to \mathscr{C}(f)$ a chain map defined for an n-symmetric pair $(f : C \to D, \delta\varphi, \varphi)$.

$\varphi_f = \mathrm{ev}_l(\delta\varphi, \varphi) \simeq (\delta\varphi_0, f\varphi_0) : \mathscr{C}(f)^{n-*} \to D$ a chain map defined for an n-symmetric pair $(f : C \to D, \delta\varphi, \varphi)$.

∂^S the symmetric boundary construction, produces for an n-symmetric chain complex an $(n-1)$-symmetric Poincaré chain complex usually denoted $(\partial C, \partial\varphi)$ (see B21).

- $(\partial C, \partial\varphi)$ the symmetric boundary of an n-symmetric chain complex obtained from algebraic surgery on the pair $(0 \to C, \varphi, 0)$, i.e. $\partial C = \Sigma^{-1}\mathscr{C}(\varphi_0)$, $\partial\varphi = \mathcal{S}^{-1}e^{\%}(\varphi)$ where $e : C \to \mathscr{C}(\varphi_0)$ is the inclusion (see B21 for more details).
  - $\partial C := \Sigma^{-1}\mathscr{C}(\varphi_0)$ the boundary chain complex.
  - $\partial\varphi := \mathcal{S}^{-1}e^{\%}(\varphi)$ the symmetric boundary structure.
    - $e : C \to \mathscr{C}(\varphi_0)$ the inclusion; with a map $\alpha : C \to D$ as subscript e_α denotes the inclusion $D \to \mathscr{C}(\alpha)$.

$\Rightarrow$ $\partial^S_{\to}$ the relative symmetric boundary construction, produces for an n-symmetric pair an $(n-1)$-symmetric Poincaré pair usually denoted $(\partial f : \partial C \to \partial_f D, \partial(\delta\varphi, \varphi))$.

- $(\partial f : \partial C \to \partial_{f^*} D, \partial_{f^*}\delta\varphi, \partial\varphi)$ the symmetric boundary of an $(n+1)$-symmetric pair $(f : C \to D, \delta\varphi, \varphi)$ not necessarily Poincaré. It is an n-symmetric Poincaré pair with
  - $(\partial C, \partial\varphi)$ the symmetric boundary of (C, φ)
  - $\partial_{f^*} D = \mathscr{C}(\begin{pmatrix} \delta\varphi_0 \\ \varphi_0 f^* \end{pmatrix} : D^{n+1-*} \to \mathscr{C}(f))$
  - $\partial f = \begin{pmatrix} f & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix} : \partial C = C_{r+1} \oplus C^{n-r-1} \to D_{r+1} \oplus D^{n-r} \oplus C^{n-r-1} = \partial_{f^*} D.$

∂^Q the quadratic boundary construction, produces for an n-quadratic chain complex $(C^!, \psi^!)$ an $(n-1)$-quadratic Poincaré chain complex denoted by $(\partial C^!, \partial\psi^!)$ (see B23).

- $(\partial C, \partial\psi)$ is the quadratic boundary of an n-quadratic chain complex, (see B23).
  - $\partial C := \Sigma^{-1}\mathscr{C}((1+t)(\varphi)_0)$ the boundary chain complex.

└ $\partial\psi := \binom{1+t}{\mathbb{S}}^{-1}(\mathbb{S}^{-1}e^{\%}(\varphi), e_{\%}(\psi))$ the quadratic boundary structure.

⇒ $\partial^Q_{\to}$ the relative quadratic boundary construction, produces for an n-quadratic pair an $(n-1)$-quadratic Poincaré pair usually denoted $(\partial f\colon \partial C \to \partial_f D, \partial(\delta\psi,\psi))$.

⇒ $\partial^Q_{\mathbb{G}}\colon L_n(\mathbb{B}\langle 1\rangle, \mathbb{G}\langle 1\rangle) \to L_{n-1}(\mathbb{G}\langle 1\rangle, \mathbb{L}\langle 1\rangle) = \mathbb{S}_n(X)$ the boundary map induced by the boundary construction ∂^Q.

⇒ $\partial^Q_{\mathbb{Z}\pi}\colon L_n(\mathbb{Z}\pi) \to \mathbb{S}_n(X)$ the boundary map obtained from $\partial^Q_{\mathbb{G}}$ by using the assembly isomorphism $L_n(\mathbb{B}\langle 1\rangle, \mathbb{G}\langle 1\rangle) \cong L_n(\mathbb{Z}\pi)$ (see 1221).

∂^N the quadratic boundary construction, produces for an n-normal chain complex $(C, (\varphi, \gamma, \chi))$ an $(n-1)$-quadratic Poincaré chain complex denoted $(\partial C, \partial\psi)$; induces a map $NL^n(R) \to L_{n-1}(R)$ (see B24).

⇒ $\partial^N_{\to}$ the relative quadratic boundary construction, produces for an n-normal pair an $(n-1)$-quadratic Poincaré pair usually denoted $(\partial f\colon \partial C \to \partial_f D, \partial(\delta\psi,\psi))$.

⇒ $\partial^N_{\mathbb{B}}\colon NL^n(\mathbb{B}\langle 0\rangle, \mathbb{B}\langle 1\rangle) = H_n(X; \mathbf{NL}^\bullet\langle 1/2\rangle) \to L_{n-1}(\mathbb{B}\langle 1\rangle, \mathbb{L}\langle 1\rangle) = H_{n-1}(X; \mathbf{L}_\bullet\langle 1\rangle)$ the boundary map induced by the boundary construction ∂^N.

⇒ $\partial^N_{\mathbb{G}}\colon NL^n(\mathbb{B}\langle 0\rangle, \mathbb{G}\langle 1\rangle) = VL^n(X) \to L_{n-1}(\mathbb{G}\langle 1\rangle, \mathbb{L}\langle 1\rangle) = \mathbb{S}_n(X)$ the boundary map induced by the boundary construction ∂^N.

$\partial\mathrm{gn}^{\mathbf{L}_\bullet}_{\mathbb{Z}\pi}(Y)$ the quadratic boundary signature for an n-dimensional normal space (Y, ν, ρ) is
B27→p.155 an $(n-1)$-quadratic chain complex $(\partial C', \partial\psi')$ in $L_{n-1}(\mathbb{Z}\pi)$ (see B27).

├ $\partial C' = \Sigma^{-1}\mathscr{C}(\gamma_Y)$ the chain complex of the quadratic boundary signature $\partial\mathrm{gn}^{\mathbf{L}_\bullet}_{\mathbb{Z}\pi}$.

├ $\partial\psi' = \binom{1+t}{\mathbb{S}}^{-1}(\partial\varphi, \mathrm{con}^{\psi^!}_{\Gamma_Y}(u^*_{\nu_Y}))$ the $(n-1)$-quadratic structure produced by the quadratic boundary signature $\partial\mathrm{gn}^{\mathbf{L}_\bullet}_{\mathbb{Z}\pi}$.

⇒ $\partial\mathrm{gn}^{\mathbf{L}_\bullet}_{\to}(Y, B)$ the quadratic boundary pair signature for an n-dimensional pair of normal spaces (Y, B) is the $(n-1)$-quadratic Poincaré pair $(\partial C(B) \to \partial_+ C(Y), (\delta\psi, \psi))$ obtained by using the boundary construction and the spectral quadratic construction (see B28).

Algebraic bordism categories

$\boxed{\Lambda = (\mathbb{A}, \mathbb{C}, \mathbb{P}, (T, e))}$ an algebraic bordism category, usually denoted by Λ or $(\mathbb{A}, \mathbb{C}, \mathbb{P})$, consists of full additive subcategories $\mathbb{P} \subseteq \mathbb{C} \subseteq \mathbb{B}(\mathbb{A})$ where $\mathbb{P}$ is closed under weak equivalences and mapping cones, i.e.

- $\mathscr{C}(f : C \to D) \in \mathbb{P}$ for any chain map f in $\mathbb{P}$,

and additionally any $C \in \mathbb{C}$ satisfies

- $\mathscr{C}(\mathrm{id} : C \to C) \in \mathbb{P}$,
- $\mathscr{C}(e(C) : T^2(C) \xrightarrow{\simeq} C) \in \mathbb{P}$.

$\boxed{\Lambda(\mathbb{Z}) = (\mathbb{A}(\mathbb{Z}), \mathbb{B}(\mathbb{Z}), \mathbb{C}(\mathbb{Z}))}$ denotes the algebraic bordism category with

- $\mathbb{A}(\mathbb{Z})$ the category of finitely generated free left $\mathbb{Z}$-modules,
- $\mathbb{B}(\mathbb{Z})$ the bounded chain complexes in $\mathbb{A}(\mathbb{Z})$,
- $\mathbb{C}(\mathbb{Z})$ the contractible chain complexes of $\mathbb{B}(\mathbb{Z})$.

$\boxed{F : \Lambda \to \Lambda'}$ a functor of algebraic bordism categories is a covariant functor of additive categories, such that

- $F(B) \in \mathbb{B}'$ for every $B \in \mathbb{B}$
- $F(C) \in \mathbb{P}'$ for every $C \in \mathbb{P}$
- for every $A \in \mathbb{A}$ there is a natural chain map $G(A) : T'F(A) \to FT(A)$ such that

$$\begin{array}{ccc} T'FT(A) & \xrightarrow{GT(A)} & FT^2(A) \\ {\scriptstyle T'G(A)}\downarrow & & \downarrow{\scriptstyle Fe(A)} \\ T'^2F(A) & \xrightarrow{e'F(A)} & F(A) \end{array}$$

commutes and $\mathscr{C}(G(A)) \in \mathbb{P}'$.

$\boxed{(C, \lambda) \text{ in } \Lambda}$ an n-dimensional structured chain complex in $\Lambda = (\mathbb{A}, \mathbb{C}, \mathbb{P})$, i.e. a chain complex $C \in \mathbb{C}$ with an n-dimensional $\mathbb{P}$-Poincaré structure λ.

$\boxed{f : (C \to D, \delta\lambda, \lambda) \text{ in } \Lambda}$ a structured pair with $C, D \in \mathbb{B}$, λ is $\mathbb{P}$-Poincaré and $\mathscr{C}(\delta\lambda_0, \lambda_0 f^*) \in \mathbb{P}$.

$\boxed{\lambda}$ placeholder for a symmetric φ resp. quadratic ψ or normal structure (φ, γ, χ). We call a pair (C, λ) a *structured chain complex* and a triple $(C \to D, \delta\lambda, \lambda)$ a *structured pair*.

$\boxed{\lambda_0}$ stands for φ_0 in the symmetric and normal case and for $(1 + t)(\psi_0)$ in the quadratic case.

$\boxed{\mathbb{P}\text{-Poincaré}}$ is what a structured complex (C, λ) is called if $\partial C := \Sigma^{-1}\mathscr{C}(\lambda_0) \in \mathbb{P}$.

$\boxed{L^n(\Lambda), L_n(\Lambda), NL^n(\Lambda)}$ the cobordism groups of n-dimensional symmetric, quadratic, and normal chain complexes in Λ respectively.

$\boxed{Cobordism}$ of n-dimensional structured chain complexes in Λ:

$(C, \lambda) \sim (C', \lambda') \iff$ there is an $(n + 1)$-dimensional structured pair $(C \oplus C' \to D, \delta\lambda, \lambda \oplus -\lambda')$ in Λ.

$L^n(F), L_n(F), NL^n(F)$ n-dimensional relative L-groups consisting, up to cobordism, of pairs $((C,\lambda),(F(C)\to D,\delta\lambda,\lambda))$ where (C,λ) is an $(n-1)$-dimensional structured chain complex in Λ and $(F(C)\to D,\delta\lambda,\lambda)$ an n-dimensional structured pair in Λ'.

$\Lambda_L X$ for $\Lambda=(\mathbb{A},\mathbb{C},\mathbb{P})$ denotes $(\mathbb{A}_*X,\mathbb{C}_LX,\mathbb{P}_LX,(T_*,e_*))$, the X-mosaicked algebraic bordism category of $\Lambda=(\mathbb{A},\mathbb{C},\mathbb{P})$ with *local Poincaré* duality.

$\Lambda_G X$ for $\Lambda=(\mathbb{A},\mathbb{C},\mathbb{P})$ denotes $(\mathbb{A}_*X,\mathbb{C}_LX,\mathbb{P}_GX,(T_*,e_*))$, the X-mosaicked algebraic bordism category of $\Lambda=(\mathbb{A},\mathbb{C},\mathbb{P})$ with *global Poincaré* duality.

$\Lambda_N X$ for $\Lambda=(\mathbb{A},\mathbb{C},\mathbb{P})$ denotes $(\mathbb{A}_*X,\mathbb{C}_LX,\mathbb{C}_LX,(T_*,e_*))$, the X-mosaicked algebraic bordism category of $\Lambda=(\mathbb{A},\mathbb{C},\mathbb{P})$ with *no Poincaré* duality.

$\mathbb{A}_*X$ additive category of X-based objects in $\mathbb{A}$, i.e.

$$\mathrm{Obj}_{\mathbb{A}_*X}=\Big\{\sum_{\sigma\in X}M_\sigma \mid M_\sigma\in\mathbb{A}, M_\sigma=0 \text{ except for finitely many } \sigma\Big\},$$

$$\mathrm{Mor}_{\mathbb{A}_*X}\Big(\sum_{\widetilde{\sigma}\in X}M_{\widetilde{\sigma}},\sum_{\widetilde{\tau}\in X}N_{\widetilde{\tau}}\Big)=\Big\{\sum_{\substack{\tau\geq\sigma\\ \sigma,\tau\in X}}f_{\tau,\sigma}\mid (f_{\tau,\sigma}\colon M_\sigma\to N_\tau)\in\mathrm{Mor}_{\mathbb{A}}(M_\sigma,N_\tau)\Big\},$$

where X is a simplicial complex.

$\mathbb{C}_LX:=\{C \text{ chain complex in } \mathbb{A}_*X \mid C(\sigma)\in\mathbb{C} \text{ for all } \sigma\in X\}$ for a category $\mathbb{C}$ of chain complexes in $\mathbb{A}$.

$C(\sigma)$ denotes for a chain complex

$$C\colon \ldots\longrightarrow\sum_{\sigma\in X}(C_n)_\sigma\xrightarrow{\sum(f_n)_{\tau,\sigma}}\sum_{\sigma\in X}(C_{n-1})_\sigma\xrightarrow{\sum(f_{n-1})_{\tau,\sigma}}\sum_{\sigma\in X}(C_{n-2})_\sigma\xrightarrow{\sum(f_{n-2})_{\tau,\sigma}}\ldots$$

in $\mathbb{A}_*X$ or $\mathbb{A}^*X$ the chain complex in $\mathbb{A}$ given by

$$C(\sigma)\colon\ldots\longrightarrow(C_n)_\sigma\xrightarrow{(f_n)_{\sigma,\sigma}}(C_{n-1})_\sigma\xrightarrow{(f_{n-1})_{\sigma,\sigma}}(C_{n-2})_\sigma\xrightarrow{(f_{n-2})_{\sigma,\sigma}}\ldots$$

$\mathbb{C}_GX:=\{C \text{ chain complex in } \mathbb{A}_*X \mid A(C)\in\mathbb{C}\}$ for a category $\mathbb{C}$ of chain complexes in $\mathbb{A}$.

T_* defined for a chain duality $T\colon\mathbb{A}\to\mathbb{B}(\mathbb{A})$ as the mosaicked chain duality $\mathbb{A}_*X\to\mathbb{B}(\mathbb{A}_*X)$ with $(T_*(\sum_{\sigma\in X}M_\sigma))_r(\tau)=(T(\bigoplus_{\tau\leq\tilde{\tau}}M_{\tilde{\tau}}))_{r-|\tau|}$.

$\mathbb{Z}_*X$ short for $\mathbb{A}(\mathbb{Z})_*X$, the additive category of X-based free $\mathbb{Z}$-modules with 'non-decreasing' morphisms $\sum\limits_{\tau\geq\sigma}f_{\tau,\sigma}\colon\sum\limits_{\sigma\in X}M_\sigma\to\sum\limits_{\tau\in X}N_\tau$ where $(f_{\tau,\sigma}\colon M_\sigma\to N_\tau)$ are $\mathbb{Z}$-module morphism.

$\mathbb{A}(R)_*X$ is the additive category of X-based free R-modules.

$\Lambda^L X$ for $\Lambda = (\mathbb{A}, \mathbb{C}, \mathbb{P})$ denotes $(\mathbb{A}^*X, \mathbb{C}^L X, \mathbb{P}^L X, (T^*, e^*))$, the X-mosaicked algebraic bordism category of $\Lambda = (\mathbb{A}, \mathbb{C}, \mathbb{P})$ with *local Poincaré* duality.

$\mathbb{A}^*X$ additive category of X-based objects in $\mathbb{A}$, i.e.

$$\mathrm{Obj}_{\mathbb{A}^*X} = \Big\{ \sum_{\sigma \in X} M_\sigma \mid M_\sigma \in \mathbb{A}, M_\sigma = 0 \text{ except for finitely many } \sigma \Big\},$$

$$\mathrm{Mor}_{\mathbb{A}^*X}\big(\sum_{\widetilde{\sigma}\in X} M_{\widetilde{\sigma}}, \sum_{\widetilde{\tau}\in X} N_{\widetilde{\tau}}\big) = \Big\{ \sum_{\substack{\tau \le \sigma \\ \sigma,\tau \in X}} f_{\tau,\sigma} \mid (f_{\tau,\sigma} \colon M_\sigma \to N_\tau) \in \mathrm{Mor}_{\mathbb{A}}(M_\sigma, N_\tau) \Big\},$$

where X is a simplicial complex.

$\mathbb{A}(R)^*X$ is the additive category of X-based free R-modules.

$\mathbb{C}^L X := \{C \text{ chain complex in } \mathbb{A}^*X \mid C(\sigma) \in \mathbb{C} \text{ for all } \sigma \in X\}$ for a category $\mathbb{C}$ of chain complexes in $\mathbb{A}$.

T^* defined for a chain duality $T \colon \mathbb{A} \to \mathbb{B}(\mathbb{A})$ as the mosaicked chain duality $\mathbb{A}^*X \to \mathbb{B}(\mathbb{A}^*X)$ with $(T^*(\sum_{\sigma\in X} M_\sigma))_r(\tau) = (T(\bigoplus_{\tau \ge \widetilde{\tau}} M_{\widetilde{\tau}}))_{r+|\tau|}$.

$\mathbb{Z}^*X$ short for $\mathbb{A}(\mathbb{Z})^*X$, the additive category of X-based free $\mathbb{Z}$-modules with 'non-increasing' morphisms $\sum_{\tau\le\sigma} f_{\tau,\sigma} \colon \sum_{\widetilde{\sigma}\in X} M_{\widetilde{\sigma}} \to \sum_{\widetilde{\tau}\in X} N_{\widetilde{\tau}}$ where $(f_{\tau,\sigma} \colon M_\sigma \to N_\tau)$ are $\mathbb{Z}$-module morphism

Simplicial constructions and Delta sets

Δ^n the standard n-simplex with ordered simplices $0 < 1 < \ldots < n$.

$\partial\Delta^{m+1}$ the boundary of the standard simplex.

$D(\sigma, K)$ dual cell of a simplex σ in a simplicial complex K is the subcomplex of the barycentric subdivision K' defined by

$$D(\sigma, K) = \{\widehat{\sigma}_0\widehat{\sigma}_1 \ldots \widehat{\sigma}_r \mid \sigma \le \sigma_0 < \sigma_1 < \ldots < \sigma_r\}.$$

$K^{(k)}$ the set of k-simplices of a simplicial complex K.

Σ^m a simplicial complex dual to $\partial\Delta^{m+1}$ with one k-simplex σ^* for every $(m-k)$-simplex σ in $\partial\Delta^{m+1}$ and face maps

$$\partial_i \colon (\Sigma^m)^{(k)} := \Big\{\sigma^* \mid \sigma \in (\partial\Delta^{m+1})^{(m-k)}\Big\} \to (\Sigma^m)^{(k-1)}; \quad \sigma^* \mapsto (\delta_i\sigma)^* \qquad (0 \le i \le k)$$

i.e. if the $(m-k)$-simplex σ is spanned by the vertices $\{0, 1, \ldots, m+1\} \setminus \{j_0, \ldots, j_k\}$ than σ^* is spanned by the vertices $\{j_0, \ldots, j_k\}$ and $\partial_i(\sigma^*) = (\delta_i\sigma)^* = (\sigma \cup \{j_i\})^* = \sigma^* \setminus \{j_i\}$.

└ $\boxed{\sigma^* \in \Sigma^m}$ the dual k-simplex with $\partial_i \sigma^* = (\delta_i \sigma)^*$ for a $(m-k)$-simplex $\sigma \in \partial\Delta^{m+1}$.

$\boxed{|K| * |L|}$ the join of two topological spaces X and Y obtained from $X \times I \times Y$ by identifying $x \times 0 \times Y$ with x for all $x \in X$ and $X \times 1 \times y$ with y for all $y \in Y$ [Whi50, p. 202, III]. Thus each point of $X * Y$ lies on a unique line segment joining a point of X to a point Y.

$\boxed{K'}$ the first barycentric subdivision of a simplicial complex K.

└ $\boxed{\hat{\sigma}}$ the vertex in K' given by the barycenter of the simplex $\sigma \in K$.

$\boxed{L \div K := \{\sigma' \in L' \mid \text{ no vertex of } \sigma' \text{ lies in } K'\} = \bigcup_{\sigma \in L, \sigma \notin K} D(\sigma, L) \subset L'}$ the supplement of a subcomplex K of a simplicial complex L, i.e. the subcomplex of L' spanned by all of the vertices of $L' - K'$.

$\boxed{\overline{K} := K \div \partial\Delta^{m+1}}$ the supplement of K embedded into $\partial\Delta^{m+1}$ for $m \in \mathbb{N}$ large enough.

$\boxed{\Phi \colon (\partial\Delta^{m+1})' \xrightarrow{\cong} (\Sigma^m)'}$ an isomorphism of simplicial complexes that maps dual cells in $\partial\Delta^{m+1}$ to simplices in Σ^m. For more details see 1311.

$\boxed{X^* := (\Sigma^m / \Phi(\overline{X}))'}$ a simplicial S-dual of X (see 1312).

$\boxed{X[\sigma]}$ is defined for a map $r \colon X \to K$ to a simplicial complex as the preimage of the dual cell $D(\sigma, K)$ after making r transverse. If X is a simplicial complex itself, choose r to be the identity. The subdivision $X = \bigcup_{\sigma \in K} X[\sigma]$ is called a K-dissection of X.

$\boxed{\widehat{f}_\Delta := \bigcup_{\sigma \in X} \widehat{f}[\sigma] \colon M[\sigma] \to X[\sigma]}$ the decomposition of a degree one normal map $\widehat{f} \colon M \to X$ into degree one normal maps $\widehat{f}[\sigma] = \widehat{f}|_{\widehat{f}^{-1}(X[\sigma])}$ of $(n - |\sigma|)$-dimensional manifold $(m - |\sigma|)$-ads.

$\boxed{M_{\Delta^k}}$ manifold k-ad consisting of a manifold M and submanifolds $\partial_0 M, \dots, \partial_k M$ such that $\partial_0 M \cap \dots \cap \partial_k M = \emptyset$.

$\boxed{(\widehat{f}[\sigma], \partial\widehat{f}[\sigma]) = ((\overline{f}[\sigma], f[\sigma]), (\partial\overline{f}[\sigma], \partial f[\sigma]))}$ an n-dimensional degree one normal map

$$\begin{array}{ccc} (\nu_M|_{M[\sigma]}, \nu_M|_{\partial M[\sigma]}) & \xrightarrow{(\overline{f}, \partial\overline{f})} & (\nu_X|_{X[\sigma]}, \nu_X|_{\partial X[\sigma]}) \\ \downarrow & & \downarrow \\ (M[\sigma], \partial M[\sigma]) := (f^{-1}, \partial f^{-1})(X[\sigma], \partial X[\sigma]) & \xrightarrow{(f, \partial f)} & (X[\sigma], \partial X[\sigma]), \end{array}$$

denoted $(f[\sigma], \partial f[\sigma]) \colon (M[\sigma], \partial M[\sigma]) \to (X[\sigma], \partial X[\sigma])$ for short, from an $(n - |\sigma|)$-dimensional manifold with boundary to an $(n - |\sigma|)$-dimensional normal pair obtained from a degree one normal map $\widehat{f}$ after making f transverse to a K-dissection $\bigcup_{\sigma \in K} X[\sigma]$ of X .

$\boxed{\Delta\text{-set}}$ a simplicial set without degeneracies, i.e. a functor $\Delta \to \mathbf{Sets}$, where Δ is the category of finite sets $\{0, \dots, n\}$, $n \geq 0$ and order-preserving injections and $\mathbf{Sets}$ the category of sets and functions.

$\boxed{\Delta\text{-map } f\colon K \to L}$ a natural transformation, i.e. a collection $\{f_n\colon K^{(n)} \to L^{(n)} \mid n \geq 0\}$ such that $\partial_i \circ f_n = \partial_i \circ f_{n+1}$.

$\boxed{\text{Kan}}$ is what a Δ-set X is called if every map $\Lambda^n_i \to X$ extends to a map $\Delta^n \to X$; this property is necessary to do homotopy theory on Δ-sets .

$\boxed{K_+}$ for a Δ-set K the pointed Δ-set with n-simplices $K^{(n)} \cup \{\emptyset\}$ $\quad (n \geq 0)$.

$\boxed{\Lambda^n_i := \Delta^n - ((\Delta^n)^{(n)} \cup \partial_i\Delta^n)}$ the subcomplex of Δ^n obtained by removing the interior of Δ^n and a single face of Δ^n.

$\boxed{\mathbf{E}}$ an Ω-spectrum of pointed Kan Δ-sets, like $\mathbf{L}^\bullet(\Lambda), \mathbf{L}_\bullet(\Lambda), \mathbf{NL}^\bullet(\Lambda)$, is a sequence $\mathbf{E}_n$ of pointed Kan Δ-sets together with homotopy equivalences $\mathbf{E}_n \xrightarrow{\simeq} \Omega\mathbf{E}_{n+1}$. Note that the indexing is reversed compared to the usual convention.

$\boxed{\Omega K := K^{S^1}}$ the loop Δ-set; can be expressed as the Δ-set with n-simplices $\{\sigma \in K^{(n+1)} \mid \partial_0\partial_1 \ldots \partial_n\sigma = \emptyset \in K^{(0)}, \partial_{n+1}\sigma = \emptyset \in K^{(n)}\}$.

$\boxed{S^n}$ the pointed Δ-set with base simplices in all dimensions and only one additional simplex in dimension n.

$\boxed{K^L}$ the function Δ-set with n-simplices the Δ-maps $K \otimes \Delta^n \to L$; face maps are induced by $\partial_i\colon \Delta^n \to \Delta^{n-1}$.

$\boxed{K \otimes L}$ the geometric product Δ-set with one p-simplex for each equivalence class of triples

$$(m\text{-simplex } \sigma \in K, n\text{-simplex } \tau \in L, p\text{-simplex } \rho \in \Delta^m \times \Delta^n),$$

subject to the equivalence relation generated by $(\sigma, \tau, \rho) \sim (\sigma', \tau', \rho')$ if there exist Δ-maps $f\colon \Delta^m \to \Delta^{m'}, g\colon \Delta^n \to \Delta^{n'}$ such that $\sigma = f^*\sigma', \tau = g^*\tau', (f \times g)_*(\rho) = \rho'$.

$\boxed{K \times L}$ the product for ordered simplicial complexes with 0-simplices $K^{(0)} \times L^{(0)}$ and vertices $(a_0, b_0), \ldots, (a_n, b_n)$ span an n-simplex if and only if $a_0 \leq \ldots \leq a_n, b_0 \leq \ldots \leq b_n$ and $(a_r, b_r) \neq (a_{r+1}, b_{r+1})$.

$\boxed{\pi_n(\mathbf{E}) := \pi_{n+k}(\mathbf{E}_{-k})}$ for $n, k \in \mathbb{Z}, n + k \geq 0$.

$\boxed{\pi_n(K) = [\partial\Delta^{n+1}, K]}$ the pointed homotopy groups; can be expressed as the set of equivalence classes $\{\sigma \in K^{(n)} \mid \partial_i\sigma = \emptyset, 0 \leq i \leq n\}$ with $\sigma \sim \tau$ if there exists $\rho \in K^{(n+1)}$ with faces $\partial_0\rho = \sigma$, $\partial_1\rho = \tau$ and $\emptyset$ otherwise. The composition is defined by $\sigma_0 \cdot \sigma_1 = \tau$ if there is a $\rho \in K^{(n+1)}$ with faces σ_0, σ_1, τ and $\emptyset$ otherwise.

└ homotopy of Δ-maps a homotopy between two Δ-maps $f_0, f_1 \colon K \to L$ is a Δ-map $h\colon K \otimes \Delta^1 \to L$ with $h(\sigma \otimes i) = f_i(\sigma) \in L^{(n)}$ $(\sigma \in K^{(n)}, i = 0, 1)$.

$K \wedge L = K \otimes L/(K \otimes \emptyset_L \cup \emptyset_K \otimes L)$ the smash product of pointed Δ-sets.

$|K| := (\coprod_{n\geq 0} \Delta^n \times K^{(n)}/_\sim)$ the geometric realization of a simplicial complex K with $\sim$ the equivalence relation generated by $(a, \partial_i\sigma) \sim (\delta_i a, \sigma)$ $(a \in \Delta^{n-1}, \sigma \in K^{(n)})$ with $\delta_i \colon \Delta^{n-1} \to \Delta^n$ $(0 \leq i \leq n)$ the inclusion of the i-th face.

Spectra

$\mathbf{E}$ an Ω ring spectrum of Kan Δ-sets, e.g. $\mathbf{NL}^\bullet$ or $\mathbf{L}^\bullet$.

$\mathbf{E}_\otimes$ the component of $1 \in \mathbb{Z}$ for an Ω ring spectrum $\mathbf{E}$ with $\pi_0(\mathbf{E}) = \mathbb{Z}$, e.g. $\mathbf{L}^\otimes\langle 0\rangle$ or $\mathbf{NL}^\otimes\langle 1/2\rangle$.

$H_n(K, \mathbf{E}) := \pi_n(K_+ \wedge \mathbf{E}) = \lim_k \pi_{n+k}(K_+ \wedge \mathbf{E}_{-k})$ the $\mathbf{E}$-homology groups for a locally finite Δ-set K.

$H^n(K, \mathbf{E}) := \pi_{-n}(\mathbf{E}^{K_+}) = [K_+, \mathbf{E}_{-n}]$ the $\mathbf{E}$-cohomology groups for a locally finite Δ-set K.

$\mathbf{L}^\bullet(\Lambda), \mathbf{L}_\bullet(\Lambda), \mathbf{NL}^\bullet(\Lambda)$ Ω-spectra of pointed Kan Δ-sets defined for an algebraic bordism category Λ by

$$\mathbf{L}^\bullet(\Lambda) = \left\{\mathbf{L}^n(\Lambda) \quad | \; n \in \mathbb{Z}, \mathbf{L}^n(\Lambda)^{(k)} = \{n\text{-dim. } (C, \varphi) \text{ in } \Lambda^L\Delta^k\}\right\},$$
$$\mathbf{L}_\bullet(\Lambda) = \left\{\mathbf{L}_n(\Lambda) \quad | \; n \in \mathbb{Z}, \mathbf{L}_n(\Lambda)^{(k)} = \{n\text{-dim. } (C, \psi) \text{ in } \Lambda^L\Delta^k\}\right\},$$
$$\mathbf{NL}^\bullet(\Lambda) = \left\{\mathbf{NL}^n(\Lambda) \mid n \in \mathbb{Z}, \mathbf{NL}^n(\Lambda)^{(k)} = \{n\text{-dim. } (C, (\varphi, \gamma, \chi)) \text{ in } \Lambda^L\Delta^{k+n}\}\right\};$$

face maps are induced by face inclusions $\partial_i \colon \Delta^{k-1} \to \Delta^k$, base point is the 0-chain complex.

└ $\Lambda^L X$ for $\Lambda = (\mathbb{A}, \mathbb{C}, \mathbb{P})$ denotes $(\mathbb{A}^*X, \mathbb{C}^L X, \mathbb{P}^L X, (T^*, e^*))$, the X-mosaicked algebraic bordism category of $\Lambda = (\mathbb{A}, \mathbb{C}, \mathbb{P})$ with *local Poincaré* duality.

├ $\mathbb{A}^*X$ additive category of X-based objects in $\mathbb{A}$, i.e.

$$\text{Obj}_{\mathbb{A}^*X} = \Big\{ \sum_{\sigma\in X} M_\sigma \mid M_\sigma \in \mathbb{A}, M_\sigma = 0 \text{ except for finitely many } \sigma\Big\},$$
$$\text{Mor}_{\mathbb{A}^*X}\Big(\sum_{\widetilde{\sigma}\in X} M_{\widetilde{\sigma}}, \sum_{\widetilde{\tau}\in X} N_{\widetilde{\tau}}\Big) = \Big\{ \sum_{\substack{\tau\leq\sigma\\ \sigma,\tau\in X}} f_{\tau,\sigma} \mid (f_{\tau,\sigma} \colon M_\sigma \to N_\tau) \in \text{Mor}_{\mathbb{A}}(M_\sigma, N_\tau)\Big\},$$

where X is a simplicial complex.

└ $\mathbb{C}^L X := \{C \text{ chain complex in } \mathbb{A}^*X \mid C(\sigma) \in \mathbb{C} \text{ for all } \sigma \in X\}$ for a category $\mathbb{C}$ of chain complexes in $\mathbb{A}$.

$\boxed{\mathbf{L}^\bullet}$ short for $\mathbf{L}^\bullet(\Lambda(\mathbb{Z})) = \mathbf{L}^\bullet(\mathbb{A}(\mathbb{Z}), \mathbb{B}(\mathbb{Z}), \mathbb{C}(\mathbb{Z}))$

$\boxed{\mathbf{L}_\bullet}$ short for $\mathbf{L}_\bullet(\Lambda(\mathbb{Z})) = \mathbf{L}_\bullet(\mathbb{A}(\mathbb{Z}), \mathbb{B}(\mathbb{Z}), \mathbb{C}(\mathbb{Z}))$

$\boxed{\mathbf{NL}^\bullet}$ short for $\mathbf{NL}^\bullet(\mathbb{A}(\mathbb{Z}), \mathbb{B}(\mathbb{Z}), \mathbb{B}(\mathbb{Z}))$

$\boxed{\mathbf{NL}/\mathbf{L}^\bullet = \text{Fiber}(J\colon \mathbf{L}^\bullet \to \mathbf{NL}^\bullet)}$

$\boxed{\mathbf{L}^\bullet\langle 0\rangle}$ short for $\mathbf{L}^\bullet(\Lambda(\mathbb{Z})\langle 0\rangle)$

$\boxed{\mathbf{L}_\bullet\langle 1\rangle}$ short for $\mathbf{L}_\bullet(\Lambda(\mathbb{Z})\langle 1\rangle)$

$\boxed{\mathbf{NL}^\bullet\langle 1/2\rangle}$ short for $\mathbf{NL}^\bullet(\Lambda(\mathbb{Z})\langle 1/2\rangle)$

- $\boxed{\Lambda(\mathbb{Z}) = (\mathbb{A}(\mathbb{Z}), \mathbb{B}(\mathbb{Z}), \mathbb{C}(\mathbb{Z}))}$ denotes the algebraic bordism category with
  - $\mathbb{A}(\mathbb{Z})$ the category of finitely generated free left $\mathbb{Z}$-modules,
  - $\mathbb{B}(\mathbb{Z})$ the bounded chain complexes in $\mathbb{A}(\mathbb{Z})$,
  - $\mathbb{C}(\mathbb{Z})$ the contractible chain complexes of $\mathbb{B}(\mathbb{Z})$.
- $\boxed{\Lambda\langle q\rangle}$ for $\Lambda = (\mathbb{A}, \mathbb{C}, \mathbb{P})$ and $q \in \mathbb{Z}$ the q-connective algebraic bordism category $(\mathbb{A}, \mathbb{C}\langle q\rangle, \mathbb{P}\langle q\rangle)$
  - $\boxed{\mathbb{C}\langle q\rangle}$ the subcategory of chain complexes of $\mathbb{C}$ that are homotopy equivalent to q-connected chain complexes.
- $\boxed{\Lambda\langle 1/2\rangle}$ denotes for $\Lambda = (\mathbb{A}, \mathbb{C}, \mathbb{P})$ the algebraic bordism category $(\mathbb{A}, \mathbb{C}\langle 0\rangle, \mathbb{P}\langle 1\rangle)$.

$\boxed{\mathbf{NL}\langle 1/2\rangle/\mathbf{L}\langle 0\rangle^\bullet := \text{Fiber}(J\colon \mathbf{L}^\bullet\langle 0\rangle \to \mathbf{NL}^\bullet\langle 1/2\rangle)}$

$\boxed{\mathbf{\Omega}_\bullet^{STOP}}$ the Ω-spectrum of Kan Δ-sets defined by

$$(\mathbf{\Omega}_n^{STOP})^{(k)} = \{(M, \partial_0 M, \ldots, \partial_k M) \mid (n+k) - \text{dimensional manifold } (k+2)\text{-ad such that } \partial_0 M \cap \ldots \cap \partial_k M = \emptyset\}.$$

The face maps $\partial_i\colon (\mathbf{\Omega}_n^{STOP})^{(k)} \to (\mathbf{\Omega}_n^{STOP})^{(k-1)}$ are given by
$\partial_i(M) = (\partial_i M,\ \partial_i M \cap \partial_0 M, \ldots, \partial_i M \cap \partial_{i-1} M,\ \partial_i M \cap \partial_{i+1} M, \ldots, \partial_i M \cap \partial_k M)$.

$\boxed{\mathbf{\Omega}_\bullet^{N}}$ the Ω-spectrum of Kan Δ-sets defined by

$$(\mathbf{\Omega}_n^{N})^{(k)} = \{(X_{\Delta^k}, \nu, \rho) \mid (n+k) - \text{dimensional normal space } (k+2)\text{-ad, i.e.}$$
$$X_{\Delta^k} = (X, \partial_0 X, \ldots, \partial_k X) \text{ s.t. } \partial_0 X \cap \ldots \cap \partial_k X = \emptyset,$$
$$\nu\colon X \to \text{BSG}(r) \text{ an } (r-1)\text{-spherical fibration},$$
$$\rho\colon \Delta^{n+k+r} \to \text{Th}(\nu) \text{ s.t. } \rho(\partial_i \Delta^{n+k+r}) \subset \text{Th}(\nu|_{\partial_i X}) \quad \}$$

The face maps $\partial_i\colon (\mathbf{\Omega}_n^{N})^{(k)} \to (\mathbf{\Omega}_n^{N})^{(k-1)}$ are given by

$\partial_i(X) = (\partial_i X,\ \partial_i X \cap \partial_0 X, \ldots, \partial_i X \cap \partial_{i-1} X,\ \partial_i X \cap \partial_{i+1} X, \ldots, \partial_i X \cap \partial_k X)$.

$\boxed{\Sigma^{-1}\mathbf{\Omega}_\bullet^{\mathrm{N,STOP}}}$ the Ω-spectrum of Δ-sets obtained as the fiber of canonical the map of spectra $\mathbf{\Omega}_\bullet^{STOP} \to \mathbf{\Omega}_\bullet^{N}$.

$\boxed{\mathbf{T}(\nu)}$ denotes the Thom spectrum of a spherical fibration ν.

$\boxed{\mathbf{MSG}}$ the Thom spectrum of the universal stable $\mathbb{Z}$-oriented spherical fibrations over BSG with the k-th space the Thom space $\mathbf{MSG}(k) = \mathrm{Th}(\gamma_{\mathrm{SG}}(k))$ of the universal k-dimensional spherical fibration $\gamma_{\mathrm{SG}}(k)$ over $\mathrm{BSG}(k)$.

$\boxed{\mathbf{MSTOP}}$ the Thom spectrum of the universal stable $\mathbb{Z}$-oriented topological bundles over the classifying space BSTOP with the k-th space the Thom space $\mathbf{MSTOP}(k) = \mathrm{Th}(\gamma_{\mathrm{STOP}}(k))$ of the universal k-dimensional bundle $\gamma_{\mathrm{STOP}}(k)$ over BSTOP.

$\boxed{\mathbf{MS(G/TOP)}}$ the fiber of $J\colon \mathbf{MSTOP} \to \mathbf{MSG}$.

Orientations

$\boxed{u^G(\beta) \in H^k(\mathrm{Th}(\beta); \mathbf{MSG})}$ the canonical $\mathbf{MSG}$-orientation of β which is a map on the Thom spaces $\mathrm{Th}(\beta) \to \mathrm{Th}(\gamma_{SG})$ induced by the classifying map of a k-dimensional $\mathbb{Z}$-oriented spherical fibration $\beta\colon X \to \mathrm{BSG}(k)$.

$\boxed{u^T(\alpha) \in H^k(\mathrm{Th}(\alpha); \mathbf{MSTOP})}$ the canonical $\mathbf{MSTOP}$-orientation of α which is a map on the Thom spaces $\mathrm{Th}(\alpha) \to \mathrm{Th}(\gamma_{STOP})$ induced by the classifying map of a k-dimensional $\mathbb{Z}$-oriented topological bundle $\alpha\colon X \to \mathrm{BSG}(k)$.

$\boxed{u^{G/T}(\nu, h) \in H^k(\mathrm{Th}(\nu_X); \mathbf{MS(G/TOP)})}$ the preferred lift of $u^T(\nu)$ for a bundle reduction ν of the Spivak normal fibration ν_X, determined by the homotopy $h\colon \mathrm{Th}(\nu_X) \times [0,1] \to \mathbf{MSG}$ between $J(\nu)$ and $J(\nu_X)$.

$\boxed{u^{G/T}((\nu, h), (\nu_0, h_0)) \in H^k(\mathrm{Th}(\nu_X); \mathbf{MS(G/TOP)})}$ the preferred lift of $u^T(\nu) - u^T(\nu_0)$ for two bundle reductions ν, ν_0 of the Spivak normal fibration ν_X. The lift is obtained from the homotopy $h_0 \cup h\colon \mathrm{Th}(\nu_X) \times [0,1] \to \mathbf{MSG}$ between $J(\nu)$ and $J(\nu_0)$. If X is a manifold with a preferred topological bundle $\overline{\nu}_X$, define $u^{G/T}(\nu) = u^{G/T}(\nu, \overline{\nu}_X)$.

$\boxed{u^{\mathbf{E}}(\nu)}$ an $\mathbf{E}$-orientation of a $\mathbb{Z}$-oriented spherical fibration $\nu\colon X \to \mathrm{BSG}(k)$ is an element $u^{\mathbf{E}}(\nu) \in H^k(\mathrm{Th}(\nu); \mathbf{E})$ such that $u^{\mathbf{E}}(\nu)$ restricts to a generator of $H^k(\mathrm{Th}(\nu_x); \mathbf{E})$ for each fiber ν_x of ν.

$\boxed{u^{\mathbf{E}}(\nu)}$ an $\mathbf{E}$-*orientation* of a $\mathbb{Z}$-oriented spherical fibration $\nu\colon X \to \mathrm{BSG}(k)$ for a ring spectrum $\mathbf{E}$, is a homotopy class of maps $u^{\mathbf{E}}(\nu)\colon \mathbf{T}(\nu) \to \mathbf{E}$ such that for each $x \in X$ the restriction $u^{\mathbf{E}}(\nu)_x\colon \mathbf{T}(\nu_x) \to \mathbf{E}$ to the fiber ν_x of ν over x represents a generator of $\mathbf{E}^*(\mathbf{T}(\nu_x)) \cong \mathbf{E}^*(S^k)$ which under the Hurewicz homomorphism $\mathbf{E}^*(\mathbf{T}(\nu_x)) \to H^*(\mathbf{T}(\nu_X); \mathbb{Z})$ maps to the chosen $\mathbb{Z}$-orientation.

$u^{\mathbf{L}^\bullet}(\alpha) \in H^k(\mathrm{Th}(\alpha);\mathbf{L}^\bullet)$ the canonical $\mathbf{L}^\bullet$-orientation for a k-dimensional $\mathbb{Z}$-oriented topological bundle $\alpha\colon X \to \mathrm{BSTOP}(k)$ is given by the composition

$$\mathrm{Th}(\alpha) \xrightarrow{u^T(\alpha)} \mathbf{MSTOP}(k) \xrightarrow{\widetilde{c}^{-1}} \mathbf{\Omega}^{STOP}_{-k} \xrightarrow{\mathrm{sgn}^{\mathbf{L}^\bullet}_{\Omega}} \mathbf{L}^{-k}\langle 0\rangle.$$

$\widetilde{c}$ the transversality homotopy equivalence $\widetilde{c}\colon \mathbf{\Omega}^{STOP}_\bullet \simeq \mathbf{MSTOP}$.

$u^{\mathbf{NL}^\bullet}(\beta) \in H^k(\mathrm{Th}(\beta);\mathbf{NL}^\bullet)$ the canonical $\mathbf{NL}^\bullet$-orientation for a k-dimensional $\mathbb{Z}$-oriented spherical fibration $\beta\colon X \to \mathrm{BSG}(k)$ given by the composition

$$\mathrm{Th}(\beta) \xrightarrow{u^G(\beta)} \mathbf{MSG}(k) \xrightarrow{c^{-1}} \mathbf{\Omega}^{N}_{-k} \xrightarrow{\mathrm{sgn}^{\mathbf{NL}^\bullet}_{\Omega}} \mathbf{NL}^{-k}\langle 1/2\rangle.$$

c the transversality homotopy equivalence $c\colon \mathbf{\Omega}^{N}_\bullet \simeq \mathbf{MSG}$.

$u^{\mathbf{NL}/\mathbf{L}^\bullet}(\alpha,h) = (u^{\mathbf{NL}^\bullet}(h), u^{\mathbf{L}^\bullet}(\alpha) - u^{\mathbf{L}^\bullet}(\overline{\nu}_X)) \in H^k(\mathrm{Th}(\alpha);\mathbf{NL}\langle 1/2\rangle/\mathbf{L}\langle 0\rangle^\bullet)$
the canonical $\mathbf{NL}\langle 1/2\rangle/\mathbf{L}\langle 0\rangle^\bullet$-orientation given by the composition

$$\mathrm{Th}(\alpha) \xrightarrow{u^{G/T}(\alpha)} \mathbf{MS}(\mathbf{G}/\mathbf{TOP})(k) \to \Sigma^{-1}\mathbf{\Omega}^{\mathrm{N,STOP}}_{-k} \xrightarrow{\mathrm{sgn}^{\mathbf{NL}/\mathbf{L}^\bullet}_{\Omega}} \mathbf{NL}\langle 1/2\rangle/\mathbf{L}\langle 0\rangle^{\cdot}$$

for a k-dimensional $\mathbb{Z}$-oriented topological bundle $\alpha\colon X \to \mathrm{BSTOP}(k)$ together with a homotopy $h\colon J(\alpha) \simeq J(\overline{\nu}_X)$

BSTOP the classifying space of stable $\mathbb{Z}$-oriented topological bundles.

BSG the classifying space of stable $\mathbb{Z}$-oriented spherical fibrations.

BSO the classifying space of stable $\mathbb{Z}$-oriented vector bundles.

γ_{SG} the universal spherical fibration over the classifying space BSG.

γ_{STOP} the universal bundle over the classifying space BSTOP.

B**E**G the classifying space of spherical fibrations with **E**-orientation; a map $X \to \mathrm{B}\mathbf{E}\mathrm{G}$ is given by a pair $(\nu, u^{\mathbf{E}})$ with ν a spherical fibration and $u^{\mathbf{E}}$ an **E**-orientation.

Signatures

$\mathrm{con}^{\varphi}_X\colon C(X) \to W^{\%}(C(X))$ a chain map called symmetric construction; defined for a topological space X.

⇒ $\mathrm{con}^{\delta\varphi,\varphi}_{X,A}\colon \mathscr{C}(j) \to \mathscr{C}(j^{\%})$ a chain map called relative symmetric construction; defined for a pair of spaces $j\colon A \to X$

$\mapsto$ $\boxed{\mathrm{con}_X^{\varphi_X}\colon C(X)\to W^{\%}(C(X))}$ a chain map over $\mathbb{Z}_*X$ called the mosaicked symmetric construction; defined for ointed CW complexes X together with a map $r\colon X\to K$ to a simplicial
A17→p.126 complex K (see A17).

$\boxed{\mathrm{con}_F^{\psi}\colon C(X)\to W_{\%}(C(Y))}$ a chain map called quadratic construction; defined for a stable map $F\colon \Sigma^pX\to\Sigma^pY$ of pointed topological spaces X,Y.

$\boxed{\mathrm{con}_{\delta F,F}^{\delta\psi,\psi}\colon \mathscr{C}(j)\to W_{\%}(\mathscr{C}(i_{\%}))}$ a chain map called relative quadratic construction; defined for a stable map $(\delta F,F)\colon(\Sigma^pX,\Sigma^pA)\to(\Sigma^pY,\Sigma^pB)$ of pairs of pointed topological spaces $j\colon A\to X$ and $i\colon B\to Y$.

$\boxed{\mathrm{con}_F^{\psi_K}\colon C(X)\to W_{\%}(\mathscr{C}(i_{\%}))}$ a chain map over $\mathbb{Z}_*X$ called the mosaicked quadratic construction; defined for a stable map $F\colon\Sigma^pX\to\Sigma^pY$ of pointed CW complexes together with a
A29→p.136 map $r\colon X\to K$ to a simplicial complex K (see A29).

$\boxed{\mathrm{con}_F^{\psi^!}\colon \widetilde{C}(X)_{p+*}\to W_{\%}(\mathscr{C}(f))}$ a chain map called the spectral quadratic construction; defined for a semi-stable map $F\colon X\to\Sigma^pY$ where $f\colon\widetilde{C}(X)_{p+*}\to\widetilde{C}(\Sigma^pY)_{p+*}\simeq\widetilde{C}(Y)$ is the chain map induced by F.

$\mapsto$ $\boxed{\mathrm{con}_{G,F}^{(\delta\psi^!,\psi^!)}\colon\Sigma^{-p}\widetilde{C}(X,A)\to\mathscr{C}(j,i)_{\%}}$ a chain map called relative spectral quadratic con-
A25→p.133 struction; defined for a semi-stable map of pairs $(G,F)\colon\to(X,A)\to\Sigma^p(Y,B)$ (see A25).

$\boxed{\mathrm{con}_\nu^{\gamma}\colon\tilde{C}^k(\mathrm{Th}(\nu))\to\widehat{W}^{\%}(C(X)^{-*})_0}$ a chain map called chain bundle construction; defined for a k-dimensional spherical fibration ν.

$\boxed{\mathrm{sgn}_{\mathbb{Z}\pi}^{\mathbf{L}^\bullet}(X)=(C,\varphi)=(C(\widetilde{X}),\mathrm{con}_{X\ X}^{\varphi}([X]))\in L^n(\mathbb{Z}\pi)}$ an n-symmetric chain complex, defined for an n-dimensional Poincaré space X, called the symmetric signature.

$\mapsto$ $\boxed{\mathrm{sgn}_{\rightarrow}^{\mathbf{L}^\bullet}(X,B)}$ relative symmetric signature

$\mapsto$ $\boxed{\mathrm{sgn}_X^{\mathbf{L}^\bullet}(X):=\mathrm{sgn}_\Omega^{\mathbf{L}^\bullet}([X]_f^{\mathbf{\Omega}_\bullet^{STOP}})\in H_n(X;\mathbf{L}^\bullet)\cong L^n(\Lambda_LX)}$ the mosaicked symmetric signature for a Poincaré space X with a degree one normal map $\widehat{f}\colon M\to X$.

$\boxed{\mathrm{sgn}_{\mathbb{Z}\pi}^{\mathbf{L}_\bullet}(\widehat{f})=(C,\psi)=(\mathscr{C}(^!),e_{\%}\,\mathrm{con}_F^{\psi}([X]))\in L_n(\mathbb{Z}\pi)}$ an n-quadratic chain complex, called quadratic signature, where $F\colon\Sigma^p\widetilde{X}_+\to\Sigma^p\widetilde{M}$ is the stable map obtained from the degree one normal map $\widehat{f}$ by equivariant S-duality (see A26).

$\boxed{f^!\colon C(\widetilde{X})\to C(\widetilde{M})}$ the Umkehr map of a degree one normal map $\widehat{f}\colon M\to X$ of Poincaré spaces M and X. We obtain a stable equivariant map $F\colon\Sigma^k\widetilde{X}_+\to\Sigma^k\widetilde{M}_+$ for some $k\in\mathbb{N}$ and define $f^!$ as the composition $C(\widetilde{X})\xrightarrow{\Sigma_X}\Sigma^{-k}C(\Sigma^k\widetilde{X}_+)\xrightarrow{F}\Sigma^{-k}C(\Sigma^k\widetilde{M}_+)\xrightarrow{\Sigma_X^{-1}}C(\widetilde{M})$.

$\boxed{\mathrm{sgn}^{\mathbf{L}_\bullet}_{\#}(\widehat{g}) = (C^!, \psi^!)}$ the quadratic signature for a degree one normal map $\widehat{g}\colon N \to Y$ from a Poincaré space N to a normal space (Y, ν, ρ).

- $\boxed{C^! = \mathscr{C}(g^!)}$ the chain complex produced by the quadratic signature $\mathrm{sgn}^{\mathbf{L}_\bullet}_{\#}(\widehat{g})$.
  - $\boxed{g^!}$ the chain map given by the composition $C(Y)^{n-*} \xrightarrow{g^*} C(N)^{n-*} \xrightarrow{(\varphi_N)_0} C(N)$.
- $\boxed{\psi^! = \mathrm{con}^{\psi^!}_{\Gamma^!}(u^*_{\nu_Y}) \in W_{\%}(\mathscr{C}(g^!))_n}$ the n-quadratic structure produced by the spectral quadratic construction, part of the quadratic signature $\mathrm{sgn}^{\mathbf{L}_\bullet}_{\#}(\widehat{g})$.
- ⇒ $\boxed{\mathrm{sgn}^{\mathbf{L}_\bullet}_{\rightarrow}(\delta\widehat{g}, \widehat{g}) = (G^!\colon C^! \to D^!, \delta\psi^!, \psi^!)}$ the quadratic pair signature for a degree one normal map $(\delta\widehat{g}, \widehat{g})\colon (N, A) \to (Y, B)$ from a Poincaré pair (N, A) to a normal pair (Y, B) (see A28).
  - $\boxed{D^! := \mathscr{C}(g^!_i)}$ the chain complex obtained used for the quadratic pair signature.
    - $\boxed{g^!_i\colon C(Y)^{n-*} \to \mathscr{C}(i)}$ the chain map given by the composition $\varphi_i \circ \delta g^*$ for a map $i\colon B \to Y$ the inclusion.
      - $\boxed{\varphi_f = \mathrm{ev}_l(\delta\varphi, \varphi) \simeq (\delta\varphi_0, f\varphi_0)\colon \mathscr{C}(f)^{n-*} \to D}$ a chain map defined for an n-symmetric pair $(f\colon C \to D, \delta\varphi, \varphi)$.
- ⇒ $\boxed{\mathrm{sgn}^{\mathbf{L}_\bullet}_{X}(\widehat{f}) \in L_n(\Lambda_G X)}$ the mosaicked quadratic signature over X constructed in the following way: make f transverse to the dual cells $D(\sigma, K)$, then each σ-component is defined as the quadratic pair signature $\mathrm{sgn}^{\mathbf{L}_\bullet}_{\rightarrow}(\widehat{f}[\sigma], \partial\widehat{f}[\sigma])$.

$\boxed{\mathrm{sgn}^{\mathbf{NL}^\bullet}_{\mathbb{Z}\pi}(Y) = (C, (\varphi, \gamma, \chi)) \in NL^n(\mathbb{Z}\pi)}$ for an n-dimensional normal space (Y, ν, ρ) is an n-normal chain complex, called normal signature (see A33).

- ⇒ $\boxed{\mathrm{sgn}^{\mathbf{NL}^\bullet}_{\rightarrow}(Y, B)}$ relative normal signature
- ⇒ $\boxed{\mathrm{sgn}^{\mathbf{NL}^\bullet}_{X}(X) := \mathrm{sgn}^{\mathbf{NL}^\bullet}_{\Omega}([X]^{\mathbf{\Omega}^N_\bullet}) \in H_n(X; \mathbf{NL}^\bullet) \cong NL^n(\Lambda_N X)}$ the X-mosaicked normal signature defined here only for an n-dimensional Poincaré space X.
- ⇒ $\boxed{\mathrm{sgn}^{VL}_{X}(X) \in VL^n(X)}$ defined for a Poincaré space X as the normal signature $\mathrm{sgn}^{\mathbf{NL}^\bullet}_{X}(X)$.

$\boxed{\mathrm{sgn}^{\mathbf{NL}/\mathbf{L}^\bullet}_{\rightarrow}(Y, X) = (f\colon C \to D, (\delta\varphi, \delta\gamma, \delta\chi), \varphi) \in L(J)^n}$ the normal/symmetric pair signature, defined for a pair of (normal, Poincaré)-spaces (Y, X) by $(D, (\delta\varphi, \delta\gamma, \delta\chi)) = \mathrm{sgn}^{\mathbf{NL}^\bullet}_{\mathbb{Z}\pi}(Y)$ and $(C, \varphi) = \mathrm{sgn}^{\mathbf{L}^\bullet}_{\mathbb{Z}\pi}(X)$.

- ⇒ $\boxed{\mathrm{sgn}^{\mathbf{NL}/\mathbf{L}^\bullet}_{X}(\widehat{f}) := \mathrm{sgn}^{\mathbf{NL}/\mathbf{L}^\bullet}_{\Omega}([\widehat{f}]^{\Sigma^{-1}\mathbf{\Omega}^{N,STOP}_\bullet}) \in H_n(M'; \mathbf{NL}/\mathbf{L}^\bullet)}$ the mosaicked normal/symmetric signature over X defined for a degree one normal map $\widehat{f}\colon M \to M'$ between manifolds (see 2221).

$\boxed{[K]^{\mathbf{E}} \in H_n(K;\mathbf{E})}$ an n-dimensional $\mathbf{E}$-cycle of a simplicial complex $K \subset \partial\Delta^{m+1}$ defined by a collection $\left\{[K]^{\mathbf{E}}(\sigma) \in \mathbf{E}_{n-m}^{(m-|\sigma|)} \mid \sigma \in K\right\}$ such that $\partial_i[K]^{\mathbf{E}}(\sigma) = [K]^{\mathbf{E}}(\delta_i\sigma)$ if $\delta_i\sigma \in K$ and $\emptyset$ otherwise.

$\boxed{[X^*]^{\mathbf{E}} : X^* \to \mathbf{E}}$ the simplicial map defined by $\mathbf{E}$-cycle $[X]^{\mathbf{E}}$ via $\sigma^* \mapsto [X]^{\mathbf{E}}(\sigma)$.

$\boxed{[X]^{\mathbf{\Omega}^N_\bullet} \in H_n(X;\mathbf{\Omega}^N_\bullet)}$ an n-dimensional $\mathbf{\Omega}^N_\bullet$-cycle which assigns to each $\sigma \in X$ a $(m-n)$-dimensional normal $(m-|\sigma|+2)$-ad $(X[\sigma],\nu(\sigma),\rho(\sigma))$ as constructed in 132.

$\boxed{[\widehat{f}]^{\Sigma^{-1}\mathbf{\Omega}^{\mathrm{N,STOP}}_\bullet} \in H_n(X;\Sigma^{-1}\mathbf{\Omega}^{\mathrm{N,STOP}}_\bullet)}$ a $\mathbf{\Omega}^N_\bullet$-cobordism class of $\mathbf{\Omega}^{STOP}_\bullet$-cycle for a degree one normal map $\widehat{f}: M \to M'$ which assigns an $(m-|\sigma|)$-ad $(W(\sigma),\nu_{\overline{f}(\sigma)},\rho(\overline{f}(\sigma)),M(\sigma) \amalg M(\sigma))$ to each $\sigma \in M'$ (see 2221).

$\boxed{[\widehat{f},\widehat{f}']^{\Sigma^{-1}\mathbf{\Omega}^{\mathrm{N,STOP}}_\bullet} \in H_n(X;\Sigma^{-1}\mathbf{\Omega}^{\mathrm{N,STOP}}_\bullet)}$ a $\mathbf{\Omega}^N_\bullet$-cobordism class of $\mathbf{\Omega}^{STOP}_\bullet$-cycle defined for degree one normal map with target Poincaré spaces (see 222 for more details).

$\boxed{[M]^{\mathbf{\Omega}^{STOP}_\bullet} \in H_n(M;\mathbf{\Omega}^{STOP}_\bullet)}$ an n-dimensional $\mathbf{\Omega}^{STOP}_\bullet$-cycle which assigns to each $\sigma \in M$ the $(m-n)$-dimensional manifold $(m-|\sigma|+2)$-ad $M(\sigma) = D(\sigma,K)(\sigma,M)$.

$\boxed{[X]_f^{\mathbf{\Omega}^{STOP}_\bullet} \in H_n(X;\mathbf{\Omega}^{STOP}_\bullet)}$ an n-dimensional $\mathbf{\Omega}^{STOP}_\bullet$-cycle which assigns to each $\sigma \in X \subset \partial\Delta^{m+1}$ the $(m-n)$-dimensional manifold $(m-|\sigma|+2)$-ad $M[\sigma] \coloneqq [X]_f^{\mathbf{\Omega}^{STOP}_\bullet}(\sigma) \coloneqq f^{-1}(D(\sigma,X)) \in (\mathbf{\Omega}^{STOP}_{m-n})^{(m-|\sigma|)}$ which is a normal cobordism of manifolds.

$\boxed{[X]^{\mathbf{\Omega}^{STOP}_\bullet} \in H_n(X;\mathbf{\Omega}^{STOP}_\bullet)}$ an n-dimensional $\mathbf{\Omega}^{STOP}_\bullet$-cycle which assigns to each $\sigma \in X \subset \partial\Delta^{m+1}$ the $(m-n)$-dimensional manifold $(m-|\sigma|+2)$-ad $M[\sigma] \coloneqq [X]^{\mathbf{\Omega}^{STOP}_\bullet}(\sigma) \coloneqq pr^{-1}(D(\sigma,X)) \in (\mathbf{\Omega}^{STOP}_{m-n})^{(m-|\sigma|)}$ using a simplicial Pontrjagin-Thom construction for $pr: \Sigma^m \to \Sigma^m/\Phi(\overline{X}) \simeq \mathrm{Sing}(\mathrm{Th}(\nu_X))$.

$\boxed{[X]^{\mathbf{\Omega}^N_\bullet}}$ Let X be an n-dimensional finite Poincaré simplicial complex with the associated n-dimensional normal space (X,ν_X,ρ_X) and an embedding $X \subset \partial\Delta^{m+1}$. We use the *geometric normal signature of X over X* $\mathrm{sgn}^{\mathbf{\Omega}}_X$ as an element of $H_n(X;\mathbf{\Omega}^N_\bullet)$, but we construct and think of it as an element of $H^{m-n}(\Sigma^m,\bar{X};\mathbf{\Omega}^N_\bullet)$:

$$\begin{array}{rcl} [X]^{\mathbf{\Omega}^N_\bullet} : (\Sigma^m,\bar{X}) & \to & \mathbf{\Omega}^N_{n-m}, \\ \sigma & \mapsto & \begin{cases} (X(\sigma),\nu(\sigma),\rho(\sigma)) \in (\mathbf{\Omega}^N_{n-m})^{(m-|\sigma|)} & \sigma \in \Sigma^m \\ \emptyset & \sigma \in \bar{X} \end{cases} \end{array}$$

where

$$\begin{array}{rcl} X(\sigma) & = & |D(\sigma,X)| \\ \nu(\sigma) & = & \nu_X \circ \mathrm{incl} : X(\sigma) \subset X \to \mathrm{BSG}(m-n-1) \end{array}$$

and is given by collapsing $\partial\Delta^{m+1}$.

$\boxed{\mathrm{sgn}^{\mathbf{L}^\bullet}_{\Omega} : \mathbf{\Omega}^{STOP}_\bullet \to \mathbf{L}^\bullet\langle 0\rangle}$ the symmetric signature map defined for a k-simplex by $X_{\Delta^k} \mapsto \mathrm{sgn}^{\mathbf{L}^\bullet}_{\Delta^k}(X_{\Delta^k})$.

$\boxed{\mathrm{sgn}^{\mathbf{L}_\bullet}_{\Omega} : \Sigma^{-1}\mathbf{\Omega}^{\mathrm{N,STOP}}_\bullet \to \mathbf{L}_\bullet\langle 1\rangle}$ the quadratic signature map given by the normal/symmetric signature map and the identification $\mathbf{NL}\langle 1/2\rangle/\mathbf{L}\langle 0\rangle^\bullet \simeq \mathbf{L}_\bullet\langle 1\rangle$.

$\boxed{\mathrm{sgn}^{\mathbf{NL}^\bullet}_{\Omega} : \mathbf{\Omega}^{N}_\bullet \to \mathbf{NL}^\bullet\langle 1/2\rangle}$ the normal signature map; based on the normal signature $\mathrm{sgn}^{\mathbf{NL}^\bullet}_{\mathbb{Z}\pi}$.

$\boxed{\mathrm{sgn}^{\mathbf{NL}/\mathbf{L}^\bullet}_{\Omega} : \Sigma^{-1}\mathbf{\Omega}^{\mathrm{N,STOP}}_\bullet \to \mathbf{NL}\langle 1/2\rangle/\mathbf{L}\langle 0\rangle^\bullet}$ the normal/symmetric signature map induced by the maps $\mathrm{sgn}^{\mathbf{NL}^\bullet}_{\Omega}$ and $\mathrm{sgn}^{\mathbf{L}^\bullet}_{\Omega}$ and the fibration sequence of 161.

$\boxed{\mathrm{sgn}^{\mathbf{L}^\bullet}_{B}}$ Let $\alpha : X \to \mathrm{BSTOP}$ a topological bundle, then we define the composition $\mathrm{sgn}^{\mathbf{L}^\bullet}_{B} \circ \alpha : X \to \mathrm{B}\mathbf{NL}^\bullet\langle 1/2\rangle\mathrm{G}$ to be the map which is represented by the pair $(\alpha, u^{\mathbf{L}^\bullet}(\alpha))$.

$\boxed{\mathrm{sgn}^{\mathbf{NL}^\bullet}_{B}}$ Let $\beta : X \to \mathrm{BSG}$ a spherical fibration, then we define the composition $\mathrm{sgn}^{\mathbf{NL}^\bullet}_{B} \circ \beta : X \to \mathrm{B}\mathbf{NL}^\bullet\langle 1/2\rangle\mathrm{G}$ to be the map which is represented by the pair $(\alpha, u^{\mathbf{NL}^\bullet}(\beta))$.

The total surgery obstruction

$\boxed{s(X) := \partial^N_{\mathbb{G}}\,\mathrm{sgn}^{VL}_X(X) \in \mathbb{S}_n(X)}$ the total surgery obstruction

- $\boxed{\mathbb{S}_n(X) := L_{n-1}(\mathbb{Z}_*X, \mathbb{G}\langle 1\rangle, \mathbb{L}\langle 1\rangle)}$ the n-dimensional *structure group* of X. An element in $\mathbb{S}_n(X)$ is represented by an 1-connective $(n-1)$-quadratic chain complex in $\mathbb{Z}_*X$ which is globally contractible and locally Poincaré.
- $\boxed{VL^n(X) := NL^n(\mathbb{Z}_*X, \mathbb{B}\langle 0\rangle, \mathbb{G}\langle 1\rangle)}$ the n-dimensional *visible L-group* of X. An element in $VL^n(X)$ is represented by an n-dimensional 0-connective normal chain complex in $\mathbb{Z}_*X$ whose underlying symmetric structure is locally 0-connective and globally Poincaré.
  - $\boxed{\mathbb{Z}_*X}$ short for $\mathbb{A}(\mathbb{Z})_*X$, the additive category of X-based free $\mathbb{Z}$-modules with 'non-decreasing' morphisms $\sum\limits_{\tau\geq\sigma} f_{\tau,\sigma} : \sum\limits_{\sigma\in X} M_\sigma \to \sum\limits_{\tau\in X} N_\tau$ where $(f_{\tau,\sigma} : M_\sigma \to N_\tau)$ are $\mathbb{Z}$-module morphism.
  - $\boxed{\mathbb{B}\langle q\rangle \subset \mathbb{B}}$ for $q \geq 0$ is the subcategory of chain complexes of $\mathbb{B}$ which are homotopy equivalent to q-connected chain complexes.
    - $\boxed{\mathbb{B}}$ short for $\mathbb{B}(\mathbb{Z})_L X = \mathbb{B}(\mathbb{Z}_*X)$, the X-based bounded chain complexes of free $\mathbb{Z}$-modules.
  - $\boxed{\mathbb{G}\langle q\rangle \subset \mathbb{G}}$ the subcategory of chain complexes of $\mathbb{G}$ which are homotopy equivalent to q-connected chain complexes.

$\mathbb{G}$ short for $\mathbb{B}(\mathbb{Z})_G X := \{C \in \mathbb{B}(\mathbb{Z}) \mid A(C) \simeq *\}$, the *globally contractible* objects of $\mathbb{B}(\mathbb{Z})$.

$\mathbb{L}\langle q\rangle \subset \mathbb{L}$ the subcategory of chain complexes of $\mathbb{L}$ which are homotopy equivalent to q-connected chain complexes.

$\mathbb{L}$ short for $\mathbb{B}(\mathbb{Z})_L X := \{C \in \mathbb{B}(\mathbb{Z}) \mid C(\sigma) \simeq *$ for all $\sigma \in X\}$, the *locally contractible* chain complexes of $\mathbb{B}$.

$t(X) := I(s(X))$

$I\colon \mathbb{S}_n(X) \to H_{n-1}(X; \mathbf{L}_\bullet\langle 1\rangle)$ group homomorphism, see room 12.

$A\colon H_n(X; \mathbf{L}_\bullet(\mathbb{Z})) \to L_n(\mathbb{Z}\pi)$ the assembly map defined by $\sum_{\sigma\in X} C(\sigma) \mapsto \oplus_{\widetilde{\sigma}\in\widetilde{X}} C(p(\widetilde{\sigma}))$ where $p\colon \widetilde{X} \to X$ is the universal covering.

$\Gamma_Y := S^{-1}(\widetilde{\Delta} \circ \rho)\colon \mathrm{Th}(\nu)^* \to \Sigma^p Y_+$ the semi-stable map obtained for an n-dimensional normal space (Y, ν, ρ) with an N-dimensional S-dual $\mathrm{Th}(\nu)^*$ of its Thom space and $p = N - (n + k)$.

Surgery dictionary

Hotel	KMM	Ranicki	Weiss	Definition
t	T	T	T	generator of $\mathbb{Z}_2$
T	T	T	T	chain duality
$W^{\%}C$	$W^{\%}C$	$W^{\%}C$	$W\&C$	$\mathrm{Hom}(W, C \otimes C)$
$f^{\%}$	$f^{\%}$	$f^{\%}$	$f^{\rightarrow}$	$W^{\%}C \to W^{\%}(D)$ induced by $f\colon C \to D$
$f_{\%}$	$f_{\%}$	$f_{\%}$		$W_{\%}C \to W_{\%}(D)$ induced by $f\colon C \to D$
$\widehat{f}^{\%}$	$\widehat{f}^{\%}$	$\widehat{f}^{\%}$	$f^{\rightarrow}$	$\widehat{W}^{\%}C \to \widehat{W}^{\%}(D)$ induced by $f\colon C \to D$
			$f^{\leftarrow}$	$\widehat{W}^{\%}(D^{-*}) \to \widehat{W}^{\%}(C^{-*})$ induced by $f\colon C \to D$
Σ	Σ		Σ	algebraic suspension $\Sigma C_k := C_{k-1}$, $\Sigma d_k = -d_{k-1}$
		Σ		algebraic suspension with $\Sigma d_k = (-)^k d_{k-1}$
		S		algebraic suspension with $\Sigma d_k = d_{k-1}$
Σ	Σ	Σ		geometric suspension $\Sigma X := X \wedge S^1$
Σ_X		Σ_X		suspension chain equivalence $C(X) \to \Sigma^{-1}C(\Sigma X)$
$\mathcal{S}$	S	S	$\mathfrak{S}$	suspension map $W^{\%}C \to \Sigma^{-1}W^{\%}(\Sigma C))$
S	S	$\underline{S}$		S-duality isomorphism $[Y, Z] \xrightarrow{\cong} [S^N, X \wedge Y]$
		$\overline{S}$		skew-suspension map $Q^n(C, \varepsilon) \to Q^{n+2}(SC, -\varepsilon)$
$\mathscr{C}(f)$	$\mathcal{C}(f)$	$C(f)$	$\mathrm{Cone}(f)$	algebraic mapping cone
$\mathbb{B}(A)$	$\mathbb{B}(A)$	$\mathbb{B}(A)$	$\mathscr{C}_A$	category of f.g. projective bounded left A-modules

Hotel	KMM	Ranicki	Definition
con_X^{φ}	φ	φ_X	symmetric construction
con_F^{ψ}	ψ	ψ_F	quadratic construction
$\mathrm{con}_\nu^{\gamma}$	γ_ν	θ_X	chain bundle (hyperquadratic resp.) construction
$\mathrm{sgn}_R^{\mathbf{L}^\bullet}$	$\mathrm{sign}_R^{\mathbf{L}^\bullet}$	σ^*	symmetric signature in $L^n(R)$
$\mathrm{sgn}_R^{\mathbf{L}_\bullet}$	$\mathrm{sign}_R^{\mathbf{L}_\bullet}$	σ_*	quadratic signature in $L_n(R)$
		$\widehat{\sigma}^*$	hyperquadratic signature in $\widehat{L}^n(R)$
$\mathrm{sgn}_R^{\mathbf{NL}^\bullet}$	$\mathrm{sign}_R^{\mathbf{NL}^\bullet}$	$\widehat{\sigma}^*$	normal signature $NL^n(R)$
$\mathbf{L}, \mathbf{NL}$	$\mathbf{L}, \mathbf{NL}$	$\mathbb{L}, \mathbb{NL}$	L-spectra
$(\mathbb{A}, \mathbb{C}, \mathbb{P})$	$(\mathbb{A}, \mathbb{B}, \mathbb{C})$	$(\mathbb{A}, \mathbb{B}, \mathbb{C})$	algebraic bordism category
$\mathbb{Z}_* X$	$\mathbb{Z}_*(X)$	$\mathbb{A}(\mathbb{Z})_*(X)$	additive category of $\mathbb{Z}$-modules over X
$\mathbb{B}$	$\mathbb{B}$	$\mathbb{B}(\mathbb{Z}, X)$	category of bounded chain complexes in $\mathbb{Z}_* X$
$\mathbb{L}$	$\mathbb{C}$	$\mathbb{C}(\mathbb{Z})_*(X)$	locally contractible chain complexes in $\mathbb{Z}_* X$
$\mathbb{G}$	$\mathbb{D}$	$\mathbb{C}(\mathbb{Z}, X)$	globally contractible chain complexes in $\mathbb{Z}_* X$
$\Lambda_L X$	$\Lambda(\mathbb{Z})_*(X)$	$\Lambda(\mathbb{Z})_*(X)$	algebraic bordism category with local Poincaré duality
$\Lambda_G X$	$\Lambda(\mathbb{Z})(X)$	$\Lambda(\mathbb{Z}, X)$	algebraic bordism category with global Poincaré duality
$\Lambda_N X$	$\widehat{\Lambda}(\mathbb{Z})(X)$	$\widehat{\Lambda}(\mathbb{Z}, X)$	algebraic bordism category with no Poincaré duality
$\mathrm{sgn}_X^{\mathbf{L}^\bullet}$	$\mathrm{sign}_X^{\mathbf{L}^\bullet}$	σ^*	symmetric signature in $L^n(\Lambda_L X)$
$\mathrm{sgn}_X^{\mathbf{L}_\bullet}$	$\mathrm{sign}_X^{\mathbf{L}_\bullet}$	σ_*	quadratic signature in $L_n(\Lambda_L X)$
$\mathrm{sgn}_X^{\mathbf{NL}^\bullet}$	$\mathrm{sign}_X^{\mathbf{NL}^\bullet}$	$\widehat{\sigma}^*$	normal signature in $NL^n(\Lambda_N X)$
sgn_X^{VL}	sign_X^{VL}	σ^*	visible symmetric in signature in $VL^n(X) = NL^n(\Lambda_G X)$
σ	σ	σ	simplex
σ^*	σ^*	σ^*	dual simplex
$\widehat{\sigma}$	$\widehat{\sigma}$	$\widehat{\sigma}$	barycenter of a simplex

Table of notations

Symbols

A

B

C

Zusammenfassung

Diese Arbeit beschäftigt sich mit dem totalen Chirurgiehindernis und der dafür notwendigen algebraischen Variante der Chirurgietheorie für topologische Mannigfaltigkeiten.

Die grundlegende Fragestellung in der Chirurgietheorie ist, wann ein CW-Komplex homotopieäquivalent zu einer Mannigfaltigkeit ist. Der ursprüngliche Chirurgieansatz, wie er von Browder, Sullivan, Novikov und Wall entwickelt wurde, verwendet dafür geometrische Operationen auf Mannigfaltigkeiten, die den Homotopietyp der Mannigfaltigkeiten verändern. Unter bestimmten Voraussetzungen kann man für einen CW-Komplex X potentielle Kandidaten für homotopieäquivalente Mannigfaltigkeiten finden. Ob so ein Kandidat dann tatsächlich mittels chirurgischer Eingriffe an den Homotopietyp von X angepasst werden kann, wird von einem algebraischen Hindernis erfasst.

Die algebraische Chirurgie, die von Andrew Ranicki entwickelt wurde, definiert ein algebraisches Äquivalent der geometrischen Chirurgieoperationen, so dass man von Anfang an direkt zur Algebra übergehen kann. Es ermöglicht den gesamten Chirurgieprozess einschließlich aller zusätzlichen Voraussetzungen, die in der geometrischen Variante nötig sind, in einem einzelnen algebraischen Hindernis zusammenzufassen, dem totalen Chirurgiehindernis. Das totale Chirurgiehindernis ist für einen endlichen n-dimensionalen Poincaré CW-Komplex X definiert und zwar als ein Element $s(X)$ in einer abelschen Gruppe $\mathbb{S}_n(X)$ mit der Eigenschaft, dass es für $n \geq 5$ nur dann verschwindet, wenn X homotopieäquivalent zu einer geschlossenen n-dimensionalen topologischen Mannigfaltigkeit ist. Dieses Resultat geht im Wesentlichen auf Andrew Ranicki zurück sowie auf Beiträge von Michael Weiss und erstreckt sich über diverse, teilweise sehr umfangreiche Veröffentlichungen aus den 80er und 90er Jahren.

Die vorliegende Arbeit leistet folgende Beiträge zu diesem Theorem und der zugrundeliegenden Theorie:

1. Sie trägt die unterschiedlichen Quellen zu einem in sich geschlossenen Werk zusammen, in dem das totale Chirurgiehindernis konstruiert und seine Eigenschaften bewiesen werden. An einigen Stellen in der Literatur sind Beweise von einigen Behauptungen nur ansatzweise oder überhaupt nicht ausgeführt. Wir liefern hier mehr Details mit dem Ziel bestehende Zweifel an der Theorie auszuräumen. Dazu gehören die folgenden Punkte:
   – Wir zeigen, dass für eine normale Grad-eins-Abbildung $\widehat{f}\colon M \to X$ die quadratische Signatur $\mathrm{sgn}^{\mathbf{L}_\bullet}_{\mathbb{Z}\pi}(\widehat{f})$ und der quadratische Rand des (normalen, Poincaré) Paares des Abbildungzylinders von f den gleichen quadratischen Kettenkomplex erzeugen. Das wird in [Ran81, p.622] behauptet, aber dort wird mit Proposition 7.4.1 lediglich eine Grundidee des Beweises gegeben.
   – Wir konstruieren die normale Signature $\mathrm{sgn}^{\mathbf{NL}^\bullet}_X$ für einen Poincaré-Komplex X, deren Existenz in [Ran92, Example 9.12] behauptet wurde (siehe auch [Ran13, Errata for p.103]).
   – Wir identifizieren in Beweis 16 bestimmte induzierte Abbildungen mit der Chirurgiehindernisabbildung. Dies wurde in [Ran79, p.291] behauptet ohne Details auszuführen.

Die Aussage von 16 findet sich auch in [Ran92, Prop. 16.1], wo der Beweis zwar skizziert, aber für weitere Details auf [Ran79] verwiesen wird.

– Wir haben versucht die expliziten Matrizen der Originalliteratur zu vermeiden und soweit wie möglich einen koordinatenfreien Zugang zu diversen algebraischen Konstruktionen zu entwickeln. Der Ansatz, der dabei für die algebraische Chirurgie und die Randkonstruktionen in B verwendet wurde, basiert auf Ideen von Jacob Lurie [**?**].

Diese Resultate sind in Zusammenarbeit mit Tibor Macko und Adam Mole enstanden und wurden im *Münster Journal of Mathematics* bereits veröffentlicht [KMM13].

2. Außerdem versucht diese Arbeit den sehr umfangreichen Beweis leichter nachvollziehbar zu präsentieren. Dafür wurde bewusst vom konventionellen Aufbau mathematischer Arbeiten abgewichen und aufbauend auf Ideen von Uri Leron [Ler83] Alternativen gesucht, auch mit dem Ziel die vielfältigen Möglichkeiten zu nutzen, die eine computergestütze Präsentation eröffnen. In Hinblick auf die Promotionsordnung mussten allerdings papiertaugliche Kompromisse gefunden werden. Das Ergebnis versucht vor allem zwei Schwierigkeiten eines mathematischen Schriftstückes zu begegnen, nämlich dass

   a) ein, insbesondere in diesem Fall, hochgradig verzweigter Inhalt, durch ein lineares Medium vermittelt werden muss und dass

   b) je nach Leser und Leseabsicht sehr unterschiedliche Anforderungen an die nötige und erwünschte Detailtiefe gestellt werden.

   Der Beweis ist deshalb in mehrere Levels unterteilt, die zunehmend mehr (technische) Details liefern. Definitionen und Erläuterungen sind jeweils getrennt von den eigentlichen Beweisen aufgeführt. In der elektronischen pdf- und web-Version sind diese Abschnitte und Levels alle durch Links verknüpft. Für die Papierversion wurde versucht mit einem ausgeklügelten Nummerierungssystem die Beziehungen der einzelnen Beweisteile transparent zu gestalten und eine Anordnung zu finden, die die Umblätter- und Sucharbeit minimiert. Das ist auch mit ein Grund für den Umfang dieser Arbeit, da viele Definitionen und Aussagen wiederholt werden, wenn sie gebraucht werden.

Die Arbeit ist wie folgt aufgebaut: Die Lobby, enthält eine allgemeine Einführung in die Thematik; in der Rezeption wird das Theorem über das totale Chirurgiehindernis präsentiert und genauer ausgeführt wie der Beweis bzw. das Hotel aufgebaut ist. Das Hotel hat vier Stockwerke und einen Keller. Das erste Stockwerk beweist das Theorem und verwendet dafür Aussagen, die in den darüber liegenden Stockwerken bewiesen werden. Im Keller sind die grundlegenden Aussagen und Konzepte ausgeführt, die überall im Hotel verwendet werden.

Zu jedem Stockwerk gibt es einen Fahrstuhl, der eine Einführung in das folgende Stockwerk gibt. Jedes Stockwerk ist in mehrere Zimmer unterteilt. Jedes Zimmer beweist eine Aussage, die im darunterliegenden Stockwerk verwendet wurde, wobei die Zimmernummer genauer bezeichnet wie diese Aussage im Beweis verankert ist. Mit jedem Stockwerk wird eine Ziffer angehängt, so dass zum Beispiel die dreistellige Zimmernummer 342 bedeutet, dass diese Ausage im dritten Stock bewiesen wird und im zweiten Stock für Aussage Nummer 34 verwendet wird. Jedes Zimmer hat neben dem eigentlichen Beweis üblicherweise einen vorangehenden Portier-Abschnitt und einen nachfolgenden Zimmerservice-Abschnitt. Der Portier führt in das Zimmer ein, indem er die Beweisidee zusammenfasst und Referenzen auflistet. Der Zimmerservice liefert für alle im jeweiligen Zimmer verwendeten Begriffe die Definitionen.

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