Thin Viscous Films on Curved Geometries

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Dedicated to the Light of my Eyes

Abstract

The topic of this thesis is the evolution of thin viscous films on curved substrates. Using techniques from differential geometry, namely the *exterior calculus of differential forms*, and from optimization theory, in particular the theory of *saddle point problems* and the *shape calculus*, we reduce a variational form of the Stoke equations, which govern the flow, to a two dimensional *optimization problem with a PDE constraint* on the substrate. This reduction is analogous to the lubrication approximation of the classic thin film equation. We study the well-posedness of a, suitably regularised, version of this reduced model of the flow, using variational techniques. Furthermore, we study the well-posedness and convergence of time- and space-discrete versions of the model. The time discretization is based on the idea of the *natural time discretization of a gradient flow*, whereas the spatial discretization is done via suitably chosen finite element spaces. Finally, we present a particular implementation of the discrete scheme on *subdivision surfaces*, together with relevant numerical results.

Contents

Introduction

In recent years, the investigation of the dynamics of liquid thin films has attracted increased attention in the field of physics, engineering and mathematics. In many applications in materials science and biology, liquid thin films do not reside on a flat Euclidean domain but on curved surfaces (Howell[How03], Roy, Roberts and Simpson[RRS02], Schwartz and Weidner[SW95], Wang[Wan84]). Examples are the spreading of liquid coatings on surfaces, the surfactant-driven thin film flow on the interior of the lung alveoli (Xu *et al.*[XLLZ06]) and the tear film on the cornea of the eye (Braun *et al.*[BUM$^+$]). The evolution of the film thickness is often of greater interest than the actual velocity or pressure field within the fluid volume. In that case, a lubrication approximation dating back already to Reynolds[Rey86] allows us to replace the governing Navier-Stokes and moving free boundary model by with an evolution model expressed solely in terms of the film height or a related quantity. For a thin film deposited on a planar substrate, and in the limit of vanishing thickness-to-length ratio, one can derive through the well-known lubrication theory (Oron, Davis and Bankoff[ODB97]) a limit model in the form of a fourth order nonlinear parabolic problem for the evolution of the film height h (Bernis and Friedman[BF90], Bertozzi and Pugh[BP96], Bernis[Ber95], Beretta, Bertsch and Dal Paso[BBDP95]). We refer to Oron, Davis, and Bankoff[ODB97] for the derivation of the model and to Myers[Mye98] for an overview of the mathematical treatment of surface-tension-driven thin fluid films. A recent review by Craster and Matar[CM09] discusses the dynamics and stability of thin liquid films involving external forcing, thermal effects and intermolecular forces.

Already in '84, Wang[Wan84] presented a lubrication model for the evolution of a thin film flowing down a curved surface. Schwartz and Weidner[SW95] discussed the additional forcing effect due to the surface curvature. A lubrication model for the dynamics of the film, in the form of a PDE for the evolution of the film thickness, has been derived by Roy, Roberts and Simpson[RRS02]. Unlike the case of a flat substrate, their lubrication model is an approximation of the Navier-Stokes equations, rather than the limit model for vanishing film thickness. The approximation is based on a second order expansion in ϵ, where ϵ is the scale ratio between the characteristic height of the film and the characteristic length of the surface. Roberts and Li[RL06] extended this model to include inertial effects, by adding an evolution law for the average lateral velocity. In Thiffeault and Kamhawi[TK06] gravity-driven thin film flows on curved substrates are studied from a dynamical systems point of view. A related gravity-driven shallow water model on curved geometries, namely topographic maps, was investigated by Boutounet *et al.*[BCNV08] Kalliadasis and Bielarz[KB00] directly applied a thin film model on topographic maps to analyze the impact of topological features on the formation of capillary ridges. Jensen *et al.*[JCK04] studied the flow of a thin, homogeneous liquid layer induced by a sudden change in the shape of the substrate. Thin film flow on moving curved surfaces was investigated by Howell[How03], who explored the behavior for large, non-uniform curvature, whose gradient dominates the flow and leads in the limit to a hyperbolic equation with shock formation at specific regions of the substrate. The flow of a thin film on a flat, but non-linearly stretching, sheet was discussed by Santra and Dandapat[SD09].

There are two main challenges in modelling the thin film flow on a *curved* substrate. The first one is that, contrary to the flat case, the anisotropic nature of the mobility can not be ignored and therefore it needs to be taken as a tensor, rather than a scalar, function of the film thickness. The second difficulty is that the free energy of the film is dominated by curvature- and gravity-driven transport-like terms, whereas the surface tension-driven Dirichlet energy is a first order correction. Since the regularizing effects of the Dirichlet energy are vital to the proper modelling of the problem, we can not limit ourselves to a leading order approximation, as in the classic lubrication approximation. The first chapter of the thesis lays down the foundations for simultaneously dealing with both of these issues. We use the *exterior calculus of differential forms* (presented with particular emphasis in physical applications in Frankel[Fra04]) to explore the differential calculus of curved thin structures. The main result (Prop. 1.55) is a set of expressions for the gradient, curl and divergence that feature

1. a decomposition into normal-tangential components,
2. a natural expansion into terms of different order in the thickness parameter ϵ, and
3. transparent inclusion of the effects of both the scalar and tensor curvatures of the substrate.

In the second chapter, we combine the results of the first chapter with tools from the *variational theory of saddle point problems* (as developed by Brezzi[Bre74] and Babuška[Bab73]) and *shape calculus* (as presented in Sokolowski and Zolésio[SZ92]) to reduce an appropriate variational form of the Stokes equations to a two-dimensional variational model for the evolution of the thickness of the film on the substrate. The reduced model, which takes the form of a PDE-constrained minimization problem, is accurate to first order in the thickness parameter ϵ. Moreover, it describes a gradient flow for the free energy of the film in a suitable metric derived from the mobility. In this way it preserves an important physical property of the original non-reduced problem, i.e. the creeping flow of the viscous fluid, which also has interesting applications in the analysis of these types of problems (as shown by, among others, Otto[Ott98], Giacomelli and Otto[GO02, GO03], Mattes *et al.*[MMS09] and Slepčev[Sle09]).

Convergent numerical discretizations of thin film flow were investigated for instance by Zhornitskaya and Bertozzi[ZB00] using an entropy-consistent finite difference scheme, and independently by Grün and Rumpf[GR00] based on a related finite element approach. A numerical discretization of surfactant spreading on liquid thin films was proposed and analyzed by Barrett *et al.*[BGN03]. For the discretization of the thin film equation on curved substrates, Roy, Roberts and Simpson[RRS02] used a straightforward finite difference approximation of the fourth order PDE with implicit treatment of the higher order terms and a small ratio of time step to spatial grid size to cope with the stiffness of the problem. Schwartz and Weidner also applied a semi-implicit finite difference scheme and Myers *et al.*[MCC02] used a semi-implicit finite volume type approach with a flux splitting. In his doctoral thesis[Nem12], Nemadjieu developed a finite volume scheme for the discretization of transport-diffusion problems on moving hypersurfaces, and applied it to the evolution of surfactant-driven thin film flows on moving surfaces. A level set implementation of the model in Roy, Roberts and Simpson[RRS02] was proposed by Greer *et al.*[GBS06]. To ensure the stability of the proposed schemes in all these cases, the time-step size has to be chosen very small. A variational time discretization of the underlying gradient flow structure offers an attractive alternative and in particular allows for large time steps. For planar surfaces and thin coatings consisting of a resin and a solvent component, such a scheme has already been investigated by Dohmen *et al.*[DGOR07]. Düring *et al.*[DMM10] also derived a numerical scheme for the nonlinear fourth order Derrida-Lebowitz-Speer-Spohn equation, using the gradient flow structure induced by the underlying Wasserstein-type transport problem. Glasner[Gla05] used a Galerkin discretization of a variational model, related to the reduced model of Chap. 2, to study the movement of the contact lines of thin films on planar substrates.

The purpose of the third chapter then is to derive and study an appropriate discretization of the reduced model of the second chapter. Our main tool is again the variational theory of saddle point problems. The well-posedness of this type of problems depends on the coercivity of the objective function and an inf-sup condition for the constraint. We introduce appropriate regularizations of the mobility and the PDE-constraint, that meet these conditions, and proceed to study the properties of the regularized model. Furthermore, we apply the aforementioned natural time discretization concept to the regularized model, and study the well-posedness and convergence of the resulting semi-discrete scheme. At this point it should be noted that in Rumpf and V.[VR13] we also presented a numerical scheme for the evolution of thin film flow on curved substrates, based on the reduced (non-regularized) model of the second chapter. The scheme was built on the aforementioned natural time discretization of the gradient flow and on discrete exterior calculus for the spatial discretization, resulting in a finite volume type scheme. In this thesis we turn to the finite elements methodology instead, ending the third chapter by stating and proving a general convergence result for the Galerkin discretization of the regularized model over appropriate finite element spaces. Finally, the fourth chapter deals with the numerical implementation of the resulting FEM scheme.

The main challenge is the need for an H^2-conforming set of basis functions over the substrate surface. We show how this can be done over *subdivision surfaces* (as defined by Catmull and Clark[CC78]), where such a basis exists essentially by construction, and conclude by presenting a number of numerical tests for the scheme. For the sake of comparison, we have also included certain numerical results from [VR13] in figures 4.1 and 4.2.

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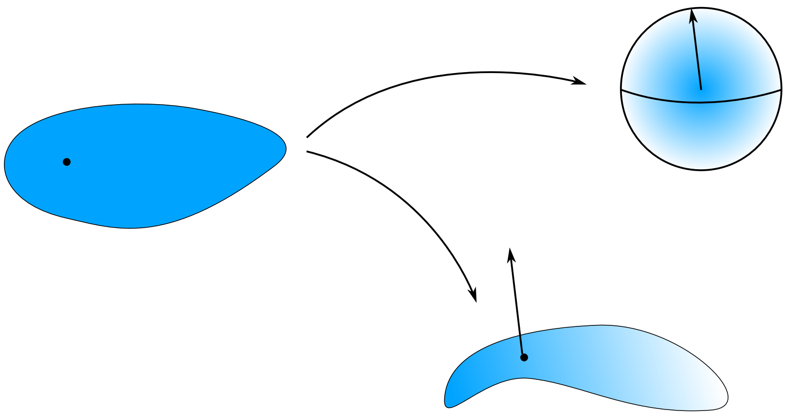
1. Exterior Calculus on Thin Domains



1.1. Outline

In this chapter, we develop a *calculus of thin structures* (films, shells), i.e. subsets of $\mathbb{R}^3$ which are made of points that lie within a small distance of a generating surface. We present decompositions of objects in the thin structure in terms of *tangential and normal components* defined on the generating surface and express the action of various differential and integral operators in terms of these components.

The section 1.2 introduces the cylindrical manifold K, which is a convenient representation of a "thickened" surface. Sections 1.3 - 1.6 are a concise presentation of exterior calculus on K, building towards Sec. 1.7, where we express the classic differential operators on K using differential forms. Proposition 1.55, the decomposition of the differential operators (grad, div, curl) in tangential and normal components, is the main result of the chapter. The rest of the chapter deals with tensor calculus, necessary for handling the vector gradient ∇v of the velocity in Chapter 2. Section 1.8 presents basic definitions and results for second-order tensors in $\mathbb{R}^n$ and in particular the vector gradient.

Our treatment is based on the *exterior calculus of differential forms* and follows notation and definitions from Frankel[Fra04], Do Carmo[DC94] and Bishop and Goldberg[BG80]. We use the *Einstein summation convention*, where the same index repeated (usually, but not necessarily, as a super- and subscript) is summed over all its possible values. Bold symbols, like $\mathbf{s}$ and $\mathbf{N}$, denote vectors in $\mathbb{R}^3$.

1.2. The cylindrical manifold K

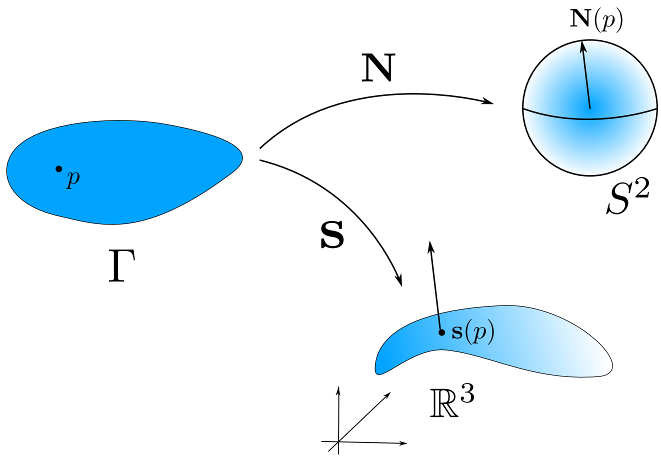


Figure 1.1.: *Embedding of the generating surface.* Embedding $\mathbf{s}$ of Γ in $\mathbb{R}^3$ and the corresponding Gauss map $\mathbf{N}$.

We consider a 2-dimensional manifold Γ and an *embedding* $\mathbf{s} : \Gamma \to \mathbb{R}^3$ (see fig. 1.1). We assume that the embedding is *isometric*, that is the metric g_Γ on Γ is the *pull-back metric*,

$$g_\Gamma(u, v) = \langle d\mathbf{s}(u), d\mathbf{s}(v)\rangle, \quad \forall u, v \in T_p\Gamma, \tag{1.1}$$

where $d\mathbf{s} : T_p\Gamma \to T_{\mathbf{s}(p)}\mathbf{s}\Gamma$ is the *differential (pushforward)* of the embedding $\mathbf{s}$ and $\langle\cdot,\cdot\rangle$ is the euclidean inner product in $\mathbb{R}^3$. Furthermore, we assume that the surface $\mathbf{s}\Gamma \subset \mathbb{R}^3$ is compact, orientable and smooth enough so that the *Gauss map* $\mathbf{N} : \Gamma \to S^2$, which maps points $p \in \Gamma$ to unit normals $\mathbf{N}(p)$, and its differential $d\mathbf{N}$ exist. Because the tangent plane $T_{\mathbf{s}(p)}\mathbf{s}\Gamma$ is perpendicular to $\mathbf{N}(p)$, it can be naturally identified with the tangent space $T_{\mathbf{N}(p)}S^2$, and therefore the differential $d\mathbf{N} : T_p\Gamma \to T_{\mathbf{N}(p)}S^2 \cong T_{\mathbf{s}(p)}\mathbf{s}\Gamma$ induces in a natural way a linear mapping $S : T_p\Gamma \to T_p\Gamma$, called the *shape operator* (or Weingarten map), such that

$$d\mathbf{s}(Su) = -d\mathbf{N}(u), \quad \forall u \in T_p\Gamma. \tag{1.2}$$

The negative sign on the right hand side is due to convention. Considering the shape operator as a (1,1) type tensor, it is a classic result in the differential geometry of surfaces (see §8.2 in [Fra04]) that S is *self-adjoint* (see §1.3). We can identify the tensor invariants $\operatorname{tr} S$ and $\det S$ with the *mean curvature* H and *Gaussian curvature* G of Γ resp.:

$$\operatorname{tr} S =: H, \quad \det S =: G. \tag{1.3}$$

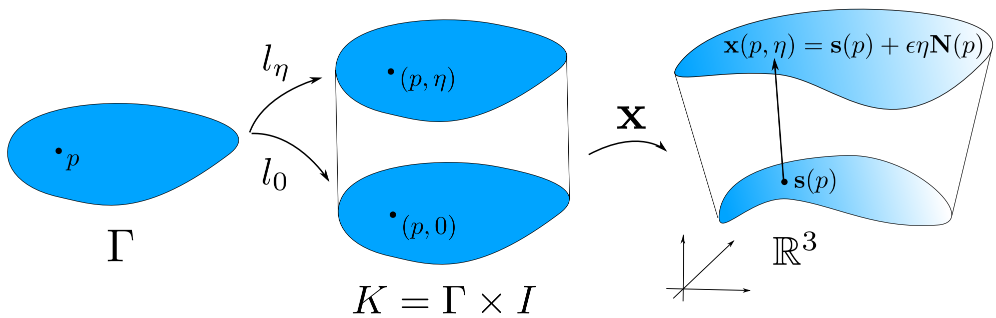


Figure 1.2.: *Embedding of the cylindrical manifold.* Embedding $\mathbf{x}$ of the cylindrical manifold $K = \Gamma \times I$, $I \subset \mathbb{R}$, in $\mathbb{R}^3$ and the 1-parameter family of lifts l_η of Γ into K.

Now we consider the Cartesian product $\Gamma \times I$, where $I \subset \mathbb{R}$ and $0 \in I$, which we map into $\mathbb{R}^3$ via the following mapping (see fig. 1.2):

$$\mathbf{x}(p, \eta) := \mathbf{s}(p) + \epsilon\eta\mathbf{N}(p), \quad \forall p \in \Gamma, \eta \in I \subset \mathbb{R}. \tag{1.4}$$

The *cylindrical manifold* K is the Cartesian product $\Gamma \times I$ with the *pull-back metric*

$$g(u,v) = \langle d\mathbf{x}(u), d\mathbf{x}(v)\rangle, \quad \forall u,v \in T_{(p,\eta)}K. \tag{1.5}$$

Note that this is **not** the natural product metric $g_{M\times N}((u_1,u_2),(v_1,v_2)) = g_M(u_1,v_1) + g_N(u_2,v_2)$, for $(u_1,u_2),(v_1,v_2) \in TM \times TN \cong TM \times N$. Indeed, it is exactly the discrepancy between the product metric $g_{\Gamma\times\mathbb{R}}$ and the pull-back metric g that necessitates a large part of the machinery that we develop in this chapter. We also define the 1-parameter family of *lifts* $l_\eta : p \in \Gamma \mapsto (p,\eta) \in K$ for $\eta \in I$.

The tangent space $T_{(p,\eta)}K$ admits a natural decomposition in terms of *tangential* and *normal* components in the following sense:

Proposition 1.1 (Decomposition of tangent vectors)**.** *For any $u \in T_{(p,\eta)}K$, there exist unique $u_\Gamma \in T_p\Gamma$ and $u_n \in \mathbb{R}$, such that $u = u_\Gamma + u_n\partial\eta$.*

Proof. The tangent space $T_{(p,\eta)}K$ of the product manifold $K = \Gamma \times I$ at a point p is naturally isomorphic to the product $T_p\Gamma \times T_\eta I$, which is in turn isomorphic to the direct sum $T_p\Gamma \oplus T_\eta I$. Let $\{\partial x_1, \partial x_2\}$ be a basis of $T_p\Gamma$ and let $\partial\eta$ be the base vector of the 1-dimensional vector space $T_\eta I \cong \mathbb{R}$, then for any $u \in T_{(p,\eta)}K$ we have $u = u^1\partial x_1 + u^2\partial x_2 + u^3\partial\eta$. Identifying $u_\Gamma := u^1\partial x_1 + u^2\partial x_2$ and $u_n := u^3$ gives us the desired decomposition.

□

Using this decomposition, we can rewrite the metric g in a more useful form:

Proposition 1.2 (Metric g of K)**.** *For any $u,v \in T_{(p,\eta)}K$,*

$$g(u,v) = g_\Gamma(\Lambda_\eta u_\Gamma, \Lambda_\eta v_\Gamma) + \epsilon^2 u_n v_n, \tag{1.6}$$

where $\Lambda_\eta := \mathrm{id} - \epsilon\eta S$ is a self-adjoint tensor.

Proof. From the definition $\mathbf{x}(p,\eta) := \mathbf{s}(p) + \epsilon\eta\mathbf{N}(p)$, we can write the pushforward $d\mathbf{x}(u)$ as

$$\begin{aligned}
d\mathbf{x}(u_\Gamma + u_n\partial\eta) &= d\mathbf{s}(u_\Gamma) + \epsilon\eta\, d\mathbf{N}(u_\Gamma) + \epsilon u_n\mathbf{N}_p \\
&= d\mathbf{s}(u_\Gamma) - \epsilon\eta\, d\mathbf{s}(Su_\Gamma) + \epsilon u_n\mathbf{N}_p \\
&= d\mathbf{s}((\mathrm{id} - \epsilon\eta S)u_\Gamma) + \epsilon u_n\mathbf{N}_p \\
&= d\mathbf{s}(\Lambda_\eta u_\Gamma) + \epsilon u_n\mathbf{N}_p.
\end{aligned}$$

The tensor Λ_η is self-adjoint, because the identity tensor id and the shape tensor S are self-adjoint (see §1.3). Recalling that $\mathbf{N}_p$ is a unit vector and that it is perpendicular to

$d\mathbf{s}(\cdot)$, the metric is then

$$\begin{aligned} g(u,v) &= \langle d\mathbf{x}(u),\, d\mathbf{x}(v)\rangle \\ &= \langle d\mathbf{s}(\Lambda_\eta u_\Gamma) + \epsilon u_n \mathbf{N}_p,\, d\mathbf{s}(\Lambda_\eta v_\Gamma) + \epsilon v_n \mathbf{N}_p\rangle \\ &= \langle d\mathbf{s}(\Lambda_\eta u_\Gamma),\, d\mathbf{s}(\Lambda_\eta v_\Gamma)\rangle + \epsilon^2 u_n v_n \langle \mathbf{N}_p, \mathbf{N}_p\rangle \\ &= g_\Gamma(\Lambda_\eta u_\Gamma, \Lambda_\eta v_\Gamma) + \epsilon^2 u_n v_n. \end{aligned}$$

□

The difference with the product metric $g_{\Gamma\times\mathbb{R}}(u,v) = g_\Gamma(u_\Gamma, v_\Gamma) + u_n v_n$ is now clear.

The following result sets a limit on how large η can be, given ϵ and the curvature of Γ:

Proposition 1.3 (Positive definiteness of g)**.** *The metric g is positive definite, if g_Γ is positive definite and $\lambda_\eta \neq 0$, where $\lambda_\eta := \det(\Lambda_\eta)$.*

Proof. Let $u \neq 0 \Rightarrow u_\Gamma \neq 0$ or $u_n \neq 0$ be an arbitrary vector in TK. If $u_\Gamma \neq 0$, then $\det \Lambda_\eta \neq 0 \Rightarrow \Lambda_\eta u_\Gamma \neq 0$ and so $g_\Gamma(\Lambda_\eta u_\Gamma, \Lambda_\eta u_\Gamma) > 0$, since g_Γ is pos. definite. It follows that $g(u,u) = g_\Gamma(\Lambda_\eta u_\Gamma, \Lambda_\eta u_\Gamma) + \epsilon^2 u_n^2 > 0$. If $u_\Gamma = 0$, then $u_n \neq 0$ and so $g(u,u) = \epsilon^2 u_n^2 > 0$. In any case $g(u,u) > 0$ and so g is positive definite. □

Lemma 1.4 (Scale factor λ_η)**.** *If $\Lambda_\eta = \mathrm{id} - \epsilon\eta S$, then $\det(\Lambda_\eta) = 1 - \epsilon\eta H + \epsilon^2\eta^2 G$, where $H = \mathrm{tr}\, S$ and $G = \det S$.*

Proof. Let κ_1, κ_2 be the eigenvalues of S, with corresponding eigenvectors σ_1, σ_2. Then σ_1 and σ_2 are also eigenvectors of Λ_η with corresponding eigenvalues $\lambda_\alpha := 1 - \epsilon\eta\kappa_\alpha$, since $\Lambda_\eta \sigma_\alpha = \sigma_\alpha - \epsilon\eta S\sigma_\alpha = \sigma_\alpha - \epsilon\eta\kappa_\alpha\sigma_\alpha = \lambda_\alpha\sigma_\alpha$. It follows that $\det\Lambda_\eta = \lambda_1\lambda_2 = (1-\epsilon\eta\kappa_1)(1-\epsilon\eta\kappa_2) = 1-\epsilon\eta(\kappa_1+\kappa_2)+\epsilon^2\eta^2\kappa_1\kappa_2 = 1-\epsilon\eta\,\mathrm{tr}\, S+\epsilon^2\eta^2 \det S = 1-\epsilon\eta H+\epsilon^2\eta^2 G$. □

Remark 1.5. *It follows that **the manifold** $K = \Gamma \times I$ **is well defined only when** $\lambda_\eta \neq 0$ **for all** $\eta \in I$. Noting that $\Lambda_\eta = \mathrm{id} + \mathrm{O}(\epsilon)$, we will make the stronger assumption that Λ_η **is positive definite everywhere in** K, i.e. $\lambda_1, \lambda_2 > 0$.*

The *differential p-forms* $\Omega^p(K)$ also admit a decomposition in tangential and normal components. Consider the following characterization of the basis of the space $\Omega^p(M)$ of p-forms on a manifold M:

Proposition 1.6 (Basis of $\Omega^p(M)$)**.** *Let $\{dx^1, \ldots, dx^n\}$ be a (local) basis of the cotangent bundle T^*M of an n-dimensional manifold M. Then the $\binom{n}{p}$-dimensional space $\Omega^p(M^n)$ admits the (local) basis $\{dx^1 \wedge \ldots \wedge dx^p, \ldots, dx^I, \ldots, dx^{n-p+1} \wedge \ldots \wedge dx^n\}$, where $I \in \mathcal{I}_p^n$ denotes an ordered subset $i_1 < \ldots < i_p$ of the indices $\{1, \ldots, n\}$ and $dx^I := dx^{i_1} \wedge \ldots \wedge dx^{i_p}$, and the base forms are in lexicographic order.*

Proof. We prove this inductively:

For $p = 1$, the space of 1-forms $\Omega^1(M) \equiv T^*M$ has indeed dimension $n = \binom{n}{1}$ and the set $\{dx^1, \ldots, dx^n\}$ has the desired form and is a basis by assumption.

Assume that the space $\Omega^p(M)$ has dimension $\binom{n}{p}$ and the p-forms $\{dx^1 \wedge \ldots \wedge dx^p, \ldots, dx^I, \ldots, dx^{n-p+1} \wedge \ldots \wedge dx^n\}$ form a basis. By definition, every $(p+1)$-form in the space $\Omega^{p+1}(M)$ is the wedge product of a p-form and a 1-form. It follows that it suffices to consider all the possible wedge products between forms in the basis of $\wedge^p M$ and T^*M:

$$dx^I \wedge dx^q \in \{dx^1 \wedge \ldots \wedge dx^p, \ldots, dx^I, \ldots, dx^{n-p+1} \wedge \ldots \wedge dx^n\} \wedge \{dx^1, \ldots, dx^n\}.$$

If $q \in I$ then $dx^I \wedge dx^q = 0$. If $q \notin I$, then we can repeatedly apply the property $\omega \wedge \psi = -\psi \wedge \omega$ of the wedge property to sort the indices of the form:

$$\begin{aligned} dx^I \wedge dx^q = dx^{i_1} \wedge \ldots \wedge dx^{i_p} \wedge dx^q &= -dx^{i_1} \wedge \ldots \wedge dx^{i_{p-1}} \wedge dx^q \wedge dx^{i_p} \\ &= \ldots = (-1)^r dx^{i_1} \wedge \ldots \wedge dx^{i_{p-r}} \wedge dx^q \wedge \ldots \wedge dx^{i_p} \end{aligned}$$

until $i_{p-r} < q$ or we run out of indices, in which case $r = p$. If q is larger than any index in I, and so no juxtapositions are necessary, we let $r = 0$. It follows that in any case, there exists an integer $r \geq 0$ and a order set of $p+1$ indices $\tilde{I} \in \mathcal{I}^n_{p+1}$ so that $dx^I \wedge dx^q = (-1)^r dx^{\tilde{I}}$. The form $dx^{\tilde{I}}$ is exactly an element of the proposed basis and, given that the elements of the proposed basis are obviously products of p-forms and 1-forms, this completes the proof.

□

Proposition 1.7 (Decomposition of p-forms in $\Omega(K)$)**.** *For any $\omega \in \Omega^p(K)$, $p \geq 1$, there exist unique $\omega_\Gamma \in \Omega^p(\Gamma)$ and $\omega_n \in \Omega^{p-1}(\Gamma)$ such that*

$$\omega = \omega_\Gamma + \omega_n \wedge d\eta. \tag{1.7}$$

Proof. If $\{dx^1, dx^2\}$ is a basis for the 1-forms of Γ, we can extend it to a basis $\{dx^1, dx^2, dx^3 \equiv d\eta\}$ for T^*K. Using the previous result, we know that $\{dx^1 \wedge \ldots \wedge dx^p, \ldots, dx^I, \ldots, dx^{n-p+1} \wedge \ldots \wedge dx^n\}$, with $n = 3$, is a basis for $\Omega^p(K)$. The set of indices $\mathcal{I}^3_p$ can be partitioned in two subsets, the subset of indices $\mathcal{I}^2_p \subset \mathcal{I}^3_p$ which do not include the index 3 and therefore the corresponding base form does not include dx^3, and the subset of indices $\mathcal{I}^3_p \setminus \mathcal{I}^2_p$ which do include 3 and therefore the corresponding p-forms can be written as $dx^I = dx^{\tilde{I}} \wedge dx^3$ with $\tilde{I} \in \mathcal{I}^2_{p-1}$. It follows that an arbitrary p-form $\omega \in \Omega^p(K)$ can be decomposed as

$$\omega = \sum_{I \in \mathcal{I}^3_p} \omega_I dx^I = \sum_{I \in \mathcal{I}^2_p} \omega_I dx^I + \sum_{\tilde{I} \in \mathcal{I}^2_{p-1}} \omega_{\tilde{I}}\, dx^{\tilde{I}} \wedge dx^3 = \sum_{I \in \mathcal{I}^2_p} \omega_I dx^I + \left(\sum_{\tilde{I} \in \mathcal{I}^2_{p-1}} \omega_{\tilde{I}} dx^{\tilde{I}} \right) \wedge d\eta.$$

Identifying $\omega_\Gamma := \sum_{I\in\mathcal{I}^2_p} \omega_I dx^I \in \Omega^p(\Gamma)$ and $\omega_n := \sum_{\tilde{I}\in\mathcal{I}^2_{p-1}} \omega_{\tilde{I}} dx^{\tilde{I}} \in \Omega^{p-1}(\Gamma)$ yields the desired result.

□

A special case is the decomposition of the *volume form* vol_K *of* K. The volume form vol_M of an (orientable) Riemannian manifold M can be defined as follows:

Definition 1.8 (Volume form of a manifold)**.** *Let M be an n-dimensional Riemannian manifold with metric g_M, and $\{dx^1, \dots, dx^n\}$ a (local) basis of the cotangent bundle T^*M. Then we define its volume form as the diff. n-form* $\mathrm{vol}_M := \sqrt{\det g_M}\, dx^1 \wedge \dots dx^n$, *where* $\det g_M$ *is understood as the determinant of the matrix representation of the metric tensor $(g_M)_{ij} = g_M(\partial x_i, \partial x_j)$ in the basis $\{\partial x_1, \dots, \partial x_n\}$.*

Proposition 1.9 (Volume forms)**.** *The volume forms of Γ and K satisfy*

$$\mathrm{vol}_K = \epsilon\, \lambda_\eta\, \mathrm{vol}_\Gamma \wedge d\eta, \tag{1.8}$$

where $\lambda_\eta = 1 - \epsilon\eta H + \epsilon^2\eta^2 G$ as in Lemma 1.4.

Proof. Expressed in the basis $\{\partial x_1, \partial x_2\}$, the metric of Γ is $(g_\Gamma)_{\alpha\beta} = g_\Gamma(\partial x_\alpha, \partial x_\beta)$. Likewise, expressed in the basis $\{\partial x_1, \partial x_2, \partial x_3 \equiv \partial\eta\}$, the metric of K is $g_{\alpha\beta} = g_\Gamma(\Lambda_\eta \partial x_\alpha, \Lambda_\eta \partial x_\beta)$, $g_{\alpha 3} = g_{3\alpha} = g_\Gamma(\partial\eta, \Lambda_\eta \partial x_\alpha) = 0$ and $g_{33} = g_\Gamma(\partial\eta, \partial\eta) = \epsilon^2$. We can write the 3×3 matrix of g in block form as

$$g = \begin{pmatrix} \Lambda_\eta^T g_\Gamma \Lambda_\eta & 0 \\ 0 & \epsilon^2 \end{pmatrix}$$

It follows that $\det g = \epsilon^2 \det(\Lambda_\eta^T g_\Gamma \Lambda_\eta) = \epsilon^2 \det(\Lambda_\eta)^2 \det(g_\Gamma) \Rightarrow \sqrt{\det g} = \epsilon\, |\det(\Lambda_\eta)|\, \sqrt{\det g_\Gamma} = \epsilon\lambda_\eta \sqrt{\det g_\Gamma}$. From the definition of vol_M above, we have

$$\begin{aligned} \mathrm{vol}_K = \sqrt{\det g}\, dx_1 \wedge dx_2 \wedge dx_3 &= \epsilon\lambda_\eta \sqrt{\det g_\Gamma}\, dx_1 \wedge dx_2 \wedge d\eta \\ &= \epsilon\lambda_\eta \left(\sqrt{\det g_\Gamma}\, dx_1 \wedge dx_2\right) \wedge d\eta = \epsilon\lambda_\eta\, \mathrm{vol}_\Gamma \wedge d\eta \end{aligned}$$

□

1.3. Musical isomorphisms

On a (finite-dimensional) manifold M, there is a natural correspondence between the vectors in the tangent space T_xM, $x \in M$, and the 1-forms in the cotangent (dual) space T_x^*M. If we consider a basis $\{\partial x_1, \dots, \partial x_n\}$ of T_xM, then ∂x_i is mapped to the dual base 1-form dx^i defined by $dx^i(\partial x_j) = \delta_{ij}$, $1 \le j \le n$. The 1-forms are linear functionals on T_xM, and likewise the vectors are linear functionals on T_x^*M, i.e.

$T_xM \cong T_x^{**}M$. This does not hold when the manifold M is infinitely-dimensional, in which case $T_xM \subset T_x^{**}M$. Note that

$$u(\omega) = \omega(u), \quad \forall u \in T_xM,\, \omega \in T_x^*M.$$

We can write the action of 1-forms and vectors from the cylindrical manifold K on each other in component form:

Proposition 1.10 (Action of vectors and 1-forms on K)**.** *For a vector $v \in T_{(p,\eta)}K$ and a 1-form $\omega \in T^*_{(p,\eta)}K$,*

$$u(\omega) = \omega(u) = \omega_\Gamma(u_\Gamma) + \omega_n u_n. \tag{1.9}$$

Proof. The action of the 1-form $d\eta$ on tangential vectors $u_\Gamma \in T_pK$ and the action of tangential 1-forms $\omega_\Gamma \in T_p^*K$ on the vector $\partial\eta$ vanish by definition:

$$\omega_\Gamma(\partial\eta) = d\eta(u_\Gamma) = 0, \quad \forall u_\Gamma \in T_pK,\, \omega_\Gamma \in T_p^*K.$$

Furthermore, $d\eta(\partial\eta) = 1$. From the linearity of 1-forms and vectors (as operators) we have then:

$$\begin{aligned}
\omega(u) &= (\omega_\Gamma + \omega_n d\eta)(u_\Gamma + u_n\partial\eta) \\
&= \omega_\Gamma(u_\Gamma) + u_n\omega_\Gamma(\partial\eta) + \omega_n d\eta(u_\Gamma) + \omega_n u_n d\eta(\partial\eta) \\
&= \omega_\Gamma(u_\Gamma) + \omega_n u_n.
\end{aligned}$$

□

When M is a *Riemannian* manifold, the metric g_M induces a second mapping between vectors and 1-forms on M. We can "flatten a vector", in the following sense:

Definition 1.11 (Flat op. $\flat$)**.** *For a vector $u \in T_xM$, there is a unique 1-form $u^\flat \in T_x^*M$, such that*

$$u^\flat(v) = g_M(u, v), \quad \forall v \in T_xM. \tag{1.10}$$

Conversely, we can "sharpen a 1-form", in the following sense:

Definition 1.12 (Sharp op. $\sharp$)**.** *For a 1-form $\omega \in T_x^*M$, there is a unique vector $\omega^\sharp \in T_xM$, such that*

$$g_M(\omega^\sharp, v) = \omega(v), \quad \forall v \in T_xM. \tag{1.11}$$

The sharp and the flat are inverses of each other:

Lemma 1.13. *For any $u \in T_xM$ and $\omega \in T_x^*M$,*

$$(u^\flat)^\sharp = u \text{ and } (\omega^\sharp)^\flat = \omega. \tag{1.12}$$

Proof. For any $v \in T_xM$, from the definitions of $\flat$ and $\sharp$, we have

$$g_M((u^\flat)^\sharp, v) = u^\flat(v) = g_M(u, v) \Rightarrow g_M((u^\flat)^\sharp - u, v) = 0.$$

From the positive definitiness of the metric, this implies that $(u^\flat)^\sharp - u = 0 \Rightarrow (u^\flat)^\sharp = u$. Likewise, for any $v \in T_xM$, we have

$$(\omega^\sharp)^\flat(v) = g_M(\omega^\sharp, v) = \omega(v).$$

Two elements of a dual space V^*, whose action coincides on any element of the primal space V, must be equal and so $(\omega^\sharp)^\flat = \omega$.

$\square$

The inverse g_M^{-1} of the metric is a bilinear form on T^*M:

Definition 1.14 (Inverse metric g_M^{-1})**.** *For any $\omega, \psi \in T_x^*M$*

$$g_M^{-1}(\omega, \psi) = g_M(\omega^\sharp, \psi^\sharp) \tag{1.13}$$

or equivalently, for any $u, v \in T_xM$

$$g_M^{-1}(u^\flat, v^\flat) := g_M(u, v). \tag{1.14}$$

This definition is justified by the following lemma:

Lemma 1.15. *For any $u \in T_xM$ and $\omega \in T_x^*M$, the following two propositions are equivalent:*

1. *$g_M(u, \cdot) = \omega$, in the sense that $\forall v \in T_xM$, $g_M(u, v) = \omega(v)$,*
2. *$g_M^{-1}(\omega, \cdot) = u$, in the sense that $\forall \psi \in T_x^*M$, $g_M^{-1}(\omega, \psi) = u(\psi)$.*

Proof. (1. $\Rightarrow$ 2.) The proposition 1. is equivalent to $\omega = u^\flat$. We then have

$$u(\psi) = \psi(u) = g_M(\psi^\sharp, u) = g_M(\psi^\sharp, (u^\flat)^\sharp) = g_M^{-1}(\psi, u^\flat) = g_M^{-1}(\psi, \omega) = g_M^{-1}(\omega, \psi).$$

(2. $\Rightarrow$ 1.) Likewise,

$$\begin{aligned} g_M(u, v) = g_M(u, (v^\flat)^\sharp) = v^\flat(u) = u(v^\flat) \\ = g_M^{-1}(\omega, v^\flat) = g_M(\omega^\sharp, (v^\flat)^\sharp) = g_M(\omega^\sharp, v) = \omega(v). \end{aligned}$$

$\square$

Now consider a type (1,1) tensor $A \in T^1_1(T_xM)$, i.e. a bilinear map $A : T_xM \times T^*_xM \to \mathbb{R}$. The tensor can be thought of as a linear mapping on vectors $A : u \in T_xM \mapsto A(u, \cdot) \in T^{**}_xM \equiv T_xM$, or equivalently as a linear mapping on 1-forms $A : \omega \in T^*_xM \mapsto A(\cdot, \omega) \in T^*_xM$. We use the same symbol for all three mappings, with the exact meaning inferred from the arguments. The equivalence between the three interpretations of A can be expressed then as

$$A(u, \omega) = \omega(Au) = (A\omega)(u), \quad \forall u \in T_xM, \omega \in T^*_xM. \tag{1.15}$$

Note that the action of the *identity tensor* id is simply the application of vectors and 1-forms on each other:

$$\mathrm{id}(u, \omega) := \omega(u) = u(\omega), \quad \forall u \in T_xM, \omega \in T^*_xM. \tag{1.16}$$

Using the musical isomorphisms, we can define the *adjoint of* A as follows:

Definition 1.16 (Adjoint tensor A^*)**.** *For a type (1,1) tensor $A \in T^1_1(T_xM)$, the adjoint A^* is the type (1,1) tensor defined by*

$$A^*(u, \omega) := A(\omega^\sharp, u^\flat), \quad \forall u \in T_xM, \omega \in T^*_xM. \tag{1.17}$$

Proposition 1.17 (Properties of A^*)**.** *The adjoint tensor $A^* \in T^1_1(T_xM)$ satisfies the following properties:*

1. $A^{**} = A$
2. $g_M(Au, v) = g_M(u, A^*v)$ *and* $g^{-1}_M(A\omega, \psi) = g^{-1}_M(\omega, A^*\psi)$
3. $Au^\flat = (A^*u)^\flat$ *and* $A\omega^\sharp = (A^*\omega)^\sharp$

for any $u, v \in T_xM$ and $\omega, \psi \in T^*_xM$.

Proof.

1. For any $u \in T_xM, \omega \in T^*_xM$, $A^{**}(u, \omega) = A^*(\omega^\sharp, u^\flat) = A((u^\flat)^\sharp, (\omega^\sharp)^\flat) = A(u, \omega) \Rightarrow A^{**} = A$.
2. $g_M(Au, v) = v^\flat(Au) = A(u, v^\flat) = A^*(v, u^\flat) = u^\flat(A^*v) = g_M(u, A^*v)$. Likewise for $g^{-1}_M(A\omega, \psi) = g^{-1}_M(\omega, A^*\psi)$.
3. For any $v \in T_xM$, $(A^*u)^\flat(v) = g_M(A^*u, v) = g_M(u, Av) = u^\flat(Av) = (Au^\flat)(v) \Rightarrow Au^\flat = (A^*u)^\flat$. Likewise for $A\omega^\sharp = (A^*\omega)^\sharp$.

□

We are particularly interested in *self-adjoint tensors*:

Definition 1.18 (Self-adjoint tensor)**.** *A tensor $A \in T^1_1(T_xM)$ is called self-adjoint when $A^* = A$.*

The following properties of A^* follow immediately.

Lemma 1.19. *If $A \in T^1_1(T_xM)$ is a self-adjoint tensor, then*

1. *if A is invertible, then A^{-1} is also self-adjoint,*
2. *$g_M(Au, v) = g_M(u, Av)$ and $g_M^{-1}(A\omega, \psi) = g_M^{-1}(\omega, A\psi)$,*
3. *$(Au)^\flat = Au^\flat$ and $(A\omega)^\sharp = A\omega^\sharp$,*

for any $u, v \in T_xM$ and $\omega, \psi \in T_x^*M$.

Proof. For 1., we will show that $(A^{-1}u)^\flat = A^{-1}(u^\flat)$. Indeed

$$A(A^{-1}u)^\flat = (AA^{-1}u)^\flat = u^\flat.$$

2. and 3. are a direct application of the corresponding properties of adjoint tensors for $A^* = A$.

□

Turning our attention to the cylindrical manifold K, we can use the decomposition (1.6) of the metric g to derive decompositions of $\sharp$ and $\flat$ on K in terms of the corresponding operators $\sharp_\Gamma$, $\flat_\Gamma$ on Γ:

Proposition 1.20 (Flat and sharp on K)**.** *For $u \in T_{(p,\eta)}K$ and $\omega \in T_{(p,\eta)}K$, we have*

$$u^\flat = (u_\Gamma + u_n\partial\eta)^\flat = (\Lambda_\eta^2 u_\Gamma)^{\flat_\Gamma} + \epsilon^2 u_n d\eta \tag{1.18}$$

and

$$\omega^\sharp = (\omega_\Gamma + \omega_n d\eta)^\sharp = (\Lambda_\eta^{-2}\omega_\Gamma)^{\sharp_\Gamma} + \epsilon^{-2}\omega_n\partial\eta. \tag{1.19}$$

Proof. For an arbitrary $v \in T_{(p,\eta)}K$,

$$\begin{aligned}
\left((\Lambda_\eta^2 u_\Gamma)^{\flat_\Gamma} + \epsilon^2 u_n d\eta\right)(v) &= (\Lambda_\eta^2 u_\Gamma)^{\flat_\Gamma}(v_\Gamma) + \epsilon^2 u_n v_n && \text{(from (1.9))}\\
&= g_\Gamma(\Lambda_\eta^2 u_\Gamma, v_\Gamma) + \epsilon^2 u_n v_n\\
&= g_\Gamma(\Lambda_\eta u_\Gamma, \Lambda_\eta v_\Gamma) + \epsilon^2 u_n v_n\\
&= g(u, v) && \text{(from (1.6))}\\
&= u^\flat(v).
\end{aligned}$$

Since v was arbitrary, this proves (1.18).

For (1.19), and keeping in mind that Λ_η is self-adjoint, we have

$$\begin{aligned} g((\Lambda_\eta^{-2}\omega_\Gamma)^{\sharp_\Gamma} + \epsilon^{-2}\omega_n\partial\eta,\ v) &= g_\Gamma(\Lambda_\eta(\Lambda_\eta^{-2}\omega_\Gamma)^{\sharp_\Gamma}, \Lambda_\eta v_\Gamma) + \epsilon^2(\epsilon^{-2}\omega_n)v_n \quad &&\text{(from (1.6))} \\ &= g_\Gamma(\Lambda_\eta^2(\Lambda_\eta^{-2}\omega_\Gamma)^{\sharp_\Gamma}, v_\Gamma) + \omega_n v_n \\ &= g_\Gamma((\Lambda_\eta^2\Lambda_\eta^{-2}\omega_\Gamma)^{\sharp_\Gamma}, v_\Gamma) + \omega_n v_n \\ &= g_\Gamma(\omega_\Gamma^{\sharp_\Gamma}, v_\Gamma) + \omega_n v_n \\ &= \omega_\Gamma(v_\Gamma) + \omega_n v_n \\ &= \omega(v) &&\text{(from (1.9))} \\ &= g(\omega^\flat, v). \end{aligned}$$

Since $g((\Lambda_\eta^{-2}\omega_\Gamma)^{\sharp_\Gamma} + \epsilon^{-2}\omega_n\partial\eta - \omega^\sharp,\ v) = 0$ for any v and g is positive definite, this indeed proves (1.19).

□

The inverse metric g^{-1} on K can also be written in terms of components, like the metric g in (1.6):

Proposition 1.21 (Inverse metric g^{-1} on K)**.** *For $\omega, \psi \in T_{(p,\eta)}K$,*

$$g^{-1}(\omega, \psi) = g_\Gamma^{-1}(\Lambda_\eta^{-1}\omega_\Gamma, \Lambda_\eta^{-1}\psi_\Gamma) + \epsilon^{-2}\omega_n\psi_n. \tag{1.20}$$

Proof. We use the decomposition (1.19) of the sharp:

$$\begin{aligned} g^{-1}(\omega, \psi) &= g(\omega^\sharp, \psi^\sharp) \\ &= g_\Gamma(\Lambda_\eta(\Lambda_\eta^{-2}\omega_\Gamma)^{\sharp_\Gamma}, \Lambda_\eta(\Lambda_\eta^{-2}\psi_\Gamma)^{\sharp_\Gamma}) + \epsilon^2(\epsilon^{-2}\omega_n)(\epsilon^{-2}\psi_n) \\ &= g_\Gamma((\Lambda_\eta^{-1}\omega_\Gamma)^{\sharp_\Gamma}, (\Lambda_\eta^{-1}\psi_\Gamma)^{\sharp_\Gamma}) + \epsilon^{-2}\omega_n\psi_n \\ &= g_\Gamma^{-1}(\Lambda_\eta^{-1}\omega_\Gamma, \Lambda_\eta^{-1}\psi_\Gamma) + \epsilon^{-2}\omega_n\psi_n. \end{aligned}$$

□

The metric g and the inverse metric g^{-1} give us norms on vectors and 1-forms on K resp.:

Definition 1.22 (Norms on TK and T^*K)**.** *The norm on TK, which is induced by the metric g, is for a vector $u \in T_{(p,\eta)}K$ given by*

$$|u|^2 := g(u, u) = |\Lambda_\eta u_\Gamma|_\Gamma^2 + \epsilon^2 u_n^2. \tag{1.21}$$

Likewise, for a 1-form $\omega \in T^*_{(p,\eta)}K$ the norm is given by

$$|\omega|^2 := g^{-1}(\omega, \omega) = |\Lambda_\eta^{-1}\omega_\Gamma|_\Gamma^2 + \epsilon^{-2}\omega_n^2. \tag{1.22}$$

1.4. Hodge star

The *Hodge star* of a manifold M^n is a bijective linear operator $\star$ which maps p-forms to $(n-p)$-forms. The Hodge star can be defined by extending the 1-form inner product $\langle \omega, \psi \rangle_M := g_M^{-1}(\omega, \psi)$ to p-forms and then, for a p-form $\omega \in \Omega^p(M^n)$, identifying $\star\omega$ as the unique $(n-p)$-form which satisfies

$$\psi \wedge \star\omega = \langle \psi, \omega \rangle_M \operatorname{vol}_M, \quad \forall \psi \in \Omega^p(M). \tag{1.23}$$

The extension of the inner product can be done as follows:

Definition 1.23 (Inner product in $\Omega(M^n)$)**.** *On a Riemannian manifold M^n with metric $g_M(\cdot,\cdot)$, we define the inner products $\langle \cdot,\cdot \rangle_M : \Omega^p(M^n) \times \Omega^p(M^n) \to \mathbb{R}$ between p-forms, as follows:*

1. *The inner product $\langle \cdot,\cdot \rangle_M$ is bilinear.*
2. *For 1-forms $\omega, \psi \in \Omega^1(M^n)$, $\langle \omega, \psi \rangle_M := g_M^{-1}(\omega, \psi)$.*
3. *For 1-forms $\omega_1, \ldots, \omega_p$ and $\psi_1, \ldots, \psi_p$,*

$$\langle \omega_1 \wedge \ldots \wedge \omega_p, \psi_1 \wedge \ldots \wedge \psi_p \rangle_M := \det\left(\langle \omega_i, \psi_j \rangle_M\right)$$

For alternative (equivalent) definitions of the Hodge star, see §14.1 in [Fra04], §2.22 in [BG80] and the exercises of §1 in [DC94]. For the needs of this section, the following properties of the Hodge star are sufficient:

Definition 1.24 (Properties of Hodge star $\star$)**.** *On a Riemannian manifold M^n,*

1. *The Hodge star operator $\star$ is linear.*
2. *For $\omega \in \Omega^p(M^n)$, $\quad \star\star\omega = (-1)^{p(n-p)}\omega$.*
3. *For $\omega, \psi \in \Omega^p(M^n)$, $\quad \omega \wedge \star\psi = \psi \wedge \star\omega$.*
4. *For 1-forms $\omega, \psi \in \Omega(M^n)$, $\quad \omega \wedge \star\psi = g_M^{-1}(\omega, \psi) \operatorname{vol}_M$.*

On the generating surface Γ, we have the following results for the Hodge star $\star_\Gamma$:

Proposition 1.25 (Hodge star $\star_\Gamma$ on Γ)**.** *On the 2-dimensional manifold Γ,*

1. $\forall f \in \Omega^0(\Gamma), \quad \star_\Gamma f = f \operatorname{vol}_\Gamma$.
2. $\forall \omega, \psi \in \Omega^1(\Gamma), \quad \omega \wedge \star_\Gamma \psi = g_\Gamma^{-1}(\omega, \psi) \operatorname{vol}_\Gamma$ *and* $\star_\Gamma \star_\Gamma \omega = -\omega$.
3. $\forall \omega \in \Omega^2(\Gamma), \quad (\star_\Gamma \omega) \operatorname{vol}_\Gamma = \omega$.

4. The adjoint of $\star_\Gamma$, taken as a type (1,1) tensor, is $-\star_\Gamma$.

5. $(\star_\Gamma \omega)^{\sharp_\Gamma} = - \star_\Gamma \omega^{\sharp_\Gamma}$ and $(\star_\Gamma u)^{\flat_\Gamma} = - \star_\Gamma u^{\flat_\Gamma}$.

Proof. 1. and 2. follow immediately from the general properties of $\star$ above.

For 3., note that every 2-form ω in Γ can be written in a unique manner as a multiple of the volume form vol_Γ. So if $\omega = \alpha \,\mathrm{vol}_\Gamma$, then $\star_\Gamma \omega = \alpha$ and indeed $(\star_\Gamma \omega)\,\mathrm{vol}_\Gamma = \alpha \,\mathrm{vol}_\Gamma = \omega$.

For 4., $\star_\Gamma$ as a linear mapping from 1-forms to 1-forms can be expanded to act on vectors via the relation $(\star_\Gamma u)(\omega) = u(\star_\Gamma \omega)$. It suffices then to show that $g_\Gamma(\star_\Gamma u, v) = g_\Gamma(u, - \star_\Gamma v)$. Indeed,

$$\begin{aligned} g_\Gamma(\star_\Gamma u, v)\,\mathrm{vol}_\Gamma = v^\flat(\star_\Gamma u)\,\mathrm{vol}_\Gamma = (\star_\Gamma v^\flat)(u)\,\mathrm{vol}_\Gamma = g_\Gamma^{-1}(\star_\Gamma v^\flat, u^\flat)\,\mathrm{vol}_\Gamma = \star_\Gamma v^\flat \wedge \star_\Gamma u^\flat \\ = - \star_\Gamma u^\flat \wedge \star_\Gamma v^\flat = \ldots = -g_\Gamma(\star_\Gamma v, u)\,\mathrm{vol}_\Gamma = g_\Gamma(u, - \star_\Gamma v)\,\mathrm{vol}_\Gamma \,. \end{aligned}$$

We will show later (Cor. 1.52) that $\star_\Gamma u$ is the pullback of the cross product $\mathbf{u} \times \mathbf{N}$ from $\mathbb{R}^3$ to Γ.

5. follows then from the properties of adjoint tensors.

□

On the cylindrical manifold K, the Hodge star $\star$ can be decomposed as follows:

Proposition 1.26 (Hodge star $\star$ on K)**.** *On the 3-dimensional manifold K,*

1. *For 0-forms $f \in \Omega^0(K)$,*
$$\star f = \epsilon \, \lambda_\eta \star_\Gamma f \wedge d\eta. \tag{1.24}$$

2. *For 1-forms $\omega \in \Omega^1(K)$,*
$$\star \omega = \star_\Gamma(\epsilon \, \lambda_\eta \Lambda_\eta^{-2} \omega_\Gamma) \wedge d\eta + \epsilon^{-1} \lambda_\eta \omega_n \,\mathrm{vol}_\Gamma, \tag{1.25}$$
where $\omega_\Gamma \in \Omega^1(\Gamma)$ and $\omega_n \in \Omega^0(\Gamma)$.

3. *For 2-forms $\omega \in \Omega^2(K)$,*
$$\star \omega = -\epsilon^{-1} \lambda_\eta^{-1} \Lambda_\eta^2 (\star_\Gamma \omega_n) + \epsilon \, \lambda_\eta^{-1} (\star_\Gamma \omega_\Gamma) d\eta, \tag{1.26}$$
where $\omega_\Gamma \in \Omega^2(\Gamma)$ and $\omega_n \in \Omega^1(\Gamma)$.

4. *For 3-forms $\omega \in \Omega^3(K)$,*
$$\star \omega = \star(\omega_n \wedge d\eta) = \epsilon^{-1} \lambda_\eta^{-1} \star_\Gamma \omega_n, \tag{1.27}$$
where $\omega_n \in \Omega^2(\Gamma)$.

Proof. 1. $\star f = f \operatorname{vol}_K = f \epsilon \,\lambda_\eta \operatorname{vol}_\Gamma \wedge d\eta = \epsilon\,\lambda_\eta (f \operatorname{vol}_\Gamma) \wedge d\eta = \epsilon\,\lambda_\eta \star_\Gamma f \wedge d\eta$.

2. We verify the property (1.23):

$$
\begin{aligned}
&(\omega_\Gamma + \omega_n d\eta) \wedge \left\{\star_\Gamma(\epsilon\,\lambda_\eta \Lambda_\eta^{-2}\psi_\Gamma) \wedge d\eta + \epsilon^{-1}\lambda_\eta \psi_n \operatorname{vol}_\Gamma\right\} \\
=& \,\omega_\Gamma \wedge \star_\Gamma(\epsilon\,\lambda_\eta \Lambda_\eta^{-2}\psi_\Gamma) \wedge d\eta + \omega_n d\eta \wedge (\epsilon^{-1}\lambda_\eta \psi_n \operatorname{vol}_\Gamma) \\
=& \,g_\Gamma^{-1}(\omega_\Gamma, \epsilon\,\lambda_\eta \Lambda_\eta^{-2}\psi_\Gamma) \operatorname{vol}_\Gamma \wedge d\eta + \epsilon^{-1}\lambda_\eta \omega_n \psi_n d\eta \wedge \operatorname{vol}_\Gamma \\
=& \,g_\Gamma^{-1}(\omega_\Gamma, \Lambda_\eta^{-2}\psi_\Gamma)(\epsilon\,\lambda_\eta \operatorname{vol}_\Gamma \wedge d\eta) + \epsilon^{-2}\omega_n\psi_n(\epsilon\,\lambda_\eta \operatorname{vol}_\Gamma \wedge d\eta) \\
=& \left\{g_\Gamma^{-1}(\Lambda_\eta^{-1}\omega_\Gamma, \Lambda_\eta^{-1}\psi_\Gamma) + \epsilon^{-2}\omega_n\psi_n\right\}(\epsilon\,\lambda_\eta \operatorname{vol}_\Gamma \wedge d\eta) \\
=& \,g^{-1}(\omega, \psi) \operatorname{vol}_K .
\end{aligned}
$$

We have used the fact that $\omega_\Gamma \wedge \operatorname{vol}_\Gamma = 0$, for any $\omega_\Gamma \in \Omega^1(\Gamma)$, and that $d\eta \wedge \operatorname{vol}_\Gamma = \operatorname{vol}_\Gamma \wedge d\eta$.

3. Because $\star$ is bijective, it suffices to validate that for any 2-form $\omega \in \Omega^2(K)$, $\star\star\omega = (-1)^{2(3-2)}\omega = \omega$. If we set

$$
\psi_\Gamma = -\epsilon^{-1}\lambda_\eta^{-1}\Lambda_\eta^2(\star_\Gamma \omega_n), \quad \psi_n = \epsilon\,\lambda_\eta^{-1}(\star_\Gamma \omega_\Gamma),
$$

we need to show that $\star(\psi_\Gamma + \psi_n d\eta) = \omega$. Indeed,

$$
\begin{aligned}
\star(\psi_\Gamma + \psi_n d\eta) &= \star_\Gamma(\epsilon\,\lambda_\eta \Lambda_\eta^{-2}\psi_\Gamma) \wedge d\eta + \epsilon^{-1}\lambda_\eta \psi_n \operatorname{vol}_\Gamma \\
&= \star_\Gamma(-\star_\Gamma \omega_n) \wedge d\eta + (\star_\Gamma \omega_\Gamma) \operatorname{vol}_\Gamma \\
&= \omega_n \wedge d\eta + \omega_\Gamma \\
&= \omega.
\end{aligned}
$$

4. Again, it suffices to show that $\star\star\omega = (-1)^{3(3-3)}\omega = \omega$. Indeed,

$$
\begin{aligned}
\star\,(\epsilon^{-1}\lambda_\eta^{-1}\star_\Gamma \omega_n) &= (\epsilon^{-1}\lambda_\eta^{-1}\star_\Gamma \omega_n) \operatorname{vol}_K \\
&= (\epsilon^{-1}\lambda_\eta^{-1}\star_\Gamma \omega_n)(\epsilon\,\lambda_\eta \operatorname{vol}_\Gamma \wedge d\eta) = (\star_\Gamma \omega_n) \operatorname{vol}_\Gamma \wedge d\eta = \omega_n \wedge d\eta,
\end{aligned}
$$

where we used the fact that, as a 2-form on Γ, ω_n satisfies $(\star_\Gamma \omega_n) \operatorname{vol}_\Gamma = \omega_n$.

□

1.5. Exterior & Lie derivatives and the interior product

We start this section by considering a class of operators that act on differential forms and satisfy certain distribution laws with respect to the wedge product.

Definition 1.27 (Derivations and Antiderivations)**.** *A* derivation *is an operator $D : \Omega(M^n) \to \Omega(M^n)$ acting on the differential forms of a manifold M^n, which is additive and satisfies a Leibniz rule with respect to the wedge product:*

$$D(\omega \wedge \psi) = D(\omega) \wedge \psi + \omega \wedge D(\psi) \tag{1.28}$$

for all $\omega \in \Omega^p(M^n)$ and $\psi \in \Omega^q(M^n)$, $0 \leq p, q \leq n$.

Likewise, an antiderivation *is an additive operator $A : \Omega(M^n) \to \Omega(M^n)$, which satisfies*

$$A(\omega \wedge \psi) = A(\omega) \wedge \psi + (-1)^p \, \omega \wedge A(\psi) \tag{1.29}$$

instead.

An (anti-)derivation is uniquely determined by its action on the 0- and 1-forms:

Proposition 1.28 ((Anti-)derivations on $\Omega(M^n)$)**.** *An operator $\hat{L} : \Omega^0(M^n) \cup \Omega^1(M^n) \to \Omega(M^n)$ can be uniquely expanded to an (anti-)derivation $L : \Omega(M^n) \to \Omega(M^n)$.*

Proof. Follows immediately from Prop. 1.6, which dictates that any $\omega^p \in \Omega(M^n)$ can be written as $\sum_{I \in \mathcal{I}_p^n} \omega_I \, dx^{i_1} \wedge \ldots \wedge dx^{i_p}$, where $\omega_I \in \Omega^0(M^n)$. Indeed, if L is a derivation then

$$\begin{aligned}
L(\omega^p) &= L\Big(\sum_{I \in \mathcal{I}_p^n} \omega_I \, dx^{i_1} \wedge \ldots \wedge dx^{i_p}\Big) \\
&= \sum_{I \in \mathcal{I}_p^n} L(\omega_I \, dx^{i_1} \wedge \ldots \wedge dx^{i_p}) \\
&= \sum_{I \in \mathcal{I}_p^n} L(\omega_I \wedge dx^{i_1} \wedge \ldots \wedge dx^{i_p}) \\
&= \sum_{I \in \mathcal{I}_p^n} \big\{ L(\omega_I) \wedge (dx^{i_1} \wedge \ldots \wedge dx^{i_p}) + \omega_I \wedge L(dx^{i_1} \wedge \ldots \wedge dx^{i_p}) \big\} \\
&= \ldots \\
&= \sum_{I \in \mathcal{I}_p^n} \big\{ L(\omega_I) \wedge dx^{i_1} \wedge \ldots \wedge dx^{i_p} + \omega_I \wedge L(dx^{i_1}) \wedge \ldots \wedge dx^{i_p} + \ldots \\
&\qquad\qquad + \omega_I \wedge dx^{i_1} \wedge \ldots \wedge L(dx^{i_p}) \big\} \\
&= \sum_{I \in \mathcal{I}_p^n} \big\{ \hat{L}(\omega_I) \wedge dx^{i_1} \wedge \ldots \wedge dx^{i_p} + \omega_I \wedge \hat{L}(dx^{i_1}) \wedge \ldots \wedge dx^{i_p} + \ldots \\
&\qquad\qquad + \omega_I \wedge dx^{i_1} \wedge \ldots \wedge \hat{L}(dx^{i_p}) \big\}
\end{aligned}$$

Likewise for an antiderivation

$$\begin{aligned}
L(\omega^p) &= \sum_{I\in\mathcal{I}_p^n} L(\omega_I \wedge dx^{i_1} \wedge \ldots \wedge dx^{i_p}) \\
&= \sum_{I\in\mathcal{I}_p^n} \big\{L(\omega_I) \wedge (dx^{i_1} \wedge \ldots \wedge dx^{i_p}) + (-1)^0 \omega_I \wedge L(dx^{i_1} \wedge \ldots \wedge dx^{i_p})\big\} \\
&= \sum_{I\in\mathcal{I}_p^n} \big\{L(\omega_I) \wedge dx^{i_1} \wedge \ldots \wedge dx^{i_p} + \omega_I \wedge L(dx^{i_1}) \wedge (dx^{i_2} \wedge \ldots \wedge dx^{i_p}) \\
&\qquad\qquad + (-1)^1 \omega_I \wedge dx^{i_1} \wedge L(dx^{i_2} \wedge \ldots \wedge dx^{i_p})\big\} \\
&= \ldots \\
&= \sum_{I\in\mathcal{I}_p^n} \big\{L(\omega_I) \wedge dx^{i_1} \wedge \ldots \wedge dx^{i_p} + \ldots \\
&\qquad\qquad + (-1)^{k-1} \omega_I \wedge dx^{i_1} \wedge \ldots \wedge L(dx^{i_k}) \wedge \ldots \wedge dx^{i_p} + \ldots\big\} \\
&= \sum_{I\in\mathcal{I}_p^n} \big\{\hat{L}(\omega_I) \wedge dx^{i_1} \wedge \ldots \wedge dx^{i_p} + \ldots \\
&\qquad\qquad + (-1)^{k-1} \omega_I \wedge dx^{i_1} \wedge \ldots \wedge \hat{L}(dx^{i_k}) \wedge \ldots \wedge dx^{i_p} + \ldots\big\}
\end{aligned}$$

□

First, we use the proposition above to generalize the natural coupling between vectors and 1-forms to general p-forms:

Definition 1.29 (Interior product)**.** *For a vector $v = v^i \partial x_i \in TM^n$, the* interior product *$i_v : \Omega^p(M^n) \to \Omega^{p-1}(M^n)$ is the unique antiderivation which satisfies*

$$i_v f = 0, \quad f \in \Omega^0(M^n) \tag{1.30a}$$

$$i_v dx^i = dx^i(v) = v^i \tag{1.30b}$$

Corollary 1.30 (Interior product of 1-forms)**.** *For any 1-form $\omega \in \Omega^1(M^n)$ and vector $v \in TM^n$,*

$$i_v \omega = \omega(v) \tag{1.31}$$

Proof. If, in terms of components, $v = v^i \partial x_i$ and $\omega = \omega_i dx^i$ then $i_v \omega = i_v(\omega_i dx^i) = i_v(\omega_i \wedge dx^i) = (i_v \omega_i) \wedge dx^i + (-1)^0 \omega_i \wedge i_v dx^i = 0 \wedge dx^i + \omega_i \wedge v^i = \omega_i v^i = \omega(v)$.

□

Corollary 1.31 (Linearity of i_v)**.** *For vectors $u, v \in TM^n$, $a, b \in \mathbb{R}$ and any differential form $\omega \in \Omega(M^n)$,*

$$i_{au+bv}\omega = a\, i_u \omega + b\, i_v \omega \tag{1.32}$$

Proof. The operator $ai_u + bi_v$ is easily shown to be an antiderivation, and furthermore $a\, i_u f + b\, i_v f = 0 = i_{au+bv} f$, for any $f \in \Omega^0(M^n)$. It remains to verify the equality in the 1-form case, where indeed $i_{au+bv}\omega = \omega(au + bv) = a\,\omega(u) + b\,\omega(v) = a\, i_u\omega + b\, i_v\omega$, for any $\omega \in \Omega^1(M^n)$.

□

Corollary 1.32 (Interior product and $\star$). *For any vector $v \in TM^n$,*

$$i_v\, \mathrm{vol}_M = \star v^\flat \tag{1.33}$$

Proof. It suffices to show that $v^\flat \wedge i_v vol_M = |v^\flat|_M^2\, \mathrm{vol}_M$ (see Prop. 1.24). Indeed, since $v^\flat \wedge \mathrm{vol}_M = 0$, we have $i_v(v^\flat \wedge \mathrm{vol}_M) = 0 \Rightarrow i_v v^\flat \wedge \mathrm{vol}_M - v^\flat \wedge i_v\, \mathrm{vol}_M = 0 \Rightarrow v^\flat \wedge i_v\, \mathrm{vol}_M = v^\flat(v)\, \mathrm{vol}_M = |v^\flat|_M^2\, \mathrm{vol}_M$.

□

The exterior derivative is the "natural" derivative for p-forms, since it generalizes the notion of the differential of a scalar function:

Definition 1.33 (Exterior derivative). *The* exterior derivative $d : \Omega^p(M^n) \to \Omega^{p+1}(M^n)$ *is the unique antiderivation which satisfies*

$$df = \frac{\partial f}{\partial x_i} dx^i, \quad f \in \Omega^0(M^n) \tag{1.34a}$$

$$d(dx^i) = 0 \tag{1.34b}$$

Proposition 1.34 (Closed forms). *The exterior derivative vanishes over*

1. exact forms, *i.e. forms $\omega \in \Omega^p(M^n)$ such that there exists $\psi \in \Omega^{p+1}(M^n)$ with $d\psi = \omega$,*

$$d(d\omega) = 0, \quad \omega \in \Omega^p(M^n) \tag{1.35}$$

2. *forms of maximal degree $\omega \in \Omega^n(M^n)$, and in particular the volume form*

$$d\, \mathrm{vol}_M = 0 \tag{1.36}$$

Proof. First we show that $d^2 := d \circ d$ is a derivation. Indeed, it is additive, $d^2(\omega + \psi) = d(d\omega + d\psi) = d^2\omega + d^2\psi$. Furthermore, for $\omega \in \Omega^p(M^n)$ and $\psi \in \Omega^q(M^n)$,

$$\begin{aligned} d^2(\omega \wedge \psi) &= d\,(d\omega \wedge \psi + (-1)^p \omega \wedge \psi) \\ &= d^2\omega \wedge \psi + (-1)^{p+1} d\omega \wedge d\psi + (-1)^p d\omega \wedge d\psi + (-1)^p \omega \wedge d^2\psi \\ &= d^2\omega \wedge \psi + (-1)^p \omega \wedge d^2\psi \end{aligned}$$

As a derivation, it is sufficient to show that d^2 vanishes on 0- and 1-forms. Since $d^2(dx^i) = d(d(dx^i)) = 0$, it remains to show that $d^2 f = 0$. We have

$$
\begin{aligned}
d^2 f = d(df) = d(\frac{\partial f}{\partial x_i} dx^i) = d(\frac{\partial f}{\partial x_i}) \wedge dx^i + \frac{\partial f}{\partial x_i} \wedge d(dx^i) &= \frac{\partial^2 f}{\partial x_i \partial x_j} dx^j \wedge dx^i \\
&= -\frac{\partial^2 f}{\partial x_i \partial x_j} dx^i \wedge dx^j = -\frac{\partial^2 f}{\partial x_j \partial x_i} dx^i \wedge dx^j = -d^2 f
\end{aligned}
$$

which implies that $d^2 f = 0$.

A form of maximal degree $\omega \in \Omega^n(M^n)$ can always be written in the form $\omega = \alpha\, dx^1 \wedge \ldots dx^n$, $\alpha \in \Omega^0(M^n)$. Using the same reasoning as in the antiderivation part of the proof of Prop. 1.28, we can show that

$$
\begin{aligned}
d\omega &= d(\alpha\, dx^1 \wedge \ldots dx^n) \\
&= d\alpha \wedge dx^1 \wedge \ldots dx^n \, + \, \ldots \, + \, (-1)^{k-1} \alpha\, dx^1 \wedge \ldots \wedge d(dx^{i_k}) \wedge \ldots \wedge dx^n \, + \, \ldots \\
&= d\alpha \wedge dx^1 \wedge \ldots dx^n \\
&= (\frac{\partial \alpha}{\partial x_i} dx^i) \wedge dx^1 \wedge \ldots \wedge dx^i \wedge \ldots \wedge dx^n \\
&= 0
\end{aligned}
$$

□

The Lie derivative generalizes the notion of the directional derivative:

Definition 1.35 (Lie derivative)**.** *For a vector field $v = v^i \partial x_i \in TM^n$, the* Lie derivative *$\mathcal{L}_v : \Omega^p(M^n) \to \Omega^p(M^n)$ is the unique derivation which satisfies*

$$
\mathcal{L}_v f = df(v) = \frac{\partial f}{\partial x_i} v^i, \quad f \in \Omega^0(M^n) \tag{1.37a}
$$

$$
\mathcal{L}_v dx^i = dv^i \tag{1.37b}
$$

Corollary 1.36 (Cartan formula)**.** *For a vector field $v \in TM^n$ and a differential form $\omega \in \Omega(M^n)$,*

$$
\mathcal{L}_v \omega = d i_v \omega + i_v d\omega \tag{1.38}
$$

Proof. We will show that $di_v + i_v d$ is an antiderivation like $\mathcal{L}_v$ and that their action on 0- and 1-forms coincides. Then from Prop. 1.27 they are equal. The additivity of $di_v + i_v d$

follows from the additivity of d and i_v. Furthermore, for $\omega \in \Omega^p(M^n)$ and $\psi \in \Omega^q(M^n)$,

$$\begin{aligned}
(di_v + i_v d)(\omega \wedge \psi) &= di_v(\omega \wedge \psi) + i_v d(\omega \wedge \psi) \\
&= d\left(i_v\omega \wedge \psi + (-1)^p \omega \wedge i_v\psi\right) + i_v\left(d\omega \wedge \psi + (-1)^p \omega \wedge d\psi\right) \\
&= (di_v\omega) \wedge \psi + (-1)^{p-1} i_v\omega \wedge d\psi + (-1)^p d\omega \wedge i_v\psi + (-1)^{2p}\omega \wedge (di_v\psi) \\
&\quad + (i_v d\omega) \wedge \psi + (-1)^{p-1} d\omega \wedge i_v\psi + (-1)^p i_v\omega \wedge d\psi + (-1)^{2p}\omega \wedge (i_v d\psi) \\
&= (di_v\omega) \wedge \psi + \omega \wedge (di_v\psi) + (i_v d\omega) \wedge \psi + \omega \wedge (i_v d\psi) \\
&= (di_v\omega + i_v d\omega) \wedge \psi + \omega \wedge (di_v\psi + i_v d\psi)
\end{aligned}$$

For 0-forms, we have $di_v f + i_v df = d(0) + df(v) = \mathcal{L}_v f$, and for base 1-forms $di_v(dx^i) + i_v d(dx^i) = dv^i + i_v(0) = \mathcal{L}_v dx^i$.

□

We turn our attention to the cylindrical manifold K, by extending the partial derivative $\frac{\partial}{\partial\eta}$ to act on p-forms:

Definition 1.37 (Normal derivative). *The normal derivative $\frac{\partial}{\partial\eta} : \Omega(K) \to \Omega(K)$ is the unique derivation which satisfies*

$$\frac{\partial}{\partial\eta}(f) = \frac{\partial f}{\partial\eta}, \quad f \in \Omega^0(K) \tag{1.39}$$

$$\frac{\partial}{\partial\eta}(dx^i) = 0 \tag{1.40}$$

For any p-form $\omega \in \Omega^p(M^n)$, $\mathcal{L}_{\partial\eta}\omega = \frac{\partial\omega}{\partial\eta}$, and if furthermore $\omega = \omega_I\, dx^{i_1} \wedge \ldots \wedge dx^{i_p}$, then $\frac{\partial\omega}{\partial\eta} = \frac{\partial\omega_I}{\partial\eta}\, dx^{i_1} \wedge \ldots \wedge dx^{i_p}$.

Proof. Since $\mathcal{L}_\eta$ and $\frac{\partial}{\partial\eta}$ are both derivations, it is sufficient to check their action on 0- and 1-forms. For a 0-form $f \in \Omega^0(K)$, $\mathcal{L}_{\partial\eta} f = df(\partial\eta) = \frac{\partial f}{\partial\eta}$ and for 1-forms, $\mathcal{L}_{\partial\eta}(dx^i) = d(\delta_{i3}) = 0 = \frac{\partial}{\partial\eta}(dx^i)$.

If $\omega = \omega_I dx^{i_1} \wedge \ldots \wedge dx^{i_p}$, then

$$\begin{aligned}
\frac{\partial\omega}{\partial\eta} &= \frac{\partial}{\partial\eta}\left(\omega_I\, dx^{i_1} \wedge \ldots \wedge dx^{i_p}\right) \\
&= \frac{\partial\omega_I}{\partial\eta}\, dx^{i_1} \wedge \ldots \wedge dx^{i_p} + \omega_I\, \frac{\partial dx^{i_1}}{\partial\eta} \wedge \ldots \wedge dx^{i_p} + \ldots + \omega_I\, dx^{i_1} \wedge \ldots \wedge \frac{\partial dx^{i_p}}{\partial\eta} \\
&= \frac{\partial\omega_I}{\partial\eta}\, dx^{i_1} \wedge \ldots \wedge dx^{i_p}
\end{aligned}$$

□

Recall that any p-form $\omega \in \Omega^p(K)$ admits a (unique) decomposition $\omega = \omega_\Gamma + \omega_n \wedge d\eta$ into tangential $\omega_\Gamma \in \Omega^p(\Gamma)$ and normal $\omega_n \in \Omega^{p-1}(\Gamma)$ parts (Prop. 1.7).

Proposition 1.38 (Exterior derivative of K)**.** *For any $\omega = \omega_\Gamma + \omega_n \wedge d\eta \in \Omega^p(K)$,*

$$\begin{aligned} d\omega &= d_\Gamma \omega + d\eta \wedge \frac{\partial \omega}{\partial \eta} \\ &= d_\Gamma \omega_\Gamma + \left(d_\Gamma \omega_n + (-1)^p \frac{\partial \omega_\Gamma}{\partial \eta} \right) \wedge d\eta \end{aligned} \tag{1.41}$$

More specifically,

$$df = d_\Gamma f + \frac{\partial f}{\partial \eta} d\eta \,, \qquad f \in \Omega^0(K) \tag{1.42a}$$

$$d\omega = d_\Gamma \omega_\Gamma + \left(d_\Gamma \omega_n - \frac{\partial \omega_\Gamma}{\partial \eta} \right) \wedge d\eta \,, \quad \omega \in \Omega^1(K) \tag{1.42b}$$

$$d\omega = \left(d_\Gamma \omega_n + \frac{\partial \omega_\Gamma}{\partial \eta} \right) \wedge d\eta \,, \qquad \omega \in \Omega^2(K) \tag{1.42c}$$

$$d\omega = 0 \,, \qquad \omega \in \Omega^3(K) \tag{1.42d}$$

Proof. First we show that $d\omega = d_\Gamma \omega + d\eta \wedge \frac{\partial \omega}{\partial \eta}$. Indeed, the operator $d_\Gamma + d\eta \wedge \frac{\partial}{\partial \eta}$ is an antiderivation like d, since it is clearly additive and

$$\begin{aligned} d_\Gamma(\omega \wedge \psi) + d\eta \wedge \frac{\partial}{\partial \eta}(\omega \wedge \psi) &= d_\Gamma \omega \wedge \psi + (-1)^p \omega \wedge d_\Gamma \psi + d\eta \wedge \left(\frac{\partial \omega}{\partial \eta} \wedge \psi + \omega \wedge \frac{\partial \psi}{\partial \eta} \right) \\ &= \left(d_\Gamma \omega + d\eta \wedge \frac{\partial \omega}{\partial \eta} \right) \wedge \psi + (-1)^p \omega \wedge \left(d_\Gamma \psi + d\eta \wedge \frac{\partial \psi}{\partial \eta} \right) \end{aligned}$$

For any 0-form $f \in \Omega^0(K)$, $df = \frac{\partial f}{\partial x_i} dx^i = \frac{\partial f}{\partial x_\alpha} dx^\alpha + \frac{\partial f}{\partial \eta} d\eta = d_\Gamma f + d\eta \wedge \frac{\partial f}{\partial \eta}$, and likewise $d(dx^i) = 0 = d_\Gamma(dx^i) + d\eta \wedge \frac{\partial dx^i}{\partial \eta}$. It follows that $d = d_\Gamma + d\eta \wedge \frac{\partial}{\partial \eta}$.

Applying d to $\omega_\Gamma + \omega_n \wedge d\eta$, we have

$$\begin{aligned} d(\omega_\Gamma + \omega_n \wedge d\eta) &= d\omega_\Gamma + d\omega_n \wedge d\eta + (-1)^{p-1} \omega_n \wedge d(d\eta) = d\omega_\Gamma + d\omega_n \wedge d\eta \\ = \left(d_\Gamma \omega_\Gamma + d\eta \wedge \frac{\partial \omega_\Gamma}{\partial \eta} \right) &+ \left(d_\Gamma \omega_n + d\eta \wedge \frac{\partial \omega_n}{\partial \eta} \right) \wedge d\eta = d_\Gamma \omega_\Gamma + d\eta \wedge \frac{\partial \omega_\Gamma}{\partial \eta} + d_\Gamma \omega_n \wedge d\eta \\ &= d_\Gamma \omega_\Gamma + \left((-1)^p \frac{\partial \omega_\Gamma}{\partial \eta} + d_\Gamma \omega_n \right) \wedge d\eta \end{aligned}$$

The expressions (1.42a) - (1.42d) are a direct corollary of this, plus the fact that $d_\Gamma \omega_\Gamma = 0$ for $\omega_\Gamma \in \Omega^2(\Gamma)$ and $d\omega = 0$ for $\omega \in \Omega^3(K)$ due to Prop. 1.34.

□

Proposition 1.39 (Interior product of K). *For any tangential vector $v_\Gamma \in T\Gamma$ and any tangential p-form $\omega_\Gamma \in \Omega^p(\Gamma)$,*

$$i_{v_\Gamma} d\eta = i_{\partial\eta}\omega_\Gamma = 0 \tag{1.43}$$

For a vector $v = v_\Gamma + v_n\partial\eta \in TK$ and a p-form $\omega = \omega_\Gamma + \omega_n \wedge d\eta \in \Omega^p(K)$,

$$i_v\omega = \big(i_{v_\Gamma}\omega_\Gamma + (-1)^{p-1}v_n\omega_n\big) + i_{v_\Gamma}\omega_n \wedge d\eta \tag{1.44}$$

More specifically,

$$\begin{aligned}
&i_v f = 0\,, && f \in \Omega^0(K) && (1.45a)\\
&i_v\omega = i_{v_\Gamma}\omega_\Gamma + v_n\omega_n\,, && \omega \in \Omega^1(K) && (1.45b)\\
&i_v\omega = (i_{v_\Gamma}\omega_\Gamma - v_n\omega_n) + (i_{v_\Gamma}\omega_n)d\eta\,, && \omega \in \Omega^2(K) && (1.45c)\\
&i_v\omega = v_n\omega_n + i_{v_\Gamma}\omega_n \wedge d\eta\,, && \omega \in \Omega^3(K) && (1.45d)
\end{aligned}$$

Proof. The property (1.43) follows directly from $i_v\omega = \omega(v)$, when ω is a 1-form, and the fact that $dx^\alpha(\partial\eta) = d\eta(\partial x^\alpha) = 0$ for the tangential base vectors/1-forms.

From the linearity of i_v with respect to v (Cor. 1.31) and the decompositions of v and ω in tangential/normal components, we have

$$\begin{aligned}
i_v\omega &= i_{v_\Gamma + v_n\partial\eta}\omega\\
&= i_{v_\Gamma}\omega + v_n i_{\partial\eta}\omega\\
&= i_{v_\Gamma}(\omega_\Gamma + \omega_n \wedge d\eta) + v_n i_{\partial\eta}(\omega_\Gamma + \omega_n \wedge d\eta)\\
&= i_{v_\Gamma}\omega_\Gamma + i_{v_\Gamma}\omega_n \wedge d\eta + (-1)^{p-1}\omega_n \wedge i_{v_\Gamma}\partial\eta + v_n\left(i_{\partial\eta}\omega_\Gamma + i_{\partial\eta}\omega_n \wedge d\eta + (-1)^{p-1}\omega_n \wedge i_{\partial\eta}d\eta\right)\\
&= i_{v_\Gamma}\omega_\Gamma + i_{v_\Gamma}\omega_n \wedge d\eta + (-1)^{p-1}v_n\omega_n
\end{aligned}$$

The formulas (1.45a) - (1.45d) are a direct application of the general formula.

□

The decomposition of $\mathcal{L}_v\omega$ in tangential and normal components can be calculated as needed by the decompositions of i_v and d with the help of Cartan's formula $\mathcal{L}_v = di_v + i_v d$.

1.6. Pullback and pushforward

In this section we study the following question: given a map $\phi : M^m \to N^n$ between two manifolds, what is the relation between the differential forms and their operators on the two manifolds? We are particularly interested of course in the case of the embeddings $\mathbf{s} : \Gamma \to \mathbb{R}^3$ and $\mathbf{x} : K \to \mathbb{R}^3$ from Sec. 1.2.

Definition 1.40 (Pullback)**.** *Given an immersion $\phi : M^m \to N^n$ between two manifolds, the* pullback $\phi^* : \Omega(N^n) \to \Omega(M^m)$ *is the unique additive operator that satisfies the following properties:*

$$(\phi^* f)(p) = f(\phi(p))\,, \qquad p \in M, f \in \Omega^0(N) \tag{1.46a}$$

$$\phi^*(d\omega) = d(\phi^*\omega)\,, \qquad \omega \in \Omega(N) \tag{1.46b}$$

$$\phi^*(\omega \wedge \psi) = (\phi^*\omega) \wedge (\phi^*\psi)\,, \quad \omega, \psi \in \Omega(N) \tag{1.46c}$$

Proof. We need to show that the pullback is well-defined. Indeed, the first property defines its action on the 0-forms of the manifold N^n. Since the base 1-forms are of the form $dx^i := d(x_i)$ where $x_i \in \Omega^0(N^n)$ is a coordinate function, the second property fixes the action of ϕ^* on the base 1-forms of N^n. Finally, the third property extends the definition to wedge products of the form $\omega_I \wedge dx^{i_1} \wedge \ldots \wedge dx^{i_p}$ and by additivity to all of $\Omega(N^n)$.

□

Definition 1.41 (Pushforward)**.** *Given an immersion $\phi : M^m \to N^n$ between two manifolds, the* pushforward $d\phi(u) \in T_{\phi(p)}N$ *of a vector $u \in T_pM$ is the (unique) vector that satisfies*

$$(\phi^*\omega)(u) = \omega(d\phi(u)) \tag{1.47}$$

for any 1-form $\omega \in T^*N^n$.

Proof. We can show that the pushforward is well-defined by deriving an explicit expression for it. Applying (1.47) with $d\phi(u) = v^j \partial x_j$ and $\omega = dx^i$, we have

$$(\phi^* dx^i)(u) = dx^i(v^j \partial x_j) \;\Rightarrow\; (\phi^* dx^i)(u) = v^i$$

and so $d\phi(u) = (\phi^* dx_i)(u)\, \partial x_i$.

Then for an arbitrary 1-form $\omega \in T^*N (\equiv \Omega^1(N^n))$, and using the properties of the pullback,

$$\omega(d\phi(u)) = \omega(v^i \partial x_i) = v^i \omega_i = (\phi^* dx_i)(u)\, \omega_i = (\omega_i\, \phi^* dx_i)(u) = (\phi^*(\omega_i dx^i))(u) = (\phi^*\omega)(u)$$

which shows that $v^i \partial x_i$ is indeed $d\phi(u)$.

□

Based on the fact that the vectors in the tangent space of a manifold M can be understood as directional derivatives acting on functions $f : M \to \mathbb{R}$, the following lemma shows us that the pushforward $d\phi$ is indeed the differential of ϕ.

Corollary 1.42 (Differential)**.** *For any $f \in \Omega^0(N)$ and $u \in TM$,*

$$df(d\phi(u)) = d(\phi^* f)(u) \tag{1.48}$$

Proof. We apply (1.47) with $\omega = df$, to get $df(d\phi(u)) = (\phi^* df)(u) = d(\phi^* f)(u)$, using the second property of Def. 1.40.

□

Proposition 1.43 (Maps & musical isomorphisms)**.** *Let $\phi : M^m \to N^n$ be an isometric immersion between two manifolds, so that the metric g_M of the manifold M^m is the pullback of g_N under ϕ, i.e. $g_M(u,v) = g_N(d\phi(u), d\phi(v))$ for all $u, v \in TM$. Then for any $u \in TM$ and $v \in TN$,*

$$v = d\phi(u) \Rightarrow u^\flat = \phi^*(v^\flat) \tag{1.49}$$

and for any $\omega \in TM^*$ and $\psi \in TN^*$,

$$\omega = \phi^*(\psi) \Rightarrow \psi^\sharp = d\phi(\omega^\sharp) \tag{1.50}$$

If furthermore $m = n$, then the following diagram commutes:

$$\begin{tikzcd} TM \arrow[r, "d\phi"] \arrow[d, shift right, "\flat"'] & TN \arrow[d, shift right, "\flat"'] \\ T^*M \arrow[u, shift right, "\sharp"'] & T^*N \arrow[l, "\phi^*"] \arrow[u, shift right, "\sharp"'] \end{tikzcd} \tag{1.51}$$

Proof. By definition, $u^\flat \in T^*M$ is the 1-form that satisfies $u^\flat(w) = g_M(u,w)$ for all $w \in TM$. If $v = d\phi(u)$ then $(\phi^* v^\flat)(w) = v^\flat(d\phi(w)) = g_N(v, d\phi(w)) = g_N(d\phi(u), d\phi(w)) = g_M(u,w)$, and so $\phi^*(v^\flat) = u^\flat$.

Since ϕ is an immersion, the linear map $d\phi : TM \to TN$ is injective. If furthermore $\dim M = \dim N \Rightarrow \dim TM = \dim TN$, then $d\phi$ is necessarily bijective. It follows that the three upper arrows in the diagram represent isomorphisms. The statement (1.49) (read as $\sharp \circ \phi^* \circ \flat = d\phi$) shows then that ϕ^* is exactly the mapping that renders the diagram commutative.

□

Lemma 1.44 (Maps & volume forms)**.** *Let $\phi : M^m \to N^n$ be an isometric embedding between two manifolds, so that $\phi(M) \subset N$ is a submanifold (with the metric induced by g_N, and the orientation induced by ϕ). Then*

$$\mathrm{vol}_M = \phi^*(\mathrm{vol}_{\phi(M)})$$

If furthermore $m = n$, then $\mathrm{vol}_M = \pm\phi^*(\mathrm{vol}_N)$, depending on whether ϕ preserves or reverses the orientation.

Proof. Let $\mathcal{B} = \{\partial x_1, \dots, \partial x_m\}$ be a basis of T_xM. Since ϕ is an embedding, the vectors $\mathrm{d}\phi(\mathcal{B}) = \{\sigma_1, \dots, \sigma_m\}$, $\sigma_i := d\phi(\partial x_i)$, form a basis of the tangent space $T_{\phi(x)}\phi(M)$.

Furthermore, the induced metric $g_{\phi(M)}$, expressed in the basis $d\phi(\mathcal{B})$, has the same matrix with the metric g_M expressed in the basis $\mathcal{B}$:

$$(g_{\phi(M)})_{ij} = g_{\phi(M)}(\sigma_i, \sigma_j) = g_N(d\phi(\partial x_i), d\phi(\partial x_j)) = g_M(\partial x_i, \partial x_j) = (g_M)_{ij}$$

and so, in particular, $\phi^*(\det(g_{\phi(M)})) = \det(g_M)$. Furthermore, let $\{\omega^1, \ldots, \omega^m\}$ be the dual basis of $\phi(M)$, i.e. $\omega^i(\sigma_j) = \delta_{ij}$. Then $\phi^*\omega^i = dx^i$, since $(\phi^*\omega^i)(\partial x_j) = \omega^i(d\phi(\partial x_j)) = \omega^i(\sigma_j) = \delta_{ij}$. It follows that for the volume forms we have

$$\begin{aligned}\phi^*(\mathrm{vol}_{\phi(M)}) &= \phi^*(\sqrt{\det(g_{\phi(M)})}\,\omega^1 \wedge \ldots \wedge \omega^m) \\ &= \phi^*\left(\sqrt{\det(g_{\phi(M)})}\right)\phi^*\omega^1 \wedge \ldots \wedge \phi^*\omega^m = \sqrt{\det(g_M)}\,dx^1 \wedge \ldots \wedge dx^m = \mathrm{vol}_M\end{aligned}$$

When $\dim M = \dim N$, the basis $d\phi(\mathcal{B})$ spans the entire tangent space $T_{\phi(x)}N$. If the orientation of the induced basis $d\phi(\mathcal{B})$ agrees with the orientation of N, then $\phi^*(\mathrm{vol}_N) = \mathrm{vol}_M$, otherwise $\phi^*(\mathrm{vol}_N) = -\mathrm{vol}_M$.

□

Corollary 1.45 (Maps & Hodge star). *Let $\phi : M^m \to N^m$ be a local (orientation-preserving) diffeomorphism. Then*

$$\star(\phi^*\omega) = \phi^*(\star\omega) \tag{1.52}$$

for any $\omega \in \Omega(N)$.

Proof. Follows immediately from the fact that $(\phi^*\omega)\wedge\phi^*(\star\omega) = \phi^*(\omega\wedge\star\omega) = \phi^*(|\omega|^2_N \mathrm{vol}_N) = |\phi^*\omega|^2_M \mathrm{vol}_M$, since $|\phi^*\omega|^2_M = g_M^{-1}(\phi^*\omega, \phi^*\omega) = g_N^{-1}(\omega, \omega) = |\omega|^2_N$.

□

Lemma 1.46 (Maps & int. product). *Let $\phi : M^m \to N^n$ be an isometric immersion between two manifolds. Then*

$$i_u(\phi^*\omega) = \phi^*(i_{d\phi(u)}\omega) \tag{1.53}$$

for any $u \in TM$ and $\omega \in \Omega(N)$.

Proof. The two parts of the equation, taken as operators on $\omega \in \Omega(N)$, are additive. Since all p-forms, for $p > 1$, are sums of wedge products of forms of smaller degree, we can prove this inductively. For 0-forms $f \in \Omega^0(N)$, we have $i_u(\phi^*f) = 0 = \phi^*(i_{d\phi(u)}f)$, and for 1-forms $\omega \in \Omega^1(N)$, $i_u(\phi^*\omega) = (\phi^*\omega)(u) = \phi^*(\omega(d\phi(u))) = \phi^*(i_{d\phi(u)}\omega)$. Then we assume the equality has been proven for all forms in $\Omega^q(N)$ for $q < p$. Then we consider the wedge product $\omega = \omega_1 \wedge \omega_2$, with $\omega_1 \in \Omega^q(N)$ and $\omega_2 \in \Omega^{p-q}(N)$, $0 < p < q$, we have

$$\begin{aligned} i_u(\phi^*\omega) &= i_u(\phi^*(\omega_1 \wedge \omega_2)) = i_u(\phi^*\omega_1 \wedge \phi^*\omega_2) \\ &= i_u(\phi^*\omega_1) \wedge (\phi^*\omega_2) + (-1)^q(\phi^*\omega_1) \wedge i_u(\phi^*\omega_2) \\ &= \phi^*(i_{d\phi(u)}\omega_1) \wedge (\phi^*\omega_2) + (-1)^q(\phi^*\omega_1) \wedge \phi^*(i_{d\phi(u)}\omega_2) \\ &= \phi^*\left((i_{d\phi(u)}\omega_1) \wedge \omega_2 + (-1)^q\omega_1 \wedge (i_{d\phi(u)}\omega_2)\right) \\ &= \phi^*(i_{d\phi(u)}(\omega_1 \wedge \omega_2)) \\ &= \phi^*(i_{d\phi(u)}\omega) \end{aligned}$$

and this generalizes to general p-forms due to additivity.

□

Corollary 1.47 (Maps & Lie derivative)**.** *Let $\phi : M^m \to N^n$ be an isometric immersion between two manifolds. Then*

$$\mathcal{L}_u(\phi^*\omega) = \phi^*(\mathcal{L}_{d\phi(u)}\omega) \tag{1.54}$$

for any $u \in TM$ and $\omega \in \Omega(N)$.

Proof. Follows directly from the Cartan formula 1.36.

□

1.7. Vector calculus with forms

In this section, we combine the results of all the previous sections, in order to derive formulas for the classic differential operators $\operatorname{grad}, \operatorname{div}, \operatorname{curl}$ on the cylindrical manifold K. Recall that K represents a thin volume around the curved hypersurface Γ (fig. 1.2). It makes sense therefore to look for formulas that express these differential operators on K in terms of:

1. differential operators $\operatorname{grad}_\Gamma, \operatorname{div}_\Gamma, \operatorname{curl}_\Gamma$ on Γ,
2. tangential and normal components,
3. thickness parameter ϵ
4. curvature related tensors S and $\Lambda_\eta = \operatorname{id} - \epsilon\eta S$.

First, we define the differential operators $\operatorname{grad}, \operatorname{div}, \operatorname{curl}$ on the manifolds Γ and K in terms of exterior calculus operators:

Definition 1.48 (Diff. operators on Γ). *For scalar fields $f \in \Omega^0(\Gamma)$ and vector fields $u \in T\Gamma$ on Γ, we define the following differential operators:*

$$\operatorname{grad}_\Gamma f := (d_\Gamma f)^{\sharp_\Gamma} \ \in T\Gamma, \tag{1.55}$$
$$\operatorname{div}_\Gamma u := \star_\Gamma d_\Gamma \star_\Gamma u^{\flat_\Gamma} \ \in \Omega^0(\Gamma), \tag{1.56}$$
$$\operatorname{curl}_\Gamma u := \star_\Gamma d_\Gamma u^{\flat_\Gamma} \ \in \Omega^0(\Gamma). \tag{1.57}$$

Definition 1.49 (Diff. operators on K). *For scalar fields $f \in \Omega^0(K)$ and vector fields $u \in TK$ on K, we define the following differential operators:*

$$\operatorname{grad} f := (df)^\sharp \ \in TK, \tag{1.58}$$
$$\operatorname{div} u := \star d \star u^\flat \ \in \Omega^0(K), \tag{1.59}$$
$$\operatorname{curl} u := (\star du^\flat)^\sharp \ \in TK. \tag{1.60}$$

Definition 1.50 (Cross product on K and Γ). *For two vector fields $u_\Gamma, v_\Gamma \in T\Gamma$,*

$$u_\Gamma \times_\Gamma v_\Gamma := i_{v_\Gamma}(\star_\Gamma u_\Gamma^{\flat_\Gamma}) \in \Omega^0(\Gamma) \tag{1.61}$$

and for vector fields $u, v \in TK$,

$$u \times v := (i_v \star u^\flat)^\sharp \in \Omega^1(\Gamma) \tag{1.62}$$

These definitions are justified by the following two results:

Proposition 1.51 (Vector calculus on $\mathbb{R}^3$). *For scalar fields $f : \mathbb{R}^3 \to \mathbb{R}$ and vector fields $\mathbf{u}, \mathbf{v}$ on $\mathbb{R}^3$,*

$$i_\mathbf{v}\mathbf{u}^\flat = \langle \mathbf{u}, \mathbf{v} \rangle \tag{1.63}$$
$$(i_\mathbf{v} \star \mathbf{u}^\flat)^\sharp = \mathbf{u} \times \mathbf{v} \tag{1.64}$$
$$(df)^\sharp = \nabla f \tag{1.65}$$
$$\star d \star \mathbf{u}^\flat = \nabla \cdot \mathbf{u} \tag{1.66}$$
$$(\star d\mathbf{u}^\flat)^\sharp = \nabla \times \mathbf{u} \tag{1.67}$$

Proof. For the scalar product of $\mathbf{u} = u_x\mathbf{x} + u_y\mathbf{y} + u_z\mathbf{z}$ and $\mathbf{v} = v_x\mathbf{x} + v_y\mathbf{y} + v_z\mathbf{z}$, we have

$$\begin{aligned} i_\mathbf{v}\mathbf{u}^\flat &= i_\mathbf{v}(u_x dx + u_y dy + u_z dz) \\ &= u_x\, i_\mathbf{v}dx + u_y\, i_\mathbf{v}dy + u_z\, i_\mathbf{v}dz = u_x v_x + u_y v_y + u_z v_z = \langle \mathbf{u}, \mathbf{v} \rangle \end{aligned}$$

The 2-form $\star\mathbf{u}^\flat$ is $\star(u_x dx + u_y dy + u_z dz) = u_x dy \wedge dz + u_y dz \wedge dx + u_z dx \wedge dy$, and so for the cross product

$$\begin{aligned}
i_{\mathbf{v}} \star \mathbf{u}^\flat &= i_{\mathbf{v}}(u_x dy \wedge dz + u_y dz \wedge dx + u_z dx \wedge dy) \\
&= u_x(i_{\mathbf{v}} dy) \wedge dz - u_x dy \wedge (i_{\mathbf{v}} dz) + u_y(i_{\mathbf{v}} dz) \wedge dx - u_y dz \wedge (i_{\mathbf{v}} dx) \\
&\qquad + u_z(i_{\mathbf{v}} dx) \wedge dy - u_z dx \wedge (i_{\mathbf{v}} dy) \\
&= (u_y v_z - u_z v_y)dx - (u_x v_z - u_z v_x)dy + (u_x v_y - u_y v_x)dz \\
&= (\mathbf{u} \times \mathbf{v})^\flat
\end{aligned}$$

For the gradient of a scalar f, we have

$$(df)^\sharp = \left(\frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz\right)^\sharp = \frac{\partial f}{\partial x}\mathbf{x} + \frac{\partial f}{\partial y}\mathbf{y} + \frac{\partial f}{\partial z}\mathbf{z} = \nabla f.$$

For the divergence of the vector field $\mathbf{u}$, we have

$$\begin{aligned}
\star d \star \mathbf{u}^\flat &= \star d \star (u_x dx + u_y dy + u_z dz) \\
&= \star d(u_x dy \wedge dz + u_y dz \wedge dx + u_z dx \wedge dy) \\
&= \star \left(\frac{\partial u_x}{\partial x}\, dx \wedge dy \wedge dz + \frac{\partial u_y}{\partial y}\, dy \wedge dz \wedge dx + \frac{\partial u_z}{\partial z}\, dz \wedge dx \wedge dy\right) \\
&= \left(\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z}\right) \star (dx \wedge dy \wedge dz) \\
&= \frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z} \\
&= \nabla \cdot \mathbf{u}.
\end{aligned}$$

Finally, for the curl of the vector field , we have

$$\begin{aligned}
\star d\mathbf{u}^\flat &= \star d(u_x dx + u_y dy + u_z dz) \\
&= \star \left(\frac{\partial u_x}{\partial y} dy \wedge dx + \frac{\partial u_x}{\partial z} dz \wedge dx + \frac{\partial u_y}{\partial x} dx \wedge dy\right. \\
&\qquad \left. + \frac{\partial u_y}{\partial z} dz \wedge dy + \frac{\partial u_z}{\partial x} dx \wedge dz + \frac{\partial u_z}{\partial y} dy \wedge dz\right) \\
&= \star \left\{ \left(\frac{\partial u_y}{\partial x} - \frac{\partial u_x}{\partial y}\right) dx \wedge dy + \left(\frac{\partial u_x}{\partial z} - \frac{\partial u_z}{\partial x}\right) dz \wedge dx + \left(\frac{\partial u_z}{\partial y} - \frac{\partial u_y}{\partial z}\right) dy \wedge dz \right\} \\
&= \left(\frac{\partial u_y}{\partial x} - \frac{\partial u_x}{\partial y}\right) dz + \left(\frac{\partial u_x}{\partial z} - \frac{\partial u_z}{\partial x}\right) dy + \left(\frac{\partial u_z}{\partial y} - \frac{\partial u_y}{\partial z}\right) dx \\
&= (\nabla \times \mathbf{u})^\flat
\end{aligned}$$

$\square$

This gives us a useful characterization of the Hodge star $\star_\Gamma u_\Gamma$ of a tangential vector:

Corollary 1.52 ($\star_\Gamma u_\Gamma$ is a cross product)**.** *Let $u_\Gamma \in T\Gamma$ and $\mathbf{u}_\Gamma = d\mathbf{s}(u_\Gamma)$. Then*

$$d\mathbf{s}(\star_\Gamma u_\Gamma) = \mathbf{u}_\Gamma \times \mathbf{N} \tag{1.68}$$

where $\mathbf{N}$ is the unit normal of Γ.

Proof. First we consider the following cross product on K:

$$\begin{aligned} u_\Gamma \times \partial\eta &= (-i_{u_\Gamma} \star \partial\eta^\flat)^\sharp = (-i_{u_\Gamma} \star (\epsilon^2 d\eta))^\sharp = -\epsilon^2 (i_{u_\Gamma}(\epsilon^{-1}\lambda_\eta \operatorname{vol}_\Gamma))^\sharp \\ &= -\epsilon\lambda_\eta (i_{u_\Gamma} \operatorname{vol}_\Gamma)^\sharp = -\epsilon\lambda_\eta (\star_\Gamma u_\Gamma^{\flat_\Gamma})^\sharp = -\epsilon\lambda_\eta \Lambda_\eta^{-2} (\star_\Gamma u_\Gamma^{\flat_\Gamma})^{\sharp_\Gamma} = \epsilon\lambda_\eta\Lambda_\eta^{-2} \star_\Gamma u_\Gamma \end{aligned}$$

and so $\star_\Gamma u_\Gamma = \epsilon^{-1}\lambda_\eta^{-1}\Lambda_\eta^2(u_\Gamma \times \partial\eta)$. Noting that $d\mathbf{x}(u)|_{\eta=0} = d\mathbf{s}(u_\Gamma) + \epsilon u_n \mathbf{N}$ and $\Lambda_0 = id$, $\lambda_0 = 1$, we have

$$\begin{aligned} d\mathbf{s}(\star_\Gamma u_\Gamma) &= d\mathbf{x}(\epsilon^{-1}\lambda_\eta^{-1}\Lambda_\eta^2(u_\Gamma \times \partial\eta))|_{\eta=0} = \epsilon^{-1} d\mathbf{x}(u_\Gamma \times \partial\eta)|_{\eta=0} \\ &= \epsilon^{-1} d\mathbf{x}(u_\Gamma) \times d\mathbf{x}(\partial\eta)|_{\eta=0} = \epsilon^{-1} d\mathbf{s}(u_\Gamma) \times (\epsilon\mathbf{N}) = \mathbf{u}_\Gamma \times \mathbf{N} \end{aligned}$$

□

Corollary 1.53 ($\operatorname{curl}_\Gamma$ and $\operatorname{div}_\Gamma$)**.** *Let $u_\Gamma \in T\Gamma$. Then*

$$\operatorname{curl}_\Gamma(\star_\Gamma u_\Gamma) = -\operatorname{div}_\Gamma u_\Gamma \tag{1.69}$$

Proof.

$$\operatorname{curl}_\Gamma(\star_\Gamma u_\Gamma) = \star_\Gamma d_\Gamma (\star_\Gamma u_\Gamma)^\flat_\Gamma = \star_\Gamma d_\Gamma(-\star_\Gamma (u_\Gamma^{\flat_\Gamma})) = -\star_\Gamma d_\Gamma \star_\Gamma u_\Gamma^{\flat_\Gamma} = -\operatorname{div}_\Gamma u_\Gamma$$

□

Corollary 1.54 (Vector calculus on K)**.** *For $f \in \Omega^0(\mathbb{R}^3)$ and $u, v \in TK$, let $f_K = \mathbf{x}^* f \in \Omega^0(K)$ and $\mathbf{u} = d\mathbf{x}(u)$, $\mathbf{v} = d\mathbf{x}(v)$. Then*

$$\mathbf{x}^*(\langle \mathbf{u}, \mathbf{v} \rangle) = i_v u^\flat \tag{1.70}$$

$$\mathbf{u} \times \mathbf{v} = d\mathbf{x}(u \times v) \tag{1.71}$$

$$\nabla f = d\mathbf{x}(\operatorname{grad} f_K) \tag{1.72}$$

$$\mathbf{x}^*(\nabla \cdot \mathbf{u}) = \operatorname{div} u \tag{1.73}$$

$$\nabla \times \mathbf{u} = d\mathbf{x}(\operatorname{curl} u) \tag{1.74}$$

Proof. For the first relation, we have

$$\mathbf{x}^*(\langle \mathbf{u}, \mathbf{v} \rangle) = \mathbf{x}^*(i_{\mathbf{v}} \mathbf{u}^\flat) = \mathbf{x}^*(i_{d\mathbf{x}(v)} \mathbf{u}^\flat) = i_v \mathbf{x}^*(\mathbf{u}^\flat) = i_v \mathbf{x}^*(d\mathbf{x}(u)^\flat) = i_v u^\flat$$

using the properties of pushforward and pullback from the previous section (in particular Prop. 1.43 and Lem. 1.46) and the fact that $\mathbf{x} : K \to \mathbb{R}^3$ is an orientation-preserving diffeomorphism. For the second relation we can use the same reasoning, together with the commutativity of $\mathbf{x}^*$ and $\star$, to show that $\mathbf{x}^*(i_{\mathbf{v}} \star \mathbf{u}^\flat) = i_v \star u^\flat \Rightarrow \mathbf{u} \times \mathbf{v} = (i_{\mathbf{v}} \star \mathbf{u}^\flat)^\sharp = d\mathbf{x}((i_v \star u^\flat)^\sharp) = d\mathbf{x}(u \times v)$. The rest of the relations are a simple application of the commutativity of the pullback of an isometry with the exterior calculus operators. □

Using the results of the previous sections, we can write the vector operators of K in terms of operators on Γ:

Proposition 1.55 (Decomposition of vec. ops on K)**.** *For scalar fields $f \in \Omega^0(K)$ and vector fields $u, v \in TK$ on K, we have:*

$$\langle u, v \rangle_K = \langle \Lambda_\eta u_\Gamma, \Lambda_\eta v_\Gamma \rangle_\Gamma + \epsilon^2 u_n v_n \tag{1.75}$$

$$u \times v = \epsilon\, \lambda_\eta \Lambda_\eta^{-2} (v_n \star_\Gamma u_\Gamma - u_n \star_\Gamma v_\Gamma) + \epsilon^{-1} \lambda_\eta (u_\Gamma \times_\Gamma v_\Gamma)\, \partial\eta \tag{1.76}$$

where $\langle u, v \rangle_K := g(u, v)$ and $\langle u_\Gamma, v_\Gamma \rangle_\Gamma := g_\Gamma(u_\Gamma, v_\Gamma)$. For the differential operators, we have

$$\operatorname{grad} f = \Lambda_\eta^{-2} \operatorname{grad}_\Gamma f + \epsilon^{-2} \frac{\partial f}{\partial \eta}\, \partial\eta, \tag{1.77}$$

$$\operatorname{div} u = \lambda_\eta^{-1} \operatorname{div}_\Gamma(\lambda_\eta u_\Gamma) + \lambda_\eta^{-1} \frac{\partial}{\partial \eta}(\lambda_\eta u_n), \tag{1.78}$$

$$\operatorname{curl} u = \epsilon^{-1} \lambda_\eta^{-1} \star_\Gamma \left(\epsilon^2 \operatorname{grad}_\Gamma u_n - \frac{\partial}{\partial \eta}(\Lambda_\eta^2 u_\Gamma) \right) + \epsilon^{-1} \lambda_\eta^{-1} \operatorname{curl}_\Gamma(\Lambda_\eta^2 u_\Gamma) \partial\eta. \tag{1.79}$$

Proof. For the scalar product, relation (1.75) is simply (1.6) rewritten using the $\langle \cdot, \cdot \rangle_{K/\Gamma}$

notation. For the cross product, we have

$$\begin{aligned}
u \times v &= (i_v \star u^\flat)^\sharp \\
&= \Big(i_v \star \big((\Lambda_\eta^2 u_\Gamma)^{\flat_\Gamma} + \epsilon^2 u_n d\eta\big)\Big)^\sharp \\
&= \Big(i_v(\epsilon\lambda_\eta u_n \operatorname{vol}_\Gamma + \star_\Gamma (\epsilon\lambda_\eta u_\Gamma)^{\flat_\Gamma} \wedge d\eta)\Big)^\sharp \\
&= \epsilon\lambda_\eta \Big(i_v(u_n \operatorname{vol}_\Gamma + (\star_\Gamma u_\Gamma^{\flat_\Gamma}) \wedge d\eta)\Big)^\sharp \\
&= \epsilon\lambda_\eta \Big(i_{v_\Gamma}(u_n \operatorname{vol}_\Gamma) - v_n \star_\Gamma u_\Gamma^{\flat_\Gamma} + (i_{v_\Gamma} \star_\Gamma u_\Gamma^{\flat_\Gamma})\, d\eta\Big)^\sharp \\
&= \epsilon\lambda_\eta \Big(u_n \star_\Gamma v_\Gamma^{\flat_\Gamma} - v_n \star_\Gamma u_\Gamma^{\flat_\Gamma} + (u_\Gamma \times_\Gamma v_\Gamma)\, d\eta\Big)^\sharp \\
&= \epsilon\lambda_\eta \Big(u_n \Lambda_\eta^{-2} (\star_\Gamma v_\Gamma^{\flat_\Gamma})^{\sharp_\Gamma} - v_n \Lambda_\eta^{-2} (\star_\Gamma u_\Gamma^{\flat_\Gamma})^{\sharp_\Gamma} + \epsilon^{-2}(u_\Gamma \times_\Gamma v_\Gamma)\, \partial\eta\Big) \\
&= \epsilon\lambda_\eta \Lambda_\eta^{-2}(v_n \star_\Gamma u_\Gamma - u_n \star_\Gamma v_\Gamma) + \epsilon^{-1}\lambda_\eta (u_\Gamma \times_\Gamma v_\Gamma)\, \partial\eta
\end{aligned}$$

For the gradient,

$$\begin{aligned}
\operatorname{grad} f = (df)^\flat = \left(d_\Gamma f + \frac{\partial f}{\partial \eta} d\eta\right)^\flat &= \left(\Lambda_\eta^{-2} d_\Gamma f\right)^{\flat_\Gamma} + \epsilon^{-2} \frac{\partial f}{\partial \eta}\, \partial\eta \\
&= \Lambda_\eta^{-2} \left(d_\Gamma f\right)^{\flat_\Gamma} + \epsilon^{-2} \frac{\partial f}{\partial \eta}\, \partial\eta = \Lambda_\eta^{-2} \operatorname{grad}_\Gamma f + \epsilon^{-2} \frac{\partial f}{\partial \eta}\, \partial\eta.
\end{aligned}$$

For the divergence,

$$\begin{aligned}
\operatorname{div} u &= \star d \star u^\flat \\
&= \star d \star \Big((\Lambda_\eta^2 u_\Gamma)^{\flat_\Gamma} + \epsilon^2 u_n d\eta\Big) \\
&= \star d \Big\{\epsilon\lambda_\eta u_n \operatorname{vol}_\Gamma + \star_\Gamma (\epsilon\lambda_\eta u_\Gamma)^{\flat_\Gamma} \wedge d\eta\Big\} \\
&= \star \left\{\left(d_\Gamma \star_\Gamma (\epsilon\lambda_\eta u_\Gamma)^{\flat_\Gamma} + \frac{\partial}{\partial\eta}(\epsilon\lambda_\eta u_n \operatorname{vol}_\Gamma)\right) \wedge d\eta\right\} \\
&= \star \left\{\left(\epsilon\, d_\Gamma \star_\Gamma (\lambda_\eta u_\Gamma)^{\flat_\Gamma} + \epsilon \frac{\partial}{\partial\eta}(\lambda_\eta u_n) \operatorname{vol}_\Gamma + \epsilon\, \lambda_\eta u_n \frac{\partial}{\partial\eta} \operatorname{vol}_\Gamma\right) \wedge d\eta\right\} \\
&= \epsilon^{-1}\lambda_\eta^{-1} \star_\Gamma \left\{\epsilon\, d_\Gamma \star_\Gamma (\lambda_\eta u_\Gamma)^{\flat_\Gamma} + \epsilon \frac{\partial}{\partial\eta}(\lambda_\eta u_n) \operatorname{vol}_\Gamma\right\} \\
&= \lambda_\eta^{-1} \star_\Gamma d_\Gamma \star_\Gamma (\lambda_\eta u_\Gamma)^{\flat_\Gamma} + \lambda_\eta^{-1} \frac{\partial}{\partial\eta}(\lambda_\eta u_n) \star_\Gamma \operatorname{vol}_\Gamma \\
&= \lambda_\eta^{-1} \operatorname{div}_\Gamma(\lambda_\eta u_\Gamma) + \lambda_\eta^{-1} \frac{\partial}{\partial\eta}(\lambda_\eta u_n).
\end{aligned}$$

We used the fact that $\frac{\partial}{\partial\eta}\operatorname{vol}_\Gamma = 0$, since $\operatorname{vol}_\Gamma$ is independent of η.

For the curl, we have

$$\begin{aligned}
\star du^\flat &= \star d\left\{(\Lambda_\eta^2 u_\Gamma)^{\flat_\Gamma} + \epsilon^2 u_n d\eta\right\} \\
&= \star\left\{d_\Gamma(\Lambda_\eta^2 u_\Gamma)^{\flat_\Gamma} + \left(d_\Gamma(\epsilon^2 u_n) - \frac{\partial}{\partial\eta}(\Lambda_\eta^2 u_\Gamma)^{\flat_\Gamma}\right)\wedge d\eta\right\} \\
&= -\epsilon^{-1}\lambda_\eta^{-1}\Lambda_\eta^2 \star_\Gamma \left(d_\Gamma(\epsilon^2 u_n) - \frac{\partial}{\partial\eta}(\Lambda_\eta^2 u_\Gamma)^{\flat_\Gamma}\right) + \epsilon\lambda_\eta^{-1}\star_\Gamma d_\Gamma(\Lambda_\eta^2 u_\Gamma)^{\flat_\Gamma} d\eta \\
&= -\epsilon^{-1}\lambda_\eta^{-1}\Lambda_\eta^2 \star_\Gamma \left(\epsilon^2 \operatorname{grad}_\Gamma u_n - \frac{\partial}{\partial\eta}(\Lambda_\eta^2 u_\Gamma)\right)^{\flat_\Gamma} + \epsilon\lambda_\eta^{-1}\star_\Gamma d_\Gamma(\Lambda_\eta^2 u_\Gamma)^{\flat_\Gamma} d\eta
\end{aligned}$$

and so

$$\begin{aligned}
\operatorname{curl} u = (\star du^\flat)^\sharp &= -\epsilon^{-1}\lambda_\eta^{-1}\left(\star_\Gamma\left(\epsilon^2 \operatorname{grad}_\Gamma u_n - \frac{\partial}{\partial\eta}(\Lambda_\eta^2 u_\Gamma)\right)^{\flat_\Gamma}\right)^{\sharp_\Gamma} + \epsilon^{-2}\epsilon\lambda_\eta^{-1}\operatorname{curl}_\Gamma(\Lambda_\eta^2 u_\Gamma)d\eta \\
&= \epsilon^{-1}\lambda_\eta^{-1}\star_\Gamma\left(\epsilon^2 \operatorname{grad}_\Gamma u_n - \frac{\partial}{\partial\eta}(\Lambda_\eta^2 u_\Gamma)\right) + \epsilon^{-1}\lambda_\eta^{-1}\operatorname{curl}_\Gamma(\Lambda_\eta^2 u_\Gamma)d\eta
\end{aligned}$$

where we used the property $(\star_\Gamma u^\flat)^\sharp = -\star_\Gamma u$ (see Prop. 1.25).

□

1.8. Tensor algebra and tensor calculus in $\mathbb{R}^n$

To be able to express in a mathematical way the notion of *viscosity* and its effects on fluid motion, we need certain basic facts about tensors in Euclidean spaces.

Definition 1.56 (Frame). *A* frame *in $\mathbb{R}^n$ is a set of n (smooth) vector fields $\{\mathbf{e}_1, \dots, \mathbf{e}_n\}$ over (a subset of) $\mathbb{R}^n$ which are linearly independent at every point.*

Definition 1.57 (Dual Frame). *If $\{\mathbf{e}_1, \dots, \mathbf{e}_n\}$ is a frame on $\mathbb{R}^n$, its* dual frame *$\{\mathbf{e}^1, \dots, \mathbf{e}^n\}$ is the unique frame that satisfies the condition $\langle \mathbf{e}_i, \mathbf{e}^j\rangle = \delta_{ij}$.*

Definition 1.58 (Tensor product). *Given two vectors $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$, the* tensor product *$\mathbf{u}\otimes\mathbf{v}$ is the second-order tensor whose action on vectors is $(\mathbf{u}\otimes\mathbf{v})(\mathbf{w}) = \langle\mathbf{v}, \mathbf{w}\rangle\mathbf{u}$.*

Lemma 1.59 (Decomposition of 2nd-order tensors). *Any second-order tensor L in $\mathbb{R}^n$ can be written as a (finite) sum of tensor products $L = \mathbf{u}_i \otimes \mathbf{v}_i$.*

Proof. Selecting a basis $\{\mathbf{e}_1, \dots, \mathbf{e}_n\}$ of $\mathbb{R}^n$, with its associated dual basis $\{\mathbf{e}^1, \dots, \mathbf{e}^n\}$, we claim that $L = (L\mathbf{e}_i)\otimes\mathbf{e}^i$. Indeed, for any vector $\mathbf{e}_j$ in the basis $((L\mathbf{e}_i)\otimes\mathbf{e}^i)(\mathbf{e}_j) = \langle\mathbf{e}^i, \mathbf{e}_j\rangle L\mathbf{e}_i = \delta_{ij}L\mathbf{e}_i = L\mathbf{e}_j$, and hence by linearity $((L\mathbf{e}_i)\otimes\mathbf{e}^i)(\mathbf{v}) = L\mathbf{v}$ for any $\mathbf{v}\in\mathbb{R}^n$.

□

Definition 1.60 (Vector gradient)**.** *For a smooth vector field* $\mathbf{v}$ *in* $\mathbb{R}^n$*, we define its* vector gradient *as the tensor* $\nabla\mathbf{v} := D_{\mathbf{e}_i}\mathbf{v}\otimes\mathbf{e}^i$ *where* $\{\mathbf{e}_1,\dots,\mathbf{e}_n\}$ *is a frame (not necessarily orthonormal or even orthogonal) on* $\mathbb{R}^n$*,* $D_{\mathbf{u}}\mathbf{v}(\mathbf{x}) := \lim_{t\to 0}\frac{\mathbf{v}(\mathbf{x}+t\mathbf{u})-\mathbf{v}(\mathbf{x})}{t}$ *is the directional derivative and* $\{\mathbf{e}^1,\dots,\mathbf{e}^n\}$ *is the dual frame.*

The vector gradient is well-defined, i.e. independent of the choice of frame, because of the following two lemmas:

Lemma 1.61 (Linearity of $D_{\mathbf{u}}\mathbf{v}$ in $\mathbf{u}$)**.** *For* $\lambda,\mu\in\mathbb{R}$*,* $\mathbf{a},\mathbf{b}\in\mathbb{R}^n$ *and* $\mathbf{v}$ *a smooth vector field in* $\mathbb{R}^n$*,*

$$D_{\lambda\mathbf{a}+\mu\mathbf{b}}\mathbf{v} = \lambda D_{\mathbf{a}}\mathbf{v} + \mu D_{\mathbf{b}}\mathbf{v}$$

Proof. Using the Taylor expansion of $\mathbf{v}$ around $\mathbf{x}\in\mathbf{R}^n$, we can show that $\mathbf{v}(\mathbf{x}+t\mathbf{u}) - \mathbf{v}(\mathbf{x}) = t\,L(\mathbf{u}) + \mathrm{O}(t^2)$ where L is a *linear* function of $\mathbf{u}$. The linearity of the directional derivative follows.

□

Lemma 1.62 (Action of vector gradient)**.** *For a vector* $\mathbf{u}\in\mathbb{R}^n$ *and a smooth vector field* $\mathbf{v}$ *in* $\mathbb{R}^n$*,*

$$\nabla\mathbf{v}(\mathbf{u}) = D_{\mathbf{u}}\mathbf{v}$$

Proof. We write $\mathbf{u}$ in terms of the frame as $\mathbf{u} = u^i\mathbf{e}_i$, and note that $\langle\mathbf{u},\mathbf{e}^i\rangle = \langle u^j\mathbf{e}_j,\mathbf{e}^i\rangle = u^j\delta_{ij} = u^i$. Then $\nabla\mathbf{v}(\mathbf{u}) = (D_{\mathbf{e}_i}\mathbf{v}\otimes\mathbf{e}^i)(\mathbf{u}) = \langle\mathbf{e}^i,\mathbf{u}\rangle D_{\mathbf{e}_i}\mathbf{v} = u^i D_{\mathbf{e}_i}\mathbf{v}$ and with the help of the previous lemma, $u^i D_{\mathbf{e}_i}\mathbf{v} = D_{u^i\mathbf{e}_i}\mathbf{v} = D_{\mathbf{u}}\mathbf{v}$.

□

It is useful to write the directional derivative $D_{\mathbf{u}}\mathbf{v}$ in terms of the classic differential operators (∇f, $\nabla\cdot\mathbf{v}\equiv\operatorname{div}\mathbf{v}$, $\nabla\times\mathbf{v}$) of $\mathbb{R}^3$. First we prove two identities which involve the symmetric and antisymmetric parts of the directional derivative (considered as a bilinear form in $\mathbf{u}$ and $\mathbf{v}$):

Lemma 1.63 (Gradient of a dot product)**.** *For two smooth vector fields* $\mathbf{u},\mathbf{v}$ *in* $\mathbb{R}^3$*,*

$$\nabla\langle\mathbf{u},\mathbf{v}\rangle = D_{\mathbf{v}}\mathbf{u} + D_{\mathbf{u}}\mathbf{v} + \mathbf{u}\times(\nabla\times\mathbf{v}) + \mathbf{v}\times(\nabla\times\mathbf{u}) \tag{1.80}$$

Proof. Using column notation for the vectors, expressed in Cartesian coordinates, we have

$$\begin{aligned}
\mathbf{u}\times(\nabla\times\mathbf{v})+\mathbf{v}\times(\nabla\times\mathbf{u}) &= \begin{pmatrix}u_x\\u_y\\u_z\end{pmatrix}\times\begin{pmatrix}\frac{\partial v_z}{\partial y}-\frac{\partial v_y}{\partial z}\\ \frac{\partial v_x}{\partial z}-\frac{\partial v_z}{\partial x}\\ \frac{\partial v_y}{\partial x}-\frac{\partial v_x}{\partial y}\end{pmatrix}+\begin{pmatrix}v_x\\v_y\\v_z\end{pmatrix}\times\begin{pmatrix}\frac{\partial u_z}{\partial y}-\frac{\partial u_y}{\partial z}\\ \frac{\partial u_x}{\partial z}-\frac{\partial u_z}{\partial x}\\ \frac{\partial u_y}{\partial x}-\frac{\partial u_x}{\partial y}\end{pmatrix}\\
&=\begin{pmatrix}u_y(\frac{\partial v_y}{\partial x}-\frac{\partial v_x}{\partial y})-u_z(\frac{\partial v_x}{\partial z}-\frac{\partial v_z}{\partial x})+v_y(\frac{\partial u_y}{\partial x}-\frac{\partial u_x}{\partial y})-v_z(\frac{\partial u_x}{\partial z}-\frac{\partial u_z}{\partial x})\\ u_z(\frac{\partial v_z}{\partial y}-\frac{\partial v_y}{\partial z})-u_x(\frac{\partial v_y}{\partial x}-\frac{\partial v_x}{\partial y})+v_z(\frac{\partial u_z}{\partial y}-\frac{\partial u_y}{\partial z})-v_x(\frac{\partial u_y}{\partial x}-\frac{\partial u_x}{\partial y})\\ u_x(\frac{\partial v_x}{\partial z}-\frac{\partial v_z}{\partial x})-u_y(\frac{\partial v_z}{\partial y}-\frac{\partial v_y}{\partial z})+v_x(\frac{\partial u_x}{\partial z}-\frac{\partial u_z}{\partial x})-v_y(\frac{\partial u_z}{\partial y}-\frac{\partial u_y}{\partial z})\end{pmatrix}\\
&=\begin{pmatrix}u_y\frac{\partial v_y}{\partial x}+u_z\frac{\partial v_z}{\partial x}+v_y\frac{\partial u_y}{\partial x}+v_z\frac{\partial u_z}{\partial x}\\ u_z\frac{\partial v_z}{\partial y}+u_x\frac{\partial v_x}{\partial y}+v_z\frac{\partial u_z}{\partial y}+v_x\frac{\partial u_x}{\partial y}\\ u_x\frac{\partial v_x}{\partial z}+u_y\frac{\partial v_y}{\partial z}+v_x\frac{\partial u_x}{\partial z}+v_y\frac{\partial u_y}{\partial z}\end{pmatrix}-\begin{pmatrix}u_y\frac{\partial v_x}{\partial y}+u_z\frac{\partial v_x}{\partial z}+v_y\frac{\partial u_x}{\partial y}+v_z\frac{\partial u_x}{\partial z}\\ u_z\frac{\partial v_y}{\partial z}+u_x\frac{\partial v_y}{\partial x}+v_z\frac{\partial u_y}{\partial z}+v_x\frac{\partial u_y}{\partial x}\\ u_x\frac{\partial v_z}{\partial x}+u_y\frac{\partial v_z}{\partial y}+v_x\frac{\partial u_z}{\partial x}+v_y\frac{\partial u_z}{\partial y}\end{pmatrix}\\
&=\begin{pmatrix}\frac{\partial u_y v_y}{\partial x}+\frac{\partial u_z v_z}{\partial x}+\frac{\partial u_x v_x}{\partial x}\\ \frac{\partial u_z v_z}{\partial y}+\frac{\partial u_x v_x}{\partial y}+\frac{\partial u_y v_y}{\partial y}\\ \frac{\partial u_x v_x}{\partial z}+\frac{\partial u_y v_y}{\partial z}+\frac{\partial u_z v_z}{\partial z}\end{pmatrix}-\begin{pmatrix}u_x\frac{\partial v_x}{\partial x}+v_x\frac{\partial u_x}{\partial x}+u_y\frac{\partial v_x}{\partial y}+u_z\frac{\partial v_x}{\partial z}+v_y\frac{\partial u_x}{\partial y}+v_z\frac{\partial u_x}{\partial z}\\ u_y\frac{\partial v_y}{\partial y}+v_y\frac{\partial u_y}{\partial y}+u_z\frac{\partial v_y}{\partial z}+u_x\frac{\partial v_y}{\partial x}+v_z\frac{\partial u_y}{\partial z}+v_x\frac{\partial u_y}{\partial x}\\ u_z\frac{\partial v_z}{\partial z}+v_z\frac{\partial u_z}{\partial z}+u_x\frac{\partial v_z}{\partial x}+u_y\frac{\partial v_z}{\partial y}+v_x\frac{\partial u_z}{\partial x}+v_y\frac{\partial u_z}{\partial y}\end{pmatrix}\\
&=\begin{pmatrix}\frac{\partial u_x v_x+u_y v_y+u_z v_z}{\partial x}\\ \frac{\partial u_x v_x+u_y v_y+u_z v_z}{\partial y}\\ \frac{\partial u_x v_x+u_y v_y+u_z v_z}{\partial z}\end{pmatrix}-\begin{pmatrix}u_x\frac{\partial v_x}{\partial x}+u_y\frac{\partial v_x}{\partial y}+u_z\frac{\partial v_x}{\partial z}\\ u_x\frac{\partial v_y}{\partial x}+u_y\frac{\partial v_y}{\partial y}+u_z\frac{\partial v_y}{\partial z}\\ u_x\frac{\partial v_z}{\partial x}+u_y\frac{\partial v_z}{\partial y}+u_z\frac{\partial v_z}{\partial z}\end{pmatrix}-\begin{pmatrix}v_x\frac{\partial u_x}{\partial x}+v_y\frac{\partial u_x}{\partial y}+v_z\frac{\partial u_x}{\partial z}\\ v_x\frac{\partial u_y}{\partial x}+v_y\frac{\partial u_y}{\partial y}+v_z\frac{\partial u_y}{\partial z}\\ v_x\frac{\partial u_z}{\partial x}+v_y\frac{\partial u_z}{\partial y}+v_z\frac{\partial u_z}{\partial z}\end{pmatrix}\\
&=\nabla\langle\mathbf{u},\mathbf{v}\rangle-D_{\mathbf{u}}\mathbf{v}-D_{\mathbf{v}}\mathbf{u}
\end{aligned}$$

□

Lemma 1.64 (Curl of a cross product)**.** *For two smooth vector fields* $\mathbf{u},\mathbf{v}$ *in* $\mathbb{R}^3$,

$$\nabla\times(\mathbf{u}\times\mathbf{v})=D_{\mathbf{v}}\mathbf{u}-D_{\mathbf{u}}\mathbf{v}+(\nabla\cdot\mathbf{v})\mathbf{u}-(\nabla\cdot\mathbf{u})\mathbf{v}\tag{1.81}$$

Proof. Using column notation for the vectors, expressed in Cartesian coordinates, we have

$$
\begin{aligned}
\nabla \times (\mathbf{u} \times \mathbf{v}) &= \nabla \times \left(\begin{pmatrix} u_x \\ u_y \\ u_z \end{pmatrix} \times \begin{pmatrix} v_x \\ v_y \\ v_z \end{pmatrix} \right) = \nabla \times \begin{pmatrix} u_y v_z - u_z v_y \\ u_z v_x - u_x v_z \\ u_x v_y - u_y v_x \end{pmatrix} \\
&= \begin{pmatrix} \frac{\partial u_x v_y - u_y v_x}{\partial y} - \frac{\partial u_z v_x - u_x v_z}{\partial z} \\ \frac{\partial u_y v_z - u_z v_y}{\partial z} - \frac{\partial u_x v_y - u_y v_x}{\partial x} \\ \frac{\partial u_z v_x - u_x v_z}{\partial x} - \frac{\partial u_y v_z - u_z v_y}{\partial y} \end{pmatrix} = \begin{pmatrix} \frac{\partial u_x v_y}{\partial y} + \frac{\partial u_x v_z}{\partial z} \\ \frac{\partial u_y v_z}{\partial z} + \frac{\partial u_y v_x}{\partial x} \\ \frac{\partial u_z v_x}{\partial x} + \frac{\partial u_z v_y}{\partial y} \end{pmatrix} - \begin{pmatrix} \frac{\partial u_y v_x}{\partial y} + \frac{\partial u_z v_x}{\partial z} \\ \frac{\partial u_z v_y}{\partial z} + \frac{\partial u_x v_y}{\partial x} \\ \frac{\partial u_x v_z}{\partial x} + \frac{\partial u_y v_z}{\partial y} \end{pmatrix} \\
&= \begin{pmatrix} \frac{\partial u_x v_y}{\partial y} + \frac{\partial u_x v_z}{\partial z} + \frac{\partial u_x v_x}{\partial x} \\ \frac{\partial u_y v_z}{\partial z} + \frac{\partial u_y v_x}{\partial x} + \frac{\partial u_y v_y}{\partial y} \\ \frac{\partial u_z v_x}{\partial x} + \frac{\partial u_z v_y}{\partial y} + \frac{\partial u_z v_z}{\partial z} \end{pmatrix} - \begin{pmatrix} \frac{\partial u_x v_x}{\partial x} + \frac{\partial u_y v_x}{\partial y} + \frac{\partial u_z v_x}{\partial z} \\ \frac{\partial u_y v_y}{\partial y} + \frac{\partial u_z v_y}{\partial z} + \frac{\partial u_x v_y}{\partial x} \\ \frac{\partial u_z v_z}{\partial z} + \frac{\partial u_x v_z}{\partial x} + \frac{\partial u_y v_z}{\partial y} \end{pmatrix} \\
&= \begin{pmatrix} v_x \frac{\partial u_x}{\partial x} + v_y \frac{\partial u_x}{\partial y} + v_z \frac{\partial u_x}{\partial z} \\ v_x \frac{\partial u_y}{\partial x} + v_y \frac{\partial u_y}{\partial y} + v_z \frac{\partial u_y}{\partial z} \\ v_x \frac{\partial u_z}{\partial x} + v_y \frac{\partial u_z}{\partial y} + v_z \frac{\partial u_z}{\partial z} \end{pmatrix} + \begin{pmatrix} (\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}) u_x \\ (\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}) u_y \\ (\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}) u_z \end{pmatrix} \\
&\quad - \begin{pmatrix} u_x \frac{\partial v_x}{\partial x} + u_y \frac{\partial v_x}{\partial y} + u_z \frac{\partial v_x}{\partial z} \\ u_x \frac{\partial v_y}{\partial x} + u_y \frac{\partial v_y}{\partial y} + u_z \frac{\partial v_y}{\partial z} \\ u_x \frac{\partial v_z}{\partial x} + u_y \frac{\partial v_z}{\partial y} + u_z \frac{\partial v_z}{\partial z} \end{pmatrix} - \begin{pmatrix} (\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z}) v_x \\ (\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z}) v_y \\ (\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z}) v_z \end{pmatrix} \\
&= D_{\mathbf{v}} \mathbf{u} + (\nabla \cdot \mathbf{v}) \mathbf{u} - D_{\mathbf{u}} \mathbf{v} - (\nabla \cdot \mathbf{u}) \mathbf{v}
\end{aligned}
$$

□

From these we can derive the desired expression:

Proposition 1.65 (Directional vector derivative)**.** *For two smooth vector fields* $\mathbf{u}, \mathbf{v}$ *in* $\mathbb{R}^3$*,*

$$D_{\mathbf{u}} \mathbf{v} = \frac{1}{2} \left\{ \nabla \times (\mathbf{v} \times \mathbf{u}) - (\nabla \cdot \mathbf{u}) \mathbf{v} + (\nabla \cdot \mathbf{v}) \mathbf{u} + \nabla \langle \mathbf{v}, \mathbf{u} \rangle - \mathbf{v} \times (\nabla \times \mathbf{u}) - \mathbf{u} \times (\nabla \times \mathbf{v}) \right\} \quad (1.82)$$

Proof. Subtract (1.81) from (1.80) and solve for $D_{\mathbf{u}} \mathbf{v}$.

□

For second-order tensors in $\mathbb{R}^n$, like $\nabla \mathbf{v}$, we will refer to the adjoint with respect to the Euclidean dot-product (see Prop. 1.17) as the *transpose*.

Definition 1.66. *For a second-order tensor* L *in* $\mathbb{R}^n$*, the* transpose L^T *is the (unique) tensor that satisfies the relation* $\langle L\mathbf{u}, \mathbf{v} \rangle = \langle \mathbf{u}, L^T \mathbf{v} \rangle$*, for all* $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$*.*

Lemma 1.67 (Transpose of tensor product)**.** *For the tensor product* $(\mathbf{u} \otimes \mathbf{v})^T = \mathbf{v} \otimes \mathbf{u}$*, for all* $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$*.*

Proof. For any $\mathbf{p}, \mathbf{q} \in \mathbb{R}^n$, $\langle \mathbf{p}, (\mathbf{u} \otimes \mathbf{v})(\mathbf{q}) \rangle = \langle \mathbf{v}, \mathbf{q} \rangle \langle \mathbf{p}, \mathbf{u} \rangle = \langle \mathbf{q}, (\mathbf{v} \otimes \mathbf{u})(\mathbf{p}) \rangle$. □

Definition 1.68 (Symmetric and antisymmetric parts of a tensor)**.** *We define the* symmetric part L_S *and* antisymmetric part L_A *of a second-order tensor* L *as* $L^S := \frac{1}{2}(L + L^T)$ *and* $L^A := \frac{1}{2}(L - L^T)$*. It follows that* $L = L^S + L^A$*.*

Definition 1.69 (Contraction of tensors)**.** *The contraction* $L : K \in \mathbb{R}$ *of two tensors is the unique bilinear tensor function, whose action on tensor products is given by*

$$(\mathbf{u} \otimes \mathbf{v}) : (\mathbf{p} \otimes \mathbf{q}) := \langle \mathbf{u}, \mathbf{p} \rangle \langle \mathbf{v}, \mathbf{q} \rangle$$

The definition then is extended to general 2nd-order tensors using Lemma 1.59 and the bilinearity of the contraction. In particular, $L : (\mathbf{p} \otimes \mathbf{q}) = \langle \mathbf{p}, L\mathbf{q} \rangle$ *.*

Lemma 1.70 (Contraction and symmetry)**.** *For two second-order tensors* L, K*,*

$$L^S : K = L : K^S = L^S : K^S$$

Proof. It is a direct result of the symmetry of the contraction and the fact that $L^A : K^S = 0$. Indeed, in the special case where $K = \mathbf{p} \otimes \mathbf{q}$, we have

$$\begin{aligned} 4L^A : K^S = 2L^A : (\mathbf{p} \otimes \mathbf{q} + \mathbf{q} \otimes \mathbf{p}) &= (L - L^T) : (\mathbf{p} \otimes \mathbf{q} + \mathbf{q} \otimes \mathbf{p}) \\ &= \langle \mathbf{p}, L\mathbf{q} \rangle + \langle \mathbf{q}, L\mathbf{p} \rangle - \langle \mathbf{p}, L^T\mathbf{q} \rangle - \langle \mathbf{q}, L^T\mathbf{p} \rangle \\ &= \langle L^T\mathbf{p}, \mathbf{q} \rangle + \langle L^T\mathbf{q}, \mathbf{p} \rangle - \langle \mathbf{p}, L^T\mathbf{q} \rangle - \langle \mathbf{q}, L^T\mathbf{p} \rangle = 0 \end{aligned}$$

and the general result $L^A : K^S = 0$ follows by linearity and Lemma 1.59. □

Definition 1.71 (Identity tensor)**.** *The* identity tensor I *is defined by its action* $I(\mathbf{u}) = \mathbf{u}$ *on vectors. A possible decomposition in tensor products, for a given basis* $\mathbf{e}_i$*, is* $I = \mathbf{e}_i \otimes \mathbf{e}^i$*.*

Definition 1.72 (Trace of tensor)**.** *The trace* $\mathrm{tr}(L)$ *of a second-order tensor* L *is its contraction with the identity tensor,* $\mathrm{tr}(L) := I : L$*. In particular,* $\mathrm{tr}(\mathbf{u} \otimes \mathbf{v}) = \langle \mathbf{u}, \mathbf{v} \rangle$*.*

Definition 1.73 (Divergence of a vector)**.** *For a smooth vector field* $\mathbf{v}$ *on* $\mathbb{R}^n$ *and a frame* $\{\mathbf{e}_1, \dots, \mathbf{e}_n\}$*, we identify the* divergence *of* $\mathbf{v}$ *with the trace of its (vector) gradient, i.e.*

$$\operatorname{div} \mathbf{v} = \mathrm{tr}(\nabla \mathbf{v}) = \langle D_{\mathbf{e}_i} \mathbf{v}, \mathbf{e}^i \rangle$$

Note that in a fixed frame, such that $D_{\mathbf{e}_i} \mathbf{e}_j = 0$*, we have* $\operatorname{div} \mathbf{v} = \langle D_{\mathbf{e}_i}(v^j \mathbf{e}_j), \mathbf{e}^i \rangle = (D_{\mathbf{e}_i} v^j) \langle \mathbf{e}_j, \mathbf{e}^i \rangle = (D_{\mathbf{e}_i} v^j) \delta_{ij} = D_{\mathbf{e}_i} v^i$*, which matches the classic definition.*

Definition 1.74 (Divergence of a second-order tensor)**.** *We define the* divergence *of a tensor product* $\mathbf{p} \otimes \mathbf{q}$ *as*

$$\operatorname{div}(\mathbf{p} \otimes \mathbf{q}) := (D_{\mathbf{e}_i}\mathbf{q} \otimes \mathbf{p} + \mathbf{q} \otimes D_{\mathbf{e}_i}\mathbf{p})(\mathbf{e}^i) = \nabla\mathbf{q}(\mathbf{p}) + \operatorname{div}(\mathbf{p})\mathbf{q}$$

The extension to a general second-order tensor L is uniquely determined by linearity and Lemma 1.59.

Proposition 1.75 (Leibniz rule for divergence of tensor-vector product)**.** *For a tensor field L and a vector field $\mathbf{v}$ in $\mathbb{R}^n$,*

$$\operatorname{div}(L\mathbf{v}) = \langle \operatorname{div} L, \mathbf{v}\rangle + L^T : \nabla\mathbf{v}$$

Proof. For $L = \mathbf{p} \otimes \mathbf{q}$,

$$\begin{aligned}
\operatorname{div}((\mathbf{p} \otimes \mathbf{q})(\mathbf{v})) =& \operatorname{tr}(\nabla((\mathbf{p} \otimes \mathbf{q})(\mathbf{v}))) \\
=& \langle D_{\mathbf{e}_i}((\mathbf{p} \otimes \mathbf{q})(\mathbf{v})), \mathbf{e}^i\rangle \\
=& \langle (D_{\mathbf{e}_i}\mathbf{p} \otimes \mathbf{q})(\mathbf{v}) + (\mathbf{p} \otimes D_{\mathbf{e}_i}\mathbf{q})(\mathbf{v}) + (\mathbf{p} \otimes \mathbf{q})(D_{\mathbf{e}_i}\mathbf{v}), \mathbf{e}^i\rangle \\
=& (D_{\mathbf{e}_i}\mathbf{p} \otimes \mathbf{q} + \mathbf{p} \otimes D_{\mathbf{e}_i}\mathbf{q}) : (\mathbf{v} \otimes \mathbf{e}^i) + (\mathbf{p} \otimes \mathbf{q}) : (\mathbf{e}^i \otimes D_{\mathbf{e}_i}\mathbf{v}) \\
=& \langle (D_{\mathbf{e}_i}\mathbf{q} \otimes \mathbf{p} + \mathbf{q} \otimes D_{\mathbf{e}_i}\mathbf{p})(\mathbf{e}^i), \mathbf{v}\rangle + (\mathbf{q} \otimes \mathbf{p}) : (D_{\mathbf{e}_i}\mathbf{v} \otimes \mathbf{e}^i) \\
=& \langle \operatorname{div}(\mathbf{p} \otimes \mathbf{q}), \mathbf{v}\rangle + (\mathbf{q} \otimes \mathbf{p}) : \nabla\mathbf{v}
\end{aligned}$$

Note that we have used the following property of the directional derivative

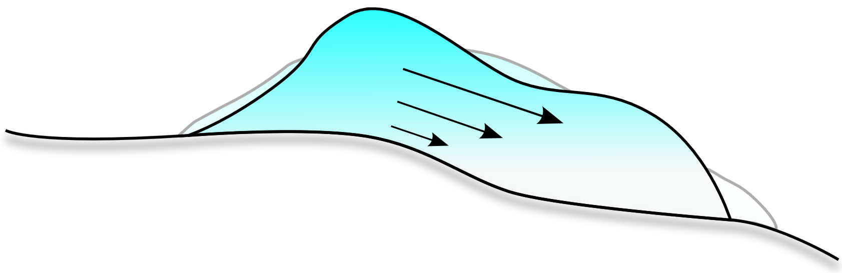
$$D_{\mathbf{e}_i}((\mathbf{p} \otimes \mathbf{q})(\mathbf{v})) = (D_{\mathbf{e}_i}\mathbf{p} \otimes \mathbf{q})(\mathbf{v}) + (\mathbf{p} \otimes D_{\mathbf{e}_i}\mathbf{q})(\mathbf{v}) + (\mathbf{p} \otimes \mathbf{q})(D_{\mathbf{e}_i}\mathbf{v})$$

which follows from the fact that $(\mathbf{p} \otimes \mathbf{q})(\mathbf{v}) = \langle \mathbf{q}, \mathbf{v}\rangle\mathbf{p}$, which is linear in all three of its arguments.

The general result follows then by the linearity and Lemma 1.59.

□

2. A Reduced Model of Thin Film Motion



2.1. Outline

In this chapter we develop a variational model for the flow of a thin viscous liquid film on a stationary curved surface under the influence of gravity and surface tension, in a process analogous to the *lubrication approximation* that yields the classic thin film equation in the flat case. In [ODB97], one can find a presentation of the lubrication approximation with various extensions and generalizations, whereas in [GO03] the authors present a rigorous lubrication approximation of the Darcy flow. The work that is perhaps most relevant is [RRS02], where the lubrication approximation is derived in the case of a curved substrate via the centre manifold technique. The 4th order PDE derived there can indeed be shown to be equivalent to the Euler-Lagrange equation of the reduced model that we arrive at in this chapter (see Rem. 2.56).

We begin the chapter with a concise presentation of the theory of *constrained optimization problems* in Sec. 2.2, essential given that our model for the flow will be stated as one. The key result is the *Brezzi splitting theorem* 2.10, which gives us sufficient conditions for the well-posedness of such a problem. Section 2.3 collects results from the theory of *shape calculus*, which will provide the bridge between the fluid mechanics on one hand and the geometry of the problem on the other. In particular, propositions 2.23 and 2.24 describe how domain and boundary integrals vary as a domain (the thin film) is deformed by a vector field (the velocity field of the fluid).

In Sec. 2.4, we present a variational form of the Stokes equations[1], which govern the flow of the film, and use shape calculus to reveal the gradient flow nature of the problem. Section 2.5 pursues the shape calculus connection further, to reveal that the deformation of the free surface under the velocity field can be described with a PDE for the height parameter (Prop. 2.38). Sections 2.6 and 2.7 present reduced forms of the free energy and rate-of-dissipation functionals. The reduction is essentially done by taking asymptotic expansions with respect to the thickness parameter ϵ and dropping the $\mathrm{O}(\epsilon^2)$ terms.

Corollary 2.47, which is the main result of Sec. 2.7, shows that the dissipation functional is of the form $\int_\Gamma \int_0^h f(\frac{\partial v_\Gamma}{\partial \eta})\, d\eta \,\mathrm{vol}_\Gamma$, where $\frac{\partial v_\Gamma}{\partial \eta}$ is the normal derivative of the tangential velocity, i.e. the dissipation is dominated by the shear stress. In Sec. 2.8, we find an optimal velocity profile which minimizes the shear stress, and therefore the dissipation itself. This completes the reduction of the problem to one which is stated exclusively on the substrate Γ, in the sense that the solution of the reduced problem can be expanded into a nearly optimal solution of the Stokes equations (in their variational form), as Prop. 2.53 of Sec. 2.9 shows.

[1] It is worth noting that the there is no such variational formulation for the full Navier Stokes equations.

2.2. Constrained optimization and saddle-point problems

Given our stated purpose of constructing a variational model for the flow of thin viscous films, we need certain results from the calculus of variations. More specifically, we are interested in the well-posedness of the solutions of certain saddle point problems, developed mainly by Brezzi [Bre74] and Babuška [Bab73] in order to study finite element methods in mixed spaces. We follow the presentation in [GF09].

First we look at the existence and uniqueness of solutions to a certain class of *unconstrained minimization problems.*

Definition 2.1 (Continuity)**.** *Let X, Y be Hilbert spaces. The bilinear form $b : X \times Y \to \mathbb{R}$ is* continuous *iff*

$$\exists M > 0, \forall u \in X, v \in Y : |b(u,v)| \leq M \|u\|_X \|v\|_Y \tag{2.1}$$

The continuity constant $\|b\|$ *is the smallest constant M that satisfies* (2.1)*.*

Definition 2.2 (Coercivity)**.** *Let X be a Hilbert space. The bilinear form $a : X \times X \to \mathbb{R}$ is (X-)coercive iff*

$$\exists \alpha > 0, \forall u \in X : a(u,u) \geq \alpha \|u\|_X^2 \tag{2.2}$$

The largest such α is the coercivity constant *of a.*

Theorem 2.3 (Lax-Milgram)**.** *Let X be a Hilbert space and $a : X \times X \to \mathbb{R}$ a continuous and coercive bilinear form. Then for any $e \in X'$, there exists a unique $u \in X$ such that*

$$a(u,v) = e(v),\ \forall v \in X \tag{2.3}$$

Furthermore,

$$\|u\|_X \leq \frac{\|e\|_{X'}}{\alpha} \tag{2.4}$$

Lemma 2.4. *If $\alpha \geq 0$ then $\frac{\lambda^2}{2}\alpha + \lambda\beta \geq 0$, $\forall \lambda \in \mathbb{R}$, if and only if $\beta = 0$.*

Proof. If $\beta = 0$, then $\frac{\lambda^2}{2}\alpha \geq 0$ by assumption. For the other direction, let $f(\lambda) := \frac{\lambda^2}{2}\alpha + \lambda\beta$, which is convex and so the single critical point $f'(\lambda) = 0 \Rightarrow \lambda = -\frac{\beta}{\alpha}$ is a minimum. It follows that $f(\lambda) \geq 0$, $\forall \lambda \in \mathbb{R} \Leftrightarrow f(-\frac{\beta}{\alpha}) = -\frac{\beta^2}{\alpha} \geq 0$, which implies $\beta = 0$. □

Proposition 2.5 (Unconstrained minimization)**.** *Let X be a Hilbert space and $a : X \times X \to \mathbb{R}$ a symmetric and positive (i.e. $a(u,u) \geq 0$ for all $u \in X$) bilinear form. For a fixed $e \in X'$, the following statements are equivalent:*

1. *$u \in X$ is such that $a(u,v) = e(v)$, $\forall v \in X$*

2. $u \in X$ is a minimizer on X of the functional

$$J(v) := \frac{1}{2}a(v,v) - e(v) \tag{2.5}$$

If moreover $a(\cdot,\cdot)$ is continuous and coercive, the minimizer exists and is unique, i.e.

$$u = \operatorname*{argmin}_{v \in X} J(v) \tag{2.6}$$

Proof. The mapping $(\lambda, v) \in \mathbb{R} \times X \mapsto u + \lambda v =: \tilde{v} \in X$ can be easily shown to be onto. It follows that

$$\begin{aligned}
2. &\Leftrightarrow \forall \tilde{v} \in X : J(u) \leq J(\tilde{v}) \\
&\Leftrightarrow \forall \lambda \in \mathbb{R}, v \in X : J(u) \leq J(u + \lambda v) \\
&\Leftrightarrow \forall \lambda \in \mathbb{R}, v \in X : \frac{\lambda^2}{2} a(v,v) + \lambda(a(u,v) - e(v)) \geq 0 \\
&\Leftrightarrow \forall v \in X : a(u,v) - e(v) = 0 \\
&\Leftrightarrow 1.
\end{aligned}$$

using lemma 2.4. If the bilinear form a is coercive, then it satisfies the conditions of the Lax-Milgram theorem (Thm. 2.3). Hence there exists a unique $u \in X$ that satisfies 1. and therefore 2., i.e. it is a minimizer.

□

Now we turn our attention to *constrained* minimization problems.

Proposition 2.6 (Constrained minimization I)**.** *Let X, Q be Hilbert spaces, $a : X \times X \to \mathbb{R}$ and $b : X \times Q \to \mathbb{R}$ two continuous bilinear forms. For $(e, g) \in X' \times Q'$, we consider the* constrained minimization problem

$$\min_{v \in Z_g} J(v) \tag{2.7a}$$

$$Z_g = \{v \in X \,|\, b(v,q) = g(q), \forall q \in Q\} \subset X \tag{2.7b}$$

where $J(v) = \frac{1}{2}a(v,v) - e(v)$ is the functional (2.5)*. If $Z_g \neq \emptyset$ and $a(\cdot,\cdot)$ is symmetric and Z-coercive, where $Z := Z_0 = \{v \in X \,|\, b(v,q) = 0, \forall q \in Q\}$, then there exists a unique minimizer $u \in X$.*

Proof. First we show that Z is a Hilbert space. Given that it is a linear subspace of the Hilbert space X, it is enough to show that it is closed. From the continuity of $b(\cdot,\cdot)$, we can show that the map $B : X \to Q'$ defined as $(Bv)(q) = b(v,q)$, $\forall q \in Q$, is continuous:

$$\|Bv\|_{Q'} = \sup_{q \in Q \setminus \{0\}} \frac{|b(v,q)|}{\|q\|_Q} \leq \|b\| \|v\|_X$$

Then $Z \subset X$ is closed as the inverse image of the closed set $\{0\} \subset Q'$ under the continuous map B.

Since Z_g is not empty, we fix an arbitrary $u_g \in Z_g$ and consider the (unconstrained) minimization problem

$$\min_{v \in Z} \left\{ \frac{1}{2} a(v,v) - (e(v) - a(u_g, v)) \right\} =: \tilde{J}(v)$$

which is equivalent to the constrained problem (2.7), in the sense that

$$u = \operatorname*{argmin}_{v \in Z_g} J(v) \Leftrightarrow u - u_g = \operatorname*{argmin}_{v \in Z} \tilde{J}(v)$$

Indeed, the map $v \in Z \mapsto u_g + v = \tilde{v} \in Z_g$ is a bijection, and so $J(u) \le J(\tilde{v}), \forall \tilde{v} \in Z_g$ is equivalent to

$$\begin{aligned}
&\forall \tilde{v} \in Z_g : J(u) \le J(\tilde{v}) \\
\Leftrightarrow &\forall v \in Z : J(u) \le J(u_g + v) \\
\Leftrightarrow &\forall v \in Z : \frac{1}{2} a(u,u) - e(u) \le \frac{1}{2} a(u_g + v, u_g + v) - e(u_g + v) \\
\Leftrightarrow &\forall v \in Z : \frac{1}{2} a(u - u_g, u - u_g) - \big(e(u - u_g) + a(u_g, u - u_g)\big) \le \tilde{J}(v) \\
\Leftrightarrow &\forall v \in Z : \tilde{J}(u - u_g) \le \tilde{J}(v)
\end{aligned}$$

Applying Prop. 2.5 to the unconstrained problem, we conclude that there is a unique minimizer $u - u_g \in Z$ and therefore $u \in Z_g$ is the unique minimizer to the constrained problem.

□

Remark 2.7. *Although the previous proposition gives us existence and uniqueness of the solution to the constrained minimization problem* (2.7), *it is not satisfactory in the sense that it does not give us an a priori way to control* $\|u\|_X$ *in terms of* $\|e\|_{X'}$ *and* $\|g\|_{Q'}$, *like the Lax-Milgram theorem does in the unconstrained case. The following proposition gives us an alternative to Lax-Milgram that will enable us to do so.*

Proposition 2.8 (Ladyzhenskaya-Babuška-Brezzi)**.** *Let* X, Q *be Hilbert spaces and* $b : X \times Q \to \mathbb{R}$ *a continuous bilinear form. If* $b(\cdot,\cdot)$ *satisfies the* inf-sup condition

$$\inf_{q \in Q \setminus \{0\}} \sup_{v \in X \setminus \{0\}} \frac{b(v,q)}{\|v\|_X \|q\|_Q} \ge \beta \tag{2.8}$$

for some $\beta > 0$, *then*

1. *For any $g \in Q'$, there exists a unique $u_g \in X$ such that*

$$\langle u_g, v\rangle_X = 0, \qquad \forall v \in Z \tag{2.9a}$$
$$b(u_g, q) = g(q), \qquad \forall q \in Q \tag{2.9b}$$
$$\|u_g\|_X \leq \beta^{-1}\|g\|_{Q'} \tag{2.9c}$$

2. *For any $h \in X'$, such that $\forall v \in Z, h(v) = 0$, there exists a unique $p_h \in Q$ such that*

$$b(v, p_h) = h(v), \qquad \forall v \in X \tag{2.10a}$$
$$\|p_h\|_Q \leq \beta^{-1}\|h\|_{X'} \tag{2.10b}$$

Proof. This is a corollary of the *closed range theorem* for Banach spaces. See [GF09]. □

Remark 2.9. *Prop. 2.8 can be read as a version of the Riesz representation theorem for the bilinear form b. Indeed, if we consider the orthogonal $Z^\perp := \{u \in X \mid \langle v, u\rangle_X = 0, \forall v \in Z\}$ and polar $Z^\circ := \{h \in X' \mid h(v) = 0, \forall v \in Z\}$ complements of Z, the proposition states that*

- *Any $g \in Q'$ can be represented as $b(u, \cdot)$ for a unique $u \in Z^\perp$.*
- *Any $h \in Z^\circ \subset X'$ can be represented as $b(\cdot, p)$ for a unique $p \in Q$.*

Theorem 2.10 (Brezzi splitting theorem)**.** *Let X, Q be Hilbert spaces and $a : X \times X \to \mathbb{R}$ and $b : X \times Q \to \mathbb{R}$ two continuous bilinear forms. If $a(\cdot,\cdot)$ is Z-coercive and $b(\cdot,\cdot)$ satisfies the Ladyzhenskaya-Babuška-Brezzi condition* (2.8)*, then the* saddle point problem

$$a(u, v) + b(v, p) = e(v)\,, \quad \forall v \in X \tag{2.11a}$$
$$b(u, q) = g(q)\,, \quad \forall q \in Q \tag{2.11b}$$

has a unique solution $(u, p) \in X \times Q$ for any $(e, g) \in X' \times Q'$.

Moreover, the unique solution (u, p) satisfies the following bounds

$$\|u\|_X \leq \frac{1}{\alpha}\|e\|_{X'} + \frac{1}{\beta}\left(1 + \frac{\|a\|}{\alpha}\right)\|g\|_{Q'} \tag{2.12a}$$
$$\|p\|_Q \leq \frac{1}{\beta}\left(1 + \frac{\|a\|}{\alpha}\right)\|e\|_{X'} + \frac{\|a\|}{\beta^2}\left(1 + \frac{\|a\|}{\alpha}\right)\|g\|_{Q'} \tag{2.12b}$$

Proof. From part 1. of Prop. 2.8, there exists unique $u_g \in Z^\perp$, such that $b(u_g, q) = g(q)$, for all $q \in Q$. We define $\tilde{e} \in Z'$ as $\tilde{e}(v) = e(v) - a(u_g, v)$, $\forall v \in Z$. Since $a(\cdot,\cdot)$ is continuous and Z-coercive, from the Lax-Milgram theorem (Thm. 2.3) there exists a unique $u_e \in Z$ such that $a(u_e, v) = \tilde{e}(v)$, for all $v \in Z$. We let $u := u_g + u_e$ and consider the linear functional $h \in X'$ with $h(v) = e(v) - a(u, v)$, $\forall v \in X$. Then $h \in Z^\circ$, since

$$h(v) = e(v) - a(u_g + u_e, v) = e(v) - a(u_g, v) - a(u_e, v) = e(v) - a(u_g, v) - \tilde{e}(v) = 0$$

for all $v \in Z$. From part 2. of Prop. 2.8, there exists unique $p \in Q$ such that $b(v, p) = h(v)$, for all $v \in X$. By construction then

$$\begin{aligned} b(v,p) &= h(v) \Rightarrow a(u,v) + b(v,p) = e(v),\ \forall v \in X \\ b(u,q) &= b(u_g, q) + b(u_e, q) = g(q) + 0,\ \forall q \in Q \end{aligned}$$

which proves that (u, p) is a solution to the saddle point problem (2.11).

Let (u', p') be another solution, then

$$\begin{aligned} a(u-u', v) + b(v, p-p') &= 0, \forall v \in X \\ b(u-u', q) &= 0, \forall q \in Q \end{aligned}$$

The second equation implies immediately that $u - u' \in Z$. Choosing then $v = u - u'$ in the first equation gives us $a(u-u', u-u') = 0 \Rightarrow u - u' = 0$ by the $Z-$coercivity of $a(\cdot,\cdot)$. The first equation becomes $b(v, p-p') = 0, \forall v \in X$, whose solution $p - p' = 0$ is unique by part 2. of Prop. 2.8. It follows that $(u', p') = (u, p)$, i.e. the solution is unique.

To prove the bounds (2.12), we retrace our steps in the first part of this proof. The definition of u_g with the help of part 1. of Prop. 2.8 gives us immediately $\|u_g\|_X \leq \beta^{-1}\|g\|_{Q'}$. The definition of u_e with the help of the Lax-Milgram thm. gives us $\|u_e\|_X \leq \alpha^{-1}\|\tilde{e}\|_{Z'}$. By definition,

$$\|\tilde{e}\|_{Z'} = \sup_{v\in Z\setminus\{0\}} \frac{|\tilde{e}(v)|}{\|v\|_X} \leq \sup_{v\in X\setminus\{0\}} \frac{|\tilde{e}(v)|}{\|v\|_X} = \sup_{v\in X\setminus\{0\}} \frac{|e(v) - a(u_g, v)|}{\|v\|_X} \leq \|e\|_{X'} + \|a\|\|u_g\|_X$$

Combining these inequalities yields $\|u\|_X \leq \|u_e\|_X + \|u_g\|_X \leq \frac{1}{\alpha}\|e\|_{X'} + \frac{1}{\beta}\left(1 + \frac{\|a\|}{\alpha}\right)\|g\|_{Q'}$. Likewise, the definition of u_g with the help of part 2. of Prop. 2.8 gives us $\|p\|_Q \leq \beta^{-1}\|h\|_{X'}$, and $\|h\|_{X'} = \sup_{v\in X\setminus\{0\}} \dfrac{|e(v) - a(u,v)|}{\|v\|_X} \leq \|e\|_{X'} + \|a\|\|u\|_X$, which yield the second bound.

□

Remark 2.11. *If we replace the Z-coercivity of $a(\cdot,\cdot)$ with*

$$\exists \alpha > 0 : \inf_{u\in Z\setminus\{0\}} \sup_{v\in Z\setminus\{0\}} \frac{a(u,v)}{\|u\|_X\|v\|_X} \geq \alpha \tag{2.13a}$$

$$a(u,v) = 0, \forall u \in Z \Rightarrow v = 0 \tag{2.13b}$$

then, by Nečas theorem, Thm. 2.10 gives necessary and sufficient conditions for the existence and uniqueness of the solution.

Definition 2.12 (Lagrangian)**.** *The functional* $\mathcal{L} : X \times Q \to \mathbb{R}$

$$\mathcal{L}(v, q) := J(v) + b(v, q) - g(q) \tag{2.14}$$

is called the Lagrangian *of the optimization problem* (2.7)*, with* Lagrangian multiplier q*. A pair* $(u, p) \in X \times Q$ *is a* saddle point *of* $\mathcal{L}$ *when*

$$\mathcal{L}(u, q) \leq \mathcal{L}(u, p) \leq \mathcal{L}(v, p),\ \forall (v, q) \in X \times Q \tag{2.15}$$

Proposition 2.13 (Constrained minimization II)**.** *Let* X*,* Q *be Hilbert spaces,* $a : X \times X \to \mathbb{R}$ *and* $b : X \times Q \to \mathbb{R}$ *two continuous bilinear forms, with* $a(\cdot, \cdot)$ *symmetric and positive. For fixed* $(e, g) \in X' \times Q'$*, the pair* $(u, p) \in X \times Q$ *is a solution of saddle point problem* (2.11) *if and only if it is a saddle point of the Lagrangian* (2.14)*.*

Moreover, if $a(\cdot, \cdot)$ *is* Z*-coercive and* $b(\cdot, \cdot)$ *satisfies the Ladyzhenskaya-Babuška-Brezzi condition* (2.8)*, then* (u, p) *exists and is unique and* $u = \operatorname{argmin}_{v \in Z_g} J(v)$*.*

Proof. As in the proof of Prop. 2.5, the mapping $(\lambda, v) \in \mathbb{R} \times X \mapsto u + \lambda v =: \tilde{v} \in X$ is onto and therefore

$$\begin{aligned}
&\forall \tilde{v} \in X : \mathcal{L}(u, p) \leq \mathcal{L}(\tilde{v}, p) \\
\Leftrightarrow\ &\forall \lambda \in \mathbb{R}, v \in X : \mathcal{L}(u, p) \leq \mathcal{L}(u + \lambda v, p) \\
\Leftrightarrow\ &\forall \lambda \in \mathbb{R}, v \in X : J(u) + b(u, p) - g(p) \leq J(u + \lambda v) + b(u + \lambda v, p) - g(p) \\
\Leftrightarrow\ &\forall \lambda \in \mathbb{R}, v \in X : \frac{\lambda^2}{2} a(v, v) + \lambda\big(a(u, v) + b(v, p) - e(v)\big) \geq 0 \\
\Leftrightarrow\ &\forall v \in X : a(u, v) + b(v, p) - e(v) = 0
\end{aligned}$$

by lemma 2.4. Likewise,the mapping $(\lambda, q) \in \mathbb{R} \times Q \mapsto p + \lambda q =: \tilde{q} \in Q$ is onto and therefore

$$\begin{aligned}
&\forall \tilde{q} \in Q : \mathcal{L}(u, \tilde{q}) \leq \mathcal{L}(\tilde{u}, p) \\
\Leftrightarrow\ &\forall \lambda \in \mathbb{R}, q \in Q : \mathcal{L}(u, p + \lambda q) \leq \mathcal{L}(u, p) \\
\Leftrightarrow\ &\forall \lambda \in \mathbb{R}, q \in Q : J(u) + b(u, p + \lambda q) - g(p + \lambda q) \leq J(u) + b(u, p) - g(p) \\
\Leftrightarrow\ &\forall \lambda \in \mathbb{R}, q \in Q : \lambda\big(g(q) - b(u, q)\big) \geq 0 \\
\Leftrightarrow\ &\forall q \in Q : g(q) - b(u, q) = 0
\end{aligned}$$

where we used lemma 2.4 again (with $\alpha = 0$). Combined, these prove the equivalence

$$\forall (v, q) \in X \times Q : \mathcal{L}(u, q) \leq \mathcal{L}(u, p) \leq \mathcal{L}(v, p) \ \Leftrightarrow \ \begin{cases} a(u, v) + b(v, p) = e(v), & \forall v \in X \\ b(u, q) = g(q), & \forall q \in Q \end{cases}$$

When $a(\cdot,\cdot)$ is Z-coercive and $b(\cdot,\cdot)$ satisfies the LBB condition, the Brezzi splitting theorem 2.10 implies the existence and uniqueness of the saddle point (u,p). Furthermore, for any $v \in Z_g \subset X$:

$$\mathcal{L}(u,p) \leq \mathcal{L}(v,p) \Rightarrow J(u) + b(u,p) - g(p) \leq J(v) + b(v,p) - g(p) \Rightarrow J(u) \leq J(v)$$

since $u, v \in Z_g \Rightarrow b(u,p) - g(p) = b(v,p) - g(p) = 0$. Hence, u is a minimizer of J over Z_g. In fact, since the conditions of Prop. 2.6 are met, u is the *unique* minimizer of J over Z_g.

□

Corollary 2.14 (Dual energy)**.** *We define the* dual energy $J^* : Q \to \mathbb{R} \cup \{-\infty\}$ *as*

$$J^*(q) := \inf_{v \in X} \mathcal{L}(v,q) \tag{2.16}$$

If (u,p) is a saddle point of $\mathcal{L}$, then p is a solution of the dual optimization (maximization) problem $\max\limits_{q \in Q} J^*(q)$.

Proof. For an arbitrary $q \in Q$, $J^*(q) = \inf_{v \in X} \mathcal{L}(v,q) \leq \mathcal{L}(u,q) \leq \mathcal{L}(u,p)$ and since $\mathcal{L}(u,p) \leq \mathcal{L}(v,p), \forall v \in X \Rightarrow \mathcal{L}(u,p) \leq \inf_{v \in X} \mathcal{L}(v,p) = J^*(p)$, we conclude that $J^*(q) \leq J^*(p)$, i.e. p is a maximizer of J^* over Q.

□

Remark 2.15. *The key intuition behind the results of this section is that the optimization theory in the finite-dimensional Euclidean spaces $\mathbb{R}^n$ [NW99] is built on top of the* fundamental theorem of algebra*, which correlates the linear subspaces* $\operatorname{Im} B$,$\operatorname{Ker} B$, $\operatorname{Im} B^T$, $\operatorname{Ker} B$ *associated with a map $B \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$. Hilbert spaces have many of the same basic properties, specifically with respect to the properties of orthogonal complements and orthogonal projections. Combined with maps induced by bilinear forms $b : X \times Y \to \mathbb{R}$ which satisfy the inf-sup property of Thm. 2.8, and so satisfy a version of the fundamental theorem of algebra, they give us an infinite-dimensional setting where many of the proofs from the Euclidean case carry over practically intact.*

Finally, we present certain results related to the *approximation of optimization problems.* Recall Céa's lemma, which is fundamental to the analysis of the Galerkin method:

Lemma 2.16 (Céa)**.** *Let X be a Hilbert space, $a : X \times X \to \mathbb{R}$ a continuous and coercive bilinear form and $e \in X'$. If X_h is a finite-dimensional subspace of X and*

$$a(u,v) = e(v), \quad \forall v \in X \tag{2.17a}$$

$$a(u_h,v_h) = e(v_h), \quad \forall v_h \in X_h \tag{2.17b}$$

then

$$\|u - u_h\|_X \leq \frac{\|a\|}{\alpha} \inf_{v_h \in X_h} \|u - v_h\|_X \tag{2.18}$$

The following result then has the same relation to Brezzi's splitting theorem (Thm. 2.10), that Céa's lemma (Lem. 2.16) has to the Lax-Milgram theorem (Thm. 2.3):

Theorem 2.17 (Approximation of saddle point problems)**.** *Let X, Q be Hilbert spaces, $a(\cdot,\cdot)$ and $b(\cdot,\cdot)$ bilinear forms satisfying the conditions of the Brezzi splitting theorem (Thm. 2.10), and let $(u,p) \in X \times Q$ be the unique solution of the saddle point problem (2.11) for given $(e,g) \in X' \times Q'$. If $X_h \subset X$ and $Q_h \subset Q$ are finite-dimensional subspaces, and*

1. *the bilinear form $a(\cdot,\cdot)$ is Z_h-coercive with coercivity constant α_h, where $Z_h := \{v_h \in X_h \,|\, b(v_h, q_h) = 0, \forall q_h \in Q_h\}$*
2. *the bilinear form $b(\cdot,\cdot)$ satisfies the LBB condition over $X_h \times Q_h$ with constant β_h*

then the saddle point problem

$$a(u_h, v_h) + b(v_h, p_h) = e(v_h)\,, \quad \forall v_h \in X_h \tag{2.19a}$$
$$b(u_h, q_h) = g(q_h)\,, \quad \forall q_h \in Q_h \tag{2.19b}$$

has a unique solution, which satisfies the bounds

$$\|u - u_h\|_X \le \left(1 + \frac{\|a\|}{\alpha_h}\right)\left(1 + \frac{\|b\|}{\beta_h}\right) \inf_{v_h \in X_h} \|u - v_h\|_X + \frac{\|b\|}{\alpha_h} \inf_{q_h \in Q_h} \|p - q_h\|_Q \tag{2.20a}$$

$$\begin{aligned} \|p - p_h\|_Q \le \frac{\|a\|}{\beta_h}\left(1 + \frac{\|a\|}{\alpha_h}\right)\left(1 + \frac{\|b\|}{\beta_h}\right) \inf_{v_h \in X_h} \|u - v_h\|_X \\ + \left(1 + \frac{\|b\|}{\beta_h} + \frac{\|a\|\|b\|}{\alpha_h \beta_h}\right) \inf_{q_h \in Q_h} \|p - q_h\|_Q \end{aligned} \tag{2.20b}$$

Proof. See section II.2.2 of [BF91].

□

2.3. Shape calculus

In this section, we present certain basic definitions and results from the theory of *shape optimization*, studied in more detail in [SZ92].

Definition 2.18 (Shape functional)**.** *We call the collection of subsets of $\mathbb{R}^3$*

$$\begin{aligned} \mathcal{O} \equiv \mathcal{O}^k(D) := \{\Omega \subseteq D \subset \mathbb{R}^3 \,|\, \Omega \text{ open, bounded and regular} \\ \text{with Lipschitz, piecewise } C^k \text{ boundary}\} \end{aligned} \tag{2.21}$$

the shape space*, where $k \ge 1$ and the fixed $D \in \mathcal{O}$ is the* hold-all set*. A* shape functional *is simply a function $J : \mathcal{O} \to \mathbb{R}$.*

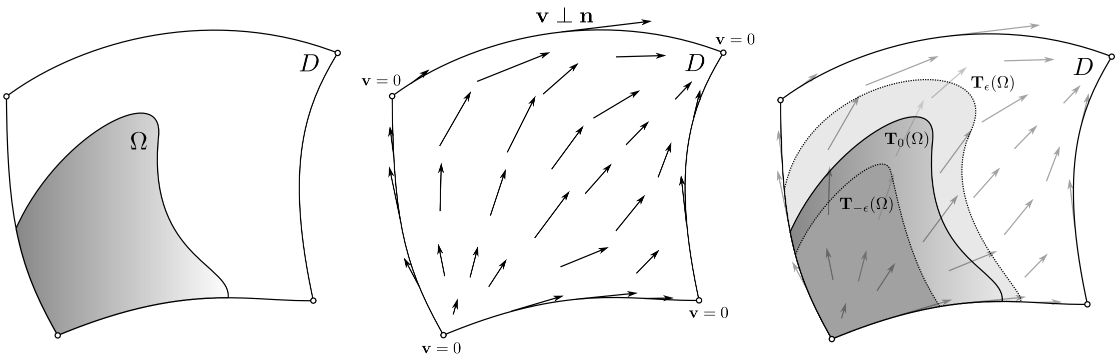


Figure 2.1.: *Shape calculus.* Transformation of a set Ω in the shape space $\mathcal{O}(D)$ under an admissible velocity field $\mathbf{v}$. Note that the velocity field vanishes at the four corner points, which are in ∂D_s, and it is tangential in the rest of the boundary ∂D.

Definition 2.19 (Regular set)**.** *A* regular *set A is equal to* the closure $\overline{\operatorname{int}(A)}$ of its interior*. In geometrical terms, and for $A \subset \mathbb{R}^3$, neither the set nor its complement have any 'thin' components, like isolated points or curves.*

Theorem 2.20 (Admissible velocities)**.** *Let $\mathcal{O}^k(D)$ be a shape space with hold-all set D. Since the boundary of D is Lipschitz, the normal $\mathbf{n}$ is defined a.e. and so we can partition it into two sets $\partial D = \partial D_\mathbf{n} \cup \partial D_s$, with $\mathbf{n}$ defined everywhere on $\partial D_\mathbf{n}$ and ∂D_s a null set. Let*

$$\mathcal{V} \equiv \mathcal{V}^k(D) := \{\mathbf{v} \in C^k(\overline{D}, \mathbb{R}^3) \,|\, \langle \mathbf{v}, \mathbf{n} \rangle = 0 \text{ on } \partial D_\mathbf{n}, \mathbf{v} = 0 \text{ on } \partial D_s\} \tag{2.22}$$

be the set of admissible velocity fields *. Then for any time-dependent velocity field $\mathbf{v}_t \equiv \mathbf{v}(t, \cdot) \in C([0,T), \mathcal{V})$, there exists a time interval I, $0 \in I \subseteq [0,T)$, and a one-parameter family of homeomorphisms $\mathbf{T}_t : \overline{D} \to \overline{D}$, $t \in I$, such that*

$$\mathbf{T}_t, \mathbf{T}_t^{-1} \in C^k(\overline{D}, \overline{D}) \tag{2.23a}$$

$$\frac{\partial}{\partial t}\mathbf{T}_t(\mathbf{x}) = \mathbf{v}_t(\mathbf{T}_t(\mathbf{x})), \quad \forall (t, \mathbf{x}) \in I \times \overline{D} \tag{2.23b}$$

$$\mathbf{T}_0(\mathbf{x}) = \mathbf{x}, \quad \forall \mathbf{x} \in \overline{D} \tag{2.23c}$$

Furthermore, the image $\Omega_t := \mathbf{T}_t(\Omega) \in \mathcal{O}^k(D)$, for any $\Omega \in \mathcal{O}^k(D)$ and $t \in I$.

Proof. See §2.10 of [SZ92].

□

Definition 2.21 (Shape derivative)**.** *We define the* Eulerian (semi-)derivative *of the shape functional* $J : \mathcal{O} \to \mathbb{R}$ *at* Ω *in the direction* $\mathbf{v}_t \in C([0,T),\mathcal{V})$*, as the limit*

$$J'(\Omega)(\mathbf{v}_t) := \lim_{t \to 0^+} \frac{J(\mathbf{T}_t(\Omega)) - J(\Omega)}{t} \tag{2.24}$$

where $\mathbf{T}_t$ *is the transformation of Thm. 2.20. A functional* $J : \mathcal{O} \to \mathbb{R}$ *is* shape-differentiable *at an* $\Omega \in \mathcal{O}$*, iff* $J'(\Omega)(\mathbf{v}_t)$ *exists for all directions* $\mathbf{v}_t \in C([0,T),\mathcal{V})$ *and the mapping* $J'(\Omega)$ *is linear and continuous. We define the* shape derivative of J at Ω in the direction $\mathbf{v} \in \mathcal{V}$ *as* $J'(\Omega)(\mathbf{v}) := J'(\Omega)(\mathbf{v}_t)$*, for any* $\mathbf{v}_t \in C([0,T),\mathcal{V})$ *with* $\mathbf{v}_0 = \mathbf{v}$*.*

Proof. The shape derivative is well-defined, because the continuity of $J'(\Omega)(\mathbf{v}_t)$ implies that its value depends only on $\mathbf{v}_0$ (Prop. 2.21 in [SZ92]).

□

Remark 2.22. *The notions of shape-differentiability and the shape derivative are very similar to the* Gâteaux differentiability *and the* Fréchet derivative *respectively on Banach spaces. The difference is that the shape space* $\mathcal{O}$*, unlike the Banach spaces, is not a vector space and therefore affine perturbations of the form "*$\Omega \mapsto \Omega + t\,\Psi$*" of a shape* Ω *in the direction of another shape* Ψ *are meaningless.*

Instead, we consider a separate set of directions, the velocity fields $\mathbf{v}$ *and their associated transformations* $\mathbf{T}_t$*, and take perturbations of the form* $\Omega \mapsto \mathbf{T}_t(\Omega)$*. Notice the similarity with the case of manifolds, where all paths that yield the same directional derivative are bundled together and identified with the same tangent vector.*

Proposition 2.23 (Shape derivative of domain integrals)**.** *Let* $\mathcal{O} \equiv \mathcal{O}^1(D)$ *be a shape space and* $\phi \in W^{1,1}(D)$*. Then the shape functional*

$$J(\Omega) := \int_\Omega \phi \, dV \tag{2.25}$$

is shape differentiable for any $\Omega \in \mathcal{O}$ *and its shape derivative in the direction* $\mathbf{v} \in \mathcal{V}^1(D)$ *is*

$$J'(\Omega)(\mathbf{v}) = \int_\Omega \operatorname{div}(\phi\,\mathbf{v}) \, dV = \int_{\partial\Omega} \phi \langle \mathbf{v}, \mathbf{n} \rangle \, da \tag{2.26}$$

Proof. See §2.16 in [SZ92].

□

Proposition 2.24 (Shape derivative of boundary intergals)**.** *Let* $\mathcal{O} \equiv \mathcal{O}^1(D)$ *be a shape space and* $\psi \in W^{2,1}(D)$*. Then the shape functional*

$$J(\Omega) := \int_\Gamma \psi \, da, \quad \Gamma \equiv \partial\Omega \tag{2.27}$$

is shape differentiable for any $\Omega \in \mathcal{O}$ *and its shape derivative in the direction* $\mathbf{v} \in \mathcal{V}^1(D)$ *is*

$$J'(\Omega)(\mathbf{v}) = \int_\Gamma \big(\langle \nabla\psi, \mathbf{v}\rangle + \psi \operatorname{div}_\Gamma \mathbf{v}\big)\, da \tag{2.28}$$

If we restrict ourselves to $\Omega \in \mathcal{O}^2(D)$*, i.e. sets with piecewise* C^2 *boundary, and a* $\psi \in H^{3/2}(D)$*, then the shape functional* (2.27) *is shape differentiable in the direction* $\mathbf{v} \in \mathcal{V}^2(D)$ *with shape derivative*

$$J'(\Omega)(\mathbf{v}) = \sum_{i\in\mathcal{I}} \left(\int_{\Gamma_i} \big(\frac{\partial\psi}{\partial n} - H\psi\big)\langle \mathbf{v}, \mathbf{n}\rangle\, da + \int_{\partial\Gamma_i} \langle \psi\, \mathbf{v}, \boldsymbol{\nu}_i\rangle\, dl \right) \tag{2.29}$$

where $\Gamma = \bigcup_{i\in\mathcal{I}} \Gamma_i$ *is the partition of* $\partial\Omega$ *in* C^2 *segments,* $H \in L^\infty(\Gamma)$ *is the mean curvature, and* $\boldsymbol{\nu}_i$ *is the outward pointing conormal of* Γ_i*.*

Proof. The shape derivative (2.28) is derived in Prop. 2.50 of [SZ92]. Likewise, (2.29) is derived in Prop. 3.16 in [SZ92]. Note that the difference in the sign of the $H\psi$ term is due to a different sign in the definition of the curvature, i.e. for us spheres have negative curvature, but in [SZ92] positive.

□

Lemma 2.25 (Ω-supported velocities)**.** *For a set* $\Omega \in \mathcal{O} \equiv \mathcal{O}^k(D)$*, we define the set of* Ω-supported admissible velocities $\mathcal{V}^k(\Omega; D) \subset \mathcal{V}^k(D; D) \equiv \mathcal{V}^k(D)$ *as*

$$\mathcal{V}^k(\Omega; D) := \{\mathbf{v} \in C^k(\overline{\Omega}, \mathbb{R}^3) \mid \langle \mathbf{v}, \mathbf{n}\rangle = 0 \text{ on } \partial\Omega \cap \partial D_{\mathbf{n}}, \mathbf{v} = 0 \text{ on } \partial\Omega \cap \partial D_s\} \tag{2.30}$$

A shape functional $J : \mathcal{O} \to \mathbb{R}$ *is shape-differentiable at an* $\Omega \in \mathcal{O}$ *in the direction* $\mathbf{v} \in \mathcal{V}^k(\Omega; D)$ *with shape derivative* $J'(\Omega)(\mathbf{v})$*, iff* $J'(\Omega)(\overline{\mathbf{v}}) = J'(\Omega)(\mathbf{v})$ *for any* $\overline{\mathbf{v}} \in \mathcal{V}^k(D)$ *with* $\overline{\mathbf{v}}|_{\overline{\Omega}} = \mathbf{v}$*. The shape derivatives of Prop. 2.23 and 2.24 hold then if we substitute* $\mathbf{v} \in \mathcal{V}^k(D)$ *with* $\mathbf{v} \in \mathcal{V}^k(\Omega; D)$*.*

Proof. A direct consequence of the fact that the shape derivatives (2.26) and (2.28), (2.29) do not depend on the values of $\mathbf{v}$ outside of $\overline{\Omega}$.

□

2.4. Variational form of the Stokes equations

We consider an incompressible (Newtonian) viscous fluid occupying a volume $\Omega(t) \subset \mathbb{R}^3$, bounded by a stationary surface S, the *substrate*, and the *free boundary* $F(t)$. The fluid flows under the influence of a time-independent body force $\mathbf{f}$ and the surface tension. If we assume that we are in the quasistatic regime, i.e. the *Reynolds number* is small, the fluid evolves according to the *Stokes equations* (see fig. 2.2).

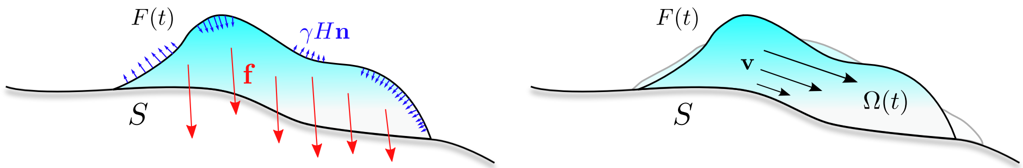


Figure 2.2.: *Viscous fluid on curved surface.* Sketch of a fluid occupying a volume $\Omega(t) \subset \mathbb{R}^3$, with boundary $\partial\Omega(t) = S \cup F(t)$, flowing with a velocity $\mathbf{v}$ under the influence of a body force $\mathbf{f}$ and the surface tension $\gamma H \mathbf{n}$.

Definition 2.26 (Hilbert spaces on domain Ω)**.** *Let $\Omega \subset \mathbb{R}^3$ be an open, bounded and connected domain with Lipschitz boundary $\partial\Omega = S \cup F$. We define the following Hilbert spaces:*

1. *Square-integrable functions on Ω:*
   - $L^2(\Omega) := \{q : \Omega \to \mathbb{R} \,|\, \|q\|_{L^2(\Omega)} < \infty\}$ *with the norm* $\|q\|_{L^2(\Omega)} := \left(\int_\Omega q^2 \, dV\right)^{1/2}$
   - $L^2_0(\Omega) := \{q \in L^2(\Omega) \,|\, \int_\Omega q \, dV = 0\} \subset L^2(\Omega)$ *with the* $\|\cdot\|_{L^2(\Omega)}$ *norm*
   - $L^2(\Omega, \mathbb{R}^3) := \{\mathbf{v} : \Omega \to \mathbb{R}^3 \,|\, \|\mathbf{v}\|_{L^2(\Omega,\mathbb{R}^3)} < \infty\}$ *with the norm* $\|\mathbf{v}\|_{L^2(\Omega,\mathbb{R}^3)} := \left(\int_\Omega |\mathbf{v}|^2 \, dV\right)^{1/2}$
   - $L^2(\Omega, \mathbb{R}^{3\times3}) := \{A : \Omega \to \mathbb{R}^{3\times3} \,|\, \|A\|_{L^2(\Omega,\mathbb{R}^{3\times3})} < \infty\}$ *with norm* $\|A\|_{L^2(\Omega,\mathbb{R}^{3\times3})} := \left(\int_\Omega A : A \, dV\right)^{1/2}$
2. *Square-integrable functions on $\partial\Omega$:*
   - $L^2(\partial\Omega) := \{q : \partial\Omega \to \mathbb{R} \,|\, \|q\|_{L^2(\partial\Omega)} < \infty\}$ *with the norm* $\|q\|_{L^2(\partial\Omega)} := \left(\int_{\partial\Omega} q^2 \, da\right)^{1/2}$
   - $L^2(\partial\Omega, \mathbb{R}^3) := \{\mathbf{v} : \partial\Omega \to \mathbb{R}^3 \,|\, \|\mathbf{v}\|_{L^2(\partial\Omega)} < \infty\}$ *with the norm* $\|\mathbf{v}\|_{L^2(\partial\Omega)} := \left(\int_{\partial\Omega} |\mathbf{v}|^2 \, da\right)^{1/2}$
3. *Sobolev spaces:*
   - $H^1(\Omega) := \{q \in L^2(\Omega) \,|\, \nabla q \in L^2(\Omega, \mathbb{R}^3)\}$ *with the norm*
   $$\|q\|_{H^1(\Omega)} := \left(\|q\|^2_{L^2(\Omega)} + \|\nabla q\|^2_{L^2(\Omega,\mathbb{R}^3)}\right)^{1/2}$$
   - $H^1(\Omega, \mathbb{R}^3) := \{\mathbf{v} \in L^2(\Omega, \mathbb{R}^3) \,|\, \nabla\mathbf{v} \in L^2(\Omega, \mathbb{R}^{3\times3})\}$ *with norm* $\|\mathbf{v}\|_{H^1(\Omega,\mathbb{R}^3)} := \left(\|\mathbf{v}\|^2_{L^2(\Omega,\mathbb{R}^3)} + \|\nabla\mathbf{v}\|^2_{L^2(\Omega,\mathbb{R}^{3\times3})}\right)^{1/2}$
   - $H^1_0(\Omega, \mathbb{R}^3) := \{\mathbf{v} \in H^1(\Omega, \mathbb{R}^3) \,|\, \bar{\mathbf{v}} = 0 \text{ on } \partial\Omega\}$ *with the* $\|\cdot\|_{H^1(\Omega,\mathbb{R}^3)}$ *norm, where* $\mathbf{v} \in H^1(\Omega, \mathbb{R}^3) \mapsto \bar{\mathbf{v}} \in L^2(\partial\Omega, \mathbb{R}^3)$ *is the trace of* $\mathbf{v}$ *on* $\partial\Omega$

- $H^1_S(\Omega, \mathbb{R}^3) := \{\mathbf{v} \in H^1(\Omega, \mathbb{R}^3) \,|\, \bar{\mathbf{v}} = 0 \text{ on } S \subset \partial\Omega\}$ *with the* $\|\cdot\|_{H^1(\Omega,\mathbb{R}^3)}$ *norm*
- $H^{-1}(\Omega, \mathbb{R}^3)$ *the dual of* $H^1_0(\Omega, \mathbb{R}^3)$ *with norm*

$$\|\theta\|_{H^{-1}(\Omega,\mathbb{R}^3)} := \sup_{\mathbf{v}\in H^1_0(\Omega,\mathbb{R}^3)\setminus\{0\}} \frac{|\theta(\mathbf{v})|}{\|\mathbf{v}\|_{H^1(\Omega,\mathbb{R}^3)}}$$

Note that the trace theorem *states that*[2] $\|\bar{\mathbf{v}}\|_{L^2(\partial\Omega,\mathbb{R}^3)} \lesssim \|\mathbf{v}\|_{H^1(\Omega,\mathbb{R}^3)}$, *for any* $\mathbf{v} \in H^1(\Omega, \mathbb{R}^3)$.

Proposition 2.27 (Variational form of Stokes equations)**.** *Let* $\Omega \subset \mathbb{R}^3$ *be an open, bounded and connected domain with Lipschitz boundary* $\partial\Omega = S \cup F$, *so that its unit normal* $\mathbf{n}$ *is defined a.e., and smooth enough so that the mean curvature* $H \in L^2(\partial\Omega)$.

Consider the functional

$$R(\mathbf{v}) := \frac{1}{2}\int_\Omega 2\mu\,\mathcal{E}(\mathbf{v}) : \mathcal{E}(\mathbf{v})\, dV - \int_\Omega \langle \mathbf{f}, \mathbf{v}\rangle\, dV - \int_F \langle \gamma H\mathbf{n}, \mathbf{v}\rangle\, da \tag{2.31}$$

where $\mathbf{v} \in H^1_S(\Omega, \mathbb{R}^3)$ *is the velocity field,* $\mathcal{E}(\mathbf{v}) := \frac{1}{2}(\nabla\mathbf{v} + \nabla\mathbf{v}^T)$ *is the symmetric part of* $\nabla\mathbf{v}$, *the viscosity* μ *and surface tension* γ *are constants and the force* $\mathbf{f} \in L^2(\Omega, \mathbb{R}^3)$. *The constrained minimization problem*

$$\min_{\mathbf{v}\in Z} R(\mathbf{v}) \tag{2.32a}$$

$$Z := \left\{\mathbf{v} \in H^1_S(\Omega, \mathbb{R}^3) \,|\, -\int_\Omega q \operatorname{div}\mathbf{v}\, dV = 0, \forall q \in L^2(\Omega)\right\} \tag{2.32b}$$

has a unique solution $(\mathbf{v}, p) \in H^1_S(\Omega, \mathbb{R}^3) \times L^2_0(\Omega)$. *The pair* $(\mathbf{v}, p)$ *satisfies a weak form of the Stokes equations (together with the appropriate boundary condition on the free boundary* F*):*

$$\operatorname{div}\sigma + \mathbf{f} = 0 \quad \text{in } B \tag{2.33a}$$

$$\sigma\mathbf{n} = \gamma H\mathbf{n} \quad \text{on } F \tag{2.33b}$$

where $\sigma = -pI + 2\mu\,\mathcal{E}(\mathbf{v})$ *is the stress tensor. Moreover,*

$$\|\mathbf{v}\|_{H^1(\Omega,\mathbb{R}^3)} + \|p\|_{L^2(\Omega)} \lesssim \|\mathbf{f}\|_{L^2(\Omega,\mathbb{R}^3)} + \gamma\|H\|_{L^2(\partial\Omega)} \tag{2.34}$$

Proof. We can write the optimization problem (2.32) in a form suitable for application

[2] The notation $a \lesssim b \Leftrightarrow \exists C > 0 : a \leq Cb$.

of Prop. 2.13 by letting

$$
\begin{aligned}
X \times Q &= H^1_S(\Omega, \mathbb{R}^3) \times L^2_0(\Omega) \\
a(\mathbf{v}, \mathbf{u}) &= 2\mu \int_\Omega \mathcal{E}(\mathbf{v}) : \mathcal{E}(\mathbf{u})\, dV \\
e(\mathbf{u}) &= \int_\Omega \langle \mathbf{f}, \mathbf{u} \rangle\, dV + \int_F \langle \gamma H \mathbf{n}, \mathbf{u} \rangle\, da \\
b(\mathbf{u}, q) &= -\int_\Omega q \operatorname{div} \mathbf{u}\, dV \\
g(q) &= 0
\end{aligned}
$$

We need to show that the bilinear forms are continuous, and moreover that $a(\cdot,\cdot)$ is Z-coercive and $b(\cdot,\cdot)$ satisfies the LBB condition (2.8).

- The continuity of $a(\cdot,\cdot)$ follows from the fact that $\int_\Omega A : B\, dV$, for $A, B \in L^2(\Omega, \mathbb{R}^{3\times 3})$, is the inner product that corresponds to the norm $\|A\|_{L^2(\Omega,\mathbb{R}^{3\times 3})}$. Using this, together with the properties of tensor contraction from Section 1.8, we can show that

  $$|a(\mathbf{v}, \mathbf{u})| \le 2\mu\, \|\nabla \mathbf{u}\|_{L^2(\Omega,\mathbb{R}^{3\times 3})} \|\nabla \mathbf{v}\|_{L^2(\Omega,\mathbb{R}^{3\times 3})} \le 2\mu\, \|\mathbf{v}\|_{H^1(\Omega,\mathbb{R}^3)} \|\mathbf{u}\|_{H^1(\Omega,\mathbb{R}^3)}$$

  and so $\|a\| \le 2\mu$.

- For the continuity of $b(\cdot,\cdot)$, we note that all the partial derivatives of $\mathbf{u}$ are dominated by $\|\nabla \mathbf{u}\|_{L^2(\Omega,\mathbb{R}^{3\times 3})}$ and so $\|\operatorname{div} \mathbf{u}\|_{L^2(\Omega)} \le 3\|\nabla \mathbf{u}\|_{L^2(\Omega,\mathbb{R}^{3\times 3})}$. It follows that

  $$|b(\mathbf{u}, q)| \le \|q\|_{L^2(\Omega)} \|\operatorname{div} \mathbf{u}\|_{L^2(\Omega)} \le 3\|q\|_{L^2(\Omega)} \|\mathbf{u}\|_{H^1(\Omega,\mathbb{R}^3)}$$

  and so $\|b\| \le 3$.

- The coercivity of $a(\cdot,\cdot)$ is a rather technical result, studied extensively in the context of *linear elasticity*. The coercivity is equivalent to *Korn's first inequality* [Kor09], which states that there exists $K > 0$ such that $K\|\mathcal{E}(\mathbf{v})\|_{L^2(\Omega,\mathbb{R}^{3\times 3})} \ge \|\mathbf{v}\|_{H^1(\Omega,\mathbb{R}^3)}$ for all $\mathbf{v} \in H^1_S(\Omega, \mathbb{R}^3)$. The key point is that $\mathbf{v}$ vanishes on a set $S \subset \partial\Omega$ (with non-zero measure), so that the velocities which correspond to *rigid body motions* are exempt from the space. Note that

  $$a(\mathbf{v}, \mathbf{v}) = 2\mu \|\mathcal{E}(\mathbf{v})\|^2_{L^2(\Omega,\mathbb{R}^{3\times 3})} \ge \frac{2\mu}{K^2} \|\mathbf{v}\|^2_{H^1(\Omega,\mathbb{R}^3)}$$

  and so the coercivity constant $\alpha \ge 2\mu K^{-2}$.

- The compliance of $b(\cdot,\cdot)$ with the LBB condition is also a non-trivial result. The key theorem is that for an open bounded and connected domain Ω with a Lipschitz boundary, the (weak) gradient $\nabla : L^2_0(\Omega) \to H^{-1}(\Omega, \mathbb{R}^3)$ has closed range [GR86] (note that we have limited the range to functions with zero mean value). The adjoint operator $-\operatorname{div} : H^1_0(\Omega, \mathbb{R}^3) \to L^2_0(\Omega)$ can then be shown to be surjective [GF09], which leads to the following inf-sup condition:

$$\exists \beta > 0 : \inf_{q \in L^2_0(\Omega)\setminus\{0\}} \sup_{\mathbf{v} \in H^1_0(\Omega,\mathbb{R}^3)\setminus\{0\}} \frac{-\int_\Omega q \operatorname{div} \mathbf{v} \, dV}{\|q\|_{L^2(\Omega)} \|\mathbf{v}\|_{H^1(\Omega,\mathbb{R}^3)}} \geq \beta$$

Since $H^1_0(\Omega, \mathbb{R}^3) \subset H^1_S(\Omega, \mathbb{R}^3)$, the supremum over $H^1_S(\Omega, \mathbb{R}^3)$ is greater or equal to the supremum over $H^1_0(\Omega, \mathbb{R}^3)$, and so the inf-sup condition holds for $H^1_S(\Omega, \mathbb{R}^3)$ too, with the same β.

Prop. 2.13 then shows that there exists a unique solution $(\mathbf{v}, p) \in X \times Q = H^1_S(\Omega, \mathbb{R}^3) \times L^2_0(\Omega)$ to the saddle point problem (2.11), which in this case takes the form

$$\int_\Omega (2\mu\, \mathcal{E}(\mathbf{v}) : \mathcal{E}(\mathbf{u}) - p \operatorname{div} \mathbf{u}) \, dV = \int_\Omega \langle \mathbf{f}, \mathbf{u} \rangle \, dV + \int_F \langle \gamma H \mathbf{n}, \mathbf{u} \rangle \, da, \quad \forall \mathbf{u} \in H^1_S(\Omega, \mathbb{R}^3)$$
$$-\int_\Omega q \operatorname{div} \mathbf{v} \, dV = 0, \quad \forall q \in L^2_0(\Omega)$$

To derive a strong form of these equations, we assume that everything is smooth. The second equation is then equivalent to $\operatorname{div} \mathbf{v} = 0$, i.e. the *incompressibility* condition. We use the tensor properties from Section 1.8, in particular Lem. 1.70 and Prop. 1.75, to integrate by parts: let $\sigma := -pI + 2\mu\, \mathcal{E}(\mathbf{v})$ be the (symmetric) *stress tensor*,

$$\begin{aligned}\int_\Omega (2\mu\, \mathcal{E}(\mathbf{v}) : \mathcal{E}(\mathbf{u}) - p \operatorname{div} \mathbf{u}) \, dV &= \int_\Omega \sigma : \nabla \mathbf{u} \, dV \\ &= \int_{\partial\Omega} \langle \sigma \mathbf{u}, \mathbf{n} \rangle \, da - \int_\Omega \langle \operatorname{div} \sigma, \mathbf{u} \rangle \, dV = \int_{\partial\Omega} \langle \mathbf{u}, \sigma \mathbf{n} \rangle \, da - \int_\Omega \langle \operatorname{div} \sigma, \mathbf{u} \rangle \, dV \\ &= \int_F \langle \sigma \mathbf{n}, \mathbf{u} \rangle \, da - \int_\Omega \langle \operatorname{div} \sigma, \mathbf{u} \rangle \, dV\end{aligned}$$

since $\mathbf{u}|_S = 0$, and so the first equation becomes

$$\int_\Omega \langle \operatorname{div} \sigma + \mathbf{f}, \mathbf{u} \rangle \, dV = \int_F \langle \sigma \mathbf{n} - \gamma H \mathbf{n}, \mathbf{u} \rangle \, da$$

for arbitrary test function $\mathbf{u}$. The volume integral yields exactly the Stokes equations (2.33a), whereas the surface integral yields the free surface boundary condition (2.33b).

Finally, the bound (2.34) is a direct application of the bounds (2.12) with $\|g\|_{Q'} = 0$ and

$$\|e\|_{X'} = \sup_{\mathbf{u} \in H^1_S(\Omega,\mathbb{R}^3)\setminus\{0\}} \frac{\left| \int_\Omega \langle \mathbf{f}, \mathbf{u} \rangle \, dV + \int_F \langle \gamma H \mathbf{n}, \mathbf{u} \rangle \, da \right|}{\|\mathbf{u}\|_{H^1(\Omega,\mathbb{R}^3)}} \lesssim \|\mathbf{f}\|_{L^2(\Omega,\mathbb{R}^3)} + \gamma \|H\|_{L^2(\partial\Omega)}$$

since

- $\left|\int_\Omega \langle \mathbf{f}, \mathbf{u}\rangle \, dV\right| \leq \|\mathbf{f}\|_{L^2(\Omega,\mathbb{R}^3)} \|\mathbf{u}\|_{L^2(\Omega,\mathbb{R}^3)} \leq \|\mathbf{f}\|_{L^2(\Omega,\mathbb{R}^3)} \|\mathbf{u}\|_{H^1(\Omega,\mathbb{R}^3)}$
- $\left|\int_F \langle \gamma H \mathbf{n}, \mathbf{u}\rangle \, da\right| \leq \|\gamma H \mathbf{n}\|_{L^2(\partial\Omega,\mathbb{R}^3)} \|\bar{\mathbf{u}}\|_{L^2(\partial\Omega,\mathbb{R}^3)} \lesssim \gamma \|H\|_{L^2(\partial\Omega)} \|\mathbf{u}\|_{H^1(\Omega,\mathbb{R}^3)}$, using the trace theorem and that $\mathbf{n}$ is a unit vector.

□

Remark 2.28. *The classic variational form of the Stokes equations involves the functional $\int_\Omega \nabla\mathbf{v} : \nabla\mathbf{v} \, dV$ instead of $\int_\Omega \mathcal{E}(\mathbf{v}) : \mathcal{E}(\mathbf{v}) \, dV$, which is more common in variational models for linear elasticity. See §2 of Chapter 1 in [Tem77] for a detailed presentation. The functional* (2.31) *has a direct physical interpretation: the quadratic part is the* rate of energy dissipation *due to internal viscous friction and the linear part is the* mechanical work *done by the external body and surface tension forces against the displacement of the moving fluid. See §3.1 and §3.3 in [Poz97] for a discussion of the energy integral balance inside a fluid. Proposition 2.27 can then be read as the* minimum energy dissipation principle *which characterizes Stokes flow, as shown by Helmholtz (see §6.1 in [Poz97]). For the thermodynamical side of the topic, the work of Onsager [Ons31] on irreversible quasi-static processes is relevant.*

Remark 2.29. *When the domain Ω is not connected, but has multiple components Ω_i that are well separated from each other, i.e. $d(\Omega_i, \Omega_j) >\geq \delta$ for some $\delta > 0$, then we can essentialy apply Prop. 2.27 to each component separately. The reason is that the problem has no long-range interactions, like point-to-point forces (for instance electrostatic) or boundary forces induced by the solution of some elliptic problem in the exterior of Ω (as in the Stefan problem), that would couple the various components. The difficulty lies then in the case where two components are tangential, as would happen during a* pinch-off *or a* merge *of two droplets.*

Proposition 2.30 (Free energy functional)**.** *We assume that there exists a $D \subset \mathbb{R}^3$ such that the set Ω of Prop. 2.27 is in the shape space $\mathcal{O}^2(D)$. Furthermore, we assume that the substrate $S \subset \partial D$, and that the free boundary F is a single C^2 component of $\partial\Omega$. If the body force is conservative, that is $\mathbf{f} = -\nabla\phi$ for some potential $\phi \in H^1(D)$, then the* free energy functional $E : \mathcal{O} \to \mathbb{R}$

$$E(\Omega) := \int_\Omega \phi \, dV + \gamma \int_F da \tag{2.35}$$

is shape-differentiable at any $\Omega \in \mathcal{O}$ for any $\mathbf{v}$ in

$$Z(\Omega; D) := \{\mathbf{v} \in C^2(\overline{\Omega}, \mathbb{R}^3) \,|\, \mathbf{v} = 0 \text{ on } \partial D \cap \partial\Omega, \operatorname{div}\mathbf{v} = 0\} \subset \mathcal{V}^2(\Omega; D) \cap Z \tag{2.36}$$

and its shape derivative is

$$E'(\Omega)(\mathbf{v}) = \int_\Omega \langle \nabla\phi, \mathbf{v}\rangle \, dV - \int_F \langle \gamma H \mathbf{v}, \mathbf{n}\rangle \, da = -\int_\Omega \langle \mathbf{f}, \mathbf{v}\rangle \, dV - \int_F \langle \gamma H \mathbf{n}, \mathbf{v}\rangle \, da \tag{2.37}$$

Proof. Consider a $\mathbf{v} \in Z(\Omega; D)$. Then clearly $\mathbf{v} \in \mathcal{V}^2(\Omega; D)$, since it is in C^2 and it satisfies the appropriate boundary conditions (see (2.22)) on $\partial D \cap \partial\Omega$. Since Ω is bounded, $C^2(\overline{\Omega}, \mathbb{R}^3) \subset H^1(\Omega, \mathbb{R}^3)$ and moreover $\mathbf{v} \in H^1_S(\Omega, \mathbb{R}^3)$, given that $\mathbf{v} = 0$ on $\partial D \cap \partial\Omega \supset S$. Finally, $\operatorname{div} \mathbf{v} = 0$ and so $\mathbf{v} \in Z$. We conclude that indeed $Z(\Omega; D) \subset \mathcal{V}^2(\Omega; D) \cap Z$.

Having established that $Z(\Omega; D)$ is a set of Ω-admissible velocities, we can apply the propositions 2.23 and 2.24, via the lemma 2.25, to prove that $E'(\Omega)(\mathbf{v})$ is as claimed. Since the measure $|D| < +\infty$, for the scalar potential $\phi \in H^1(D) \subset W^{1,1}(D)$, so Prop. 2.23 is applicable. For $J_1(\Omega) := \int_\Omega \phi \, dV$, we have then

$$J_1'(\Omega)(\mathbf{v}) = \int_\Omega \operatorname{div}(\phi\,\mathbf{v})\, dV = \int_\Omega \left(\langle \nabla\phi, \mathbf{v}\rangle + \phi \operatorname{div} \mathbf{v} \right) dV = -\int_\Omega \langle \mathbf{f}, \mathbf{v}\rangle \, dV$$

since $\mathbf{f} = -\nabla\phi$ and $\operatorname{div} \mathbf{v} = 0$. Likewise $\psi := \gamma = \mathit{const} \Rightarrow \psi \in H^{3/2}(D)$ and both the domain and the velocities are of class C^2, therefore Prop. 2.24 is applicable. More specifically, (2.29) holds: if $J_2 := \int_{\partial\Omega} \gamma \, da$ then

$$J_2'(\Omega)(\mathbf{v}) = \sum_{i \in \mathcal{I}} \left(\int_{\Gamma_i} \langle -\gamma H \mathbf{v}, \mathbf{n}\rangle \, da + \int_{\partial\Gamma_i} \langle \gamma\, \mathbf{v}, \boldsymbol{\nu}_i \rangle \, dl \right)$$

Since $S \subset \partial D \cap \partial\Omega$ and $\mathbf{v} = 0$ on $\partial D \cap \partial\Omega$, the contributions of all the C^2 components Γ_i vanish, except for F itself. But since F is the single component that is not in $\partial D \cap \partial\Omega$, its boundary $\partial F \subset \partial D \cap \partial\Omega$ and so $\mathbf{v} = 0$ there. It follows that $\int_{\partial F} \langle \psi\, \mathbf{v}, \boldsymbol{\nu}\rangle \, dl = 0$ and so $J_2'(\Omega)(\mathbf{v}) = -\int_F \langle \gamma H \mathbf{v}, \mathbf{n}\rangle \, da$. Since $E(\Omega) = J_1(\Omega) + J_2(\Omega)$, adding the two shape derivatives yields the desired result.

□

Corollary 2.31 (Stokes flow as a gradient flow)**.** *The variational form* (2.32) *of the Stokes equations, restricted to* $Z(\Omega; D)$*, is equivalent to*

$$\mathbf{v} = \operatorname*{argmin}_{\mathbf{u} \in Z(\Omega; D)} \left\{ \frac{1}{2} a(\mathbf{u}, \mathbf{u}) + E'(\Omega)(\mathbf{u}) \right\} \quad \Rightarrow \quad a(\mathbf{v}, \mathbf{u}) = -E'(\Omega)(\mathbf{u}),\ \forall \mathbf{u} \in Z(\Omega; D) \tag{2.38}$$

where $a(\mathbf{v}, \mathbf{u}) = \int_\Omega 2\mu \mathcal{E}(\mathbf{v}) : \mathcal{E}(\mathbf{u}) \, dV$*. This is the* gradient flow *for the shape functional* E *that corresponds to the inner product on* $Z(\Omega; D)$ *induced by the symmetric & coercive bilinear form* $a(\cdot, \cdot)$*.*

Proof. For the coercivity of the bilinear form $a(\cdot, \cdot)$ see the proof of Prop. 2.27.

□

Proposition 2.32 (Dimensionless form)**.** *Let* $\mathbf{L}$ *and* $\boldsymbol{\Phi}$ *be the characteristic length and energy density (per unit volume) scale respectively. Then there exist constants* $\mathbf{V}, \mathbf{R}, \mathbf{E}$

(with dimensions of velocity, power and energy resp.) and a dimensionless constant ζ *so that*

$$R(\mathbf{v}) = \mathbf{R}\,\tilde{R}(\tilde{\mathbf{v}}) \quad \textit{and} \quad E(\Omega) = \mathbf{E}\,\tilde{E}(\Omega) \tag{2.39}$$

where

$$\tilde{R}(\tilde{\mathbf{v}}) := \frac{1}{2}\tilde{a}(\tilde{\mathbf{v}},\tilde{\mathbf{v}}) + \tilde{E}'(\Omega)(\tilde{\mathbf{v}}) \tag{2.40a}$$

$$\tilde{a}(\tilde{\mathbf{v}},\tilde{\mathbf{u}}) := \int_\Omega 2\mathcal{E}(\tilde{\mathbf{v}}) : \mathcal{E}(\tilde{\mathbf{u}})\,dV(\tilde{\mathbf{x}}) \tag{2.40b}$$

$$\tilde{E}(\Omega) := \int_\Omega \zeta\tilde{\phi}\,dV(\tilde{\mathbf{x}}) + \int_F da(\tilde{\mathbf{x}}) \tag{2.40c}$$

and $\tilde{\mathbf{x}} := \mathbf{L}^{-1}\mathbf{x}$*,* $\tilde{\mathbf{v}}(\tilde{\mathbf{x}}) := \mathbf{V}^{-1}\mathbf{v}(\mathbf{x})$ *and* $\tilde{\phi}(\tilde{\mathbf{x}}) := \mathbf{\Phi}^{-1}\phi(\mathbf{x})$*.*

Proof. Starting from the non-dimensionless functional E and substituting non-dimensionless with dimensionless variables, we have

$$E(\Omega) = \mathbf{\Phi}\mathbf{L}^3 \cdot \int_\Omega \tilde{\phi}\,dV(\tilde{\mathbf{x}}) + \gamma\mathbf{L}^2 \cdot \int_F da(\tilde{\mathbf{x}}) = \gamma\mathbf{L}^2 \cdot \left(\frac{\mathbf{\Phi}\mathbf{L}}{\gamma} \cdot \int_\Omega \tilde{\phi}\,dV(\tilde{\mathbf{x}}) + \int_F da(\tilde{\mathbf{x}})\right)$$

which gives us immediately $\mathbf{E} = \gamma\mathbf{L}^2$ and $\zeta = \frac{\mathbf{\Phi}\mathbf{L}}{\gamma}$.
Likewise for R:

$$R(\mathbf{v}) = \mu\mathbf{V}^2\mathbf{L} \cdot \frac{1}{2}\int_{B(\tilde{t})} 2\mathcal{E}(\tilde{\mathbf{v}}) : \mathcal{E}(\tilde{\mathbf{v}})\,dV(\tilde{\mathbf{x}}) + \gamma\mathbf{V}\mathbf{L} \cdot \left(\zeta\int_\Omega \langle\nabla\tilde{\phi},\tilde{\mathbf{v}}\rangle\,dV(\tilde{\mathbf{x}}) - \int_F \langle\tilde{H}\mathbf{n},\tilde{\mathbf{v}}\rangle\,da(\tilde{\mathbf{x}})\right)$$

Setting $\mathbf{V} = \frac{\gamma}{\mu}$ gives us $\mu\mathbf{V}^2\mathbf{L} = \gamma\mathbf{V}\mathbf{L} = \mathbf{R}$. It follows that the dimensional constants $(\mathbf{V},\mathbf{R},\mathbf{E}) = (\frac{\gamma}{\mu}, \frac{\gamma^2\mathbf{L}}{\mu}, \gamma\mathbf{L}^2)$ and the dimensionless $\zeta = \frac{\mathbf{\Phi}\mathbf{L}}{\gamma}$ satisfy the statement.

□

Definition 2.33 (Gravitational potential)**.** *Let* ρ *be the density of the fluid and* g *the gravitational acceleration. We make the simplifying assumption that the* z *Cartesian coordinate represents the altitude, and so define the gravitational potential as* $\phi_g := \rho g z$*.*

In the case where $\phi = \phi_g$*, the potential scales like* $\mathbf{\Phi} = \rho g\mathbf{L}$ *and so the constant* $\zeta = \frac{\mathbf{\Phi}\mathbf{L}}{\gamma} = \frac{\rho g\mathbf{L}^2}{\gamma}$*, which is the so-called* Bond number*.*

□

In the material that follows we will focus on the dimensionless form of the problem with the gravitational potential, while at the same dropping the tilda notation, i.e.

$$R(\mathbf{v}) = \frac{1}{2}a(\mathbf{v},\mathbf{v}) + E'(\Omega)(\mathbf{v}) \tag{2.41a}$$

$$a(\mathbf{v},\mathbf{u}) = \int_\Omega 2\mathcal{E}(\mathbf{v}) : \mathcal{E}(\mathbf{u})\, dV \tag{2.41b}$$

$$E(\Omega) = \int_\Omega \zeta z\, dV + \int_F da \tag{2.41c}$$

2.5. Flow in thin domains

In this section, as well as the next one, we will apply the results of shape calculus from Sec. 2.3 on the manifolds Γ and K from Chap. 1 (see fig. 2.3).

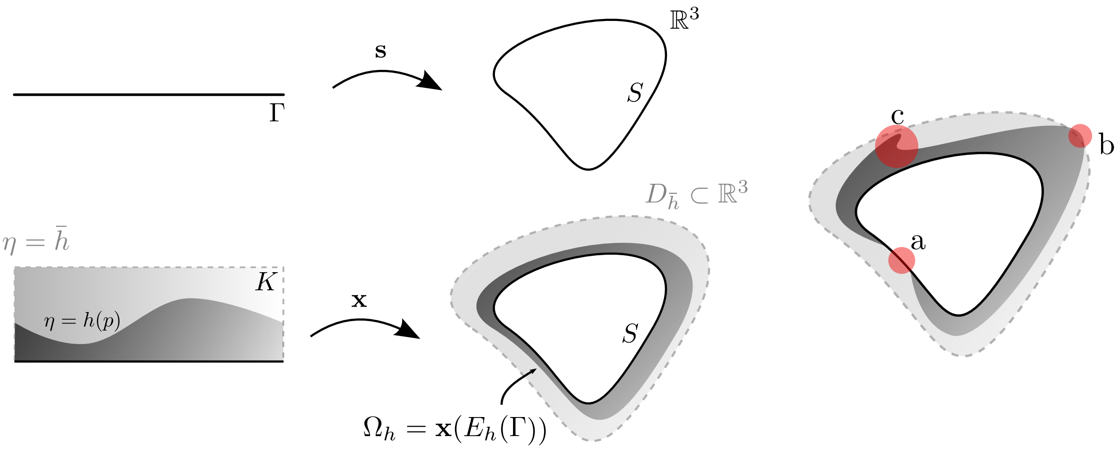


Figure 2.3.: *Thin domains.* Given an embedding $\mathbf{s} : \Gamma \to S$ of the manifold Γ onto the substrate $S \subset \mathbb{R}^3$, we consider a certain class of "thin" shapes Ω_h over S, which are derived from a height-field $h : \Gamma \to \mathbb{R}$. This fails when a) $h = 0$ (dewetting), b) h is larger than an upper bound $\bar{h}$ (see Rem. 1.5), or c) h is multivalued (folding).

Definition 2.34 (Flow and extrusion on Γ)**.** *Let $h : \Gamma \to \mathbb{R}$, with $h(p) > 0$ for all $p \in \Gamma$. We define the (normal)* flow *of $U \subset \Gamma$ as $\phi_h(U) := \{(p, h(p)) \in K \,|\, p \in U\}$ and the (normal)* extrusion *of U as $E_h(U) := \{(p, \tau h(p)) \in K \,|\, p \in U, \tau \in (0,1)\}$.*

Remark 2.35. *In the remainder, we will assume that the zero forms in $\Omega^0(\Gamma)$ and $\Omega^0(K)$ are at least in $C^2(\Gamma)$ or $C^2(K)$ respectively when considered as functions, unless* *specified otherwise. Likewise for the coefficients ω_I of p-forms $\omega_I dx^I$ and v^i of tangent vectors $v^i \partial x_i$.*

Definition 2.36 (Thin shapes over Γ)**.** *We assume that the embedding $\mathbf{s}(\Gamma) \subset \mathbb{R}^3$ of the 2-manifold Γ is a compact and connected surface of class C^2. We define the set $\mathcal{O}_\Gamma$ of* thin shapes on Γ *as*

$$\mathcal{O}_\Gamma := \{\Omega_h \in \mathcal{O}^2(D_{\bar{h}}) \,|\, \Omega_h = \mathbf{x}(E_h(\Gamma)),\ \textit{for some } h \in \Omega^0(\Gamma), 0 < h \leq \bar{h}\} \tag{2.42}$$

where the hold-all set $D_{\bar{h}} = \mathbf{x}(E_{\bar{h}}(\Gamma))$ for a constant $h(p) = \bar{h} > 0$ for all $p \in \Gamma$, small enough so that the conditions of Remark 1.5 are satisfied and, furthermore, the boundary $\partial D_{\bar{h}}$ is piecewise C^2.

Lemma 2.37 (Admissible velocities in K)**.** *Let $\mathbf{v} \in C^2(\overline{D_{\bar{h}}}, \mathbb{R}^3)$ and $v = v_\Gamma + v_n \partial\eta \in TK$ with $\mathbf{v} = d\mathbf{x}(v)$. Then $\mathbf{v} \in Z(D_{\bar{h}}; D_{\bar{h}}) = \{\mathbf{v} \in C^2(\overline{D_{\bar{h}}}, \mathbb{R}^3) \,|\, \mathbf{v} = 0$ on $\partial D_{\bar{h}}, \operatorname{div} \mathbf{v} = 0\}$, if and only if*

$$v(p,0) = 0, \qquad \forall p \in \Gamma \tag{2.43a}$$

$$v(p,H) = 0, \qquad \forall p \in \Gamma \tag{2.43b}$$

$$\frac{\partial}{\partial \eta}(\lambda_\eta v_n) + \operatorname{div}_\Gamma(\lambda_\eta v_\Gamma) = 0, \quad \forall (p,\eta) \in \Gamma \times (0,H) \tag{2.43c}$$

or equivalently

$$v_\Gamma(p,0) = 0, \qquad \forall p \in \Gamma \tag{2.44a}$$

$$v_\Gamma(p,H) = 0, \qquad \forall p \in \Gamma \tag{2.44b}$$

$$v_n = -\lambda_\eta^{-1} \int_0^\eta \operatorname{div}_\Gamma(\lambda_\xi v_\Gamma)\, d\xi, \quad \forall (p,\eta) \in \Gamma \times (0,H) \tag{2.44c}$$

$$\int_0^H \operatorname{div}_\Gamma(\lambda_\xi v_\Gamma)\, d\xi = 0, \qquad \forall p \in \Gamma \tag{2.44d}$$

Proof. From Cor. 1.54, we have that $\mathbf{v} = d\mathbf{x} \Rightarrow \mathbf{x}^*(\operatorname{div} \mathbf{v}) = \operatorname{div} v$. Since $\mathbf{x}$ is a diffeomorphism, it follows that the pullback $\mathbf{x}^*$ is a bijection and so $\operatorname{div} \mathbf{v} = 0 \Leftrightarrow \operatorname{div} v = 0 \Leftrightarrow \frac{\partial}{\partial \eta}(\lambda_\eta v_n) + \operatorname{div}_\Gamma(\lambda_\eta v_\Gamma) = 0$, from Prop. 1.55. Likewise, the pushforward $d\mathbf{x}$ is also a bijection, and so $\mathbf{v} = 0$ on $\partial D_{\bar{h}} \Leftrightarrow v(p,0) = v(p,H) = 0$ for any $p \in \Gamma$, since $\mathbf{x}(\Gamma \times \{0, H\}) = \partial D_{\bar{h}}$.

We conclude that $\mathbf{v} \in Z(D_{\bar{h}}; D_{\bar{h}})$ is equivalent to the first set of conditions. The second set of conditions comes simply from solving (2.43c) as an ODE w.r.t. η with initial condition $v_n(p,0) = 0$.

□

Proposition 2.38 (Transport of the Ω_h)**.** *Let $\mathbf{v} \in C([0,T), Z(D_{\bar{h}}; D_{\bar{h}}))$ and $v = v_\Gamma + v_n \partial\eta$ be time-dependent velocities, such that $d\mathbf{x}(v) = \mathbf{v}$ for all $t \in [0,T)$. Let $\mathbf{T}_t : \overline{D_{\bar{h}}} \to$*

$\overline{D_{\bar{h}}}$ be the corresponding transformation (Thm. 2.20) and $\Omega_{h_0} \in \mathcal{O}_\Gamma$. If $h_0 \in C^1(\Gamma)$, then there exists a $0 < T' \leq T$ such that

$$\mathbf{T}_t(\Omega_{h_0}) = \Omega_h, \quad \forall t \in [0, T') \tag{2.45}$$

where $h \equiv h(t,p)$, $0 < h \leq \bar{h}$, is the solution of the initial value problem

$$\lambda_h \frac{\partial h}{\partial t} + \mathrm{div}_\Gamma \left(\int_0^h \lambda_\eta v_\Gamma \, d\eta \right) = 0 \tag{2.46a}$$

$$h(0, \cdot) = h_0 \tag{2.46b}$$

Proof. Consider the functions $f(p,\eta) = \eta - h_0(p)$ and $f_t(p,\eta) = (f \circ \Pi_t)(p,\eta)$, $\Pi_t := \mathbf{x}^{-1} \circ \mathbf{T}_t^{-1} \circ \mathbf{x}$, in $C^1(K)$. The function $f \equiv f_0$ satisfies the conditions of the *implicit function theorem* on the level set $f = 0$, since $\frac{\partial f}{\partial \eta} = 1 \neq 0$, and indeed it is trivial to show that the level set $f = 0$ is a graph of the form $\eta = h_0(p)$, that is $f(p,\eta) = 0 \Leftrightarrow \eta = h_0(p) \Leftrightarrow \mathbf{x}(p,\eta) \in F_{h_0}$. Because the maps $\mathbf{x}$ and $\mathbf{T}_t$ are homomorphisms and because of the continuity of $\mathbf{T}_t$ in time, we can show that there exists a time interval $[0, T')$, so that for any $t \in [0, T')$ the function f_t also satisfies the implicit function theorem on the level set $f_t = 0$. Hence there exists a function $h_t(p)$ such that $\eta = h_t(p) \Leftrightarrow f_t(p,\eta) = 0 \Leftrightarrow f(\Pi_t(p,\eta)) = 0 \Leftrightarrow \mathbf{x}(\Pi_t(p,\eta)) \in F_{h_0} \Leftrightarrow \mathbf{x}(p,\eta) \in \mathbf{T}_t(F_{h_0})$. We conclude that $\mathbf{T}_t(F_{h_0}) = F_{h_t}$ and we identify $h(t,\cdot) := h_t$. Because $\mathbf{v} = 0$ on the substrate S, it follows that it is invariant under the $\mathbf{T}_t$, i.e. $\mathbf{T}_t(S) = S$. Then $\mathbf{T}_t(\partial\Omega_{h_0}) = \mathbf{T}_t(F_{h_0} \cup S) = F_{h_t} \cup S = \partial\Omega_{h_t}$ and so by the continuity of $\mathbf{T}_t$ we have that $\mathbf{T}_t(\Omega_{h_0}) = \Omega_{h_t}$.

Consider an arbitrary $p_0 \in \Gamma$. The point $\mathbf{x}(p_0, h_0(p_0))$ belongs to the free boundary F_{h_0} and so there exists $p \equiv p(t) \in \Gamma$ such that $\mathbf{T}_t(\mathbf{x}(p_0, h_0(p_0))) = \mathbf{x}(p(t), h(t, p(t))) \equiv \mathbf{x}(p, h(p))$. Taking the time derivative of both sides, we have

$$\begin{aligned}
\frac{\partial}{\partial t}\mathbf{T}_t(\mathbf{x}(p_0, h_0(p_0))) = \frac{\partial}{\partial t}\mathbf{x}(p, h(p)) \;&\Rightarrow\; \mathbf{v}(\mathbf{x}(p, h(p))) = d\mathbf{x}\left(\frac{\partial}{\partial t}(p, h(p))\right) \\
&\Rightarrow\; d\mathbf{x}(v(p, h(p))) = d\mathbf{x}\left(\dot{p} + \big(\frac{\partial h}{\partial t} + d_\Gamma h(\dot{p})\big)\partial\eta\right) \\
&\Rightarrow\; v(p, h(p)) = \dot{p} + \big(\frac{\partial h}{\partial t} + d_\Gamma h(\dot{p})\big)\partial\eta
\end{aligned}$$

where $\dot{p} \in T\Gamma$ is the velocity of the curve $p(t)$ on Γ. Since $v(p, h(p)) = v_\Gamma(p, h(p)) + v_n(p, h(p))$, we conclude that $v_{\Gamma,h} \equiv v_\Gamma(p, h(p)) = \dot{p}$ and $v_{n,h} \equiv v_n(p, h(p)) = \frac{\partial h}{\partial t} + d_\Gamma h(\dot{p})$, and therefore

$$\frac{\partial h}{\partial t} = v_{n,h} - d_\Gamma h(v_{\Gamma,h})$$

Note the following *differentiation under the integral sign* rule for diff. forms:

$$
\begin{aligned}
d_\Gamma \left(\int_0^{h(p)} \omega(p,\eta)\, d\eta \right) &= dx^\alpha \wedge \frac{\partial}{\partial x_\alpha} \left(\int_0^{h(p)} \omega(p,\eta)\, d\eta \right) \\
&= dx^\alpha \wedge \left(\frac{\partial h}{\partial x_\alpha} \omega_h + \int_0^h \frac{\partial \omega_\eta}{\partial x^\alpha}\, d\eta \right) = \left(\frac{\partial h}{\partial x_\alpha} dx^\alpha \right) \wedge \omega_h + \int_0^h dx^\alpha \wedge \frac{\partial \omega_\eta}{\partial x^\alpha}\, d\eta \\
&= d_\Gamma h \wedge \omega_h + \int_0^h d_\Gamma \omega_\eta \, d\eta
\end{aligned}
$$

Applying this identity to the 1-form $\omega(p,\eta) = \star_\Gamma (\lambda_\eta v_{\Gamma,\eta})^{\flat_\Gamma}$, we have

$$
\begin{aligned}
& d_\Gamma \left(\int_0^h \star_\Gamma (\lambda_\eta v_{\Gamma,\eta})^{\flat_\Gamma}\, d\eta \right) = d_\Gamma h \wedge \star_\Gamma (\lambda_h v_{\Gamma,h})^{\flat_\Gamma} + \int_0^h d_\Gamma \star_\Gamma (\lambda_\eta v_{\Gamma,\eta})^{\flat_\Gamma}\, d\eta \\
\Rightarrow\ & d_\Gamma \star_\Gamma \left(\int_0^h \lambda_\eta v_{\Gamma,\eta}\, d\eta \right)^{\flat_\Gamma} = d_\Gamma h (\lambda_h v_{\Gamma,h})\, \mathrm{vol}_\Gamma + \int_0^h d_\Gamma \star_\Gamma (\lambda_\eta v_{\Gamma,\eta})^{\flat_\Gamma}\, d\eta \\
\Rightarrow\ & \star_\Gamma d_\Gamma \star_\Gamma \left(\int_0^h \lambda_\eta v_{\Gamma,\eta}\, d\eta \right)^{\flat_\Gamma} = d_\Gamma h (\lambda_h v_{\Gamma,h}) + \star_\Gamma \int_0^h d_\Gamma \star_\Gamma (\lambda_\eta v_{\Gamma,\eta})^{\flat_\Gamma}\, d\eta \\
\Rightarrow\ & \star_\Gamma d_\Gamma \star_\Gamma \left(\int_0^h \lambda_\eta v_{\Gamma,\eta}\, d\eta \right)^{\flat_\Gamma} = \lambda_h\, d_\Gamma h (v_{\Gamma,h}) + \int_0^h \star_\Gamma d_\Gamma \star_\Gamma (\lambda_\eta v_{\Gamma,\eta})^{\flat_\Gamma}\, d\eta \\
\Rightarrow\ & \mathrm{div}_\Gamma \left(\int_0^h \lambda_\eta v_{\Gamma,\eta}\, d\eta \right) = \lambda_h\, d_\Gamma h (v_{\Gamma,h}) + \int_0^h \mathrm{div}_\Gamma (\lambda_\eta v_{\Gamma,\eta})\, d\eta
\end{aligned}
$$

Since $\mathbf{v} \in Z(D_{\bar{h}}; D_{\bar{h}})$ for every $t \in [0, T')$, it satisfies Lem. 2.37. Taking (2.44c) with $\eta = h(p)$ (and exchanging the dummy variable ξ with η), we have $v_{n,h} = -\lambda_h^{-1} \int_0^h \mathrm{div}_\Gamma (\lambda_\eta v_{\Gamma,\eta})\, d\eta$ and so

$$
\begin{aligned}
\lambda_h \frac{\partial h}{\partial t} &= \lambda_h v_{n,h} - \lambda_h\, d_\Gamma h (v_{\Gamma,h}) \\
&= - \int_0^h \mathrm{div}_\Gamma (\lambda_\eta v_{\Gamma,\eta})\, d\eta - \lambda_h\, d_\Gamma h (v_{\Gamma,h}) = - \mathrm{div}_\Gamma \left(\int_0^h \lambda_\eta v_{\Gamma,\eta}\, d\eta \right)
\end{aligned}
$$

□

2.6. Reduced energy

In this section, we calculate asymptotic expansions of the gravitational and surface parts of the free energy functional using the shape calculus of Sec. 2.3.

Definition 2.39 (Landau notation)**.** *For a function $f \equiv f(\epsilon)$ of a real (non-negative) variable ϵ, we define the notation $\mathrm{O}(\epsilon^p)$, as*

$$f = \mathrm{O}(\epsilon^p) \Leftrightarrow \limsup_{\epsilon \to 0^+} \frac{|f|}{\epsilon^p} < +\infty \tag{2.47}$$

In particular, $f = \mathrm{O}(\epsilon^0)$ implies the existence of an upper bound $|f| \le C$ independent of *ϵ. Note also the following corollary:*

$$f = g + \mathrm{O}(\epsilon^p) \Leftrightarrow \exists C, \epsilon' > 0 : \, |f - g| \le C\epsilon^p, \, \forall \epsilon \in (0, \epsilon') \tag{2.48}$$

Lemma 2.40 (Reduced gravitational energy)**.** *The volume integral of z over Ω_h can be approximated as*

$$\int_{\Omega_h} z \, dV = \epsilon \int_\Gamma \Big(hz_\Gamma + \frac{\epsilon}{2} h^2 (\mathbf{N}_z - Hz_\Gamma) + \mathrm{O}(\epsilon^2) \Big) \operatorname{vol}_\Gamma \tag{2.49}$$

where $z_\Gamma = \mathbf{s}^*(z)$ is the pullback of z onto Γ and $\mathbf{N}_z := \langle \mathbf{N}, \mathbf{z} \rangle$ is the vertical component of the normal $\mathbf{N}$ of the substrate.

Proof. We have $\Omega_h = \mathbf{x}(E_h(\Gamma))$ and so

$$\int_{\mathbf{x}(E_h(\Gamma))} z \, dV = \int_{E_h(\Gamma)} z_K \operatorname{vol}_K = \int_{E_h(\Gamma)} z_K \epsilon \lambda_\eta \operatorname{vol}_\Gamma \wedge d\eta = \epsilon \int_\Gamma \Big(\int_0^h z_K \lambda_\eta \, d\eta \Big) \operatorname{vol}_\Gamma$$

where $z_K = \mathbf{x}^*(z)$ is the pull-back of z onto K. Furthermore, $\frac{\partial z_K}{\partial \eta} = \langle \nabla z, \frac{\partial \mathbf{x}}{\partial \eta} \rangle = \epsilon \langle \nabla z, \mathbf{N} \rangle = \epsilon \, \mathbf{N}_z$, and so we can use the Taylor expansion $z_K = z_\Gamma + \epsilon \eta \mathbf{N}_z + \mathrm{O}(\epsilon^2)$, since $z_K|_{\eta=0} \equiv z_\Gamma$. It follows that $\int_0^h z_K \lambda_\eta \, d\eta = \int_0^h (z_\Gamma + \epsilon\eta \mathbf{N}_z + \mathrm{O}(\epsilon^2))(1 - \epsilon \eta H + \mathrm{O}(\epsilon^2)) \, d\eta = \int_0^h (z_\Gamma + \epsilon \eta \mathbf{N}_z - \epsilon \eta H z_\Gamma) \, d\eta + \mathrm{O}(\epsilon^2) = hz_\Gamma + \frac{\epsilon}{2} h^2 (\mathbf{N}_z - Hz_\Gamma) + \mathrm{O}(\epsilon^2)$ and so

$$\int_{E_h(\Gamma)} z_K \operatorname{vol}_K = \epsilon \int_\Gamma \Big(hz_\Gamma + \frac{\epsilon}{2} h^2 (\mathbf{N}_z - Hz_\Gamma) + \mathrm{O}(\epsilon^2) \Big) \operatorname{vol}_\Gamma$$

$\square$

Lemma 2.41 (Reduced surface energy)**.** *The area of the free surface F_h can be approximated as*

$$\int_{F_h} da = \int_\Gamma \operatorname{vol}_\Gamma + \epsilon \int_\Gamma \Big(-hH + \epsilon h^2 G + \frac{\epsilon}{2} |\operatorname{grad}_\Gamma h|_\Gamma^2 + \mathrm{O}(\epsilon^2) \Big) \operatorname{vol}_\Gamma \tag{2.50}$$

Proof. Since $\phi_h(\Gamma)$ is the image of Γ under the map ϕ_h, we can consider Γ with the pull-back metric $g_{\phi_h(\Gamma)}(u_\Gamma, v_\Gamma) := g(d\phi_h(u_\Gamma), d\phi_h(v_\Gamma))$, $u_\Gamma, v_\Gamma \in T\Gamma$, in which case

$\int_{\mathbf{x}(\phi_h(\Gamma))} da = \int_\Gamma \mathrm{vol}_{\phi_h(\Gamma)}$. More specifically, $\phi_h(p) = (p, h(p)) \Rightarrow d\phi_h(u_\Gamma) = u_\Gamma + d_\Gamma h(u_\Gamma)\,\partial\eta$ and so

$$\begin{aligned} g_{\phi_h(\Gamma)}(u_\Gamma, v_\Gamma) &= g(d\phi_h(u_\Gamma), d\phi_h(v_\Gamma)) \\ &= g(u_\Gamma + d_\Gamma h(u_\Gamma)\,\partial\eta, v_\Gamma + d_\Gamma h(v_\Gamma)\,\partial\eta) = g_\Gamma(\Lambda_h u_\Gamma, \Lambda_h v_\Gamma) + \epsilon^2 d_\Gamma h(u_\Gamma) d_\Gamma h(v_\Gamma) \end{aligned}$$

In matrix form, and following the proof of Prop. 1.9, $(g_{\phi_h(\Gamma)})_{\alpha\beta} = g_\Gamma(\Lambda_h \partial x_\alpha, \Lambda_h \partial x_\beta) + \epsilon^2 d_\Gamma h(\partial x_\alpha) d_\Gamma h(\partial x_\beta) = (\Lambda_h g_\Gamma \Lambda_h^T)_{\alpha\beta} + \epsilon^2 \frac{\partial h}{\partial x_\alpha}\frac{\partial h}{\partial x_\beta}$ and so $g_{\phi_h(\Gamma)} = \Lambda_h g_\Gamma \Lambda_h^T + \epsilon^2 \partial h\,\partial h^T$, where $(\partial h)_\alpha := \frac{\partial h}{\partial x_\alpha}$ is the column vector of the partial derivatives of h. Using Lemma 2.42, we get $\det(g_{\phi_h(\Gamma)}) = \big(1 + \epsilon^2 \partial h^T (\Lambda_h g_\Gamma \Lambda_h^T)^{-1} \partial h\big) \det(\Lambda_h g_\Gamma \Lambda_h^T)$. Since $\Lambda_h = \mathrm{id} - \epsilon h S \Rightarrow \Lambda_h^{-1} = \mathrm{id} + \mathrm{O}(\epsilon)$, it follows that $\partial h^T (\Lambda_h g_\Gamma \Lambda_h^T)^{-1} \partial h = \partial h^T g_\Gamma^{-1} \partial h + \mathrm{O}(\epsilon)$. But

$$\begin{aligned} \partial h^T g_\Gamma^{-1} \partial h = g_\Gamma^{-1}(\partial h_\alpha dx^\alpha, \partial h_\beta dx^\beta) &= g_\Gamma^{-1}(d_\Gamma h, d_\Gamma h) \\ &= g_\Gamma((d_\Gamma h)^{\sharp_\Gamma}, (d_\Gamma h)^{\sharp_\Gamma}) = |\mathrm{grad}_\Gamma h|_\Gamma^2 \end{aligned}$$

and furthermore, $\det(\Lambda_h g_\Gamma \Lambda_h^T) = \lambda_h^2 \det(g_\Gamma)$, hence

$$\begin{aligned} \mathrm{vol}_{\phi_h(\Gamma)} = \sqrt{\det(g_{\phi_h(\Gamma)})}\, dx^1 \wedge dx^2 &= \sqrt{(1 + \epsilon^2 |\mathrm{grad}_\Gamma h|^2)\lambda_h^2 \det(g_\Gamma)}\, dx^1 \wedge dx^2 \\ &= \left(\lambda_h + \frac{\epsilon^2}{2}|\mathrm{grad}_\Gamma h|_\Gamma^2 + \mathrm{O}(\epsilon^3)\right) \sqrt{\det(g_\Gamma)}\, dx^1 \wedge dx^2 \end{aligned}$$

and so

$$\begin{aligned} \int_{F_h} da = \int_{\mathbf{x}(\phi_h(\Gamma))} da &= \int_\Gamma \mathrm{vol}_{\phi_h(\Gamma)} \\ &= \int_\Gamma \left(1 - \epsilon h H + \epsilon^2 h^2 G + \frac{\epsilon^2}{2}|\mathrm{grad}_\Gamma h|_\Gamma^2 + \mathrm{O}(\epsilon^3)\right) \mathrm{vol}_\Gamma \\ &= \int_\Gamma \mathrm{vol}_\Gamma + \epsilon \int_\Gamma \left(-hH + \epsilon h^2 G + \frac{\epsilon}{2}|\mathrm{grad}_\Gamma h|_\Gamma^2 + \mathrm{O}(\epsilon^2)\right) \mathrm{vol}_\Gamma \end{aligned}$$

□

Lemma 2.42 (Rank-one update of det)**.** *Let $A \in \mathbb{R}^{n\times n}$ be invertible and $u, v \in \mathbb{R}^n$. Then*

$$\det(A + uv^T) = (1 + v^T A^{-1} u) \det(A)\,.$$

Proof. The factorization

$$\begin{pmatrix} I + uv^T & u \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} I & 0 \\ -v^T & 1 \end{pmatrix} \begin{pmatrix} I & u \\ 0 & 1 + v^T u \end{pmatrix} \begin{pmatrix} I & 0 \\ v^T & 1 \end{pmatrix}$$

gives us $\det(I + uv^T) = 1 + v^T u$. Then $\det(A + uv^T) = \det(A)\det(I + A^{-1}uv^T) = \det(A)(1 + v^T A^{-1} u)$.

□

Proposition 2.43 (Reduced free energy)**.** *Let $\Omega_h \in \mathcal{O}_\Gamma$. If we let*

$$E_\epsilon(h) := \int_\Gamma \left\{ (\zeta z_\Gamma - H)h + \frac{\epsilon}{2}(\zeta \mathbf{N}_z - \zeta H z_\Gamma + 2G)h^2 + \frac{\epsilon}{2}|\mathrm{grad}_\Gamma\, h|_\Gamma^2 \right\} \mathrm{vol}_\Gamma \qquad (2.51)$$

then $E(\Omega_h) = \int_\Gamma da + \epsilon\, E_\epsilon(h) + \mathrm{O}(\epsilon^3)$. Furthermore, if $\mathbf{v} \in Z(\Omega_h; D_{\bar{h}})$ with $\mathbf{v} = d\mathbf{x}(v_\Gamma + v_n \partial\eta)$ then

$$E'(\Omega_h)(\mathbf{v}) = \epsilon\, E'_\epsilon(h)(\dot{h}) + \mathrm{O}(\epsilon^3) \qquad (2.52)$$

$$\lambda_h \dot{h} + \mathrm{div}_\Gamma \left(\int_0^h \lambda_\eta v_\Gamma \, d\eta \right) = 0 \qquad (2.53)$$

and

$$E'_\epsilon(h)(\dot{h}) = \int_\Gamma \left\{ P(h)\dot{h} + \epsilon \langle \mathrm{grad}_\Gamma\, h, \mathrm{grad}_\Gamma\, \dot{h} \rangle_\Gamma \right\} \mathrm{vol}_\Gamma \qquad (2.54)$$

$$P(h) := (\zeta z_\Gamma - H) + \epsilon(\zeta \mathbf{N}_z - \zeta H z_\Gamma + 2G)h \qquad (2.55)$$

is the Gâteaux derivative of E_ϵ at h in the direction of $\dot{h}$.

Proof. The relation $E(\Omega_h) = \int_\Gamma da + \epsilon\, E_\epsilon(h) + \mathrm{O}(\epsilon^3)$ is a direct application of Lem. 2.40 and Lem. 2.41.

We consider an extended velocity $\hat{\mathbf{v}}(t) \in C([0,T), Z(D_{\bar{h}}; D_{\bar{h}}))$ with $\hat{\mathbf{v}}(0)|_{\Omega_h} = \mathbf{v}$. Then by definition

$$\begin{aligned} E'(\Omega_h)(\hat{\mathbf{v}}) = \lim_{t\to 0^+} \frac{E(\mathbf{T}_t(\Omega_h)) - E(\Omega_h)}{t} &= \lim_{t\to 0^+} \frac{E(\Omega_{h(t)}) - E(\Omega_h)}{t} \\ &= \lim_{t\to 0^+} \frac{\epsilon(E_\epsilon(h(t)) - E_\epsilon(h)) + \mathrm{O}(\epsilon^3)}{t} = \epsilon\, E'_\epsilon(h)(h'(0)) + \mathrm{O}(\epsilon^3) \end{aligned}$$

where $h(t)$ is the heightfield associated to $\hat{\mathbf{v}}$ by Prop. 2.38., and so is a solution of the initial value problem $\lambda_h \frac{\partial h}{\partial t} + \mathrm{div}_\Gamma \left(\int_0^h \lambda_\eta \hat{v}_\Gamma \, d\eta \right) = 0$ with $h(0) = h$.

We identify $\dot{h} = h'(0)$ and note that it is well-defined, because it only depends on the values of $\hat{\mathbf{v}}$ at $t = 0$ and $\mathbf{x} \in \Omega_h$ and so is independent of the actual extension.

□

2.7. Reduced dissipation

Here we calculate an asymptotic expansion of the dissipation functional. We make extensive use of the exterior and tensor calculus results of Chapter 1.

Lemma 2.44 (Directional derivatives on K)**.** *Let* $\mathbf{u} = d\mathbf{x}(u)$ *and* $\mathbf{v} = d\mathbf{x}(v)$, $u, v \in TK$. *Then* $D_{\mathbf{u}}\mathbf{v} = d\mathbf{x}(D_u v)$ *where*

$$D_u v := \frac{1}{2}\Big\{ \operatorname{curl}(v \times u) - (\operatorname{div} u)v + (\operatorname{div} v)u + \operatorname{grad}\langle v, u\rangle_K - v \times \operatorname{curl} u - u \times \operatorname{curl} v \Big\} \quad (2.56)$$

Proof. A straightforward application of the pushforward/pullback properties of Cor. 1.54 on expression (1.82).

□

Lemma 2.45 (Decomposition of $D_{\partial x_i} v$)**.** *For the directional derivative* $D_{\partial x_i} v$ *of* $v \in TK$ *in the direction of the basis vectors* ∂x_i*, we have*

$$D_{\partial x_\alpha} v = \frac{1}{2}\left(v_{\Gamma,\alpha} + v_{n,\alpha}\,\partial\eta\right) \quad (2.57)$$

$$D_{\partial x_3} v = \frac{1}{2}\left(v_{\Gamma,3} + v_{n,3}\,\partial\eta\right) \quad (2.58)$$

where

$$v_{\Gamma,\alpha} = \mathrm{O}(\epsilon^0) \quad (2.59a)$$

$$v_{n,\alpha} = -\epsilon^{-2}\langle v_\Gamma, \frac{\partial}{\partial\eta}(\Lambda_\eta^2 \partial x_\alpha)\rangle_\Gamma + \mathrm{O}(\epsilon^0) = O(\epsilon^{-1}) \quad (2.59b)$$

$$v_{\Gamma,3} = \frac{\partial v_\Gamma}{\partial\eta} + \Lambda_\eta^{-2}\frac{\partial}{\partial\eta}(\Lambda_\eta^2 v_\Gamma) = \mathrm{O}(\epsilon^0) \quad (2.59c)$$

$$v_{n,3} = 2\frac{\partial v_n}{\partial\eta} = \mathrm{O}(\epsilon^0) \quad (2.59d)$$

Proof. We look at the terms of (2.56) in turn and apply the expansions of Prop. 1.55:

- For the first term, we have

$$\begin{aligned}\operatorname{curl}(v \times u) &= \operatorname{curl}\left(\epsilon\lambda_\eta\Lambda_\eta^{-2}(u_n \star_\Gamma v_\Gamma - v_n \star_\Gamma u_\Gamma) + \epsilon^{-1}\lambda_\eta(v_\Gamma \times_\Gamma u_\Gamma)\partial\eta\right)\\ &= \lambda_\eta^{-1} \star_\Gamma \left(\operatorname{grad}_\Gamma(\lambda_\eta(v_\Gamma \times_\Gamma u_\Gamma)) - \frac{\partial}{\partial\eta}(\lambda_\eta u_n \star_\Gamma v_\Gamma - \lambda_\eta v_n \star_\Gamma u_\Gamma)\right)\\ &\qquad + \lambda_\eta^{-1}\operatorname{curl}_\Gamma(\lambda_\eta u_n \star_\Gamma v_\Gamma - \lambda_\eta v_n \star_\Gamma u_\Gamma)\,\partial\eta\end{aligned}$$

  Setting $u_\Gamma = \partial x_\alpha, u_n = 0$ and $u_\Gamma = 0, u_n = 1$ and splitting in tangential/normal components, gives us the following contributions to (2.59):

$$\begin{aligned}
v_{\Gamma,\alpha} &\leftarrow \lambda_\eta^{-1} \star_\Gamma \left(\operatorname{grad}_\Gamma(\lambda_\eta(v_\Gamma \times_\Gamma \partial x_\alpha)) + \frac{\partial}{\partial\eta}(\lambda_\eta v_n \star_\Gamma \partial x_\alpha) \right) \\
v_{n,\alpha} &\leftarrow -\lambda_\eta^{-1} \operatorname{curl}_\Gamma(\lambda_\eta v_n \star_\Gamma \partial x_\alpha) \\
v_{\Gamma,3} &\leftarrow -\lambda_\eta^{-1} \star_\Gamma \frac{\partial}{\partial\eta}(\lambda_\eta \star_\Gamma v_\Gamma) \\
v_{n,3} &\leftarrow \lambda_\eta^{-1} \operatorname{curl}_\Gamma(\lambda_\eta \star_\Gamma v_\Gamma)
\end{aligned}$$

- For the second and third terms, we have

$$\begin{aligned}
-(\operatorname{div} u)v + (\operatorname{div} v)u = -\lambda_\eta^{-1} &\left(\operatorname{div}_\Gamma(\lambda_\eta u_\Gamma) + \frac{\partial}{\partial\eta}(\lambda_\eta u_n) \right)(v_\Gamma + v_n \partial\eta) \\
&+ \lambda_\eta^{-1} \left(\operatorname{div}_\Gamma(\lambda_\eta v_\Gamma) + \frac{\partial}{\partial\eta}(\lambda_\eta v_n) \right)(u_\Gamma + u_n \partial\eta)
\end{aligned}$$

which contributes

$$\begin{aligned}
v_{\Gamma,\alpha} &\leftarrow -\lambda_\eta^{-1} \operatorname{div}_\Gamma(\lambda_\eta \partial x_\alpha) v_\Gamma + \lambda_\eta^{-1} \left(\operatorname{div}_\Gamma(\lambda_\eta v_\Gamma) + \frac{\partial}{\partial\eta}(\lambda_\eta v_n) \right) \partial x_\alpha \\
v_{n,\alpha} &\leftarrow -\lambda_\eta^{-1} \operatorname{div}_\Gamma(\lambda_\eta \partial x_\alpha) v_n \\
v_{\Gamma,3} &\leftarrow -\lambda_\eta^{-1} \frac{\partial\lambda_\eta}{\partial\eta} v_\Gamma \\
v_{n,3} &\leftarrow -\lambda_\eta^{-1} \frac{\partial\lambda_\eta}{\partial\eta} v_n + \lambda_\eta^{-1} \left(\operatorname{div}_\Gamma(\lambda_\eta v_\Gamma) + \frac{\partial}{\partial\eta}(\lambda_\eta v_n) \right)
\end{aligned}$$

- For the fourth term, we have

$$\begin{aligned}
&\operatorname{grad}\langle v, u\rangle_K = \operatorname{grad}\left(\langle \Lambda_\eta v_\Gamma, \Lambda_\eta u_\Gamma\rangle_\Gamma + \epsilon^2 v_n u_n\right) \\
&= \Lambda_\eta^{-2} \operatorname{grad}_\Gamma(\langle \Lambda_\eta v_\Gamma, \Lambda_\eta u_\Gamma\rangle_\Gamma + \epsilon^2 v_n u_n) + \epsilon^{-2} \frac{\partial}{\partial\eta}(\langle \Lambda_\eta v_\Gamma, \Lambda_\eta u_\Gamma\rangle_\Gamma)\partial\eta + \frac{\partial}{\partial\eta}(v_n u_n)\partial\eta
\end{aligned}$$

which contributes

$$\begin{aligned}
v_{\Gamma,\alpha} &\leftarrow \Lambda_\eta^{-2} \operatorname{grad}_\Gamma(\langle \Lambda_\eta v_\Gamma, \Lambda_\eta \partial x_\alpha\rangle_\Gamma) \\
v_{n,\alpha} &\leftarrow \epsilon^{-2} \frac{\partial}{\partial\eta}(\langle \Lambda_\eta v_\Gamma, \Lambda_\eta \partial x_\alpha\rangle_\Gamma) \\
v_{\Gamma,3} &\leftarrow \epsilon^2 \Lambda_\eta^{-2} \operatorname{grad}_\Gamma v_n \\
v_{n,3} &\leftarrow \frac{\partial v_n}{\partial\eta}
\end{aligned}$$

- For the last two terms, we have

$$
\begin{aligned}
-u\times\operatorname{curl} v &= -u\times\left\{\epsilon^{-1}\lambda_\eta^{-1}\star_\Gamma\left(\epsilon^2\operatorname{grad}_\Gamma u_n - \frac{\partial}{\partial\eta}(\Lambda_\eta^2 u_\Gamma)\right) + \epsilon^{-1}\lambda_\eta^{-1}\operatorname{curl}_\Gamma(\Lambda_\eta^2 u_\Gamma)\partial\eta\right\} \\
&= -\Lambda_\eta^{-2}\left(\operatorname{curl}_\Gamma(\Lambda_\eta^2 v_\Gamma)\star_\Gamma u_\Gamma - u_n\star_\Gamma\star_\Gamma(\epsilon^2\operatorname{grad}_\Gamma v_n - \frac{\partial}{\partial\eta}(\Lambda_\eta^2 v_\Gamma))\right) \\
&\qquad - \epsilon^{-2}\left(u_\Gamma\times_\Gamma\star_\Gamma(\epsilon^2\operatorname{grad}_\Gamma v_n - \frac{\partial}{\partial\eta}(\Lambda_\eta^2 v_\Gamma))\right)\partial\eta \\
&= -\Lambda_\eta^{-2}\left(\operatorname{curl}_\Gamma(\Lambda_\eta^2 v_\Gamma)\star_\Gamma u_\Gamma + u_n(\epsilon^2\operatorname{grad}_\Gamma v_n - \frac{\partial}{\partial\eta}(\Lambda_\eta^2 v_\Gamma))\right) \\
&\qquad + \epsilon^{-2}\langle u_\Gamma, \epsilon^2\operatorname{grad}_\Gamma v_n - \frac{\partial}{\partial\eta}(\Lambda_\eta^2 v_\Gamma)\rangle_\Gamma\,\partial\eta
\end{aligned}
$$

where we used the fact that $u_\Gamma\times_\Gamma\star_\Gamma v_\Gamma = -i_{u_\Gamma}\star_\Gamma(\star_\Gamma v_\Gamma)^{\flat_\Gamma} = i_{u_\Gamma}\star_\Gamma\star_\Gamma(v_\Gamma^{\flat_\Gamma}) = -i_{u_\Gamma}v_\Gamma^{\flat_\Gamma} = -\langle u_\Gamma, v_\Gamma\rangle_\Gamma$. Likewise

$$
\begin{aligned}
-v\times\operatorname{curl} u &= -\Lambda_\eta^{-2}\left(\operatorname{curl}_\Gamma(\Lambda_\eta^2 u_\Gamma)\star_\Gamma v_\Gamma + v_n(\epsilon^2\operatorname{grad}_\Gamma u_n - \frac{\partial}{\partial\eta}(\Lambda_\eta^2 u_\Gamma))\right) \\
&\qquad + \epsilon^{-2}\langle v_\Gamma, \epsilon^2\operatorname{grad}_\Gamma u_n - \frac{\partial}{\partial\eta}(\Lambda_\eta^2 u_\Gamma)\rangle_\Gamma\,\partial\eta
\end{aligned}
$$

The contributions of these terms are then

$$
\begin{aligned}
&v_{\Gamma,\alpha}\leftarrow -\Lambda_\eta^{-2}\operatorname{curl}_\Gamma(\Lambda_\eta^2 v_\Gamma)\star_\Gamma\partial x_\alpha - \Lambda_\eta^{-2}\operatorname{curl}_\Gamma(\Lambda_\eta^2\partial x_\alpha)\star_\Gamma v_\Gamma + v_n\Lambda_\eta^{-2}\frac{\partial}{\partial\eta}(\Lambda_\eta^2\partial x_\alpha) \\
&v_{n,\alpha}\leftarrow -\epsilon^{-2}\langle\partial x_\alpha, \frac{\partial}{\partial\eta}(\Lambda_\eta^2 v_\Gamma)\rangle_\Gamma + \langle\partial x_\alpha, \operatorname{grad}_\Gamma v_n\rangle_\Gamma - \epsilon^{-2}\langle v_\Gamma, \frac{\partial}{\partial\eta}(\Lambda_\eta^2\partial x_\alpha)\rangle_\Gamma \\
&v_{\Gamma,3}\leftarrow \Lambda_\eta^{-2}\frac{\partial}{\partial\eta}(\Lambda_\eta^2 v_\Gamma) - \epsilon^2\Lambda_\eta^{-2}\operatorname{grad}_\Gamma v_n \\
&v_{n,3}\leftarrow 0
\end{aligned}
$$

Collecting all the $v_{\Gamma,3}$ contributions, we have

$$\begin{aligned}
v_{\Gamma,3} &= -\lambda_\eta^{-1} \star_\Gamma \frac{\partial}{\partial\eta}(\lambda_\eta \star_\Gamma v_\Gamma) - \lambda_\eta^{-1}\frac{\partial\lambda_\eta}{\partial\eta} v_\Gamma + \epsilon^2\Lambda_\eta^{-2}\operatorname{grad}_\Gamma v_n \\
&\qquad\qquad\qquad\qquad + \Lambda_\eta^{-2}\frac{\partial}{\partial\eta}(\Lambda_\eta^2 v_\Gamma) - \epsilon^2\Lambda_\eta^{-2}\operatorname{grad}_\Gamma v_n \\
&= -\lambda_\eta^{-1} \star_\Gamma \star_\Gamma \frac{\partial\lambda_\eta v_\Gamma}{\partial\eta} - \lambda_\eta^{-1}\frac{\partial\lambda_\eta}{\partial\eta} v_\Gamma + \Lambda_\eta^{-2}\frac{\partial}{\partial\eta}(\Lambda_\eta^2 v_\Gamma) \\
&= \lambda_\eta^{-1}\frac{\partial\lambda_\eta v_\Gamma}{\partial\eta} - \lambda_\eta^{-1}\frac{\partial\lambda_\eta}{\partial\eta} v_\Gamma + \Lambda_\eta^{-2}\frac{\partial}{\partial\eta}(\Lambda_\eta^2 v_\Gamma) \\
&= \lambda_\eta^{-1}\left(\lambda_\eta\frac{\partial v_\Gamma}{\partial\eta}\right) + \Lambda_\eta^{-2}\frac{\partial}{\partial\eta}(\Lambda_\eta^2 v_\Gamma) \\
&= \frac{\partial v_\Gamma}{\partial\eta} + \Lambda_\eta^{-2}\frac{\partial}{\partial\eta}(\Lambda_\eta^2 v_\Gamma)
\end{aligned}$$

Likewise, collecting all the $v_{n,\alpha}$ contributions, we have

$$\begin{aligned}
v_{n,\alpha} &= -\lambda_\eta^{-1}\operatorname{curl}_\Gamma(\lambda_\eta v_n \star_\Gamma \partial x_\alpha) - \lambda_\eta^{-1}\operatorname{div}_\Gamma(\lambda_\eta \partial x_\alpha) v_n + \epsilon^{-2}\frac{\partial}{\partial\eta}(\langle\Lambda_\eta v_\Gamma, \Lambda_\eta \partial x_\alpha\rangle_\Gamma) \\
&\qquad - \epsilon^{-2}\langle \partial x_\alpha, \frac{\partial}{\partial\eta}(\Lambda_\eta^2 v_\Gamma)\rangle_\Gamma + \langle\partial x_\alpha, \operatorname{grad}_\Gamma v_n\rangle_\Gamma - \epsilon^{-2}\langle v_\Gamma, \frac{\partial}{\partial\eta}(\Lambda_\eta^2\partial x_\alpha)\rangle_\Gamma \\
&= \epsilon^{-2}\frac{\partial}{\partial\eta}(\langle\Lambda_\eta v_\Gamma, \Lambda_\eta\partial x_\alpha\rangle_\Gamma) - \epsilon^{-2}\langle\partial x_\alpha, \frac{\partial}{\partial\eta}(\Lambda_\eta^2 v_\Gamma)\rangle_\Gamma - \epsilon^{-2}\langle v_\Gamma, \frac{\partial}{\partial\eta}(\Lambda_\eta^2\partial x_\alpha)\rangle_\Gamma + \mathrm{O}(\epsilon^0) \\
&= \epsilon^{-2}\frac{\partial}{\partial\eta}(\langle\Lambda_\eta^2 v_\Gamma, \partial x_\alpha\rangle_\Gamma) - \epsilon^{-2}\langle\partial x_\alpha, \frac{\partial}{\partial\eta}(\Lambda_\eta^2 v_\Gamma)\rangle_\Gamma - \epsilon^{-2}\langle v_\Gamma, \frac{\partial}{\partial\eta}(\Lambda_\eta^2\partial x_\alpha)\rangle_\Gamma + \mathrm{O}(\epsilon^0) \\
&= \epsilon^{-2}\langle\frac{\partial}{\partial\eta}(\Lambda_\eta^2 v_\Gamma), \partial x_\alpha\rangle_\Gamma - \epsilon^{-2}\langle\partial x_\alpha, \frac{\partial}{\partial\eta}(\Lambda_\eta^2 v_\Gamma)\rangle_\Gamma - \epsilon^{-2}\langle v_\Gamma, \frac{\partial}{\partial\eta}(\Lambda_\eta^2\partial x_\alpha)\rangle_\Gamma + \mathrm{O}(\epsilon^0) \\
&= -\epsilon^{-2}\langle v_\Gamma, \frac{\partial}{\partial\eta}(\Lambda_\eta^2\partial x_\alpha)\rangle_\Gamma + \mathrm{O}(\epsilon^0) \\
&= -\epsilon^{-2}\langle v_\Gamma, \frac{\partial\Lambda_\eta^2}{\partial\eta}\partial x_\alpha\rangle_\Gamma + \mathrm{O}(\epsilon^0) = -\epsilon^{-2}\langle v_\Gamma, (-2\epsilon S + 2\epsilon^2\eta S^2)\partial x_\alpha\rangle_\Gamma + \mathrm{O}(\epsilon^0) = \mathrm{O}(\epsilon^{-1})
\end{aligned}$$

Finally, collecting all the $v_{n,3}$ contributions, we have

$$\begin{aligned}
v_{n,3} &= \lambda_\eta^{-1}\operatorname{curl}_\Gamma(\lambda_\eta \star_\Gamma v_\Gamma) - \lambda_\eta^{-1}\frac{\partial\lambda_\eta}{\partial\eta} v_n + \lambda_\eta^{-1}\operatorname{div}_\Gamma(\lambda_\eta v_\Gamma) + \lambda_\eta^{-1}\frac{\partial}{\partial\eta}(\lambda_\eta v_n) + \frac{\partial v_n}{\partial\eta} \\
&= \lambda_\eta^{-1}\left(\operatorname{curl}_\Gamma(\star_\Gamma(\lambda_\eta v_\Gamma)) + \operatorname{div}_\Gamma(\lambda_\eta v_\Gamma)\right) \\
&\qquad\qquad\qquad - \lambda_\eta^{-1}\frac{\partial\lambda_\eta}{\partial\eta} v_n + \lambda_\eta^{-1}\left(\frac{\partial\lambda_\eta}{\partial\eta} v_n + \lambda_\eta\frac{\partial v_n}{\partial\eta}\right) + \frac{\partial v_n}{\partial\eta} \\
&= 2\frac{\partial v_n}{\partial\eta}
\end{aligned}$$

since $\mathrm{curl}_\Gamma(\star_\Gamma(\lambda_\eta v_\Gamma)) = -\,\mathrm{div}_\Gamma(\lambda_\eta v_\Gamma)$ due to Cor. 1.53.

□

The following result, that the energy dissipation is to leading order the result of friction due to shear stress, lies at the core of the approximation. Compare with eq. (28) in [RRS02].

Proposition 2.46 (Dissipation and shear stress). *If* $\mathbf{v} = d\mathbf{x}(v_\Gamma + v_n\partial\eta)$ *then*

$$\int_{\Omega_h} (\nabla\mathbf{v} + \nabla\mathbf{v}^T) : \nabla\mathbf{v}\, dV = \epsilon^{-2}\int_{E_h(\Gamma)} \left(|\frac{\partial v_\Gamma}{\partial\eta}|_K^2 + \mathrm{O}(\epsilon^2)\right) \mathrm{vol}_K \tag{2.60}$$

Proof. From the definition of the velocity gradient $\nabla\mathbf{v}$ (Def. 1.60) and the tensor contraction $A : B$ (Def. 1.69), we have

$$\begin{aligned}(\nabla\mathbf{v} + \nabla\mathbf{v}^T) : \nabla\mathbf{v} &= (D_{\mathbf{e}_i}\mathbf{v}\otimes\mathbf{e}^i + \mathbf{e}^i\otimes D_{\mathbf{e}_i}\mathbf{v}) : (D_{\mathbf{e}_j}\mathbf{v}\otimes\mathbf{e}^j)\\ &= \langle D_{\mathbf{e}_i}\mathbf{v}, D_{\mathbf{e}_j}\mathbf{v}\rangle\langle\mathbf{e}^i,\mathbf{e}^j\rangle + \langle D_{\mathbf{e}_i}\mathbf{v},\mathbf{e}^j\rangle\langle D_{\mathbf{e}_j}\mathbf{v},\mathbf{e}^i\rangle\end{aligned}$$

Given that $\mathbf{v} = d\mathbf{x}(v)$, $\mathbf{e}_i = d\mathbf{x}(\partial x_i)$ and $\mathbf{e}^i = d\mathbf{x}(\sigma^i)$, with $\sigma^i := (dx^i)^\sharp$, we have

$$\begin{aligned}\mathbf{x}^*((\nabla\mathbf{v} + \nabla\mathbf{v}^T) : \nabla\mathbf{v}) &= \langle D_{\partial x_i}v, D_{\partial x_j}v\rangle_K\langle\sigma^i,\sigma^j\rangle_K + \langle D_{\partial x_i}v,\sigma^j\rangle_K\langle D_{\partial x_j}v,\sigma^i\rangle_K\\ &= \langle D_{\partial x_\alpha}v, D_{\partial x_\beta}v\rangle_K\langle\sigma^\alpha,\sigma^\beta\rangle_K + |D_{\partial x_3}v|^2|\sigma^3|_K^2 + \langle D_{\partial x_\alpha}v,\sigma^\beta\rangle_K\langle D_{\partial x_\beta}v,\sigma^\alpha\rangle_K\\ &\quad + 2\langle D_{\partial x_\alpha}v,\sigma^3\rangle_K\langle D_{\partial x_3}v,\sigma^\alpha\rangle_K + |\langle D_{\partial x_3}v,\sigma^3\rangle_K|^2\\ &= \frac{1}{4}\Big\{ \left(\langle v_{\Gamma,\alpha}, v_{\Gamma,\beta}\rangle_K + \epsilon^2 v_{n,\alpha}v_{n,\beta}\right)\langle\sigma^\alpha,\sigma^\beta\rangle_K + \epsilon^{-2}\left(|v_{\Gamma,3}|_K^2 + \epsilon^2|v_{n,3}|^2\right)\\ &\qquad + \langle v_{\Gamma,\alpha},\sigma^\beta\rangle_K\langle v_{\Gamma,\beta},\sigma^\alpha\rangle_K + 2\,v_{n,\alpha}\,\langle v_{\Gamma,3},\sigma^\alpha\rangle_K + |v_{n,3}|^2\Big\}\end{aligned}$$

where we used that $|\sigma^3|_K^2 = |(d\eta)^\sharp|_K^2 = |\epsilon^{-2}\partial\eta|_K^2 = \epsilon^{-2}$.

From lemma 2.45, we know that all the v-terms are $\mathrm{O}(\epsilon^0)$, except for $v_{n,\alpha}$ which is $\mathrm{O}(\epsilon^{-1})$, and hence $\mathbf{x}^*((\nabla\mathbf{v} + \nabla\mathbf{v}^T) : \nabla\mathbf{v}) = \frac{\epsilon^{-2}}{4}|v_{\Gamma,3} + \epsilon^2 v_{n,\alpha}\sigma^\alpha|_K + \mathrm{O}(\epsilon^0)$. Then

$$\begin{aligned}v_{\Gamma,3} + \epsilon^2 v_{n,\alpha}\sigma^\alpha &= \frac{\partial v_\Gamma}{\partial\eta} + \Lambda_\eta^{-2}\frac{\partial}{\partial\eta}(\Lambda_\eta^2 v_\Gamma) - \langle v_\Gamma, \frac{\partial}{\partial\eta}(\Lambda_\eta^2\partial x_\alpha)\rangle_\Gamma\,\sigma^\alpha\\ &= \frac{\partial v_\Gamma}{\partial\eta} + \Lambda_\eta^{-2}\frac{\partial}{\partial\eta}(\Lambda_\eta^2 v_\Gamma) - \langle\frac{\partial\Lambda_\eta^2}{\partial\eta}v_\Gamma,\partial x_\alpha\rangle_\Gamma\,\sigma^\alpha\\ &= \frac{\partial v_\Gamma}{\partial\eta} + \Lambda_\eta^{-2}\frac{\partial}{\partial\eta}(\Lambda_\eta^2 v_\Gamma) - \langle\Lambda_\eta^{-2}\frac{\partial\Lambda_\eta^2}{\partial\eta}v_\Gamma,\partial x_\alpha\rangle_K\,\sigma^\alpha\\ &= \frac{\partial v_\Gamma}{\partial\eta} + \Lambda_\eta^{-2}\frac{\partial}{\partial\eta}(\Lambda_\eta^2 v_\Gamma) - \Lambda_\eta^{-2}\frac{\partial\Lambda_\eta^2}{\partial\eta}v_\Gamma\\ &= \frac{\partial v_\Gamma}{\partial\eta} + \Lambda_\eta^{-2}\left(\Lambda_\eta^2\frac{\partial v_\Gamma}{\partial\eta}\right) = 2\frac{\partial v_\Gamma}{\partial\eta}\end{aligned}$$

where we used the fact that for any tangential vector v, $\langle v, \partial x_\alpha \rangle_K \sigma^\alpha = \langle v_\beta \sigma^\beta, \partial x_\alpha \rangle_K \sigma^\alpha = v_\beta \delta_{\beta\alpha} \sigma^\alpha = v_\alpha \sigma^\alpha = v$. Finally,

$$\int_{\mathbf{x}(E_h(\Gamma))} (\nabla \mathbf{v} + \nabla \mathbf{v}^T) : \nabla \mathbf{v} \, dV = \int_{E_h(\Gamma)} \mathbf{x}^*((\nabla \mathbf{v} + \nabla \mathbf{v}^T) : \nabla \mathbf{v} \, dV)$$
$$= \int_{E_h(\Gamma)} \mathbf{x}^*((\nabla \mathbf{v} + \nabla \mathbf{v}^T) : \nabla \mathbf{v}) \, \mathrm{vol}_K = \epsilon^{-2} \int_{E_h(\Gamma)} \left(|\frac{\partial v_\Gamma}{\partial \eta}|_K^2 + \mathrm{O}(\epsilon^2) \right) \mathrm{vol}_K$$

□

Corollary 2.47 (Reduced dissipation). *If* $\mathbf{v} = d\mathbf{x}(v_\Gamma + v_n \partial\eta)$, *then*

$$a(\mathbf{v}, \mathbf{v}) = \epsilon^{-1} \left(a_\epsilon(v_\Gamma, v_\Gamma) + \mathrm{O}(\epsilon^2) \right) \tag{2.61}$$

where

$$a_\epsilon(v_\Gamma, v_\Gamma) := \int_\Gamma \left(\int_0^h \lambda_\eta |\Lambda_\eta \frac{\partial v_\Gamma}{\partial \eta}|_\Gamma^2 \, d\eta \right) \mathrm{vol}_\Gamma \tag{2.62}$$

Proof. Recall that $a(\mathbf{v}, \mathbf{v}) = \int_{\Omega_h} 2\mathcal{E}(\mathbf{v}) : \mathcal{E}(\mathbf{v}) \, dV$. We will show that (2.61) is equivalent to (2.60).

For the left hand side, we have $2\mathcal{E}(\mathbf{v}) = \nabla \mathbf{v} + \nabla \mathbf{v}^T$ by definition, and then $(\nabla \mathbf{v} + \nabla \mathbf{v}^T) : \nabla \mathbf{v} = 2\mathcal{E}(\mathbf{v}) : \mathcal{E}(\mathbf{v})$ by Lem 1.70.

For the right hand side, we look at the right hand side of (2.60):

$$\epsilon^{-2} \int_{E_h(\Gamma)} |\frac{\partial v_\Gamma}{\partial \eta}|_K^2 \, \mathrm{vol}_K = \epsilon^{-2} \int_{E_h(\Gamma)} |\Lambda_\eta \frac{\partial v_\Gamma}{\partial \eta}|_\Gamma^2 \, \mathrm{vol}_K$$
$$= \epsilon^{-2} \int_{E_h(\Gamma)} |\Lambda_\eta \frac{\partial v_\Gamma}{\partial \eta}|_\Gamma^2 \, (\epsilon \lambda_\eta \, \mathrm{vol}_\Gamma \wedge d\eta)$$
$$= \epsilon^{-1} \int_\Gamma \left(\int_0^h \lambda_\eta |\Lambda_\eta \frac{\partial v_\Gamma}{\partial \eta}|_\Gamma^2 \, d\eta \right) \mathrm{vol}_\Gamma$$

□

Proposition 2.48 (Scaling). *Let* $\Omega \equiv \Omega_h \in \mathcal{O}_\Gamma$ *and* $\mathbf{v} = d\mathbf{x}(v_\Gamma + v_n \partial\eta) \in Z(\Omega_h; D_{\bar{h}})$. *Then there exist* $p, q \in \mathbb{Z}$, *such that the rescaled functional* $\hat{R}(\mathbf{v}) := \epsilon^{-q} R(\epsilon^p \mathbf{v})$ *is*

$$\hat{R}(\mathbf{v}) = R_\epsilon(v_\Gamma) + \mathrm{O}(\epsilon^2) = \mathrm{O}(\epsilon^0) \tag{2.63}$$

where

$$R_\epsilon(v_\Gamma) := \frac{1}{2} a_\epsilon(v_\Gamma, v_\Gamma) + E'_\epsilon(h)(\dot{h}) \tag{2.64a}$$

$$\textit{with } \lambda_h \dot{h} + \mathrm{div}_\Gamma \left(\int_0^h \lambda_\eta v_\Gamma \, d\eta \right) = 0 \tag{2.64b}$$

Proof. Using Prop. 2.43 and Cor. 2.47, we have

$$\begin{aligned} R(\epsilon^p \mathbf{v}) &= \frac{1}{2} a(\epsilon^p \mathbf{v}, \epsilon^p \mathbf{v}) + E'(\Omega_h)(\epsilon^p \mathbf{v}) \\ &= \frac{1}{2} \epsilon^{2p} a(\mathbf{v}, \mathbf{v}) + \epsilon^p E'(\Omega_h)(\mathbf{v}) \\ &= \epsilon^{2p-1}\big(\frac{1}{2} a_\epsilon(v_\Gamma, v_\Gamma) + \mathrm{O}(\epsilon^2)\big) + \epsilon^{p+1}\big(E'(h)(\dot{h}) + \mathrm{O}(\epsilon^2)\big) \end{aligned}$$

where $\dot{h}$ is a solution of $\lambda_h \dot{h} + \operatorname{div}_\Gamma \left(\int_0^h \lambda_\eta v_\Gamma \, d\eta \right) = 0$. Choosing $p = 2$, gives us $\epsilon^{2p-1} = \epsilon^{p+1} = \epsilon^3$, and therefore (2.63) holds with $q = 3$.

□

Remark 2.49. *Note that the scaling of the previous result implies a 'long' time scale $t \sim e^{-2}$ for the problem. From a physics point of view, the time scale comes from the balance between the energy dissipation rate which has units of energy/time and scales like ϵ^{-1} (Cor. 2.47), and the free energy (the non-constant part) which scales like ϵ (Prop. 2.43).*

2.8. Optimal velocity profile

The key insight behind this section is that we can calculate an optimal velocity profile $v_\Gamma^*(\eta)$ such that, for a fixed total flux $f = \int_0^h \lambda_\eta v_\Gamma \, d\eta$, the shear stress $\int_0^h \lambda_\eta |\Lambda_\eta \frac{\partial v_\Gamma}{\partial \eta}|_\Gamma^2 \, d\eta$ is minimized. In the flat case, this leads to the well-known parabolic velocity profile of the lubrication approximation.

Proposition 2.50 (Optimal velocity profile)**.** *Let $p \in \Gamma$ be a fixed point, and $h \in \mathbb{R}$ with $0 < h < H$. There exists a tangential tensor function Π_η, $\eta \in [0, h]$, such that for any tangential vector $f \in T_p\Gamma$*

$$v_\Gamma^* := \Pi_\eta f = \operatorname*{argmin}_{v_\Gamma \in V_f} \frac{1}{2} \int_0^h \lambda_\eta |\Lambda_\eta \frac{\partial v_\Gamma}{\partial \eta}|_\Gamma^2 \, d\eta \tag{2.65}$$

$$V_f := \{ v_\Gamma \in C^2([0,h], T_p\Gamma) \,|\, v_\Gamma|_{\eta=0} = 0, \int_0^h \lambda_\eta v_\Gamma \, d\eta = f \} \tag{2.66}$$

Furthermore,

$$\frac{1}{2} \langle f, M_h^{-1} f \rangle_\Gamma = \frac{1}{2} \int_0^h \lambda_\eta |\Lambda_\eta \frac{\partial v_\Gamma^*}{\partial \eta}|_\Gamma^2 \, d\eta = \min_{v_\Gamma \in V_f} \frac{1}{2} \int_0^h \lambda_\eta |\Lambda_\eta \frac{\partial v_\Gamma}{\partial \eta}|_\Gamma^2 \, d\eta \tag{2.67}$$

where the tensor

$$M_h := \int_0^h \lambda_\eta \int_0^\eta \lambda_\xi^{-1} \left(\int_\xi^h \lambda_{\bar{\xi}} \, d\bar{\xi} \right) \Lambda_\xi^{-2} \, d\xi \, d\eta \tag{2.68}$$

is positive definite (see Lem. 2.51).

Proof. We rewrite the problem in the standard form of Section 2.2 (with $e = 0$):

$$\min_{v_\Gamma \in Z_g} \frac{1}{2} a(v_\Gamma, v_\Gamma)$$
$$Z_g = \{v_\Gamma \in X \mid b(v_\Gamma, q) = g(q),\ \forall q \in Q\}$$

where

$$\begin{aligned}
X \times Q &= \{v_\Gamma \in C^2([0,h], T_p\Gamma) \mid v_\Gamma|_{\eta=0} = 0\} \times T_\Gamma \\
a(v_\Gamma, u_\Gamma) &= \int_0^h \lambda_\eta \langle \Lambda_\eta \frac{\partial v_\Gamma}{\partial \eta}, \Lambda_\eta \frac{\partial u_\Gamma}{\partial \eta} \rangle_\Gamma \, d\eta \\
b(v_\Gamma, q) &= \int_0^h \lambda_\eta \langle v_\Gamma, q \rangle_\Gamma \, d\eta \\
g &= f^{\flat_\Gamma} \in T_p^*\Gamma \equiv Q'
\end{aligned}$$

Note that the weak form of the constraint $b(v_\Gamma, q) = g(q) \Rightarrow \langle \int_0^h \lambda_\eta v_\Gamma \, d\eta - f, q \rangle_\Gamma = 0$ for all $q \in T_p\Gamma$, which is equivalent to $\int_0^h \lambda_\eta v_\Gamma \, d\eta = f$. We conclude that $Z_g \equiv V_f$.

We will show that the pair (v_Γ^*, q^*) given by

$$\begin{aligned}
P_\eta &:= -\int_0^\eta \lambda_\xi^{-1} \left(\int_\xi^h \lambda_{\bar\xi} \, d\bar\xi \right) \Lambda_\xi^{-2} \, d\xi \ \Rightarrow\ M_h = -\int_0^h \lambda_\eta P_\eta \, d\eta \\
q^* &:= -M_h^{-1} f \in Q \\
v_\Gamma^* &:= P_\eta q^* \in X
\end{aligned}$$

is a solution to the saddle point problem

$$\begin{aligned}
a(v_\Gamma^*, v_\Gamma) + b(v_\Gamma, q^*) &= 0\,, \quad \forall v_\Gamma \in X \\
b(v_\Gamma^*, q) &= g(q)\,, \quad \forall q \in Q
\end{aligned}$$

First we need to establish that (v_Γ^*, q^*) is well-defined. Recall that the tensor Λ_η is assumed positive definite (see Rem. 1.5) and so $\lambda_\eta = \det(\Lambda_\eta) > 0$ and the inverse tensor Λ_η^{-1} exists (and is also positive definite). It follows that the tensor P_η is well-defined and a smooth function of η. Combined with the fact that $P_0 = 0$ (see next paragraph), we conclude that indeed $v_\Gamma^* \in X$. For the invertibility of M_h, see Lem. 2.51.

The tensor P_η is defined so that the following properties hold:

$$\begin{aligned}
P_0 &= -\int_0^0 \ldots \, d\xi = 0 \\
\frac{\partial P_\eta}{\partial \eta} &= -\lambda_\eta^{-1} \left(\int_\eta^h \lambda_{\bar\xi} \, d\bar\xi \right) \Lambda_\eta^{-2} \Rightarrow \frac{\partial P_\eta}{\partial \eta}\Big|_{\eta=h} = 0 \\
\frac{\partial}{\partial \eta}\big(\lambda_\eta \Lambda_\eta^2 \frac{\partial P_\eta}{\partial \eta}\big) &= \lambda_\eta
\end{aligned}$$

For the first equation of the saddle point problem then, we have for an arbitrary $v_\Gamma \in X$:

$$\begin{aligned}
a(v_\Gamma^*, v_\Gamma) &= \int_0^h \lambda_\eta \langle \Lambda_\eta \frac{\partial v_\Gamma^*}{\partial \eta}, \Lambda_\eta \frac{\partial v_\Gamma}{\partial \eta} \rangle_\Gamma \, d\eta \\
&= \int_0^h \langle \lambda_\eta \Lambda_\eta^2 \frac{\partial v_\Gamma^*}{\partial \eta}, \frac{\partial v_\Gamma}{\partial \eta} \rangle_\Gamma \, d\eta \\
&= \Big[\langle \lambda_\eta \Lambda_\eta^2 \frac{\partial v_\Gamma^*}{\partial \eta}, v_\Gamma \rangle_\Gamma \Big]_0^h - \int_0^h \langle \frac{\partial}{\partial \eta} \big(\lambda_\eta \Lambda_\eta^2 \frac{\partial v_\Gamma^*}{\partial \eta} \big), v_\Gamma \rangle_\Gamma \, d\eta \\
&= \Big[\langle \lambda_\eta \Lambda_\eta^2 \frac{\partial P_\eta}{\partial \eta} q^*, v_\Gamma \rangle_\Gamma \Big]_0^h - \int_0^h \langle \frac{\partial}{\partial \eta} \big(\lambda_\eta \Lambda_\eta^2 \frac{\partial P_\eta}{\partial \eta} \big) q^*, v_\Gamma \rangle_\Gamma \, d\eta \\
&= -\int_0^h \langle \lambda_\eta q^*, v_\Gamma \rangle_\Gamma \, d\eta \\
&= -\int_0^h \lambda_\eta \langle q^*, v_\Gamma \rangle \, d\eta \\
&= -b(v_\Gamma, q^*)
\end{aligned}$$

The square bracket is zero at $\eta = 0$ because $v_\Gamma = 0$ there, and also at $\eta = h$ because $\frac{\partial P_\eta}{\partial \eta} = 0$ there.

The second equation of the saddle point problem is straightforward to verify:

$$\begin{aligned}
b(v_\Gamma^*, q) = \int_0^h \lambda_\eta \langle v_\Gamma^*, q \rangle_\Gamma \, d\eta &= \int_0^h \lambda_\eta \langle P_\eta q^*, q \rangle_\Gamma \, d\eta \\
&= \langle \big(\int_0^h \lambda_\eta P_\eta \, d\eta \big) q^*, q \rangle_\Gamma = \langle -M_h q^*, q \rangle_\Gamma = \langle f, q \rangle_\Gamma = g(q)
\end{aligned}$$

Since the bilinear form $a(\cdot, \cdot)$ is symmetric and positive, the solution (v_Γ^*, q^*) of the saddle point problem gives also a, not necessarily unique, minimizer (see the proof[3] of Prop. 2.13), in the sense that $a(v_\Gamma^*, v_\Gamma^*) \leq a(v_\Gamma, v_\Gamma)$, $\forall v_\Gamma \in Z_g = V_f$. Finally using the saddle point equations, we have

$$\int_0^h \lambda_\eta |\tfrac{\partial}{\partial \eta}(\Lambda_\eta v_\Gamma)|_\Gamma^2 \, d\eta = a(v_\Gamma, v_\Gamma) \geq a(v_\Gamma^*, v_\Gamma^*) = -b(v_\Gamma^*, q^*) = \langle M_\eta q^*, q^* \rangle_\Gamma = \langle f, M_h^{-1} f \rangle_\Gamma$$

for any $v_\Gamma \in V_f$. We conclude that the statement holds with $\Pi_\eta := -P_\eta M_h^{-1}$.

□

Lemma 2.51. *The tensor M_h is positive definite and self-adjoint.*

[3] Although the space X is *not* a Hilbert space, the part of the proof of Prop. 2.13 which shows that a solution of the saddle point problem is a minimizer, is purely algebraic and does not depend on whether X is complete or not.

Proof. Let $v \in T_p\Gamma$ be an eigenvector of the shape operator S with corresponding eigenvalue κ. We will show that it is also an eigenvector of Λ_η and of M_h. Indeed, $\Lambda_\eta = (\mathrm{id} - \epsilon\eta S)v = (1 - \epsilon\eta\kappa)v$ and so v is an eigenvector of Λ_η with corresponding eigenvalue $\mu_\eta := 1 - \epsilon\eta\kappa > 0$ (since Λ_η is pos. definite). For M_h, and keeping in mind that v is constant, we have

$$\begin{aligned}
M_h v &= \left(\int_0^h \lambda_\eta \int_0^\eta \lambda_\xi^{-1} \left(\int_\xi^h \lambda_{\bar\xi}\, d\bar\xi\right) \Lambda_\xi^{-2}\, d\xi\, d\eta\right) v \\
&= \int_0^h \lambda_\eta \int_0^\eta \lambda_\xi^{-1} \left(\int_\xi^h \lambda_{\bar\xi}\, d\bar\xi\right) \Lambda_\xi^{-2} v\, d\xi\, d\eta \\
&= \int_0^h \lambda_\eta \int_0^\eta \lambda_\xi^{-1} \left(\int_\xi^h \lambda_{\bar\xi}\, d\bar\xi\right) \mu_\xi^{-2} v\, d\xi\, d\eta \\
&= \left(\int_0^h \lambda_\eta \int_0^\eta \lambda_\xi^{-1} \left(\int_\xi^h \lambda_{\bar\xi}\, d\bar\xi\right) \mu_\xi^{-2}\, d\xi\, d\eta\right) v
\end{aligned}$$

It follows that v is an eigenvector of M_h and the coefficient is the corresponding eigenvalue. Since all the integrands are strictly positive, the triple integral is strictly positive and so the corresponding eigenvalue is strictly positive.

Because the tensor S is self-adjoint, its eigenvectors span the entire tangent space $T_p\Gamma$ and so M_h can not have any extra eigenvectors. We conclude that all the eigenvalues of M_h are strictly positive, and hence it is positive definite.

To show that M_h is self-adjoint, we consider the following product for arbitrary $u, v \in T\Gamma$:

$$\begin{aligned}
\langle u, M_h v\rangle_\Gamma = \int_0^h \lambda_\eta \int_0^\eta \lambda_\xi^{-1} \left(\int_\xi^h \lambda_{\bar\xi}\, d\bar\xi\right) \langle u, \Lambda_\xi^{-2} v\rangle_\Gamma\, d\xi\, d\eta = \\
\int_0^h \lambda_\eta \int_0^\eta \lambda_\xi^{-1} \left(\int_\xi^h \lambda_{\bar\xi}\, d\bar\xi\right) \langle \Lambda_\xi^{-2} u, v\rangle_\Gamma\, d\xi\, d\eta = \langle M_h u, v\rangle_\Gamma
\end{aligned}$$

□

Corollary 2.52 (Flux-based dissipation). *Let $\Omega_h \in \mathcal{O}_\Gamma$ with $0 < h < H$. Then for any $\mathbf{v} = d\mathbf{x}(v_\Gamma + v_n \partial\eta) \in Z(\Omega_h; D_{\bar h})$*

$$f = \int_0^h \lambda_\eta v_\Gamma \Rightarrow R_h(f) \le R_\epsilon(v_\Gamma) \tag{2.69}$$

where

$$R_h(f) := \frac{1}{2} a_h(f, f) + E'_\epsilon(h)(\dot h) \tag{2.70}$$

$$\lambda_h \dot h + \mathrm{div}_\Gamma f = 0 \tag{2.71}$$

$$a_h(f, f) := \int_\Gamma \langle f, M_h^{-1} f\rangle_\Gamma \,\mathrm{vol}_\Gamma \tag{2.72}$$

Proof. Integrating both sides of (2.67) over Γ gives us $a_h(f,f) \le a_\epsilon(v_\Gamma, v_\Gamma)$ and $R_h(f) \le R_\epsilon(v_\Gamma)$ follows directly.

□

2.9. The reduced model

This final section pulls everything together, to arrive at the promised reduced model for the thin viscous film flow on a curved substrate.

Proposition 2.53 (Approximate Stokes flow). *Let $\Omega_h \in \mathcal{O}_\Gamma$ with $0 < h < H$, and let the optimal velocity $\mathbf{v}^\dagger$ and flux $f^\star$ be the solutions of the following optimization problems:*

$$\mathbf{v}^\dagger := \operatorname*{argmin}_{\mathbf{v} \in Z(\Omega_h; D_{\bar{h}})} \hat{R}(\mathbf{v}) \tag{2.73}$$

$$f^* := \operatorname*{argmin}_{f \in T\Gamma} R_h(f) \tag{2.74}$$

Using the optimal flux, we can construct a nearly optimal velocity

$$\mathbf{v}^* = d\mathbf{x}(v_\Gamma^* + v_n^* \partial\eta) \tag{2.75a}$$

$$v_\Gamma^* = \Pi_\eta f^* \tag{2.75b}$$

$$v_n^* = -\lambda_\eta \int_0^\eta \operatorname{div}_\Gamma(\lambda_\xi v_\Gamma^*)\, d\xi \tag{2.75c}$$

for the Stokes flow, in the sense that

$$\hat{R}(\mathbf{v}^*) = \hat{R}(\mathbf{v}^\dagger) + \mathrm{O}(\epsilon^2) \tag{2.76}$$

Proof. From the optimal velocity $\mathbf{v}^\dagger = d\mathbf{x}(v_\Gamma^\dagger + v_n^\dagger \partial\eta)$, we extract the corresponding flux $f^\dagger = \int_0^h \lambda_\eta v_\Gamma^\dagger \, d\eta$. Then,

$$\begin{aligned}
\hat{R}(\mathbf{v}^\dagger) &\le \hat{R}(\mathbf{v}^*) && \text{(by (2.73))}\\
&\le R_\epsilon(v_\Gamma^*) + C\epsilon^2 && \text{(by (2.63) \& (2.48))}\\
&= R_h(f^*) + C\epsilon^2 && \text{(by Prop. 2.50 \& Cor. 2.52)}\\
&\le R_h(f^\dagger) + C\epsilon^2 && \text{(by (2.74))}\\
&\le R_\epsilon(v_\Gamma^\dagger) + C\epsilon^2 && \text{(by Cor. 2.52)}\\
&\le \hat{R}(\mathbf{v}^\dagger) + C'\epsilon^2 && \text{(by (2.63) \& (2.48))}
\end{aligned}$$

for ϵ small enough, and therefore $\hat{R}(\mathbf{v}^\dagger) \le \hat{R}(\mathbf{v}^*) \le \hat{R}(\mathbf{v}^\dagger) + C'\epsilon^2 \Rightarrow |\hat{R}(\mathbf{v}^\dagger) - \hat{R}(\mathbf{v}^*)| \le C'\epsilon^2 \Rightarrow \hat{R}(\mathbf{v}^*) = \hat{R}(\mathbf{v}^\dagger) + \mathrm{O}(\epsilon^2)$.

□

Proposition 2.54 (Conservative form)**.** *Consider the functional*

$$\mathcal{R}_u(f) := \frac{1}{2}\alpha_u(f,f) + \mathcal{E}'(u)(\dot{u}) \tag{2.77a}$$

$$\dot{u} + \operatorname{div}_\Gamma f = 0 \tag{2.77b}$$

$$\alpha_u(f,f) := \int_\Gamma \langle f, \mathcal{M}_u^{-1} f\rangle_\Gamma \operatorname{vol}_\Gamma \tag{2.77c}$$

$$\mathcal{E}(u) := \int_\Gamma \left\{ W_1 u + \frac{\epsilon}{2} W_2 u^2 + \frac{\epsilon}{2} |\operatorname{grad}_\Gamma u|_\Gamma^2 \right\} \operatorname{vol}_\Gamma \tag{2.77d}$$

where $\mathcal{M}_u := \frac{u^3}{3} + \epsilon \frac{u^4}{6}(H \operatorname{id} + S)$*,* $W_1 := \zeta z_\Gamma - H$ *and* $W_2 := \zeta \mathbf{N}_z - H^2 + 2G$*.*

If we let $u(h) := \int_0^h \lambda_\eta \, d\eta = h - \frac{\epsilon}{2} h^2 H + \frac{\epsilon^2}{3} h^3 G$*, then*

$$R_h(f) = \mathcal{R}_{u(h)}(f) + \mathrm{O}(\epsilon^2) \tag{2.78}$$

for any $f \in T\Gamma$*.*

Proof. First we look at the mobility tensor, taking into account that $\Lambda_\eta = 1 - \epsilon\eta S$ and $\lambda_\eta = 1 - \epsilon\eta H + \mathrm{O}(\epsilon^2)$,

$$\begin{aligned} M_h = \int_0^h \lambda_\eta \int_0^\eta \lambda_\xi^{-1} \left(\int_\xi^h \lambda_{\bar{\xi}} \, d\bar{\xi} \right) \Lambda_\xi^{-2} \, d\xi \, d\eta \\ = \frac{h^3}{3} + \epsilon \frac{h^4}{6}(S - 2H \operatorname{id}) + \mathrm{O}(\epsilon^2) = \mathcal{M}_{u(h)} + \mathrm{O}(\epsilon^2) \end{aligned}$$

and so $a_h(f,f) = \alpha_{u(h)}(f,f) + \mathrm{O}(\epsilon^2)$.

For the energy, a direct calculation gives us $\mathcal{E}(u(h)) = E_\epsilon(h) + \mathrm{O}(\epsilon^2)$. From $\dot{u} + \operatorname{div}_\Gamma f = 0$ and $\lambda_h \dot{h} + \operatorname{div}_\Gamma f = 0$, we deduce that $\dot{u} = \lambda_h \dot{h}$, and so

$$\begin{aligned} \mathcal{E}'(u(h))(\dot{u}) &= \int_\Gamma \{ (W_1 + \epsilon W_2 u)\dot{u} + \epsilon \langle \operatorname{grad}_\Gamma u(h), \operatorname{grad}_\Gamma \dot{u} \rangle_\Gamma \} \operatorname{vol}_\Gamma \\ &= \int_\Gamma \left\{ \big((\zeta z_\Gamma - H) + \epsilon(\zeta \mathbf{N}_z - \zeta H z_\Gamma + 2G)h \big) \dot{h} + \epsilon \langle \operatorname{grad}_\Gamma h, \operatorname{grad}_\Gamma \dot{h} \rangle_\Gamma \right\} \operatorname{vol}_\Gamma + \mathrm{O}(\epsilon^2) \\ &= E_h'(h)(\dot{h}) + \mathrm{O}(\epsilon^2) \end{aligned}$$

The approximation (2.78) follows immediately.

□

Corollary 2.55 (Conservative approximation)**.** *If*

$$f^* := \operatorname*{argmin}_{f \in T\Gamma} R_h(f) \tag{2.79}$$

$$f_u^* := \operatorname*{argmin}_{f \in T\Gamma} \mathcal{R}_{u(h)}(f) \tag{2.80}$$

then

$$R_h(f_u^*) = R_h(f^*) + \mathrm{O}(\epsilon^2) \tag{2.81}$$

Proof. Using the same reasoning as in the proof of 2.53, we have

$$R_h(f^*) \leq R_h(f_u^*) \leq R_{u(h)}(f_u^*) + C\epsilon^2 \leq R_{u(h)}(f^*) + C\epsilon^2 \leq R_h(f^*) + C'\epsilon^2$$

which proves that indeed $R_h(f_u^*) = R_h(f^*) + \mathrm{O}(\epsilon^2)$.

□

Remark 2.56. *The reduction of the Stokes flow equations to the conservative variational model of Prop. 2.54 is essentially the* lubrication approximation*, well-known in the case of a flat inclined substrate. The reduced model is identical to the one derived in [VR13], in which paper the Euler-Lagrange equation of the variational model* (2.77)*:*

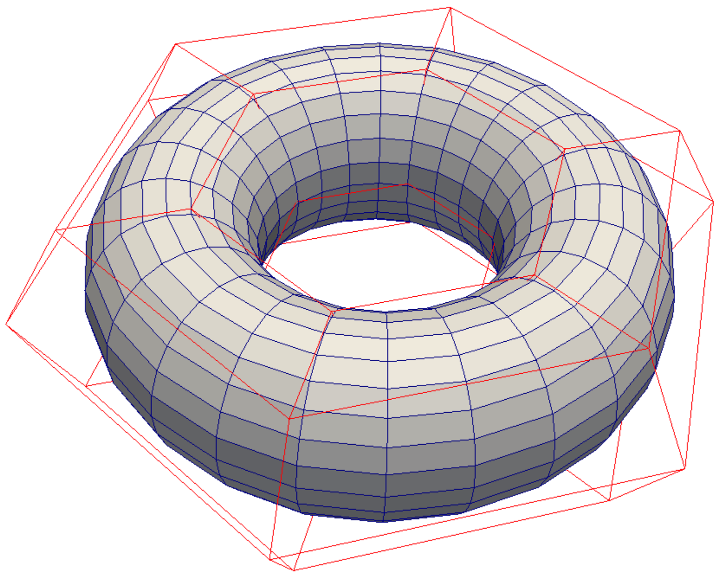
$$\dot{u} - \mathrm{div}_\Gamma\left(\mathcal{M}_u \,\mathrm{grad}_\Gamma P(u)\right) = 0 \tag{2.82a}$$

$$\mathcal{M}_u = \frac{u^3}{3} + \epsilon\frac{u^4}{6}(H\,\mathrm{id} + S) \tag{2.82b}$$

$$P(u) = (\zeta z_\Gamma - H) + \epsilon(\zeta \mathbf{N}_z - H^2 + 2G)u - \epsilon\Delta_\Gamma u \tag{2.82c}$$

is also shown to be equivalent to the 4th order PDE derived in [RRS02].

3. Evolution and Variational Discretization of the Model



3.1. Outline

In this chapter we use tools from functional analysis and the calculus of variations to study the properties of the reduced model (as derived in the previous chapter and presented in Prop. 2.54), as well as to discretize it in time and space. The development and analysis of numerical schemes for lubrication-type fourth-order parabolic equations of the form $u_t + \operatorname{div}(M(u) \operatorname{grad} \Delta u) = 0$, in various dimensions and for various mobilities $M(u)$, is the subject of numerous publications; using finite differences in [ZB00], finite elements in [BBG98] and finite volumes (as well as finite elements) in [GR00]. The analysis of these schemes relies heavily on so-called *entropy estimates* of the form $\frac{d}{dt} \int \mathcal{S}(u) = -\int |\Delta u|^2$, for an appropriate entropy function $S(u)$ (which depends on the mobility). In combination with energy estimates, they can be used to show non-negativity of the (continuous and discrete) solutions for suitable initial data, as well as convergence of the discrete to the continuous solutions in the appropriate norms.

Unfortunately, in the case of a curved substrate the corresponding PDE (2.82) does not admit such an estimate. We opt instead to apply the results of constrained optimization (Sec. 2.2) directly on the variational model (2.77). There are two issues with the model of Prop. 2.54, which make its analysis with the methods of Sec. 2.2 problematic. The first one is that the bilinear form $\langle f, \mathcal{M}_u^{-1}, f \rangle_{L^2}$ is not bounded and coercive for extreme values of u, as the mobility $\mathcal{M}_u$ vanishes or becomes unbounded respectively. After certain preliminary definitions and results in Sec. 3.2, we deal with this problem in Sec. 3.3, where we present a regularized mobility and study its essential properties. The second problem is that the PDE constraint $\langle \dot{u}, q \rangle_{L^2} - \langle f, \operatorname{grad}_\Gamma q \rangle_{L^2} = 0$ does not satisfy the inf-sup-condition (2.8) for functions in $H^1(\Gamma)$. We rectify this by adding a diffusive term and consider the regularized PDE $\langle \dot{u}, q \rangle_{L^2} + \lambda \langle \operatorname{grad}_\Gamma \dot{u}, \operatorname{grad}_\Gamma q \rangle_{L^2} - \langle f, \operatorname{grad}_\Gamma q \rangle_{L^2} = 0$ instead. In Sec. 3.4, we show the well-posedness of the regularized variational model in Cor. 3.16. We also show local existence of solutions in time in Prop. 3.17, global existence in Cor. 3.18 and finally uniqueness of the solution in Cor. 3.19. It is interesting to note that although the regularization was meant to allow us to show well-posedness in $H^1(\Gamma)$, we can actually show higher-regularity in $H^2(\Gamma)$ under appropriate conditions (Prop. 3.15).

In Sec. 3.5 we discretize the problem in time. We establish local well-posedness (Prop. 3.23) and higher regularity (Prop. 3.25) results, under appropriate bounds for τ. Then we use these to establish the existence and uniqueness of the solution of the time-discrete scheme (Lem. 3.27), higher regularity for initial data in $H^2(\Gamma)$ (Lem. 3.29) and finally its convergence to the solution of the continuous problem in Prop. 3.30. Finally, in Sec. 3.6, we present a proof of the convergence of a Galerkin-type approximation under general assumptions for the finite element spaces used to discretize the problem in space.

3.2. Preliminaries

Definition 3.1 (Essential infimum/supremum)**.** *Let $u \in \Omega^0(\Gamma)$. We define the* essential infimum and supremum *of f as*

$$\operatorname{ess\,sup} u := \inf\{C \in \mathbb{R} \mid \int_\Gamma \chi_{u>C} \operatorname{vol}_\Gamma = 0\} \tag{3.1}$$

$$\operatorname{ess\,inf} u := \sup\{C \in \mathbb{R} \mid \int_\Gamma \chi_{u<C} \operatorname{vol}_\Gamma = 0\} \tag{3.2}$$

where $\chi_{u>C}$ is the characteristic function *of the set $\{p \in \Gamma \mid u(p) > C\}$ (and likewise for $\chi_{u<C}$).*

Definition 3.2 (Function spaces on Γ)**.** *Let Γ be a (C^∞) smooth, compact, connected and orientable 2-manifold without boundary, such that $\int_\Gamma \operatorname{vol}_\Gamma < \infty$. We define the following L^p spaces on Γ:*

- *$u \in L^p(\Gamma)$, for $1 \leq p < \infty$, iff $u \in \Omega^0(\Gamma)$ and $\int_\Gamma |u|^p \operatorname{vol}_\Gamma < +\infty$. We define the norms*

$$\|u\|_{L^p(\Gamma)} := \left(\int_\Gamma |u|^p \operatorname{vol}_\Gamma\right)^{1/p} \tag{3.3}$$

  and, specifically for $p = 2$, the inner product

$$\langle u, v\rangle_{L^2(\Gamma)} := \int_\Gamma uv \operatorname{vol}_\Gamma \tag{3.4}$$

- *$u \in L^\infty(\Gamma)$, iff $u \in \Omega^0(\Gamma)$ and $\|u\|_{L^\infty(\Gamma)} < +\infty$ where*

$$\|u\|_{L^\infty(\Gamma)} := \operatorname{ess\,sup}|u| \tag{3.5}$$

- *$v \in L^2(T\Gamma)$, iff $v \in T\Gamma$ and $\int_\Gamma |v|_\Gamma^2 \operatorname{vol}_\Gamma < +\infty$. We define the norm*

$$\|v\|_{L^2(T\Gamma)} := \left(\int_\Gamma |v|_\Gamma^2 \operatorname{vol}_\Gamma\right)^{1/2} \tag{3.6}$$

  and the inner product

$$\langle v, w\rangle_{L^2(T\Gamma)} := \int_\Gamma \langle v, w\rangle_\Gamma \operatorname{vol}_\Gamma \tag{3.7}$$

Furthermore, we define the following Sobolev spaces on Γ:

- *$u \in H^1(\Gamma)$, iff $u \in L^2(\Gamma)$ and $\operatorname{grad}_\Gamma u \in L^2(T\Gamma)$, with norm*

$$\|u\|_{H^1(\Gamma)} := \left(\|u\|_{L^2(\Gamma)}^2 + \|\operatorname{grad}_\Gamma u\|_{L^2(T\Gamma)}^2\right)^{1/2} \tag{3.8}$$

- $v \in H_{\mathrm{div}}(T\Gamma)$, *iff* $v \in L^2(T\Gamma)$ *and* $\mathrm{div}_\Gamma v \in L^2(\Gamma)$, *with norm*

$$\|v\|_{H_{\mathrm{div}}(T\Gamma)} := \left(\|v\|^2_{L^2(T\Gamma)} + \|\mathrm{div}_\Gamma v\|^2_{L^2(\Gamma)}\right)^{1/2} \tag{3.9}$$

- $u \in H^2(\Gamma)$, *iff* $u \in H^1(\Gamma)$ *and* $\mathrm{grad}_\Gamma u \in H_{\mathrm{div}}(T\Gamma)$, *with norm*

$$\|u\|_{H^2(\Gamma)} := \left(\|u\|^2_{L^2(\Gamma)} + \|\mathrm{grad}_\Gamma u\|^2_{L^2(T\Gamma)} + \|\Delta_\Gamma u\|^2_{L^2(\Gamma)}\right)^{1/2} \tag{3.10}$$

where $\Delta_\Gamma u := \mathrm{div}_\Gamma \, \mathrm{grad}_\Gamma f$ *is the* Laplace-Beltrami operator *on* Γ.

We denote the associated inner products with $\langle \cdot, \cdot \rangle_{L^2(\Gamma)}$*, etc. Furthermore, we will omit the domain specification* $\ldots(\Gamma)$ *and* $\ldots(T\Gamma)$ *whenever a space appears in a subscript, for instance* $\|\cdot\|_{H^1}$ *instead of* $\|\cdot\|_{H^1(\Gamma)}$.

Lemma 3.3 (L^∞ functions)**.** *Let* $u \in L^\infty(\Gamma)$*. Then*

1. $\|u^p\|_{L^\infty} = \|u\|^p_{L^\infty}$ *for any* $p > 0$*, and so* $u^p \in L^\infty(\Gamma)$
2. *for any* $f \in L^2(\Gamma)$,

$$-\|u\|_{L^\infty}\|f\|^2_{L^2} \le \underline{u}\,\|f\|^2_{L^2} \le \langle f, uf \rangle_{L^2} \le \overline{u}\,\|f\|^2_{L^2} \le \|u\|_{L^\infty}\|f\|^2_{L^2} \tag{3.11}$$

   where $\underline{u} := \operatorname{ess\,inf} u$ *and* $\overline{u} := \operatorname{ess\,sup} u$

3. *for any* $f \in L^2(\Gamma)$, $\|uf\|_{L^2} \le \|u\|_{L^\infty}\|f\|_{L^2}$ *and so* $uf \in L^2(\Gamma)$

2. and 3. are also true for functions in $L^2(T\Gamma)$.

Proof. For the first statement, we note that $\operatorname{ess\,sup}|u| \ge 0$ and, since the function x^p is monotone for $x \ge 0$ and $p > 0$, $|u| > C \Leftrightarrow |u|^p > C^p$ and so $\chi_{|u|>C} = \chi_{|u|^p>C^p}$. It follows that $\operatorname{ess\,sup}(|u|^p) = (\operatorname{ess\,sup}|u|)^p$.

For the second and third statement, the inequalities follow directly from the definition of the L^∞ norm and the basic inequality $\operatorname{ess\,inf} u \cdot \int_\Gamma f \,\mathrm{vol}_\Gamma \le \int_\Gamma uf \,\mathrm{vol}_\Gamma \le \operatorname{ess\,sup} u \cdot \int_\Gamma f \,\mathrm{vol}_\Gamma$, which holds for any non-negative measurable function f.

□

Remark 3.4. *We will make use of the* Sobolev embedding theorem (second part) *for compact Riemannian manifolds (Thm. 2.20 in [Aub98]). More specifically, the fact that the Sobolev space* $H^2(\Gamma)$ *on the 2-dimensional compact Riemannian manifold* Γ *can be continuously embedded in the Hölder space* $C^{0,\alpha}(\Gamma)$ *for any* $\alpha \in (0,1)$*, i.e. there exists a constant* $C_\alpha > 0$ *such that*

$$\|u\|_{C^{0,\alpha}} \le C_\alpha \|u\|_{H^2}, \quad \forall u \in H^2(\Gamma) \tag{3.12}$$

where $\|u\|_{C^{0,\alpha}} := \sup_{p\in\Gamma}|u(p)| + \sup_{p\neq q}\frac{|u(p)-u(q)|}{d(p,q)^\alpha}$ *(*$d(p,q)$ *is the infimum of the lengths of all the curves from* p *to* q*). This inequality gives us immediately that*

$$\|u\|_{L^\infty} \le \sup_{p\in\Gamma}|u(p)| \le C_\infty\|u\|_{H^2} \tag{3.13}$$

where $C_\infty := \inf_{0<\alpha<1} C_\alpha < +\infty$*. We will in fact use a stronger statement, a direct corollary of the Sobolev embedding theorem known as* Agmon's inequality *(Lem. 13.3 in [Agm65]): since* $\dim\Gamma = 2$*, there exists a constant* $C_K > 0$ *such that*

$$\|u\|^2_{L^\infty} \le C_K^2\|u\|_{H^2}\|u\|_{L^2} \tag{3.14}$$

for any $u \in H^2(\Gamma)$*.*

Definition 3.5 (Bochner spaces)**.** *Let* X *be one of the spaces* $L^2(\Gamma), L^2(T\Gamma)$ *or* $H^1(\Gamma)$*, and* $T > 0$*. We define the following spaces of functions from* $[0,T]$ *to* X*:*

- $u \in L^p(0,T;X)$*, iff* $\|u\|_{L^p(0,T;X)} := \left(\int_0^T \|u(t)\|_X^p\,dt\right)^{1/p} < \infty$*.*
- $u \in L^\infty(0,T;X)$*, iff* $\|u\|_{L^\infty(0,T;X)} := \operatorname{ess\,sup}_{0\le t\le T}\|u(t)\|_X < \infty$*.*
- $u \in C([0,T];X)$*, iff* $u : [0,T] \to X$ *is a continuous function, with norm* $\|u\|_{C([0,T];X)} := \max_{0\le t\le T}\|u(t)\|_X < \infty$*.*
- $u \in C^1([0,T];X)$*, iff* $u \in C([0,T];X)$ *and* $u_t \in C([0,T];X)$*.*
- $u \in H^1(0,T;X)$*, iff* $u \in L^2(0,T;X)$ *and there exists* $u' \in L^2(0,T;X)$ *such that* $\int_0^T (u(t)\psi'(t) + \dot u(t)\psi(t))\,dt = 0$*, for all* $\psi \in C_c^\infty(0,T)$*. The space is equipped with the norm* $\|u\|_{H^1(0,T;X)} := \left(\|u(t)\|^2_{L^2(0,T;X)} + \|u'(t)\|^2_{L^2(0,T;X)}\right)^{1/2}$*.*

See section 4.9 in [Eva02] for more details.

Lemma 3.6 (A Gronwall-type inequality)**.** *Let* $u_0 \in X$*, where* X *is a Banach space, and* $\dot u \in L^2(0,T;X)$*. Then the function*

$$u(t) := u_0 + \int_0^t \dot u(s)\,ds, \quad t \in [0,T] \tag{3.15}$$

is in $C([0,T];X)$ *with*

$$\|u\|_{C([0,T];X)} \le \|u_0\|_X + T^{1/2}\|\dot u\|_{L^2(0,T;X)} \tag{3.16}$$

and

$$\|u\|_{L^2(0,T;X)} \le T^{1/2}\|u\|_{C([0,T];X)} \le T^{1/2}\|u_0\|_X + T\|\dot u\|_{L^2(0,T;X)} \tag{3.17}$$

Furthermore, $u \in H^1(0,T;X)$ *with* $u' = \dot u$ *and* $u(0) = u_0$*. Finally, if there exist constants* $\alpha \ge 0$ *and* $\beta > 0$ *such that* $\|\dot u(t)\|_X \le \alpha + \beta\|u(t)\|_X$ *for a.e.* $t \in (0,T)$*, then*

$$\|u\|_{C([0,T];X)} \le (\|u_0\|_X + \alpha T)\,e^{\beta T} \tag{3.18}$$

3.3. Regularization of the mobility

As noted in the introductory section, lubrication-type equations of the type that we wish to study here present technical difficulties when the mobility $M(u)$ vanishes as $u \to 0$. Furthermore, the associated numerical schemes require that the discrete solutions remain non-negative. This is achieved by regularizing the mobility, for instance by taking $M_{\epsilon,s}(u) := \frac{u^s M(u)}{u^s + \epsilon M(u)}$ with $\epsilon \ll 1$ and $s > 0$ as in [ZB00], and/or enforcing the non-negativity of u, via Lagrange multipliers as in [BBG98] or by a CFL-type control of the time-step as in [GR00]. In this work on the other hand, the purpose of the regularization is not to preserve non-negativity, but rather to ensure that the mobility remains coercive and bound as an operator and that the prerequisites of the Brezzi splitting theorem (Thm. 2.10) are met.

Definition 3.7 (Bounded and coercive operators)**.** *An operator $F \in \mathcal{L}(L^2(\Gamma), L^2(\Gamma))$ is* bounded with constant C, *if* $\forall u \in L^2(\Gamma) : \|Fu\| \leq C\|u\|_{L^2}$, *and* coercive with constant c, *when* $\forall u \in L^2(\Gamma) : \langle u, Fu\rangle_{L^2} \geq c\|u\|^2_{L^2}$. *Corresponding definitions are valid for operators in* $L^2(T\Gamma)$.

Lemma 3.8 (Mobility)**.** *Let $u \in L^\infty(\Gamma)$, with $\|u\|_{L^\infty} =: \overline{u} > 0$, and bounded away from zero, in the sense that $\underline{u} := \operatorname{ess\,inf} u > 0$. Furthermore, let $H \in L^\infty(\Gamma)$ and $S \in \mathcal{L}(L^2(T\Gamma), L^2(T\Gamma))$ be self-adjoint and bounded with constant $\|S\|$. Then the operator*

$$\mathcal{M}_u f := \frac{u^3}{3} f + \epsilon \frac{u^4}{6}(Hf + Sf) \tag{3.19}$$

(with $\epsilon > 0$) is also self-adjoint and bounded with constant $M := \dfrac{\overline{u}^3}{3} + \epsilon \dfrac{\overline{u}^4}{6}(\|H\|_{L^\infty} + \|S\|)$. If moreover

$$\|u\|_{L^\infty} < \left\{ \frac{\epsilon}{2}(\|H\|_{L^\infty} + \|S\|) \right\}^{-1} \tag{3.20}$$

then

1. *$\mathcal{M}_u$ is coercive with constant* $\mu := \dfrac{\underline{u}^3}{3}\left(1 - \epsilon \dfrac{\overline{u}}{2}(\|H\|_{L^\infty} + \|S\|)\right)$.
2. *$\mathcal{M}_u$ is invertible.*
3. *$\mathcal{M}_u^{-1}$ is self-adjoint, bounded with constant μ^{-1} and coercive with constant μM^{-2}.*

Proof. It is straightforward to verify that $\langle f, \mathcal{M}_u g\rangle_{L^2} = \langle \mathcal{M}_u f, g\rangle_{L^2}$ and therefore $\mathcal{M}_u$ is self-adjoint. Using Lem. 3.3, we have for the boundedness

$$\|\mathcal{M}_u f\|_{L^2} = \|\frac{u^3}{3} f + \epsilon \frac{u^4}{6}(Hf + Sf)\|_{L^2} \leq \left(\frac{\overline{u}^3}{3} + \epsilon \frac{\overline{u}^4}{6}(\|H\|_{L^\infty} + \|S\|)\right)\|f\|_{L^2} = M\|f\|_{L^2}$$

and likewise for the coercivity

$$
\begin{aligned}
\langle f, \mathcal{M}_u f\rangle_{L^2} = \langle f, \frac{u^3}{3} f + \epsilon \frac{u^4}{6}(Hf + Sf)\rangle_{L^2} &= \langle f, \frac{u^3}{3}\left(f + \epsilon \frac{u}{6}(Hf + Sf)\right)\rangle_{L^2} \\
&\geq \frac{u^3}{3}\left(1 - \epsilon \frac{\overline{u}}{2}(\|H\|_{L^\infty} + \|S\|)\right)\|f\|_{L^2}^2 = \mu \|f\|_{L^2}^2
\end{aligned}
$$

since the quantity in parentheses is positive due to assumption (3.20).

For any $f \in L^2(T\Gamma)$, we define $f' \equiv \mathcal{M}_u^{-1} f \in L^2(T\Gamma)$ to be the unique solution of

$$
\langle g, \mathcal{M}_u f'\rangle_{L^2} = \langle g, f\rangle_{L^2}, \, \forall g \in L^2(T\Gamma)
$$

This is well defined, because the bilinear form $\langle \cdot, \mathcal{M}_u \cdot\rangle_{L^2}$ is coercive (with coercivity constant μ) and continuous, and therefore the Lax-Milgram theorem 2.3 ensures that the solution exists and is unique. Moreover, the bound (2.4) implies that $\|\mathcal{M}_u^{-1} f\|_{L^2} \leq \frac{1}{\mu}\|f\|_{L^2}$, since the norm of the functional $\langle \cdot, f\rangle_{L^2}$ (as an element of the dual space) is equal to the norm of f by the Riesz representation theorem. It follows that $\mathcal{M}_u^{-1}$ is indeed bounded with constant μ^{-1}.

For the coercivity of $\mathcal{M}_u^{-1}$, we note that if $f' = \mathcal{M}_u^{-1} f$ then $\langle f', \mathcal{M}_u f'\rangle_{L^2} \geq \mu \|f'\|_{L^2}^2 \geq \mu M^{-2} \|\mathcal{M}_u f'\|_{L^2}^2$ (since $\|\mathcal{M}_u f'\|_{L^2} \leq M \|f'\|_{L^2}$) and so $\langle \mathcal{M}_u^{-1} f, f\rangle_{L^2} \geq \mu M^{-2} \|f\|_{L^2}^2$. Finally, $\mathcal{M}_u^{-1}$ is also self-adjoint, because

$$
\langle f, \mathcal{M}_u^{-1} g\rangle_{L^2} = \langle \mathcal{M}_u f', \mathcal{M}_u^{-1}(\mathcal{M}_u g')\rangle_{L^2} = \langle \mathcal{M}_u f', g'\rangle_{L^2} = \langle f', \mathcal{M}_u g'\rangle_{L^2} = \langle \mathcal{M}_u^{-1} f, g\rangle_{L^2}
$$

for arbitrary $f, g \in L^2(T\Gamma)$.

□

Definition 3.9 (Truncated ramp function)**.** *We consider the following piecewise-cubic (spline) function $\rho_{m,M} : \mathbb{R} \to \mathbb{R}$:*

$$
\rho_{m,M}(x) := \begin{cases} m, & x \leq 0 \\ m + \dfrac{x^2}{4m}, & x \in (0, 2m] \\ x, & x \in (2m, M - m] \\ \dfrac{2m(x+M) - (M-x)^2 - m^2}{4m}, & x \in (M - m, M + m] \\ M, & x > M + m \end{cases} \tag{3.21}
$$

with parameters $m \geq 0$ and $M \geq 3m$. The spline function is a smoothed version of the function $\rho(x) := \min(\max(m, x), M)$ (see Fig. 3.1), and satisfies the following properties:

1. *$m \leq \rho_{m,M}(x) \leq M$, for all $x \in \mathbb{R}$.*

2. *$\rho_{m,M} \in C^1(\mathbb{R})$ with $0 \leq \rho'_{m,M}(x) \leq 1$, for all $x \in \mathbb{R}$, and so*

$$|\rho_{m,M}(x) - \rho_{m,M}(y)| \leq |x - y| \tag{3.22}$$

for any $x, y \in \mathbb{R}$.

3. *$\rho'_{m,M}$ is Lipschitz-continuous,*

$$|\rho'_{m,M}(x) - \rho'_{m,M}(y)| \leq L_\rho |x - y| \tag{3.23}$$

for any $x, y \in \mathbb{R}$, with Lipschitz constant $L_\rho = \frac{1}{2m}$.

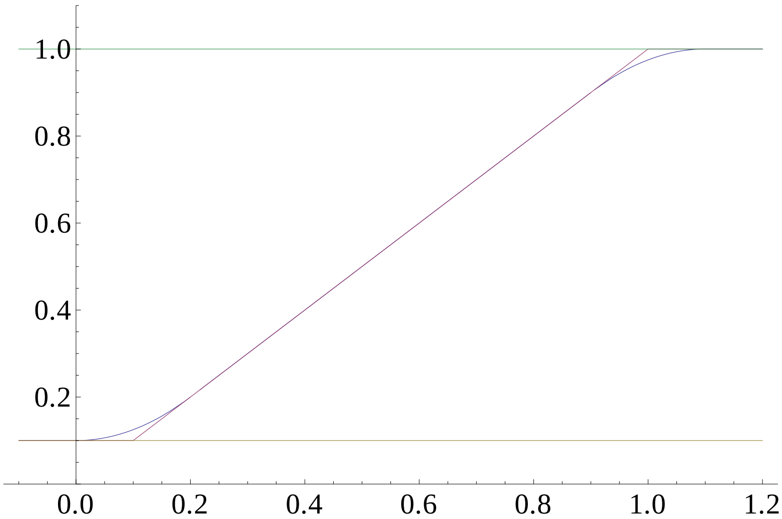


Figure 3.1.: *Truncated ramp function.* The truncated ramp function $\min(\max(x, m), M)$ and its spline approximation $\rho_{m,M}(x)$ (for $m = .1$ and $M = 1$).

Corollary 3.10 (Regularized mobility)**.** *Let $u \in L^2(\Gamma)$, $H \in L^\infty(\Gamma)$ and the self-adjoint operator $S \in \mathcal{L}(L^2(T\Gamma), L^2(T\Gamma))$ be bounded with constant $\|S\|$. Consider the operator*

$$\mathcal{M}_{[u]_\lambda} f = \frac{[u]_\lambda^3}{3} f + \frac{\epsilon [u]_\lambda^4}{6}(Hf + Sf) \tag{3.24}$$

where

$$[u]_\lambda := \rho_{\lambda, (\Lambda+\lambda)^{-1}}(u) \tag{3.25}$$

$$\Lambda := \frac{\epsilon}{2}(\|H\|_{L^\infty} + \|S\|) \geq 0 \tag{3.26}$$

and $\epsilon > 0$, $0 < \lambda < (1 + \Lambda)^{-1} \leq 1$. The operator is also self-adjoint and furthermore

1. *$\mathcal{M}_{[u]_\lambda}$ is bounded with constant* $M_\lambda = \dfrac{2}{3(\Lambda+\lambda)^3}$.

2. *$\mathcal{M}_{[u]_\lambda}$ is coercive with constant* $\mu_\lambda = \dfrac{\lambda^4}{3(\Lambda+\lambda)}$.

3. *$\mathcal{M}_{[u]_\lambda}$ is invertible.*

4. *$\mathcal{M}^{-1}_{[u]_\lambda}$ is self-adjoint, bounded with constant μ_λ^{-1} and coercive with constant $\mu_\lambda M_\lambda^{-2}$.*

Proof. It is a direct application of the boundedness of the truncated ramp function and the Lem. 3.8 with

$$u \equiv [u]_\lambda \;\Rightarrow\; 0 < \lambda \leq \underline{u} \leq \overline{u} \leq (\Lambda+\lambda)^{-1} < \infty$$

Note that $\overline{u} < \Lambda^{-1} \Rightarrow \overline{u}\Lambda < 1$. We get immediately that $\mathcal{M}_{[u]_\lambda}$ is self-adjoint and bounded with constant

$$M = \frac{\overline{u}^3}{3} + \epsilon\frac{\overline{u}^4}{6}(\|H\|_{L^\infty} + \|S\|) = \frac{\overline{u}^3}{3}(1+\overline{u}\Lambda) < \frac{2}{3(\Lambda+\lambda)^3} = M_\lambda$$

The inequality $\overline{u}\Lambda < 1$ is equivalent to condition (3.20), and so the operator is coercive with constant

$$\begin{aligned}\mu = \frac{\underline{u}^3}{3}\left(1 - \epsilon\frac{\overline{u}}{2}(\|H\|_{L^\infty} + \|S\|)\right) &= \frac{\underline{u}^3}{3}(1-\overline{u}\Lambda)\\ &\geq \frac{\lambda^3}{3}\left(1-\frac{\Lambda}{\Lambda+\lambda}\right) = \frac{\lambda^4}{3(\Lambda+\lambda)} = \mu_\lambda\end{aligned}$$

Furthermore, the operator is invertible and the claimed properties of the inverse follow directly.

□

Lemma 3.11 (Mobility on $H_{\mathrm{div}}(T\Gamma)$)**.** *Let $H \in L^\infty(\Gamma)\cap H^1(\Gamma)$ and $S \in \mathcal{L}(L^2(T\Gamma), L^2(T\Gamma))$ such that $\|Sf\|_{H_{\mathrm{div}}} \leq \|S\|_{H_{\mathrm{div}}}\|f\|_{H_{\mathrm{div}}}$, $\forall f \in H_{\mathrm{div}}(T\Gamma)$.*

1. *If $u \in H^1(\Gamma)$, then*

$$\|\mathcal{M}_{[u]_\lambda} f\|_{H_{\mathrm{div}}} \leq C_{\mathcal{M}}(\|u\|_{H^1})\,\|f\|_{H_{\mathrm{div}}} \tag{3.27}$$

for any $f \in H_{\mathrm{div}}(T\Gamma)$. The function $C_{\mathcal{M}}(\cdot)$ is a polynomial with non-negative coefficients.

2. *If $u, w \in H^2(\Gamma)$, then*

$$\begin{aligned}\|\mathcal{M}_{[u]_\lambda} f - \mathcal{M}_{[w]_\lambda} f\|_{H_{\mathrm{div}}} &\le L'_{\mathcal{M}}(\|u\|_{H^1}, \|w\|_{H^1})\|f\|_{H_{\mathrm{div}}}\left(\|u-w\|_{H^1} + \|u-w\|_{L^\infty}\right)\\ &\le L_{\mathcal{M}}(\|u\|_{H^1}, \|w\|_{H^1})\|f\|_{H_{\mathrm{div}}}\|u-w\|_{H^2} \quad (3.28)\end{aligned}$$

where $L'_{\mathcal{M}}(\cdot,\cdot)$ is a symmetric bivariate polynomial with non-negative coefficients and $L_{\mathcal{M}} := (1 + C_\infty)L'_{\mathcal{M}}$ (see Rem. 3.4 for the constant C_∞).

Proof. We present first certain small results about functions in $L^\infty(\Gamma) \cap H^1(\Gamma)$ and H_{div}:

- If $w \in L^\infty(\Gamma) \cap H^1(\Gamma)$ and $g \in H_{\mathrm{div}}(T\Gamma)$, then

$$\|wg\|_{H_{\mathrm{div}}} \le (\|w\|_{L^\infty} + \|\mathrm{grad}_\Gamma\, w\|_{L^2})\|g\|_{H_{\mathrm{div}}} \le (\|w\|_{L^\infty} + \|w\|_{H^1})\|g\|_{H_{\mathrm{div}}}$$

  since $\|wg\|_{L^2} + \|\mathrm{div}_\Gamma(wg)\|_{L^2} \le \|w\|_{L^\infty}\|g\|_{L^2} + \|\mathrm{grad}_\Gamma\, w\|_{L^2}\|g\|_{L^2} + \|w\|_{L^\infty}\|\mathrm{div}_\Gamma\, g\|_{L^2}$.

- If $u \in H^1(\Gamma)$, then $\|\mathrm{grad}_\Gamma[u]_\lambda\|_{L^2} \le \|\mathrm{grad}_\Gamma\, u\|_{L^2}$. Indeed $[\cdot]_\lambda \equiv \rho_{\lambda,(\Lambda+\lambda)^{-1}}$ is continuously differentiable, and so the chain rule $\mathrm{grad}_\Gamma[u]_\lambda = \rho'_{\lambda,(\Lambda+\lambda)^{-1}}(u)\,\mathrm{grad}_\Gamma\, u$ holds. The bound comes then from the fact that $|\rho'_{\lambda,(\Lambda+\lambda)^{-1}}(u)| \le 1$ (see Def. 3.9). Moreover, it follows that for $p \ge 1$:

$$\|\mathrm{grad}_\Gamma[u]_\lambda^p\|_{L^2} \le p\|[u]_\lambda\|_{L^\infty}^{p-1}\|\mathrm{grad}_\Gamma[u]_\lambda\|_{L^2} \le p(\Lambda+\lambda)^{-(p-1)}\|u\|_{H^1}$$

- Combining the previous two results, we have that if $u \in H^1(\Gamma)$ and $g \in H_{\mathrm{div}}(T\Gamma)$, then for $p \ge 1$:

$$\|[u]_\lambda^p g\|_{H_{\mathrm{div}}} \le P_p(\|u\|_{H^1})\|g\|_{H_{\mathrm{div}}}$$

  where $P_p(x) := (\Lambda+\lambda)^{-p}\left(1 + p(\Lambda+\lambda)x\right)$ is a first order polynomial with non-negative coefficients.

Using these, we can prove the stated inequalities:

1. Applying these results on $\mathcal{M}_{[u]_\lambda} f$, we have:

$$\begin{aligned}\|\mathcal{M}_{[u]_\lambda} f\|_{H_{\mathrm{div}}} &\le \frac{1}{3}\|[u]_\lambda^3 f\|_{H_{\mathrm{div}}} + \frac{\epsilon}{6}\|[u]_\lambda^4(Hf + Sf)\|_{H_{\mathrm{div}}}\\ &\le \frac{1}{3}\big(P_3(\|u\|_{H^1}) + \Lambda' P_4(\|u\|_{H^1})\big)\|f\|_{H_{\mathrm{div}}}\end{aligned}$$

   where $\Lambda' := \frac{\epsilon}{2}\left(\|H\|_{L^\infty} + \|H\|_{H^1} + \|S\|_{H_{\mathrm{div}}}\right)$. The desired bound (3.27) follows with $C_{\mathcal{M}} := \frac{1}{3}(P_3 + \Lambda' P_4)$.

2. Likewise,

$$
\begin{aligned}
&\|\mathcal{M}_{[u]_\lambda} f - \mathcal{M}_{[w]_\lambda} f\|_{H_{\mathrm{div}}} \\
&\leq \frac{1}{3}\|([u]_\lambda^3 - [w]_\lambda^3) f\|_{H_{\mathrm{div}}} + \frac{\epsilon}{6}\|([u]_\lambda^4 - [w]_\lambda^4)(Hf + Sf)\|_{H_{\mathrm{div}}} \\
&= \frac{1}{3}\|([u]_\lambda - [w]_\lambda)([u]_\lambda^2 + [u]_\lambda [w]_\lambda + [w]_\lambda^2) f\|_{H_{\mathrm{div}}} \\
&\qquad + \frac{\epsilon}{6}\|([u]_\lambda - [w]_\lambda)([u]_\lambda + [w]_\lambda)([u]_\lambda^2 + [w]_\lambda^2)(Hf + Sf)\|_{H_{\mathrm{div}}} \\
&\leq \big(\|[u]_\lambda - [w]_\lambda\|_{L^\infty} + \|\mathrm{grad}_\Gamma [u]_\lambda - \mathrm{grad}_\Gamma [w]_\lambda\|_{L^2}\big) Q(\|u\|_{H^1}, \|w\|_{H^1}) \|f\|_{H_{\mathrm{div}}}
\end{aligned}
$$

where $Q(x,y) := \frac{1}{3}\left(P_2(x) + P_1(x)P_1(y) + P_2(y)\right) + \frac{\Lambda'}{3}\left(P_1(x) + P_1(y)\right)\left(P_2(x) + P_2(y)\right)$. Moreover, given the properties (Def. 3.9) of $[\cdot]_\lambda \equiv \rho_{\lambda,(\Lambda+\lambda)^{-1}}$ and $[\cdot]_\lambda' \equiv \rho'_{\lambda,(\Lambda+\lambda)^{-1}}$,

$$
\forall u, w \in \mathbb{R} : |[u]_\lambda - [w]_\lambda| \leq |u - w| \;\Rightarrow\; \forall u, w \in L^\infty(\Gamma) : \|[u]_\lambda - [w]_\lambda\|_{L^\infty} \leq \|u - w\|_{L^\infty}
$$

and

$$
\begin{aligned}
&\|\mathrm{grad}_\Gamma [u]_\lambda - \mathrm{grad}_\Gamma [w]_\lambda\|_{L^2} = \|[u]_\lambda' \,\mathrm{grad}_\Gamma u - [w]_\lambda' \,\mathrm{grad}_\Gamma w\|_{L^2} \\
&\leq \frac{1}{2}\|[u]_\lambda' \,\mathrm{grad}_\Gamma u - [u]_\lambda' \,\mathrm{grad}_\Gamma w\|_{L^2} + \frac{1}{2}\|[u]_\lambda' \,\mathrm{grad}_\Gamma w - [w]_\lambda' \,\mathrm{grad}_\Gamma w\|_{L^2} \\
&\qquad + \frac{1}{2}\|[u]_\lambda' \,\mathrm{grad}_\Gamma u - [w]_\lambda' \,\mathrm{grad}_\Gamma u\|_{L^2} + \frac{1}{2}\|[w]_\lambda' \,\mathrm{grad}_\Gamma u - [w]_\lambda' \,\mathrm{grad}_\Gamma w\|_{L^2} \\
&\leq \frac{1}{2}\left(\|[u]_\lambda'\|_{L^\infty} + \|[w]_\lambda'\|_{L^\infty}\right) \|\mathrm{grad}_\Gamma (u - w)\|_{L^2} \\
&\qquad + \frac{1}{2}\|[u]_\lambda' - [w]_\lambda'\|_{L^\infty} \left(\|\mathrm{grad}_\Gamma u\|_{L^2} + \|\mathrm{grad}_\Gamma w\|_{L^2}\right) \\
&\leq \|\mathrm{grad}_\Gamma (u - w)\|_{L^2} + \frac{1}{4\lambda}\left(\|\mathrm{grad}_\Gamma u\|_{L^2} + \|\mathrm{grad}_\Gamma w\|_{L^2}\right) \|u - w\|_{L^\infty} \\
&\leq \|u - w\|_{H^1} + \frac{1}{4\lambda}\left(\|u\|_{H^1} + \|w\|_{H^1}\right) \|u - w\|_{L^\infty}
\end{aligned}
$$

It follows that

$$
\begin{aligned}
&\|[u]_\lambda - [w]_\lambda\|_{L^\infty} + \|\mathrm{grad}_\Gamma [u]_\lambda - \mathrm{grad}_\Gamma [w]_\lambda\|_{L^2} \\
&\qquad\qquad \leq \|u - w\|_{H^1} + \left(1 + \frac{1}{4\lambda}\left(\|u\|_{H^1} + \|w\|_{H^1}\right)\right) \|u - w\|_{L^\infty} \\
&\qquad\qquad\qquad\qquad \leq R(\|u\|_{H^1}, \|w\|_{H^1}) \left(\|u - w\|_{H^1} + \|u - w\|_{L^\infty}\right)
\end{aligned}
$$

where $R(x,y) := 1 + \frac{1}{4\lambda}(x + y)$. The bounds (3.28) follow immediately with $L'_{\mathcal{M}}(x,y) := R(x,y)Q(x,y)$ and, via (3.13), with $L_{\mathcal{M}} = (1 + C_\infty)L'_{\mathcal{M}}$.

□

3.4. Well-posedness of the model

Before we study the regularized model itself, we need to show certain auxiliary results.

Proposition 3.12 (Regularized optimization problem)**.** *Consider the regularized optimization problem*

$$\min_{(f,\dot{u})\in L^2(T\Gamma)\times H^1(\Gamma)} \left\{\frac{1}{2}\langle f, \mathcal{M}_{[u]_\lambda}^{-1} f\rangle_{L^2} + E((f,\dot{u}))\right\} \tag{3.29a}$$

$$\langle \dot{u}, q\rangle_{L^2} + \lambda\langle \operatorname{grad}_\Gamma \dot{u}, \operatorname{grad}_\Gamma q\rangle_{L^2} - \langle f, \operatorname{grad}_\Gamma q\rangle_{L^2} = R(q),\ \forall q \in H^1(\Gamma) \tag{3.29b}$$

where $E \in X'$, $R \in Q'$ are continuous linear functionals over $X = L^2(T\Gamma) \times H^1(\Gamma)$ and $Q = H^1(\Gamma)$ respectively. If $u \in H^1(\Gamma)$ and the assumptions of Cor. 3.10 are met, then there exists a unique solution $(f_\lambda, \dot{u}_\lambda) \in L^2(T\Gamma) \times H^1(\Gamma)$ with a unique multiplier $p_\lambda \in H^1(\Gamma)$. Furthermore, there exists a constant C such that

$$\|\dot{u}_\lambda\|_{H^1} + \|f_\lambda\|_{L^2} + \|p_\lambda\|_{H^1} \le C\left(\|E\|_{X'} + \|R\|_{Q'}\right) \tag{3.30}$$

Proof. We will apply Prop. 2.13 with $X = L^2(T\Gamma) \times H^1(\Gamma)$ with norm $\|(f,\dot{u})\|_X = \left(\|f\|_{L^2}^2 + \|\dot{u}\|_{H^1}^2\right)^{1/2}$, $Q = H^1(\Gamma)$ and

$$\begin{aligned}
a((f,\dot{u}),(f',\dot{u}')) &:= \langle f, \mathcal{M}_{[u]_\lambda}^{-1} f'\rangle_{L^2}\\
e((f,\dot{u})) &:= E((f,\dot{u}))\\
b((f,\dot{u}),q) &:= \langle \dot{u}, q\rangle_{L^2} + \lambda\langle \operatorname{grad}_\Gamma \dot{u}, \operatorname{grad}_\Gamma q\rangle_{L^2} - \langle f, \operatorname{grad}_\Gamma q\rangle_{L^2}\\
g(q) &:= R(q)
\end{aligned}$$

for any $f, f' \in L^2(T\Gamma)$ and $\dot{u}, \dot{u}', q \in H^1(\Gamma)$. For the parameter λ, we will assume that it meets the conditions of Cor. 3.10, i.e. $0 < \lambda < (1+\Lambda)^{-1} \le 1$, so that the conclusions of the corollary are valid. We verify that the conditions of Prop. 2.13 are met:

- $a(\cdot,\cdot)$ *is continuous:*

  $$|a((f,\dot{u}),(f',\dot{u}'))| \le |\langle f, \mathcal{M}_{[u]_\lambda}^{-1} f'\rangle_{L^2}| \le \mu_\lambda^{-1}\|f\|_{L^2}\|f'\|_{L^2} \le \mu_\lambda^{-1}\|(f,\dot{u})\|_X\|(f',\dot{u}')\|_X$$

  where we used the fact that $\|f\|_{L^2} \le \left(\|f\|_{L^2}^2 + \|\dot{u}\|_{H^1}^2\right)^{1/2} = \|(f,\dot{u})\|_X$ and likewise for f'. Note that $\|a\| = \mu_\lambda^{-1}$.

- $b(\cdot,\cdot)$ *is continuous:*

  $$\begin{aligned}
  |b((f,\dot{u}),q)| &\le |\langle \dot{u}, q\rangle_{L^2}| + \lambda|\langle \operatorname{grad}_\Gamma \dot{u}, \operatorname{grad}_\Gamma q\rangle_{L^2}| + |\langle f, \operatorname{grad}_\Gamma q\rangle_{L^2}|\\
  &\le \|\dot{u}\|_{L^2}\|q\|_{L^2} + \lambda\|\operatorname{grad}_\Gamma \dot{u}\|_{L^2}\|\operatorname{grad}_\Gamma q\|_{L^2} + \|f\|_{L^2}\|\operatorname{grad}_\Gamma q\|_{L^2}\\
  &\le (2+\lambda)\|(f,\dot{u})\|_X\|q\|_{H^1}
  \end{aligned}$$

  and so $\|b\| = 2 + \lambda$.

- $b(\cdot,\cdot)$ *satisfies the LBB condition:* We note that for an arbitrary $q \in H^1(\Gamma)$, $q \neq 0$, the pair $(-\operatorname{grad}_\Gamma q, q) \in L^2(T\Gamma) \times H^1(\Gamma) = X$. Then

$$b((-\operatorname{grad}_\Gamma q, q), q) = \|q\|_{L^2}^2 + \lambda\|\operatorname{grad}_\Gamma q\|_{L^2}^2 + \|\operatorname{grad}_\Gamma q\|_{L^2}^2 \geq \frac{1}{2}(\|q\|_{L^2}^2 + 2\|\operatorname{grad}_\Gamma q\|_{L^2}^2) = \frac{1}{2}\|(-\operatorname{grad}_\Gamma q, q)\|_X^2$$

and so

$$\sup_{p \in X \setminus \{0\}} \frac{b(p,q)}{\|p\|_X\|q\|_Q} \geq \frac{b((-\operatorname{grad}_\Gamma q, q), q)}{\|(-\operatorname{grad}_\Gamma q, q)\|_X\|q\|_{H^1}} \geq \frac{b((-\operatorname{grad}_\Gamma q, q), q)}{\|(-\operatorname{grad}_\Gamma q, q)\|_X^2} = \frac{1}{2}$$

since $\|(-\operatorname{grad}_\Gamma q, q)\|_X \geq \|q\|_{H^1}$. It follows that $b(\cdot,\cdot)$ indeed satisfies the LBB condition (2.8) with constant $\beta = \frac{1}{2}$.

- $a(\cdot,\cdot)$ *is Z-coercive:* The pair $(f, \dot{u}) \in Z$, when $b((f,\dot{u}), q) = 0$ for all $q \in Q$. Noting that $\dot{u} \in H^1(\Gamma) = Q$, we have $b((f,\dot{u}),\dot{u}) = 0 \Rightarrow \|\dot{u}\|_{L^2}^2 + \lambda\|\operatorname{grad}_\Gamma \dot{u}\|_{L^2} = \langle f, \operatorname{grad}_\Gamma \dot{u}\rangle_{L^2}$ and therefore

$$\|f - \lambda \operatorname{grad}_\Gamma \dot{u}\|_{L^2}^2 \geq 0 \Rightarrow \|f\|_{L^2}^2 - 2\lambda\langle f, \operatorname{grad}_\Gamma \dot{u}\rangle_{L^2} + \lambda^2\|\operatorname{grad}_\Gamma \dot{u}\|_{L^2}^2 \geq 0$$
$$\Rightarrow \|f\|_{L^2}^2 \geq 2\lambda\|\dot{u}\|_{L^2}^2 + \lambda^2\|\operatorname{grad}_\Gamma \dot{u}\|_{L^2} \Rightarrow \|f\|_{L^2}^2 \geq \lambda^2\|\dot{u}\|_{H^1}^2$$

since $\lambda < 1 \Rightarrow \lambda^2 < 2\lambda$. The coercivity follows:

$$\begin{aligned} a((f,\dot{u}),(f,\dot{u})) &= \langle f, \mathcal{M}_{[u]_\lambda}^{-1} f\rangle_{L^2} \geq \mu_\lambda M_\lambda^{-2}\|f\|_{L^2}^2 \\ &= \mu_\lambda M_\lambda^{-2}\left(\frac{\lambda^2}{1+\lambda^2}\|f\|_{L^2}^2 + \frac{1}{1+\lambda^2}\|f\|_{L^2}^2\right) \\ &\geq \mu_\lambda M_\lambda^{-2}\frac{\lambda^2}{1+\lambda^2}\left(\|f\|_{L^2}^2 + \|\dot{u}\|^2\right) = \alpha\|(f,\dot{u})\|_X^2 \end{aligned}$$

with $\alpha := \dfrac{\mu_\lambda M_\lambda^{-2}\lambda^2}{1+\lambda^2}$.

The bound (3.30) follows directly from the bounds (2.12) and the fact that the constants $\alpha, \beta, \|a\|, \|b\|$ depend only on λ.

□

Lemma 3.13 (Auxiliary problem I)**.** *Let $e \in L^2(\Gamma)$ and $0 < \lambda < 1$. Then the problem*

$$\langle u, \theta\rangle_{L^2} + \lambda\langle \operatorname{grad}_\Gamma u, \operatorname{grad}_\Gamma \theta\rangle_{L^2} = -\langle e, \theta\rangle_{L^2}, \quad \forall \theta \in H^1(\Gamma) \tag{3.31}$$

has a unique solution $u \in H^1(\Gamma)$. Moreover, there exists a constant $\gamma_\lambda > 0$ such that

$$\|u\|_{H^2} \leq \gamma_\lambda\|e\|_{L^2} \tag{3.32}$$

and so $u \in H^2(\Gamma)$.

Proof. First we use the Lax-Milgram theorem 2.3 to show that a unique solution exists in $H^1(\Gamma)$. Defining the symmetric bilinear form $\alpha_\lambda(u,w) := \langle u, w\rangle_{L^2} + \lambda\langle \operatorname{grad}_\Gamma u, \operatorname{grad}_\Gamma w\rangle_{L^2}$, it is straightforward to verify that the conditions of the theorem are met, in particular that α_λ is coercive in $H^1(\Gamma)$ with constant $\alpha = \lambda$. It follows that there is indeed a unique solution $u \in H^1(\Gamma)$, such that $\|u\|_{H^1} \leq \lambda^{-1}\|e\|_{L^2}$. Choosing an arbitrary test function $\theta \in C^\infty(\Gamma)$, we have

$$\langle u, \theta\rangle_{L^2} + \lambda\langle \operatorname{grad}_\Gamma u, \operatorname{grad}_\Gamma \theta\rangle_{L^2} = -\langle e, \theta\rangle_{L^2} \Rightarrow \langle u, \Delta_\Gamma \theta\rangle_{L^2} = \langle \lambda^{-1}(u+e), \theta\rangle_{L^2}$$

This proves that $\Delta_\Gamma u = \lambda^{-1}(u+e) \in L^2$ (in the sense of distributions) and so

$$\|u\|_{H^2} \leq \|u\|_{H^1} + \|\Delta_\Gamma u\|_{L^2} \leq \|u\|_{H^1} + \lambda^{-1}(\|e\|_{L^2} + \|u\|_{L^2}) \leq (1+\lambda^{-1})\|u\|_{H^1} + \lambda^{-1}\|e\|_{L^2}$$

which gives us the desired bound with $\gamma_\lambda = \lambda^{-1}(2+\lambda^{-1})$.

$\square$

Lemma 3.14 (Auxiliary problem II)**.** *If $u \in H^1(\Gamma)$ and the assumptions of Lem. 3.11 are met, the problem*

$$\langle f, \mathcal{M}^{-1}_{[u]_\lambda} g\rangle_{L^2} = \langle j, g\rangle_{L^2}, \quad \forall g \in L^2(T\Gamma) \tag{3.33}$$

has a unique solution $f \in L^2(T\Gamma)$ for any $j \in H_{\mathrm{div}}$, which satisfies the bound

$$\|f\|_{H_{\mathrm{div}}} \leq \|\mathcal{M}_{[u]_\lambda} j\|_{H_{\mathrm{div}}} \leq \gamma_{\mathcal{M}}\|j\|_{H_{\mathrm{div}}} \tag{3.34}$$

where $\gamma_{\mathcal{M}} := C_{\mathcal{M}}(\|u\|_{H^1})$.

Proof. Because of the properties of the (regularized) mobility (Cor. 3.10), the equation is equivalent to $\mathcal{M}^{-1}_{[u]_\lambda} f = j \Rightarrow f = \mathcal{M}_{[u]_\lambda} j$. The bound follows by Lem. 3.11.

$\square$

Proposition 3.15 (H^2-regularity)**.** *Let $u \in H^2(\Gamma)$ and the assumptions of Lem. 3.11 be met. If there exist $e \in L^2(\Gamma)$, $j \in H_{\mathrm{div}}(T\Gamma)$ and $r \in L^2(\Gamma)$ such that*

$$E((g,\theta)) = \langle e, \theta\rangle_{L^2} + \langle j, g\rangle_{L^2}, \quad \forall (g,\theta) \in L^2(T\Gamma) \times H^1(\Gamma) \tag{3.35a}$$

$$R(q) = \langle r, q\rangle_{L^2}, \quad \forall q \in H^1(\Gamma) \tag{3.35b}$$

then the unique solution $(f, \dot{u}, p)$ of Prop. 3.12 satisfies the bound

$$\begin{aligned}\|\dot{u}\|_{H^2} + \|f\|_{H_{\mathrm{div}}} + \|p\|_{H^2} &\leq \gamma_\lambda^2\gamma_{\mathcal{M}}\|e\|_{L^2} + \gamma_\lambda\|\mathcal{M}_{[u]_\lambda} j\|_{H_{\mathrm{div}}} + \gamma_\lambda\|r\|_{L^2} \\ &\leq \gamma_\lambda^2\gamma_{\mathcal{M}}\|e\|_{L^2} + \gamma_\lambda\gamma_{\mathcal{M}}\|j\|_{H_{\mathrm{div}}} + \gamma_\lambda\|r\|_{L^2}\end{aligned} \tag{3.36}$$

where $\gamma_{\mathcal{M}} := C_{\mathcal{M}}(\|u\|_{H^1})$.

Proof. The solution of Prop. 3.12 satisfies the saddle point equations

$$\begin{gathered}\langle f, \mathcal{M}_{[u]_\lambda}^{-1} g\rangle_{L^2} + \langle \theta, p\rangle_{L^2} + \lambda\langle \operatorname{grad}_\Gamma \theta, \operatorname{grad}_\Gamma p\rangle_{L^2} - \langle g, \operatorname{grad}_\Gamma p\rangle_{L^2} = -E((g,\theta)) \\ \langle \dot{u}, q\rangle_{L^2} + \lambda\langle \operatorname{grad}_\Gamma \dot{u}, \operatorname{grad}_\Gamma q\rangle_{L^2} - \langle f, \operatorname{grad}_\Gamma q\rangle_{L^2} = R(q)\end{gathered}$$

for all $(g, \theta) \in L^2(T\Gamma) \times H^1(\Gamma) =: X$ and $q \in H^1(\Gamma) =: Q$. Under the assumptions of this proposition, these can be rewritten as the equivalent system

$$\begin{gathered}\langle f, \mathcal{M}_{[u]_\lambda}^{-1} g\rangle_{L^2} = \langle g, \operatorname{grad}_\Gamma p\rangle_{L^2} - \langle j, g\rangle_{L^2} \\ \langle \theta, p\rangle_{L^2} + \lambda\langle \operatorname{grad}_\Gamma \theta, \operatorname{grad}_\Gamma p\rangle_{L^2} = -\langle e, \theta\rangle_{L^2} \\ \langle \dot{u}, q\rangle_{L^2} + \lambda\langle \operatorname{grad}_\Gamma \dot{u}, \operatorname{grad}_\Gamma q\rangle_{L^2} = \langle f, \operatorname{grad}_\Gamma q\rangle_{L^2} + \langle r, q\rangle_{L^2}\end{gathered}$$

for all $g \in L^2(T\Gamma)$, $\theta \in H^1(\Gamma)$ and $q \in H^1(\Gamma)$ respectively.

Applying Lem. 3.13 to the second equation, gives us immediately that $p \in H^2(\Gamma)$ with

$$\|p\|_{H^2} \le \gamma_\lambda \|e\|_{L^2}$$

Then by definition $\operatorname{grad}_\Gamma p \in H_{\operatorname{div}}(T\Gamma)$ and, since by assumption $j \in H_{\operatorname{div}}(T\Gamma)$ too, applying Lem. 3.14 to the first equation yields

$$\|f\|_{H_{\operatorname{div}}} \le \|\mathcal{M}_{[u]_\lambda} (\operatorname{grad}_\Gamma p - j)\|_{H_{\operatorname{div}}} \le \gamma_{\mathcal{M}} \|p\|_{H^2} + \|\mathcal{M}_{[u]_\lambda} j\|_{H_{\operatorname{div}}}$$

Finally, given that $f \in H_{\operatorname{div}}(T\Gamma)$, the right hand side of the third equation is equal to $\langle -\operatorname{div}_\Gamma f + r, q\rangle_{L^2}$, and applying Lem. 3.13 once more, gives us

$$\|\dot{u}\|_{H^2} \le \gamma_\lambda \|-\operatorname{div}_\Gamma f + r\|_{L^2} \le \gamma_\lambda \left(\|f\|_{H_{\operatorname{div}}} + \|r\|_{L^2}\right)$$

Chaining the three inequalities and adding everything together, we eventually arrive at the desired bound.

□

Now we can focus on the model itself:

Corollary 3.16 (The regularized variational model)**.** *Consider the optimization problem*

$$\min_{(f,\dot{u}) \in L^2(T\Gamma) \times H^1(\Gamma)} \left\{ \frac{1}{2} \langle f, \mathcal{M}_{[u]_\lambda}^{-1} f\rangle_{L^2} + \mathcal{E}'(u)(\dot{u}) \right\} \tag{3.37a}$$

$$\langle \dot{u}, q\rangle_{L^2} + \lambda\langle \operatorname{grad}_\Gamma \dot{u}, \operatorname{grad}_\Gamma q\rangle_{L^2} = \langle f, \operatorname{grad}_\Gamma q\rangle_{L^2}, \ \forall q \in H^1(\Gamma) \tag{3.37b}$$

with

$$\mathcal{E}'(u)(\theta) := \langle W_1, \theta\rangle_{L^2} + \epsilon\langle W_2 u, \theta\rangle_{L^2} + \epsilon\langle \operatorname{grad}_\Gamma u, \operatorname{grad}_\Gamma \theta\rangle_{L^2} \tag{3.38}$$

and $W_1 \in L^2(\Gamma)$*,* $W_2 \in L^\infty(\Gamma)$*.*

1. *If $u \in H^1(\Gamma)$ and the assumptions of Cor. 3.10 are met, then there exists a unique solution $(f, \dot{u}) \in L^2(T\Gamma) \times H^1(\Gamma)$ with a unique multiplier $p \in H^1(\Gamma)$. Furthermore, there exist constants $\alpha_\lambda, \beta_\lambda > 0$ such that*
$$\|\dot{u}\|_{H^1} + \|f\|_{L^2} + \|p\|_{H^1} \leq \alpha_\lambda + \beta_\lambda \|u\|_{H^1} \tag{3.39}$$
2. *If, moreover, $u \in H^2(\Gamma)$ and the assumptions of Lem. 3.11 are met then the solution satisfies the bound*
$$\|\dot{u}\|_{H^2} + \|f\|_{H_{\mathrm{div}}} + \|p\|_{H^2} \leq A_\lambda(\|u\|_{H^1}) + B_\lambda(\|u\|_{H^1})\|u\|_{H^2} \tag{3.40}$$
where A_λ, B_λ are monotonically increasing functions.

Proof.

1. The key observation is that the optimization problem (3.37) is the problem (3.29) with $E((g, \theta)) = \mathcal{E}'(u)(\theta)$ and $R = 0$. Then
$$\begin{aligned} |E((g, \theta))| &= |\mathcal{E}'(u)(\theta)| \\ &\leq \|W_1\|_{L^2}\|\theta\|_{L^2} + \epsilon\|W_2\|_{L^\infty}\|u\|_{L^2}\|\theta\|_{L^2} + \epsilon\|u\|_{H^1}\|\theta\|_{H^1} \\ &\leq (\|W_1\|_{L^2} + \epsilon(1 + \|W_2\|_{L^\infty})\|u\|_{H^1})\,\|\theta\|_{H^1} \\ &\leq (\|W_1\|_{L^2} + \epsilon(1 + \|W_2\|_{L^\infty})\|u\|_{H^1})\,\|(g, \theta)\|_X \end{aligned}$$
and so $\|E\|_{X'} \leq \|W_1\|_{L^2} + \epsilon(1 + \|W_2\|_{L^\infty})\|u\|_{H^1}$. The bound (3.39) follows with $\alpha_\lambda := C\|W_1\|_{L^2}$ and $\beta_\lambda := C\epsilon(1 + \|W_2\|_{L^\infty})$.
2. When $u \in H^2(\Gamma)$, the functional $\mathcal{E}'(u)$ can be written as $\mathcal{E}'(u)(\theta) = \langle W_1 + \epsilon W_2 u - \epsilon\Delta_\Gamma u, \theta\rangle_{L^2}$, and so we can set $e := W_1 + \epsilon W_2 u - \epsilon\Delta_\Gamma u \in L^2(\Gamma)$. Together with $j = 0$ and $r = 0$, we can apply Prop. 3.15 to derive the bound
$$\begin{aligned} \|\dot{u}\|_{H^2} + \|f\|_{H_{\mathrm{div}}} + \|p\|_{H^2} &\leq \gamma_\lambda^2\gamma_{\mathcal{M}}\|e\|_{L^2} \\ &\leq \gamma_\lambda^2\gamma_{\mathcal{M}}\,(\|W_1\|_{L^2} + \epsilon(1 + \|W_2\|_{L^\infty})\|u\|_{H^2}) \end{aligned}$$
Recalling that $\gamma_{\mathcal{M}} = C_{\mathcal{M}}(\|u\|_{H^1})$ is an increasing function of $\|u\|_{H^1}$, and since the other constants are positive, it follows that $A_\lambda := \gamma_\lambda^2\|W_1\|_{L^2}\gamma_{\mathcal{M}}$ and $B_\lambda := \epsilon\gamma_\lambda^2(1 + \|W_2\|_{L^\infty})\gamma_{\mathcal{M}}$ are also increasing functions of $\|u\|_{H^1}$.

□

Proposition 3.17 (Local existence of solutions)**.** *Let $u_0 \in H^2(\Gamma)$ and assume that the conditions of Lem. 3.11 are met. Then, for small enough $\tau > 0$, there exists $u \in C([0, \tau]; H^2(\Gamma)) \cap H^1(0, \tau; H^2(\Gamma))$ and $f \in L^2(0, \tau; H_{\mathrm{div}}(T\Gamma))$, $\dot{u} \in L^2(0, \tau; H^2(\Gamma))$, $p \in L^2(0, \tau; H^2(\Gamma))$ such that*

1. *$u' = \dot{u}$ and $u(0) = u_0$*
2. *for almost all $t \in (0, \tau)$:*

$$\langle f(t), \mathcal{M}^{-1}_{[u(t)]_\lambda} g\rangle_{L^2} = \langle g, \operatorname{grad}_\Gamma p(t)\rangle_{L^2} \tag{3.41a}$$

$$\langle \theta, p(t)\rangle_{L^2} + \lambda\langle \operatorname{grad}_\Gamma \theta, \operatorname{grad}_\Gamma p(t)\rangle_{L^2} = -\mathcal{E}'(u(t))(\theta) \tag{3.41b}$$

$$\langle \dot{u}(t), q\rangle_{L^2} + \lambda\langle \operatorname{grad}_\Gamma \dot{u}(t), \operatorname{grad}_\Gamma q\rangle_{L^2} = \langle f(t), \operatorname{grad}_\Gamma q\rangle_{L^2} \tag{3.41c}$$

   for all $(g, \theta, q) \in L^2(T\Gamma) \times H^1(\Gamma) \times H^1(\Gamma)$

3. *the following bounds hold:*

$$\|u\|_{C([0,\tau];H^1)} \leq (\|u_0\|_{H^1} + \alpha_\lambda \tau)e^{\beta_\lambda \tau} =: M \tag{3.42a}$$

$$\|u\|_{C([0,\tau];H^2)} \leq (\|u_0\|_{H^2} + A_\lambda(M)\tau)e^{B_\lambda(M)\tau} \tag{3.42b}$$

More specifically, there exists a (monotone increasing) function $C(U, T)$, such that if $\|u_0\|_{H^2} \leq U$, $\tau \leq T$ and $\tau < C(U, T)^{-1}$ then the conclusions of this proposition hold.

Proof. Given u_0, we define a sequence of $u^{(k)}$, $k \geq 0$, as follows:

- $u^{(0)}(t) = u_0$, $t \in [0, \tau]$
- $(f^{(k)}(t), \dot{u}^{(k)}(t), p^{(k)}(t))$ is the unique saddle point of the problem (3.37) with $u = u^{(k)}(t)$, for almost all $t \in (0, \tau)$
- $u^{(k+1)}(t) = u_0 + \int_0^t \dot{u}^{(k)}(s)\, ds$, $t \in [0, \tau]$

We will show that, for a small enough τ, the sequence $u^{(k)}$ converges to a fixed point:

1. ($u_0 \in H^1(\Gamma) \Rightarrow u^{(k)} \in C([0, \tau], H^1(\Gamma))$): If $u_0 \in H^1(\Gamma)$, then $u^{(0)} \in C([0, \tau]; H^1(\Gamma))$. We assume that $u^{(k)} \in C([0, \tau]; H^1(\Gamma))$. Then for almost all $t \in (0, \tau)$, $(f^{(k)}(t), \dot{u}^{(k)}(t))$ and $p^{(k)}(t)$ are the saddle point of the problem (3.37) with $u = u^{(k)}(t) \in H^1(\Gamma)$, and so satisfy the bound $\|\dot{u}^{(k)}(t)\|_{H^1} + \|f^{(k)}(t)\|_{L^2} + \|p^{(k)}(t)\|_{H^1} \leq \alpha_\lambda + \beta_\lambda \|u^{(k)}(t)\|_{H^1}$. This implies that $\|\dot{u}^{(k)}(t)\|_{H^1} \leq \alpha_\lambda + \beta_\lambda\|u^{(k)}(t)\|_{H^1}$ and so

$$\begin{aligned}
\int_0^\tau \|\dot{u}^{(k)}(t)\|^2_{H^1}\, dt &\leq \int_0^\tau \left(\alpha_\lambda + \beta_\lambda \|u^{(k)}(t)\|_{H^1}\right)^2 dt \\
&= \tau\alpha_\lambda^2 + 2\beta_\lambda \int_0^\tau \|u^{(k)}(t)\|_{H^1}\, dt + \beta_\lambda^2 \int_0^\tau \|u^{(k)}(t)\|^2_{H^1}\, dt \\
&\leq \tau\alpha_\lambda^2 + 2\beta_\lambda \tau^{1/2}\|u^{(k)}\|_{L^2(0,\tau;H^1)} + \beta_\lambda^2 \|u^{(k)}\|^2_{L^2(0,\tau;H^1)} \\
&= \left(\tau^{1/2}\alpha_\lambda + \beta_\lambda \|u^{(k)}\|_{L^2(0,\tau;H^1)}\right)^2
\end{aligned}$$

It follows that

$$\|\dot{u}^{(k)}\|_{L^2(0,\tau;H^1)} \leq \tau^{1/2}\alpha_\lambda + \beta_\lambda \|u^{(k)}\|_{L^2(0,\tau;H^1)} \leq \tau^{1/2}\left(\alpha_\lambda + \beta_\lambda \|u^{(k)}\|_{C([0,\tau];H^1)}\right)$$

and thus $\dot{u}^{(k)} \in L^2(0,\tau;H^1(\Gamma))$ and, by identical arguments, $f^{(k)} \in L^2(0,\tau;L^2(T\Gamma))$ and $p^{(k)} \in L^2(0,\tau;H^1(\Gamma))$. Since $\dot{u}^{(k)} \in L^2(0,\tau;H^1(\Gamma))$, we can apply Lem. 3.6 to deduce that $u^{(k+1)} \in C([0,\tau];H^1(\Gamma))$ too. By induction then, $u^{(k)} \in C([0,\tau];H^1(\Gamma))$ for all $k \geq 0$.

2. ($u^{(k)}$ bounded in $C([0,\tau],H^1(\Gamma))$): The application of Lem. 3.6 in the previous step gives us in addition the bound

$$\begin{aligned}\|u^{(k+1)}\|_{C([0,\tau];H^1)} &\leq \|u_0\|_{H^1} + \tau^{1/2}\|\dot{u}^{(k)}\|_{L^2(0,\tau;H^1)} \\ &\leq \|u_0\|_{H^1} + \tau\left(\alpha_\lambda + \beta_\lambda \|u^{(k)}\|_{C([0,\tau];H^1)}\right)\end{aligned}$$

We will inductively show that if $\tau\beta_\lambda < 1$, then

$$\|u^{(k)}\|_{C([0,\tau];H^1)} \leq \frac{\|u_0\|_{H^1} + \tau\alpha_\lambda}{1-\tau\beta_\lambda} =: R_1$$

for any $k \geq 0$. For $k = 0$, the inequality is clearly true, since $\alpha_\lambda \geq 0$ and $0 \leq \tau\beta_\lambda < 1$. Assuming that it holds for k, we have

$$\begin{aligned}\|u^{(k+1)}\|_{C([0,\tau];H^1)} &\leq \|u_0\|_{H^1} + \tau\left(\alpha_\lambda + \beta_\lambda \|u^{(k)}\|_{C([0,\tau];H^1)}\right) \\ &\leq \|u_0\|_{H^1} + \tau\alpha_\lambda + \tau\beta_\lambda \frac{\|u_0\|_{H^1} + \tau\alpha_\lambda}{1-\tau\beta_\lambda} = \frac{\|u_0\|_{H^1} + \tau\alpha_\lambda}{1-\tau\beta_\lambda}\end{aligned}$$

and so the bound holds for all k.

3. ($u_0 \in H^2(\Gamma) \Rightarrow u^{(k)} \in C([0,\tau];H^2(\Gamma))$): If $u_0 \in H^2(\Gamma) \Rightarrow u^{(0)} \in C([0,\tau];H^2(\Gamma))$, we can reuse the reasoning of step 1. to show that if $u^{(k)} \in C([0,\tau];H^2(\Gamma))$ then $\dot{u}^{(k)} \in L^2(0,\tau;H^2(\Gamma))$ and so $u^{(k+1)} \in C([0,\tau];H^2(\Gamma))$, which by induction gives us that indeed $u^{(k)} \in C([0,\tau];H^2(\Gamma))$ for all $k \geq 0$. The key difference is that here the second part of Cor. 3.16 is applicable (since $u^{(k)}(t) \in H^2(\Gamma)$ and the assumptions of Lem. 3.11 hold), and gives us the bound

$$\begin{aligned}&\|\dot{u}^{(k)}(t)\|_{H^2} + \|f^{(k)}(t)\|_{H_{\mathrm{div}}} + \|p^{(k)}(t)\|_{H^2} \\ &\qquad \leq A_\lambda(\|u^{(k)}(t)\|_{H^1}) + B_\lambda(\|u^{(k)}(t)\|_{H^1})\,\|u^{(k)}(t)\|_{H^2} \\ &\qquad \leq \alpha_R + \beta_R\|u^{(k)}(t)\|_{H^2}\end{aligned}$$

where $\alpha_R := A_\lambda(R_1)$ and $\beta_R := B_\lambda(R_1)$. The second inequality follows form the bound of step 2 and the fact that the functions A_λ and B_λ are monotonically increasing.

4. ($u^{(k)}$ bounded in $C([0,\tau];H^2(\Gamma))$): Likewise, we can retrace the reasoning of step 2 to show that if $\tau\beta_R < 1$, then

$$\|u^{(k)}\|_{C([0,\tau];H^2)} \leq \frac{\|u_0\|_{H^2} + \tau\alpha_R}{1-\tau\beta_R} =: R_2$$

for any $k \geq 0$.

5. ($u^{(k)}$ is a Cauchy sequence): Going back to step 1., we recall that for $k \geq 0$ and for almost all $t \in (0,\tau)$, $(f^{(k)}(t), \dot{u}^{(k)}(t))$ and $p^{(k)}(t)$ is a saddle point of the problem (3.37) with $u = u^{(k)}(t) \in H^2(\Gamma)$. We take the difference of the saddle point equations for k and for $k-1$ and, after some manipulations, we arrive at the system of equations

$$\begin{aligned}
&\langle \delta f, \mathcal{M}^{-1}_{[u^{(k)}]_\lambda} g\rangle_{L^2} - \langle g, \operatorname{grad}_\Gamma \delta p\rangle_{L^2} \\
&\qquad\qquad = -\langle \mathcal{M}^{-1}_{[u^{(k)}]_\lambda} f^{(k-1)} - \mathcal{M}^{-1}_{[u^{(k-1)}]_\lambda} f^{(k-1)}, g\rangle_{L^2} \\
&\qquad\qquad = -\langle \mathcal{M}^{-1}_{[u^{(k)}]_\lambda} \mathcal{M}_{[u^{(k-1)}]_\lambda} \operatorname{grad}_\Gamma p^{(k-1)} - \operatorname{grad}_\Gamma p^{(k-1)}, g\rangle_{L^2} \\
&\langle \theta, \delta p\rangle_{L^2} + \lambda \langle \operatorname{grad}_\Gamma \theta, \operatorname{grad}_\Gamma \delta p\rangle_{L^2} = -\langle \epsilon W_2\, \delta u - \epsilon \Delta_\Gamma \delta u, \theta\rangle_{L^2} \\
&\langle \delta \dot{u}, q\rangle_{L^2} + \lambda \langle \operatorname{grad}_\Gamma \delta \dot{u}, \operatorname{grad}_\Gamma q\rangle_{L^2} - \langle \delta f, \operatorname{grad}_\Gamma q\rangle_{L^2} = 0
\end{aligned}$$

where $\delta u := u^{(k)}(t) - u^{(k-1)}(t)$, $\delta \dot{u} := \dot{u}^{(k)}(t) - \dot{u}^{(k-1)}(t)$, $\delta f := f^{(k)}(t) - f^{(k-1)}(t)$ and $\delta p := p^{(k)}(t) - p^{(k-1)}(t)$. The second form of the right-hand side of the first equation follows from the saddle point equation $\langle f^{(k-1)}, \mathcal{M}^{-1}_{[u^{(k-1)}]_\lambda} g\rangle_{L^2} = \langle g, \operatorname{grad}_\Gamma p^{(k-1)}\rangle_{L^2}$, which $f^{(k-1)}$ satisfies. The equations above are exactly the saddle point equations of an optimization problem of the form (3.29), which furthermore satisfies the conditions for H^2-regularity (Prop. 3.15) with

$$\begin{aligned}
e &= \epsilon W_2\, \delta u - \epsilon \Delta_\Gamma \delta u \\
j &= \mathcal{M}^{-1}_{[u^{(k)}]_\lambda} \mathcal{M}_{[u^{(k-1)}]_\lambda} \operatorname{grad}_\Gamma p^{(k-1)} - \operatorname{grad}_\Gamma p^{(k-1)} \\
r &= 0
\end{aligned}$$

We estimate the norms $\|e\|_{L^2} \leq \epsilon(1 + \|W_2\|_{L^\infty})\|\delta u\|_{H^2}$ and, due to Lem 3.11,

$$\begin{aligned}
\|\mathcal{M}_{[u^{(k)}]_\lambda} j\|_{H_{\mathrm{div}}} &= \|\mathcal{M}_{[u^{(k-1)}]_\lambda} \operatorname{grad}_\Gamma p^{(k-1)} - \mathcal{M}_{[u^{(k)}]_\lambda} \operatorname{grad}_\Gamma p^{(k-1)}\|_{H_{\mathrm{div}}} \\
&\leq L_{\mathcal{M}}(\|u^{(k)}\|_{H^1}, \|u^{(k-1)}\|_{H^1}) \|p^{(k-1)}\|_{H^2} \|u^{(k)} - u^{(k-1)}\|_{H^2} \\
&\leq L_{\mathcal{M}}(R_1, R_1) \left(\alpha_R + \beta_R \|u^{(k-1)}\|_{H^2}\right) \|\delta u\|_{H^2} \\
&\leq L_{\mathcal{M}}(R_1, R_1) \left(\alpha_R + \beta_R R_2\right) \|\delta u\|_{H^2}
\end{aligned}$$

where we used the various bounds from the previous steps together with the fact that $L_{\mathcal{M}}$ is an increasing function. Combining with the bound (3.36) (with $r = 0$), we have

$$\begin{aligned}\|\delta\dot{u}\|_{H^2} + \|\delta f\|_{H_{\mathrm{div}}} + \|\delta p\|_{H^2} &\leq \gamma_\lambda^2 C_{\mathcal{M}}(\|u^{(k)}\|_{H^1})\|e\|_{L^2} + \gamma_\lambda \|\mathcal{M}_{[u^{(k)}]_\lambda} j\|_{H_{\mathrm{div}}} \\ &\leq L_R \|\delta u\|_{H^2}\end{aligned}$$

where $L_R \equiv L_R(R_1, R_2) := \epsilon\gamma_\lambda^2 C_{\mathcal{M}}(R_1)(1+\|W_2\|_{L^\infty})+\gamma_\lambda L_{\mathcal{M}}(R_1, R_1)\,(\alpha_R + \beta_R R_2)$ is an increasing function of both R_1 and R_2, and depends only on the various parameters and on u_0.

Now we are ready to show that the $u^{(k)}$ constitute a Cauchy sequence . We have by construction

$$u^{(k+1)}(t) - u^{(k)}(t) = \int_0^t \left(\dot{u}^{(k)}(s) - \dot{u}^{(k-1)}(s)\right)\, ds, \quad t \in [0, \tau]$$

and therefore the function $u^{(k+1)} - u^{(k)} \in C([0, \tau]; H^2(\Gamma))$ satisfies the conditions of Lem. 3.6 with $u_0 = 0$ and $\dot{u} = \dot{u}^{(k)} - \dot{u}^{(k-1)} \in L^2(0, \tau; H^2(\Gamma))$. It follows that

$$\begin{aligned}\|u^{(k+1)} - u^{(k)}\|_{L^2(0,\tau;H^2)} &\leq \tau\|\dot{u}^{(k)} - \dot{u}^{(k-1)}\|_{L^2(0,\tau;H^2)} \\ &= \tau\left(\int_0^\tau \|\delta\dot{u}\|_{H^2}^2\right)^{1/2} \leq \tau L_R \left(\int_0^\tau \|\delta u\|_{H^2}^2\right)^{1/2} \\ &= \tau L_R \|u^{(k)} - u^{(k-1)}\|_{L^2(0,\tau;H^2)}\end{aligned}$$

If $\tau L_R < 1$, then the sequence is indeed Cauchy.

6. (existence of a fixed point): Since the sequence $u^{(k)}$ is Cauchy in the complete space $L^2(0, \tau; H^2(\Gamma))$, it converges to a $u \in L^2(0, \tau; H^2(\Gamma))$. Moreover, u is a *fixed point* of the iteration $u^{(k+1)}(t) = u_0 + \int_0^t \dot{u}^{(k)}(s)\, ds$ and so, as can be shown by walking through the various steps above one more time, there exist $f \in L^2(0, \tau; H_{\mathrm{div}}(T\Gamma))$, $p \in L^2(0, \tau; H^2(\Gamma))$, $\dot{u} \in L^2(0, \tau; H^2(\Gamma))$ such that for almost all $t \in (0, \tau)$ the triplet $(f(t), \dot{u}(t), p(t))$ is the unique saddle point of the problem (3.37) with $u = u(t)$. Hence, by Lem. 3.6, $u \in C([0, \tau]; H^2(\Gamma))$ with $u' = \dot{u}$ and $u(0) = u_0$.

From the properties of the saddle points of (3.37) follows that $\|\dot{u}\|_{H^1} \leq \alpha_\lambda + \beta_\lambda\|u(t)\|_{H^1}$ for almost all $t \in (0, T)$ and so, by (3.18) of Lem. 3.6, the fixed point $u(t) = u_0 + \int_0^t \dot{u}(s)\, ds$ satisfies the bound

$$\|u\|_{C([0,\tau];H^1)} \leq (\|u_0\|_{H^1} + \alpha_\lambda\tau)e^{\beta_\lambda\tau} =: M$$

The conditions for H^2-regularity are also met, and so $\|\dot{u}(t)\|_{H^2} \leq A_\lambda(\|u(t)\|_{H^1}) + B_\lambda(\|u(t)\|_{H^1})\|u(t)\|_{H^2} \leq A_\lambda(M) + B_\lambda(M)\|u(t)\|_{H^2}$ for almost all $t \in (0, T)$. Applying again Lem. 3.6, yields the second bound

$$\|u\|_{C([0,\tau];H^2)} \leq (\|u_0\|_{H^2} + A_\lambda(M)\tau)e^{B_\lambda(M)\tau}$$

Finally, for the precise bound on τ, we note that if $\|u_0\|_{H^2} \leq U$ (and so $\|u_0\|_{H^1} \leq U$) and $\tau \leq T$, we have that $R_1 \leq U+\alpha_\lambda T =: \bar{R}_1(U,T)$ and $R_2 \leq U+A_\lambda(\bar{R}_1)T =: \bar{R}_2(U,T)$. Defining $C(U,T) := \max\big(\beta_\lambda, B_\lambda(\bar{R}_1), L_R(\bar{R}_1, \bar{R}_2)\big)$, it is straightforward to verify that when $\tau < C(U,T)^{-1}$ then $\tau\beta_\lambda$ (step 2), $\tau B_\lambda(R_1)$ (step 4) and $\tau L_R(R_1, R_2)$ (step 6) are indeed all less than 1.

□

Corollary 3.18 (Global existence of solutions)**.** *Let $u_0 \in H^2(\Gamma)$ and assume that the conditions of Lem. 3.11 are met. Then for any $T > 0$, there exist $u \in C([0,T]; H^2(\Gamma)) \cap H^1(0,T; H^2(\Gamma))$ and $f \in L^2(0,T; H_{\mathrm{div}}(T\Gamma))$, $\dot{u} \in L^2(0,T; H^2(\Gamma))$, $p \in L^2(0,T; H^2(\Gamma))$ so that the conclusions of Prop. 3.17 are valid over the entire interval $[0,T]$. In particular, the following a priori bounds hold:*

$$\|u\|_{C([0,T];H^1)} \leq (\|u_0\|_{H^1} + \alpha_\lambda T)e^{\beta_\lambda T} =: M_1 \tag{3.43a}$$

$$\|u\|_{C([0,T];H^2)} \leq (\|u_0\|_{H^2} + A_\lambda(M_1)T)e^{B_\lambda(M_1)T} =: M_2 \tag{3.43b}$$

Proof. We will show that we can partition the interval $[0,T]$ in a finite number of subintervals, small enough so that the local existence result (Prop. 3.17) applies. Let $\tau := \frac{T}{N}$ with $N \in \mathbb{N}$ large enough so that $\tau < C(M_2, T)^{-1}$. The aforementioned partition is then $0 = t_0 < t_1 = \tau < \ldots < t_N = \tau N \equiv T$.

For the first interval $[t_0, t_1] = [0, \tau]$, we have that $\|u(t_0)\|_{H^2} = \|u_0\|_{H^2} \leq M_2$ (and of course $\tau \leq T$). Since moreover $\tau < C(M_2, T)^{-1}$ by construction, there exists a local solution $u \in C([0,\tau]; H^2(\Gamma))$ with

$$\begin{aligned}
&\|u\|_{C([0,\tau];H^1)} \leq (\|u_0\|_{H^1} + \alpha_\lambda \tau)e^{\beta_\lambda \tau} =: M \leq M_1 \\
&\|u\|_{C([0,\tau];H^2)} \leq (\|u_0\|_{H^2} + A_\lambda(M)\tau)e^{B_\lambda(M)\tau} \leq (\|u_0\|_{H^2} + A_\lambda(M_1)T)\, e^{B_\lambda(M_1)T} = M_2
\end{aligned}$$

and so $\|u(t_1)\|_{H^2} \equiv \|u(\tau)\|_{H^2} \leq \|u\|_{C([0,\tau];H^2)} \leq M_2$.

For the second interval $[t_1, t_2]$, we have again that $\|u(t_1)\|_{H^2} \leq M_2$ and $t_2 - t_1 = \tau \leq T$ with $\tau < C(M_2, T)^{-1}$. Applying the local existence result to the shifted function $\tilde{u}(t) = u(t + t_1)$, $t \in [0, \tau]$, yields a local solution $u(t) = u(t_1) + \int_{t_1}^t \dot{u}(s)\, ds \in C([t_1, t_2]; H^2(\Gamma))$. This expands the local solution of the first interval in a continuous manner to a single solution $u \in C([t_0, t_2]; H^2(\Gamma))$ which by Lem. 3.6 satisfies the bounds

$$\begin{aligned}
&\|u\|_{C([t_0,t_2];H^1)} \leq (\|u_0\|_{H^1} + 2\alpha_\lambda \tau)e^{2\beta_\lambda \tau} =: M \leq M_1 \\
&\|u\|_{C([t_0,t_2];H^2)} \leq (\|u_0\|_{H^2} + 2A_\lambda(M)\tau)e^{2B_\lambda(M)\tau} \leq (\|u_0\|_{H^2} + A_\lambda(M_1)T)\, e^{B_\lambda(M_1)T} = M_2
\end{aligned}$$

Then $\|u(t_2)\|_{H^2} \leq \|u\|_{C([t_0,t_2];H^2)} \leq M_2$ and we can continue the process to the next interval. At the interval $[t_k, t_{k+1}]$, and given that $\|u(t_k)\|_{H^2} \leq M_2$, we use the local existence result to expand the solution to $C([t_0, t_{k+1}]; H^2(\Gamma))$ with bounds which ensure that $\|u(t_{k+1})\|_{H^2} \leq M_2$. Note that M_1 and M_2, which we defined in an *ad hoc* manner, are exactly the bounds for $\|u\|_{C([0,T];H^1)}$ and $\|u\|_{C([0,T];H^2)}$ that we get at the final subinterval.

□

Corollary 3.19 (Uniqueness of the solution). *Let $u_0 \in H^2(\Gamma)$ and $T > 0$, and assume that the conditions of Lem. 3.11 are met. If $u_1, u_2 \in C([0,T]; H^2(\Gamma)) \cap H^1(0,T; H^2(\Gamma))$, with corresponding $f_i \in L^2(0,T; H_{\mathrm{div}}(T\Gamma))$, $\dot{u}_i \in L^2(0,T; H^2(\Gamma))$, $p_i \in L^2(0,T; H^2(\Gamma))$, are solutions in the sense of Cor. 3.18, then $u_1 = u_2$.*

Proof. Using exactly the same reasoning as in step 5. of the proof of Prop. 3.17, we can show that for almost all $t \in (0,T)$,

$$\|\dot{u}_1(t) - \dot{u}_2(t)\|_{H^2} + \|f_1(t) - f_2(t)\|_{H_{\mathrm{div}}} + \|p_1(t) - p_2(t)\|_{H^2} \leq L_R(M_1, M_2)\|u_1(t) - u_2(t)\|_{H^2}$$

Note that we used the bounds (3.43). Then for the function $\delta u := u_1 - u_2$ we have that $\delta u(t) = \int_0^t \delta\dot{u}(s)\, ds$, $t \in [0,T]$, where $\delta\dot{u} := \dot{u}_1 - \dot{u}_2 \in L^2(0,T; H^2(\Gamma))$. The Lem. 3.6 is applicable and, since we just showed that $\|\delta\dot{u}(t)\|_{H^2} \leq L_R(M_1, M_2)\|\delta u(t)\|_{H^2}$ for almost all $t \in (0,T)$, the bound (3.18) gives us

$$\|\delta u\|_{C([0,T];H^2)} \leq (\|\delta u(0)\|_{H^2} + 0)\; e^{L_R(M_1,M_2)T} = 0$$

and so $\delta u = 0 \Rightarrow u_1 = u_2$.

□

Corollary 3.20 (Volume conservation). *Let u be the solution of Cor. 3.18. Then the total volume* $\mathrm{vol}(u) := \int_\Gamma u \,\mathrm{vol}_\Gamma$ *is conserved.*

Proof. Let $V(t) := \mathrm{vol}(u(t))$. Then

$$V'(t) = \frac{d}{dt}\mathrm{vol}(u(t)) = \frac{d}{dt}\int_\Gamma u(t)\,\mathrm{vol}_\Gamma = \int_\Gamma \dot{u}(t)\,\mathrm{vol}_\Gamma = \langle \dot{u}(t), 1\rangle_{L^2}$$

Setting $q = 1$ in (3.41c) gives us exactly that $V'(t) = \langle \dot{u}(t), 1\rangle_{L^2} = 0$ for almost all $t \in (0,T)$, and so $V(t) = V_0 + \int_0^t V'(s)\, ds = V_0$.

□

Corollary 3.21 (Energy reduction). *Let u be the solution of Cor. 3.18, then the free energy $\mathcal{E}(u)$ is non-increasing. More specifically,*

$$\frac{d}{dt}\mathcal{E}(u(t)) \leq -\mu_\lambda \|\mathrm{grad}_\Gamma\, p(t)\|_{L^2}^2 \tag{3.44}$$

where μ_λ is the coercivity constant of $\mathcal{M}_{[u]_\lambda}$ (which is independent of u).

Proof. Setting $g = \mathcal{M}_{[u(t)]_\lambda} \operatorname{grad}_\Gamma p(t)$ in (3.41a), $\theta = \dot{u}(t)$ in (3.41b) and $q = p(t)$ in (3.41c), we get the system (dropping the time dependency for clarity)

$$\begin{aligned}
\langle f, \operatorname{grad}_\Gamma p\rangle_{L^2} &= \langle \operatorname{grad}_\Gamma p, \mathcal{M}_{[u]_\lambda} \operatorname{grad}_\Gamma p\rangle_{L^2} \\
\langle \dot{u}, p\rangle_{L^2} + \lambda\langle \operatorname{grad}_\Gamma \dot{u}, \operatorname{grad}_\Gamma p\rangle_{L^2} &= -\mathcal{E}'(u)(\dot{u}) \\
\langle \dot{u}, p\rangle_{L^2} + \lambda\langle \operatorname{grad}_\Gamma \dot{u}, \operatorname{grad}_\Gamma p\rangle_{L^2} &= \langle f, \operatorname{grad}_\Gamma p\rangle_{L^2}
\end{aligned}$$

This directly gives $\mathcal{E}'(u)(\dot{u}) = -\langle \operatorname{grad}_\Gamma p, \mathcal{M}_{[u]_\lambda} \operatorname{grad}_\Gamma p\rangle_{L^2}$ for almost all $t \in (0, T)$. The inequality (3.44) follows from the coercivity of $\mathcal{M}_{[u]_\lambda}$ and the fact that $\frac{d}{dt}\mathcal{E}(u) = \mathcal{E}'(u)(\dot{u})$. □

3.5. Time discretization

Recall from the local existence result (Prop. 3.17) that the solution to our problem with initial condition $u(0) = u_0$ over an interval $[0, \tau]$ is determined by the following equations:

1. $u(t) = u_0 + \int_0^t \dot{u}(s)\, ds$, $t \in [0, \tau]$
2. for almost all $t \in (0, T)$, $(f(t), \dot{u}(t), p(t))$ is a solution of

$$\begin{aligned}
&\langle f(t), \mathcal{M}^{-1}_{[u(t)]_\lambda} g\rangle_{L^2} = \langle g, \operatorname{grad}_\Gamma p(t)\rangle_{L^2}, \quad \forall g \in L^2(T\Gamma) \\
&\langle p(t), \theta\rangle_{L^2} + \lambda\langle \operatorname{grad}_\Gamma p(t), \operatorname{grad}_\Gamma \theta\rangle_{L^2} = -\mathcal{E}'(u(t))(\theta), \quad \forall \theta \in H^1(\Gamma) \\
&\langle \dot{u}(t), q\rangle_{L^2} + \lambda\langle \operatorname{grad}_\Gamma \dot{u}(t), \operatorname{grad}_\Gamma q\rangle_{L^2} = \langle f(t), \operatorname{grad}_\Gamma q\rangle_{L^2}, \quad \forall q \in H^1(\Gamma)
\end{aligned}$$

We consider the following *time-discrete scheme*:

1. $u_\tau(t) = u_0 + \int_0^t \dot{u}_\tau(s)\, ds = u_0 + t\, \dot{u}_\tau$, $t \in [0, \tau]$
2. $(f_\tau, \dot{u}_\tau, p_\tau)$ are constant in time and solve the system:

$$\begin{aligned}
&\langle f_\tau, \mathcal{M}^{-1}_{[u_0]_\lambda} g\rangle_{L^2} = \langle g, \operatorname{grad}_\Gamma p_\tau\rangle_{L^2}, \quad \forall g \in L^2(T\Gamma) \\
&\langle p_\tau, \theta\rangle_{L^2} + \lambda\langle \operatorname{grad}_\Gamma p_\tau, \operatorname{grad}_\Gamma \theta\rangle_{L^2} = -\mathcal{E}'(u_0 + \tau\, \dot{u}_\tau)(\theta), \quad \forall \theta \in H^1(\Gamma) \\
&\langle \dot{u}_\tau, q\rangle_{L^2} + \lambda\langle \operatorname{grad}_\Gamma \dot{u}_\tau, \operatorname{grad}_\Gamma q\rangle_{L^2} = \langle f_\tau, \operatorname{grad}_\Gamma q\rangle_{L^2}, \quad \forall q \in H^1(\Gamma)
\end{aligned}$$

Note the semi-implicit nature of the scheme, since the rate of change of the energy $\mathcal{E}'(u(t))$ is discretized with $u(t) = u_\tau(\tau)$ (implicit discretization), whereas the mobility $\mathcal{M}_{[u(t)]_\lambda}$ is discretized with $u(t) = u_\tau(0)$ (explicit discretization).

Remark 3.22. *The motivation for this time-discrete scheme is the notion of the* natural time discretization *of a gradient flow, as presented in [Ott01] for the porous medium* *equation. For a gradient flow of the (weak) form $a(\dot{u}, v) = -E'(u)(v)$, the natural time discretization (from t^k to $t^{k+1} = t^k + \tau$) is*

$$u^{k+1} = \operatorname*{argmin}_u \left\{ \frac{1}{2\tau} \operatorname{dist}^2(u^k, u) + E(u) \right\}$$
$$\textit{with } \operatorname{dist}^2(u_1, u_2) = \min_{\substack{u(0)=u_1 \\ u(1)=u_2}} \int_0^1 a(u'(s), u'(s))\, ds$$

Limiting ourselves to linear paths $u(s) = u_1 + s(u_2 - u_1)$ and reparametrizing from the interpolation parameter s to the physical time t, we get the approximation

$$\dot{u} = \operatorname*{argmin}_v \left\{ \frac{\tau}{2} a(v, v) + E(u^k + \tau v) \right\}$$
$$u^{k+1} = u^k + \tau \dot{u}$$

on which the time-discrete scheme is based. See §3.1 of [VR13] for more details.

Before we focus on the time-discrete scheme above, we will first prove two results for a more general optimization problem:

Proposition 3.23 (A time-discrete optimization problem)**.** *Consider the optimization problem*

$$\min_{\substack{(f,\dot{u}) \in \\ L^2(T\Gamma) \times H^1(\Gamma)}} \left\{ \frac{1}{2} \langle f, \mathcal{M}^{-1}_{[u_0]_\lambda} f \rangle_{L^2} + \frac{\tau\epsilon}{2} \langle \dot{u}, W_2 \dot{u} \rangle_{L^2} + \frac{\tau\epsilon}{2} \|\operatorname{grad}_\Gamma \dot{u}\|^2_{L^2} + E((f, \dot{u})) \right\} \tag{3.45a}$$

$$\langle \dot{u}, q \rangle_{L^2} + \lambda \langle \operatorname{grad}_\Gamma \dot{u}, \operatorname{grad}_\Gamma q \rangle_{L^2} - \langle f, \operatorname{grad}_\Gamma q \rangle_{L^2} = R(q),\ \forall q \in H^1(\Gamma) \tag{3.45b}$$

where $u_0 \in H^1(\Gamma)$, $W_1 \in L^2(\Gamma)$ and $W_2 \in L^\infty(\Gamma)$, and $E \in X', R \in Q'$ are continuous linear functionals on $X := L^2(T\Gamma) \times H^1(\Gamma)$ and $Q := H^1(\Gamma)$ respectively. We also assume that the conditions of Cor. 3.10 are met, and that there exist positive constants $\overline{W}_2, K > 0$ such that $\|W_2\|_{L^\infty} \leq \overline{W}_2$ and $\tau\epsilon \leq K$.

1. *If τ is small enough, so that*
$$K < \frac{\mu_\lambda M_\lambda^{-2}}{|\overline{W}_2|^2} \tag{3.46}$$
then there exists a unique solution $(f_\tau, \dot{u}_\tau) \in L^2(T\Gamma) \times H^1(\Gamma)$ with a unique multiplier $p_\tau \in H^1$, and a constant $C_\tau > 0$, such that
$$\|\dot{u}_\tau\|_{H^1} + \|f_\tau\|_{L^2} + \|p_\tau\|_{H^1} \leq C_\tau \left(\|E\|_{X'} + \|R\|_{Q'} \right) \tag{3.47}$$

2. *If τ is small enough, so that*

$$K < \frac{\lambda^2 \mu_\lambda M_\lambda^{-2}}{\overline{W}_2} \tag{3.48}$$

then there exists a unique solution $(f_\tau, \dot{u}_\tau) \in L^2(T\Gamma) \times H^1(\Gamma)$ with a unique multiplier $p_\tau \in H^1$, and a constant $C_\lambda > 0$, independent of τ, *such that*

$$\|\dot{u}_\tau\|_{H^1} + \|f_\tau\|_{L^2} + \|p_\tau\|_{H^1} \le C_\lambda \left(\|E\|_{X'} + \|R\|_{Q'} \right) \tag{3.49}$$

Proof. As in the proof of Prop. 3.12, we will apply Prop. 2.13 with $X = L^2(T\Gamma) \times H^1(\Gamma)$ with norm $\|(f, \dot{u})\|_X = \left(\|f\|_{L^2}^2 + \|\dot{u}\|_{H^1}^2 \right)^{1/2}$, $Q = H^1(\Gamma)$ and

$$\begin{aligned}
a((f, \dot{u}), (f', \dot{u}')) &:= \langle f, \mathcal{M}_{[u_0]_\lambda}^{-1} f' \rangle_{L^2} + \tau\epsilon \langle \dot{u}, W_2 \dot{u}' \rangle_{L^2} + \tau\epsilon \langle \operatorname{grad}_\Gamma \dot{u}, \operatorname{grad}_\Gamma \dot{u}' \rangle_{L^2} \\
e((f, \dot{u})) &:= E((f, \dot{u})) \\
b((f, \dot{u}), q) &:= \langle \dot{u}, q \rangle_{L^2} + \lambda \langle \operatorname{grad}_\Gamma \dot{u}, \operatorname{grad}_\Gamma q \rangle_{L^2} - \langle f, \operatorname{grad}_\Gamma q \rangle_{L^2} \\
g(q) &:= R(q)
\end{aligned}$$

for any $f, f' \in L^2(T\Gamma)$ and $\dot{u}, \dot{u}', q \in H^1(\Gamma)$. We verify that the conditions of the proposition are met:

- *$a(\cdot, \cdot)$ is continuous:*

$$\begin{aligned}
|a((f, \dot{u}), (f', \dot{u}'))| &\le \mu_\lambda^{-1} \|f\|_{L^2} \|f'\|_{L^2} \\
&\quad + \tau\epsilon \|W_2\|_{L^\infty} \|\dot{u}\|_{L^2} \|\dot{u}'\|_{L^2} + \tau\epsilon \|\operatorname{grad}_\Gamma \dot{u}\|_{L^2} \|\operatorname{grad}_\Gamma \dot{u}'\|_{L^2} \\
&\le \max(\mu_\lambda^{-1}, K(1 + \|W_2\|_{L^\infty})) \|(f, \dot{u})\|_X \|(f', \dot{u}')\|_X
\end{aligned}$$

  since $\|f\|_{L^2} \|f'\|_{L^2} + \|\dot{u}\|_{H^1} \|\dot{u}'\|_{H^1} \le \left(\|f\|_{L^2}^2 + \|\dot{u}\|_{H^1}^2 \right)^{1/2} \left(\|f'\|_{L^2}^2 + \|\dot{u}'\|_{H^1}^2 \right)^{1/2}$ by Hölder's inequality. The continuity constant is $\|a\| = \max(\mu_\lambda^{-1}, K(1 + \|W_2\|_{L^\infty}))$.

- *$b(\cdot, \cdot)$ is continuous:*

$$\begin{aligned}
|b((f, \dot{u}), q)| &\le \|\dot{u}\|_{L^2} \|q\|_{L^2} + \lambda \|\operatorname{grad}_\Gamma \dot{u}\|_{L^2} \|\operatorname{grad}_\Gamma q\|_{L^2} + \|f\|_{L^2} \|\operatorname{grad}_\Gamma q\|_{L^2} \\
&\le (2 + \lambda) \|(f, \dot{u})\|_X \|q\|_{H^1}
\end{aligned}$$

  and the continuity constant is $\|b\| = 2 + \lambda$.

- *$b(\cdot, \cdot)$ satisfies the LBB condition:* We note that for an arbitrary $q \in H^1(\Gamma)$, $q \neq 0$, the pair $(-\operatorname{grad}_\Gamma q, q) \in L^2(T\Gamma) \times H^1(\Gamma) = X$. Then

$$\begin{aligned}
b((-\operatorname{grad}_\Gamma q, q), q) &= \|q\|_{L^2}^2 + \lambda \|\operatorname{grad}_\Gamma q\|_{L^2}^2 + \|\operatorname{grad}_\Gamma q\|_{L^2}^2 \\
&\ge \frac{1}{2} (\|q\|_{L^2}^2 + 2 \|\operatorname{grad}_\Gamma q\|_{L^2}^2) = \frac{1}{2} \|(-\operatorname{grad}_\Gamma q, q)\|_X^2
\end{aligned}$$

and so

$$\sup_{p\in X\setminus\{0\}} \frac{b(p,q)}{\|p\|_X\|q\|_Q} \geq \frac{b((-\operatorname{grad}_\Gamma q,q),q)}{\|(-\operatorname{grad}_\Gamma q,q)\|_X\|q\|_{H^1}} \geq \frac{b((-\operatorname{grad}_\Gamma q,q),q)}{\|(-\operatorname{grad}_\Gamma q,q)\|_X^2} = \frac{1}{2}$$

since $\|(-\operatorname{grad}_\Gamma q,q)\|_X \geq \|q\|_{H^1}$. It follows that $b(\cdot,\cdot)$ indeed satisfies the LBB condition (2.8) with constant $\beta = \frac{1}{2}$.

- *$a(\cdot,\cdot)$ is Z-coercive, case 1.:* From the coercivity of $\mathcal{M}^{-1}_{[u_0]}$, we have

$$\begin{aligned} a((f,\dot{u}),(f,\dot{u})) &\geq \mu_\lambda M_\lambda^{-2}\|f\|^2_{L^2} - \tau\epsilon\|W_2\|_{L^\infty}\|\dot{u}\|^2_{L^2} + \tau\epsilon\|\operatorname{grad}_\Gamma \dot{u}\|^2_{L^2} \\ &\geq \mu_\lambda M_\lambda^{-2}\|f\|^2_{L^2} - 2\tau\epsilon\overline{W}_2\|\dot{u}\|^2_{L^2} + \tau\epsilon\|\operatorname{grad}_\Gamma \dot{u}\|^2_{L^2} + \tau\epsilon\overline{W}_2\|\dot{u}\|^2_{L^2} \end{aligned}$$

  Substituting $q = \dot{u}$ into the constraint, we get

$$\|\dot{u}\|^2_{L^2} + \lambda\|\operatorname{grad}_\Gamma \dot{u}\|^2_{L^2} = \langle f, \operatorname{grad}_\Gamma \dot{u}\rangle_{L^2} \Rightarrow \|\dot{u}\|^2_{L^2} \leq \|f\|_{L^2}\|\operatorname{grad}_\Gamma\dot{u}\|_{L^2}$$

  and so

$$\begin{aligned} &\mu_\lambda M_\lambda^{-2}\|f\|^2_{L^2} - 2\tau\epsilon\overline{W}_2\|\dot{u}\|^2_{L^2} + \tau\epsilon\|\operatorname{grad}_\Gamma \dot{u}\|^2_{L^2} \\ &\qquad \geq \mu_\lambda M_\lambda^{-2}\|f\|^2_{L^2} - 2\tau\epsilon\overline{W}_2\|f\|_{L^2}\|\operatorname{grad}_\Gamma\dot{u}\|_{L^2} + \tau\epsilon\|\operatorname{grad}_\Gamma \dot{u}\|^2_{L^2} \\ &\qquad = \begin{pmatrix} \|f\|_{L^2} \\ \|\operatorname{grad}_\Gamma \dot{u}\|_{L^2} \end{pmatrix}^T \begin{pmatrix} \mu_\lambda M_\lambda^{-2} & -\tau\epsilon\overline{W}_2 \\ -\tau\epsilon\overline{W}_2 & \tau\epsilon \end{pmatrix} \begin{pmatrix} \|f\|_{L^2} \\ \|\operatorname{grad}_\Gamma \dot{u}\|_{L^2} \end{pmatrix} \\ &\qquad\qquad \geq \lambda_1(\|f\|^2_{L^2} + \|\operatorname{grad}_\Gamma \dot{u}\|^2_{L^2}) \end{aligned}$$

  where λ_1 is the smallest eigenvalue of the 2×2 matrix, let us call it A. The condition (3.46) implies immediately that $\det A > 0$ and, given that $\operatorname{tr} A = \mu_\lambda M_\lambda^{-2} + \tau\epsilon > 0$, it follows that the two eigenvalues[1] of A are both positive. Hence $\lambda_1 > 0$ and we arrive at the coercivity inequality

$$a((f,\dot{u}),(f,\dot{u})) \geq \lambda_1(\|f\|^2_{L^2} + \|\operatorname{grad}_\Gamma \dot{u}\|^2_{L^2}) + \tau\epsilon\overline{W}_2\|\dot{u}\|^2_{L^2} \geq \alpha_\tau\|(f,\dot{u})\|^2_X$$

  with the coercivity constant $\alpha_\tau := \min(\lambda_1, \tau\epsilon\overline{W}_2) > 0$.

- *$a(\cdot,\cdot)$ is Z-coercive, case 2.:* Using the same reasoning as in the corresponding part of the proof of Prop. 3.12, we can show that $\|f\|^2_{L^2} \geq \lambda^2\|\dot{u}\|^2_{H^1}$. By the coercivity of $\mathcal{M}^{-1}_{[u_0]_\lambda}$ and with the help of the condition (3.48), we have

$$\begin{aligned} a((f,\dot{u}),(f,\dot{u})) &\geq \mu_\lambda M_\lambda^{-2}\|f\|^2_{L^2} - \tau\epsilon\|W_2\|_{L^\infty}\|\dot{u}\|^2_{L^2} + \tau\epsilon\|\operatorname{grad}_\Gamma \dot{u}\|^2_{L^2} \\ &\geq \mu_\lambda M_\lambda^{-2}\|f\|^2_{L^2} - \tau\epsilon\overline{W}_2\|\dot{u}\|^2_{H^1} \\ &\geq (1-\sigma)\mu_\lambda M_\lambda^{-2}\|f\|^2_{L^2} + \sigma\mu_\lambda M_\lambda\|f\|^2_{L^2} - \tau\epsilon\overline{W}_2\|\dot{u}\|^2_{H^1} \\ &\geq (1-\sigma)\mu_\lambda M_\lambda^{-2}\|f\|^2_{L^2} + (\sigma\mu_\lambda M_\lambda\lambda^2 - K\overline{W}_2)\|\dot{u}\|^2_{H^1} \\ &\geq (1-\sigma)\mu_\lambda M_\lambda^{-2}\|f\|^2_{L^2} + (\sigma - \Sigma)\lambda^2\mu_\lambda M_\lambda^{-2}\|\dot{u}\|^2_{H^1} \end{aligned}$$

[1] A is symmetric and so the eigenvalues are real.

where $\Sigma = \frac{K\overline{W}_2}{\mu_\lambda M_\lambda^{-2}\lambda^2} < 1$ by the assumption (3.48). Choosing $\sigma = \frac{1+\lambda^2\Sigma}{1+\lambda^2}$, gives us $(1-\sigma) = (\sigma - \Sigma)\lambda^2 = \frac{(1-\Sigma)\lambda^2}{1+\lambda^2}$, and so we arrive at the coercivity inequality

$$a((f,\dot{u}),(f,\dot{u})) \geq \alpha_\lambda \left(\|f\|_{L^2}^2 + \|\dot{u}\|_{H^1}^2\right) = \alpha_\lambda \|(f,\dot{u})\|_X^2$$

where the coercivity constant $\alpha_\lambda := \mu_\lambda M_\lambda^{-2}\frac{(1-\Sigma)\lambda^2}{1+\lambda^2}$ is strictly positive, since $\Sigma < 1$, and is independent of τ.

We conclude that the problem admits a unique solution $(f_\tau, \dot{u}_\tau)$ with a unique Lagrange multiplier p_τ. By the bounds (2.12) of the Brezzi splitting theorem, there exists a function $C(\alpha, \|a\|, \beta, \|b\|)$ such that

$$\|\dot{u}_\tau\|_{H^1} + \|f_\tau\|_{L^2} + \|p_\tau\|_{H^1} \leq C(\alpha, \|a\|, \beta, \|b\|)\left(\|E\|_{X'} + \|R\|_{Q'}\right)$$

and, hence, $C_\lambda := C(\alpha_\lambda, \|a\|, \beta, \|b\|)$ and $C_\tau := C(\alpha_\tau, \|a\|, \beta, \|b\|)$ respectively. Finally, the independence of C_λ from τ follows from the independence of α_λ and the other constants from τ.

□

Remark 3.24. *As will be shown later, the time-discrete scheme corresponds to* $E((f,\dot{u})) = \mathcal{E}'(u_0)(\theta)$ *and* $R = 0$. *The first case of Prop. 3.23 shows that, for small enough* τ, *the regularizing effect of the* Dirichlet energy *term* $\frac{\tau\epsilon}{2}\|\operatorname{grad}_\Gamma \dot{u}\|_{L^2}^2$, *combined with the constraint, gives us the desired coercivity in* H^1. *The bound* (3.46), *especially after noting that it can be improved by using* $\max(\mu, \mu_\lambda)$ *and* $\min(M, M_\lambda)$ *(with* μ, M *as in 3.8) instead of* μ_λ, M_λ, *gives us a useful upper bound for* τ *for numerical applications.*

The problem is that the coercivity constant $\alpha_\tau \to 0$ *as* $\tau \to 0$ *and so* $C_\tau \to +\infty$ *in* (3.47) *and the sequence of solutions* $(f_\tau, \dot{u}_\tau)$ *is unbounded, making a convergence proof difficult. This is where the regularization becomes necessary. For a fixed* $\lambda > 0$ *and for small enough* τ *so that the condition* (3.48) *holds, we get coercivity with a coercivity constant* $\alpha_\lambda > 0$ *independent of* τ. *This shows us that the* $(f_\tau, \dot{u}_\tau)$ *indeed remain bounded as* $\tau \to 0$.

Proposition 3.25 (H^2-regularity of the time-discrete opt. problem)**.** *We assume that there exist* $e \in L^2(\Gamma)$, $j \in H_{\text{div}}(\Gamma)$ *and* $r \in L^2(\Gamma)$ *such that*

$$E((f,\dot{u})) = \langle j, f\rangle_{L^2} + \langle e, \dot{u}\rangle_{L^2}$$
$$R(q) = \langle r, q\rangle_{L^2}$$

If furthermore the assumptions of Lem. 3.11 are met then, for small enough τ, *the unique solution* $(f_\tau, \dot{u}_\tau, p_\tau)$ *of Prop. 3.23 satisfies a bound of the form*

$$\|\dot{u}_\tau\|_{H^2} + \|f_\tau\|_{H_{\text{div}}} + \|p_\tau\|_{H^2} \\ \leq C_e(\tau, \|u_0\|_{H^1})\|e\|_{L^2} + C_j(\tau, \|u_0\|_{H^1})\|j\|_{H_{\text{div}}} + C_r(\tau, \|u_0\|_{H^1})\|r\|_{L^2} \quad (3.51)$$

where the C_e, C_j, C_r *are monotonically increasing in both arguments. A sufficient condition on* τ *is*

$$\tau L_{\dot{u}} < 1 \tag{3.52}$$

where $L_{\dot{u}} := \epsilon\gamma_\lambda^2(1 + \|W_2\|_{L^\infty})C_{\mathcal{M}}(\|u_0\|_{H^1})$.

Proof. Under these conditions, the saddle point system of the optimization problem (3.45) is equivalent to

$$\begin{aligned}
&\langle f_\tau, \mathcal{M}^{-1}_{[u_0]_\lambda} g\rangle_{L^2} - \langle g, \operatorname{grad}_\Gamma p_\tau\rangle_{L^2} = -\langle j, g\rangle_{L^2} \quad \forall g \in L^2(T\Gamma)\\
&\langle p_\tau, \theta\rangle_{L^2} + \lambda\langle \operatorname{grad}_\Gamma p_\tau, \operatorname{grad}_\Gamma \theta\rangle_{L^2} + \tau\epsilon\langle W_2\dot{u}_\tau, \theta\rangle_{L^2} + \tau\epsilon\langle \operatorname{grad}_\Gamma \dot{u}_\tau, \operatorname{grad}_\Gamma \theta\rangle_{L^2}\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad = -\langle e, \theta\rangle_{L^2}, \quad \forall \theta \in H^1(\Gamma)\\
&\langle \dot{u}_\tau, q\rangle_{L^2} + \lambda\langle \operatorname{grad}_\Gamma \dot{u}_\tau, \operatorname{grad}_\Gamma q\rangle_{L^2} - \langle f_\tau, \operatorname{grad}_\Gamma q\rangle_{L^2} = \langle r, q\rangle_{L^2}, \quad \forall q \in H^1(\Gamma)
\end{aligned}$$

The solution $(f_\tau, \dot{u}_\tau, p_\tau)$ of this system is a *fixed point* of the following iteration:

- $(f_\tau^{(0)}, \dot{u}_\tau^{(0)}, p_\tau^{(0)}) = (0, 0, 0)$
- $(f_\tau^{(k)}, \dot{u}_\tau^{(k)}, p_\tau^{(k)})$ is the unique solution of

$$\begin{aligned}
&\langle f_\tau^{(k)}, \mathcal{M}^{-1}_{[u_0]_\lambda} g\rangle_{L^2} = \langle g, \operatorname{grad}_\Gamma p_\tau^{(k)}\rangle_{L^2} + \langle j, g\rangle_{L^2} \quad \forall g \in L^2(T\Gamma)\\
&\langle p_\tau^{(k)}, \theta\rangle_{L^2} + \lambda\langle \operatorname{grad}_\Gamma p_\tau^{(k)}, \operatorname{grad}_\Gamma \theta\rangle_{L^2}\\
&\qquad = -\langle e, \theta\rangle_{L^2} - \tau\epsilon\langle W_2\dot{u}_\tau^{(k-1)}, \theta\rangle_{L^2} - \tau\epsilon\langle \operatorname{grad}_\Gamma \dot{u}_\tau^{(k-1)}, \operatorname{grad}_\Gamma \theta\rangle_{L^2}, \quad \forall \theta \in H^1(\Gamma)\\
&\langle \dot{u}_\tau^{(k)}, q\rangle_{L^2} + \lambda\langle \operatorname{grad}_\Gamma \dot{u}_\tau^{(k)}, \operatorname{grad}_\Gamma q\rangle_{L^2} = \langle f_\tau^{(k)}, \operatorname{grad}_\Gamma q\rangle_{L^2} + \langle r, q\rangle_{L^2}, \quad \forall q \in H^1(\Gamma)
\end{aligned}$$

This decouples the right-hand side of the second equation and allows us to apply the same technique as in step 5 of the proof of Prop. 3.17 to show that $\dot{u}_\tau^{(k)}$ is a Cauchy sequence in $H^2(\Gamma)$.

First we show that the sequence $(f_\tau^{(k)}, \dot{u}_\tau^{(k)}, p_\tau^{(k)})$ is indeed in $H_{\mathrm{div}}(T\Gamma) \times H^2(\Gamma) \times H^2(\Gamma)$. We note that $\dot{u}_\tau^{(0)} = 0 \in H^2(\Gamma)$, and assume that $\dot{u}_\tau^{(k-1)} \in H^2(\Gamma)$. Applying Lem. 3.13 to the second equation yields the inequality

$$\|p_\tau^{(k)}\|_{H^2} \leq \gamma_\lambda\|e + \tau\epsilon W_2\dot{u}_\tau^{(k-1)} - \tau\epsilon\Delta_\Gamma\dot{u}_\tau^{(k-1)}\|_{L^2} \leq \gamma_\lambda\left(\|e\|_{L^2} + \tau\gamma_{\dot{u}}\|\dot{u}_\tau^{(k-1)}\|_{H^2}\right)$$

where $\gamma_{\dot{u}} := \epsilon(1 + \|W_2\|_{L^\infty})$, and so $p^{(k)} \in H^2(\Gamma)$. Next, by applying Lem. 3.14 to the first equation, we get

$$\|f_\tau^{(k)}\|_{H_{\mathrm{div}}} \leq C_{\mathcal{M}}(\|u_0\|_{H^1})\|\operatorname{grad}_\Gamma p_\tau^{(k)} + j\|_{H_{\mathrm{div}}} \leq \gamma_{\mathcal{M}}\left(\|p_\tau^{(k)}\|_{H^2} + \|j\|_{H_{\mathrm{div}}}\right)$$

with $\gamma_{\mathcal{M}} := C_{\mathcal{M}}(\|u_0\|_{H^1})$, and therefore $f_\tau^{(k)} \in H_{\text{div}}(T\Gamma)$. Finally, by applying Lem. 3.13 to the third equation, we get

$$\|\dot{u}_\tau^{(k)}\|_{H^2} \leq \gamma_\lambda \|\text{div}_\Gamma f_\tau^{(k)} - r\|_{L^2} \leq \gamma_\lambda \left(\|f_\tau^{(k)}\|_{H_{\text{div}}} + \|r\|_{L^2}\right)$$

By induction, it follows that $\dot{u}_\tau^{(k)} \in H^2(\Gamma)$, for all $k \in \mathbb{N}$.

Now, as in step 5 of the proof of Prop. 3.17, we subtract the equations for k from the equations for $k+1$. Noting that the $\langle j, g\rangle_{L^2}$, $\langle e, \theta\rangle_{L^2}$ and $\langle r, q\rangle_{L^2}$ terms cancel out, and by applying the same chain of lemmas, we get the set of inequalities

$$\begin{aligned}
\|p_\tau^{(k+1)} - p_\tau^{(k)}\|_{H^2} &\leq \gamma_\lambda \tau \gamma_{\dot{u}} \|\dot{u}_\tau^{(k)} - \dot{u}_\tau^{(k-1)}\|_{H^2} \\
\|f_\tau^{(k+1)} - f_\tau^{(k)}\|_{H_{\text{div}}} &\leq \gamma_{\mathcal{M}} \|p_\tau^{(k+1)} - p_\tau^{(k)}\|_{H^2} \\
\|\dot{u}_\tau^{(k+1)} - \dot{u}_\tau^{(k)}\|_{H^2} &\leq \gamma_\lambda \|f_\tau^{(k+1)} - f_\tau^{(k)}\|_{H_{\text{div}}} \\
\Rightarrow \|\dot{u}_\tau^{(k+1)} - \dot{u}_\tau^{(k)}\|_{H^2} &\leq \tau L_{\dot{u}} \|\dot{u}_\tau^{(k)} - \dot{u}_\tau^{(k-1)}\|_{H^2}
\end{aligned}$$

with $L_{\dot{u}} := \gamma_\lambda^2 \gamma_{\dot{u}} \gamma_{\mathcal{M}} = \epsilon \gamma_\lambda^2 (1 + \|W_2\|_{L^\infty}) C_{\mathcal{M}}(\|u_0\|_{H^1})$. If τ is small enough, so that $\tau L_{\dot{u}} < 1$, the sequence of $\dot{u}_\tau^{(k)}$ is Cauchy and it converges to a fixed point $\dot{u}_\tau \in H^2(\Gamma)$ with associated f_τ and p_τ that satisfy the saddle point equations. Starting from $\|p_\tau\|_{H^2}$ and chaining the associated inequalities, we have:

$$\begin{aligned}
\|p_\tau\|_{H^2} \leq \gamma_\lambda \left(\|e\|_{L^2} + \tau \gamma_{\dot{u}} \|\dot{u}_\tau\|_{H^2}\right) &\leq \gamma_\lambda \|e\|_{L^2} + \tau \gamma_{\dot{u}} \gamma_\lambda^2 \left(\|r\|_{L^2} + \|f_\tau\|_{H_{\text{div}}}\right) \\
&\leq \gamma_\lambda \|e\|_{L^2} + \tau \gamma_{\dot{u}} \gamma_\lambda^2 \|r\|_{L^2} + \tau L_{\dot{u}} \|j\|_{H_{\text{div}}} + \tau L_{\dot{u}} \|p_\tau\|_{H^2}
\end{aligned}$$

Doing the same for $\|f_\tau\|_{H^2}$ and $\|\dot{u}_\tau\|_{H^2}$ and summing the inequalities, yields the bound

$$(1 - \tau L_{\dot{u}}) \left(\|\dot{u}_\tau\|_{H^2} + \|f_\tau\|_{H_{\text{div}}} + \|p_\tau\|_{H^2}\right) \leq \gamma_e \|e\|_{L^2} + \gamma_j \|j\|_{H_{\text{div}}} + \gamma_r \|r\|_{L^2}$$

with $\gamma_e := \gamma_\lambda(1 + \gamma_{\mathcal{M}}(1 + \gamma_\lambda))$, $\gamma_j := \tau L_{\dot{u}} + \gamma_{\mathcal{M}}(1 + \gamma_\lambda)$ and $\gamma_r := \tau L_{\dot{u}} + \gamma_\lambda(1 + \tau \gamma_\lambda \gamma_{\dot{u}})$. Dividing by $1 - \tau L_{\dot{u}}$ gives us the desired bound.

□

Now, we can study the time-discrete scheme itself:

Corollary 3.26 (Inner opt. problem of the time-discrete scheme)**.** *Consider the optimization problem*

$$\min_{\substack{(f,\dot{u}) \in \\ L^2(T\Gamma) \times H^1(\Gamma)}} \left\{ \frac{1}{2} \langle f, \mathcal{M}_{[u_0]_\lambda}^{-1} f\rangle_{L^2} + \frac{\tau\epsilon}{2} \langle \dot{u}, W_2 \dot{u}\rangle_{L^2} + \frac{\tau\epsilon}{2} \|\text{grad}_\Gamma \dot{u}\|_{L^2}^2 + \mathcal{E}'(u_0)(\dot{u}) \right\} \tag{3.53a}$$

$$\langle \dot{u}, q\rangle_{L^2} + \lambda \langle \text{grad}_\Gamma \dot{u}, \text{grad}_\Gamma q\rangle_{L^2} - \langle f, \text{grad}_\Gamma q\rangle_{L^2} = 0, \ \forall q \in H^1(\Gamma) \tag{3.53b}$$

with

$$\mathcal{E}'(u_0)(\theta) := \langle W_1, \theta\rangle_{L^2} + \epsilon \langle W_2 u_0, \theta\rangle_{L^2} + \epsilon \langle \text{grad}_\Gamma u_0, \text{grad}_\Gamma \theta\rangle_{L^2} \tag{3.54}$$

and $W_1 \in L^2(\Gamma)$*,* $W_2 \in L^\infty(\Gamma)$*.*

1. *If $u_0 \in H^1(\Gamma)$, the assumptions of Cor. 3.10 are met and τ satisfies either* (3.46) *or* (3.48)*, then there exists a unique solution $(f_\tau, \dot{u}_\tau) \in L^2(T\Gamma) \times H^1(\Gamma)$ with a unique multiplier $p_\tau \in H^1(\Gamma)$. Furthermore, there exist constants $\alpha_\tau, \beta_\tau > 0$ such that*
$$\|\dot{u}_\tau\|_{H^1} + \|f_\tau\|_{L^2} + \|p_\tau\|_{H^1} \leq \alpha_\tau + \beta_\tau \|u_0\|_{H^1} \tag{3.55}$$

2. *If in addition $u_0 \in H^2(\Gamma)$, the assumptions of Lem. 3.11 are met and τ satisfies the condition* (3.52)*, then the solution also satisfies the bound*
$$\|\dot{u}_\tau\|_{H^2} + \|f_\tau\|_{H_{\mathrm{div}}} + \|p_\tau\|_{H^2} \leq A_\tau(\tau, \|u_0\|_{H^1}) + B_\tau(\tau, \|u_0\|_{H^1})\|u_0\|_{H^2} \tag{3.56}$$
where A_τ, B_τ are monotonically increasing functions.

Proof.

1. Exactly the same reasoning as in part 1 of the proof of Cor. 3.16, with Prop. 3.23 instead of Prop. 3.12, leads to the desired bound with $(\alpha_\tau, \beta_\tau) = (C\|W_1\|_{L^2}, C\epsilon(1 + \|W_2\|_{L^\infty}))$ and $C = C_\tau$ or $C = C_\lambda$, depending on which condition τ satisfies.

2. Again, following the same reasoning as in part 2 of the proof of Cor. 3.16, with Prop. 3.25 instead of Prop. 3.15, gives us the second bound with $A_\tau = \|W_1\|_{L^2} C_e$ and $B_\tau = \epsilon(1 + \|W_2\|_{L^\infty})C_e$.

□

Lemma 3.27 (Well-posedness of the time-discrete scheme in H^1)**.** *Assume that $u_0 \in H^1(\Gamma)$, $W_1 \in L^2(\Gamma)$ and $W_2 \in L^\infty(\Gamma)$, and that the assumptions of Cor. 3.10 are met. Let $0 = t_0 < t_1 < \ldots < t_N = T$ be a partition of the interval $[0,T]$ into subintervals of length $\tau_k := t_k - t_{k-1}$, with $\tau := \max_{0<k\leq N} \tau_k$ small enough so that either of the bounds* (3.46) *or* (3.48) *of Prop. 3.23 is satisfied.*

Then there exists a unique $u_\tau \in C([0,T]; H^1(\Gamma)) \cap H^1(0,T; H^1(\Gamma))$, piecewise-linear in time, with associated $f_\tau \in L^2(0,T; L^2(T\Gamma))$, $\dot{u}_\tau \in L^2(0,T; H^1(\Gamma))$ and $p_\tau \in L^2(0,T; H^1(\Gamma))$, piecewise-constant in time, such that

1. $u_\tau^0 := u_\tau(0) = u_0$

2. *in every subinterval $[t_{k-1}, t_k]$:*
$$u_\tau(t) = u_\tau(t_{k-1}) + \int_{t_{k-1}}^{t} \dot{u}_\tau(s)\, ds = u_\tau^{k-1} + (t - t_{k-1})\dot{u}_\tau^k \tag{3.57}$$
and
$$\langle f_\tau^k, \mathcal{M}^{-1}_{[u_\tau^{k-1}]_\lambda} g\rangle_{L^2} = \langle g, \mathrm{grad}_\Gamma\, p_\tau^k\rangle_{L^2}, \quad \forall g \in L^2(T\Gamma) \tag{3.58a}$$
$$\langle p_\tau^k, \theta\rangle_{L^2} + \lambda\langle \mathrm{grad}_\Gamma\, p_\tau^k, \mathrm{grad}_\Gamma\, \theta\rangle_{L^2} = -\mathcal{E}'(u_\tau^{k-1} + \tau_k\, \dot{u}_\tau^k)(\theta), \quad \forall \theta \in H^1(\Gamma) \tag{3.58b}$$
$$\langle \dot{u}_\tau^k, q\rangle_{L^2} + \lambda\langle \mathrm{grad}_\Gamma\, \dot{u}_\tau^k, \mathrm{grad}_\Gamma\, q\rangle_{L^2} = \langle f_\tau^k, \mathrm{grad}_\Gamma\, q\rangle_{L^2}, \quad \forall q \in H^1(\Gamma) \tag{3.58c}$$

3. if τ satisfies (3.48), then the bound

$$\|u_\tau\|_{C([0,T];H^1)} \le \left(\|u_0\|_{H^1} + \frac{\alpha_\tau}{\beta_\tau}\right) e^{\beta_\tau T} - \frac{\alpha_\tau}{\beta_\tau} =: M_T \tag{3.59}$$

holds with $(\alpha_\tau, \beta_\tau) := \big(C_\lambda \|W_1\|_{L^2}, \epsilon\, C_\lambda(1 + \|W_2\|_{L^\infty})\big)$ independent of τ.

Proof. We note that the system of equations (3.58) is the saddle point system of the problem (3.53) of Cor. 3.26, with $u_0 = u_\tau^{k-1}$ and $\tau = \tau_k$. Starting from the first interval and proceeding inductively, we see that given $u_\tau^{(k-1)} \in H^1(\Gamma)$ and because τ_k is small enough, there exists a unique tuple $(f_\tau^k, \dot{u}_\tau^k, p_\tau^k) \in L^2(T\Gamma) \times H^1(\Gamma) \times H^1(\Gamma)$, from which a unique $u_\tau^k = u_\tau^{k-1} + \tau_k \dot{u}_\tau^{k-1} \in H^1(\Gamma)$ can be constructed. The key point that allows this construction is that the right-hand side of the bounds (3.46) and (3.48) does not depend on u, and so can be used to estimate an *a priori* uniform bound for the time steps τ_k.

For the bound (3.59), we will first show that

$$\|u_\tau^k\|_{H^1} \le \left(\|u_0\|_{H^1} + \frac{\alpha_\tau}{\beta_\tau}\right) e^{\beta_\tau t_k} - \frac{\alpha_\tau}{\beta_\tau}$$

For $k = 0$, so that $u_\tau^0 = u_0$ and $t_0 = 0$, the inequality is straightforward to verify. For the subinterval $[t_{k-1}, t_k]$, we assume that the inequality holds for $k-1$, and recall that, since $\tau_k \le \tau$ satisfies the bound (3.48), the inequality (3.55) implies that

$$\|\dot{u}_\tau^k\|_{H^1} + \|f_\tau^k\|_{L^2} + \|p_\tau^k\|_{H^1} \le \alpha_\tau + \beta_\tau \|u_\tau^{k-1}\|_{H^1}$$

with $(\alpha_\tau, \beta_\tau) = \big(C_\lambda \|W_1\|_{L^2}, \epsilon\, C_\lambda(1 + \|W_2\|_{L^\infty})\big)$ independent of τ. Then, with the help of the inequality $1 + x \le e^x$,

$$\begin{aligned}
\|u_\tau^k\|_{H^1} &= \|u_\tau^{k-1} + \tau_k\, \dot{u}_\tau^k\|_{H^1} \le \|u_\tau^{k-1}\|_{H^1} + \tau_\kappa \|\dot{u}_\tau^k\|_{H^1} \\
&\le (1 + \beta_\tau \tau_k)\|u_\tau^{k-1}\|_{H^1} + \alpha_\tau \tau_k \\
&\le (1 + \beta_\tau \tau_k)\left\{\left(\|u_0\|_{H^1} + \frac{\alpha_\tau}{\beta_\tau}\right) e^{\beta_\tau t_{k-1}} - \frac{\alpha_\tau}{\beta_\tau}\right\} + \alpha_\tau \tau_k \\
&\le \left(\|u_0\|_{H^1} + \frac{\alpha_\tau}{\beta_\tau}\right) e^{\beta_\tau \tau_k} e^{\beta_\tau t_{k-1}} - (1 + \beta_\tau \tau_k)\frac{\alpha_\tau}{\beta_\tau} + \alpha_\tau \tau_k \\
&= \left(\|u_0\|_{H^1} + \frac{\alpha_\tau}{\beta_\tau}\right) e^{\beta_\tau (t_{k-1} + \tau_k)} - \left(\frac{\alpha_\tau}{\beta_\tau} + \alpha_\tau \tau_k\right) + \alpha_\tau \tau_k \\
&= \left(\|u_0\|_{H^1} + \frac{\alpha_\tau}{\beta_\tau}\right) e^{\beta_\tau t_k} - \frac{\alpha_\tau}{\beta_\tau}
\end{aligned}$$

By induction, the inequality holds for all $0 \le k \le N$. Then, since

- $t_k \le T \Rightarrow e^{\beta_\tau t_k} \le e^{\beta_\tau T}$ for all $0 \le k \le N$,

- $u_\tau(t)$ is a linear combination of the u_τ^k for all $t \in [0,T]$,

we have that

$$\|u_\tau\|_{C([0,T];H^1)} \leq \max_{0\leq k\leq N} \|u_\tau^k\|_{H^1} \leq \left(\|u_0\|_{H^1} + \frac{\alpha_\tau}{\beta_\tau}\right) e^{\beta_\tau T} - \frac{\alpha_\tau}{\beta_\tau}$$

□

The following corrolary, which is a time-discrete equivalent of Cor. 3.21, justifies the use of the natural time discretization by ensuring that the free energy does not increase between time steps:

Corollary 3.28 (Energy reduction)**.** *Let u_τ and $(f_\tau, \dot{u}_\tau, p_\tau)$ be as in Lemma 3.27. Then in every subinterval $[t_{k-1}, t_k]$,*

$$\begin{aligned} \mathcal{E}(u_\tau^k) - \mathcal{E}(u_\tau^{k-1}) \leq -\frac{\tau_k}{2}\langle \operatorname{grad}_\Gamma p_\tau^k, \mathcal{M}_{[u_\tau^{k-1}]_\lambda} \operatorname{grad}_\Gamma p_\tau^k\rangle_{L^2} \leq -\frac{\tau_k}{2}\mu_\lambda \|p_\tau^k\|_{H^1}^2 \\ \textit{where } \mathcal{E}(u) := \langle W_1, u\rangle_{L^2} + \frac{\epsilon}{2}\langle W_2\, u, u\rangle_{L^2} + \frac{\epsilon}{2}\|\operatorname{grad}_\Gamma u\|_{L^2}^2 \end{aligned} \tag{3.60}$$

Proof. By construction, $u_\tau^k = u_\tau^{k-1} + \tau_k \dot{u}_\tau^k$ and so it is straightforward using the definitions of $\mathcal{E}$ and $\mathcal{E}'$ to verify that

$$\mathcal{E}(u_\tau^k) = \mathcal{E}(u_\tau^{k-1}) + \tau_k \left\{\mathcal{E}'(u_\tau^{k-1})(\dot{u}_\tau^k) + \frac{\tau_k \epsilon}{2}\langle W_2\, \dot{u}_\tau^k, \dot{u}_\tau^k\rangle_{L^2} + \frac{\tau_k\epsilon}{2}\|\operatorname{grad}_\Gamma \dot{u}_\tau^k\|_{L^2}^2\right\}$$

We have already shown in the proof of Lem. 3.27 that the $(f_\tau^k, \dot{u}_\tau^k)$ are the solution of the optimization problem

$$\min_{\substack{(f,\dot{u})\in \\ L^2(T\Gamma)\times H^1(\Gamma)}} \left\{\frac{1}{2}\langle f, \mathcal{M}^{-1}_{[u_\tau^{k-1}]_\lambda} f\rangle_{L^2} + \frac{\tau_k\epsilon}{2}\langle \dot{u}, W_2\dot{u}\rangle_{L^2} + \frac{\tau_k\epsilon}{2}\|\operatorname{grad}_\Gamma \dot{u}\|_{L^2}^2 + \mathcal{E}'(u_\tau^{k-1})(\dot{u})\right\}$$

$$\langle \dot{u}, q\rangle_{L^2} + \lambda\langle \operatorname{grad}_\Gamma \dot{u}, \operatorname{grad}_\Gamma q\rangle_{L^2} - \langle f, \operatorname{grad}_\Gamma q\rangle_{L^2} = 0,\ \forall q \in H^1(\Gamma)$$

Observing that the pair $(f, \dot{u}) = (0,0)$ satisfies the constraint, we deduce immediately that

$$\begin{aligned} & J((f_\tau^k, \dot{u}_\tau^k)) \leq J((0,0)) \\ &\Rightarrow \frac{1}{2}\langle f_\tau^k, \mathcal{M}^{-1}_{[u_\tau^{k-1}]_\lambda} f_\tau^k\rangle_{L^2} + \frac{\tau_k\epsilon}{2}\langle \dot{u}_\tau^k, W_2\dot{u}_\tau^k\rangle_{L^2} + \frac{\tau_k\epsilon}{2}\|\operatorname{grad}_\Gamma \dot{u}_\tau^k\|_{L^2}^2 + \mathcal{E}'(u_\tau^{k-1})(\dot{u}_\tau^k) \leq 0 \\ &\quad \Rightarrow \mathcal{E}'(u_\tau^{k-1})(\dot{u}_\tau^k) + \frac{\tau_k\epsilon}{2}\langle W_2\, \dot{u}_\tau^k, \dot{u}_\tau^k\rangle_{L^2} + \frac{\tau_k\epsilon}{2}\|\operatorname{grad}_\Gamma \dot{u}_\tau^k\|_{L^2}^2 \leq -\frac{1}{2}\langle f_\tau^k, \mathcal{M}^{-1}_{[u_\tau^{k-1}]_\lambda} f_\tau^k\rangle_{L^2} \end{aligned}$$

This, combined with the first equality and the coercivity of $\mathcal{M}_{[u_\tau^{k-1}]_\lambda}$, yield the desired inequality, if we take into account that the first equation of the saddle point system (3.58) is equivalent to $f_\tau^k = \mathcal{M}_{[u_\tau^{k-1}]_\lambda} \operatorname{grad}_\Gamma p_\tau^k$.

□

Lemma 3.29 (H^2-regularity of the time-discrete scheme)**.** *Assume that $u_0 \in H^2(\Gamma)$, the assumptions of Lem. 3.11 are met, and that $\tau > 0$ is small enough so that* (3.48) *holds. Then for*

$$\tau < \overline{\tau} := \left\{\epsilon\gamma_\lambda^2(1 + \|W_2\|_{L^\infty})C_{\mathcal{M}}(M_T)\right\}^{-1} \tag{3.61}$$

the unique solution of Lem. 3.27 is $u_\tau \in C([0,T];H^2(\Gamma)) \cap H^1(0,T;H^2(\Gamma))$, with associated $f_\tau \in L^2(0,T;H_{\text{div}}(T\Gamma))$, $\dot{u}_\tau \in L^2(0,T;H^2(\Gamma))$ and $p_\tau \in L^2(0,T;H^2(\Gamma))$, and satisfies the bound

$$\|u_\tau\|_{C([0,T];H^2)} \leq \left(\|u_0\|_{H^2} + \frac{\alpha'_\tau}{\beta'_\tau}\right) e^{\beta'_\tau T} - \frac{\alpha'_\tau}{\beta'_\tau} =: M'_T \tag{3.62}$$

where $\alpha'_\tau := C_e(\overline{\tau}, M_T)\alpha_\tau$ and $\beta'_\tau := C_e(\overline{\tau}, M_T)\beta_\tau$.

Proof. At the subinterval $[t_{k-1}, t_k]$, the bound (3.59) implies that

$$\|u_\tau^{k-1}\|_{H^1} \leq M_T \Rightarrow \tau_k \leq \tau < \overline{\tau} \leq \left(\epsilon\gamma_\lambda^2(1 + \|W_2\|_{L^\infty})C_{\mathcal{M}}(\|u_\tau^{k-1}\|_{H^1})\right)^{-1}$$

and therefore τ_k satisfies the condition (3.52). If $u_\tau^{k-1} \in H^2(\Gamma)$ then part 2 of Cor. 3.26 is applicable, and so $f_\tau^k \in H_{\text{div}}$ and $\dot{u}_\tau^k, p_\tau^k \in H^2(\Gamma)$. Moreover

$$\|\dot{u}_\tau^k\|_{H^2} \leq A_\tau(\tau_k, \|u_\tau^{k-1}\|_{H^1}) + B_\tau(\tau_k, \|u_\tau^{k-1}\|_{H^1})\|u_\tau^{k-1}\|_{H^2} \leq \alpha'_\tau + \beta'_\tau\|u_\tau^{k-1}\|_{H^2}$$

with $\alpha'_\tau := A_\tau(\overline{\tau}, M_T)$ and $\beta'_\tau := B_\tau(\overline{\tau}, M_T)$ independent of τ_k. Starting from $u_\tau^0 = u_0 \in H^2(\Gamma)$, we can proceed inductively to show that $u_\tau^k \in H^2(\Gamma)$, for all $0 \leq k \leq N$, and so $u_\tau(t) \in H^2(\Gamma)$ for all $t \in [0,T]$. Finally, using the bound for $\|\dot{u}_\tau^k\|_{H^2}$ above and working exactly like in the proof of Lem. 3.27, we can derive the bound (3.62).

□

Proposition 3.30 (Convergence of the time-discrete scheme)**.** *Let $u_0 \in H^2(\Gamma)$ and assume that the conditions of Lem. 3.11 are satisfied. Let u, with the associated $(f, \dot{u}, p)$, be the (continuous) solution of Cor. 3.18, whereas u_τ, with the associated $(f_\tau, \dot{u}_\tau, p_\tau)$, a (time-discrete) solution as in Lem. 3.27. If τ is small enough so that Lem. 3.29 holds, then*

$$\|u - u_\tau\|_{C([0,T];H^2)} = \mathrm{O}(\tau) \tag{3.63}$$

Proof. We define the error as

$$\varepsilon(t) := u(t) - u_\tau(t) = \int_0^t (\dot{u}(s) - \dot{u}_\tau)\, ds, \quad t \in [0,T]$$

We analyse first the local error in the subinterval $[t_{k-1}, t_k]$. Recall:

- the saddle point equations for the continuous problem at time $t \in [t_{k-1}, t_k]$:

$$\begin{aligned}
&\langle f(t), \mathcal{M}^{-1}_{[u(t)]_\lambda} g\rangle_{L^2} = \langle g, \operatorname{grad}_\Gamma p(t)\rangle_{L^2}, \quad \forall g \in L^2(T\Gamma)\\
&\langle p(t), \theta\rangle_{L^2} + \lambda \langle \operatorname{grad}_\Gamma p(t), \operatorname{grad}_\Gamma \theta\rangle_{L^2} = -\langle e(u(t)), \theta\rangle_{L^2}, \quad \forall \theta \in H^1(\Gamma)\\
&\langle \dot u(t), q\rangle_{L^2} + \lambda \langle \operatorname{grad}_\Gamma \dot u(t), \operatorname{grad}_\Gamma q\rangle_{L^2} = \langle f(t), \operatorname{grad}_\Gamma q\rangle_{L^2}, \quad \forall q \in H^1(\Gamma)
\end{aligned}$$

  where $e(u) := W_1 + \epsilon W_2 u - \epsilon \Delta_\Gamma u \in L^2(\Gamma)$, for $u \in H^2(\Gamma)$.

- the saddle point equations for the time-discrete scheme in the subinterval $[t_{k-1}, t_k]$:

$$\begin{aligned}
&\langle f^k_\tau, \mathcal{M}^{-1}_{[u^{k-1}_\tau]_\lambda} g\rangle_{L^2} = \langle g, \operatorname{grad}_\Gamma p^k_\tau\rangle_{L^2}, \quad \forall g \in L^2(T\Gamma)\\
&\langle p^k_\tau, \theta\rangle_{L^2} + \lambda \langle \operatorname{grad}_\Gamma p^k_\tau, \operatorname{grad}_\Gamma \theta\rangle_{L^2} = -\langle e(u^{k-1}_\tau + \tau_k \dot u^k_\tau), \theta\rangle_{L^2}, \quad \forall \theta \in H^1(\Gamma)\\
&\langle \dot u^k_\tau, q\rangle_{L^2} + \lambda \langle \operatorname{grad}_\Gamma \dot u^k_\tau, \operatorname{grad}_\Gamma q\rangle_{L^2} = \langle f^k_\tau, \operatorname{grad}_\Gamma q\rangle_{L^2}, \quad \forall q \in H^1(\Gamma)
\end{aligned}$$

- the bounds:

$$\begin{aligned}
\|u\|_{C([0,T];H^1)} &\le (\|u_0\|_{H^1} + \alpha_\lambda T) e^{\beta_\lambda T} =: M\\
\|u_\tau\|_{C([0,T];H^1)} &\le \left(\|u_0\|_{H^1} + \frac{\alpha_\tau}{\beta_\tau}\right) e^{\beta_\tau T} - \frac{\alpha_\tau}{\beta_\tau} =: M_T\\
\|u_\tau\|_{C([0,T];H^2)} &\le \left(\|u_0\|_{H^2} + \frac{\alpha'_\tau}{\beta'_\tau}\right) e^{\beta'_\tau T} - \frac{\alpha'_\tau}{\beta'_\tau} =: M'_T
\end{aligned}$$

  and, for any $1 \le k \le N$,

$$\begin{aligned}
&\tau_k \le \tau < \overline{\tau}\\
&\|\dot u^k_\tau\|_{H^2} + \|f^k_\tau\|_{H_{\mathrm{div}}} + \|p^k_\tau\|_{H^2} \le \alpha'_\tau + \beta'_\tau \|u^{k-1}_\tau\|_{H^2} \le \alpha'_\tau + \beta'_\tau M'_T =: M''_\tau
\end{aligned}$$

  Note that all the constants, despite the notation, are independent of τ.

As in part 5 of the proof of Prop. 3.17, we take the difference of the two sets of equations. After some manipulation, we arrive at the following system:

$$\begin{aligned}
\langle \delta f, \mathcal{M}^{-1}_{[u(t)]_\lambda} g\rangle_{L^2} - \langle g, \operatorname{grad}_\Gamma \delta p\rangle_{L^2} &= -\langle \mathcal{M}^{-1}_{[u(t)]_\lambda} f^k_\tau - \mathcal{M}^{-1}_{[u^{k-1}_\tau]_\lambda} f^k_\tau, g\rangle_{L^2}\\
&= -\langle \mathcal{M}^{-1}_{[u(t)]_\lambda} \mathcal{M}_{[u^{k-1}_\tau]_\lambda} \operatorname{grad}_\Gamma p^k_\tau - \operatorname{grad}_\Gamma p^k_\tau, g\rangle_{L^2}, \quad \forall g \in L^2(T\Gamma)\\
\langle \delta p, \theta\rangle_{L^2} + \lambda \langle \operatorname{grad}_\Gamma \delta p, \operatorname{grad}_\Gamma \theta\rangle_{L^2} &= -\langle e(u(t)) - e(u^{k-1}_\tau + \tau_k \dot u^k_\tau), \theta\rangle_{L^2}, \quad \forall \theta \in H^1(\Gamma)\\
\langle \delta \dot u, q\rangle_{L^2} + \lambda \langle \operatorname{grad}_\Gamma \delta \dot u, \operatorname{grad}_\Gamma q\rangle_{L^2} - \langle \delta f, \operatorname{grad}_\Gamma q\rangle_{L^2} &= 0, \quad \forall q \in H^1(\Gamma)
\end{aligned}$$

where $\delta f := f(t) - f^k_\tau$, $\delta p := p(t) - p^k_\tau$ and $\delta \dot u := \dot u(t) - \dot u^k_\tau = \dot\varepsilon(t)$. The second form of the right-hand side of the first equation comes from the saddle point equation

$\langle f_\tau^k, \mathcal{M}^{-1}_{[u_\tau^{k-1}]_\lambda} g\rangle_{L^2} = \langle g, \operatorname{grad}_\Gamma p_\tau^k\rangle_{L^2}$. Prop. 3.15 is applicable to this saddle point system, with

$$\begin{aligned} e &:= e(u(t)) - e(u_\tau^{k-1} + \tau_k \dot{u}_\tau^k) \\ &= \epsilon\, W_2(u(t) - u_\tau^{k-1} - \tau_k \dot{u}_\tau^k) - \epsilon\, \Delta_\Gamma (u(t) - u_\tau^{k-1} - \tau_k \dot{u}_\tau^k) \\ j &:= \mathcal{M}^{-1}_{[u(t)]_\lambda} \mathcal{M}_{[u_\tau^{k-1}]_\lambda} \operatorname{grad}_\Gamma p_\tau^k - \operatorname{grad}_\Gamma p_\tau^k \\ r &:= 0 \end{aligned}$$

Then, using the bounds above, we have

$$\begin{aligned} \|e\|_{L^2} &= \|e(u(t)) - e(u_\tau^{k-1} + \tau_k \dot{u}_\tau^k)\|_{L^2} \\ &\le \epsilon(1 + \|W_2\|_{L^\infty}) \|u(t) - u_\tau^{k-1} - \tau_k \dot{u}_\tau^k\|_{H^2} \\ &\le \epsilon(1 + \|W_2\|_{L^\infty}) \left(\|u(t) - u_\tau(t)\|_{H^2} + \|(t - t^{k-1}) \dot{u}_\tau^k - \tau_k \dot{u}_\tau^k\|_{H^2} \right) \\ &\le \epsilon(1 + \|W_2\|_{L^\infty}) \left(\|\varepsilon(t)\|_{H^2} + \tau_k \|\dot{u}_\tau^k\|_{H^2} \right) \\ &\le \epsilon(1 + \|W_2\|_{L^\infty}) \left(\|\varepsilon(t)\|_{H^2} + \tau M_T'' \right) \end{aligned}$$

and, with the help of Lem. 3.11,

$$\begin{aligned} \|\mathcal{M}_{[u(t)]_\lambda} j\|_{H_{\mathrm{div}}} &= \|\mathcal{M}_{[u_\tau^{k-1}]_\lambda} \operatorname{grad}_\Gamma p_\tau^k - \mathcal{M}_{[u(t)]_\lambda} \operatorname{grad}_\Gamma p_\tau^k\|_{H_{\mathrm{div}}} \\ &\le L_{\mathcal{M}}(\|u_\tau^{k-1}\|_{H^1}, \|u(t)\|_{H^1}) \|\operatorname{grad}_\Gamma p_\tau^k\|_{H_{\mathrm{div}}} \|u(t) - u_\tau^{k-1}\|_{H^2} \\ &\le L_{\mathcal{M}}(M_T, M) \|p_\tau^k\|_{H^2} \left(\|u(t) - u_\tau(t)\|_{H^2} + \|(t - t^{k-1}) \dot{u}_\tau^k\|_{H^2} \right) \\ &\le L_{\mathcal{M}}(M_T, M) M_T'' \left(\|\varepsilon(t)\|_{H^2} + \tau M_T'' \right) \end{aligned}$$

The bound (3.36) gives us

$$\begin{aligned} \|\delta \dot{u}\|_{H^2} \le \gamma_\lambda^2 C_{\mathcal{M}}(\|u(t)\|_{H^1}) \|e\|_{L^2} + \gamma_\lambda \|\mathcal{M}_{[u(t)]_\lambda} j\|_{H_{\mathrm{div}}} \\ \Rightarrow \|\dot{\varepsilon}(t)\|_{H^2} \le C_\varepsilon \left(\|\varepsilon(t)\|_{H^2} + \tau M_T'' \right) \end{aligned}$$

with $C_\varepsilon := \epsilon \gamma_\lambda^2 C_{\mathcal{M}}(M)(1 + \|W_2\|_{L^\infty}) + \gamma_\lambda L_{\mathcal{M}}(M_T, M) M_T''$, independent of τ. Since this bound holds for almost all $t \in (0, T)$, a direct application of Lem. 3.6 gives us for the global error

$$\|\varepsilon\|_{C([0,T];H^2)} \le \left(\|\varepsilon(0)\|_{H^2} + \tau C_\varepsilon M_T'' T \right) e^{C_\varepsilon T} = \tau\, C_\varepsilon M_T'' T e^{C_\varepsilon T}$$

which is indeed $\mathrm{O}(\tau)$ for fixed T.

□

3.6. Galerkin approximation

Finally, in this section we study a Galerkin-type spatial approximation of the time-discrete problem (3.53). Under fairly general assumptions for the finite element spaces used, we can show convergence of $\dot{u}$ and p in the H^1 norm and of f in the L^2 norm. Furthermore, using a variation of the *Nitsche-Aubin* duality argument, we can show an improved error estimate for $\dot{u}$ and p in the L^2 norm.

Definition 3.31 (Finite element spaces)**.** *We consider a family of triangulations $\mathcal{T}$ of Γ, and assume that for each triangulation $T_h \in \mathcal{T}$ there exist two finite-dimensional subspaces V_h and W_h such that:*

1. $W_h \subset H_{\mathrm{div}}(T\Gamma) \subset L^2(T\Gamma)$
2. $V_h \subset H^1(\Gamma)$ *and* $\mathrm{grad}_\Gamma : V_h \to W_h \subset H_{\mathrm{div}}(T\Gamma) \Rightarrow V_h \subset H^2(\Gamma)$
3. *there exists a constant $C_\mathcal{T} > 0$ and a projection $\Pi_h : H^2(\Gamma) \to V_h$, such that for any $u \in H^2(\Gamma)$,*

$$\|u - \Pi_h u\|_{L^2} \leq C_\mathcal{T} h^2 \|u\|_{H^2} \tag{3.64a}$$
$$\|u - \Pi_h u\|_{H^1} \leq C_\mathcal{T} h \|u\|_{H^2} \tag{3.64b}$$
$$\|u - \Pi_h u\|_{H^2} \leq C_\mathcal{T} \|u\|_{H^2} \tag{3.64c}$$

   and so, by Agmon's inequality (3.14),

$$\|u - \Pi_h u\|_{L^\infty} \leq C_\mathcal{T} C_K\, h \|u\|_{H^2} \tag{3.64d}$$

4. *there exists (abusing the notation and using the same constant $C_\mathcal{T} > 0$) a projection $\Pi_h : H_{\mathrm{div}}(T\Gamma) \to W_h$, such that for any $f \in H_{\mathrm{div}}(\Gamma)$,*

$$\|f - \Pi_h f\|_{L^2} \leq C_\mathcal{T} h \|f\|_{H_{\mathrm{div}}} \tag{3.65a}$$
$$\|f - \Pi_h f\|_{H_{\mathrm{div}}} \leq C_\mathcal{T} \|f\|_{H_{\mathrm{div}}} \tag{3.65b}$$

5. *there exists a constant $C'_\mathcal{T}$, such that for any $u_h \in V_h$ and any $f_h \in W_h$,*

$$\|u_h\|_{H^1} \leq C'_\mathcal{T} h^{-1} \|u_h\|_{L^2} \tag{3.66a}$$
$$\|f_h\|_{H_{\mathrm{div}}} \leq C'_\mathcal{T} h^{-1} \|f_h\|_{L^2} \tag{3.66b}$$

   and so

$$\begin{aligned} \|u_h\|_{H^2} &\leq \sqrt{\|u_h\|_{H^1}^2 + \|\mathrm{grad}_\Gamma u_h\|_{H_{\mathrm{div}}}^2} \\ &\leq C'_\mathcal{T} h^{-1} \sqrt{\|u_h\|_{L^2}^2 + \|\mathrm{grad}_\Gamma u_h\|_{L^2}^2} = C'_\mathcal{T} h^{-1} \|u_h\|_{H^1} \end{aligned} \tag{3.66c}$$

We will furthermore assume that the estimates above are uniform, i.e. the constants $C_{\mathcal{T}}, C'_{\mathcal{T}}$ *are the same for all* $T_h \in \mathcal{T}$.

Remark 3.32. *In the following result, we study a space- and time-discrete form of the problem. For brevity, we use h-subscripts to denote the relevant variables, although they depend on* τ *too, and therefore a more appropriate subscript would involve both parameters, writing for instance* $\dot{u}_{\tau,h}$ *instead of* $\dot{u}_h$.

Proposition 3.33 (Galerkin approx. of the time-discrete opt. problem)**.** *Consider the optimization problem*

$$\min_{(f_h,\dot{u}_h)\in W_h\times V_h} \left\{ \frac{1}{2}\langle f_h, \mathcal{M}^{-1}_{[u_0]_\lambda} f_h\rangle_{L^2} + \frac{\tau\epsilon}{2}\langle \dot{u}_h, W_2\dot{u}_h\rangle_{L^2} + \frac{\tau\epsilon}{2}\|\operatorname{grad}_\Gamma \dot{u}_h\|^2_{L^2} + \mathcal{E}'(u_0)(\dot{u}) \right\} \tag{3.67a}$$

$$\langle \dot{u}_h, q_h\rangle_{L^2} + \lambda\langle \operatorname{grad}_\Gamma \dot{u}_h, \operatorname{grad}_\Gamma q_h\rangle_{L^2} = \langle f_h, \operatorname{grad}_\Gamma q_h\rangle_{L^2},\ \forall q_h \in V_h \tag{3.67b}$$

with $u_0 \in H^2(\Gamma)$ *and* τ *small enough so that the conditions of part 2. of Cor. 3.26 are met. There exists a unique solution* $(f_h, \dot{u}_h) \in W_h \times V_h$ *with a unique multiplier* $p_h \in V_h$*, such that*

$$\|\dot{u}_h - \dot{u}_\tau\|_{L^2} + \|p_h - p_\tau\|_{L^2} = \mathrm{O}(h^2) \tag{3.68a}$$

$$\|\dot{u}_h - \dot{u}_\tau\|_{L^\infty} + \|p_h - p_\tau\|_{L^\infty} = \mathrm{O}(h) \tag{3.68b}$$

$$\|\dot{u}_h - \dot{u}_\tau\|_{H^1} + \|f_h - f_\tau\|_{L^2} + \|p_h - p_\tau\|_{H^1} = \mathrm{O}(h) \tag{3.68c}$$

$$\|\dot{u}_h - \dot{u}_\tau\|_{H^2} + \|f_h - f_\tau\|_{H_{\mathrm{div}}} + \|p_h - p_\tau\|_{H^2} = \mathrm{O}(h^0) \tag{3.68d}$$

where $(f_\tau, \dot{u}_\tau)$ *and* p_τ *is the solution and multiplier resp. of the problem* (3.53)*. Recall that the notation* $\mathrm{O}(h^0)$ *denotes the existance of an upper bound independent of* h.

Proof.

1. For the proof of the third estimate, the key point is that the conditions of Thm. 2.17 are satisfied with constants independent of T_h. Indeed, the continuity constants $\|a\|$, $\|b\|$ can be easily shown to be the same as in Prop. 3.23, and moreover:
   - $a(\cdot,\cdot)$ *is* Z_h*-coercive:* Going back to the corresponding part of the proof of Prop. 3.23 ("$a(\cdot,\cdot)$ *is* Z*-coercive, case 1/2.*"), we see that it hinges on the fact that $\dot{u} \in H^1(\Gamma) \equiv Q$, i.e. that $\dot{u}$ is in the same space with the multiplier. This is also true in this case, since $\dot{u}_h \in V_h \equiv Q_h$. The rest works unchanged (modulo the substitutions $\dot{u} \to \dot{u}_h$, $f \to f_h$, etc.) and so $a(\cdot,\cdot)$ is indeed Z_h-coercive *with the same coercivity constant* $\alpha_h = \alpha$.
   - $b(\cdot,\cdot)$ *satisfies the LBB condition over* $X_h \times Q_h$*:* Likewise, looking at the corresponding part of the proof of Prop. 3.23, we see that it is based on the

fact that for a $q \in Q$, the pair $(-\operatorname{grad}_\Gamma q, q) \in L^2(T\Gamma) \times H^1(\Gamma) \equiv X$. This holds here as well, since for a $q_h \in Q_h \equiv V_h$, the pair $(-\operatorname{grad}_\Gamma q_h, q_h) \in W_h \times V_h \equiv X_h$ (see the note at the end of Def. 3.31). The rest works again essentially unchanged, and so $b(\cdot,\cdot)$ satisfies the LBB condition over $X_h \times Q_h$ *with the same constant* $\beta_h = \beta$.

We conclude then that the solution $(f_h, \dot{u}_h)$ and multiplier p_h of the approximate problem exist and are unique and the bound

$$\begin{aligned}
\|\dot{u}_h - \dot{u}_\tau\|_{H^1} + &\|f_h - f_\tau\|_{L^2} + \|p_h - p_\tau\|_{H^1} \\
&\leq C_h \left(\inf_{\theta_h \in V_h} \|\dot{u}_\tau - \theta_h\|_{H^1} + \inf_{g_h \in W_h} \|f_\tau - g_h\|_{L^2} + \inf_{q_h \in V_h} \|p_\tau - q_h\|_{H^1} \right) \\
&\qquad\qquad \leq C_h C_{\mathcal{T}}\, h \left(\|\dot{u}_\tau\|_{H^2} + \|f_\tau\|_{H_{\mathrm{div}}} + \|p_\tau\|_{H^2} \right)
\end{aligned}$$

follows directly from the bounds (2.20), with $C_h \equiv C_h(\alpha, \|a\|, \beta, \|b\|)$ independent of T_h, and the approximation estimates (3.64) and (3.65).

2. For the fourth estimate, we have

$$\begin{aligned}
\|\dot{u}_h - \dot{u}_\tau\|_{H^2} &\leq \|\dot{u}_\tau - \Pi_h \dot{u}_\tau\|_{H^2} + \|\Pi_h \dot{u}_\tau - \dot{u}_h\|_{H^2} \\
&\leq C_{\mathcal{T}} \|\dot{u}_\tau\|_{H^2} + C'_{\mathcal{T}} h^{-1} \|\Pi_h \dot{u}_\tau - \dot{u}_h\|_{H^1} \\
&\leq C_{\mathcal{T}} \|\dot{u}_\tau\|_{H^2} + C'_{\mathcal{T}} h^{-1} \left(\|\Pi_h \dot{u}_\tau - \dot{u}_\tau\|_{H^1} + \|\dot{u}_\tau - \dot{u}_h\|_{H^1} \right) \\
&\leq C_{\mathcal{T}} \|\dot{u}_\tau\|_{H^2} + C'_{\mathcal{T}} h^{-1} \left(C_{\mathcal{T}} h \|\dot{u}_\tau\|_{H^2} + \|\dot{u}_\tau - \dot{u}_h\|_{H^1} \right) \\
&\leq C_{\mathcal{T}}(1 + C'_{\mathcal{T}}) \|\dot{u}_\tau\|_{H^2} + C_h C_{\mathcal{T}} C'_{\mathcal{T}} \left(\|\dot{u}_\tau\|_{H^2} + \|f_\tau\|_{H_{\mathrm{div}}} + \|p_\tau\|_{H^2} \right)
\end{aligned}$$

and likewise for $\|f_h - f_\tau\|_{H_{\mathrm{div}}}$ and $\|p_h - p_\tau\|_{H^2}$. Summing the three inequalities together, we get

$$\|\dot{u}_h - \dot{u}_\tau\|_{H^2} + \|f_h - f_\tau\|_{H_{\mathrm{div}}} + \|p_h - p_\tau\|_{H^2} \leq C'_h \left(\|\dot{u}_\tau\|_{H^2} + \|f_\tau\|_{H_{\mathrm{div}}} + \|p_\tau\|_{H^2} \right)$$

where $C'_h := C_{\mathcal{T}}(1 + C'_{\mathcal{T}}) + 3 C_h C_{\mathcal{T}} C'_{\mathcal{T}}$.

3. Finally, we will prove the estimate for $\|\dot{u}_\tau - \dot{u}_h\|_{L^2} + \|p_\tau - p_h\|_{L^2}$ using a variation of the *Nitsche-Aubin duality argument*. Taking the difference of the saddle point equations for the two problems, we get the following *orthogonality* equations for the residuals $\delta\dot{u} := \dot{u}_h - \dot{u}_\tau$, $\delta f := f_h - f_\tau$ and $\delta p := p_h - p_\tau$:

$$\begin{aligned}
&\langle \delta f, \mathcal{M}^{-1}_{[u_0]_\lambda} g_h \rangle_{L^2} - \langle g_h, \operatorname{grad}_\Gamma \delta p \rangle_{L^2} = 0 \\
&\langle \delta p, \theta_h \rangle_{L^2} + \lambda \langle \operatorname{grad}_\Gamma \delta p, \operatorname{grad}_\Gamma \theta_h \rangle_{L^2} + \tau\epsilon \langle W_2\, \delta\dot{u}, \theta_h \rangle_{L^2} + \tau\epsilon \langle \operatorname{grad}_\Gamma \delta\dot{u}, \operatorname{grad}_\Gamma \theta_h \rangle_{L^2} = 0 \\
&\langle \delta\dot{u}, q_h \rangle_{L^2} + \lambda \langle \operatorname{grad}_\Gamma \delta\dot{u}, \operatorname{grad}_\Gamma q_h \rangle_{L^2} - \langle \delta f, \operatorname{grad}_\Gamma q_h \rangle_{L^2} = 0
\end{aligned}$$

for any $g_h \in W_h$, $\theta_h \in V_h$ and $q_h \in V_h$. Using the bilinear forms $a_\lambda(u,v) := \langle u,v\rangle_{L^2} + \lambda\langle \operatorname{grad}_\Gamma u, \operatorname{grad}_\Gamma v\rangle_{L^2}$ and $a_\tau(u,v) := \tau\epsilon\langle W_2\, u, v\rangle_{L^2} + \tau\epsilon\langle \operatorname{grad}_\Gamma u, \operatorname{grad}_\Gamma v\rangle_{L^2}$, we can shorten this to

$$\begin{aligned}
&\langle \delta f, \mathcal{M}^{-1}_{[u_0]_\lambda} g_h\rangle_{L^2} - \langle g_h, \operatorname{grad}_\Gamma \delta p\rangle_{L^2} = 0\\
&a_\lambda(\delta p, \theta_h) + a_\tau(\delta\dot u, \theta_h) = 0\\
&a_\lambda(\delta\dot u, q_h) - \langle \delta f, \operatorname{grad}_\Gamma q_h\rangle_{L^2} = 0
\end{aligned}$$

In the *duality* part of the argument, we define the quantities $\psi \in L^2(T\Gamma)$, $\pi \in H^1(\Gamma)$ and $\omega \in H^1(\Gamma)$, as the solution of the saddle point system:

$$\begin{aligned}
&\langle \psi, \mathcal{M}^{-1}_{[u_0]_\lambda} g\rangle_{L^2} - \langle g, \operatorname{grad}_\Gamma \pi\rangle_{L^2} = \langle h^2\,\delta f, g\rangle_{L^2}\\
&\langle \pi, \theta\rangle_{L^2} + \lambda\langle \operatorname{grad}_\Gamma \pi, \operatorname{grad}_\Gamma \theta\rangle_{L^2} + \tau\epsilon\langle W_2\,\omega, \theta\rangle_{L^2} + \tau\epsilon\langle \operatorname{grad}_\Gamma \omega, \operatorname{grad}_\Gamma \theta\rangle_{L^2} = \langle \delta\dot u, \theta\rangle_{L^2}\\
&\langle \omega, q\rangle_{L^2} + \lambda\langle \operatorname{grad}_\Gamma \omega, \operatorname{grad}_\Gamma q\rangle_{L^2} - \langle \psi, \operatorname{grad}_\Gamma q\rangle_{L^2} = \langle \delta p, q\rangle_{L^2}
\end{aligned}$$

or equivalently

$$\begin{aligned}
&\langle \psi, \mathcal{M}^{-1}_{[u_0]_\lambda} g\rangle_{L^2} - \langle g, \operatorname{grad}_\Gamma \pi\rangle_{L^2} = \langle h^2\,\delta f, g\rangle_{L^2}\\
&a_\lambda(\pi, \theta) + a_\tau(\omega, \theta) = \langle \delta\dot u, \theta\rangle_{L^2}\\
&a_\lambda(\omega, q) - \langle \psi, \operatorname{grad}_\Gamma q\rangle_{L^2} = \langle \delta p, q\rangle_{L^2}
\end{aligned}$$

for any $g \in L^2(T\Gamma)$, $\theta \in H^1(\Gamma)$ and $q \in H^1(\Gamma)$. The well-posedness of (ψ, π, ω) is guaranteed by Prop. 3.23, with $E((g,\theta)) = -\langle h^2\delta f, g\rangle_{L^2} - \langle \delta\dot u, \theta\rangle_{L^2}$ and $R(q) = -\langle \delta p, q\rangle_{L^2}$. Furthermore, the conditions of Prop. 3.25 are met with $(j,e,r) = (-h^2\delta f, -\delta\dot u, \delta p)$, and so by (3.51):

$$\|\omega\|_{H^2} + \|\psi\|_{H_{\mathrm{div}}} + \|\pi\|_{H^2} \le C_d(\tau, \|u_0\|_{H^1})\left(\|\delta\dot u\|_{L^2} + h^2\|\delta f\|_{H_{\mathrm{div}}} + \|\delta p\|_{L^2}\right)$$

where $C_d = \max(C_e, C_j, C_r)$. Substituting $(g,\theta,q) = (\delta f, \delta\dot u, \delta p)$ into the dual saddle point system and summing, we get

$$\begin{aligned}
\|\delta\dot u\|^2_{L^2} + {}& h^2\|\delta f\|^2_{L^2} + \|\delta p\|^2_{L^2}\\
&= \langle \psi, \mathcal{M}^{-1}_{[u_0]_\lambda} \delta f\rangle_{L^2} - \langle \delta f, \operatorname{grad}_\Gamma \pi\rangle_{L^2} + a_\lambda(\pi, \delta\dot u)\\
&\quad + a_\tau(\omega, \delta\dot u) + a_\lambda(\omega, \delta p) - \langle \psi, \operatorname{grad}_\Gamma \delta p\rangle_{L^2}\\
&= \langle \psi - \Pi_h\psi, \mathcal{M}^{-1}_{[u_0]_\lambda} \delta f\rangle_{L^2} - \langle \delta f, \operatorname{grad}_\Gamma(\pi - \Pi_h\pi)\rangle_{L^2} + a_\lambda(\pi - \Pi_h\pi, \delta\dot u)\\
&\quad + a_\tau(\omega - \Pi_h\omega, \delta\dot u) + a_\lambda(\omega - \Pi_h\omega, \delta p) - \langle \psi - \Pi_h\psi, \operatorname{grad}_\Gamma \delta p\rangle_{L^2}\\
&\quad + \big\{\langle \Pi_h\psi, \mathcal{M}^{-1}_{[u_0]_\lambda} \delta f\rangle_{L^2} - \langle \delta f, \operatorname{grad}_\Gamma \Pi_h\pi\rangle_{L^2} + a_\lambda(\Pi_h\pi, \delta\dot u)\\
&\qquad + a_\tau(\Pi_h\omega, \delta\dot u) + a_\lambda(\Pi_h\omega, \delta p) - \langle \Pi_h\psi, \operatorname{grad}_\Gamma \delta p\rangle_{L^2}\big\}\\
&= \langle \psi - \Pi_h\psi, \mathcal{M}^{-1}_{[u_0]_\lambda} \delta f\rangle_{L^2} - \langle \delta f, \operatorname{grad}_\Gamma(\pi - \Pi_h\pi)\rangle_{L^2} + a_\lambda(\pi - \Pi_h\pi, \delta\dot u)\\
&\quad + a_\tau(\omega - \Pi_h\omega, \delta\dot u) + a_\lambda(\omega - \Pi_h\omega, \delta p) - \langle \psi - \Pi_h\psi, \operatorname{grad}_\Gamma \delta p\rangle_{L^2}
\end{aligned}$$

since the expression in the brackets is the sum of the left hand sides of the three orthogonality equations (with $g_h = \Pi_h\psi \in W_h$, $\theta_h = \Pi_h\omega \in V_h$ and $q_h = \Pi_h\pi \in V_h$). We note that the bilinear forms a_λ, a_τ are coercive in $H^1(\Gamma)$ with constants $\alpha_1 := \lambda$ and $\alpha_2 := \tau\epsilon(1+\|W_2\|_{L^\infty})$ respectively, and the operator $\mathcal{M}^{-1}_{[u_0]_\lambda}$ is likewise coercive in $L^2(T\Gamma)$ with constant $\alpha_3 := \mu_\lambda M_\lambda^{-2}$. It follows that

$$
\begin{aligned}
&\|\delta\dot{u}\|^2_{L^2} + h^2\|\delta f\|^2_{L^2} + \|\delta p\|^2_{L^2} \\
&\qquad \le \alpha_3\|\psi - \Pi_h\psi\|_{L^2}\|\delta f\|_{L^2} + \|\pi - \Pi_h\pi\|_{H^1}\|\delta f\|_{L^2} + \alpha_1\|\pi - \Pi_h\pi\|_{H^1}\|\delta\dot{u}\|_{H^1} \\
&\qquad\qquad + \alpha_2\|\omega - \Pi_h\omega\|_{H^1}\|\delta\dot{u}\|_{H^1} + \alpha_1\|\omega - \Pi_h\omega\|_{H^1}\|\delta p\|_{H^1} + \|\psi - \Pi_h\psi\|_{L^2}\|\delta p\|_{H^1} \\
&\qquad \le \alpha_4\left(\|\omega - \Pi_h\omega\|_{H^1} + \|\psi - \Pi_h\psi\|_{L^2} + \|\pi - \Pi_h\pi\|_{H^1}\right)\left(\|\delta\dot{u}\|_{H^1} + \|\delta f\|_{L^2} + \|\delta p\|_{H^1}\right) \\
&\qquad \le \alpha_4 C_{\mathcal{T}} h\left(\|\omega\|_{H^2} + \|\psi\|_{H_{\mathrm{div}}} + \|\pi\|_{H^2}\right)\left(\|\delta\dot{u}\|_{H^1} + \|\delta f\|_{L^2} + \|\delta p\|_{H^1}\right) \\
&\qquad \le C''_h h^2\left(\|\delta\dot{u}\|_{L^2} + h^2\|\delta f\|_{H_{\mathrm{div}}} + \|\delta p\|_{L^2}\right)\left(\|\dot{u}_\tau\|_{H^2} + \|f_\tau\|_{H_{\mathrm{div}}} + \|p_\tau\|_{H^2}\right)
\end{aligned}
$$

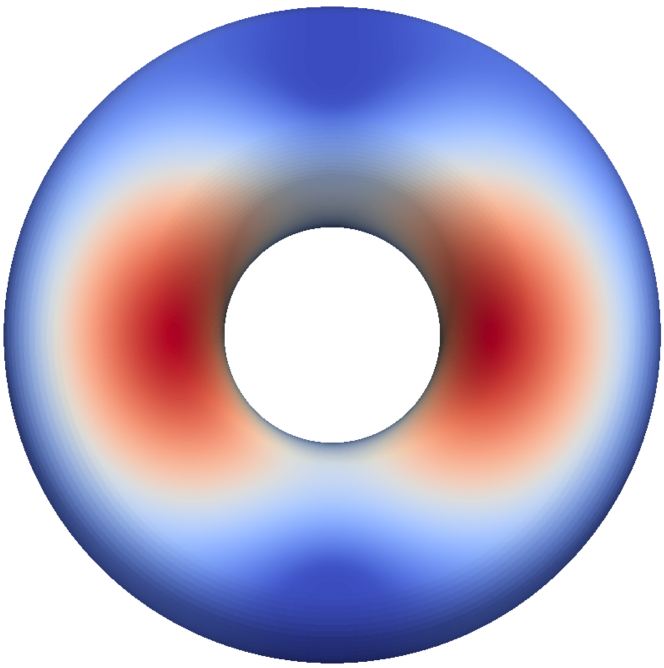
where $\alpha_4 := \max(1, \alpha_1, \alpha_2, \alpha_3)$ and $C''_h \equiv C''_h(\tau, \|u_0\|_{H^1}) := \alpha_4 C^2_{\mathcal{T}} C_h\, C_d(\tau, \|u_0\|_{H^1})$. Letting $K := \|\dot{u}_\tau\|_{H^2} + \|f_\tau\|_{H_{\mathrm{div}}} + \|p_\tau\|_{H^2}$ and given that we have already shown that $\|\delta f\|_{H_{\mathrm{div}}} \le C'_h K$, we deduce that

$$
\begin{aligned}
&\|\delta\dot{u}\|^2_{L^2} + \|\delta p\|^2_{L^2} \le C''_h K h^2(\|\delta\dot{u}\|_{L^2} + \|\delta p\|_{L^2}) + C''_h C'_h K^2 h^4 \\
&\Rightarrow \left(\|\delta\dot{u}\|_{L^2} - \frac{C''_h K h^2}{2}\right)^2 + \left(\|\delta p\|_{L^2} - \frac{C''_h K h^2}{2}\right)^2 \le \frac{C''^2_h K^2 h^4}{2} + C''_h C'_h K^2 h^4 \\
&\qquad\qquad\qquad\qquad\qquad\qquad \Rightarrow \|\delta\dot{u}\|_{L^2} + \|\delta p\|_{L^2} \le \alpha_5 K h^2
\end{aligned}
$$

where $\alpha_5 := C''_h + \left(\frac{1}{2}C''^2_h + C''_h C'_h\right)^{1/2}$. This yields the first inequality. The L^∞ estimate follows then directly by (3.14).

□

4. Numerical Implementation with Subdivision Surfaces



4.1. Introduction and outline

In this chapter, we turn our attention to numerical solutions of the problem. In [VR13], we presented numerical solutions of the (non-regularized) problem (see fig. 4.2 and fig. 4.1), using the natural time discretization of the previous chapter combined with a space discretization based on *discrete exterior calculus*. Discrete exterior calculus is a method for deriving discrete versions of PDEs on simplicial meshes, originally used for the stable approximation of the Navier Stokes equations ([DKT04] and [Hir03]). The core of the method is the association of different types of quantities with different parts of the mesh. In three dimensions for instance, intensive scalar quantities, like pressure, are represented by discrete 0-forms (values on nodes), vector quantities are represented by discrete 1- and 2-forms (circulation over edges and fluxes through tet faces resp.), and extensive quantities, like fluid mass, are represented by discrete 3-forms (total quantity inside tet). When applied to fluid dynamics, this naturally leads to schemes of the finite volume type.

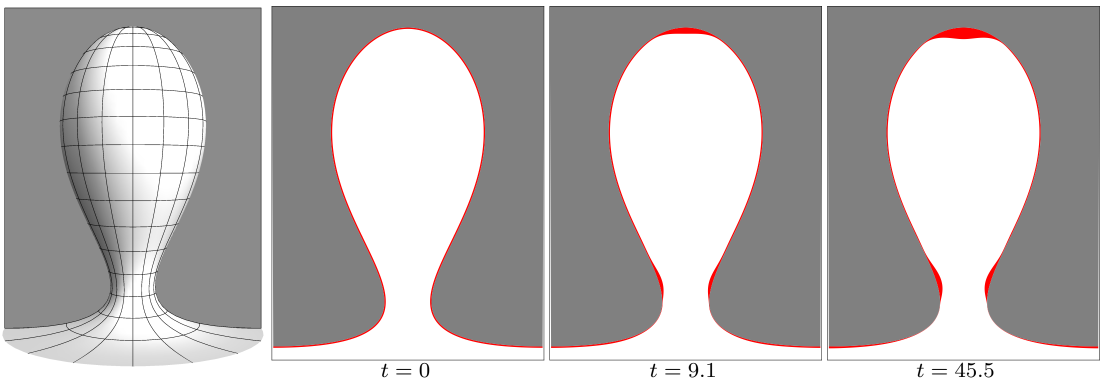


Figure 4.1.: Different time steps of the evolution of a (initially uniform) thin film inside a rotational symmetric cavity *(left)* with a graph of the mass concentration in red.

The numerical scheme of [VR13] conforms, to a certain degree, to the analysis of the previous chapter. The scheme is built on the natural time discretization of the gradient flow (Rem. 3.22), and therefore the free energy is non-increasing between time steps (as per Cor. 3.28). Furthermore, within the framework of the discrete exterior calculus, the primary variable u and the dual variable q are chosen to be in the space of *discrete 0-forms* Ω_h^0, whereas the flux f is taken to be in the space of *discrete 1-forms* Ω_h^1. Since

1. u and q are in the same space, and
2. their gradient (or more precisely their exterior derivative) is in the same space with the flux f,

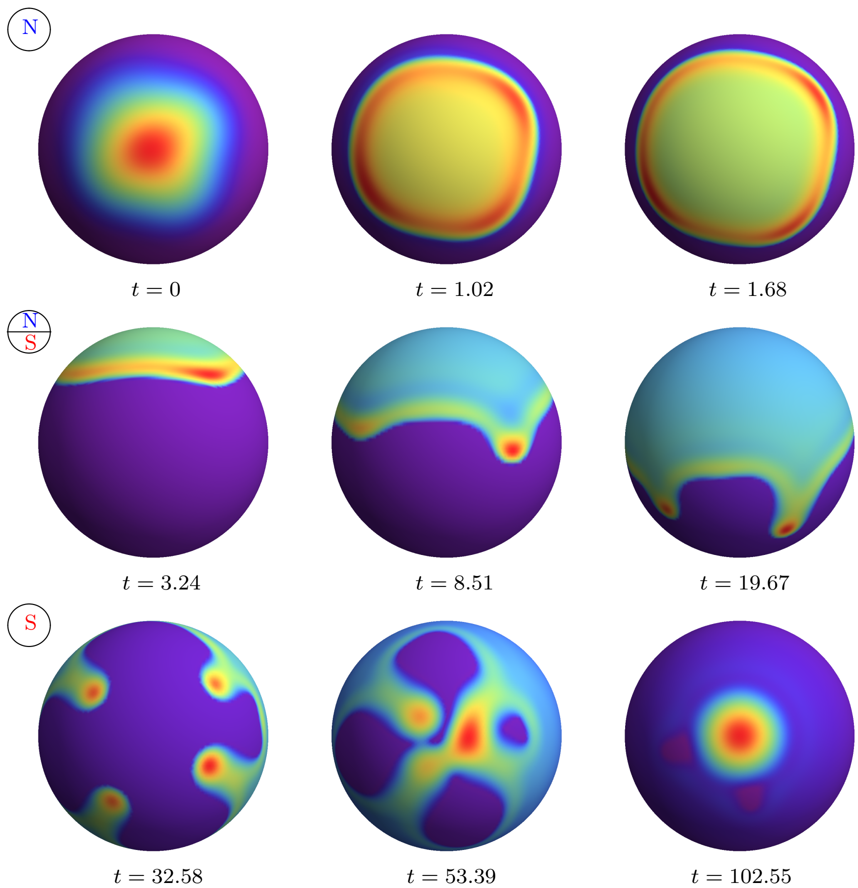


Figure 4.2.: The fingering evolution of a droplet on a sphere is displayed at different times (*top row*, north pole view; *middle row*, equatorial view; *bottom row*, south pole view). The mass concentration is color-coded as .

we can show the well-posedness of the scheme (with a reasoning similar to part 1 of the proof of Prop. 3.33), with a bound for τ independent of the spatial discretization.

The numerical scheme presented in this chapter is based on the *finite elements method*, instead of discrete exterior calculus, and rectifies two issues with the scheme of [VR13]: on one hand the lack of regularization, especially of the mobility, and on the other hand the use of a non-conforming function space for the space discretization. In section 4.2, we take the fully discrete variational problem of Prop. 3.33 and reduce it to a matrix form suitable for numerical calculations. The main challenge is the specification of an H^2-conforming space of basis functions, which we do in Sec. 4.3 with the help of *subdivision surfaces*. Finally, in Sec. 4.4 we present a number of numerical convergence tests for the scheme.

4.2. Galerkin system

We start with the saddle point equations of the Galerkin approximation 3.67:

$$\begin{aligned}
&\langle f_h, \mathcal{M}^{-1}_{[u_h^k]_\lambda} g_h\rangle_{L^2} - \langle g_h, \operatorname{grad}_\Gamma p_h\rangle_{L^2} = 0 \\
&\langle \theta_h, p_h\rangle_{L^2} + \lambda\langle \operatorname{grad}_\Gamma \theta_h, \operatorname{grad}_\Gamma p_h\rangle_{L^2} + \tau\epsilon\langle \dot{u}_h, W_2\theta_h\rangle_{L^2} + \tau\epsilon\langle \operatorname{grad}_\Gamma \dot{u}_h, \operatorname{grad}_\Gamma \theta_h\rangle_{L^2} \\
&\qquad\qquad = -\langle W_1, \theta_h\rangle_{L^2} - \epsilon\langle u_h^k, W_2\theta_h\rangle_{L^2} - \epsilon\langle \operatorname{grad}_\Gamma u_h^k, \operatorname{grad}_\Gamma \theta_h\rangle_{L^2} \\
&\langle \dot{u}_h, q_h\rangle_{L^2} + \lambda\langle \operatorname{grad}_\Gamma \dot{u}_h, \operatorname{grad}_\Gamma q_h\rangle_{L^2} = \langle f_h, \operatorname{grad}_\Gamma q_h\rangle_{L^2}
\end{aligned}$$

for any test functions $(\theta_h, g_h, q_h) \in V_h \times W_h \times V_h$. We recall that

$$\begin{aligned}
W_1 &= \zeta z_\Gamma - H \\
W_2 &= \zeta \mathbf{N}_z - H^2 + 2G \\
\mathcal{M}_{[u]_\lambda} f &= \frac{[u]_\lambda^3}{3} f + \frac{\epsilon [u]_\lambda^4}{6}(Hf + Sf)
\end{aligned}$$

where z_Γ and $\mathbf{N}_z$ are the altitude (z-coordinate) and vertical component of the surface normal $\mathbf{N}$ resp. and S, H, G are the shape operator and the mean and Gaussian curvatures resp. For the definition of $[\cdot]_\lambda$, see Cor. 3.10.

The first saddle point equation gives us immediately

$$\langle f_h, \mathcal{M}^{-1}_{[u_h^k]_\lambda} g_h\rangle_{L^2} = \langle g_h, \operatorname{grad}_\Gamma p_h\rangle_{L^2} \quad \Rightarrow \quad f_h = \mathcal{M}_{[u_h^k]} \operatorname{grad}_\Gamma p_h$$

and so, eliminating f_h, we get the equivalent system:

$$\begin{aligned}
&\tau\epsilon\langle \dot{u}_h, W_2\theta_h\rangle_{L^2} + \tau\epsilon\langle \operatorname{grad}_\Gamma \dot{u}_h, \operatorname{grad}_\Gamma \theta_h\rangle_{L^2} + \langle \theta_h, p_h\rangle_{L^2} + \lambda\langle \operatorname{grad}_\Gamma \theta_h, \operatorname{grad}_\Gamma p_h\rangle_{L^2} \\
&\qquad\qquad = -\langle W_1, \theta_h\rangle_{L^2} - \epsilon\langle u_h^k, W_2\theta_h\rangle_{L^2} - \epsilon\langle \operatorname{grad}_\Gamma u_h^k, \operatorname{grad}_\Gamma \theta_h\rangle_{L^2} \\
&\langle \dot{u}_h, q_h\rangle_{L^2} + \lambda\langle \operatorname{grad}_\Gamma \dot{u}_h, \operatorname{grad}_\Gamma q_h\rangle_{L^2} - \langle \operatorname{grad}_\Gamma p_h, \mathcal{M}_{[u_h^k]_\lambda} \operatorname{grad}_\Gamma q_h\rangle_{L^2} = 0
\end{aligned}$$

for any test functions $(\theta_h, q_h) \in V_h \times V_h$.

If $\{\phi_i\}_{1\leq i\leq N}$ is a basis of V_h, we can write the equations in (symmetric indefinite) matrix form as follows:

$$\begin{pmatrix} \tau\epsilon(M_W + L) & M + \lambda L \\ M + \lambda L & -L_{\mathcal{M}} \end{pmatrix} \begin{pmatrix} \dot{\mathbf{u}} \\ \mathbf{p} \end{pmatrix} = - \begin{pmatrix} \mathbf{w} + \epsilon\,\mathbf{e} \\ 0 \end{pmatrix} \tag{4.3}$$

where $\dot{\mathbf{u}}, \mathbf{p} \in \mathbb{R}^N$ are the coefficients of $\dot{u}_h$ and p_h in the basis, the matrices are all in $\mathbb{R}^{N\times N}$ and defined as

$$M_{ij} = \langle \phi_i, \phi_j \rangle_{L^2} \tag{4.4a}$$

$$(M_W)_{ij} = \langle \phi_i, W_2\phi_j \rangle_{L^2} \tag{4.4b}$$

$$L_{ij} = \langle \operatorname{grad}_\Gamma \phi_i, \operatorname{grad}_\Gamma \phi_j \rangle_{L^2} \tag{4.4c}$$

$$(L_{\mathcal{M}})_{ij} = \langle \operatorname{grad}_\Gamma \phi_i, \mathcal{M}_{[u_h^k]_\lambda} \operatorname{grad}_\Gamma \phi_j \rangle_{L^2} \tag{4.4d}$$

and the vectors $\mathbf{w}, \mathbf{e} \in \mathbb{R}^N$ are defined as

$$\mathbf{w}_i = \langle W_1, \phi_i \rangle_{L^2} \tag{4.4e}$$

$$\mathbf{e}_i = \langle u_h^k, W_2\phi_i \rangle_{L^2} + \langle \operatorname{grad}_\Gamma u_h^k, \operatorname{grad}_\Gamma \phi_i \rangle_{L^2} \tag{4.4f}$$

The system (4.3) is of *saddle point type*, since the coefficient matrix is of the 2×2 block form $\left(\begin{smallmatrix} A & B^T \\ B & -C \end{smallmatrix}\right)$. There exists extensive literature on the effective, direct or iterative, solution of linear systems of this form (see [BGL05]).

4.3. H^2-conforming elements on subdivision surfaces

The main decision that one needs to make while implementing a numerical scheme based on the system (4.3), is the choice of basis functions ϕ_i. The analysis of Sec. 3.6 strongly encourages the use of H^2*-conforming* basis functions, so that $V_h \subset H^2(\Gamma)$. The construction of such a basis is in general a non-trivial problem. Our approach is to limit our attention to *subdivision surfaces* ([CC78] and [Sta98]), where a natural H^2-conforming basis exists by construction.

A subdivision surface Γ_S is determined by a set of *control nodes* connected in a mesh, together with a *subdivision scheme* which prescribes a process of adding new nodes and refining the mesh. The subdivision surface is defined as the limit of the subdivision process. We are interested in particular in *Catmull-Clark* subdivision surfaces, which are defined by quadrilateral control meshes. See fig. 4.3 for the subdivision scheme and fig. 4.4 for some examples. The Catmull-Clark subdivision surfaces are known to be C^2 everywhere, except in certain exceptional points where they are C^1. These exceptional points correspond to control nodes with degree $\neq 4$.

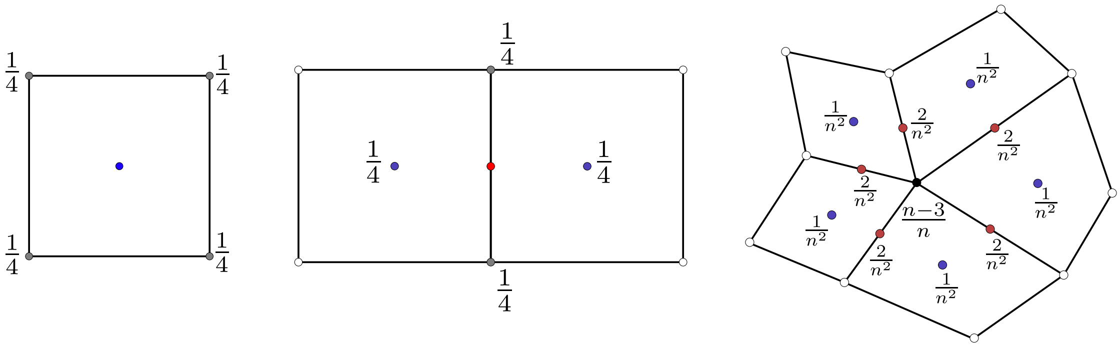


Figure 4.3.: *Catmull-Clark subdivision scheme.* First we calculate new center nodes *(left)*, then new edge nodes *(center)* and finally new positions for the vertex nodes *(right)*. All the nodes are linear combinations of previously calculated nodes, with the weights shown. Each quad is then subdivided into 4 smaller ones.

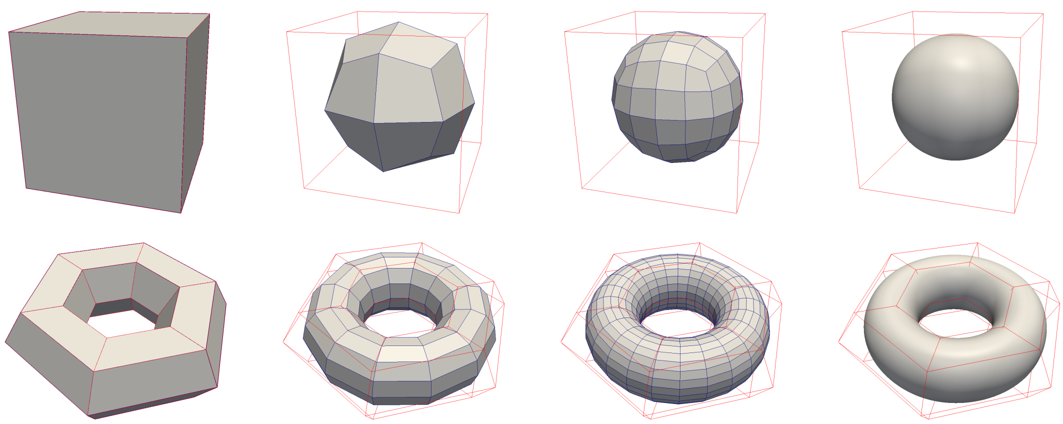


Figure 4.4.: *Catmull-Clark subdivision surfaces.* Control mesh, first and second application of the Catmull-Clark subdivision algorithm and the limit surface, for a cubic *(top row)* and a toroidal *(bottom row)* control mesh.

The weights of the Catmull-Clark scheme have been chosen so that over any regular quad (all control nodes have degree 4) the subdivision surface is a single *bicubic b-spline patch* (see fig. 4.5). If $\mathbf{x}_j \in \mathbb{R}^3$, $j = 1, \dots, 16$, are the coordinates of the 16 control nodes, the subdivision surface can be parametrized over the regular quad as $\mathbf{x}(u,v) = \sum_{j=1}^{16} \phi_j(u,v)\mathbf{x}_j$. The 16 (local) basis functions $\phi(u,v)$ are defined as products of the one-dimensional cubic b-spline basis $\psi_j(t)$, $j \in \{0,1,2,3\}$:

$$\phi_{4\alpha+\beta+1}(u,v) := \psi_\beta(u)\psi_\alpha(v) \tag{4.5a}$$

$$\psi_0(t) := \frac{1}{6}(1-t)^3 \tag{4.5b}$$

$$\psi_1(t) := \frac{1}{6}(3t^3 - 6t^2 + 4) \tag{4.5c}$$

$$\psi_2(t) := \frac{1}{6}(-3t^3 + 3t^2 + 3t + 1) \tag{4.5d}$$

$$\psi_3(t) := \frac{1}{6}t^3 \tag{4.5e}$$

It is worth noting that a single application of the Catmull-Clark subdivision scheme to the control mesh yields a refined control mesh that defines the *same* subdivision surface. In particular, over regular quads this is equivalent to the well-known *refinement property* of bicubic b-splines, which allows one to add control nodes without actually changing the surface.

At exceptional quads, the subdivision surface can not be parametrized by a single b-spline patch. It can be partitioned though into a nested pattern of smaller quads, centered around the exceptional node, each one of which can be covered by such a patch (see fig. 4.6 and [Sta98]). The 16 control nodes of each subpatch are a product of one or more iterations of the Catmull-Clark subdivision scheme, and therefore can be written as a linear combination of the N control nodes ($N = 14$ for a quad with a node of degree 3) of the larger, exceptional quad. It follows that for each subpatch, we can determine a *subdivision matrix* $S \in \mathbb{R}^{16\times N}$, such that the position $\mathbf{x}_i$ of the 16 control nodes of the subpatch, as a function of the control nodes $\mathbf{X}_j$ of the quad, is $\mathbf{x}_i = \sum_{i=1}^{N} S_{ij}\mathbf{X}_j$.

We can combine these subpatches into a parametrisation $\mathbf{x}(u,v)$, $(u,v) \in [0,1]^2 \setminus \{(1,1)\}$, of the quad (minus the exceptional node[1]), as follows. If we identify each of the four nodes of the exceptional quad with the corners of the unit square, so that the exceptional node is matched with $(1,1)$, then each one of the subpatches corresponds in a natural way (see again fig. 4.6) to a subset of the unit square of the form $P^p_{k,l} := [2^{-p}(k-1), 2^{-p}\,k] \times [2^{-p}(l-1), 2^{-p}\,l]$, for a $p \geq 1$ and $1 \leq k \leq 2^p$, $1 \leq l \leq 2^p$ but $(k,l) \neq (2^p, 2^p)$. If we denote by $S^p_{i,j} \in \mathbb{R}^{16\times N}$ the corresponding subdivision matrix,

[1]For a more complete treatment, which covers also the exceptional node, see [Sta98].

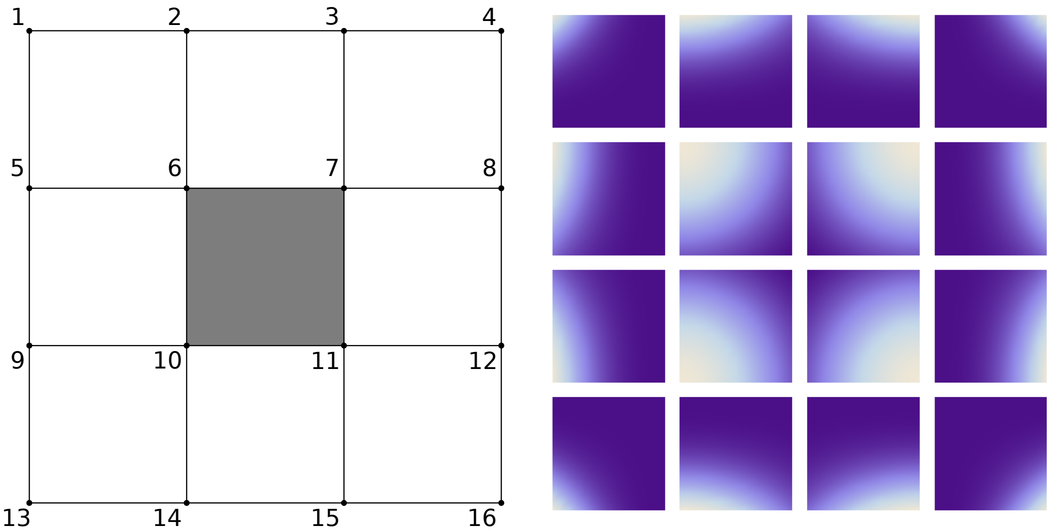


Figure 4.5.: *Basis functions over a regular quad.* The subdivision surface over a regular quad (all nodes have degree 4) is simply a *bicubic B-spline patch.* The shaded patch has 16 control nodes, numbered on the left. The corresponding basis functions are shown on the right.

then we can write the parametrisation as

$$\mathbf{x}(u,v) := \sum_{j=1}^{N}\sum_{i=1}^{16}(S^p_{k,l})_{ij}\,\phi_i(2^p u-(k-1),2^p v-(l-1))\,\mathbf{X}_j, \quad \text{if } (u,v)\in P^p_{k,l} \tag{4.6}$$

From this definition, we can extract the following N (local) basis functions:

$$\Phi_j(u,v) := \sum_{i=1}^{16}(S^p_{k,l})_{ij}\,\phi_i(2^p u-(k-1),2^p v-(l-1)), \quad \text{if } (u,v)\in P^p_{k,l} \tag{4.7}$$

which correspond to the control nodes of the quad (in the order implied by the definition of the subdivision matrix). It is important to note that the subdivision matrix does *not* depend on the location of the control nodes, only on their connectivity. As a result, for any type of exceptional quad (based on the degree of the exceptional node alone) we can precompute the subdivision matrix of any subpatch.

To summarize, for any control node we can evaluate its basis function over any quad that it influences. If the quad is regular, the basis function there is simply one of the bicubic b-spline basis functions; if the quad is exceptional, we can always work on a subpatch, where the basis function is also (locally) a bicubic b-spline according to (4.7). These basis functions can be shown, as a result of the aforementioned refinement

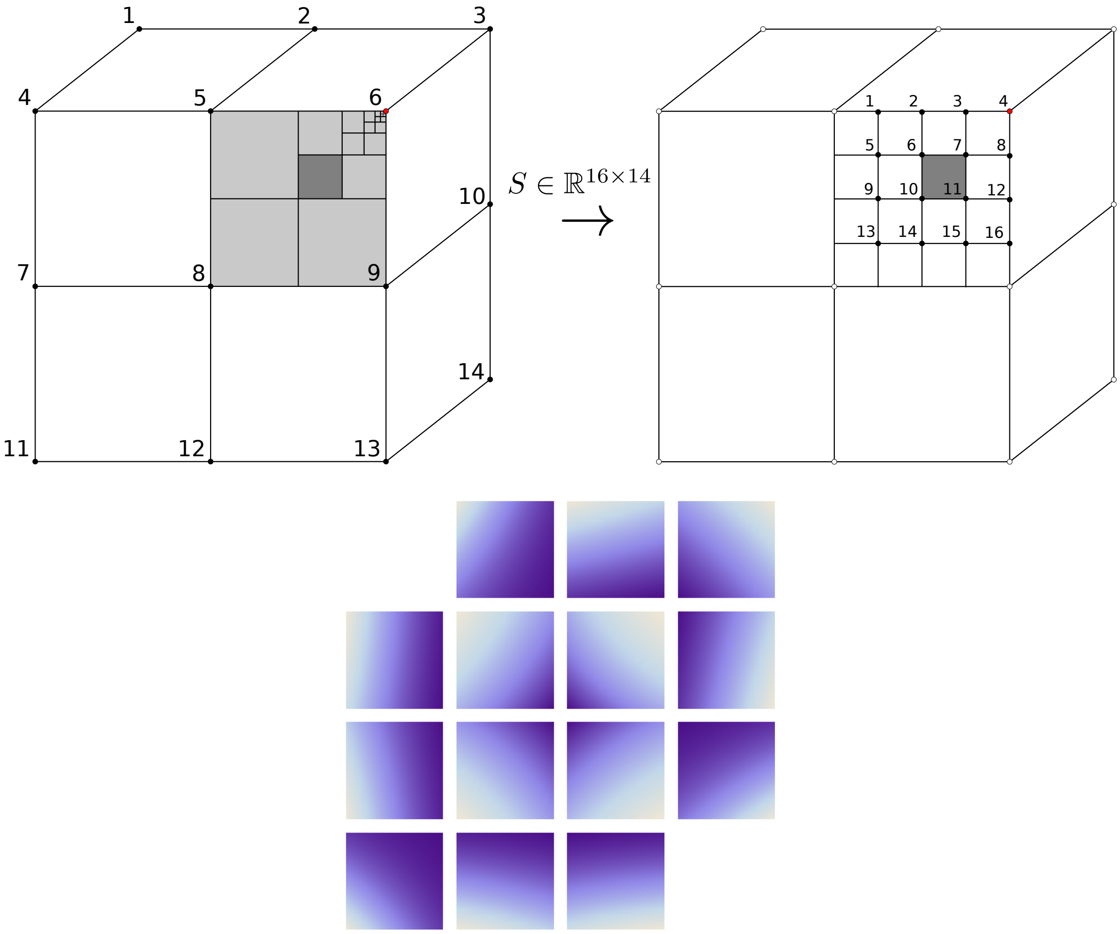


Figure 4.6.: *Basis functions over an exceptional quad.* The subdivision surface over an exceptional quad (a single node with degree $\neq 4$, in this case node 6 with degree 3) can be tiled with a self-similar pattern of patches, so that the subdivision surface is a bicubic b-spline (see fig. 4.5) over each one of them. For any such tile, we can calculate the basis functions there via a *subdivision matrix* S, which is derived from the weights of the Catmull-Clark scheme (see fig. 4.3).

property, to be C^1-continuous everywhere, like the subdivision surface Γ_S itself. It follows that they are in $H^2(\Gamma_S)$ and can be used as an H^2-conforming finite element basis.

4.4. Convergence tests on level sets

In this section we present certain numerical results based on the basis functions of 4.3 and the Galerkin system of 4.2. In each case, we consider a level set of the form $\Gamma_{\psi=c} := \{\mathbf{x} \in \mathbb{R}^3 \,|\, \psi(\mathbf{x}) = c\}$, approximated by (a sequence of) subdivision surfaces Γ_S (see fig. 4.7). We use the basis functions ϕ_i defined in the previous section, together with a 4-point Gauss-Legendre numerical quadrature rule (see fig. 4.8) for the calculation of $\langle\cdot,\cdot\rangle_{L^2}$ products over the subdivision surfaces. The curvature-related quantities are *not* estimated using the subdivision surface,[2] but are instead calculated directly using the level set ψ. At a point $\mathbf{x} \in \mathbb{R}^3$ in the neighborhood of the level set $\Gamma_{\psi=c}$, the shape operator of the level set $\Gamma_{\psi=\psi(\mathbf{x})}$ can be evaluated as $S = -|\nabla\psi|^{-1} P\, \nabla^2\psi\, P$ with curvatures $H = \operatorname{tr} S$ and $G = \frac{1}{2}\left((\operatorname{tr} S)^2 - \operatorname{tr}(S^2)\right)$, with normal $\mathbf{N} = \frac{\nabla\psi}{|\nabla\psi|}$ and tangential projection $P = \mathrm{id} - \mathbf{N} \otimes \mathbf{N}$.

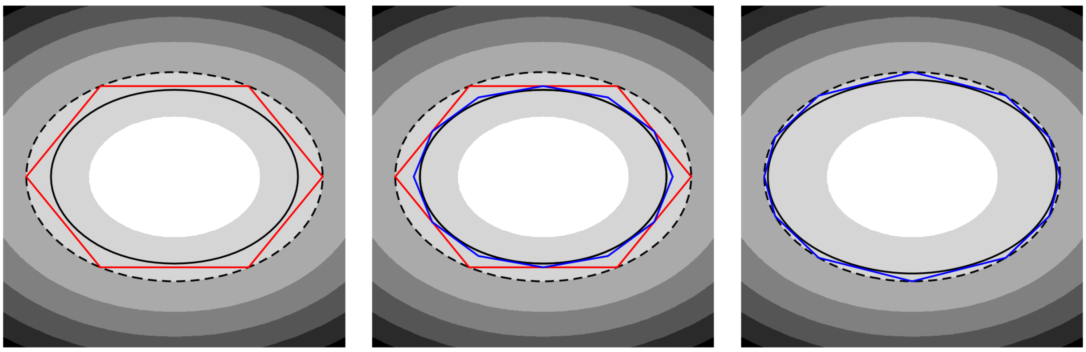


Figure 4.7.: *Spline approximation of a level set.* The level set $\psi(\mathbf{x}) = c$ of a function can be approximated by a spline whose control nodes lie on the level set (*left*; control polygon in red, level set in dashed and spline in solid line). Approximations of higher accuracy can be achieved by adding extra control nodes (*center*; refined control polygon in blue, yields the same spline), and then projecting them onto the level set (*right*; projected control polygon and resulting spline). Although pictured here for curves in $\mathbb{R}^2$, the same procedure works for subdivision surfaces in $\mathbb{R}^3$.

[2]It is known that the curvature of the subdivision surfaces is only L^2, and in fact can be unbounded at the exceptional nodes. Since we need H to be in L^∞, this is not smooth enough.

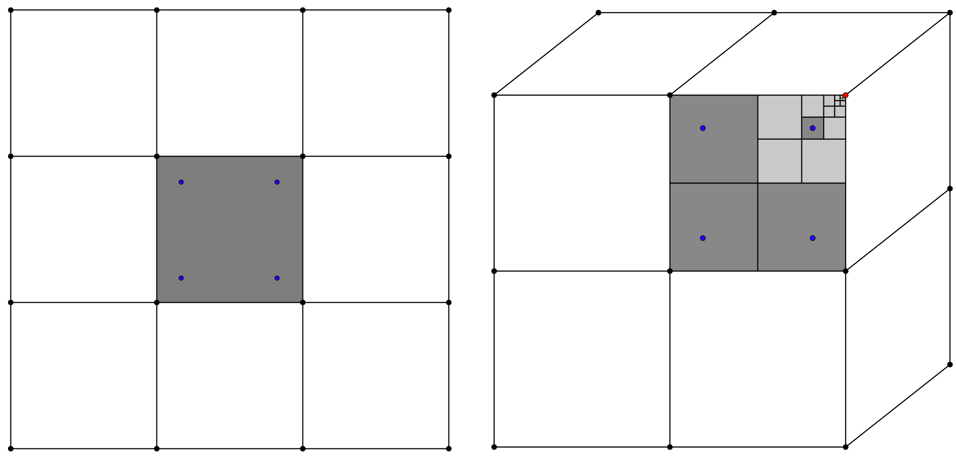


Figure 4.8.: *Gauss-Legendre quadrature points.* We approximate integrals over quads of the sudivision surface with the 4-point Gauss-Legendre quadrature rule, $(u_i, v_i) = \left(\frac{1}{2}(1 \pm 3^{-1/2}), \frac{1}{2}(1 \pm 3^{-1/2})\right) \in [0,1]^2$ with weights $w_i = \frac{1}{4}$. At exceptional quads, each quadrature point lies on one of the b-spline subpatches and can be handled using local coordinates, as per (4.7).

The numerical tests are presented in figures 4.9 - 4.12. In each case we specify the level set $\psi(\mathbf{x}) = c$ that we take as substrate, together with the initial condition (as a function $u_0(\mathbf{x})$ on $\mathbb{R}^3$). To test the estimates of Prop. 3.33, i.e. the accuracy of the Galerkin approximation, we consider a sequence of control meshes and their associated subdivision surfaces, which approximate the level set with various number of elements. For each of the meshes, we solve the Galerkin system (4.3) for $\dot{u}$ and p, and compare to the solution of the Galerkin system for the finest mesh. We do this in various norms and show the results in log-log plots. We do observe convergence in the L^∞-, L^2- and H^1-norms and, once there are enough elements to resolve the fluid distribution and the geometry of the substrate, we also observe the quadratic accuracy in the L^2-norm as expected by the estimates of Prop. 3.33. Note that in the log-log plots of the figures 4.9-4.12, this shows up as *linear* slope, since the x-axis represents $N_{\text{elements}} \sim h^{-2}$. To test the convergence of the time discretization (Prop. 3.30), we take a fixed final time T and evolve the initial condition up to that time, using a fixed mesh and a sequence of diminishing time steps τ. Again we compare with the solution $u_{t=T}$ given by the smallest τ in various norms, and collect the results in a log-log plot. Numerical convergence is again observed in agreement with Prop. 3.30, particularly once the time step τ is small enough to satisfy a CFL-type condition with respect to the velocity of the spreading fluid on the substrate.

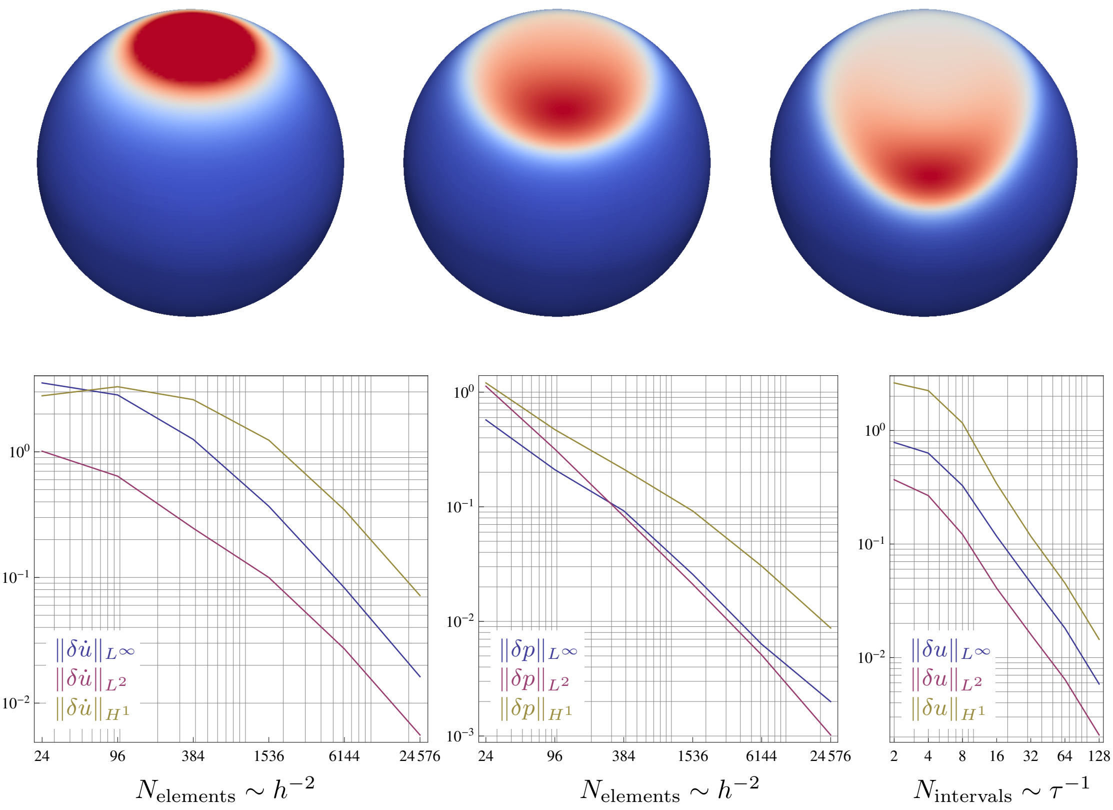


Figure 4.9.: *Droplet on sphere.* Level set $\psi(\mathbf{x}) = 1$ of $\psi(\mathbf{x}) = |\mathbf{x}|^2$. Initial condition $u_0(\mathbf{x}) = \exp(-\frac{|\mathbf{x}-\mathbf{C}|^2}{d^2})$ with $\mathbf{C} = (-\frac{1}{2}, 0, \frac{\sqrt{3}}{2})$ and $d = 0.4$. Constants $(\zeta, \epsilon, \lambda, \tau) = (10, 0.05, 0.005, 0.01)$. Convergence of $\dot{u}$ and p (*bottom left and center*) was tested using control meshes with 24, 96, 384, 1536, 6144 and 24576 elements, compared to a mesh with 98304 elements. Convergence of u (*bottom right*) was tested by solving up to time $T = 1$ with 24576 elements, using 2, 4,…, 128 equal time subintervals, compared to 256 subintervals. The evolution of the droplet is pictured for $t = 0, 0.39, 1$ (*top row*; view from the side). High concentration in red, low in blue.

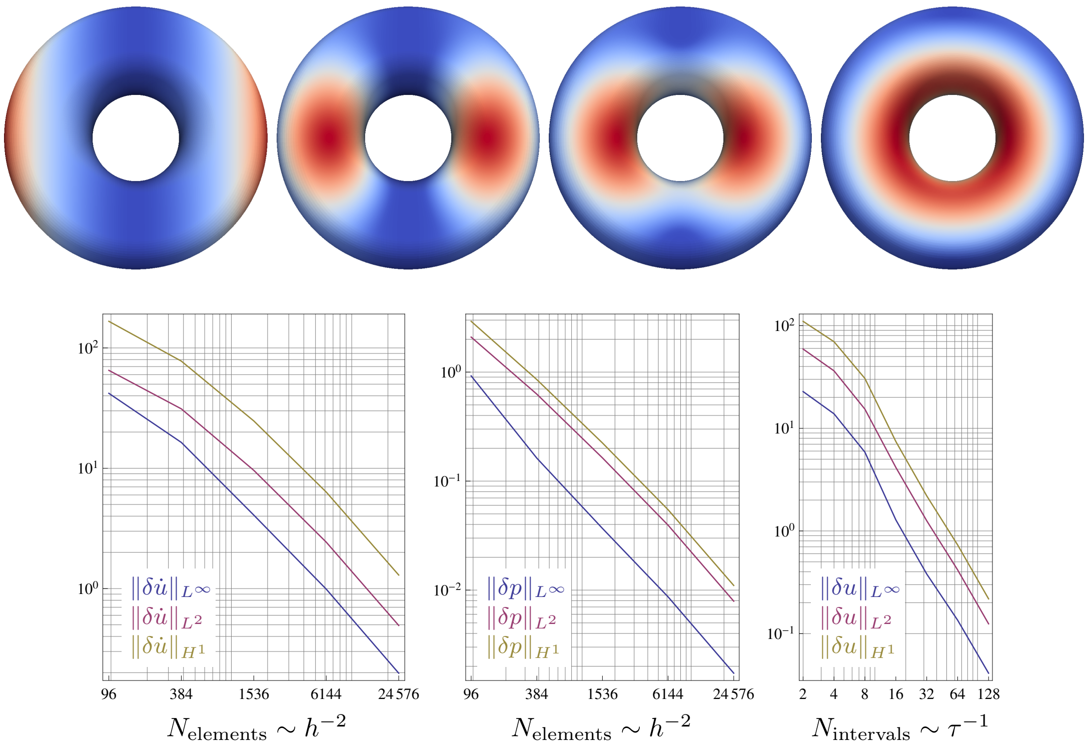


Figure 4.10.: *Droplets on torus.* Level set $\psi(\mathbf{x}) = 0$ of $\psi(\mathbf{x}) = (\sqrt{x^2+y^2} - R)^2 + z^2 - r^2$ with $R = 2$ and $r = 1$. Initial condition $u_0(\mathbf{x}) = x^2 + 0.1$ and constants $(\zeta, \epsilon, \lambda, \tau) = (1, 0.05, 0.005, 0.01)$. Convergence of $\dot{u}$ and p (*bottom left and center*) was tested using control meshes with 96, 384, 1536, 6144 and 24576 elements, compared to a mesh with 98304 elements. Convergence of u (*bottom right*) was tested by solving up to time $T = 1$ with 24576 elements, using 2, 4,..., 128 equal time subintervals, compared to 256 subintervals. The evolution of the droplets is pictured for $t = 0, 0.2, 0.45, 1$ (*top row*; view from below). High concentration in red, low in blue.

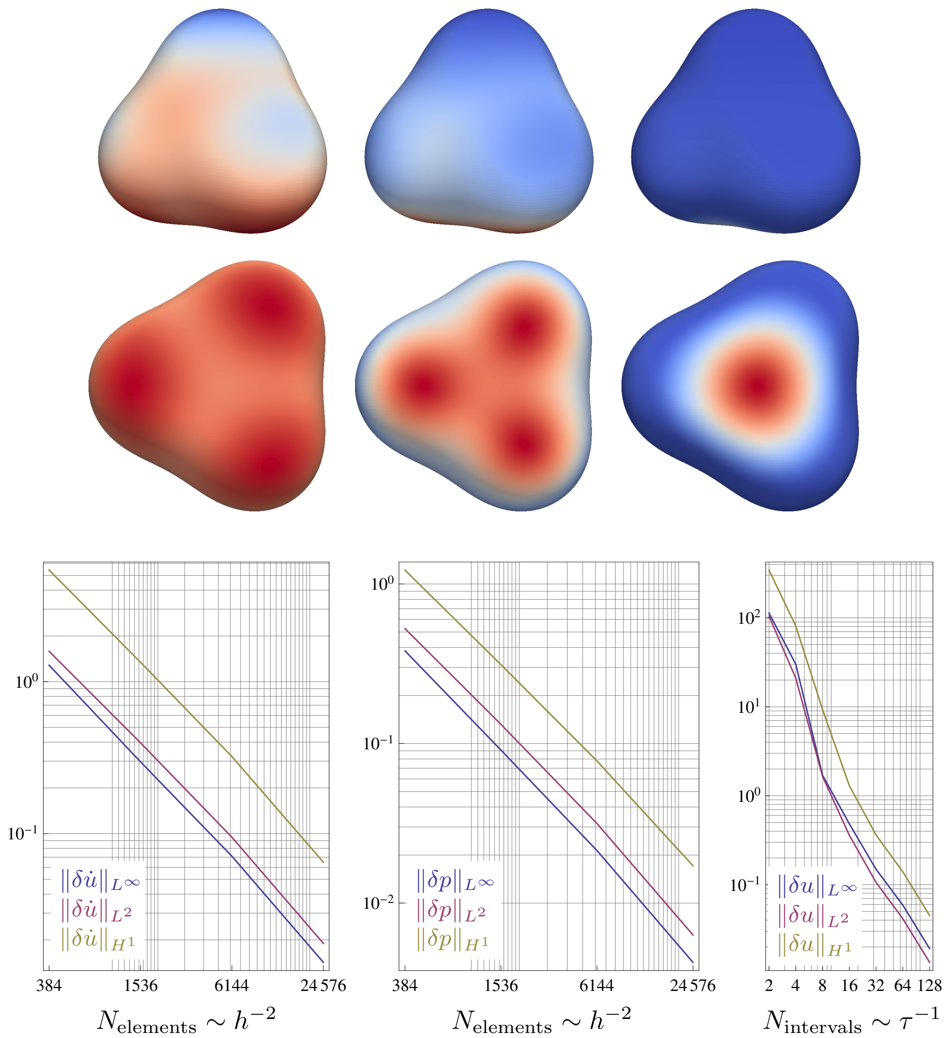


Figure 4.11.: *Uniform coating on tetrahedral surface.* Level set $\psi(\mathbf{x}) = 0.4$ of $\psi(\mathbf{x}) = \sum_{i=1}^{4} \exp(-\frac{|\mathbf{x}-\mathbf{C}_i|^2}{d^2})$ with $d = 0.6$, $\mathbf{C}_1 = (0, 0, 0.61)$, $\mathbf{C}_2 = (-0.29, -0.5, -0.20)$, $\mathbf{C}_3 = (-0.29, 0.5, -0.20)$ and $\mathbf{C}_4 = (0.58, 0, -0.20)$. Uniform intial condition $u_0(\mathbf{x}) = 1$ and constants $(\zeta, \epsilon, \lambda, \tau) = (10, 0.05, 0.005, 0.05)$. Convergence of $\dot{u}$ and p (*bottom left and center*) was tested using control meshes with 384, 1536, 6144 and 24576 elements, compared to a mesh with 98304 elements. Convergence of u (*bottom right*) was tested by solving up to time $T = 0.25$ with 24576 elements, using 2, 4,. . . , 128 equal time subintervals, compared to 256 subintervals. The evolution of the film is pictured for $t = 0.001, 0.05, 0.25$ (view from the side in *top row*, view from below in *middle row*). High concentration in red, low in blue.

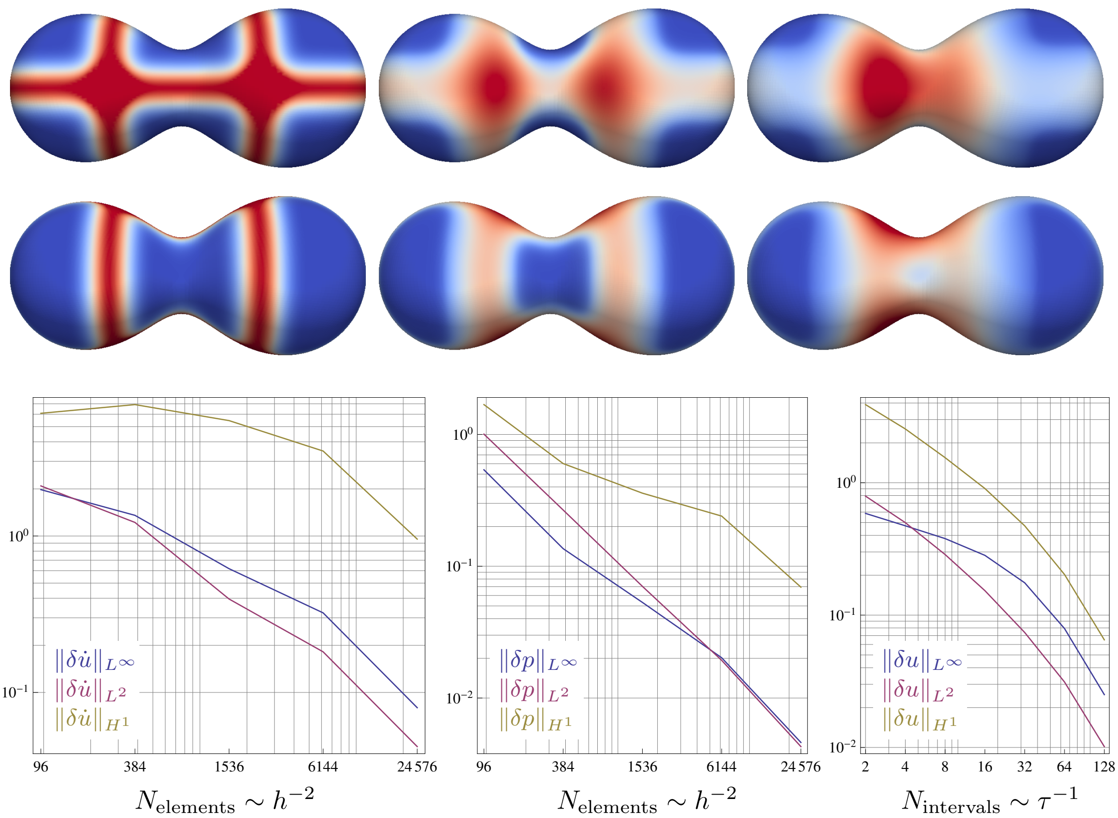


Figure 4.12.: *Bands on dumbbell.* Level set $\psi(\mathbf{x}) = 0.5$ of $\psi(\mathbf{x}) = \sum_{i=1}^{2} A_i \exp(-|\mathbf{x} - \mathbf{C}_i|^2)$ with $A_1 = 0.95$, $\mathbf{C}_1 = (-1.1, 0, 0)$, $A_2 = 1.05$ and $\mathbf{C}_2 = (1.1, 0, 0)$. Initial condition $u_0(\mathbf{x}) = \rho_{0.01,1}(\exp(-\frac{z^2}{d^2}) + \exp(-\frac{(x-0.75)^2}{d^2}) + \exp(-\frac{(x+0.75)^2}{d^2}))$ with $d = 0.2$ (see Def. 3.9 for ρ). Constants $(\zeta, \epsilon, \lambda, \tau) = (0, 0.1, 0.005, 0.05)$, i.e. *no gravity.* Convergence of $\dot{u}$ and p (*bottom left and center*) was tested using control meshes with 96, 384, 1536, 6144 and 24576 elements, compared to a mesh with 98304 elements. Convergence of u (*bottom right*) was tested by solving up to time $T = 12$ with 24576 elements, using 2, 4,. . . , 128 equal time subintervals, compared to 256 subintervals. The evolution of the bands is pictured for $t = 0, 2.34, 12$ (view from the side in *top row*, view from the top in *middle row*). High concentration in red, low in blue.

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