Homotopy-Theoretic Studies of Khovanov-Rozansky Homology

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Für meine Eltern

"I believe that the justification of art is the internal combustion it ignites in the hearts of men..."

— Glenn Gould

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Danksagung

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0. Einleitung

Die vorliegende Dissertation untersucht die von Khovanov und Rozansky definierte Khovanov-Rozansky-Homologie [KR08a], eine Invariante orientierter Knoten und Verschlingungen im Raum, mit Methoden der Homotopietheorie. Sie gliedert sich in zwei Teile, einen Teil „Knotentheoretische Aspekte“, in dem der Fokus auf der konkreten Anwendung homotopietheoretischer Techniken in der Khovanov-Rozansky-Homologie liegt, und einen Teil „Homotopietheoretische Aspekte“, in dem die Techniken unabhängig und im Streben nach größtmöglicher Allgemeinheit untersucht werden. Detaillierte Beschreibungen der Inhalte der beiden Teile finden sich jeweils an deren Anfang.

In dieser Einleitung beschreibe ich zunächst in den voneinander unabhängigen Abschnitten A, B, C den knotentheoretischen, algebraischen und homotopietheoretischen Kontext der Arbeit und skizziere anschließend in Abschnitt D die untersuchten Probleme und Ergebnisse der Arbeit. Leser, die mit den Grundlagen zu Quanteninvarianten und/oder Singularitätenkategorien und/oder Homotopietheorie vertraut sind, können die entsprechenden Abschnitte gefahrlos überspringen.

A. Quanteninvarianten und die Idee der Kategorifizierung

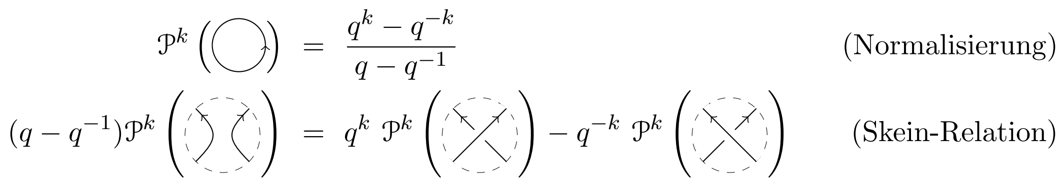
Eine allgemeinverständliche Einführung in die Probleme und Methoden der Knotentheorie ohne Voraussetzung mathematischen Hintergrunds findet sich im Anhang auf S. 277. Für einführende Literatur siehe zum Beispiel [Lic97; Rol76].

Sei $k \geq 2$ eine ganze Zahl. Die in dieser Arbeit untersuchte *Khovanov-Rozansky-Homologie* $\mathcal{KR}^k$ [KR08a] ist eine Verfeinerung der *Quanten-*$\mathfrak{sl}(k)$*-Knoteninvariante* $\mathcal{P}^k$ im folgenden Sinne: Während $\mathcal{P}^k$ einer orientierten Verschlingung L ein Laurent-Polynom $\mathcal{P}^k(L) \in \mathbb{Z}[q^{\pm 1}]$ in einer Variablen q zuordnet, so ist der Wert von $\mathcal{KR}^k$ an L ein Laurent-Polynom $\mathcal{KR}^k(L) \in \mathbb{Z}[q^{\pm 1}, t^{\pm 1}]$ in *zwei* Variablen q und t, welches bei Ersetzung von t durch -1 in $\mathcal{P}^k(L)$ übergeht:

$$\mathcal{KR}^k(L)|_{t=-1} \;=\; \mathcal{P}^k(L) \qquad \text{(Kat)}$$

Insbesondere werden je zwei durch $\mathcal{P}^k$ unterschiedene Knoten auch durch $\mathcal{KR}^k$ unterschieden, was die Bezeichnung von $\mathcal{KR}^k$ als Verfeinerung von $\mathcal{P}^k$ rechtfertigt; hingegen gibt es Paare von Knoten, die durch $\mathcal{KR}^k$ unterschieden werden, nicht aber durch $\mathcal{P}^k$, sodass sogar eine echte Verfeinerung vorliegt. Eine Invariante mit der Eigenschaft (Kat) heißt auch *Kategorifizierung* von $\mathcal{P}^k$; wir werden die Motivation für diese Terminologie weiter unten beleuchten.

Die Quanten-$\mathfrak{sl}(k)$-Invariante $\mathcal{P}^k$ ist eine Verallgemeinerung des im Falle $k = 2$ erhaltenen *Jones-Polynoms* $\mathcal{P}^2$ [Jon85] und durch die beiden Eigenschaften



eindeutig bestimmt; dass $\mathcal{P}^k$ eine Knoteninvariante ist bedeutet ferner, dass

$$\mathcal{P}^k(L) = \mathcal{P}^k(L'), \qquad \text{wenn } L \text{ und } L' \text{ äquivalente Knoten beschreiben.} \qquad \text{(Invariante)}$$

Die Notation in (Skein-Relation) ist hierbei so zu verstehen, dass sich die drei Anwendungen von $\mathcal{P}^k$ auf Knotendiagramme beziehen, die außerhalb des gestrichelten Bereichs identisch sind und sich innerhalb unterscheiden mögen wie angegeben. Abbildung 1 und Abbildung 2 vermitteln einen Eindruck von Berechnungen von Werten von $\mathcal{P}^k$ unter Verwendung der Eigenschaften (Normalisierung), (Skein-Relation) und (Invariante). Die *Existenz* von $\mathcal{P}^k$ kann über die im folgenden Absatz kurz skizzierte Reshitikhin-Turaev-Konstruktion eingesehen werden, die eine generische planare Projektion von L als diagrammatische Schreibweise für einen Morphismus von Moduln über der Quantengruppe $\mathcal{U}_q(\mathfrak{sl}(k))$ von $\mathfrak{sl}(k)$ interpretiert (daher auch der Name Quanten-$\mathfrak{sl}(k)$-Invariante).

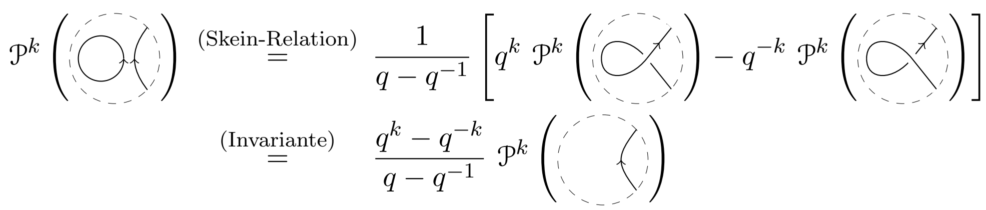


Abbildung 1. Verhalten von $\mathcal{P}^k$ unter disjunkter Vereinigung mit Unknoten

Die Quantengruppe $\mathcal{U}_q(\mathfrak{sl}(k))$ ist eine über dem Körper $\mathbb{C}(q)$ der rationalen Funktionen über $\mathbb{C}$ definierte Deformation der universellen einhüllenden Algebra $\mathcal{U}(\mathfrak{sl}(k))$ von $\mathfrak{sl}(k)$. Sie ist abermals eine Hopf-Algebra, die zwar nicht kokommutativ ist (und für die daher die naive Vertauschungsabbildung $V \otimes W \to W \otimes V$ nicht $\mathcal{U}_q(\mathfrak{sl}(k))$-linear ist), die aber für endlich-dimensionale Moduln V, W dennoch spezielle „Zopfungs"-Isomorphismen $\Theta_{V,W} : V \otimes W \cong W \otimes V$ von $\mathcal{U}_q(\mathfrak{sl}(k))$-Moduln zulässt, welche zwar nicht mehr $\Theta_{V,W} \circ \Theta_{W,V} =$

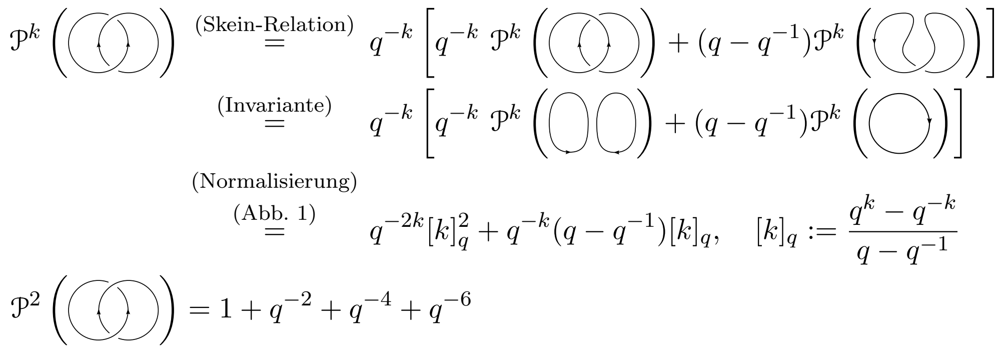


Abbildung 2. Berechnung der Quanten-$\mathfrak{sl}(k)$-Invariante des Hopf-Links

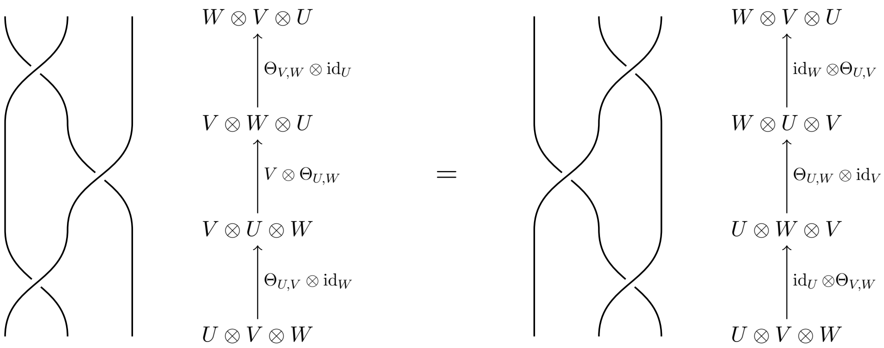


Abbildung 3. Die Zopf-Relation für die Zopfungs-Isomorphismen in $\mathcal{U}_q(\mathfrak{sl}(k))$-mod

$\mathrm{id}_{V\otimes W}$, aber immer noch die Zopf-Relationen erfüllen – siehe Abbildung 3. Sie erlauben ferner eine Identifikation von Links- und Rechtsdualen ${}^*V \cong V^*$.

Dies vorausgesetzt verläuft die Reshitikhin-Turaev-Konstruktion wie folgt (für Details siehe [Kas95, Kapitel XIV], insbesondere [Kas95, Theorem XIV.5.1]): Zuächst wird jedem nach unten bzw. oben gerichteten Strang die Vektordarstellung V von $\mathcal{U}_q(\mathfrak{sl}(k))$ (dem Analogon der $\mathfrak{sl}(k)$-Darstellung auf $\mathbb{C}^k$ durch Matrix-Vektor-Multiplikation) bzw. ihr Dual V^* zugeordnet, und jeder horizontalen Konkatenation von Strängen das Tensorprodukt der entsprechenden Moduln, wobei das leere Tensorprodukt als die triviale Darstellung von $\mathcal{U}_q(\mathfrak{sl}(k))$ im Grundkörper $\mathbb{C}(q)$ zu verstehen ist. Im zweiten Schritt wird die gewählte planare Projektion von L in Bausteine zerlegt und jeder von diesen als Morphismus interpretiert wie beispielhaft in Abbildung 4 angegeben. Schließlich werden die den Bausteinen zugeordneten Morphismen durch Tensorieren und Verketten zu einem Endomorphismus des trivialen Moduls $\mathbb{C}(q)$ zusammengesetzt; dieser ist durch Multiplikation mit einem Skalar in $\mathbb{C}(q)$ gegeben, und dieser Skalar ist bis auf eine abschließende

Normalisierung gleich $\mathcal{P}^k(L)$. Zwei Spezialfälle sind von besonderem Interesse:

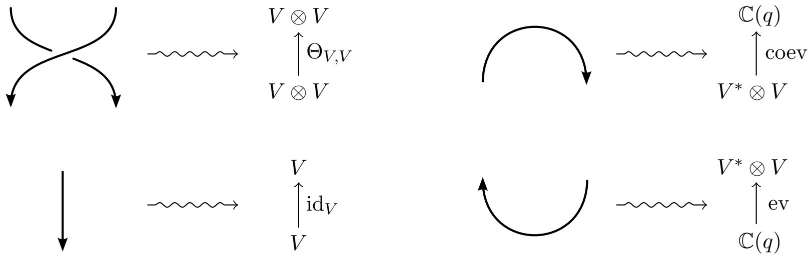


Abbildung 4. Planare Knotendiagramme als Morphismen in $\mathcal{U}_q(\mathfrak{sl}(k))$-mod.

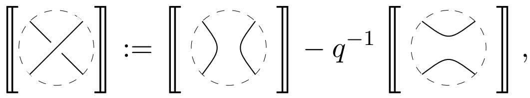
– Für Zöpfe in n Strängen erhalten wir eine Interpretation in $\mathrm{End}_{\mathcal{U}_q(\mathfrak{sl}(k))}(V^{\otimes n})$. Bezeichnet Br_n die Zopf-Gruppe auf n Strängen und $\mathbb{C}(q)[\mathrm{Br}_n]$ ihre Gruppenalgebra über $\mathbb{C}(q)$, so ist der resultierende Algebra-Homomorphismus $\mathbb{C}(q)[\mathrm{Br}_n] \to \mathrm{End}_{\mathcal{U}_q(\mathfrak{sl}(k))}(V^{\otimes n})$ surjektiv und faktorisiert über die *(Iwahori-)Hecke-Algebra*

$$\mathcal{H}_n(q) \;:=\; \mathbb{C}(q)[\mathrm{Br}_n]/(\mathrm{T}_i^2 = (q^2-1)\,\mathrm{T}_i + q^2\,\mathrm{T}_e);$$

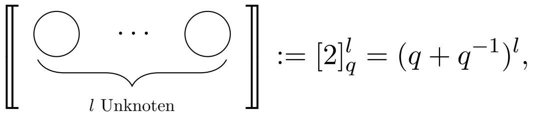
das Abschließen von Zöpfen entspricht dem Anwenden einer $\mathbb{C}(q)$-linearen Spurform $\mathcal{H}_n(q) \to \mathbb{C}(q)$, einer Spezialisierung der zwei-Variablen Ocneanu-Spur, über die das HOMFLYPT-Polynom [Fre+85] konstruiert werden kann; siehe z.B. [KT08, §4.4].

– Im Fall $k = 2$ ist $V \cong V^*$ in $\mathcal{U}_q(\mathfrak{sl}(2))$-mod, und die Zopfung $\Theta_{V,V} : V \otimes V \xrightarrow{\cong} V \otimes V$ und ihr Inverses sind jeweils $\mathbb{C}(q)$-Linearkombinationen der Identität und der Verkettung $V \otimes V \to \mathbb{C}(q) \to V \otimes V$ von Evaluation und Koevaluation. Alle für die Konstruktion von $\mathcal{P}^2$ relevanten Morphismen sind daher bereits in der freien, $\mathbb{C}(q)$-linearen monoidalen Kategorie über einem selbstdualen Objekt definiert, welche folgende konkrete Beschreibung zulässt, siehe [FY89, Theorem 4.1.1]: Objekte sind nicht-negative ganze Zahlen $\mathbf{0}, \mathbf{1}, \ldots$, Morphismen $\mathbf{n} \to \mathbf{m}$ formale $\mathbb{C}(q)$-Linearkombinationen planarer, unorientierter und kreuzungsfreier (n,m)-Tangles bis auf Isotopie (siehe Abbildung 5), und die monoidale Struktur ist auf Objekten durch Addition und auf Morphismen durch horizontale Konkatenation gegeben. Herausteilen der in der Reshitikhin-Turaev-Interpretation gültigen Relation $\bigcirc = (q + q^{-1})\,\mathrm{id}_{\mathbf{0}}$ in $\mathrm{End}(\mathbf{0})$ ergibt schließlich die sog. Temperley-Lieb-Kategorie $\mathcal{TL}$, und man erhält die diagrammatische Konstruktion des

Jones-Polynoms über die $\mathcal{TL}(\mathbf{0},\mathbf{0}) \cong \mathbb{C}(q)$-wertige Kauffman-Klammer $[\![\ ]\!]$,



(K1)



(K2)

($[\![\ ]\!]$ hängt nicht von der Orientierung des betrachteten Knotendiagramms ab) und anschließende Normalisierung um den Faktor $q^{n_-(L)-n_+(L)}$, wobei $n_+(L)$ und $n_-(L)$ die Anzahl der „positiven“ und „negativen“ Kreuzungen  und  im gewählten planaren Diagramm sind. Die Berechnung von $\mathcal{P}^2$  über die Kauffman-Klammer (im Einklang mit dem Ergebnis aus Abbildung 2) findet sich in Abbildung 6.

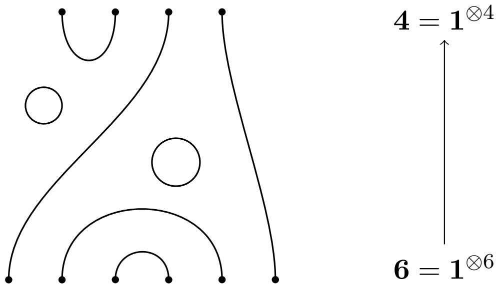


Abbildung 5. Ein $(6,4)$-Tangle als Morphismus $\mathbf{1}^{\otimes 6} \to \mathbf{1}^{\otimes 4}$ in der freien monoidalen Kategorie über einem selbst-dualen Objekt $\mathbf{1}$

Die obige Definition von Kategorifizierung rührt nun daher, dass man eine Verfeinerung der Invariante $\mathcal{P}^k$ anstreben kann, indem man die in der Konstruktion von $\mathcal{P}^k$ involvierten Strukturen – z.B. die Temperley-Lieb-Kategorie $\mathcal{TL}$ oder die Hecke-Algebren $\mathcal{H}_n(q)$ mitsamt ihrer Spurformen – kategorifiziert, und dass das Ergebnis der erfolgreichen Umsetzungen dieser Idee bisher häufig eine zwei-Variablen-Invariante mit der oben genannten Eigenschaft (Kat) war. Der hierbei verwendete Begriff von „Kategorifizieren einer mathematischen Struktur“ ist nicht präzise definiert, meint aber beispielsweise im Falle algebraischer Strukturen oft die Konstruktion abelscher oder triangulierter Kategorien (respektive Funktoren zwischen oder monoidalen Strukturen auf ihnen), die bei Anwenden der Grothendieck-Konstruktion K_0 in die gegebenen Moduln (respektive ihre Abbildungen oder Ringstrukturen) übergehen. Informell kann man sagen, man realisiere eine gegebene algebraische Struktur als die unterliegende Kombinatorik einer geeigneten kategoriellen Struktur.

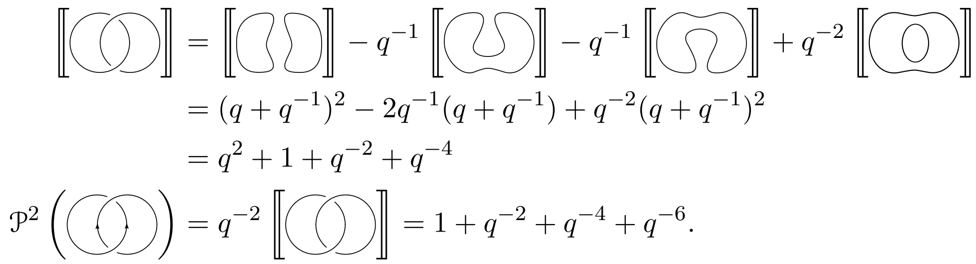


Abbildung 6. Berechnung von $\mathcal{P}^2$ ( ) über die Kauffman-Klammer

Das erste und einfachste Beispiel einer Kategorifizierung ist die in [Kho00] eingeführte *Khovanov-Homologie* $\mathcal{KH}$, eine Kategorifizierung des Jones-Polynoms $\mathcal{P}^2$, die auf der obigen Konstruktion von $\mathcal{P}^2$ über die Kauffman-Klammer $[\![\ ]\!]$ beruht. Für die Definition von $\mathcal{KH}$ versteht man den Ring der Laurent-Polynome als Dekategorifizierung der monoidalen Kategorie der endlich-dimensionalen graduierten Vektorräume – bis auf Isomorphie ist ein endlicher-dimensionaler graduierter Vektorraum durch die Dimensionen seiner homogenen Anteile bestimmt, welche in einem ganzzahligen Laurent-Polynom kodiert werden können, und das Tensorieren graduierter Vektorräume entspricht unter dieser Zuordnung der Multiplikation von Laurent-Polynomen – und realisiert die beim Auflösen der Definition (K1) von $[\![\ ]\!]$ entstehende alternierende Summe von Laurent-Polynomen als Euler-Charakteristik eines geeigneten Komplexes graduierter Vektorräume. Die Kodierung der Kohomologie dieses Komplexes als Laurent-Polynom in zwei Variablen – jeweils eine für die interne und die kohomologische Graduierung – ist dann eine Kategorifizierung von $\mathcal{P}^2$ im Sinne von (Kat).

Ebenso wie Khovanov-Homologie beruht die in dieser Arbeit untersuchte Khovanov-Rozansky-Homologie $\mathcal{KR}^k$ in ihrer ursprünglichen Konstruktion auf einem grafischen Kalkül zur Berechnung der Quanten-$\mathfrak{sl}(k)$-Invariante, der in [MOY98] eingeführt wurde. Ähnlich der zweischrittigen Konstruktion des Jones-Polynoms $\mathcal{P}^2$ durch die Kauffman-Klammer (K1), (K2) wird im MOY-Kalkül ein orientiertes, planares Knotendiagramm zunächst durch Auflösen seiner Kreuzungen ⤧ und ⤨ in eine formale alternierende Summe sog. MOY-Graphen überführt, von denen anschließend jeder einzelne einer fixen Kombinatorik folgend zu einem Laurent-Polynom ausgewertet wird; siehe [KR08a, Figures 2 & 3]. Der wesentliche Schritt in der Konstruktion von $\mathcal{KR}^k$ ist nun das Kategorifizieren der MOY-Graphen und ihrer Auswertungskombinatorik durch Tensorprodukte von Komplexen von *Matrixfaktorisierungen.*

B. Matrixfaktorisierungen und Singularitätenkategorien

Definition. *Sei A ein kommutativer Ring und $w \in A$ beliebig, genannt Potential.*

(i) *Eine* lineare Faktorisierung *vom Typ (A, w) ist ein Diagramm $M^0 \xrightarrow{f} M^1 \xrightarrow{g} M^0$ von A-Moduln und A-Modulhomomorphismen mit $fg = w \cdot id_{M^1}$ und $gf = w \cdot id_{M^0}$.*

(ii) *Eine* Matrixfaktorisierung *vom Typ (A, w) ist eine lineare Faktorisierung vom Typ (A, w) wie in (i), mit der Eigenschaft, dass M^0 und M^1 projektive A-Moduln sind.*

Für $M^0 = M^1 = A^n$ sind f, g durch Matrizen $X, Y \in \mathrm{Mat}_{n\times n}(A)$ gegeben, die $XY = YX = w \cdot \mathrm{id}_n$ erfüllen – daher der Name. Beispielsweise bilden die Matrizen

$$X := \begin{pmatrix} y & -x & 0 \\ 0 & y & -x \\ x & 0 & y^3 \end{pmatrix} \qquad \text{und} \qquad Y := \begin{pmatrix} y^4 & xy^3 & x^2 \\ -x^2 & y^4 & xy \\ -xy & -x^2 & y^2 \end{pmatrix}$$

eine Matrixfaktorisierung vom Typ $(\mathbb{Z}[x, y], x^3 + y^5)$. Weiterhin können Matrixfaktorisierungen als 2-periodische Komplexe projektiver A-Moduln verstanden werden, in denen die Komplex-Bedingung $\delta^2 = 0$ zu $\delta^2 = w \cdot \mathrm{id}$ abgewandelt wurde, und dieser Analogie folgend definiert man eine *Homotopiekategorie von Matrixfaktorisierungen* $\underline{\mathrm{MF}}(A, w)$; die volle Unterkategorie von $\underline{\mathrm{MF}}(A, w)$ bestehend aus den Matrixfaktorisierungen, deren Komponenten M^0 und M^1 zusätzlich endlich erzeugt sind, sei mit $\underline{\mathrm{MF}}^b(A, w)$ bezeichnet.

Ursprünglich wurden Matrixfaktorisierungen von Eisenbud in [Eis80] als elementare Beschreibung der *Singularitätenkategorien* von Hyperflächen eingeführt, deren verschiedene Definitionen wir zunächst motivieren und wiederholen wollen:

Nach dem Auslander-Buchsbaum-Serre'schen Regularitätskriterium [BH93, Theorem 2.2.7] ist ein Noetherscher lokaler Ring $(R, \mathfrak{m})$ mit Restklassenkörper $k := R/\mathfrak{m}$ genau dann regulär im geometrischen Sinn (d.h. $\dim_k \mathfrak{m}/\mathfrak{m}^2 = \dim R$) wenn jeder endlich erzeugte R-Modul eine endliche projektive Auflösung besitzt. Das eröffnet folgende Möglichkeiten, die „Singularität“ von R kategoriell zu erfassen:

– Da ein R-Modul genau dann endliche projektive Dimension hat, wenn er – als Komplex aufgefasst – quasi-isomorph zu einem beschränkten Komplex projektiver R-Moduln ist, kann ein Maß für die „Singularität“ von R wie folgt erklärt werden:

  Definition [Buc86, Definition 1.2.2]**.** *Der Verdier-Quotient*

  $$\mathbf{D}_{\mathrm{sg}}(R) \;:=\; \mathbf{D}^b(R\text{-mod}) \;/\; \mathrm{Perf}(R)$$

  wird die Singularitätenkategorie von R *genannt.*

Hierbei ist $\mathbf{D}^b(R\text{-mod})$ die beschränkte derivierte Kategorie von R-mod und $\mathrm{Perf}(R)$ die volle Unterkategorie bestehend aus jenen Komplexen, die in $\mathbf{D}^b(R\text{-mod})$ isomorph zu einem beschränkten Komplex endlich erzeugter, projektiver R-Moduln sind.

Ein Noetherscher lokaler Ring R somit somit genau dann regulär, wenn $\mathbf{D}_{\mathrm{sg}}(R) = 0$.

– Ist R Gorenstein, d.h. Noethersch und von endlicher injektiver Dimension als Links- und Rechtsmodul über sich selbst, so heißt ein endlich erzeugter R-Modul M *maximal Cohen-Macaulay* (MCM), wenn $\mathrm{Ext}^k_R(M, R) = 0$ für alle $k > 0$ [Buc86, Definition 4.2.1]. Unter den Moduln von endlicher projektiver Dimension sind dies die Projektiven, und so ist auch folgende Kategorie ein Maß der Singularität von R:

  Definition. *Für einen Gorenstein-Ring R ist die* stabile Kategorie maximaler Cohen-Macaulay-Moduln $\underline{\mathrm{MCM}}(R)$ *definiert als Quotient additiver Kategorien*

$$\underline{\mathrm{MCM}}(R) \;:=\; \mathrm{MCM}(R)\;/\;\mathrm{proj}(R).$$

  Ihre Objekte sind demnach MCM-Moduln über R, und ihre Morphismen sind Morphismen von R-Moduln modulo jene, die durch projektive R-Moduln faktorisieren.

– Jeder nach oben beschränkte, azyklische Komplex projektiver R-Moduln ist zusammenziehbar, eine Folge aus der klassischen Äquivalenz $\mathbf{K}^-(\mathrm{Proj}(R)) \xrightarrow{\cong} \mathbf{D}^-(R\text{-Mod})$ (siehe dazu auch den folgenden Abschnitt C). Für unbeschränkte Komplexe ist das nicht mehr richtig, wie z.B. der Komplex

$$\ldots \xrightarrow{\varepsilon} \Bbbk[\varepsilon]/(\varepsilon^2) \xrightarrow{\varepsilon} \Bbbk[\varepsilon]/(\varepsilon^2) \xrightarrow{\varepsilon} \ldots$$

  über $\Bbbk[\varepsilon]/(\varepsilon^2)$ zeigt. Ist jedoch die projektive Dimension von R-Moduln nach oben beschränkt, so ist in der Tat auch jeder azyklische Komplex projektiver R-Moduln zusammenziehbar. Als drittes Maß für die Singularität von R bietet sich also $\mathbf{K}_{\mathrm{ac}}(\mathrm{proj}(R))$ an, die Homotopiekategorie azyklischer Komplexe endlich erzeugter, projektiver R-Moduln.

Im Fall von Gorenstein-Ringen sind all diese Kandidaten äquivalent:

Theorem [Buc86, Theorem 4.4.1]**.** *Für einen Gorenstein-Ring R bestehen folgende Äquivalenzen von Kategorien:*

$$\begin{array}{ccc} & \mathbf{K}_{\mathrm{ac}}(\mathrm{proj}(R)) & \\ {}^{Q^0}\swarrow & & \searrow^{\iota^0 \circ Q^0} \\ \underline{\mathrm{MCM}}(R) & \xrightarrow[\iota^0]{} & \mathbf{D}_{\mathrm{sg}}(R) \end{array} \tag{B}$$

Hierbei ist Q^0 der 0-te Syzygienfunktor gegeben durch $(X, \partial) \mapsto \operatorname{coker}(X^{-1} \xrightarrow{\partial^{-1}} X^0)$, und ι^0 die Einbettung, die einen R-Modul als Komplex konzentriert im Grad 0 auffasst.

Für eine Hyperfläche $R = S/(w)$ ist $\underline{\mathrm{MF}}^b(S, w)$ eine weitere Beschreibung für $\mathbf{D}_{\mathrm{sg}}(R)$:

Theorem ([Eis80], siehe auch [Yos90, Theorem 7.4])**.** *Für einen Noetherschen regulären lokalen Ring $(S, \mathfrak{m})$ und $w \in \mathfrak{m}$ besteht eine Äquivalenz von Kategorien:*

$$\underline{\mathrm{MF}}^b(S, w) \xrightarrow{\simeq} \underline{\mathrm{MCM}}(S/(w)), \quad (M^0 \xrightarrow{f} M^1 \xrightarrow{g} M^0) \longmapsto \operatorname{coker}(g). \tag{E}$$

Definition. *Der* Stabilisierungsfunktor *ist definiert als die Komposition*

$$R\text{-mod} \xrightarrow{\iota^0} \mathbf{D}^b(R\text{-mod}) \xrightarrow{\text{can}} \mathbf{D}_{\mathrm{sg}}(R). \tag{Stab}$$

Der Name kommt daher, dass im Falle eines Gorenstein-Ringes die Eigenschaft eines R-Moduls, MCM zu sein, eine stabile Eigenschaft ist in dem Sinne, dass zum einen für jeden R-Modul M jede hinreichend hohe Syzygie $\Omega^k(M)$ MCM ist, und dass zum anderen der Syzygienfunktor $\Omega := \Omega^1$ auf $\underline{\mathrm{MCM}}(R)$ eingeschränkt eine Äquivalenz ist, deren Inverses wir mit Σ bezeichnen – die Unterkategorie der MCM-Moduln ist also der bzgl. der Syzygienbildung „stabile“ Bereich von R-mod, und der durch (Stab) induzierte Funktor $R\text{-mod} \to \mathbf{D}_{\mathrm{sg}}(R) \cong \underline{\mathrm{MCM}}(R)$ ist durch $\Sigma^k \Omega^k$ für $k \gg 0$ gegeben, also eine „Projektion“ in ebendiesen stabilen Bereich (formal ist $\Sigma^k \Omega^k : \underline{R\text{-mod}} \to \underline{\mathrm{MCM}}(R)$ für $k \gg 0$ ein Rechtsadjungierter zur Einbettung $\underline{\mathrm{MCM}}(R) \subset \underline{R\text{-mod}}$).

C. Modellkategorien

Modellkategorien wurden 1967 in [Qui67] von Quillen als formaler Rahmen für axiomatische Homotopietheorie eingeführt. Ein Grundproblem besteht darin, für eine gegebene Kategorie $\mathscr{C}$ zusammen mit einer Klasse W *schwacher Äquivalenzen* ein Verständnis für die *Homotopiekategorie* $\mathrm{Ho}(\mathscr{C}, \mathrm{W}) := \mathscr{C}[\mathrm{W}^{-1}]$ zu erlangen, der durch formales Invertieren der Morphismen aus W erhaltenen Lokalisierung von $\mathscr{C}$ nach W (bereits die Frage, ob $\mathscr{C}[\mathrm{W}^{-1}]$ eine lokal kleine Kategorie im mengentheoretischen Sinne ist, ist nicht trivial). Prominente Beispiele sind der Fall $\mathscr{C} = \mathrm{Top}$ der Kategorie aller topologischen Räume versehen mit der Klasse $\mathrm{W} = \mathrm{weq}$ der schwachen Homotopieäquivalenzen (jenen stetigen Abbildungen, die auf allen Homotopiegruppen zu sämtlichen Basispunkten Bijektionen induzieren), oder die Kategorie $\mathscr{C} = \mathrm{Ch}_{\geq 0}(R\text{-Mod})$ der in nicht-negativen Graden konzentrierten Kettenkomplexe über einem Ring R versehen mit der Klasse $\mathrm{W} = \mathrm{qis}$ der Quasi-Isomorphismen (jenen Abbildungen, die auf sämtlichen Homologiegruppen Isomorphismen induzieren). Bereits in diesen klassischen Beispielen lässt sich die fundamentale Beobachtung machen, dass die *Lokalisierung* $\mathrm{Ho}(\mathscr{C}, \mathrm{W})$ – die a priori *sämtliche* Objekte von $\mathscr{C}$ involviert, im Fall von $\mathscr{C} = \mathrm{Top}$ also z.B. auch die Hawaiianischen Ohrringe – jeweils eine Beschreibung als *Quotient* einer geeigneten Unterkategorie „gutartiger" Objekte in $\mathscr{C}$ zulässt: Im Fall topologischer Räume ist beispielsweise $\mathrm{Top}[\mathrm{weq}^{-1}]$ äquivalent zur Kategorie $\underline{\mathrm{CW}}$ der CW-Komplexe mit stetigen Abbildungen bis auf Homotopie, und im Fall von Kettenkomplexen ist die *derivierte Kategorie* $\mathbf{D}_{\geq 0}(R) := \mathrm{Ch}_{\geq 0}(R)[\mathrm{qis}^{-1}]$ äquivalent zur Kategorie $\mathbf{K}_{\geq 0}(\mathrm{Proj}(R))$ der Komplexe projektiver R-Moduln mit Morphismen bis auf Homotopie (insbesondere ist die Homotopiekategorie in diesem Fall in der Tat eine lokal kleine Kategorie).

Eine *Modellstruktur* $\mathcal{M}$ (siehe Definition II.2.1.1 und generell [Hov99]) auf $\mathscr{C}$ beinhaltet nun neben W noch ein zusätzliches Datum welches grundsätzlich solch eine Beschreibung der Lokalisierung $\mathrm{Ho}\,\mathcal{M} := \mathscr{C}[\mathrm{W}^{-1}]$ als Quotient $\mathscr{C}_{\mathrm{cf}}/_\sim$ einer Unterkategorie $\mathscr{C}_{\mathrm{cf}} \subset \mathscr{C}$ von „Modellen" nach einer geeigneten „Homotopie"-Relation $\sim$ auf deren Morphismenmengen erlaubt [Hov99, Theorem 1.2.10]. Neben der daraus folgenden lokalen Kleinheit von $\mathscr{C}[\mathrm{W}^{-1}]$ gewährleistet die Modellstruktur ferner einen flexiblen Formalismus von Homotopielimiten, -Kolimiten (siehe z.B. [Gro13]) sowie derivierten Funktoren, und stiftet nicht zuletzt einen Funktor $\mathscr{C} \to \mathrm{Ho}\,\mathcal{M} \to \mathscr{C}_{\mathrm{cf}}/_\sim$ in Verallgemeinerung der im Fall topologischer Räume erhaltenen CW-Approximation der im Fall von Komplexen von R-Moduln erhaltenen projektiven Auflösung, und der im Fall einer geeigneten Modellstruktur auf R-Mod erhaltenen Stabilisierung (Stab). Das Beispiel topologischer Räume wird in [Hov99, §2.4, insb. Theorem 2.4.19] diskutiert, das Beispiel von Kettenkomplexen in [Qui67, §II.4, insb. Rem. 5 in II.4.11], siehe auch [DS95, Theorem 7.2].

Ist $\mathcal{T}$ eine Kategorie und $\mathcal{M}$ eine Modellstruktur auf einer Kategorie $\mathscr{C}$ mit $\mathrm{Ho}\,\mathcal{M} \cong \mathcal{T}$, so sagen wir, $\mathcal{M}$ sei *ein Modell für* $\mathcal{T}$. Wie eben gesehen ist die Konstruktion von Modellen zu $\mathcal{T} = \mathscr{C}[\mathrm{W}^{-1}]$ bei vorgegebenem $\mathscr{C}$ und W von Interesse, aber auch der umgekehrte Fall, in dem $\mathcal{T}$ als Quotientenkategorie vorgegeben ist und wir über das Modell $\mathcal{M}$ eine Beschreibung von $\mathcal{T}$ als Lokalisierung $\mathscr{C}[\mathrm{W}^{-1}]$ anstreben, spielt eine Rolle: Beispielsweise kann es in konkreten Berechnungen von Nutzen sein, ein Objekt aus der Unterkategorie $\mathscr{C}_{\mathrm{cf}}$ durch ein schwach äquivalentes Objekt aus $\mathscr{C}$ zu ersetzen (sofern man weiß, dass dies das Ergebnis der Berechnung nicht verändert). Ein klassisches Beispiel dafür ist der Tor_*^R-Funktor über einem kommutativen Ring R: A priori ist $\mathrm{Tor}_*^R(M, N)$ als Homologie des Tensorprodukts $P \otimes_R Q$ für projektive Auflösungen $P, Q \in \mathrm{Ch}_{\geq 0}(\mathrm{Proj}(R))$ von M bzw. N erklärt, aber zur Berechnung können auch entweder $M \otimes_R Q$ oder $P \otimes_R N$ verwendet werden – generell ist es erlaubt, in einem Tensorprodukt $P \otimes_R Q$ mit $P, Q \in \mathrm{Ch}_{\geq 0}(\mathrm{Proj}(R))$ einen der beiden Faktoren durch einen quasi-isomorphen (d.h. in $\mathbf{D}_{\geq 0}(R)$ isomorphen) Komplex nicht notwendig projektiver Moduln zu ersetzen, ohne die Homologie zu verändern; eine Aussage, die ohne Kenntnis des Begriffs des Quasi-Isomorphismus und der derivierten Kategorie nicht einmal zu formulieren wäre.

Dieses elementare Beispiel wird hier auch deshalb so betont, weil die analoge Frage für Matrixfaktorisierungen zentral für diese Arbeit und schwieriger ist als für Komplexe.

D. Skizze der Probleme und Ergebnisse

Wir beschreiben nun die wesentlichen Resultate dieser Arbeit. Detailliertere Beschreibungen der Inhalte beider Teile finden sich jeweils in deren Einleitungen.

Überblick. Wie zuvor erwähnt arbeitet die ursprüngliche Definition von Khovanov-Rozansky-Homologie mit Tensorprodukten von Matrixfaktorisierungen, welche man in Analogie zu 2-periodischen Komplexen projektiver Moduln sehen kann. Dieser Vergleich mit der klassischen Situation von Komplexen legt die Frage nahe, ob man, ebenso wie die Homotopiekategorie $\mathbf{K}_{\geq 0}(\mathrm{Proj}(R))$ projektiver Komplexe von einer Modellstruktur auf ganz $\mathrm{Ch}_{\geq 0}(R)$ herrührt, die Homotopiekategorie der Matrixfaktorisierungen $\underline{\mathrm{MF}}(A,w)$ als Homotopiekategorie einer geeigneten Modellstruktur auf der Kategorie $\mathrm{LF}(A,w)$ aller linearen Faktorisierungen realisieren kann. *Motivation* für die Suche nach solch einer Modellstruktur ist in Hinblick auf Khovanov-Rozansky-Homologie der Wunsch, ein flexibleres Arbeiten mit Matrixfaktorisierungen zu ermöglichen, so wie es auch die Beschreibung von $\mathbf{K}_{\geq 0}(\mathrm{Proj}(R))$ als $\mathbf{D}_{\geq 0}(R)$ für Komplexe erlaubt – insbesondere ist ein deriviertes Tensorprodukt linearer Faktorisierungen erstrebenswert, das wie in der klassischen Situation durch Auflösen nur eines Tensorfaktors berechnet werden kann. *Problematisch* ist bei der Suche nach solch einer Modellstruktur zunächst schlicht, dass es auf Grund von $\delta^2 \neq 0$ keinen Begriff von Homologie und somit keinen kanonischen Begriff von Quasi-Isomorphismus gibt, in Bezug auf den man $\mathrm{LF}(A,w)$ lokalisieren könnte.

Wie in Abschnitt B beschrieben ist die Homotopiekategorie von Matrixfaktorisierungen ein Beispiel für eine Singularitätenkategorie, und ein wesentlicher Teil der Arbeit betrifft nun generell die Konstruktion und den Vergleich verschiedener Modelle für Singularitätenkategorien differentiell graduierter Ringe. Sämtliche Modellstrukturen sind dabei auf abelschen Kategorien definiert und mit der abelschen exakten Struktur verträglich, und ein Großteil des zweiten Teils dieser Arbeit untersucht diese *abelschen Modellkategorien* im Allgemeinen; insbesondere werden zwei Resultate bereitgestellt, die das Lokalisieren und den Beweis der kofasernden Erzeugtheit solcher Modellstrukturen erleichtern. Weite Teile dieser Untersuchungen sind bereits in [Bec14] erschienen.

Singuläre Modellstrukturen. Unser erstes Resultat stellt eine Reihe kofasernd erzeugter, abelscher Modellstrukturen bereit, darunter auch zwei Modelle für eine „große" Variante der Singularitätenkategorie $\mathbf{D}_{\mathrm{sg}}(R)$, wie wir weiter unten sehen werden:

Resultat (Proposition II.2.3.6, Definition II.4.1.2, Proposition II.4.2.1)**.** *Für einen differentiell graduierten Ring A trägt die Kategorie A*-Mod *der (differentiell graduierten)*

A-Moduln die folgenden kofasernd erzeugten, abelschen Modellstrukturen:

(i) *Die* standard injektive und gemischte injektive Modellstruktur $\mathcal{M}^{\mathrm{inj}}(A)$ *bzw.* ${}^m\mathcal{M}^{\mathrm{inj}}(A)$ *mit Homotopiekategorien* $\mathrm{Ho}(\mathcal{M}^{\mathrm{inj}}(A)) \cong \mathrm{Ho}({}^m\mathcal{M}^{\mathrm{inj}}(A)) \cong \mathbf{D}(A)$.

(ii) *Die* standard projektive und gemischte projektive Modellstruktur $\mathcal{M}^{\mathrm{proj}}(A)$ *bzw.* ${}^m\mathcal{M}^{\mathrm{proj}}(A)$ *mit Homotopiekategorien* $\mathrm{Ho}(\mathcal{M}^{\mathrm{proj}}(A)) \cong \mathrm{Ho}({}^m\mathcal{M}^{\mathrm{proj}}(A)) \cong \mathbf{D}(A)$.

(iii) *Die* kontraderivierte Modellstruktur $\mathcal{M}^{\mathrm{ctr}}(A)$ *mit* $\mathrm{Ho}(\mathcal{M}^{\mathrm{ctr}}(A)) \cong \mathbf{K}(A\text{-Mod}_{\mathrm{proj}})$.

(iv) *Die* koderivierte Modellstruktur $\mathcal{M}^{\mathrm{co}}(A)$ *mit* $\mathrm{Ho}(\mathcal{M}^{\mathrm{co}}(A)) \cong \mathbf{K}(A\text{-Mod}_{\mathrm{inj}})$.

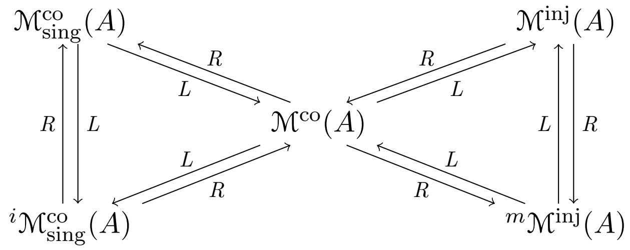
(v) *Die* gemischte bzw. projektive singuläre kontraderivierte Modellstruktur $\mathcal{M}^{\mathrm{ctr}}_{\mathrm{sing}}(A)$ *bzw.* ${}^p\mathcal{M}^{\mathrm{ctr}}_{\mathrm{sing}}(A)$ *mit* $\mathrm{Ho}(\mathcal{M}^{\mathrm{ctr}}_{\mathrm{sing}}(A)) \cong \mathrm{Ho}({}^p\mathcal{M}^{\mathrm{ctr}}_{\mathrm{sing}}(A)) \cong \mathbf{K}_{\mathrm{ac}}(A\text{-Mod}_{\mathrm{proj}})$.

(vi) *Die* gemischte bzw. injektive singuläre koderivierte Modellstruktur $\mathcal{M}^{\mathrm{co}}_{\mathrm{sing}}(A)$ *bzw.* ${}^i\mathcal{M}^{\mathrm{co}}_{\mathrm{sing}}(A)$ *mit* $\mathrm{Ho}(\mathcal{M}^{\mathrm{co}}_{\mathrm{sing}}(A)) \cong \mathrm{Ho}({}^i\mathcal{M}^{\mathrm{co}}_{\mathrm{sing}}(A)) \cong \mathbf{K}_{\mathrm{ac}}(A\text{-Mod}_{\mathrm{inj}})$.

Die Modellstrukturen (iii) und (iv) sind auch für gekrümmte dg Ringe definiert.

Hierbei bezeichnen $A\text{-Mod}_{\mathrm{proj}}$ und $A\text{-Mod}_{\mathrm{inj}}$ die Unterkategorien von $A\text{-Mod}$ bestehend aus jenen A-Moduln, die nach Ignorieren des Differentials projektiv bzw. injektiv sind; ist beispielsweise A ein Ring R aufgefasst als dg Ring konzentriert im kohomologischen Grad 0, so ist $A\text{-Mod} = \mathrm{Ch}(R)$, $A\text{-Mod}_{\mathrm{proj}} = \mathrm{Ch}(\mathrm{Proj}(R))$ und $A\text{-Mod}_{\mathrm{inj}} = \mathrm{Ch}(\mathrm{Inj}(R))$. Die Homotopiekategorien $\mathrm{Ho}(A\text{-Mod}_{\mathrm{inj}})$ bzw. $\mathrm{Ho}(A\text{-Mod}_{\mathrm{proj}})$ der koderivierten bzw. kontraderivierten Modellstrukturen werden *koderivierte* bzw. *kontraderivierte* Kategorie genannt. Sie wurden in [Pos11] untersucht.

Resultat (Proposition II.4.2.8 und Korollare II.4.2.6, II.4.2.7)**.** *Die injektiven und koderivierten Modellstrukturen stehen über folgenden „Schmetterling“ in Verbindung, in dem* L / R *Links- bzw. Rechts-Quillen Funktoren bezeichnen:*



Bei Übergang zu den Homotopiekategorien entsteht daraus das Recollement

$$\mathbf{K}_{\mathrm{ac}}(A\text{-Mod}_{\mathrm{inj}}) \rightleftarrows \mathbf{K}(A\text{-Mod}_{\mathrm{inj}}) \rightleftarrows \mathbf{D}(A). \qquad (\infty)$$

Analoges gilt für die projektive und kontraderivierte Modellstruktur. Für Noethersche Ringe (aufgefasst als dg Ringe im Grad 0) wurde (∞) bereits in [Kra05] konstruiert; die Resultate von loc.cit. gelten generell im Kontext lokal Noetherscher Grothendieck-Kategorien $\mathscr{A}$ mit kompakt erzeugter derivierte Kategorie $\mathbf{D}(\mathscr{A})$, selbst wenn nicht notwendig genügend Projektive vorliegen. Diese Allgemeinheit ist mit den hier vorgestellten Methoden bisher nicht greifbar; siehe dazu Abschnitt II.4.3.

Technisches Herzstück der Konstruktionen sind Theorem II.3.1.2 und Proposition II.3.2.3 zur expliziten Beschreibung von Bousfield-Lokalisierungen im Kontext abelscher Modellstrukturen, sowie folgende Proposition zum Beweis kofasernder Erzeugtheit:

Resultat (Theorem II.B.11)**.** *Sei $U : \mathscr{B} \to \mathscr{A}$ ein kostetiger, monadischer Funktor zwischen Grothendieck-Kategorien, und $\mathcal{F} \subset \mathscr{A}$ eine dekonstruierbare Klasse. Dann ist auch $U^*(\mathcal{F}) := \{X \in \mathscr{B} \mid U(X) \in \mathcal{F}\}$ dekonstruierbar.*

Dieses Resultat lässt sich z.B. auf Vergissfunktoren anwenden. Beispielsweise kann man das Vergessen $\mathrm{Ch}(R) \to \mathrm{Ch}(\mathbb{Z})$ der Modulstruktur betrachten und auf diese Weise die Dekonstruierbarkeit von $\mathrm{Acyc}(R)$ auf die von $\mathrm{Acyc}(\mathbb{Z})$ zurückführen. Ein anderes Beispiel ist das Vergessen $\mathrm{Ch}(R) \to R\text{-}\mathrm{Mod}^{\mathbb{Z}}$ des Differentials, über das die Dekonstruierbarkeit von $\mathrm{Ch}(\mathrm{Proj}(R))$ auf die von $\mathrm{Proj}(R)$ in R-Mod zurückgeführt werden kann.

Die Existenz obiger Modellstrukturen hat interessante Konsequenzen: Beispielsweise zeigt die Existenz und kofasernde Erzeugtheit der koderivierten Modellstruktur auf $\mathrm{Ch}(R\text{-}\mathrm{Mod})$, dass die Homotopiekategorie $\mathbf{K}(\mathrm{Inj}(R))$ injektiver R-Moduln wohlerzeugt ist (Proposition II.2.2.10) und damit z.B. insbesondere beliebige Koprodukte hat. Dieses Resultat wurde mit anderen Methoden kürzlich auch von Neeman [Nee14] bewiesen.

Der Name „singuläre Modellstruktur“ ist durch folgendes Theorem gerechtfertigt:

Theorem [Kra05]**.** *Für einen Noetherschen Ring R ist $\mathbf{D}_{\mathrm{sg}}(R)$ bis auf direkte Summanden äquivalent zur Kategorie der kompakten Objekte in der kompakt erzeugten Homotopiekategorie $\mathbf{K}_{\mathrm{ac}}(\mathrm{Inj}(R))$ azyklischer Komplexe injektiver R-Moduln.*

In diesem Sinne ist also die singuläre koderivierte Modellstruktur auf $\mathrm{Ch}(R)$ ein Modell für die „große“ Singularitätenkategorie $\mathbf{K}_{\mathrm{ac}}(\mathrm{Inj}(R)) \supset \mathbf{D}_{\mathrm{sg}}(R)$. Ein Modell für $\mathbf{D}_{\mathrm{sg}}(R)$ selbst ist hingegen nicht zu erwarten, denn $\mathbf{D}_{\mathrm{sg}}(R)$ ist auf Grund der Beschränkung auf endlich erzeugte Moduln essentiell klein, wohingegen die Homotopiekategorien der hier betrachteten Modellstrukturen stets Nullobjekte sowie beliebige Koprodukte und Produkte haben und damit nur dann essentiell klein sind, wenn sie trivial sind (in einer essentiell kleinen Kategorie $\mathcal{C}$ existiert $\sup_{X,Y\in\mathcal{C}} |\mathcal{C}(X,Y)|$, hingegen ist $\{|\mathcal{C}(X^{(\kappa)},Y)| = |\mathcal{C}(X,Y)|^{\kappa}\}_{\kappa\in\mathrm{Set}}$ für $|\mathcal{C}(X,Y)| > 1$ unbeschränkt – essentiell kleine Kategorien mit beliebigen Koprodukten und Produkten sind daher bis auf Äquivalenz die vollständigen Verbände).

Beispiele. In Abschnitt II.5 untersuchen wir die singulären Modellstrukturen in einigen Beispielen mit dem Ziel, die klassischen Äquivalenzen (B) und (E) zwischen den verschiedenen Beschreibungen der Singularitätenkategorie auf die Ebene der Modellkategorien hochzuheben. Da die Varianten der Singularitätenkategorie aus Abschnitt B ebenso wie $\mathbf{D}_{\mathrm{sg}}(R)$ sämtlich essentiell klein sind, sind hier als Anreicherungen der klassischen Äquivalenzen abermals nur Quillen-Äquivalenzen zu erwarten, deren auf den Homotopiekategorien induzierte Äquivalenzen sich zu den klassischen Äquivalenzen einschränken.

Als erste Anreicherung – im gerade beschriebenen Sinne – einer klassischen Äquivalenz erhalten wir folgendes Resultat, das die Äquivalenz $\underline{\mathrm{MCM}}(R) \cong \mathbf{K}_{\mathrm{ac}}(\mathrm{proj}(R))$ hochhebt:

Resultat (Theorem II.5.1.5)**.** *Für einen Gorenstein-Ring R besteht eine links-Quillen-Äquivalenz*

$$Q^0 : {}^{p}\mathcal{M}^{\mathrm{ctr}}_{\mathrm{sing}}(R) \xrightarrow{\ \simeq\ } \mathcal{M}^{\mathrm{G\text{-}proj}}(R)$$

zwischen der projektiven singulären kontraderivierten Modellstruktur und Hoveys Gorenstein-projektiver Modellstruktur $\mathcal{M}^{\mathrm{G\text{-}proj}}(R)$ *auf* R-Mod *(Proposition II.2.1.5).*

Auf den Homotopiekategorien stiftet dies eine Äquivalenz $\mathbf{K}_{\mathrm{ac}}(\mathrm{Proj}(R)) \cong \underline{\mathrm{G\text{-}proj}}(R)$, welche sich zur Äquivalenz $\mathbf{K}_{\mathrm{ac}}(\mathrm{proj}(R)) \cong \underline{\mathrm{MCM}}(R)$ einschränkt; hierbei bezeichnet $\underline{\mathrm{G\text{-}proj}}(R)$ die stabile Kategorie der Gorenstein-projektiven R-Moduln, den Analoga zu MCM Moduln im nicht notwendig endlich erzeugten Fall.

Als Nächstes betrachten wir die Äquivalenz $\underline{\mathrm{MF}}^b(S,w) \cong \underline{\mathrm{MCM}}(S/(w))$. Für einen geeigneten zu (S,w) assoziierten $\mathbb{Z}/2\mathbb{Z}$-graduierten gekrümmenten dg Ring S_w (siehe Abschnitt II.5.2) ist $\mathrm{LF}(S,w) \cong S_w$-Mod, wobei $\mathrm{MF}(S,w)$ zu S_w-$\mathrm{Mod}_{\mathrm{proj}}$ korrespondiert, sodass wir insbesondere auf $\mathrm{LF}(S,w)$ eine kontraderivierte Modellstruktur $\mathcal{M}^{\mathrm{ctr}}\,\mathrm{LF}(S,w)$ mit $\mathrm{Ho}(\mathcal{M}^{\mathrm{ctr}}\,\mathrm{LF}(S,w)) \cong \underline{\mathrm{MF}}(S,w)$ erhalten.

Resultat (Theorem II.5.3.2)**.** *Für einen regulären Noetherschen lokalen Ring* $(S,\mathfrak{m})$*,* $w \in \mathfrak{m}\backslash\{0\}$ *und* $R := S/(w)$ *besteht eine Quillen-Äquivalenz* $\mathcal{M}^{\mathrm{ctr}}\,\mathrm{LF}(S,w) \cong \mathcal{M}^{\mathrm{G\text{-}proj}}(R)$*.*

Auf den Homotopiekategorien liefert dies eine Äquivalenz $\underline{\mathrm{MF}}(S,w) \cong \underline{\mathrm{G\text{-}proj}}(R)$, die sich zu Eisenbuds Äquivalenz $\underline{\mathrm{MF}}^b(S,w) \cong \underline{\mathrm{MCM}}(R)$ einschränkt.

Zuletzt kann $\iota^0 \circ Q^0 : \mathbf{K}_{\mathrm{ac}}(\mathrm{proj}(R)) \cong \mathbf{D}_{\mathrm{sg}}(R)$ aus (B) wie folgt hochgehoben werden:

Resultat (Propositionen II.2.3.14 und II.4.4.5)**.** *Für einen Gorenstein-Ring R bestehen Quillen-Äquivalenzen*

$$\mathcal{M}^{\mathrm{ctr}}(R) \rightleftarrows \mathcal{M}^{\mathrm{co}}(R), \quad {}^{p}\mathcal{M}^{\mathrm{ctr}}_{\mathrm{sing}}(R) \rightleftarrows \mathcal{M}^{\mathrm{co}}_{\mathrm{sing}}(R) \quad \textit{und} \quad \mathcal{M}^{\mathrm{ctr}}_{\mathrm{sing}}(R) \rightleftarrows {}^{i}\mathcal{M}^{\mathrm{co}}_{\mathrm{sing}}(R). \qquad (\ddagger)$$

Der Satz gilt wörtlich auch für Gorensteinsche dg-Ringe. Die Äquivalenz $\mathcal{M}^{\mathrm{ctr}}(R) \rightleftarrows \mathcal{M}^{\mathrm{co}}(R)$ induziert eine Äquivalenz $\mathbf{K}(\mathrm{Proj}(R)) \rightleftarrows \mathbf{K}(\mathrm{Inj}(R))$, eine in [IK06] studierte „große“ Variante der Grothendieck-Dualität für Gorenstein-Ringe im folgenden Sinne: $\mathbf{K}(\mathrm{Proj}(R)) \to \mathbf{K}(\mathrm{Inj}(R))$ ist konkret durch $- \otimes_R I^*$ für eine injektive Auflösung I^* von ${}_R R$ gegeben (siehe Beispiel II.2.3.12), und es bestehen Äquivalenzen $\mathbf{K}(\mathrm{Proj}(R))^c \cong \mathbf{D}^b(R^{\mathrm{op}}\text{-mod})^{\mathrm{opp}}$ [Jør05, Theorem 3.2] sowie $\mathbf{K}(\mathrm{Inj}(R))^c \cong \mathbf{D}^b(R\text{-mod})$ [Kra05, Proposition 2.3] bzgl. derer sich obige Äquivalenz zur klassischen Grothendieck-Dualität $\mathbf{R}\,\mathrm{Hom}_R(-,R) = \mathrm{Hom}_R(-,I^*) : \mathbf{D}^b(R\text{-mod})^{\mathrm{opp}} \cong \mathbf{D}^b(R^{\mathrm{op}}\text{-mod})$ einschränkt [IK06, Einleitung]. Die anderen beiden Äquivalenzen zeigen, dass sich $\mathbf{K}(\mathrm{Proj}(R)) \rightleftarrows \mathbf{K}(\mathrm{Inj}(R))$ zu einer Äquivalenz $\mathbf{K}_{\mathrm{ac}}(\mathrm{Proj}(R)) \rightleftarrows \mathbf{K}_{\mathrm{ac}}(\mathrm{Inj}(R))$ einschränkt; ferner ist das Diagramm

$$\begin{array}{ccc}
\mathbf{K}_{\mathrm{ac}}(\mathrm{Proj}(R)) & \xrightarrow{\cong} & \mathbf{K}_{\mathrm{ac}}(\mathrm{Inj}(R)) \\
\uparrow & & \uparrow \\
\mathbf{K}_{\mathrm{ac}}(\mathrm{proj}(R)) & \xrightarrow{\cong} & \mathbf{D}_{\mathrm{sg}}(R)
\end{array}$$

bis auf Shift kommutativ (Proposition II.5.1.8). In diesem Sinne reichern die Quillen-Äquivalenzen aus (‡) die klassische Äquivalenz $\mathbf{K}_{\mathrm{ac}}(\mathrm{proj}(R)) \cong \mathbf{D}_{\mathrm{sg}}(R)$ an.

Anwendungen auf Khovanov-Rozansky-Homologie. Als Beispiel einer kontraderivierten Modellstruktur liefern obige Überlegungen insbesondere einen Begriff von schwacher Äquivalenz zwischen linearer Faktorisierungen zugehörig zu einer Modellstruktur $\mathcal{M}^{\mathrm{ctr}}\,\mathrm{LF}(S,w)$ auf der Kategorie $\mathrm{LF}(S,w)$ aller linearen Faktorisierungen vom Typ (S,w), deren Homotopiekategorie $\mathbf{D}^{\mathrm{ctr}}\,\mathrm{LF}(S,w) := \mathrm{Ho}(\mathcal{M}^{\mathrm{ctr}}\,\mathrm{LF}(S,w))$ äquivalent zur Homotopiekategorie $\underline{\mathrm{MF}}(S,w)$ von Matrixfaktorisierungen ist. Bezüglich der in der Konstruktion von $\mathcal{KR}^k$ auftauchenden Matrixfaktorisierungen machen wir nun folgende

Resultat. *Die in der Konstruktion von $\mathcal{KR}^k$ zu den Bausteinen ⤲, ⤱ und↑ assoziierten Komplexe von Matrixfaktorisierungen sind termweise schwach äquivalent zu elementaren Rouquier Komplexen von Soergel Bimoduln.*

Diese Beobachtung bringt die Darstellungstheorie ins Spiel, in der Soergel Bimoduln und Rouquier Komplexe als Kategorifizierungen der Hecke-Algebra und der Zopfgruppe eine tragende Rolle spielen [Soe07; Rou06; EW14] und auch in der Konstruktion von Kategorifzierungen der Quanten-$\mathfrak{sl}(k)$-Invariante über die Bernstein-Gelfand-Gelfand Kategorie $\mathcal{O}$ zentral sind [Str05; MS09; Sus07].

Unser nächsts Resultat betrifft die Verträglichkeit des Tensorprodukts linearer Faktorisierungen mit der kontraderivierten Modellstruktur $\mathcal{M}^{\mathrm{ctr}}\,\mathrm{LF}(S,w)$.

Resultat. *Es gibt ein kontraderiviertes Tensorprodukt linearer Faktorisierungen*

$$- \otimes_S^{\mathbf{L}} - : \mathbf{D}^{\mathrm{ctr}}\,\mathrm{LF}(S,w) \times \mathbf{D}^{\mathrm{ctr}}\,\mathrm{LF}(S,w') \longrightarrow \mathbf{D}^{\mathrm{ctr}}\,\mathrm{LF}(S,w+w'),$$

welches jedoch i.A. nicht *durch Auflösen nur eines Tensorfaktors berechnet werden kann.*

Siehe Bemerkung I.4.2.2 für ein Beispiel dafür, dass i.A. beide Faktoren in $- \otimes_S^{\mathbf{L}} -$ durch Matrixfaktorisierungen aufgelöst werden müssen, sowie eine modellkategorielle Analyse des Unterschieds zur klassischen Situation von Komplexen.

In Abschnitt I.4.2 entwickeln wir jedoch Kriterien, die in den für die Konstruktion von $\mathcal{KR}^k$ relevanten Spezialfällen das Berechnen von $-\otimes_S^{\mathbf{L}}-$ durch Auflösen nur eines Faktors ermöglichen. Dadurch lässt sich schließlich zeigen, dass – in Bezug auf die kontraderivierte Modellstruktur $\mathcal{M}^{\mathrm{ctr}}\,\mathrm{LF}(A,w)$ – die in der Konstruktion von $\mathcal{KR}^k$ auftauchenden, zu Zöpfen assoziierten Komplexe von Matrixfaktorisierungen termweise schwach äquivalent sind zu den entsprechenden Rouquier Komplexen von Soergel Bimoduln. Das Abschließen von Zöpfen stellt sich als eine Variante von Hochschild-Homologie heraus:

Resultat (Theorem I.5.3)**.** *Sei L eine als Abschluss des Zopfes β dargestellte, orientierte Verschlingung im Raum und $\mathcal{CCKR}^k(\beta)$ der durch die Khovanov-Rozansky-Konstruktion zu β assoziierte Komplex von Matrixfaktorisierungen.*

(i) *Es besteht (bis auf Shift) eine kanonische, termweise schwache Äquivalenz zwischen Komplexen linearer Faktorisierungen $\mathcal{CCKR}^k(\beta) \simeq \mathcal{RC}_{\mathbb{Q}}(\beta)$.*

(ii) *Die Invariante $\mathcal{KR}^k(L)$ ist (bis auf Normalisierung) gegeben durch das Poincaré-Polynom der stabilen Hochschild-Homologie des Rouquier Komplexes zu β, d.h.*

$$\mathcal{KR}^k(L) = \sum_{i,j\in\mathbb{Z}} \dim_{\mathbb{Q}} \mathrm{H}^i \left[{}^k\,\mathrm{sHH}_t^{\mathbb{A}^n_{\mathbb{Q}}/\mathbb{Q}}\, \mathcal{RC}_{\mathbb{Q}}(\beta)_j\right] a^i q^j. \tag{KR}$$

Hierbei ist $\mathcal{RC}_{\mathbb{Q}}(\beta)$ der über $\mathbb{Q}$ definierte Rouquier Komplex von Soergel Bimoduln zu β (siehe Anhang I.A), und ${}^k\,\mathrm{sHH} := \Delta \otimes^{\mathbf{L}} -$ bezeichnet die k-stabile Hochschild-Homologie (Definition I.4.5.1), eine Variante klassischer Hochschild-Homologie gegeben durch das kontraderivierte Tensorprodukt mit der Diagonalen in $\mathbf{D}^{\mathrm{ctr}}\,\mathrm{LF}$. Da Soergel Bimoduln eine Kategorifizierung der Hecke-Algebra bilden und Hochschild-Homologie auf Grund seiner Symmetrieeigenschaften (Proposition I.6.2.1) als kategorielles Analogon der Spur verstanden werden kann, stellt das obige Ergebnis eine Kategorifizerung der in Abschnitt A skizzierten Beschreibung von $\mathcal{P}^k$ über Spurformen auf Hecke-Algebren dar.

Die Beschreibung von $\mathcal{KR}^k$ in (KR) wurde auf anderem Wege und mit einer ad-hoc Definition stabiler Hochschild-Homologie bereits in [Web07, Theorem 2.7] bewiesen. Die hier vorgestellte Theorie stellt einen konzeptionellen Unterbau für dieses Resultat und das Studium stabiler Hochschild Homologie dar. Für einen genauen Vergleich zwischen unserem Ansatz und dem in [Web07] siehe die Einleitung zu Teil I.

Die Untersuchung von $\mathcal{KR}^k$ über die kontraderivierte Kategorie ermöglicht zusätzlich auch einen direkten Nachweis davon, dass (KR) eine Invariante ist:

Resultat (siehe Theorem I.6.1.1 und Korollar I.6.1.4)**.** *Für jeden kommutativen Grundring* $\Bbbk$ *mit* $k+1 \in \Bbbk^\times$ *ist* k*-stabile Hochschild-Homologie von Rouquier Komplexen von Soergel Bimoduln eine Invariante orientierter Verschlingungen.*

Der vorgestellte Beweis ist insbesondere unabhängig von der Originalarbeit [KR08a] und umgeht explizite Rechnungen unter Verwendung bekannter Resultate zur Kombinatorik der Soergel Bimoduln.

Zuletzt untersuchen wir noch den entgegengesetzten Fall $k+1=0$ im Grundring $\Bbbk$:

Resultat (Theorem I.6.1.5)**.** *Im Fall* $k+1=0$ *in* $\Bbbk$ *liefert eine alternative Normalisierung* k*-stabiler Hochschild-Homologie von Rouquier Komplexen ebenfalls eine Invariante.*

Die verwendete Normalisierung entspricht dabei der Normalisierung, die auch Rouquier [Rou12] beim direkten Nachweis darüber verwendet, dass klassische Hochschild-Homologie von Rouquier Komplexen eine (dreifach graduierte) Invariante orientierter Verschlingungen ist, die mit der dreifach graduierten Invariante [KR08b] von Khovanov und Rozansky übereinstimmt. Die hier vorgestellten Methoden liefern einen alternativen Beweis für dieses Resultat, und zeigen ferner:

Resultat (Proposition I.6.5.7, Korollar I.6.5.8)**.** *Im Fall* $k+1=0$ *in* $\Bbbk$ *degeneriert die Spektralsequenz zwischen* k*-stabiler und klassischer Hochschild-Homologie auf der ersten Seite, sodass Hochschild-Homologie kanonisch mit dem assoziierten Graduierten einer Filtrierung auf* k*-stabiler Hochschild-Homologie identifiziert werden kann.*

Weitere Resultate und detailliertere Beschreibungen der Ergebnisse dieser Arbeit finden sich in Einleitung I.1 zum knotentheoretischen Teil I und Einleitung II.1 zum homotopietheoretischen Teil II.

0. Introduction

This thesis studies Khovanov-Rozansky homology, an invariant of oriented links introduced by Khovanov and Rozansky in [KR08a], through methods of homotopy theory. It is divided in two parts: a first part “Knot Theoretic Aspects” in which the focus lies on concrete applications of homotopy theoretic techniques to Khovanov-Rozansky homology, and a second part “Homotopy Theoretic Aspects”, in which these techniques are studied independently and with the goal of largest possible generality. Detailed descriptions of the contents of the two parts are given in their introductions.

In the first three Sections A, B, C of this introduction, I introduce the reader to the knot theoretic, algebraic and homotopy theoretic context of this work. Afterwards, in Section D, I sketch the results of the thesis. Readers familiar with the basics of quantum invariants and/or singularity categories and/or homotopy theory may safely skip the respective sections.

A. Quantum invariants and the idea of categorification

We assume that the reader is familiar with the idea of constructing invariants of knots; for introductions to knot theory, see e.g. [Lic97; Rol76].

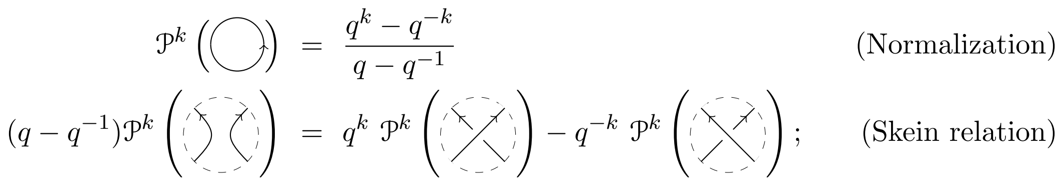
Let $k \geq 2$ be an integer. The *Khovanov-Rozansky homology* $\mathcal{KR}^k$ [KR08a] studied in this work is a refinement of the classical *quantum-*$\mathfrak{sl}(k)$ *invariant* $\mathcal{P}^k$ in the following sense: While $\mathcal{P}^k$ assigns to an oriented link L a Laurent polynomial $\mathcal{P}^k(L) \in \mathbb{Z}[q^{\pm 1}]$ in a single variable q, $\mathcal{KR}^k$ assigns to it a Laurent polynomial $\mathcal{KR}^k(L) \in \mathbb{Z}[q^{\pm 1}, t^{\pm 1}]$ in *two* variables q and t in such a way that specializing t to -1 recovers $\mathcal{P}^k(L)$:

$$\mathcal{KR}^k(L)|_{t=-1} \;=\; \mathcal{P}^k(L) \qquad \text{(Cat)}$$

In particular, any two knots distinguished by $\mathcal{P}^k$ are also distinguished by $\mathcal{KR}^k$, which justifies calling $\mathcal{KR}^k$ a refinement of $\mathcal{P}^k$; since there are indeed knots which $\mathcal{KR}^k$ distinguishes while $\mathcal{P}^k$ does not, $\mathcal{KR}^k$ is even a proper refinement of $\mathcal{P}^k$. An invariant having property (Cat) is also called *categorification* of $\mathcal{P}^k$; we will discuss the motivation for this terminology below.

The quantum-$\mathfrak{sl}(k)$ invariant $\mathcal{P}^k$ is a generalization of the Jones polynomial [Jon85] –

obtained in case $k = 2$ – which is determined by the following two properties:



Further, $\mathcal{P}^k$ being an invariant of links means that

$$\mathcal{P}^k(L) = \mathcal{P}^k(L'), \quad \text{if } L \text{ and } L' \text{ describe equivalent links.} \qquad \text{(Invariant)}$$

The notation in (Skein relation) means that the three applications of $\mathcal{P}^k$ concern link diagrams that agree outside the dashed area and differ inside in the way indicated in the figure; Figure 1 and Figure 2 give an impression of how to calculate values of $\mathcal{P}^k$ using the properties (Normalization), (Skein relation) and (Invariant). The *existence* of $\mathcal{P}^k$ can be demonstrated using the Reshitikhin-Turaev construction, to be explained in more detail in the following paragraph, which interprets a generic planar projection of L as a diagrammatic denotation of a morphism of modules over the quantum group $\mathcal{U}_q(\mathfrak{sl}(k))$ (therefore also the name quantum-$\mathfrak{sl}(k)$ invariant).

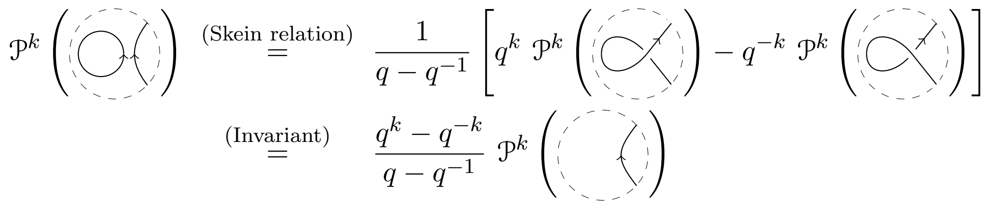


Figure 1. Behavior of $\mathcal{P}^k$ under disjoint union with unknots

The quantum group $\mathcal{U}_q(\mathfrak{sl}(k))$ is a deformation of the universal enveloping algebra $\mathcal{U}(\mathfrak{sl}(k))$ that is defined over the field $\mathbb{C}(q)$ of complex rational functions. It is again a Hopf algebra, which albeit not being cocommutative (and for which the naive flip map $V \otimes W \to W \otimes V$ is therefore not linear) still admits special "braiding" isomorphisms $\Theta_{V,W} : V \otimes W \to W \otimes V$ of $\mathcal{U}_q(\mathfrak{sl}(k))$-modules: These are no longer involutive, i.e. $\Theta_{V,W} \circ \Theta_{W,V} \neq \mathrm{id}_{V \otimes W}$, but still satisfy the braid relations – see Figure 3. Further, they allow for an identification of left and right duals, ${}^*V \cong V^*$.

Taking this for granted, the Reshitikhin-Turaev goes as follows (for details, see [Kas95, Chapter XIV], in particular [Kas95, Theorem XIV.5.1]): Firstly, to any downward resp. upward directed strand one assigns the vector representation V of $\mathcal{U}_q(\mathfrak{sl}(k))$ (the analogue of the $\mathfrak{sl}(k)$ representation on $\mathbb{C}^k$ by matrix-vector multiplication) resp. its dual V^*, and to any horizontal concatenation of strands the tensor products of the respective modules,

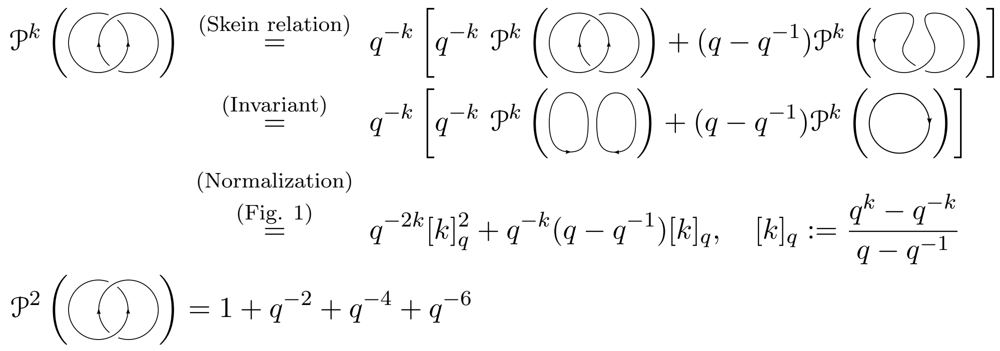


Figure 2. Computation of the quantum-$\mathfrak{sl}(k)$ invariant of the Hopf link

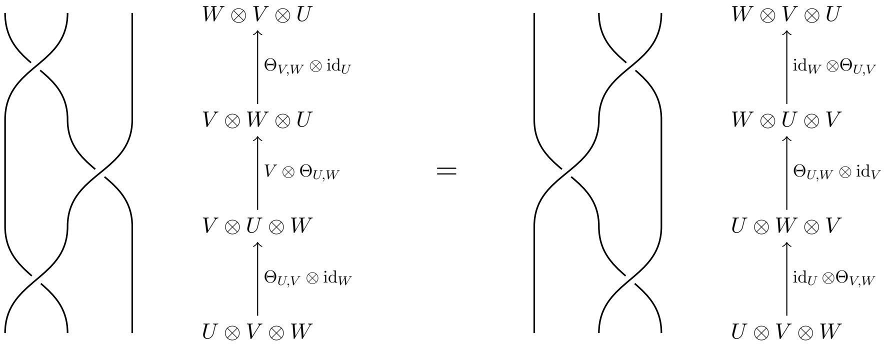


Figure 3. The braid relation for the the braiding isomorphisms in $\mathcal{U}_q(\mathfrak{sl}(k))$-mod

the empty tensor product being interpreted as the trivial representation of $\mathcal{U}_q(\mathfrak{sl}(k))$ in the base field $\mathbb{C}(q)$. Secondly, the chosen planar projection of L is decomposed into simple pieces each of which is interpreted as a morphism in $\mathcal{U}_q(\mathfrak{sl}(k))$-mod as exemplified in Figure 4. Finally, the morphisms attached to the simple pieces are put together, through tensoring and composition, to give an endomorphism of the trivial module $\mathbb{C}(q)$; this morphism is then given by multiplication with a scalar in $\mathbb{C}(q)$, and this scalar is, up to a final normalization, the value $\mathcal{P}^k(L)$. Two special cases are of interest:

– For braids on n strands, we obtain an interpretation in $\mathrm{End}_{\mathcal{U}_q(\mathfrak{sl}(k))}(V^{\otimes n})$. Denoting Br_n the braid group on n strands and $\mathbb{C}(q)[\mathrm{Br}_n]$ its group algebra over $\mathbb{C}(q)$, the resulting algebra homomorphism $\mathbb{C}(q)[\mathrm{Br}_n] \to \mathrm{End}_{\mathcal{U}_q(\mathfrak{sl}(k))}(V^{\otimes n})$ is surjective and factors through the *(Iwahori-)Hecke algebra*

$$\mathcal{H}_n(q) \;:=\; \mathbb{C}(q)[\mathrm{Br}_n]/(\mathrm{T}_i^2 = (q^2-1)\,\mathrm{T}_i + q^2\,\mathrm{T}_e);$$

moreover, closing braids corresponds to applying a $\mathbb{C}(q)$-linear trace $\mathcal{H}_n(q) \to \mathbb{C}(q)$,

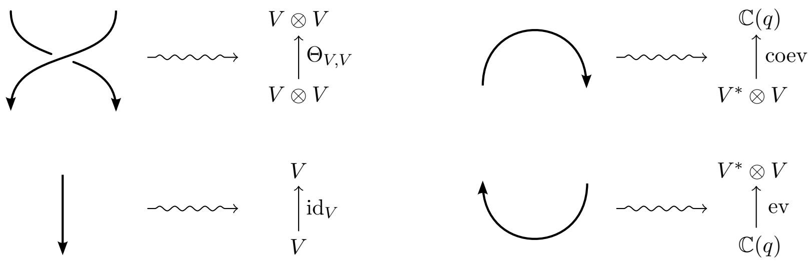
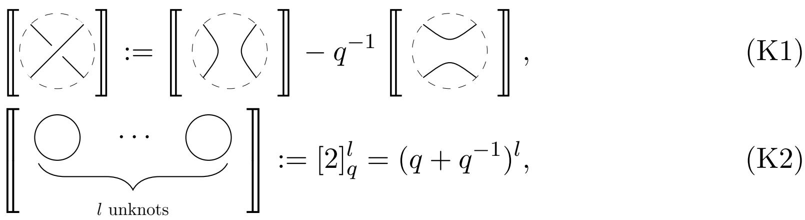


Figure 4. Planar knot diagrams as morphisms $\mathcal{U}_q(\mathfrak{sl}(k))$-mod.

a specialization of the two-variable Ocneanu trace through which the HOMFLYPT polynomial [Fre+85] can be constructed [KT08, §4.4].

– In case $k = 2$ we have $V^* \cong V$, and the braiding isomorphism $\Theta_{V,V} : V \otimes V \xrightarrow{\cong} V \otimes V$ and its inverse are both linear combinations over $\mathbb{C}(q)$ of the identity and the composition $V \otimes V \to \mathbb{C}(q) \to V \otimes V$ of evaluation and coevaluation. All morphisms relevant for the construction of $\mathcal{P}^2$ are therefore already contained in the free $\mathbb{C}(q)$-linear monoidal category over a self-dual object, and this admits the following explicit description [FY89, Theorem 4.1.1]: Objects are natural numbers $\mathbf{0}, \mathbf{1}, \ldots$, morphisms $\mathbf{n} \to \mathbf{m}$ are formal $\mathbb{C}(q)$-linear combinations of planar, unoriented and crossingless (n, m)-tangles up to isotopy (see Figure 5), and the monoidal structure is given by addition on objects and by horizontal concatenation on morphisms. Passing to the quotient by the relation $\bigcirc = (q + q^{-1})\,\mathrm{id}_{\mathbf{0}}$ in $\mathrm{End}(\mathbf{0})$ (which holds in the Reshitikhin-Turaev interpretation) finally yields the Temperley-Lieb category $\mathcal{TL}$, and one obtains the diagrammatic construction of the Jones polynomial through the $\mathcal{TL}(\mathbf{0}, \mathbf{0}) \cong \mathbb{C}(q)$-valued Kauffman bracket $[\![\ ]\!]$,



($[\![\ ]\!]$ does not depend on the orientation of the given knot diagram) and a final normalization by the factor $q^{n_-(L) - n_+(L)}$, where $n_+(L)$ and $n_-(L)$ are the numbers of "positive" resp. "negative" crossings ⤱ resp. ⤲ in the chosen planar diagram. The computation of $\mathcal{P}^2$ (Hopf link) via the Kauffman bracket (in accordance with the results of Figure 2) is shown in Figure 6.

The above definition of categorification is motivated by the fact that one might seek for a refinement of the invariant $\mathcal{P}^k$ by categorifying the structures involved in its construction

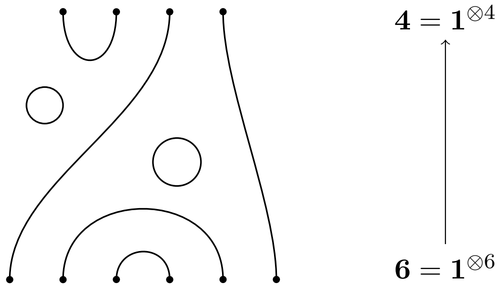


Figure 5. A $(6,4)$-tangle as a morphism $\mathbf{1}^{\otimes 6} \to \mathbf{1}^{\otimes 4}$ in the free monoidal category over a self-dual object $\mathbf{1}$

– e.g. the Temperley-Lieb category or the Hecke algebra $\mathcal{H}_n(q)$ with its trace forms – and that the result of the successful attempts in doing so were often two variable invariants with the above property (Cat). The notion of "categorification of a mathematical structure" used here is not precisely defined, but for algebraic structures it often means the construction of abelian or triangulated categories (respectively functors between them or monoidal structures on them) which yield the given modules (resp. morphisms or ring structures) on application of the Grothendieck construction K_0. Informally, one might say that one realizes a given algebraic structure as the underlying combinatorics of some categorical structure.

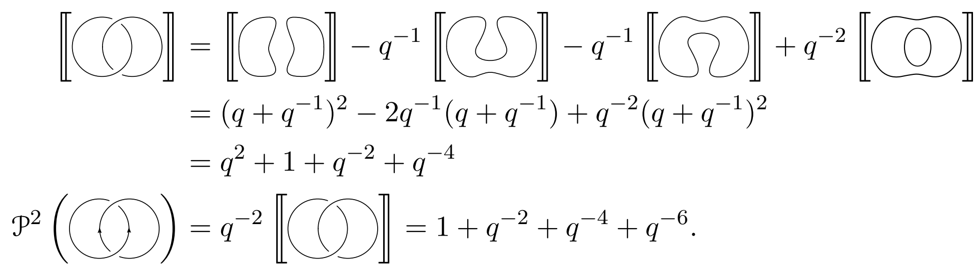


Figure 6. Computation of $\mathcal{P}^2$ ( ) via the Kauffman bracket

The first and simplest example of a categorification is Khovanov homology $\mathcal{KH}$, introduced in [Kho00]. It is a categorification of the Jones polynomial that is based on the above construction of $\mathcal{P}^2$ via the Kauffman bracket $[\![\ ]\!]$. For the definition of $\mathcal{KH}$ one understands the ring of integer Laurent polynomials as the decategorification of the monoidal category of finite-dimensional graded vector spaces – up to isomorphism, any finite-dimensional graded vector space is determined by the dimensions of its homogeneous components that can be coded in an integer Laurent polynomial, and tensoring graded vector spaces corresponds to multiplication of Laurent polynomials under this assignment – and realizes the alternating sum arising when repeatedly applying (K1) as the Euler characteristic of a suitable complex of graded vector spaces. The coding of the cohomology of this complex as a Laurent polynomial in two variables – one for the internal and one for the cohomological grading – then yields the desired categorification $\mathcal{KH}$ of $\mathcal{P}^2$ in the sense of (Cat).

Like Khovanov homology, the original construction of Khovanov-Rozansky homology $\mathcal{KR}^k$ studied in this work is also based on a graphical calculus for the calculation of quantum-$\mathfrak{sl}(k)$ invariant $\mathcal{P}^k$, introduced in [MOY98]. Similar to the two-step construction of the Jones polynomial $\mathcal{P}^2$ via the Kauffman bracket (K1), (K2), in the MOY-calculus the positive and negative crossings $\overcrossing$ and $\undercrossing$ in a planar oriented link diagram are resolved into a formal alternating sum of so-called MOY-graphs, each of which is afterwards evaluated to a Laurent polynomial following a fixed combinatorics; see [KR08a, Figures 2 & 3]. The essential step in the construction of $\mathcal{KR}^k$ is the categorification of MOY-graphs and their evaluation combinatorics through tensor products of *matrix factorizations*.

B. Matrix factorizations and singularity categories

Definition. *Let A be a commutative ring and $w \in A$ be arbitrary, called potential.*

(i) *A* linear factorization *of type (A, w) is a diagram $M^0 \xrightarrow{f} M^1 \xrightarrow{g} M^0$ of A-modules and A-module homomorphisms with $fg = w \cdot id_{M^1}$ and $gf = w \cdot id_{M^0}$.*

(ii) *A* matrix factorization *of type (A, w) is a linear factorization of type (A, w) as in (i) with the additional property that M^0 and M^1 are projective A-modules.*

For $M^0 = M^1 = A^n$ the morphisms f, g are given by matrices $X, Y \in \mathrm{Mat}_{n\times n}(A)$ satisfying $XY = YX = w \cdot \mathrm{id}_n$ – therefore the name. For example, the matrices

$$X := \begin{pmatrix} y & -x & 0 \\ 0 & y & -x \\ x & 0 & y^3 \end{pmatrix} \qquad \text{and} \qquad Y := \begin{pmatrix} y^4 & xy^3 & x^2 \\ -x^2 & y^4 & xy \\ -xy & -x^2 & y^2 \end{pmatrix}$$

form a matrix factorization of type $(\mathbb{Z}[x, y], x^3 + y^5)$. Further, matrix factorizations can be viewed as 2-periodic complexes of projective A-modules with the complex-condition $\delta^2 = 0$ being replaced by $\delta^2 = w \cdot \mathrm{id}$. Following this analogy one defines a *homotopy category of matrix factorizations* $\underline{\mathrm{MF}}(A, w)$; its full subcategory consisting of those matrix factorizations having finitely generated components M^0 and M^1 is denoted $\underline{\mathrm{MF}}^b(A, w)$.

Originally, matrix factorizations were introduced by Eisenbud [Eis80] as elementary descriptions of *singularity categories* of hypersurfaces, the various definitions of which we want to recall in the following:

By the Auslander-Buchsbaum-Serre regularity criterion [BH93, Theorem 2.2.7], a Noetherian local ring $(R, \mathfrak{m})$ with residue field $k := R/\mathfrak{m}$ is regular in the geometric sense (i.e. $\dim_k \mathfrak{m}/\mathfrak{m}^2 = \dim R$) if and only if every finitely generated R-module has a finite projective resolution. This opens the following possibilities for defining "categorical measures" for the singularity of R:

- An R-module has finite projective dimension if and only if, when considered as a complex, it is quasi-isomorphic to a bounded complex of projective R-modules. Therefore, a measure for the singularity of R can be introduced as follows:

  Definition [Buc86, Definition 1.2.2]**.** *The Verdier quotient*

  $$\mathbf{D}_{\mathrm{sg}}(R) \ := \ \mathbf{D}^b(R\text{-mod}) \ / \ \mathrm{Perf}(R)$$

  is called the singularity category *of R.*

  Here $\mathbf{D}^b(R\text{-mod})$ denotes the bounded derived category of R-mod and $\mathrm{Perf}(R)$ its full subcategory consisting of those complexes which are isomorphic, in $\mathbf{D}^b(R)$, to a bounded complex of finitely generated projective R-modules.

  A Noetherian local ring R is therefore regular if and only if $\mathbf{D}_{\mathrm{sg}}(R) = 0$.

- If R is Gorenstein, i.e. Noetherian and of finite injective dimension both as a left and as a right module over itself, a finitely generated R-module M is called *maximal Cohen-Macaulay* (MCM) if $\mathrm{Ext}^k_R(M, R) = 0$ for all $k > 0$ [Buc86, Definition 4.2.1]. Among the modules of finite projective dimension, these are precisely the projective modules, and hence the following category is a measure for the singularity of R, too:

  Definition. *For a Gorenstein ring R the* stable category of maximal Cohen-Macaulay modules $\underline{\mathrm{MCM}}(R)$ *is the quotient of additive categories*

  $$\underline{\mathrm{MCM}}(R) \ := \ \mathrm{MCM}(R) \ / \ \mathrm{proj}(R).$$

  Its objects are therefore MCM modules over R, and morphisms are morphisms of R-modules modulo such which factor through a projective R-module.

- Every bounded above and acyclic complex of projective R-modules is contractible, a consequence of the classical equivalence $\mathbf{K}^-(\mathrm{Proj}(R)) \xrightarrow{\simeq} \mathbf{D}^-(R\text{-mod})$ (see also Section C). For unbounded complexes this is no longer true, as e.g. the complex

  $$\ldots \xrightarrow{\varepsilon} \Bbbk[\varepsilon]/(\varepsilon^2) \xrightarrow{\varepsilon} \Bbbk[\varepsilon]/(\varepsilon^2) \xrightarrow{\varepsilon} \ldots$$

  over $\Bbbk[\varepsilon]/(\varepsilon^2)$ shows. If, however, the projective dimensions of R-modules are bound from above, then indeed *any* acyclic complex of projectives is contractible. As a third measure for the singularity of R we therefore arrive at considering $\mathbf{K}_{\mathrm{ac}}(\mathrm{proj}(R))$, the homotopy category of acyclic complexes of finitely generated, projective R-modules.

In the case of Gorenstein rings, all these candidates are equivalent:

Theorem [Buc86, Theorem 4.4.1]**.** *For a Gorenstein ring R there are the following equivalences of categories:*

$$\begin{array}{ccc} & \mathbf{K}_{\mathrm{ac}}(\mathrm{proj}(R)) & \\ {\scriptstyle Q^0}\swarrow & & \searrow{\scriptstyle \iota^0 \circ Q^0} \\ \underline{\mathrm{MCM}}(R) & \xrightarrow[\iota^0]{} & \mathbf{D}_{\mathrm{sg}}(R) \end{array} \tag{B}$$

Here Q^0 is the 0-th syzygy functor given by $(X, \partial) \mapsto \mathrm{coker}(X^{-1} \xrightarrow{\partial^{-1}} X^0)$, and ι^0 is the embedding considering an R-module as a complex concentrated in degree 0.

In the special case of a hypersurface $R = S/(w)$, the homotopy category of matrix factorizations $\underline{\mathrm{MF}}^b(S, w)$ is yet another description:

Theorem ([Eis80], see also [Yos90, Theorem 7.4])**.** *For a Noetherian regular local ring $(S, \mathfrak{m})$ and $w \in \mathfrak{m}$ there is an equivalence of categories:*

$$\underline{\mathrm{MF}}^b(S, w) \xrightarrow{\simeq} \underline{\mathrm{MCM}}(S/(w)), \quad (M^0 \xrightarrow{f} M^1 \xrightarrow{g} M^0) \longmapsto \mathrm{coker}(g). \tag{E}$$

Definition. *The* stabilization functor *is defined as the composition*

$$R\text{-mod} \xrightarrow{\iota^0} \mathbf{D}^b(R\text{-mod}) \xrightarrow{\mathrm{can}} \mathbf{D}_{\mathrm{sg}}(R). \tag{Stab}$$

The name stems from the following: If R is Gorenstein, the property of an R-module of being MCM is stable in the sense that firstly any R-module M becomes MCM after passage to a sufficiently high syzygy $\Omega^k M$, and that secondly the syzygy functor $\Omega := \Omega^1$ is an equivalence when restricted to $\underline{\mathrm{MCM}}(R)$, whose inverse we denote Σ. The subcategory of MCM modules over R is therefore the "stable range" of R-mod with respect to taking syzygies, and the functor $R\text{-mod} \to \mathbf{D}_{\mathrm{sg}}(R) \cong \underline{\mathrm{MCM}}(R)$ induced by (Stab) is explicitly given by $\Sigma^k\Omega^k$ for $k \gg 0$, hence a "projection" onto this stable range (formally, $\Sigma^k\Omega^k : \underline{R\text{-mod}} \to \underline{\mathrm{MCM}}(R)$, $k \gg 0$, is a right adjoint to $\underline{\mathrm{MCM}}(R) \hookrightarrow \underline{R\text{-mod}}$).

C. Model categories

Model categories were introduced in 1967 by Quillen [Qui67] as a framework for axiomatic homotopy theory. One basic problem is the following: Given a category $\mathscr{C}$ equipped with a class W of *weak equivalences*, obtain an understanding of the *homotopy category* $\mathrm{Ho}(\mathscr{C}, \mathrm{W}) := \mathscr{C}[\mathrm{W}^{-1}]$ obtained from $\mathscr{C}$ by formally inverting all morphisms in W (even the question as to whether this category is locally small in the set theoretic sense is nontrivial). Prominent examples are the case of the category $\mathscr{C} = \mathrm{Top}$ of all topological spaces equipped with the class $\mathrm{W} = \mathrm{weq}$ of weak homotopy equivalences (those continuous maps inducing bijections on all homotopy groups with respect to all base points), or the category $\mathscr{C} = \mathrm{Ch}_{\geq 0}(R\text{-Mod})$ of non-negatively graded chain complexes of R-modules equipped with the class $\mathrm{W} = \mathrm{qis}$ of quasi-isomorphisms (those maps inducing isomorphisms on all homology groups). Already in these classical examples one can make the fundamental observation that the *localization* $\mathrm{Ho}(\mathscr{C}, \mathrm{W})$ – which a priori involves *all* objects of $\mathscr{C}$, e.g. in case $\mathscr{C} = \mathrm{Top}$ also the Hawaiian earrings – admits a description as a *quotient* of a suitable subcategory of $\mathscr{C}$ of "well-behaved" objects: E.g., in case of $(\mathrm{Top}, \mathrm{weq})$ the *homotopy category of spaces* $\mathrm{Top}[\mathrm{weq}^{-1}]$ is equivalent to the category $\underline{\mathrm{CW}}$ of CW-complexes with continuous maps up to homotopy, and in the case of chain complexes the *derived category* $\mathbf{D}_{\geq 0}(R) := \mathrm{Ch}_{\geq 0}(R)[\mathrm{qis}^{-1}]$ is equivalent to the category $\mathbf{K}_{\geq 0}(\mathrm{Proj}(R))$ of complexes of projective R-modules with morphisms up to homotopy. In particular, the homotopy category is indeed a locally small category in these examples.

A *model structure* $\mathcal{M}$ (see Definition II.2.1.1 and generally [Hov99]) on $\mathscr{C}$ is an additional datum to W generally allowing for a description of the localization $\mathrm{Ho}\,\mathcal{M} := \mathscr{C}[\mathrm{W}^{-1}]$ as a quotient $\mathscr{C}_{\mathrm{cf}}/_\sim$ of a subcategory $\mathscr{C}_{\mathrm{cf}} \subset \mathscr{C}$ of "models" by a suitable "homotopy relation" $\sim$ on its morphism spaces [Hov99, Theorem 1.2.10]. Apart from the local smallness of $\mathscr{C}[\mathrm{W}^{-1}]$, a model structure also ensures a flexible formalism of homotopy limits and colimits (see e.g. [Gro13]) as well as derived functors, and last not least induces a functor $\mathscr{C} \to \mathrm{Ho}\,\mathcal{M} \to \mathscr{C}_{\mathrm{cf}}/_\sim$ generalizing the CW-approximation (obtained in the case of topological spaces), the projective resolution (obtained in the case of chain complexes) and the stabilization functor (Stab) (obtained from a suitable model structure on R-Mod). The example of topological spaces is discussed in [Hov99, §2.4, in particular Theorem 2.4.19], the example of chain complexes in [Qui67, §II.4, in particular Rem. 5 in II.4.11], see also [DS95, Theorem 7.2].

If $\mathcal{T}$ is a category and $\mathcal{M}$ is a model structure on some category $\mathscr{C}$ such that $\mathrm{Ho}\,\mathcal{M} \cong \mathcal{T}$, we shall say that $\mathcal{M}$ is a *model for* $\mathcal{T}$. As seen above, the construction of models for $\mathcal{T} = \mathscr{C}[\mathrm{W}^{-1}]$ with $\mathscr{C}$ and W prescribed is of interest, but also the opposite case where $\mathcal{T}$ is given as a quotient category and in which we seek for a description of $\mathcal{T}$ as a localization $\mathscr{C}[\mathrm{W}^{-1}]$ plays a role: For example, it can be useful in explicit calculations to replace an object from the subcategory $\mathscr{C}_{\mathrm{cf}}$ by a weakly equivalent one from $\mathscr{C}$ (assuming one knows that this won't change the outcome of the calculation). A classical example for this is the definition of the Tor_*^R functor over a commutative ring R: A priori, $\mathrm{Tor}_*^R(M, N)$ is defined as the homology of $P \otimes_R Q$ for projective resolutions $P, Q \in \mathrm{Ch}_{\geq 0}(\mathrm{Proj}(R))$ of M resp. N, but one can also use either $M \otimes_R Q$ or $P \otimes_R N$ instead – generally, it is allowed to replace one of the two factors in a tensor product $P \otimes_R Q$ with $P, Q \in \mathrm{Ch}_{\geq 0}(\mathrm{Proj}(R))$

by a quasi-isomorphic complex (not necessarily with projective components) without changing the homology; a statement that wouldn't even be possible to formulate without knowing about the notion of quasi-isomorphism and the derived category. We emphasize this very elementary example because the analogous question for matrix factorization is central for this work and more difficult to handle than for ordinary complexes.

D. Sketch of problems and results

We now describe the main results of this thesis. Detailed descriptions of the contents of its two parts are contained in their respective introductions.

Overview. As mentioned above, the original definition of Khovanov-Rozansky homology works with tensor products of matrix factorizations which can be seen in analogy to 2-periodic complexes of projective modules. This comparison with the classical situation of complexes suggests the question as to whether the homotopy category of matrix factorizations $\underline{\mathrm{MF}}(A, w)$ admits a description as the homotopy category of a suitable model structure on the category $\mathrm{LF}(A, w)$ of all linear factorizations – just as $\mathbf{K}_{\geq 0}(\mathrm{Proj}(R))$ was obtained from a suitable model structure on the whole of $\mathrm{Ch}_{\geq 0}(R)$. With a view towards Khovanov-Rozansky homology, the *motivation* for finding such a model structure is the aim of allowing for a more flexible work with matrix factorizations, just as the description of $\mathbf{K}_{\geq 0}(\mathrm{Proj}(R))$ as $\mathbf{D}_{\geq 0}(R)$ does for complexes – in particular, it is desirable to have a derived tensor product of linear factorization which, as in the classical situation, can be computed by resolution of a single factor only. The *problem* is simply that because of $\delta^2 \neq 0$ there is no notion of homology and therefore no canonical notion of quasi-isomorphism with respect to which one might localize $\mathrm{LF}(A, w)$.

As described in Section B, the homotopy category of matrix factorizations is an example of a singularity category, and a substantial amount of this work concerns the construction of various models for singularity categories of differential graded rings in general. All model structure we shall encounter are defined on abelian categories and are compatible with the abelian exact structure, and a large part of the second part of this thesis studies these *abelian model categories* in general; in particular, two results are presented which allow the localization and proof of cofibrant generation of such model structures. The bulk of these studies has already been published in [Bec14].

Singular model structures. Our first result provides a number of cofibrantly generated abelian model structures, among those two models for a "big" variant of the singularity category $\mathbf{D}_{\mathrm{sg}}(R)$, as we shall see below:

Result (Proposition II.2.3.6, Definition II.4.1.2, Proposition II.4.2.1)**.** *For a differential graded ring A the category A-*Mod *of (differential graded) A-modules admits the following* *cofibrantly generated, abelian model structures:*

(i) *The* standard injective and mixed injective model structures $\mathcal{M}^{\mathrm{inj}}(A)$ *and* ${}^m\mathcal{M}^{\mathrm{inj}}(A)$, *with homotopy categories* $\mathrm{Ho}(\mathcal{M}^{\mathrm{inj}}(A)) \cong \mathrm{Ho}({}^m\mathcal{M}^{\mathrm{inj}}(A)) \cong \mathbf{D}(A)$.

(ii) *The* standard projective and mixed projective model structures $\mathcal{M}^{\mathrm{proj}}(A)$ *and* ${}^m\mathcal{M}^{\mathrm{proj}}(A)$, *with homotopy categories* $\mathrm{Ho}(\mathcal{M}^{\mathrm{proj}}(A)) \cong \mathrm{Ho}({}^m\mathcal{M}^{\mathrm{proj}}(A)) \cong \mathbf{D}(A)$.

(iii) *The* contraderived model structure $\mathcal{M}^{\mathrm{ctr}}(A)$ *with* $\mathrm{Ho}(\mathcal{M}^{\mathrm{ctr}}(A)) \cong \mathbf{K}(A\text{-}\mathrm{Mod}_{\mathrm{proj}})$.

(iv) *The* coderived model structure $\mathcal{M}^{\mathrm{co}}(A)$ *with* $\mathrm{Ho}(\mathcal{M}^{\mathrm{co}}(A)) \cong \mathbf{K}(A\text{-}\mathrm{Mod}_{\mathrm{inj}})$.

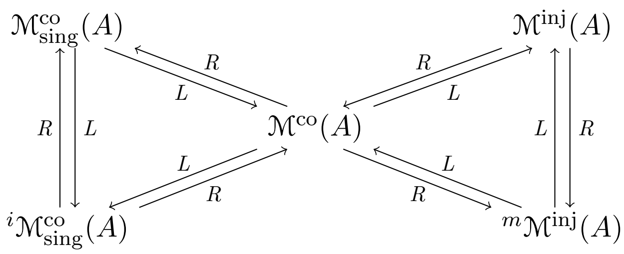
(v) *The* mixed and projective singular contraderived model structures $\mathcal{M}^{\mathrm{ctr}}_{\mathrm{sing}}(A)$ *and* ${}^p\mathcal{M}^{\mathrm{ctr}}_{\mathrm{sing}}(A)$, *with* $\mathrm{Ho}(\mathcal{M}^{\mathrm{ctr}}_{\mathrm{sing}}(A)) \cong \mathrm{Ho}({}^p\mathcal{M}^{\mathrm{ctr}}_{\mathrm{sing}}(A)) \cong \mathbf{K}_{\mathrm{ac}}(A\text{-}\mathrm{Mod}_{\mathrm{proj}})$.

(vi) *The* mixed and injective singular coderived model structures $\mathcal{M}^{\mathrm{co}}_{\mathrm{sing}}(A)$ *and* ${}^i\mathcal{M}^{\mathrm{co}}_{\mathrm{sing}}(A)$ *with* $\mathrm{Ho}(\mathcal{M}^{\mathrm{co}}_{\mathrm{sing}}(A)) \cong \mathrm{Ho}({}^i\mathcal{M}^{\mathrm{co}}_{\mathrm{sing}}(A)) \cong \mathbf{K}_{\mathrm{ac}}(A\text{-}\mathrm{Mod}_{\mathrm{inj}})$.

The model structures (iii) and (iv) are also defined for curved dg rings.

Here, $A\text{-}\mathrm{Mod}_{\mathrm{proj}}$ and $A\text{-}\mathrm{Mod}_{\mathrm{inj}}$ denote the subcategories of $A\text{-}\mathrm{Mod}$ consisting of those A-modules which are projective resp. injective after forgetting the differential; for example, if A is an ordinary ring R considered as a dg ring concentrated in degree 0, then $A\text{-}\mathrm{Mod}_{\mathrm{proj}} = \mathrm{Ch}(\mathrm{Proj}(R))$ and $A\text{-}\mathrm{Mod}_{\mathrm{inj}} = \mathrm{Ch}(\mathrm{Inj}(R))$. The homotopy categories $\mathrm{Ho}(A\text{-}\mathrm{Mod}_{\mathrm{inj}})$ resp. $\mathrm{Ho}(A\text{-}\mathrm{Mod}_{\mathrm{proj}})$ of the coderived resp. contraderived model structures are called *coderived* resp. *contraderived* categories. They were studied in [Pos11].

Result (Proposition II.4.2.8 and Corollaries II.4.2.6, II.4.2.7)**.** *The injective and coderived model structures are connected via the following "butterfly", in which* L / R *denote left resp. right Quillen functors:*



On passage to the homotopy categories this becomes the recollement

$$\mathbf{K}_{\mathrm{ac}}(A\text{-}\mathrm{Mod}_{\mathrm{inj}}) \leftrightarrows \mathbf{K}(A\text{-}\mathrm{Mod}_{\mathrm{inj}}) \leftrightarrows \mathbf{D}(A). \qquad (\infty)$$

An analogous result holds for the projective and contraderived model structures. For Noetherian rings (considered as dg rings in degree 0) the recollement (∞) was already studied in [Kra05]; the results of loc.cit. are valid in the general context of locally Noetherian Grothendieck categories $\mathscr{A}$ with compactly generated derived category $\mathbf{D}(\mathscr{A})$, even if they do not possess enough projectives. This generality is currently not available through the methods introduced here; see Section II.4.3.

The technical heart of the constructions are Theorem II.3.1.2 and Proposition II.3.2.3 on the explicit description of Bousfield localizations in the context of abelian model structures, as well as the following proposition for the proof of their cofibrant generation:

Result (Theorem II.B.11)**.** *Suppose that $U : \mathscr{B} \to \mathscr{A}$ is a cocontinuous, monadic functor between Grothendieck categories, and that $\mathcal{F} \subset \mathscr{A}$ is a deconstructible class. Then the class $U^*(\mathcal{F}) := \{X \in \mathscr{B} \mid U(X) \in \mathcal{F}\}$ is deconstructible, too.*

For example, this result applies to various forgetful functors. E.g., one can consider the functor $\mathrm{Ch}(R) \to \mathrm{Ch}(\mathbb{Z})$ forgetting the module structure, thereby reducing the deconstructibility of $\mathrm{Acyc}(R)$ to the deconstructibility of $\mathrm{Acyc}(\mathbb{Z})$. Another example is the functor $\mathrm{Ch}(R) \to R\text{-}\mathrm{Mod}^{\mathbb{Z}}$ forgetting the differential, through which the deconstructibility of $\mathrm{Ch}(\mathrm{Proj}(R))$ can be deduced from the deconstructibility of $\mathrm{Proj}(R)$.

The existence of the above model structure has interesting consequences: For example, the existence and cofibrant generation of the coderived model structure on $\mathrm{Ch}(R\text{-}\mathrm{Mod})$ shows that its homotopy category, the homotopy category $\mathbf{K}(\mathrm{Inj}(R))$ of complexes of injective R-modules, is a well-generated triangulated category (Proposition II.2.2.10), therefore in particular admitting arbitrary coproducts. This result was recently proved by Neeman [Nee14] using different methods.

The name "singular model structures" is justified by the following theorem:

Theorem [Kra05]**.** *For a Noetherian ring R the singularity category $\mathbf{D}_{\mathrm{sg}}(R)$ is, up to direct summands, equivalent to the subcategory of compact objects in the compactly generated homotopy category $\mathbf{K}_{\mathrm{ac}}(\mathrm{Inj}(R))$ of acyclic complexes of injective R-modules.*

In this sense, the singular coderived model structure on $\mathrm{Ch}(R)$ is a model for the "big" singularity category $\mathbf{K}_{\mathrm{ac}}(\mathrm{Inj}(R)) \supset \mathbf{D}_{\mathrm{sg}}(R)$. A model for $\mathbf{D}_{\mathrm{sg}}(R)$ itself is not to be expected, as $\mathbf{D}_{\mathrm{sg}}(R)$ is essentially small due to the restriction to finitely generated modules, while the homotopy categories of model structures considered here are always pointed and admit arbitrary limits and colimits, and are therefore essentially small only if they are trivial (in an essentially small category $\mathcal{C}$ the supremum $\sup_{X,Y\in\mathcal{C}} |\mathcal{C}(X,Y)|$ exists, while $\{|\mathcal{C}(X^{(\kappa)}, Y)| = |\mathcal{C}(X,Y)|^{\kappa}\}_{\kappa\in\mathrm{Set}}$ is unbounded for $|\mathcal{C}(X,Y)| > 1$ – essentially small categories with arbitrary products and coproducts are therefore the complete lattices, up to equivalence).

Examples. In II.5 we study the singular model structures in several examples with the goal of lifting the classical equivalences (B) and (E) to the level of model categories. Like $\mathbf{D}_{\mathrm{sg}}(R)$, its variants introduced in Section B are all essentially small, and so we may again only expect Quillen equivalences of model categories whose induced equivalences on the homotopy categories restrict to the classical equivalences.

As a first enhancement of a classical equivalence – in the sense just described – we obtain the following result, lifting the equivalence $\underline{\mathrm{MCM}}(R) \cong \mathbf{K}_{\mathrm{ac}}(\mathrm{proj}(R))$:

Result (Theorem II.5.1.5)**.** *For a Gorenstein ring R, there is a left Quillen equivalence*

$$Q^0 : {}^{p}\mathcal{M}^{\mathrm{ctr}}_{\mathrm{sing}}(R) \xrightarrow{\simeq} \mathcal{M}^{\mathrm{G\text{-}proj}}(R)$$

*between the projective singular contraderived model structure and Hovey's Gorenstein projective model structure $\mathcal{M}^{\mathrm{G\text{-}proj}}(R)$ on R-*Mod *(Proposition II.2.1.5).*

On the level of homotopy categories, this induces an equivalence $\mathbf{K}_{\mathrm{ac}}(\mathrm{Proj}(R)) \cong \underline{\mathrm{G\text{-}proj}}(R)$ restricting to $\mathbf{K}_{\mathrm{ac}}(\mathrm{proj}(R)) \cong \underline{\mathrm{MCM}}(R)$; here $\underline{\mathrm{G\text{-}proj}}(R)$ denotes the stable category of Gorenstein projective R-modules, the possibly infinitely generated analogues of maximal Cohen-Macaulay modules (see the paragraph preceding Proposition II.2.1.7).

Next we consider the equivalence $\underline{\mathrm{MF}}^b(S,w) \cong \underline{\mathrm{MCM}}(S/(w))$. For a suitable $\mathbb{Z}/2\mathbb{Z}$-graded curved dg ring S_w attached to (S,w) (see Section II.5.2) we have $\mathrm{LF}(S,w) \cong S_w$-Mod, with $\mathrm{MF}(S,w)$ corresponding to S_w-$\mathrm{Mod}_{\mathrm{proj}}$, so we obtain a contraderived model structure $\mathcal{M}^{\mathrm{ctr}}\,\mathrm{LF}(S,w)$ on $\mathrm{LF}(S,w)$ such that $\mathrm{Ho}(\mathcal{M}^{\mathrm{ctr}}\,\mathrm{LF}(S,w)) \cong \underline{\mathrm{MF}}(S,w)$.

Result (Theorem II.5.3.2)**.** *For a Noetherian regular local ring $(S,\mathfrak{m})$, $w \in \mathfrak{m} \setminus \{0\}$ and $R := S/(w)$ there is a Quillen equivalence $\mathcal{M}^{\mathrm{ctr}}\,\mathrm{LF}(S,w) \cong \mathcal{M}^{\mathrm{G\text{-}proj}}(R)$.*

On the homotopy categories this yields an equivalence $\underline{\mathrm{MF}}(S,w) \cong \underline{\mathrm{G\text{-}proj}}(R)$ restricting to Eisenbud's equivalence $\underline{\mathrm{MF}}^b(S,w) \cong \underline{\mathrm{MCM}}(R)$.

Finally we lift the equivalence $\iota^0 \circ Q^0 : \mathbf{K}_{\mathrm{ac}}(\mathrm{proj}(R)) \cong \mathbf{D}_{\mathrm{sg}}(R)$ from (B):

Result (Theorem II.2.3.14, Proposition II.4.4.5)**.** *For a Gorenstein ring R there are Quillen equivalences*

$$\mathcal{M}^{\mathrm{ctr}}(R) \rightleftarrows \mathcal{M}^{\mathrm{co}}(R), \quad {}^{p}\mathcal{M}^{\mathrm{ctr}}_{\mathrm{sing}}(R) \rightleftarrows \mathcal{M}^{\mathrm{co}}_{\mathrm{sing}}(R) \quad \textit{and} \quad \mathcal{M}^{\mathrm{ctr}}_{\mathrm{sing}}(R) \rightleftarrows {}^{i}\mathcal{M}^{\mathrm{co}}_{\mathrm{sing}}(R). \qquad (\ddagger)$$

The proposition also holds verbatimly for Gorenstein dg rings. The equivalence $\mathcal{M}^{\mathrm{ctr}}(R) \rightleftarrows \mathcal{M}^{\mathrm{co}}(R)$ induces an equivalence $\mathbf{K}(\mathrm{Proj}(R)) \rightleftarrows \mathbf{K}(\mathrm{Inj}(R))$ which was already studied in [IK06] and which constitutes a "big" variant of Grothendieck duality for Gorenstein rings in the following sense: Concretely, $\mathbf{K}(\mathrm{Proj}(R)) \to \mathbf{K}(\mathrm{Inj}(R))$ is given by $- \otimes_R I^*$ for an injective resolution I^* of ${}_RR$ (see Example II.2.3.12), and there are equivalences $\mathbf{K}(\mathrm{Proj}(R))^c \cong \mathbf{D}^b(R^{\mathrm{op}}\text{-mod})^{\mathrm{opp}}$ [Jør05, Theorem 3.2] as well as $\mathbf{K}(\mathrm{Inj}(R))^c \cong \mathbf{D}^b(R\text{-mod})$ [Kra05, Proposition 2.3] with respect to which it restricts to classical Grothendieck duality $\mathbf{R}\operatorname{Hom}_R(-,R) = \operatorname{Hom}_R(-,I^*) : \mathbf{D}^b(R\text{-mod})^{\mathrm{opp}} \cong \mathbf{D}^b(R^{\mathrm{op}}\text{-mod})$ [IK06, Introduction]. The other two Quillen equivalences show that we also obtain an equivalence $\mathbf{K}_{\mathrm{ac}}(\mathrm{Proj}(R)) \rightleftarrows \mathbf{K}_{\mathrm{ac}}(\mathrm{Inj}(R))$; finally, the diagram

$$\begin{array}{ccc} \mathbf{K}_{\mathrm{ac}}(\mathrm{Proj}(R)) & \xrightarrow{\ \cong\ } & \mathbf{K}_{\mathrm{ac}}(\mathrm{Inj}(R)) \\ \uparrow & & \uparrow \\ \mathbf{K}_{\mathrm{ac}}(\mathrm{proj}(R)) & \xrightarrow{\ \cong\ } & \mathbf{D}_{\mathrm{sg}}(R) \end{array}$$

is commutative up to shift (Proposition II.5.1.8). In this sense, the Quillen equivalences from (‡) lift the classical equivalence $\mathbf{K}_{\mathrm{ac}}(\mathrm{proj}(R)) \cong \mathbf{D}_{\mathrm{sg}}(R)$.

Applications to Khovanov-Rozansky homology. As a special case of a contraderived category the above considerations in particular yield a notion of weak equivalence for linear factorizations belonging to a model structure $\mathcal{M}^{\mathrm{ctr}}\,\mathrm{LF}(S,w)$ on the category $\mathrm{LF}(S,w)$ of all linear factorizations of type (S,w), the homotopy category $\mathbf{D}^{\mathrm{ctr}}\,\mathrm{LF}(S,w) := \mathrm{Ho}(\mathcal{M}^{\mathrm{ctr}}\,\mathrm{LF}(S,w))$ of which is equivalent to the homotopy category $\underline{\mathrm{MF}}(S,w)$ of matrix factorizations. Regarding the matrix factorizations occurring in the construction of $\mathcal{KR}^k$, we now make the following

Result. *The complexes of matrix factorizations assigned to the elementary pieces ⤧, ⤨ and↑ in the construction of $\mathcal{KR}^k$ are termwise weakly equivalent to elementary Rouquier complexes of Soergel bimodules.*

This observation gets representation theory into the game: There, Soergel bimodules and Rouquier complexes play a crucial role as categorifications of the Hecke algebra and the braid group [Soe07; Rou06; EW14] and are also central in the categorification of the quantum-$\mathfrak{sl}(k)$ invariant $\mathcal{P}^k$ via the BGG category $\mathcal{O}$ [Str05; MS09; Sus07].

Our next observation concerns the compatibility of the tensor product of linear factorizations with the contraderived model structure $\mathcal{M}^{\mathrm{ctr}}\,\mathrm{LF}(S,w)$:

Result. *There is a contraderived tensor product of linear factorizations*

$$-\otimes_S^{\mathbf{L}} - : \mathbf{D}^{\mathrm{ctr}}\,\mathrm{LF}(S,w) \times \mathbf{D}^{\mathrm{ctr}}\,\mathrm{LF}(S,w') \longrightarrow \mathbf{D}^{\mathrm{ctr}}\,\mathrm{LF}(S,w+w');$$

however, it can in general not *be computed through resolution of a single factor only.*

See Remark I.4.2.2 for an example witnessing that in general both factors in $-\otimes_S^{\mathbf{L}} -$ need to be resolved by matrix factorizations, as well as a model categorical analysis of the difference with the classical situation for complexes.

In Section I.4.2 we shall nonetheless develop criteria allowing for the computation of $-\otimes_S^{\mathbf{L}}-$ through resolution of a single factor only, which in particular apply to the matrix factorizations occurring in the construction of $\mathcal{KR}^k$. This finally allows for proving that – referring to the contraderived model structure $\mathcal{M}^{\mathrm{ctr}}\,\mathrm{LF}(S,w)$ – the complexes of matrix factorizations assigned to braids in the construction of $\mathcal{KR}^k$ are termwise weakly equivalent to the respective Rouquier complexes of Soergel bimodules. Further, braid closure turns out to correspond to a variant of Hochschild homology:

Result (Theorem I.5.3)**.** *Let L be an oriented link presented as the closure of a braid β, and let $\mathcal{CCKR}^k(\beta)$ be the complex of matrix factorizations assigned to β in the Khovanov-Rozansky construction.*

(i) *Up to shift, there is a canonical, termwise weak equivalence of complexes of linear factorizations $\mathcal{CCKR}^k(\beta) \simeq \mathcal{RC}_{\mathbb{Q}}(\beta)$.*

(ii) *The invariant $\mathcal{KR}^k(L)$ is (up to normalization) given by the Poincaré polynomial of the stable Hochschild homology of the Rouquier complex attached to β, i.e.*

$$\mathcal{KR}^k(L) = \sum_{i,j\in\mathbb{Z}} \dim_{\mathbb{Q}} \mathrm{H}^i \left[{}^k\,\mathrm{sHH}_t^{\mathbb{A}^n_{\mathbb{Q}}/\mathbb{Q}}\, \mathcal{RC}_{\mathbb{Q}}(\beta)_j \right] a^i q^j. \tag{KR}$$

Here $\mathcal{RC}_{\mathbb{Q}}(\beta)$ is the Rouquier complex of Soergel bimodules over $\mathbb{Q}$ attached to β (see Appendix I.A), and ${}^k\,\mathrm{sHH} := \Delta \otimes^{\mathbf{L}} -$ denotes the k-stable Hochschild homology (Definition I.4.5.1), a variant of ordinary Hochschild homology given by the contraderived tensor product with the diagonal in $\mathbf{D}^{\mathrm{ctr}}\,\mathrm{LF}(S,w)$. As firstly Soergel bimodules are a categorification of the Hecke algebra, and secondly Hochschild homology can – in view of its symmetry properties (Proposition I.6.2.1) – be conceived as a categorical analogue of the trace, the above result categorifies the description of $\mathcal{P}^k$ via traces on Hecke algebras outlined in Section A.

The description of $\mathcal{KR}^k$ in (KR) was already established in [Web07, Theorem 2.7] by different methods and using an ad-hoc definition of stable Hochschild homology. The theory presented here forms a conceptual ground for this result and the study of stable Hochschild homology. For a precise comparison between our approach and the one in [Web07] we refer to the introduction of Part I.

Further, the study of $\mathcal{KR}^k$ through the contraderived category allows for a direct proof of the fact that (KR) is an invariant of links:

Result (Theorem I.6.1.1, Corollary I.6.1.4)**.** *For any commutative ground ring $\Bbbk$ with $k+1 \in \Bbbk^\times$, the k-stable Hochschild homology of Rouquier complexes of Soergel bimodules is an invariant of oriented links.*

In particular, the proof presented here is independent of the original work [KR08a] and circumvents explicit calculations by invoking known results about the combinatorics of Soergel bimodules.

Finally, we study the opposite case $k+1=0$ in the ground ring $\Bbbk$:

Result (Theorem I.6.1.5)**.** *In case $k+1=0$ in $\Bbbk$ a different normalization of k-stable Hochschild homology of Rouquier complexes defines an invariant of oriented links, too.*

The normalization to use matches the one used by Rouquier in [Rou12] in his direct proof of the fact that ordinary Hochschild homology of Rouquier complexes is a (triply graded) invariant of oriented links, agreeing with the triply graded invariant defined by Khovanov and Rozansky in [KR08b]. The methods presented here give an alternative proof of this result, and also show:

Result (Proposition I.6.5.7, Corollary I.6.5.8)**.** *In case $k+1=0$ in $\Bbbk$ the spectral sequence between k-stable and ordinary Hochschild homology degenerates on the E_1-page. In particular, classical Hochschild homology can canonically be identified with the associated graded of the canonical filtration on k-stable Hochschild homology.*

Further results and more detailed descriptions of the results of this thesis are contained in the introduction to the knot theoretic Part I and in the introduction to the homotopy theoretic Part II.

Part I.

Knot Theoretic Aspects

I.1. Introduction to part I

In [KR08a] and [KR08b], Khovanov and Rozansky constructed categorifications $\mathcal{KR}$ resp. $\mathcal{KR}^k$ of the HOMFLYPT polynomial resp. its quantum group $\mathfrak{sl}(k)$ specialization using *matrix factorizations*, a tool known from commutative algebra and singularity categories.

Later, in [Kho07] Khovanov realized that given the closure L of a braid β, the value $\mathcal{KR}(L)$ can be obtained as the *Hochschild homology* of the *Rouquier complex* $\mathcal{RC}_{\mathbb{Q}}(\beta)$ of *Soergel bimodules* attached to β [Rou06]. Rouquier [Rou12] complemented this by introducing the notion of a 2-*Markov trace* as a categorification of the classical concept of a Markov trace and gave a direct proof of the fact the Hochschild homology of Rouquier complexes is indeed a link invariant.

Concerning $\mathcal{KR}^k$, in the spirit of [Kho07] Webster [Web07] introduced a variant of Hochschild homology depending on k, called *k-stable Hochschild homology* in the following, and proved that given L and β as above, the value $\mathcal{KR}^k(L)$ can be obtained by applying k-stable Hochschild homology to $\mathcal{RC}(\beta)$. The *first major goal* of this part is to complement Webster's result, just as Rouquier [Rou12] did for $\mathcal{KR}$, by showing directly that k-stable Hochschild homology of Rouquier complexes is invariant under the two Markov moves, hence gives a link invariant.

However, paving the way for that, we first promote several techniques, notably *curved mixed complexes*, the *stabilization functor* and *model structures*, as convenient tools to study matrix factorizations, and reprove Webster's result in this conceptual framework – the details will be explained below. Although not stated explicitly there, these concepts are already contained implicitly in [Web07]. We thereby hope to convince the reader that the model categorical point of view, introduced in [Pos11] and further developed in Part II of this work, comes up very naturally in the study of Khovanov-Rozansky homology and matrix factorizations in general.

To describe our results in more detail we fix some notation. In the following, the cohomological grading and shift will be denoted $|-| = |-|^c$ and Σ^n, respectively, while the grading and shift internal to rings and modules (the "q-grading") will be denoted $|-|_q$ and $\langle n\rangle$. The underlined component in a complex is always the one sitting in cohomological degree 0.

We begin with an ad-hoc definition of stable Hochschild homology for polynomial rings as found in [Web07]; a more conceptual definition will be presented in Section I.4.5. Let

$\mathbb{A}^n_{\mathbb{Q}} := \mathbb{Q}[x_1,\ldots,x_n]$ be the polynomial ring, considered as a $\mathbb{Z}$-graded algebra by $|x_i|_q = 2$, $\mathbb{A}^n_{\mathbb{Q}}$-Mod be the category of graded $\mathbb{A}^n_{\mathbb{Q}}$-modules and $\widehat{\mathbb{A}}^n_{\mathbb{Q}} = \mathbb{Q}[x_1,\ldots,x_n,y_1,\ldots,y_n]$ be the enveloping algebra of $\widehat{\mathbb{A}}^n_{\mathbb{Q}}$. Further, we denote

$$P_\Delta := \left(\bigwedge^* \bigoplus_{i=1}^n \Sigma e_i \widehat{\mathbb{A}}^n_{\mathbb{Q}}\langle -2\rangle, \mathrm{d} := \sum_{i=1}^n (x_i - y_i)\frac{\partial}{\partial e_i}\right) \cong \bigotimes_{i=1}^n \left(\widehat{\mathbb{A}}^n_{\mathbb{Q}}\langle -2\rangle \xrightarrow{x_i - y_i} \underline{\widehat{\mathbb{A}}^n_{\mathbb{Q}}}\right)$$

the Koszul resolution of the diagonal $\mathbb{A}^n_{\mathbb{Q}}$-module $\Delta^n_{\mathbb{Q}} := \widehat{\mathbb{A}}^n_{\mathbb{Q}}/(x_i - y_i)$. Then, given a graded $\mathbb{A}^n_{\mathbb{Q}}$-bimodule M, recall that its ordinary Hochschild homology $\mathrm{HH}^{\mathbb{A}^n_{\mathbb{Q}}}_*(M) = \mathrm{H}^{-*}\left(M \otimes^{\mathbf{L}}_{\widehat{\mathbb{A}}^{\mathbb{Q}}_n} \Delta^n_{\mathbb{Q}}\right)$ can be computed as the homology of $P_\Delta \otimes_{\widehat{\mathbb{A}}^n_{\mathbb{Q}}} M$. In contrast, the definition of *stable* Hochschild homology not only depends on the $\mathbb{Q}$-algebra $\mathbb{A}^n_{\mathbb{Q}}$ and the bimodule M but requires the extra datum of some fixed homogeneous element $w \in \mathbb{A}^n_{\mathbb{Q}}$ called *potential.* Given w, we write $\widehat{w} = w\otimes 1 - 1\otimes w = \sum_i (x_i - y_i)u_i$ for some $u_i \in \widehat{\mathbb{A}}^n_{\mathbb{Q}}$, and put $s := \sum_i u_i(e_i \wedge -)$, an operator on P_Δ which is of cohomological degree -1 and q-degree $d := |w|_q$. Together with d and s, P_Δ becomes a *$\widehat{w}$-curved mixed complex* over $\widehat{\mathbb{A}}^n_{\mathbb{Q}}$ in the sense that $\mathrm{d}^2 = 0$, $s^2 = 0$ and $\mathrm{d}\,s + s\,\mathrm{d} = \widehat{w}$; this generalizes the well-known notion of mixed complexes, see e.g. [Lod98, §2.5.13], which is recovered when $w = 0$. Moreover, given such an an ordinary mixed complex (X, d, s) (again with $|s|_q = d$) we can form its *(total) cyclic homology* $\mathrm{HC}(X, \mathrm{d}, s)$ as the cohomology of $\mathrm{d}+s$ acting on $\bigoplus_{i\in\mathbb{Z}} X^i\langle \frac{id}{2}\rangle$ (the shift makes $\mathrm{d}+s$ homogeneous). As a special case of that, given M a graded $\mathbb{A}^n_{\mathbb{Q}}$-bimodule with $w.m = m.w$ for all $m \in M$, we have the following definition of w-stable Hochschild homology, see e.g. [Web07, Paragraph following Theorem 2.7]:

Definition. *The* total stable Hochschild homology *of M with respect to w is defined as*

$${}^w\,\mathrm{sHH}^{\mathbb{A}^n_{\mathbb{Q}}/\mathbb{Q}}_t(M) \;:=\; \mathrm{HC}\left(P_\Delta \otimes_{\widehat{\mathbb{A}^n_{\mathbb{Q}}}} M, \mathrm{d}\otimes \mathrm{id}_M, s\otimes \mathrm{id}_M\right).$$

In the case where $w = w_n := \sum_i x_i^{k+1}$ – the potential relevant for the construction of $\mathcal{KR}^k$ – we denote ${}^w\,\mathrm{sHH}$ by ${}^k\,\mathrm{sHH}$ and call it *k-stable Hochschild homology.*

To state the main result of [Web07], we need to introduce Soergel bimodules and Rouquier complexes. For fixed n, the category $\mathcal{SBM}_{\mathbb{Q}}(n)$ of *Soergel bimodules* [Soe07] (over $\mathbb{Q}$) is the smallest subcategory of $\mathbb{A}^n_{\mathbb{Q}}$-bimod containing the diagonal bimodule $\Delta^n_{\mathbb{Q}}$ as well as the bimodules $B^{n,i}_{\mathbb{Q}} := \mathbb{A}^n_{\mathbb{Q}} \otimes_{(\mathbb{A}^n_{\mathbb{Q}})^{(i,i+1)}} \mathbb{A}^n_{\mathbb{Q}}$ and which is closed under the operations of internal shifting and taking finite direct sums, summands and tensor products; here, for $1 \le i < n$ we denote $(\mathbb{A}^n_{\mathbb{Q}})^{(i,i+1)} \subset \mathbb{A}^n_{\mathbb{Q}}$ the polynomials invariant under exchanging $x_i \leftrightarrow x_{i+1}$. Further, denote $\mathrm{K}^{\oplus}_0(\mathcal{SBM}_{\mathbb{Q}}(n))$ the split Grothendieck $\mathbb{Z}[q^{\pm 1}]$-algebra of $\mathcal{SBM}_{\mathbb{Q}}(n)$ (with multiplication given by tensor product $\otimes_{\mathbb{A}^n_{\mathbb{Q}}}$ and with q acting via $[M] \mapsto [M\langle 1\rangle]$), and by $\mathrm{H}_n(q)$ the *Hecke algebra* of $\mathfrak{S}_n$, i.e. the quotient of the $\mathbb{Z}[q^{\pm 1}]$-group algebra of the n-strand Artin braid group Br_n on generators $\mathrm{T}_1,\ldots,\mathrm{T}_{n-1}$ by the additional quadratic relations $\mathrm{T}_i^2 = (q^2-1)\mathrm{T}_i + q^2\mathrm{T}_e$. Finally, we denote the unit element of $\mathrm{H}_n(q)$ by T_e. A fundamental Theorem of Soergel [Soe07] then states that the assignments

$$\underline{\mathrm{H}}_i := q^{-1}(\mathrm{T}_e + \mathrm{T}_i) \longmapsto [B_{\mathbb{Q}}^{n,i}\langle 1\rangle], \quad \mathrm{T}_e \longmapsto [\Delta_{\mathbb{Q}}^n]$$

extend uniquely to a well-defined homomorphism of $\mathbb{Z}[q^{\pm 1}]$-algebras

$$\mathrm{H}_n(q) \to \mathrm{K}_0^{\oplus}(\mathcal{SBM}_{\mathbb{Q}}(n));$$

in other words, the combinatorics of tensor products of Soergel bimodules is captured by the Hecke algebra. However, there is no $X \in \mathcal{SBM}_{\mathbb{Q}}(n)$ with $[X] = \mathrm{T}_i$ – it is only the *Kazhdan-Lusztig basis* elements that lift to (indecomposable) Soergel bimodules. In contrast, one may obtain a categorification of the *braid group* using *complexes* of Soergel bimodules; this idea originally appeared in Lie theory in the context of shuffling functors on the BGG category $\mathcal{O}$ (see e.g. [MS07, Theorem 1]), and in our situation was worked out by Rouquier in [Rou06], where he constructed for each braid β a complex $\mathcal{RC}_{\mathbb{Q}}(\beta)$ of Soergel bimodules, now called the *Rouquier complex* of β, such that for two braids β, β' one has isomorphisms $\mathcal{RC}_{\mathbb{Q}}(\beta) \otimes_{\mathbb{A}_{\mathbb{Q}}^n} \mathcal{RC}_{\mathbb{Q}}(\beta') \cong \mathcal{RC}_{\mathbb{Q}}(\beta\beta')$ in the homotopy category $\mathrm{Ho}(\mathbb{A}_{\mathbb{Q}}^n\text{-bimod})$ of complexes of graded $\mathbb{A}_{\mathbb{Q}}^n$-bimodules (he even produces a *strong* categorification of the braid group, but we won't need that here).

The main theorem of [Web07] now describes $\mathcal{KR}^k$ in terms of stable Hochschild homology of Rouquier complexes:

Theorem [Web07, Theorem 2.7]**.** *Let $\mathcal{KR}^k$ be Khovanov-Rozansky's categorification of the quantum $\mathfrak{sl}(k)$ link invariant, defined over the rational numbers $\mathbb{Q}$. Then, given a link L isotopic to the closure of a braid β on n strands and writhe $w(\beta)$,*

$$\mathcal{KR}^k(L) \;=\; (a^{-1}q^{k+1})^{w(\beta)} \sum_{i,j\in\mathbb{Z}} \dim_{\mathbb{Q}} \mathrm{H}^i \left[{}^k\,\mathrm{sHH}_t^{\mathbb{A}_{\mathbb{Q}}^n/\mathbb{Q}}\, \mathcal{RC}_{\mathbb{Q}}(\beta))_j\right] a^i q^j \in \mathbb{Z}[a^{\pm 1}, q^{\pm 1}],$$

the Poincaré polynomial of the cohomology of the complex obtained by applying total k-stable Hochschild homology termwise to the Rouquier complex $\mathcal{RC}_{\mathbb{Q}}(\beta)$ of β.

The proof is based on two key insights, which we want to comment on separately. Throughout the rest of this introduction, $k \geq 2$ is fixed, and we put $w_n := \sum_i x_i^{k+1} \in \mathbb{A}_{\mathbb{Q}}^n$.

The *first key insight* states, in our language that we shall explain below: Given a braid β on n strands, the i-th matrix factorization term in $\mathcal{KR}^k(\beta)$ is the folding of a bounded $\widehat{w_n}$-curved mixed complex (X, d, s) of $\mathbb{A}_{\mathbb{Q}}^n$-bimodules with the property that (X, d) is an $\widehat{\mathbb{A}}_{\mathbb{Q}}^n$-free resolution of the i-th Soergel bimodule term of the Rouquier complex $\mathcal{RC}_{\mathbb{Q}}(\beta)$.

Fix a $\mathbb{Z}$-graded, commutative ring A and a homogeneous potential $w \in A$ of q-degree d. Then a graded *w-curved mixed complex* is a triple $X = (X, \mathrm{d}, s)$ where X is $\mathbb{Z}$-graded

graded A-module together with A-linear differentials d, s of cohomological degrees $+1$ and -1 and q-degrees 0 and d, respectively, such that $\mathrm{d}\,s + s\,\mathrm{d} = w$. Denote the category of such by $\mathrm{MC}(A, w)$. Curved mixed complexes appear in [Web07] as "($\mathbb{Z}$-graded) matrix factorizations"; however, the morphisms are required to be degree-preserving only modulo 2 there, whence the resulting category is equivalent to usual ($\mathbb{Z}/2\mathbb{Z}$-graded) matrix factorizations. We propose to see the passage from $\mathbb{Z}$-grading to $\mathbb{Z}/2\mathbb{Z}$-grading as a folding procedure: for $X \in \mathrm{MC}(A, w)$, its *folding* $\mathrm{fold}^{\Pi}(X)$ is defined as

$$X^{\mathrm{even}} \xrightarrow{\mathrm{d}+s} X^{\mathrm{odd}} \xrightarrow{\mathrm{d}+s} X^{\mathrm{even}}, \quad X^{\mathrm{even}} := \prod_{n\in\mathbb{Z}} X^{2n}\langle -nd\rangle, \quad X^{\mathrm{odd}} := \prod_{n\in\mathbb{Z}} X^{2n-1}\langle -nd\rangle;$$

since $(d+s)^2 = d^2 + ds + sd + s^2 = w$ by definition, it is a *linear factorization of type* (A, w), i.e. a matrix factorization of type (A, w) with possibly non-free components. We denote the category of linear factorizations of type (A, w) by $\mathrm{LF}(A, w)$.

The interesting point that we want to add here is that the procedure of folding a free resolution of a module is actually *functorial* and *monoidal*:

Proposition (see Section I.4.4)**.** *The folding functor* fold^{Π} *maps quasi-isomorphisms of bounded A-free w-curved mixed complexes to homotopy equivalences of matrix factorizations. In particular, for* $\mathrm{gl.dim}(A\text{-Mod}) < \infty$*, taking the folding of bounded A-free resolutions of bounded w-curved mixed complexes therefore yields a well-defined functor*

$$\mathbf{R}\,\mathrm{fold}^{\Pi} : \mathbf{D}^b\,\mathrm{MC}(A, w) \longrightarrow \underline{\mathrm{MF}}(A, w),$$

and moreover $\mathbf{R}\,\mathrm{fold}^{\Pi}$ *commutes with (derived) tensor product.*

This implies that sending a graded $A/(w)$-module to the folding of a bounded A-free resolution of it as a w-curved mixed complex is *functorial* and in particular independent of the choice of resolution up to canonical homotopy equivalence: namely, it is given by

$$A/(w)\text{-Mod} \longrightarrow \mathbf{D}^b\,\mathrm{MC}(A, w) \xrightarrow{\mathbf{R}\,\mathrm{fold}^{\Pi}} \underline{\mathrm{MF}}(A, w).$$

In case A is regular local with maximal ideal $\mathfrak{m}$ and $w \in \mathfrak{m}\setminus\{0\}$, this functor fits with the categories and functors classically attached to the hypersurface singularity $R := A/(w)$ [Buc86] in the sense that there is a diagram commutative up to isomorphism:

$$\begin{array}{ccccc}
\mathbf{D}^b\,\mathrm{MC}(A, w) & & \xrightarrow{\quad\mathbf{R}\,\mathrm{fold}^{\Pi}\quad} & & \underline{\mathrm{MF}}(A, w) \\
\cong\uparrow & & & & \downarrow\cong\ \mathrm{coker} \\
\mathbf{D}^b(R) & \xrightarrow[\mathrm{can}]{} & \mathbf{D}^b_{\mathrm{sg}}(R) \equiv \mathbf{D}^b(R)/\,\mathrm{Perf}(R) & \xleftarrow[\mathrm{can}]{\cong} & \underline{\mathrm{MCM}}(R)
\end{array}$$

With these definitions, the first insight reads as follows: For a braid β, the matrix factorization terms in $\mathcal{KR}^k(\beta)$ are canonically homotopy equivalent to the images of the Soergel bimodules terms in $\mathcal{RC}_{\mathbb{Q}}(\beta)$ under the derived folding functor $\mathbf{R}\,\mathrm{fold}^{\Pi}$.

The functoriality and monoidality not only prove the independence of stabilization of the choice of free resolution, but furthermore imply (together with Soergel's theorem on the combinatorics of Soergel bimodules) that the terms of $\mathcal{KR}^k(\beta)$ obey the Hecke algebra relations, a fact which required a substantial amount of work in [KR08a] where it was checked directly.

The *second key insight* is [Web07, Theorem 1.2]: Given an $A/(w)$-module M and a *bounded* A-free resolution $P \to M$ of M as a w-curved mixed complex, then the projection $\mathrm{fold}^{\Pi}(P) \to M$ is a *near isomorphism* [Web07, Definition 2], i.e. turns into a classical quasi-isomorphism upon tensoring with any matrix factorization of the opposite potential $-w$. Together with the first key insight, this yields Webster's theorem.

Note that the issue of boundedness is crucial here and should be added in [Web07, Proposition 1.1, Theorem 1.2], as otherwise it can happen for non-regular rings that the spectral sequences used in [Web07] fail to converge, even though they collapse; see Remark I.4.4.4. This is, however, of no trouble concerning the application to Khovanov-Rozansky homology, as all complexes arising there are bounded with regular base rings.

We propose to think of near isomorphisms as slight weakenings of *weak equivalences* in the natural *model category enhancement* of $\underline{\mathrm{MF}}(A, w)$ as studied in [Pos11] and Part II of this work: Due to $\delta^2 \neq 0$, when working with linear factorizations there is no obvious way to speak about acyclicity. However, it is still possible to *totalize* chain complexes of linear factorizations like one does for ordinary (bi)complexes; building on the fact that the totalization of a short exact sequence of ordinary chain complexes is always acyclic, is therefore reasonable to think of totalizations of short exact sequences of linear factorizations as proposals for "acyclic" linear factorizations. Cutting it short and leaving the details for later (see Section I.4.1), pursuing this idea results inter alia in the definition of the *contraderived model structure* $\mathcal{M}^{\mathrm{ctr}}\,\mathrm{LF}(A, w)$ on the category of linear factorizations, with homotopy category canonically equivalent to $\underline{\mathrm{MF}}(A, w)$. The basic facts about this model structure are the following:

(i) If $w = 0$, then any weak equivalence in $\mathcal{M}^{\mathrm{ctr}}\,\mathrm{LF}(A, w)$ is a quasi-isomorphism. If $\mathrm{gl.\,dim}(A\text{-mod}) < \infty$, then the converse holds, so we recover the classical projective model structure for the 2-periodic derived category of A.

(ii) The folding $\mathrm{fold}^{\Pi} : \mathrm{MC}^b(A, w) \to \mathrm{LF}(A, w)$ maps quasi-isomorphisms to weak equivalences and hence descends naively to a functor $\mathbf{R}\,\mathrm{fold}^{\Pi} : \mathbf{D}^b\,\mathrm{MC}(A, w) \to \mathbf{D}^{\mathrm{ctr}}\,\mathrm{LF}(A, w)$; specifically, there is also a contraderived category $\mathbf{D}^{\mathrm{ctr}}\,\mathrm{MC}(A, w)$ of curved mixed complexes receiving a canonical identity functor from $\mathbf{D}^b\,\mathrm{MC}(A,w)$, and fold^{Π} maps weak equivalences of curved mixed complexes to contraderived weak equivalences of linear factorizations. This strengthens the above proposition on foldings of bounded, A-free curved mixed complexes. See Section I.4.4.

(iii) The tensor product of linear factorizations is a Quillen bifunctor $\mathcal{M}^{\mathrm{ctr}}(A,w) \times \mathcal{M}^{\mathrm{ctr}}(A,w') \to \mathcal{M}^{\mathrm{ctr}}(A,w+w')$, giving a derived tensor product on the homotopy categories. Unlike in the situation for ordinary derived categories, this derived tensor product has to be computed using resolutions in *both* arguments; however, if $\mathrm{gl.dim}(A\text{-mod}) < \infty$, resolution of a single argument suffices. See Section I.4.2.

In view of (ii), we can conclude from our formulation of the first insight that the complex of matrix factorizations attached to a braid is canonically termwise weakly equivalent to the Rouquier complex attached to the braid. Moreover, from (i) and (iii) we see that weak equivalences are near-isomorphisms, giving the second insight. All in all, this results in the following theorem; for the notation, see Section I.5:

Theorem I.5.3. *Given an oriented link L presented as the closure of an n-strand braid word β with writhe $w(\beta)$, there is a canonical termwise contraderived weak equivalence of complexes of linear factorizations of type $(\widehat{\mathbb{A}}^n_{\mathbb{Q}}, \widehat{w}_n)$*

$$\mathcal{CCKR}^k(\beta) \xrightarrow{\simeq} \Sigma^{w(\beta)}{}_{\widehat{w}_n}\mathcal{RC}_{\mathbb{Q}}(\beta)\langle -(k+1)w(\beta)\rangle,$$

where $\mathcal{RC}_{\mathbb{Q}}(\beta)$ is the Rouquier complex of β defined over $\mathbb{Q}$. Moreover, there is a canonical isomorphism of complexes of graded $\mathbb{Q}$-vector spaces

$$\mathcal{CKR}^k(L) \xrightarrow{\simeq} \Sigma^{w(\beta)}\ {}^{k}\,\mathrm{sHH}^{\mathbb{A}^n_{\Bbbk}}\left[\mathcal{RC}_{\mathbb{Q}}(\beta)\right]\langle -(k+1)w(\beta)\rangle \tag{I.1.1}$$

and hence the Khovanov-Rozansky homology $\mathcal{KR}^k(L) \in \mathbb{Z}[a^{\pm 1}, q^{\pm 1}]$ is given by

$$\mathcal{KR}^k(L) = (a^{-1}q^{k+1})^{w(\beta)} \sum_{i,j\in\mathbb{Z}} \dim_{\mathbb{Q}} \mathrm{H}^i\left[{}^{k}\,\mathrm{sHH}_t^{\mathbb{A}^n_{\mathbb{Q}}/\mathbb{Q}}\,\mathcal{RC}_{\mathbb{Q}}(\beta)_j\right] a^i q^j.$$

Further, given the contraderived category of linear factorizations together with its contraderived tensor product, we propose the following conceptual definition of stable Hochschild homology; see Section I.4.5.

Definition I.4.5.1. *Let A be a commutative $\Bbbk$-algebra, $w \in A$ and M be an A-bimodule with $\widehat{w}m = 0$ for all $m \in M$. The w-*stable Hochschild homology *is defined as*

$${}^{w}\,\mathrm{sHH}_*^{A/\Bbbk}(M) \ := \ \mathrm{H}^*\left[{}_{-\widehat{w}}\Delta \otimes^{\mathbf{L}}_{\widehat{A}} {}_{\widehat{w}}M\right];$$

here $\widehat{w} := w\otimes 1 - 1\otimes w \in \widehat{A} := A\otimes_{\Bbbk} A$, and ${}_{-\widehat{w}}\Delta$ and ${}_{\widehat{w}}M$ denote the linear factorizations of potential $-\widehat{w}$ resp. $\widehat{w}$ given by Δ resp. M in cohomological degree 0.

The complex (I.1.1) makes sense with $\mathbb{Q}$ replaced by an arbitrary commutative ring $\Bbbk$, and in Section I.6 we give a direct proof that it is an invariant of oriented links if $k+1 \in \Bbbk^\times$; slightly stronger, we have the following (for the notation, see Section I.6):

Theorem I.6.1.1. *Let $\Bbbk$ be a commutative ring with $k+1 \in \Bbbk^\times$. Then, for an n-strand braid word β with writhe $w(\beta)$, the complex*

$$\mathcal{CKR}^k_{\Bbbk}(\beta) \;:=\; \Sigma^{-w(\beta)k}\, \mathrm{sHH}_t^{\mathbb{A}^n_{\Bbbk}/\Bbbk}\,[\mathcal{RC}_{\Bbbk}(\beta)]\,\langle (k+1)w(\beta)\rangle$$

has $\Bbbk$-free components of finite rank. Moreover, its isomorphism class in $\mathrm{Ho}^b(\Bbbk\text{-Mod})$ *is invariant under the Markov moves, and hence it defines an invariant of oriented links.*

The action of the polynomial ring can be incorporated by considering *ordered* links; see Theorem I.6.1.3. The theorem complements Rouquier's direct proof in [Rou12] of the fact that Hochschild homology of Rouquier complexes gives an invariant of oriented links. If $k+1=0$ in $\Bbbk$, a different normalization still yields an invariant of oriented links; this is discussed in Section I.6.5, in particular Theorem I.6.5.6. The normalization matches the one in Rouquier's theorem, and in fact the latter can also be deduced from our approach.

Structure: In Section I.2.1 we introduce the basic definitions around linear factorizations and curved mixed complexes, and Sections I.2.2 and I.2.3 discuss tensor products and the folding functor, respectively. Chapter I.3 quickly recalls the original definition of Khovanov-Rozansky homology from [KR08a]. In Chapter I.4 we introduce the contraderived categories of linear factorizations and of curved mixed complexes and study their derived tensor products as well as the derived folding functor. As an application of this framework, the conceptual definition of stable Hochschild homology is presented in Section I.4.5 and compared to ordinary Hochschild homology in Section I.4.6. Chapter I.5 concerns the description of Khovanov-Rozansky homology in terms of stable Hochschild homology of Rouquier complexes, in particular containing Theorem I.5.3. Chapter I.6 is devoted to the direct proof of the fact that stable Hochschild homology of Rouquier complexes gives an invariant of oriented links: Section I.6.1 gives an overview, Section I.6.2 treats invariance under the first Markov move, while Sections I.6.3, I.6.4 and I.6.5 treat the second Markov move, distinguishing between those statements that do not depend on the value of $k+1$ in $\Bbbk$, those which need $k+1 \in \Bbbk^\times$ and those which work only for $k+1=0$ in $\Bbbk$. In Chapter I.7 we discuss some further variations on KR-homology: In Section I.7.1, we argue that many technical difficulties of the previous sections can be avoided by working with comodules instead of modules, in Section I.7.2 we check directly that indeed $\mathcal{KR}^k$ is a categorification of quantum $\mathfrak{sl}(k)$-invariant; in Sections I.7.3 and I.7.4 we describe an approximation of Khovanov-Rozansky homology by a combinatorial cut-and-join formalism, and in Section I.7.6 we introduce equivariant and deformed KR-homology. Finally, Appendix I.A recalls basic definitions and statements about knots and their presentations as well as Soergel bimodules and Rouquier complexes.

I.2. Linear Factorizations & Curved Mixed Complexes

In this chapter we recall linear factorizations, curved mixed complexes and their connection via the folding functor. The basic definitions are also contained in Section II.5.2 in Part II on homotopy theoretic aspects, but for completeness and the need of an additional internal grading, we give them here as well.

We fix a torsion-free abelian grading group Ω (e.g. $\Omega = \{e\}$ or $\Omega = \mathbb{Z}$) and let A be an Ω-graded ring with $w \in A$ central and homogeneous of q-degree $d := |w|_q$. We also assume that $\frac{d}{2}$ exists in Ω. If $w = 0$ we might write 0_d to emphasize that it is to be considered as of q-degree d. By an A-module or an $A/(w)$-module we will always mean an Ω-graded A resp. $A/(w)$-module, and we do not stress this further in the notation ?-Mod for the module categories. The shift in internal grading is denoted $\langle - \rangle$.

I.2.1. Basic definitions

Definition I.2.1.1. *A* linear factorization *of type* (A, w) *is a* $\mathbb{Z}/2\mathbb{Z}$*-graded* A*-module* $M = M^0 \oplus M^1$ *with an odd* A*-linear endomorphism* $\delta = (\delta^0, \delta^1)$, $\delta^0 : M^0 \to M^1$, $\delta^1 : M^1 \to M^0$ *such that* $|\delta^1|_q = 0$, $|\delta^0|_q = d$ *and* $\delta^2 = w \cdot \mathrm{id}_M$. *It is called a* matrix factorization *if, moreover,* M^0, M^1 *are projective as* A*-modules.*

We will often denote (M, δ) by $\delta^1 : M^1 \rightleftarrows M^0 : \delta^0$ or by $M^0 \xrightarrow{\delta^1} M^1 \xrightarrow{\delta^0} M^0$. The categories of linear factorization resp. matrix factorizations of type (A, w) are denoted $\mathrm{LF}(A, w)$ resp. $\mathrm{MF}(A, w)$. As for ordinary complexes, one can define the notion of *homotopy* between morphisms of linear factorizations, giving rise to *homotopy categories* $\underline{\mathrm{LF}}(A, w)$ resp. $\underline{\mathrm{MF}}(A, w)$ of linear factorizations resp. matrix factorizations; the morphism spaces in these categories will sometimes be denoted by $[-, -]$.

As explained in Section II.5.2 of Part II, a linear factorization of type (A, w) is the same as a $\mathbb{Z}/2\mathbb{Z}$-graded curved dg module over the $\mathbb{Z}/2\mathbb{Z}$-graded curved dg ring A_w given by $A_w^{\overline{0}} = A$, $A_w^{\overline{1}} = 0$, with trivial differential and curvature term $w \in A_w^{\overline{2}} = A$ (see II.2.3.1 for the definition of curved dg rings and their modules); in particular $\mathrm{LF}(A, w)$ is a Grothendieck category with enough projectives.

Definition I.2.1.2. *Let M and N be linear factorizations of type (A, w).*

(i) *We denote ΣM the* shift *of M, given by*

$$\Sigma M \ := \ -\delta^0 : M^0 \rightleftarrows M^1\langle d\rangle : -\delta^1;$$

it is again a linear factorization of type (A, w).

(ii) *If $f : M \to N$ is a morphism of linear factorizations, we denote* Cone(f) *its* cone

$$\mathrm{Cone}(f) \ := \ \begin{pmatrix} \delta^1_M & f^0 \\ 0 & -\delta^0_N \end{pmatrix} : M^1 \oplus N^0 \rightleftarrows M^0 \oplus N^1\langle d\rangle : \begin{pmatrix} \delta^0_M & f^1 \\ 0 & -\delta^1_N \end{pmatrix};$$

it is a again a linear factorization of type (A, w).

(iii) *In the special case of $w = 0$ we define the* 0*-th* cohomology H^0 *of M as*

$$\mathrm{H}^0 M \ := \ \ker \delta^0 / \operatorname{im} \delta^1$$

and put $\mathrm{H}^i M := \mathrm{H}^0 \Sigma^i M$ for general $i \in \mathbb{Z}$.

Note that $\Sigma^2 = \langle d\rangle$, so $\mathrm{H}^{i+2l} M = \mathrm{H}^i M\langle ld\rangle$, and that $\mathrm{H}^1 M = (\ker \delta^1 / \operatorname{im} \delta^0)\langle d\rangle$. In case $\Omega = \mathbb{Z}$ and d is even we define the *total cohomology* as $\mathrm{H}^t M := \mathrm{H}^0 M \oplus \mathrm{H}^1 M\langle -\frac{d}{2}\rangle$, i.e. $\mathrm{H}^t M = (\ker \delta^0 / \operatorname{im} \delta^1) \oplus (\ker \delta^1 / \operatorname{im} \delta^0)\langle \frac{d}{2}\rangle$.

Remark I.2.1.3. The normalization in H^t is convenient since $\mathrm{H}^t \Sigma M = \mathrm{H}^t M\langle \frac{d}{2}\rangle$. Moreover, it corresponds to the total cohomology with no q-shifts involved in the definition of linear factorizations from [KR08a], where $|\delta^0|_q = |\delta^1|_q = \frac{d}{2}$: For a linear factorization $\delta^1 : M^1 \rightleftarrows M^0 : \delta^0$ of type (A, w), associate to it the 2-periodic w-curved complex

$$\widetilde{M} \quad := \quad \ldots \to M^0 \to M^1\langle \tfrac{d}{2}\rangle \to \underline{M^0} \to M^1\langle \tfrac{d}{2}\rangle \to \ldots$$

whose differentials are of q-degree $\frac{d}{2}$ and where the underlined entry M^0 is in cohomological degree 0. Then, if $w = 0$, we have $\mathrm{H}^t(M) = \mathrm{H}^0(\widetilde{M}) \oplus \mathrm{H}^{-1}(\widetilde{M})$. We will use this construction later to apply classical spectral sequences for complexes to compute the cohomology groups of some linear factorizations. Also, note that given a (q-degree preserving) morphism $f : M \to N$ of linear factorizations, we have $\widetilde{\mathrm{Cone}(f)} = \mathrm{Cone}(\widetilde{M}\langle \frac{d}{2}\rangle \to \widetilde{N})$, where on the right hand side we mean the ordinary cone of a morphism of complexes, and the shift of M is necessary to make $M\langle \frac{d}{2}\rangle \to N$ of q-degree $\frac{d}{2}$, as are the differentials of $\widetilde{M}$ and $\widetilde{N}$. A similar compatibility holds for totalizations of chain complexes of linear factorizations. The construction $\widetilde{(-)}$ will only be used in Section I.7.3. ♢

Example I.2.1.4. If M is an $A/(w)$-module then $0 \rightleftarrows M$ is a linear factorization of type (A, w). For ease of notation, we will abbreviate this factorization by ${}_wM$ or even M if there is no chance of confusion. ♢

Example I.2.1.5. For any A-module M there are two induced linear factorizations $1 : M \rightleftarrows M : w$ and $w : M\langle -|w|_q\rangle \rightleftarrows M : 1$ of type (A, w), both of which are both contractible (i.e. isomorphic to 0 in $\underline{\mathrm{LF}}(A, w)$). ♢

Example I.2.1.6. For x, y homogeneous and central, $\{x, y\} := A \xrightarrow{y} A\langle -|x|_q\rangle \xrightarrow{x} A$ is a matrix factorization of type (A, xy), called the *elementary Koszul factorization* associated to x, y. ♢

Definition I.2.1.7. *A* w-curved mixed complex *is a triple* $X = (X, \mathrm{d}, s)$ *where* X *is* $\mathbb{Z}$*-graded* A*-module together with* A*-linear differentials* d, s *of cohomological degrees* $+1$ *and* -1 *and* q*-degrees* 0 *and* d*, respectively, such that* $\mathrm{d}\, s + s\, \mathrm{d} = w$*.*

We denote the category of w-curved mixed complexes by $\mathrm{MC}(A, w)$; as noted in Section II.5.2 it is isomorphic to the category of dg modules over the *Koszul-Algebra* $K(A, w) := A[s]/(s^2)$, given by $|s|^c = -1$, $|s|_q = d$ and $\mathrm{d}(s) = w$, hence is a Grothendieck category with enough projectives, too.

Remark I.2.1.8. A 0-curved mixed complex (X, d, s) is called a *mixed complex*, see e.g. [Lod98, §2.5.13]. Its *(total) cyclic homology* $\mathrm{HC}(X)$ is the homology of $\mathrm{d} + s$ acting on $\bigoplus_{n\in\mathbb{Z}} X^n \left\langle -\frac{nd}{2}\right\rangle$, which is the same as the total homology in the sense of Definition I.2.1.2 of the linear factorization $\mathrm{fold}^{\oplus} X$ to be defined in Definition I.2.3.1 below. ♢

Example I.2.1.9. Any linear factorization $g : M^1 \rightleftarrows M^0 : f$ of type (A, w) can be considered as a w-curved mixed complex concentrated in cohomological degrees -1 and 0. In particular, any $A/(w)$-module M can be viewed as a w-curved mixed complex concentrated in degree 0, which we denote ${}_wM$; this conflicts with the definition of ${}_wM$ as a linear factorization, but it will be clear from the context whether we want to consider ${}_wM$ as a w-curved mixed complex or as a linear factorization. Analogously, we may consider the elementary Koszul factorization from Example I.2.1.6 as a w-curved mixed complex, which we also denote $\{x, y\}$, with the same notational caveat. ♢

Example I.2.1.10. Let $(X, \mathrm{d}, \cdot)$ be a $\mathbb{Z}$-graded dg algebra over A, and fix an arbitrary element $x \in X^{-1}$ satisfying $x^2 = 0$ (this is automatic if $(X, \cdot)$ is graded-commutative and 2 is invertible in A). Then $(X, \mathrm{d}, s := x \cdot -)$ is a curved mixed complex with curvature $w := \mathrm{d}(x)$. This will be used in Examples I.4.4.6 and I.4.4.7 later. ♢

Example I.2.1.11. For (X, d, s) a w-curved mixed complex and $n \in \mathbb{Z}$, the *truncation*

$$\tau_{\geq n} X \quad := \quad \ldots \to 0 \to X^n / \mathrm{im}(\mathrm{d}^{n-1}) \to X^{n+1} \to \ldots$$

inherits the structure of a w-curved mixed complex from X. ♢

Definition I.2.1.12. *If A is a commutative ring, $w \in A$, and $X \in \mathrm{LF}(A, w)$ (resp. $X \in \mathrm{MC}(A, w)$), the multiplication by some $a \in A$ defines an endomorphism of X which we call the* external action *of a on X. By definition, it is natural in X and can hence be viewed as an element of the center of* $\mathrm{LF}(A, w)$ *(resp.* $\mathrm{MC}(A, w)$*). In particular, the image of X under any additive functor inherits an external action of A.*

I.2.2. Tensor products

We assume from now on that A is commutative.

Definition I.2.2.1. *Let $M = (M, \delta)$ and $N = (N, \delta')$ be linear factorizations of type (A, w) and (A, w'), respectively. Then the* tensor product *$M \otimes_A N$ is defined as*

$$M^0 \otimes_A N^1 \;\oplus\; M^1 \otimes_A N^0 \underset{\begin{pmatrix} \mathrm{id} \otimes \delta' & \delta \otimes \mathrm{id} \\ \delta \otimes \mathrm{id} & -\mathrm{id} \otimes \delta' \end{pmatrix}}{\overset{\begin{pmatrix} \mathrm{id} \otimes \delta' & \delta \otimes \mathrm{id} \\ \delta \otimes \mathrm{id} & -\mathrm{id} \otimes \delta' \end{pmatrix}}{\rightleftarrows}} M^0 \otimes_A N^0 \;\oplus\; M^1 \otimes_A N^1\langle d \rangle.$$

It is a linear factorization of type $(A, w + w')$ and a matrix factorization if both M and N were such.

Example I.2.2.2. Let $\underline{x} = (x_1, \ldots, x_n)$, $\underline{y} = (y_1, \ldots, y_n)$ be two sequences of homogeneous elements of constant q-degree. Then the *Koszul factorization* $\{\underline{x}, \underline{y}\}$ is defined as the tensor product of the Koszul factorizations $\{x_i, y_i\}$, see Example I.2.1.6. It is a matrix factorization of type $(A, \sum_i x_i y_i)$. ♢

Similarly, if $(X, \mathrm{d}, \delta) \in \mathrm{MC}(A, w)$ and $(Y, \mathrm{d}', \delta') \in \mathrm{MC}(A, w')$, then the usual tensor product complex $(X, \mathrm{d}) \otimes_A (Y, \mathrm{d}')$ turns into a $(w + w')$-curved mixed complex when equipped with the additional differential $\delta \otimes \mathrm{id} + \mathrm{id} \otimes \delta'$; note that there is no shift in q-grading involved here. In particular, in the context of the previous Example I.2.2.2 we also have a Koszul curved mixed complex $\{\underline{x}, \underline{y}\}$ by taking the tensor product the $\{x_i, y_i\}$ in the category of curved mixed complexes.

I.2.3. The folding functor

We adapt the definition of the folding functors from Section II.5.2 to the graded setting.

Definition I.2.3.1. *Let (X, d, s) be a curved mixed complex of type (A, w).*

(i) *The* folding via products $\mathrm{fold}^{\Pi}(X)$ *of* X *is the linear factorization of type* (A, w)

$$\mathrm{fold}^{\Pi}(X) \;:=\; \prod_{n\in\mathbb{Z}} X^{2n}\langle -nd\rangle \xrightarrow{\mathrm{d}+s} \prod_{n\in\mathbb{Z}} X^{2n-1}\langle -nd\rangle \xrightarrow{\mathrm{d}+s} \prod_{n\in\mathbb{Z}} X^{2n}\langle -nd\rangle.$$

(ii) *The* folding via sums $\mathrm{fold}^{\oplus}(X)$ *of* X *is the linear factorization of type* (A, w)

$$\mathrm{fold}^{\oplus}(X) \;:=\; \bigoplus_{n\in\mathbb{Z}} X^{2n}\langle -nd\rangle \xrightarrow{\mathrm{d}+s} \bigoplus_{n\in\mathbb{Z}} X^{2n-1}\langle -nd\rangle \xrightarrow{\mathrm{d}+s} \bigoplus_{n\in\mathbb{Z}} X^{2n}\langle -nd\rangle.$$

We have $(\mathrm{d}+s)^2 = w$*, so* fold^{Π} *and* $\mathrm{fold}^{\oplus}$ *constitute functors* $\mathrm{MC}(A, w) \to \mathrm{LF}(A, w)$.

Fact I.2.3.2. *For curved mixed complexes* X, Y *of type* (A, w) *and* (A, w')*, respectively, there is a canonical isomorphism of linear factorizations of type* $(A, w + w')$*:*

$$\mathrm{fold}^{\oplus}(X \otimes_A Y) \;\cong\; \mathrm{fold}^{\oplus}(X) \otimes_A \mathrm{fold}^{\oplus}(Y).$$

In particular, given any two sequences $\underline{x} = (x_1, \ldots, x_n)$ and $\underline{y} = (y_1, \ldots, y_n)$ in A, $\mathrm{fold}^{\oplus}$ maps the Koszul curved mixed complex $\{\underline{x}, \underline{y}\}$ to the corresponding Koszul factorization.

I.3. Khovanov-Rozansky homology

Notation I.3.1. In every section concerned with Khovanov-Rozansky homology, k will always denote a once and for all fixed natural number ≥ 2. $\Diamond$

In this section, we review the original construction of $\mathcal{KR}^k$ from [KR08a]. Given an oriented link L, the construction of $\mathcal{KR}^k(L) \in \mathbb{Z}[q^{\pm 1}, t^{\pm 1}]$ is structured as follows:

(i) Firstly, one chooses a triple-point free projection of L onto the plane.

(ii) Secondly, one cuts the chosen projection into pieces each of which looks like an unknotted single strand $\uparrow$ or one of the two crossings ⤱ or ⤲, and assigns a variable to any point where a cut was made.

(iii) Thirdly, to each of the pieces just obtained one associates a certain complex of $\mathbb{Z}$-graded matrix factorizations to be given below, the ground ring being the rational polynomial ring over the variables attached to the open ends of the piece.

(iv) Finally, one takes the tensor product of all these complexes to obtain a complex $\mathcal{CCKR}^k(L)$ of matrix factorizations of potential 0. Taking total cohomology in each component, one gets a complex $\mathcal{CKR}^k(L)$ of graded vector spaces, and $\mathcal{KR}^k(L) \in \mathbb{Z}[a^{\pm 1}, q^{\pm 1}]$ is defined as the graded Poincaré series of the cohomology of $\mathcal{CKR}^k(L)$.

We provide the details below, but we need to fix some important notation first.

Notation I.3.2. For any commutative ring $\Bbbk$, we denote $\mathbb{A}^n_\Bbbk := \Bbbk[x_1, \ldots, x_n]$ the polynomial ring in n variables over $\Bbbk$, graded over $\Omega := \mathbb{Z}$ by $|x_i|_q = 2$. We denote $\widehat{\mathbb{A}}^n_\Bbbk := \Bbbk[x_1, \ldots, x_n, y_1, \ldots, y_n] = \mathbb{A}^n_\Bbbk \otimes_\Bbbk \mathbb{A}^n_\Bbbk$ its enveloping algebra. Further, we define the *Khovanov-Rozansky potentials* $w_n := \sum_i x_i^{k+1} \in \mathbb{A}^n_\Bbbk$ and $\widehat{w_n} := \sum_i x_i^{k+1} - \sum_i y_i^{k+1} \in \widehat{\mathbb{A}}^n_\Bbbk$; recall that the integer $k \geq 2$ is fixed once and for all.

The following bimodules are central for the construction of $\mathcal{KR}^k$ and the rest of this work: Firstly, the *diagonal* $\mathbb{A}^n_\Bbbk$*-bimodule*, denoted $\Delta^n_\Bbbk$. Secondly, for $1 \leq i < n$, the i-th *twisted diagonal* $\mathbb{A}^n_\Bbbk$*-bimodule* $X^{n,i}_\Bbbk$, given by ${}_{\mathbb{A}^n_\Bbbk}\mathbb{A}^n_\Bbbk$ as a left $\mathbb{A}^n_\Bbbk$-module and with the right $\mathbb{A}^n_\Bbbk$-action twisted by the automorphism exchanging x_i and x_{i+1}. The symbol X is meant to be suggestive here: in $X^{2,1}_\Bbbk$, considered as a $\widehat{\mathbb{A}}^2_\Bbbk$-module, the "lower right" variable y_2 acts like the "upper left" variable x_1, the same for y_1 and x_2. Finally, the most prominent bimodule in the construction of Khovanov-Rozansky homology is, for $1 \leq i < n$, the i-th *elementary Soergel bimodule* $B_{\Bbbk}^{n,i}$ over $\mathbb{A}_{\Bbbk}^n$, defined as $B_{\Bbbk}^{n,i} := \mathbb{A}_{\Bbbk}^n \otimes_{(\mathbb{A}_{\Bbbk}^n)^{(i,i+1)}} \mathbb{A}_{\Bbbk}^n$, where $(\mathbb{A}_{\Bbbk}^n)^{(i,i+1)} \subset \Bbbk[x_1, \ldots, x_n]$ is the subring of polynomials invariant under exchanging x_i and x_{i+1}. Pictorially, $B_{\Bbbk}^{2,1}$ is often written as χ: not all polynomials might go from bottom to top, but only the symmetric ones fitting through the "bottleneck" in the middle.

Note that since $(\mathbb{A}_{\Bbbk}^2)^{\mathfrak{S}_2} = \Bbbk[x_1+x_2, x_1x_2]$, we have $B_{\Bbbk}^{2,1} = \widehat{\mathbb{A}}_{\Bbbk}^2/(\underline{e})$ with $\underline{e} := (x_1+x_2-y_1-y_2, x_1x_2-y_1y_2)$. Moreover, the image of $\widehat{w_2}$ vanishes in $B_{\Bbbk}^{2,1}$, so that we can choose homogeneous $\underline{u} = (u_1, u_2)$ with $\widehat{w_2} = (x_1+x_2-y_1-y_2)u_1+(x_1x_2-y_1y_2)u_2$. The choice of u_1 and u_2 is not canonical, but will not affect the output of the construction, as we shall see in Example I.4.4.6. Finally, let $\underline{d} = (d_1, d_2) := (x_1 - y_1, x_2 - y_2)$ and $\underline{f} = (f_1, f_2) := (u_k(x_1, y_1), u_k(x_2, y_2))$ for $u_k(x, y) := \frac{x^{k+1}-y^{k+1}}{x-y} = x^k + x^{k-1}y + \cdots + xy^{k-1} + y^k$. $\Diamond$

We continue with the original construction of $\mathcal{KR}^k$, which works over $\Bbbk := \mathbb{Q}$. Following [KR08a], we abbreviate $\mathcal{CKR}^k\,(\uparrow\uparrow) := \{\underline{d}, \underline{f}\}$ and $\mathcal{CKR}^k\,(\chi) := \{\underline{e}, \underline{u}\}$, and define

$$\mathcal{CCKR}^k\,(\uparrow) := \ldots \to 0 \to \underline{\{d_1, f_1\}} \to 0 \to \ldots \tag{I.3.1}$$

$$\mathcal{CCKR}^k\,(\times) := \ldots \to 0 \to \mathcal{CKR}^k\,(\chi)\,\langle 1-k\rangle \xrightarrow{\chi_1} \underline{\mathcal{CKR}^k\,(\uparrow\uparrow)}\langle 1-k\rangle \to 0 \to \ldots \tag{I.3.2}$$

$$\mathcal{CCKR}^k\,(\times) := \ldots \to 0 \to \underline{\mathcal{CKR}^k\,(\uparrow\uparrow)}\langle k-1\rangle \xrightarrow{\chi_0} \mathcal{CKR}^k\,(\chi)\,\langle k+1\rangle \to 0 \to \ldots; \tag{I.3.3}$$

the first is a complex of matrix factorizations of type $(\widehat{\mathbb{A}}_{\mathbb{Q}}^1, \widehat{w_1})$, while the second and third are complexes of matrix factorizations of type $(\widehat{\mathbb{A}}_{\mathbb{Q}}^2, \widehat{w_2})$; in all cases, the underlined component is the one sitting in cohomological degree zero. The maps χ_0 and χ_1 are given by explicit matrices in [KR08a, Section following Proposition 27]; for now, their explicit form is not important, and an alternative description will be given later; see Section I.5.

As sketched in (iv) above, with these definitions the Khovanov-Rozansky homology $\mathcal{KR}^k(L)$ of a link is obtained as follows: First, take the tensor product $\mathcal{CCKR}^k(L)$ of various copies of the complexes $\mathcal{CCKR}^k\,(\uparrow)$, $\mathcal{CCKR}^k\,(\times)$ and $\mathcal{CCKR}^k\,(\times)$, one for each the unknotted strand resp. crossing in a chosen planar projection of the given link L – this results in a bounded complex of matrix factorizations of potential 0, defined over a large polynomial ring. Then, take total cohomology termwise to get $\mathcal{CKR}^k(L)$ – a bounded complex of graded $\mathbb{Q}$-vector spaces (the action of the polynomial ring is ignored). Finally, take as $\mathcal{KR}^k(L)$ the Poincaré polynomial of $\mathcal{CKR}^k(L)$. The mnemonic for $\mathcal{CCKR}$, $\mathcal{CKR}$ and $\mathcal{KR}$ is that the number of $\mathcal{C}$'s indicate how many differential are still present.

Note, however, that from the description above it is not yet clear that $\mathcal{KR}^k(L)$ is well-defined, since the components of $\mathcal{CCKR}^k(L)$ are infinite-dimensional. It turns out, however, that their cohomology is finite-dimensional, so that $\mathcal{CKR}^k(L)$ is indeed a bounded complex of finite-dimensional graded vector spaces, and the definition of $\mathcal{KR}^k(L)$ a posteriori makes sense; we will give a conceptual proof of this fact in Corollary I.4.1.11.

I.4. Some homotopy theory

In this section we give a brief account on derived and contraderived categories of (curved) differential graded modules over (curved) differential graded rings. The focus is on intuition and not on giving an exhaustive treatment; for the latter, see the original work of Positselski [Pos11], or Part II of this work for details concerning model categorical aspects as well as the case of linear factorizations. We have intentionally chosen to give the basic definitions and results here at the cost of a substantial overlap with Part II, but at the benefit of making the present part on knot theory mostly self-contained.

We keep the convention of Chapter I.2 that all rings Ω-graded for an abelian grading group Ω, called the internal or the q-grading. In addition to this q-grading, (c)dg rings will be cohomologically graded by some grading group Γ, equipped with a distinguished element $1 \in \Gamma$ and a sign homomorphism $|-| : \Gamma \to \mathbb{Z}/2\mathbb{Z}$ with $|1| = 1$. See II.2.3.1 for details and the notions of (c)dg rings. For us Γ will always be either $\mathbb{Z}$ or $\mathbb{Z}/2\mathbb{Z}$.

I.4.1. (Contra)derived categories

Given a dg ring A, the category A-Mod of A-modules comes equipped with the distinguished class qis of *quasi-isomorphisms*. Formally adding inverses to all these quasi-isomorphisms yields the well-known *derived category* $\mathbf{D}(A) := A\text{-Mod}[\text{qis}^{-1}]$. As is often the case when taking a localization (see e.g. [Hov99, Theorem 1.2.10] for the fundamental theorem on model categories or for a statement in the setting of triangulated categories), it also admits a description as a quotient of a subcategory of A-Mod: Recall that an A-module is called *semi-free* if it admits an exhaustive and bounded below increasing filtration by submodules with free filtration quotients; it is *cofibrant* if it is a summand of a semi-free A-module, and we denote $A\text{-Mod}_{\text{cf}}$ the full subcategory of cofibrant A-modules.

Theorem I.4.1.1 [Hov99, Theorems 1.2.10, 2.3.11]**.** *$A\text{-Mod}_{\text{cf}} \to A\text{-Mod} \to \mathbf{D}(A)$ factors through an equivalence $\text{Ho}(A\text{-Mod}_{\text{cf}}) \cong \mathbf{D}(A)$.*

We'd like to have a derived categories for modules over *curved* dg rings, too. However, since $\delta^2 \neq 0$, there is no notion of quasi-isomorphism or acyclic object, hence no obvious candidate for what the derived category should be.

The following beautiful idea due to Positselski [Pos11] finds a remedy: As in the case of ordinary complexes, short exact sequences of modules over cdg rings can be *totalized*, and it is known that for ordinary complexes such a totalization is always acyclic. It is therefore reasonable (and, as it turns out, very powerful) to think of totalizations of short exact sequences of cdg modules as being acyclic. Moreover, contractible A-modules should be considered to be acyclic, too, and the class of acyclic objects should be closed under summands and satisfy the 2-out-of-3 property in A-Mod, meaning that if two terms in a short exact sequence of A-modules belong to it, then so should the third. The smallest subcategory $\mathcal{W}^{\text{abs}}$ of A-Mod meeting these requirements is called the class of *absolutely acyclic* A-modules and is a candidate for the acyclic objects in A-Mod.

But there's another approach: considering the category of linear factorizations, we hope to find, in the end, a notion of weak equivalences which yields $\underline{\text{MF}}(A, w)$ after passage to the localization. Experience from triangulated categories [Kra10c, Proposition 4.9.1(5)] tells us that then the class of acyclic objects should be the class of those linear factorizations $X \in \text{LF}(A, w)$ such that $[M, X] = 0$ for all $M \in \text{MF}(A, w)$. For a general cdg ring A, one might therefore take the class of acyclics to be class of those $X \in A$-Mod such that $[M, X] = 0$ for all $X \in A\text{-Mod}_{\text{proj}}$, where the latter denotes the class of A-modules whose underlying $A^\sharp$-modules are projective. Call these modules *contraacyclic* and denote $\mathcal{W}^{\text{ctr}}$ the class of contraacyclic modules.

Comparing the two approaches, the exactness of homomorphism complex functor $\text{Hom}^*_{A\text{-Mod}}(M, -)$ in case $M \in A\text{-Mod}_{\text{proj}}$ shows that totalizations of short exact sequences are always contraacyclic and that $\mathcal{W}^{\text{ctr}}$ satisfies the 2-out-of-3 property; it is also clear that $\mathcal{W}^{\text{ctr}}$ contains all contractible A-modules and that it is closed under summands. Hence, the class $\mathcal{W}^{\text{ctr}}$ of contraacyclics contains the class $\mathcal{W}^{\text{abs}}$ of absolutely acyclic modules. Note, however, that $\mathcal{W}^{\text{ctr}}$ is also closed under products.

Proposition I.4.1.2 see [Pos11, §3.6, §3.8]**.** *Let A be a cdg ring and $\mathcal{W}^{\text{ctr}}$ as above.*

(i) *If in $A^\sharp$-Mod countable products of projectives have finite projective dimension, then $\mathcal{W}^{\text{ctr}}$ is the smallest class of A-modules containing both the contractible A-modules and totalizations of short exact sequences of A-modules, and which satisfies the 2-out-of-3 property and is closed under taking products and summands.*

(ii) *If* $\text{gl.dim}(A^\sharp\text{-Mod}) < \infty$*, part (i) is true without the requirement of being closed under products (hence, it follows from the other conditions), i.e. the classes $\mathcal{W}^{\text{abs}}$ of absolutely acyclic and $\mathcal{W}^{\text{ctr}}$ of contraacyclic A-modules coincide.*

Definition I.4.1.3. *A morphism of A-modules is called a* (contraderived) weak equivalence *if it can be written as a composition of morphisms each of which is either a monomorphism with projective cokernel or an epimorphism with contraacyclic kernel. Denote* $\text{we}^{\text{ctr}} \subset \text{Mor}(A\text{-Mod})$ *the class of contraderived weak equivalences.*

The following Definition I.4.1.4 and Theorem I.4.1.5 originate from Positselski's foundational article [Pos11] and are stated there under the condition of Proposition I.4.1.2(a) in the language of triangulated categories. Here we work with the abelian model categorical formulation treated developed in Part I of this work.

Definition I.4.1.4. $\mathbf{D}^{\mathrm{ctr}}(A) := A\text{-Mod}[(\mathrm{we}^{\mathrm{ctr}})^{-1}]$, *the* contraderived category *of* A.

What makes this reasonable is the following theorem analogous to Theorem I.4.1.1:

Theorem I.4.1.5. *The canonical functor* $A\text{-Mod}_{\mathrm{proj}} \to A\text{-Mod}[(\mathrm{we}^{\mathrm{ctr}})^{-1}] = \mathbf{D}^{\mathrm{ctr}}(A)$ *factors through an equivalence of categories* $\mathrm{Ho}(A\text{-Mod}_{\mathrm{proj}}) \cong \mathbf{D}^{\mathrm{ctr}}(A)$.

Proof. Under mild assumptions on A, this is described in [Pos11, §3.3 and §3.8]. For a general ring, we use our results from Part II: By Proposition II.2.3.6 there is a hereditary abelian model structure on A-Mod, the *contraderived model structure*, in which $\mathrm{we}^{\mathrm{ctr}}$ is the class of weak equivalences, the cofibrants are $A\text{-Mod}_{\mathrm{proj}}$ and where everything is fibrant. The claim then follows from Proposition II.2.1.21. □

Remark I.4.1.6. By Corollary II.C.1.3, the class of contraderived weak equivalences can also be described as the smallest class closed under composition that contains monomorphisms with contraacyclic cokernel and epimorphisms with contraacyclic kernel. ◊

Remark I.4.1.7. It follows from Theorem I.4.1.5 that the contraderived category $\mathbf{D}^{\mathrm{ctr}}(A)$ inherits a canonical triangulated structure from $\mathrm{Ho}(A\text{-Mod}_{\mathrm{proj}})$, and an explicit description is given in the Appendix II.C. For now, it suffices to remark that given any short exact sequence $0 \to X \xrightarrow{f} Y \to Z \to 0$ of A-modules, the canonical morphism $\mathrm{Cone}(f) \to Z$ is an isomorphism in $\mathbf{D}^{\mathrm{ctr}}(A)$, and that the resulting morphism $Z \leftarrow \mathrm{Cone}(f) \to \Sigma X$ in $\mathbf{D}^{\mathrm{ctr}}(A)$ yields a distinguished triangle $X \to Y \to Z \to \Sigma X$. The triangle constructed in Example I.4.1.9 below is a special case of this. ◊

Definition I.4.1.8. *Since* $\mathrm{LF}(A, w)$ *and* $\mathrm{MC}(A, w)$ *can be described as categories of (curved) modules over suitable (curved) dg algebras (see the beginning of Section I.2.1), Definition I.4.1.4 in particular yields a* contraderived categories of linear factorizations $\mathbf{D}^{\mathrm{ctr}}\,\mathrm{LF}(A, w)$ *and a* contraderived category of curved mixed complexes $\mathbf{D}^{\mathrm{ctr}}\,\mathrm{MC}(A, w)$.

In the case of linear factorizations, Theorem I.4.1.5 says that the canonical functor

$$\mathrm{MF}(A, w) \to \mathrm{LF}(A, w) \to \mathbf{D}^{\mathrm{ctr}}\,\mathrm{LF}(A, w)$$

factors through an equivalence $\underline{\mathrm{MF}}(A, w) \xrightarrow{\cong} \mathbf{D}^{\mathrm{ctr}}\,\mathrm{LF}(A, w)$. This realizes the homotopy category of matrix factorizations as a localization of $\mathrm{LF}(A, w)$, thereby providing flexibility in using matrix factorizations, as we are no longer forced to work with projective modules all the time.

Applied to curved mixed complexes, Theorem I.4.1.5 proves that

$$\mathrm{MC}(A, w)_{\mathrm{proj}} \to \mathrm{MC}(A, w) \to \mathbf{D}^{\mathrm{ctr}}\,\mathrm{MC}(A, w)$$

factors through an equivalence $\underline{\mathrm{MC}}(A, w)_{\mathrm{proj}} \cong \mathbf{D}^{\mathrm{ctr}}\,\mathrm{MC}(A, w)$, where $\mathrm{MC}(A, w)_{\mathrm{proj}}$ denotes the class of w-curved mixed complexes (X, d, s) for which all components of X are A-projective and where (X, s) is contractible (this is equivalent to X being projective over $K(A, w)^{\sharp}$, see Lemma II.2.3.3).

Example I.4.1.9. We consider linear factorizations of type (A, w). For any short exact sequence $0 \to X \xrightarrow{\alpha} Y \xrightarrow{\beta} Z \to 0$ of $A/(w)$-modules there is a short exact sequence

$$\begin{array}{ccccccccc}
0 & \longrightarrow & X & \xrightarrow{\alpha} & Y & \xrightarrow{\beta} & Z & \longrightarrow & 0 \\
 & & {\scriptstyle 1}\uparrow\downarrow{\scriptstyle 0} & & {\scriptstyle \alpha}\uparrow\downarrow{\scriptstyle 0} & & \uparrow\downarrow & & \\
0 & \longrightarrow & X & \xrightarrow{1} & X & \longrightarrow & 0 & \longrightarrow & 0
\end{array}$$

of linear factorizations of type (A, w). Since the left linear factorization is contractible, we infer that the right morphism $(X \rightleftarrows Y) \to (0 \rightleftarrows Z)$ is a contraderived weak equivalence, hence an isomorphism in $\mathbf{D}^{\mathrm{ctr}}\,\mathrm{LF}(A, w)$. Forming the composition

$${}_wZ \;=\; (0 \rightleftarrows Z) \xleftarrow{\;\simeq\;} (X \rightleftarrows Y) \longrightarrow (X \rightleftarrows 0) \;=\; \Sigma\,{}_wX,$$

the resulting triangle ${}_wX \to {}_wY \to {}_wZ \to \Sigma_w X$ is distinguished (see Remark I.4.1.7). $\Diamond$

In case A is a (non-curved) dg algebra, it is natural to ask what the relation between contraderived weak equivalences and the classical weak equivalences – the quasi-isomorphisms – is. As $\mathrm{H}^i \cong [\Sigma^{-i}A, -]$, any contraacyclic A-module is also acyclic, hence any contraderived weak equivalence is a quasi-isomorphism; the converse, however, is not true, see e.g. Remark I.4.2.2 below. Still, one has the following positive result:

Proposition I.4.1.10 [Pos11, Theorem 3.4]**.** *Let A be a cohomologically $\mathbb{Z}$-graded dg algebra concentrated in non-positive degrees. Then any bounded above acyclic A-module is contraacyclic. In other words, the identity on objects yields a well-defined functor $\mathbf{D}^{-}(A) \to \mathbf{D}^{\mathrm{ctr}}(A)$. Moreover, a bounded acyclic A-module is absolutely acyclic.*

We end this section with a remark on how to apply the contraderived model structure to find smaller models for linear factorizations up to summands. Suppose X is a linear factorization of type (A, w) and $t \in A$ is X-regular, i.e. $X \xrightarrow{t\cdot -} X$ is injective. Then we have the short exact sequence $0 \to X \xrightarrow{t\cdot -} X \to X/tX \to 0$ in $\mathrm{LF}(A, w)$, which by Corollary II.C.1.2 yields a distinguished triangle $X \xrightarrow{t\cdot -} X \to X/tX \to \Sigma X$ in

$\mathbf{D}^{\mathrm{ctr}}\,\mathrm{LF}(A,w)$. Now, if t happens to act nullhomotopically on X, this distinguished triangle degenerates to a split short exact sequence $0 \to X \to X/tX \to \Sigma X \to 0$ in $\mathbf{D}^{\mathrm{ctr}}\,\mathrm{LF}(A,w)$, and hence $X/tX \cong X \oplus \Sigma X$ (non-canonically). This has the following interesting and well-known consequence: Suppose A, A', A'' are commutative $\Bbbk$-algebras for a field $\Bbbk$, with potentials $w \in A, w' \in A', w'' \in A''$, respectively. Further, suppose that $A' = \Bbbk[x_1, \ldots, x_n]$ and $\dim_{\Bbbk} A'/(\partial_{x_i} w') < \infty$, and that X, Y are finite rank matrix factorizations of type $(A \otimes_{\Bbbk} A', w' - w)$ and $(A' \otimes_{\Bbbk} A'', w'' - w')$, respectively. Then the restriction of $X \otimes_{A'} Y \in \mathrm{MF}(A \otimes_{\Bbbk} A' \otimes_{\Bbbk} A'', w'' - w)$ to $A \otimes_{\Bbbk} A''$ as well as its n-fold shift are summands, up to homotopy, of the finite rank matrix factorization $X \otimes_{A'} A'/(\partial_{x_i} w') \otimes_{A'} Y \in \mathrm{MF}(A \otimes_{\Bbbk} A'', w'' - w)$. This follows by successively applying the above observation to the A'-regular sequence $\partial_{x_1} w', \ldots, \partial_{x_n} w'$.

Corollary I.4.1.11. *Khovanov-Rozansky homology $\mathcal{KR}^k$ is finite-dimensional.*

We want to emphasize that the above observation is by no means new, and that in [CM14] the authors even give a beautiful explicit formula for the idempotent on $X \otimes_{A'} A'/(\partial_{x_i} w') \otimes_{A'} Y \in \mathrm{MF}(A \otimes_{\Bbbk} A'', w'' - w)$ that projects onto the summand $\Sigma^n X \otimes_{A'} Y$ in $\mathbf{D}^{\mathrm{ctr}}\,\mathrm{LF}(A \otimes_{\Bbbk} A'', w'' - w)$. However, we feel that the above proof is a nice and quick application of the model structure on $\mathrm{LF}(A,w)$, which is why we included it here.

I.4.2. Contraderived tensor product of linear factorizations

Fix a commutative ring A and potentials $w, w' \in A$. The tensor product of two matrix factorizations of type (A,w) and (A,w') is a matrix factorization of type $(A, w+w')$. Pulling back the resulting functor $\underline{\mathrm{MF}}(A,w) \times \underline{\mathrm{MF}}(A,w') \to \underline{\mathrm{MF}}(A, w+w')$ along $\underline{\mathrm{MF}}(A,?) \cong \mathbf{D}^{\mathrm{ctr}}\,\mathrm{LF}(A,?)$ yields the *contraderived tensor product*

$$- \otimes_A^{\mathbf{L}} - :\ \mathbf{D}^{\mathrm{ctr}}\,\mathrm{LF}(A,w) \times \mathbf{D}^{\mathrm{ctr}}\,\mathrm{LF}(A,w') \longrightarrow \mathbf{D}^{\mathrm{ctr}}\,\mathrm{LF}(A, w+w').$$

Explicitly, for linear factorizations $X \in \mathrm{LF}(A,w)$ and $Y \in \mathrm{LF}(A,w')$ their derived tensor product $X \otimes_A^{\mathbf{L}} Y$ is given by $PX \otimes_A PY$, where PX resp. PY are matrix factorizations of type (A,w) resp. (A,w') that are contraderived weakly equivalent to X resp. Y.

Remark I.4.2.1. The tensor product of linear factorizations is a Quillen adjunction in two variables [Hov99, Definition 4.2.1] with respect the contraderived model structure on $\mathrm{LF}(A,?)$, and $- \otimes_A^{\mathbf{L}} -$ is its derived functor. $\Diamond$

Remark I.4.2.2. In contrast to the case of ordinary derived categories, the contraderived tensor product of linear factorizations *cannot* be computed by resolution of only one argument! In other words, the analogue of the statement that K-projective complexes are K-flat is *false* in the contraderived setting. It might be illuminating to the reader (at least it was to the author) to recall how the latter is usually proved, as it turns out that it involves an interplay between the standard projective and the standard injective model structures on chain complexes, which are both based on the class of quasi-isomorphisms as the weak equivalences: If R is a commutative ring, $P \in \mathrm{Ch}(R)$ is K-projective and $A \in \mathrm{Acyc}(R)$ is acyclic, then for any K-injective complex I we have that $\text{dg-Hom}_R(P \otimes_R A, I) \cong \text{dg-Hom}_R(P, \text{dg-Hom}_R(A, I))$ is acyclic by the definition of K-projectivity and K-injectivity; hence, taking I to be some injective cogenerator of R-Mod, we infer that $\mathrm{Hom}_R(\mathrm{H}^*(P \otimes_R A), I) = 0$, so $P \otimes_R A$ is acyclic. In our contraderived situation, however, there is, in general, no injective model structure on linear factorizations sharing the same set of weak equivalences as the contraderived one! There is a *coderived* model structure on linear factorizations, but its weakly trivial objects are the *coacyclic* linear factorizations, those X where $[X, I] = 0$ for all linear factorizations I with injective components, and as we shall recall in a second, the notion of coacyclicity and contraacyclicity differ in general. This prevents a nice interplay between projective and injective model structure possible for the classical derived category to work in the contraderived/coderived setting.

We recall Example II.2.3.16 over $A = \Bbbk[\varepsilon]/(\varepsilon^2)$, $\Bbbk$ a field:

$$\begin{aligned}
Y &:= \ldots \xrightarrow{\varepsilon} A \xrightarrow{\varepsilon} A \xrightarrow{\varepsilon} A \xrightarrow{\varepsilon} A \xrightarrow{\varepsilon} A \xrightarrow{\varepsilon} \ldots \\
X &:= \ldots \xrightarrow{\varepsilon} A \xrightarrow{\varepsilon} A \xrightarrow{\varepsilon} \Bbbk \xrightarrow{0} 0 \xrightarrow{0} 0 \xrightarrow{0} \ldots \\
Z &:= \ldots \xrightarrow{0} 0 \xrightarrow{0} 0 \xrightarrow{0} \Bbbk \xrightarrow{\varepsilon} A \xrightarrow{\varepsilon} A \xrightarrow{\varepsilon} \ldots
\end{aligned}$$

Here $\Bbbk$ lives in cohomological degree 0. Since X is acyclic and bounded above, it is contraacyclic; similarly, since Z is acyclic and bounded below, it is coacyclic. See [Pos11, Section 3.4] for both statements. However, Y is neither co- nor contraacyclic, since if it were, we would have $[Y, Y] = 0$ (since Y has projective-injective components), meaning that Y was contractible, which is not true ($Y \otimes_A \Bbbk$ has nonzero cohomology). Since the classes of coacyclics and contraacyclics both satisfy the 2-out-of-3 property and we have a short exact sequence $0 \to X \to Y \to Z \to 0$, we conclude that X is contraacyclic but not coacyclic, while Z is coacyclic but not contraacyclic. By the dual of Proposition I.4.1.2, see Proposition II.2.3.13 in Part II of this work, the class of contraacyclics is therefore not closed under coproducts. Also, the tensor product $X \otimes_A Y$ is not acyclic, and hence $0 \cong X \otimes_A^{\mathbf{L}} Y$ cannot be computed naively in $\mathbf{D}^{\mathrm{ctr}}\,\mathrm{LF}(A, 0)$ although $Y \in \mathrm{MF}(A, 0)$: First, note that $X \otimes_A Y$ is the cone of the canonical morphism $\Sigma^{-1} X_{\leq -1} \otimes_A Y \to \Bbbk \otimes_A Y$. Next, $X_{\leq -1}$ is K-flat since it is bounded above and has A-free components, so $X_{\leq -1} \otimes_A Y$ is acyclic. Hence $X \otimes_A Y$ is quasi-isomorphic to $\Bbbk \otimes_A Y$, which has nonzero cohomology.

Through folding, this example can be transferred to the $\mathbb{Z}/2\mathbb{Z}$-graded complexes which constitute $\mathbf{D}^{\mathrm{ctr}}\,\mathrm{LF}(A, 0)$, although one has to be careful which variants of folding to use:

Namely, we put $X' := \mathrm{fold}^{\Pi} X$ and $Z' := \mathrm{fold}^{\oplus} Z$, the linear factorizations of type $(A, 0)$ obtained from X and Z by taking products and coproducts by parity, respectively. Then the adjunctions $\mathrm{fold}^{\oplus} \vdash \widetilde{(-)} \vdash \mathrm{fold}^{\Pi}$ show that X' is contraacyclic while Z' is coacyclic. The analogue of Y is the "semi-infinite folding" $Y' := \mathrm{fold}^{\sqsupset} Y := \varinjlim_{n\to\infty} \mathrm{fold}^{\Pi}(\tau_{\leq n} Y)$. Its terms are isomorphic to $A((t^{-1})) \cong \Bbbk((t^{-1})) \otimes_{\Bbbk} A$ and hence A-projective; moreover, since projective, flat and injective modules over A coincide, they are also A-injective. However, Y' is neither co- nor contraacyclic, since if it were, we could as above infer that $0 = [Y', Y']$, contradicting $\mathrm{H}^*(Y' \otimes_A \Bbbk) \neq 0$. As again there is a short exact sequence $0 \to X' \to Y' \to Z' \to 0$, we deduce that X' is not coacyclic while Z' is not contraacyclic. Also, Y' fits into a short exact sequence $0 \to \bigoplus_n \mathrm{fold}^{\Pi}(\tau_{\leq 2n} Y) \to \bigoplus_{n\in\mathbb{Z}} \mathrm{fold}^{\Pi}(\tau_{\leq 2n} Y) \to Y' \to 0$, and since $\mathrm{fold}^{\Pi}(\tau_{\leq 2n} Y) \cong \mathrm{fold}^{\Pi}(\tau_{\leq 0} Y) \cong X'$ for all $n \in \mathbb{Z}$, we deduce that the $\mathbb{Z}$-fold coproduct of the contraacyclic complex X' is not contraacyclic; in particular, the contraderived tensor product $X' \otimes^{\mathbf{L}}_A (0 \rightleftarrows A^{(\mathbb{Z})})$ cannot be computed naively. See also Remark II.2.3.9 from Part II. ◊

In many cases, resolution of a single factor suffices:

Proposition I.4.2.3. *Let $X \in \mathrm{MF}(A, w)$ and $C \in \mathrm{LF}(A, w')$ be contraacyclic. Suppose further that one of the following conditions is satisfied:*

(i) *X has finite rank components.*

(ii) *C is absolutely acyclic.*

Then $X \otimes_A C$ is contraacyclic, and even absolutely acyclic in case (ii). .

Proof. If X has finite rank components, then $X \otimes_A C \cong \mathrm{Hom}_A(X^{\vee}, C)$ with $X^{\vee} = \mathrm{Hom}_A(X, A) \in \mathrm{LF}(A, -w)$ being the component-wise A-dual of X. Hence, if $Y \in \mathrm{MF}(A, w + w')$ we have $[Y, X \otimes_A C] \cong [Y \otimes_A X^{\vee}, C]$, and the last group is trivial since $Y \otimes_A X^{\vee} \in \mathrm{MF}(A, w')$. As Y was arbitrary, this shows that $X \otimes_A C$ is contraacyclic.

If we assume that C is absolutely acyclic, then $X \otimes_A C$ is absolutely acyclic, too, since $X \otimes_A -$ is exact, preserves contractibles and commutes with totalizations. □

Corollary I.4.2.4. *If $\mathrm{gl.dim}(A\text{-Mod}) < \infty$, the contraderived tensor product over A can be computed by resolution in one factor only.*

In Khovanov-Rozansky homology, the tensor product of matrix factorizations is usually not taken over the whole ring but only over a polynomial subring. We therefore consider the following slightly more general situation: Suppose we are given homomorphisms $A \leftarrow R \to B$ of commutative rings and potentials $w \in A$ and $w' \in B$. Then there are *relative* contraderived tensor products

$$- \otimes^{\mathbf{L}}_R - : \ \mathbf{D}^{\mathrm{ctr}} \mathrm{LF}(A, w) \times \mathbf{D}^{\mathrm{ctr}} \mathrm{LF}(B, w') \longrightarrow \mathbf{D}^{\mathrm{ctr}} \mathrm{LF}(A \otimes_R B, w \otimes 1 + 1 \otimes w')$$

obtained by pulling back the ordinary tensor product along the equivalences of Theorem I.4.1.5. Analogously to Proposition I.4.2.3, one can prove the following result:

Proposition I.4.2.5. *Let $X \in \mathrm{LF}(A, w)$ have R-projective components, $C \in \mathrm{LF}(B, w')$ be contraacyclic, and assume further that one of the following conditions is satisfied:*

(i) *The components of X have finite rank over R.*

(ii) *C is absolutely acyclic (e.g. $\mathrm{gl.dim}(B\text{-Mod}) < \infty$).*

Then $X \otimes_R A$ is contraacyclic, and even absolutely acyclic in case (ii).

In particular, if A is R-projective and $\mathrm{gl.dim}(B\text{-Mod}) < \infty$, the contraderived tensor product $-\otimes^{\mathbf{L}}_R-$ can be computed by choice of an A-free resolution of the first factor only, and the same holds for the B-factor in case B is R-projective and $\mathrm{gl.dim}(A\text{-Mod}) < \infty$. E.g. these requirements are met for inclusions of polynomial rings $\Bbbk[x_1, \ldots, x_n] \hookrightarrow \Bbbk[x_1, \ldots, x_{n+m}]$ over fields that we encountered in the construction of KR-homology. If both A and B are R-projective and of finite global dimension, one can even work with R-free resolutions of a single factor:

Corollary I.4.2.6. *Suppose A, B are R-free of finite global dimension and that $X \in \mathrm{LF}(A, w)$, $Y \in \mathrm{LF}(B, w')$. Then the canonical morphism $X \otimes^{\mathbf{L}}_R Y \to X \otimes_R Y$ is an isomorphism in $\mathbf{D}^{\mathrm{ctr}}\,\mathrm{LF}(A \otimes_R B, w \otimes 1 + 1 \otimes w')$ if either X or Y has R-free components.*

Proof. Suppose X has R-free components and let $\alpha : X' \to X$ and $\beta : Y' \to Y$ be A-free resp. B-free resolutions of X resp. Y, respectively. Then, by Proposition I.4.2.5 (using $\mathrm{gl.dim}(B\text{-Mod}) < \infty$), $X \otimes \beta : X \otimes_R Y' \to X \otimes_R Y$ is a contraderived weak equivalence, and by Proposition I.4.2.5 again (this time using $\mathrm{gl.dim}(A\text{-Mod}) < \infty$ and the fact that B is free over R) also $\alpha \otimes Y' : X' \otimes Y' \to X' \otimes Y$ is a contraderived weak equivalence. Hence, also the composition $\alpha \otimes \beta : X \otimes^{\mathbf{L}}_R Y \cong X' \otimes_R Y' \to X \otimes_R Y$ is a contraderived weak equivalence, as claimed. □

Remark I.4.2.7. Note that in general the relative contraderived tensor product cannot even be computed by R-projective resolutions of both factors: Indeed, taking e.g. $B = R = \Bbbk$ to be a field and $w' := 0$, $-\otimes^{\mathbf{L}}_{\Bbbk}- : \mathbf{D}^{\mathrm{ctr}}\,\mathrm{LF}(A, w) \times \mathbf{D}^{\mathrm{ctr}}\,\mathrm{LF}(\Bbbk, 0) \to \mathbf{D}^{\mathrm{ctr}}\,\mathrm{LF}(A, w)$ cannot be computed naively, since this would force the class of contraacyclics in $\mathrm{LF}(A, w)$ to be closed under self-coproducts – however, as we have seen in Remark I.4.2.2, this is wrong for example in case $A = \Bbbk[\varepsilon]/(\varepsilon^2)$. ◊

I.4.3. Contraderived tensor product of mixed complexes

We keep the notation from the previous section. Pulling back the ordinary tensor product of curved mixed complexes along the equivalences $\underline{\mathrm{MC}}(A, ?)_{\mathrm{proj}} \cong \mathbf{D}^{\mathrm{ctr}}\,\mathrm{MC}(A, ?)$, there is a *contraderived tensor product of curved mixed complexes*

$$- \otimes_A^{\mathbf{L}} - : \ \mathbf{D}^{\mathrm{ctr}} \, \mathrm{MC}(A, w) \times \mathbf{D}^{\mathrm{ctr}} \, \mathrm{MC}(A, w') \longrightarrow \mathbf{D}^{\mathrm{ctr}} \, \mathrm{MC}(A, w + w').$$

Proposition I.4.3.1. *The contraderived tensor product of two bounded above curved mixed complexes can be computed by choice of a bounded above A-projective resolution of one of the factors.*

Proof. This follows from Proposition I.4.1.10 together with the fact that tensoring with a bounded above, A-projective complex preserves quasi-isomorphisms. □

Definition I.4.3.2. *We denote* $\mathbf{D}^b \, \mathrm{MC}(A, w)$ *and* $\mathbf{D}^+ \, \mathrm{MC}(A, w)$ *the full subcategories of* $\mathbf{D}^{\mathrm{ctr}} \, \mathrm{MC}(A, w)$ *consisting of those* $X \in \mathbf{D}^{\mathrm{ctr}} \, \mathrm{MC}(A, w)$ *which are isomorphic to bounded resp. bounded above w-curved mixed complexes. Further, we denote* $\mathbf{D}^b_{\mathrm{f}} \, \mathrm{MC}(A, w) \subset \mathbf{D}^b \, \mathrm{MC}(A, w)$ *the full subcategory consisting of those X which are isomorphic to a bounded and A-free w-curved mixed complex (not necessarily of finite rank).*

Any bounded above w-curved mixed complex receives a surjective quasi-isomorphism from a bounded above, semi-free (hence A-free) w-curved mixed complex, so by Proposition I.4.1.10 some $X \in \mathbf{D}^{\mathrm{ctr}} \, \mathrm{MC}(A, w)$ belongs to $\mathbf{D}^+ \, \mathrm{MC}(A, w)$ if and only if it is isomorphic in $\mathbf{D}^{\mathrm{ctr}} \, \mathrm{MC}(A, w)$ to a bounded above A-free w-curved mixed complex. Next, using Proposition I.4.1.10 again together with the fact that curved mixed complexes can be truncated below (Example I.2.1.11), $X \in \mathbf{D}^+ \, \mathrm{MC}(A, w)$ belongs to $\mathbf{D}^b \, \mathrm{MC}(A, w)$ if and only if its cohomology is bounded. Finally, the difference between $\mathbf{D}^b_{\mathrm{f}}(A, w)$ and $\mathbf{D}^b(A, w)$ is described in the following proposition:

Proposition I.4.3.3. *A w-curved mixed complex* $X \in \mathbf{D}^b(A, w)$ *belongs to* $\mathbf{D}^b_{\mathrm{f}}(A, w)$ *if and only if its underlying complex of A-modules is quasi-isomorphic to a* bounded *complex of projective A-modules.*

Proof. Recall that any contraderived weak equivalence is a quasi-isomorphism, so the identity on objects yields a well-defined functor $\mathbf{D}^{\mathrm{ctr}}(A, w) \to \mathbf{D}(A)$. In particular, if $X \in \mathbf{D}^b_{\mathrm{f}}(A, w)$, then X, now considered as an object in $\mathbf{D}(A)$, is isomorphic to a bounded complex of projective A-modules.

Conversely, suppose that the complex of A-modules underlying X is quasi-isomorphic to a bounded complex of projective A-modules, and pick a surjective quasi-isomorphism $\pi : P \to X$ with P a bounded above and semi-free w-curved mixed complex. Then, considering P as a complex of A-modules, it is also quasi-isomorphic to a bounded complex of projective A-modules, and hence its high enough syzygies are A-projective. Truncating P (see Example I.2.1.11) therefore yields a bounded and A-projective w-curved mixed complex quasi-isomorphic to M, as required. □

We note the following special cases of Proposition I.4.3.3:

Corollary I.4.3.4. *Given an $A/(w)$-module M, the w-curved mixed complex ${}_wM$ belongs to $\mathbf{D}^b_{\mathrm{f}}(A,w)$ if and only if M is of finite projective dimension as an A-module.*

Corollary I.4.3.5. *If* $\mathrm{gl.dim}(A\text{-Mod}) < \infty$*, then* $\mathbf{D}^b(A,w) = \mathbf{D}^b_{\mathrm{f}}(A,w)$.

We also have a relative contraderived tensor product

$$- \overset{\mathbf{L}}{\underset{R}{\otimes}} - : \mathbf{D}^{\mathrm{ctr}}\,\mathrm{MC}(A,w) \times \mathbf{D}^{\mathrm{ctr}}\,\mathrm{MC}(B,w') \to \mathbf{D}^{\mathrm{ctr}}\,\mathrm{MC}(A \underset{R}{\otimes} B, w\otimes 1 + 1\otimes w'), \quad \text{(I.4.1)}$$

and analogously to Proposition I.4.3.1 we get:

Proposition I.4.3.6. *If A and B are R-free, the relative contraderived tensor product of two bounded above curved mixed complexes can be computed by choice of a bounded above, R-projective resolution of one of the factors. In particular,* (I.4.1) *restricts to*

$$\begin{aligned} \mathbf{D}^{+}\,\mathrm{MC}(A,w) \times \mathbf{D}^{+}\,\mathrm{MC}(B,w') &\longrightarrow \mathbf{D}^{+}\,\mathrm{MC}(A\otimes_R B, w\otimes 1 + 1\otimes w'), \\ \mathbf{D}^b_{\mathrm{f}}\,\mathrm{MC}(A,w) \times \mathbf{D}^b_{\mathrm{f}}\,\mathrm{MC}(B,w') &\longrightarrow \mathbf{D}^b_{\mathrm{f}}\,\mathrm{MC}(A\otimes_R B, w\otimes 1 + 1\otimes w'). \end{aligned} \quad \text{(I.4.2)}$$

I.4.4. Derived folding

Proposition I.4.4.1 (see Proposition II.5.2.7 in Part II)**.** *The folding by products functor* $\mathrm{fold}^{\Pi} : \mathrm{MC}(A,w) \to \mathrm{LF}(A,w)$ *preserves contraderived weak equivalences, hence descends naively to a well-defined functor*

$$\mathbf{R}\,\mathrm{fold}^{\Pi} : \mathbf{D}^{\mathrm{ctr}}\,\mathrm{MC}(A,w) \to \mathbf{D}^{\mathrm{ctr}}\,\mathrm{LF}(A,w).$$

Corollary I.4.4.2. *Given a quasi-isomorphism $f : X \to Y$ of bounded above w-curved mixed complexes, its folding* $\mathrm{fold}^{\Pi}(f) : \mathrm{fold}^{\Pi}(X) \to \mathrm{fold}^{\Pi}(Y)$ *is a contraderived weak equivalence of linear factorizations. Moreover, the folding of a bounded and acyclic w-curved mixed complex is even absolutely acyclic.*

Proof. By Proposition I.4.1.10 the morphism f under consideration is a contraderived weak equivalence, hence so is $\mathrm{fold}^{\Pi}(f)$ by Proposition I.4.4.1. For the second part, it suffices to note that again by Proposition I.4.1.10 we have that any bounded and acyclic w-curved mixed complex is absolutely acyclic, and that fold^{Π} preserves such since it is exact, maps contractible w-curved mixed complexes to contractible linear factorizations and commutes with totalizations. □

Remark I.4.4.3. Assuming that X, Y are bounded and A-projective in Corollary I.4.4.2 we obtain that $\mathrm{fold}^{\Pi}(f)$ is a contraderived weak equivalence of matrix factorizations, hence a homotopy equivalence by the fundamental Theorem I.4.1.5. Let us give a down-to-earth proof for this statement. It suffices to show that $\mathrm{fold}^{\Pi}(X)$ is contractible if (X, d, s) is an A-projective, bounded and acyclic w-curved mixed complex. Denote $(\mathrm{End}(X, \mathrm{d}), \partial = [\mathrm{d}, -])$ the endomorphism complex of (X, d) and pick an A-linear contraction $s_1 : X \to \Sigma^{-1} X$ of (X, d), i.e. $s_1 \in \mathrm{End}^{-1}(X, \mathrm{d})$ and $\partial(s_1) = \mathrm{id}_X$; such s_1 exists since X is bounded above, acyclic and A-projective. Then, we construct inductively a family of A-linear maps $s_n : X \to \Sigma^{-2n+1} X$ of q-degree $(n-1)d$ satisfying $\partial(s_{n+1}) = s s_n + s_n s$ for all $n \geq 1$; this in turn is possible by the acyclicity of $(\mathrm{End}(X, \mathrm{d}), \partial)$ and

$$\partial(s s_n + s_n s) = w s_n + s(s s_{n-1} + s_{n-1} s) - (s s_{n-1} + s_{n-1} s)s - w s_n = 0.$$

Then, a short calculation shows that the s_n constitute a contraction for $\mathrm{fold}^{\Pi}(X)$. ♢

Remark I.4.4.4. It is crucial that we are working with folding by *products* instead of *sums* here: For example, consider the following (0-curved) mixed complex (X, d, s) over the ring of dual numbers $A := \Bbbk[\varepsilon]/(\varepsilon^2)$:

$$\ldots \underset{\varepsilon}{\overset{\varepsilon}{\rightleftarrows}} A \underset{\varepsilon}{\overset{\varepsilon}{\rightleftarrows}} A \underset{0}{\overset{\pi}{\rightleftarrows}} \underline{\Bbbk} \underset{0}{\overset{0}{\rightleftarrows}} 0 \underset{0}{\overset{0}{\rightleftarrows}} \ldots,$$

Since (X, d, s) is bounded above and d-acyclic, it is contraacyclic by Proposition I.4.1.10; still, its folding by sums $\mathrm{fold}^{\oplus} X$ is not even acyclic – hence a fortiori not contraacyclic either – since the cycle $1 \in \Bbbk = X^0 \subset (\mathrm{fold}^{\oplus} X)^0$ is not a boundary. In $\mathrm{fold}^{\Pi} X$, however, it bounds the element $(1, -1, 1, \ldots) \in X^1 \times X^3 \times \ldots = (\mathrm{fold}^{\Pi} X)^1$. Note also that such examples can only exist over non-regular rings, as for regular rings, every contraacyclic module is absolutely acyclic (Proposition I.4.1.2), and absolute acyclicity is preserved by $\mathrm{fold}^{\oplus}$. ♢

Corollary I.4.4.5. *Let $X \in \mathrm{LF}(A, w)$, and let $\pi : P \to X$ be a resolution of X, considered a w-curved mixed complex, by a bounded and A-projective w-curved mixed complex. Then* $\mathrm{fold}^{\Pi}(\pi) : \mathrm{fold}^{\Pi}(P) \to \mathrm{fold}^{\Pi}(X) = X$ *is a contraderived weak equivalence, i.e. a resolution of X by a matrix factorization in* $\mathbf{D}^{\mathrm{ctr}} \mathrm{LF}(A, w)$. *In fact,* $\mathrm{Cone}(\mathrm{fold}^{\Pi}(\pi))$ *is even absolutely acyclic.*

Example I.4.4.6. Suppose $w = \sum_i x_i y_i$ for sequences $\underline{x}$ and $\underline{y}$, with $\underline{x}$ regular. Then the Koszul resolution $P = \bigwedge^* \bigoplus_{i=1}^n \Sigma e_i A$, equipped with differentials $\mathrm{d} = \sum_i x_i \frac{\partial}{\partial e_i}$ and $s = \sum_i y_i(- \wedge e_i)$, is a resolution of $A/(\underline{x})$ in $\mathbf{D}^b \mathrm{MC}(A, w)$. Hence, upon folding we get that the canonical morphism of linear factorizations $\{\underline{x}, \underline{y}\} \longrightarrow (0 \rightleftarrows A/(\underline{x}))$ is a contraderived weak equivalence, hence a resolution of $A/(\underline{x})$ by a (finite rank) matrix factorization in $\mathbf{D}^{\mathrm{ctr}}\,\mathrm{LF}(A,w)$. In particular, using Theorem I.4.1.5 we see that up to homotopy equivalence of matrix factorizations, $\{\underline{x},\underline{y}\}$ does not depend on the choice of $\underline{y}$; this explains why the choices made in the construction of Khovanov-Rozansky homology in Section I.3 do not affect the outcome of the construction.

This example will also be used in Section I.4.5 to follow to construct the Koszul model for the stabilization of the diagonal. ♢

Example I.4.4.7. The use of the folding functor is not limited to bounded curved mixed complexes. Namely, it can also be used to construct the *Bar model* of the diagonal that was introduced in [CM12, Section 2.3], and which we now recall. See also [Lod98] for the relevant properties of the Bar resolution.

Suppose $\Bbbk$ is a field, A is a $\Bbbk$-algebra, and put $\overline{A} := \mathrm{coker}(\Bbbk \to A)$. Then the diagonal A-A-bimodule Δ admits the *augmented reduced Bar complex* $(\mathbb{B}_*(A), \mathrm{d}_{\mathbb{B}}) \to \Delta$ as an A-A-projective resolution, given by $\mathbb{B}_n(A) := A \otimes_{\Bbbk} \overline{A}^{\otimes n} \otimes_{\Bbbk} A$ and differential $\mathrm{d}_{\mathbb{B}}$,

$$a_0 \otimes a_1 \otimes \cdots \otimes a_n \quad \longmapsto \quad \sum_{i=0}^{n-1} (-1)^i a_0 \otimes \cdots \otimes a_{i-1} \otimes a_i a_{i+1} \otimes a_{i+2} \otimes \cdots \otimes a_n.$$

Moreover, there is a *shuffle product* $\times$ on $\mathbb{B}_*(A)$ that turns it into a graded-commutative dg algebra over $A \otimes_{\Bbbk} A$. Now, by the acyclicity of $\mathbb{B}_*(A) \to \Delta$, any $x \in A \otimes_{\Bbbk} A = \mathbb{B}_0(A)$ satisfying $\mu(x) = 0$ admits some $\widetilde{x} \in \mathbb{B}_1(A)$ with $\mathrm{d}_{\mathbb{B}}(\widetilde{x}) = x$, and since the shuffle square vanishes on $\mathbb{B}_1(A)$, Example I.2.1.10 shows that $(\mathbb{B}_*(A), \mathrm{d}_{\mathbb{B}}, \widetilde{x} \times -)$ is a bounded above x-curved mixed complex quasi-isomorphic to ${}_x\Delta$. Applying Corollary I.4.4.2 shows that $\mathrm{fold}^{\Pi}(\mathbb{B}_*(A), \mathrm{d}_{\mathbb{B}}, \widetilde{x} \times -) \to {}_x\Delta$ is a contraderived weak equivalence of linear factorizations of type $(A \otimes_{\Bbbk} A, x)$. In particular, if $x = \widehat{w} := w \otimes 1 - 1 \otimes w$ for some $w \in A$, and taking $\widetilde{x} := 1 \otimes w \otimes 1$, this can be used to construct a model for ${}_{\widehat{w}}\Delta$. ♢

We discuss the compatibility of folding with relative contraderived tensor products.

Proposition I.4.4.8. *Let A and B be free as R-modules, and let $X \in \mathbf{D}^b\,\mathrm{MC}(A,w)$ and $Y \in \mathbf{D}^b\,\mathrm{MC}(B,w')$. Further, assume one of the following two conditions holds:*

(i) *$X \in \mathbf{D}^b_{\mathrm{f}}(A,w)$ and $Y \in \mathbf{D}^b_{\mathrm{f}}(B,w')$; e.g., this is automatic if A and B are of finite global dimension.*

(ii) *$R = A$ and X is isomorphic in $\mathbf{D}^{\mathrm{ctr}}\,\mathrm{MC}(A,w)$ to a bounded w-curved mixed complex which is componentwise free of finite rank over A.*

Then there is a canonical isomorphism in $\mathbf{D}^{\mathrm{ctr}}\,\mathrm{LF}(A \otimes_R B, w+w')$:

$$\mathbf{R}\,\mathrm{fold}^{\Pi}\left(X \otimes^{\mathbf{L}}_R Y\right) \;\cong\; \mathbf{R}\,\mathrm{fold}^{\Pi}(X) \otimes^{\mathbf{L}}_R \mathbf{R}\,\mathrm{fold}^{\Pi}(Y).$$

Proof. Since $\mathrm{fold}^{\Pi} = \mathrm{fold}^{\oplus}$ for bounded curved mixed complexes, and because $\mathrm{fold}^{\oplus}$ commutes with tensor products, it suffices to show that in both cases both contraderived tensor products can be computed naively.

In case (i) we may assume that X is bounded and A-projective and that Y is bounded and B-projective. Then the right hand contraderived tensor product can be computed naively by definition, while for the left hand side, we use Proposition I.4.3.1.

In case (ii) we may assume that X is bounded and A-free of finite rank. Then again $X \otimes_A^{\mathbf{L}} Y$ can be computed naively by Proposition I.4.3.1, and the contraderived tensor product $\mathrm{fold}^{\Pi} X \otimes_A^{\mathbf{L}} \mathrm{fold}^{\Pi} Y$ can be computed naively by Proposition I.4.2.5 since $\mathrm{fold}^{\Pi} X$ is free of finite rank over A. □

Remark I.4.4.9. The statement of Proposition I.4.4.8 might seem odd because of its use of the folding by products instead of folding by sums, which always commutes with sums. See Section I.7.1.1 for more on this point. ◊

I.4.5. Stable Hochschild homology

Let $\Bbbk$ be a commutative base ring, A be a commutative $\Bbbk$-algebra, $w \in A$, and let M be a $\Bbbk$-symmetric A-A-bimodule such that $w.m = m.w$ for all $w \in M$. Further, let Δ be the diagonal A-A-bimodule, and denote $\widehat{w} := w \otimes 1 - 1 \otimes w \in \widehat{A} := A \otimes_{\Bbbk} A$. Recall also our grading convention from the beginning of this chapter, and put $d := |w|_q$.

Definition I.4.5.1. *The* w-stable Hochschild homology *of M is defined as*

$$ {}^{w}\,\mathrm{sHH}_k^{A/\Bbbk}(M) \;:=\; \mathrm{H}^k \left[{}_{-\widehat{w}}\Delta \otimes_{\widehat{A}}^{\mathbf{L}} {}_{\widehat{w}}M \right], $$

the cohomology of the derived tensor product of Δ and M, considered as linear factorizations of type $(\widehat{A}, -\widehat{w})$ and $(\widehat{A}, \widehat{w})$, respectively. We also put

$$ {}^{w}\,\mathrm{sHH}_t^{A/\Bbbk}(M) := \mathrm{H}^t \left[{}_{-\widehat{w}}\Delta \otimes_{\widehat{A}}^{\mathbf{L}} {}_{\widehat{w}}M \right], $$

the total w-stable Hochschild homology *(recall the normalization described in I.2.1).*

Classical Hochschild homology of bimodules can be calculated by tensoring with a chosen resolution for the diagonal bimodule. Since, as we have seen in Remark I.4.2.2, contraderived tensor products of linear factorizations can in general not be computed by resolution of a single factor only, we have to make additional assumptions to get a similar description of stable Hochschild homology. Consider the following conditions:

(i) Δ admits a resolution by a finite rank matrix factorization of type $(\widehat{A}, -\widehat{w})$.

(ii) $\mathrm{gl.dim}(\widehat{A}) < \infty$.

If either (i) or (ii) holds, Proposition I.4.2.3 implies that ${}^{w}\,\mathrm{sHH}_*^{A/\Bbbk}(M)$ can be computed as the cohomology of $P_\Delta \otimes_{\widehat{A}} {}_{\widehat{w}}M \in \mathrm{LF}(\widehat{A}, 0)$, where $P_\Delta \to {}_{-\widehat{w}}\Delta$ is a resolution of Δ considered as a linear factorization of type $(\widehat{A}, -\widehat{w})$ by a matrix factorization P_Δ, arbitrary in case (ii) and to be assumed of finite rank in case (i). As a special case of this, consider $A := \Bbbk[x_1, \ldots, x_n] = \mathbb{A}^n_\Bbbk$, and choose $u_i \in \widehat{A}$ such that $\widehat{w} = \sum_i (x_i - y_i)u_i$. By Example I.4.4.6, we may then take $P_\Delta = \{x_i - y_i, -u_i\}$, which shows that our definition of stable Hochschild homology agrees with the ad-hoc definition from the introduction. In particular, this enables us to compute the stable Hochschild homology of the diagonal over an affine space:

Fact I.4.5.2. *For any $w \in \mathbb{A}^n_\Bbbk$, the stable Hochschild homology ${}^{w}\,\mathrm{sHH}_*^{\mathbb{A}^n_\Bbbk}(\Delta)$ is canonically isomorphic to the Koszul homology of the sequence $\partial_{x_1} w, \ldots, \partial_{x_n} w$, with cohomological grading reduced modulo 2.*

Proof. In the notation from the previous paragraph, we may take

$$u_i := \frac{w(x_1, \ldots, x_{i-1}, x_i, y_{i+1}, \ldots, y_n) - w(x_1, \ldots, x_{i-1}, y_i, y_{i+1}, \ldots, y_n)}{x_i - y_i},$$

so

$$\mathrm{sHH}_*^{\mathbb{A}^n_\Bbbk/\Bbbk}(\Delta) = \mathrm{H}^t\left(P_\Delta \otimes_{\mathbb{A}^n_\Bbbk} {}_{w}\Delta\right) = \mathrm{H}^t\left(\{x_i - y_i, -u_i\} \otimes_{\mathbb{A}^n_\Bbbk} {}_{w}\Delta\right).$$

Considering $\{x_i - y_i, -u_i\} \otimes_{\mathbb{A}^n_\Bbbk} {}_{w}\Delta$ as a matrix factorization of type $(\mathbb{A}^n_\Bbbk, 0)$ it is the Koszul factorization $\{0, u_i|_{x_i = y_i}\}$, and since $u_i|_{x_i=y_i} = \partial_i w(x_1, \ldots, x_n)$, the claim follows. ☐

Notation I.4.5.3. For $A = \mathbb{A}^n_\Bbbk$, $w = \sum_i x_i^{k+1}$, we write ${}^{k}\,\mathrm{sHH}_*^{\mathbb{A}^n_\Bbbk/\Bbbk}(M)$ and ${}^{k}\,\mathrm{sHH}_t^{\mathbb{A}^n_\Bbbk}(M)$ for ${}^{w}\,\mathrm{sHH}_*^{A/\Bbbk}(M)$ resp. ${}^{w}\,\mathrm{sHH}_t^{A/\Bbbk}(M)$ and call it the *k-stable Hochschild homology.* ♢

Remark I.4.5.4. The definition of the w-stable Hochschild homology of some $A/(w)$-module M is slightly asymmetric: one also could have defined it as the cohomology of ${}_{\widehat{w}}\Delta \otimes^{\mathbf{L}}_{\widehat{A}} {}_{-\widehat{w}}M$. This is, however, not a serious issue, as both candidates turn out to be canonically isomorphic as we shall see now.

For any potential $w \in A$ we have the *sign-change* isomorphism $\sigma : \mathrm{LF}(A, w) \to \mathrm{LF}(A, -w)$, sending a linear factorization $\delta^1 : M^1 \rightleftarrows M^0 : \delta^0$ of type (A, w) to the linear factorization $-\delta^1 : M^1 \rightleftarrows M^0 : \delta^0$ of type $(A, -w)$. Moreover, it is compatible with tensor products in the sense that, given another potential $w' \in A$, the diagram

$$\begin{array}{ccc} \mathrm{LF}(A, w) \times \mathrm{LF}(A, w') & \xrightarrow{- \otimes_A -} & \mathrm{LF}(A, w + w') \\ \downarrow{\scriptstyle \sigma \times \sigma} & & \downarrow{\scriptstyle \sigma} \\ \mathrm{LF}(A, -w) \times \mathrm{LF}(A, -w') & \xrightarrow[- \otimes_A -]{} & \mathrm{LF}(A, -(w + w')) \end{array}$$

commutes up to canonical isomorphism of functors (though not strictly). This also descends to the contraderived categories of linear factorizations equipped with the contraderived tensor product. In particular, given $A/(w)$-modules X, Y, there is a canonical isomorphism

$$\sigma\left({}_wX \otimes^{\mathbf{L}}_A {}_{-w}Y\right) \cong {}_{-w}X \otimes^{\mathbf{L}}_A {}_wY$$

in $\mathbf{D}^{\mathrm{ctr}}\,\mathrm{LF}(A,0)$, hence also $\mathrm{H}^*\left({}_wX \otimes^{\mathbf{L}}_A {}_{-w}Y\right) \cong {}_{-w}X \otimes^{\mathbf{L}}_A {}_wY$ canonically, since $\sigma : \mathbf{D}^{\mathrm{ctr}}\,\mathrm{LF}(A,0) \to \mathbf{D}^{\mathrm{ctr}}\,\mathrm{LF}(A,0)$ doesn't affect H^*. In particular, the two candidates for the definition of stable Hochschild homology are canonically isomorphic.

Note, however, that in contrast to classical cohomologically $\mathbb{Z}$-graded complexes, the sign change automorphism $\sigma : \mathbf{D}^{\mathrm{ctr}}\,\mathrm{LF}(A,0) \to \mathbf{D}^{\mathrm{ctr}}\,\mathrm{LF}(A,0)$ is in general *not* isomorphic to the identity functor: For example, taking $A = \Bbbk[\varepsilon]/(\varepsilon^2)$ with $\Bbbk$ a field of characteristic $\neq 2$, the matrix factorizations $X := (\varepsilon : A \rightleftarrows A : \varepsilon)$ and $\sigma X = (-\varepsilon : A \rightleftarrows A : \varepsilon)$ are not contraderived weakly equivalent: if they where, they would be homotopy equivalent by Theorem I.4.1.5, but $[X, \sigma X] = 0$ as a short calculation shows. ◊

Proposition I.4.5.5. *Assume that one of the following holds:*

(i) Δ *admits a bounded, finite rank $\widehat{A}$-free resolution as a $-\widehat{w}$-curved mixed complex.*

(ii) $\mathrm{gl.dim}(\widehat{A}\text{-Mod}) < \infty$.

Then there is a canonical isomorphism in $\mathbf{D}^{\mathrm{ctr}}\,\mathrm{LF}(\widehat{A},0)$*:*

$${}_{-\widehat{w}}\Delta \otimes^{\mathbf{L}}_{\widehat{A}} {}_{\widehat{w}}M \cong \mathbf{R}\,\mathrm{fold}^{\Pi}\left({}_{-\widehat{w}}\Delta \otimes^{\mathbf{L}}_{\widehat{A}} {}_{\widehat{w}}M\right)$$

Proof. This is a special case of Proposition I.4.4.8. □

I.4.6. Ordinary versus stable Hochschild homology

In this section we describe the relation between ordinary and w-stable Hochschild homology. We keep the notation from the beginning of the previous Section I.4.5.

Under the assumptions of Proposition I.4.5.5 the total w-stable Hochschild homology ${}^w\,\mathrm{sHH}_t^{A/\Bbbk}(X)$ of an $\widehat{A}/(\widehat{w})$-module X can be computed as the total cyclic homology (in the sense of Remark I.2.1.8) of a bounded mixed complex $P_\Delta \otimes_{\widehat{A}} {}_{\widehat{w}}X \in \mathbf{D}^b\,\mathrm{MC}(\widehat{A}, 0_d)$, the underlying complex of which computes the ordinary Hochschild homology $\mathrm{HH}_*^{A/\Bbbk}(X)$ of X; here $P_\Delta \to {}_{-\widehat{w}}\Delta$ is a suitable resolution of the diagonal. The ordinary and total cyclic homology associated with a bounded mixed complex are related through a converging spectral sequence which is a special case of the spectral sequence of a double complex. Instead of recalling the latter in full generality, we decided to describe the structure that arises in the application to mixed complexes directly. In particular, this section does not assume any knowledge about spectral sequences.

Suppose $X \in \mathrm{MC}(A, 0_d)$, i.e. (X, d, s) is a bounded 0-curved mixed complex over A with s of q-degree d. Our goal is to define successive approximations to the total cyclic homology $\mathrm{HC}(X)$ – the homology of $\bigoplus_n X^n \left\langle -\frac{nd}{2} \right\rangle$ with respect to the homogeneous differential $\mathrm{d} + s$ of degree $\frac{d}{2}$ – starting from the ordinary ($\mathbb{Z}$-graded) homology $\mathrm{H}^*(X, \mathrm{d})$ of (X, d). For that, given an integer $t \geq 1$ we call an element $x \in X^{\leq n} := \bigoplus_{k \leq n} X^k \left\langle -\frac{kd}{2} \right\rangle$ an *approximative n-cycle of order t* if $(\mathrm{d} + s)(x) \in X^{\leq n+1-t}$; it is called an *approximative n-boundary of order t* if there exists some $y \in X^{\leq n-1+(t-1)}$ such that $x \equiv (\mathrm{d} + s)(y)$ modulo $X^{\leq n-1}$. The A-submodules of $X^{\leq n}$ consisting of approximative n-cycles of order t resp. approximative n-boundaries of order t are denoted Z^n_t and B^n_t, respectively. Their definition expresses the idea that, given an element $x = \sum_k x_k \in X^{\leq n}$, instead of requiring $\mathrm{d}\, x_{k-1} + s x_{k+1}$ to vanish for *all* $k \in \mathbb{Z}$ (which would mean that x is an honest $(\mathrm{d} + s)$-cycle) one can define weakenings by requiring the vanishing of $\mathrm{d}\, x_{k-1} + s x_{k+1}$ for *some* k only, and indeed $x \in \mathrm{Z}^n_t$ is equivalent to requiring it for $k > n + 1 - t$. Also note that, once t is large enough, the notions of approximative n-cycle of order t and actual $(\mathrm{d} + s)$-cycle in $X^{\leq n}$ coincide since X is bounded. Back to the definitions, the quotient $(\mathrm{Z}^n_t + X^{\leq n-1}) / \mathrm{B}^n_t \cong \mathrm{Z}^n_t / (\mathrm{B}^n_t \cap \mathrm{Z}^n_t)$ is called the *n-th approximative cyclic homology of order t* of X and is denoted $\mathrm{HC}^n_t(X)$. Finally, we denote $\mathrm{HC}^{\leq n}(X) \subset \mathrm{HC}(X)$ the submodule of $\mathrm{HC}(X)$ consisting of those cohomology classes that can be represented by a cycle from $X^{\leq n}$.

Proposition I.4.6.1. *The following properties hold:*

(i) *There is a canonical isomorphism* $\mathrm{HC}^n_1(X) \cong \mathrm{H}^n(X, \mathrm{d}) \left\langle -\frac{nd}{2} \right\rangle$.

(ii) *The restriction of* $\mathrm{d} + s$ *to* Z^n_t *induces a differential* $\mathrm{d}^n_t : \mathrm{HC}^n_t(X) \to \mathrm{HC}^{n+1-t}_t(X)$ *of q-degree* $\frac{d}{2}$*, satisfying* $\ker(\mathrm{d}^n_t) / \mathrm{im}(\mathrm{d}^{n-1+t}_t) \cong \mathrm{HC}^n_{t+1}(X)$*. In particular,* d^n_1 *vanishes, hence* $\mathrm{HC}^n_2(X) \cong \mathrm{H}^n(X, \mathrm{d}) \left\langle -\frac{nd}{2} \right\rangle$ *canonically, too, and*

$$\mathrm{d}^n_2 : \mathrm{H}^n(X, \mathrm{d}) \left\langle -\tfrac{nd}{2} \right\rangle \cong \mathrm{HC}^n_2 \to \mathrm{HC}^{n-1}_2 \cong \mathrm{H}^{n-1}(X, \mathrm{d}) \left\langle -\tfrac{(n-1)d}{2} \right\rangle$$

is the differential induced by s.

(iii) *For* $t \gg 0$ *there is a canonical isomorphism* $\mathrm{HC}^n_t(X) \cong \mathrm{HC}^{\leq n}(X) / \mathrm{HC}^{\leq n-1}(X)$.

Moreover, the the formation of approximative cycles, boundaries and homologies, as well as the differentials d^n_t *and the isomorphisms in (ii) are functorial in* (X, d, s) *with respect to morphisms of mixed complexes.*

Proof. This amounts to (hopefully) carefully writing out the definitions. In the following, the groups H^n, Z^n etc. will be taken with respect to X. (i) The embedding $X^n \left\langle \frac{nd}{2} \right\rangle \rightarrowtail X^{\leq n}$ maps the d-cycles Z^n into the approximative n-cycles Z^n_1 of order 1 and the d-boundaries into the approximative n-boundaries B_1^n of order 1, hence gives rise to a well-defined map $\psi^n : \mathrm{H}^n \to \mathrm{HC}_1^n = (\mathrm{Z}_1^n + X^{\leq n-1}) / \mathrm{B}_1^n$. We prove that this map is injective and surjective. For injectivity, suppose that $x \in \mathrm{Z}^n$ is an approximative n-boundary of order 1. Then, by definition there exists some $y \in X^{\leq n-1}$ such that $(\mathrm{d}+s)(y) \equiv x$ modulo $X^{\leq n-1}$. In particular, we conclude $x = \mathrm{d}\, y_{n-1}$, so x is a d-boundary. For surjectivity, suppose that $x \in \mathrm{Z}_1^n \subset X^{\leq n}$ is an approximative n-cycle of order 1. Then $0 = [(\mathrm{d}+s)(x)]_{n+1} = \mathrm{d}\, x_n$, so $x_n \in \mathrm{Z}^n$ defines a cohomology class in $[x_n] \in \mathrm{H}^n$. Also $x - x_n \in X^{\leq n-1} \subset \mathrm{B}_1^n$, so $[x] = [x_n] = \psi^n([x_n])$ in HC_1^t, proving surjectivity. (ii) If $x \in \mathrm{Z}_n^t$, then by definition we have $(\mathrm{d}+s)(x) \in X^{\leq n+1-t}$. Also, $(\mathrm{d}+s)(x) \in \ker(\mathrm{d}+s)$ since $\mathrm{d}+s$ is a differential, so $(\mathrm{d}+s)(x) \in \mathrm{Z}_t^{n+1-t}$ represents an approximative $(n+1-t)$-th cohomology class of order t. Further, if $x \in \mathrm{Z}_t^n \cap \mathrm{B}_t^n$, there exists $y \in X^{\leq n-1+(t-1)}$ such that $x = (\mathrm{d}+s)(y) + x'$ for some $x' \in X^{\leq n-1} = X^{\leq (n+1-t)-1+(t-1)}$, so $(\mathrm{d}+s)(x) = (\mathrm{d}+s)(x') \in \mathrm{B}_t^{n+1-t}$. This proves that $\mathrm{d}+s$ induces a well-defined differential $\mathrm{d}_t^n : \mathrm{HC}_t^n \to \mathrm{HC}_t^{n+1-t}$, and we prove next that there are canonical isomorphisms $\psi_t^n : \ker(\mathrm{d}_t^n) / \operatorname{im}(\mathrm{d}_t^{n-1+t}) \cong \mathrm{H}_{t+1}^n$. For this, note that given $x \in \mathrm{Z}_t^n$, we have $\mathrm{d}_t^n([x]) = 0$ if and only if $(\mathrm{d}+s)(x) \in \mathrm{B}_t^{n+1-t}$, i.e. if and only if there exists some $y \in X^{\leq n-1}$ such that $(\mathrm{d}+s)(x) \equiv (\mathrm{d}+s)(y)$ modulo $X^{\leq n-t}$, which in turn is equivalent to $x \in \mathrm{Z}_{t+1}^n + X^{\leq n-1}$. Since moreover $\mathrm{B}_t^n \subseteq \mathrm{B}_{t+1}^n$, it follows that the identity on representatives induces a well-defined map $\tilde{\psi}_t^n : \mathrm{HC}_t^n \supseteq \ker(\mathrm{d}_t^n) \to (\mathrm{Z}_{t+1}^n + X^{\leq n-1}) / \mathrm{B}_{t+1}^n \cong \mathrm{HC}_{t+1}^n$. Since $\mathrm{Z}_{t+1}^n \subseteq \mathrm{Z}_t^n$, it is clear that this map is surjective; also, if $x \in \mathrm{Z}_t^n$ represents $[x] \in \ker(\mathrm{d}_t^n)$, we have $\tilde{\psi}_t^n([x]) = 0$ in HC_{t+1}^n if and only if $x \in \mathrm{B}_{t+1}^n$, i.e. if and only if there is some $y \in X^{\leq n-1+t}$ such that $x \equiv (\mathrm{d}+s)(y)$ modulo $X^{\leq n-1}$. By definition, any such y belongs to Z_t^{n-1+t}, so we conclude $\ker(\tilde{\psi}_t^n) = \operatorname{im}(\mathrm{d}_t^{n-1+t})$, and hence $\tilde{\psi}_t^n$ induces an isomorphism $\psi_t^n : \ker(\mathrm{d}_t^n) / \operatorname{im}(\mathrm{d}_t^{n-1+t}) \cong \mathrm{H}_{t+1}^n$ as claimed. Finally for (iii) note that since X is bounded we have $\mathrm{Z}_t^n = \ker(\mathrm{d}+s) \cap X^{\leq n}$ and $\mathrm{B}_t^n = \operatorname{im}(\mathrm{d}+s) \cap X^{\leq n} + X^{\leq n-1}$ for $t \gg 0$, and hence $\mathrm{HC}_t^n = \mathrm{Z}_t^n / \mathrm{B}_t^n \cap \mathrm{Z}_t^n = \ker(\mathrm{d}+s) \cap X^{\leq n} / (\operatorname{im}(\mathrm{d}+s) \cap X^{\leq n} + \ker(\mathrm{d}+s) \cap X^{\leq n-1}) \cong \mathrm{HC}^{\leq n} / \mathrm{HC}^{\leq n-1}$ as claimed. □

Remark I.4.6.2. The additional $\mathbb{Z}/2\mathbb{Z}$-grading on $\mathrm{HC}_t(X)$ that we neglected in the above construction of HC causes all odd differentials d_{2t+1}^n on HC to vanish. ♢

We want to consider the above construction as a functor on the bounded derived category $\mathbf{D}^b\,\mathrm{MC}(A, 0_d)$ of mixed complexes. First, we define the appropriate target category by abstracting from the properties of HC we established in Proposition I.4.6.1:

Definition I.4.6.3. *A* spectral complex *(of q-degree d) over A is a tuple*

$$\mathrm{E} = ((\mathrm{E}_t^n, \mathrm{d}_t^n, \psi_t^n)_{n \in \mathbb{Z}, t \geq 1}, (\mathrm{E}_\infty^{\leq n}, \psi_\infty^n)_{n \in \mathbb{Z}})$$

consisting of the following data:

(i) *A* $t \geq 1$*-indexed family of bounded* A*-complexes* $(\mathrm{E}_t^n, \mathrm{d}_t^n : \mathrm{E}_t^n \to \mathrm{E}_t^{n+1-2t}\langle \frac{d}{2}\rangle)_{n\in\mathbb{Z}}$.

(ii) *Isomorphisms* $\psi_t^n : \mathrm{H}^n(\mathrm{E}_t^*) := \ker(\mathrm{d}_t^n)/\operatorname{im}(\mathrm{d}_t^{n-1+2t}) \cong \mathrm{E}_{t+1}^n$ *for* $n \in \mathbb{Z}$ *and* $t \geq 1$.

(iii) *A* $\mathbb{Z}$*-filtered* A*-module* $(\mathrm{E}_\infty^{\leq n})_{n\in\mathbb{Z}} = (\cdots \subseteq \mathrm{E}_\infty^{\leq n} \subseteq \mathrm{E}_\infty^{\leq n+1} \subseteq \cdots)$.

(iv) *Isomorphisms of* A*-modules* $\psi_\infty^n : \mathrm{E}_\infty^{\leq n} / \mathrm{E}_\infty^{\leq n-1} \cong \varinjlim_t \mathrm{E}_t^n$ *for all* $n \in \mathbb{Z}$.

Here, the colimit $\varinjlim_t \mathrm{E}_t^n$ *is taken over the* $\mathbb{Z}$*-diagram* $t \mapsto \mathrm{E}_t^n$ *whose transition maps* $\mathrm{E}_t^n \to \mathrm{E}_{t+1}^n$ *are defined as* 0 *if* $\mathrm{d}_t^n \neq 0$ *and as* $\mathrm{E}_t^n = \mathrm{Z}_t^n \twoheadrightarrow \mathrm{H}^n(\mathrm{E}_t^*, \mathrm{d}_t^*) \cong \mathrm{E}_{t+1}^n$ *if* $\mathrm{d}_t^n = 0$. *Note that this diagram is eventually constant since the* E_t^n *are uniformly bounded in* n *and the differential* d_t^n *has cohomological degree* $1-2t$.

The A*-module underlying the filtered module* $(\mathrm{E}_\infty^{\leq n})_{n\in\mathbb{Z}}$ *is called the* limit *of* E.

Given spectral complexes E and F, a *morphism of spectral complexes* $f : \mathrm{E} \to \mathrm{F}$ is a family of A-linear homomorphisms $f_t^n : \mathrm{E}_t^n \to \mathrm{F}_t^n$ together with a homomorphism of $\mathbb{Z}$-filtered A-modules $(\mathrm{E}_\infty^{\leq n})_{n\in\mathbb{Z}} \to (\mathrm{F}_\infty^{\leq n})_{n\in\mathbb{Z}}$ that are compatible with the differentials d and the isomorphisms ψ. The resulting category of spectral complexes is denoted $\mathrm{SCh}(A)$. Assigning to a spectral complex E its *underlying complex* $(\mathrm{E}_1^*, \mathrm{d}_1^*)$ defines a forgetful functor $\mathrm{SCh}(A) \to \mathrm{Ch}_*(A)$ from spectral complexes to (homologically graded) chain complexes over A, which is faithful and reflects isomorphisms.

Proposition I.4.6.1 says that $X \mapsto \mathrm{HC}(X)$ defines a functor from bounded mixed complexes to spectral complexes. Next we check that it factors over the derived category:

Fact I.4.6.4. *Mapping a bounded mixed complex* (X, d, s) *to the spectral complex* $\mathrm{HC}(X)$ *descends naively to a functor* $\mathrm{HC} : \mathbf{D}^b\,\mathrm{MC}(A, 0_d) \to \mathrm{SCh}(A)$.

Proof. We have to check that for a quasi-isomorphism (with respect to d) of mixed complexes $X \to Y$ the induced morphism of spectral complexes $\mathrm{HC}(X) \to \mathrm{HC}(Y)$ is an isomorphism, and this follows from $\mathrm{HC}_1^*(-) \cong \mathrm{H}^*(-)$ and the fact that the forgetful functor $\mathrm{SCh}(A) \to \mathrm{Ch}_*(A)$ reflects isomorphisms. □

Proposition I.4.6.5. *Suppose the hypothesis of Proposition I.4.5.5 are met, and let* $M \in \widehat{A}/(\widehat{w})$-Mod. *Consider the following spectral complex over* A*:*

$$ {}^{w}\,\mathrm{SHH}^{A/\Bbbk}(M) \;:=\; \mathrm{HC}({}_{-\widehat{w}}\Delta \otimes^{\mathbf{L}}_{\widehat{A}} {}_{\widehat{w}}M) \tag{I.4.3}$$

It has the following properties:

(i) *The* n*-th component of the underlying complex of* ${}^{w}\,\mathrm{SHH}^{A/\Bbbk}(M)$ *is canonically isomorphic to the shifted ordinary Hochschild homology group* $\mathrm{HH}_{-n}^{A/\Bbbk}(M)\left\langle -\frac{nd}{2}\right\rangle$.

(ii) *The limit of* ${}^{w}\,\mathrm{SHH}^{A/\Bbbk}(M)$ *is canonically isomorphic to* ${}^{w}\,\mathrm{sHH}_t^{A/\Bbbk}(M)$*, the total* w*-stable Hochschild homology of* M *over* A*.*

Corollary I.4.6.6. *Under the hypothesis of Proposition I.4.5.5, the* w*-stable Hochschild homology* ${}^{w}\,\mathrm{sHH}_t^{A/\Bbbk}(M)$ *of an* $\widehat{A}/(\widehat{w})$*-module* M *admits a natural, bounded* $\mathbb{Z}$*-filtration.*

Finally, we consider degeneration of the spectral complex $\mathrm{HC}(X)$:

Fact I.4.6.7. *For a bounded mixed complex* $X \in \mathrm{MC}(A, 0_d)$*, the following are equivalent:*

(i) $\mathrm{HC}(X)$ *is degenerate, i.e. all differentials* d_t^n *for* $n \in \mathbb{Z}$ *and* $t \geq 1$ *vanish.*

(ii) *Any* d*-cycle* $x \in X^n$ *extends to a* $(\mathrm{d}+s)$*-cycle* $\tilde{x} := x + x' \in X^{\leq n}$ *for* $x' \in X^{\leq n-1}$*.*

In this case, $[x] \mapsto [\tilde{x}]$ *defines an isomorphism* $\mathrm{H}^n(X, \mathrm{d})\left\langle -\frac{nd}{2}\right\rangle \cong \mathrm{HC}^{\leq n}(X)/\,\mathrm{HC}^{\leq n-1}(X)$*.*

Proof. Consider an approximative n-cycle $x \in \mathrm{Z}_t^n$ of order t. We have seen in the proof of Proposition I.4.6.1 that $\mathrm{d}_t^n([x]) = 0$ is equivalent to $x \in \mathrm{Z}_{t+1}^n + X^{\leq n-1}$. Hence, the vanishing of d_t^n is equivalent to every approximative n-cycle of order t having an approximative n-cycle of order $t+1$ in its $X^{\leq n-1}$-coset. Since some $x \in X^{\leq n}$ is an approximative n-cycle of order 1 if and only if $x_n \in X^n$ is a d-cycle, and since cycles of order t agree with actual $(d+s)$-cycles for $t \gg 0$, the equivalence of (i) and (ii) follows. The last claim follows by observing that in the chain of isomorphisms

$$\begin{aligned}
\mathrm{H}^n(X,\mathrm{d})\langle -\tfrac{nd}{2}\rangle \quad &\cong \quad \mathrm{HC}_1^n(X) = \mathrm{H}^n(\mathrm{HC}_1^*(X), \mathrm{d}_1^* \equiv 0) \\
&\cong \quad \mathrm{HC}_2^n(X) = \mathrm{H}^n(\mathrm{HC}_1^*(X), \mathrm{d}_2^* \equiv 0) \\
&\;\;\vdots \\
&\cong \quad \mathrm{HC}_t^n(X) \cong \mathrm{HC}^{\leq n}(X)/\,\mathrm{HC}^{\leq n-1}(X), \quad t \gg 0:
\end{aligned}$$

each isomorphism is given on representatives by extending of an approximative cycle of order t to an approximative cycle of order $t+1$, modulo $X^{\leq n-1}$. □

I.5. Khovanov-Rozansky homology as stable Hochschild homology

In this section we prove that $\mathfrak{sl}(k)$ Khovanov-Rozansky homology, as defined in Section I.3, can be described in terms of k-stable Hochschild homology of Rouquier complexes of Soergel bimodules. This is analogous to Khovanov's result [Kho07, Theorem 1] describing triply-graded Khovanov-Rozansky homology from [KR08b] in terms of ordinary Hochschild homology of Rouquier complexes.

Recall Notations I.3.1 and I.3.2 from Section I.3; in particular, $\Bbbk := \mathbb{Q}$, k is a fixed integer ≥ 2, $w_n := \sum_i x_i^{k+1} \in \mathbb{A}^n_\Bbbk := \Bbbk[x_1, \ldots, x_n]$ and $\widehat{w_n} := \sum_i x_i^{k+1} - y_i^{k+1} \in \widehat{\mathbb{A}}^n_\Bbbk := \Bbbk[x_1, \ldots, x_n, y_1, \ldots, y_n]$. Also, recall the diagonal bimodule $\Delta^n_\Bbbk$ over $\mathbb{A}^n_\Bbbk$, the antidiagonal $X_\Bbbk$ over $\mathbb{A}^2_\Bbbk$ and the elementary Soergel bimodule $B_\Bbbk$ over $\mathbb{A}^2_\Bbbk$.

The matrix factorizations considered in the definition of Khovanov-Rozansky homology $\mathcal{KR}^k$ are Koszul factorizations associated to regular sequences, see Section I.3. From Example I.2.2.2 we therefore deduce canonical contraderived weak equivalences $\mathcal{CKR}^k(\uparrow\uparrow) \to {}_{\widehat{w}_2}\Delta^2_\Bbbk$ and $\mathcal{CKR}^k(\nwarrow\!\!\!\!\nearrow) \to {}_{\widehat{w}_2}B_\Bbbk$ in the contraderived category $\mathbf{D}^{\mathrm{ctr}}\,\mathrm{LF}(\widehat{\mathbb{A}}^2_\Bbbk, \widehat{w_2})$. Moreover, by looking at the explicit matrices (U_0, U_1) and (V_0, V_1) defining the maps χ_0 and χ_1 in [KR08a, Section following Proposition 27] (the relevant entries being the ones in the upper left corner) we see that the two diagrams

$$\begin{array}{ccc} \mathcal{CKR}^k(\uparrow\uparrow)\langle k-1\rangle & \xrightarrow{\chi_0} & \mathcal{CKR}^k(\nwarrow\!\!\!\!\nearrow)\langle k+1\rangle \\ \simeq\downarrow & & \downarrow\simeq \\ {}_{\widehat{w}_2}\Delta^2_\Bbbk & \xrightarrow[1\mapsto x_1-y_2]{} & {}_{\widehat{w}_2}B_\Bbbk\langle 1\rangle \end{array} \qquad \begin{array}{ccc} \mathcal{CKR}^k(\nwarrow\!\!\!\!\nearrow)\langle 1-k\rangle & \xrightarrow{\chi_1} & \mathcal{CKR}^k(\uparrow\uparrow)\langle 1-k\rangle \\ \simeq\downarrow & & \downarrow\simeq \\ {}_{\widehat{w}_2}B_\Bbbk & \xrightarrow[\mathrm{can}]{} & {}_{\widehat{w}_2}\Delta^2_\Bbbk \end{array}$$

commute, so that we get canonical termwise contraderived weak equivalences of complexes of linear factorizations of type $(\widehat{\mathbb{A}}^1_\mathbb{Q}, \widehat{w}_1)$ resp. $(\widehat{\mathbb{A}}^2_\mathbb{Q}, \widehat{w}_2)$:

$$\begin{aligned} \mathcal{CCKR}^k(\uparrow) &\xrightarrow{\simeq} \Big(\ldots \to 0 \to \underline{{}_{\widehat{w}_1}\Delta^1_\Bbbk} \to 0 \to \ldots\Big) \\ \mathcal{CCKR}^k(\nwarrow\!\!\!\!\nearrow) &\xrightarrow{\simeq} \Big(\ldots \to 0 \to {}_{\widehat{w}_2}B_\Bbbk\langle 1-k\rangle \xrightarrow{\mathrm{can}} \underline{{}_{\widehat{w}_2}\Delta^2_\Bbbk\langle 1-k\rangle} \to 0 \to \ldots\Big) \qquad \text{(I.5.1)} \\ \mathcal{CCKR}^k(\nwarrow\!\!\!\!\nearrow) &\xrightarrow{\simeq} \Big(\ldots \to 0 \to \underline{{}_{\widehat{w}_2}\Delta^2_\Bbbk\langle k-1\rangle} \xrightarrow{1\mapsto x_1-y_2} {}_{\widehat{w}_2}B_\Bbbk\langle k+1\rangle \to 0 \to \ldots\Big) \end{aligned}$$

Next we consider complexes of matrix factorizations associated to braid words β on n strands, but first some notation:

Notation I.5.1. For $i \in \{1, 2, \ldots, n-1\}$ and $\varepsilon \in \{\pm 1\}$ we denote s_i^ε the positive and negative crossings of the i-th and $i+1$-th strands; a *braid word* β on n strands is then by definition a formal concatenation of the s_i^ε, i.e. $\beta = s_{i_1}^{\varepsilon_1} \ldots s_{i_l}^{\varepsilon_l}$ for some $i_1, \ldots, i_l \in \{1, 2, \ldots, n-1\}$ and $\varepsilon_1, \ldots, \varepsilon_l \in \{\pm 1\}$. It can be viewed as part of a planar oriented link diagram, hence associated to it is a complex $\mathcal{CCKR}^k(\beta)$ of matrix factorization as described in Section I.3. The number $\sum_j \varepsilon_j$ is called the *writhe* of β and is denoted $w(\beta)$. Further (see also Definition I.A.2.1), we denote $B_{\Bbbk}^{n,i}$ the $\mathbb{A}_{\Bbbk}^n$-bimodule given as the tensor product of the diagonal $\Bbbk[x_1, \ldots, \widehat{x_i}, \widehat{x_{i+1}}, x_{i+2}, \ldots, x_n]$-bimodule and the $\Bbbk[x_i, x_{i+1}]$-bimodule $B_{\Bbbk}^2$; note that it is free of rank 2 both as a left and as a right $\mathbb{A}_{\Bbbk}^n$-module. Finally, we define complexes $G_+^{n,i}$ and $G_-^{n,i}$ of $\mathbb{A}_{\Bbbk}^n$-bimodules via

$$G_+^{n,i} := \ldots \to 0 \to B_{\Bbbk}^{n,i}\langle 1-k\rangle \xrightarrow{\text{can}} \underline{\Delta_{\Bbbk}^n\langle 1-k\rangle} \to 0 \to \ldots \tag{I.5.2}$$

$$G_-^{n,i} := \ldots \to 0 \to \underline{\Delta_{\Bbbk}^n\langle k-1\rangle} \xrightarrow{1 \mapsto x_i - y_{i+1}} B_{\Bbbk}^{n,i}\langle k+1\rangle \to 0 \to \ldots \tag{I.5.3}$$

where the underlined component is the one sitting in cohomological degree zero. Note that all $\mathbb{A}_{\Bbbk}^n$-bimodules we consider here are modules over $\mathbb{A}_{\Bbbk}^n \otimes_{(\mathbb{A}_{\Bbbk}^n)^{\mathfrak{S}_n}} \mathbb{A}_{\Bbbk}^n$, so that in particular we can consider them as linear factorizations of type $(\widehat{\mathbb{A}}_{\Bbbk}^n, \widehat{w}_n)$. $\Diamond$

The following is our first main result; its second part is [Web07, Theorem 2.7].

Theorem I.5.2. *For a braid word $\beta = s_{i_1}^{\varepsilon_1} \cdots s_{i_l}^{\varepsilon_l}$ there is a canonical termwise contraderived weak equivalence of complexes of linear factorizations of type $(\widehat{\mathbb{A}}_{\mathbb{Q}}^n, \widehat{w}_n)$:*

$$\mathcal{CCKR}^k(\beta) \xrightarrow{\simeq} {}_{\widehat{w}_n}G(\beta) := {}_{\widehat{w}_n}\left[G_{\varepsilon_1}^{n,i_1} \otimes_{\mathbb{A}_{\Bbbk}^n} \ldots \otimes_{\mathbb{A}_{\Bbbk}^n} G_{\varepsilon_l}^{n,i_l}\right].$$

Moreover, denoting L the closure of the braid represented by β, we have

$$\begin{aligned} \mathcal{CKR}^k(L) &\cong {}^k\,\mathrm{sHH}_t^{\mathbb{A}_{\mathbb{Q}}^n/\mathbb{Q}}\left[G(\beta)\right], \\ \mathcal{KR}^k(L) &= \sum_{i,j\in\mathbb{Z}} \dim_{\mathbb{Q}} \mathrm{H}^i\left[{}^k\,\mathrm{sHH}_t^{\mathbb{A}_{\mathbb{Q}}^n/\mathbb{Q}}\, G(\beta)_j\right] a^i q^j \in \mathbb{Z}[a^{\pm 1}, q^{\pm 1}], \end{aligned}$$

where in the second line we mean isomorphism of complexes of graded $\mathbb{Q}$-vector spaces.

Proof. This follows from the contraderived equivalences (I.5.1) and Corollary I.4.2.6 on the relative contraderived tensor product, using the fact that the entries of the complexes $G_{\pm}^{n,i}$, hence also the entries of any iterated tensor product of the $G_{\pm}^{n,i}$, are free both as left and as right $\mathbb{A}_{\mathbb{Q}}^n$-modules. Note that all rings we are working with are polynomial rings over $\Bbbk = \mathbb{Q}$ and in particular of finite global dimension; the technical difficulties discussed in the previous sections do therefore not occur. $\square$

For a braid word β as above, the complex $G(\beta)$ is closely related to the *Rouquier complex* $\mathcal{RC}_{\mathbb{Q}}(\beta)$ of Soergel bimodules attached to β [Rou12, §3.2], the latter being defined as the tensor product $F^{n,i_1}_{\varepsilon_1} \otimes_{\mathbb{A}^n_{\Bbbk}} \ldots \otimes_{\mathbb{A}^n_{\Bbbk}} F^{n,i_l}_{\varepsilon_l}$, where

$$F^{n,i}_{+} = \Big(\ldots \to 0 \to \underline{B^{n,i}_{\Bbbk}\langle 2\rangle} \xrightarrow{\text{can}} \Delta^n_{\Bbbk}\langle 2\rangle \to 0 \to \ldots\Big) = \Sigma^{-1} G^{n,i}_{+}\langle k+1\rangle \qquad \text{(I.5.4)}$$

and

$$F^{n,i}_{-} = \Big(\ldots \to 0 \to \Delta^n_{\Bbbk}\langle -2\rangle \xrightarrow{1\mapsto x_i - y_{i+1}} \underline{B^{n,i}_{\Bbbk}} \to 0 \to \ldots\Big) \;=\; \Sigma G^{n,i}_{-}\langle -(k+1)\rangle. \quad \text{(I.5.5)}$$

See Appendix I.A. Note however that we slightly deviate from [Rou12] in that we are working with unreduced Soergel bimodules here, with q-degree doubled compared to [Rou12]. Recalling from Notation I.5.1 the writhe $w(\beta) := \sum_j \varepsilon_j$ of a braid word $\beta = s^{\varepsilon_1}_{i_1} \ldots s^{\varepsilon_l}_{i_l}$, we may restate Theorem I.5.2 as follows:

Theorem I.5.3. *Given an oriented link L presented as the closure of an n-strand braid word β with writhe $w(\beta)$, there is a canonical termwise contraderived weak equivalence of complexes of linear factorizations of type $(\widehat{\mathbb{A}}^n_{\mathbb{Q}}, \widehat{w}_n)$*

$$\mathcal{CCKR}^k(\beta) \xrightarrow{\simeq} \Sigma^{w(\beta)}{}_{\widehat{w}_n}\mathcal{RC}_{\mathbb{Q}}(\beta)\langle -(k+1)w(\beta)\rangle,$$

where $\mathcal{RC}_{\mathbb{Q}}(\beta)$ is the Rouquier complex of β defined over $\mathbb{Q}$. Moreover, there is a canonical isomorphism of complexes of graded $\mathbb{Q}$-vector spaces

$$\mathcal{CKR}^k(L) \xrightarrow{\simeq} \Sigma^{w(\beta)}\ {}^k\,\mathrm{sHH}^{\mathbb{A}^n_{\Bbbk}}\left[\mathcal{RC}_{\mathbb{Q}}(\beta)\right]\langle -(k+1)w(\beta)\rangle$$

and hence the Khovanov-Rozansky homology $\mathcal{KR}^k(L) \in \mathbb{Z}[a^{\pm 1}, q^{\pm 1}]$ is given by

$$\mathcal{KR}^k(L) = (a^{-1}q^{k+1})^{w(\beta)} \sum_{i,j\in\mathbb{Z}} \dim_{\mathbb{Q}} \mathrm{H}^i\left[{}^k\,\mathrm{sHH}_t^{\mathbb{A}^n_{\mathbb{Q}}/\mathbb{Q}}\,\mathcal{RC}_{\mathbb{Q}}(\beta)_j\right] a^i q^j. \qquad \text{(I.5.6)}$$

Remark I.5.4. For the construction of the complex of matrix factorizations $\mathcal{CCKR}^k(\beta)$ it is nowhere important that the base is a field or even $\mathbb{Q}$, and the statements of Theorem I.5.2 continue to hold with $\mathbb{Q}$ being replaced by an arbitrary commutative base ring $\Bbbk$. Although the base rings $\mathbb{A}^n_{\Bbbk}$ are no longer of finite global dimension in this case (at least absolutely, see also Section I.6.6), one can check that part (ii) of Proposition I.4.2.5 can always be applied to the tensor products in question; this is because all Soergel bimodules on n strands have finite projective dimension over $\widehat{\mathbb{A}}^n_{\Bbbk}$.

This observation is independent of the question whether $\mathcal{CCKR}^k(\beta)$ gives rise to a knot invariant for bases different from $\mathbb{Q}$. This will be studied in the next Chapter I.6. $\Diamond$

I.6. Khovanov-Rozansky homology via Markov moves

I.6.1. Introduction

The aim of the following sections is to study the following questions:

(i) Can one check directly that (I.5.6) defines an invariant of oriented links?

(ii) Does (I.5.6) define an invariant of oriented links for other bases than $\mathbb{Q}$?

(iii) What is the relation between these invariants defined over different bases?

Concerning the first question, it is known that assigning to a braid word β the Rouquier complex $\mathcal{RC}_{\mathbb{Q}}(\beta)$ yields an invariant of braids in the set of homotopy types of complexes of $\mathbb{A}^n_{\mathbb{Q}}$-bimodules [Rou06, Proposition 3.4]; moreover, in view of the second question it is interesting to note that this is still true for any commutative base ring $\mathbb{k}$, see Proposition I.A.2.8. The connection to oriented links is established through the classical Theorems of Alexander and Markov: Alexander's Theorem [KT08, Theorem 2.3] says that any oriented link is isotopic to the closure of a braid as depicted in Figure I.6.1.1, while by Markov's Theorem [KT08, Theorem 2.8] two braids give rise to isotopic oriented links upon braid closure if and only if they can be connected by iterated application of the operations in Figures I.6.1.2 and I.6.1.3 called *Markov moves*; see also Section I.A.1 for both theorems. Therefore, showing that (I.5.6) gives rise to an invariant of oriented links reduces to proving its invariance under the two Markov moves.

While the invariance of (I.5.6) under the first Markov move is simple to prove, its behavior under the second Markov move depends on the invertibility of $k+1$ in the base ring $\mathbb{k}$. We begin by describing the situation where $k+1 \in \mathbb{k}^\times$.

Theorem I.6.1.1. *Let $\mathbb{k}$ be a commutative ring with $k+1 \in \mathbb{k}^\times$. Then, for an n-strand braid word β with writhe $w(\beta)$, the complex*

$$\mathcal{CKR}^k_{\mathbb{k}}(\beta) \;:=\; \Sigma^{-w(\beta)k}\, \mathrm{sHH}_t^{\mathbb{A}^n_{\mathbb{k}}/\mathbb{k}}\, [\mathcal{RC}_{\mathbb{k}}(\beta)]\, \langle (k+1)w(\beta) \rangle \tag{I.6.1.1}$$

has $\mathbb{k}$-free components of finite rank. Moreover, its isomorphism class in the homotopy category $\mathrm{Ho}^b(\mathbb{k}\text{-Mod})$ is invariant under the Markov moves, and hence it defines an invariant of oriented links which we also denote $\mathcal{CKR}^k_{\mathbb{k}}$.

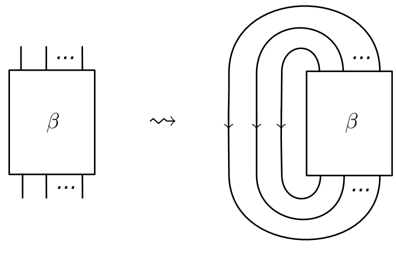


Figure I.6.1.1. Assigning an oriented link to a braid through *braid closure*

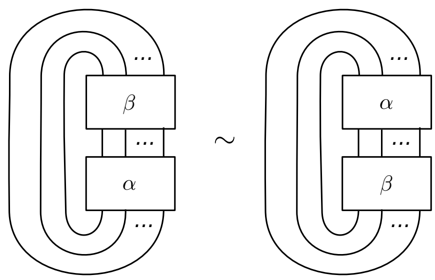


Figure I.6.1.2. The first Markov move

In words, (I.6.1.1) denotes the complex obtained from the Rouquier complex $\mathcal{RC}_{\Bbbk}(\beta)$ over $\Bbbk$ by firstly applying k-stable Hochschild homology ${}^{k}\,\mathrm{sHH}_t^{\mathbb{A}^n_{\Bbbk}/\Bbbk} : \widehat{\mathbb{A}}^n_{\Bbbk}/(\widehat{w_n})\text{-Mod} \to \Bbbk\text{-Mod}$ componentwise and then shifting the cohomological resp. internal degree by $-w(\beta)$ resp. $(k+1)w(\beta)$. Note that the complexes (I.6.1.1) are naturally complexes of modules over the polynomial ring $\mathbb{A}^n_{\Bbbk}$, but that we neglect this action in the formulation of Theorem I.6.1.1. We can, however, incorporate the action when considering invariants of *ordered* oriented links (see Definition I.A.1.1):

Definition I.6.1.2. *Let β be an* ordered *l-component braid on n strands in the sense that the components of its closure L (the cycles of the permutation underlying β) are labelled $1, 2, \ldots, l$. Assume further that the i-th component of L contains the $n(i)$-th strand of β for some choice of $n(i) \in \{1, 2, \ldots, n\}$. Then the* i-th external action *on $\mathcal{CKR}^k_{\Bbbk}(\beta)$ from* (I.6.1.1) *is defined as the external action of $x_{n(i)} \in \widehat{\mathbb{A}}^n_{\Bbbk}$ (see Definition I.2.1.12), viewed as an endomorphism of $\mathcal{CKR}^k_{\Bbbk}(\beta)$ in the homotopy category* $\mathrm{Ho}^b(\Bbbk\text{-Mod})$*. For $i > l$, we define the i-th external action to be* 0*.*

Definition I.6.1.2 is independent both on whether we count the strands of a braid at the top or bottom and on the particular choice of the $n(i)$; this follows from Proposition I.A.2.11 and the fact that the external left and right actions of $\mathbb{A}^n_{\Bbbk}$ on the k-stable

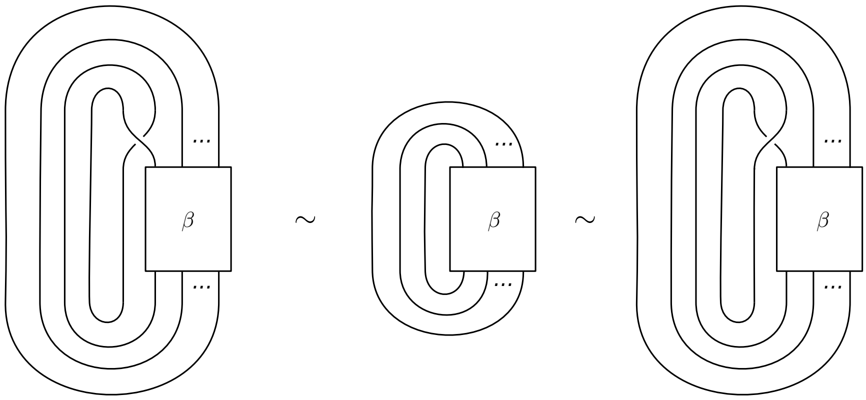


Figure I.6.1.3. The second Markov move

Hochschild homology of any $\widehat{\mathbb{A}}^n_{\Bbbk}/(\widehat{w}_n)$-module coincide. In this way, $\mathcal{CKR}^k_{\Bbbk}(\beta)$ can be viewed as an object of the following category $\Bbbk[l_1, l_2, \ldots]$-Mod-$\mathrm{Ho}^b(\Bbbk\text{-Mod})$: Its objects are pairs $(X, \{l_i\}_{i\in\mathbb{N}})$ consisting of an object X of $\mathrm{Ho}^b(\Bbbk\text{-Mod})$ together with an $\mathbb{N}$-indexed family of pairwise commuting endomorphisms $l_i : X \to X$, and morphisms $(X, \{l_i\}) \to (X', \{l_i'\})$ are morphisms $X \to X'$ in $\mathrm{Ho}^b(\Bbbk\text{-Mod})$ intertwining the l_i and l_i'.

Theorem I.6.1.3. *Consider associating to an ordered braid β the complex $\mathcal{CKR}^k_{\Bbbk}(\beta)$ from Theorem I.6.1.1 together with their external actions in the sense of Definition I.6.1.2. Then, up to isomorphism in* $\Bbbk[l_1, l_2, \ldots]$-Mod-$\mathrm{Ho}^b(\Bbbk\text{-Mod})$*, this assignment is invariant under the ordered Markov moves.*

With the caveat of Remark I.A.1.8 one may therefore view the $\mathcal{CKR}^k_{\Bbbk}(-)$ together with its external actions as an invariant of ordered, oriented links.

In case $\Bbbk$ is a field, taking cohomology in (I.6.1.1) yields a numerical invariant of oriented links which agrees with $\mathcal{KR}^k$ for $\Bbbk = \mathbb{Q}$ by Theorem I.5.3:

Corollary I.6.1.4. *Under the hypotheses of Theorem I.6.1.1, the Laurent polynomial*

$$\mathcal{KR}^k_{\Bbbk}(\beta) \;:=\; (a^{-1}q^{k+1})^{w(\beta)} \sum_{i,j\in\mathbb{Z}} \dim_{\Bbbk} \mathrm{H}^i \left[{}^k\,\mathrm{sHH}_t^{\mathbb{A}^n_{\Bbbk}/\Bbbk}\, \mathcal{RC}_{\Bbbk}(\beta)_j\right] a^i q^j \in \mathbb{Z}[a^{\pm 1}, q^{\pm 1}]$$

is well-defined and invariant under the Markov moves, hence a $\mathbb{Z}[a^{\pm 1}, q^{\pm 1}]$-valued invariant of oriented links, denoted $\mathcal{KR}^k_{\Bbbk}$. Moreover $\mathcal{KR}^k_{\mathbb{Q}} = \mathcal{KR}^k$.

In case $k+1 = 0$ in $\Bbbk$, the situation is different. To state the result, following [Rou12] we denote by $\Bbbk\text{-Mod}^{\frac{1}{2}}$ the category of $\frac{1}{2}\mathbb{Z}$-graded $\Bbbk$-modules, and by $\mathrm{Ho}^b_{\frac{1}{2}}\left(\Bbbk\text{-Mod}^{\frac{1}{2}}\right)$ their homotopy category of cohomologically $\frac{1}{2}\mathbb{Z}$-graded complexes. We then have the following theorem:

Theorem I.6.1.5. *Assume that $k+1=0$ in $\mathbb{k}$. Then assigning to a braid word β on n strands and writhe $w(\beta)$ the isomorphism class of*

$$\mathcal{CKR}^k_{\mathbb{k}}(\beta) \;:=\; \Sigma^{\frac{n+w(\beta)}{2}}\, {}^k\,\mathrm{sHH}_t^{\mathbb{A}^n_{\mathbb{k}}/\mathbb{k}}\,[\mathcal{RC}_{\mathbb{k}}(\beta)]\left\langle -\tfrac{(k+1)(n+w(\beta))}{2}\right\rangle$$

in $\mathrm{Ho}^b_{\frac{1}{2}}\left(\mathbb{k}\text{-}\mathrm{Mod}^{\frac{1}{2}}\right)$ defines an invariant of oriented links, denoted $\mathcal{CKR}^k_{\mathbb{k}}$.

The normalization in Theorem I.6.1.5 matches the one in Rouquier's theorem [Rou12, Theorem 4.9] describing triply graded Khovanov-Rozansky homology in terms of *ordinary* Hochschild homology of Soergel bimodules. This is no coincidence:

Theorem I.6.1.6. *Suppose $k+1=0$ in $\mathbb{k}$, and let $M\in\mathcal{SBM}_{\mathbb{k}}(n)$ be a Soergel bimodule over $\mathbb{k}$. Then there is a canonical isomorphism*

$$\mathrm{gr}\left[{}^k\,\mathrm{sHH}_t^{\mathbb{A}^n_{\mathbb{k}}/\mathbb{k}}(M)\right] \;\cong\; \left(\mathrm{HH}_j^{\mathbb{A}^n_{\mathbb{k}}/\mathbb{k}}(M)\left\langle \tfrac{jd}{2}\right\rangle\right)_{j\in\mathbb{Z}}, \qquad \text{(I.6.1.2)}$$

where ${}^k\,\mathrm{sHH}_t^{\mathbb{A}^n_{\mathbb{k}}/\mathbb{k}}(M)$ is equipped with the $\mathbb{Z}$-filtration defined in Corollary I.4.6.6.

Theorem I.6.1.5 is a consequence of Rouquier's Theorem [Rou12, Theorem 4.9] and Theorem I.6.1.6. However, the methods used to prove Theorems I.6.1.1 and I.6.1.6 yield another proof of Rouquier's Theorem and are independent of [Rou12].

The results stated above are established using the contraderived categories of curved mixed complexes and of linear factorizations; the arguments are based both on explicit calculations on the one hand and on general properties of abelian model structures and triangulated categories on the other hand. To avoid overlap, we formulate basic results that are independent on the value of $k+1$ in $\mathbb{k}$ in the first two Sections I.6.2 and I.6.3, and consider the characteristics of the contrasting situations $k+1\in\mathbb{k}^\times$ and $k+1=0$ in the Sections I.6.4 and I.6.5, respectively.

I.6.2. The first Markov move

The invariance of classical and k-stable Hochschild homology under the first Markov move is established in the following proposition. Throughout, $\mathbb{k}$ is a commutative ring.

Proposition I.6.2.1. *Let A be a commutative $\mathbb{k}$-algebra which is free as an $\mathbb{k}$-module, and denote $\widehat{A}:=A\otimes_{\mathbb{k}}A$ its $\mathbb{k}$-enveloping algebra. Further, let $w\in A$ and let $X,Y\in\widehat{A}/(\widehat{w})\text{-}\mathrm{Mod}$ be free both as left and right A-modules. Then there exists a canonical and natural isomorphism in $\mathbf{D}^{\mathrm{ctr}}\,\mathrm{MC}(\mathbb{k},0)$*

$$_{-\widehat{w}}\Delta\otimes^{\mathbf{L}}_{\widehat{A}}\,{}_{\widehat{w}}(X\otimes_A Y)\cong\;{}_{-\widehat{w}}\Delta\otimes^{\mathbf{L}}_{\widehat{A}}\,{}_{\widehat{w}}(Y\otimes_A X). \qquad \text{(I.6.2.3)}$$

In particular, there is a canonical isomorphism between classical Hochschild homologies:

$$\mathrm{HH}_*^{A/\Bbbk}(X \otimes_A Y) \;\cong\; \mathrm{HH}_*^{A/\Bbbk}(Y \otimes_A X) \tag{I.6.2.4}$$

If one of the conditions in Proposition I.4.5.5 is met, (I.6.2.3) *induces an isomorphism:*

$$^{w}\,\mathrm{sHH}_*^{A/\Bbbk}(X \otimes_A Y) \;\cong\; {}^{w}\,\mathrm{sHH}_*^{A/\Bbbk}(Y \otimes_A X) \tag{I.6.2.5}$$

Proof. First, choose quasi-isomorphisms $P \to {}_{\widehat{w}}X$ and $Q \to {}_{\widehat{w}}Y$ for P, Q bounded above, $\widehat{A}$-free $\widehat{w}$-curved mixed complexes. The freeness of X, Y as left and right A-modules as well as the freeness of A over $\Bbbk$ then imply that the maps $P \otimes_A Q \to {}_{\widehat{w}}(X \otimes_A Y)$ and $Q \otimes_A P \to {}_{\widehat{w}}(Y \otimes_A X)$ are also resolutions by bounded above, $\widehat{A}$-free $\widehat{w}$-curved mixed complexes. By Proposition I.4.3.1 they can therefore be used to compute the contraderived tensor products in question, establishing the isomorphism (I.6.2.3):

$$\begin{aligned} {}_{-\widehat{w}}\Delta \otimes^{\mathbf{L}}_{\widehat{A}} {}_{\widehat{w}}(X \otimes_A Y) &\cong {}_{-\widehat{w}}\Delta \otimes_{\widehat{A}} (P \otimes_A Q) && \text{(I.6.2.6)} \\ &\cong {}_{-\widehat{w}}\Delta \otimes_{\widehat{A}} (Q \otimes_A P) && \text{(I.6.2.7)} \\ &\cong {}_{-\widehat{w}}\Delta \otimes_{\widehat{A}} {}_{\widehat{w}}(Y \otimes_A X) \end{aligned}$$

Here (I.6.2.7) is justified by Fact I.6.2.2 below. It remains to explain how to deduce from this the isomorphisms (I.6.2.4) and (I.6.2.5). Firstly, (I.6.2.4) follows from (I.6.2.3) by taking cohomology, since the complexes underlying the curved mixed complexes in (I.6.2.6) also compute the ordinary Hochschild homology of $X \otimes_A Y$ and $Y \otimes_A X$ over $\widehat{A}$. Secondly, the isomorphism (I.6.2.5) follows from (I.6.2.3) by applying $\mathbf{R}\,\mathrm{fold}^{\Pi}$ and taking cohomology, using Proposition I.4.5.5. □

Fact I.6.2.2. *For A-bimodules X, Y the flip defines a natural isomorphism of $\Bbbk$-modules*

$$\Delta \otimes_{\widehat{A}} (X \otimes_A Y) \cong \Delta \otimes_{\widehat{A}} (Y \otimes_A X). \tag{I.6.2.8}$$

Remark I.6.2.3. Note, however, that the flip map is not $\widehat{A}$-linear with respect to the canonical $\widehat{A}$-actions on both sides of (I.6.2.8). Instead, the naturality of the isomorphism implies that the outer A-action on $X \otimes_A Y$ on the left hand side gets identified with the inner A-action on $Y \otimes_A X$ on the right hand side, and vice versa. ◊

I.6.3. Generalities about the second Markov move

Consider the left hand side of Figure I.6.3.4: In order to understand the effect of the second Markov move (Figure I.6.1.3) on classical and k-stable Hochschild homology of

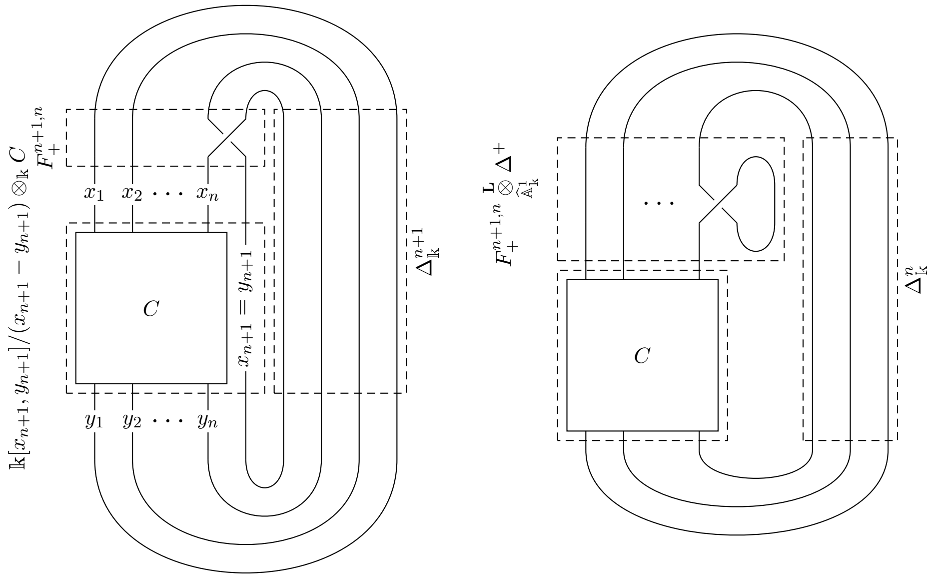


Figure I.6.3.4. The second Markov move

Rouquier complexes we have to study, for a Rouquier complex C on n strands (Definition I.A.2.7), the relation between ${}_{-\widehat{w}_n}\Delta^n_{\mathbb{k}} \otimes^{\mathbf{L}}_{\widehat{\mathbb{A}}^n_{\mathbb{k}}} {}_{\widehat{w}_n}C$ (computing the k-stable Hochschild homology of C) and the following complex:

$$_{-\widehat{w}_{n+1}}\Delta^{n+1}_{\mathbb{k}} \overset{\mathbf{L}}{\underset{\widehat{\mathbb{A}}^{n+1}_{\mathbb{k}}}{\otimes}} {}_{\widehat{w}_{n+1}}\left[F^{n+1,n}_{\pm} \otimes_{\mathbb{A}^{n+1}_{\mathbb{k}}} (\mathbb{k}[x_{n+1}, y_{n+1}]/(x_{n+1} - y_{n+1}) \otimes_{\mathbb{k}} C)\right] \tag{I.6.3.9}$$

This complex (with values in $\mathbf{D}^{\mathrm{ctr}}\,\mathrm{MC}(\widehat{\mathbb{A}}^{n+1}_{\mathbb{k}}, 0)$) is obtained from the Rouquier complex

$$F^{n+1,n}_{\pm} \otimes_{\mathbb{A}^{n+1}_{\mathbb{k}}} (\mathbb{k}[x_{n+1}, y_{n+1}]/(x_{n+1} - y_{n+1}) \otimes_{\mathbb{k}} C) \tag{I.6.3.10}$$

of $\widehat{\mathbb{A}}^{n+1}_{\mathbb{k}}/(\widehat{w}_{n+1})$-modules by applying the functor ${}_{-\widehat{w}_{n+1}}\Delta^{n+1}_{\mathbb{k}} \otimes^{\mathbf{L}}_{\widehat{\mathbb{A}}^{n+1}_{\mathbb{k}}} {}_{\widehat{w}_{n+1}}(-)$ componentwise. Taking homology componentwise in (I.6.3.9) gives rise to the componentwise classical Hochschild homology of (I.6.3.10), while applying $\mathbf{R}\,\mathrm{fold}^{\Pi}$ componentwise and taking componentwise homology afterwards in (I.6.3.9) gives rise to the componentwise k-stable Hochschild homology of (I.6.3.10) by Proposition I.4.5.5.

The strategy is to first formalize the right hand side of Figure I.6.3.4 by introducing a "partial trace functor" $\mathrm{Tr}_{\mathbb{Z}}$ corresponding to closure of a single strand only (Definition I.6.3.3) and to prove afterwards the isomorphism between the formalizations of both sides of Figure I.6.3.4 (Lemma I.6.3.2 and Corollary I.6.3.5). With this done, the main problem is to study the effect of $\mathrm{Tr}_{\mathbb{Z}}$ and its $\mathbb{Z}/2\mathbb{Z}$-graded variant $\mathrm{Tr}_{\mathbb{Z}_2}$ (Definition I.6.4.1) on the elementary Rouquier complexes $F_{\pm}^{n+1,n}$, which will be done in Propositions I.6.3.8, I.6.4.4 and I.6.5.1 below. Altogether, the effect of the second Markov move on k-stable Hochschild homology will finally be described in Corollary I.6.4.14 in case $k+1 \in \Bbbk^{\times}$ and in Corollary I.6.5.4 in case $k+1=0$.

Notation I.6.3.1. We write F for the elementary positive Rouquier complex $F_{+}^{n+1,n}$, see (I.5.4). Further, we write Δ for the diagonal $\mathbb{A}_{\Bbbk}^{n+1}$-bimodule $\Delta_{\Bbbk}^{n+1}$, B for the elementary Soergel $\mathbb{A}_{\Bbbk}^{n+1}$-bimodule $B_{\Bbbk}^{n+1,n}$ and X for the twisted diagonal $\mathbb{A}_{\Bbbk}^{n+1}$-bimodule $X_{\Bbbk}^{n+1,n}$; see Notation I.3.2. Finally, we write $\mathbb{A}_{\Bbbk}^{+}$ resp. $\widehat{\mathbb{A}}_{\Bbbk}^{+}$ for $\Bbbk[x_{n+1}]$ resp. $\Bbbk[x_{n+1}, y_{n+1}]$, Δ^{+} for the diagonal $\mathbb{A}_{\Bbbk}^{+}$-bimodule and $\widehat{w}_{+} \in \widehat{\mathbb{A}}_{\Bbbk}^{+}$ for $\widehat{w}_{n+1}-\widehat{w}_{n} = x_{n+1}^{k+1} - y_{n+1}^{k+1}$. In this notation, (I.6.3.9) becomes ${}_{-\widehat{w}_{n+1}}\Delta \otimes^{\mathbf{L}}_{\widehat{\mathbb{A}}_{\Bbbk}^{n+1}} {}_{\widehat{w}_{n+1}}\left[F \otimes_{\mathbb{A}_{\Bbbk}^{n+1}} (\Delta^{+} \otimes_{\Bbbk} C)\right]$. $\Diamond$

By definition (I.5.4) the complex F can be written as $\Sigma^{-1}\operatorname{Cone}(\underline{B} \to \underline{\Delta})\langle 2\rangle$, hence

$$F \underset{\mathbb{A}_{\Bbbk}^{n+1}}{\otimes} \left(\Delta^{+} \otimes_{\Bbbk} C\right) \cong \Sigma^{-1}\operatorname{Cone}\left[B \underset{\mathbb{A}_{\Bbbk}^{n+1}}{\otimes} \left(\Delta^{+} \otimes_{\Bbbk} C\right) \to \Delta \underset{\mathbb{A}_{\Bbbk}^{n+1}}{\otimes} \left(\Delta^{+} \otimes_{\Bbbk} C\right)\right]\langle 2\rangle. \quad \text{(I.6.3.11)}$$

Since ${}_{-\widehat{w}_{n+1}}\Delta \otimes^{\mathbf{L}}_{\widehat{\mathbb{A}}_{\Bbbk}^{n+1}} {}_{\widehat{w}_{n+1}}(-)$ is additive and hence preserves cones, we are therefore naturally lead to study, for some fixed $\widehat{\mathbb{A}}_{\Bbbk}^{n+1}/(\widehat{w}_{n+1})$-module D, the functor

$$\mathrm{M}_{\mathbb{Z}}^{D} := {}_{-\widehat{w}_{n+1}}\Delta \underset{\widehat{\mathbb{A}}_{\Bbbk}^{n+1}}{\overset{\mathbf{L}}{\otimes}} {}_{\widehat{w}_{n+1}}\left[D \underset{\mathbb{A}_{\Bbbk}^{n+1}}{\otimes} \left(\Delta^{+} \underset{\Bbbk}{\otimes} -\right)\right] : \widehat{\mathbb{A}}_{\Bbbk}^{n}/(\widehat{w}_{n})\text{-Mod} \to \mathbf{D}^{\mathrm{ctr}}\operatorname{MC}(\widehat{\mathbb{A}}_{\Bbbk}^{n+1}, 0).$$

If D is free as a right $\mathbb{A}_{\Bbbk}^{n+1}$-module, then Proposition I.4.3.1 applied twice gives

$$_{\widehat{w}_{n+1}}\left[D \otimes_{\mathbb{A}_{\Bbbk}^{n+1}} \left(\Delta^{+} \otimes_{\Bbbk} -\right)\right] \;\cong\; {}_{\widehat{w}_{n+1}}D \otimes^{\mathbf{L}}_{\mathbb{A}_{\Bbbk}^{n+1}} \left({}_{\widehat{w}_{+}}\Delta^{+} \otimes^{\mathbf{L}}_{\Bbbk} {}_{\widehat{w}_{n}}(-)\right),$$

so that we may also study the functor $\mathbf{D}^{+}\operatorname{MC}(\widehat{\mathbb{A}}_{\Bbbk}^{n}, \widehat{w}_{n}) \to \mathbf{D}^{+}\operatorname{MC}(\widehat{\mathbb{A}}_{\Bbbk}^{n+1}, 0)$

$${}_{-\widehat{w}_{n+1}}\Delta \otimes^{\mathbf{L}}_{\widehat{\mathbb{A}}_{\Bbbk}^{n+1}} \left[{}_{\widehat{w}_{n+1}}D \otimes^{\mathbf{L}}_{\mathbb{A}_{\Bbbk}^{n+1}} \left({}_{\widehat{w}_{+}}\Delta^{+} \otimes^{\mathbf{L}}_{\Bbbk} -\right)\right].$$

This reduces to the study of the object ${}_{-\widehat{w}_{+}}\Delta^{+} \otimes^{\mathbf{L}}_{\widehat{\mathbb{A}}_{\Bbbk}^{+}} {}_{\widehat{w}_{n+1}}D$ by the following lemma which formalizes the "equality" of both sides in Figure I.6.3.4:

Lemma I.6.3.2. *For any $C \in \mathbf{D}^{+}\operatorname{MC}(\widehat{\mathbb{A}}_{\Bbbk}^{n}, \widehat{w}_{n})$ and any $D \in \mathbf{D}^{+}\operatorname{MC}(\widehat{\mathbb{A}}_{\Bbbk}^{n+1}, \widehat{w}_{n+1})$ there is a canonical and natural isomorphism in $\mathbf{D}^{\mathrm{ctr}}\operatorname{MC}(\widehat{\mathbb{A}}_{\Bbbk}^{n+1}, \widehat{w}_{n})$:*

$${}_{-\widehat{w}_{+}}\Delta^{+} \otimes^{\mathbf{L}}_{\widehat{\mathbb{A}}_{\Bbbk}^{+}} \left[D \otimes^{\mathbf{L}}_{\mathbb{A}_{\Bbbk}^{n+1}} \left({}_{\widehat{w}_{1}}\Delta^{+} \otimes^{\mathbf{L}}_{\Bbbk} C\right)\right] \;\cong\; \left[{}_{-\widehat{w}_{+}}\Delta^{+} \otimes^{\mathbf{L}}_{\widehat{\mathbb{A}}_{\Bbbk}^{+}} D\right] \otimes^{\mathbf{L}}_{\mathbb{A}_{\Bbbk}^{n}} C. \quad \text{(I.6.3.12)}$$

This induces a canonical isomorphism

$${}_{-\widehat{w}_{n+1}}\Delta \underset{\widehat{\mathbb{A}}_{\Bbbk}^{n+1}}{\overset{\mathbf{L}}{\otimes}} \left[D \otimes^{\mathbf{L}}_{\mathbb{A}_{\Bbbk}^{n+1}} \left({}_{\widehat{w}_{+}}\Delta^{+} \otimes^{\mathbf{L}}_{\Bbbk} -\right)\right] \cong {}_{-\widehat{w}_{n}}\Delta_{\Bbbk}^{n} \underset{\widehat{\mathbb{A}}_{\Bbbk}^{n}}{\overset{\mathbf{L}}{\otimes}} \left[\left({}_{-\widehat{w}_{+}}\Delta^{+} \underset{\widehat{\mathbb{A}}_{\Bbbk}^{+}}{\overset{\mathbf{L}}{\otimes}} D\right) \underset{\mathbb{A}_{\Bbbk}^{n}}{\overset{\mathbf{L}}{\otimes}} -\right] \quad \text{(I.6.3.13)}$$

of functors $\mathbf{D}^{+}\operatorname{MC}(\widehat{\mathbb{A}}_{\Bbbk}^{n}, \widehat{w}_{n}) \to \mathbf{D}^{+}\operatorname{MC}(\widehat{\mathbb{A}}_{\Bbbk}^{n+1}, 0)$.

Proof. We may assume that C is bounded above and $\widehat{\mathbb{A}}^n_{\Bbbk}$-free and that D is bounded above and $\widehat{\mathbb{A}}^{n+1}_{\Bbbk}$-free. Then all tensor products in (I.6.3.12) can be computed naively (Proposition I.4.3.1) and (I.6.3.12) is clear. Finally (I.6.3.13) follows from (I.6.3.12) and the isomorphism of functors $\mathbf{D}^+ \operatorname{MC}(\widehat{\mathbb{A}}^{n+1}_{\Bbbk}, \widehat{w}_{n+1}) \to \mathbf{D}^{\mathrm{ctr}} \operatorname{MC}(\widehat{\mathbb{A}}^{n+1}_{\Bbbk}, 0)$

$$_{-\widehat{w}_{n+1}}\Delta \otimes^{\mathbf{L}}_{\widehat{\mathbb{A}}^{n+1}_{\Bbbk}} (-) \cong {}_{-\widehat{w}_n}\Delta^n_{\Bbbk} \otimes^{\mathbf{L}}_{\widehat{\mathbb{A}}^n_{\Bbbk}} \left({}_{-\widehat{w}_+}\Delta^+ \otimes^{\mathbf{L}}_{\widehat{\mathbb{A}}^+_{\Bbbk}} (-) \right)$$

which in turn follows from $\Delta \cong \Delta^n_{\Bbbk} \otimes_{\Bbbk} \Delta^+$ and Proposition I.4.3.6. □

Lemma I.6.3.2 motivates the definition of the following "partial trace" functor:

Definition I.6.3.3. *We denote by $\operatorname{Tr}^{n+1}_{\mathbb{Z}}$ (for "trace") the functor*

$$_{-\widehat{w}_+}\Delta^+ \otimes^{\mathbf{L}}_{\widehat{\mathbb{A}}^+_{\Bbbk}} - : \mathbf{D}^{\mathrm{ctr}} \operatorname{MC}(\widehat{\mathbb{A}}^{n+1}_{\Bbbk}, \widehat{w}_{n+1}) \to \mathbf{D}^{\mathrm{ctr}} \operatorname{MC}(\widehat{\mathbb{A}}^{n+1}_{\Bbbk}, \widehat{w}_n)$$

and by

$$\mathbb{V}^{n+1}_{\mathbb{Z}} : \mathbf{D}^{\mathrm{ctr}} \operatorname{MC}(\widehat{\mathbb{A}}^{n+1}_{\Bbbk}, \widehat{w}_n) \to \mathbf{D}^{\mathrm{ctr}} \operatorname{MC}(\widehat{\mathbb{A}}^n_{\Bbbk}, \widehat{w}_n)$$

the functor forgetting the variable x_{n+1}.

Remark I.6.3.4. We did see many unexpected technicalities, but for any homomorphism of commutative rings $\varphi : A \to B$ and any $w \in A$, there is indeed a well-defined, naively computable forgetful functor $\mathbf{D}^{\mathrm{ctr}} \operatorname{MC}(B, \varphi(w)) \to \mathbf{D}^{\mathrm{ctr}} \operatorname{MC}(A, w)$ as a special case of Proposition II.2.3.18 from Part I. ◊

We summarize the discussion around Lemma I.6.3.2:

Corollary I.6.3.5. *Let $D \in \widehat{\mathbb{A}}^{n+1}_{\Bbbk}/(\widehat{w}_{n+1})$-Mod be free as a right $\mathbb{A}^{n+1}_{\Bbbk}$-module. Then there is a canonical isomorphism of functors $\widehat{\mathbb{A}}^n_{\Bbbk}/(\widehat{w}_n)$-Mod $\to \mathbf{D}^{\mathrm{ctr}} \operatorname{MC}(\widehat{\mathbb{A}}^{n+1}_{\Bbbk}, 0)$*

$$\mathrm{M}^D_{\mathbb{Z}} \equiv {}_{-\widehat{w}_{n+1}}\Delta \overset{\mathbf{L}}{\underset{\widehat{\mathbb{A}}^{n+1}_{\Bbbk}}{\otimes}} {}_{\widehat{w}_{n+1}}\left[D \otimes_{\mathbb{A}^{n+1}_{\Bbbk}} \left(\Delta^+ \otimes_{\Bbbk} -\right)\right] \cong {}_{-\widehat{w}_n}\Delta^n_{\Bbbk} \overset{\mathbf{L}}{\underset{\widehat{\mathbb{A}}^n_{\Bbbk}}{\otimes}} \left[\operatorname{Tr}^{n+1}_{\mathbb{Z}} {}_{\widehat{w}_{n+1}}D \overset{\mathbf{L}}{\underset{\mathbb{A}^n_{\Bbbk}}{\otimes}} {}_{\widehat{w}_n}-\right]$$

Notation I.6.3.6. By abuse of notation but in support of readability, we will, in the rest of this as well as the next section, often omit the subscript $_w(-)$ for the embedding of $A/(w)$-modules into $\operatorname{MC}(A, w)$ and $\operatorname{LF}(A, w)$; for example, we will write X, B and Δ instead of $_{\widehat{w}_{n+1}}X$, $_{\widehat{w}_{n+1}}B$ and $_{\widehat{w}_{n+1}}\Delta$, respectively. ◊

In view of Corollary I.6.3.5, in order to understand the effect of the second Markov move on classical and k-stable Hochschild homology we have to understand the objects $\operatorname{Tr}^{n+1}_{\mathbb{Z}} B$ and $\operatorname{Tr}^{n+1}_{\mathbb{Z}} \Delta$ as well as the morphisms $\operatorname{Tr}^{n+1}_{\mathbb{Z}} B \to \operatorname{Tr}^{n+1}_{\mathbb{Z}} \Delta$ and $\operatorname{Tr}^{n+1}_{\mathbb{Z}} \Delta\langle -2\rangle \to \operatorname{Tr}^{n+1}_{\mathbb{Z}} B$ obtained from the morphisms $B \to \Delta$ and $\Delta\langle -2\rangle \to B$ occurring in $F^{n+1,n}_{\pm}$ by applying $\operatorname{Tr}^{n+1}_{\mathbb{Z}}$. This will be done in Propositions I.6.3.8, I.6.4.4 and I.6.5.1. We begin by recalling the relation between B, Δ and X on the abelian level:

Fact I.6.3.7 (see Fact I.A.2.9)**.** *The $\mathbb{A}_{\Bbbk}^{n+1}$-bimodules Δ, B and X are free both as left and as right $\mathbb{A}_{\Bbbk}^{n+1}$-modules and fit into short exact sequences in $\widehat{\mathbb{A}}_{\Bbbk}^{n+1}$*-Mod

$$0 \to X\langle -2\rangle \xrightarrow{1\mapsto x_n - y_n = y_{n+1} - x_{n+1}} B \xrightarrow{\text{can}} \Delta \to 0,$$
$$0 \to \Delta\langle -2\rangle \xrightarrow{1\mapsto x_n - y_{n+1} = y_n - x_{n+1}} B \xrightarrow{\text{can}} X \to 0.$$

Moreover, $\ker(B \xrightarrow{\text{can}} \Delta) \cap \ker(B \xrightarrow{\text{can}} X) = \{0\}$.

Proposition I.6.3.8. *There are canonical distinguished triangles in $\mathbf{D}^{\mathrm{ctr}}\,\mathrm{MC}(\widehat{\mathbb{A}}_{\Bbbk}^{n+1}, \widehat{w}_n)$:*

$$\mathrm{Tr}_{\mathbb{Z}}^{n+1} X\langle -2\rangle \to \mathrm{Tr}_{\mathbb{Z}}^{n+1} B \to \mathrm{Tr}_{\mathbb{Z}}^{n+1} \Delta \to \Sigma\,\mathrm{Tr}_{\mathbb{Z}}^{n+1} X\langle -2\rangle \tag{I.6.3.14}$$
$$\mathrm{Tr}_{\mathbb{Z}}^{n+1} \Delta\langle -2\rangle \to \mathrm{Tr}_{\mathbb{Z}}^{n+1} B \to \mathrm{Tr}_{\mathbb{Z}}^{n+1} X \to \Sigma\,\mathrm{Tr}_{\mathbb{Z}}^{n+1} \Delta\langle -2\rangle. \tag{I.6.3.15}$$

Moreover, $\mathbb{V}_{\mathbb{Z}}^{n+1} : \mathbf{D}^{\mathrm{ctr}}\,\mathrm{MC}(\widehat{\mathbb{A}}_{\Bbbk}^{n+1}, \widehat{w}_n) \to \mathbf{D}^{\mathrm{ctr}}\,\mathrm{MC}(\widehat{\mathbb{A}}_{\Bbbk}^{n}, \widehat{w}_n)$ annihilates the morphism $\mathrm{Tr}_{\mathbb{Z}}^{n+1} X\langle -2\rangle \to \mathrm{Tr}_{\mathbb{Z}}^{n+1} B$, so that (I.6.3.14) *splits after applying $\mathbb{V}_{\mathbb{Z}}^{n+1}$.*

Proof. For ease of notation, we restrict to the (essential) case $n = 1$. The existence of the triangles (I.6.3.14) and (I.6.3.15) follows from Fact I.6.3.7 and the fact that short exact sequences in abelian categories equipped with hereditary abelian model structures yield *canonical* connecting morphisms turning them into distinguished triangles in the associated homotopy categories; see Remark I.4.1.7 and Appendix II.C.

It remains to show that the image of $X\langle -2\rangle \to B$, $1 \mapsto x_1 - y_1$ under $\mathbb{V}_{\mathbb{Z}}^2\,\mathrm{Tr}_{\mathbb{Z}}^2$ vanishes. Justified by Proposition I.4.3.6, we may realize $\mathrm{Tr}_{\mathbb{Z}}^2 X$ and $\mathrm{Tr}_{\mathbb{Z}}^2 B$ as the (underived) tensor product over $\widehat{\mathbb{A}}_{\Bbbk}^{+} = \Bbbk[x_2, y_2]$ of X resp. B with the resolution

$$\widehat{\mathbb{A}}_{\Bbbk}^{+}\langle -2\rangle \underset{u_k(x_2,y_2)}{\overset{y_2 - x_2}{\rightleftarrows}} \widehat{\mathbb{A}}_{\Bbbk}^{+}, \quad u_k(x_2, y_2) := \frac{x_2^{k+1} - y_2^{k+1}}{x_2 - y_2} = \sum_{i=0}^{k} x_2^i y_2^{k-i},$$

of ${}_{-\widehat{w}_+}\Delta^+$ in $\mathbf{D}^{\mathrm{ctr}}\,\mathrm{MC}(\widehat{\mathbb{A}}_{\Bbbk}^{+}, -\widehat{w}_+)$, so the morphism under consideration is given by

$$\begin{array}{ccc}
X = \Bbbk[x_1, y_1]\langle -4\rangle & \underset{u_k(x_2,y_2)}{\overset{y_2 - x_2 = x_1 - y_1}{\rightleftarrows}} & \underline{X = \Bbbk[x_1, y_1]\langle -2\rangle} \\
\big\downarrow{\scriptstyle x_1 - y_1} & \swarrow{\scriptstyle 1 \mapsto 1} \text{ (dashed)} & \big\downarrow{\scriptstyle x_1 - y_1} \\
B\langle -2\rangle = \widehat{\mathbb{A}}_{\Bbbk}^{2} \Big/ \begin{pmatrix} x_1 + x_2 - y_1 - y_2, \\ x_1x_2 - y_1y_2 \end{pmatrix} \langle -2\rangle & \underset{u_k(x_2,y_2)}{\overset{y_2 - x_2 = x_1 - y_1}{\rightleftarrows}} & \underline{B = \widehat{\mathbb{A}}_{\Bbbk}^{2} \Big/ \begin{pmatrix} x_1 + x_2 - y_1 - y_2, \\ x_1x_2 - y_1y_2 \end{pmatrix}},
\end{array}$$

and the dashed arrow provides an $\widehat{\mathbb{A}}_{\Bbbk}^{1}$-linear nullhomotopy. □

Corollary I.6.3.9. *There are (pointwise) distinguished triangles of functors*

$$\mathrm{M}^X_{\mathbb{Z}}\langle -2\rangle \to \mathrm{M}^B_{\mathbb{Z}} \to \mathrm{M}^\Delta_{\mathbb{Z}} \to \Sigma\,\mathrm{M}^X_{\mathbb{Z}}\langle -2\rangle, \tag{I.6.3.16}$$

$$\mathrm{M}^\Delta_{\mathbb{Z}}\langle -2\rangle \to \mathrm{M}^B_{\mathbb{Z}} \to \mathrm{M}^X_{\mathbb{Z}} \to \Sigma\,\mathrm{M}^\Delta_{\mathbb{Z}}\langle -2\rangle. \tag{I.6.3.17}$$

Further, $\mathbb{V}^{n+1}_{\mathbb{Z}} : \mathbf{D}^{\mathrm{ctr}}\,\mathrm{MC}(\widehat{\mathbb{A}}^{n+1}_{\Bbbk}, \widehat{w}_n) \to \mathbf{D}^{\mathrm{ctr}}\,\mathrm{MC}(\widehat{\mathbb{A}}^{n}_{\Bbbk}, \widehat{w}_n)$ *annihilates* $\mathrm{M}^X_{\mathbb{Z}}\langle -2\rangle \to \mathrm{M}^B_{\mathbb{Z}}$*, so that* (I.6.3.16) *splits when composed with* $\mathbb{V}^{n+1}_{\mathbb{Z}}$.

Proof. This follows from Proposition I.6.3.8 and Corollary I.6.3.5. □

The behavior of the second sequence (I.6.3.15) under $\mathbb{V}^{n+1}_{\mathbb{Z}}$ depends on the value of $k+1$ and will be studied in the next sections. For use in these sections, we note the following identity; recall $u_k(x,y) := \frac{x^{k+1}-y^{k+1}}{x-y} = \sum_i x^i y^{k-i}$.

Lemma I.6.3.10. *In* $B = \Bbbk[x_1,x_2,y_1,y_2]/(x_1+x_2-y_1-y_2, x_1x_2-y_1y_2)$ *we have*

$$u_k(x_1,y_1) = (k+1)(x_1-y_2)\sum_j x_1^j x_2^{k-1-j} + u_k(x_2,y_2). \tag{I.6.3.18}$$

Proof. By Fact I.6.3.7 it suffices to check (I.6.3.18) in $\Delta = \Bbbk[x_1,x_2,y_1,y_2]/(x_1-y_1, x_2-y_2)$ and $X = \Bbbk[x_1,x_2,y_1,y_2]/(x_1-y_2, x_2-y_1)$, where it is clear. □

I.6.4. The generic case: $k+1$ invertible in $\Bbbk$

In this section we assume that $k+1 \in \Bbbk^\times$ and prove Theorem I.6.1.1. The crucial step lies in understanding the behavior of (I.6.3.15) under application of $\mathbb{V}^{n+1}_{\mathbb{Z}}$; as it turns out, it splits only on the level of linear factorizations, so we need to introduce the variants of $\mathrm{Tr}^{n+1}_{\mathbb{Z}}$ and $\mathbb{V}^{n+1}_{\mathbb{Z}}$ in that context. We keep Notations I.6.3.1 and I.6.3.6.

Definition I.6.4.1. *We denote by* $\mathrm{Tr}^{n+1}_{\mathbb{Z}_2}$ *the functor*

$$_{-\widehat{w}_+}\Delta^+ \otimes^{\mathbf{L}}_{\widehat{\mathbb{A}}^+_{\Bbbk}} - : \mathbf{D}^{\mathrm{ctr}}\,\mathrm{LF}(\widehat{\mathbb{A}}^{n+1}_{\Bbbk}, \widehat{w}_{n+1}) \to \mathbf{D}^{\mathrm{ctr}}\,\mathrm{LF}(\widehat{\mathbb{A}}^{n+1}_{\Bbbk}, \widehat{w}_n)$$

and by

$$\mathbb{V}^{n+1}_{\mathbb{Z}_2} : \mathbf{D}^{\mathrm{ctr}}\,\mathrm{LF}(\widehat{\mathbb{A}}^{n+1}_{\Bbbk}, \widehat{w}_n) \to \mathbf{D}^{\mathrm{ctr}}\,\mathrm{LF}(\widehat{\mathbb{A}}^{n}_{\Bbbk}, \widehat{w}_n)$$

the functor forgetting the variable x_{n+1}.

Remark I.6.4.2. As in Remark I.6.3.4, note that $\mathbf{D}^{\mathrm{ctr}}\,\mathrm{LF}(\widehat{\mathbb{A}}^{n+1}_{\Bbbk}, \widehat{w}_n) \to \mathbf{D}^{\mathrm{ctr}}\,\mathrm{LF}(\widehat{\mathbb{A}}^{n}_{\Bbbk}, \widehat{w}_n)$ is well-defined and naively computable by Proposition II.2.3.18 from Part II. ♢

Proposition I.6.4.3. *There are canonical isomorphisms of functors*

$$\mathbf{R}\,\mathrm{fold}^\Pi \circ \mathrm{Tr}^{n+1}_{\mathbb{Z}} \cong \mathrm{Tr}^{n+1}_{\mathbb{Z}_2} \circ \mathbf{R}\,\mathrm{fold}^\Pi : \ \mathbf{D}^b\,\mathrm{MC}(\widehat{\mathbb{A}}^{n+1}_{\Bbbk}, \widehat{w}_{n+1}) \to \mathbf{D}^{\mathrm{ctr}}\,\mathrm{LF}(\widehat{\mathbb{A}}^{n}_{\Bbbk}, \widehat{w}_n),$$

$$\mathbf{R}\,\mathrm{fold}^\Pi \circ \mathbb{V}^{n+1}_{\mathbb{Z}} \cong \mathbb{V}^{n+1}_{\mathbb{Z}_2} \circ \mathbf{R}\,\mathrm{fold}^\Pi : \ \mathbf{D}^{\mathrm{ctr}}\,\mathrm{MC}(\widehat{\mathbb{A}}^{n+1}_{\Bbbk}, \widehat{w}_n) \to \mathbf{D}^{\mathrm{ctr}}\,\mathrm{LF}(\widehat{\mathbb{A}}^{n}_{\Bbbk}, \widehat{w}_n).$$

Proof. For Tr, this follows from Proposition I.4.4.8, while for $\mathbb{V}$ it holds since $\mathbf{R}\,\mathrm{fold}^{\Pi}$ and $\mathbb{V}$ are naively computable (Remarks I.6.3.4, I.6.4.2 and Proposition I.4.4.1). □

Proposition I.6.4.4. *There are distinguished triangles in* $\mathbf{D}^{\mathrm{ctr}}\,\mathrm{LF}(\widehat{\mathbb{A}}^{n+1}_{\Bbbk}, \widehat{w}_n)$

$$\mathrm{Tr}^{n+1}_{\mathbb{Z}_2} X\langle -2\rangle \to \mathrm{Tr}^{n+1}_{\mathbb{Z}_2} B \to \mathrm{Tr}^{n+1}_{\mathbb{Z}_2} \Delta \to \Sigma\,\mathrm{Tr}^{n+1}_{\mathbb{Z}_2} X\langle -2\rangle$$
$$\mathrm{Tr}^{n+1}_{\mathbb{Z}_2} \Delta\langle -2\rangle \to \mathrm{Tr}^{n+1}_{\mathbb{Z}_2} B \to \mathrm{Tr}^{n+1}_{\mathbb{Z}_2} X \to \Sigma\,\mathrm{Tr}^{n+1}_{\mathbb{Z}_2} \Delta\langle -2\rangle.$$

which split after applying $\mathbb{V}^{n+1}_{\mathbb{Z}_2} : \mathbf{D}^{\mathrm{ctr}}\,\mathrm{LF}(\widehat{\mathbb{A}}^{n+1}_{\Bbbk}, \widehat{w}_n) \to \mathbf{D}^{\mathrm{ctr}}\,\mathrm{LF}(\widehat{\mathbb{A}}^{n}_{\Bbbk}, \widehat{w}_n)$*:*

(i) $\mathbb{V}^{n+1}_{\mathbb{Z}_2}$ *annihilates* $\mathrm{Tr}^{n+1}_{\mathbb{Z}_2} X\langle -2\rangle \to \mathrm{Tr}^{n+1}_{\mathbb{Z}_2} B$*.*

(ii) $\mathbb{V}^{n+1}_{\mathbb{Z}_2}$ *annihilates* $\mathrm{Tr}^{n+1}_{\mathbb{Z}_2} B \to \mathrm{Tr}^{n+1}_{\mathbb{Z}_2} X$*.*

Strengthening (ii), there is a canonical commutative diagram in $\mathbf{D}^{\mathrm{ctr}}\,\mathrm{LF}(\widehat{\mathbb{A}}^{n}_{\Bbbk}, \widehat{w}_n)$

$$\begin{array}{ccc}
\mathbb{V}^{n+1}_{\mathbb{Z}_2}\mathrm{Tr}^{n+1}_{\mathbb{Z}_2} X & \longrightarrow & \Sigma\mathbb{V}^{n+1}_{\mathbb{Z}_2}\mathrm{Tr}^{n+1}_{\mathbb{Z}_2} \Delta\langle -2\rangle \\
\cong\downarrow & & \downarrow\cong \\
{}_{\widehat{w}_n}\Delta^n_{\Bbbk} & \xrightarrow{(k+1)\sum_{i=0}^{k-1} x_{n+1}^i x_n^{k-1-i}} & \Bbbk[x_{n+1}]/(x_{n+1}^k)\otimes_{\Bbbk} {}_{\widehat{w}_n}\Delta^n_{\Bbbk}\langle 2k-2\rangle
\end{array} \tag{I.6.4.19}$$

in which the induced external actions (see Definition I.2.1.12) of x_{n+1} *on the lower row are given by the external action of* x_n *in case of* ${}_{\widehat{w}_n}\Delta^n_{\Bbbk}$ *and by the multiplication by* x_{n+1} *on the tensor factor* $\Bbbk[x_{n+1}]/(x_{n+1}^k)$ *in case of* $\Bbbk[x_{n+1}]/(x_{n+1}^k)\otimes_{\Bbbk} {}_{\widehat{w}_n}\Delta^n_{\Bbbk}\langle 2k-2\rangle$*.*

Remark I.6.4.5. We are referring to the *canonical* connecting morphisms $\Delta \to \Sigma X\langle -2\rangle$ and $X \to \Sigma\Delta\langle -2\rangle$ induced by the short exact sequences from Fact I.6.3.7 here. ◊

Proof of Proposition I.6.4.4. The existence of the triangles as well as the vanishing of $\mathrm{Tr}^{n+1}_{\mathbb{Z}_2} X\langle -2\rangle \to \mathrm{Tr}^{n+1}_{\mathbb{Z}_2} B$ under $\mathbb{V}^{n+1}_{\mathbb{Z}_2}$ follow from Propositions I.6.3.8 and I.6.4.3.

The vanishing of $\mathrm{Tr}^{n+1}_{\mathbb{Z}_2} B \to \mathrm{Tr}^{n+1}_{\mathbb{Z}_2} X$ under $\mathbb{V}^{n+1}_{\mathbb{Z}_2}$ follows from the commutative diagram (I.6.4.19) since its lower row is a split monomorphism and the composition of any two consecutive arrows in a distinguished triangle is zero. For the construction of (I.6.4.19), we again restrict to the essential case $n = 1$ and explain the left and right vertical isomorphisms first. Concerning the left one, since X is free over $\widehat{\mathbb{A}}^{+}_{\Bbbk}$ the factorization $\mathrm{Tr}^2_{\mathbb{Z}_2} X = {}_{-\widehat{w}_+}\Delta^+ \otimes^{\mathbf{L}}_{\widehat{\mathbb{A}}^+_{\Bbbk}} {}_{\widehat{w}_2}X$ can be computed naively (apply Propositions I.4.4.8 and I.4.3.6), so $\mathrm{Tr}^2_{\mathbb{Z}_2} X = \widehat{\mathbb{A}}^2_{\Bbbk}/(x_1 = x_2 = y_1 = y_2)$. To identify the right hand side in (I.6.4.19) we compute $\mathrm{Tr}^2_{\mathbb{Z}_2}\Delta$ through the resolution $\Bbbk[x_2, y_2]\langle -2\rangle \underset{u_k(x_2,y_2)}{\overset{y_2-x_2}{\rightleftarrows}} \Bbbk[x_2, y_2]$ of ${}_{-\widehat{w}_+}\Delta^+$ as above (justified by Proposition I.6.4.3),

$$\mathbb{V}^2_{\mathbb{Z}_2}\mathrm{Tr}^2_{\mathbb{Z}_2}\Delta \;\cong\; \Delta\langle -2\rangle \underset{(k+1)x_2^k}{\overset{0}{\rightleftarrows}} \Delta \tag{I.6.4.20}$$
$$\xrightarrow{\simeq}\; \Bbbk[x_2]/(x_2^k)\otimes_{\Bbbk}\Delta^1_{\Bbbk}\langle -2\rangle \underset{0}{\overset{0}{\rightleftarrows}} 0$$

where the last map is a contraderived weak equivalence since it is an epimorphism with contractible kernel. Shifting the q-grading by -2 and applying Σ (which also involves a shift in q-grading by $2(k+1)$) yields the desired second vertical isomorphism

$$\Sigma \mathbb{V}^2_{\mathbb{Z}_2} \operatorname{Tr}^2_{\mathbb{Z}_2} \Delta\langle -2\rangle \cong \Bbbk[x_2]/(x_2^k) \otimes_{\Bbbk\ \widehat{w}_1} \Delta^1_{\Bbbk}\langle 2k-2\rangle \cong \Delta^1_{\Bbbk} \oplus \Delta^1_{\Bbbk}\langle 2\rangle \oplus \cdots \oplus \Delta^1_{\Bbbk}\langle 2(k-1)\rangle.$$

We now turn to the proof of commutativity of (I.6.4.19). For this, recall from Corollary II.C.1.5 that $X \to \Sigma\Delta\langle -2\rangle$ is given by the roof

$$\begin{array}{ccccc} & & \Delta\langle -2\rangle & & \\ & \swarrow & {\scriptstyle 0}\upharpoonleft\downharpoonright{\scriptstyle x_1-y_2} & \searrow & \\ 0 & & & & \Delta\langle -2\rangle \\ {\scriptstyle 0}\upharpoonleft\downharpoonright{\scriptstyle 0} & & B & & {\scriptstyle 0}\upharpoonleft\downharpoonright{\scriptstyle 0} \\ & \swarrow & & \searrow & \\ X & & & & 0 \end{array}$$

Checking the commutativity of (I.6.4.19) therefore unravels to show that for any $0 \le i \le k-1$ certain two maps of the the following type coincide in $\mathbf{D}^{\mathrm{ctr}}\operatorname{LF}(\widehat{\mathbb{A}}^1_{\Bbbk}, \widehat{w}_1)$:

$$\operatorname{Cone}\left(\begin{array}{ccc} \Delta\langle -4\rangle & \underset{(k+1)x_2^k}{\overset{0}{\rightleftarrows}} & \Delta\langle -2\rangle \\ \Big\downarrow{\scriptstyle x_1-y_2} & & \Big\downarrow{\scriptstyle x_1-y_2} \\ B\langle -2\rangle & \underset{u_k(x_2,y_2)}{\overset{y_2-x_2}{\rightleftarrows}} & B \end{array}\right) \longrightarrow \left(0 \underset{0}{\overset{0}{\rightleftarrows}} \Delta^1_{\Bbbk}\langle 2i\rangle\right) \qquad \text{(I.6.4.21)}$$

Firstly, the composition in (I.6.4.19) along the lower left corner is given by projecting onto the lower right summand B in (I.6.4.21) and composing with the canonical map

$$B \to X \to \widehat{\mathbb{A}}^2_{\Bbbk}/(x_1 = x_2 = y_1 = y_2) = \Delta^1_{\Bbbk} \xrightarrow{(k+1)x_1^i} \Delta^1_{\Bbbk}\langle 2i\rangle.$$

Secondly, the composition in (I.6.4.19) along the upper right corner is given by projecting first onto the upper left summand $\Delta\langle 2k-2\rangle$ in (I.6.4.21) (note the shift in q-degree involved in taking cones) and then onto $\Delta^1_{\Bbbk}\langle 2i\rangle$ by picking out the coefficient of $x_2^{k-1-i} = y_2^{k-1-i}$ of an element of $\Delta = \widehat{\mathbb{A}}^2_{\Bbbk}/(x_1 - y_1, x_2 - y_2)$.

We claim that a homotopy H between these two maps is given by projection onto the lower left component $B\langle -2\rangle$ in (I.6.4.21) and by composing it with the following $\widehat{\mathbb{A}}^1_{\Bbbk}$-linear map $B\langle -2\rangle \xrightarrow{\sigma} \Delta\langle -4\rangle \to \Delta^1_{\Bbbk}\langle 2i\rangle$: The second factor $\Delta\langle -4\rangle \to \Delta^1_{\Bbbk}\langle 2i\rangle$ is again the map picking the coefficient of $x_2^{k-1-i} = y_2^{k-1-i}$ (considered as a map $\Delta \to \Delta^1_{\Bbbk}$, this has degree $-2(k-1-i)$, so it has degree $-2(k-1-i)-4-2i = -(2k+2)$ as a map $\Delta\langle -4\rangle \to \Delta^1_{\Bbbk}$, as required for a homotopy). The first factor $\sigma : B\langle -2\rangle \to \Delta\langle -4\rangle$ is the $\widehat{\mathbb{A}}^1_{\Bbbk}$-linear splitting of $\Delta\langle -4\rangle \to B\langle -2\rangle$, $1 \mapsto x_1 - y_2$, that corresponds via the short exact sequence $0 \to \Delta\langle -2\rangle \to B \to X \to 0$ to the $\widehat{\mathbb{A}}^1_{\Bbbk}$-linear splitting $1 \mapsto 1$ of the projection $B \to X$ (note that X is free of rank 1 over $\widehat{\mathbb{A}}^1_{\Bbbk}$). It follows from the identity $u_k(x_2, y_2) = -(k+1)(x_1 - y_2)\sum_j x_1^j x_2^{k-1-j} + u_k(x_1, y_1)$ in B, established in Lemma I.6.3.10, that H indeed is a homotopy we require. □

With Proposition I.6.4.4 at hand we can now finish the study of the second Markov move. First, recall from Section I.6.3 the functor

$$\mathrm{M}^D_{\mathbb{Z}} \equiv {}_{-\widehat{w}_{n+1}}\Delta \overset{\mathbf{L}}{\underset{\widehat{\mathbb{A}}^{n+1}_{\Bbbk}}{\otimes}} {}_{\widehat{w}_{n+1}} \left[D \underset{\mathbb{A}^{n+1}_{\Bbbk}}{\otimes} \left(\Delta^+ \underset{\Bbbk}{\otimes} - \right) \right] : \widehat{\mathbb{A}}^n_{\Bbbk}/(\widehat{w}_n)\text{-Mod} \to \mathbf{D}^{\mathrm{ctr}}\,\mathrm{MC}(\widehat{\mathbb{A}}^{n+1}_{\Bbbk}, 0)$$

associated to an $\widehat{\mathbb{A}}^{n+1}_{\Bbbk}/(\widehat{w}_{n+1})$-module D. It arose naturally from the study of the second Markov move, and in case D is free as a right $\mathbb{A}^{n+1}_{\Bbbk}$-module Corollary I.6.3.5 allowed for a description of $\mathrm{M}^D_{\mathbb{Z}}$ in terms of the partial trace functor $\mathrm{Tr}^D_{\mathbb{Z}}$:

$$\mathrm{M}^D_{\mathbb{Z}} \;\cong\; {}_{-\widehat{w}_n}\Delta^n_{\Bbbk} \overset{\mathbf{L}}{\underset{\widehat{\mathbb{A}}^n_{\Bbbk}}{\otimes}} \left[\mathrm{Tr}^{n+1}_{\mathbb{Z}} \; {}_{\widehat{w}_{n+1}} D \overset{\mathbf{L}}{\underset{\mathbb{A}^n_{\Bbbk}}{\otimes}} {}_{\widehat{w}_n} - \right] \tag{I.6.4.22}$$

Passing to the $\mathbb{Z}_2$-graded setting by applying $\mathbf{R}\,\mathrm{fold}^{\Pi}$ we obtain a functor

$$\mathrm{M}^D_{\mathbb{Z}_2} \;:=\; \mathbf{R}\,\mathrm{fold}^{\Pi} \circ \mathrm{M}^D_{\mathbb{Z}} : \; \widehat{\mathbb{A}}^n_{\Bbbk}/(\widehat{w}_n)\text{-Mod} \to \mathbf{D}^{\mathrm{ctr}}\,\mathrm{LF}(\widehat{\mathbb{A}}^{n+1}_{\Bbbk}, 0)$$

which in the cases we are interested in admits a description in terms of the $\mathbb{Z}/2\mathbb{Z}$-graded partial trace functor $\mathrm{Tr}_{\mathbb{Z}_2}$ analogous to (I.6.4.22) from Corollary I.6.3.5:

Lemma I.6.4.6. *For any $D \in \widehat{\mathbb{A}}^{n+1}_{\Bbbk}/(\widehat{w}_{n+1})$-Mod there is a canonical isomorphism*

$$\mathrm{M}^D_{\mathbb{Z}_2} \equiv \mathbf{R}\,\mathrm{fold}^{\Pi} \circ \mathrm{M}^D_{\mathbb{Z}} \;\cong\; {}_{-\widehat{w}_{n+1}}\Delta_{n+1} \overset{\mathbf{L}}{\underset{\widehat{\mathbb{A}}^{n+1}_{\Bbbk}}{\otimes}} {}_{\widehat{w}_{n+1}} \left[D \underset{\mathbb{A}^{n+1}_{\Bbbk}}{\otimes} \left(\Delta^+ \underset{\Bbbk}{\otimes} - \right) \right]. \tag{I.6.4.23}$$

If moreover D is a free right $\mathbb{A}^{n+1}_{\Bbbk}$-module and of finite projective dimension over $\widehat{\mathbb{A}}^{n+1}_{\Bbbk}$, and if Q is an $\widehat{\mathbb{A}}^n_{\Bbbk}/(\widehat{w}_n)$-module and of finite projective dimension over $\widehat{\mathbb{A}}^n_{\Bbbk}$, then

$$\mathrm{M}^D_{\mathbb{Z}_2} Q \cong {}_{-\widehat{w}_n}\Delta^n_{\Bbbk} \overset{\mathbf{L}}{\underset{\widehat{\mathbb{A}}^n_{\Bbbk}}{\otimes}} \left[\mathrm{Tr}^{n+1}_{\mathbb{Z}_2} \; {}_{\widehat{w}_{n+1}} D \overset{\mathbf{L}}{\underset{\mathbb{A}^n_{\Bbbk}}{\otimes}} Q \right]. \tag{I.6.4.24}$$

Proof. The isomorphism (I.6.4.23) follows from Proposition I.4.5.5. For (I.6.4.24), note that since Q is of finite projective dimension over $\widehat{\mathbb{A}}^n_{\Bbbk}$ we have ${}_{\widehat{w}_n}Q \in \mathbf{D}^b_{\mathrm{f}}(\widehat{\mathbb{A}}^n_{\Bbbk}, \widehat{w}_n)$ by Corollary I.4.3.4, and analogously ${}_{\widehat{w}_{n+1}}D \in \mathbf{D}^b_{\mathrm{f}}(\widehat{\mathbb{A}}^{n+1}_{\Bbbk}, \widehat{w}_{n+1})$. Since the contraderived tensor product (and in particular the partial trace functor $\mathrm{Tr}_{\mathbb{Z}}$) preserves the decorated categories $\mathbf{D}^b_{\mathrm{f}}(?,?)$ by (I.4.2) and furthermore commutes with $\mathbf{R}\,\mathrm{fold}^{\Pi}$ when restricted to these categories (Proposition I.4.4.8), applying $\mathbf{R}\,\mathrm{fold}^{\Pi}$ to (I.6.4.22) and using $\mathbf{R}\,\mathrm{fold}^{\Pi} \circ \mathrm{Tr}^{n+1}_{\mathbb{Z}} \cong \mathrm{Tr}^{n+1}_{\mathbb{Z}_2}$ (Proposition I.6.4.3) gives (I.6.4.24). □

Corollary I.6.4.7. *For $M \in \widehat{\mathbb{A}}^n_{\Bbbk}/(\widehat{w}_n)$-Mod of finite projective dimension over $\widehat{\mathbb{A}}^n_{\Bbbk}$, there are natural distinguished triangles in $\mathbf{D}^{\mathrm{ctr}}\,\mathrm{LF}(\widehat{\mathbb{A}}^{n+1}_{\Bbbk}, 0)$*

$$\begin{aligned}
&\mathrm{M}^X_{\mathbb{Z}_2} M\langle -2\rangle \to \mathrm{M}^B_{\mathbb{Z}_2} M \to \mathrm{M}^\Delta_{\mathbb{Z}_2} M \to \Sigma\, \mathrm{M}^X_{\mathbb{Z}_2} M\langle -2\rangle,\\
&\mathrm{M}^\Delta_{\mathbb{Z}_2} M\langle -2\rangle \to \mathrm{M}^B_{\mathbb{Z}_2} M \to \mathrm{M}^X_{\mathbb{Z}_2} M \to \Sigma\, \mathrm{M}^\Delta_{\mathbb{Z}_2} M\langle -2\rangle.
\end{aligned}$$

which split after applying $\mathbb{V}^{n+1}_{\mathbb{Z}_2} : \mathbf{D}^{\mathrm{ctr}}\,\mathrm{LF}(\widehat{\mathbb{A}}^{n+1}_{\Bbbk}, \widehat{w}_n) \to \mathbf{D}^{\mathrm{ctr}}\,\mathrm{LF}(\widehat{\mathbb{A}}^{n}_{\Bbbk}, \widehat{w}_n)$:

(i) *$\mathbb{V}^{n+1}_{\mathbb{Z}_2}$ annihilates $\mathrm{M}^X_{\mathbb{Z}_2} M\langle -2\rangle \to \mathrm{M}^B_{\mathbb{Z}_2} M$.*

(ii) *$\mathbb{V}^{n+1}_{\mathbb{Z}_2}$ annihilates $\mathrm{M}^B_{\mathbb{Z}_2} M \to \mathrm{M}^X_{\mathbb{Z}_2} M$.*

Strengthening (ii), there is a commutative diagram in $\mathbf{D}^{\mathrm{ctr}}\,\mathrm{LF}(\widehat{\mathbb{A}}^{n}_{\Bbbk}, 0)$

$$\begin{array}{ccc}
\mathbb{V}^{n+1}_{\mathbb{Z}_2} \mathrm{M}^X_{\mathbb{Z}_2} M & \longrightarrow & \Sigma\mathbb{V}^{n+1}_{\mathbb{Z}_2} \mathrm{M}^\Delta_{\mathbb{Z}_2} M\langle -2\rangle \\
\cong\downarrow & & \downarrow\cong \\
-\widehat{w}_n \Delta^n_{\Bbbk} \otimes^{\mathbf{L}}_{\widehat{\mathbb{A}}^n_{\Bbbk}} \widehat{w}_n M & \xrightarrow{(k+1)\sum_{i=0}^{k-1} x^i_{n+1} x_n^{k-1-i}} & \Bbbk[x_{n+1}]/(x^k_{n+1}) \otimes_{\Bbbk} \left(-\widehat{w}_n \Delta^n_{\Bbbk} \otimes^{\mathbf{L}}_{\widehat{\mathbb{A}}^n_{\Bbbk}} \widehat{w}_n M\right)\langle 2k-2\rangle
\end{array}$$

in which the induced external actions of x_{n+1} are as in Proposition I.6.4.4.

Proof. This follows from Proposition I.6.4.4 and Lemma I.6.4.6. □

Composing $\mathrm{M}^D_{\mathbb{Z}_2}$ with $\mathrm{H}^* : \mathbf{D}^{\mathrm{ctr}}\,\mathrm{LF}(\widehat{\mathbb{A}}^{n+1}_{\Bbbk}, 0) \to \widehat{\mathbb{A}}^{n+1}_{\Bbbk}$-Mod we get functors $\mathrm{T}^D_* : \widehat{\mathbb{A}}^n_{\Bbbk}$-Mod $\to \widehat{\mathbb{A}}^{n+1}_{\Bbbk}$-Mod; they actually take values in symmetric $\mathbb{A}^{n+1}_{\Bbbk}$-bimodules, so we may consider them as functors $\mathrm{T}^D_* : \widehat{\mathbb{A}}^n_{\Bbbk}/(\widehat{w}_n)$-Mod $\to \mathbb{A}^{n+1}_{\Bbbk}$-Mod.

Proposition I.6.4.8. *Let Q be an $\widehat{\mathbb{A}}^n_{\Bbbk}/(\widehat{w}_n)$-module which is of finite projective dimension over $\widehat{\mathbb{A}}^n_{\Bbbk}$. Then there are exact sequences of $\mathbb{A}^{n+1}_{\Bbbk}$-modules, splitting over $\mathbb{A}^n_{\Bbbk}$-Mod:*

$$0 \to \mathrm{T}^B_* Q \to \mathrm{T}^\Delta_* Q \to \mathrm{T}^X_{*+1} Q\langle -2\rangle \to 0 \tag{I.6.4.25}$$

$$0 \to \mathrm{T}^X_* Q \to \mathrm{T}^\Delta_{*+1} Q\langle -2\rangle \to \mathrm{T}^B_{*+1} Q \to 0 \tag{I.6.4.26}$$

Moreover, there are natural isomorphisms

$$\mathrm{T}^X_* Q \cong (\rho^* \circ {}^k\mathrm{sHH}^{\mathbb{A}^n_{\Bbbk}/\Bbbk}_*)(Q) \quad \textit{and} \quad \mathrm{T}^\Delta_* Q \cong \Bbbk[x_{n+1}]/(x^k_{n+1}) \otimes_{\Bbbk} {}^k\mathrm{sHH}^{\mathbb{A}^n_{\Bbbk}/\Bbbk}_{*+1}(Q)\langle -2\rangle$$

fitting into a commutative diagram

$$\begin{array}{ccc}
\mathrm{T}^X_* Q & \longrightarrow & \mathrm{T}^\Delta_{*+1} Q\langle -2\rangle \\
\cong\downarrow & & \downarrow\cong \\
(\rho^* \circ {}^k\mathrm{sHH}^{\mathbb{A}^n_{\Bbbk}/\Bbbk}_*)(Q) & \xrightarrow{(k+1)\sum_{i=0}^{k-1} x^i_{n+1} x_n^{k-1-i}} & \Bbbk[x_{n+1}]/(x^k_{n+1}) \otimes_{\Bbbk} {}^k\mathrm{sHH}^{\mathbb{A}^n_{\Bbbk}/\Bbbk}_*(Q)\langle 2k-2\rangle,
\end{array}$$

where $\rho_n : \mathbb{A}^{n+1}_{\Bbbk} \twoheadrightarrow \mathbb{A}^n_{\Bbbk}$ is given by $x_{n+1} \mapsto x_n$ and $x_i \mapsto x_i$ for $i \le n$. Note that in the second vertical isomorphism we used ${}^k\mathrm{sHH}^{\mathbb{A}^n_{\Bbbk}/\Bbbk}_{*+2} = {}^k\mathrm{sHH}^{\mathbb{A}^n_{\Bbbk}/\Bbbk}_*\langle 2k+2\rangle$.

Proof. This follows from Corollary I.6.4.7 by applying H^*. □

Corollary I.6.4.9. *For any Soergel bimodule $M \in \mathcal{SBM}_{\Bbbk}(n)$ on n strands over $\Bbbk$, the total k-stable Hochschild homology ${}^k\,\mathrm{sHH}_t^{\mathbb{A}^n_{\Bbbk}/\Bbbk}(M)$ is free of finite rank over $\Bbbk$.*

Proof. We apply the induction principle for Soergel bimodules (Proposition I.A.2.6) to the property expressing that the total k-stable Hochschild homology ${}^k\,\mathrm{sHH}_t^{\mathbb{A}^n_{\Bbbk}/\Bbbk}(M)$ of a Soergel bimodule M on n strands is free of finite rank over $\Bbbk$: The validity of properties (i), (ii) and (iv) from the induction principle is clear, while properties (iii) and (v) follow from Propositions I.6.2.1 and I.6.4.8 on the effect of first and second Markov move on k-stable Hochschild homology, respectively. □

Corollary I.6.4.10. *For any homomorphism $\Bbbk \to \Bbbk'$ of commutative $\mathbb{Z}[\frac{1}{k+1}]$-algebras and any Soergel bimodule $M \in \mathcal{SBM}_{\Bbbk}(n)$ over $\Bbbk$, the canonical base change homomorphism*

$$ {}^k\,\mathrm{sHH}_t^{\mathbb{A}^n_{\Bbbk}/\Bbbk}(M) \otimes_{\Bbbk} \Bbbk' \longrightarrow {}^k\,\mathrm{sHH}_t^{\mathbb{A}^n_{\Bbbk'}/\Bbbk'}(M \otimes_{\Bbbk} \Bbbk') $$

is an isomorphism.

Proof. It suffices to treat the case $\Bbbk = \mathbb{Z}[\frac{1}{k+1}]$. For this, recall from Section I.4.5 that ${}^k\,\mathrm{sHH}_t^{\mathbb{A}^n_{\Bbbk}/\Bbbk}(M)$ can be computed as the total cyclic homology of the bounded mixed complex $C := P_\Delta \otimes_{\widehat{\mathbb{A}}^n_{\Bbbk}\ \widehat{w}_n} M$, where P_Δ is the Koszul resolution of the diagonal $\mathbb{A}^n_{\Bbbk}$-bimodule Δ as a curved mixed complex of type $(\widehat{\mathbb{A}}^n_{\Bbbk}, -\widehat{w}_n)$. Similarly, ${}^k\,\mathrm{sHH}_t^{\mathbb{A}^n_{\Bbbk'}/\Bbbk'}(M \otimes_{\Bbbk} \Bbbk')$ is computed as the cyclic cohomology of $C \otimes_{\Bbbk} \Bbbk'$, and the base change morphism under consideration is the canonical morphism $\mathrm{HC}(C) \otimes_{\Bbbk} \Bbbk' \to \mathrm{HC}(C \otimes_{\Bbbk} \Bbbk')$. Since M is free over $\Bbbk$, so are all components of C, and since $\Bbbk = \mathbb{Z}[\frac{1}{k+1}]$ is a principal ideal domain, this is inherited to the $(\mathrm{d}+s)$-boundaries. Finally, Corollary I.6.4.9 shows that also $\mathrm{HC}(C)$ is free over $\Bbbk$, so Lemma I.6.4.11 below gives the claim (viewing $\mathrm{HC}(C)$ as the cohomology of a 2-periodic complex over $\Bbbk$). □

Lemma I.6.4.11. *Let $\Bbbk$ be a commutative ring and (C^*, δ^*) be a chain complex over $\Bbbk$ such that C^k, $B^k = \mathrm{im}(\delta^{k-1})$ and $\mathrm{H}^k\,C$ are projective for all k. Then for any $\Bbbk$-module M the morphism $(\mathrm{H}^k\,C) \otimes_R M \to \mathrm{H}^k(C \otimes_R M)$ is an isomorphism.*

Corollary I.6.4.12. *For a homomorphism $\Bbbk \to \Bbbk'$ of commutative $\mathbb{Z}[\frac{1}{k+1}]$-algebras and an n-strand braid word β, there is a canonical base change isomorphism in $\mathrm{Ch}^b(\widehat{\mathbb{A}}^n_{\Bbbk}\text{-Mod})$:*

$$ \mathcal{CKR}^k_{\Bbbk}(\beta) \otimes_{\Bbbk} \Bbbk' \xrightarrow{\ \cong\ } \mathcal{CKR}^k_{\Bbbk'}(\beta) \qquad \text{(I.6.4.27)} $$

Proof. This follows from Corollaries I.6.4.9 and I.6.4.10. □

For the next corollary, we need the notion of a relative derived category:

Definition I.6.4.13. *Let A/B be an extension of rings. The* relative derived category $\mathbf{D}_B(A)$ *is the localization of the category* $\mathrm{Ch}(A)$ *of chain complexes over* A *at the class of those morphisms that map to homotopy equivalences under the forgetful functor* $\mathrm{Ch}(A) \to \mathrm{Ch}(B)$.

By definition, the forgetful functor $\mathrm{Ch}(A) \to \mathrm{Ch}(B)$ gives rise to a well-defined functor $\mathbf{D}_B(A) \to \mathrm{Ho}(B)$. Further, if $0 \to X \to Y \to Z \to 0$ is a short exact sequence in $\mathrm{Ch}(A)$ which splits over B, then the canonical morphisms $X \to \mathrm{Cone}(Y \to Z)$ and $\mathrm{Cone}(X \to Y) \to Z$ are homotopy equivalences over B, hence isomorphisms in $\mathbf{D}_B(A)$. This is used in the proof of the following corollary:

Corollary I.6.4.14. *For a complex S of $\widehat{\mathbb{A}}^n_{\Bbbk}/(\widehat{w}_n)$-modules whose components are of finite projective dimension over $\widehat{\mathbb{A}}^n_{\Bbbk}$ there are canonical isomorphisms in $\mathbf{D}_{\mathbb{A}^n_{\Bbbk}}(\mathbb{A}^{n+1}_{\Bbbk})$:*

$$ {}^k\,\mathrm{sHH}_*^{\mathbb{A}^{n+1}_{\Bbbk}/\Bbbk}\left[F_+^{n+1,n} \otimes_{\mathbb{A}^{n+1}_{\Bbbk}} \left(\Delta^+ \otimes_{\Bbbk} S\right)\right] \cong \rho_n^* \circ \Sigma^{-1}\left({}^k\,\mathrm{sHH}_{*+1}^{\mathbb{A}^n_{\Bbbk}/\Bbbk}\, S\right), \tag{I.6.4.28}$$

$$ {}^k\,\mathrm{sHH}_*^{\mathbb{A}^{n+1}_{\Bbbk}/\Bbbk}\left[F_-^{n+1,n} \otimes_{\mathbb{A}^{n+1}_{\Bbbk}} \left(\Delta^+ \otimes_{\Bbbk} S\right)\right] \cong \rho_n^* \circ \Sigma\left({}^k\,\mathrm{sHH}_{*-1}^{\mathbb{A}^n_{\Bbbk}/\Bbbk}\, S\right) \tag{I.6.4.29}$$

Here ρ_n is as in Proposition I.6.4.8.

Proof. By (I.6.3.11) we have

$$ {}^k\,\mathrm{sHH}_*^{\mathbb{A}^{n+1}_{\Bbbk}/\Bbbk}\left[F_+^{n+1,n} \otimes_{\mathbb{A}^{n+1}_{\Bbbk}} (\Bbbk[x_{n+1}] \otimes_{\Bbbk} S)\right] \cong \Sigma^{-1}\,\mathrm{Cone}\left[\mathrm{T}_*^B\, S \to \mathrm{T}_*^{\Delta}\, S\right]\langle 2\rangle,$$

so (I.6.4.28) follows from Proposition I.6.4.8 and the paragraph preceding this corollary. Similarly, we have $F_-^{n+1,n} = \mathrm{Cone}\,(\underline{\Delta}\langle -2\rangle \to \underline{B})$, so

$$ {}^k\,\mathrm{sHH}_*^{\mathbb{A}^{n+1}_{\Bbbk}/\Bbbk}\left[F_-^{n+1,n} \otimes_{\mathbb{A}^{n+1}_{\Bbbk}} (\Bbbk[x_{n+1}] \otimes_{\Bbbk} S)\right] \cong \mathrm{Cone}\left[\mathrm{T}_*^{\Delta}\, S\langle -2\rangle \to \mathrm{T}_*^B\, S\right],$$

which together with (I.6.4.26) gives (I.6.4.29). □

Proof of Theorems I.6.1.1 and I.6.1.3. We consider the statement of Theorem I.6.1.1 first. The invariance of (I.6.1.1) under the two Markov moves (up to isomorphism in $\mathrm{Ho}^b(\Bbbk\text{-Mod})$) follows from Proposition I.6.2.1 and Corollary I.6.4.14, noting that the writhe $w(\beta)$ increases resp. decreases by 1 under a positive resp. negative second Markov move; recall also that $\mathrm{H}^t \circ \Sigma = \mathrm{H}^t\langle k+1\rangle$. The claim that the components of (I.6.1.1) are $\Bbbk$-free of finite rank was proved in Corollary I.6.4.9.

The slight strengthening provided by Theorem I.6.1.3 (taking into account the action of $\mathbb{A}^n_{\Bbbk}$ in case of ordered links) follows from Corollary I.6.4.14 and Remark I.6.2.3. □

I.6.5. The degenerate case: $k+1=0$ in $\Bbbk$

Next we treat the case $k+1=0$ in $\Bbbk$. Here, the behavior of the sequence (I.6.3.15) in Proposition I.6.3.8 under application of $\mathbb{V}_{\mathbb{Z}}^{n+1}$ differs from the one described for $k+1 \in \Bbbk^{\times}$ in Proposition I.6.4.4. Still, a slight variation of Proposition I.6.4.4 is true (Proposition I.6.5.1) and is even provable on the $\mathbb{Z}$-graded level of curved mixed complexes where it results in Rouquier's Theorem [Rou12, Theorem 4.9] stating that ordinary Hochschild homology of Soergel bimodules is a (triply graded) invariant of oriented links. Passing to the $\mathbb{Z}/2\mathbb{Z}$-graded level, we again obtain that doubly graded Khovanov-Rozansky homology is an invariant of oriented links, this time however with a different normalization, matching the one in Rouquier's Theorem. This observation is explained in Proposition I.6.5.7 and Corollary I.6.5.8 where we show that for $k+1=0$ in $\Bbbk$ and any Soergel bimodule M, the spectral sequence from ordinary to k-stable Hochschild homology of M degenerates on the E_1-page, so that the filtration quotients of the natural filtration of k-stable Hochschild homology of M are canonically isomorphic to ordinary Hochschild homology. In particular, this shows that for any prime p the triply graded Khovanov-Rozansky homology $\mathcal{KR}$ can be described as doubly graded Khovanov-Rozansky homology $\mathcal{KR}^p_{\mathbb{F}_p}$, defined via matrix factorizations over $\mathbb{F}_p$ with potential $\sum_i x_i^p$.

We keep Notations I.6.3.1 and I.6.3.6.

Proposition I.6.5.1. *There are distinguished triangles in* $\mathbf{D}^{\mathrm{ctr}}\,\mathrm{MC}(\widehat{\mathbb{A}}^{n+1}_{\Bbbk}, \widehat{w}_n)$

$$\begin{gathered}\mathrm{Tr}^{n+1}_{\mathbb{Z}} X\langle -2\rangle \to \mathrm{Tr}^{n+1}_{\mathbb{Z}} B \to \mathrm{Tr}^{n+1}_{\mathbb{Z}} \Delta \to \Sigma\ \mathrm{Tr}^{n+1}_{\mathbb{Z}} X\langle -2\rangle \\ \mathrm{Tr}^{n+1}_{\mathbb{Z}} \Delta\langle -2\rangle \to \mathrm{Tr}^{n+1}_{\mathbb{Z}} B \to \mathrm{Tr}^{n+1}_{\mathbb{Z}} X \to \Sigma\ \mathrm{Tr}^{n+1}_{\mathbb{Z}} \Delta\langle -2\rangle.\end{gathered} \tag{I.6.5.30}$$

which split after applying $\mathbb{V}^{n+1}_{\mathbb{Z}} : \mathbf{D}^{\mathrm{ctr}}\,\mathrm{MC}(\widehat{\mathbb{A}}^{n+1}_{\Bbbk}, \widehat{w}_n) \to \mathbf{D}^{\mathrm{ctr}}\,\mathrm{MC}(\widehat{\mathbb{A}}^{n}_{\Bbbk}, \widehat{w}_n)$*:*

(i) $\mathbb{V}^{n+1}_{\mathbb{Z}}$ *annihilates* $\mathrm{Tr}^{n+1}_{\mathbb{Z}} X\langle -2\rangle \to \mathrm{Tr}^{n+1}_{\mathbb{Z}} B$*.*

(ii) $\mathbb{V}^{n+1}_{\mathbb{Z}}$ *annihilates* $\mathrm{Tr}^{n+1}_{\mathbb{Z}} X \to \Sigma\ \mathrm{Tr}^{n+1}_{\mathbb{Z}} \Delta\langle -2\rangle$*.*

Proof. The existence of the triangles as well as the vanishing of $\mathrm{Tr}^{n+1}_{\mathbb{Z}} X\langle -2\rangle \to \mathrm{Tr}^{n+1}_{\mathbb{Z}} B$ under $\mathbb{V}^{n+1}_{\mathbb{Z}}$ were already proved in Proposition I.6.3.8. It therefore suffices to prove the vanishing of $\mathrm{Tr}^{n+1}_{\mathbb{Z}} X \to \Sigma\,\mathrm{Tr}^{n+1}_{\mathbb{Z}} \Delta\langle -2\rangle$ under $\mathbb{V}^{n+1}_{\mathbb{Z}}$, and again we restrict to the essential case $n=1$.

To prove the vanishing of $\mathrm{Tr}^2_{\mathbb{Z}} X \to \Sigma\ \mathrm{Tr}^2_{\mathbb{Z}} \Delta\langle -2\rangle$ it suffices to check that its successor $\mathbb{V}^2_{\mathbb{Z}}\mathrm{Tr}^2_{\mathbb{Z}}\Delta_2\langle -2\rangle \to \mathbb{V}^2_{\mathbb{Z}}\mathrm{Tr}^2_{\mathbb{Z}} B$ in the distinguished triangle (I.6.5.30) is a split monomorphism. From Lemma I.6.3.10 we know that $u_k(x_1,y_1) = u_k(x_2,y_2)$ in B, so the morphism in question is given by

$$\begin{array}{ccccccccccc}
\ldots & \rightleftarrows & 0 & \rightleftarrows & \Delta_2\langle -4\rangle & \underset{(k+1)x_2^k=0}{\overset{0}{\rightleftarrows}} & \Delta_2\langle -2\rangle & \rightleftarrows & 0 & \rightleftarrows & \ldots \\
 & & \downarrow & & \downarrow{\scriptstyle x_1-x_2+y_1-y_2} & & \downarrow{\scriptstyle x_1-x_2+y_1-y_2} & & \downarrow & & \\
\ldots & \rightleftarrows & 0 & \rightleftarrows & B\langle -2\rangle & \underset{u_k(x_1,y_1)=u_k(x_2,y_2)}{\overset{y_2-x_2=x_1-y_1}{\rightleftarrows}} & B & \rightleftarrows & 0 & \rightleftarrows & \ldots
\end{array}$$

A splitting over $\widehat{\mathbb{A}}^1_{\Bbbk}$ is provided $\sigma : B \to \Delta_2\langle -2\rangle$ described at the end of the proof of Proposition I.6.4.4, as by definition σ vanishes on $u_k(x_1, y_1)$ and equals the identity when composed with $\Delta_2\langle -2\rangle \to B$, $1 \mapsto x_1 - x_2 + y_1 - y_2$. □

Corollary I.6.5.2. *There are (pointwise) distinguished triangles of functors*

$$\mathrm{M}^X_{\mathbb{Z}}\langle -2\rangle \to \mathrm{M}^B_{\mathbb{Z}} \to \mathrm{M}^\Delta_{\mathbb{Z}} \to \Sigma\,\mathrm{M}^X_{\mathbb{Z}}\langle -2\rangle, \tag{I.6.5.31}$$
$$\mathrm{M}^\Delta_{\mathbb{Z}}\langle -2\rangle \to \mathrm{M}^B_{\mathbb{Z}} \to \mathrm{M}^X_{\mathbb{Z}} \to \Sigma\,\mathrm{M}^\Delta_{\mathbb{Z}}\langle -2\rangle. \tag{I.6.5.32}$$

which split on composition with $\mathbb{V}^{n+1}_{\mathbb{Z}} : \mathbf{D}^{\mathrm{ctr}}\,\mathrm{MC}(\widehat{\mathbb{A}}^{n+1}_{\Bbbk}, \widehat{w}_n) \to \mathbf{D}^{\mathrm{ctr}}\,\mathrm{MC}(\widehat{\mathbb{A}}^n_{\Bbbk}, \widehat{w}_n)$*:*

(i) $\mathbb{V}^{n+1}_{\mathbb{Z}}$ *annihilates* $\mathrm{M}^X_{\mathbb{Z}}\langle -2\rangle \to \mathrm{M}^B_{\mathbb{Z}}$*.*

(ii) $\mathbb{V}^{n+1}_{\mathbb{Z}}$ *annihilates* $\mathrm{M}^X_{\mathbb{Z}} \to \Sigma\,\mathrm{M}^\Delta_{\mathbb{Z}}\langle -2\rangle$*.*

With Proposition I.6.5.1 and Corollary I.6.5.2 at hand, we can now derive the desired link invariant as in Section I.6.4. Defining $\mathrm{M}^D_{\mathbb{Z}_2} := \mathbf{R}\,\mathrm{fold}^\Pi \circ \mathrm{M}^D_{\mathbb{Z}}$ as in (I.6.4.22) and putting $\mathrm{T}^D_* := \mathrm{H}^* \circ \mathrm{M}^D_{\mathbb{Z}_2}$, the proof of Proposition I.6.4.8 carries over to give the following – note, however, that (I.6.5.34) differs from (I.6.4.26):

Proposition I.6.5.3. *Let Q be an $\widehat{\mathbb{A}}^n_{\Bbbk}$-module of finite projective dimension. Then there are canonical exact sequences of $\mathbb{A}^{n+1}_{\Bbbk}$-modules which split over $\mathbb{A}^n_{\Bbbk}$*-Mod*:*

$$0 \to \mathrm{T}^B_* \to \mathrm{T}^\Delta_* \to \mathrm{T}^X_{*+1}\langle -2\rangle \to 0 \tag{I.6.5.33}$$
$$0 \to \mathrm{T}^\Delta_*\langle -2\rangle \to \mathrm{T}^B_* \to \mathrm{T}^X_* \to 0 \tag{I.6.5.34}$$

Moreover, there are canonical isomorphisms

$$\mathrm{T}^\Delta_*\,Q \cong \Delta^+_{\Bbbk} \otimes_{\Bbbk} ({}^k\,\mathrm{sHH}^{\mathbb{A}^n_{\Bbbk}/\Bbbk}_*(Q) \oplus {}^k\,\mathrm{sHH}^{\mathbb{A}^n_{\Bbbk}/\Bbbk}_{*+1}(Q)\langle -2\rangle), \qquad \mathrm{T}^X_*\,Q \cong (\rho_n^* \circ {}^k\,\mathrm{sHH}^{\mathbb{A}^n_{\Bbbk}/\Bbbk}_*)(Q),$$

where $\rho_n : \mathbb{A}^{n+1}_{\Bbbk} \twoheadrightarrow \mathbb{A}^n_{\Bbbk}$ is given by $x_{n+1} \mapsto x_n$ and $x_i \mapsto x_i$ for $i \leq n$.

From this we deduce the analogue of Corollary I.6.4.14 – note again how (I.6.5.36) differs from (I.6.4.29) in the case $k+1 \in \Bbbk^\times$, and recall Definition I.6.4.13 of the relative derived category $\mathbf{D}_{\mathbb{A}^n_{\Bbbk}}(\mathbb{A}^{n+1}_{\Bbbk})$.

Corollary I.6.5.4. *For a complex S of $\widehat{\mathbb{A}}^n_{\mathbb{k}}/(\widehat{w}_n)$-modules whose components are of finite projective dimension over $\widehat{\mathbb{A}}^n_{\mathbb{k}}$ there are canonical isomorphisms in $\mathbf{D}_{\mathbb{A}^n_{\mathbb{k}}}(\mathbb{A}^{n+1}_{\mathbb{k}})$:*

$$ {}^{k}\,\mathrm{sHH}_*^{\mathbb{A}^{n+1}_{\mathbb{k}}/\mathbb{k}}\left[F_+^{n+1,n}\otimes_{\mathbb{A}^{n+1}_{\mathbb{k}}}\left(\Delta^+\otimes_{\mathbb{k}} S\right)\right]\cong\rho_n^*\circ\Sigma^{-1}\left({}^{k}\,\mathrm{sHH}_{*+1}^{\mathbb{A}^{n}_{\mathbb{k}}/\mathbb{k}}\,S\right), \tag{I.6.5.35}$$

$$ {}^{k}\,\mathrm{sHH}_*^{\mathbb{A}^{n+1}_{\mathbb{k}}/\mathbb{k}}\left[F_-^{n+1,n}\otimes_{\mathbb{A}^{n+1}_{\mathbb{k}}}\left(\Delta^+\otimes_{\mathbb{k}} S\right)\right]\cong\rho_n^*\circ\left({}^{k}\,\mathrm{sHH}_{*}^{\mathbb{A}^{n}_{\mathbb{k}}/\mathbb{k}}\,S\right) \tag{I.6.5.36}$$

Here ρ_n is as in Proposition I.6.5.3.

Proof of Theorem I.6.1.5. The theorem follows directly from Corollary I.6.5.4, noting that the expression $\frac{n+w(\beta)}{2}$ increases by one under a positive second Markov move and stays invariant under a negative second Markov move. □

The invariant from Theorem I.6.1.5 is not the strongest possible in case $k+1=0$ in $\mathbb{k}$. Namely, in contrast to the case $k+1\in\mathbb{k}^\times$, the key Proposition I.6.5.1 already holds on the $\mathbb{Z}$-graded level, so there is no need pass to linear factorizations through folding. Instead, we could replace T^D_* with $\mathrm{S}^D_*:=\mathrm{H}^*\circ\mathrm{M}^D_{\mathbb{Z}}$ (computing ordinary Hochschild homology) in Proposition I.6.5.3 and Corollary I.6.5.4. Since ordinary Hochschild homology is itself $\mathbb{Z}$-graded, this results in triply graded link homology with values in $\frac{1}{2}\mathbb{Z}$-graded complexes of $\mathbb{Z}$-graded $\mathbb{k}$-modules that are equipped with an additional $\frac{1}{2}\mathbb{Z}$-grading coming from Hochschild homology; the shift of this grading is denoted $\{-\}$.

Theorem I.6.5.5. *Assume that $k+1=0$ in $\mathbb{k}$. Then assigning to a braid word β on n strands and writhe $w(\beta)$ the isomorphism class of*

$$\Sigma^{\frac{n+w(\beta)}{2}}\,\mathrm{HH}_*^{\mathbb{A}^n_{\mathbb{k}}/\mathbb{k}}\left[\mathcal{RC}_{\mathbb{k}}(\beta)\right]\left\{-\frac{n+w(\beta)}{2}\right\} \tag{I.6.5.37}$$

in the homotopy category $\mathrm{Ho}^b_{\frac{1}{2}}\left(\mathbb{k}\text{-}\mathrm{Mod}^{\mathbb{Z}}\right)$ of $\frac{1}{2}\mathbb{Z}$-graded complexes of $\frac{1}{2}\mathbb{Z}\times\mathbb{Z}$-graded $\mathbb{k}$-modules, defines an invariant of oriented links.

However, the invariant described in Theorem I.6.5.5 does not involve the parameter k anymore – in fact, the whole proof goes through for arbitrary $\mathbb{k}$ when replacing w with the zero potential, in which case we'd arrive at Theorem I.6.5.5 without any assumptions on $\mathbb{k}$; in case $\mathbb{k}=\mathbb{C}$ this recovers Rouquier's Theorem [Rou12, Theorem 4.9].

A stronger link invariant which allows to recover both Theorems I.6.1.5 and I.6.5.5 can be constructed by sticking with $\mathrm{M}^D_{\mathbb{Z}}$ and not taking cohomology as follows: Denote $\mathrm{Ho}^b_{\frac{1}{2}\mathbb{Z}}\left(\mathbf{D}^b_{\frac{1}{2}\mathbb{Z}}\,\mathrm{MC}(\mathbb{k},0)\right)$ the $\frac{1}{2}\mathbb{Z}$-graded bounded homotopy category of the (additive) derived category $\mathbf{D}^b_{\frac{1}{2}\mathbb{Z}}\,\mathrm{MC}(\mathbb{k},0)$ of $\frac{1}{2}\mathbb{Z}$-graded mixed complexes, and let $\{-\}$ be the automorphism of $\mathrm{Ho}^b_{\frac{1}{2}\mathbb{Z}}\left(\mathbf{D}^b_{\frac{1}{2}\mathbb{Z}}\,\mathrm{MC}(\mathbb{k},0)\right)$ which is given by applying the shift functor of $\mathbf{D}^b_{\frac{1}{2}\mathbb{Z}}\,\mathrm{MC}(\mathbb{k},0)$ componentwise. We then have the following:

Theorem I.6.5.6. *Assume that $k+1=0$ in $\Bbbk$. Then assigning to a braid word β on n strands and writhe $w(\beta)$ the isomorphism class of*

$$\Sigma^{\frac{n+w(\beta)}{2}} {}_{-\widehat{w}_n}\Delta^n_{\Bbbk} \otimes^{\mathbf{L}}_{\widehat{\mathbb{A}}^n_{\Bbbk}} {}_{\widehat{w}_n}\left[\mathcal{RC}_{\Bbbk}(\beta)\right]\left\{-\frac{n+w(\beta)}{2}\right\} \tag{I.6.5.38}$$

in $\mathrm{Ho}^b_{\frac{1}{2}\mathbb{Z}}\left(\mathbf{D}^b_{\frac{1}{2}\mathbb{Z}}\,\mathrm{MC}(\Bbbk,0)\right)$ defines an invariant of oriented links.

It is no coincidence that in case $k+1=0$ in $\Bbbk$ we can derive an invariant of oriented links with both ordinary and k-stable Hochschild homology:

Proposition I.6.5.7. *Suppose $\Bbbk$ is a commutative ring with $k+1=0$ in $\Bbbk$, and let M be a Soergel bimodule on n strands over $\Bbbk$. Then the spectral complex ${}^{w}\,\mathrm{SHH}^{\mathbb{A}^n_{\Bbbk}/\Bbbk}(M)$ from Proposition I.4.6.5, relating $\mathrm{HH}^{\mathbb{A}^n_{\Bbbk}/\Bbbk}_*(M)$ and ${}^{k}\,\mathrm{sHH}^{\mathbb{A}^n_{\Bbbk}/\Bbbk}_*(M)$, is degenerate.*

Proof. We use the induction principle for Soergel bimodules from Proposition I.A.2.6 and check its assumptions (i)-(v) in what follows. To begin, assumption (i) asserts that the spectral complex ${}^{w}\,\mathrm{SHH}^{\mathbb{A}^1_{\Bbbk}/\Bbbk}(\Delta^1_{\Bbbk})$ is degenerate. Computing ${}_{-\widehat{w}_1}\Delta^1_{\Bbbk} \otimes^{\mathbf{L}}_{\widehat{\mathbb{A}}^1_{\Bbbk}} -$ via the Koszul resolution of $\Delta^1_{\Bbbk}$, it is the spectral complex associated to the mixed complex

$$\{x_1-y_1, u_k(x_1,y_1)\} \otimes_{\widehat{\mathbb{A}}^1_{\Bbbk}} {}_{\widehat{w}_1}\Delta^1_{\Bbbk} \;=\; \Delta^1_{\Bbbk}\langle -2\rangle \underset{u_k(x_1,y_1)}{\overset{y_1-x_1}{\rightleftarrows}} \Delta^1_{\Bbbk}$$

over $\widehat{\mathbb{A}}^1_{\Bbbk}$. The d-differential of this mixed complex vanishes since $x_1=y_1$ in $\Delta^1_{\Bbbk}$, and moreover $u_k(x_1,y_1)=(k+1)x_1^k=0$ in $\Delta^1_{\Bbbk}$ since $k+1=0$ in $\Bbbk$ – hence, by Proposition I.4.6.1 the first differential d^*_1 of ${}^{k}\,\mathrm{SHH}^{\mathbb{A}^1_{\Bbbk}/\Bbbk}(\Delta^1_{\Bbbk})$ vanishes. All the higher differentials d^n_t for $t>1$ vanish for degree reasons, so we conclude that ${}^{k}\,\mathrm{SHH}^{\mathbb{A}^1_{\Bbbk}/\Bbbk}(\Delta^1_{\Bbbk})$ is degenerate. Next, conditions (ii) and (iv) in the induction principle follow from the two observations that firstly HC commutes with internal degree shift and direct sums, and that secondly the subcategory of degenerate spectral complexes is closed under internal shift, sums and summands. Property (iii) asserts that ${}^{k}\,\mathrm{SHH}^{\mathbb{A}^n_{\Bbbk}/\Bbbk}(M\otimes_{\mathbb{A}^n_{\Bbbk}} N)$ is degenerate if and only if ${}^{k}\,\mathrm{SHH}^{\mathbb{A}^n_{\Bbbk}/\Bbbk}(N\otimes_{\mathbb{A}^n_{\Bbbk}} M)$ is, and this follows from the definition (I.4.3) of SHH together with Proposition I.6.2.1 implying that ${}_{-\widehat{w}_n}\Delta^n_{\Bbbk} \otimes^{\mathbf{L}}_{\widehat{\mathbb{A}}^n_{\Bbbk}} {}_{\widehat{w}_n}(M\otimes_{\Bbbk} N) \cong {}_{-\widehat{w}_n}\Delta^n_{\Bbbk} \otimes^{\mathbf{L}}_{\widehat{\mathbb{A}}^n_{\Bbbk}} {}_{\widehat{w}_n}(N\otimes_{\Bbbk} M)$ in $\mathbf{D}^b\,\mathrm{MC}(\widehat{\mathbb{A}}^n_{\Bbbk},0)$. Finally, condition (v) asserts that if ${}^{k}\,\mathrm{SHH}^{\mathbb{A}^n_{\Bbbk}}(M)$ is degenerate, then so are ${}^{k}\,\mathrm{SHH}^{\mathbb{A}^{n+1}_{\Bbbk}}(M\otimes_{\Bbbk}\Bbbk[x_{n+1},y_{n+1}]/(x_{n+1}-y_{n+1}))$ and ${}^{k}\,\mathrm{SHH}^{\mathbb{A}^{n+1}_{\Bbbk}}((M\otimes_{\Bbbk}\Bbbk[x_{n+1},y_{n+1}]/(x_{n+1}-y_{n+1}))\otimes_{\mathbb{A}^{n+1}_{\Bbbk}} B^{n+1,n}_{\Bbbk})$. These are by definition the spectral complexes associated to $\mathrm{M}^{\Delta^{n+1}_{\Bbbk}}_{\mathbb{Z}}(M)$ and $\mathrm{M}^{B^{n+1,n}_{\Bbbk}}_{\mathbb{Z}}(M)$, so the claim follows from Corollary I.6.5.2 and the observation that degeneracy of a spectral complexes is preserved and reflected under restriction of scalars. □

Corollary I.6.5.8. *Suppose $\Bbbk$ is a commutative ring with $k+1=0$ in $\Bbbk$, and let M be a Soergel bimodule on n strands over $\Bbbk$. Then there is a canonical isomorphism*

$$\operatorname{gr}\left[{}^k\operatorname{sHH}^{\mathbb{A}^n_\Bbbk/\Bbbk}(M)\right] \;\cong\; \left(\operatorname{HH}_j^{\mathbb{A}^n_\Bbbk/\Bbbk}(M)\left\langle\tfrac{jd}{2}\right\rangle\right)_{j\in\mathbb{Z}},$$

where ${}^k\operatorname{sHH}^{\mathbb{A}^n_\Bbbk/\Bbbk}(M)$ is equipped with the $\mathbb{Z}$-filtration defined in Corollary I.4.6.6.

In light of Corollary I.6.5.8, the numerical contents of Theorems I.6.1.5 and I.6.5.5 are therefore indeed the same.

Remark I.6.5.9. Recall that there are two nested homotopy theories involved in the construction of Khovanov-Rozansky homology: we are concerned with complexes whose components are themselves linear factorizations. By now, we have developed and made use of a proper model categorical understanding of the "inner layer", the homotopy category of matrix factorizations, but we are still lacking a homotopical understanding for the "outer layer", the complexes with values in matrix factorizations. In the previous sections, however, we did not pursue this question, and instead contented ourselves with working with ad-hoc definitions of the relative derived category $\mathbf{D}_B(A)$ and the nested homotopy category $\operatorname{Ho}^b_{\frac{1}{2}\mathbb{Z}}\left(\mathbf{D}^b_{\frac{1}{2}\mathbb{Z}}\operatorname{MC}(\Bbbk,0)\right)$. See the following Section I.6.6 for remarks on working relative to a base ring, Section II.4 on model categorical enhancements for relative derived categories, and Section II.C.3 for general considerations on how to extend model structures on an abelian category $\mathscr{A}$ to model structures on its category $\operatorname{Ch}(\mathscr{A})$ of chain complexes. ◊

I.6.6. Avoiding technicalities I: Working relative to $\Bbbk$

When studying the contraderived tensor product of linear factorizations, we observed that in general it cannot be computed by resolution of a single factor only, but that it can if the base ring has finite global dimension. We applied this for example in rewriting classical Khovanov-Rozansky homology in terms of stable Hochschild homology of Soergel bimodules in Theorem I.5.2, since there the base rings where polynomial rings over $\mathbb{Q}$, hence of finite global dimension. However, we couldn't apply it in the previous sections although we were working with polynomial rings there, too, since we didn't assume anything about the base ring $\Bbbk$ over which the polynomial rings where taken. On the other hand, apart from the question as to whether $k+1$ is invertible in $\Bbbk$ or not, the base $\Bbbk$ didn't play a role, so one might hope to reestablish the convenient case of finite global dimension by replacing the "absolute" context of $\mathbb{A}^n_\Bbbk$-modules by relative homological algebra of the extension $\mathbb{A}^n_\Bbbk/\Bbbk$. This would in particular involve developing the necessary homotopy theory in the framework of exact model structures [Gil11]; we leave the investigation to what extend this can be done as a possibility for further study.

I.7. Variations

I.7.1. Avoiding technicalities II: Working with comodules

I.7.1.1. Reflection & Motivation

Our main motivation for studying the folding functors (Definition I.2.3.1) between curved mixed complexes and linear factorization was the following observation: any quasi-isomorphism of bounded above curved mixed complexes is a contraderived equivalence and the folding by products fold^{Π} preserves such. This allowed for the construction of contraderived equivalences of linear factorizations via resolutions by quasi-isomorphisms in the category of curved mixed complexes and folding these afterwards (Corollary I.4.4.5) – for example, we obtained the Koszul resolutions of quotients by regular sequences this way, see Example I.4.4.6. This method does not mainly rely on the existence of the folding by products functor as a right Quillen functor $\mathrm{fold}^{\Pi} : \mathcal{M}^{\mathrm{ctr}}\,\mathrm{MC}(A,w) \to \mathcal{M}^{\mathrm{ctr}}\,\mathrm{LF}(A,w)$, but instead on the fact that its right derived functor $\mathbf{R}\,\mathrm{fold}^{\Pi}$ can be computed naively, i.e. without taking resolutions (everything is fibrant in $\mathcal{M}^{\mathrm{ctr}}\,\mathrm{MC}(A,w)$). In contrast, even though the folding by sums functor $\mathrm{fold}^{\oplus}$ is also compatible with the model structures in that it is a left Quillen functor $\mathrm{fold}^{\oplus} : \mathcal{M}^{\mathrm{ctr}}\,\mathrm{MC}(A,w) \to \mathcal{M}^{\mathrm{ctr}}\,\mathrm{LF}(A,w)$, the computation of its left derived $\mathbf{L}\,\mathrm{fold}^{\oplus}$ involves taking cofibrant resolutions, i.e. resolutions by $K(A,w)^{\sharp}$-free $K(A,w)$-modules, and is therefore not suitable for implementing the above idea of constructing contraderived equivalences in $\mathrm{LF}(A,w)$.

On the other hand we where also facing a different problem for which the use of $\mathrm{fold}^{\oplus}$ instead of fold^{Π} would have been much more convenient, namely the compatibility of tensor products with folding. Being forced to work with fold^{Π} by the reasons explained in the previous paragraph we had to study when also $\mathbf{R}\,\mathrm{fold}^{\Pi}$ commutes with contraderived tensor products. This lead us to restricting firstly to bounded curved mixed complexes, for which $\mathrm{fold}^{\oplus} \cong \mathrm{fold}^{\Pi}$, and secondly to rings of finite global dimension, for which the contraderived tensor product can be computed by resolutions of a single factor only, something we saw is not possible for arbitrary rings and constitutes a very inconvenient deficiency of contraderived categories.

Finally, we name two milder inconveniences of contraderived categories of rings, though they didn't cause difficulties in the application to Khovanov-Rozansky homology: Firstly, even though the contraderived category exists for any (cdg) ring A, Positselski's explicit description of contraderived weak equivalences in terms of totalizations (Proposition I.4.1.2) is only available under the assumption that countable products of projective $A^\sharp$-modules have finite projective dimension over $A^\sharp$; similarly, the coderived weak equivalences admit an explicit description only if direct sums of injective $A^\sharp$-modules are of finite injective dimension (e.g., if $A^\sharp$ is Noetherian). Secondly, Grothendieck duality between contraderived and coderived category (see for example Proposition II.2.3.14 in Part II) is only available in some situations, e.g. if the ring under consideration is Gorenstein.

All these technical problems where proved by Positselski to disappear for the coderived and contraderived categories of comodules and contramodules over corings, as we recall now. Fix a (cdg) coring C over a field $\Bbbk$. Then, in [Pos11, §4.2,4.4] Positselski defines the class of coacyclic C-comodules, introduces the coderived category $\mathbf{D}^{\text{co}}(C\text{-coMod})$ of C-comodules by localizing at coacyclic C-comodules and shows that – without any further assumptions – the coderived category is equivalent to the homotopy category $\text{Ho}(C\text{-coMod}_{\text{inj}})$ of those C-comodules whose underlying $C^\sharp$-comodules are injective. The reason why, in contrast to the context of modules, one does not need further assumptions here is that, somewhat surprisingly, direct sums of injective comodules are always injective. Dually, Positselski defines contraacyclic C-contramodules, introduces the contraderived category $\mathbf{D}^{\text{ctr}}(C\text{-ctrMod})$ of C-contramodules and proves it to be equivalent to the homotopy category $\text{Ho}(C\text{-ctrMod}_{\text{proj}})$ of those C-contramodules whose underlying $C^\sharp$-contramodules are projective; again, no further assumptions are needed since products of projective contramodules are always projective. Grothendieck duality between coderived and contraderived categories of modules over Gorenstein rings transfers to what Positselski calls the comodule-contramodule-correspondence [Pos11, Theorem 5.2] $\mathbf{D}^{\text{ctr}}(C\text{-ctrMod}) \cong \mathbf{D}^{\text{co}}(C\text{-coMod})$ and which holds in complete generality. Finally, the cotensor product of a fibrant and a coacyclic C-comodule is coacyclic, giving rise to a coderived cotensor product functor that can be computed through fibrant resolution of a single factor; see [Pos11, §4.7].

The goal of this section is to indicate how one can define the analogues of matrix factorizations, curved mixed complexes, folding, and finally Khovanov-Rozansky homology in the setting of comodules.

I.7.1.2. Generalities on coalgebras and comodules

For convenience of the reader we recall in this section some basic definitions and facts about coalgebras and their comodules. For an extensive treatment, see e.g. [BW03] – our focus here lies on providing a feeling for the categorical properties of the category of comodules over a coalgebra.

In the following, we restrict to coalgebras defined over a field $\Bbbk$; over arbitrary commutative base rings, the theory of coalgebras and their comodules is more difficult to develop. A *coalgebra C over* $\Bbbk$ is a $\Bbbk$-vector space equipped with comultiplication and counit maps $\Delta : C \to C \otimes_{\Bbbk} C$ and $\eta : C \to \Bbbk$, respectively, which are coassociative and counital in the obvious way. Given a vector space V over $\Bbbk$, the structure of a *left C-comodule* on V is a $\Bbbk$-linear map $\Delta_M : M \to M \otimes_{\Bbbk} C$ which is coassociative and counital with respect to Δ and η. If (M, Δ_M) and (N, Δ_N) are two such C-comodules, a $\Bbbk$-linear morphism $\varphi : M \to N$ is called C-linear if $(\mathrm{id}_C \otimes \varphi) \circ \Delta_M = \Delta_N \circ \varphi$. The resulting category $C\text{-coMod}$ of left C-comodules admits a forgetful functor $C\text{-coMod} \to \Bbbk\text{-Mod}$ to $\Bbbk$-vector spaces, with the *cofree comodule functor* $V \mapsto (C \otimes_{\Bbbk} V, \Delta \otimes \mathrm{id}_V)$ as its right adjoint: $\mathrm{Hom}_{C\text{-coMod}}(M, C \otimes_{\Bbbk} V) \cong \mathrm{Hom}_{\Bbbk\text{-Mod}}(M, V)$. In particular, $C\text{-coMod} \to \Bbbk\text{-Mod}$ preserves colimits – indeed all small colimits exist in $C\text{-coMod}$ and can be computed naively on the level of underlying $\Bbbk$-modules – and the cofree comodule functor preserves products, so $\prod_i C \otimes_{\Bbbk} V_i \cong C \otimes_{\Bbbk} \prod_i V_i$ in $C\text{-coMod}$ by means of the projections $C \otimes_{\Bbbk} \prod_i V_i \to C \otimes_{\Bbbk} V_{i_0}$. The latter already shows that products in $C\text{-coMod}$ cannot be computed naively on the underlying $\Bbbk$-vector spaces. Note that, formally, $C \otimes_{\Bbbk} - : \Bbbk\text{-Mod} \to \Bbbk\text{-Mod}$ carries the structure of an exact comonad, and $C\text{-coMod}$ is the category of coalgebras over that comonad – these notions are dual to the notions of monad and algebra over a monad which we recall in Definition II.B.1 of Part II. The above-mentioned existence of colimits in $C\text{-coMod}$ then follows from the proof of Lemma II.B.3, and this lemma also shows that $C\text{-coMod}$ is abelian, with $C\text{-coMod} \to \Bbbk\text{-Mod}$ preserving and reflecting exactness. Together with the existence of colimits this shows that $C\text{-coMod}$ is an AB5-category. Finally, any C-comodule is the directed union of its finite-dimensional sub-C-comodules, so a representative set of isomorphism classes of finite-dimensional C-comodules forms a generating set of Noetherian objects for $C\text{-coMod}$, which is therefore a locally Noetherian Grothendieck category. Note that apart from the fact that $\Bbbk$ is a field we didn't assume anything for that. That $C\text{-coMod}$ is locally Noetherian implies for example that direct sums of injective C-comodules are again injective, which also follows directly from the observation that firstly the injective C-comodules are precisely the summands of the cofree C-comodules (any C-comodule M embeds into the cofree C-comodule $C \otimes_{\Bbbk} M$ by means of its coaction $\Delta : M \to C \otimes_{\Bbbk} M$) and that secondly these are stable under coproducts, $\bigoplus_i C \otimes_{\Bbbk} V_i \cong C \otimes_{\Bbbk} \bigoplus_i V_i$. Moreover, as a Grothendieck category $C\text{-coMod}$ possesses arbitrary small products, but we emphasize again that they are not computed naively on the level of the underlying $\Bbbk$-vector spaces. Instead, they can be constructed as follows: First, recall that the $\Bbbk$-dual C^* of a $\Bbbk$-coalgebra C inherits the structure of a $\Bbbk$-algebra, and that any C-comodule M is naturally a C^*-module via $\varphi \cdot m := ((\varphi \otimes \mathrm{id}_M) \circ \Delta_M)(m)$. This construction extends to a fully faithful functor $C\text{-coMod} \hookrightarrow C^*\text{-Mod}$ realizing $C\text{-coMod}$ as a coreflective subcategory of C^*-Mod (i.e., there exists a right adjoint C^*-Mod $\to C$-coMod to C-coMod $\hookrightarrow C^*$-Mod), and in particular products in C-coMod can be constructed by taking products in C^*-Mod and applying the adjoint C^*-Mod $\to C$-coMod afterwards; see [BW03, Theorem 4.3].

Example I.7.1.1. We consider the following classical example, see e.g. [BW03, Exercise 8.12(1)]: For C take the $\Bbbk$-coalgebra $\Bbbk\{x_0, x_1, \ldots\}$ with comultiplication $\Delta(x_n) := \sum_{p+q=n} x_p \otimes x_q$ and counit $\eta(x_n) = \delta_{n,0}$. Then $C^* \cong \Bbbk[\![t]\!]$ with $t \in \Bbbk[\![t]\!]$ corresponding to the functional $x_n \mapsto \delta_{n,1}$, and C-coMod $\hookrightarrow \Bbbk[\![t]\!]$-Mod identifies C-coMod with the full subcategory $\Bbbk[\![t]\!]$-Tor of torsion modules, i.e. those $\Bbbk[\![t]\!]$-modules M in which for any $m \in M$ there exists some $n \geq 0$ such that $t^n m = 0$: Namely, if $M \in C$-coMod and $m \in M$ with $\Delta(m) = \sum_{i=1}^{n} x_i \otimes m_i$, then $t^k m = 0$ for $k > n$ with respect to the $\Bbbk[\![t]\!]$-action on M. Conversely, if M is a torsion $\Bbbk[\![t]\!]$-module, $m \mapsto \sum_{i\geq 0} x_i \otimes (t^i.m)$ yields a well-defined C-coaction on M giving rise to the original $\Bbbk[\![t]\!]$-action under C-coMod $\hookrightarrow \Bbbk[\![t]\!]$-Mod. The coreflection $\Bbbk[\![t]\!]$-Mod $\to \Bbbk[\![t]\!]$-Tor is given by $M \mapsto M_{\mathrm{tor}} := \{m \in M \mid \exists n \geq 0 : t^n m = 0\}$, and hence the product in $\Bbbk[\![t]\!]$-Tor of a family $\{M_i\}_i$ of torsion modules can be constructed as $(\prod_i M_i)_{\mathrm{tor}}$. However, this product is not exact: for any $n \geq 0$, the morphism $p_n : \Bbbk[\![t]\!]/(t^n) \to \Bbbk$ annihilating t is an epimorphism, but the product of the family $(p_n)_{n\geq 0}$ in $\Bbbk[\![t]\!]$-Tor is $(\prod_n \Bbbk[\![t]\!]/(t^n))_{\mathrm{tor}} \to \Bbbk^{\mathbb{N}}$, which is not surjective. ◊

Example I.7.1.2. Example I.7.1.1 extends to n variables by considering the $\Bbbk$-coalgebra $C := \Bbbk\{x_{\mathbf{i}}\}$, with $\mathbf{i}$ running over multisubsets of $\{1, 2, \ldots, n\}$ and with the comultiplication given by $\Delta(x_{\mathbf{i}}) := \sum_{\mathbf{k}\cup\mathbf{l}=\mathbf{i}} x_{\mathbf{k}} \otimes x_{\mathbf{l}}$. Its $\Bbbk$-dual is canonically isomorphic to the power series ring $\Bbbk[\![t_1, \ldots, t_n]\!]$, and the image of the embedding C-coMod $\hookrightarrow \Bbbk[\![t_1, \ldots, t_n]\!]$-Mod consists of those $\Bbbk[\![t_1, \ldots, t_n]\!]$-modules M which are supported at the maximal ideal $\mathfrak{m} = (t_1, \ldots, t_n)$, i.e. for which for any $m \in M$ there exists some $k \gg 0$ such that $\mathfrak{m}^k m = \{0\}$. Moreover, the regular C-comodule C corresponds to the injective hull $E(\Bbbk)$ of the residue field $\Bbbk$, and hence the finitely cogenerated C-comodules – those which embed into a cofree comodule $\Bbbk^l \otimes_{\Bbbk} C$ for some l – correspond to the Artinian $\Bbbk[\![t_1, \ldots, t_n]\!]$-modules. Finally, note that the latter are in Matlis duality $\mathrm{Hom}_{\Bbbk[\![t_1,\ldots,t_n]\!]}(-, E(k))$ with the finitely generated $\Bbbk[\![t_1, \ldots, t_n]\!]$, see e.g. [BH93, Theorem 3.2.13]. ◊

Given a left C-comodule N and a right C-comodule M, the *cotensor product* $M \boxtimes_C N \in \Bbbk$-Mod is defined as the difference kernel of $M \otimes_{\Bbbk} N \rightrightarrows M \otimes_{\Bbbk} C \otimes_{\Bbbk} N$. In general, this is only a $\Bbbk$-vector space, but if C is cocommutative, left and right C-comodule structures on a $\Bbbk$-vector space are in canonical bijection and the induced functor $- \boxtimes_C - : C$-coMod $\times C$-coMod $\to \Bbbk$-Mod canonically lifts to a functor with values in C-coMod and constitutes a monoidal structure on C-coMod. For example, if

$M = C \otimes_{\Bbbk} V$ is cofree over V, then $M \boxtimes_C N \cong V \otimes_{\Bbbk} N$ with the C-coaction given by $\mathrm{id}_V \otimes \Delta_N$. In particular, cofree C-comodules are coflat in the sense that cotensoring with them preserves exactness, and moreover the cotensor product of two cofree C-comodules $C \otimes_{\Bbbk} V$ and $C \otimes_{\Bbbk} W$ is the cofree C-comodule $C \otimes_{\Bbbk} (V \otimes_{\Bbbk} W)$ over $V \otimes_{\Bbbk} W$

All definitions carry over to Ω-graded $\Bbbk$-coalgebras for a grading group Ω, in particular:

Example I.7.1.3. The polynomial $\Bbbk$-coalgebra C from Example I.7.1.2 is naturally $\Omega := \mathbb{Z}$-graded by $|x_{\mathbf{i}}| := \sum_j i_j$ if $\mathbf{i} = \{i_1, \ldots, i_k\}$. Its graded $\Bbbk$-dual is naturally isomorphic to the polynomial $\Bbbk$-algebra $\Bbbk[t_1, \ldots, t_n]$, $\mathbb{Z}$-graded by $|t_i| = 1$, and since the homogeneous components of C and C^* are finite-dimensional over $\Bbbk$, we also have $C = \mathrm{Hom}_{\Bbbk}(\Bbbk[t_1, \ldots, t_n], \Bbbk)$ as $\Bbbk[t_1, \ldots, t_n]$-modules. Consequently (see [BH93, Proposition 3.6.16] for details) $C = E(\Bbbk)$ is an injective hull of the residue field $\Bbbk$ considered in degree 0, and graded Matlis duality $\mathrm{Hom}_{\Bbbk[t_1,\ldots,t_n]}(-, E(\Bbbk))$ is naturally isomorphic to the graded $\Bbbk$-dual functor $\mathrm{Hom}_{\Bbbk}(-, \Bbbk)$ as is witnessed by the adjunction between the coinduction $\mathrm{Hom}_{\Bbbk}(\Bbbk[t_1, \ldots, t_n], -)$ and the forgetful functor from graded $\Bbbk[t_1, \ldots, t_n]$-modules to graded $\Bbbk$-modules. $\diamondsuit$

I.7.1.3. Matrix cofactorizations and comixed curved complexes

We want to define the analogues of linear factorizations and curved mixed complexes in the setting of coalgebras. Throughout, we fix a grading group Ω and an Ω-graded, cocommutative $\Bbbk$-coalgebra C, analogous to the Ω-graded base ring A we fixed when we defined linear factorizations in Section I.2.1. The analogue of the potential $w \in A$ of q-degree $d \in \Omega$ is a functional $\omega \in (C_{-d})^*$ called copotential, and given a C-comodule M, the analog of $\cdot w : M \to M$ is the coaction map $*\omega : M \xrightarrow{\Delta_M} C \otimes_{\Bbbk} M \xrightarrow{\omega \otimes \mathrm{id}_M} \Bbbk \otimes_{\Bbbk} M \cong M$, i.e. the multiplication by ω in the canonical C^*-module structure on M.

Definition I.7.1.4. *A linear cofactorization of type (C, φ) is a $\mathbb{Z}/2\mathbb{Z}$-graded C-comodule $M = M^0 \oplus M^1$ with an odd C-colinear endomorpism $\delta = (\delta^0, \delta^1)$, $\delta^0 : M^0 \to M^1$, $\delta^1 : M^1 \to M^0$ such that $|\delta^1|_q = d$, $|\delta^0|_q = 0$ and $\delta^2 = *\omega$. It is called a* matrix cofactorization *if, moreover, M^0, M^1 are injective as C-comodules.*

Note the difference in the grading convention for δ^0 and δ^1 compared to Definition I.2.1.1 of linear factorizations. The categories of linear cofactorizations and matrix cofactorizations of type (C, ω) are denoted $\mathrm{cLF}(C, \omega)$ and $\mathrm{cMF}(C, \omega)$, respectively. Applying C-coMod $\hookrightarrow C^*$-Mod componentwise and shifting the degree 1 component by d shows that $\mathrm{cLF}(C, \omega)$ is equivalent to the full subcategory of $\mathrm{LF}(C^*, \omega)$ consisting of those linear factorizations whose terms belong to the essential image of the embedding

C-coMod $\hookrightarrow C^*$-Mod. Note, however, that under this embedding matrix cofactorizations do not map to matrix factorizations, but that instead $\mathrm{Hom}_{C^*\text{-Mod}}(C,C) \cong \mathrm{Hom}_{C\text{-coMod}}(C,C) \cong C^*$ shows that the image of finite corank matrix cofactorizations of type (C,ω) are in contravariant duality $(-)^\vee := \mathrm{Hom}_{C^*\text{-Mod}}(-,C)$ with finite rank matrix factorizations of type (C^*,ω). Explicitly, this duality maps $M^0 \xrightarrow{\delta^0} M^1 \xrightarrow{\delta^1} M^0$ to $(M^0)^\vee \xrightarrow{(\delta^1)^\vee} (M^1)^\vee \xrightarrow{(\delta^0)^\vee} (M^0)^\vee$, fitting with the different grading conventions.

Example I.7.1.5. In the setting of Example I.7.1.3, we see that the above duality is the restriction of Matlis duality (which is the same as taking the $\Bbbk$-dual as we have seen) between linear cofactorizations with finitely cogenerated components and linear factorizations with finitely generated components. ◊

Example I.7.1.6. This is analogous to Example I.2.1.4: If M is a C-comodule such that $M \xrightarrow{*\omega} M$ is the zero map, then M can be considered as a linear cofactorization of type (C,ω) concentrated in degree 0, denoted ${}_\omega M$. ◊

We now define the analogues of elementary Koszul factorizations from Example I.2.1.6:

Example I.7.1.7. If $\alpha \in (C_{-a})^*$ and $\beta \in (C_{-b})^*$, then $C \xrightarrow{*\alpha} C\langle a\rangle \xrightarrow{*\beta} C$ is a matrix cofactorization of type $(C,\alpha\beta)$ (the product $\alpha\beta$ is taken in C^*), called the *elementary Koszul cofactorization* associated with (α,β) and denoted $\{\alpha,\beta\}$. It is $(-)^\vee$-dual to the elementary Koszul factorization from Example I.2.1.6. ◊

As for linear factorizations, the cotensor product of comodules induces a cotensor product of linear cofactorizations adding their copotentials, $-\boxtimes_C - : \mathrm{cLF}(C,\omega_0) \times \mathrm{cLF}(C,\omega_1) \to \mathrm{LF\,co}(C,\omega_0{+}\omega_1)$. Since the cotensor product of two injective C-comodules is injective again, $-\boxtimes_C-$ also restricts to a functor between categories of matrix cofactorizations. We now define the analogues of Koszul factorizations from Example I.2.2.2:

Example I.7.1.8. If $\omega = \sum_i \alpha_i\beta_i$ for sequences $\underline{\alpha} = (\alpha_1,\ldots,\alpha_n)$ and $\underline{\beta} = (\beta_1,\ldots,\beta_n)$ of homogeneous elements, then $\{\underline{\alpha},\underline{\beta}\} := \boxtimes_{i=1}^n\{\alpha_i,\beta_i\}$ is a matrix cofactorization of type $(C,\sum_i\alpha_i\beta_i)$ called the Koszul cofactorization associated with $(\underline{\alpha},\underline{\beta})$. It is $(-)^\vee$-dual to the Koszul factorization associated to $(\underline{\alpha},\underline{\beta})$ from Example I.2.2.2. ◊

Example I.7.1.9. As a special case of Example I.7.1.8 we get analogues of the matrix factorizations $\mathcal{KR}^k(\uparrow\uparrow)$ and $\mathcal{KR}^k(\times)$ defined in Section I.3 by reading their definitions in the context of matrix cofactorizations. ◊

Definition I.7.1.10. *A* cocurved mixed complex of type (C,ω) *is a triple* $X = (X,\mathrm{d},s)$ *where* X *is a* $\mathbb{Z}$*-graded* C*-comodule together with* C*-colinear differentials* d, s *of cohomological degree* $+1$ *and* -1 *and q-degrees* 0 *and* d*, respectively, such that* $\mathrm{d}\,s + s\,\mathrm{d} = *\omega$*.*

The category of cocurved mixed complexes of type (C,ω) is denoted $\mathrm{cMC}(C,\omega)$. Again, it inherits a cotensor product adding the cocurvatures from C-coMod, and we have an embedding $\mathrm{cLF}(C;\omega) \hookrightarrow \mathrm{cMC}(C,\omega)$ by considering a linear cofactorization as a cocurved mixed complex considered in cohomological degrees 0 and 1. In particular, any C-comodule M for which $M \xrightarrow{*\omega} M$ is the zero map can be considered as a cocurved mixed complex ${}_\omega M$ considered in degree 0, we have elementary Koszul cocurved mixed complexes $\{\alpha,\beta\}$ of type $(C,\alpha\beta)$ associated to pairs $\alpha \in (C_{-a})^*$ and $\beta \in (C_{-b})^*$, and general Koszul cocurved mixed complexes associated to finite sequences $\underline{\alpha}, \underline{\beta}$ are defined by taking the cotensor product of the $\{\alpha_i,\beta_i\}$. Finally, there is the folding by sums functor $\mathrm{fold}^\oplus : \mathrm{cMC}(C,\omega) \to \mathrm{cLF}(C,\omega)$ defined by

$$\mathrm{fold}^\oplus(X) \; := \; \bigoplus_{n\in\mathbb{Z}} X^{2n}\langle -nd\rangle \xrightarrow{\mathrm{d}+s} \bigoplus_{n\in\mathbb{Z}} X^{2n+1}\langle -nd\rangle \xrightarrow{\mathrm{d}+s} \bigoplus_{n\in\mathbb{Z}} X^{2n}\langle -nd\rangle;$$

this is the same as in Definition I.2.3.1 up to a shift by d in the internal grading of the degree 1 component. Again, $\mathrm{fold}^\oplus$ commutes with cotensor products and maps Koszul cocurved mixed complexes to the corresponding Koszul cofactorizations. In contrast to the classical situation for modules, we will not need the analogue of the folding by products functor fold^Π at all (luckily, as it is no longer exact, for example).

I.7.1.4. Some homotopy theory

We briefly sketch the definition of coderived categories of linear cofactorizations and cocurved mixed complexes, following [Pos11, §4]. As for algebras, these exist for categories of comodules over arbitrary curved differential graded (cdg) $\Bbbk$-coalgebras and we can construct them within the framework of abelian model categories.

In the following, let $(\Gamma, |\cdot|)$ be a grading group as in Notation II.2.3.1. A *curved differential graded (cdg)* $\Bbbk$*-coalgebra* is a Γ-graded $\Bbbk$-coalgebra C together with a map $\mathrm{d} : C \to \Sigma C$ of Γ-graded $\Bbbk$-modules and an element $\omega \in (C^{-2})^*$ such that $\mathrm{d}^2(x) = x{*}\omega - \omega{*}x$ for all $x \in C$. Given such a cdg $\Bbbk$-coalgebra C, a *(cdg) comodule* over C is a Γ-graded comodule M over the Γ-graded $\Bbbk$-coalgebra $C^\sharp$ underlying C together with a degree 1 endomorphism $\mathrm{d} : M \to \Sigma M$ which is a coderivation with respect to the coaction of C and satisfies $\mathrm{d}^2 = \omega * -$. The forgetful functor C-coMod $\to C^\sharp$-coMod is cocontinuous and monadic and has exact left and right adjoints $G^\pm$ as in Proposition II.2.3.2 from Part II. In particular, C-coMod is a locally Noetherian Grothendieck category as follows from Lemma II.B.3 and the fact that $C^\sharp$-coMod is locally Noetherian Grothendieck. Finally, denote $C\text{-coMod}_{\mathrm{inj}}$ the class of those C-comodules whose underlying $C^\sharp$-comodules are injective. Then, analogously to Proposition II.2.3.6 from Part II, we have:

Proposition I.7.1.11. *Let $\Bbbk$ be a field and C be a Γ-graded cdg $\Bbbk$-coalgebra. Then there exists a unique abelian model structure $\mathcal{M}^{\mathrm{co}}(C) = (\mathcal{C}, \mathcal{W}, \mathcal{F})$ on C*-coMod *with $\mathcal{C} = C$*-coMod *and $\mathcal{F} = C\text{-coMod}_{\mathrm{inj}}$. Moreover, $\mathcal{M}^{\mathrm{co}}(C)$ is cofibrantly generated.*

What's more, Positselski's constructive description of coacyclic modules – which was available in the ring-theoretic context only under some Noetherianness assumption on the ring (see Proposition II.2.3.13 from Part II) – holds without any conditions on C:

Proposition I.7.1.12. *In the context of Proposition I.7.1.11 the class of coacyclic cdg C-comodules is the smallest thick subcategory of C*-coMod *which is closed under coproducts and contains the totalizations of short exact sequences of C-comodules as well as all contractible C-comodules.*

Proof. See [Pos11, §3.6 and §4.4]. □

In Examples I.7.1.13, I.7.1.14 we fix a cocommutative $\Bbbk$-coalgebra C and $\omega \in C^*$.

Example I.7.1.13. The following is analogous to the paragraph following Definition I.2.1.1: Linear cofactorizations of type (C, ω) can be identified with comodules over the $\mathbb{Z}/2\mathbb{Z}$-graded curved $\Bbbk$-coalgebra C_ω given by $(C_\omega)^{\overline{0}} := C_\omega$, $(C_\omega)^{\overline{1}} := 0$ and cocurvature ω. In particular, there is a unique cofibrantly generated projective abelian model structure on $\mathrm{cLF}(C, \omega)$ with $\mathrm{cMF}(C, \omega)$ as the class of fibrant objects. Its homotopy category will be called the *coderived category of linear cofactorizations*, denoted $\mathbf{D}^{\mathrm{co}}\,\mathrm{cLF}(C, \omega)$. ◊

Example I.7.1.14. The following is analogous to the paragraph following Definition I.2.1.7: cocurved mixed complexes of type (C, ω) are comodules over the $\mathbb{Z}$-graded Koszul-coalgebra $K(C, \omega)$ that we describe now: Firstly, the $\mathbb{Z}$-graded $\Bbbk$-module underlying $K(C, \omega)$ is given by $K(C, \omega)^0 := C$, $K(C, \omega)^1 := C$ and $K(C, \omega)^i := 0$ for $i \neq 0, 1$. Secondly, the comultiplication $\Delta : K(C, \omega) \to K(C, \omega) \otimes_\Bbbk K(C, \omega)$ is given by

$$\Delta^0 := K(C,\omega)^0 = C \xrightarrow{\Delta} C \otimes_\Bbbk C = K(C,\omega)^0 \otimes_\Bbbk K(C,\omega)^0 = (K(C,\omega) \otimes_\Bbbk K(C,\omega))^0,$$

$$\begin{aligned}\Delta^1 &:= K(C,\omega)^1 = C \xrightarrow{\begin{pmatrix}\Delta & \Delta\end{pmatrix}} (C \otimes_\Bbbk C) \oplus (C \otimes_\Bbbk C) \\ &= \left(K(C,\omega)^1 \otimes_\Bbbk K(C,\omega)^0\right) \oplus \left(K(C,\omega)^0 \otimes_\Bbbk K(C,\omega)^1\right) \subset (K(C,\omega) \otimes_\Bbbk K(C,\omega))^1,\end{aligned}$$

and finally, the differential is given by $\mathrm{d}^0 : K(C,\omega)^0 = C \xrightarrow{*\omega} C = K(C,\omega)^1$. We elaborate on $K(C, \omega)$-coMod $\cong \mathrm{cMC}(C, \omega)$; it is only a matter of writing out definitions, but since working with comodules doesn't seem to be very common, we give some details: First, we claim that if X is a $\mathbb{Z}$-graded $\Bbbk$-module, endowing it with the structure of a comodule over $K(C, \omega)^\sharp$ is equivalent to providing each component of X with the structure of a C-comodule and to choosing a C-colinear map $s : X \to \Omega X$ with $s^2 = 0$.

For this, suppose X is endowed with a $K(C,\omega)^\sharp$-comodule structure. Then, since the projection of $K(C,\omega)^\sharp$ onto its degree 0 component C is a map of $\mathbb{Z}$-graded $\Bbbk$-coalgebras, any component of X inherits a C-comodule structure by composing the coaction of $K(C,\omega)^\sharp$ with $K(C,\omega)^\sharp \to C$; this is analogous to restriction of module structures to subrings. Further, the coassociativity of the $K(C,\omega)^\sharp$-coaction implies that $X \to K(C,\omega)^\sharp \otimes_\Bbbk X \to K(C,\omega)^1 \otimes_\Bbbk \Omega X = C \otimes_\Bbbk \Omega X$ is C-colinear (here the right hand side is endowed with the cofree C-comodule structure), hence of the form $X \xrightarrow{s} \Omega X \xrightarrow{\Delta} C \otimes_\Bbbk \Omega X$ for some unique $\Bbbk$-linear $s : X \to \Omega X$. Since $\Delta : \Omega X \to C \otimes_\Bbbk \Omega X$ is C-colinear and injective, it follows that $s : X \to \Omega X$ is even C-colinear. The coaction of $K(C,\omega)^\sharp$ is therefore given by the composition

$$X \xrightarrow{(\Delta\ s)} (C \underset{\Bbbk}{\otimes} X) \oplus \Omega X \xrightarrow{(\mathrm{id}\ \ \Delta\otimes\mathrm{id})} (C \underset{\Bbbk}{\otimes} X) \oplus (C \underset{\Bbbk}{\otimes} \Omega X) = K(C,\omega)^\sharp \underset{\Bbbk}{\otimes} X, \tag{I.7.1.1}$$

the coassociativity of which implies that $s^2 = 0$. Conversely, any family of C-comodule structures on the components on X together with a C-colinear map $s : X \to \Omega X$ satisfying $s^2 = 0$ gives rise to a $K(C,\omega)^\sharp$-coaction through (I.7.1.1). Finally, given such a $K(C,\omega)^\sharp$-comodule X, a $\Bbbk$-linear map $\mathrm{d} : X \to \Sigma X$ is a $K(C,\omega)^\sharp$-coderivation if and only if $\mathrm{d}\, s + s\, \mathrm{d} = *\omega$ (details omitted), finishing the sketch of $K(C,\omega)\text{-coMod} \cong \mathrm{cMC}(C,\omega)$. In particular, we get a projective abelian model structure $\mathcal{M}^{\mathrm{co}}\,\mathrm{cMC}(C,\omega)$ on $\mathrm{cMC}(C,\omega)$ in which the fibrant objects are those ω-cocurved mixed complexes (X, d, s) for which (X, s) is an injective $K(C,\omega)^\sharp$-module, i.e. (by the analogue of Lemma II.2.3.3 from Part II for cdg $\Bbbk$-coalgebras) for which (X, s) is contractible with injective components. The homotopy category of $\mathcal{M}^{\mathrm{co}}\,\mathrm{cMC}(C,\omega)$ will be called the *coderived category of ω-cocurved mixed complexes*, denoted $\mathbf{D}^{\mathrm{co}}\,\mathrm{cMC}(C,\omega)$. Analogously to Proposition I.4.1.10 we have that bounded below acyclic cocurved mixed complexes are coacyclic, see [Pos11, Theorem 4.3.1]. ♢

Now we can ripe the fruits of our detour to comodules. To begin, the cotensor product functors for linear cofactorizations and cocurved mixed complexes give rise to functors

$$- \boxtimes^{\mathbf{R}}_C - : \mathbf{D}^{\mathrm{co}}\,\mathrm{cLF}(C,\omega) \times \mathbf{D}^{\mathrm{co}}\,\mathrm{cLF}(C,\omega') \to \mathbf{D}^{\mathrm{co}}\,\mathrm{cLF}(C,\omega+\omega'), \tag{I.7.1.2}$$

$$- \boxtimes^{\mathbf{R}}_C - : \mathbf{D}^{\mathrm{co}}\,\mathrm{cMC}(C,\omega) \times \mathbf{D}^{\mathrm{co}}\,\mathrm{cMC}(C,\omega') \to \mathbf{D}^{\mathrm{co}}\,\mathrm{cMC}(C,\omega+\omega'), \tag{I.7.1.3}$$

defined by taking fibrant resolutions in one of the two factors. This is well-defined since, as explained in [Pos11, §4.7], the coflatness of cofree comodules and the explicit description of coacyclic comodules from Proposition I.7.1.12 show that the cotensor product of a coacyclic C-comodule with a $C^\sharp$-injective C-comodule is again coacyclic.

Next, we come to the main point why we switched to comodules: the folding by sums functor can be computed naively *and* commutes with coderived tensor products:

Proposition I.7.1.15. *The folding by sums functor* $\mathrm{fold}^{\oplus} : \mathrm{cMC}(C,\omega) \to \mathrm{cLF}(C,\omega)$ *is a left Quillen functor* $\mathcal{M}^{\mathrm{co}}\,\mathrm{cMC}(C,\omega) \to \mathcal{M}^{\mathrm{co}}\,\mathrm{cLF}(C,\omega)$*. In particular, it descends naively to a functor* $\mathbf{L}\,\mathrm{fold}^{\oplus} : \mathbf{D}^{\mathrm{co}}\,\mathrm{cMC}(C,\omega) \to \mathbf{D}^{\mathrm{co}}\,\mathrm{cLF}(C,\omega)$*.*

Proof. We have to show that $\mathrm{fold}^{\oplus}$ preserves cofibrations and trivial cofibrations. By definition, the cofibrations in $\mathcal{M}^{\mathrm{co}}\,\mathrm{cMC}(C,\omega)$ and $\mathcal{M}^{\mathrm{co}}\,\mathrm{cLF}(C,\omega)$ are just the monomorphisms, while the trivial cofibrations are the monomorphisms with coacyclic cokernel. Since $\mathrm{fold}^{\oplus}$ is exact it is therefore sufficient to show that it maps coacyclic cocurved mixed complexes to coacyclic linear cofactorizations, which in turn follows from the explicit description of coacyclic comodules from Proposition I.7.1.12 and the observation that $\mathrm{fold}^{\oplus}$ commutes both with coproducts and with totalizations. □

As in the case of modules, Koszul cocurved complexes and Koszul cofactorizations can often be described in terms of single comodules, up to coderived equivalence. For this, we need the analogue of the notion of a regular sequence in the context of comodules:

Definition I.7.1.16. *Let* C *be a cocommutative* $\Bbbk$*-coalgebra and* M *be a* C*-comodule. An element* $\alpha \in C^*$ *is called* M-coregular *if* $M \xrightarrow{*\alpha} M$ *is surjective. Inductively, a sequence* $\alpha_1, \ldots, \alpha_n \in C^*$ *is* M*-coregular if either* $n = 0$ *or if* α_1 *is* M*-coregular and* $\alpha_2, \ldots, \alpha_n$ *is coregular for the* C*-comodule* $\ker(M \xrightarrow{*\alpha_1} M)$*.*

Remark I.7.1.17. Definition I.7.1.16 is in agreement with the notion of coregular sequences introduced in [Tan04, §2] for Artinian modules: namely, if C is the $\Bbbk$-coalgebra from Example I.7.1.2 the dual of which is the power series ring $\Bbbk[\![t_1, \ldots, t_n]\!]$, then a sequence $\alpha_1, \ldots, \alpha_n \in C^* = \Bbbk[\![t_1, \ldots, t_n]\!]$ is C-coregular in the sense of Definition I.7.1.16 if and only if it is coregular for the Artinian C^*-module $C \cong E(\Bbbk) \cong \mathrm{H}^n_{\mathfrak{m}}(\Bbbk[\![t_1, \ldots, t_n]\!])$ in the sense of [Tan04]. ♢

Fact I.7.1.18. *If in the situation of Definition I.7.1.16 the sequence* $\underline{\alpha} = (\alpha_1, \ldots, \alpha_n)$ *is* M*-coregular and if* $\mathrm{K}(M;\underline{\alpha}) := \bigotimes_{i=1}^{n} \left(\underline{M} \xrightarrow{*\alpha_i} M\right)$ *is the* Koszul complex *of* $\underline{\alpha}$*, then the canonical morphism* $\bigcap_{i=1}^{n} (M \xrightarrow{*\alpha_i} M) \to \mathrm{K}(M;\underline{\alpha})$ *is a quasi-isomorphism.*

Proposition I.7.1.19. *Let* C *be a cocommutative* $\Bbbk$*-coalgebra,* $\underline{\alpha} = (\alpha_1, \ldots, \alpha_n)$ *be a* C*-coregular sequence in* C^* *and* $\underline{\beta} = (\beta_1, \ldots, \beta_n)$ *be any sequence in* C^**. Then, putting* $\omega := \sum_i \alpha_i\beta_i \in C^*$*, the canonical morphism* $_{\omega}\left[\bigcap_{i=1}^{n} \left(C \xrightarrow{*\alpha_i} C\right)\right] \to \{\underline{\alpha}, \underline{\beta}\}$ *is an isomorphism in* $\mathbf{D}^{\mathrm{co}}\,\mathrm{cLF}(C,\omega)$ *and* $\mathbf{D}^{\mathrm{co}}\,\mathrm{cMC}(C,\omega)$*.*

I.7.1.5. Khovanov-Rozanky homology via comodules

With the above preparation the construction of Khovanov-Rozansky homology in the context of comodules proceeds along the same lines as in Section I.3, with the differences that firstly we are now working over polynomial $\Bbbk$-coalgebras instead of polynomial rings over $\Bbbk$ and that secondly we are using cotensor products of the Koszul cofactorizations from Example I.7.1.9 instead of tensor products of Koszul factorizations.

Proposition I.7.1.20. *The above variant of Khovanov-Rozansky homology defined via comodules agrees with ordinary Khovanov-Rozansky homology* $\mathcal{KR}^k_{\Bbbk}$ *up to* $q \leftrightarrow q^{-1}$.

Proof. First, note that we may slightly alter the construction as follows without affecting the outcome: Instead of defining the Koszul (co)factorizations assigned to each elementary piece of the chosen planar oriented link diagram (that is, each crossing or unknotted strand) over the "local" polynomial (co)algebra with variables indexed by the open ends of the current elementary piece only, and (co)tensoring these together along their common variables afterwards, we may instead work solely with the "global" polynomial (co)algebra with variables indexed over all cutting points and define and (co)tensor all Koszul (co)factorizations over this big (co)algebra. While this does not change the resulting complex of matrix (co)factorizations in either of the two constructions, it shows that the complex of matrix cofactorizations we obtain in the construction using comodules is – when viewed as a complex of linear factorizations – the componentwise Matlis dual, again with respect to the global polynomial algebra, of the complex of matrix factorizations obtained in the original construction. Since Matlis duality is exact, it follows that also the vertical-then-horizontal cohomology comodules in the comodule construction of Khovanov-Rozansky homology are the Matlis duals of the vertical-then-horizontal cohomology modules in the ordinary construction of Khovanov-Rozansky homology. Since Matlis duality preserves the length of modules (which here agrees with the $\Bbbk$-dimension) and changes the sign of the grading, the claim follows. □

As intended by our change to comodules, it is now simple and technically refreshingly pleasant to prove the analogue of Theorem I.5.2, namely that, when defined via comodules, Khovanov-Rozansky homology can be described via the analogue of stable Hochschild homology of Soergel bimodules in the context of comodules.

Definition I.7.1.21. *Let* C *be a cocommutative* $\Bbbk$*-coalgebra and let* $\widehat{C} := C \otimes_{\Bbbk} C$ *be its enveloping* $\Bbbk$*-coalgebra. Further, let* $\omega \in C^*$ *and denote* $\widehat{\omega} := \omega \otimes \eta - \eta \otimes \omega$*, where* $\eta : C \to \Bbbk$ *is the counit of* C*. Finally, let* M *be a* C*-bicomodule such that* $*\widehat{\omega}$ *vanishes on* M*. Then the* ω-stable Hochschild homology *of* M *is defined as*

$$ {}^{\omega}\,\mathrm{sHH}_*^{C/\Bbbk}(M) \;:=\; \mathrm{H}^*\left[{}_{-\omega}\Delta \boxtimes^{\mathbf{R}}_{\widehat{C}} {}_{\omega}M\right]. $$

Here Δ is the diagonal $\widehat{C}$-comodule given by $\Delta = C$ as $\Bbbk$-modules, and with $\widehat{C}$-coaction

$$\Delta = C \xrightarrow{(\Delta\otimes\mathrm{id})\circ\Delta} C \otimes_{\Bbbk} C \otimes_{\Bbbk} C = \widehat{C} \otimes_{\Bbbk} \Delta.$$

We next define the analogues of Soergel bimodules in the context of comodules. For that, we denote by ${}^{\mathrm{co}}\mathbb{A}^n_{\Bbbk}$ the polynomial $\Bbbk$-coalgebra in n variables (see Example I.7.1.2) and by ${}^{\mathrm{co}}\widehat{\mathbb{A}}^n_{\Bbbk}$ its enveloping $\Bbbk$-coalgebra, so that naturally $({}^{\mathrm{co}}\mathbb{A}^n_{\Bbbk})^* \cong \mathbb{A}^n_{\Bbbk}$ and $({}^{\mathrm{co}}\widehat{\mathbb{A}}^n_{\Bbbk})^* \cong \widehat{\mathbb{A}}^n_{\Bbbk}$.

Definition I.7.1.22. *For $n \geq 2$, $1 \leq i < n$ we define the ${}^{\mathrm{co}}\widehat{\mathbb{A}}^n_{\Bbbk}$-comodule $C^{n,i}_{\Bbbk}$ as*

$$C^{n,i}_{\Bbbk} \;:=\; \ker\left[{}^{\mathrm{co}}\widehat{\mathbb{A}}^n_{\Bbbk} \xrightarrow{*(x_i+x_{i+1}-y_i-y_{i+1})} {}^{\mathrm{co}}\widehat{\mathbb{A}}^n_{\Bbbk}\right] \cap \ker\left[{}^{\mathrm{co}}\widehat{\mathbb{A}}^n_{\Bbbk} \xrightarrow{*(x_ix_{i+1}-y_iy_{i+1})} {}^{\mathrm{co}}\widehat{\mathbb{A}}^n_{\Bbbk}\right].$$

If $\boldsymbol{i} = (i_1, \ldots, i_k)$ is a sequence in $\{1, 2, \ldots, n-1\}$, we define the ${}^{\mathrm{co}}\widehat{\mathbb{A}}^n_{\Bbbk}$-comodule $C^{n,\boldsymbol{i}}_{\Bbbk}$ as

$$C^{n,\boldsymbol{i}}_{\Bbbk} \;:=\; C^{n,i_1}_{\Bbbk} \underset{{}^{\mathrm{co}}\mathbb{A}^n_{\Bbbk}}{\boxtimes} C^{n,i_2}_{\Bbbk} \underset{{}^{\mathrm{co}}\mathbb{A}^n_{\Bbbk}}{\boxtimes} \cdots \underset{{}^{\mathrm{co}}\mathbb{A}^n_{\Bbbk}}{\boxtimes} C^{n,i_k}_{\Bbbk}.$$

Fact I.7.1.23. *$C^{n,\boldsymbol{i}}_{\Bbbk}$ is the Matlis dual of $B^{n,\boldsymbol{i}}_{\Bbbk}$ with respect to $\widehat{\mathbb{A}}^n_{\Bbbk}$.*

Proof. First, note that we may either consider $C^{n,\mathbf{i}}_{\Bbbk}$ and $B^{n,\mathbf{i}}_{\Bbbk}$ as (co)modules over ${}^{\mathrm{co}}\widehat{\mathbb{A}}^n_{\Bbbk}$ resp. $\widehat{\mathbb{A}}^n_{\Bbbk}$ or as (co)modules over the "global" polynomial $\Bbbk$-(co)algebra involving all variables, i.e. including those over which the (co)tensor product defining $C^{n,\mathbf{i}}_{\Bbbk}$ resp. $B^{n,\mathbf{i}}_{\Bbbk}$ was taken – these are $n \cdot (|\mathbf{i}| + 1)$ in total. For the statement, however, the choice of base (co)algebra is irrelevant, since we have seen in Example I.7.1.3 that Matlis duality is naturally isomorphic to $\Bbbk$-duality. We may therefore prove the claim for the global polynomial $\Bbbk$-algebra – denote it $\mathbb{A} := \Bbbk[x^{(k)}_i \mid 1 \leq k \leq |\mathbf{i}| + 1,\ 1 \leq i \leq n]$. Over $\mathbb{A}$, the right exactness of the tensor product shows that $B^{n,\mathbf{i}}_{\Bbbk}$ can be written as the quotient of $\mathbb{A}$ by the elements $x^{(k)}_{i_k} + x^{(k)}_{i_k+1} - x^{(k+1)}_{i_k} - x^{(k+1)}_{i_k+1}$ and $x^{(k)}_{i_k} x^{(k)}_{i_k+1} - x^{(k+1)}_{i_k} x^{(k+1)}_{i_k+1}$ for $1 \leq k \leq |\mathbf{i}|$, and hence the $\Bbbk$-dual of $B^{n,\mathbf{i}}_{\Bbbk}$ is isomorphic to the intersection of the kernels of the multiplication maps by these elements in the $\Bbbk$-dual $\mathrm{Hom}_{\Bbbk}(\mathbb{A}, \Bbbk)$ of $\mathbb{A}$. This in turn is isomorphic to $C^{n,\mathbf{i}}_{\Bbbk}$ by the left exactness of the cotensor product. □

One can now define the analogues of Rouquier complexes using the comodules $C^{n,\mathbf{i}}_{\Bbbk}$, and prove – without the technical difficulties encountered when working with modules – that firstly the Khovanov-Rozansky construction, when carried out via comodules, can be described as w_n-stable Hochschild homology of Rouquier complexes of comodules, and that secondly the latter is invariant under the two Markov moves and hence gives rise to an invariant of oriented links. We omit precise formulations and details here as they wouldn't be of any additional insight anymore.

I.7.1.6. Koszul duality for matrix factorizations

Describing Khovanov-Rozansky homology in terms of comodules opens the possibility to view it from the perspective of Koszul duality for linear (co)factorizations [Pos11; Tu14]. In this section, we will quickly recall the relevant background and afterwards describe which questions one might pursue in this direction.

Let C be a cdg coalgebra over a field $\Bbbk$, and let $w : \Bbbk \to C$ be a $\Bbbk$-linear section of the counit $\varepsilon : C \to \Bbbk$. Further, denote $\mathrm{Cob}_w(C)$ the associated *cobar cdg algebra* in the sense of [Pos11, §6.1]. Positselski proves the following beautiful Koszul duality statements:

Theorem I.7.1.24 [Pos11, Theorem 6.7]**.** *For a cdg coalgebra C and a section $w : \Bbbk \to C$ of its counit, there is a commutative diagram of equivalences of triangulated categories*

$$\begin{array}{ccccc} \mathbf{D}^{\mathrm{co}}(\mathrm{Cob}_w(C)\text{-Mod}) & = & \mathbf{D}^{\mathrm{abs}}(\mathrm{Cob}_w(C)\text{-Mod}) & = & \mathbf{D}^{\mathrm{ctr}}(\mathrm{Cob}_w(C)\text{-Mod}) \\ \cong\downarrow & & & & \cong\downarrow \\ \mathbf{D}^{\mathrm{co}}(C\text{-coMod}) & & \overset{\cong}{\rightleftarrows} & & \mathbf{D}^{\mathrm{ctr}}(C\text{-ctrMod}). \end{array} \tag{I.7.1.4}$$

The lower equivalences are the comodule-contramodule-correspondence *[Pos11, §5].*

Theorem I.7.1.25 [Pos11, Theorem 6.4, Corollary 6.7]**.** *If in Theorem I.7.1.24 the cdg coalgebra C is conilpotent with coaugmentation $w : \Bbbk \to C$, then* $\mathrm{Cob}_w(C)$ *is a dg algebra and* $\mathbf{D}(A)$ *coincides with* $\mathbf{D}^{\mathrm{ctr}}(A) = \mathbf{D}^{\mathrm{abs}}(A) = \mathbf{D}^{\mathrm{co}}(A)$*. In particular, the commutative diagram of equivalences* (I.7.1.4) *persists with the upper row enlarged by* $\mathbf{D}(A)$*.*

In [Tu14], Tu uses the homological perturbation lemma to give alternative and explicit proofs for the above results. Further, for a polynomial coalgebra $C := \Bbbk[x_1, \ldots, x_n]$ and any power series $w \in (x_1, \ldots, x_n)^2 \subset \Bbbk[\![x_1, \ldots, x_n]\!] = C^*$ without constant or linear terms, he considers Koszul duality for the $\mathbb{Z}/2\mathbb{Z}$-graded conilpotent cdg coalgebra C_ω that we introduced in Example I.7.1.13, and the comodules over which we have seen to be the linear cofactorizations of type (C, ω). In this case, the cohomology algebra of $\mathrm{Cob}(C_\omega)$ is isomorphic to the exterior algebra $\bigwedge^* \Bbbk^n$ [Tu14, Corollary 4.5], and hence $\bigwedge^* \Bbbk^n$ inherits an A_∞-structure from $\mathrm{Cob}(C_\omega)$ such that, as an A_∞-algebra, it is quasi-isomorphic $\mathrm{Cob}(C_\omega)$.

Now, it would be interesting to study firstly whether this A_∞-structure can be made explicit in case of the Khovanov-Rozansky potential $\sum_i x_i^{k+1}$, and secondly if one can gain additional insight into Khovanov-Rozansky homology by applying the above Koszul duality to rewrite our description of Khovanov-Rozanksy homology via comodules in terms of A_∞-modules. We leave this as a possibility for further study.

I.7.2. Khovanov-Rozansky homology categorifies quantum $\mathfrak{sl}(k)$-invariant

The following proposition is of no surprise and for $\Bbbk = \mathbb{Q}$ known from [KR08a], but for completeness we include a proof here.

Proposition I.7.2.1. *For $\Bbbk$ a field with $\operatorname{char}\Bbbk \nmid k+1$, $\mathcal{KR}^k_\Bbbk|_{a=-1}$ agrees with the quantum invariant $\mathcal{P}^k(q^{-1})$ for links labeled by the vector representation of $\mathcal{U}_q(\mathfrak{sl}(k))$.*

Proof. The quantum $\mathfrak{sl}(k)$-invariant $\mathcal{P}^k$ is determined by its value $\mathcal{P}^k(\bigcirc) = [k]_q$ on the unknot as well as the Skein relation $q^k\mathcal{P}^k(L_{\slashoverback}) - q^{-k}\mathcal{P}^k(L_{\backoverslash}) = (q-q^{-1})L_{\uparrow\uparrow}$; it is therefore sufficient to show that $\mathcal{KR}^k_\Bbbk|_{a=-1}$ satisfies $\mathcal{KR}^k_\Bbbk(\bigcirc)|_{a=-1} = [k]_{q^{-1}} = [k]_q$ and

$$q^k\ \mathcal{KR}^k_\Bbbk(L_{\backoverslash})|_{a=-1} - q^{-k}\ \mathcal{KR}^k_\Bbbk(L_{\slashoverback})|_{a=-1} = (q-q^{-1})\mathcal{KR}^k_\Bbbk(L_{\uparrow\uparrow})|_{a=-1}. \tag{I.7.2.5}$$

By definition, $\mathcal{KR}^k_\Bbbk(\bigcirc)$ is the Poincaré polynomial of ${}^k\operatorname{sHH}_t^{\mathbb{A}^1_\Bbbk/\Bbbk}(\Delta)$. By a short calculation, see e.g. (I.6.4.20), we find that ${}^k\operatorname{sHH}_1^{\mathbb{A}^1_\Bbbk/\Bbbk}(\Delta) \cong \Bbbk\langle 2k\rangle \oplus \Bbbk\langle 2k-2\rangle \oplus \cdots \oplus \Bbbk$ while ${}^k\operatorname{sHH}_0^{\mathbb{A}^1_\Bbbk/\Bbbk}(\Delta) = 0$, hence

$${}^k\operatorname{sHH}_t^{\mathbb{A}^1_\Bbbk/\Bbbk}(\Delta) = \Bbbk\langle k-1\rangle \oplus \Bbbk\langle k-3\rangle \oplus \cdots \oplus \Bbbk\langle -k+3\rangle \oplus \Bbbk\langle -k-1\rangle,$$

proving $\mathcal{KR}^k_\Bbbk(\bigcirc)|_{a=-1} = [k]_q$ (recall Definition I.2.1.2 for the shifting in H^1 and H^t).

For the Skein relation, note first that for any oriented link L presented as the closure of a braid β on n strands we have

$$\begin{aligned}\mathcal{KR}^k_\Bbbk(L)|_{a=-1} &= (-q)^{(k+1)w(\beta)} \sum_{i,j\in\mathbb{Z}} (-1)^i \dim_\Bbbk \mathrm{H}^i\left[{}^k\operatorname{sHH}_t^{\mathbb{A}^n_\Bbbk/\Bbbk}(\mathcal{RC}_\Bbbk(\beta))_j\right] q^j \\ &= (-q)^{(k+1)w(\beta)} \sum_{i,j\in\mathbb{Z}} (-1)^i \dim_\Bbbk {}^k\operatorname{sHH}_t^{\mathbb{A}^n_\Bbbk/\Bbbk}(\mathcal{RC}^i_\Bbbk(\beta))_j\ q^j. \end{aligned} \tag{I.7.2.6}$$

since taking cohomology doesn't change the Euler characteristic. By definition of $\mathcal{RC}_\Bbbk(\beta)$ (see the paragraph preceding Theorem I.5.3), the right hand side in (I.7.2.6) can be computed as a "state sum" as follows: Resolve any crossing in β by $\uparrow\uparrow$ or a wide edge X, obtaining $2^{n_+(\beta)+n_-(\beta)}$ braid-like diagrams Γ consisting of unknotted strands or wide edges only. In any resolution, keep track of local weights $-q^{-2}$ and q^{-2} (resp. $-q^2$ and 1) for resolutions of positive (resp. negative) crossings by $\uparrow\uparrow$ and X, respectively, and define the weight $w(\Gamma)$ of the resolution Γ as the product of its local weights. Then, denoting the Soergel bimodule corresponding to Γ by $B^{n,\Gamma}_\Bbbk$ (see Notation I.A.2.2),

$$\sum_{i,j\in\mathbb{Z}} (-1)^i \dim_\Bbbk {}^w\operatorname{sHH}_t^{\mathbb{A}^n_\Bbbk/\Bbbk}(\mathcal{RC}^i_\Bbbk(\beta))_j\ q^j = \sum_\Gamma \dim_\Bbbk {}^k\operatorname{sHH}_t^{\mathbb{A}^n_\Bbbk/\Bbbk}(B^{n,\Gamma}_\Bbbk)_j w(\Gamma) q^j.$$

Temporarily denoting this term by $Q(\beta)$, we therefore have

$$qQ(\beta_{\backoverslash}) - q^{-1}Q(\beta_{\slashoverback}) = (q-q^{-1})Q(\beta_{\uparrow\uparrow})$$

if $\beta_{\backoverslash}, \beta_{\slashoverback}$ and $\beta_{\uparrow\uparrow}$ are braids that differ in only one crossing in a way according to their subscript. Taking into account the normalization by $(-q)^{(k+1)w(\beta)}$ we obtain (I.7.2.5). □

I.7.3. Towards a description of KR homology in terms of Frobenius algebras

In this section, we assume that $\Bbbk$ is a commutative ring such that $k+1 \in \Bbbk^\times$. Then, we have already seen that the value of the unknot under $\mathcal{KR}^k_\Bbbk$ is given by

$$^k\,\mathrm{sHH}_t^{\mathbb{A}^1_\Bbbk/\Bbbk}(\Delta) \quad\cong\quad \Bbbk[x]/(x^k)\langle k-1\rangle,$$

the Frobenius algebra associated to the potential x^{k+1}. In particular, for $k=2$ we recover the algebra of dual numbers $\Bbbk[x]/(x^2)$ underlying Khovanov homology [Kho00] (see also [BN02; BN05]). Now, it was a folklore result for some time and recently proved rigorously in [Hug14] that $\mathcal{KR}^2$ coincides with Khovanov homology, and in view of the beautiful and simple original construction of the latter, completely in terms of the Frobenius algebra $\mathbb{Z}[x]/(x^2)$ and its structure maps, it is natural to ask if and to what extend also Khovanov-Rozansky homology is "controlled" by the Frobenius algebra $\Bbbk[x]/(x^k)$. This is what we study in the present and the next section. We let $A := \Bbbk[x]/(x^k)$ and denote the multiplication and comultiplication maps of A by $\mu : A\otimes_\Bbbk A \to A$ and $\Delta : A \to A\otimes_\Bbbk A\langle 2k-2\rangle$, respectively; the comultiplication Δ is determined by $\Delta(1) = \sum_{i=0}^{k-1} x^i \otimes x^{k-1-i}$ and the property of being a morphism of A-bimodules.

Although Proposition I.6.4.4 is sufficient for understanding the effect of the second Markov move on stable Hochschild homology, the commutative diagram (I.6.4.19) in its proof actually shows how the comultiplication of the Frobenius algebra A is hidden in Khovanov-Rozansky homology; we keep Notation I.6.3.1:

Proposition I.7.3.1. *There is a commutative diagram*

$$\begin{array}{ccc} ^k\,\mathrm{sHH}_t^{\mathbb{A}^2_\Bbbk/\Bbbk}(X) & \xrightarrow{\ \delta\ } & ^k\,\mathrm{sHH}_t^{\mathbb{A}^2_\Bbbk/\Bbbk}(\Delta)\langle k-1\rangle \\ \cong\downarrow & & \downarrow\cong \\ A\langle k-1\rangle & \xrightarrow[(k+1)\cdot\Delta]{} & A\otimes_\Bbbk A\,\langle 3k-3\rangle \end{array} \tag{I.7.3.7}$$

where δ is the connecting homomorphism assigned to $0\to\Delta\langle -2\rangle \to B \to X \to 0$.

For the shifts, recall that $^k\,\mathrm{sHH}_t \circ\Sigma = {}^k\,\mathrm{sHH}_t\langle k+1\rangle$ and $^k\,\mathrm{sHH}_t^{\mathbb{A}^1_\Bbbk/\Bbbk}(\Delta) \cong A\langle k-1\rangle$.

We also have the following dual statement:

Proposition I.7.3.2. *There is a commutative diagram*

$$\begin{array}{ccc} ^k\,\mathrm{sHH}_t^{\mathbb{A}^2_\Bbbk/\Bbbk}(\Delta) & \xrightarrow{\ \delta\ } & ^k\,\mathrm{sHH}_t^{\mathbb{A}^2_\Bbbk/\Bbbk}(X)\langle k-1\rangle \\ \cong\downarrow & & \downarrow\cong \\ A\otimes_\Bbbk A\,\langle 2k-2\rangle & \xrightarrow[\mu]{} & A\langle 2k-2\rangle \end{array} \tag{I.7.3.8}$$

where δ is the connecting homomorphism assigned to $0 \to X\langle -2\rangle \to B \to \Delta \to 0$.

Proof. As in Proposition I.6.4.4 we will show the stronger statement that

$$\begin{array}{ccc} \mathbb{V}_2 \operatorname{Tr}_2 \Delta & \xrightarrow{\quad\delta\quad} & \Sigma\mathbb{V}_2 \operatorname{Tr}_2 X\langle -2\rangle \\ \cong\downarrow & & \downarrow\cong \\ \Sigma\left(\Delta_1\langle -2\rangle \oplus \Delta_1\langle -4\rangle \oplus \cdots \oplus \Delta_1\langle -2k\rangle\right) & \xrightarrow[\left(\mathrm{id}\ \ \cdot x_1\ \ \ldots\ \ \cdot x_1^{k-1}\right)]{} & \Sigma\Delta_1\langle -2\rangle \end{array}$$

commutes, where the vertical isomorphisms are analogous to the ones used in Proposition I.6.4.4. To start, Proposition II.C.1.1 implies that the connecting homomorphism δ is given by the following roof; recall $u_k(x,y) = \frac{x^{k+1}-y^{k+1}}{x-y} = x^k + x^{k-1}y + \cdots + xy^{k-1} + y^k$:

$$\operatorname{Cone}\left(\begin{array}{ccc} X\langle -4\rangle & \overset{y_2-x_2=x_1-y_1}{\underset{u_k(x_2,y_2)=u_k(x_1,y_1)}{\rightleftarrows}} & X\langle -2\rangle \\ \downarrow x_1-y_1 & & \downarrow x_1-y_1 \\ B\langle -2\rangle & \overset{y_2-x_2=x_1-y_1}{\underset{u_k(x_2,y_2)}{\rightleftarrows}} & B \end{array}\right) =: C$$

$$\begin{array}{ccc} \swarrow \alpha, \simeq & & \searrow \sigma \\ \left(\Delta\langle -2\rangle \overset{0}{\underset{(k+1)x_2^k}{\rightleftarrows}} \Delta\right) & & \Sigma\left(X\langle -4\rangle \overset{y_2-x_2=x_1-y_1}{\underset{u_k(x_2,y_2)=u_k(x_1,y_1)}{\rightleftarrows}} X\langle -2\rangle\right) \\ \simeq\downarrow\beta & & \rho\downarrow\simeq \\ \Sigma\Big(\Delta_1\langle -2\rangle \oplus \Delta_1\langle -4\rangle \oplus \cdots \oplus \Delta_1\langle -2k\rangle\Big) & \dashrightarrow & \Sigma\Delta_1 \end{array}$$

After base change along the resolution

$$\begin{array}{c} \bigoplus_{i=1}^{k}\left(\Bbbk[x_1,y_1]\langle -2\rangle \overset{u_k(x_1,y_1)}{\underset{x_1-y_1}{\rightleftarrows}} \Bbbk[x_1,y_1]\langle 2k-2\rangle\right)\langle -2i\rangle \\ \downarrow\gamma \\ \Sigma\Big(\Delta_1\langle -2\rangle \oplus \Delta_1\langle -4\rangle \oplus \cdots \oplus \Delta_1\langle -2k\rangle\Big) \end{array}$$

the composition $\beta \circ \alpha$ can be split by maps

$$\bigoplus_{i=1}^{k}\left(\Bbbk[x_1,y_1]\langle -2\rangle \overset{u_k(x_1,y_1)}{\underset{x_1-y_1}{\rightleftarrows}} \Bbbk[x_1,y_1]\langle 2k-2\rangle\right)\langle -2i\rangle \xrightarrow{\tau=(\tau_1,\ldots,\tau_k)} C$$

given as follows: τ_i sends the identity element $1 \in \Bbbk[x_1,y_1]\langle 2k-2i-2\rangle$ sitting in even cohomological degree to $x_1^i \in X\langle -4\rangle$ in the upper left corner of C, and the identity element $1 \in \Bbbk[x_1, y_1]\langle -2-2i\rangle$ sitting in odd cohomological degree to the sum of $x_1^i \in B\langle -2\rangle$ in the upper left corner of C and $x_1^i \in X\langle -2\rangle$ in the upper right corner of C; due to the shift in q-grading by $2k+2$ applied to the upper left corner $X\langle -4\rangle$ when forming the cone C, this is indeed degree preserving. Moreover, a quick check shows that τ is indeed a morphism of linear factorizations splitting such that $\beta \circ \alpha \circ \tau = \gamma$, as claimed. This shows that $\delta : \Sigma\left(\Delta_1\langle -2\rangle \oplus \Delta_1\langle -4\rangle \oplus \cdots \oplus \Delta_1\langle -2k\rangle\right) \to \Sigma\Delta_1\langle -2\rangle$ can also be described by the roof $(\rho \circ \sigma \circ \tau) \circ \gamma^{-1}$:

$$\begin{array}{ccc} & \displaystyle\bigoplus_{i=1}^{k}\left(\Bbbk[x_1, y_1]\langle -2\rangle \underset{x_1 - y_1}{\overset{u_k(x_1,y_1)}{\rightleftarrows}} \Bbbk[x_1, y_1]\langle 2k-2\rangle \right)\langle -2i\rangle & \\ \swarrow^{\gamma} & & \searrow^{\rho\circ\sigma\circ\tau} \\ \Sigma\left(\Delta_1\langle -2\rangle \oplus \Delta_1\langle -4\rangle \oplus \cdots \oplus \Delta_1\langle -2k\rangle\right) & \xrightarrow{\begin{pmatrix}\mathrm{id} & \cdot x_1 & \ldots & \cdot x_1^{k-1}\end{pmatrix}} & \Sigma\Delta_1\langle -2\rangle \end{array}$$

Now the dashed arrow makes the diagram commute, so the claim follows. □

We can use the above observations to approximate the k-stable Hochschild homology of Rouquier complexes through a Khovanov-homology like formalism with the Frobenius algebra $\Bbbk[x]/(x^k)$ in place of $\Bbbk[x]/(x^2)$, and this is what we describe in this and the next section. Similar results where also obtained independently by Abel-Rozansky [AR14].

Notation I.7.3.3. In the following, we denote $\mathrm{Co}(f)$ the (non-canonical, non-functorial) cone of a morphism $f : X \to Y$ in a triangulated category $\mathcal{T}$, the choice of a distinguished triangle $X \to Y \to \mathrm{Co}(f) \to \Sigma X$ implicit. In contrast, we reserve the notation $\mathrm{Cone}(f)$ for situations where $\mathcal{T}$ arises as the homotopy category of some category $\mathcal{C}$ equipped with an explicit functorial cone construction Cone (recall e.g. Definition I.2.1.2 in the case of linear factorizations) and where f is presented as a morphism in $\mathcal{C}$. ♢

The idea is the following: Firstly, the components of $\mathcal{CCKR}^k_{\Bbbk}$ are contraderived tensor products of copies of ${}_{\widehat{w}_2}B$ and ${}_{\widehat{w}_1}\Delta$. Secondly, there are isomorphisms in $\mathbf{D}^{\mathrm{ctr}}\,\mathrm{LF}(\widehat{\mathbb{A}}^2_{\Bbbk}, \widehat{w}_2)$,

$${}_{\widehat{w}_2}B \cong \mathrm{Co}\left(\Sigma^{-1}{}_{\widehat{w}_2}\Delta \xrightarrow{\Sigma^{-1}\delta^+} {}_{\widehat{w}_2}X\langle -2\rangle\right) \cong \mathrm{Co}\left(\Sigma^{-1}{}_{\widehat{w}_2}X \xrightarrow{\Sigma^{-1}\delta^-} {}_{\widehat{w}_1}\Delta\langle -2\rangle\right), \quad \text{(I.7.3.9)}$$

with $\delta^{\pm}$ the connecting morphisms associated to the short exact sequences

$$0 \to X\langle -2\rangle \to B \to \Delta \to 0 \qquad \text{and} \qquad 0 \to \Delta\langle -2\rangle \to B \to X \to 0$$

from Fact I.A.2.9; more precisely, whatever choices have been made for $\mathrm{Co}(\Sigma^{-1}\delta^{\pm})$ – for example, they might be the functorial cones of representatives of $\delta^{\pm}$ as morphisms of linear factorizations – the isomorphisms (I.7.3.9) are (non-canonical) choices for the dashed arrows in the following morphisms of distinguished triangles in $\mathbf{D}^{\mathrm{ctr}}\,\mathrm{LF}(\widehat{\mathbb{A}}^2_{\Bbbk}, \widehat{w}_2)$:

$$\begin{array}{ccccccc}
{}_{\widehat{w}_2}X\langle -2\rangle & \longrightarrow & {}_{\widehat{w}_2}B & \longrightarrow & {}_{\widehat{w}_2}\Delta & \xrightarrow{\delta^+} & \Sigma\, {}_{\widehat{w}_2}\Delta\langle -2\rangle \\
\| & & \downarrow \cong & & \| & & \| \\
{}_{\widehat{w}_2}X\langle -2\rangle & \longrightarrow & \mathrm{Co}(\Sigma^{-1}\delta^+) & \longrightarrow & {}_{\widehat{w}_2}\Delta & \xrightarrow{\delta^+} & \Sigma\, {}_{\widehat{w}_2}X\langle -2\rangle
\end{array}$$

$$\begin{array}{ccccccc}
{}_{\widehat{w}_2}\Delta\langle -2\rangle & \longrightarrow & {}_{\widehat{w}_2}B & \longrightarrow & {}_{\widehat{w}_2}X & \xrightarrow{\delta^-} & \Sigma\, {}_{\widehat{w}_2}\Delta\langle -2\rangle \\
\| & & \downarrow \cong & & \| & & \| \\
{}_{\widehat{w}_2}\Delta\langle -2\rangle & \longrightarrow & \mathrm{Co}(\Sigma^{-1}\delta^-) & \longrightarrow & {}_{\widehat{w}_2}X & \xrightarrow{\delta^-} & \Sigma\, {}_{\widehat{w}_2}\Delta\langle -2\rangle
\end{array}$$

Now, one might ask to what extend the components of $\mathcal{CCKR}^k_{\Bbbk}$ can be obtained from the commutative cube-shaped diagrams in $\mathbf{D}^{\mathrm{ctr}}\,\mathrm{LF}(\Bbbk, 0)$ obtained by taking the contraderived tensor product of the respective copies of the morphisms $\delta^{\pm}$. This is reminiscent of the following general problem: Given a tensor triangulated category $(\mathcal{T}, \otimes)$ [Bal05, Definition 1.1] and morphisms $\varphi_i : X_i \to Y_i$ in $\mathcal{T}$, $i = 1, \ldots, n$, can one compute (the isomorphism class of) $\bigotimes_i \mathrm{Co}(\varphi_i)$ from the commutative n-cube $\bigotimes \varphi_i$ in $\mathcal{T}$? Example I.7.3.5 below shows that the answer is *no* already in the simple case of the derived category of a field, but we introduce some notation first:

Notation I.7.3.4. We denote $[1]$ the poset $\{0 < 1\}$ considered as a category, i.e. $[1]$ is the category with objects 0, 1 and a unique non-identity morphism $0 \to 1$. For an integer $n \geq 2$, we denote $[1]^n$ the n-th power of $[1]$, i.e. the category associated to the poset $\{0 < 1\}^n$, and call it the *commutative n-cube*.

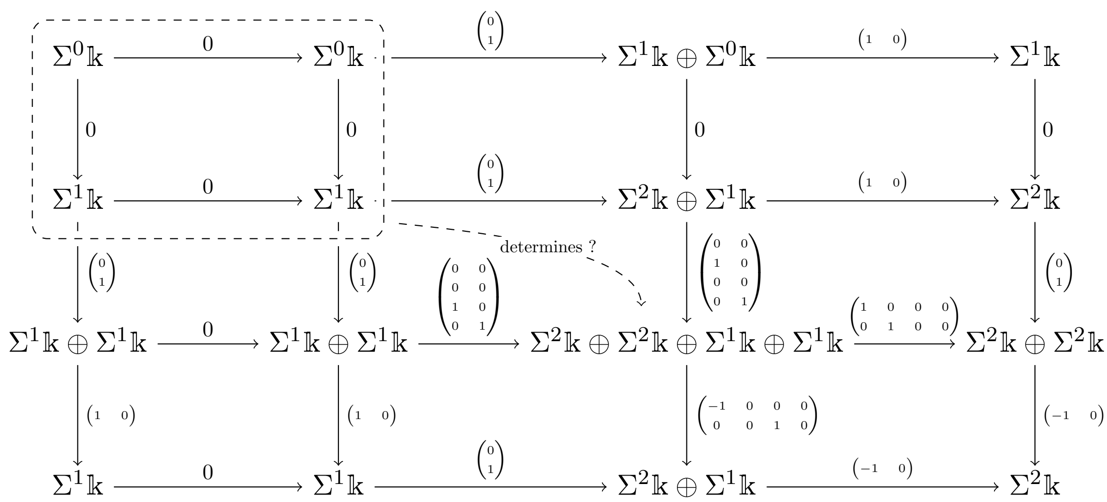
Given a category $\mathcal{C}$, the functor category $\mathcal{C}^{[1]}$ is the category of morphisms in $\mathcal{C}$, and more generally $\mathcal{C}^{[1]^n}$ is the category of commutative n-cubes in $\mathcal{C}$. If $\alpha : [1]^n \to \mathcal{C}$ is a commutative n-cube in $\mathcal{C}$ and $F : \mathcal{C} \to \mathcal{D}$ is a functor, we use the composition notation $F \circ \alpha$ to denote the commutative n-cube in $\mathcal{D}$ obtained by applying F to the vertices and edges of α. If $\mathcal{C}$ is additive, any commutative n-cube $\alpha : [1]^n \to \mathcal{C}$ yields a complex $\mathrm{C}^*(\alpha)$ in $\mathcal{C}$, with $\mathrm{C}^k(\alpha)$ the coproduct of the $\alpha(\varepsilon)$ with $\varepsilon \in [1]^n$ satisfying $\sum_i \varepsilon_i = n + k$, and with the differential given by an alternating sum over the relevant edges of α. For example, if $\alpha : [1] \to \mathcal{C}$ is a morphism, then $\mathrm{C}^*(\alpha) = \ldots \to 0 \to \alpha(0) \to \underline{\alpha(1)} \to 0 \to \ldots$. If $\mathcal{C}$ is abelian, we denote $\mathrm{H}^*(\alpha)$ the cohomology of $\mathrm{C}^*(\alpha)$.

Further, we write $\Bbbk[1]^n$ for the path algebra of $[1]^n$ over $\Bbbk$, so that $\Bbbk[1]^n\text{-Mod} \cong \Bbbk\text{-Mod}^{[1]^n}$. If $(\mathcal{C}, \otimes)$ is a monoidal category and α and β are commutative n-cubes resp. m-cubes in $\mathcal{C}$, their tensor product $\alpha \otimes \beta : [1]^{n+m} = [1]^n \times [1]^m \to \mathcal{C} \times \mathcal{C} \xrightarrow{\otimes} \mathcal{C}$ is a commutative $n + m$-cube. In particular, given morphisms $\varphi_i : X_i \to Y_i$ in $\mathcal{C}$, we can form their tensor product $\bigotimes_i \varphi_i$ as a commutative n-cube in $\mathcal{C}$. ◊

Example I.7.3.5. Let $\Bbbk$ be a field and consider the zero morphisms $\varphi_1 : \Sigma^0\Bbbk \xrightarrow{0} \Sigma^0\Bbbk$ and $\varphi_2 : \Sigma^0\Bbbk \xrightarrow{0} \Sigma^1\Bbbk$ in its derived category $\mathbf{D}(\Bbbk\text{-Mod}) = \mathrm{Ho}(\Bbbk\text{-Mod})$. The tensor product of the induced distinguished triangles

$$\Sigma^0\Bbbk \xrightarrow{0} \Sigma^0\Bbbk \xrightarrow{\binom{0}{1}} \mathrm{Co}(\varphi_1) = \Sigma^1\Bbbk \oplus \Sigma^0\Bbbk \xrightarrow{(1\ \ 0)} \Sigma^1\Bbbk,$$
$$\Sigma^0\Bbbk \xrightarrow{0} \Sigma^1\Bbbk \xrightarrow{\binom{0}{1}} \mathrm{Co}(\varphi_2) = \Sigma^1\Bbbk \oplus \Sigma^1\Bbbk \xrightarrow{(1\ \ 0)} \Sigma^1\Bbbk$$

gives the diagram



with distinguished rows and columns, and where all squares are commutative except the lower right one, which anticommutes. We ask whether $\mathrm{Co}(\varphi_1) \otimes \mathrm{Co}(\varphi_2)$ at position $(3, 3)$ is uniquely determined by this property and the dashed commutative square in the upper left corner (note that by [May01, Lemma 2.6] any choice for the dashed square admits an extension to a 4×4-diagram as above). This is not the case, since already

$$\begin{array}{ccc} \Sigma^0\Bbbk & \longrightarrow & 0 \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & \Sigma^1\Bbbk, \end{array} \tag{I.7.3.10}$$

a summand of the dashed square, admits the two different extensions

$$\begin{array}{ccccccc}
\Sigma^0\mathbb{k} & \longrightarrow & 0 & \longrightarrow & \Sigma^1\mathbb{k} & = & \Sigma^1\mathbb{k} \\
\downarrow & & \downarrow & & \downarrow{\scriptstyle 0} & & \downarrow \\
0 & \longrightarrow & \Sigma^1\mathbb{k} & = & \Sigma^1\mathbb{k} & \longrightarrow & 0 \\
\downarrow & & \| & & \downarrow{\scriptstyle \begin{pmatrix}0\\1\end{pmatrix}} & & \downarrow \\
\Sigma^1\mathbb{k} & \xrightarrow{0} & \Sigma^1\mathbb{k} & \xrightarrow{\begin{pmatrix}0\\1\end{pmatrix}} & \Sigma^1\mathbb{k}\oplus\Sigma^2\mathbb{k} & \xrightarrow{\begin{pmatrix}1 & 0\end{pmatrix}} & \Sigma^2\mathbb{k} \\
\| & & \downarrow & & \downarrow{\scriptstyle \begin{pmatrix}-1 & 0\end{pmatrix}} & & \| \\
\Sigma^1\mathbb{k} & \longrightarrow & 0 & \longrightarrow & \Sigma^2\mathbb{k} & = & \Sigma^2\mathbb{k}
\end{array} \tag{I.7.3.11}$$

$$\begin{array}{ccccccc}
\Sigma^0\mathbb{k} & \longrightarrow & 0 & \longrightarrow & \Sigma^1\mathbb{k} & = & \Sigma^1\mathbb{k} \\
\downarrow & & \downarrow & & \| & & \downarrow \\
0 & \longrightarrow & \Sigma^1\mathbb{k} & = & \Sigma^1\mathbb{k} & \longrightarrow & 0 \\
\downarrow & & \| & & \downarrow & & \downarrow \\
\Sigma^1\mathbb{k} & = & \Sigma^1\mathbb{k} & \longrightarrow & 0 & \longrightarrow & \Sigma^2\mathbb{k} \\
\| & & \downarrow & & \downarrow & & \| \\
\Sigma^1\mathbb{k} & \longrightarrow & 0 & \longrightarrow & \Sigma^2\mathbb{k} & = & \Sigma^2\mathbb{k}.
\end{array} \tag{I.7.3.12}$$

The structural explanation for such kind of phenomena is the following: In order to form the iterated cone of a commutative square like (I.7.3.10) in $\mathbf{D}(\mathbb{k}\text{-Mod}) = \mathrm{Ho}(\mathbb{k}\text{-Mod})$ – which amounts to a *homotopy commutative* square of chain complexes over $\mathbb{k}$ – we need to lift the square to a *strictly commutative* (also called *coherent* [GS14, §2]) square of chain complexes over $\mathbb{k}$ first, and only then we can apply the usual cone construction. Any choice of a lift gives rise to a 4×4-diagram as above. Now, a strictly commutative square of chain complexes over $\mathbb{k}$ is the same as a chain complex over $\mathbb{k}\text{-Mod}^{\square}$, the category of representation over $\mathbb{k}$ of the commutative square $\square := [1]^2$, and the functor $\mathbf{D}(\mathbb{k}\text{-Mod}^{\square}) \to \mathbf{D}(\mathbb{k}\text{-Mod})^{\square} \cong (\mathbb{k}\text{-Mod}^{\square})^{\mathbb{Z}}$ forgetting the coherence is equivalent to the cohomology functor H^*. Hence, finding a coherent lift of (I.7.3.10) amounts to realizing it as the cohomology of a chain complex over $\mathbb{k}\text{-Mod}^{\square}$. Once such a lift has been chosen, there is a canonical iterated cone, and we are left with the problem of uniqueness (in the derived category) of complexes with prescribed cohomology.

For hereditary categories like $\mathbb{k}\text{-Mod}^{[1]}$ any complex is non-canonically isomorphic to its cohomology, accounting for the fact that cones are unique up to non-canonical isomorphism. The category $\mathbb{k}\text{-Mod}^{\square}$ of representations of $\square$, is however not hereditary, and in fact the square (I.7.3.10) can be realized as the cohomology of both

$$\begin{array}{ccccccccccc}
\cdots & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \Bbbk & \longrightarrow & 0 & \longrightarrow & \cdots \\
\cdots & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \cdots \\
\cdots & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \cdots \\
\cdots & \longrightarrow & 0 & \longrightarrow & \Bbbk & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \cdots
\end{array} \qquad \text{(I.7.3.13)}$$

and

$$\begin{array}{ccccccccccc}
\cdots & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \Bbbk & \longrightarrow & 0 & \longrightarrow & \cdots \\
\cdots & \longrightarrow & 0 & \longrightarrow & \Bbbk & = & \Bbbk & \longrightarrow & 0 & \longrightarrow & \cdots \\
\cdots & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \cdots \\
\cdots & \longrightarrow & 0 & \longrightarrow & \Bbbk & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \cdots .
\end{array} \qquad \text{(I.7.3.14)}$$

Note that these complexes stem from the two middle terms of a trivial and a non-trivial length two extension between the simple $\square$-modules corresponding to the source and the sink of the square, respectively, the non-trivial extension witnessing that $\Bbbk\text{-Mod}^{\square}$ is not hereditary. Also, the diagrams (I.7.3.11) and (I.7.3.12) presented above are obtained from (I.7.3.13) and (I.7.3.14) by taking the iterated cone, respectively.

The above indicates that in order to canonically define homotopy limits and colimits of shape I in a triangulated category $\mathcal{T}$ like $\mathbf{D}(\Bbbk\text{-Mod})$, we need to be given suitable categories of "coherent I-diagrams in $\mathcal{T}$" in addition to the naive category $\mathcal{T}^I$ of "homotopy commutative" I-diagrams in $\mathcal{T}$; in the example of $\mathcal{T} = \mathbf{D}(\Bbbk\text{-Mod})$, such a replacement is $\mathbf{D}(\Bbbk\text{-Mod}^I)$. The datum of a category together with suitable categories of coherent I-diagrams over it is abstracted in the notion of a *derivator*; see the paragraph preceding Proposition II.C.1.6 of Part II for a quick motivation, as well as [Gro13] for an introduction to derivators and [Gro12] for the study of monoidal derivators. $\lozenge$

We return to Khovanov-Rozansky homology: So far we observed that every component of $\mathcal{CCKR}^k_{\Bbbk}(\beta)$, call it Z, is a contraderived tensor product of linear factorizations each of which can be written as a cone of the connecting morphisms

$$\Sigma^{-1}\delta^+ : \Sigma^{-1}{}_{\widehat{w}_2}X \to {}_{\widehat{w}_2}\Delta\langle -2\rangle \qquad \text{and} \qquad \Sigma^{-1}\delta^- : \Sigma^{-1}{}_{\widehat{w}_2}\Delta \to {}_{\widehat{w}_2}X\langle -2\rangle.$$

However, we saw in the previous example that one shouldn't expect the commutative n-cube in $\mathbf{D}^{\text{ctr}}\,\text{LF}(\Bbbk, 0)$ obtained by tensoring the respective copies of $\delta^{\pm}$ to be sufficient to recover Z. Instead, one needs to choose representatives of the $\delta^{\pm}$ by morphisms of matrix factorizations (or at least such linear factorizations allowing for the contraderived tensor product to be computed naively) and consider their tensor product as a coherent (i.e., strictly commutative) n-cube of linear factorizations of potential 0, that is, as an object of $\mathbf{D}^{\mathrm{ctr}}\,\mathrm{LF}(\Bbbk[1]^n, 0)$. Then Z can be recovered as the iterated cone of this coherent n-cube. Still, concerning the cohomology of Z, the underlying "incoherent" n-cube in $\mathbf{D}^{\mathrm{ctr}}\,\mathrm{LF}(\Bbbk, 0)$ can be viewed as an approximation to it as we shall explain now.

Recall Notation I.7.3.4 and Remark I.2.1.3; in particular, if M is a linear factorization of potential 0_d, then $\widetilde{M}$ is a 2-periodic complex such that $\mathrm{H}^n(\widetilde{M}) = \mathrm{H}^0(M)$ for n even and $\mathrm{H}^n(\widetilde{M}) = \mathrm{H}^1(M)\langle -\frac{d}{2}\rangle$ for n odd.

Fact I.7.3.6. *Let S be a $\mathbb{Z}$-graded commutative ring and $w_1, \ldots, w_n \in S$ homogeneous potentials of degree d such that $\sum_{i=1}^{n} w_i = 0$. Moreover, let $\varphi_i : X_i \to Y_i$ be morphisms in $\mathrm{LF}(S, w_i)$, and put $Z := \bigotimes_{i=1}^{n} \mathrm{Cone}(\varphi_i)$. Then there is a cohomological spectral sequence*

$$\mathrm{E}_1^{p,q} \cong \mathrm{C}^p \left[\mathrm{H}^q \circ \widetilde{(-)} \circ \bigotimes_{i=1}^{n} \varphi_i \right] \left\langle -\tfrac{pd}{2} \right\rangle \qquad \Longrightarrow \qquad \mathrm{E}_\infty^n \cong \mathrm{H}^n(\widetilde{Z}) \tag{I.7.3.15}$$

in the category of graded S-modules, with differentials of q-degree $\frac{d}{2}$.

Here *cohomological* means that the differentials have the form $\mathrm{d}_t^{p,q} : \mathrm{E}_t^{p,q} \to \mathrm{E}_t^{p+t,q+1-t}$. Since H^q is additive, the $\mathrm{E}_1^{p,q}$-term in (I.7.3.15) can also be described as the p-th component of the complex of S-modules obtained by termwise taking the q-th cohomology in the complex $\mathrm{C}^*\left(\widetilde{(-)} \circ \bigotimes_{i=1}^{n} \varphi_i\right)$ of 2-periodic complexes associated to the commutative n-cube $\widetilde{(-)} \circ \bigotimes_{i=1}^{n} \varphi_i$, shifted by $-\frac{pd}{2}$. In particular, $\mathrm{E}_1^{p,q}$ is 2-periodic in q.

Proof of Fact I.7.3.6. This is a special case of the spectral sequence of a bicomplex: Recall from Remark I.2.1.3 how to pass from a linear factorization of potential 0_d to a 2-periodic complex, as well as the compatibility of this operation with taking cohomology and totalizations. In particular, as displayed in Figure I.7.3.1, the complex of linear factorizations $\mathrm{C}^*\left(\bigotimes_i \varphi_i\right)$ of potential $(A, 0_d)$ gives rise to a horizontally 2-periodic bicomplex with horizontal and vertical differentials of q-degree $\frac{d}{2}$, the totalization of which is the 2-periodic complex $\widetilde{Z}$ associated to $Z = \bigotimes_i \mathrm{Cone}(\varphi_i)$. The spectral sequence associated to the vertical filtration on this totalization has the desired properties. □

Fact I.7.3.7. *The spectral sequence from Fact I.7.3.6 is natural.*

By *natural* we mean the following: Pick $1 \leq r \neq s \leq n$ arbitrary and consider the modified datum of morphisms $\varphi'_1, \ldots, \varphi'_{n-1}$ obtained from $\varphi_1, \ldots, \varphi_n$ by removing the s-th morphism φ_s and replacing the r-th morphism φ_r by $\varphi_r \otimes X_s : X_r \otimes \Sigma X_s \to Y_r \otimes \Sigma X_s$.

$$
\begin{array}{ccccc}
\vdots & & \vdots & & \vdots \\
\downarrow & & \downarrow & & \downarrow \\
M^{0,-2} & \to & M^{1,-2} & \to & M^{0,-2} \\
\downarrow & & \downarrow & & \downarrow \\
M^{0,-1} & \to & M^{1,-1} & \to & M^{0,-1} \\
\downarrow & & \downarrow & & \downarrow \\
M^{0,0} & \longrightarrow & M^{1,0} & \longrightarrow & M^{0,0} \\
\downarrow & & \downarrow & & \downarrow \\
0 & & 0 & & 0
\end{array}
\quad \rightsquigarrow \quad
\begin{array}{ccccccccc}
& & \vdots & & \vdots & & \vdots & & \\
& & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \to & M^{1,-2}\langle\frac{3d}{2}\rangle & \to & M^{0,-2}\langle d\rangle & \to & M^{1,-2}\langle\frac{3d}{2}\rangle & \to & \cdots \\
& & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & M^{1,-1}\langle d\rangle & \longrightarrow & M^{0,-1}\langle\frac{d}{2}\rangle & \longrightarrow & M^{1,-1}\langle d\rangle & \longrightarrow & \cdots \\
& & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & M^{1,0}\langle\frac{d}{2}\rangle & \longrightarrow & M^{0,0} & \longrightarrow & M^{0,0}\langle\frac{d}{2}\rangle & \longrightarrow & \cdots \\
& & \downarrow & & \downarrow & & \downarrow & & \\
& & 0 & & 0 & & 0 & &
\end{array}
$$

Figure I.7.3.1. Passing from a complex of linear factorizations of type $(A, 0_d)$ to a horizontally 2-periodic bicomplex with differentials of q-degree $\frac{d}{2}$.

Further, let $Z' := \bigotimes_i \mathrm{Cone}(\varphi_i')$. Then Fact I.7.3.6 induces another spectral sequence E'

$$\mathrm{E}'^{p,q}_1 \cong \mathrm{C}^p \left[\mathrm{H}^q \circ \widetilde{(-)} \circ \bigotimes_{i=1}^{n-1} \varphi_i' \right] \left\langle -\tfrac{pd}{2} \right\rangle \implies \mathrm{E}^n_\infty \cong \mathrm{H}^n(\widetilde{Z'})$$

and there is a canonical morphism of spectral sequences

$$\Sigma^{-1,0}\,\mathrm{E} \to \mathrm{E}', \quad \text{with } \Sigma^{-1,0}\,\mathrm{E} \text{ is defined by } (\Sigma^{-1,0}\,\mathrm{E})^{p,q}_t := \mathrm{E}^{p-1,q}_t,$$

which on the E_1-page is the one induced by the canonical morphism of complexes $\Sigma^{-1}\,\mathrm{C}^*(\bigotimes_i \varphi_i) \to \mathrm{C}^*(\bigotimes_i \varphi_i')$, and in the limit the one induced by the morphism $Z \to Z'$ coming from the canonical morphism $\mathrm{Cone}(\varphi_i) \to \Sigma X_i$. Similarly, if one takes $\varphi_r' := \varphi_r \otimes Y_s : X_r \otimes Y_s \to Y_r \otimes Y_s$ one gets a canonical morphism $\mathrm{E}' \to \mathrm{E}$ which on the E_1-page and the limit coincides with the natural ones.

The previous naturality applies in the situation encountered in Khovanov-Rozansky homology. The differentials of $\mathcal{CKR}^k_{\Bbbk}(\beta)$ are induced by the morphisms ${}_{\widehat{w}_2}B \to {}_{\widehat{w}_2}\Delta$ and ${}_{\widehat{w}_2}\Delta\langle -2\rangle \to {}_{\widehat{w}_2}B$, which one depending on the sign of the crossing whose resolution is to be changed – see (I.5.1). If we present ${}_{\widehat{w}_2}B$ as $\mathrm{Co}(\Sigma^{-1}{}_{\widehat{w}_2}\Delta \xrightarrow{\Sigma^{-1}\delta^+} {}_{\widehat{w}_2}X\langle -2\rangle)$, then the morphism ${}_{\widehat{w}_2}B \to {}_{\widehat{w}_2}\Delta$ corresponds to the morphism $\mathrm{Co}(\Sigma^{-1}\delta^+) \to {}_{\widehat{w}_2}\Delta$ that's part of the choice of cone; similarly, if we present ${}_{\widehat{w}_2}B$ as $\mathrm{Co}(\Sigma^{-1}\delta^-)$, the morphism ${}_{\widehat{w}_2}\Delta\langle -2\rangle \to {}_{\widehat{w}_2}B$ corresponds to the morphism ${}_{\widehat{w}_2}\Delta\langle -2\rangle \to \mathrm{Co}(\delta^-)$. We conclude:

Fact I.7.3.8. *Let D be an oriented link diagram D. Then the components of $\mathcal{CKR}^k_{\Bbbk}(D)$ can be realized as the limit terms of (horizontal) shifts of spectral sequences associated* *to representatives of the connecting morphisms $\delta^{\pm}$ in such a way that the differentials on $\mathcal{CKR}^k_{\Bbbk}(D)$ canonically lift to morphisms of spectral sequences.*

Next we study the E_1-pages of the spectral sequences converging to the components of $\mathcal{CKR}^k_{\Bbbk}(D)$ in more detail. By definition (see (I.7.3.15)), they are the complexes assigned to the termwise cohomology of the cubes obtained by tensoring the connecting morphisms $\delta^{\pm}$, and it will turn out that each such cohomology cube has a combinatorial description very similar to Khovanov's original cube used in the definition of Khovanov-homology [Kho00]: To begin, recall that the cohomology of a circular contraderived tensor product of copies of ${}_{\widehat{w}_1}\Delta^1_{\Bbbk}$ vanishes in even cohomological degree and is given by a shift of $A := \Bbbk[x]/(x^k)$ in odd degree (see e.g. (I.6.4.20)). Consequently, the cohomology of a closed (i.e. potential cancelling) contraderived tensor product of copies of the factorizations ${}_{\widehat{w}_1}\Delta^1_{\Bbbk}$, ${}_{\widehat{w}_2}\Delta^2_{\Bbbk}$, $\Sigma\,{}_{\widehat{w}_2}\Delta^2_{\Bbbk}\langle -\rangle$, ${}_{\widehat{w}_2}X$ and $\Sigma\,{}_{\widehat{w}_2}X\langle -2\rangle$ (as occurring at the vertices of the commutative cube in question) is also concentrated in a single cohomological parity, and in the parity where it does not vanish it is given by an internal shift of $A^{\otimes n}$, where n the number of circles obtained by depicting the constituents of the tensor product as unknotted strands $\uparrow$, $\uparrow\uparrow$ or crossings $\times$, respectively. The details on the cohomological and internal shifts are still to be made precise; for now we only note that all tensor products arising have their cohomology in the *same* parity: Replacing a tensor factor ${}_{\widehat{w}_2}X\langle -2\rangle$ (the codomain of $\Sigma^{-1}\delta^{+}$) by $\Sigma^{-1}{}_{\widehat{w}_2}\Delta$ (the domain of $\Sigma^{-1}\delta^{+}$) results both in a change in the number of circles and in a cohomological shift by one; the parity of cohomology therefore doesn't change. The reasoning for the other connecting homomorpihsm $\Sigma^{-1}\delta^{-}$ is similar, and so we deduce:

Fact I.7.3.9. *In any commutative cube in $\mathbf{D}^{\mathrm{ctr}}\,\mathrm{LF}(\Bbbk, 0)$ obtained by taking a contraderived tensor product of copies of the connecting morphisms $\Sigma^{-1}\delta^{\pm}$, all vertices have cohomology in the same parity.*

In particular, in the spectral sequence associated to a coherent cube obtained by tensoring presentations φ_i of $\Sigma^{-1}\delta^{\pm}$ as morphisms of matrix factorizations, all even differentials vanish, and it reduces to a spectral complex of degree $2k+2$ (Definition I.4.6.3)

$$\mathrm{E}_1^n \;:=\; \mathrm{C}^{-n}\left[\mathrm{H}^t \circ \bigotimes_i \varphi_i\right]\langle n(k+1)\rangle \;\Longrightarrow\; \mathrm{E}_\infty^n \;:=\; \mathrm{H}^t(Z).$$

The cohomology of the components of $\mathcal{CCKR}^k_{\Bbbk}$ is also concentrated in a single parity:

Proposition I.7.3.10. *For any oriented link diagram D, the cohomology of the components of $\mathcal{CCKR}^k_{\Bbbk}(D)$ is concentrated in the number of* Seifert circles *of D, i.e. the number of circles in the diagram arising from D by replacing each crossing by the uncrossing.*

Proof. This is already stated in the original article [KR08a] and a consequence of the categorifications [KR08a, Proposition 29–32] of the MOY-relations [KR08a, Figure 3]. Restricting to closures of braid diagrams it would alternatively follow from a restriction of the induction principle for Soergel bimodules (Proposition I.A.2.6) to Bott-Samelson bimodules, with the crucial assumption (v) provided by Proposition I.6.4.8. □

Fact I.7.3.9 and Proposition I.7.3.10 together give the following degeneracy criterion:

Proposition I.7.3.11. *Consider a spectral complex* E *associated to a coherent cube of matrix factorizations of potential* 0 *obtained by tensoring matrix factorization representatives of copies of the connecting morphisms* $\Sigma^{-1}\delta^{\pm}$*. Then the following are equivalent:*

(i) E *degenerates at the* E_2*-page.*

(ii) *The terms* E_2^n *are non-zero only in a single parity of* n*.*

Finally, Propositions I.7.3.2 and I.7.3.1 show that the edges of the cohomology cubes approximating the terms of $\mathcal{CKR}_{\Bbbk}^k$ are either multiplications or comultiplications of the Frobenius algebra A, depending on whether replacing a crossing $\times$ by the uncrossing (or vice versa) results in splitting or merging of a circle. Therefore, the E_1-pages in the spectral complexes approximating the components of $\mathcal{CKR}_{\Bbbk}^k$ admit a purely combinatorial description that does not involve any matrix factorizations, and we will elaborate on this in the next section.

We may summarize the considerations of the present section as follows. Recall from Definition I.4.6.3 the notion of a spectral complex.

Theorem I.7.3.12. *For any oriented link diagram* D*, there is a complex* X *of spectral complexes of* q*-degree* $2k+2$ *with the following properties:*

(i) *The Khovanov-Rozansky complex* $\mathcal{CKR}_{\Bbbk}^k(D)$ *is canonically isomorphic to the complex obtained from* X *by passing to the limit terms of its components.*

(ii) *The bicomplex obtained from* X *by passing to the* E_1*-pages of its components is canonically isomorphic to the bicomplex* $\langle D\rangle_{\Bbbk[x]/(x^k)}$ *defined in Section I.7.4 below, with its* i*-th row* q*-shifted by* $i(k+1)$*.*

(iii) *The components of* X *degenerate at the* E_2*-page if and only if their* E_2*-pages are concentrated in a single parity. In this case, the complex of* E_2*-pages is isomorphic to the associated graded of the filtration on* $\mathcal{CKR}_{\Bbbk}^k(D)$ *induced by (i).*

Proof. We already observed in Fact I.7.3.8 that $\mathcal{CKR}_{\Bbbk}^k(D)$ can be realized as the complex of limit terms of a complex of spectral sequences obtained from Fact I.7.3.6 by presenting a copy of ${}_{\widehat{w}_2}B$ involved in a component of $\mathcal{CKR}_{\Bbbk}^k(D)$ as $\mathrm{Co}(\Sigma^{-1}\delta^{+})$ if it belonged to a positive crossing in D, and as $\mathrm{Co}(\Sigma^{-1}\delta^{-})$ if it belonged to a negative crossing; further, to be able to lift the morphisms ${}_{\widehat{w}_2}B \to {}_{\widehat{w}_2}\Delta$ assigned to positive crossings, we needed to lower the homological grading in the resulting spectral complex by the number of copies of ${}_{\widehat{w}_2}B$ that belonged to a positive crossing. By Fact I.7.3.9, this complex of spectral sequences actually reduces to a complex of spectral complexes, call it X, satisfying property (i), and Proposition I.7.3.11 ensures property (iii).

Of course, we can't argue on point (ii) as long as we haven't defined $\langle - \rangle_{\Bbbk[x]/(x^k)}$ (which will be done in the next section), but we shall in the rest of the proof describe the complex of E_1-pages associated to X in detail, in particular paying attention to the precise grading shifts that we neglected so far. Once the definition of $\langle - \rangle_{\Bbbk[x]/(x^k)}$ has been given, property (ii) will then be clear. The reader may want to jump ahead to the definition of $\langle - \rangle_{\Bbbk[x]/(x^k)}$ and read the rest of this proof afterwards.

Recall from (I.5.1) that $\mathcal{CCKR}^k_{\Bbbk}(D)$ is constructed by associating to each positive crossing ⤲ a copy of the complex $\ldots \to 0 \to {}_{\widehat{w}_2}B\langle 1-k\rangle \to \underline{{}_{\widehat{w}_2}\Delta\langle 1-k\rangle} \to 0 \to \ldots$, to each negative crossing ⤱ a copy of the complex $\ldots \to 0 \to \underline{{}_{\widehat{w}_2}\Delta\langle k-1\rangle} \to {}_{\widehat{w}_2}B\langle k+1\rangle \to 0 \to \ldots$, and by taking the contraderived tensor products of these complexes. Therefore, up to the homological downshift by the number of copies of ${}_{\widehat{w}_2}B\langle 1-k\rangle$ coming from a positive crossing, the spectral complexes in X are those associated to tensor products of representatives of the morphisms $\Sigma^{-1}{}_{\widehat{w}_2}\Delta\langle 1-k\rangle \to {}_{\widehat{w}_2}X\langle -(k+1)\rangle$ (in case of a tensor factor ${}_{\widehat{w}_2}B\langle 1-k\rangle$ coming from a positive crossing) and $\Sigma^{-1}{}_{\widehat{w}_2}X\langle k+1\rangle \to {}_{\widehat{w}_2}\Delta\langle k-1\rangle$ (in case of a factor ${}_{\widehat{w}_2}B\langle k+1\rangle$ coming from a negative crossing). In particular, their E_1-pages consist of the termwise total cohomology of the tensor product of these morphisms considered as complexes, where $\Sigma^{-1}{}_{\widehat{w}_2}\Delta\langle 1-k\rangle \to {}_{\widehat{w}_2}X\langle -(k+1)\rangle$ is considered as concentrated in homological degree 0 and -1 and where $\Sigma^{-1}{}_{\widehat{w}_2}X\langle k+1\rangle \to {}_{\widehat{w}_2}\Delta\langle k-1\rangle$ is considered as concentrated in homological degree 1 and 0. Now, consider a constituent of this tensor product obtained by choosing $n_{+,\Delta}$ times the term $\Sigma^{-1}{}_{\widehat{w}_2}\Delta\langle 1-k\rangle$, $n_{+,X}$ times the term ${}_{\widehat{w}_2}X\langle -(k+1)\rangle$ and $n_{\Delta,-}$ times the term ${}_{\widehat{w}_2}\Delta\langle k-1\rangle$; further, denote n the number of circles in the diagram obtained by depicting each choice of Δ by $||$ and each choice of X by $\times$. Then, since the total cohomology of a circular contraderived tensor product of copies of ${}_{\widehat{w}_1}\Delta$ is $\Bbbk[x]/(x^k)\langle k-1\rangle$ and since $\mathrm{H}^t \circ \Sigma = \mathrm{H}^t\langle k+1\rangle$, the total cohomology of the constituent under consideration is given by

$$\Big[\Bbbk[x]/(x^k)\langle k-1\rangle\Big]^{\otimes n}\langle 2k\cdot n_{+,\Delta} - (k+1)\cdot n_{+,X} + (k-1)\cdot n_{-,\Delta}\rangle\,; \qquad \text{(I.7.3.16)}$$

there is no term involving the number of occurrences of $\Sigma^{-1}{}_{\widehat{w}_2}X\langle k+1\rangle$, since in any such the shift by $-(k+1)$ coming from the effect of Σ^{-1} on the total cohomology is annihilated by the positive shift by $k+1$. Hence, denoting $n_+ := n_{+,\Delta} + n_{+,X}$ the number of copies of ${}_{\widehat{w}_2}B$ coming from a positive crossing, (I.7.3.16) can be rewritten as

$$\left[\Bbbk[x]/(x^k)\langle k-1\rangle\right]^{\otimes n}\langle -(k+1)\cdot n_+\rangle\,\langle (k-1)\cdot(n_{-,\Delta}-n_{+,\Delta})\rangle\,.$$

Once the definition of $\langle - \rangle_{\Bbbk[x]/(x^k)}$ is given in the next section, this will verify (ii). □

Remark I.7.3.13. The whole point of this section was that in our study of the components of $\mathcal{CKR}^k_{\Bbbk}(D)$ we needed to work with representatives of the connecting morphisms $\delta^{\pm}$ on the level of linear factorizations, and that viewing their tensor product as an "incoherent" cube in $\mathbf{D}^{\mathrm{ctr}}\,\mathrm{LF}(\Bbbk,0)$ was not sufficient. In contrast, note that the Khovanov-Rozansky complex $\mathcal{CKR}^k_{\Bbbk}(D)$ itself arises from such an incoherent cube, namely the one obtained from the morphisms ${}_{\widehat{w}_2}B \to {}_{\widehat{w}_2}\Delta$ and ${}_{\widehat{w}_2}\Delta\langle -2\rangle \to {}_{\widehat{w}_2}B$, and that here we *do not* want to consider the iterated cone of the coherent lift of this cube: As the cones of the above morphisms are shifts of ${}_{\widehat{w}_2}X$, we'd otherwise end up with the trivial invariant assigning to any link with l the tensor power $(\Bbbk[x]/(x^k)\langle k-1\rangle)^{\otimes l}$. As a special case of Fact I.7.3.6, we deduce the existence of a spectral complex from Khovanov-Rozansky homology to this trivial link invariant. ♢

I.7.4. A cut-and-join formalism approximating Khovanov-Rozansky homology

This section is completely independent of the rest of the thesis.

Fix a commutative, $\mathbb{Z}$-graded Frobenius algebra A over a field $\Bbbk$. In this section we describe a combinatorial, Khovanov-homology like construction of a bicomplex $\langle D\rangle_A$ of graded vector spaces over $\Bbbk$ out of a planar, oriented link diagram D. It is *not* a link invariant in any sense, but in case $A = \Bbbk[x]/(x^k)$ and $k+1 \in \Bbbk^{\times}$ instead constitutes an approximation to Khovanov-Rozansky homology $\mathcal{KR}^k_{\mathbb{Q}}$ in the sense explained in Theorem I.7.3.12. Sometimes, one can tell from this approximation that it must already coincide with Khovanov-Rozansky homology, as is the case e.g. for the Hopf link and trefoil knot, which we shall consider in detail – it would be interesting to see more examples of this degeneration phenomenom, since computing Khovanov-Rozansky homology is very in difficult in general, see [CM14]. While of no use as a knot invariant on its own, we found it interesting to observe that there is combinatorial formalism similar to the one in Khovanov homology inherent in Khovanov-Rozansky homology $\mathcal{KR}^k$.

I.7.4.1. Conventions

The underlined entry in a complex is the one at position 0; the doubly underlined entry in a bicomplex is the one at position $(0,0)$. Differentials always *lower* the degree (in contrast to the cohomological convention used in the rest of the thesis) and entries not listed are meant to be zero. When using indices, the first index always denotes the horizontal and second index denotes the vertical degree. Square brackets are a short-hand for totalization of bicomplexes, where any ordinary complex is considered a bicomplex concentrated in horizontal degree 0, and where any ordinary object is considered a bicomplex concentrated in bidegree $(0, 0)$. The base of totalization is indicated by double underlining. Note that this does not cause any ambiguity since any bicomplex is the totalization of its terms, each considered as a bicomplex in degree $(0, 0)$, and with the base of totalization set to the entry at degree $(0, 0)$. A superscript t denotes transposition of a bicomplex. Finally, an *upward* shift in any grading (either internal, horizontal homological or vertical homological) by $k \in \mathbb{Z}$ is denoted $\{k\}_?$, with the subscript indicating which grading is meant; in particular, $\{k\}_q := \langle -k \rangle$ in the notation of the rest of the thesis.

The category of bounded bicomplexes of finite-dimensional graded vector spaces over $\Bbbk$ is denoted $\mathrm{Ch}^2(\Bbbk)$, and we will often view it as the category of (horizontal) chain complexes over the category of (vertical) chain complexes, $\mathrm{Ch}^2(\Bbbk) \cong \mathrm{Ch}^{\mathrm{hor}}(\mathrm{Ch}^{\mathrm{ver}}(\Bbbk))$.

Definition I.7.4.1. *Let $f : X \to Y$ be a morphism of bicomplexes.*

(i) *We say that f is a* termwise vertical equivalence *if for all $k \in \mathbb{Z}$ the induced morphisms $f_{k,*} : X_{k,*} \to Y_{k,*}$ between the vertical complexes at horizontal position k are homotopy equivalences. This is equivalent to f being mapped to an isomorphism under* $\mathrm{Ch}^2(\Bbbk) \cong \mathrm{Ch}^{\mathrm{hor}}(\mathrm{Ch}^{\mathrm{ver}}(\Bbbk)) \to \mathrm{Ch}^{\mathrm{hor}}(\mathrm{Ho}^{\mathrm{ver}}(\Bbbk))$.

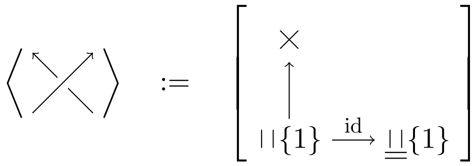
(ii) *We say that f is a* horizontal equivalence *if it is mapped to an isomorphism under* $\mathrm{Ch}^2(\Bbbk) \cong \mathrm{Ch}^{\mathrm{hor}}(\mathrm{Ch}^{\mathrm{ver}}(\Bbbk)) \to \mathrm{Ho}^{\mathrm{hor}}(\mathrm{Ch}^{\mathrm{ver}}(\Bbbk))$.

The localization of $\mathrm{Ch}^2(\Bbbk)$ *at the union of the classes of termwise vertical and horizontal equivalences is denoted* $\mathrm{Ch}^2_{\text{v-h}}(\Bbbk)$. *The isomorphism class of a bicomplex in* $\mathrm{Ch}^2_{\text{v-h}}(\Bbbk)$ *is called its* vertical-then-horizontal homotopy type.

The reason for this terminology is that the functor assigning to a bicomplex its vertical-then-horizontal homology factors through $\mathrm{Ch}^2_{\text{v-h}}(\Bbbk)$. Slightly more generally, the same is true for the functor assigning to a bicomplex its spectral sequence associated to the horizontal filtration on its total complex, starting from the E_2-page.

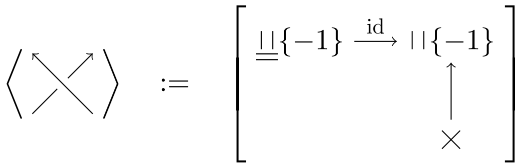
I.7.4.2. Construction

Step 1: To any positive crossing ⤧ in D, assign the bicomplex



(I.7.4.17)

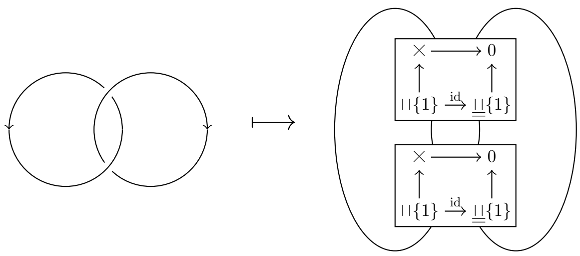
while to a negative crossing ⤧ we assign the bicomplex



(I.7.4.18)

For the moment we want to ask for the reader's permission for not explaining in what categories these bicomplexes live, treating their terms as formal symbols with formal shift; we shall extract an honest bicomplex of vector spaces in Step 3 below.

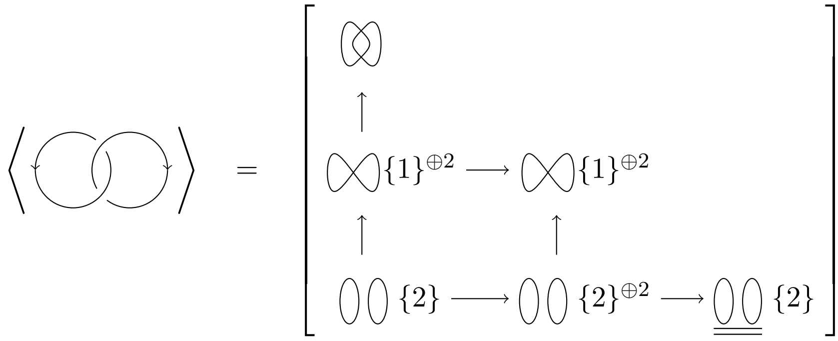
It might be helpful to imagine the bicomplexes as replacing the crossings they where assigned to in the diagram D, resulting in a diagram of unknotted, unoriented strands and bicomplexes. As an example, for the Hopf link we get the following:



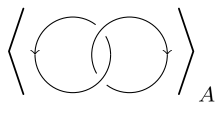
(I.7.4.19)

Step 2: Take the formal tensor product $\langle D\rangle$ of all the elementary bicomplexes assigned to the crossings of D. Concretely, this means the following: Firstly, go through all crossings in D and pick for each such crossing one non-zero entry from the elementary bicomplex assigned to it in Step 1; if n is the number of crossings, there are 3^n ways of doing this. Then, replace all crossings by the entries just chosen, resulting in a diagram of possibly intersecting circles, together with a formal shift. This diagram is one constituent of the desired tensor product $\langle D\rangle$, and its position is the sum of all the positions of the entries chosen from each crossing. Usually, there will be more than one diagram occupying a single position in the bicomplex, and in this case the corresponding entry is the formal sum of the diagrams demanding that place. In our example of the

Hopf link, the resulting bicomplex has $3^2 = 9$ terms and looks as follows:



Step 3: Finally, we construct the bicomplex $\langle D\rangle_A$ of graded vector spaces. For this, first go through the diagrams in the entries of $\langle D\rangle$ and replace each of them by $\left(A\left\{-\frac{a}{2}\right\}_q\right)^{\otimes n}$, where n is the number of closed circles occurring in the diagram and $a \in \mathbb{N}$ is the top degree in which A lives (the Gorenstein parameter of A). Further, interpret the formal shift $\{k\}$ as $\left\{\frac{ka}{2}\right\}_q$, and the formal sum of diagrams as the direct sum of graded vector spaces. With this done, it remains to specify the maps on the arrows. Any two diagrams matched by a vertical arrow differ precisely by one crossing $\times$ being replaced by the uncrossing $||$ or vice versa, resulting in two closed circles being merged into one or one circle being split into two. In the first case, assign the multiplication $\mu : A \otimes A \to A$ to the arrow, while in the second case, assign to it the comultiplication $\Delta : A \to A \otimes A$. Since μ is grading preserving, while Δ increases the degree by a, it will turn out (Proposition I.7.4.5) that the resulting maps are actually grading preserving. Horizontal arrows are linear combinations of formal identities, and we interpret them as the corresponding linear combinations of identities on the respective tensor powers of A. For the Hopf link, we get the following:



$$\left\langle \cdot \right\rangle_A = \left[\begin{array}{ccccc} A^{\otimes 2}\{-a\}_q & & & & \\ \uparrow{\scriptstyle\begin{pmatrix}\Delta & \Delta\end{pmatrix}} & & & & \\ A\oplus A & \xrightarrow{\text{id}} & A\oplus A & & \\ \uparrow{\scriptstyle\begin{pmatrix}\mu \\ -\mu\end{pmatrix}} & & \uparrow{\scriptstyle\mu} & & \\ A^{\otimes 2} & \xrightarrow{\begin{pmatrix}1 \\ -1\end{pmatrix}} & A^{\otimes 2}\oplus A^{\otimes 2} & \xrightarrow{\begin{pmatrix}1 & 1\end{pmatrix}} & \underline{\underline{A^{\otimes 2}}} \end{array}\right] \tag{I.7.4.20}$$

Definition I.7.4.2. *For an oriented link diagram D and a graded Frobenius algebra A, we denote by $\mathrm{H}_{**}(D; A)$ the vertical-then-horizontal homology $\mathrm{H}^{\mathrm{hor}}(\mathrm{H}^{\mathrm{ver}}(\langle D\rangle_A))$ of $\langle D\rangle_A$ and call it the* A-homology *of D. We denote its Poincaré polynomial by*

$$\mathrm{P}(D; A) := \sum_{a,b,c\in\mathbb{Z}} \dim_{\Bbbk} \mathrm{H}_{ab}(D)_c \; q^c t^a s^b \in \mathbb{Z}[q^{\pm 1}, s^{\pm 1}, t^{\pm 1}].$$

Theorem I.7.4.3. *Let $k \geq 2$ and let $\Bbbk$ be a field such that $k+1 \in \Bbbk^\times$. Further, let D be an oriented link diagram, and assume the following:*

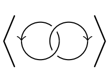
(i) *The vertical homology $\mathrm{H}^{\mathrm{ver}}_{t,s}(\langle D \rangle_{\Bbbk[x]/(x^k)})$ is concentrated in a single parity of s only.*

(ii) *If $\mathrm{H}_{t,s}(D; \Bbbk[x]/(x^k)) \neq 0$, then $\mathrm{H}_{t-1,s-n}(D; \Bbbk[x]/(x^k)) = 0$ for all $n > 0$.*

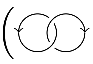
Then $\mathrm{P}(D; \Bbbk[x]/(x^k))\big|_{s=q^{-(k+1)}} = \mathcal{KR}^k_\Bbbk(L)$ for the link L presented by D.

Proof. Using assumption (i), part (ii) of Theorem I.7.3.12 shows that the vertical-then-horizontal homology $\mathrm{H}_{**}(D; \Bbbk[x]/(x^k))$ of $\langle D \rangle_A$ is the homology of the associated graded of a filtration on $\mathcal{KR}^k_\Bbbk(D)$, up to a q-shift by $-(k+1)d$ in s-degree d. Hence $\mathrm{H}_{**}(D; \Bbbk[x]/(x^k))$ is, up to the same shift as well as some reindexing, the E_2-page of a spectral sequence converging to the homology of $\mathcal{KR}^k_\Bbbk(D)$, and assumption (ii) ensures that this spectral sequence degenerates at the E_2-page. □

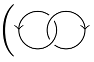
Example I.7.4.4. We compute the A-homology of the presentation  of the Hopf link. From (I.7.4.20) one sees that the vertical homology of $\langle$$\rangle_A$ is given by

$\mathrm{H}^{\mathrm{ver}}\Big(\Big\langle$  $\Big\rangle_A\Big) \cong$
$$\begin{bmatrix} \mathrm{coker}(\Delta)\{-a\}_q & 0 & 0 \\ 0 & 0 & 0 \\ \ker(\mu) \xrightarrow{\begin{pmatrix} 1 & -1 \end{pmatrix}^t} & \ker(\mu)^{\oplus 2} \xrightarrow{\begin{pmatrix} \mathrm{incl} & \mathrm{incl} \end{pmatrix}} & \underline{\underline{A^{\otimes 2}}} \end{bmatrix}$$

and we conclude that the A-homology of  is

$\mathrm{H}_{**}\Big($  $; A\Big) \cong$
$$\begin{bmatrix} \mathrm{coker}(\Delta)\{-a\}_q & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \underline{\underline{A}} \end{bmatrix}$$

with the Poincaré polynomial

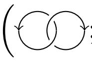
$\mathrm{P}\Big($  $; A\Big) = s^{-2}t^2q^{-a}[\mathrm{coker}(\Delta)]_q + [A]_q;$

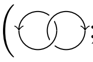
here $[V]_q := \sum_{n \in \mathbb{Z}} \dim_\Bbbk V_n q^n \in \mathbb{Z}[q^{\pm 1}]$ denotes the graded dimension of a graded vector space $V = \{V_n\}_{n \in \mathbb{Z}}$. In particular, the assumptions of Theorem I.7.4.3 are fulfilled, and hence we should recover the Khovanov-Roansky homology of the Hopf link when specializing to $A := \mathbb{Q}[x]/(x^k)$.

Let's double check this: We have $\deg(x) = 2$, $a = 2k-2$ and $[A]_q = q^{k-1}[k]_q$, where $[k]_q := \frac{q^k - q^{-k}}{q - q^{-1}}$ is the k-th quantum number. Further,

$$[\mathrm{coker}(\Delta)]_q = [A^{\otimes 2}]_q - q^a[A]_q = q^{2k-2}[k]_q^2 - q^{3k-3}[k]_q,$$

so we get

P( $; \mathbb{Q}[x]/(x^k)) = s^{-2}t^2[k]_q([k]_q - q^{k-1}) + q^{k-1}[k]_q$

P( $; \mathbb{Q}[x]/(x^k))\Big|_{s=q^{-(k+1)}} = t^2q^{2k+2}[k]_q([k]_q - q^{k-1}) + q^{k-1}[k]_q.$

We have $q^{2k} = (q^2-1)q^{k-1}[k]_q + 1$, hence $q^{3k+1}[k]_q = q^{2k+2}[k]_q^2 - q^{2k}[k]_q^2 + q^{k+1}[k]_q$, so

P( $; \mathbb{Q}[x]/(x^k))\Big|_{s=q^{-(k+1)}} = t^2q^{2k}[k]_q^2 - t^2q^{k+1}[k]_q + q^{k-1}[k]_q, \qquad \text{(I.7.4.21)}$

in accordance with the computation [CM14, p.509] of $\mathcal{KR}^k$(). ◊

We still have to check that $\langle - \rangle_A$ is indeed a bicomplexes of graded vector spaces:

Proposition I.7.4.5. *For any oriented link diagram D and any commutative, positively graded Frobenius algebra over a field $\Bbbk$, $\langle D\rangle_A$ is a bicomplex of graded vector spaces and grading preserving maps over $\Bbbk$.*

Proof. Firstly, we check that all differentials in $\langle D\rangle_A$ are grading preserving; since all horizontal differentials are linear combinations of projections and inclusions, we only need to consider the vertical ones. If the replacement of a crossing $\times$ by a shifted uncrossing $||\{-1\}$ results in splitting one circle into two, the corresponding terms in $\langle D\rangle_A$ are of the form $X\otimes A\{-\frac{a}{2}\}_q$ and $X\otimes A\otimes A\{-a\}_q\{-\frac{a}{2}\}_q = X\otimes A\otimes A\{-\frac{3}{2}a\}_q$ for some X, respectively, and the vertical differential is given by $\mathrm{id}_X\otimes\Delta$ by definition – since Δ has degree a, it is therefore degree-preserving. If replacing $\times$ by $||$ merges two circles into one, the terms have the form $X\otimes A\otimes A\{-a\}_q$ and $X\otimes A\{-\frac{a}{2}\}_q\{-\frac{a}{2}\}_q = X\otimes A\{-a\}_q$, and the vertical differential is given by $\mathrm{id}_X\otimes\mu$ – again, this is degree-preserving since the multiplication μ is. This treats the vertical differentials arising from the elementary bicomplex (I.7.4.18). The two cases arising from (I.7.4.17), where some shifted uncrossing $||\{1\}$ is replaced by the crossing $\times$, are similar.

Secondly, we have to check that horizontal and vertical differentials anticommute and square to zero. We restrict to convince ourselves that the vertical differential squares to zero. Due to Koszul signs, this amounts to showing that whenever at two fixed places in a diagram of closed circles we exchange $\times$ and $||$, the resulting two vertical differentials commute. Apart from trivial commutativity relations such as those in Figure I.7.4.2, the constraints on μ and Δ occurring this way are precisely the associativity of μ (Figure I.7.4.3), the coassociativity of Δ (Figure I.7.4.3 read backwards), and finally the Frobenius condition (Figure I.7.4.4). □

Notation I.7.4.6. Considering an oriented link diagram D as a graph with the crossings as 4-valent vertices, the bicomplex $\langle D\rangle_A$ is naturally a bicomplex of $A^{\otimes e}$-modules, where e is the number of edges of D. The copy of A associated to an edge α is denoted A_α. ◊

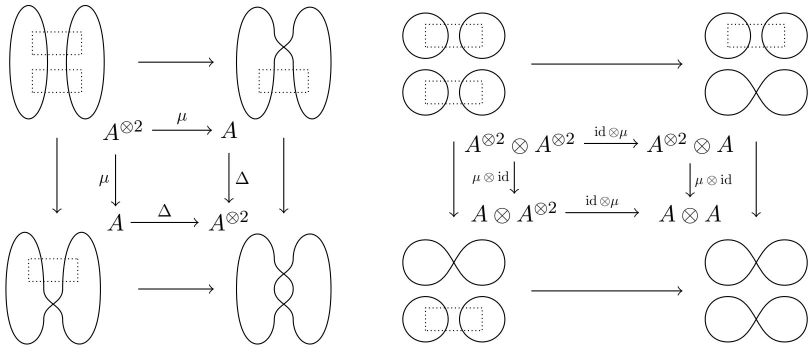


Figure I.7.4.2. Some trivial commutativity relations

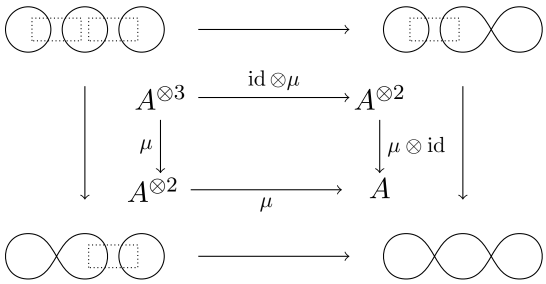


Figure I.7.4.3. Associativity constraint on $\mu : A \otimes A \to A$

I.7.4.3. The cancelling spectral sequence

In this section we collect some facts about $\langle - \rangle_A$ and make some more definitions, preparing for ground for the example of the trefoil to come. The first observation concerns the total homology of $\langle - \rangle_A$. Since the 0-th rows in the elementary bicomplexes (I.7.4.17) and (I.7.4.18) are contractible, the horizontal homology of $\langle D \rangle_A$ is concentrated at (t, s)-bidegree $(-1, 1) \cdot \mathrm{w}(D)$, where it is given by $(A\{-\frac{a}{2}\}_q)^{\otimes \sharp D}$. Here, $\sharp D$ is the number of components of the link represented by D, and $\mathrm{w}(D)$ is the writhe of D, i.e. the number of positive crossings minus the number of negative crossings. The two spectral sequences from horizontal-then-vertical and vertical-then-horizontal homology to the total homology therefore yield the following result.

Proposition I.7.4.7. *Let D be an oriented link diagram.*

(i) *The total homology of $\langle D \rangle_A$ is canonically isomorphic to $(A\{-\frac{a}{2}\}_q)^{\otimes \sharp D}$, concentrated in degree* 0.

(ii) *There is a spectral sequence $\{\mathrm{E}_2^{p,q}\}$ starting on the second sheet $\mathrm{E}_2^{p,q} \cong \mathrm{H}_{p,q}(D; A)$*

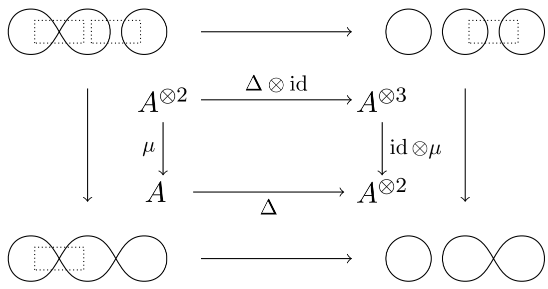


Figure I.7.4.4. Frobenius condition

and converging to $A\{-\frac{a}{2}\}_q^{\otimes \sharp D}$ *concentrated in degree* 0.

We will call the spectral sequence from Proposition I.7.4.7 the *cancelling spectral sequence*, as it reduces the homology of any knot to a single copy of A. In the example of the Hopf link the sequence already degenerates at the E_2-page, and $A^{\otimes 2}\{-a\}_q \cong \mathrm{coker}(\Delta)\{-a\}_q \oplus A$. However, as we will see already in the example of the trefoil knot to be discussed in the next section, this is not true in general.

Before coming to the next section, we introduce some more notation.

Notation I.7.4.8. For ease of notation we extend the definition of $\langle - \rangle$ by

$$\langle \times_+ \rangle := \left[\,\underline{\underline{\sqcup\sqcup}} \longrightarrow \times\{-1\} \right]^t, \quad \langle \times_- \rangle := \left[\times\{1\} \longrightarrow \underline{\underline{\sqcup\sqcup}} \right]^t, \tag{I.7.4.22}$$

so that the bicomplexes (I.7.4.17) and (I.7.4.18) assigned to positive and negative crossings can also respectively be written as follows (recall our conventions for $[-]$ in I.7.4.1):

$$\langle \overline{\times} \rangle = \left[\langle \times_+ \rangle \{1\} \longrightarrow \underline{\underline{\sqcup\sqcup}}\{1\} \right], \quad \langle \underline{\times} \rangle = \left[\underline{\underline{\sqcup\sqcup}}\{-1\} \longrightarrow \langle \times_- \rangle \{-1\} \right]. \tag{I.7.4.23}$$

The pieces $\times_+$ and $\times_-$ are called *positive and negative wide edges*, respectively. Note that there are the following canonical morphisms between them,

$$W_\pm : \langle \times_+ \rangle \to \langle \times_- \rangle, \qquad \text{and} \qquad W_\mp : \langle \times_- \rangle \{-2\}_s\{-2\} \to \langle \times_+ \rangle. \tag{I.7.4.24}$$

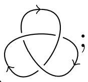
$\Diamond$

To summarize: We defined a bracket $\langle - \rangle$ taking as argument any planar, not necessarily closed, oriented link diagram that may additionally contain the signed 4-valent vertices $\times_+$ and $\times_-$. This bracket takes values in formal bicomplexes of planar, unoriented strands (possibly intersecting and possibly with boundary), together with formal sums and shifts, but we haven't been precise about the meaning of these diagrams. We have seen, however, that the decorated bracket $\langle - \rangle_A$ produces an honest bicomplex of graded vector spaces from a *closed* planar oriented link diagram, which again might also contain the 4-valent vertices $\times_+$ and $\times_-$.

I.7.4.4. The homology of the trefoil

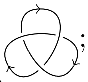
The purpose of this section is to give the reader an impression of what kind of structure can arise when computing the A-homology of a link diagram, with the concrete goal of establishing the following result:

Proposition I.7.4.9. *The A-homology of the presentation*  *of the trefoil is given by*

$\mathrm{H}_{**}\Big($  $; A\Big) =$

$$\begin{bmatrix} \ker(\mu\circ\Delta)\{\tfrac{a}{2}\}_q & \operatorname{coker}(\mu\circ\Delta)\{-\tfrac{a}{2}\}_q & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \underline{\underline{A\{\tfrac{a}{2}\}_q}} \end{bmatrix}$$

E_3

$\mathrm{P}\Big($  $; A\Big) =$

$$[\ker(\mu\circ\Delta)]_q q^{\frac{a}{2}} s^{-2} t^3 + [\operatorname{coker}(\mu\circ\Delta)]_q q^{-\frac{a}{2}} s^{-2} t^2 + [A]_q q^{\frac{a}{2}} s^0 t^0.$$

The only nontrivial differential in the cancelling spectral sequence is at the E_3*-page, where it is the natural inclusion* $\ker(\mu\circ\Delta)\{\frac{a}{2}\}_q \hookrightarrow A\{\frac{a}{2}\}_q$ *with cokernel* $\operatorname{im}(\mu\circ\Delta\{\frac{a}{2}\}_q) \subset A\{-\frac{a}{2}\}_q$*. In particular, the sequence is constant starting from*

$$\begin{bmatrix} \operatorname{coker}(\mu\circ\Delta)\{-\tfrac{a}{2}\}_q & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \underline{\underline{\operatorname{im}\left(A\{\tfrac{a}{2}\}_q \xrightarrow{\mu\circ\Delta} A\{-\tfrac{a}{2}\}_q\right)}} \end{bmatrix},$$

showing the associated graded of the following filtration of the limit term $A\{-\frac{a}{2}\}_q$*:*

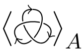
$$0 \subseteq \operatorname{im}(\mu\circ\Delta\{\tfrac{a}{2}\}_q) \subseteq A\{-\tfrac{a}{2}\}_q$$

In particular, the hypothesis (ii) of Theorem I.7.4.3 is fulfilled, and we'll see during the proof of Proposition I.7.4.9 that so is hypothesis (i). Let's double check again that for $A = \mathbb{Q}[x]/(x^k)$ the result of Proposition I.7.4.9 indeed agrees with previous computations of Khovanov-Rozansky homology:

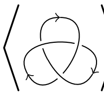
We have $\ker(\mu\circ\Delta) = (x)$, $\operatorname{im}(\mu\circ\Delta) = (x^{k-1})$, hence $[\ker(\mu\circ\Delta)]_q = [A]_q - 1 = q^k[k-1]_q$ and $[\operatorname{im}(\mu\circ\Delta)]_q = [A]_q - q^{2k-2} = q^{k-2}[k-1]_q$; a short calculation then finally gives

$\mathrm{P}\Big($  $; \mathbb{Q}[x]/(x^k)\Big)\big|_{s=q^{-(k+1)}} = q^{4k+1}[k-1]_q t^3 + q^{2k+1}[k-1]_q t^2 + q^{2k-2}[k]_q$

in accordance with [CM14, p.509]. In the case of integral coefficients, i.e. $A = \mathbb{Z}[x]/(x^k)$, we obtain an additional copy of $\mathbb{Z}/k\mathbb{Z}$ at (q,t)-degree $q^{2k+3}[k-1]_q t^2$, which for $k = 2$ is in accordance with the integral Khovanov homology of the trefoil [BNMa]. Also, note that for $A = (\mathbb{Z}/k\mathbb{Z})[x]/(x^k)$ the composition $\mu \circ \Delta$ vanishes, affecting the link homology and matching observations [Kho03, §4] for Khovanov homology computed over $\mathbb{Z}/2\mathbb{Z}$.

Proposition I.7.4.9 is a consequence of the following description of $\langle$  $\rangle_A$:

Proposition I.7.4.10. *There is a horizontal equivalence (Definition I.7.4.1):*

$\Big\langle$  $\Big\rangle_A \simeq$

$$\left[\begin{array}{llll} A\{-\frac{a}{2}\}_q & & & \\ \uparrow \mu & & & \\ A^{\otimes 2}\{-\frac{a}{2}\}_q & \xrightarrow{\text{id}} A^{\otimes 2}\{-\frac{a}{2}\}_q & & \\ & \quad\uparrow \Delta & & \\ & A\{\frac{a}{2}\}_q & \xrightarrow{\text{id}} A\{\frac{a}{2}\}_q & \\ & & \quad\uparrow \mu & \\ & & A^{\otimes 2}\{\frac{a}{2}\}_q & \xrightarrow{\text{id}} \underline{\underline{A^{\otimes 2}}}\{\frac{a}{2}\}_q \end{array}\right] \qquad \text{(I.7.4.25)}$$

Given Proposition I.7.4.10 and the explicit description of the spectral sequence of a bicomplex (similar to the explicit description of the spectral complex associated to a mixed complex given in Section I.4.6), Proposition I.7.4.9 follows. The proof of Proposition I.7.4.10 will occupy the rest of this section, and we will establish some isomorphisms along the way that might also be useful for further calculations. To begin, the rough structure of the bicomplex $\Big\langle$  $\Big\rangle_A$ is shown in Figure I.7.4.5. In the following, we will

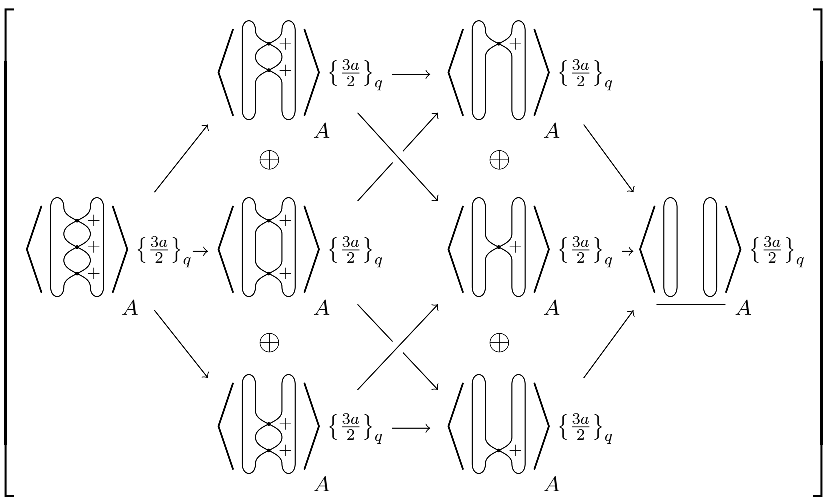
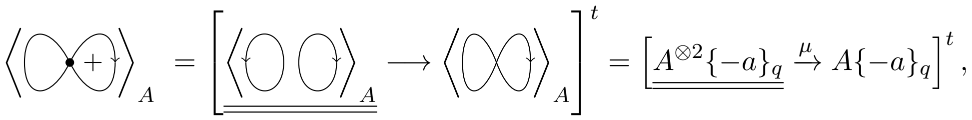


Figure I.7.4.5. Structure of the bicomplex associated to 

go through its different t-degrees and simplify the corresponding columns, in particular noting that with respect to the s-grading their cohomology is always concentrated in a single parity.

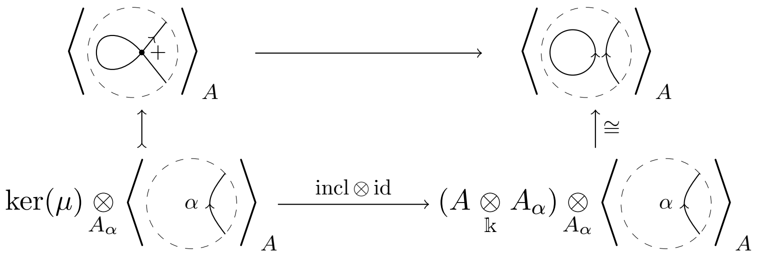
The t-degree 0 is by definition just $A^{\otimes 2}\{\frac{3}{2}a\}_q\{-a\}_q$ concentrated in homological degree 0, so the vertical homology is $A^{\otimes 2}\{\frac{a}{2}\}_q$. For the t-degree 1, we have



whose vertical homology is therefore canonically isomorphic to $\ker(\mu)\{-a\}_q$. This identifies the vertical homology of (I.7.4.26) in t-degree 1 with $\ker(\mu)^{\oplus 3}\{\frac{a}{2}\}_q$, with the morphism to the vertical homology $A^{\otimes 2}\{\frac{a}{2}\}_q$ in t-degree 0 being given by the inclusions.

Before going on to the t-degree 2, we pause to study in general what happens when one adds a positive or negative wide edge to a diagram, looped at one side. The proofs are not difficult and omitted. Recall Notation I.7.4.6.

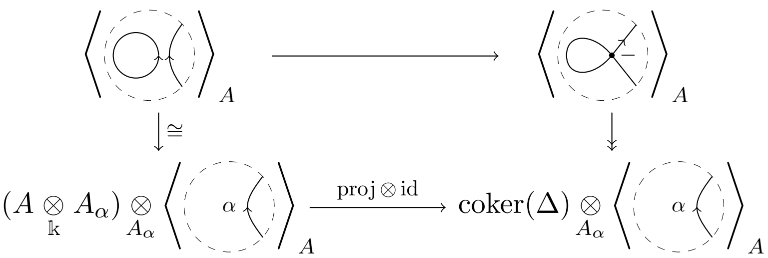
Fact I.7.4.11. *There is a commutative diagram of bicomplexes*



(I.7.4.27)

where the right vertical map is an isomorphism of bicomplexes and the left vertical map is a monomorphism which is a termwise vertical equivalence.

Fact I.7.4.12. *There is a commutative diagram of bicomplexes*

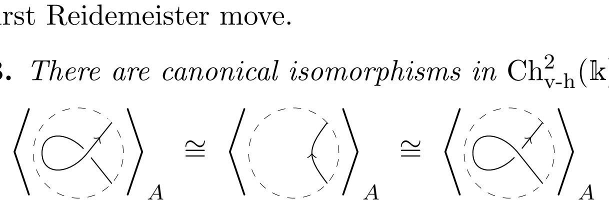


(I.7.4.28)

where the left vertical map is an isomorphism of bicomplexes and the right vertical map is an epimorphism which is a termwise vertical equivalence.

The previous facts show the invariance of the vertical-then-horizontal homotopy type of $\langle - \rangle_A$ under the first Reidemeister move.

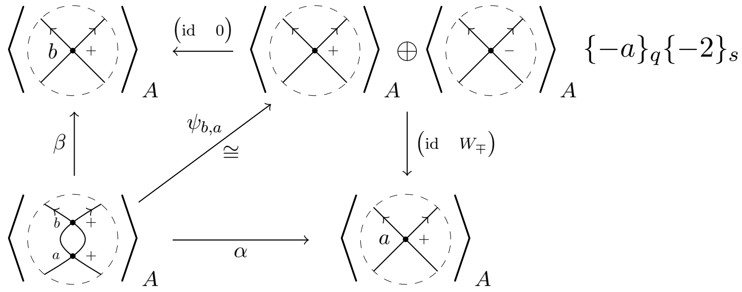
Corollary I.7.4.13. *There are canonical isomorphisms in* $\mathrm{Ch}^2_{\text{v-h}}(\Bbbk)$*:*



(I.7.4.29)

Next we study the terms of t-degree 2 in (I.7.4.26).

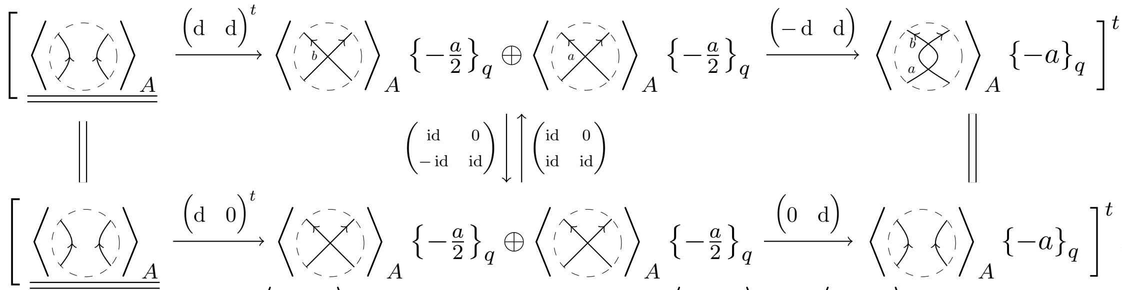
Proposition I.7.4.14. *There exists a canonical isomorphism* $\psi_{b,a}$ *of bicomplexes making the following diagram of bicomplexes commute:*

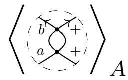
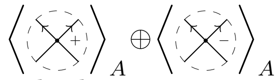
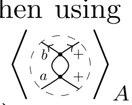


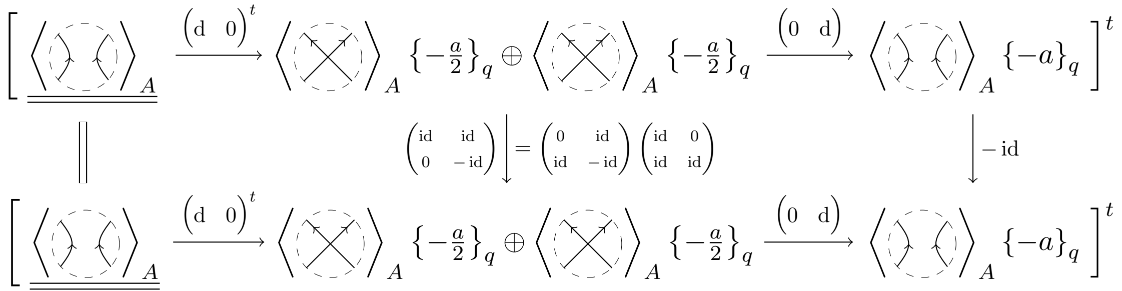
(I.7.4.30)

Similarly, there is a canonical isomorphism $\psi_{a,b}$ *such that* $\beta = \begin{pmatrix}\mathrm{id} & W_{\mp}\end{pmatrix} \circ \psi_{a,b}$ *and* $\alpha = \begin{pmatrix}\mathrm{id} & 0\end{pmatrix} \circ \psi_{a,b}$, *and* $\psi_{a,b} \circ \psi_{b,a}^{-1} = \psi_{b,a} \circ \psi_{a,b}^{-1} = \begin{pmatrix}\mathrm{id} & W_{\mp} \\ 0 & -\mathrm{id}\end{pmatrix}$.

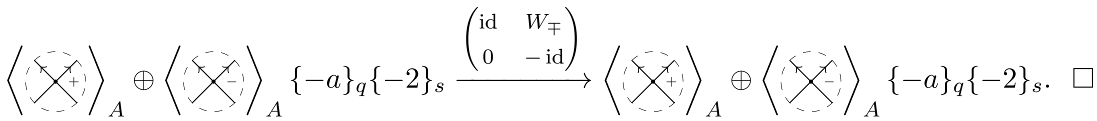
Proof. Ordering $b < a$ to determine Koszul signs, the isomorphism $\psi_{b,a}$ is given by



Here, the first row is  and the second equals  $\{-a\}_q\{-2\}_s$. A quick check shows that the diagram (I.7.4.30) is indeed commutative. The isomorphism $\psi_{a,b}$ can be obtained likewise when using the ordering $a < b$ and exchanging a and b everywhere; in the realization of  as the first row above, it is therefore given by $\begin{pmatrix}\mathrm{id} & 0 \\ -\mathrm{id} & \mathrm{id}\end{pmatrix}\begin{pmatrix}0 & \mathrm{id} \\ \mathrm{id} & 0\end{pmatrix} = \begin{pmatrix}0 & \mathrm{id} \\ \mathrm{id} & -\mathrm{id}\end{pmatrix}$ as the middle vertical map, and by $-\mathrm{id}$ as the right vertical map (due to the change of Koszul signs). Hence $\psi_{a,b} \circ \psi_{b,a}^{-1}$ is given by

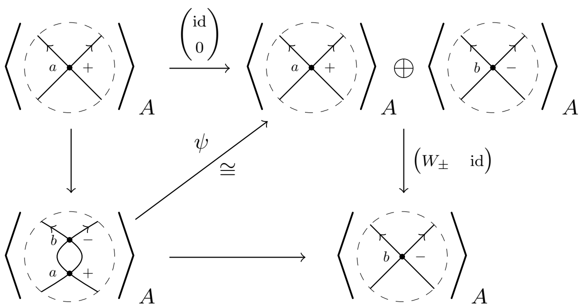


which by definition (I.7.4.24) of $W_{\mp}$ is the same as



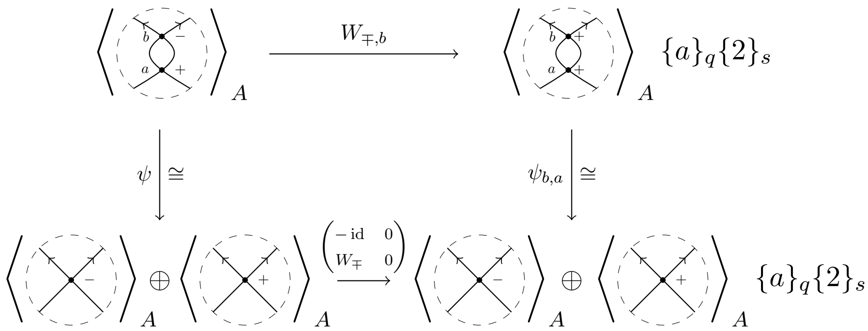
The morphisms $\psi_{a,b}$ resp. $\psi_{b,a}$ are called *simplifications of (a,b) based at a resp. b.* For a combination of a positive and a negative wide edge, we have:

Proposition I.7.4.15. *There exists a canonical isomorphism ψ making the following diagram of bicomplexes commute:*



(I.7.4.31)

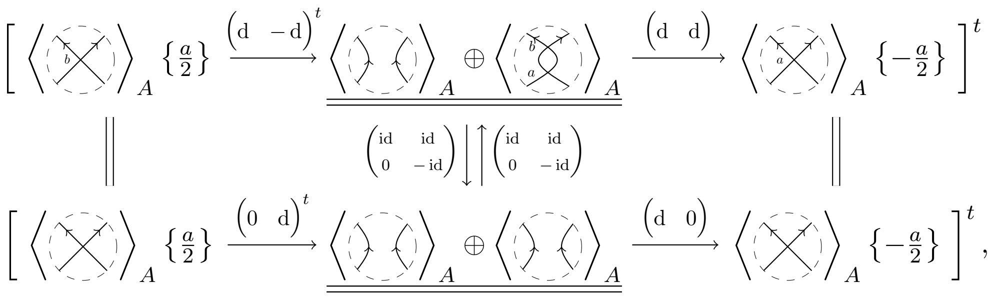
It is compatible with the simplifications of double positive wide edges studied in Proposition I.7.4.14 in the sense that the following diagram commutes:

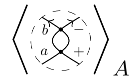
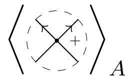
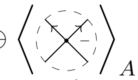


(I.7.4.32)

For the simplification $\psi_{a,b}$ we have $\psi_{a,b} W_{\mp,b} \psi^{-1} = \begin{pmatrix} -\mathrm{id} & 0 \\ -W_{\pm} & 0 \end{pmatrix}$.

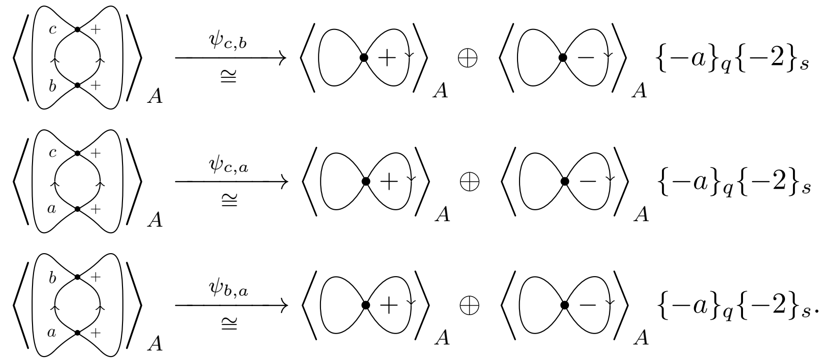
Proof. Again ordering $b < a$ to determine signs, the isomorphism ψ is given by



where the first row is  and the second equals  $\oplus$ . One can then quickly check the commutativity of (I.7.4.31), and given the explicit formula for $\psi_{b,a}$ from the proof of Proposition I.7.4.14 one can also check the commutativity of (I.7.4.32). Finally, by Proposition I.7.4.14 we have $\psi_{a,b} \circ \psi_{b,a}^{-1} = \begin{pmatrix} \mathrm{id} & W_\mp \\ 0 & -\mathrm{id} \end{pmatrix}$, so

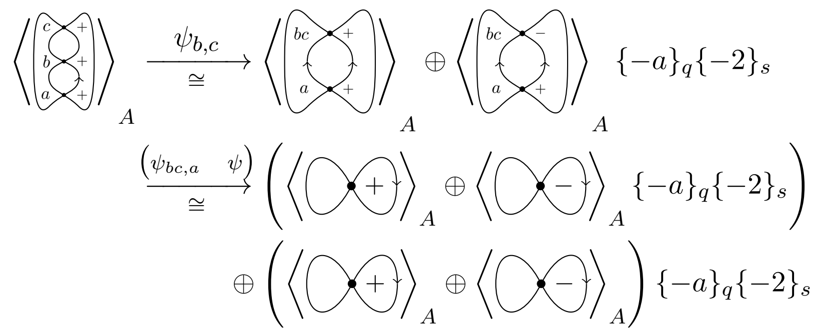
$$\psi_{a,b} W_{\mp,b} \psi^{-1} = \begin{pmatrix} \mathrm{id} & W_\mp \\ 0 & -\mathrm{id} \end{pmatrix} \psi_{b,a} W_{\mp,b} \psi^{-1} = \begin{pmatrix} \mathrm{id} & W_\mp \\ 0 & -\mathrm{id} \end{pmatrix} \begin{pmatrix} -\mathrm{id} & 0 \\ W_\pm & 0 \end{pmatrix} = \begin{pmatrix} -\mathrm{id} & 0 \\ -W_\pm & 0 \end{pmatrix}. \qquad \square$$

With these rules at hand, we can now compute the t-degrees 2 and 3 in (I.7.4.26). For the terms in t-degree 2, we have the isomorphisms from Proposition I.7.4.14,



(I.7.4.33)

Note that it is very important here to specify the precise isomorphism, i.e. to be clear about the base of simplification: we always choose to be the upper of the two positive wide edges under consideration. For the t-degree 3, we get the following chain of isomorphisms, where again we use simplification based at the upper edge:



(I.7.4.34)

The isomorphisms (I.7.4.33) and (I.7.4.34) allow to simplify the columns in the bicomplex (I.7.4.26); in particular, we see that they all have vertical cohomology concentrated in a single parity, that is, condition (i) of Theorem I.7.4.3 is fulfilled. Next, the various compatibilities between the simplifications that we studied above allow one to also work out the differentials. We omit the calculations here, but when using the ordering $a < b <$

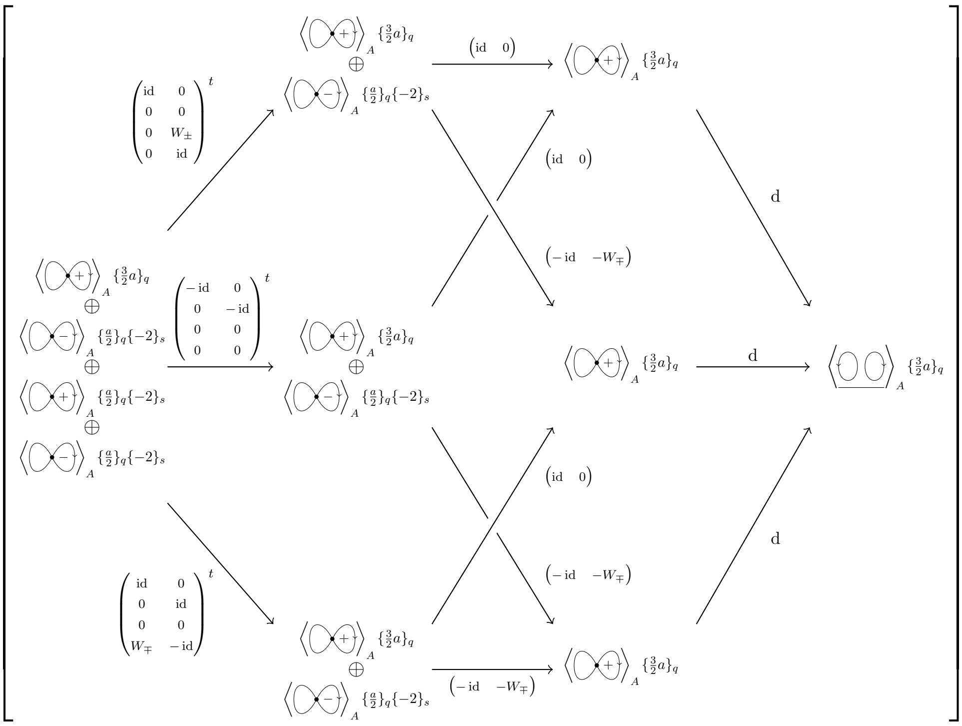


Figure I.7.4.6. The bicomplex associated to the presentation of the trefoil

c to determine Koszul signs, $\langle \cdots \rangle_A$ turns out to be isomorphic to the bicomplex shown in Figure I.7.4.6. Via row and column operations as well as Gaussian elimination (e.g. [BNBS14, Lemma 3.2]) – the details of which we again omit – this bicomplex reduces to

$$\langle \cdots \rangle_A \{\tfrac{a}{2}\}_q\{-2\}_s \xrightarrow{W_\pm} \langle \cdots \rangle_A \{\tfrac{a}{2}\}_q\{-2\}_s \xrightarrow{W_\mp} \langle \cdots \rangle_A \{\tfrac{3}{2}a\}_q \xrightarrow{\mathrm{d}} \underline{\langle \cdots \rangle_A}\{\tfrac{3}{2}a\}_q$$

which by (I.7.4.22) is (I.7.4.25), finishing the proof of Propositions I.7.4.9 and I.7.4.10.

I.7.5. Integrating the Frobenius algebra of an isolated hypersurface singularity

In this section, we assume $\Bbbk$ to be a field.

Statements analogous to Propositions I.7.3.1 and I.7.3.2 are also true in $\mathfrak{so}(2n)$ KR-homology: In [KR07, Proposition 4.4] it is shown that properly defined saddle morphisms between the stabilized diagonal and the stabilized twisted diagonal yield multiplication and comultiplication of the Frobenius algebra $\Bbbk[x,y]/(y^2+(2n+1)x^{2n}, xy)$ underlying their link homology. Also, Proposition I.7.3.1 is in accordance with [KR07, Lemma A.3].

In view of these results, one might wonder whether for *any* quasi-homogeneous potential $w \in \Bbbk[x_1, \ldots, x_s]$ with isolated singularity one can integrate multiplication and comultiplication of the associated Frobenius algebra $A = \Bbbk[x_1, \ldots, x_s]/(\partial_{x_i} w)$ to saddle morphisms between the stabilizations of the diagonal Δ resp. twisted diagonal X,

$$\begin{aligned} \Delta &:= \Bbbk[x_{1,*}, x_{2,*}, y_{1,*}, y_{2,*} \mid * = 1, \ldots, s]/(x_{1,*} = y_{1,*},\ x_{2,*} = y_{2,*}), \\ X &:= \Bbbk[x_{1,*}, x_{2,*}, y_{1,*}, y_{2,*} \mid * = 1, \ldots, s]/(x_{1,*} = y_{2,*},\ x_{2,*} = y_{1,*}). \end{aligned}$$

In type A, the saddle morphisms originated from the fundamental extensions $0 \to \Delta\langle -2\rangle \to B \to X \to 0$ and $0 \to X\langle -2\rangle \to B \to \Delta \to 0$. These can be generalized, but we need to fix some notation first.

Notation I.7.5.1. We denote by l_i the degree of the variable x_i and by d the total degree of the potential w under consideration. Also, recall that s is the dimension of the ambient space. These numbers relate to the the top dimension a of A (called a-invariant or Gorenstein parameter) by $a = sd - 2\sum_i l_i$ [BH93, Example 3.6.10, Corollary 3.6.14, Theorem 3.6.19]. For example, in case of the type A potential $w = x^{k+1}$ we have $a = 1 \cdot (2k+2) - 2 \cdot 2 = 2k - 2$, while for the type D potential $w = xy^2 + x^{2n+1}$ we get $a = 2(4n+2) - 2(2+2n) = 4n$. Further, we put

$$\dot{\mathrm{X}} := \Bbbk[x_{1,*}, x_{2,*}, y_{1,*}, y_{2,*} \mid * = 1, \ldots, s]/(x_{1,*} = y_{1,*} = x_{2,*} = y_{2,*}),$$

the quotient of Δ by the Δ-regular sequence $x_{1,*} - y_{2,*}$, or equivalently the quotient of X by the X-regular sequence $x_{1,*} - y_{1,*}$. Finally, we denote $\mathrm{K}(M; \underline{x})$ the Koszul complex associated to a module M and a sequence of elements $\underline{x}$, and recall our previous notation

$$\widehat{w}_2 := w(x_{1,*}) + w(x_{2,*}) - w(y_{1,*}) - w(y_{2,*}) \in \Bbbk[x_{1,*}, x_{2,*}, y_{1,*}, y_{2,*}] =: S. \qquad \diamondsuit$$

Proposition I.7.5.2. *There are canonical isomorphisms of $S/(\widehat{w}_2)$-modules:*

$$\mathrm{Ext}^s_{S/(\widehat{w}_2)}(X, \Delta) \xleftarrow[\cong]{\widetilde{\alpha}} \dot{\mathrm{X}} \left\langle \sum\nolimits_i l_i \right\rangle \xrightarrow[\cong]{\widetilde{\beta}} \mathrm{Ext}^s_{S/(\widehat{w}_2)}(\Delta, X)$$

determined by the following:

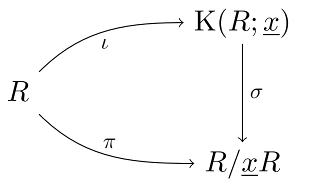
(i) *$\widetilde{\alpha}$ sends $1 \in \dot{\mathrm{X}}\langle \sum_i l_i \rangle$ to the pullback along the projection $X \twoheadrightarrow \dot{\mathrm{X}}$ of the augmented Koszul complex $\mathrm{K}(\Delta; x_{1,*} - y_{2,*}) \twoheadrightarrow \dot{\mathrm{X}}$.*

(ii) *$\widetilde{\beta}$ sends $1 \in \dot{\mathrm{X}}\langle \sum_i l_i \rangle$ to the pullback along the projection $\Delta \twoheadrightarrow \dot{\mathrm{X}}$ of the augmented Koszul complex $\mathrm{K}(X; x_{1,*} - y_{1,*}) \twoheadrightarrow \dot{\mathrm{X}}$.*

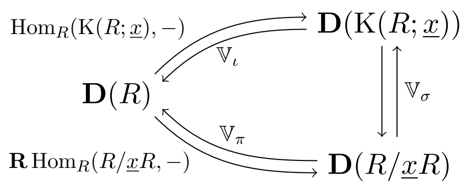
Proof. If one only cares about the existence of the isomorphisms, one can apply the following result [BH93, Lemma 3.1.16]: If N is a module over a commutative ring R

and $x \in R$ is both R-regular and N-regular, then for any R/xR-module M there is a canonical isomorphism $\operatorname{Ext}_R^{*+1}(M,N) \cong \operatorname{Ext}^*_{R/xR}(M, N/xN)$; the proof given in loc.cit. goes by showing that both sides are effaceable δ-functors sharing their base term for $* = 0$. Here we will instead employ the language of derived categories.

Suppose R is a commutative ring and $\underline{x} = x_1, \ldots, x_n \in R$ is a regular sequence with Koszul dg algebra $\mathrm{K}(R;\underline{x})$. We then have a commutative diagram of dg rings



where the right vertical map is a quasi-isomorphism by the regularity of $\underline{x}$. Associated to this diagram of dg rings there is a diagram of adjunctions



where the right vertical adjunction is an adjoint equivalence. Suppose now that M is an $R/\underline{x}R$-module and that N is an R-module such that $\underline{x}$ is N-regular. Then

$$\begin{aligned}\operatorname{Hom}_{\mathbf{D}(R)}(\mathbb{V}_\pi M, \Sigma^k N) &\cong \operatorname{Hom}_{\mathbf{D}(R)}(\mathbb{V}_\iota \mathbb{V}_\sigma M, \Sigma^k N) \\ &\cong \operatorname{Hom}_{\mathbf{D}(\mathrm{K}(R;\underline{x}))}(\mathbb{V}_\sigma M, \Sigma^k \operatorname{Hom}_R(\mathrm{K}(R;\underline{x}), N)) \\ &\cong \operatorname{Hom}_{\mathbf{D}(\mathrm{K}(R;\underline{x}))}(\mathbb{V}_\sigma M, \Sigma^{k-n} \mathbb{V}_\sigma(N/\underline{x}N)) \\ &\cong \operatorname{Hom}_{\mathbf{D}(R/\underline{x}R)}(M, \Sigma^{k-n} N/\underline{x}N);\end{aligned} \tag{I.7.5.35}$$

here we used in the third step that, by definition of regularity, the projection

$$\operatorname{Hom}_R(\mathrm{K}(R;\underline{x}), N) \to \Sigma^{-n} N/\underline{x}N$$

is a quasi-isomorphism of $\mathrm{K}(R;\underline{x})$-modules, and in the last step that $\mathbb{V}_\sigma$ is fully faithful.

This previous considerations can be carried out in the graded setting as well, giving

$$\operatorname{Hom}_{\mathbf{D}(R)}(\mathbb{V}_\pi M, \Sigma^k N) \cong \operatorname{Hom}_{\mathbf{D}(R/\underline{x}R)}(M, \Sigma^{k-n} N/\underline{x}N) \left\langle \sum\nolimits_i l_i \right\rangle.$$

We claim that these results are applicable in our situation, where $R = S/(\widehat{w}_2)$, $\underline{x} := x_{1,*} - y_{2,*}$, $M := X$ and $N := \Delta$. Indeed, the regularity of $\underline{x}$ in Δ is clear, and the regularity of $\underline{x}$ in $S/(\widehat{w}_2)$ follows since for (graded) local rings, regularity is invariant under permutation, and $x_{1,*} - y_{2,*}, \widehat{w}_2$ is regular in S. Hence we may go through the chain of isomorphisms (I.7.5.35) to get the desired isomorphism

$$\widetilde{\alpha} : \mathrm{Ext}^s_{S/(\widehat{w}_2)}(X, \Delta) \cong \mathrm{Hom}_{S/(\widehat{w}_2, x_{1,*}-y_{2,*})}(X, \mathrm{X}) \left\langle \sum\nolimits_i |x_i| \right\rangle \cong \mathrm{X} \left\langle \sum\nolimits_i l_i \right\rangle .$$

For its explicit description, note that by the naturality in M of the isomorphisms involved in (I.7.5.35) it suffices to show that for $M := N/\underline{x}N$ the morphism $N/\underline{x}N \to \Sigma^k N$ corresponding to the identity on $N/\underline{x}N$ under (I.7.5.35) is given by the roof $N/\underline{x}N \leftarrow \mathrm{Hom}_R(K(\underline{x}), \Sigma^k N) \to \mathrm{Hom}_R(R, \Sigma^k N) = \Sigma^k N$, which follows by carefully making the isomorphisms in (I.7.5.35) explicit.

The existence and explicit description of the isomorphism $\widetilde{\beta}$ proceeds along the same lines, exchanging X and Δ and taking $\underline{x} := x_{1,*} - y_{1,*}$. □

Definition I.7.5.3. *The morphisms*

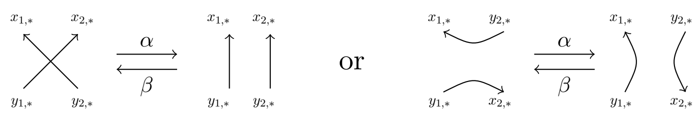
$$\alpha : {}_{\widehat{w}_2}X \longrightarrow \Sigma^k{}_{\widehat{w}_2}\Delta\langle -\sum\nolimits_i l_i\rangle, \quad \beta : {}_{\widehat{w}_2}\Delta \longrightarrow \Sigma^k{}_{\widehat{w}_2}X\langle -\sum\nolimits_i l_i\rangle \tag{I.7.5.36}$$

in $\mathbf{D}^{\mathrm{ctr}}\,\mathrm{LF}(S, \widehat{w}_2)$ *induced via Proposition II.C.2.2 by the morphisms*

$$\widetilde{\alpha}(1) \in \mathrm{Ext}^k_{S/(\widehat{w}_2)}\left(X, \Delta\left\langle -\sum\nolimits_i l_i\right\rangle\right), \quad \widetilde{\beta}(1) \in \mathrm{Ext}^k_{S/(\widehat{w}_2)}\left(\Delta, X\left\langle -\sum\nolimits_i l_i\right\rangle\right)$$

from Proposition I.7.5.2 are called saddle morphisms.

Graphically one may imagine α and β as



explaining the name saddle morphisms.

Conjecture I.7.5.4. *When taking stable Hochschild homology the saddle morphisms* α *and* β *give the multiplication and comultiplication maps of the canonical Frobenius algebra structure on* $A = \Bbbk[x_1, \ldots, x_s]/(\partial_{x_i} w)$*, respectively.*

We do not expect an explicit approach to fit here, as already the case of the type A potential x^{k+1}, treated in Propositions I.6.4.4 and I.7.3.2, was computationally involved. Instead, one might try to relate the above algebraically constructed saddle morphisms to the saddle morphisms formally deduced from the properties of the bicategory of Landau-Ginzburg models $\mathcal{LG}$. More precisely, we expect α and β from above to coincide with the unit and counit 2-morphisms associated to ${}_{\widehat{w}_2}\Delta$ when the latter is viewed as a 1-morphism $(\Bbbk, 0) \to (\widehat{\mathbb{A}}^2_\Bbbk, \widehat{w}_2)$ in $\mathcal{LG}$. Moreover, this 1-morphism should itself be the unit 1-morphism as part of a duality between the objects $(\widehat{\mathbb{A}}^1_\Bbbk, \widehat{w}_1)$ and $(\widehat{\mathbb{A}}^1_\Bbbk, -\widehat{w}_1)$ in a hypothetical pivotal monoidal structure on the bicategory $\mathcal{LG}$. The definition and a thorough discussion of adjoints in $\mathcal{LG}$ can be found in [CM12].

I.7.6. Equivariant and deformed Khovanov-Rozansky homology

The construction of Khovanov-Rozansky homology in case $k+1 \in \Bbbk^\times$ remains valid if we allow $\Bbbk$ itself to be $\mathbb{Z}$-graded and replace the elementary potential $w = x^{k+1}$ by $w(a_1,\ldots,a_k) = x^{k+1} + a_k x^k + \cdots + a_1 x$ for $a_i \in \Bbbk$ of q-degree $|a_i|_q = 2(k+1-i)$. Let's indicate the required modifications for this case:

(i) A resolution of the diagonal $\Bbbk[x,y]$-module by a matrix factorization of type $(\Bbbk[x,y], \widehat{w}_1(a_1,\ldots,a_k))$ is given by

$$\Bbbk[x,y]\langle -2\rangle \underset{u_k(x,y)+a_k u_{k-1}(x,y)+\ldots+a_2(x+y)+a_1}{\overset{x-y}{\rightleftarrows}} \Bbbk[x,y],$$

where as usual $u_i(x,y) := \frac{x^{i+1}-y^{i+1}}{x-y}$.

(ii) The value of the unknot is given by the Jacobian algebra $\Bbbk[x]/(\partial_x w(a_1,\ldots,a_k))$ of $w(a_1,\ldots,a_k)$, i.e. by

$$\Bbbk[x]/((k+1)x^k + k a_k x^{k-1} + \ldots + 2a_2 x + a_1) \cong \Bbbk \oplus \Bbbk\langle -2\rangle \oplus \cdots \oplus \Bbbk\langle -2k+2\rangle.$$

(iii) Proposition I.6.4.4 remains true and is proved in the same way; the only difference is that the commutative diagram (I.6.4.19) needs to be replaced by

$$\begin{array}{ccc} \mathbb{V}^{n+1}_{\mathbb{Z}_2}\operatorname{Tr}^{n+1}_{\mathbb{Z}_2} X & \longrightarrow & \Sigma\mathbb{V}^{n+1}_{\mathbb{Z}_2}\operatorname{Tr}^{n+1}_{\mathbb{Z}_2}\Delta\langle -2\rangle \\ \downarrow\cong & & \downarrow\cong \\ {}_{\widehat{w}_n}\Delta^n_{\Bbbk} & \xrightarrow{\sum_{i=0}^{k-1}\psi_i x^i_{n+1}} & \Bbbk[x_{n+1}]/(x^k_{n+1}) \otimes_{\Bbbk} {}_{\widehat{w}_n}\Delta^n_{\Bbbk}\langle 2k-2\rangle \end{array}$$

$$\psi_i := (k+1)x_n^{k-1-i} + k a_k x_n^{k-2-i} + \ldots + (i+3)a_{i+3}x_n + (i+2)a_{i+2};$$

the lower horizontal map encodes the comultiplication in the Frobenius algebra structure on the Jacobian algebra $\Bbbk[x]/((k+1)x^k + k a_1 x^{k-1} + \ldots + a_k)$ induced by the non-degenerate trace form

$$f = b_{k-1}x^{k-1} + \ldots + b_1 x + b_0 \longmapsto \operatorname{Res}_{\Bbbk[x]/\Bbbk}\begin{bmatrix} f\ \mathrm{d}\,x \\ \partial_x w(a_1,\ldots,a_k)\end{bmatrix} = \frac{b_{k-1}}{k+1}.$$

This leads to the following theorem, analogous to Theorem I.6.1.1:

Theorem I.7.6.1. *Let $\Bbbk$ be a $\mathbb{Z}$-graded $\mathbb{Z}[\frac{1}{k+1}]$-algebra and $a_i \in \Bbbk_{2(k+1-i)}$ for $i = 1, \ldots, k$. Then, for an n-strand braid word β with writhe $w(\beta)$, the complex*

$$\Sigma^{-w(\beta)} \ {}^{w_n(\underline{a})}\,\mathrm{sHH}_t^{\mathbb{A}^n_{\Bbbk}} [\mathcal{RC}_{\Bbbk}(\beta)] \langle (k+1)w(\beta)\rangle \tag{I.7.6.37}$$

has $\Bbbk$-free components of finite rank. Moreover, its isomorphism class in $\mathrm{Ho}^b(\Bbbk\text{-Mod})$ is invariant under the Markov moves, hence gives rise to an invariant of oriented links. Here, the potential $w_n(\underline{a}) \in \mathbb{A}^n_{\Bbbk}$ for the stable Hochschild homology is given by

$$w_n(\underline{a}) \ := \ \sum_{i=1}^{n} \left(x_i^{k+1} + a_k x_i^k + \cdots + a_1 x_i \right).$$

In particular, we can define equivariant Khovanov-Rozansky homology as follows:

Definition I.7.6.2. *Consider the following two special cases of Theorem I.7.6.1:*

(i) *For $\Bbbk = \Bbbk_{\mathrm{eq}} := \mathbb{Z}[\frac{1}{k+1}][\mathsf{a}_1, \ldots, \mathsf{a}_k]$ with $|\mathsf{a}_i|_q := 2(k+1-i)$ and $a_i := \mathsf{a}_i$, we call the resulting invariant the* equivariant Khovanov-Rozansky homology*, denoted $\mathcal{CKR}^k_{\mathrm{eq}}$.*

(ii) *For $\Bbbk = \widetilde{\Bbbk}_{\mathrm{eq}} := \mathbb{Z}[\frac{1}{(k+1)!}][\mathsf{z}_1, \ldots, \mathsf{z}_k]$ with $|\mathsf{z}_i|_q = 2$ and $a_i := \frac{k+1}{i} e_{k+1-i}(\mathsf{z}_1, \ldots, \mathsf{z}_k)$, with e_i being the elementary symmetric polynomials, we call it the* extended equivariant Khovanov-Rozansky homology*, denoted $\widetilde{\mathcal{CKR}}^k_{\mathrm{eq}}$.*

Note that the normalization in the definition of $\widetilde{\mathcal{CKR}}^k_{\mathrm{eq}}$ is chosen such that the value of the unknot is given by $\mathbb{Z}[\frac{1}{(k+1)!}][\mathsf{z}_1, \ldots, \mathsf{z}_k][T]/(T+\mathsf{z}_1)\cdots(T+\mathsf{z}_k)$.

Our definition of equivariant Khovanov-Rozansky homology is in terms of stable Hochschild homology; originally, it was introduced by Krasner [Kra10a] by mimicking the original construction of Khovanov-Rozansky homology. The arguments of Section I.5 carry over to show that, up to normalization, the two definitions agree.

We now turn to the Lee-Gornik type deformations [Lee05; Gor08] of Khovanov-Rozansky homology obtained by allowing the a_i above to be scalars. In this case, the construction of KR would still go through without changes; however, it would also completely ignore the q-grading on the base ring as well as the gradability of Soergel bimodules, merely resulting in a singly-graded invariant, not only lacking an internal q-grading but also the weaker q-filtration which is to be expected from Lee-Gornik's original constructions. Instead, we shall in Definition I.7.6.4 introduce (extended) deformed Khovanov-Rozansky homology as a base change of equivariant Khovanov-Rozansky homology along reduction of the a_i resp. z_i.

Notation I.7.6.3. If $\Bbbk$ is a $\mathbb{Z}$-filtered ring, we denote $\Bbbk$-Filt the category of *filtered $\Bbbk$-modules* and filtered $\Bbbk$-linear homomorphisms. While not abelian in general, $\Bbbk$-Filt is an additive category, hence its bounded homotopy category $\mathrm{Ho}^b(\Bbbk\text{-Filt})$ is defined. Further, denoting $\mathrm{gr}\,\Bbbk$ the associated $\mathbb{Z}$-graded ring of $\Bbbk$, assigning to a filtered $\Bbbk$-module M its associated $\mathbb{Z}$-graded $\mathrm{gr}\,\Bbbk$-module $\mathrm{gr}\,M$ yields an additive functor $\mathrm{gr} : \Bbbk\text{-Filt} \to \mathrm{gr}\,\Bbbk\text{-Mod}$, hence a functor $\mathrm{Ho}^b(\Bbbk\text{-Filt}) \to \mathrm{Ho}^b(\mathrm{gr}\,\Bbbk\text{-Mod})$. Conversely, if the $\mathbb{Z}$-filtration on $\Bbbk$ happens to come from a $\mathbb{Z}$-grading, we have $\mathrm{gr}\,\Bbbk = \Bbbk$, and there are canonical functors $\Bbbk\text{-Mod} \to \Bbbk\text{-Filt}$ and $\mathrm{Ho}^b(\Bbbk\text{-Mod}) \to \mathrm{Ho}^b(\Bbbk\text{-Filt})$ right inverse to the functors induced by taking the associated graded. Finally, given a morphism $\Bbbk \to \Bbbk'$ of $\mathbb{Z}$-filtered rings, there are functors $- \otimes_{\Bbbk} \Bbbk' : \Bbbk\text{-Filt} \to \Bbbk'\text{-Filt}$ and $- \otimes_{\Bbbk} \Bbbk' : \mathrm{Ho}^b(\Bbbk\text{-Filt}) \to \mathrm{Ho}^b(\Bbbk'\text{-Filt})$. $\Diamond$

Definition I.7.6.4. *Consider the following specializations of* $\mathcal{CKR}^k_{\mathrm{eq}}$ *and* $\widetilde{\mathcal{CKR}}^k_{\mathrm{eq}}$*:*

(i) *For any* $\mathbb{Z}[\frac{1}{k+1}]$*-algebra* $\Bbbk$ *and elements* $a_1, \ldots, a_k \in \Bbbk$ *we define the* deformed Khovanov-Rozansky homology $\mathcal{CKR}^k_{\Bbbk}(a_1, \ldots, a_k)$ with parameters a_i *as the base change* $\mathcal{CKR}^k_{\mathrm{eq}} \otimes_{\Bbbk_{\mathrm{eq}}} \Bbbk$ *of* $\mathcal{CKR}^k_{\mathrm{eq}}$ *along* $\Bbbk_{\mathrm{eq}} \to \Bbbk$*,* $\mathsf{a}_i \mapsto a_i$*.*

(ii) *For any* $\mathbb{Z}[\frac{1}{(k+1)!}]$*-algebra* $\Bbbk$ *and elements* $z_1, \ldots, z_k \in \Bbbk$ *we define* extended deformed Khovanov-Rozansky homology $\widetilde{\mathcal{CKR}}^k_{\Bbbk}(z_1, \ldots, z_k)$ with parameters a_i *as the base change* $\widetilde{\mathcal{CKR}}^k_{\mathrm{eq}} \otimes_{\widetilde{\Bbbk}_{\mathrm{eq}}} \Bbbk$ *of* $\widetilde{\mathcal{CKR}}^k_{\mathrm{eq}}$ *along* $\widetilde{\Bbbk}_{\mathrm{eq}} \to \Bbbk$*,* $\mathsf{z}_i \mapsto z_i$*.*

Here $\Bbbk$ *is viewed as a* $\mathbb{Z}$*-filtered ring by* $\Bbbk_{\leq -1} = 0$ *and* $\Bbbk_{\leq 0} = \Bbbk$*, and both* $\mathcal{CKR}^k_{\Bbbk}(a_1, \ldots, a_k)$ *and* $\widetilde{\mathcal{CKR}}^k_{\Bbbk}(a_1, \ldots, a_k)$ *are considered as link invariants with values in* $\mathrm{Ho}^b(\Bbbk\text{-Filt})$*. To be precise, they are obtained from* $\mathcal{CKR}^k_{\mathrm{eq}}$ *resp.* $\widetilde{\mathcal{CKR}}^k_{\mathrm{eq}}$ *by respective composition with*

$$\mathrm{Ho}^b(\Bbbk_{\mathrm{eq}}\text{-Mod}) \longrightarrow \mathrm{Ho}^b(\Bbbk_{\mathrm{eq}}\text{-Filt}) \xrightarrow{-\otimes_{\Bbbk_{\mathrm{eq}}} \Bbbk} \mathrm{Ho}^b(\Bbbk\text{-Filt}),$$

$$\mathrm{Ho}^b(\widetilde{\Bbbk}_{\mathrm{eq}}\text{-Mod}) \longrightarrow \mathrm{Ho}^b(\widetilde{\Bbbk}_{\mathrm{eq}}\text{-Filt}) \xrightarrow{-\otimes_{\widetilde{\Bbbk}_{\mathrm{eq}}} \Bbbk} \mathrm{Ho}^b(\Bbbk\text{-Filt}).$$

The homology of a complex of filtered modules is naturally filtered. In particular, if $\Bbbk$ is a field, we can deduce numerical variants of deformed Khovanov-Rozansky homology as follows:

Definition I.7.6.5. *Let* $\Bbbk$ *be a field with* $k+1 \in \Bbbk^\times$*, and let* $a_1, \ldots, a_k \in \Bbbk$ *be arbitrary. Then, given an oriented link* L*, we put*

$$\mathcal{KR}^k_{\Bbbk}(a_1, \ldots, a_k)(L) := \sum_{i,j \in \mathbb{Z}} \dim_{\Bbbk} \left(\mathrm{gr}\, \mathrm{H}^i \left[\mathcal{CKR}^k_{\Bbbk}(a_1, \ldots, a_k)(L) \right] \right)_j a^i q^j \in \mathbb{Z}[a^{\pm 1}, q^{\pm 1}].$$

If moreover $(k+1)! \in \Bbbk^\times$ *and* $z_1, \ldots, z_k \in \Bbbk$*, we further put*

$$\widetilde{\mathcal{KR}}^k_{\Bbbk}(z_1, \ldots, z_k)(L) := \sum_{i,j \in \mathbb{Z}} \dim_{\Bbbk} \left(\mathrm{gr}\, \mathrm{H}^i \left[\widetilde{\mathcal{CKR}}^k_{\Bbbk}(z_1, \ldots, z_k)(L) \right] \right)_j a^i q^j \in \mathbb{Z}[a^{\pm 1}, q^{\pm 1}].$$

On the other hand, we can pass to the associated graded on the level of deformed complexes $\mathcal{CKR}^k_{\Bbbk}(a_1,\ldots,a_k)(L)$ and $\widetilde{\mathcal{CKR}}^k_{\Bbbk}(z_1,\ldots,z_k)(L)$ and take cohomology afterwards. As we shall see now, this yields ordinary undeformed Khovanov-Rozansky homology.

Fact I.7.6.6. *Let $\Bbbk$ be a $\mathbb{Z}$-graded ring with homogeneous elements $\mathsf{a}_1,\ldots,\mathsf{a}_n\in\Bbbk$ of strictly positive degree and elements $a_1,\ldots,a_n\in\Bbbk$ of degree 0.*

(i) *There is a unique isomorphism of graded rings $\varphi:\operatorname{gr}\Bbbk/(\mathsf{a}_i-a_i)\cong\Bbbk/(\mathsf{a}_i)$ such that for any $n\in\mathbb{Z}$, the following diagram commutes:*

$$\begin{array}{ccccc} & & \Bbbk_k & & \\ & \swarrow_{\text{can}} & & \searrow^{\text{can}} & \\ (\operatorname{gr}\Bbbk/(\mathsf{a}_i-a_i))_k & & \xrightarrow{\ \varphi\ } & & (\Bbbk/(\mathsf{a}_i))_k \end{array}$$

In particular, given a graded $\Bbbk$-module X, we can consider $\operatorname{gr}(X/(\mathsf{a}_i-a_i)X)$ as a graded $\Bbbk/(\mathsf{a}_i)$-module.

(ii) *Let X be any graded $\Bbbk$-module. Then there is a unique isomorphism of graded $\Bbbk/(\mathsf{a}_i)$-modules $\psi:\operatorname{gr}(X/(\mathsf{a}_i-a_i)X)\cong X/(\mathsf{a}_i)X$ such that the following commutes:*

$$\begin{array}{ccccc} & & \Bbbk_k & & \\ & \swarrow_{\text{can}} & & \searrow^{\text{can}} & \\ (\operatorname{gr}X/(\mathsf{a}_i-a_i)X)_k & & \xrightarrow{\ \psi\ } & & (X/(\mathsf{a}_i)X)_k \end{array}$$

To summarize, the following commutes up to canonical, natural isomorphism:

$$\begin{array}{ccccc} & & \Bbbk\text{-Mod} & & \\ & \swarrow_{\text{can}} & & \searrow^{\text{can}} & \\ \Bbbk/(\mathsf{a}_i)\text{-Mod}\cong\operatorname{gr}(\Bbbk/(\mathsf{a}_i-a_i))\text{-Mod} & & \xleftarrow{\ \operatorname{gr}\ } & & \Bbbk/(\mathsf{a}_i-a_i)\text{-Filt} \end{array}$$

Proposition I.7.6.7. *Let $\Bbbk$ be a (non-graded) $\mathbb{Z}[\frac{1}{k+1}]$-algebra and let $a_1,\ldots,a_k\in\Bbbk$ be arbitrary. Then, given any oriented link L, there is an isomorphism in the homotopy category $\operatorname{Ho}^b(\Bbbk\text{-Mod})$:*

$$\operatorname{gr}\left[\mathcal{CKR}^k_{\Bbbk}(a_1,\ldots,a_k)(L)\right]\ \cong\ \mathcal{CKR}^k_{\Bbbk}(L)$$

The analogous statement holds for extended deformed Khovanov-Rozansky homology. As a special case of the spectral sequences of a filtered complex, we therefore obtain:

Corollary I.7.6.8. *For a $\mathbb{Z}[\frac{1}{k+1}]$-algebra $\Bbbk$ and $a_1,\ldots,a_k\in\Bbbk$, there is a canonical spectral sequence in the category of $\Bbbk$-modules:*

$$\mathrm{E}_2^{p,q}\cong\mathrm{H}^{q-p}\left[\mathcal{CKR}^k_{\Bbbk}(L)\right]_{-p}\implies\mathrm{E}_\infty^{p,q}\cong\operatorname{gr}\left[\mathrm{H}^{q-p}\left(\mathcal{CKR}^k_{\Bbbk}(a_1,\ldots,a_k)(L)\right)\right]_{-p}.$$

The analogous statement holds for extended deformed Khovanov-Rozansky homology.

I.A. Basic definitions

I.A.1. Links and knots

We begin by recalling the analytic definition of links, see e.g. [KT08, §2.1.1]:

Definition I.A.1.1. *A* link *is a smooth, compact,* 1*-dimensional embedded submanifold of* $\mathbb{R}^3$*, usually denoted* L*. The* components *of* L *are the connected components of its underlying topological space, and* L *is called a* knot *if there is only one such, i.e. if* $L \cong \mathbb{S}^1$*. Similarly, an* orientation *of* L *is an orientation of its underlying manifold, and* L *is said to be* oriented *if an orientation has been fixed. Finally,* L *is said to be* ordered *if a linear ordering on its components is fixed.*

Equivalence of (oriented, ordered) links is defined through ambient isotopy in $\mathbb{R}^3$:

Definition I.A.1.2. *Two links* $L, L' \subseteq \mathbb{R}^3$ *are called* equivalent *if there is a smooth family* $\{F_s : \mathbb{R}^3 \to \mathbb{R}^3\}_{s\in[0,1]}$ *of diffeomorphisms of* $\mathbb{R}^3$ *such that* $F_0 = \mathrm{id}$ *and* $F_1(L) = L'$*. Taking orientation and ordering into account, we say that two oriented (resp. ordered, resp. ordered and oriented) links* L, L' *are* equivalent as oriented (resp. ordered, resp. ordered and oriented) links *if* $\{F_s\}_s$ *can be chosen such that* $F_1|_L : L \cong L'$ *preserves the orientation of* L *and* L' *(resp. their ordering, resp. their ordering and orientation).*

Next we recall the combinatorial presentations of links via link diagrams and braids.

Definition I.A.1.3. *A* planar, oriented link diagram *is an oriented graph planarly embedded in* $\mathbb{R}^2$ *with two types of* $(2,2)$*-valent vertices that are depicted as* ⤱ *and* ⤲*.*

Any planar, oriented link diagram gives rise to an equivalence class of oriented link in the way suggested by its depiction. The following theorem of Reidemeister describes combinatorially when two planar, oriented link diagrams give rise to equivalent oriented links; it is among the most fundamental results of knot theory:

Theorem I.A.1.4 [Rei27]**.** *Any oriented link is equivalent to one induced by a planar, oriented link diagram. Two planar, oriented link diagrams represent equivalent links if and only if they can be transformed into each other through the following operations:*

(i) *Isotopies of planarly embedded, oriented graphs.*

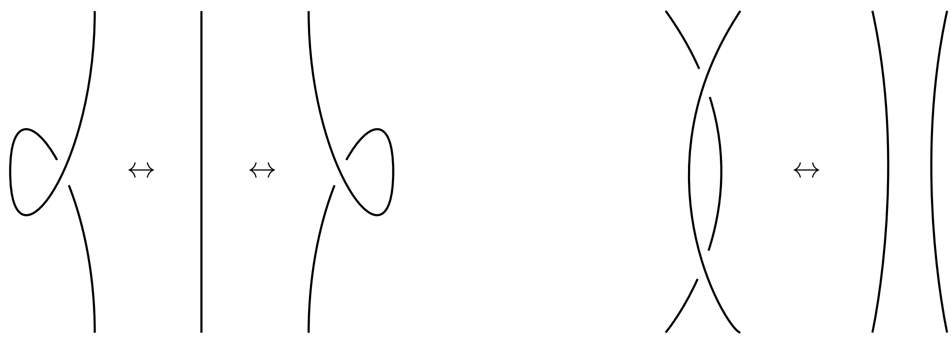


Figure I.A.1.1. The first and second Reidemeister moves

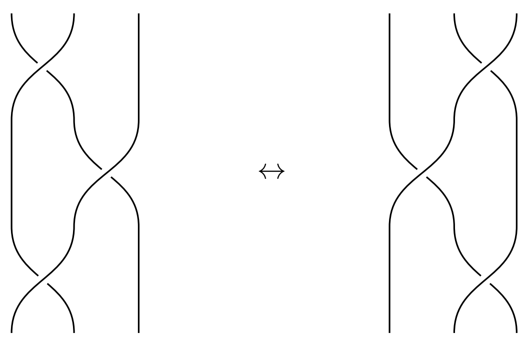


Figure I.A.1.2. The third, braid-like Reidemeister move

(ii) *Oriented versions of* Reidemeister moves *as in Figures I.A.1.1 and I.A.1.2.*

Next, we consider the presentation of links as closures of braids.

Definition I.A.1.5. *For an integer $n \geq 1$, the* Artin braid group on n strands Br_n *is the group given by generators $\sigma_1, \ldots, \sigma_{n-1}$ and relations $\sigma_i\sigma_j = \sigma_j\sigma_i$ for $1 \leq i, j < n$ with $|i-j| > 1$ and $\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$ for $1 \leq i < n$.*

Assigning to σ_i the twisting of the i-th and $i+1$-th strand in a row of n strands as depicted in Figure I.A.1.3 gives rise to a bijection between elements of the braid group and isotopy classes of braids on n strands, see e.g. [KT08, Theorem 1.6]. Moreover, as depicted in Figure I.6.1.1, any braid gives rise to an oriented link through *braid closure*.

Just as Reidemeister's Theorem described oriented links through planar, oriented link diagrams up to the Reidemeister moves, the following Theorems of Alexander and Markov describe links in terms of braids up to the *Markov moves*:

Figure I.A.1.3. Topological meaning of the braid generators σ_i and σ_i^{-1}

Theorem I.A.1.6 (Alexander's Theorem [Ale23], see also [KT08, Theorem 2.3])**.** *Any oriented link is equivalent to the closure of a braid.*

Theorem I.A.1.7 (Markov's Theorem [Mar36], see also [KT08, Theorem 2.8])**.** *Two braids give equivalent oriented links upon braid closure (Figure I.6.1.1) if and only if they can transformed into each other via the* Markov moves *(Figures I.6.1.2 and I.6.1.3).*

Hence, invariants of oriented links can be identified with invariants of braids that are additionally invariant under the two Markov moves. For example, we constructed Khovanov-Rozansky homology this way in Section I.6.

Remark I.A.1.8. We expect the variant of Markov's Theorem for *ordered* oriented links to hold as well: Suppose $\beta \in \mathrm{Br}_n$ and $\gamma \in \mathrm{Br}_m$ are *ordered* braids in the sense that the sets of cycles of their underlying permutations are equipped with a linear ordering; in particular, this also orders the braid closures of β and γ. Then we expect the braid closures of β and γ to be equivalent as *ordered*, oriented links precisely if the ordered braids β and γ can be transformed into each other through the ordered versions of the first and second Markov move. This is used in Theorem I.6.1.3, where we construct an invariant of ordered braids that is also invariant under the ordered Markov moves, hence presumably descends to an invariant of ordered, oriented links. ◊

I.A.2. Soergel bimodules and Rouquier complexes

In this section we quickly define type A Soergel bimodules and Rouquier complexes for arbitrary commutative base rings and recall some statements about their combinatorics relevant for us. We keep Notation I.3.2, i.e. $\mathbb{k}$ is a commutative base ring, $n \in \mathbb{N}$ is fixed and $\mathbb{A}^n_{\mathbb{k}} = \mathbb{k}[x_1, \ldots, x_n]$ denotes the polynomial ring on n variables, with $\mathbb{k}$-enveloping algebra $\widehat{\mathbb{A}}^n_{\mathbb{k}} = \mathbb{k}[x_1, \ldots, x_n, y_1, \ldots, y_n]$.

Definition I.A.2.1. *The* category of Bott-Samelson bimodules $\mathcal{BSM}_{\mathbb{k}}(n)$ *is defined as the smallest full and additive subcategory of the category* $\widehat{\mathbb{A}}^n_{\mathbb{k}}$-Mod *of* $\mathbb{Z}$*-graded* $\mathbb{A}^n_{\mathbb{k}}$*-bimodules having the following properties:*

(i) $\mathcal{BSM}_{\Bbbk}(n)$ *contains the diagonal* $\mathbb{A}^n_{\Bbbk}$*-bimodule* $\Delta^n_{\Bbbk}$.

(ii) $\mathcal{BSM}_{\Bbbk}(n)$ *contains the elementary Soergel bimodules* $B^{n,i}_{\Bbbk} := \mathbb{A}^n_{\Bbbk} \otimes_{(\mathbb{A}^n_{\Bbbk})^{(i,i+1)}} \mathbb{A}^n_{\Bbbk}$, *where* $1 \leq i < n$ *and where* $(\mathbb{A}^n_{\Bbbk})^{(i,i+1)}$ *denotes the polynomials invariant under exchange of the variables* x_i *and* x_{i+1}.

(iii) $\mathcal{BSM}_{\Bbbk}(n)$ *is closed under one-sided tensor product, shifts and finite sums.*

The category of Soergel bimodules $\mathcal{SBM}_{\Bbbk}(n)$ *is the idempotent completion of* $\mathcal{BSM}_{\Bbbk}(n)$, *i.e. the full subcategory of* $\widehat{\mathbb{A}}^n_{\Bbbk}$-Mod *consisting of summands of objects of* $\mathcal{BSM}_{\Bbbk}(n)$.

Notation I.A.2.2. For $n \geq 1$ we denote the set of finite sequences in $\{1, 2, \dots, n-1\}$ by Seq_n. We naturally embed $\mathrm{Seq}_n \hookrightarrow \mathrm{Seq}_{n+1}$, and and we denote $* : \mathrm{Seq}_n \times \mathrm{Seq}_m \to \mathrm{Seq}_{n+m}$ the concatenation. Finally, if $\mathbf{i} = (i_1, \dots, i_k) \in \mathrm{Seq}_n$, then we denote $B^{n,\mathbf{i}}_{\Bbbk} := B^{n,i_1}_{\Bbbk} \otimes_{\mathbb{A}^n_{\Bbbk}} \cdots \otimes_{\mathbb{A}^n_{\Bbbk}} B^{n,i_k}_{\Bbbk}$ the Bott-Samelson bimodule attached to **i**. ♢

Bott-Samelson and Soergel bimodules are defined for arbitrary Coxeter systems, and starting with Soergel's paper [Soe07] there is by now a very rich literature about their combinatorics, see e.g. [Wil11; EK10; EW13; Eli13]. For us, the following is sufficient:

Proposition I.A.2.3. *In the category* $\widehat{\mathbb{A}}^n_{\Bbbk}$-Mod *one has the following isomorphisms:*

$$B^{n,i}_{\Bbbk} \underset{\mathbb{A}^n_{\Bbbk}}{\otimes} B^{n,i}_{\Bbbk} \cong B^{n,i}_{\Bbbk} \oplus B^{n,i}_{\Bbbk}\langle -2 \rangle \tag{I.A.2.1}$$

$$B^{n,i}_{\Bbbk} \underset{\mathbb{A}^n_{\Bbbk}}{\otimes} B^{n,j}_{\Bbbk} \cong B^{n,j}_{\Bbbk} \underset{\mathbb{A}^n_{\Bbbk}}{\otimes} B^{n,i}_{\Bbbk} \tag{I.A.2.2}$$

$$[B^{n,i}_{\Bbbk} \underset{\mathbb{A}^n_{\Bbbk}}{\otimes} B^{n,i+1}_{\Bbbk} \underset{\mathbb{A}^n_{\Bbbk}}{\otimes} B^{n,i}_{\Bbbk}] \oplus B^{n,i+1}_{\Bbbk} \cong [B^{n,i+1}_{\Bbbk} \underset{\mathbb{A}^n_{\Bbbk}}{\otimes} B^{n,i}_{\Bbbk} \underset{\mathbb{A}^n_{\Bbbk}}{\otimes} B^{n,i+1}_{\Bbbk}] \oplus B^{n,i}_{\Bbbk} \tag{I.A.2.3}$$

The relations (I.A.2.1), (I.A.2.2) and (I.A.2.3) are categorified versions of the relations defining the type A Hecke algebra in terms of its Kazhdan-Lusztig generators:

Definition I.A.2.4. *The* generic type A (Iwahori-)Hecke algebra $\mathcal{H}_n(q)$ *is defined as the* $\mathbb{Z}[q^{\pm 1}]$*-algebra with generators* $\mathrm{T}_1, \mathrm{T}_2, \dots, \mathrm{T}_{n-1}$ *and relations*

$$\mathrm{T}_i\mathrm{T}_j = \mathrm{T}_j\mathrm{T}_i \qquad \textit{for all } i, j = 1, 2, \dots, n-1 \textit{ s.t. } |i-j| > 1 \tag{I.A.2.4}$$

$$\mathrm{T}_i\mathrm{T}_{i+1}\mathrm{T}_i = \mathrm{T}_{i+1}\mathrm{T}_i\mathrm{T}_{i+1} \qquad \textit{for all } i = 1, 2, \dots, n-2$$

$$\mathrm{T}_i^2 = (q^2 - 1)\mathrm{T}_i + q^2\mathrm{T}_e$$

Here $\mathrm{T}_e := 1$. *Further, we put* $\underline{\mathrm{H}}_i := q^{-1}(\mathrm{T}_e + \mathrm{T}_i)$ *for* $1 \leq i < n$.

Alternatively, $\mathcal{H}_n(q)$ is the quotient of the group algebra $\mathbb{Z}[q^{\pm 1}][\mathrm{Br}_n]$ of the Artin braid group Br_n (Definition I.A.1.5) over $\mathbb{Z}[q^{\pm 1}]$ by the relation $\sigma_i^2 = (q^2 - 1)\sigma_i + q^2 e$.

In terms of the $\underline{\mathrm{H}}_i$, the relations (I.A.2.4) transform into $\underline{\mathrm{H}}_i\underline{\mathrm{H}}_j = \underline{\mathrm{H}}_j\underline{\mathrm{H}}_i$ for all $|i-j| \geq 2$, $\underline{\mathrm{H}}_i\underline{\mathrm{H}}_{i+1}\underline{\mathrm{H}}_i + \underline{\mathrm{H}}_{i+1} = \underline{\mathrm{H}}_{i+1}\underline{\mathrm{H}}_i\underline{\mathrm{H}}_{i+1} + \underline{\mathrm{H}}_i$ for all $1 \leq i < n-1$, and $\underline{\mathrm{H}}_i^2 = (q+q^{-1})\underline{\mathrm{H}}_i$ for all $1 \leq i < n$, resembling the relations (I.A.2.1), (I.A.2.2) and (I.A.2.3). Denoting $\mathrm{K}^{\oplus}(\mathcal{SBM}_{\Bbbk}(n))$ the split Grothendieck group of $\mathcal{SBM}_{\Bbbk}(n)$, considered as a $\mathbb{Z}[q^{\pm 1}]$-module via $q.[M] := [M\langle 1\rangle]$, the following formalizes this observation:

Fact I.A.2.5. *There is a unique $\mathbb{Z}[q^{\pm 1}]$-algebra homomorphism*

$$\mathcal{H}_n(q) \to \mathrm{K}^{\oplus}(\mathcal{SBM}_{\Bbbk}(n)), \quad \underline{\mathrm{H}}_i \longmapsto [B_{\Bbbk}^{n,i}\langle 1\rangle]. \tag{I.A.2.5}$$

Over $\Bbbk = \mathbb{C}$, the morphism (I.A.2.5) is even an isomorphism, mapping the Kazhdan-Lusztig basis of $\mathcal{H}_n(q)$ to the basis of $\mathrm{K}^{\oplus}(\mathcal{SBM}_{\Bbbk}(n))$ consisting of the classes of the indecomposable Soergel bimodules – see [Soe07].

Proposition I.A.2.3 yields the following induction principle for Soergel bimodules:

Proposition I.A.2.6. *Suppose that for each $n \geq 1$ we are given a property $\mathcal{P}_n(-)$ of isomorphism classes of Soergel bimodules on n strands over $\Bbbk$, and assume that*

(i) $\mathcal{P}_1(\Delta_{\Bbbk}^1)$ *holds,*

(ii) *if $M \in \mathcal{SBM}_{\Bbbk}(n)$ then $\mathcal{P}_n(M)$ holds if and only if $\mathcal{P}_n(M\langle 1\rangle)$ holds,*

(iii) *if $M, N \in \mathcal{SBM}_{\Bbbk}(n)$ then $\mathcal{P}_n(M \otimes_{\mathbb{A}_{\Bbbk}^n} N)$ holds if and only if $\mathcal{P}_n(N \otimes_{\mathbb{A}_{\Bbbk}^n} M)$ holds,*

(iv) *if $M, N \in \mathcal{SBM}_{\Bbbk}(n)$ then $\mathcal{P}_n(M \oplus N)$ holds if and only if $\mathcal{P}_n(M)$ and $\mathcal{P}_n(N)$ hold,*

(v) *if $M \in \mathcal{SBM}_{\Bbbk}(n)$ and $\mathcal{P}_n(M)$ holds, then the following hold, too:*

$$\mathcal{P}_{n+1}\left(M \otimes_{\Bbbk} \Bbbk[x_{n+1}, y_{n+1}]/(x_{n+1} - y_{n+1})\right) \tag{I.A.2.6}$$

$$\mathcal{P}_{n+1}\left((M \otimes_{\Bbbk} \Bbbk[x_{n+1}, y_{n+1}]/(x_{n+1} - y_{n+1})) \otimes_{\mathbb{A}_{\Bbbk}^{n+1}} B_{\Bbbk}^{n+1,n}\right) \tag{I.A.2.7}$$

Then $\mathcal{P}_n(M)$ holds for any $n \geq 1$ and any $M \in \mathcal{SBM}_{\Bbbk}(n)$.

Proof. By assumption (iv) it suffices to show that for any $n \geq 1$ and any sequence $\mathbf{i} \in \mathrm{Seq}_n$ the property $\mathcal{P}_n(B_{\Bbbk}^{n,\mathbf{i}})$ holds. For ease of notation, we write $\mathcal{P}_n(\mathbf{i})$ instead of $\mathcal{P}_n(B_{\Bbbk}^{n,\mathbf{i}})$. We prove $\mathcal{P}_n(\mathbf{i})$ for all $\mathbf{i}$ following an induction scheme introduced in [Wu08, Proof of Proposition 3.6] and slightly modified in [Ras06, §4.2], namely we proceed by induction on the complexity $c(\mathbf{i}) := n + \sum_j i_j \geq 1$ of $\mathbf{i}$. In the base case $c(\mathbf{i}) = 1$ we have $n = 1$ and $\mathbf{i} = \emptyset$, hence $B_{\Bbbk}^{n,\mathbf{i}} = \Delta_{\Bbbk}^1$ and $\mathcal{P}_1$ holds by assumption (i).

For the induction step, relations (I.A.2.2),(I.A.2.1) and (I.A.2.3) together with our assumptions (i)-(v) on $\mathcal{P}_n$ imply the following rules for reasoning about $\mathcal{P}_n$:

(i′) If $\mathbf{i} = \mathbf{i}' * \mathbf{i}''$, then $\mathcal{P}_n(\mathbf{i})$ holds if and only if $\mathcal{P}_n(\mathbf{i}'' * \mathbf{i}')$ holds.

(ii′) $\mathbf{i} = \mathbf{i}' * (l, l') * \mathbf{i}''$, $|l - l'| \geq 2$, then $\mathcal{P}_n(\mathbf{i})$ holds provided $\mathcal{P}_n(\mathbf{i}' * (l', l) * \mathbf{i}'')$ holds.

(iii′) If $\mathbf{i} = \mathbf{i}' * (l, l-1, l) * \mathbf{i}''$ then $\mathcal{P}_n(\mathbf{i})$ holds provided all of $\mathcal{P}_n(\mathbf{i}' * (l-1, l, l-1) * \mathbf{i}'')$, $\mathcal{P}_n(\mathbf{i}' * (l) * \mathbf{i}'')$ and $\mathcal{P}_n(\mathbf{i}' * (l-1) * \mathbf{i}'')$ hold.

(iv′) If $\mathbf{i} = \mathbf{i}' * (l, l) * \mathbf{i}''$, then $\mathcal{P}_n(\mathbf{i})$ holds provided $\mathcal{P}_n(\mathbf{i}' * (l) * \mathbf{i}'')$ holds.

(v′) If $\mathbf{i} \in \mathrm{Seq}_{n-1}$ then $\mathcal{P}_n(\mathbf{i})$ holds provided $\mathcal{P}_{n-1}(\mathbf{i})$ holds.

(vi′) If $\mathbf{i} \in \mathrm{Seq}_{n-1}$ then $\mathcal{P}_n(\mathbf{i} * (n-1))$ holds if $\mathcal{P}_{n-1}(\mathbf{i})$ holds.

Note that rules (iii′)-(vi′) reduce the goal of proving $\mathcal{P}_n(\mathbf{i})$ to a finite number of goals of the form $\mathcal{P}_{n'}(\mathbf{i}')$ with $\mathbf{i}'$ of lower complexity than $\mathbf{i}$. By induction hypothesis, it is therefore sufficient to show that rules (i′) and (ii′) always allow for modifying $\mathbf{i}$ in such a way that one of the rules (iii′)-(vi′) become applicable.

If (v′) is not applicable, then $\mathbf{i}$ must contain the index $n-1$ at least once. If it contains is precisely once, an application of (i′) allows for assuming that $\mathbf{i} = \mathbf{i}' * (n-1)$ with $\mathbf{i}' \in \mathrm{Seq}_{n-1}$, and then rule (vi′) applies. If $n-1$ is contained at least twice in $\mathbf{i}$, then we may write $\mathbf{i} = \mathbf{i}' * (n-1) * \mathbf{i}'' * (n-1) * \mathbf{i}'''$ with $\mathbf{i}'' \in \mathrm{Seq}_{n-1}$. If $\mathbf{i}'' \in \mathrm{Seq}_{n-2}$, then applying rule (ii′) we may replace $\mathbf{i}$ by $\mathbf{i}' * \mathbf{i}'' * (n-1, n-1) * \mathbf{i}'''$ and rule (iv′) becomes applicable. If $\mathbf{i}''$ contains $n-2$ precisely once, then applying rule (ii′) we may replace $\mathbf{i}$ by a sequence containing a subsequence of the form $(n-1, n-2, n-1)$, and rule (iii′) becomes applicable. If $\mathbf{i}''$ contains $n-2$ at least twice, we may repeat this procedure for $\mathbf{i}''$, resulting in an iterated application of rule (ii′) on $\mathbf{i}''$ after which one of the rules (iii′)-(vi′) become applicable. As rules (ii′)-(vi′) are local, they also apply to our original sequence $\mathbf{i}$ containing $\mathbf{i}''$ as well, and we are done. □

Definition I.A.2.7. *For a braid word $\beta = s_{i_1}^{\varepsilon_1} \cdots s_{i_k}^{\varepsilon_k}$ on n strands the* Rouquier complex *$\mathcal{RC}_{\Bbbk}(\beta)$ of β over $\Bbbk$ is defined as $\mathcal{RC}_{\Bbbk}(\beta) := F_{\varepsilon_1}^{n,i_1} \otimes_{\mathbb{A}_{\Bbbk}^n} \cdots \otimes_{\mathbb{A}_{\Bbbk}^n} F_{\varepsilon_k}^{n,i_k}$, where*

$$F_+^{n,i} := \left(\ldots \to 0 \to \underline{B_{\Bbbk}^{n,i}\langle 2\rangle} \xrightarrow{\text{can}} \Delta_{\Bbbk}^n\langle 2\rangle \to 0 \to \ldots \right), \quad \textit{and} \tag{I.A.2.8}$$

$$F_-^{n,i} := \left(\ldots \to 0 \to \Delta_{\Bbbk}^n\langle -2\rangle \xrightarrow{1 \mapsto x_i - y_{i+1}} \underline{B_{\Bbbk}^{n,i}} \to 0 \to \ldots \right). \tag{I.A.2.9}$$

As usual, the underlined component is the one sitting in cohomological degree 0.

Proposition I.A.2.8. *The Rouquier complexes satisfy the braid relations up to homotopy, i.e. we have isomorphisms in $\mathrm{Ho}^b(\widehat{\mathbb{A}}_{\Bbbk}^n\text{-Mod})$:*

$$\mathcal{RC}_{\Bbbk}(s_i) \underset{\mathbb{A}_{\Bbbk}^n}{\otimes} \mathcal{RC}_{\Bbbk}(s_i^{-1}) \cong \left(\ldots \to 0 \to \underline{\Delta_{\Bbbk}^n} \to 0 \to \ldots \right)$$

$$\mathcal{RC}_{\Bbbk}(s_i) \underset{\mathbb{A}_{\Bbbk}^n}{\otimes} \mathcal{RC}_{\Bbbk}(s_{i+1}) \underset{\mathbb{A}_{\Bbbk}^n}{\otimes} \mathcal{RC}_{\Bbbk}(s_i) \cong \mathcal{RC}_{\Bbbk}(s_{i+1}) \underset{\mathbb{A}_{\Bbbk}^n}{\otimes} \mathcal{RC}_{\Bbbk}(s_i) \underset{\mathbb{A}_{\Bbbk}^n}{\otimes} \mathcal{RC}_{\Bbbk}(s_{i+1}).$$

Proof. This was proved in [Kra10b] using the diagrammatic calculus of Soergel bimodules developed in [EK10] for type A and in [EW13] for arbitrary Coxeter systems. $\square$

In particular, up to isomorphism in $\mathrm{Ho}^b(\widehat{\mathbb{A}}^n_{\Bbbk}\text{-Mod})$, we can consider the Rouquier complexes as invariants of n-strand braids instead of n-strand braid words. As we shall see below, passing to the derived category this invariant cannot distinguish s_i and s_i^{-1} anymore, and in fact it reduces to the obvious invariant $\pi \mapsto X^{n,\pi}_{\Bbbk}$ of permutations in $\widehat{\mathbb{A}}^n_{\Bbbk}$-Mod. Here, for a permutation $\pi \in \mathfrak{S}_n$ we denote $X^{n,\pi}_{\Bbbk}$ the π-twisted $\mathbb{A}^n_{\Bbbk}$-bimodule, given by $\mathbb{A}^n_{\Bbbk}$ as a right module over $\mathbb{A}^n_{\Bbbk}$, and with the left action twisted by the automorphism of $\mathbb{A}^n_{\Bbbk}$, $x_i \mapsto x_{\pi(i)}$. In particular, we recover the twisted diagonal bimodule $X^{n,i}_{\Bbbk}$ from Notation I.3.2 in case $\pi = (i, i+1)$.

Fact I.A.2.9. *The $\mathbb{A}^n_{\Bbbk}$-bimodules $\Delta^n_{\Bbbk}$, $B^{n,i}_{\Bbbk}$ and $X^{n,i}_{\Bbbk}$ are free both as left and as right $\mathbb{A}^n_{\Bbbk}$-modules and fit into short exact sequences in $\widehat{\mathbb{A}}^n_{\Bbbk}$-Mod*

$$0 \to X^{n,i}_{\Bbbk}\langle -2\rangle \xrightarrow{1\mapsto x_i - y_i = y_{i+1} - x_{i+1}} B^{n,i}_{\Bbbk} \xrightarrow{\quad\text{can}\quad} \Delta^n_{\Bbbk} \to 0, \tag{I.A.2.10}$$

$$0 \to \Delta^n_{\Bbbk}\langle -2\rangle \xrightarrow{1\mapsto x_i - y_{i+1} = y_i - x_{i+1}} B^{n,i}_{\Bbbk} \xrightarrow{\quad\text{can}\quad} X^{n,i}_{\Bbbk} \to 0. \tag{I.A.2.11}$$

Moreover, $\ker(B^{n,i}_{\Bbbk} \xrightarrow{\text{can}} \Delta^n_{\Bbbk}) \cap \ker(B^{n,i}_{\Bbbk} \xrightarrow{\text{can}} X^{n,i}_{\Bbbk}) = \{0\}$.

Proof. We may assume $n = 2$ and $i = 1$, so that $X^{n,i}_{\Bbbk} = \Bbbk[x_1, x_2, y_1, y_2]/(x_1 - y_2, x_2 - y_1)$, $\Delta^n_{\Bbbk} = \Bbbk[x_1, x_2, y_1, y_2]/(x_1 - y_1, x_2 - y_2)$ and $B^{n,i}_{\Bbbk} = \Bbbk[x_1, x_2, y_1, y_2]/(x_1 + x_2 - y_1 - y_2, x_1x_2 - y_1y_2)$; we abbreviate these modules X, Δ and B, respectively.

To begin, it is a short calculation to check that $\alpha : X\langle -2\rangle \xrightarrow{1\mapsto x_1 - y_1 = y_2 - x_2} B$ and $\beta : \Delta\langle -2\rangle \xrightarrow{1\mapsto x_1 - y_2 = y_1 - x_2} B$ are well-defined. Further, α is injective since its composition with the projection $B \xrightarrow{\text{can}} X$ equals multiplication by $x_1 - x_2$ in X, which is injective; similarly, β is injective since its composition with the projection $B \xrightarrow{\text{can}} \Delta$ equals multiplication by $x_1 - x_2$ in Δ, which is injective. Hence the sequences (I.A.2.10) and (I.A.2.11) are short exact, and since Δ and X are free both as left and right $\mathbb{A}^2_{\Bbbk}$-modules, B inherits this property. Finally, we have

$$\ker(B \xrightarrow{\text{can}} \Delta) \cap \ker(B \xrightarrow{\text{can}} X) = \mathrm{im}(X\langle -2\rangle \xrightarrow{\alpha} B) \cap \ker(B \xrightarrow{\text{can}} X) = \{0\}$$

since, as we have already observed, $\text{can} \circ \alpha : X\langle -2\rangle \to X$ is injective. $\square$

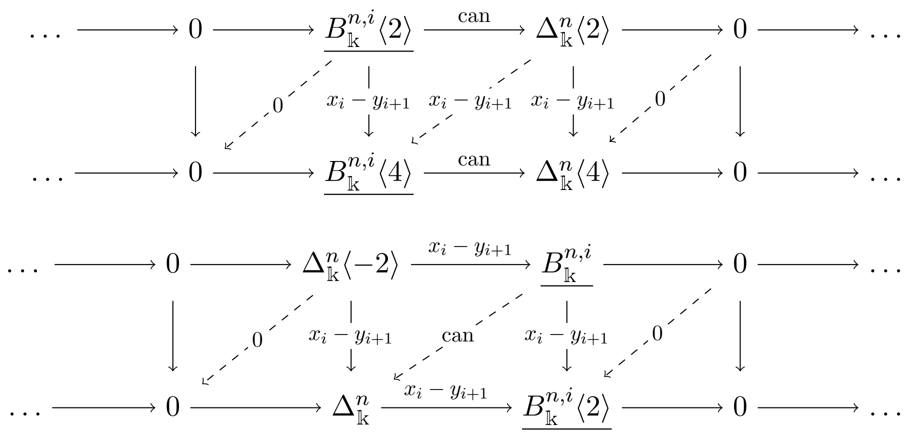
Corollary I.A.2.10. *For a braid word β on n strands with underlying permutation $\pi \in \mathfrak{S}_n$, there is a canonical isomorphism in the derived category $\mathbf{D}^b(\widehat{\mathbb{A}}^n_{\Bbbk}\text{-Mod})$*

$$\mathcal{RC}_{\Bbbk}(\beta) \quad\cong\quad \Big(\ldots \to 0 \to \underline{X^{n,\pi}_{\Bbbk}} \to 0 \to \ldots\Big).$$

The external action of $\widehat{\mathbb{A}}^n_{\Bbbk}$ on $\mathcal{RC}_{\Bbbk}(\beta)$ can't distinguish $s_i^{\pm 1}$ either:

Proposition I.A.2.11. *Let β be an n-strand braid with underlying permutation $\pi \in \mathfrak{S}_n$, and let $\mathcal{RC}_{\Bbbk}(\beta)$ be its associated Rouquier complex. Then, for any $1 \leq i \leq n$ the external actions of $x_i \in \widehat{\mathbb{A}}^n_{\Bbbk}$ and $y_{\pi(i)} \in \widehat{\mathbb{A}}^n_{\Bbbk}$ on $\mathcal{RC}_{\Bbbk}(\beta)$ coincide as morphisms in* $\mathrm{Ho}^b(\widehat{\mathbb{A}}^n_{\Bbbk}\text{-Mod})$.

Proof. It suffices to consider $\beta = s_i^{\pm 1}$, in which case we have to show that the external actions of $x_i - y_{i+1}$ and $x_{i+1} - y_i$ on the complexes $F^{n,i}_{\pm}$ from (I.A.2.8) are nullhomotopic. Considering the external action of $x_i - y_{i+1}$ on $F^{n,i}_{\pm}$ first, the dashed arrows in



provide nullhomotopies. The remaining case of $x_{i+1} - y_i$ follows from $x_{i+1} - y_i = -(x_i - y_{i+1})$ in $B^{n,i}_{\Bbbk}$ and $\Delta^n_{\Bbbk}$. $\square$

Part II.

Homotopy Theoretic Aspects

II.1. Introduction to part II

Let R be a Noetherian ring and $\mathbf{D}_{\text{sg}}(R) = \mathbf{D}^b(R\text{-mod})/\operatorname{Perf}(R)$ its singularity category. We ask if it is possible to realize $\mathbf{D}_{\text{sg}}(R)$ as the homotopy category of a stable model category attached to R. Firstly, the singularity category is essentially small, whereas the homotopy category of a model category in the sense of [Hov99] always has arbitrary small coproducts [Hov99, Example 1.3.11]. This forces us to think first about how to define a "large" singularity category for R (admitting arbitrary small coproducts) in which $\mathbf{D}_{\text{sg}}(R)$ naturally embeds. Secondly, if this is done, we can try to find a model for this large singularity category.

Given a locally Noetherian Grothendieck category $\mathscr{A}$ with compactly generated derived category $\mathbf{D}(\mathscr{A})$, Krause [Kra05] proved that the singularity category

$$\mathbf{D}_{\text{sg}}(\mathscr{A}) \ := \ \mathbf{D}^b(\operatorname{Noeth}(\mathscr{A}))/\mathbf{D}(\mathscr{A})^{\text{c}}$$

of $\mathscr{A}$, is up to direct summands, equivalent to the subcategory of compact objects in the homotopy category $\mathbf{K}_{\text{ac}}(\operatorname{Inj}(\mathscr{A}))$ of acyclic complexes of injectives, and that there is even a recollement

$$\mathbf{K}_{\text{ac}}(\operatorname{Inj}(\mathscr{A})) \leftrightarrows \mathbf{K}(\operatorname{Inj}(\mathscr{A})) \leftrightarrows \mathbf{D}(\mathscr{A}).$$

This suggests firstly that we should attempt to construct a model for $\mathbf{K}_{\text{ac}}(\operatorname{Inj}(\mathscr{A}))$ and secondly that such a model might be obtained by localizing a suitable model for $\mathbf{K}(\operatorname{Inj}(\mathscr{A}))$ with respect to $\mathbf{D}(\mathscr{A})$, whatever this should mean precisely.

If $\mathscr{A} = R\text{-Mod}$ for a Noetherian ring R, Positselski [Pos11, Theorem 3.7] showed that $\mathbf{K}(\operatorname{Inj}(\mathscr{A}))$ is equivalent to what he calls the *coderived category* $\mathbf{D}^{\text{co}}(R)$ of R, defined as the Verdier quotient $\mathbf{K}(R)/\operatorname{Acyc}^{\text{co}}(R)$, where $\operatorname{Acyc}^{\text{co}}(R)$ is the localizing subcategory of $\mathbf{K}(R)$ generated by the total complexes of short exact sequences of complexes of R-modules; objects of $\operatorname{Acyc}^{\text{co}}(R)$ are called *coacyclic complexes*. In particular, Krause's "large" singularity category $\mathbf{K}_{\text{ac}}(\operatorname{Inj}(R))$ is equivalent to a Verdier quotient $\mathbf{D}^{\text{co}}(R)/\mathbf{D}(R)$.

All in all, the last paragraphs suggest that a model for the singularity category could be obtained by lifting the quotient $\mathbf{D}^{\text{co}}(R)/\mathbf{D}(R)$ to the world of model categories. For $\mathbf{D}(R)$ there are the well-known projective and injective models, and for $\mathbf{D}^{\text{co}}(R)$ a model has been constructed by Positselski [Pos11]. Moreover, these models are *abelian*, i.e. they are compatible with the abelian structure of Ch(R-Mod) in the sense of [Hov02, Definition 2.1]. By [Hov02, Theorem 2.2] an abelian model structure is completely determined by the classes $\mathcal{C}$, $\mathcal{W}$, $\mathcal{F}$ of cofibrant, weakly trivial and fibrant objects, respectively, and the triples $(\mathcal{C}, \mathcal{W}, \mathcal{F})$ arising in this way are precisely those for which $\mathcal{W}$ is thick and both $(\mathcal{C}, \mathcal{W} \cap \mathcal{F})$ and $(\mathcal{C} \cap \mathcal{W}, \mathcal{F})$ are complete cotorsion pairs (see Definitions II.2.1.14 and II.2.1.12 for the definition of thickness and cotorsion pairs, respectively). For example, in the injective model $\mathcal{M}^{\text{inj}}(R)$ for $\mathbf{D}(R)$, everything is cofibrant, the weakly trivial objects $\mathcal{W}^{\text{inj}}$ are the acyclic complexes and the fibrant objects $\mathcal{F}^{\text{inj}}$ are the dg injectives. In Positselski's coderived model $\mathcal{M}^{\text{co}}(R)$ for $\mathbf{D}^{\text{co}}(R)$, again everything is cofibrant, but the weakly trivial objects $\mathcal{W}^{\text{co}}$ are the coacyclic complexes (see Proposition II.2.3.6) and the fibrant objects $\mathcal{F}^{\text{co}}$ are the componentwise injective complexes of R-modules. In particular, we see that both model structures are *injective* in the sense that everything is cofibrant, and that $\mathcal{W}^{\text{co}}(R) \subset \mathcal{W}^{\text{inj}}(R)$ and $\mathcal{F}^{\text{inj}}(R) \subset \mathcal{F}^{\text{co}}(R)$.

In order to construct the desired localization, we show (Theorem II.3.1.2) that given an abelian category $\mathscr{A}$ with two injective abelian model structures $\mathcal{M}_i = (\mathscr{A}, \mathcal{W}_i, \mathcal{F}_i)$, $i = 1, 2$, satisfying $\mathcal{F}_2 \subset \mathcal{F}_1$ (hence $\mathcal{W}_1 \subset \mathcal{W}_2$), there is another new abelian model structure $\mathcal{M}_1/\mathcal{M}_2$ on $\mathscr{A}$ with $\mathcal{C} = \mathcal{W}_2$ and $\mathcal{F} = \mathcal{F}_1$ (the class $\mathcal{W}$ of weakly trivials is determined by this and described explicitly in the Proposition), called the *right localization* of $\mathcal{M}_1$ with respect to $\mathcal{M}_2$. Moreover, we show (Proposition II.3.2.3) that $\mathcal{M}_1/\mathcal{M}_2$ is a right Bousfield localization of $\mathcal{M}_1$ with respect to $\{0 \to X \mid X \in \mathcal{F}_2\}$ in the sense of [Hir03, Definition 3.3.1(2)], and that on the level of homotopy categories we get a colocalization sequence [Kra05, Definition 3.1] of triangulated categories $\text{Ho}(\mathcal{M}_2) \to \text{Ho}(\mathcal{M}_1) \to \text{Ho}(\mathcal{M}_1/\mathcal{M}_2)$.

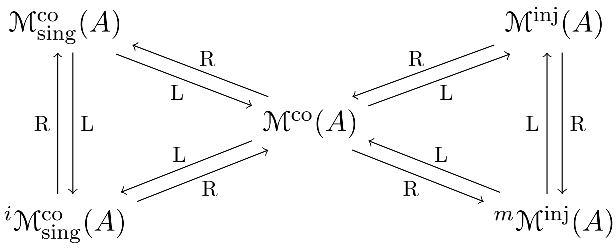
Applied to the injective model $\mathcal{M}^{\text{inj}}(R)$ for the ordinary derived category $\mathbf{D}(R)$ and Positselski's coderived model $\mathcal{M}^{\text{co}}(R)$ for the contraderived category $\mathbf{D}^{\text{co}}(R)$, we get another abelian model structure $\mathcal{M}^{\text{co}}_{\text{sing}}(R) = \mathcal{M}^{\text{co}}(R)/\mathcal{M}^{\text{inj}}(R)$ on Ch(R-Mod), called the *(absolute) singular coderived model*, where the cofibrant objects are the acyclic complexes of R-modules and the fibrant objects are the componentwise injective complexes of R-modules. In particular, $\text{Ho}(\mathcal{M}^{\text{co}}_{\text{sing}}(R)) \cong \mathbf{K}_{\text{ac}}(\text{Inj}(R))$ and there is a colocalization sequence $\mathbf{D}(R) \to \mathbf{D}^{\text{co}}(R) \cong \mathbf{K}(\text{Inj}(R)) \to \mathbf{K}_{\text{ac}}(\text{Inj}(R))$.

More generally, we construct a *relative singular coderived model* $\mathcal{M}^{\text{co}}_{\text{sing}}(A/R)$ for any morphism of dg rings $\varphi : R \to A$ as follows: first we show that the coderived model structure $\mathcal{M}^{\text{co}}(R)$ on R-Mod pulls back to a model structure $\varphi^*\mathcal{M}^{\text{co}}(R)$ on A-Mod (Proposition II.4.1.1), and then (Definition II.4.1.2) we define $\mathcal{M}^{\text{co}}_{\text{sing}}(A/R)$ as the right localization $\mathcal{M}^{\text{co}}(A)/\varphi^*\mathcal{M}^{\text{co}}(R)$. In case R is an ordinary ring of finite left-global dimension, this will be seen to be equal to the absolute singular coderived model $\mathcal{M}^{\text{co}}_{\text{sing}}(A)$ as defined above (Proposition II.2.3.17).

At this point we have succeeded in constructing models for singularity categories, but we cannot yet explain from the model categorical perspective why the sequence

$\mathbf{K}_{\mathrm{ac}}(\mathrm{Inj}(A)) \to \mathbf{K}(\mathrm{Inj}(A)) \to \mathbf{D}(A)$ is not only a localization sequence but in fact a recollement, as is known at least in the case A is an ordinary Noetherian ring by [Kra05, Proposition 3.6]. For this, we show that the absolute (it is important to restrict to the absolute case) singular model structure $\mathcal{M}^{\mathrm{co}}_{\mathrm{sing}}(A)$, which is a "mixed" model structure in the sense that usually neither everything is fibrant nor everything is cofibrant, admits a certain (Quillen equivalent) injective variant ${}^{i}\mathcal{M}^{\mathrm{co}}_{\mathrm{sing}}(A)$. The construction of this model structure is presented in Proposition II.4.2.1. The point is that while the identity on A-Mod is right Quillen $\mathcal{M}^{\mathrm{co}}(A) \to \mathcal{M}^{\mathrm{co}}_{\mathrm{sing}}(A)$ and provides a right adjoint of $\mathbf{K}_{\mathrm{ac}}(\mathrm{Inj}(A)) \to \mathbf{K}(\mathrm{Inj}(A))$, it is *left* Quillen $\mathcal{M}^{\mathrm{co}}(A) \to {}^{i}\mathcal{M}^{\mathrm{co}}_{\mathrm{sing}}(A)$, providing a left adjoint of $\mathbf{K}_{\mathrm{ac}}(\mathrm{Inj}(A)) \to \mathbf{K}(\mathrm{Inj}(A))$ and proving that $\mathbf{K}_{\mathrm{ac}}(\mathrm{Inj}(A)) \to \mathbf{K}(\mathrm{Inj}(A)) \to \mathbf{D}(A)$ is a recollement (Corollary II.4.2.6).

Moreover, we can now right-localize $\mathcal{M}^{\mathrm{inj}}(A)$ at ${}^{i}\mathcal{M}^{\mathrm{co}}_{\mathrm{sing}}(A)$ to obtain another "mixed" model structure ${}^{m}\mathcal{M}^{\mathrm{inj}}(A)$, which turns out to be another model for $\mathbf{D}(A)$ Quillen equivalent to the injective model $\mathcal{M}^{\mathrm{inj}}(A)$, explaining the existence of the left adjoint of $\mathbf{K}(\mathrm{Inj}(A)) \to \mathbf{D}(A)$. We see that the recollement $\mathbf{K}_{\mathrm{ac}}(\mathrm{Inj}(A)) \to \mathbf{K}(\mathrm{Inj}(A)) \to \mathbf{D}(A)$ unfolds to a butterfly of model structures and Quillen functors as follows (L denotes left Quillen functors and R denotes right Quillen functors). For more details on the properties of the butterfly, see Proposition II.4.2.8.



All the constructions mentioned so far also work in the projective/contraderived setting, yielding absolute and relative singular contraderived model structures on categories of modules over a dg ring, as well as a projective variant and a butterfly unfolding the recollement $\mathbf{K}_{\mathrm{ac}}(\mathrm{Proj}(A)) \to \mathbf{K}(\mathrm{Proj}(A)) \to \mathbf{D}(A)$.

We discuss two examples. Firstly, let R be a Gorenstein ring in the sense of [Buc86], i.e. R is Noetherian and of finite injective dimension both as a left and as a right module over itself. Then the 0-th cosyzygy functor $\mathrm{Ch}(R\text{-Mod}) \to R\text{-Mod}$ is a (left) Quillen equivalence between the absolute singular contraderived model $\mathcal{M}^{\mathrm{ctr}}_{\mathrm{sing}}(R)$ on $\mathrm{Ch}(R\text{-Mod})$ and Hovey's Gorenstein projective model structure on R-Mod [Hov02, Theorem 8.6]. Similarly, the 0-th syzygy functor is a (right) Quillen equivalence between the absolute singular coderived model $\mathcal{M}^{\mathrm{co}}_{\mathrm{sing}}(R)$ and Hovey's Gorenstein injective model on R-Mod. These two results are proved in Section II.5.1.

Secondly, in Section II.5.2 we consider matrix factorizations. Fix any ring S with a central element $w \in Z(S)$ and let $K_{S,w} = S[s]/(s^2)$ be the *Koszul algebra* of (S, w), i.e. $\deg(s) = -1$ and $\mathrm{d}(s) = w$. Modules over $K_{S,w}$ can be identified with complexes of S-modules X equipped with a square-zero nullhomotopy $s : X \to \Sigma^{-1}X$ for $X \xrightarrow{\cdot w} X$, i.e. they can be thought of as "curved" mixed complexes with curvature w. For any such curved mixed complex (X, d, s) we can form the sequences

$$\begin{aligned} \mathrm{fold}^{\Pi} X &:= \prod X^{\mathrm{even}} \xrightarrow{\mathrm{d}+s} \prod X^{\mathrm{odd}} \xrightarrow{\mathrm{d}+s} \prod X^{\mathrm{even}} \quad \text{and} \\ \mathrm{fold}^{\oplus} X &:= \bigoplus X^{\mathrm{even}} \xrightarrow{\mathrm{d}+s} \bigoplus X^{\mathrm{odd}} \xrightarrow{\mathrm{d}+s} \bigoplus X^{\mathrm{even}}, \end{aligned}$$

called the *folding with products* and *folding with sums* of (X, d, s). Since $\mathrm{d}\, s + s\, \mathrm{d} = w$ we see that $(\mathrm{d}+s)^2 = w$, and hence $\mathrm{fold}^{\oplus}(X)$ and $\mathrm{fold}^{\Pi}(X)$ are linear factorizations of type (S, w), i.e. matrix factorizations of type (S, w) with possibly non-free components. The category of linear factorizations $\mathrm{LF}(S, w)$ of type (S, w) is the same as the category of curved dg modules over the $\mathbb{Z}/2\mathbb{Z}$-graded curved dg ring S_w with $(S_w)^{\overline{0}} = S$, $(S_w)^{\overline{1}} = 0$ and curvature $w \in Z(S)$, and in particular it carries Positselski's contraderived model structure $\mathcal{M}^{\mathrm{ctr}}(S_w)$. We then prove that $\mathrm{fold}^{\oplus}$ and fold^{Π} are left resp. right Quillen equivalences $\mathcal{M}^{\mathrm{ctr}}_{\mathrm{sing}}(K_{S,w}/S) \to \mathcal{M}^{\mathrm{ctr}}(S_w)$ (Theorem II.5.2.13). In case S is regular local with maximal ideal $\mathfrak{m}$, $w \in \mathfrak{m} \setminus \{0\}$ and $R := S/(w)$, we also prove in Section II.5.3 that the cokernel functor $\mathrm{MF}(S, w) \to R$-Mod is a left Quillen equivalence $\mathcal{M}^{\mathrm{ctr}}(S_w) \to \mathcal{M}^{\text{G-proj}}(R)$ (Theorem II.5.3.2). This lifts Eisenbud's equivalence $\underline{\mathrm{MF}}(S, w) \cong \underline{\mathrm{MCM}}(R)$ [Eis80] to the level of model categories.

This part has two appendices. In Appendix II.B we prove that pullbacks of deconstructible classes along cocontinuous, monadic functors between Grothendieck categories are deconstructible (Proposition II.B.7), a fact which is used several times in Section II.2.3. In Appendix II.C we discuss the homotopy category of a hereditary abelian model structure $\mathcal{M}$ on $\mathscr{A}$, showing firstly that short exact sequences in $\mathscr{A}$ induce distinguished triangles in $\mathrm{Ho}(\mathcal{M})$ and secondly that the resulting assignment $\mathrm{Ext}^k_{\mathscr{A}}(X, Y) \to \mathrm{Ho}(\mathcal{M})(X, \Sigma^k Y)$ comes from a triangulated functor $\mathbf{D}(\mathscr{A}) \to \mathrm{Ho}(\mathcal{M})$; this is done by first lifting $\mathcal{M}$ to a Quillen equivalent model structure on $\mathrm{Ch}(\mathscr{A})$ and connecting the latter to a suitable model for $\mathbf{D}(\mathscr{A})$ through a butterfly of model structures on $\mathrm{Ch}(\mathscr{A})$.

Structure: In Sections II.2.1 and II.2.2 we recall the definition of abelian model categories as well as their relation to complete cotorsion pairs and deconstructible classes. In Section II.2.3 we use this relation to give self-contained constructions of the injective, projective, contraderived and coderived model structures on the category of modules over a dg ring. Next, in Section II.3.1 we prove Theorem II.3.1.2 providing a method for the construction of localizations of abelian model structures. In the intermediate Section II.3.2, which is not needed anywhere else in this work, we show that these new model structures can be described as Bousfield localizations in the classical sense (Proposition II.3.2.3). Then, in Section II.4.1 we turn to the construction of the relative and absolute singular contraderived and coderived model structures as well as their projective and injective variants, and in the central Section II.4.2 we construct the butterfly of Quillen functors lifting Krause's recollement to the level of model categories. In Section II.4.3 we discuss the possibility of extending our results to Grothendieck categories without enough projectives, and Section II.4.4 is devoted to comparing contraderived and coderived categories. Sections II.5.1, II.5.2 and II.5.3 contain the discussion of the examples of Gorenstein rings and matrix factorizations. In Appendices II.B and II.C we finally discuss pullbacks of deconstructible classes and the existence of "realization" functors $\mathbf{D}(\mathscr{A}) \to \mathrm{Ho}(\mathcal{M})$.

The question of finding and studying models for the stable derived category of a ring has been addressed independently by Daniel Bravo in his PhD thesis [Bra11]. Given a ring R, Bravo proves (in our terminology) that ${}^{i}\mathcal{M}^{\mathrm{co}}_{\mathrm{sing}}(R)$ is indeed a cofibrantly generated abelian model structure and establishes one half of the butterfly of Proposition II.4.2.8. Though he relies on Hovey's theorem on abelian model structures, too, his arguments are more direct and concrete, in particular exhibiting concrete cofibrant generators for ${}^{i}\mathcal{M}^{\mathrm{co}}_{\mathrm{sing}}(R)$ and reproving that $\mathrm{Ho}({}^{i}\mathcal{M}^{\mathrm{co}}(R)) \cong \mathbf{K}_{\mathrm{ac}}(\mathrm{Inj}(R))$ is compactly generated in case R is Noetherian. He also studies in detail the case $R = k[x,y]/(x^2, xy, y^2)$. Also, by now (note that many of the results of this part have already been published in [Bec14]) Gillespie [Gil12; Gil14a; Gil14b] has continued our study of model categorical enhancements of recollements and localizations of abelian model structures, and his results are crucial for our Appendix II.C which is not part of [Bec14].

II.2. Abelian model categories

II.2.1. Basic definitions

We begin by recalling the definition of (abelian) model structures and their homotopy categories, focusing on the abelian case.

Definition II.2.1.1. *A* model structure $\mathcal{M}$ *on a category* $\mathscr{C}$ *is a triple* $(\mathrm{Cof}, \mathrm{W}, \mathrm{Fib})$ *of classes of morphisms, called* cofibrations, weak equivalences *and* fibrations*, respectively, such that the following axioms are satisfied:*

(i) W *satisfies the* 2*-out-of-*3 *axiom, i.e. given two composable morphisms* f, g *in* $\mathcal{M}$*, if two of* f, g, gf *belong to* W*, then so does the third.*

(ii) Cof, W *and* Fib *are closed under retracts.*

(iii) *In any commutative square*

$$\begin{array}{ccc} A & \longrightarrow & X \\ {\scriptstyle f}\big\downarrow & \nearrow & \big\downarrow{\scriptstyle g} \\ B & \longrightarrow & Y \end{array}$$

the dashed arrow exists, making everything commutative, provided that either $f \in \mathrm{Cof}$ *and* $g \in \mathrm{W} \cap \mathrm{Fib}$ *or* $f \in \mathrm{Cof} \cap \mathrm{W}$ *and* $g \in \mathrm{Fib}$.

(iv) *Any morphism* f *factors as* $f = \beta \circ \alpha$ *with* $\alpha \in \mathrm{Cof}$*,* $\beta \in \mathrm{W} \cap \mathrm{Fib}$.

(v) *Any morphism* f *factors as* $f = \beta \circ \alpha$ *with* $\alpha \in \mathrm{Cof} \cap \mathrm{W}$*,* $\beta \in \mathrm{Fib}$.

A model category *is a bicomplete category (i.e. a category possessing arbitrary small limits and colimits) equipped with a model structure. Given a model category, we will sometimes drop the classes* $\mathrm{Cof}, \mathrm{W}, \mathrm{Fib}$ *from the notation.*

Notation II.2.1.2. Given a model category $(\mathscr{C}, \mathcal{M})$, an object $X \in \mathscr{C}$ is called *weakly trivial* if $0 \to X \in \mathrm{W}$ (equivalently, $X \to 0 \in \mathrm{W}$). Similarly, it is called *cofibrant* if $0 \to X \in \mathrm{Cof}$, and it is called *fibrant* if $X \to 0 \in \mathrm{Fib}$. It is called *bifibrant* if it is both fibrant and cofibrant. The classes of cofibrant, weakly trivial, and fibrant objects will be denoted $\mathcal{C}$, $\mathcal{W}$ and $\mathcal{F}$, respectively. The *homotopy category* is the localization $\mathscr{C}[\mathrm{W}^{-1}]$ and is denoted $\mathrm{Ho}(\mathcal{M})$. ◊

In this chapter we will mainly be concerned with model structures on abelian categories "compatible" with the abelian structure in the following way:

Definition II.2.1.3. *A model structure on an abelian category is called* abelian *if cofibrations equal monomorphism with cofibrant kernel and fibrations equal epimorphisms with fibrant kernel. An* abelian model category *is a bicomplete abelian category equipped with an abelian model structure.*

Remark II.2.1.4. There are other definitions of abelian model structures which seem different at first. In [Hov02] a model structure on an abelian category is said to be compatible with the abelian structure if every cofibration is a monomorphism and a morphism is a (trivial) fibration if and only if it is an epimorphism with (trivially) fibrant kernel. In [Gil11], Gillespie requires in addition that a morphism is a (trivial) cofibration if and only if it is a monomorphism with (trivially) cofibrant cokernel. The connection between these definitions is drawn in [Hov02, Proposition 4.2]: Assuming that every cofibration is a monomorphism and every fibration is an epimorphism, the four possible conditions on the characterization (trivial) (co)fibration in terms of their (co)kernels come in two pairs: Assuming that cofibrations equal monomorphisms with cofibrant cokernel is equivalent to assuming that trivial fibrations are epimorphisms with trivially fibrant kernel, and assuming that trivial cofibrations equal monomorphisms with trivially cofibrant cokernel is equivalent to assuming that fibrations are epimorphisms with fibrant kernel. In particular, our Definition II.2.1.3 is equivalent to [Hov02] is equivalent to [Gil11]. ♢

Requiring that any cofibration (resp. fibration) should be a monomorphism (resp. epimorphism) is not as automatic as it might appear at first: for example, given a ring R the standard projective model structure on $\mathrm{Ch}_{\geq 0}(R\text{-Mod})$ [Qui67] is *not* abelian since fibrations are required to be epimorphisms only in positive degrees. As a positive example, the standard injective and projective model structures on the category $\mathrm{Ch}(R\text{-Mod})$ of *unbounded* chain complexes of R-modules are abelian:

Proposition II.2.1.5 [Hov99]**.** *Let R be a ring.*

(i) *There exists a cofibrantly generated abelian model structure on* $\mathrm{Ch}(R\text{-Mod})$ *with* $\mathcal{C} = \mathrm{Ch}(R\text{-Mod})$*,* $\mathcal{W} = \mathrm{Acyc}(R\text{-Mod})$ *and* $\mathcal{F} = \text{dg-Inj}(R)$*, called the* standard injective model structure *on* $\mathrm{Ch}(R\text{-Mod})$*.*

(ii) *There exists a cofibrantly generated abelian model structure on* $\mathrm{Ch}(R\text{-Mod})$ *with* $\mathcal{F} = \mathrm{Ch}(R\text{-Mod})$*,* $\mathcal{W} = \mathrm{Acyc}(R\text{-Mod})$ *and* $\mathcal{C} = \text{dg-Proj}(R)$*, called the* standard projective model structure *on* $\mathrm{Ch}(R\text{-Mod})$*.*

Here Acyc(R) *is the class of acyclic complexes of R-modules. The standard projective and injective model structures are denoted* $\mathcal{M}^{\text{proj}}(R)$ *and* $\mathcal{M}^{\text{inj}}(R)$*.*

Proof. The existence and cofibrant generation of injective and projective model structures on Ch(R-Mod) is proved in [Hov99, Theorems 2.3.11 and 2.3.13], and [Hov99, Propositions 2.3.9 and 2.3.20] show that they are abelian. □

Another more complicated example of an abelian model structure on Ch(R-Mod) is Gillespie's flat model structure; in particular, it's an example of an abelian model structure in which both the classes of cofibrants and fibrants are non-trivial:

Theorem II.2.1.6 ([Gil04, Corollary 5.1])**.** *Let R be a ring. Then there exists a cofibrantly generated abelian model structure* $\mathcal{M}^{\text{flat}}(R)$ *on* Ch(R-Mod) *with $\mathcal{C}$ the class of dg flat complexes and $\mathcal{F}$ the class of dg cotorsion complexes. It is called the* flat model structure *on* Ch(R-Mod)*.*

Proof. Apart from the cofibrant generation, this is part of [Gil04, Corollary 5.1], and in view of [Hov02, Lemma 6.7] (see also Proposition II.2.2.9 below) Gillespie in fact also proves cofibrant generation as [Gil04, Propositions 4.9, 4.17]. □

An example of a non-trivial abelian model structure defined on a category which is not the category of chain complexes is Hovey's model for the singularity category of a Gorenstein ring. Recall that a ring R is *Gorenstein* [Buc86] if R is Noetherian and of finite injective dimension both as a left and as a right module over itself. An R-module M is called *Gorenstein projective* if it arises as the 0-th syzygy of an acyclic complex of projective R-modules, which is then called a *complete projective resolution* of M. Similarly, M is called *Gorenstein injective* if it arises as the 0-th syzygy of an acyclic complex of injective R-modules, which is then called a *complete injective resolution* of M. The classes of Gorenstein projective and Gorenstein injective R-modules are denoted G-proj(R) and G-inj(R), respectively.

Proposition II.2.1.7 [Hov02, Theorem 8.6]**.** *Let R be a Gorenstein ring.*

(i) *There exists an abelian model structure on R*-Mod*, called the* Gorenstein projective *model structure and denoted* $\mathcal{M}^{\text{G-proj}}(R)$*, with* $\mathcal{C} = \text{G-proj}(R)$*,* $\mathcal{W} = \mathcal{P}^{<\infty}(R)$ *(the modules of finite projective dimension) and* $\mathcal{F} = R\text{-Mod}$*.*

(ii) *There exists an abelian model structure on R*-Mod*, called the* Gorenstein injective *model structure and denoted* $\mathcal{M}^{\text{G-inj}}(R)$*, with* $\mathcal{C} = R\text{-Mod}$*,* $\mathcal{W} = \mathcal{I}^{<\infty}(R)$ *(the modules of finite injective dimension) and* $\mathcal{F} = \text{G-inj}(R)$*.*

Moreover, both $\mathcal{M}^{\text{G-proj}}(R)$ *and* $\mathcal{M}^{\text{G-inj}}(R)$ *are cofibrantly generated.*

Remark II.2.1.8. The definition of Gorenstein projectivity and injectivity in terms of complete projective and injective resolutions is not suitable for the proof of II.2.1.7. Instead, Hovey shows that $({}^{\perp}\mathcal{P}^{<\infty}(R), \mathcal{P}^{<\infty}(R), R\text{-Mod})$ and $(R\text{-Mod}, \mathcal{I}^{<\infty}(R), \mathcal{I}^{<\infty}(R)^{\perp})$ are cofibrantly generated abelian model structures on R-Mod and *defines* the classes G-proj(R) and G-inj(R) of Gorenstein projective resp. injective modules as ${}^{\perp}\mathcal{P}^{<\infty}(R)$ resp. $\mathcal{I}^{<\infty}(R)^{\perp}$ afterwards. Therefore, according to these definitions an R-module X is Gorenstein projective if and only if $\mathrm{Ext}^1_R(X, M) = 0$ for all $M \in \mathcal{P}^{<\infty}(R)$. By Lemma II.2.1.17 and Corollary II.2.1.19 below this is equivalent to $\mathrm{Ext}^k_R(X, M) = 0$ for all $k > 0$ and all $M \in \mathcal{P}^{<\infty}(R)$, which is also quickly seen to be equivalent to $\mathrm{Ext}^k_R(X, P) = 0$ for all projective R-modules P. In case of X finitely generated, one may even reduce to $\mathrm{Ext}^k_R(X, R) = 0$ for all $k > 0$, which is another common definition of Gorenstein projective or maximal Cohen-Macaulay modules; note, however, that in view of the Whitehead problem (see Remark II.2.2.5) this last reduction is not possible for general X.

Finally, we argue that ${}^{\perp}\mathcal{P}^{<\infty}(R)$ agrees with the class of modules admitting a complete projective resolution – the case of Gorenstein injective modules is analogous. In the one direction, if X admits a complete projective resolution then it can be written as an arbitrarily high syzygy $\Omega^j X'$ of a suitable X', hence $\mathrm{Ext}^k_R(X, M) \cong \mathrm{Ext}^{k+j}_R(X', M) = 0$ for each $M \in \mathcal{P}^{<\infty}(R) = \mathcal{I}^{<\infty}(R)$ and $j \gg 0$. In the other direction, the completeness of the cotorsion pair $({}^{\perp}\mathcal{P}^{<\infty}(R), \mathcal{P}^{<\infty}(R))$ together with ${}^{\perp}\mathcal{P}^{<\infty}(R) \cap \mathcal{P}^{<\infty}(R) = \mathrm{Proj}(R)$ show that any $X \in {}^{\perp}\mathcal{P}^{<\infty}(R)$ admits a short exact sequence $0 \to X \to P \to X' \to 0$ with $P \in \mathrm{Proj}(R)$ and $X' \in {}^{\perp}\mathcal{P}^{<\infty}(R)$ again. Repeating this procedure shows that any $X \in {}^{\perp}\mathcal{P}^{<\infty}(R)$ admits a projective resolution to the right, which can be extended to a complete projective resolution by splicing it with an ordinary projective resolution. $\Diamond$

Proposition II.2.1.9. *Let R be a Gorenstein ring. Then $\mathcal{P}^{<\infty} = \mathcal{I}^{<\infty}$, and the identity is a left Quillen equivalence $\mathcal{M}^{\text{G-proj}}(R) \rightleftarrows \mathcal{M}^{\text{G-inj}}(R)$.*

Proof. The equality $\mathcal{P}^{<\infty} = \mathcal{I}^{<\infty}$ is proved in [EJ11, Proposition 9.1.7]. The second statement is mentioned in [Hov02, Paragraphs following Theorem 8.6] without proof, so we include an argument here: By [Hov02, Lemma 5.8] (see also Corollary II.C.1.3), the weak equivalences of $\mathcal{M}^{\text{G-proj}}(R)$ (resp. $\mathcal{M}^{\text{G-inj}}(R)$) are precisely the compositions of monomorphisms and epimorphisms the kernels resp. cokernels of which have finite projective (resp. injective) dimension. Since $\mathcal{P}^{<\infty} = \mathcal{I}^{<\infty}$, we conclude that $\mathcal{M}^{\text{G-proj}}(R)$ and $\mathcal{M}^{\text{G-inj}}(R)$ have the same classes of weak equivalences, and that the derived adjunction of $\mathcal{M}^{\text{G-proj}}(R) \rightleftarrows \mathcal{M}^{\text{G-inj}}(R)$ is indeed an adjoint equivalence. $\square$

The fact that $\mathcal{P}^{<\infty} = \mathcal{I}^{<\infty}$ relies on the duality $M \leftrightarrow \mathrm{Hom}_R(M, \mathbb{Q}/\mathbb{Z})$ between left and right R-modules, which exchanges injectivity and flatness for Noetherian rings [EJ11, Theorem 3.2.10 and Corollary 3.2.17]. Very recently, Bravo, Gillespie and Hovey [BGH14] generalized this, for *any* ring R, to a duality between what they call absolutely clean

(AC) and level modules [BGH14, Theorem 2.12]. Further, they derive the notions of Gorenstein AC-injective and Gorenstein AC-projective modules [BGH14, §5 and §8], and establish the following result (generalizing Proposition II.2.1.7):

Proposition II.2.1.10 [BGH14, §5 and §8]**.** *Let R be any ring, and denote* G-AC-inj *and* G-AC-proj *the classes of Gorenstein AC-injective and Gorenstein AC-projective modules, respectively. Then the following two triples are cofibrantly generated abelian model structures on R*-Mod*:*

(i) $(\text{G-AC-proj}, \text{G-AC-proj}^{\perp}, R\text{-Mod})$

(ii) $(R\text{-Mod}, {}^{\perp}\text{G-AC-inj}, \text{G-AC-inj})$

Remark II.2.1.11. All examples of abelian model structures $\mathcal{M} = (\mathcal{C}, \mathcal{W}, \mathcal{F})$ on abelian categories $\mathscr{A}$ we will be concerned with in this work have the property that their *core* $\omega := \mathcal{C} \cap \mathcal{W} \cap \mathcal{F} = {}^{\perp}\mathcal{F} \cap \mathcal{F} = \mathcal{C} \cap \mathcal{C}^{\perp}$ is either the class $\mathcal{I}(\mathscr{A})$ of injectives or the class $\mathcal{P}(\mathscr{A})$ of projectives in $\mathscr{A}$ (such model structures are called *weakly injective* resp. *weakly projective* in [Gil12]). The reason is that all of them will either be injective or projective in the sense that $\mathcal{C} = \mathscr{A}$ or $\mathscr{F} = \mathscr{A}$, respectively, or arise from those through the localization construction for abelian model structures introduced in Chapter II.3; since injective resp. projective model structures have their core equal to $\mathcal{P}(\mathscr{A})$ resp. $\mathcal{I}(\mathscr{A})$, and the localization procedure does not change the core, this explains why we shall not encounter other cores. However, there *are* abelian model structures for which ω is different from both $\mathcal{I}(\mathscr{A})$ and $\mathcal{P}(\mathscr{A})$: For example, the *Gorenstein flat model structure* on the category of left modules over a right-coherent ring, that was very recently constructed by Gillespie [Gil14b], has the class of flat cotorsion modules as its core. We will come back to this example in Remark II.3.1.3. ◊

We return to generalities of abelian model structures. Right from its definition we know that an abelian model structure is determined by the triple of cofibrant, weakly trivial and fibrant objects. The question which such triples actually give rise to abelian model structures was solved in [Hov02] in terms of complete cotorsion pairs:

Definition II.2.1.12 [Hov02, Definition 2.3]**.** *For an abelian category $\mathscr{A}$, a* cotorsion pair *in $\mathscr{A}$ is a pair $(\mathcal{D}, \mathcal{E})$ of classes of objects such that the following hold:*

(i) $\mathcal{D} = {}^{\perp}\mathcal{E} := \{X \in \mathscr{A} \mid \operatorname{Ext}^1_{\mathscr{A}}(X, \mathcal{E}) = 0\}$.

(ii) $\mathcal{E} = \mathcal{D}^{\perp} := \{Y \in \mathscr{A} \mid \operatorname{Ext}^1_{\mathscr{A}}(\mathcal{D}, Y) = 0\}$.

In this case, we call $\mathcal{D}$ the cotorsion class *and $\mathcal{E}$ the* cotorsionfree class*. A cotorsion pair $(\mathcal{D}, \mathcal{E})$ is called* complete *if the following two conditions are satisfied:*

(3) $(\mathcal{D}, \mathcal{E})$ *has* enough projectives*, i.e. for each* $Z \in \mathscr{A}$ *there exists an exact sequence* $0 \to Y \to X \to Z \to 0$ *such that* $X \in \mathcal{D}$ *and* $Y \in \mathcal{E}$*.*

(4) $(\mathcal{D}, \mathcal{E})$ *has* enough injectives*, i.e. for each* $Z \in \mathscr{A}$ *there exists an exact sequence* $0 \to Z \to Y \to X \to 0$ *such that* $Y \in \mathcal{E}$ *and* $X \in \mathcal{D}$*.*

A cotorsion pair $(\mathcal{D}, \mathcal{E})$ *is called* resolving *if* $\mathcal{D}$ *is closed under taking kernels of epimorphisms, and it is called* coresolving *if* $\mathcal{E}$ *is closed under taking cokernels of monomorphisms. It is called* hereditary *if it is both resolving and coresolving.*

Example II.2.1.13. Denoting $\mathcal{I}$ the class of injectives, the pair $(\mathscr{A}, \mathcal{I})$ is a hereditary cotorsion pair with enough projectives. It is complete if and only if $\mathscr{A}$ has enough injectives in the usual sense. Similarly, denoting $\mathcal{P}$ the class of projectives, the pair $(\mathcal{P}, \mathscr{A})$ is a hereditary cotorsion pair with enough injectives, and it is complete if and only if $\mathscr{A}$ has enough projectives in the usual sense. ♢

Definition II.2.1.14. *A subcategory* $\mathcal{W}$ *of an abelian category* $\mathscr{A}$ *is called* thick *if it is closed under summands and if it satisfies the* 2-out-of-3 property*, i.e. whenever two out of three terms in a short exact sequence lie in* $\mathcal{W}$*, then so does the third.*

Theorem II.2.1.15 [Hov02, Theorem 2.2]**.** *Let* $\mathscr{A}$ *be a bicomplete abelian category and* $\mathcal{C}$*,* $\mathcal{W}$ *and* $\mathcal{F}$ *classes of objects in* $\mathscr{A}$*. Then the following are equivalent:*

(i) *There exists an abelian model structure on* $\mathscr{A}$ *where* $\mathcal{C}$ *is the class of cofibrant,* $\mathcal{F}$ *is the class of fibrant, and* $\mathcal{W}$ *is the class of weakly trivial objects.*

(ii) $\mathcal{W}$ *is thick and both* $(\mathcal{C}, \mathcal{F} \cap \mathcal{W})$ *and* $(\mathcal{C} \cap \mathcal{W}, \mathcal{F})$ *are complete cotorsion pairs.*

Slightly abusing the notation, given a triple $(\mathcal{C}, \mathcal{W}, \mathcal{F})$ *as above we will often denote its induced abelian model structure* $(\mathcal{C}, \mathcal{W}, \mathcal{F})$ *as well.*

We call an abelian model structure $\mathcal{M} = (\mathcal{C}, \mathcal{W}, \mathcal{F})$ *hereditary* if their associated cotorsion pairs $(\mathcal{C}, \mathcal{W} \cap \mathcal{F})$ and $(\mathcal{C} \cap \mathcal{W}, \mathcal{F})$ are hereditary. In view of the 2-out-of-3 property of $\mathcal{W}$, this is equivalent to saying that $\mathcal{C}$ is closed under taking kernels of epimorphisms and $\mathcal{F}$ is closed under taking cokernels of monomorphisms. Note that Gillespie [Gil11] even obtained a version of Theorem II.2.1.15 for exact categories endowed with model structures compatible with the exact structure. Moreover, he does *not* assume the existence of arbitrary small colimits and limits, as is done here and in [Hov99], for example.

Let us consider the extreme cases of *projective* (resp. *injective*) abelian model structures, i.e. model structures where everything is fibrant (resp. cofibrant).

Corollary II.2.1.16. *Let* $\mathscr{A}$ *be a bicomplete abelian category and* $\mathcal{C}, \mathcal{W} \subset \mathscr{A}$ *classes of objects in* $\mathscr{A}$*. Then the following are equivalent:*

(i) $(\mathcal{C}, \mathcal{W}, \mathscr{A})$ *gives rise to an abelian model structure on* $\mathscr{A}$.

(ii) $\mathscr{A}$ *has enough projectives,* $(\mathcal{C}, \mathcal{W})$ *is a complete cotorsion pair with* $\mathcal{C} \cap \mathcal{W} = \mathcal{P}(\mathscr{A})$ *and* $\mathcal{W}$ *satisfies the* 2*-out-of-*3 *property.*

Dually, for classes of objects $\mathcal{W}, \mathcal{F} \subseteq \mathscr{A}$ *the following are equivalent:*

(i) $(\mathscr{A}, \mathcal{W}, \mathcal{F})$ *gives rise to an abelian model structure on* $\mathscr{A}$.

(ii) $\mathscr{A}$ *has enough injectives,* $(\mathcal{W}, \mathcal{F})$ *is a complete cotorsion pair with* $\mathcal{W} \cap \mathcal{F} = \mathcal{I}(\mathscr{A})$ *and* $\mathcal{W}$ *satisfies the* 2*-out-of-*3 *property.*

Proof. By Theorem II.2.1.15, $(\mathcal{C}, \mathcal{W}, \mathscr{A})$ giving rise to an abelian model structure on $\mathscr{A}$ is equivalent to $\mathcal{W}$ satisfying the 2-out-of-3 property and $(\mathcal{C}, \mathcal{W} \cap \mathcal{F}) = (\mathcal{C}, \mathcal{W})$, $(\mathcal{C} \cap \mathcal{W}, \mathcal{F}) = (\mathcal{C} \cap \mathcal{W}, \mathscr{A})$ being complete cotorsion pairs. The latter means that $\mathscr{A}$ has enough projectives and $\mathcal{C} \cap \mathcal{W} = \mathcal{P}(\mathscr{A})$. The second part is dual. $\square$

We will see how complete cotorsion pairs can be constructed in the next section. Concerning the 2-out-of-3 property, the next lemma will be useful.

Lemma II.2.1.17. *Let* $(\mathcal{W}, \mathcal{F})$ *be a cotorsion pair in an abelian category* $\mathscr{A}$ *with enough injectives. Consider the following statements:*

(i) $(\mathcal{W}, \mathcal{F})$ *is coresolving.*

(ii) $\operatorname{Ext}^k_{\mathscr{A}}(W, F) = 0$ *for all* $W \in \mathcal{W}$, $F \in \mathcal{F}$ *and* $k \geq 1$.

(iii) $\mathcal{W}$ *satisfies the* 2*-out-of-*3 *property.*

Then (i)⇔(ii). If $(\mathcal{W}, \mathcal{F})$ *is complete with* $\mathcal{W} \cap \mathcal{F} = \mathcal{I}(\mathscr{A})$, *then also (ii)⇒(iii).*

Proof. (ii)⇒(i) follows from the long exact Ext-sequence. Now assume (i) holds. For $F \in \mathcal{F}$, pick an embedding $i : F \hookrightarrow I$ with $I \in \mathcal{I}(\mathscr{A}) \subset \mathcal{F}$. Then $\Sigma F := \operatorname{coker}(i) \in \mathcal{F}$ by assumption, and $\operatorname{Ext}^k_{\mathscr{A}}(-, F) \cong \operatorname{Ext}^{k-1}_{\mathscr{A}}(-, \Sigma F)$ for all $k \geq 2$. Inductively, we deduce (ii). This shows (i)⇔(ii), so it remains to show (ii)⇒(iii) in case $(\mathcal{W}, \mathcal{F})$ is complete and $\mathcal{W} \cap \mathcal{F} = \mathcal{I}(\mathscr{A})$. If $0 \to W_1 \to W_2 \to W_3 \to 0$ is a short exact sequence with at least two of the W_i belonging to $\mathcal{W}$, we have $\operatorname{Ext}^2_{\mathscr{A}}(W_i, \mathcal{F}) = 0$ for all $i = 1, 2, 3$. It is therefore sufficient to show that any $X \in \mathscr{A}$ satisfying $\operatorname{Ext}^2_{\mathscr{A}}(X, \mathcal{F}) = 0$ actually satisfies $\operatorname{Ext}^1_{\mathscr{A}}(X, \mathcal{F}) = 0$, i.e. $X \in \mathcal{W}$. For this, pick $F \in \mathcal{F}$ arbitrary and choose an exact sequence $0 \to F' \to I \to F \to 0$ with $F' \in \mathcal{F}$ and $I \in \mathcal{I}(\mathscr{A})$. Such a sequence exists since $(\mathcal{W}, \mathcal{F})$ has enough projectives, $\mathcal{F}$ is closed under extensions and $\mathcal{W} \cap \mathcal{F} = \mathcal{I}(\mathscr{A})$ by assumption. Then $\operatorname{Ext}^1_{\mathscr{A}}(X, F) \cong \operatorname{Ext}^2_{\mathscr{A}}(X, F') = 0$, and hence $X \in \mathcal{W}$. $\square$

Combining Lemma II.2.1.17 with its dual (note that (ii)$\Rightarrow$(i) did only use the existence of Ext^* and the long exact Ext^*-sequence) shows that in case $\mathscr{A}$ has enough injectives, then $(\mathcal{W}, \mathcal{F})$ being coresolving implies $(\mathcal{W}, \mathcal{F})$ being resolving. Dually, if $\mathscr{A}$ has enough projectives, then $(\mathcal{W}, \mathcal{F})$ being resolving implies $(\mathcal{W}, \mathcal{F})$ being coresolving. Restricting to complete cotorsion pairs, the existence of enough projectives or injectives is not necessary:

Proposition II.2.1.18. *Let $\mathscr{A}$ be an abelian category, $(\mathcal{X}, \mathcal{Y})$ be a complete, coresolving cotorsion pair and $\omega := \mathcal{X} \cap \mathcal{Y}$. Then $\mathcal{X}/\omega = {}^{\ddagger}(\mathcal{Y}/\omega)$, $\mathcal{Y}/\omega = (\mathcal{X}/\omega)^{\ddagger}$ in $\mathscr{A}/\omega$. Here $\mathscr{A}/\omega$, $\mathcal{X}/\omega$ and $\mathcal{Y}/\omega$ denote the stable categories and $\ddagger$ denotes the* Hom*-orthogonal (because $\perp$ is already occupied). Moreover, $(\mathcal{X}, \mathcal{Y})$ is resolving.*

Proof. Given $Y \in \mathcal{Y}$, in a sequence $0 \to Y' \to X \to Y \to 0$ with $Y' \in \mathcal{Y}$ and $X \in \mathcal{X}$ we have $X \in \mathcal{X} \cap \mathcal{Y} = \omega$ since $\mathcal{Y}$ is extension-closed. As $X \to Y$ is an $\mathcal{X}$-approximation, it follows that any map $X' \to Y$ for some other $X' \in \mathcal{X}$ factors through ω, hence vanishes in $\mathscr{A}/\omega$.

Next, let $A \in \mathscr{A}$ and pick exact sequences $0 \to Y \to X \to A \to 0$ and $0 \to X \to I \to X' \to 0$ with $X, X' \in \mathcal{X}$, $I \in \omega$ and $Y \in \mathcal{Y}$. Taking pushout yields a commutative diagram with exact rows and columns, and a bicartesian upper right square:

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & Y & \longrightarrow & X & \twoheadrightarrow & A & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & Y & \longrightarrow & I & \longrightarrow & Y' & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & X' & = & X' & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

Moreover, since $\mathcal{Y}$ is closed under taking cokernels of monomorphisms by assumption, we also have $Y' \in \mathcal{Y}$. Now, in case $A \in {}^{\ddagger}(\mathcal{Y}/\omega)$ the map $A \to Y'$ factors through an object in ω, hence through $I \to Y'$ as $Y = \ker(I \to Y') \in \mathcal{Y} \subset \omega^{\perp}$. Since the upper right square is cartesian, any such factorization $A \to I$ gives rise to a splitting of $X \to A$, and hence $A \in \mathcal{X}$. Similarly, if $A \in (\mathcal{X}/\omega)^{\ddagger}$, the map $X \to A$ factors through an object in ω, hence through $X \to I$, and since the upper right square is cocartesian, such a factorization yields a splitting of $A \to Y$, so $A \in \mathcal{Y}$.

For the last part, suppose $0 \to Z \to X \to X' \to 0$ is an exact sequence with $X, X' \in \mathcal{X}$. We want to show that $Z \in \mathcal{X}$, and by the above it is sufficient to show that any morphism

$f : Z \to Y$ factors through ω. But f extends to a morphism $g : X \to Y$ (since $X' \in \mathcal{X}$) which then factors through ω (since $X \in \mathcal{X}$). □

Corollary II.2.1.19. *A complete cotorsion pair is coresolving if and only if it is resolving. In particular, any injective/projective abelian model structure is hereditary.*

Proof. The first statement follows from Proposition II.2.1.18 combined with its dual. For the second, note that if $(\mathscr{A}, \mathcal{W}, \mathcal{F})$ is an injective abelian model structure, then $(\mathcal{W}, \mathcal{F})$ is a resolving cotorsion pair (since $\mathcal{W}$ satisfies the 2-out-of-3 property), hence hereditary by the first part. The projective case is similar. □

We now describe the homotopy category of an abelian model category.

Proposition II.2.1.20. *Let $\mathscr{A}$ be a bicomplete abelian category and $\mathcal{M} = (\mathcal{C}, \mathcal{W}, \mathcal{F})$ be an abelian model structure on $\mathscr{A}$. Then the composition $\mathcal{C} \cap \mathcal{F} \hookrightarrow \mathscr{A} \to \mathrm{Ho}(\mathcal{M})$ induces an equivalence of categories $\mathcal{C} \cap \mathcal{F}/\omega \cong \mathrm{Ho}(\mathcal{M})$, where $\omega = \mathcal{C} \cap \mathcal{W} \cap \mathcal{F}$.*

Proof. This is known – see for example [Gil11, Proposition 4.3,4.7] or [BR07, Theorem VIII.4.2] – but for completeness we give a proof here. For a general model category $\mathcal{M}$ and objects X, Y, the set $\mathcal{M}(X, Y)$ admits two natural relations $\sim_{l/r}$ of left and right homotopy, defined via cylinder and path objects, respectively. If X is cofibrant and Y is fibrant, these two relations coincide and are equivalence relations, and $\mathcal{M}(X, Y) \to \mathrm{Ho}(\mathcal{M})(X, Y)$ induces a bijection $\mathcal{M}(X, Y)/_\sim \cong \mathrm{Ho}(\mathcal{M})(X, Y)$. In particular, there is a fully-faithful functor $\mathcal{M}_{\mathrm{cf}}/_\sim \to \mathrm{Ho}(\mathcal{M})$, where $\mathcal{M}_{\mathrm{cf}}$ is the class of bifibrant objects of $\mathcal{M}$, and by the existence of fibrant and cofibrant resolutions this is even an equivalence of categories. See [Hov99, Theorem 1.2.10] for details.

To prove the claim, it is therefore sufficient to show that for $X \in \mathcal{C}$ and $Y \in \mathcal{F}$, two morphisms $f, g : X \to Y$ are right homotopic in the above sense if and only if $f - g$ factors through $\mathcal{C} \cap \mathcal{W} \cap \mathcal{F}$. For this, we construct a path object PY for Y as follows: first choose a short exact sequence $0 \to \Omega Y \to I \to Y \to 0$ with $I \in \mathcal{C} \cap \mathcal{W}$ and $\Omega Y \in \mathcal{F}$. Such a sequence exists by the completeness of the cotorsion pair $(\mathcal{C} \cap \mathcal{W}, \mathcal{F})$. Since $\mathcal{F}$ is closed under extensions, we even have $I \in \mathcal{C} \cap \mathcal{W} \cap \mathcal{F} = \omega$. Taking the pullback of $Y \oplus Y \xrightarrow{(1,-1)} Y \leftarrow I$, we get the following commutative diagram with exact rows and columns:

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & Y & \xrightarrow{\Delta} & Y \oplus Y & \xrightarrow{(1\ \ -1)} & Y & \longrightarrow & 0 \\
 & & \| & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & Y & \longrightarrow & PY & \longrightarrow & I & \longrightarrow & 0 \\
 & & & & \uparrow & & \uparrow & & \\
 & & & & \Omega Y & = & \Omega Y & & \\
 & & & & \uparrow & & \uparrow & & \\
 & & & & 0 & & 0 & &
\end{array} \qquad (*)$$

The morphism $PY \to Y \oplus Y$ is a fibration because its kernel ΩY lies in $\mathcal{F}$, and $Y \to PY$ is a trivial cofibration because its cokernel I belongs to $\omega \subset \mathcal{C} \cap \mathcal{W}$. In other words, the factorization $Y \to PY \to Y \oplus Y$ of $\Delta : Y \to Y \oplus Y$ is a path object for Y and can be used to compute the right homotopy relation. By definition of the pullback, the morphism $(f, g)^t : X \to Y \oplus Y$ factors through $PY \to Y \oplus Y$ if and only if $f - g : X \to Y$ factors through $I \to Y$. Finally, since $I \to Y$ is a ω-cover for Y (its kernel ΩY is in $\mathcal{F} = (\mathcal{C} \cap \mathcal{W})^\perp \subset \omega^\perp$), this is in turn equivalent to $f - g : X \to Y$ factoring through *some* object in ω. ☐

The homotopy category of a model category $(\mathscr{A}, \mathcal{M})$ whose underlying category $\mathscr{A}$ is abelian carries a natural pretriangulated structure in the sense of [BR07, Definition II.1.1]. This follows from [Hov99, Section 6.5] together with the fact that any cogroup object in an additive category is isomorphic to one of the form $(X, \Delta : X \to X \oplus X, 0 : X \to 0)$ and that giving some object Y a comodule structure over such a cogroup is equivalent to giving a morphism $Y \to X$. See also [Hov99, Remark 7.1.3, Theorem 7.1.6]. Concretely [Hov99, Paragraph following Definition 6.1.1], the suspension functor $\Sigma : \mathrm{Ho}(\mathcal{M}) \to \mathrm{Ho}(\mathcal{M})$ takes a cofibrant object X to the cokernel of the inclusion $X \oplus X \to \mathrm{Cyl}(X)$, where $X \oplus X \to \mathrm{Cyl}(X) \to X$ is a cylinder object for X, and the loop functor $\Omega : \mathrm{Ho}(\mathcal{M}) \to \mathrm{Ho}(\mathcal{M})$ takes a fibrant object Y to the kernel of the projection $PY \to Y \oplus Y$, where $Y \to PY \to Y \oplus Y$ is a path object for Y. If $\mathcal{M} = (\mathcal{C}, \mathcal{W}, \mathcal{F})$ is an abelian model structure, in view of the explicit construction $(*)$ of path objects in Proposition II.2.1.20 and the corresponding dual construction of cylinder objects, we conclude that given objects $X \in \mathcal{C}$ and $Y \in \mathcal{F}$ their suspension and loop objects $\Sigma X \in \mathcal{C}$, $\Omega Y \in \mathcal{F}$ can be defined by the property that they belong to exact sequences $0 \to X \to I \to \Sigma X \to 0$ and $0 \to \Omega Y \to P \to Y \to 0$ with $I \in \mathcal{W} \cap \mathcal{F}$ and $P \in \mathcal{C} \cap \mathcal{W}$. However, for $X, Y \in \mathcal{C} \cap \mathcal{F}$ it is not clear in this situation that ΣX and ΩY again belong to $\mathcal{C} \cap \mathcal{W}$, at least if $\mathcal{M}$ is not assumed to be hereditary. Hence, in this case we don't know how the pretriangulated structure on $\mathcal{C} \cap \mathcal{F}/\omega$ obtained by pulling back the pretriangulated structure on $\mathrm{Ho}(\mathcal{M})$ along the equivalence $\mathcal{C} \cap \mathcal{F}/\omega \to \mathrm{Ho}(\mathcal{M})$ of Proposition II.2.1.20 can be described explicitly. Assuming that $\mathcal{M}$ is hereditary, however, we have the following [Gil11, Proposition 5.2]:

Proposition II.2.1.21. *Let $\mathcal{M} = (\mathcal{C}, \mathcal{W}, \mathcal{F})$ be a hereditary abelian model structure on an abelian category $\mathscr{A}$ Then $\mathcal{C} \cap \mathcal{F}$, endowed with the exact structure inherited from $\mathscr{A}$, is Frobenius. Its class of projective-injective objects equals $\omega := \mathcal{C} \cap \mathcal{W} \cap \mathcal{F}$, and $\mathcal{C} \cap \mathcal{F}/\omega \to \mathrm{Ho}(\mathcal{M})$ is an equivalence of pretriangulated categories.*

Corollary II.2.1.22. *A hereditary abelian model category is stable.*

Proof of Proposition II.2.1.21. Denote $\mathscr{E}$ the class of short exact sequences in $\mathscr{A}$ with entries in $\mathcal{C} \cap \mathcal{F}$. We only check that $(\mathcal{C} \cap \mathcal{F}, \mathscr{E})$ is a Frobenius category; the remaining part involves comparing the definition of distinguished triangles in stable categories of Frobenius categories to the definition of fiber and cofiber sequences in the homotopy category of a pointed model category [Hov99, Definition 6.2.6], but we omit it.

First, we have $\mathcal{C} \cap \mathcal{F} \subset \mathcal{C} = {}^{\perp}(\mathcal{W} \cap \mathcal{F}) \subset {}^{\perp}\omega$ and similarly $\mathcal{C} \cap \mathcal{F} \subset \omega^{\perp}$, showing that any object in ω is projective-injective in $(\mathcal{C} \cap \mathcal{F}, \mathscr{E})$. Next, given $X \in \mathcal{C} \cap \mathcal{F}$, the completeness of $(\mathcal{C} \cap \mathcal{W}, \mathcal{F})$ provides a short exact sequence $0 \to X' \to I \to X \to 0$ in $\mathscr{A}$ with $X' \in \mathcal{F}$ and $I \in \mathcal{C} \cap \mathcal{W}$. As $\mathcal{C}$ is closed under taking kernels of epimorphisms by assumption and $\mathcal{F}$ is closed under taking extensions, we infer that $X' \in \mathcal{C} \cap \mathcal{F}$ and $I \in \omega$, proving that $(\mathcal{C} \cap \mathcal{F}, \mathscr{E})$ has enough projectives, and that $\mathcal{P}(\mathcal{C} \cap \mathcal{F}, \mathscr{E}) = \omega$. Similarly, using that $\mathcal{F}$ is closed under taking cokernels of monomorphisms we get that $(\mathcal{C} \cap \mathcal{F}, \mathscr{E})$ has enough injectives and $\mathcal{I}(\mathcal{C} \cap \mathcal{F}, \mathscr{E}) = \omega$, finishing the proof. □

II.2.2. Small cotorsion pairs

In the previous section we recalled the definition and properties of abelian model structures, and in particular we discussed Hovey's one-to-one correspondence between abelian model structures and pairs of compatible complete cotorsion pairs. However, we did not explain so far how one can actually construct such complete cotorsion pairs, and this is the topic of the present section. We describe how each set $\mathcal{S}$ of objects in an abelian category $\mathscr{A}$ yields a cotorsion pair in $\mathscr{A}$, called the cotorsion pair cogenerated by $\mathcal{S}$, and discuss when such cotorsion pairs are complete, our main source being [SŠ11]. We then use these results to give a handy description of classes occurring as cotorsion classes in complete cotorsion pairs cogenerated by sets in terms of generators and deconstructibility. This prepares the ground for the construction of the projective, injective, coderived and contraderived abelian model structures for modules over (curved) differential graded rings in the next section. We end with a theorem of Hovey connecting complete cotorsion pairs cogenerated by sets to cofibrantly generated abelian model categories.

Let $\mathscr{A}$ be an abelian category with small coproducts. We say that a class of objects $\mathcal{G} \subseteq \mathscr{A}$ is *generating* or that it *generates* $\mathscr{A}$ if any object in $\mathscr{A}$ is the quotient of a set-indexed coproduct of objects in $\mathcal{G}$. An object $G \in \mathscr{A}$ is called a *generator* if $\{G\}$ is generating, i.e. if any object in $\mathscr{A}$ is a quotient of $G^{\coprod I}$ for some large enough set I (for a comparison to other definitions of generators and generating sets, see [KS06, Proposition 5.2.4]). We call $\mathscr{A}$ an *(AB5)-category* if small colimits exist in $\mathscr{A}$ and if filtered colimits are exact, and we say that $\mathscr{A}$ is a *Grothendieck category* if, in addition to being (AB5), it admits a generating set of objects (or equivalently, a generator). Note that in a Grothendieck category a class of objects is generating if and only if it contains a generating set. We refer to [KS06] for generalities on Grothendieck categories. For example, any Grothendieck category possesses arbitrary small limits [KS06, Proposition 8.3.27(i)] and has enough injectives [KS06, Theorem 9.6.2].

From now on let $\mathscr{A}$ be a Grothendieck category. A cotorsion pair $(\mathcal{D}, \mathcal{E})$ in $\mathscr{A}$ is said to be *cogenerated by a set* if there exists a set $\mathcal{S} \subset \mathcal{D}$ such that $\mathcal{E} = \mathcal{S}^{\perp}$. Any set of objects $\mathcal{S}$ serves as the cogenerating set for a unique cotorsion pair, namely $({}^{\perp}(\mathcal{S}^{\perp}), \mathcal{S}^{\perp})$. Although trivial, this is a useful method for constructing cotorsion pairs. In order to get abelian model structures, however, a criterion is needed to check when cotorsion pairs cogenerated by certain sets of objects are complete, which is provided by the following proposition. The important special case of module categories was first treated in [ET01, Theorem 10] and lead to the resolution of the *flat cover conjecture* in [BEBE01]; see Example II.2.2.8 below.

Proposition II.2.2.1 [SŠ11]**.** *Let $\mathscr{A}$ be a Grothendieck category and $(\mathcal{D}, \mathcal{E})$ be a cotorsion pair cogenerated by a set. Then the following hold:*

(i) $(\mathcal{D}, \mathcal{E})$ *has enough injectives.*

(ii) $(\mathcal{D}, \mathcal{E})$ *has enough projectives if and only if $\mathcal{D}$ is generating.*

Proof. Part (i) and the implication "$\Leftarrow$" in (ii) follow from Quillen's small object argument and are explained very clearly in [SŠ11, Theorem 2.13] in the bigger generality of efficient exact categories (of which Grothendieck categories are examples by [SŠ11, Proposition 2.7]). It remains to check the implication "$\Rightarrow$" in (ii): Assuming $(\mathcal{D}, \mathcal{E})$ is complete, let $G \in \mathscr{A}$ be a generator of $\mathscr{A}$ and pick a short exact sequence $0 \to E \to D \to G \to 0$ with $E \in \mathcal{E}$ and $D \in \mathcal{D}$. Then D is a generator for $\mathscr{A}$, too, so $\mathcal{D}$ is generating. □

A cotorsion pair $(\mathcal{D}, \mathcal{E})$ is called *small* if it is cogenerated by a set and if $\mathcal{D}$ is generating. The notion of small cotorsion pairs was introduced in [Hov02, Definition 6.4] in the study of completeness of cotorsion pairs cogenerated by sets. The definition given here differs from Hovey's in that we do not assume condition (iii) of loc.cit. However, in our situation that condition (iii) is automatic by [SŠ11, Proposition 2.7]. In case our underlying category $\mathscr{A}$ has enough projectives (as for example in the cases of modules over dg rings we will be studying later) any cotorsion pair cogenerated by a set is automatically small:

Corollary II.2.2.2. *Let $\mathscr{A}$ be a Grothendieck category with enough projectives. Then any cotorsion pair cogenerated by a set is small, and in particular complete.*

Proof. Since $\mathscr{A}$ has enough projectives it admits a projective generator. In particular, the class of projectives is generating, and hence so is any cotorsion class. The second part follows from Proposition II.2.2.1. □

Proposition II.2.2.1 and Corollary II.2.2.2 allow for proving that a certain class $\mathcal{E}$ arises as the cotorsionfree part of a complete cotorsion pair. To give criteria when a class $\mathcal{D}$ arises as the cotorsion part in a complete cotorsion pair, we need a more concrete description of ${}^{\perp}(\mathcal{S}^{\perp})$ for a cogenerating set $\mathcal{S} \subseteq \mathscr{A}$. For this, we recall the notion of an $\mathcal{S}$-filtration.

Definition II.2.2.3 [Što13, Definition 1.3]**.** *Let $\mathscr{A}$ be a Grothendieck category, $\mathcal{S}$ a class of objects in $\mathscr{A}$ and $X \in \mathscr{A}$. An* $\mathcal{S}$-filtration *on X consists of an ordinal τ together with a family $\{X_\sigma\}_{\sigma \leq \tau}$ of subobjects of X such that the following hold:*

(i) $X_0 = 0$, $X_\tau = X$ *and* $X_\mu \subseteq X_\sigma$ *if* $\mu \leq \sigma \leq \tau$.

(ii) *If* $\sigma \leq \tau$ *is a limit ordinal,* $X_\sigma = \sum_{\mu<\sigma} X_\mu$.

(iii) $X_{\sigma+1}/X_\sigma$ *is isomorphic to an object in* $\mathcal{S}$ *for all* $\sigma < \tau$.

The size *of such an $\mathcal{S}$-filtration is $|\tau|$. The class of objects admitting an $\mathcal{S}$-filtration is denoted* filt-$\mathcal{S}$*, and its closure under taking summands is denoted* ${}^{\oplus}$filt-$\mathcal{S}$*. A class $\mathcal{F} \subset \mathscr{A}$ of the form $\mathcal{F} =$* filt-$\mathcal{S}$ *for some set $\mathcal{S} \subset \mathscr{A}$ is called* deconstructible.

Proposition II.2.2.4. *Let $\mathscr{A}$ be a Grothendieck category and $\mathcal{S} \subseteq \mathscr{A}$ be a set of objects. Assume that* filt-$\mathcal{S}$ *is a generating class for $\mathscr{A}$. Then* ${}^{\perp}(\mathcal{S}^{\perp}) = {}^{\oplus}$filt-$\mathcal{S}$.

Proof. This is also part of [SŠ11, Theorem 2.13]. □

Remark II.2.2.5. The related question of describing the double orthogonal $({}^{\perp}\mathcal{S})^{\perp}$ in elementary terms as in Proposition II.2.2.4 cannot be settled within standard set theory: For example, taking $\mathscr{A} = \mathbb{Z}$-Mod and $\mathcal{S} := \{\mathbb{Z}\}$, the question whether ${}^{\perp}\mathcal{S} = \mathrm{Free}_{\mathbb{Z}}$ is the *Whitehead problem*, known to be independent of ZFC+GCH by the work of Shelah

[She74; She77; She80]. For general rings, it is known [ES91, Corollary 2.2], [Trl96, Theorem 2.5] that it is consistent with ZFC+GCH to assume that no non right-perfect ring R possesses a test-module for projectivity, i.e. some $M \in R\text{-Mod}$ such that $X \in R\text{-Mod}$ is projective if and only if $\operatorname{Ext}^1_R(X, M) = 0$. Finally, concerning the completeness of $({}^\perp\mathcal{S}, ({}^\perp\mathcal{S})^\perp)$, it is known [ES03, Theorem 0.4] that it is consistent with ZFC+GCH to assume that $\mathbb{Q}$ does not have a precover with respective to $({}^\perp\{\mathbb{Z}\}, ({}^\perp\{\mathbb{Z}\})^\perp)$. ◊

Proposition II.2.2.6. *Let $\mathscr{A}$ be a Grothendieck category and let $\mathcal{D} \subseteq \mathscr{A}$ be some class of objects. Then the following are equivalent:*

(i) *$\mathcal{D}$ arises as the cotorsion part in a small cotorsion pair.*

(ii) *$\mathcal{D}$ is generating and $\mathcal{D} = {}^\oplus\operatorname{filt-}\mathcal{S}$ for a set of objects $\mathcal{S}$.*

(iii) *$\mathcal{D}$ is generating, closed under direct summands, and deconstructible.*

Proof. (i)⇒(ii) Suppose $(\mathcal{D}, \mathcal{E})$ a small cotorsion pair cogenerated by some set $\mathcal{S} \subseteq \mathcal{D}$, i.e. $\mathcal{E} = \mathcal{S}^\perp$. By definition, $\mathcal{D}$ is generating and hence we may without loss of generality assume that $\mathcal{S}$ is generating, too (otherwise enlarge $\mathcal{S}$ by a set of generators of $\mathscr{A}$ inside $\mathcal{D}$). We then get $\mathcal{D} = {}^\perp\mathcal{E} = {}^\perp(\mathcal{S}^\perp) = {}^\oplus\operatorname{filt-}\mathcal{S}$ by Proposition II.2.2.4. (ii)⇒(i): If $\mathcal{D} = {}^\oplus\operatorname{filt-}\mathcal{S}$ and $\mathcal{D}$ is generating, then so is $\operatorname{filt-}\mathcal{S}$. Hence Propositions II.2.2.4 and II.2.2.1 yield the small cotorsion pair $({}^\perp(\mathcal{S}^\perp), \mathcal{S}^\perp) = ({}^\oplus\operatorname{filt-}\mathcal{S}, \mathcal{S}^\perp) = (\mathcal{D}, \mathcal{S}^\perp)$. This shows (i)⇔(ii). (iii)⇒(ii) is clear and finally (ii)⇒(iii) follows from [Šťo13, Proposition 2.9(1)] which says that given any deconstructible class in a Grothendieck category, the class of direct summands of objects of this class is again deconstructible. □

Example II.2.2.7. Let $\mathscr{A}$ be a Grothendieck category.

(i) Suppose G is generator of $\mathscr{A}$ and let $\mathcal{S}$ be a representative set of isomorphism classes of quotients of G. Then $\mathscr{A} = \operatorname{filt-}\mathcal{S}$, so $\mathscr{A}$ is deconstructible. As $\mathscr{A}$ itself is clearly generating, we deduce from Proposition II.2.2.6 that $(\mathscr{A}, \mathcal{I}(\mathscr{A}))$ is a complete cotorsion pair, i.e. that $\mathscr{A}$ has enough injectives.

(ii) Assume that $\mathscr{A}$ has enough projectives. Then $\mathcal{P}(\mathscr{A})$ is generating, and hence the cotorsion pair $(\mathcal{P}(\mathscr{A}), \mathscr{A})$ is small. Applying Proposition II.2.2.6 shows that $\mathcal{P}(\mathscr{A})$ is deconstructible.

◊

Example II.2.2.8. If R is any ring and $\kappa \geq \max\{\aleph_0, |R|\}$ a cardinal, then the class $\operatorname{flat}(R)$ of flat left R-modules is κ-deconstructible [ET01, Lemma 1]. As it is also generating and closed under summands, we conclude from Proposition II.2.2.6 that $(\operatorname{flat}(R), \operatorname{flat}(R)^\perp)$ is a complete cotorsion pair. As said above, this was done in [BEBE01] and settled the flat cover conjecture affirmatively. ◊

We end the section by recalling that cotorsion pairs cogenerated by sets are also relevant because of their relation to the cofibrant generation of abelian model structures, as is shown in the following Theorem of Hovey.

Proposition II.2.2.9. *Let $\mathscr{A}$ be a Grothendieck category and let $\mathcal{M} = (\mathcal{C}, \mathcal{W}, \mathcal{F})$ be an abelian model structure on $\mathscr{A}$. Then the following are equivalent:*

(i) $\mathcal{M}$ *is cofibrantly generated.*

(ii) $(\mathcal{C} \cap \mathcal{W}, \mathcal{F})$ *and* $(\mathcal{C}, \mathcal{W} \cap \mathcal{F})$ *are small.*

Proof. "$\Leftarrow$" is proved in [Hov02, Lemma 6.7]. "$\Rightarrow$" is [Hov07, Lemma 3.1]; however, it is stated there without proof, so we give an argument for convenience of the reader. Suppose $\mathcal{M}$ is cofibrantly generated with a generating set of cofibrations $I \subseteq \mathrm{Cof}$ and a generating set of trivial cofibrations $J \subset \mathrm{Cof} \cap \mathrm{W}$, and put $\mathcal{S} := \{\mathrm{coker}(f) \mid f \in I\}$. As cofibrations are monomorphisms with cofibrant cokernel, we have $\mathcal{S} \subseteq \mathcal{C}$, and we claim that $\mathcal{S}^{\perp} = \mathcal{F} \cap \mathcal{W}$. Indeed, if $X \in \mathcal{S}^{\perp}$, then $X \to 0$ has the right lifting property with respect to all $f \in I$, and hence is a trivial fibration by assumption. In other words, $X \in \mathcal{W} \cap \mathcal{F}$ as claimed. Similarly one shows $\mathcal{F} = \mathcal{T}^{\perp}$ for $\mathcal{T} := \{\mathrm{coker}(g) \mid g \in J\} \subseteq \mathcal{C} \cap \mathcal{W}$. □

In particular, Proposition II.2.2.9 shows that in case $\mathscr{A}$ has enough projectives $\mathcal{M} \leftrightarrow (\mathcal{C}, \mathcal{W}, \mathcal{F})$ gives a one-to-one correspondence between cofibrantly generated abelian model structures on $\mathscr{A}$ and triples $(\mathcal{C}, \mathcal{W}, \mathcal{F})$ such that $\mathcal{W}$ is thick and both $(\mathcal{C} \cap \mathcal{W}, \mathcal{F})$ and $(\mathcal{C}, \mathcal{W} \cap \mathcal{F})$ are cotorsion pairs cogenerated by sets.

Finally, we note the following relation to the notions of combinatoriality of model categories and well-generatedness of triangulated categories:

Proposition II.2.2.10. *Let $\mathscr{A}$ be a Grothendieck category and $\mathcal{M}$ be a cofibrantly generated, hereditary abelian model structure on $\mathscr{A}$. Then $\mathcal{M}$ is combinatorial, and hence* $\mathrm{Ho}(\mathcal{M})$ *is well-generated as a triangulated category.*

Proof. By definition, a model category is combinatorial if it is cofibrantly generated and its underlying category is accessible [AR94, Definition 2.1]. As any Grothendieck abelian category is accessible ([KS06, Corollary 9.3.6]; see also the Remarks at the beginning of Appendix II.B), the first claim follows. The second is then a consequence of [Ros05, Remark 3.4(1) and Theorem 4.9] stating that the homotopy category of any combinatorial stable model category is well-generated. □

II.2.3. Four model structures on modules over a dg ring

In this section we use the results of the previous section to construct four prominent abelian model structures on the category of modules over a (curved) differential graded ring (dg rings resp. cdg rings for short): Firstly, the standard injective and projective abelian model structures for modules over a dg ring, and secondly, Positselski's coderived and contraderived abelian model structures for modules over a cdg ring.

Notation II.2.3.1. A *grading group* [Pos11, Remark preceding Section 1.2] is an abelian group Γ together with a parity homomorphism $|\cdot| : \Gamma \to \mathbb{Z}/2\mathbb{Z}$ and a distinguished element $1 \in \Gamma$ satisfying $|1| = \overline{1}$. A Γ-graded abelian group is a Γ-indexed family $X^* = \{X^k\}_{k\in\Gamma}$ of abelian groups, but we will often drop the index from the notation. We will also sometimes drop Γ from the notation, in which case it is implicitly assumed that a grading group has been fixed. Given such a Γ-graded abelian group X and some $n \in \Gamma$, we denote $\Sigma^n X = X$ the Γ-graded abelian group given by $(\Sigma^n X)^k := X^{k+n}$ and call it the *n-fold suspension* of X. We also put $\Sigma := \Sigma^1$ and $\Omega := \Sigma^{-1}$. The category of Γ-graded abelian groups has a monoidal structure given by the tensor product $(X \otimes Y)^n := \bigoplus_{p+q=n} X^p \otimes_{\mathbb{Z}} Y^q$; a Γ-*graded ring* is an algebra object in that monoidal category, and a module over such an algebra object is called a Γ-*graded module*. A Γ-*graded curved differential graded ring* (cdg ring for short) is a Γ-graded ring A together with a map $\mathrm{d} : A \to \Sigma A$ of Γ-graded abelian groups called differential and an element $w \in A^2$ such that $\mathrm{d}(w) = 0$, d satisfies the Leibniz rule and for any $x \in A$ we have $\mathrm{d}^2(x) = [w, x]$. The Γ-graded ring underlying a Γ-graded cdg ring A is denoted $A^\sharp$. For a cdg ring A, a *(cdg) module* over A is a Γ-graded module X over $A^\sharp$ together with a map $\mathrm{d} : X \to \Sigma X$ of Γ-graded abelian groups satisfying the Leibniz rule and $\mathrm{d}^2(x) = wx$ for all $x \in X$. Given such an A-module X and $n \in \Gamma$, the n-fold suspension $\Sigma^n X$ carries a natural A-module structure as follows: its differential $\mathrm{d}_{\Sigma^n X}$ is given by $\mathrm{d}_{\Sigma^n X} := (-1)^{|n|}\,\mathrm{d}_X$, and the action of some homogeneous $a \in A$ on some $x \in X$ given by $(-1)^{|a|\cdot|n|}ax$. The $A^\sharp$-module underlying X is denoted $X^\sharp$. Given two A-modules X, Y, the (Γ-indexed) complex of $A^\sharp$-linear homomorphisms $X^\sharp \to \Sigma^* Y^\sharp$ is denoted $\text{dg-Hom}^*_A(X, Y)$: for $k \in \Gamma$, its k-th component is $\mathrm{Hom}_{A^\sharp}(X^\sharp, \Sigma^k Y^\sharp)$, with differential sending $f : X^\sharp \to \Sigma^k Y^\sharp$ to $\partial_Y f - (-1)^{|k|} f \partial_X$. The k-th cohomology $\mathrm{H}^k(\text{dg-Hom}^*_A(X, Y))$ equals the set $[X, \Sigma^k Y]$ of homotopy classes of morphisms $X \to \Sigma^k Y$. Finally, we denote $A\text{-Mod}_{\mathrm{proj}}$ (resp. $A\text{-Mod}_{\mathrm{inj}}$) the class of A-modules whose underlying graded $A^\sharp$-modules are projective (resp. injective). ◊

Recall from [Pos11] the following explicit description of the adjoints of $(-)^\sharp$:

Proposition II.2.3.2 see [Pos11, Proof of Theorem 3.6]**.** *Let A be a cdg ring and define the functors $G^+, G^- : A^\sharp\text{-Mod} \to A\text{-Mod}$ as follows:*

(i) *$G^+(X) := X \oplus \Omega X$ as graded abelian groups. An element $(x, y) \in G^+(X)$ is denoted $x + \mathrm{d}(y)$. The action of some $a \in A$ on $x + \mathrm{d}(y)$ is given by $ax - (-1)^{|a|}\,\mathrm{d}(a)y + (-1)^{|a|}\,\mathrm{d}(ay)$, while the differential on $G^+(X)$ sends $x + \mathrm{d}(y)$ to $wy + \mathrm{d}(x)$.*

(ii) $G^- := \Sigma \circ G^+$.

Then there are canonical adjunctions $G^+ \dashv (-)^\sharp \dashv G^-$.

Note that if A is a dg ring (so that we can talk about homology of A-modules) the images of G^+ and G^- consist of acyclic modules. This follows immediately from the explicit description of $G^\pm$, or alternatively by using the adjunction property: $\mathrm{H}^n(G^-(X)) \cong [A, \Sigma^n G^-(X)] \cong \mathrm{Ext}^1_A(\Omega^{n-1}A, G^-(X)) \cong \mathrm{Ext}^1_{A^\sharp}(\Omega^{n-1}A^\sharp, X) = 0$, where the latter equality holds because $A^\sharp$ is projective in $A^\sharp$-Mod; as $G^+ = \Omega \circ G^-$, this also shows the acyclicity of objects in the image of G^+. Here we have used that, given a cdg ring A and $X \in A\text{-Mod}_{\mathrm{proj}}$, there is a canonical isomorphism $\mathrm{Ext}^1_A(X, -) \cong [\Omega X, -]$. Similarly, if $X \in A\text{-Mod}_{\mathrm{inj}}$, we have $\mathrm{Ext}^1_A(-, X) \cong [-, \Sigma X]$. These isomorphisms will be used very often in what follows. We will also need the following characterization of projective and injective objects in A-Mod:

Lemma II.2.3.3. *Let A be a cdg ring and X an A-module. Then X is projective in A-Mod if and only if $X^\sharp$ is projective in $A^\sharp$-Mod and X is contractible as an A-module. Similarly, X is injective in A-Mod if and only if $X^\sharp$ is injective in $A^\sharp$-Mod and X is contractible as an A-module.*

Proof. For any A-module there is a canonical epimorphism $\mathrm{Cone}(\mathrm{id}_{\Omega X}) \to X$ in A-Mod. Hence, if X is projective in A-Mod, it is a summand of $\mathrm{Cone}(\mathrm{id}_{\Omega X})$ and hence contractible as an A-module. Further, as the forgetful functor $A\text{-Mod} \to A^\sharp\text{-Mod}$ is left adjoint to the exact functor G^- (see Proposition II.2.3.2), it preserves projective objects, and hence one direction is proved. Conversely, assume that $X^\sharp$ is projective in $A^\sharp$-Mod and X is contractible as an A-module. Given another A-module Z, the projectivity of $X^\sharp$ implies that there is a canonical isomorphism $\mathrm{Ext}^1_A(X, Z) \cong [X, \Sigma Z]$, and the latter group is trivial since X is contractible. It follows that X is projective in A-Mod, as claimed.

The part on injective objects in A-Mod is similar. □

Lemma II.2.3.4. *Let A be a cdg ring and $(\mathcal{D}, \mathcal{E})$ be a cotorsion pair with $\Sigma\mathcal{D} \subseteq \mathcal{D}$.*

(i) *If $\mathcal{D} \subseteq A\text{-Mod}_{\mathrm{proj}}$, then $\mathcal{D} \cap \mathcal{E} = \mathcal{P}(A\text{-Mod})$.*

(ii) *If $\mathcal{E} \subseteq A\text{-Mod}_{\mathrm{inj}}$, then $\mathcal{D} \cap \mathcal{E} = \mathcal{I}(A\text{-Mod})$.*

Proof. We only prove (i), as the proof of (ii) is similar. Assuming $\mathcal{D} \subseteq A\text{-Mod}_{\mathrm{proj}}$, we claim that $\mathcal{D} \cap \mathcal{E} = \mathcal{P}(A\text{-Mod})$. "$\supseteq$": Clearly $\mathcal{P}(A\text{-Mod}) = {}^\perp A\text{-Mod} \subseteq {}^\perp\mathcal{E} = \mathcal{D}$. Moreover, if $X \in \mathcal{P}(A\text{-Mod})$ and $Z \in \mathcal{D} \subseteq A\text{-Mod}_{\mathrm{proj}}$, we have $\mathrm{Ext}^1_A(Z, X) \cong [Z, \Sigma X] = 0$ since X is contractible (Lemma II.2.3.3). This shows $\mathcal{P}(A\text{-Mod}) \subseteq \mathcal{D}^\perp = \mathcal{E}$, and hence $\mathcal{P}(A\text{-Mod}) \subseteq \mathcal{D} \cap \mathcal{E}$. "$\subseteq$": By Lemma II.2.3.3 and the assumption that $\mathcal{D} \subseteq A\text{-Mod}_{\mathrm{proj}}$

it suffices to show that any $X \in \mathcal{D} \cap \mathcal{D}^{\perp}$ is contractible as an A-module. Using that $\Sigma\mathcal{D} \subseteq \mathcal{D}$ by assumption, this follows from $0 = \operatorname{Ext}^1_A(\Sigma X, X) \cong [\Sigma X, \Sigma X]$. □

Proposition II.2.3.5. *For a dg ring A, the following hold:*

(i) *There exists a unique projective abelian model structure $\mathcal{M}^{\text{proj}}(A)$ on A-*Mod *with $\mathcal{W} = \operatorname{Acyc}(A)$, called the* standard projective model structure *on A-*Mod*. The class $\mathcal{C}^{\text{proj}}(A)$ of cofibrant objects in $\mathcal{M}^{\text{proj}}(A)$ is contained in A-*Mod_{proj}.

(ii) *There exists a unique injective abelian model structure $\mathcal{M}^{\text{inj}}(A)$ on A-*Mod *with $\mathcal{W} = \operatorname{Acyc}(A)$, called the* standard injective model structure *on A-*Mod*. The class $\mathcal{F}^{\text{inj}}(A)$ of fibrant objects in $\mathcal{M}^{\text{inj}}(A)$ is contained in A-*Mod_{inj}.

Moreover, $\mathcal{M}^{\text{proj}}(A)$ and $\mathcal{M}^{\text{inj}}(A)$ are cofibrantly generated. Their common homotopy category is denoted $\mathbf{D}(A)$ and called the derived category of A-modules.

Proof. (i) Let $\mathcal{S} := \{\Sigma^n A \mid n \in \Gamma\}$. For any $n \in \Gamma$ and any $X \in A$-Mod we have a canonical isomorphism $\operatorname{Ext}^1_A(\Omega^n A, X) \cong [A, \Sigma^{n+1} X] \cong \mathrm{H}^{n+1}(X)$, so it follows that $\mathcal{S}^{\perp} = \operatorname{Acyc}(A)$. Hence, by Corollary II.2.2.2, the cotorsion pair $({}^{\perp}\operatorname{Acyc}, \operatorname{Acyc})$ is complete. By Corollary II.2.1.16 and the thickness of $\operatorname{Acyc}(A)$ it remains to show that ${}^{\perp}\operatorname{Acyc} \cap \operatorname{Acyc} = \mathcal{P}(A\text{-Mod})$, so that by Lemma II.2.3.4 it suffices to show that ${}^{\perp}\operatorname{Acyc} \subseteq A\text{-Mod}_{\text{proj}}$. For this, note that for any $X \in {}^{\perp}\operatorname{Acyc}$ and any $Z \in A^{\sharp}$-Mod, we have $0 = \operatorname{Ext}^1_A(X, G^-(Z)) \cong \operatorname{Ext}^1_{A^{\sharp}}(X^{\sharp}, Z)$, so that $X^{\sharp}$ is projective in $A^{\sharp}$-Mod as claimed. Here we used that the image of G^- consists of acyclic A-modules.

(ii) By Corollary II.2.1.16 and Proposition II.2.2.6 it suffices to show that $\operatorname{Acyc}(A)$ is generating and deconstructible, and that $\operatorname{Acyc}(A) \cap \operatorname{Acyc}(A)^{\perp} = \mathcal{I}(A\text{-Mod})$. By Lemma II.2.3.3 $\mathcal{P}(A\text{-Mod}) \subseteq \operatorname{Acyc}(A)$, so $\operatorname{Acyc}(A)$ is generating. The deconstructibility of $\operatorname{Acyc}(A)$ follows from Theorem II.B.11 applied to the monadic forgetful functor : $A\text{-Mod} \to \operatorname{Ch}_{\Gamma}(\mathbb{Z})$ and the fact [Šťo13, Theorem 4.2.(2)] that $\operatorname{Acyc}(\mathbb{Z}) \subset \operatorname{Ch}_{\Gamma}(\mathbb{Z})$ is deconstructible (in loc.cit. the result is proved for $\Gamma = \mathbb{Z}$, but the arguments carry over to the case of a general grading group). Finally, the equality $\operatorname{Acyc}(A) \cap \operatorname{Acyc}(A)^{\perp} = \mathcal{I}(A\text{-Mod})$ again follows from Lemma II.2.3.4 once we've showed that for any $X \in \operatorname{Acyc}(A)^{\perp}$ its underlying $A^{\sharp}$-module $X^{\sharp}$ is injective. Indeed, if $Z \in A^{\sharp}$-Mod, we have $0 = \operatorname{Ext}^1_A(G^+(Z), X) \cong \operatorname{Ext}^1_{A^{\sharp}}(Z, X^{\sharp})$, where the first equality holds because the image of G^+ consists of acyclic A-modules.

The statement about cofibrant generation follows from Proposition II.2.2.9. □

Proposition II.2.3.6. *For a cdg ring A, the following hold:*

(i) *There exists a unique projective abelian model structure $\mathcal{M}^{\text{ctr}}(A)$ on A-*Mod *such that $\mathcal{C} = A$-*Mod_{proj}*. $\mathcal{M}^{\text{ctr}}(A)$ is called the* contraderived model structure.

(ii) *There exists a unique injective abelian model structure* $\mathcal{M}^{\mathrm{co}}(A)$ *on* A-Mod *such that* $\mathcal{F} = A\text{-Mod}_{\mathrm{inj}}$. $\mathcal{M}^{\mathrm{ctr}}(A)$ *is called the* coderived model structure*.*

Moreover, $\mathcal{M}^{\mathrm{ctr}}(A)$ *and* $\mathcal{M}^{\mathrm{co}}(A)$ *are cofibrantly generated. Their homotopy categories* Ho $\mathcal{M}^{\mathrm{ctr}}(A)$ *and* Ho $\mathcal{M}^{\mathrm{co}}(A)$ *are denoted* $\mathbf{D}^{\mathrm{ctr}}(A)$ *and* $\mathbf{D}^{\mathrm{co}}(A)$ *and are called the* coderived and contraderived categories of A-modules*, respectively.*

Proof. (i) By Corollary II.2.1.16 and Proposition II.2.2.6 we have to show that $A\text{-Mod}_{\mathrm{proj}}$ is generating and deconstructible, that $A\text{-Mod}_{\mathrm{proj}} \cap A\text{-Mod}_{\mathrm{proj}}^{\perp} = \mathcal{P}(A\text{-Mod})$ and that $A\text{-Mod}_{\mathrm{proj}}^{\perp}$ has the 2-out-of-3 property. By Lemma II.2.3.3, $\mathcal{P}(A\text{-Mod}) \subseteq A\text{-Mod}_{\mathrm{proj}}$, so $A\text{-Mod}_{\mathrm{proj}}$ is generating. For the deconstructibility of $A\text{-Mod}_{\mathrm{proj}}$, we again apply Theorem II.B.11: The forgetful functor $(-)^\sharp : A\text{-Mod} \to A^\sharp\text{-Mod}$ is monadic, for example by the explicit description of its left adjoint G^+ in Proposition II.2.3.2, and $A\text{-Mod}_{\mathrm{proj}}$ is the preimage under $(-)^\sharp$ of $\mathcal{P}(A^\sharp\text{-Mod})$, which is deconstructible by Example II.2.2.7(ii). Finally, $A\text{-Mod}_{\mathrm{proj}} \cap A\text{-Mod}_{\mathrm{proj}}^{\perp} = \mathcal{P}(A\text{-Mod})$ follows from Lemma II.2.3.4, and the 2-out-of-3 property of $A\text{-Mod}_{\mathrm{proj}}^{\perp}$ is ensured by the dual of Lemma II.2.1.17, using that $A\text{-Mod}_{\mathrm{proj}}$ is closed under kernels of epimorphisms.

(ii) By definition, an A-module X belongs to $A\text{-Mod}_{\mathrm{inj}}$ if and only if $X^\sharp \in \mathcal{I}(A^\sharp\text{-Mod})$, i.e. $0 = \mathrm{Ext}^1_{A^\sharp}(Z, X^\sharp) = \mathrm{Ext}^1_A(G^+(Z), X)$ for all $Z \in A^\sharp\text{-Mod}$. In other words, $A\text{-Mod}_{\mathrm{inj}} = G^+(A^\sharp\text{-Mod})^{\perp}$. Hence, choosing a set $\mathcal{S} \subset A^\sharp\text{-Mod}$ such that $A^\sharp\text{-Mod} = \text{filt-}\mathcal{S}$ we have $A\text{-Mod}_{\mathrm{inj}} = G^+(\mathcal{S})^{\perp}$. We conclude that $({}^{\perp}A_{\mathrm{inj}}, A_{\mathrm{inj}})$ is a complete cotorsion pair by Corollary II.2.2.2. As above, ${}^{\perp}A_{\mathrm{inj}} \cap A_{\mathrm{inj}} = \mathcal{I}(A\text{-Mod})$ follows from Lemma II.2.3.4, and the 2-out-of-3 property of ${}^{\perp}A\text{-Mod}_{\mathrm{inj}}$ follows from Lemma II.2.1.17 together with the fact that $A\text{-Mod}_{\mathrm{inj}}$ is closed under cokernels of monomorphisms.

The cofibrant generation follows from Proposition II.2.2.9. □

Corollary II.2.3.7. *Let A be a cdg ring.*

(i) *The identity is a left Quillen functor* $\mathcal{M}^{\mathrm{ctr}}(A) \to \mathcal{M}^{\mathrm{co}}(A)$*.*

(ii) *If A is a dg ring, the identity on* A-Mod *is a left Quillen functor* $\mathcal{M}^{\mathrm{proj}}(A) \to \mathcal{M}^{\mathrm{ctr}}(A)$ *and a right Quillen functor* $\mathcal{M}^{\mathrm{inj}}(A) \to \mathcal{M}^{\mathrm{co}}(A)$*.*

Proof. (i) is clear, and (ii) means that we have $\mathcal{C}^{\mathrm{proj}}(A) \subseteq A\text{-Mod}_{\mathrm{proj}}$ and $\mathcal{F}^{\mathrm{inj}}(A) \subseteq A\text{-Mod}_{\mathrm{inj}}$, which was shown in Proposition II.2.3.5. □

Following [Pos11], weakly trivial objects in $\mathcal{M}^{\mathrm{co}}(A)$ are called *coacyclic*, while weakly trivial objects in $\mathcal{M}^{\mathrm{ctr}}(A)$ are called *contraacyclic*. We denote them $\mathcal{W}^{\mathrm{co}}(A)$ and $\mathcal{W}^{\mathrm{ctr}}(A)$, respectively. If A is a dg ring, then Corollary II.2.3.7 implies that $\mathcal{W}^{\mathrm{co}}(A) \subseteq \mathrm{Acyc}(A) \supseteq \mathcal{W}^{\mathrm{ctr}}(A)$, so coacyclic and contraacyclic modules are in particular acyclic in the classical sense. In general, we can only give the following description:

Proposition II.2.3.8. *Let A be a dg ring and $X \in A\text{-Mod}$.*

(i) *X is contraacyclic if and only if for each $Z \in A\text{-Mod}_{\text{proj}}$ the homomorphism complex* $\text{dg-Hom}_A^*(Z, X)$ *is acyclic, if and only if* $[Z, X] = 0$ *for all* $Z \in A\text{-Mod}_{\text{proj}}$.

(ii) *X is coacyclic if and only if for each $Z \in A\text{-Mod}_{\text{inj}}$ the homomorphism complex* $\text{dg-Hom}_A^*(X, Z)$ *is acyclic if and only if* $[X, Z] = 0$ *for all* $Z \in A\text{-Mod}_{\text{inj}}$.

In particular, any contractible A-module is both contraacyclic and coacyclic.

Proof. (i) follows from $\text{Ext}_A^1(Z, -) \cong [\Omega Z, -]$ for $Z \in A\text{-Mod}_{\text{proj}}$ and the isomorphism $\text{H}^k[\text{dg-Hom}_A^*(X, Y)] \cong [X, \Sigma^k Y]$, and (ii) follows using $\text{Ext}_A^1(-, Z) \cong [-, \Sigma Z]$ for $Z \in A\text{-Mod}_{\text{inj}}$. ☐

Remark II.2.3.9. Proposition II.2.3.8 implies that the class $\mathcal{W}^{\text{ctr}}$ of contraacyclic modules over a cdg ring A is closed under products, while the class $\mathcal{W}^{\text{co}}$ of coacyclic modules is closed under coproducts (and in general, not vice versa, see Example II.2.3.16); still, if A is a dg algebra, then $\mathcal{W}^{\text{ctr}}$ and $\mathcal{W}^{\text{co}}$ are both contained in the class $\text{Acyc}(A)$ which *is* closed both under products and coproducts and which ensures $\text{Ho}\,\mathcal{M}^{\text{co}}(A) \neq 0 \neq \text{Ho}\,\mathcal{M}^{\text{ctr}}(A) \neq 0$ if $\text{H}(A) \neq 0$. However, if A has nonzero curvature and $\text{Acyc}(A)$ is not at our disposal, it can happen that there is no nontrivial notion of "acyclicity" for modules over a cdg ring A such that firstly all contractible A-modules are "acyclic", secondly the totalization of any short exact sequence of A-modules is "acyclic" and thirdly the class of "acyclic" modules is closed both under products and under products; this was studied in [KLN10]. For example (see [KLN10, Proposition 3.2]), if $\Bbbk$ is a field, then the initial $\mathbb{Z}$-graded cdg $\Bbbk$-algebra $A := \Bbbk[c]$, with $\deg(c) := 2$ and curvature c, does not admit any nontrivial notion of "acyclicity" having the above properties: in fact, already $\mathcal{W}^{\text{ctr}} = \mathcal{W}^{\text{co}} = A\text{-Mod}$ in this example. ◊

Lemma II.2.3.10. *Let A be a cdg ring and $\ldots \xrightarrow{p_2} X_1 \xrightarrow{p_1} X_0$ be an inverse system of contraacyclic A-modules with all p_n being epimorphisms. Then $\varprojlim X_n$ is A-contraacyclic, too. In particular, the totalization formed by taking products of any bounded above exact sequence of A-modules is contraacyclic.*

Proof. The first statement follows from the existence of a short exact sequence $0 \to \varprojlim X_n \to \prod X_n \to \prod X_n \to \varprojlim^1 X_n = 0$ (the surjectivity of p_n implies the Mittag-Leffler condition for the inverse system of the p_n) and the fact that $\mathcal{W}^{\text{ctr}}(A)$ satisfies the 2-out-of-3 property. It remains to show that the totalization $\text{Tot}^{\Pi}(X_*)$ formed by taking products of an exact, bounded above sequence of A-modules $\ldots \xrightarrow{f_3} X_2 \xrightarrow{f_2} X_1 \xrightarrow{f_1} X_0$ is contraacyclic, which is essentially a special case of the first statement: $\text{Tot}^{\Pi}(X_*)$ is the inverse limit of the totalizations of the soft truncations $0 \to X_n/\operatorname{im}(f_{n+1}) \to X_{n-1} \to$

$\ldots \to X_1 \to X_0$, which in turn are iterated extensions of contractible A-modules, hence contraacyclic by Proposition II.2.3.8. □

Dually, we have:

Lemma II.2.3.11. *Let A be a cdg ring and $X_0 \overset{f_0}{\to} X_1 \overset{f_1}{\to} \ldots$ be a direct system of coacyclic A-modules. Then $\varinjlim X_n$ is A-coacyclic, too. In particular, the totalization formed by taking sums of any bounded below exact sequence of A-modules is coacyclic.*

Proof. Again, the first statement follows from the existence of a short exact sequence $0 \to \bigoplus_n X_n \to \bigoplus_n X_n \to \varinjlim X_n \to 0$ (in contrast to the situation for inverse limits, no condition on the f_n is needed here) and the 2-out-of-3 property of $\mathcal{W}^{\mathrm{ctr}}(A)$. For the second statement, the totalization $\mathrm{Tot}^{\oplus}(X_*)$ by sums of an bounded below exact sequence of A-modules $X_0 \overset{f_1}{\to} X_1 \overset{f_2}{\to} X_2 \to \ldots$ is the direct limit of the totalizations of the soft truncations $X_0 \to X_1 \to \ldots \to X_{n-1} \to \ker(f_n) \to 0 \to \ldots$ which are iterated extensions of contractible A-modules, hence coacyclic by Proposition II.2.3.8. □

Example II.2.3.12. We give an example for the Quillen adjunction $\mathcal{M}^{\mathrm{ctr}}(A) \rightleftarrows \mathcal{M}^{\mathrm{co}}(A)$ from Corollary II.2.3.7. Suppose $A = R$ is a commutative Noetherian ring considered as a dg ring concentrated in degree 0, and let $R \overset{\eta}{\to} I^0 \to I^1 \to \ldots$ be an injective resolution of R over itself. Then the left derived functor $\mathbf{K}(\mathrm{Proj}(R)) \to \mathbf{K}(\mathrm{Inj}(R))$ of the identity left Quillen functor $\mathrm{id} : \mathcal{M}^{\mathrm{ctr}}(R) \to \mathcal{M}^{\mathrm{co}}(R)$ from Corollary II.2.3.7 is given by the tensor product $-\otimes_R I^* : \mathbf{K}(\mathrm{Proj}(R)) \to \mathbf{K}(\mathrm{Inj}(R))$: Firstly, if $X \in \mathrm{Ch}(\mathrm{Proj}(R))$ is a complex of projectives, then $X \otimes_R I^*$ is a complex of injectives since R is assumed to be Noetherian and the class of injective R-modules is therefore stable under arbitrary coproducts. Further, the cokernel of the natural map $X = X \otimes_R R \to X \otimes_R I^*$ is the totalization by sums of the bounded below and acyclic complex $X \otimes_R \mathrm{coker}(\eta) \to X \otimes_R I^1 \to \ldots$ of complexes of R-modules, and hence coacyclic by Lemma II.2.3.11 above. In other words, the embedding $X \to X \otimes_R I^*$ is a fibrant resolution of X in $\mathcal{M}^{\mathrm{co}}(R)$, and hence $X \otimes_R I^*$ represents the image of X under $\mathbf{L}\,\mathrm{id} : \mathbf{D}^{\mathrm{ctr}}(R) \to \mathbf{D}^{\mathrm{co}}(R) \cong \mathbf{K}(\mathrm{Inj}(R))$. See also Examples II.2.3.15, II.4.2.4 and II.4.4.6. ♢

In case some mild conditions on $A^\sharp$ is satisfied, Positselski gives the following description of coacyclic and contraacyclic modules:

Proposition II.2.3.13 [Pos11, Theorem 3.7, 3.8]**.** *Let A be a cdg ring.*

(i) *Suppose countable products of projective $A^\sharp$-modules have finite projective dimension. Then $\mathcal{W}^{\mathrm{ctr}}(A)$ is the smallest thick triangulated subcategory of $\mathbf{K}(A\text{-Mod})$ closed under products and containing totalizations of exact sequences of A-modules.*

(ii) *Suppose countable sums of injective $A^\sharp$-modules have finite injective dimension. Then $\mathcal{W}^{\mathrm{co}}(A)$ is the smallest thick triangulated subcategory of $\mathbf{K}(A\text{-Mod})$ closed under coproducts and containing totalizations of exact sequences of A-modules.*

For the next proposition, we call A *left Gorenstein* if the class of graded left $A^\sharp$-modules of finite projective dimension coincides with the class of graded left $A^\sharp$-modules of finite injective dimension. For example, any Gorenstein ring in the sense of Section II.5.1 is left Gorenstein when considered as a dg ring concentrated in degree 0.

Proposition II.2.3.14 [Pos11, see Section 3.9]**.** *If A is left Gorenstein, then the identity left Quillen adjunction $\mathcal{M}^{\mathrm{ctr}}(A) \rightleftarrows \mathcal{M}^{\mathrm{co}}(A)$ is a Quillen equivalence.*

Proof. The essential ideas are contained in [Pos11, Section 3.9], where the existence of an equivalence $\mathrm{Ho}(A\text{-Mod}_{\mathrm{inj}}) \cong \mathrm{Ho}(A\text{-Mod}_{\mathrm{proj}})$ is shown. For our formulation, the following observation suffices: Any $X \in A\text{-Mod}_{\mathrm{proj}}$ admits a finite resolution $0 \to X \to Y^0 \to Y^1 \to \ldots \to Y^n \to 0$ in A-Mod with $Y^i \in A\text{-Mod}_{\mathrm{inj}}$ (take a high enough truncation of an injective resolution of X in A-Mod), and consequently $X \to \mathrm{Tot}(Y^*)$ is both a contraderived and a coderived weak equivalence between X and $\mathrm{Tot}(Y^*) \in A\text{-Mod}_{\mathrm{inj}}$. Dually, any $Y \in A\text{-Mod}_{\mathrm{inj}}$ admits a finite resolution $0 \to X^n \to X^{n-1} \to \ldots \to X^0 \to Y \to 0$ with $X^i \in A\text{-Mod}_{\mathrm{proj}}$, and then $\mathrm{Tot}(X^*) \to Y$ is both a contraderived and a coderived weak equivalence between X and $\mathrm{Tot}(X^*) \in A\text{-Mod}_{\mathrm{proj}}$. □

Example II.2.3.15. In case $A = R$ is a commutative Gorenstein ring with injective resolution $D \in \mathrm{Ch}(\mathrm{Inj}(R))$ of R over itself, we know from Example II.2.3.12 that

$$\begin{array}{ccc} \mathrm{Ho}(\mathcal{M}^{\mathrm{ctr}}(R)) & \xrightarrow{\mathbf{L}\,\mathrm{id}} & \mathrm{Ho}(\mathcal{M}^{\mathrm{co}}(R)) \\ \uparrow \cong & & \uparrow \cong \\ \mathbf{K}(\mathrm{Proj}(R)) & \xrightarrow{-\otimes_R D} & \mathbf{K}(\mathrm{Inj}(R)) \end{array}$$

commutes up to isomorphism, so that by Proposition II.2.3.14 the functor

$$-\otimes_R D : \mathbf{K}(\mathrm{Proj}(R)) \longrightarrow \mathbf{K}(\mathrm{Inj}(R))$$

is an equivalence of triangulated categories. This was shown in the greater generality of commutative Noetherian rings admitting a dualizing complex in [IK06, Theorem I]. For the restriction to acyclic complexes, see Examples II.4.2.4 and II.4.4.6. Also, note that our proof differs significantly from the one in [IK06] where the strategy is to use the compact generation of $\mathbf{K}(\mathrm{Proj}(R))$ and $\mathbf{K}(\mathrm{Inj}(R))$ to reduce the claim to proving that $-\otimes_R D$ restricts to an equivalence $\mathbf{K}^c(\mathrm{Proj}(R)) \to \mathbf{K}^c(\mathrm{Inj}(R))$. ◊

However, Proposition II.2.3.14 do *not* say that the coderived and contraderived weak equivalences coincide – this is indeed false, as the following example shows. For a generalization to arbitrary non-regular Gorenstein rings, see Corollary II.5.1.4.

Example II.2.3.16. Consider the following complexes over $A = k[\varepsilon]/(\varepsilon^2)$, k a field:

$$\begin{aligned} Y &:= \ldots \xrightarrow{\varepsilon} A \xrightarrow{\varepsilon} A \xrightarrow{\varepsilon} A \xrightarrow{\varepsilon} A \xrightarrow{\varepsilon} A \xrightarrow{\varepsilon} \ldots \\ X &:= \ldots \xrightarrow{\varepsilon} A \xrightarrow{\varepsilon} A \xrightarrow{\varepsilon} k \xrightarrow{0} 0 \xrightarrow{0} 0 \xrightarrow{0} \ldots \\ Z &:= \ldots \xrightarrow{0} 0 \xrightarrow{0} 0 \xrightarrow{0} k \xrightarrow{\varepsilon} A \xrightarrow{\varepsilon} A \xrightarrow{\varepsilon} \ldots \end{aligned}$$

Here k lives in cohomological degree 0. Since X is acyclic and bounded above, it is contraacyclic; similarly, since Z is acyclic and bounded below, it is coacyclic. See [Pos11, Section 3.4] for both statements. However, Y is neither co- nor contraacyclic, for if it was, we would have $[Y, Y] = 0$ (since Y has projective-injective components), meaning that Y was contractible, which is not true ($Y \otimes_A k$ has nonzero cohomology). Since the classes of coacyclics and contraacyclics both satisfy the 2-out-of-3 property and we have a short exact sequence $0 \to X \to Y \to Z \to 0$, we conclude that X is contraacyclic but not coacyclic, while Z is coacyclic but not contraacyclic. ♢

The next proposition is contained in greater generality in [Pos11, Section 3.6]. Restricting to ordinary rings here, we give a direct proof in the setting of abelian categories.

Proposition II.2.3.17. *If R is an ordinary ring of finite left-global dimension, then $\mathcal{M}^{\mathrm{ctr}}(R) = \mathcal{M}^{\mathrm{proj}}(R)$ and $\mathcal{M}^{\mathrm{co}}(R) = \mathcal{M}^{\mathrm{inj}}(R)$.*

Proof. By Corollary II.2.3.7 we have $\mathcal{C}^{\mathrm{proj}}(R) \subseteq \mathcal{C}^{\mathrm{ctr}}(R)$, so it suffices to show the reverse inclusion, i.e. that for any $X \in \mathrm{Ch}_\Gamma(\mathrm{Proj}(R))$ we have $X \in {}^\perp\mathrm{Acyc}(R)$. Suppose first that $X \in \mathrm{Ch}_\Gamma(\mathrm{Proj}(R)) \cap \mathrm{Acyc}(R)$. Since $\mathrm{gl.dim}(R\text{-Mod}) < \infty$ by assumption, the syzygies $Z^n(X)$ of X are projective in this case, so X is contractible. By Lemma II.2.3.3, it follows that $X \in \mathcal{P}(\mathrm{Ch}_\Gamma(R)) \subseteq {}^\perp\mathrm{Acyc}(R)$ as claimed. In the general case, pick a cofibrant resolution $p : P \to X$ in $\mathcal{M}^{\mathrm{proj}}(R)$, i.e. p is an epimorphism with $K := \ker(p) \in \mathrm{Acyc}(R)$ and $P \in \mathcal{C}^{\mathrm{proj}}(R)$. As the components of X are projective, p is degree-wise split, so $K \in \mathrm{Acyc}(R) \cap \mathrm{Ch}_\Gamma(\mathrm{Proj}(R)) \subseteq {}^\perp\mathrm{Acyc}(R)$ by the first case. Moreover, applying $\text{dg-Hom}^*_R(-, Z)$ to $0 \to K \to P \to X \to 0$ for $Z \in \mathrm{Acyc}(R)$ and taking cohomology shows $[X, Z] = 0$. The proof of $\mathcal{M}^{\mathrm{co}}(R) = \mathcal{M}^{\mathrm{inj}}(R)$ is similar. □

Morphisms of dg rings induce Quillen adjunctions between the four models:

Proposition II.2.3.18. *Let $\varphi : R \to A$ be a morphism of dg rings and let $U_\varphi : A\text{-Mod} \to R\text{-Mod}$ be the forgetful functor.*

(i) *$A \otimes_R - \dashv U_\varphi$ is a Quillen adjunction $\mathcal{M}^{\mathrm{proj}}(R) \rightleftarrows \mathcal{M}^{\mathrm{proj}}(A)$.*

(ii) $A \otimes_R - \dashv U_\varphi$ *is a Quillen adjunction* $\mathcal{M}^{\mathrm{ctr}}(R) \rightleftarrows \mathcal{M}^{\mathrm{ctr}}(A)$.

(iii) $U_\varphi \dashv \text{dg-Hom}_R(A, -)$ *is a Quillen adjunction* $\mathcal{M}^{\mathrm{inj}}(A) \rightleftarrows \mathcal{M}^{\mathrm{inj}}(R)$.

(iv) $U_\varphi \dashv \text{dg-Hom}_R(A, -)$ *is a Quillen adjunction* $\mathcal{M}^{\mathrm{co}}(A) \rightleftarrows \mathcal{M}^{\mathrm{co}}(R)$.

(v) *If* $A^\sharp$ *is projective as an* $R^\sharp$*-module, then* $U_\varphi \dashv \text{dg-Hom}_R(A, -)$ *is a Quillen adjunction* $\mathcal{M}^{\mathrm{ctr}}(A) \rightleftarrows \mathcal{M}^{\mathrm{ctr}}(R)$.

Proof. Given an adjunction between model categories, checking that it is a Quillen adjunction means either to check that the left adjoint preserves (trivial) cofibrations, or, equivalently, that the right adjoint preserves (trivial) fibrations. The point here is to check the alternative which involves the parts of the model structures that we know explicitly. As an example, we check that $U_\varphi \dashv \text{dg-Hom}_R(A, -)$ is a Quillen adjunction $\mathcal{M}^{\mathrm{co}}(A) \rightleftarrows \mathcal{M}^{\mathrm{co}}(R)$ by proving that $\text{dg-Hom}_R(A, -)$ preserves (trivial) fibrations. A fibration in $\mathcal{M}^{\mathrm{co}}(R)$ is an epimorphism $f : Z \to X$ with $\ker(f) \in \mathcal{F}^{\mathrm{co}}(R) = R\text{-Mod}_{\mathrm{inj}}$. Since $\text{dg-Hom}_R(A, -)^\sharp = \text{dg-Hom}_{R^\sharp}(A^\sharp, -)$ and $\ker(f)^\sharp \in \mathcal{I}(R^\sharp\text{-Mod})$, we see that $\text{dg-Hom}_R(A, f) : \text{dg-Hom}_R(A, Z) \to \text{dg-Hom}_R(A, X)$ is an epimorphism with kernel $\text{dg-Hom}_R(A, \ker(f))$. Now, since $\text{dg-Hom}_R(A, \ker(f))^\sharp \cong \text{dg-Hom}_{R^\sharp}(A^\sharp, \ker(f)^\sharp)$ and $\text{dg-Hom}_{R^\sharp}(A^\sharp, -)$ is right adjoint to the exact functor $A^\sharp\text{-Mod} \to R^\sharp\text{-Mod}$, we get $\ker(\text{dg-Hom}_R(A, f))^\sharp \in \mathcal{I}(A^\sharp\text{-Mod})$, and hence $\ker(\text{dg-Hom}_R(A, f)) \in A\text{-Mod}_{\mathrm{inj}}$. In other words, $\text{dg-Hom}_R(A, f)$ is a fibration. Similarly, let f is a trivial fibration in $\mathcal{M}^{\mathrm{co}}(R)$. Then $\ker(f) \in \mathcal{I}(R\text{-Mod})$, so f is a split epimorphism with injective kernel. Since $\text{dg-Hom}_R(A, -)$ preserves injectives as the right adjoint to the exact functor $A\text{-Mod} \to R\text{-Mod}$, $\text{dg-Hom}_R(A, f)$ is a split epimorphism with injective kernel, too, i.e. a trivial fibration in $\mathcal{M}^{\mathrm{co}}(A)$. □

Remark II.2.3.19. The results of this section generalize to the case where we replaced our base category of abelian groups by any Grothendieck category $\mathscr{A}$ equipped with a closed symmetric monoidal tensor product $- \otimes - : \mathscr{A} \times \mathscr{A} \to \mathscr{A}$. Given a grading group Γ, the category $\mathscr{A}^\Gamma$ of Γ-indexed objects in $\mathscr{A}$ and the category $\mathrm{Ch}_\Gamma(\mathscr{A})$ of Γ-indexed complexes in $\mathscr{A}$ are again Grothendieck and inherit a closed symmetric monoidal tensor product; one can then speak about algebra objects in these categories (Γ-graded rings and Γ-graded dg rings in case $\mathscr{A} = \mathbb{Z}\text{-Mod}$), and form their categories of modules, which are again Grothendieck by Lemma II.B.3. The arguments of this section carry over to this situation and show that for any Γ-graded dg ring A over $(\mathscr{A}, \otimes)$ its category of modules carries the standard injective model structure, determined by injectivity and $\mathcal{W} = \mathrm{Acyc}(A)$, and the coderived model structure, determined by injectivity and $\mathcal{F} = A\text{-Mod}_{\mathrm{inj}}$. The only difference is that one has to argue why $\mathrm{Acyc}(A)$ and ${}^\perp A\text{-Mod}_{\mathrm{inj}}$ are generating; for example, this follows from the fact that both $\mathrm{Acyc}(A)$

and ${}^{\perp}A\text{-Mod}_{\mathrm{inj}}$ contain the class of contractible A-modules, and any A-module X is the quotient of the contractible A-module $\mathrm{Cone}(\mathrm{id}_{\Omega X})$. If $\mathscr{A}$ has enough projectives, then so do $\mathscr{A}^{\Gamma}$, $\mathrm{Ch}_{\Gamma}(\mathscr{A})$, $A^{\sharp}$-Mod and A-Mod, and we also get the standard projective and the contraderived model structure on A-Mod, determined by projectivity and $\mathcal{W} = \mathrm{Acyc}(A)$ resp. $\mathcal{C} = A\text{-Mod}_{\mathrm{proj}}$. Also, if $\mathscr{A}$ is just a Grothendieck abelian category without any additional monoidal structure, then $\mathrm{Ch}(\mathscr{A})$ carries the injective and coderived model structures $\mathcal{M}^{\mathrm{inj}}(\mathscr{A})$ and $\mathcal{M}^{\mathrm{co}}(\mathscr{A})$, respectively, and if moreover $\mathscr{A}$ has enough projectives, we also have the projective and contraderived model structures $\mathcal{M}^{\mathrm{proj}}(\mathscr{A})$ and $\mathcal{M}^{\mathrm{ctr}}(\mathscr{A})$ on $\mathrm{Ch}(\mathscr{A})$, respectively.

This generalization applies for example to the case where $\mathscr{A} = \mathrm{QCoh}(X)$ for a quasi-compact and quasi-separated scheme X (see [Mur, Proposition 66], or to $\mathscr{A} = \mathcal{O}_X$-Mod for some ringed space $(X, \mathcal{O}_X)$ (see [KS06, Theorem 18.1.6]).

See also the later Remark II.4.1.5 and Section II.4.3. ◊

II.3. Localization of abelian model structures

Let $\mathscr{A}$ be a bicomplete abelian category and $\mathcal{M}_1$, $\mathcal{M}_2$ two injective abelian model structures on $\mathscr{A}$ such that $\mathrm{id} : \mathcal{M}_2 \to \mathcal{M}_1$ is right Quillen. In this section we will construct from this datum another hereditary (usually non-injective) abelian model structure, called the right localization of $\mathcal{M}_1$ with respect to $\mathcal{M}_2$ and denoted $\mathcal{M}_1/\mathcal{M}_2$, whose homotopy category fits into a colocalization sequence with the homotopy categories of $\mathcal{M}_1$ and $\mathcal{M}_2$. The arguments in the proof are elementary homological algebra only, and in particular do not use Quillen's small object argument. Hence, we neither need to assume that the model structures we work with are cofibrantly generated, nor that the underlying bicomplete abelian category is Grothendieck. Instead, the assumptions are completely self-dual, and we get a dual left localization result for comparable pairs of projective abelian model structures. We will see in the next section that what we call localizations here are indeed Bousfield localizations in the sense of [Hir03].

II.3.1. The construction

Fact II.3.1.1. *Let $\mathscr{A}$ be an abelian category equipped with an abelian model structure $\mathcal{M} = (\mathcal{C}, \mathcal{W}, \mathcal{F})$. Then, given a morphism $f : A \to B$ the following are equivalent:*

(i) *f is a weak equivalence.*

(ii) *f factors as $A \overset{\iota}{\rightarrowtail} X \overset{p}{\twoheadrightarrow} B$ with $\operatorname{coker}(\iota) \in \mathcal{C} \cap \mathcal{W}$ and $\ker(p) \in \mathcal{F} \cap \mathcal{W}$.*

Proof. (ii)⇒(i) is clear, and (i)⇒(ii) follows from the factorization axiom. □

Fact II.3.1.1 is meant to motivate the description of $\mathcal{W}$ in the following proposition.

Theorem II.3.1.2. *Let $\mathscr{A}$ be a bicomplete abelian category and $\mathcal{M}_1 = (\mathcal{W}_1, \mathcal{F}_1)$ and $\mathcal{M}_2 = (\mathcal{W}_2, \mathcal{F}_2)$ be injective abelian model structures on $\mathscr{A}$ with $\mathcal{F}_2 \subset \mathcal{F}_1$. Then there exists a hereditary abelian model structure on $\mathscr{A}$, called the* right localization *of $\mathcal{M}_1$ with respect to $\mathcal{M}_2$ and denoted $\mathcal{M}_1/\mathcal{M}_2$, with $\mathcal{C} = \mathcal{W}_2$, $\mathcal{F} = \mathcal{F}_1$ and*

$$\begin{aligned} \mathcal{W} &:= \{X \in \mathscr{A} \mid \exists \text{ ex. seq. } 0 \to X \to A \to B \to 0 \text{ with } A \in \mathcal{F}_2, B \in \mathcal{W}_1\} \\ &= \{X \in \mathscr{A} \mid \exists \text{ ex. seq. } 0 \to A \to B \to X \to 0 \text{ with } A \in \mathcal{F}_2, B \in \mathcal{W}_1\}. \end{aligned}$$

Moreover, $X \in \mathcal{W}$ if and only if it belongs to the essential image of $\mathcal{F}_2 \to \mathrm{Ho}(\mathcal{M}_1)$.

Remark II.3.1.3. Recently, Gillespie [Gil14a] found a very interesting generalization of Theorem II.3.1.2 (which is part of the already published work [Bec14]): Instead of requiring two comparable injective abelian model structures as the input for the localization construction, Gillespie shows [Gil14a, Theorem 1.1] (see also Theorem II.C.3.13 below) that for any two complete, hereditary cotorsion pairs $(\mathcal{Q}, \widetilde{\mathcal{R}})$ and $(\widetilde{\mathcal{Q}}, \mathcal{R})$ satisfying $\widetilde{\mathcal{Q}} \subset \mathcal{Q}$ (hence $\widetilde{\mathcal{R}} \subset \mathcal{R}$) and $\mathcal{Q} \cap \widetilde{\mathcal{R}} = \widetilde{\mathcal{Q}} \cap \mathcal{R}$, there exists a unique $\mathcal{W}$ such that $(\mathcal{Q}, \mathcal{W}, \mathcal{R})$ is a hereditary abelian model structure. Its core $\mathcal{Q} \cap \mathcal{W} \cap \mathcal{F}$ agrees with the equal cores $\mathcal{Q} \cap \widetilde{\mathcal{R}} = \widetilde{\mathcal{Q}} \cap \mathcal{R}$ of the two given cotorsion pairs. In particular, this allows for the construction of abelian model structures whose cores are different from both $\mathcal{P}(\mathscr{A})$ and $\mathcal{I}(\mathscr{A})$, something which is not possible through our Theorem II.3.1.2 or its dual, Theorem II.3.1.7 below. As an example, Gillespie [Gil14b, Theorem 3.3] shows that for a right coherent ring R the category R-Mod of left R-modules carries the *Gorenstein flat* model structure, the cofibrant objects of which are the Gorenstein flat modules [Gil14b, Definition 2.3] and the trivially cofibrant objects of which are the flat modules. The core of this Gorenstein flat model structure is the class of flat cotorsion modules. For example, in case $R = \Bbbk[\![t]\!]$ with $\Bbbk$ a field this consists of the modules of the form $\Bbbk((t))^{(I)} \oplus \widehat{\Bbbk[\![t]\!]^{(J)}}$, where $\widehat{-}$ denotes t-adic completion [Eno84, Theorem 2]; in particular, it is different from $\mathcal{P}(R)$ as it contains the non-projective R-module $\Bbbk((t))$, and also different from $\mathcal{I}(R)$ as it contains the non-injective R-module $\Bbbk[\![t]\!]$. ◊

In the course of the proof of Theorem II.3.1.2 we will need the following lemmata:

Lemma II.3.1.4. *Let $\mathcal{F}$ be a Frobenius category and let $\mathcal{I}$ be its class of projective-injective objects. Then the following hold:*

(i) *Assume $\mathcal{F}$ weakly idempotent complete, i.e. every split monomorphism has a cokernel. Then, given $X, Y \in \mathcal{F}$, we have $X \cong Y$ in the stable category $\mathcal{F}/\mathcal{I}$ if and only if there exist $I, J \in \mathcal{I}$ such that $X \oplus J \cong Y \oplus I$ in $\mathcal{F}$.*

(ii) *Given an admissible short exact sequence $X \rightarrowtail Y \twoheadrightarrow Z$, there exists a canonical morphism $Z \to \Sigma X$ in the stable category $\mathcal{F}/\mathcal{I}$ such that $X \to Y \to Z \to \Sigma X$ is a distinguished triangle in $\mathcal{F}/\mathcal{I}$.*

Proof. (i) "⇐" is clear since all object in $\mathcal{I}$ are isomorphic to 0 in $\mathcal{F}/\mathcal{I}$. "⇒": Suppose $X \cong Y$ in $\mathcal{F}/\mathcal{I}$. By definition, this means that we can find $f : X \to Y$, $g : Y \to X$ such that $\mathrm{id}_Y - fg$ and $\mathrm{id}_X - gf$ respectively factor through some object in $\mathcal{I}$. Pick $p : I \to X$ and $u : X \to I$ with $I \in \mathcal{I}$ such that $\mathrm{id}_X = gf + pu$. Then $(f, u)^t : X \to Y \oplus I$ is a split monomorphism with left inverse $(g, p) : Y \oplus I$, so replacing Y by $Y \oplus I$ we may assume $gf = \mathrm{id}_X$. In this case, f is a split monomorphism, so by assumption we can choose a cokernel $k : Y \to K$ of f, and we have $s : K \to Y$ be such that $sk = \mathrm{id} - fg$. Then, picking morphisms $q : J \to Y$ and $v : Y \to J$ with $J \in \mathcal{I}$ such that $\mathrm{id}_Y = fg + qv$ we get $\mathrm{id}_K = ks = k(fg + qv)s = (kq)(vs)$. Again using the assumption that $\mathcal{F}$ is weakly idempotent complete, we conclude that K is a summand of J, and in particular $K \in \mathcal{I}$. Since $Y \cong X \oplus K$, this proves the claim.

(ii) See [Hap88, Lemma 2.7]. □

Lemma II.3.1.5 (Resolution Lemma)**.** *Let $\mathscr{A}$ be an abelian category and $(\mathcal{W}, \mathcal{F})$ be a coresolving cotorsion pair with enough injectives. Then for any short exact sequence $0 \to X_1 \to X_2 \to X_3 \to 0$ in $\mathscr{A}$ there exists a commutative diagram*

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & X_1 & \longrightarrow & X_2 & \longrightarrow & X_3 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A_1 & \longrightarrow & A_2 & \longrightarrow & A_3 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & B_1 & \longrightarrow & B_2 & \longrightarrow & B_3 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

such that $A_i \in \mathcal{F}$, $B_i \in \mathcal{W}$ and all rows and columns are exact.

Proof. Let $0 \to X_1 \to A_1 \to B_1 \to 0$ be short exact with $A_1 \in \mathcal{F}$, $B_1 \in \mathcal{W}$. Taking the pushout of $A_1 \leftarrow X_1 \to X_2$ we get a monomorphism of exact sequences

$$\begin{array}{ccccccccc}
0 & \longrightarrow & X_1 & \longrightarrow & X_2 & \longrightarrow & X_3 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A_1 & \longrightarrow & Z & \longrightarrow & X_3 & \longrightarrow & 0
\end{array}$$

whose cokernel $0 \to B_1 \to B_1 \to 0 \to 0$ is an exact sequence in $\mathcal{W}$. Replacing $0 \to X_1 \to X_2 \to X_3 \to 0$ by $0 \to A_1 \to Z \to X_3 \to 0$ we may therefore assume $A_1 = X_1 \in \mathcal{F}$ right from the beginning. In this case, choose an exact sequence $0 \to X_2 \to A_2 \to B_2 \to 0$ with $A_2 \in \mathcal{F}$, $B_2 \in \mathcal{W}$. Forming the pushout of $A_2 \leftarrow X_2 \to X_3$ we get the following commutative diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & A_1 & \longrightarrow & X_2 & \longrightarrow & X_3 & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A_1 & \longrightarrow & A_2 & \longrightarrow & Z & \longrightarrow & 0
\end{array}$$

By definition, the right square is pushout, but as $X_2 \to A_2$ is a monomorphism, it is also pullback, and hence the second row is exact. Since $\mathcal{F}$ is closed under cokernels of monomorphisms by assumption, we conclude $Z \in \mathcal{F}$. Hence we have constructed a monomorphism from $0 \to A_1 \to X_2 \to X_3 \to 0$ into a short exact sequence in $\mathcal{F}$ with cokernel $0 \to 0 \to B_2 \to B_2 \to 0$ lying in $\mathcal{W}$, as required. □

Proof of Theorem II.3.1.2. Recall from Corollary II.2.1.19 that $\mathcal{M}_1$ and $\mathcal{M}_2$ are automatically hereditary, and in particular $\mathcal{F}_1$ and $\mathcal{F}_2$ are closed under taking cokernels of monomorphisms; this will be used several times in the proof. We begin by showing that both definitions of $\mathcal{W}$ agree.

Suppose $X \in \mathscr{A}$ admits a short exact sequence $0 \to A \to B \to X \to 0$ with $A \in \mathcal{F}_2$ and $B \in \mathcal{W}_1$. Since $(\mathcal{W}_1, \mathcal{F}_1)$ is a cotorsion pair with $\mathcal{W}_1 \cap \mathcal{F}_1 = \mathcal{I}$, we can choose a short exact sequence $0 \to B \to I \to B' \to 0$ with $I \in \mathcal{I}$ and $B' \in \mathcal{W}_1$. Taking pushout, we get the following commutative diagram with exact rows and columns and bicartesian upper right square:

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & X & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A & \longrightarrow & I & \longrightarrow & A' & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & B' & = & B' & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

As $\mathcal{F}_2$ is closed under cokernels of monomorphisms, we have $A' \in \mathcal{F}_2$, and hence $0 \to X \to A' \to B'$ is our desired sequence.

Reversing the argument (using that any $A \in \mathcal{F}_2$ admits a short exact sequence $0 \to A' \to I \to A \to 0$ with $I \in \mathcal{W}_2 \cap \mathcal{F}_2 = \mathcal{I}$ and $A' \in \mathcal{F}_2$), we see that the existence of a short exact sequence $0 \to X \to A \to B \to 0$ with $A \in \mathcal{F}_2$ and $B \in \mathcal{W}_1$ also implies the existence of a short exact sequence $0 \to A' \to B' \to X \to 0$ with $A' \in \mathcal{F}_2$ and $B' \in \mathcal{W}_1$. Hence the two definitions of $\mathcal{W}$ agree.

For the thickness and the last claim, the argument goes as follows: As $(\mathcal{W}_1, \mathcal{F}_1)$ is a complete cotorsion pair, for any $X \in \mathscr{A}$ there exists an exact sequence $0 \to X \to A \to B \to 0$ with $A \in \mathcal{F}_1$ and $B \in \mathcal{W}_1$. The assignment $X \mapsto A$ defines an additive functor $\mathscr{A} \to \mathcal{F}_1/\mathcal{F}_1 \cap \mathcal{W}_1 = \mathcal{F}_1/\mathcal{I}$ (it is a short check that any morphism between objects of $\mathcal{F}_1$ factoring through an object in $\mathcal{W}_1$ actually factors through some object in $\mathcal{F}_1 \cap \mathcal{W}_1$; see also Proposition II.2.1.18) and in particular the object A from above is unique up to canonical isomorphism in $\mathcal{F}_1/\mathcal{I}$. Next, form the full subcategory $\mathcal{F}_2/\mathcal{I}$ of $\mathcal{F}_1/\mathcal{I}$ consisting of objects $\mathcal{F}_2$ (recall that passing to the stable category does not change objects). It is isomorphism closed by Lemma II.3.1.4, and using this we see that $\mathcal{W}$ equals the preimage of $\mathcal{F}_2/\mathcal{I}$ under $\mathscr{A} \to \mathcal{F}_1/\mathcal{I}$. With this description at hand, we can now prove the thickness of $\mathcal{W}$. As the functor $\mathscr{A} \to \mathcal{F}_1/\mathcal{I}$ from above is additive and $\mathcal{F}_2/\mathcal{I}$ is closed under direct summands in $\mathcal{F}_1/\mathcal{I}$, $\mathcal{W}$ is closed under direct summands, too. For the 2-out-of-3 property, note that $\mathcal{F}_2/\mathcal{I}$ is a triangulated subcategory of $\mathcal{F}_1/\mathcal{I}$, so it suffices to show that $\mathscr{A} \to \mathcal{F}_1/\mathcal{I}$ turns short exact sequences into distinguished triangles, which follows from Lemma II.3.1.4(ii) and Lemma II.3.1.5.

It remains to show that $\mathcal{M}_1/\mathcal{M}_2$ is hereditary and that $(\mathcal{C} \cap \mathcal{W}, \mathcal{F})$ and $(\mathcal{C}, \mathcal{W} \cap \mathcal{F})$ are complete cotorsion pairs. The former is true since $\mathcal{F} = \mathcal{F}_1$ is closed under cokernels of monomorphisms by assumption and $\mathcal{C} = \mathcal{W}_2$ even satisfies the 2-out-of-3 property; the latter will follow once we showed that $(\mathcal{C} \cap \mathcal{W}, \mathcal{F}) = (\mathcal{W}_1, \mathcal{F}_1)$ and $(\mathcal{C}, \mathcal{W} \cap \mathcal{F}) = (\mathcal{W}_2, \mathcal{F}_2)$, as these are complete cotorsion pairs by assumption.

$\mathcal{W} \cap \mathcal{F} = \mathcal{F}_2$: Suppose $X \in \mathcal{W} \cap \mathcal{F} = \mathcal{W} \cap \mathcal{F}_1$ and let $0 \to X \to A \to B \to 0$ be a short exact sequence with $A \in \mathcal{F}_2$ and $B \in \mathcal{W}_1$. By definition, $\operatorname{Ext}^1(\mathcal{W}_1, X) = 0$, so the sequence splits and $X \in \mathcal{F}_2$ as $\mathcal{F}_2$ is thick. This shows that $\mathcal{F}_1 \cap \mathcal{W} \subset \mathcal{F}_2$, and the reverse inclusion $\mathcal{F}_2 \subset \mathcal{F}_1 \cap \mathcal{W}$ is clear.

$\mathcal{C} \cap \mathcal{W} = \mathcal{W}_1$: Suppose $X \in \mathcal{C} \cap \mathcal{W} = \mathcal{W}_2 \cap \mathcal{W}$ and let $0 \to A \to B \to X \to 0$ be a short exact sequence with $A \in \mathcal{F}_2$ and $B \in \mathcal{W}_1$. Again, this sequence is split since $X \in {}^{\perp}\mathcal{F}_2$, so $X \in \mathcal{W}_1$. Hence $\mathcal{W}_2 \cap \mathcal{W} \subset \mathcal{W}_1$, and the reverse inclusion is clear. □

Corollary II.3.1.6. *In the situation of Theorem II.3.1.2 the sequence*

$$\operatorname{Ho}(\mathcal{M}_2) \xrightarrow{\mathbf{R}\,\mathrm{id}} \operatorname{Ho}(\mathcal{M}_1) \xrightarrow{\mathbf{R}\,\mathrm{id}} \operatorname{Ho}(\mathcal{M}_1/\mathcal{M}_2)$$

is a colocalization sequence [Kra05, Definition 3.1] of triangulated categories.

Proof. Consider the following commutative diagram

$$\begin{array}{ccccc}
\operatorname{Ho}(\mathcal{M}_1/\mathcal{M}_2) & \underset{\mathbf{R}\,\mathrm{id}}{\overset{\mathbf{L}\,\mathrm{id}}{\rightleftarrows}} & \operatorname{Ho}(\mathcal{M}_1) & \underset{\mathbf{R}\,\mathrm{id}}{\overset{\mathbf{L}\,\mathrm{id}}{\rightleftarrows}} & \operatorname{Ho}(\mathcal{M}_2) \\
\cong\big\uparrow & & \cong\big\uparrow & & \cong\big\uparrow \\
\mathcal{F}_1 \cap \mathcal{W}_2/\mathcal{I} & \overset{\mathrm{inc}}{\rightleftarrows} & \mathcal{F}_1/\mathcal{I} & \underset{\mathrm{inc}}{\rightleftarrows} & \mathcal{F}_2/\mathcal{I}.
\end{array}$$

By Theorem II.3.1.2 the kernel of $\operatorname{Ho}(\mathcal{M}_1) \to \operatorname{Ho}(\mathcal{M}_1/\mathcal{M}_2)$ equals the essential image of $\mathcal{F}_2/\mathcal{I} \to \operatorname{Ho}(\mathcal{M}_1)$, i.e. the essential image of $\mathbf{R}\,\mathrm{id} : \operatorname{Ho}(\mathcal{M}_2) \to \operatorname{Ho}(\mathcal{M}_1)$. It remains to be shown that the derived functors $\mathbf{R}\,\mathrm{id} : \operatorname{Ho}(\mathcal{M}_2) \to \operatorname{Ho}(\mathcal{M}_1)$ and $\mathbf{L}\,\mathrm{id} : \operatorname{Ho}(\mathcal{M}_1/\mathcal{M}_2) \to \operatorname{Ho}(\mathcal{M}_1)$ are fully faithful, which follows from the commutativity of the diagram and the fully faithfulness of $\mathcal{F}_2/\mathcal{I} \to \mathcal{F}_1/\mathcal{I}$ and $\mathcal{F}_1 \cap \mathcal{W}_2/\mathcal{I} \to \mathcal{F}_1/\mathcal{I}$. □

Dually, we have the following localization result for projective model structures:

Theorem II.3.1.7. *Let $\mathscr{A}$ be a bicomplete abelian category and $\mathcal{M}_1 = (\mathcal{C}_1, \mathcal{W}_1)$ and $\mathcal{M}_2 = (\mathcal{C}_2, \mathcal{W}_2)$ be projective, abelian model structures on $\mathscr{A}$ with $\mathcal{C}_2 \subset \mathcal{C}_1$. Then there exists a hereditary abelian model structure on $\mathscr{A}$, called the* left localization *of $\mathcal{M}_1$ with respect to $\mathcal{M}_2$ and denoted $\mathcal{M}_2 \backslash \mathcal{M}_1$, with $\mathcal{C} = \mathcal{C}_1$, $\mathcal{F} = \mathcal{W}_2$ and*

$$\begin{aligned} \mathcal{W} &:= \{X \in \mathscr{A} \mid \exists \text{ ex. seq. } 0 \to X \to A \to B \to 0 \text{ with } A \in \mathcal{W}_1, B \in \mathcal{C}_2\} \\ &= \{X \in \mathscr{A} \mid \exists \text{ ex. seq. } 0 \to A \to B \to X \to 0 \text{ with } A \in \mathcal{W}_1, B \in \mathcal{C}_2\}. \end{aligned}$$

Moreover, $X \in \mathcal{W}$ if and only if it belongs to the essential image of $\mathcal{C}_2 \to \mathrm{Ho}(\mathcal{M}_1)$, and there is a localization sequence of triangulated categories

$$\mathrm{Ho}(\mathcal{M}_2) \xrightarrow{\mathbf{L}\,\mathrm{id}} \mathrm{Ho}(\mathcal{M}_1) \xrightarrow{\mathbf{L}\,\mathrm{id}} \mathrm{Ho}(\mathcal{M}_2 \backslash \mathcal{M}_1).$$

Example II.3.1.8. We consider a simple example, anticipating the more general results that will be discussed later in Section II.4. Let R be a ring considered as a dg ring concentrated in cohomological degree zero. From Propositions II.2.3.5 and II.2.3.6 we get the standard projective model structure $({}^{\perp}\mathrm{Acyc}(R), \mathrm{Acyc}(R), \mathrm{Ch}(R))$ and the contraderived model structure $(\mathrm{Ch}(\mathrm{Proj}(R)), \mathcal{W}^{\mathrm{ctr}}(R), \mathrm{Ch}(R))$ on $\mathrm{Ch}(R)$. Since $\mathcal{C}^{\mathrm{proj}}(R) \subseteq \mathcal{C}^{\mathrm{ctr}}(R)$ by Corollary II.2.3.7, we can apply Theorem II.3.1.7 and get as the left localization $\mathcal{M}^{\mathrm{proj}}(R) \backslash \mathcal{M}^{\mathrm{ctr}}(R)$ the model structure $(\mathrm{Ch}(\mathrm{Proj}(R)), ?, \mathrm{Acyc}(R))$ on $\mathrm{Ch}(R)$, the homotopy category of which is $\mathbf{K}_{\mathrm{ac}}(\mathrm{Proj}(R))$. Similarly, applying Theorem II.3.1.2 we can form the right localization $\mathcal{M}^{\mathrm{co}}(R)/\mathcal{M}^{\mathrm{inj}}(R)$, i.e. the abelian model structure corresponding to the triple $(\mathrm{Acyc}(R), ?, \mathrm{Ch}(\mathrm{Inj}(R)))$, with homotopy category $\mathbf{K}_{\mathrm{ac}}(\mathrm{Inj}(R))$. In particular, we deduce that there is a colocalization sequence $\mathbf{K}_{\mathrm{ac}}(\mathrm{Inj}(R)) \to \mathbf{K}(\mathrm{Inj}(R)) \to \mathbf{D}(R)$ and a localization sequence $\mathbf{K}_{\mathrm{ac}}(\mathrm{Proj}(R)) \to \mathbf{K}(\mathrm{Proj}(R)) \to \mathbf{D}(R)$. ◊

II.3.2. Connection to Bousfield localization

In this section, we again go back to the classical language of model categories and rewrite Theorem II.3.1.2 as a statement about existence of certain right Bousfield localizations. The results of this section are not needed anywhere else and are included solely for the purpose of connecting and making explicit well-established notions and results on model categories in the case of abelian model categories.

Definition II.3.2.1 [Hir03, Definition 3.3.1(2)]**.** *Let $\mathcal{M}$ be a model category and S be a class of maps in $\mathcal{M}$. The* right Bousfield localization *of $\mathcal{M}$ with respect to S is, if it exists, the model structure $\mathbf{R}_{\mathrm{S}}\mathcal{M}$ on the category underlying $\mathcal{M}$ such that*

(i) *the class of weak equivalences of* $\mathbf{R}_{\mathrm{S}}\mathcal{M}$ *is the class of* S-*colocal equivalences,*

(ii) *the class of fibrations of* $\mathbf{R}_{\mathrm{S}}\mathcal{M}$ *is the class of fibrations of* $\mathcal{M}$*, and*

(iii) *the class of cofibrations of* $\mathbf{R}_{\mathrm{S}}\mathcal{M}$ *is determined by the left lifting property with respect to trivial fibrations.*

Definition II.3.2.2. *Let* $\mathcal{M}$ *be a model category,* K *a class of objects and* S *a class of morphisms in* $\mathcal{M}$.

(i) *A morphism* $f : A \to B$ *is called a* K-colocal equivalence *if for all* $X \in K$ *and* $k \geq 0$ *the induced map* $\mathrm{Ho}(\mathcal{M})(X, \Omega^k A) \to \mathrm{Ho}(\mathcal{M})(X, \Omega^k B)$ *is a bijection.*

(ii) *An object* $X \in \mathcal{M}$ *is called* S-colocal *if for all* $f : A \to B$ *in* S *and* $k \geq 0$ *the induced map* $\mathrm{Ho}(\mathcal{M})(X, \Omega^k A) \to \mathrm{Ho}(\mathcal{M})(X, \Omega^k B)$ *is a bijection.*

(iii) *A morphism is called a* S-colocal equivalence *if it is a colocal equivalence with respect to the class of* S-*colocal objects.*

Proposition II.3.2.3. *Let* $\mathscr{A}$ *be a bicomplete abelian category and* $\mathcal{M}_1 = (\mathcal{W}_1, \mathcal{F}_1)$ *and* $\mathcal{M}_2 = (\mathcal{W}_2, \mathcal{F}_2)$ *be injective model structures on* $\mathscr{A}$ *satisfying* $\mathcal{F}_2 \subset \mathcal{F}_1$. *Then the model structure* $\mathcal{M}_1/\mathcal{M}_2$ *described in Theorem II.3.1.2 is the right Bousfield localization of* $\mathcal{M}_1$ *with respect to* $\mathrm{S} := \{0 \to X \mid X \in \mathcal{F}_2\} \subset \mathrm{Mor}(\mathscr{A})$.

Proof. Since domain and codomain of each morphism in S are fibrant in $\mathcal{M}_1$, Proposition II.2.1.20 reveals that the class of S-colocal objects equals ${}^{\perp}(\mathcal{F}_2/\mathcal{I})$ in $\mathscr{A}/\mathcal{I}$, which is $\mathcal{W}_2/\mathcal{I}$ by Proposition II.2.1.18 applied to the cotorsion pair $(\mathcal{W}_2, \mathcal{F}_2)$.

It remains to show that the weak equivalences in $\mathcal{M}_1/\mathcal{M}_2$ are precisely the $\mathcal{W}_2$-colocal equivalences. For this, note the following:

(i) In $\mathrm{Ho}(\mathcal{M}_1)$ any morphism is isomorphic to a morphism between objects in $\mathcal{F}_1$: This follows from the fact that in $\mathrm{Ho}(\mathcal{M}_1)$ any object is isomorphic to an object in $\mathcal{F}_1$ (see Proposition II.2.1.20).

(ii) In $\mathrm{Ho}(\mathcal{M}_1)$, any morphism between objects in $\mathcal{F}_1$ is isomorphic to an epimorphism between objects in $\mathcal{F}_1$ with kernel again in $\mathcal{F}_1$: If $f : A \to B$ is (a representative of) the given morphism with $A, B \in \mathcal{F}_1$, and $0 \to B' \to I \xrightarrow{p} B \to 0$ is exact with $I \in \mathcal{I}$ and $B' \in \mathcal{F}_1$, then f is isomorphic in $\mathrm{Ho}(\mathcal{M}_1)$ to $(f, -p) : A \oplus I \to B$. Moreover, $K := \ker(f, -p) \in \mathcal{F}_1$ since it fits into the commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & B' & \longrightarrow & K & \longrightarrow & A & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & B' & \longrightarrow & I & \longrightarrow & B & \longrightarrow & 0 \end{array}$$

and $\mathcal{F}_1$ is closed under extensions.

(iii) If $f : A \to B$ is an epimorphism of objects in $\mathcal{F}_1$ and kernel $K \in \mathcal{F}_1$ as in (ii), then $X \in \mathscr{A}$ is f-colocal if and only if $(\mathscr{A}/\mathcal{I})(X, \Omega^k K) = 0$ for all $k \geq 0$: To begin, the short exact sequence $0 \to K \to A \to B \to 0$ gives rise to a triangle in $\mathrm{Ho}(\mathcal{M})$. Now the functor $\mathrm{Ho}(\mathcal{M})(X, -)$ is cohomological, i.e. turns exact triangles into long exact sequences, and hence $\mathrm{Ho}(\mathcal{M})(X, \Omega^k(f))$ is bijective for all $k \geq 0$ if and only if $\mathrm{Ho}(\mathcal{M})(X, \Omega^k K) = 0$ for all $k \geq 0$. By Proposition II.2.1.20 the latter is equivalent to $(\mathscr{A}/\mathcal{I})(X, \Omega^k K) = 0$ for all $k \geq 0$.

As $(\mathcal{W}_2/\mathcal{I})^\perp = \mathcal{F}_2/\mathcal{I}$ in $\mathscr{A}/\mathcal{I}$, steps (i)-(iii) show that the $\mathcal{W}_2$-colocal equivalences are precisely those morphisms which are isomorphic in $\mathrm{Ho}(\mathcal{M}_1)$ to epimorphism of objects in $\mathcal{F}_1$ with kernel in $\mathcal{F}_2$.

We will show that the same description applies to the weak equivalences in $\mathcal{M}_1/\mathcal{M}_2$. By Fact II.3.1.1, any weak equivalence in $\mathcal{M}_1/\mathcal{M}_2$ is the composition of a monomorphism with cokernel in $\mathcal{C} \cap \mathcal{W} = {}^\perp\mathcal{F}_1 = \mathcal{W}_1$ and an epimorphism with kernel in $\mathcal{W} \cap \mathcal{F} = \mathcal{W}_2^\perp = \mathcal{F}_2$. The former is already a weak equivalence in $\mathcal{M}_1$, hence any weak equivalence in $\mathcal{M}_1/\mathcal{M}_2$ is isomorphic to an epimorphism with kernel in $\mathcal{F}_2$ in $\mathrm{Ho}(\mathcal{M}_1)$. Let $f : B \to A$ be such an epimorphism and pick a short exact sequence $0 \to B \overset{\alpha}{\to} F \to W \to 0$ with $F \in \mathcal{F}_1$. Taking the pushout of $F \overset{\alpha}{\leftarrow} B \overset{f}{\to} A$, we get the following commutative diagram (note that the right square is also pullback):

$$
\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & K & \longrightarrow & B & \xrightarrow{f} & A & \longrightarrow & 0 \\
 & & \| & & \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} & & \\
0 & \longrightarrow & K & \longrightarrow & F & \xrightarrow[g]{} & F' & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & W & = & W & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}
$$

As α, β are weak equivalences in $\mathcal{M}_1$, f is isomorphic to g in $\mathrm{Ho}(\mathcal{M}_1)$. Moreover, as $\mathcal{F}_1$ is closed under cokernels of monomorphisms, $F' \in \mathcal{F}_1$. This shows that f is isomorphic in $\mathrm{Ho}(\mathcal{M}_1)$ to an epimorphism of objects in $\mathcal{F}_1$ with kernel in $\mathcal{F}_2$. Conversely, since any weak equivalence in $\mathcal{M}_1$ is also a weak equivalence in $\mathcal{M}_1/\mathcal{M}_2$, it is clear that any such morphism is a weak equivalence in $\mathcal{M}_1/\mathcal{M}_2$. □

II.4. The singular model structures

In this section we attach to each morphism of dg rings $\varphi : R \to A$ two "relative singular" model structures on A-Mod, a contraderived and a coderived one. Roughly, the contraderived (resp. coderived) singular model structure is obtained by pulling back the contraderived (resp. coderived) model $\mathcal{M}^{\mathrm{ctr}}(R)$ (resp. $\mathcal{M}^{\mathrm{co}}(R)$) on R-Mod to A-Mod along the right (resp. left) adjoint $U_\varphi : A\text{-Mod} \to R\text{-Mod}$, and afterwards taking the left (resp. right) localization of $\mathcal{M}^{\mathrm{ctr}}(A)$ (resp. $\mathcal{M}^{\mathrm{co}}(A)$) with respect to this pullback model structure. If R is an ordinary ring of finite left-global dimension, we will see that the relative singular contraderived and coderived model structures only depend on A, and we will call them the "absolute singular" model structures attached to A.

In general, pulling back model structures along adjoints is a nontrivial problem, so we need to justify that the above pullbacks are again abelian model structures. In our situation, the connection between abelian model structures and deconstructible classes makes this problem tractable and we give ad-hoc arguments to establish the desired pullbacks.

Recall that right (resp. left) localization of two projective (resp. injective) model structures produces abelian model structures which are neither projective nor injective in general. In particular, the (relative or absolute) singular model structures are neither projective nor injective. We will be able, however, to establish a concrete projective (resp. injective) abelian model structure on A-Mod Quillen equivalent to the singular contraderived (resp. coderived) one. This alternative description is useful for example in proving that the absolute contraderived (resp. coderived) singular model structure on $\mathrm{Ch}(R)$, for R Gorenstein, is Quillen equivalent to Hovey's Gorenstein projective (resp. Gorenstein injective) model structure on R-Mod, as well as in the construction of recollements later.

II.4.1. General definitions

Let $U : \mathcal{D} \longrightarrow \mathcal{C}$ be a functor between two categories $\mathcal{C}, \mathcal{D}$, and suppose that $\mathcal{C}$ carries a model structure $\mathcal{M}$. The *right pullback* of $\mathcal{M}$ along U is, if it exists, the model structure on $\mathcal{D}$ in which a morphism is a weak equivalence (resp. fibration) if and only if its image under U is a weak equivalence (resp. fibration) in $\mathcal{M}$, and where the cofibrations are determined by the left lifting property with respect to all trivial fibrations. Similarly, the *left pullback* of $\mathcal{M}$ along U is, if it exists, the model structure on $\mathcal{D}$ where the cofibrations (resp. weak equivalences) are the morphisms which become cofibrations (resp. weak equivalences) in $\mathcal{M}$ after application of U, and where the fibrations are determined by the right lifting property with respect to all trivial cofibrations.

Proposition II.4.1.1. *Let $\varphi : R \to A$ be a morphism of dg rings.*

(i) *The right-pullback $\varphi^*\mathcal{M}^{\mathrm{ctr}}(R)$ of $\mathcal{M}^{\mathrm{ctr}}(R)$ along U_φ exists.*

(ii) *The left-pullback $\varphi^*\mathcal{M}^{\mathrm{co}}(R)$ of $\mathcal{M}^{\mathrm{co}}(R)$ along U_φ exists.*

Moreover, both $\varphi^*\mathcal{M}^{\mathrm{ctr}}(R)$ and $\varphi^*\mathcal{M}^{\mathrm{co}}(R)$ are cofibrantly generated.

Proof. (i) It suffices to show that firstly $U_\varphi^*(\mathcal{W}^{\mathrm{ctr}}(R))$ is of the form $\mathcal{S}^\perp$ for a set $\mathcal{S} \subset A$-Mod, and secondly that $U_\varphi^*(\mathcal{W}^{\mathrm{ctr}}(R)) \cap {}^\perp U_\varphi^*(\mathcal{W}^{\mathrm{ctr}}(R)) = \mathcal{P}(A\text{-Mod})$. By Proposition II.2.3.6 $\mathcal{C}^{\mathrm{ctr}}(R)$ is deconstructible, so we may choose a set $\mathcal{T}$ such that $\mathcal{C}^{\mathrm{ctr}}(R) = \text{filt-}\mathcal{T}$. Denoting the left adjoint $A \otimes_R -$ to U_φ by F for a moment, we claim that $U_\varphi^*(\mathcal{W}^{\mathrm{ctr}}(R)) = F(\mathcal{T})^\perp$. In fact, we will even show that $\mathrm{Ext}^1_A(F(T), -) \cong \mathrm{Ext}^1_R(T, U_\varphi(-))$ for all $T \in \mathcal{T}$. Having done this, the claim follows via $F(\mathcal{T})^\perp = U_\varphi^*(\mathcal{T}^\perp) = U_\varphi^*(\mathcal{W}^{\mathrm{ctr}}(R))$. Let $Y \in A$-Mod be arbitrary and $0 \to Y \to W \xrightarrow{f} C \to 0$ be an exact sequence with $W \in \mathcal{W}^{\mathrm{ctr}}(A)$ and $C \in \mathcal{C}^{\mathrm{ctr}}(A)$. Since $F(\mathcal{T}) \subseteq \mathcal{C}^{\mathrm{ctr}}(A)$ (Proposition II.2.3.18), we get $\mathrm{Ext}^1_A(F(T), Y) \cong \mathrm{coker}\,[\mathrm{Hom}_A(F(T), f)]$. Moreover, since U_φ is exact and $U_\varphi(\mathcal{W}^{\mathrm{ctr}}(A)) \subseteq \mathcal{W}^{\mathrm{ctr}}(R)$ (Proposition II.2.3.18), computing $\mathrm{Ext}^1_A(T, U_\varphi(Y))$ using the exact sequence $0 \to U_\varphi(Y) \to U_\varphi(W) \xrightarrow{U_\varphi(f)} U_\varphi(C) \to 0$ gives $\mathrm{Ext}^1_R(T, U_\varphi(Y)) \cong \mathrm{coker}\,[\mathrm{Hom}_R(T, U_\varphi(f))]$. Now, the adjunction $F \dashv U_\varphi$ gives $\mathrm{coker}\,[\mathrm{Hom}_R(T, U_\varphi(f))] \cong \mathrm{coker}\,[\mathrm{Hom}_A(F(T), f)]$, and hence $\mathrm{Ext}^1_A(F(T), Y) \cong \mathrm{Ext}^1_R(T, U_\varphi(Y))$ for all $T \in \mathcal{T}$ and $Y \in A$-Mod. The remaining part $U_\varphi^*(\mathcal{W}^{\mathrm{ctr}}(R)) \cap {}^\perp U_\varphi^*(\mathcal{W}^{\mathrm{ctr}}(R)) = \mathcal{P}(A\text{-Mod})$ follows from Lemma II.2.3.4 since $\mathcal{W}^{\mathrm{ctr}}(A) \subseteq U_\varphi^*(\mathcal{W}^{\mathrm{ctr}}(R))$ and hence ${}^\perp U_\varphi^*(\mathcal{W}^{\mathrm{ctr}}(R)) \subseteq \mathcal{C}^{\mathrm{ctr}}(A) = A\text{-Mod}_{\mathrm{proj}}$.

(ii) We have to show that $\mathcal{K} := U_\varphi^*(\mathcal{W}^{\mathrm{co}}(R))$ is deconstructible and $\mathcal{K} \cap \mathcal{K}^\perp = \mathcal{I}(A\text{-Mod})$. The deconstructibility of $\mathcal{K}$ follows from Theorem II.B.11 together with the deconstructibility of $\mathcal{W}^{\mathrm{co}}(R)$ established in Proposition II.2.3.6. Hence $(\mathcal{K}, \mathcal{K}^\perp)$ is a complete cotorsion pair cogenerated by a set. For $\mathcal{K} \cap \mathcal{K}^\perp = \mathcal{I}(A\text{-Mod})$, first note that since $U_\varphi : \mathcal{M}^{\mathrm{co}}(A) \to \mathcal{M}^{\mathrm{co}}(R)$ is left Quillen (Proposition II.2.3.18), we have $\mathcal{K} \supseteq \mathcal{W}^{\mathrm{co}}(A)$, and hence $\mathcal{K}^\perp \subseteq \mathcal{F}^{\mathrm{co}}(A) = A\text{-Mod}_{\mathrm{inj}}$. Applying Lemma II.2.3.4 now gives $\mathcal{K} \cap \mathcal{K}^\perp = \mathcal{I}(A\text{-Mod})$ as required. □

Note that if R is an ordinary ring of finite left-global dimension, then $\mathcal{M}^{\mathrm{ctr}}(R) = \mathcal{M}^{\mathrm{proj}}(R)$ and $\mathcal{M}^{\mathrm{co}}(R) = \mathcal{M}^{\mathrm{inj}}(R)$ (Proposition II.2.3.17), and hence for any morphism $\varphi : R \to A$ of dg rings $\varphi^*\mathcal{M}^{\mathrm{ctr}}(R) = \mathcal{M}^{\mathrm{proj}}(A)$ and $\varphi^*\mathcal{M}^{\mathrm{co}}(R) = \mathcal{M}^{\mathrm{inj}}(A)$.

Definition II.4.1.2. *Let $\varphi : R \to A$ be a morphism of dg rings.*

(i) *The* relative singular coderived model structure *on* A-Mod *is the right localization* $\mathcal{M}^{\mathrm{co}}(A)/\varphi^*\mathcal{M}^{\mathrm{co}}(R)$ *in the sense of Theorem II.3.1.2 and denoted* $\mathcal{M}^{\mathrm{co}}_{\mathrm{sing}}(A/R)$.

(ii) *The* relative singular contraderived model structure *on* A-Mod *is the left localization* $\varphi^*\mathcal{M}^{\mathrm{ctr}}(R)\backslash\mathcal{M}^{\mathrm{ctr}}(A)$ *in the sense of Theorem II.3.1.7 and denoted* $\mathcal{M}^{\mathrm{ctr}}_{\mathrm{sing}}(A/R)$.

If R is a ring of finite left-global dimension (e.g. $R = \mathbb{Z}$ or $R = k$ is a field), then $\mathcal{M}^{\mathrm{ctr}\,/\,\mathrm{co}}_{\mathrm{sing}}(A) := \mathcal{M}^{\mathrm{ctr}\,/\,\mathrm{co}}_{\mathrm{sing}}(A/R)$ *does not depend on R and is called the* absolute singular contraderived resp. coderived model structure.

Proposition II.4.1.3. *Let $\varphi : R \to A$ be a morphism of dg rings. The relative singular contraderived model structure $\mathcal{M}^{\mathrm{ctr}}_{\mathrm{sing}}(A/R)$ can be described as follows:*

(i) *The class $\mathcal{C}$ of cofibrant objects equals* A-$\mathrm{Mod}_{\mathrm{proj}}$.

(ii) *The class $\mathcal{F}$ of fibrant objects is the class of A-modules whose underlying R-modules are contraacyclic.*

(iii) *The class $\mathcal{W}$ of weakly trivial objects is determined by Fact II.3.1.1.*

In particular, the fibrant objects in $\mathcal{M}^{\mathrm{ctr}}_{\mathrm{sing}}(A)$ are the acyclic A-modules.

A similar description holds for the relative singular coderived model:

Proposition II.4.1.4. *Let $\varphi : R \to A$ be a morphism of dg rings. The relative singular coderived model structure $\mathcal{M}^{\mathrm{co}}_{\mathrm{sing}}(A/R)$ can be described as follows:*

(i) *The class $\mathcal{C}$ of cofibrant objects is the class of A-modules whose underlying R-modules are coacyclic.*

(ii) *The class $\mathcal{F}$ of fibrant objects equals* A-$\mathrm{Mod}_{\mathrm{inj}}$.

(iii) *The class $\mathcal{W}$ of weakly trivial objects is determined by Fact II.3.1.1.*

In particular, the cofibrant objects in $\mathcal{M}^{\mathrm{ctr}}_{\mathrm{sing}}(A)$ are the acyclic A-modules.

Remark II.4.1.5. The construction of the relative and absolute singular coderived model structures carries over to the setting discussed in Remark II.2.3.19. ♢

II.4.2. Constructing recollements

From Proposition II.4.1.3 (resp. II.4.1.4) it is clear that $\mathcal{M}^{\mathrm{ctr}}_{\mathrm{sing}}(A)$ (resp. $\mathcal{M}^{\mathrm{co}}_{\mathrm{sing}}(A)$) is almost never projective (resp. injective). However, there is a canonical projective (resp. injective) abelian model structure which is Quillen equivalent to the absolute singular contraderived (resp. coderived) model, which we describe in this section.

Proposition II.4.2.1. *For a dg ring A, the following hold:*

(i) *There exists a projective abelian model structure ${}^{p}\mathcal{M}^{\mathrm{ctr}}_{\mathrm{sing}}(A)$ on A-Mod satisfying $\mathcal{C} = A\text{-Mod}_{\mathrm{proj}} \cap \mathrm{Acyc}(A)$.*

(ii) *There exists an injective abelian model structure ${}^{i}\mathcal{M}^{\mathrm{co}}_{\mathrm{sing}}(A)$ on A-Mod satisfying $\mathcal{F} = A\text{-Mod}_{\mathrm{inj}} \cap \mathrm{Acyc}(A)$.*

${}^{p}\mathcal{M}^{\mathrm{ctr}}_{\mathrm{sing}}(A)$ and ${}^{i}\mathcal{M}^{\mathrm{co}}_{\mathrm{sing}}(A)$ are cofibrantly generated and the identity is a left resp. right Quillen equivalence ${}^{p}\mathcal{M}^{\mathrm{ctr}}_{\mathrm{sing}}(A) \to \mathcal{M}^{\mathrm{ctr}}_{\mathrm{sing}}(A)$ resp. ${}^{i}\mathcal{M}^{\mathrm{co}}_{\mathrm{sing}}(A) \to \mathcal{M}^{\mathrm{co}}_{\mathrm{sing}}(A)$.

Proof. (i) As usual it suffices to show that ${}^{p}\mathcal{C}^{\mathrm{ctr}}_{\mathrm{sing}}(A) = A\text{-Mod}_{\mathrm{proj}} \cap \mathrm{Acyc}(A)$ is deconstructible, that ${}^{p}\mathcal{C}^{\mathrm{ctr}}_{\mathrm{sing}}(A) \cap {}^{p}\mathcal{C}^{\mathrm{ctr}}_{\mathrm{sing}}(A)^{\perp} = \mathcal{P}(A\text{-Mod})$ and that ${}^{p}\mathcal{C}^{\mathrm{ctr}}_{\mathrm{sing}}(A)^{\perp}$ has the 2-out-of-3 property. Since both $A\text{-Mod}_{\mathrm{proj}}$ and $\mathrm{Acyc}(A)$ are deconstructible by Propositions II.2.3.6 and II.2.3.5, the deconstructibility of $A\text{-Mod}_{\mathrm{proj}} \cap \mathrm{Acyc}(A)$ follows from the stability of deconstructible classes under intersections [Što13, Proposition 2.9]. The equality ${}^{p}\mathcal{C}^{\mathrm{ctr}}_{\mathrm{sing}}(A) \cap {}^{p}\mathcal{C}^{\mathrm{ctr}}_{\mathrm{sing}}(A)^{\perp} = \mathcal{P}(A\text{-Mod})$ follows from Lemma II.2.3.4, and Lemma II.2.1.17 ensures the 2-out-of-3 property since ${}^{p}\mathcal{C}^{\mathrm{ctr}}_{\mathrm{sing}}(A)$ is closed under kernels of epimorphisms. Finally, it is clear that the identity is a left Quillen functor ${}^{p}\mathcal{M}^{\mathrm{ctr}}_{\mathrm{sing}}(A) \to \mathcal{M}^{\mathrm{ctr}}_{\mathrm{sing}}(A)$; moreover, Proposition II.2.1.20 implies that it induces an equivalence on homotopy categories, hence is a Quillen equivalence.

(ii) ${}^{i}\mathcal{F}^{\mathrm{co}}_{\mathrm{sing}}(A) = A\text{-Mod}_{\mathrm{inj}} \cap \mathrm{Acyc}(A)$ is of the form $\mathcal{S}^{\perp}$ for some set $\mathcal{S}$ as this is true both for $A\text{-Mod}_{\mathrm{inj}}$ (Proposition II.2.3.6) and $\mathrm{Acyc}(A)$ (Proposition II.2.3.5). Hence $\left({}^{\perp i}\mathcal{F}^{\mathrm{co}}_{\mathrm{sing}}(A), {}^{i}\mathcal{F}^{\mathrm{co}}_{\mathrm{sing}}(A)\right)$ is a complete cotorsion pair. By Lemma II.2.3.4, we have that ${}^{i}\mathcal{F}^{\mathrm{co}}_{\mathrm{sing}}(A) \cap {}^{\perp}({}^{i}\mathcal{F}^{\mathrm{co}}_{\mathrm{sing}}(A)) = A\text{-Mod}_{\mathrm{inj}}$, and Lemma II.2.1.17 again provides the 2-out-of-3 property since ${}^{i}\mathcal{F}^{\mathrm{co}}_{\mathrm{sing}}(A)$ is closed under cokernels of monomorphisms. Finally the identity is a right Quillen equivalence ${}^{i}\mathcal{M}^{\mathrm{co}}_{\mathrm{sing}}(A) \to \mathcal{M}^{\mathrm{co}}_{\mathrm{sing}}(A)$ by Proposition II.2.1.20. □

Fact II.4.2.2. *For a dg ring A, the four singular model structures $\mathcal{M}^{\mathrm{ctr}}_{\mathrm{sing}}(A)$, ${}^{p}\mathcal{M}^{\mathrm{ctr}}_{\mathrm{sing}}(A)$, $\mathcal{M}^{\mathrm{co}}_{\mathrm{sing}}(A)$ and ${}^{i}\mathcal{M}^{\mathrm{co}}_{\mathrm{sing}}(A)$ on A-Mod are connected via the following square of identity*

Quillen adjunctions, the horizontal adjunctions being Quillen equivalences:

$$\begin{array}{ccc}
{}^{p}\mathcal{M}^{\mathrm{ctr}}_{\mathrm{sing}}(A) & \underset{\mathrm{R}}{\overset{\mathrm{L}}{\rightleftarrows}} & \mathcal{M}^{\mathrm{ctr}}_{\mathrm{sing}}(A) \\
\mathrm{L}\downarrow\uparrow\mathrm{R} & & \mathrm{L}\downarrow\uparrow\mathrm{R} \\
\mathcal{M}^{\mathrm{co}}_{\mathrm{sing}}(A) & \underset{\mathrm{R}}{\overset{\mathrm{L}}{\rightleftarrows}} & {}^{i}\mathcal{M}^{\mathrm{co}}_{\mathrm{sing}}(A)
\end{array}$$

(Here, L resp. R denote the left resp. right Quillen functors) In particular, the left vertical adjunction is a Quillen equivalence if and only if the right vertical one is.

Proof. By Proposition II.4.2.1 it only remains to be proved that ${}^{p}\mathcal{M}^{\mathrm{ctr}}_{\mathrm{sing}}(A) \rightleftarrows \mathcal{M}^{\mathrm{co}}_{\mathrm{sing}}(A)$ and $\mathcal{M}^{\mathrm{ctr}}_{\mathrm{sing}}(A) \rightleftarrows {}^{i}\mathcal{M}^{\mathrm{co}}_{\mathrm{sing}}(A)$ are Quillen adjunctions. For the former, we need to check that $\mathcal{M}^{\mathrm{co}}_{\mathrm{sing}}(A) \to {}^{p}\mathcal{M}^{\mathrm{ctr}}_{\mathrm{sing}}(A)$ preserves fibrant and trivially fibrant objects; as everything is fibrant in ${}^{p}\mathcal{M}^{\mathrm{ctr}}_{\mathrm{sing}}(A)$, this is equivalent to $\mathcal{W}^{\mathrm{co}}_{\mathrm{sing}}(A) \cap \mathcal{F}^{\mathrm{co}}_{\mathrm{sing}}(A) = \mathcal{C}^{\mathrm{co}}_{\mathrm{sing}}(A)^{\perp} = \mathrm{Acyc}(A)^{\perp}$ being contained in ${}^{p}\mathcal{W}^{\mathrm{ctr}}_{\mathrm{sing}}(A) = (\mathrm{Acyc}(A) \cap A\text{-}\mathrm{Mod}_{\mathrm{proj}})^{\perp}$, which is clear. The check for the adjunction $\mathcal{M}^{\mathrm{ctr}}_{\mathrm{sing}}(A) \rightleftarrows {}^{i}\mathcal{M}^{\mathrm{co}}_{\mathrm{sing}}(A)$ is similar. □

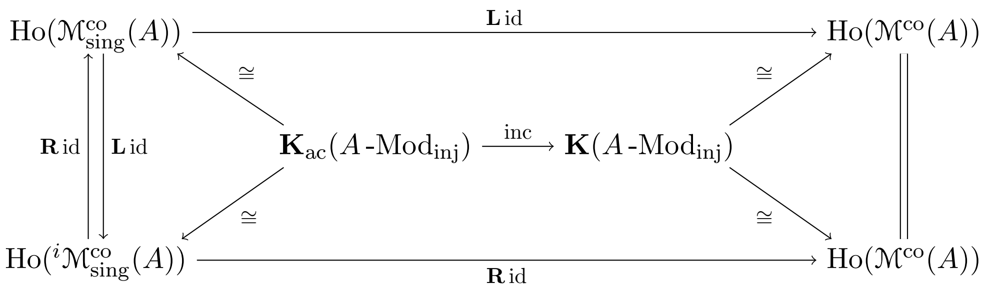
Corollary II.4.2.3. *For any dg ring A, there is a canonical adjunction*

$$\mathbf{K}_{\mathrm{ac}}(A\text{-}\mathrm{Mod}_{\mathrm{proj}}) \rightleftarrows \mathbf{K}_{\mathrm{ac}}(A\text{-}\mathrm{Mod}_{\mathrm{inj}}).$$

Example II.4.2.4. As in Example II.2.3.12 we consider the case where $A = R$ is a commutative Noetherian ring viewed as a dg ring concentrated in degree 0. Then the adjunction from Corollary II.4.2.3 has the form $\mathbf{K}_{\mathrm{ac}}(\mathrm{Proj}(R)) \rightleftarrows \mathbf{K}_{\mathrm{ac}}(\mathrm{Inj}(R))$, and the argument from Example II.2.3.12 shows that the left adjoint $\mathbf{K}_{\mathrm{ac}}(\mathrm{Proj}(R)) \to \mathbf{K}_{\mathrm{ac}}(\mathrm{Inj}(R))$ is given by $- \otimes_R I^*$, where $R \to I^*$ is an injective resolution of R over itself. ◊

Remark II.4.2.5. We do not expect a variant of Proposition II.4.2.1 to hold for the relative singular models attached to a morphism $\varphi : R \to A$ since we see no reason for $\mathcal{W}^{\mathrm{ctr}}(R)$ and $U_{\varphi}^{*}\mathcal{W}^{\mathrm{ctr}}(R)$ to be deconstructible (resp. for $\mathcal{W}^{\mathrm{co}}(R)$ and $U_{\varphi}^{*}\mathcal{W}^{\mathrm{co}}(R)$ to be of the form $\mathcal{S}^{\perp}$ for a set of objects $\mathcal{S}$). For the absolute singular models, this is different, because luckily $\mathrm{Acyc}(A)$ arises both as the cotorsionfree class in $(\mathcal{C}^{\mathrm{proj}}(A), \mathrm{Acyc}(A))$ and as the cotorsion class in $(\mathrm{Acyc}(A), \mathcal{F}^{\mathrm{inj}}(A))$. ◊

Let us pause for a moment to see what model structures are currently around, restricting to the injective case. We started out with the identity right Quillen functor $\mathcal{M}^{\mathrm{inj}}(A) \to \mathcal{M}^{\mathrm{co}}(A)$ and applied Theorem II.3.1.2 to get the right localization $\mathcal{M}^{\mathrm{co}}_{\mathrm{sing}}(A) := \mathcal{M}^{\mathrm{inj}}(A)/\mathcal{M}^{\mathrm{co}}(A)$, fitting into a colocalization sequence $\mathrm{Ho}(\mathcal{M}^{\mathrm{inj}}(A)) \to \mathrm{Ho}(\mathcal{M}^{\mathrm{co}}(A)) \to \mathrm{Ho}(\mathcal{M}^{\mathrm{co}}_{\mathrm{sing}}(A))$. Now, however, we have also constructed a model ${}^{i}\mathcal{M}^{\mathrm{co}}_{\mathrm{sing}}(A)$ for which the identity is *right* Quillen ${}^{i}\mathcal{M}^{\mathrm{co}}_{\mathrm{sing}}(A) \to \mathcal{M}^{\mathrm{co}}(A)$, and on the level of homotopy categories we have the following commutative diagram:



Note that the diagonal functors are equivalences since they are the canonical functors from the homotopy category of cofibrant and fibrant objects into the homotopy category. From this diagram we see that $\mathbf{L}\,\mathrm{id} : \mathcal{M}^{\mathrm{co}}_{\mathrm{sing}}(A) \to \mathcal{M}^{\mathrm{co}}(A)$ and $\mathbf{R}\,\mathrm{id} : {}^{i}\mathcal{M}^{\mathrm{co}}_{\mathrm{sing}}(A) \to \mathcal{M}^{\mathrm{co}}(A)$ are equivalent, and hence $\mathbf{L}\,\mathrm{id} : \mathcal{M}^{\mathrm{co}}_{\mathrm{sing}}(A) \to \mathcal{M}^{\mathrm{co}}(A)$ has a *left* adjoint while $\mathbf{R}\,\mathrm{id} : {}^{i}\mathcal{M}^{\mathrm{co}}_{\mathrm{sing}}(A) \to \mathcal{M}^{\mathrm{co}}(A)$ has a *right* adjoint. Thus:

Corollary II.4.2.6. *For any dg ring A, there is a recollement*

$$\mathbf{K}_{\mathrm{ac}}(A\text{-Mod}_{\mathrm{inj}}) \leftrightarrows \mathbf{K}(A\text{-Mod}_{\mathrm{inj}}) \leftrightarrows \mathbf{D}(A).$$

Proof. $\mathbf{K}_{\mathrm{ac}}(A\text{-Mod}_{\mathrm{inj}}) \to \mathbf{K}(A\text{-Mod}_{\mathrm{inj}}) \to \mathbf{D}(A)$ is a colocalization sequence by Corollary II.3.1.6, and by the above $\mathbf{K}_{\mathrm{ac}}(A\text{-Mod}_{\mathrm{inj}}) \to \mathbf{K}(A\text{-Mod}_{\mathrm{inj}})$ also has a left adjoint. This is all we need for a recollement. □

In case A is a Noetherian ring (considered as a dg ring concentrated in degree 0) the recollement from Corollary II.4.2.6 was constructed by Krause [Kra05, Corollary 4.3] in the more general framework of complexes over a locally Noetherian Grothendieck category with compactly generated derived category.

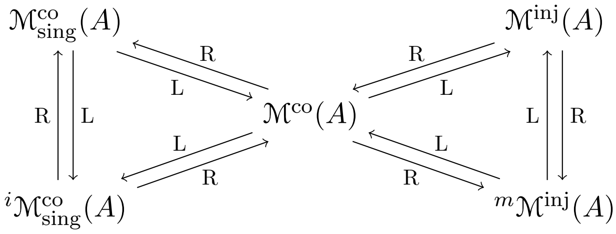
Dually, in the projective/contraderived situation we have the following recollement, which again is already known for ordinary rings by [Mur07, Theorem 5.15]:

Corollary II.4.2.7. *For any dg ring A, there is a recollement*

$$\mathbf{K}_{\mathrm{ac}}(A\text{-Mod}_{\mathrm{proj}}) \leftrightarrows \mathbf{K}(A\text{-Mod}_{\mathrm{proj}}) \leftrightarrows \mathbf{D}(A).$$

Back in the injective situation we also want to give a model categorical construction of the left adjoint of $\mathbf{K}(A\text{-Mod}_{\mathrm{inj}}) \to \mathbf{D}(A)$. For this, note that the injective version ${}^{i}\mathcal{M}^{\mathrm{co}}_{\mathrm{sing}}(A)$ of the singular coderived model structure has ${}^{i}\mathcal{F}^{\mathrm{co}}_{\mathrm{sing}}(A) \subseteq \mathcal{F}^{\mathrm{co}}(A)$; we can therefore apply Theorem II.3.1.2 to form the right localization ${}^{m}\mathcal{M}^{\mathrm{inj}}(A) := \mathcal{M}^{\mathrm{co}}(A)/{}^{i}\mathcal{M}^{\mathrm{co}}_{\mathrm{sing}}(A)$. This is the abelian model structure determined by ${}^{m}\mathcal{C}^{\mathrm{inj}}(A) =$

${}^{\perp}(\mathrm{Acyc}(A) \cap A\text{-}\mathrm{Mod}_{\mathrm{inj}})$ and ${}^{m}\mathcal{F}^{\mathrm{inj}}(A) = A\text{-}\mathrm{Mod}_{\mathrm{inj}}$, and the identity is a left Quillen functor ${}^{m}\mathcal{M}^{\mathrm{inj}}(A) \to \mathcal{M}^{\mathrm{inj}}(A)$. All in all, we get the following diagram of abelian model structures and Quillen functors on A-Mod, where L denotes left Quillen functors and R denotes right Quillen functors:



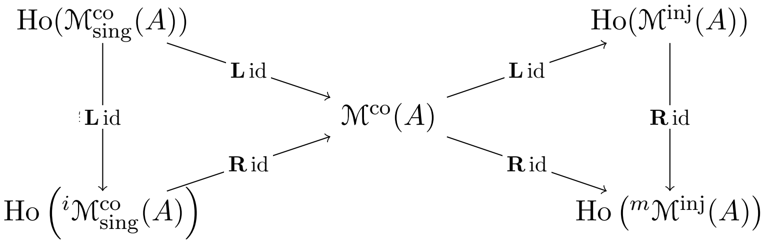
(⋈)

The properties of this diagram are summarized in the following proposition:

Proposition II.4.2.8. *Let A be a dg ring and consider the diagram* (⋈).

(⋈.i) $\mathcal{M}^{\mathrm{co}}_{\mathrm{sing}}(A) \rightleftarrows {}^{i}\mathcal{M}^{\mathrm{co}}_{\mathrm{sing}}(A)$ *and* ${}^{m}\mathcal{M}^{\mathrm{inj}}(A) \rightleftarrows \mathcal{M}^{\mathrm{inj}}(A)$ *are Quillen equivalences. More precisely, the classes of bifibrant objects in* $\mathcal{M}^{\mathrm{co}}_{\mathrm{sing}}(A)$ *and* ${}^{i}\mathcal{M}^{\mathrm{co}}_{\mathrm{sing}}(A)$ *coincide, and the classes of weak equivalences in* $\mathcal{M}^{\mathrm{inj}}(A)$ *and* ${}^{m}\mathcal{M}^{\mathrm{inj}}(A)$ *coincide.*

(⋈.ii) *The two wings in the following following diagram commute:*



(II.4.2.1)

(⋈.iii) $\mathcal{M}^{\mathrm{inj}}(A) \to \mathcal{M}^{\mathrm{co}}(A) \to \mathcal{M}^{\mathrm{co}}_{\mathrm{sing}}(A)$ *and* ${}^{i}\mathcal{M}^{\mathrm{co}}_{\mathrm{sing}}(A) \to \mathcal{M}^{\mathrm{co}}(A) \to {}^{m}\mathcal{M}^{\mathrm{inj}}(A)$ *are right localizations in the sense of Theorem II.3.1.2 – in particular:*

(⋈.iv) *The derived horizontal adjunctions in* (⋈) *realize the homotopy categories of the left and right sides as full subcategories of the homotopy category of the middle term, and the horizontal functors in* (II.4.2.1) *form an exact sequence.*

Proof. (⋈.iii) and the part of (⋈.i) concerning $\mathcal{M}^{\mathrm{co}}_{\mathrm{sing}}(A) \rightleftarrows {}^{i}\mathcal{M}^{\mathrm{co}}_{\mathrm{sing}}(A)$ hold by definition, and (⋈.iv) follows from (⋈.iii) using Corollary II.3.1.6. Consider now the adjunction ${}^{m}\mathcal{M}^{\mathrm{inj}}(A) \rightleftarrows \mathcal{M}^{\mathrm{inj}}(A)$: By Fact II.3.1.1 the weak equivalences in ${}^{m}\mathcal{M}^{\mathrm{inj}}(A)$ are compositions of monomorphisms with cokernel in ${}^{\perp}\left({}^{m}\mathcal{F}^{\mathrm{inj}}(A)\right) = \mathcal{W}^{\mathrm{co}}(A)$ and epimorphisms with kernel in $\mathrm{Acyc}(A) \cap A\text{-}\mathrm{Mod}_{\mathrm{inj}}$. In particular, any weak equivalence in ${}^{m}\mathcal{M}^{\mathrm{inj}}(A)$ is a quasi-isomorphism. Conversely, suppose $f : A \to B$ is a quasi-isomorphism and $f = g \circ h$ is a factorization of f into a trivial cofibration $h : A \to C$ followed by a fibration $g : C \to B$, both with respect to ${}^m\mathcal{M}^{\mathrm{inj}}(A)$. Then h is a monomorphism with cokernel in $\mathcal{W}^{\mathrm{co}}(A)$, so in particular it is a quasi-isomorphism. Consequently, $g : C \to B$ is both an epimorphism with kernel in A-$\mathrm{Mod}_{\mathrm{inj}}$ and a quasi-isomorphism, hence a trivial fibration in ${}^m\mathcal{M}^{\mathrm{inj}}(A)$. As the composition of g and h, we conclude that f is a weak equivalence in ${}^m\mathcal{M}^{\mathrm{inj}}(A)$, too, as claimed. Finally, (⋈.ii) follows from (⋈.i). □

The description of weak equivalences in ${}^m\mathcal{M}^{\mathrm{inj}}(\mathscr{A})$ seems worth emphasizing:

Corollary II.4.2.9. *The mixed injective model* ${}^m\mathcal{M}^{\mathrm{inj}}(A)$ *for* $\mathbf{D}(A)$ *is given by*

$${}^m\mathcal{M}^{\mathrm{inj}}(A) \;=\; ({}^{\perp}(\mathrm{Acyc}(A) \cap A\text{-}\mathrm{Mod}_{\mathrm{inj}}), \mathrm{Acyc}(A), A\text{-}\mathrm{Mod}_{\mathrm{inj}}),$$

and its weak equivalences are precisely the quasi-isomorphisms.

Remark II.4.2.10. In the projective situation the analogues of Proposition II.4.2.8 and Corollary II.4.2.9 hold for the projective/contraderived/singular model structures. ◊

Definition II.4.2.11. *A diagram of the form (⋈) of identity Quillen adjunctions having properties (⋈.i), (⋈.ii) and (⋈.iv) will be called a* butterfly of model structures.

To summarize, Proposition II.4.2.8 shows that when lifting a recollement $\mathcal{T}' \leftrightarrows \mathcal{T} \leftrightarrows \mathcal{T}''$ of triangulated categories to the world of model categories, it is likely to happen that it unfolds to a butterfly of model structures and Quillen adjunctions between them. The two adjoints both for $\mathcal{T}' \to \mathcal{T}$ and $\mathcal{T} \to \mathcal{T}''$ are then explained by the presence of two different model structures for $\mathcal{T}'$ and $\mathcal{T}''$, compensating the fact that a functor between model categories is usually either left or right Quillen, but rarely both.

II.4.3. Beyond enough projectives

When trying to generalize the results of the previous section to the setting of Remark II.2.3.19, we run into a problem: we need to know that A-$\mathrm{Mod}_{\mathrm{inj}} \cap \mathrm{Acyc}(A)$ is of the form $\mathcal{S}^{\perp}$ for some set of objects $\mathcal{S}$. If $\mathscr{A}$ has enough projectives, then $\mathrm{Acyc}(A) = \{\Sigma^k A \otimes P \mid k \in \Gamma\}^{\perp}$ for a projective generator P of $\mathscr{A}$ and hence A-$\mathrm{Mod}_{\mathrm{inj}} \cap \mathrm{Acyc}(A) = \mathcal{S}^{\perp}$ for $\mathcal{S}$ being the union of a representative set of isomorphism classes in $\{\Sigma^k A \otimes P \mid k \in \Gamma\}$, and $G^{+}(\mathcal{T})$, for a set $\mathcal{T} \subset A^{\sharp}$-Mod such that $A^{\sharp}$-Mod = filt-$\mathcal{T}$. However, without the existence of enough projectives, we don't know whether A-$\mathrm{Mod}_{\mathrm{inj}} \cap \mathrm{Acyc}(A)$ is of the form $\mathcal{S}^{\perp}$ for some set $\mathcal{S} \subset A$-Mod.

Proposition II.4.3.1. *The following are equivalent:*

(i) *There exists a set* $\mathcal{S} \subset A$-Mod *such that* $\mathrm{Acyc}(A) \cap A\text{-}\mathrm{Mod}_{\mathrm{inj}} = \mathcal{S}^{\perp}$.

(ii) *The butterfly from Proposition II.4.2.8 exists.*

(iii) *The sequence* $\mathbf{K}_{\mathrm{ac}}(A\text{-Mod}_{\mathrm{inj}}) \to \mathbf{K}(A\text{-Mod}_{\mathrm{inj}}) \to \mathbf{D}(A)$ *is a recollement.*

(iv) *There exists a set* $\mathcal{S} \subset \mathbf{K}(A\text{-Mod}_{\mathrm{inj}})$ *with* $\mathbf{K}_{\mathrm{ac}}(A\text{-Mod}_{\mathrm{inj}}) = \mathcal{S}^{\perp}$ *in* $\mathbf{K}(A\text{-Mod}_{\mathrm{inj}})$.

Proof. The implications (i)⇒(ii)⇒(iii) follow as in the previous section. For (iii)⇒(iv), note that in any recollement $\mathcal{T}' \leftrightarrows \mathcal{T} \leftrightarrows \mathcal{T}''$ of triangulated categories, we have $I(\mathcal{T}') = (Q_\lambda \mathcal{T}'')^{\perp}$, where $I : \mathcal{T}' \to \mathcal{T}$ is the given embedding and $Q_\lambda : \mathcal{T}'' \to \mathcal{T}$ is the left adjoint to the projection $Q : \mathcal{T} \to \mathcal{T}''$; see [Kra10c, Section 4]. Further, in our situation the triangulated categories $\mathcal{T}$ and $\mathcal{T}'' \cong Q_\lambda \mathcal{T}''$ are well-generated by Proposition II.2.2.10, and hence it follows from [Kra10c, Theorem from the Introduction] that $Q_\lambda \mathcal{T}''$ is generated, as a localizing subcategory of $\mathcal{T}$, by a set $\mathcal{S}$ of objects. We conclude that $I(\mathcal{T}') = (Q_\lambda \mathcal{T}'')^{\perp} = \mathcal{S}^{\perp}$, proving (iv). Finally, assume that (iv) holds, so that there exists some set $\mathcal{S} \subset A\text{-Mod}_{\mathrm{inj}}$ such that for any $X \in A\text{-Mod}_{\mathrm{inj}}$ we have $[S, X] = 0$ for all $S \in \mathcal{S}$ if and only if X is acyclic. Further, let $\mathcal{S}' \subset A\text{-Mod}$ be a set of A-modules such that $A\text{-Mod}_{\mathrm{inj}} = \mathcal{S}'^{\perp}$ (in the sense of vanishing Ext^1); such a set exists by Proposition II.2.3.6. Then $\mathrm{Acyc}(A) \cap A\text{-Mod}_{\mathrm{inj}} = (\mathcal{S} \cup \mathcal{S}')^{\perp}$, since $(\mathcal{S} \cup \mathcal{S}')^{\perp} = \mathcal{S}^{\perp} \cap \mathcal{S}'^{\perp} = \mathcal{S}^{\perp} \cap A\text{-Mod}_{\mathrm{inj}}$ and $\mathrm{Ext}^1_{A\text{-Mod}}(X, I) \cong [\Omega X, I]$ if $I \in A\text{-Mod}_{\mathrm{inj}}$. □

The same holds if we are considering ordinary complexes over a Grothendieck category $\mathscr{A}$, not necessarily equipped with a monoidal structure. In this setup, we have the following very general positive result of Krause [Kra05], applying for example to the case $\mathscr{A} = \mathrm{QCoh}(X)$ for a Noetherian scheme X.

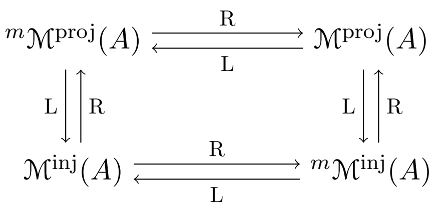
Theorem II.4.3.2 [Kra05, Corollary 4.3]**.** *The equivalent conditions of Proposition II.4.3.1 are fulfilled if* $\mathscr{A}$ *is locally Noetherian and* $\mathbf{D}(\mathscr{A})$ *is compactly generated.*

On the other hand, Neeman [Nee14] has constructed an example of a locally Noetherian Grothendieck category $\mathscr{A}$ in which the category of acyclic complexes of injectives is not stable under the naive, termwise product, and hence the inclusion $\mathbf{K}_{\mathrm{ac}}(\mathrm{Inj}(\mathscr{A})) \to \mathbf{K}(\mathrm{Inj}(\mathscr{A}))$ cannot have a left adjoint. Note, however, that it is not necessary that products of arbitrary objects in $\mathscr{A}$ are exact, as Krause's Theorem shows: In $\mathrm{QCoh}(\mathbb{P}^1_k)$, products are not exact [Kra05, Example 4.9], but nevertheless $\mathrm{QCoh}(\mathbb{P}^1_k)$ admits Krause's recollement as a consequence of Theorem II.4.3.2.

II.4.4. Comparing coderived and contraderived categories

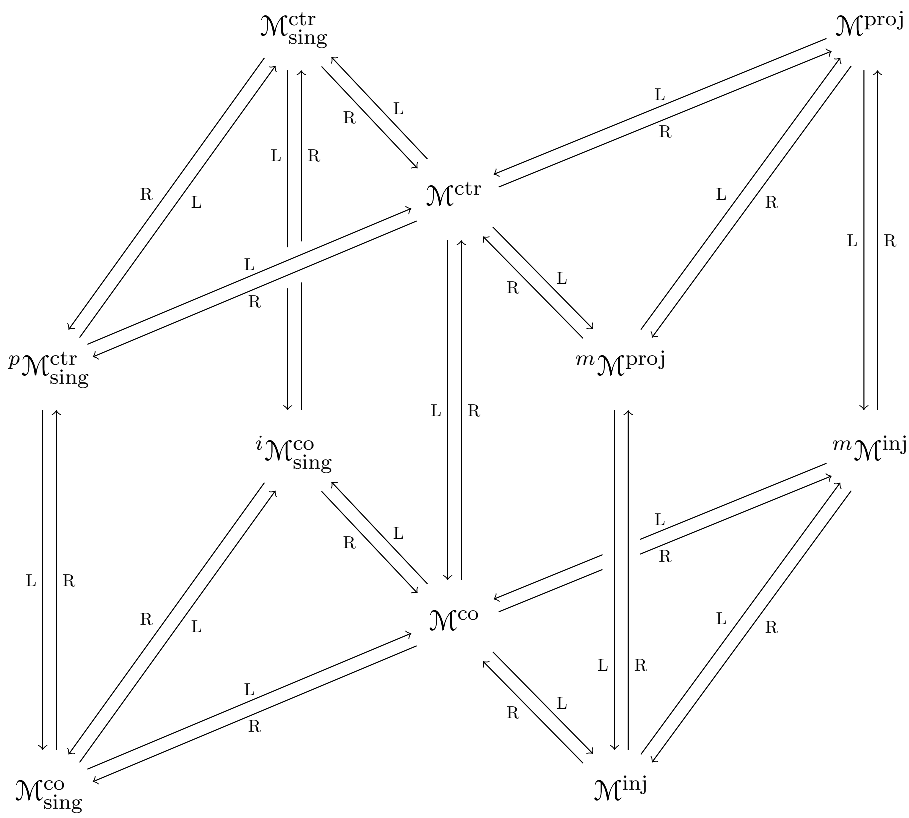
In this section we continue the comparison of injective, projective, coderived, contraderived (singular) model structures on A-Mod. Analogous to Fact II.4.2.2, we have:

Fact II.4.4.1. *For a dg ring A, the four models $\mathcal{M}^{\mathrm{proj}}(A)$, ${}^{m}\mathcal{M}^{\mathrm{proj}}(A)$, $\mathcal{M}^{\mathrm{inj}}(A)$ and ${}^{m}\mathcal{M}^{\mathrm{inj}}(A)$ on A-Mod for the ordinary derived category $\mathbf{D}(A)$ are connected via the following square of identity Quillen equivalences:*



Here, as usual L resp. R denote the left resp. right Quillen functors.

All in all, the projective and injective butterflies from Section II.4.2 together with the Quillen adjunctions from Fact II.4.2.2, Fact II.4.4.1 and Corollary II.2.3.7 form the following diagram of identity Quillen adjunctions:



Passing to homotopy categories, the wings of the injective and projective butterflies collapse, and we get the following diagram of triangulated categories:

Proposition II.4.4.2. *For any dg ring A there is an adjunction of recollements:*

$$\begin{array}{ccccc}
\mathbf{K}_{\mathrm{ac}}(A\text{-Mod}_{\mathrm{proj}}) & \overset{\mathrm{L}}{\underset{\mathrm{R}}{\overset{\longleftarrow}{\underset{\longleftarrow}{\xrightarrow{\mathrm{L=R}}}}}} & \mathbf{K}(A\text{-Mod}_{\mathrm{proj}}) & \overset{\mathrm{L}}{\underset{\mathrm{R}}{\overset{\longleftarrow}{\underset{\longleftarrow}{\xrightarrow{\mathrm{L=R}}}}}} & \mathbf{D}(A\text{-Mod}) \\
(\dagger)\ \mathrm{L}\downarrow\uparrow\mathrm{R} & & (\ddagger)\ \mathrm{L}\downarrow\uparrow\mathrm{R} & & \| \\
\mathbf{K}_{\mathrm{ac}}(A\text{-Mod}_{\mathrm{inj}}) & \overset{\mathrm{L}}{\underset{\mathrm{R}}{\overset{\longleftarrow}{\underset{\longleftarrow}{\xrightarrow{\mathrm{L=R}}}}}} & \mathbf{K}(A\text{-Mod}_{\mathrm{inj}}) & \overset{\mathrm{L}}{\underset{\mathrm{R}}{\overset{\longleftarrow}{\underset{\longleftarrow}{\xrightarrow{\mathrm{L=R}}}}}} & \mathbf{D}(A\text{-Mod}).
\end{array} \qquad \text{(II.4.4.2)}$$

In particular, the adjunction $(\dagger)$ *is an equivalence if and only if* $(\ddagger)$ *is.*

Definition II.4.4.3. *Here,* (II.4.4.2) *being an* adjunction of recollements *means:*

- *The two rows of the diagram are recollements.*
- *The vertical functors are left/right adjoints as indicated by their label L/R.*
- *Upon restriction to the left/right adjoints one gets commutative diagrams.*

Corollary II.4.4.4. *For any dg ring A, the following are equivalent:*

(i) *The Quillen adjunction $\mathcal{M}^{\mathrm{ctr}}(A) \rightleftarrows \mathcal{M}^{\mathrm{co}}(A)$ is a Quillen equivalence.*

(ii) *The Quillen adjunction ${}^{p}\mathcal{M}^{\mathrm{ctr}}_{\mathrm{sing}}(A) \rightleftarrows \mathcal{M}^{\mathrm{co}}_{\mathrm{sing}}(A)$ is a Quillen equivalence.*

(iii) *The Quillen adjunction $\mathcal{M}^{\mathrm{ctr}}_{\mathrm{sing}}(A) \rightleftarrows {}^{i}\mathcal{M}^{\mathrm{co}}_{\mathrm{sing}}(A)$ is a Quillen equivalence.*

Proof. The adjunction $\mathcal{M}^{\mathrm{ctr}}(A) \rightleftarrows \mathcal{M}^{\mathrm{co}}(A)$ is a Quillen equivalence if and only if the derived adjunction $\mathbf{K}(A\text{-Mod}_{\mathrm{proj}}) \rightleftarrows \mathbf{K}(A\text{-Mod}_{\mathrm{inj}})$ is an adjoint equivalence. By Proposition II.4.4.2, this is equivalent to $\mathbf{K}_{\mathrm{ac}}(A\text{-Mod}_{\mathrm{proj}}) \rightleftarrows \mathbf{K}_{\mathrm{ac}}(A\text{-Mod}_{\mathrm{inj}})$ being an adjoint equivalence, which is both the derived adjunction of $\mathcal{M}^{\mathrm{ctr}}_{\mathrm{sing}}(A) \rightleftarrows {}^{i}\mathcal{M}^{\mathrm{co}}_{\mathrm{sing}}(A)$ and ${}^{p}\mathcal{M}^{\mathrm{ctr}}_{\mathrm{sing}}(A) \rightleftarrows \mathcal{M}^{\mathrm{co}}_{\mathrm{sing}}(A)$. The claim follows. □

Proposition II.4.4.5. *For A left Gorenstein, the Quillen adjunctions*

$$\begin{array}{ccc}
{}^{p}\mathcal{M}^{\mathrm{ctr}}_{\mathrm{sing}}(A) & \overset{\mathrm{L}}{\underset{\mathrm{R}}{\rightleftarrows}} & \mathcal{M}^{\mathrm{ctr}}_{\mathrm{sing}}(A) \\
\mathrm{L}\downarrow\uparrow\mathrm{R} & & \mathrm{L}\downarrow\uparrow\mathrm{R} \\
\mathcal{M}^{\mathrm{co}}_{\mathrm{sing}}(A) & \overset{\mathrm{L}}{\underset{\mathrm{R}}{\rightleftarrows}} & {}^{i}\mathcal{M}^{\mathrm{co}}_{\mathrm{sing}}(A)
\end{array}$$

from Fact II.4.2.2 are Quillen equivalences.

Proof. By Proposition II.2.3.14 the Quillen adjunction $\mathcal{M}^{\mathrm{ctr}}(A) \rightleftarrows \mathcal{M}^{\mathrm{co}}(A)$ is a Quillen equivalence, which together with Corollary II.4.4.4 proves the claim. □

Example II.4.4.6. Suppose as in Example II.2.3.15 that $A = R$ is a Gorenstein ring considered as a dg ring concentrated in degree 0. Then we know from Proposition II.4.4.5 that the left derived functor $\mathbf{K}_{\mathrm{ac}}(\mathrm{Proj}(R)) \to \mathbf{K}_{\mathrm{ac}}(\mathrm{Inj}(R))$ of ${}^{p}\mathcal{M}^{\mathrm{ctr}}_{\mathrm{sing}}(R) \rightleftarrows \mathcal{M}^{\mathrm{co}}_{\mathrm{sing}}(R)$ is an adjoint equivalence. Moreover, by Proposition II.4.4.2, it is the restriction of the left derived functor $\mathbf{K}(A\text{-}\mathrm{Mod}_{\mathrm{proj}}) \to \mathbf{K}(A\text{-}\mathrm{Mod}_{\mathrm{inj}})$ of $\mathcal{M}^{\mathrm{ctr}}(R) \rightleftarrows \mathcal{M}^{\mathrm{co}}(R)$, which by Example II.2.3.12 is given by $- \otimes_R I^*$ for an injective resolution $R \to I^*$ of R over itself. Hence, we conclude that also $- \otimes_R I^* : \mathbf{K}_{\mathrm{ac}}(\mathrm{Proj}(R)) \longrightarrow \mathbf{K}_{\mathrm{ac}}(\mathrm{Inj}(R))$ is an equivalence. $\lozenge$

II.5. Examples

In this section we study the singular model structures for Gorenstein rings (Section II.5.1), for the Koszul algebra (Section II.5.2) and for hypersurfaces (Section II.5.3). The main results are Theorems II.5.1.5, II.5.2.13 and II.5.3.2, in particular lifting the equivalences $\mathbf{K}_{\mathrm{ac}}(\mathrm{Proj}(R)) \cong \underline{\mathrm{MCM}}(R)$ for a Gorenstein ring R and $\underline{\mathrm{MCM}}(R) \cong \underline{\mathrm{MF}}(S, w)$ for a hypersurface $R = S/(w)$ to the level of model categories.

II.5.1. Gorenstein rings

Let R be a Gorenstein ring, i.e. R is Noetherian and of finite injective dimension both as a left and as a right module over itself. Considering R as a dg ring concentrated in degree 0, we can form the absolute singular contraderived and coderived models $\mathcal{M}^{\mathrm{ctr}}_{\mathrm{sing}}(R)$ and $\mathcal{M}^{\mathrm{co}}_{\mathrm{sing}}(R)$ on $\mathrm{Ch}(R)$, see Definition II.4.1.2. The goal of this section is to see that they can be connected through a zig-zag of Quillen equivalences to Hovey's Gorenstein projective and injective models on R-Mod, respectively (see Proposition II.2.1.7). The "intermediate" model structures we meet along that zig-zag are the projective and injective versions ${}^{p}\mathcal{M}^{\mathrm{ctr}}_{\mathrm{sing}}(R)$ and ${}^{i}\mathcal{M}^{\mathrm{co}}_{\mathrm{sing}}(R)$ of the relative singular models introduced in Proposition II.4.2.1.

Denote σ_* resp. τ_* the brutal and soft truncation functors on categories of complexes of R-modules. Given such a complex (X, ∂), its k-th syzygy $\ker(\delta^k)$ is denoted $Z^k(X)$, and its k-th cosyzygy $\mathrm{coker}(\delta^{k-1})$ is denoted $Q^k(X)$. Given an R-module M, we denote $\iota^k(M)$ the stalk complex which has M sitting in degree k and vanishes otherwise.

Lemma II.5.1.1. *For any ring R, there are canonical adjunctions $Q^0 : \mathrm{Ch}(R) \rightleftarrows R\text{-Mod} : \iota^0$ and $\iota^0 : R\text{-Mod} \rightleftarrows \mathrm{Ch}(R) : Z^0$. Moreover, if R is Gorenstein, the adjunction $Q^0 \dashv \iota^0$ is a Quillen adjunction ${}^{p}\mathcal{M}^{\mathrm{ctr}}(R) \rightleftarrows \mathcal{M}^{\text{G-proj}}(R)$, and the adjunction $\iota^0 \dashv Z^0$ is a Quillen adjunction $\mathcal{M}^{\text{G-inj}}(R) \rightleftarrows {}^{i}\mathcal{M}^{\mathrm{co}}(R)$.*

Proof. To show that $Q^0 \dashv \iota^0$ is a Quillen adjunction ${}^{p}\mathcal{M}^{\mathrm{ctr}}_{\mathrm{sing}}(R) \rightleftarrows \mathcal{M}^{\text{G-proj}}(R)$ we have to check that Q^0 preserves cofibrations and trivial cofibrations. By Proposition II.4.2.1, a cofibration in ${}^{p}\mathcal{M}^{\mathrm{ctr}}_{\mathrm{sing}}(R)$ is a monomorphism of complexes $f : X \to Y$ such that $P := \mathrm{coker}(f)$ is an acyclic complex of projective R-modules. Given such an f, the long exact sequence in cohomology associated to the exact sequence of brutal truncations

$0 \to \sigma_{\leq 0}X \to \sigma_{\leq 0}Y \to \sigma_{\leq 0}P \to 0$ together with the acyclicity of P show that the sequence $0 \to Q^0(X) \to Q^0(Y) \to Q^0(P) \to 0$ is exact. Moreover, $Q^0(P) \in \text{G-proj}(R)$ by definition of Gorenstein projective modules, so $Q^0(f)$ is a monomorphism with Gorenstein projective cokernel, i.e. a cofibration in $\mathcal{M}^{\text{G-proj}}(R)$. Next, Q^0 preserves trivial cofibrations since these are monomorphisms with projective cokernel, and Q^0 preserves projective objects as the left adjoint to the exact functor ι^0.

The proof in the injective case is similar. □

To prove that the Quillen adjunctions from Lemma II.5.1.1 are Quillen equivalences, we need some preparation. We begin with two examples of weakly trivial objects in ${}^p\mathcal{M}^{\text{ctr}}_{\text{sing}}(R)$ and ${}^i\mathcal{M}^{\text{co}}_{\text{sing}}(R)$:

Proposition II.5.1.2. *Let R be a Gorenstein ring and $X \in \text{Ch}(R)$. Then we have $X \in {}^p\mathcal{W}^{\text{ctr}}_{\text{sing}}(R) = (\text{Acyc}(R) \cap \text{Ch}(\text{Proj}(R)))^{\perp}$ if either of the following holds:*

(i) $X \in \text{Ch}^+(\mathcal{P}^{<\infty})$.

(ii) $X \in \text{Ch}^-(R) \cap \text{Acyc}(R)$.

Similarly, we have $\text{Ch}^-(\mathcal{I}^{<\infty}) \cup (\text{Ch}^+(R) \cap \text{Acyc}(R)) \subset {}^i\mathcal{W}^{\text{co}}_{\text{sing}}(R)$.

Proof. We restrict to the projective case. Firstly, recall that for any $P \in {}^p\mathcal{C}^{\text{ctr}}_{\text{sing}}(R) = \text{Acyc}(R) \cap \text{Ch}(\text{Proj}(R))$ and any $X \in \text{Ch}(R\text{-Mod})$ we have $\text{Ext}^1_{\text{Ch}(R)}(P, X) \cong [P, \Sigma X]$. Now, if $X \in \text{Ch}^+(\mathcal{P}^{<\infty})$ then $[P, \Sigma X] = 0$ because P is acyclic, has Gorenstein projective syzygies and X consists of modules of finite projective dimension, which are injective relative to injections with Gorenstein projective cokernels. If $X \in \text{Ch}^-(R) \cap \text{Acyc}(R)$, then $[P, \Sigma X] = 0$ by the fundamental lemma of homological algebra. □

Corollary II.5.1.3. *Let R be a Gorenstein ring.*

(i) *For $X \in \text{Acyc}(R) \cap \text{Ch}(\mathcal{P}^{<\infty})$, the counit $X \to (\iota^0 \circ Q^0)(X)$ is a weak equivalence in ${}^p\mathcal{M}^{\text{ctr}}_{\text{sing}}(R)$, but vanishes in $\text{Ho}({}^i\mathcal{M}^{\text{co}}_{\text{sing}}(R))$.*

(ii) *Similarly, if $X \in \text{Acyc}(R) \cap \text{Ch}(\mathcal{I}^{<\infty})$, then the unit $(\iota^0 \circ Z^0)(X) \to X$ is a weak equivalence in ${}^i\mathcal{M}^{\text{co}}_{\text{sing}}(R)$, but vanishes in $\text{Ho}({}^p\mathcal{M}^{\text{ctr}}_{\text{sing}}(R))$.*

Proof. For the first part of (i), note that we have $\ker(X \to (\iota^0 \circ Q^0)(X)) = \tau_{\leq 0}(X) \oplus \sigma_{>0}(X)$, and both summands are weakly trivial in ${}^p\mathcal{M}^{\text{ctr}}_{\text{sing}}(R)$ by Proposition II.5.1.2. The first part of (ii) follows similarly. For the vanishing statements, first note that for $X \in \text{Acyc}(R)$ we have commutative diagram in $\text{Ch}(R)$,

$$\begin{array}{ccc} X & \twoheadrightarrow & \iota^0 Q^0 X \\ {\scriptstyle \mathrm{d} \simeq 0} \downarrow & & \downarrow {\scriptstyle \mathrm{d}\ \cong} \\ \Sigma X & \hookleftarrow & \iota^0 Z^1 X, \end{array}$$

and that the composition $\mathrm{d}: X \to \Sigma X$ in it is null-homotopic, hence vanishes in the homotopy categories $\mathrm{Ho}({}^p\mathcal{M}^{\mathrm{ctr}}_{\mathrm{sing}}(R))$ and $\mathrm{Ho}({}^i\mathcal{M}^{\mathrm{co}}_{\mathrm{sing}}(R))$. Now, if $X \in \mathrm{Acyc}(R) \cap \mathrm{Ch}(\mathcal{P}^{<\infty})$, we know from (i) that $X \to \iota^0 Q^0 X$ is an isomorphism in $\mathrm{Ho}({}^p\mathcal{M}^{\mathrm{ctr}}_{\mathrm{sing}}(R))$, hence $\iota^0 Z^1 X \to \Sigma X$ vanishes in $\mathrm{Ho}({}^p\mathcal{M}^{\mathrm{ctr}}_{\mathrm{sing}}(R))$. Up to replacing X by ΣX, this gives the vanishing statement of part (ii). Similarly, the vanishing statement of part (i) follows from the isomorphism statement of part (ii). $\square$

In particular, we see that although ${}^p\mathcal{M}^{\mathrm{ctr}}_{\mathrm{sing}}(R)$ and ${}^i\mathcal{M}^{\mathrm{co}}_{\mathrm{sing}}(R)$ are Quillen equivalent through the identity adjunction by the results of Section II.4.4, their classes of weak equivalences are *never* equal unless R is regular. The following proposition illustrates this in a way parallel to Example II.2.3.16, which discussed the case $R = k[x]/(x^2)$:

Corollary II.5.1.4. *For R Gorenstein and $X \in \mathrm{Acyc}(R) \cap \mathrm{Ch}(\mathcal{P}^{<\infty})$ we have:*

(i) *X is weakly trivial in ${}^p\mathcal{M}^{\mathrm{ctr}}_{\mathrm{sing}}(R)$ if and only if it is weakly trivial in ${}^i\mathcal{M}^{\mathrm{co}}_{\mathrm{sing}}(R)$.*

(ii) *$\tau_{\leq 0}X$ is weakly trivial in ${}^p\mathcal{M}^{\mathrm{ctr}}_{\mathrm{sing}}(R)$.*

(iii) *$\tau_{>0}X$ is weakly trivial in ${}^i\mathcal{M}^{\mathrm{co}}_{\mathrm{sing}}(R)$.*

In particular, for X not weakly trivial in ${}^p\mathcal{M}^{\mathrm{ctr}}_{\mathrm{sing}}(R)$ or ${}^i\mathcal{M}^{\mathrm{co}}_{\mathrm{sing}}(R)$ (for example, if $X \in \mathrm{Acyc}(R) \cap \mathrm{Ch}(\mathrm{Proj}(R))$ or $X \in \mathrm{Acyc}(R) \cap \mathrm{Ch}(\mathrm{Inj}(R))$ but X is not contractible), we have that $\tau_{\leq 0}X$ is weakly trivial in ${}^p\mathcal{M}^{\mathrm{ctr}}_{\mathrm{sing}}(R)$ but not weakly trivial in ${}^i\mathcal{M}^{\mathrm{co}}_{\mathrm{sing}}(R)$, while $\tau_{>0}X$ is weakly trivial in ${}^i\mathcal{M}^{\mathrm{co}}_{\mathrm{sing}}(R)$ but not weakly trivial in ${}^p\mathcal{M}^{\mathrm{ctr}}_{\mathrm{sing}}(R)$.

Proof. Parts (ii) and (iii) follow from Proposition II.5.1.2, and the last claim follows from (i)-(iii) together with the 2-out-of-3 property of the class of weakly trivial objects in any abelian model structure. We now prove (i): First, suppose $X \in \mathrm{Acyc}(R) \cap \mathrm{Ch}(\mathcal{P}^{<\infty})$ is weakly trivial in ${}^i\mathcal{M}^{\mathrm{co}}_{\mathrm{sing}}(R)$. Then, since the terms of X have finite injective dimension, as in the proof of Proposition II.2.3.14 we can find an acyclic complex of injectives I such that $X \cong I$ both in $\mathrm{Ho}({}^i\mathcal{M}^{\mathrm{co}}_{\mathrm{sing}}(R))$ and $\mathrm{Ho}({}^p\mathcal{M}^{\mathrm{ctr}}_{\mathrm{sing}}(R))$. Then $I \cong X \cong 0$ in $\mathrm{Ho}({}^i\mathcal{M}^{\mathrm{co}}_{\mathrm{sing}}(R))$, hence I is contractible as a complex, and so $X \cong I \cong 0$ in $\mathrm{Ho}({}^p\mathcal{M}^{\mathrm{ctr}}_{\mathrm{sing}}(R))$ as well, i.e. X is weakly trivial in ${}^p\mathcal{M}^{\mathrm{ctr}}_{\mathrm{sing}}(R)$. The converse is proved similarly. $\square$

Theorem II.5.1.5. *For R Gorenstein the adjunctions $Q^0 : {}^p\mathcal{M}^{\mathrm{ctr}}_{\mathrm{sing}}(R) \rightleftarrows \mathcal{M}^{\text{G-proj}}(R) : \iota^0$ and $\iota^0 : {}^i\mathcal{M}^{\mathrm{co}}_{\mathrm{sing}}(R) \rightleftarrows \mathcal{M}^{\text{G-inj}}(R) : Z^0$ from Lemma II.5.1.1 are Quillen equivalences.*

Proof. We restrict to the projective case, where we have to show the following:

(i) For each $X \in \mathrm{Acyc}(R) \cap \mathrm{Ch}(\mathrm{Proj}(R))$ the map $X \to \iota^0(Q^0(X))$ is a weak equivalence in ${}^p\mathcal{M}^{\mathrm{ctr}}_{\mathrm{sing}}(R)$.

(ii) For each $M \in R$-Mod and some (hence any) cofibrant replacement $P \to \iota^0(M)$ in ${}^p\mathcal{M}^{\text{ctr}}_{\text{sing}}(R)$, the resulting composition $Q^0(P) \to Q^0(\iota^0(M)) = M$ is a weak equivalence in $\mathcal{M}^{\text{G-proj}}(R)$.

(i) is already included in Corollary II.5.1.3. (ii): Pick a cofibrant replacement $p : K \to M$ in $\mathcal{M}^{\text{G-proj}}(R)$, i.e. p is a trivial fibration with K Gorenstein projective. As ι^0 is right Quillen, $\iota^0(p) : \iota^0(K) \to \iota^0(M)$ is a trivial fibration, too, and hence for a cofibrant replacement of $\iota^0(M)$ we may take any cofibrant replacement of $\iota^0(K)$. As $Q^0 \circ \iota^0 \cong \text{id}$, we may therefore assume M being Gorenstein projective right from the beginning. If in that case P is a complete projective resolution of M, we know from (i) that $P \to \iota^0(M)$ is a cofibrant replacement, and applying Q^0 gives the identity on M, a weak equivalence. □

Proposition II.5.1.6. *For R Gorenstein there is a zig-zag of left Quillen equivalences*

$$\mathcal{M}^{\text{ctr}}_{\text{sing}}(R) \xleftarrow{\text{id}} {}^p\mathcal{M}^{\text{ctr}}_{\text{sing}}(R) \xrightarrow{Q^0} \mathcal{M}^{\text{G-proj}}(R)$$

and a zig-zag of right Quillen equivalences

$$\mathcal{M}^{\text{co}}_{\text{sing}}(R) \xleftarrow{\text{id}} {}^i\mathcal{M}^{\text{co}}_{\text{sing}}(R) \xrightarrow{Z^0} \mathcal{M}^{\text{G-inj}}(R).$$

Remark II.5.1.7. As explained and stated around Proposition II.2.1.10, Bravo, Gillespie and Hovey [BGH14] have recently generalized the Gorenstein projective and injective model structures on modules over Gorenstein rings to the Gorenstein AC-projective and Gorenstein AC-injective model structures defined on R-Mod for an arbitrary ring R. Generalizing Proposition II.5.1.6 they also constructed [BGH14, Theorems 4.9 and 6.8] corresponding model structures on $\text{Ch}(R\text{-Mod})$ generalizing our ${}^i\mathcal{M}^{\text{co}}_{\text{sing}}(R)$ and ${}^p\mathcal{M}^{\text{ctr}}_{\text{sing}}(R)$ such that the adjunctions $Q^0 \dashv \iota^0$ and $\iota^0 \dashv Z^0$ are Quillen equivalences to the Gorenstein AC-projective resp. Gorenstein AC-injective model structures. ◊

Finally we study whether the Quillen equivalences ${}^p\mathcal{M}^{\text{ctr}}_{\text{sing}}(R) \rightleftarrows \mathcal{M}^{\text{G-proj}}(R)$ and $\mathcal{M}^{\text{G-inj}}(R) \rightleftarrows {}^i\mathcal{M}^{\text{co}}_{\text{sing}}(R)$ from the previous Proposition II.5.1.6 are compatible with the Quillen equivalence ${}^p\mathcal{M}^{\text{ctr}}_{\text{sing}}(R) \rightleftarrows {}^i\mathcal{M}^{\text{co}}_{\text{sing}}(R)$ between the coderived and contraderived singular model structures on $\text{Ch}(R)$ (see Section II.4.4) and the Quillen equivalence $\mathcal{M}^{\text{G-proj}}(R) \rightleftarrows \mathcal{M}^{\text{G-inj}}(R)$ between the Gorenstein projective and injective model structures on R-Mod:

Proposition II.5.1.8. *For R Gorenstein, consider the square of Quillen equivalences:*

$$\begin{array}{ccc} \mathcal{M}^{\text{G-inj}}(R) & \underset{\text{L}}{\overset{\text{R}}{\rightleftarrows}} & \mathcal{M}^{\text{G-proj}}(R) \\ \text{L} = \iota^0 \downarrow\uparrow \text{R} = Z^0 & & \text{R} = \iota^0 \downarrow\uparrow \text{L} = Q^0 \\ {}^i\mathcal{M}^{\text{co}}_{\text{sing}}(R) & \underset{\text{L}}{\overset{\text{R}}{\rightleftarrows}} & {}^p\mathcal{M}^{\text{ctr}}_{\text{sing}}(R) \end{array} \qquad \text{(II.5.1.1)}$$

It is not *commutative up to isomorphism when restricted to the left or right adjoints* L *resp.* R*, but there are canonical natural transformations*

$$\mathrm{id} \xRightarrow{\mathrm{prj}} \iota^0 \circ \mathrm{id} \circ Q^0 \xRightarrow{\partial} \Sigma : \ {}^p\mathcal{M}^{\mathrm{ctr}}_{\mathrm{sing}}(R) \longrightarrow {}^i\mathcal{M}^{\mathrm{co}}_{\mathrm{sing}}(R)$$

between the compositions of left adjoints, and dually canonical natural transformations

$$\Sigma^{-1} \xRightarrow{\partial} \iota^0 \circ \mathrm{id} \circ Z^0 \xRightarrow{\mathrm{inc}} \mathrm{id} : \ {}^i\mathcal{M}^{\mathrm{co}}_{\mathrm{sing}}(R) \to {}^p\mathcal{M}^{\mathrm{ctr}}_{\mathrm{sing}}(R)$$

between the compositions of right adjoints. Passing to homotopy categories we have:

(i) *The following derived transformations are isomorphisms:*

$$\mathbf{L}\iota^0 \circ \mathbf{L}\,\mathrm{id} \circ \mathbf{L}Q^0 \Longrightarrow \Sigma\mathbf{L}\,\mathrm{id} : \mathrm{Ho}({}^p\mathcal{M}^{\mathrm{ctr}}_{\mathrm{sing}}(R)) \to \mathrm{Ho}({}^i\mathcal{M}^{\mathrm{co}}_{\mathrm{sing}}(R))$$
$$\Sigma^{-1}\mathbf{R}\,\mathrm{id} \Longrightarrow \mathbf{R}\iota^0 \circ \mathbf{R}\,\mathrm{id} \circ \mathbf{R}Z^0 : \mathrm{Ho}({}^i\mathcal{M}^{\mathrm{co}}_{\mathrm{sing}}(R)) \to \mathrm{Ho}({}^p\mathcal{M}^{\mathrm{ctr}}_{\mathrm{sing}}(R))$$

(ii) *The following derived transformations vanish:*

$$\mathbf{L}\,\mathrm{id} \Longrightarrow \mathbf{L}\iota^0 \circ \mathbf{L}\,\mathrm{id} \circ \mathbf{L}Q^0 : \mathrm{Ho}({}^p\mathcal{M}^{\mathrm{ctr}}_{\mathrm{sing}}(R)) \to \mathrm{Ho}({}^i\mathcal{M}^{\mathrm{co}}_{\mathrm{sing}}(R))$$
$$\mathbf{R}\iota^0 \circ \mathbf{R}\,\mathrm{id} \circ \mathbf{R}Z^0 \Longrightarrow \mathbf{R}\,\mathrm{id} : \mathrm{Ho}({}^i\mathcal{M}^{\mathrm{co}}_{\mathrm{sing}}(R)) \to \mathrm{Ho}({}^p\mathcal{M}^{\mathrm{ctr}}_{\mathrm{sing}}(R))$$

In particular, the square of equivalences of homotopy categories induced by (II.5.1.1)

$$\begin{array}{ccc} \underline{\text{G-inj}}(R) \cong \mathrm{Ho}(\mathcal{M}^{\text{G-inj}}(R)) & \underset{\mathbf{L}\,\mathrm{id}}{\overset{\mathbf{R}\,\mathrm{id}}{\rightleftarrows}} & \mathrm{Ho}(\mathcal{M}^{\text{G-proj}}(R)) \cong \underline{\text{G-proj}}(R) \\ {\scriptstyle \mathbf{L}\iota^0}\downarrow\uparrow{\scriptstyle \mathbf{R}Z^0} & & {\scriptstyle \mathbf{R}\iota^0}\downarrow\uparrow{\scriptstyle \mathbf{L}Q^0} \\ \mathbf{K}_{\mathrm{ac}}(\mathrm{Inj}(R)) \cong \mathrm{Ho}({}^i\mathcal{M}^{\mathrm{co}}_{\mathrm{sing}}(R)) & \underset{\mathbf{L}\,\mathrm{id}}{\overset{\mathbf{R}\,\mathrm{id}}{\rightleftarrows}} & \mathrm{Ho}({}^p\mathcal{M}^{\mathrm{ctr}}_{\mathrm{sing}}(R)) \cong \mathbf{K}_{\mathrm{ac}}(\mathrm{Proj}(R)) \end{array} \qquad \text{(II.5.1.2)}$$

is not *canonically commutative up to isomorphism when restricted to the left or right derived functors, but instead any circular clockwise composition of the equivalences is canonically isomorphic to* Σ^{-1} *in the respective homotopy category.*

Proof. Unraveling the definitions, this follows from Corollary II.5.1.3. □

The commutativity of (II.5.1.2) up to shift is only a matter of convention: had we used the Quillen equivalence $\iota^1 : \mathcal{M}^{\text{G-inj}}(R) \rightleftarrows {}^i\mathcal{M}^{\mathrm{co}}_{\mathrm{sing}}(R) : Z^1$ instead of $\iota^0 \dashv Z^0$ in Theorem II.5.1.5 would have made (II.5.1.2) commutative up to canonical isomorphism. This becomes particularly clear if one considers the special case where R is self-injective. In this case we have $\mathrm{Proj}(R) = \mathrm{Inj}(R)$ and $\text{G-proj}(R) = \text{G-inj}(R) = R\text{-Mod}$, and (II.5.1.2) takes the simple form

$$\begin{array}{ccc} \underline{R\text{-Mod}} & = & \underline{R\text{-Mod}} \\ {\scriptstyle \cong}\uparrow{\scriptstyle Z^0} & & {\scriptstyle Q^0}\uparrow{\scriptstyle \cong} \\ \mathbf{K}_{\mathrm{ac}}(\mathrm{Inj}(R)) & = & \mathbf{K}_{\mathrm{ac}}(\mathrm{Proj}(R)) \end{array}$$

which commutes up to shift, $Q^0 \cong \Sigma Z^0$, because the shift ΣX of $X \in \underline{R\text{-Mod}}$ is defined by choice of a short exact sequence $0 \to X \to I \to \Sigma X \to 0$ with $I \in \mathrm{Proj}(R) = \mathrm{Inj}(R)$.

II.5.2. Curved mixed complexes

In this section we study the relative singular contraderived model structure on the category of curved mixed complexes over a ring and show that it is Quillen equivalent to the contraderived model structure on the corresponding category of linear factorizations.

Definition II.5.2.1. *Let S be a ring and $w \in Z(S)$.*

(i) *We denote $K_{S,w}$ the* Koszul-algebra *of (S, w), i.e. the $\mathbb{Z}$-graded algebra $S[s]/(s^2)$ with $\deg(s) = -1$ and differential d given by $\mathrm{d}(s) = w$.*

(ii) *We denote S_w the curved $\mathbb{Z}/2\mathbb{Z}$-graded dg ring with $(S_w)^{\overline{0}} = S$, $(S_w)^{\overline{1}} = 0$, trivial differential and curvature $w \in S = (S_w)^{\overline{2}}$.*

Fact II.5.2.2. *Let S be a ring and $w \in Z(S)$.*

(i) *A dg module over $K_{S,w}$ is a complex of S-modules with a square-zero nullhomotopy for the multiplication by w, i.e. a curved mixed complex with curvature w.*

(ii) *A curved dg module over S_w is a* linear factorization of type (S, w)*, i.e. a sequence $M^0 \xrightarrow{f} M^1 \xrightarrow{g} M^0$ of S-modules such that $fg = w \cdot \mathrm{id}_{M^1}$ and $gf = w \cdot \mathrm{id}_{M^0}$. Sometimes we abbreviate such a sequence by $g : M^1 \rightleftarrows M^0 : f$.*

We denote $\mathrm{LF}(S, w) := S_w\text{-Mod}$ the category of linear factorizations of type (S, w). The objects in the full subcategory $\mathrm{MF}(S, w) := S_w\text{-Mod}_{\mathrm{proj}}$ are called *matrix factorizations* of type (S, w); these are the linear factorizations $g : M^1 \rightleftarrows M^0 : f$ for which M^0, M^1 are projective as S-modules.

Viewing $K_{S,w}$-modules as curved mixed complexes, the cofibrant and fibrant objects in $\mathcal{M}^{\mathrm{ctr}}_{\mathrm{sing}}(K_{S,w}/S)$ are easy to describe in terms of the differentials of the mixed complex:

Proposition II.5.2.3. *Let $X = (X, \mathrm{d}, s)$ be a $K_{S,w}$-module. Then the following hold:*

(i) *X is cofibrant in $\mathcal{M}^{\mathrm{ctr}}_{\mathrm{sing}}(K_{S,w}/S)$ (or, equivalently, $\mathcal{M}^{\mathrm{ctr}}(K_{S,w})$) if and only if (X, s) is contractible and S-projective.*

(ii) *X is fibrant in $\mathcal{M}^{\mathrm{ctr}}_{\mathrm{sing}}(K_{S,w}/S)$ if and only if (X, d) is S-contraacyclic.*

(iii) *X is fibrant in $\mathcal{M}^{\mathrm{ctr}}_{\mathrm{sing}}(K_{S,w})$ if and only if (X, d) is acyclic.*

In particular, if S is semisimple, then X is cofibrant (resp. fibrant) in $\mathcal{M}^{\mathrm{ctr}}_{\mathrm{sing}}(K_{S,w}/S)$ if and only if (X, d) (resp. (X, s)) is acyclic.

Proof. (ii) and (iii) hold by definition. (i) is true by Lemma II.2.3.3, since, by definition, X is cofibrant in $\mathcal{M}^{\mathrm{ctr}}_{\mathrm{sing}}(K_{S,w}/S)$ or $\mathcal{M}^{\mathrm{ctr}}_{\mathrm{sing}}(K_{S,w})$ if and only if (X,s) is projective in $K^{\sharp}_{S,w}$-Mod $\cong \mathrm{Ch}(S)$. □

Curved mixed complexes with curvature w are connected to linear factorizations of type (S,w) via the operations of folding and stabilization:

Definition II.5.2.4. *Let S be a ring and $w \in Z(S)$. Further, let (X, d, s) be a $K_{S,w}$-module and $g : M^1 \rightleftarrows M^0 : f$ be a linear factorization of type (S,w).*

(i) *The* folding via products $\mathrm{fold}^{\Pi}(X)$ *of X is the linear factorization of type (S,w)*

$$\mathrm{fold}^{\Pi}(X) \;:=\; \prod_{n\in\mathbb{Z}} X^{2n} \xrightarrow{\mathrm{d}+s} \prod_{n\in\mathbb{Z}} X^{2n+1} \xrightarrow{\mathrm{d}+s} \prod_{n\in\mathbb{Z}} X^{2n}.$$

(ii) *The* folding via sums $\mathrm{fold}^{\oplus}(X)$ *of X is the linear factorization of type (S,w)*

$$\mathrm{fold}^{\oplus}(X) \;:=\; \bigoplus_{n\in\mathbb{Z}} X^{2n} \xrightarrow{\mathrm{d}+s} \bigoplus_{n\in\mathbb{Z}} X^{2n+1} \xrightarrow{\mathrm{d}+s} \bigoplus_{n\in\mathbb{Z}} X^{2n}.$$

(iii) *The* stable bar resolution $\underline{\mathrm{bar}}(M)$ *is the $K_{S,w}$-module given by*

$$\cdots \underset{\begin{pmatrix} 0 & 0 \\ \mathrm{id} & 0\end{pmatrix}}{\overset{\begin{pmatrix} f & w \\ -\mathrm{id} & -g\end{pmatrix}}{\rightleftarrows}} M^1 \oplus M^0 \underset{\begin{pmatrix} 0 & 0 \\ \mathrm{id} & 0\end{pmatrix}}{\overset{\begin{pmatrix} g & w \\ -\mathrm{id} & -f\end{pmatrix}}{\rightleftarrows}} M^0 \oplus M^1 \underset{\begin{pmatrix} 0 & 0 \\ \mathrm{id} & 0\end{pmatrix}}{\overset{\begin{pmatrix} f & w \\ -\mathrm{id} & -g\end{pmatrix}}{\rightleftarrows}} M^1 \oplus M^0 \underset{\begin{pmatrix} 0 & 0 \\ \mathrm{id} & 0\end{pmatrix}}{\overset{\begin{pmatrix} g & w \\ -\mathrm{id} & -f\end{pmatrix}}{\rightleftarrows}} \cdots,$$

where the terms $M^0 \oplus M^1$ live in cohomologically even degrees.

Proposition II.5.2.5. *There are canonical adjunctions $\underline{\mathrm{bar}} \dashv \mathrm{fold}^{\Pi}$, $\mathrm{fold}^{\oplus} \dashv \underline{\mathrm{bar}} \circ \Sigma$.*

Proof. Let $g : M^1 \rightleftarrows M^0 : f$ be a linear factorization of type (S,w) and $(X, \mathrm{d}, s) \in K_{S,w}$-Mod. A morphism $\underline{\mathrm{bar}}(M) \to X$ is given by a diagram

$$\begin{array}{ccccccccc}
\cdots \rightleftarrows & M^1 \oplus M^0 & \underset{\begin{pmatrix} 0 & 0 \\ \mathrm{id} & 0\end{pmatrix}}{\overset{\begin{pmatrix} g & w \\ -\mathrm{id} & -f\end{pmatrix}}{\rightleftarrows}} & M^0 \oplus M^1 & \underset{\begin{pmatrix} 0 & 0 \\ \mathrm{id} & 0\end{pmatrix}}{\overset{\begin{pmatrix} f & w \\ -\mathrm{id} & -g\end{pmatrix}}{\rightleftarrows}} & M^1 \oplus M^0 & \rightleftarrows \cdots \\
& \downarrow{\scriptstyle (\alpha_{-1}\ \alpha'_{-1})} & & \downarrow{\scriptstyle (\alpha_0\ \alpha'_0)} & & \downarrow{\scriptstyle (\alpha_1\ \alpha'_1)} & \\
\cdots \underset{s}{\overset{\mathrm{d}}{\rightleftarrows}} & X^{-1} & \underset{s}{\overset{\mathrm{d}}{\rightleftarrows}} & X^0 & \underset{s}{\overset{\mathrm{d}}{\rightleftarrows}} & X^1 & \underset{s}{\overset{\mathrm{d}}{\rightleftarrows}} \cdots
\end{array}$$

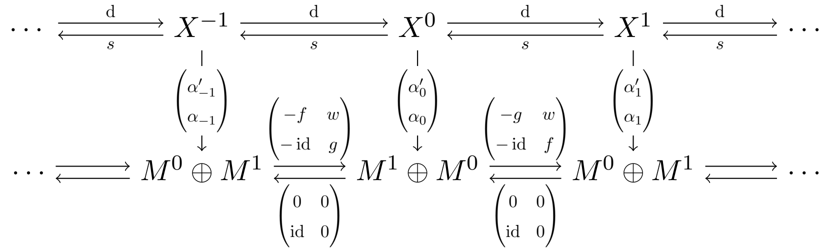
such that each square commutes both with respect to the maps pointing to the right and the ones pointing to the left. The latter is equivalent to $\alpha'_n = s\alpha_{n+1}$ for all $n \in \mathbb{Z}$, so assume this from now on. Writing ∂ in place of f and g (to avoid distinction of cases), the other commutativity constraint then writes as follows:

(i) $\alpha_n \partial - s\alpha_{n+1} = \mathrm{d}\,\alpha_{n-1}$.

(ii) $\mathrm{d}\, s\alpha_n = w\alpha_n - s\alpha_{n+1}\partial$.

The second condition follows from the first by applying $s \circ -$. Thus, the constraint on the family $\{\alpha_n\}_{n\in\mathbb{Z}}$ to yield a morphism of $K_{S,w}$-modules $\underline{\mathrm{bar}}(M) \to X$ is $\alpha\partial = (\mathrm{d} + s)\alpha$, in this in turn is equivalent to saying that $\prod \alpha_{2n}$ and $\prod \alpha_{2n+1}$ yield a morphism of linear factorizations $M \to \mathrm{fold}^{\Pi}(X)$.

Similarly, a morphism $X \to \underline{\mathrm{bar}}(M) \circ \Sigma$ is given by a diagram



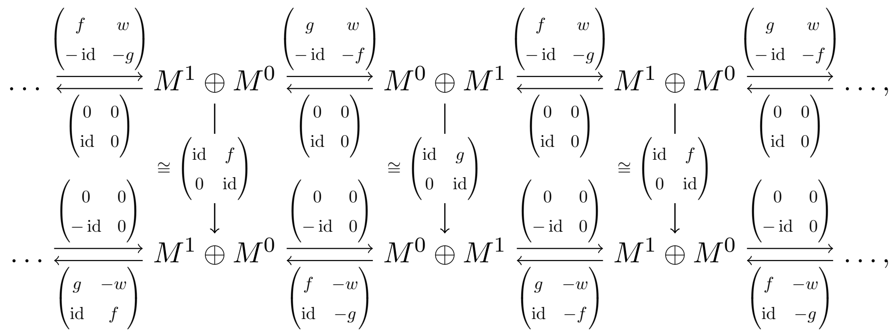
such that each square commutes both with respect to the maps pointing to the right and the ones pointing to the left. The latter is equivalent to $\alpha'_n = \alpha_{n-1}s$, and we assume this from now. Then, again writing ∂ for f and g, the other commutativity constraint writes as

(i) $w\alpha_n - \partial\alpha_{n-1}s = \alpha_n s\,\mathrm{d}$

(ii) $\partial\alpha_n - \alpha_{n-1}s = \alpha_{n+1}\,\mathrm{d}$.

The first condition follows from the second by applying $- \circ s$, and the second is equivalent to saying that $\bigoplus \alpha_{2n}$ and $\bigoplus \alpha_{2n+1}$ yield a morphism of S_w-modules $\mathrm{fold}^{\oplus}(X) \to M$. □

Remark II.5.2.6. At first sight, the asymmetry in the definition of $\underline{\mathrm{bar}}$ seems to contradict, by uniqueness of adjoints, the invariance of $\mathrm{fold}^{\oplus}$ and fold^{Π} under exchange of the two differentials d and s. However, uniqueness of adjoints holds only up to canonical isomorphism, and up to isomorphism, $\underline{\mathrm{bar}}$ *is* symmetric in d and s in the sense that for a linear factorization $g : M^1 \rightleftarrows M^0 : f$ of type (S, w), we have the following isomorphism:



◊

Proposition II.5.2.7. *For a ring S and $w \in Z(S)$, there are Quillen adjunctions:*

(i) $\underline{\mathrm{bar}} : \mathcal{M}^{\mathrm{ctr}}(S_w) \rightleftarrows \mathcal{M}^{\mathrm{ctr}}(K_{S,w}) : \mathrm{fold}^{\Pi}$

(ii) $\underline{\mathrm{bar}} : \mathcal{M}^{\mathrm{ctr}}(S_w) \rightleftarrows \mathcal{M}^{\mathrm{ctr}}_{\mathrm{sing}}(K_{S,w}) : \mathrm{fold}^{\Pi}$

(iii) $\underline{\mathrm{bar}} : \mathcal{M}^{\mathrm{ctr}}(S_w) \rightleftarrows \mathcal{M}^{\mathrm{ctr}}_{\mathrm{sing}}(K_{S,w}/S) : \mathrm{fold}^{\Pi}$

(iv) $\mathrm{fold}^{\oplus} : \mathcal{M}^{\mathrm{ctr}}(K_{S,w}) \rightleftarrows \mathcal{M}^{\mathrm{ctr}}(S_w) : \underline{\mathrm{bar}} \circ \Sigma$

(v) $\mathrm{fold}^{\oplus} : \mathcal{M}^{\mathrm{ctr}}_{\mathrm{sing}}(K_{S,w}) \rightleftarrows \mathcal{M}^{\mathrm{ctr}}(S_w) : \underline{\mathrm{bar}} \circ \Sigma$

(vi) $\mathrm{fold}^{\oplus} : \mathcal{M}^{\mathrm{ctr}}_{\mathrm{sing}}(K_{S,w}/S) \rightleftarrows \mathcal{M}^{\mathrm{ctr}}(S_w) : \underline{\mathrm{bar}} \circ \Sigma$

Proof. Because of the trivial identity left Quillen functors $\mathcal{M}^{\mathrm{ctr}}(A) \to \mathcal{M}^{\mathrm{ctr}}_{\mathrm{sing}}(A) \to \mathcal{M}^{\mathrm{ctr}}_{\mathrm{sing}}(A/R)$, (iii) & (ii) follow from (i), and (iv) & (v) follow from (vi).

For (i), we have to show that $\underline{\mathrm{bar}}$ preserves cofibrations and trivial cofibrations. By the exactness of $\underline{\mathrm{bar}}$ and the definition of an abelian model structure, it suffices to show $\underline{\mathrm{bar}}(\mathcal{C}) \subset \mathcal{C}$ and $\underline{\mathrm{bar}}(\mathcal{C} \cap \mathcal{W}) \subset \mathcal{C} \cap \mathcal{W}$. The cofibrants in $\mathcal{M}^{\mathrm{ctr}}(S_w)$ are the matrix factorizations, i.e. those $g : M^1 \rightleftarrows M^0 : f$ with M^0, M^1 projective S-modules, and the cofibrants in $\mathcal{M}^{\mathrm{ctr}}(K_{S,w})$ are the $K_{S,w}$-modules with underlying projective $K^{\sharp}_{S,w}$-modules. By definition of $\underline{\mathrm{bar}}$, the $K^{\sharp}_{S,w}$-module underlying $\underline{\mathrm{bar}}(M)$ is isomorphic to $\bigoplus_{n\in\mathbb{Z}} K^{\sharp}_{S,w} \otimes_S \Sigma^{2n} M^0 \oplus K^{\sharp}_{S,w} \otimes_S \Sigma^{2n+1} M^1$, and hence is $K^{\sharp}_{S,w}$-projective if M^0, M^1 are S-projective. This proves $\underline{\mathrm{bar}}(\mathcal{C}) \subset \mathcal{C}$. The assertion $\underline{\mathrm{bar}}(\mathcal{C}\cap\mathcal{W}) \subset \mathcal{C}\cap\mathcal{W}$ is clear because $\mathcal{C} \cap \mathcal{W} = \mathcal{P}$ for both model structures, and $\underline{\mathrm{bar}}$ preserves projectives as the left adjoint to the exact functor fold^{Π}.

For (vi), we have to show that $(\underline{\mathrm{bar}} \circ \Sigma)(\mathcal{F}) \subset \mathcal{F}$ and $(\underline{\mathrm{bar}} \circ \Sigma)(\mathcal{W} \cap \mathcal{F}) \subset \mathcal{W} \cap \mathcal{F}$. In $\mathcal{M}^{\mathrm{ctr}}(S_w)$ everything is fibrant, while in $\mathcal{M}^{\mathrm{ctr}}_{\mathrm{sing}}(K_{S,w}/S)$ the fibrants are the S-contraacyclic $K_{S,w}$-modules, so for $(\underline{\mathrm{bar}}\circ\Sigma)(\mathcal{F}) \subset \mathcal{F}$ we have to show that the image of $\underline{\mathrm{bar}}$ consists of S-contraacyclic complexes. The stable bar resolutions are even contractible as complexes of S-modules (see Remark II.5.2.6), so this follows from Proposition II.2.3.8. The other condition $(\underline{\mathrm{bar}} \circ \Sigma)(\mathcal{W} \cap \mathcal{F}) \subset \mathcal{W} \cap \mathcal{F}$ means that $\underline{\mathrm{bar}}$ maps S_w-contraacyclics to $K_{S,w}$-contraacyclics, i.e. that it maps $S_w\text{-Mod}^{\perp}_{\mathrm{proj}}$ to $K_{S,w}\text{-Mod}^{\perp}_{\mathrm{proj}}$. For this, suppose $X \in K_{S,w}\text{-Mod}$ and M is S_w-contraacyclic. Then $\mathrm{Ext}^1_{K_{S,w}}(X, (\underline{\mathrm{bar}} \circ \Sigma)(M)) \cong \mathrm{Ext}^1_{S_w}(\mathrm{fold}^{\oplus}(X), M)$, which is trivial since $\mathrm{fold}^{\oplus}(X) \in S_w\text{-Mod}_{\mathrm{proj}}$. □

Our goal is to show that the adjunctions II.5.2.7(iii) and II.5.2.7(vi) are Quillen equivalences, but before we come to the proof, we define the completed Bar resolution.

Fact II.5.2.8 [Wei94, Proposition 8.6.10]**.** *Let $F : \mathscr{A} \rightleftarrows \mathscr{B} : U$ be an adjunction between abelian categories and $\perp := FU : \mathscr{B} \to \mathscr{B}$ the associated comonad. For $X \in \mathscr{B}$ there is a canonical structure of a simplicial object on $\perp^* X := \left\{\perp^{n+1} X\right\}_{n \geq 0}$, and $U(\perp^* X)$ admits a canonical left contraction. In particular, if U is exact and faithful, then the normalized augmented chain complex $N(\perp^* X) \to X$ is acyclic.*

Corollary II.5.2.9. *Let S be a ring, A be a dg S-algebra and M an A-module. Let $\eta : S \to A$ be the structure map and $\overline{A} := \operatorname{coker}(\eta)$. Then the following augmented complex of A-modules is acyclic:*

$$\big(\ldots \to A \otimes_S \overline{A} \otimes_S \overline{A} \otimes_S M \to A \otimes_S \overline{A} \otimes_S M \to A \otimes_S M\big) \to M. \tag{II.5.2.3}$$

Definition II.5.2.10. *Let S be a ring, A be a dg S-algebra and M an A-module. The* completed Bar resolution *of M is the totalization of the augmented complex* (II.5.2.3) *formed by taking products, and is denoted $B^{\Pi} M \to M$.*

Lemma II.5.2.11. *Let S, A and M be as in Definition II.5.2.10 and let $q : B^{\Pi} M \to M$ be the completed Bar resolution. Then $\ker(q)$ is contraacyclic. In other words, the completed Bar resolution $B^{\Pi} M \to M$ is a trivial fibration in $\mathcal{M}^{\mathrm{ctr}}_{\mathrm{sing}}(A)$.*

Proof. The second statement follows from the first since the contraacyclic A-modules are precisely the trivially fibrant objects in $\mathcal{M}^{\mathrm{ctr}}_{\mathrm{sing}}(A)$. That $\ker(q)$ is contraacyclic follows from Lemma II.2.3.10 as it is the totalization by taking products of a bounded above exact sequence of A-modules. □

The following gives explicit descriptions of the functors $\underline{\mathrm{bar}} \circ \mathrm{fold}^{\Pi}$ and B^{Π}.

Lemma II.5.2.12. *Let (X, d, s) be a $K_{S,w}$-module. There are natural isomorphisms*

$$(\underline{\mathrm{bar}} \circ \mathrm{fold}^{\Pi})(X)^n \cong \prod_{k \in \mathbb{Z}} X^k \quad \text{and} \quad (B^{\Pi} X)^n \cong \prod_{k \geq n} X^k.$$

Under these isomorphisms, the $K_{S,w}$-module structure can be described as follows:

(i) d *acts on X^k as $\mathrm{d} + s - \mathrm{id}$ for $k \equiv n \pmod 2$ and as $w - \mathrm{d} - s$ otherwise.*

(ii) *s acts on X^k as id if $k \equiv n \pmod 2$ and as 0 otherwise.*

In particular, we have the following:

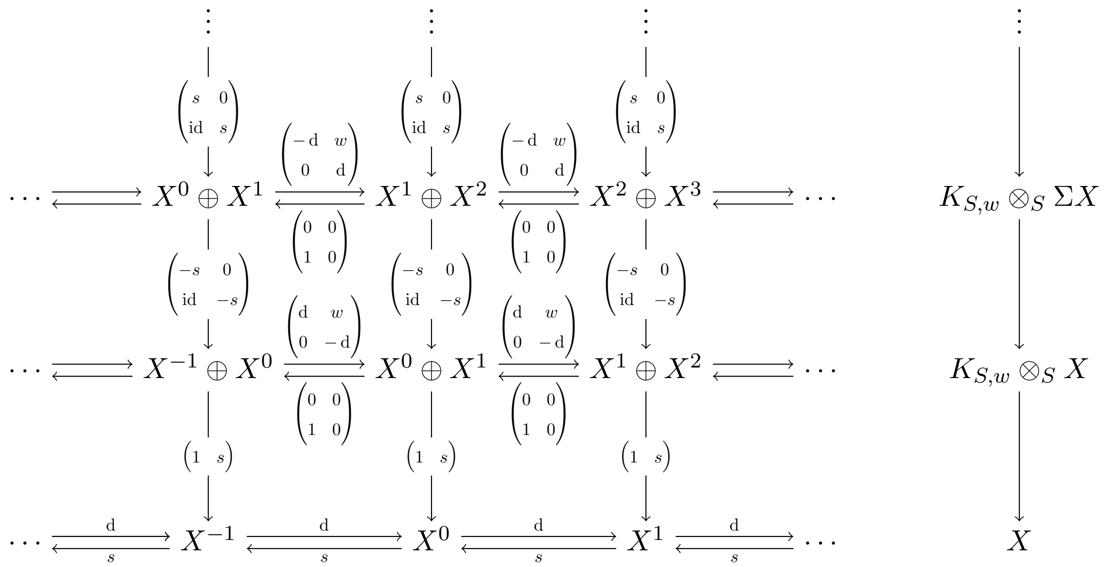
(i) *There is a canonical epimorphism of $K_{S,w}$-modules*

$$\alpha : (\underline{\mathrm{bar}} \circ \mathrm{fold}^{\Pi})(X) \longrightarrow B^{\Pi} X$$

with $\ker(\alpha)^n \cong \prod_{k<n} X^k$ and $K_{S,w}$-module structure as in (i) and (ii).

(ii) $\ker(\alpha)$ *admits a complete decreasing filtration* $\ldots \subset F_2 \subset F_1 \subset F_0 = \ker(\alpha)$ *with* $F_n/F_{n+1} \cong K_{S,w} \otimes_S \Sigma^{-2n-2} X$.

Proof. To compute $B^{\Pi}X$, note that for the unit $\eta : S \to K_{S,w}$ we have $\overline{K_{S,w}} = \operatorname{coker}(\eta) = \Sigma S$. Hence the n-th term in the augmented Bar resolution (II.5.2.3) is given by $K_{S,w} \otimes_S \Sigma^n X$, and the differential $K_{S,w} \otimes_S \Sigma^n X \to K_{S,w} \otimes_S \Sigma^{n-1} X$ maps $a \otimes x$ to $as \otimes x + (-1)^n a \otimes sx$. All in all, the Bar (bi)complex is given as follows:



By definition of the totalization, $B^{\Pi}X$ is equal to $\prod_{k\geq 0} \Sigma^k(K_{S,w} \otimes_S \Sigma^k X)$ as a $K^{\sharp}_{S,w}$-module, with differential being the sum of the differentials on the $\Sigma^k(K_{S,w} \otimes_S \Sigma^k X)$ and the maps $K_{S,w} \otimes_S \Sigma^k X \to K_{S,w} \otimes \Sigma^{k-1} X$. As $K^{\sharp}_{S,w}$-modules we have $\Sigma^k(K_{S,w} \otimes_S \Sigma^k X) \cong K_{S,w} \otimes_S \Sigma^{2k} X$ via $a \otimes x \mapsto (-1)^{k|a| + \frac{k(k+1)}{2}} a \otimes x$, and the n-th term of $\prod_{k\geq 0}(K_{S,w} \otimes_S \Sigma^{2k} X)$ is given by $\prod_{k\geq n} X^n$. Pulling back the differential on $\prod_{k\geq 0} \Sigma^k(K_{S,w} \otimes_S \Sigma^k X)$ to $\prod_{k\geq 0}(K_{S,w} \otimes_S \Sigma^{2k} X)$ via the above sign change, the resulting differential is given as $\mathrm{d} + s - \mathrm{id}$ on factors X^k with $n \equiv k \pmod 2$ and as $w - \mathrm{d} - s$ on those X^k with $k \not\equiv n \pmod 2$, as claimed.

The statement about the description of $(\underline{\mathrm{bar}} \circ \mathrm{fold}^{\Pi})(X)$ and the canonical epimorphism $\alpha : (\underline{\mathrm{bar}} \circ \mathrm{fold}^{\Pi})(X) \to B^{\Pi}X$ is clear. For the last statement about the filtration on $\ker(\alpha)$, define $F_i \subset \ker(\alpha)$ by $(F_i)^n := \prod_{k<n-2i} X^k$. Clearly this is a complete decreasing filtration, and the filtration quotient F_i/F_{i+1} is given by $(F_i/F_{i+1})^n = X^{n-2i-1} \oplus X^{n-2i-2}$. Together with the explicit description of the differential on $\ker(\alpha)$ we conclude that $F_i/F_{i+1} \cong K_{S,w} \otimes_S \Sigma^{-2i-2} X$. $\square$

Theorem II.5.2.13. *Let S be a ring and $w \in Z(S)$. Then the adjunctions*

$$\underline{\mathrm{bar}}: \quad \mathcal{M}^{\mathrm{ctr}}(S_w) \rightleftarrows \mathcal{M}^{\mathrm{ctr}}_{\mathrm{sing}}(K_{S,w}/S) : \mathrm{fold}^{\Pi}.$$

$$\mathrm{fold}^{\oplus} : \mathcal{M}^{\mathrm{ctr}}_{\mathrm{sing}}(K_{S,w}/S) \rightleftarrows \mathcal{M}^{\mathrm{ctr}}(S_w) \quad : \underline{\mathrm{bar}} \circ \Sigma.$$

are Quillen equivalences.

Proof. We already know from Proposition II.5.2.7 that the adjunctions in question are Quillen adjunctions, so it remains to check that unit and counit of the derived adjunctions are isomorphisms.

To show that the derived counit $\mathbf{L}\underline{\mathrm{bar}} \circ \mathbf{R}\,\mathrm{fold}^{\Pi} \Rightarrow \mathrm{id}$ is an isomorphism, we have to show that for fibrant $X \in \mathcal{M}^{\mathrm{ctr}}_{\mathrm{sing}}(K_{S,w})$ and a cofibrant resolution $Y \to \mathrm{fold}^{\Pi} X$ in $\mathcal{M}^{\mathrm{ctr}}(S_w)$ the morphism

$$\underline{\mathrm{bar}}(Y) \longrightarrow (\underline{\mathrm{bar}} \circ \mathrm{fold}^{\Pi})(X) \longrightarrow X$$

is a weak equivalence in $\mathcal{M}^{\mathrm{ctr}}_{\mathrm{sing}}(K_{S,w})$. By definition of a cofibrant resolution, the morphism $Y \to \mathrm{fold}^{\Pi} X$ is a trivial fibration, and hence so is $\underline{\mathrm{bar}}(Y \to \mathrm{fold}^{\Pi} X)$ by Proposition II.5.2.7(vi). Moreover, since the fibrants in $\mathcal{M}^{\mathrm{ctr}}_{\mathrm{sing}}(K_{S,w}/S)$ are the S-contraacyclic $K_{S,w}$-modules, we therefore have to show that for some S-contraacyclic $X \in K_{S,w}\text{-Mod}$ the (ordinary) counit $\varepsilon_X : (\underline{\mathrm{bar}} \circ \mathrm{fold}^{\Pi})(X) \to X$ is a weak equivalence in $\mathcal{M}^{\mathrm{ctr}}_{\mathrm{sing}}(K_{S,w}/S)$. For this, recall from Lemma II.5.2.12 that ε_X factors through the completed Bar resolution $q : B^{\Pi}X \to X$ via a canonical epimorphism $\alpha : (\underline{\mathrm{bar}} \circ \mathrm{fold}^{\Pi})(X) \to B^{\Pi}X$ described there. Since the completed Bar resolution $B^{\Pi}X \to X$ is a weak equivalence in $\mathcal{M}^{\mathrm{ctr}}_{\mathrm{sing}}(K_{S,w}/S)$ (even in $\mathcal{M}^{\mathrm{ctr}}(K_{S,w})$) by Lemma II.5.2.11, it is therefore sufficient to check that α is a weak equivalence in $\mathcal{M}^{\mathrm{ctr}}_{\mathrm{sing}}(K_{S,w}/S)$. In fact, we will show that α is even a trivial fibration, i.e. that $\ker(\alpha)$ is $K_{S,w}$-contraacyclic: First, by Lemma II.5.2.12 we know that $\ker(\alpha)$ admits a complete descending filtration with filtration quotients isomorphic to shifts of $K_{S,w} \otimes_S X$. We have $\mathrm{Hom}_S(K_{S,w}, X) \cong \mathrm{Hom}_S(K_{S,w}, S) \otimes_S X$, and since $\mathrm{Hom}_S(K_{S,w}, S) \cong \Omega K_{S,w}$ as $K_{S,w}$-S-bimodules, we get $K_{S,w} \otimes_S X \cong \Sigma\,\mathrm{Hom}_S(K_{S,w}, X)$. Since $K^{\sharp}_{S,w}$ is free over $S^{\sharp}$, Proposition II.2.3.18(v) and the assumption that X is S-contraacyclic yield that $K_{S,w} \otimes_S X$ is $K_{S,w}$-contraacyclic, too. We conclude that $\ker(\alpha)$ admits a complete descending filtration with $K_{S,w}$-contraacyclic filtration quotients; Lemma II.2.3.10 then shows that $\ker(\alpha)$ is $K_{S,w}$-contraacyclic, as claimed.

Similarly, the derived unit $\mathrm{id} \Rightarrow \mathbf{R}\,\mathrm{fold}^{\Pi} \circ \mathbf{L}\underline{\mathrm{bar}}$ being an isomorphism means that for any cofibrant linear factorization $g : M^1 \rightleftarrows M^0 : f$ and a fibrant resolution $\underline{\mathrm{bar}}(M) \to X$ in $\mathcal{M}^{\mathrm{ctr}}_{\mathrm{sing}}(K_{S,w}/S)$ the morphism

$$M \to (\mathrm{fold}^{\Pi} \circ \underline{\mathrm{bar}})(M) \to \mathrm{fold}^{\Pi}(X)$$

is a weak equivalence in $\mathcal{M}^{\mathrm{ctr}}(S_w)$. By Proposition II.5.2.7(vi) any object in the image of $\underline{\mathrm{bar}}$ is fibrant in $\mathcal{M}^{\mathrm{ctr}}_{\mathrm{sing}}(K_{S,w}/S)$, and hence we have to show that for $M \in S_w$-Mod with M^0, M^1 projective over S the unit $M \to (\mathrm{fold}^{\Pi} \circ \underline{\mathrm{bar}})(M)$ is a weak equivalence in $\mathcal{M}^{\mathrm{ctr}}(S_w)$. In fact, we will show that this is true for *any* S_w-module M.

Note that there is a canonical isomorphism $M \cong \mathrm{fold}^{\Pi}(i(M))$ where $i(M)$ is given by $g : M^1 \rightleftarrows M^0 : f$ in cohomological degrees -1 and 0, and 0 otherwise; it follows that the unit $M \to (\mathrm{fold}^{\Pi} \circ \underline{\mathrm{bar}})(M)$ is split by the composition

$$(\mathrm{fold}^{\Pi} \circ \underline{\mathrm{bar}})(M) \cong \mathrm{fold}^{\Pi}((\underline{\mathrm{bar}} \circ \mathrm{fold}^{\Pi})(i(M))) \xrightarrow{\mathrm{fold}^{\Pi}(\varepsilon_{i(M)})} \mathrm{fold}^{\Pi}(i(M)) = M.$$

Hence, in order to show that $M \to (\mathrm{fold}^{\Pi} \circ \underline{\mathrm{bar}})(M)$ is a weak equivalence in $\mathcal{M}^{\mathrm{ctr}}(S_w)$ it is therefore sufficient to show that $\mathrm{fold}^{\Pi}(\varepsilon_{i(M)})$ is a weak equivalence in $\mathcal{M}^{\mathrm{ctr}}(S_w)$, and we will show that it is even a trivial fibration. First, recall that $\varepsilon_{i(M)}$ factors through the completed Bar resolution $q : B^{\Pi}(i(M)) \to i(M)$ via the map $\alpha : \underline{\mathrm{bar}}(M) \to B^{\Pi}(i(M))$. Since q is a trivial fibration and the right Quillen functor fold^{Π} preserves trivial fibrations, this means that we only have to check that $\mathrm{fold}^{\Pi}(\alpha)$ is a trivial fibration, i.e. that $\mathrm{fold}^{\Pi}(\ker(\alpha))$ is trivially fibrant in $\mathcal{M}^{\mathrm{ctr}}(S_w)$. For this, recall from Lemma II.5.2.12 that $\mathrm{fold}^{\Pi}(\ker(\alpha))$ admits a complete decreasing filtration with filtration quotients being shifts of $\mathrm{fold}^{\Pi}(K_{S,w} \otimes_S i(M))$. $K_{S,w} \otimes_S i(M)$ is an extension of $K_{S,w} \otimes_S M^0$ and $K_{S,w} \otimes_S \Sigma M^1$, and hence $\mathrm{fold}^{\Pi}(K_{S,w} \otimes_S i(M))$ is an extension of $\mathrm{fold}^{\Pi}(K_{S,w} \otimes_S M^0)$ and $\mathrm{fold}^{\Pi}(K_{S,w} \otimes_S \Sigma M^1)$, both of which are contractible, hence contraacyclic, by Proposition II.2.3.8. Applying Lemma II.2.3.10 shows that $\mathrm{fold}^{\Pi}(\ker(\alpha))$ is S_w-contraacyclic, as claimed.

The statement that $\mathrm{fold}^{\oplus} \dashv \underline{\mathrm{bar}} \circ \Sigma$ is a Quillen equivalence follows from the first part since $\mathbf{R}(\underline{\mathrm{bar}} \circ \Sigma) = \mathbf{R}\underline{\mathrm{bar}} \circ \Sigma = \mathbf{L}\underline{\mathrm{bar}} \circ \Sigma$ is invertible and a Quillen adjunction is a Quillen equivalence if and only if its derived adjunction is an adjoint equivalence [Hov99, Proposition 1.3.13]. □

From Theorem II.5.2.13 we get the following consequence:

Corollary II.5.2.14. *There is an isomorphism*

$$\Sigma \circ \mathbf{L}\,\mathrm{fold}^{\oplus} \cong \mathbf{R}\,\mathrm{fold}^{\Pi}$$

of functors $\mathrm{Ho}(\mathcal{M}^{\mathrm{ctr}}_{\mathrm{sing}}(K_{S,w}/S)) \to \mathrm{Ho}(\mathcal{M}^{\mathrm{ctr}}(S_w))$.

Proof. By Theorem II.5.2.13 we know that $\mathbf{L}\underline{\mathrm{bar}} = \mathbf{R}\underline{\mathrm{bar}}$ is invertible, and that we have canonical adjunctions $\mathbf{L}\underline{\mathrm{bar}} \dashv \mathbf{R}\,\mathrm{fold}^{\Pi}$ and $\Sigma \circ \mathbf{L}\,\mathrm{fold}^{\oplus} \dashv \mathbf{R}\underline{\mathrm{bar}}$. □

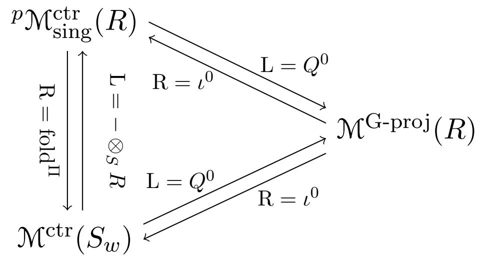
II.5.3. Hypersurfaces

Let $(S, \mathfrak{m})$ be a regular local ring and $w \in \mathfrak{m} \setminus \{0\}$. In this section, we study the singular model structure for the hypersurface $R = S/(w)$ and establish a Quillen equivalence $\mathcal{M}^{\mathrm{ctr}}(S_w) \rightleftarrows \mathcal{M}^{\mathrm{G\text{-}proj}}(R)$ lifting Eisenbud's equivalence $\underline{\mathrm{MF}}(S, w) \cong \underline{\mathrm{MCM}}(S/(w))$.

We introduce some notation first. Given a linear factorization $g : M^1 \rightleftarrows M^0 : f$, we denote $Q^0(M) := \mathrm{coker}(g)$. Since $gf = w \cdot \mathrm{id}_{M^0}$, $Q^0(M)$ is naturally a module over $S/(w) = R$, so we may consider Q^0 as a functor $\mathrm{LF}(S, w) \to R\text{-Mod}$; the assignment $\iota^0 : R\text{-Mod} \to \mathrm{LF}(S, w)$, mapping an R-module X to the linear factorization $0 \rightleftarrows X$, is canonically right adjoint to Q^0. Further, $\mathrm{fold}^{\Pi} : \mathrm{Ch}(R) \to \mathrm{LF}(S, w)$ denotes the functor mapping a complex $X \in \mathrm{Ch}(R)$ to the linear factorization $\prod_{n \in \mathbb{Z}} X^{2n+1} \rightleftarrows \prod_{n \in \mathbb{Z}} X^{2n}$, and its left adjoint $- \otimes_S R : \mathrm{LF}(S, w) \to \mathrm{Ch}(R)$ sends a linear factorization $g : M^1 \rightleftarrows M^0 : f$ to the 2-periodic complex of R-modules

$$\ldots \to M^0/wM^0 \xrightarrow{\overline{f}} M^1/wM^1 \xrightarrow{\overline{g}} M^0/wM^0 \to \ldots \quad .$$

Lemma II.5.3.1. *Let $(S, \mathfrak{m})$ be a regular local ring, $w \in \mathfrak{m} \setminus \{0\}$ and $R := S/(w)$. Then there is a diagram of Quillen adjunctions which when restricted to the left resp. right adjoints commutes up to canonical isomorphism:*



Proof. For the commutativity up to isomorphism it suffices to note that for a linear factorization $M^0 \xrightarrow{f} M^1 \xrightarrow{g} M^0$ we have $\mathrm{coker}(g) \xrightarrow{\cong} \mathrm{coker}(g \otimes_S S/(w))$ canonically since $\mathrm{im}(g) \supset \mathrm{im}(gf) = w \cdot M^0$.

It remains to show the adjunctions are Quillen adjunctions. We have already seen in Lemma II.5.1.1 that $Q^0 : {}^p\mathcal{M}^{\mathrm{ctr}}_{\mathrm{sing}}(R) \rightleftarrows \mathcal{M}^{\mathrm{G\text{-}proj}}(R) : \iota^0$ is a Quillen adjunction. As the composition of two left/right Quillen functors is again left/right Quillen, it is therefore sufficient to prove that $- \otimes_S R : \mathcal{M}^{\mathrm{ctr}}(S_w) \to {}^p\mathcal{M}^{\mathrm{ctr}}_{\mathrm{sing}}(R)$ is left Quillen, i.e. that it preserves cofibrations and trivial cofibrations.

A cofibration $\mathcal{M}^{\mathrm{ctr}}(S_w)$ is a monomorphism $\alpha : X \to Y$ of linear factorizations whose cokernel $M := \mathrm{coker}(\alpha)$ is a matrix factorization. The components of M being projective over S, such a morphism is componentwise split, and hence

$$0 \to X \otimes_S R \xrightarrow{\alpha \otimes_S R} Y \otimes_S R \to M \otimes_S R \to 0$$

is a componentwise split short exact sequence of complexes of R-modules, with $M \otimes_S R$ having R-projective components. Recalling that the cofibrations in ${}^p\mathcal{M}^{\mathrm{ctr}}_{\mathrm{sing}}(R)$ are the monomorphisms with acyclic and componentwise R-projective cokernel, for proving that $\alpha \otimes_S R$ is a cofibration in ${}^p\mathcal{M}^{\mathrm{ctr}}_{\mathrm{sing}}(R)$ it therefore remains to show that the complex

$$\ldots \to M^0/wM^0 \xrightarrow{\overline{f}} M^1/wM^1 \xrightarrow{\overline{g}} M^0/wM^0 \to \ldots$$

is acyclic. For that, suppose $\overline{x} \in M^0/wM^0$ is a cycle, i.e. $\overline{f}(\overline{x}) = \overline{f(x)} = 0$ in M^1/wM^1. Then there exists some $y \in M^1$ with $f(x) = wy = f(g(y))$, so $f(x - g(y)) = 0$, which implies $0 = gf(x-g(y)) = w(x-g(y))$. As a regular local ring, S is a domain, and since $w \neq 0$ and M^0 is projective, we conclude that $x = g(y)$, so that in particular $\overline{x} = \overline{g}(\overline{y})$ is a boundary. The equality $\ker(\overline{g}) = \mathrm{im}(\overline{f})$ is proved in the same way.

Trivial cofibrations in $\mathcal{M}^{\mathrm{ctr}}(S_w)$ and ${}^p\mathcal{M}^{\mathrm{ctr}}_{\mathrm{sing}}(R)$ are split monomorphisms with projective cokernel, and $- \otimes_S R$ preserves such as a left adjoint to the exact functor fold^{Π}. □

Theorem II.5.3.2. *Let $(S, \mathfrak{m})$ be a regular local ring, $w \in \mathfrak{m} \setminus \{0\}$ and $R := S/(w)$. Then the Quillen adjunctions from Lemma II.5.3.1 are Quillen equivalences:*

$$\begin{array}{ccc}
{}^p\mathcal{M}^{\mathrm{ctr}}_{\mathrm{sing}}(R) & \overset{\mathrm{L} = Q^0}{\underset{\mathrm{R} = \iota^0}{\rightleftarrows}} & \mathcal{M}^{\mathrm{G\text{-}proj}}(R) \\
\mathrm{R} = \mathrm{fold}^{\Pi} \downarrow \uparrow \mathrm{L} = - \otimes_S R & & \\
\mathcal{M}^{\mathrm{ctr}}(S_w) & \overset{\mathrm{L} = Q^0}{\underset{\mathrm{R} = \iota^0}{\rightleftarrows}} & \mathcal{M}^{\mathrm{G\text{-}proj}}(R)
\end{array}$$

Proof. By Theorem II.5.1.5 we know that $Q^0 : {}^p\mathcal{M}^{\mathrm{ctr}}_{\mathrm{sing}}(R) \rightleftarrows \mathcal{M}^{\mathrm{G\text{-}proj}}(R) : \iota^0$ is a Quillen equivalence, hence it suffices to show the same for $Q^0 : \mathcal{M}^{\mathrm{ctr}}(S_w) \rightleftarrows \mathcal{M}^{\mathrm{G\text{-}proj}}(R) : \iota^0$. As in Theorem II.5.1.5, we therefore have to show the following:

(i) For $M \in \mathrm{MF}(S, w)$ the unit $M \to \iota^0 Q^0(M)$ is a weak equivalence in $\mathcal{M}^{\mathrm{ctr}}(S_w)$.

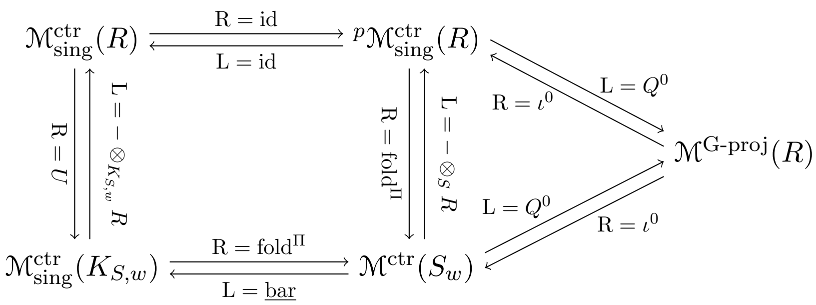
(ii) For $X \in R$-Mod and some cofibrant replacement $M \to \iota^0(X)$ of $\iota^0(X)$ in $\mathcal{M}^{\mathrm{ctr}}(S_w)$, the composition $Q^0(M) \to Q^0\iota^0(X) \to X$ is a weak equivalence in $\mathcal{M}^{\mathrm{G\text{-}proj}}(R)$.

For (i), if $g : M^1 \rightleftarrows M^0 : f$ is a matrix factorization, the unit morphism $M \to \iota^0 Q^0(M)$ fits into a short exact sequence in $\mathrm{LF}(S, w)$:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & M^1 & \xrightarrow{g} & M^0 & \xrightarrow{\mathrm{can}} & \mathrm{coker}(g) & \longrightarrow & 0 \\
 & & \uparrow 1 & & \uparrow g & & \uparrow 0 & & \\
0 & \longrightarrow & M^1 & \xrightarrow{1} & M^1 & \longrightarrow & 0 & \longrightarrow & 0 \\
 & & \uparrow w & & \uparrow f & & \uparrow 0 & & \\
0 & \longrightarrow & M^1 & \xrightarrow[g]{} & M^0 & \xrightarrow[\mathrm{can}]{} & \mathrm{coker}(g) & \longrightarrow & 0
\end{array}$$

Since the linear factorization $M^1 \xrightarrow{w} M^1 \xrightarrow{1} M^1$ is contractible, it follows that $M \to \iota^0 Q^0(M)$ is a trivial fibration in $\mathcal{M}^{\mathrm{ctr}}(S_w)$. For (ii), we may as in the proof of Theorem II.5.1.5 assume that X is Gorenstein projective, in which case we claim that X has projective dimension at most 1 over S: Indeed, if P is a projective S-module and $k \geq 2$, we have $\mathrm{Ext}^k_S(X, P) \cong \mathrm{Ext}^{k-1}_R(X, P/wP) = 0$ since P/wP is projective over $S/(w) = R$ and X is Gorenstein projective over R (for the first isomorphism, see [BH93, Lemma 3.1.16] or the proof of Proposition I.7.5.2). In particular, the first syzygy of X over S is Gorenstein projective over S, and hence projective since S is regular. This proves that X has projective dimension ≤ 1 over S, i.e. that there exists a short exact sequence $0 \to P \xrightarrow{g} Q \to X \to 0$ with P and Q projective S-modules. In such a sequence, vanishing of multiplication by w on X implies that there exists some $f : Q \to P$ with $gf = w \cdot \mathrm{id}_P$, and then $M := (g : Q \rightleftarrows P : f)$ is a matrix factorization of type (S, w) with $\mathrm{coker}(g) \cong X$. By (i), $M \to \iota^0(X)$ is a cofibrant replacement of X in $\mathcal{M}^{\mathrm{ctr}}(S_w)$, and the induced map $Q^0(M) \to Q^0\iota^0(X) = X$ is an isomorphism. □

The Quillen equivalence ${}^p\mathcal{M}^{\mathrm{ctr}}_{\mathrm{sing}}(R) \rightleftarrows \mathcal{M}^{\mathrm{ctr}}(S_w)$ from Theorem II.5.3.2 relates to the folding equivalence $\mathcal{M}^{\mathrm{ctr}}_{\mathrm{sing}}(K_{S,w}) = \mathcal{M}^{\mathrm{ctr}}_{\mathrm{sing}}(K_{S,w}/S) \rightleftarrows \mathcal{M}^{\mathrm{ctr}}(S_w)$ from Theorem II.5.2.13 (recall from Definition II.4.1.2) that over a regular base the absolute and relative singular model structures coincide) through the following diagram of Quillen adjunctions which commutes up to canonical isomorphism when restricted to the left or right adjoints:



With the exception of the left vertical adjunction $\mathcal{M}^{\mathrm{ctr}}_{\mathrm{sing}}(K_{S,w}) \rightleftarrows \mathcal{M}^{\mathrm{ctr}}_{\mathrm{sing}}(R)$ we already know all adjunctions in this diagram to be Quillen equivalences, hence so is $\mathcal{M}^{\mathrm{ctr}}_{\mathrm{sing}}(K_{S,w}) \rightleftarrows \mathcal{M}^{\mathrm{ctr}}_{\mathrm{sing}}(R)$. It would be interesting to prove this directly, and, more generally, to establish criteria for when the Quillen adjunctions

$$- \otimes_A B : \mathcal{M}^{\mathrm{ctr}}(A) \rightleftarrows \mathcal{M}^{\mathrm{ctr}}(B) : U_\varphi \quad \text{and} \quad - \otimes_A B : \mathcal{M}^{\mathrm{ctr}}_{\mathrm{sing}}(A) \rightleftarrows \mathcal{M}^{\mathrm{ctr}}_{\mathrm{sing}}(B) : U_\varphi$$

associated to a morphism of dg rings $\varphi : A \to B$ are Quillen equivalences.

II.B. Pulling back deconstructible classes

Throughout the section we use the notions of $< \kappa$-presentable objects and locally $< \kappa$-presentable categories as defined in [AR94, Definition 1.13]. Note [Što13, Section 1] that by [AR94, Remark 1.21] $< \kappa$-presentability is the same as κ-accessibility in the sense of [KS06, Definition 9.2.7], so it is legitimate to use results from loc.cit. when studying $< \kappa$-presentable objects. If $\mathcal{F} \subset \mathscr{A}$ is a class of objects in a category $\mathscr{A}$, $\mathcal{F}^{<\kappa}$ denotes the class of $< \kappa$-presentable objects in $\mathcal{F}$.

We begin by recalling the definition of a monad and its category of algebras.

Definition II.B.1. *Let $\mathcal{C}$ be a category.*

(i) *A* monad *on $\mathcal{C}$ is a triple $(\perp, \eta, \mu)$ consisting of an endofunctor $\perp : \mathcal{C} \to \mathcal{C}$ and natural transformations $\eta : \mathrm{id}_{\mathcal{C}} \to \perp$, $\mu : \perp^2 \to \perp$, such that μ and η obey the associativity and unit axioms $\mu \circ \perp\mu = \mu \circ \mu\perp$ and $\mu \circ \perp\eta = \mathrm{id}_{\perp} = \mu \circ \eta\perp$.*

(ii) *An* algebra *over $\perp$ is a pair (X, ρ) consisting of an object X of $\mathcal{C}$ and a morphism $\rho : \perp X \to X$ such that $\rho \circ \eta_X = \mathrm{id}_X$ and $\rho \circ \mu_X = \rho \circ \perp\rho$.*

The category of $\perp$-algebras is denoted $\perp$-Alg. If $\mathcal{F}$ is a class of objects in $\mathcal{C}$, then $\perp\text{-Alg}_{\mathcal{F}}$ denotes the class of $\perp$-algebras whose underlying objects belong to $\mathcal{F}$. The forgetful functor $\perp\text{-Alg} \to \mathcal{C}$ is denoted U.

Example II.B.2. The standard example of a monad is the following. If $F : \mathcal{D} \rightleftarrows \mathcal{C} : U$ is an adjunction, then $\perp := UF$ together with the unit $\eta : \mathrm{id} \to UF$ and the counit $U\varepsilon F : \perp^2 = U(FU)F \to UF$ is a monad on $\mathcal{C}$.

For example, given a dg ring A, there is the monad associated to the adjunction $G^+ : A\text{-Mod} \rightleftarrows A^\sharp\text{-Mod} : (-)^\sharp$ defined in Proposition II.2.3.2. Its category of algebras is canonically equivalent to A-Mod (i.e. $(-)^\sharp$ is a *monadic functor*). ◊

Lemma II.B.3. *Let $\perp : \mathscr{A} \to \mathscr{A}$ be a right exact monad on an abelian category $\mathscr{A}$.*

(i) *$\perp$-Alg is abelian.*

(ii) *The forgetful functor $\perp\text{-Alg} \to \mathscr{A}$ is faithful and exact.*

Suppose that, in addition, $\mathscr{A}$ is Grothendieck and $\perp$ is cocontinuous.

(i) $\perp$-Alg *is a Grothendieck category, and if G is a generator of $\mathscr{A}$ then $\perp G$ is a generator of* $\perp$-Alg.

(ii) *The forgetful functor $U : \perp\text{-Alg} \to \mathscr{A}$ is bicontinuous.*

Proof. Since $\perp$ is additive, the sum in $\mathscr{A}$ of two morphisms of $\perp$-algebras is again a morphism of $\perp$-algebras. Hence $\perp$-Alg inherits a unique preadditive structure from $\mathscr{A}$ such that $U : \perp\text{-Alg} \to \mathscr{A}$ is preadditive.

Next, let $D : I \to \perp\text{-Alg}$, $D(x) = (M(x), \rho_x)$, be a diagram such that the underlying $\mathscr{A}$-diagram $M : I \to \mathscr{A}$ has a colimit $\varinjlim M$, and assume that $\perp$ commutes with that colimit, i.e. that $\perp(\varinjlim M)$ is a colimit for $\perp M$ with respect to the maps $\perp M(x) \to \perp(\varinjlim M)$. Then there is a unique structure ρ of a $\perp$-algebra on $\varinjlim M$ such that all maps $(M(x), \rho_x)) \to (\varinjlim M, \rho)$ are morphisms of $\perp$-algebras: take as $\rho : \perp(\varinjlim M) \to \varinjlim M$ the unique map such that for each $x \in I$ the diagram

$$\begin{array}{ccc} \perp M(x) & \longrightarrow & \perp \varinjlim M \\ \downarrow{\scriptstyle \rho_x} & & \downarrow{\scriptstyle \rho} \\ M(x) & \longrightarrow & \varinjlim M \end{array}$$

This is justified by our assumption that $\perp$ commutes with $\varinjlim M$. Unit and associativity axiom also follow by using the universal property of the colimit, and hence $(\varinjlim M, \rho)$ indeed is a $\perp$-algebra. Moreover, it is straightforward to check that $(\varinjlim M, \rho)$ together with the maps $D(x) \to (\varinjlim M, \rho)$ is a colimit of D.

Similarly, if M has a limit $\varprojlim M$ in $\mathscr{A}$, then $\varprojlim M$ admits a unique structure ρ of a $\perp$-algebra such that all maps $(\varprojlim M, \rho) \to D(x)$ are morphisms of $\perp$-algebras, and, moreover, $(\varprojlim M, \rho)$ is then a limit of D in $\perp$-Alg with respect to these. Note, however, that we don't have to assume that $\perp$ commutes with $\varprojlim M$ here.

The preceding arguments show that for right-exact $\mathscr{A}$ the category $\perp$-Alg admits arbitrary finite limits and colimits and $U : \perp\text{-Alg} \to \mathscr{A}$ commutes with these. In particular, we get that $\perp$-Alg is additive, and that any morphism admits a kernel and a cokernel. Finally, since $U : \perp\text{-Alg} \to \mathscr{A}$ reflects isomorphisms, we even get that coim = im in $\perp$-Alg, and hence $\perp$-Alg is abelian.

If $\mathscr{A}$ admits arbitrary colimits and $\perp$ is cocontinuous, $\perp$-Alg also admits arbitrary colimits and $U : \perp\text{-Alg} \to \mathscr{A}$ is cocontinuous, and since U reflects isomorphisms, directed colimits are exact in $\perp$-Alg provided they are exact in $\mathscr{A}$. Similarly, if $\mathscr{A}$ admits arbitrary limits, then so does $\perp$-Alg and U preserves them. Finally, if $\mathscr{A}$ is Grothendieck with generator G, the free algebra $\perp G$ on G is a generator for $\perp$-Alg. Indeed, given

$(X, \rho) \in \perp$-Alg we can choose an epimorphism $G^{\coprod I} \to X$ by [KS06, Proposition 5.2.4]. Applying $\perp$, we get the morphism of $\perp$-algebras $\perp(G^{\coprod I}) \cong (\perp G)^{\coprod I} \to \perp X \to X$, which is an epimorphism, too, since $\perp$ is cocontinuous and $\rho : \perp X \to X$ is a split epimorphism in $\mathscr{A}$. Applying [KS06, Proposition 5.2.4] again we conclude that $\perp G$ is a generator for $\perp$-Alg, as claimed. □

Lemma II.B.4. *Let $\mathscr{A}$ be a Grothendieck category, $\perp$ be a cocontinuous monad on $\mathscr{A}$ and (X, ρ) be a $\perp$-algebra. Then the forgetful functor $U : \perp\text{-Alg} \to \mathscr{A}$ induces an injective complete lattice homomorphism*

$$\left(\mathrm{Subobj}_{\perp\text{-Alg}}(X, \rho), \Sigma, \cap\right) \longrightarrow \left(\mathrm{Subobj}_{\mathscr{A}}(X), \Sigma, \cap\right).$$

Its image consists of (the classes of) those monomorphisms $\iota : Y \hookrightarrow X$ such that the composite $\perp Y \xrightarrow{\perp\iota} \perp X \xrightarrow{\mu} X$ factors through ι.

Proof. Given an object X in a Grothendieck category and a family $\{X_i\}$ of subobjects, the intersection $\bigcap X_i$ is the limit of the diagram consisting of the inclusions $X_i \hookrightarrow X$, and the sum $\sum X_i$ is the image of the canonical map $\bigoplus X_i \to X$. Hence any bicontinuous functor between Grothendieck categories, in particular $U : \perp\text{-Alg} \to \mathscr{A}$ (Lemma II.B.3(3)), induces a complete lattice homomorphism on subobjects. The second statement is clear. □

Fact II.B.5. *Let $\perp : \mathscr{A} \to \mathscr{A}$ be a cocontinuous monad on an abelian category $\mathscr{A}$, (X, ρ) be a $\perp$-algebra and $Z \subseteq X$ a subobject of X. Then the poset of $\perp$-subalgebras of (X, ρ) containing Z has a minimal element* $\mathrm{span}_\perp Z := \mathrm{im}(\perp Z \to \perp X \to X)$.

Proof. If $Z' \subseteq X$ is a $\perp$-subalgebra of (X, ρ) with $Z \subseteq Z'$, then $\perp Z' \to \perp X \to X$ factors through Z', and hence so does $\perp Z \to \perp X \to X$. Thus $\mathrm{span}_\perp Z \subseteq Z'$.

It remains to show that $\mathrm{span}_\perp Z$ is a $\perp$-subalgebra of (X, ρ), i.e. that the composition $\perp(\mathrm{span}_\perp Z) \to \perp X \to X$ factors through $\mathrm{span}_\perp Z$. By definition of $\mathrm{span}_\perp Z$ there is a commutative diagram

$$\begin{array}{ccccc} \perp Z & \longrightarrow & \perp X & \longrightarrow & X \\ & \searrow & & \nearrow & \\ & & \mathrm{span}_\perp Z & & \end{array}$$

and hence it is sufficient to show that $\perp^2 Z \to \perp^2 X \xrightarrow{\perp\rho} \perp X \xrightarrow{\rho} X$ factors through $\mathrm{span}_\perp Z$. By associativity and naturality, this composition equals $\perp^2 Z \xrightarrow{\mu_Z} \perp Z \to \perp X \xrightarrow{\rho} X$, which factors through $\mathrm{span}_\perp Z$ by definition. □

We need the following version of the generalized Hill lemma [Što13, Theorem 2.1] as a tool for constructing filtrations.

Proposition II.B.6 Hill Lemma**.** *Let κ be an infinite regular cardinal and let $\mathscr{A}$ be a locally $< \kappa$-presentable Grothendieck category. Further, let $\mathcal{S}$ be a set of $< \kappa$-presentable objects and $X \in \text{filt-}\mathcal{S}$. Then there exists a set σ together a subset $\mathcal{L} \subseteq \mathcal{P}(\sigma)$ and a map $l : \mathcal{L} \to \text{Subobj}(X)$ such that the following hold:*

(H1) *For any family $\{S_i\} \subset \mathcal{L}$, both $\bigcup_i S_i$ and $\bigcap_i S_i$ belong to $\mathcal{L}$ again, and we have $l\left(\bigcup_i S_i\right) = \sum_i l(S_i)$ and $l\left(\bigcap_i S_i\right) = \bigcap_i l(S_i)$.*

(H2) *Given $S, T \in \mathcal{L}$ with $S \subseteq T$, $l(T)/l(S)$ admits an $\mathcal{S}$-filtration of size $|T \setminus S|$.*

(H3) *For any $< \kappa$-presentable $Z \subseteq X$ there exists some $S \in \mathcal{L}$ satisfying $|S| < \kappa$ and $Z \subseteq l(S)$.*

The Hill Lemma allows for recursive constructions of filtrations on X by first constructing continuous chains of elements in $\mathcal{L} \subset \mathcal{P}(\sigma)$ and then applying $l : \mathcal{L} \to \text{Subobj}(X)$ to these chains. The continuity of the resulting filtration is guaranteed by (H1), control over filtration quotients is given by (H2), and finally property (H3) is needed for the recursion step. This principle is illustrated in the proof of the following proposition, which is the main result of this section:

Proposition II.B.7. *Let κ be an uncountable regular cardinal and $\mathscr{A}$ be a locally $< \kappa$-presentable Grothendieck category. Assume further that $\mathcal{F} \subset \mathscr{A}$ is a class of objects and $\perp : \mathscr{A} \to \mathscr{A}$ a cocontinuous monad such that*

(i) *$\mathcal{F} = \text{filt-}\mathcal{S}$, where $\mathcal{S}$ is a representative set of $< \kappa$-presentable objects in $\mathcal{F}$,*

(ii) *$\perp$ preserves the class of $< \kappa$-presentable objects in $\mathscr{A}$.*

Then $\perp\text{-Alg}_{\mathcal{F}} = \text{filt-}(\perp\text{-Alg}_{\mathcal{S}})$. In particular, $\perp\text{-Alg}_{\mathcal{F}}$ is deconstructible.

Lemma II.B.8 see [KS06, Proposition 9.2.10]**.** *For any Grothendieck category $\mathscr{A}$ and any infinite cardinal κ, the class $\mathscr{A}^{<\kappa}$ of $< \kappa$-presentable objects is closed under the formation of $\mathscr{A}$-colimits of diagrams $I \to \mathscr{A}^{<\kappa}$ with $|\text{Mor}(I)| < \kappa$.*

Proof of Proposition II.B.7. Let $(X, \rho) \in \perp\text{-Alg}_{\mathcal{F}}$. By definition we have $X \in \mathcal{F} = \text{filt-}\mathcal{S}$, so we may apply Proposition II.B.6 to get $l : \mathcal{P}(\sigma) \supset \mathcal{L} \to \text{Subobj}(X)$ satisfying the properties (H1), (H2), (H3). By transfinite recursion, we will now define for each ordinal λ a subset $T(\lambda) \in \mathcal{L}$ such that the following hold:

(i) $l(T(\lambda))$ is a $\perp$-subalgebra of X.

(ii) $T(\lambda) \subseteq T(\mu)$ if $\lambda \leq \mu$, and $T(\lambda) \subsetneq T(\mu)$ if $\lambda < \mu$ and $l(T(\lambda)) \neq X$.

(iii) $|T(\lambda + 1) \setminus T(\lambda)| < \kappa$.

(iv) $T(\lambda) = \bigcup_{\mu<\lambda} T(\mu)$ if λ is a limit ordinal.

Start with $T(0) := \emptyset$ and assume that we are given an ordinal λ such that we already constructed $T(\mu)$ for all $\mu < \lambda$. If λ is a limit ordinal, we put $T(\lambda) := \bigcup_{\mu<\lambda} T(\mu)$, and if $\lambda = \mu + 1$ with $l(T(\mu)) = X$, we put $T(\lambda) := T(\mu)$. In case $\lambda = \mu + 1$ with $l(T) \subsetneq X$ for $T := T(\mu)$, we proceed as follows: Since $\mathscr{A}$ is locally $< \kappa$-presentable, there exists some $< \kappa$-presentable $Z \subset X$ with $Z \not\subseteq l(T)$, and by (H3) we find $Z \subset l(S_0)$ for some $S_0 \in \mathcal{L}$ with $|S_0| < \kappa$. By Lemma II.B.8, $l(S_0)$ is $< \kappa$-presentable and hence so is $\mathrm{span}_\perp\, l(S_0) = \mathrm{im}\,(\perp l(S_0) \to \perp X \to X)$. Applying (H3) again, we can find $S_1 \in \mathcal{L}$ with $|S_1| < \kappa$, $S_0 \subseteq S_1$ and $\mathrm{span}_\perp\, Z \subseteq l(S_1)$, and again $l(S_1) \in \mathscr{A}^{<\kappa}$. Continuing this way, we find a sequence $S_0 \subseteq S_1 \subseteq S_2 \subseteq \ldots$ in $\mathcal{P}(\sigma)$ with $S_i \in \mathcal{L}$, $|S_i| < \kappa$ and $\mathrm{span}_\perp\, l(S_i) \subseteq l(S_{i+1})$ for all $i \geq 0$. Put $S := \bigcup_{i\geq 0} S_i$. We then have $S \in \mathcal{L}$, $|S| < \kappa$ and $l(S) = \sum_{i\geq 0} l(S_i)$ by (H1). In particular, as $\perp$ is cocontinuous, $l(S)$ is a $\perp$-subalgebra of (X, ρ). We put $T(\lambda) := T \cup S$. This finishes the recursion step and the construction of T.

Pick λ sufficiently large such that $l(T(\lambda)) = X$ and consider the filtration $l \circ T : \{\tau \mid \tau \leq \lambda\} \to \mathrm{Subobj}(X)$ on X. By (i) all its components are $\perp$-subalgebras of X, and its successive quotients are given by $l(T(\mu + 1))/l(T(\mu))$, all of which lie in $\mathcal{S}$ by (iii) and Lemma II.B.8. Finally, since $\mathrm{Subobj}_{\perp\text{-Alg}}(X) \hookrightarrow \mathrm{Subobj}_{\mathscr{A}}(X)$ is a complete lattice homomorphism, $l \circ T$ is also continuous considered as a filtration of (X, ρ) in $\perp$-Alg. Summing up, $l \circ T$ is the desired $\perp\text{-Alg}_{\mathcal{S}}$-filtration of X. □

To give a less technical version of Proposition II.B.7 we need some generalities about $< \kappa$-presentable objects in Grothendieck categories.

Lemma II.B.9. *Let $\mathscr{A}$ be a Grothendieck category.*

(i) *For any set $\mathcal{S} \subset \mathscr{A}$ there exists some cardinal κ such that $\mathcal{S} \subseteq \mathscr{A}^{<\kappa}$.*

(ii) *For any cardinal κ the category $\mathscr{A}^{<\kappa}$ is essentially small.*

Proof. Part (i) is contained in [KS06, Theorem 9.6.1]. Part (ii) follows from [KS06, Corollary 9.3.5(i)] and the fact that $\mathscr{A}^{<\kappa} \subseteq \mathscr{A}^{<\mu}$ for $\kappa \leq \mu$. □

Lemma II.B.10. *Let $\mathscr{A}$, $\mathscr{B}$ be Grothendieck categories and $F : \mathscr{A} \to \mathscr{B}$ be a cocontinuous functor. Then there exist arbitrarily large regular cardinals κ such that F preserves $< \kappa$-presentable objects, i.e. $F(\mathscr{A}^{<\kappa}) \subseteq \mathscr{B}^{<\kappa}$.*

Proof. Let G be a generator of $\mathscr{A}$ and pick any cardinal κ such that $G \in \mathscr{A}^{<\kappa}$ and $F(G) \in \mathscr{B}^{<\kappa}$ hold. This is possible by Lemma II.B.9. Moreover, possibly after enlarging κ we get that $\mathscr{A}^{<\kappa} = \{X \in \mathscr{A} \mid |\mathrm{Hom}_{\mathscr{A}}(G, X)| < \kappa\}$ [KS06, Theorem 9.3.4] (note, however, that this characterization doesn't seem to be true for all sufficiently large, but only for a cofinal class of cardinals κ). We claim that F preserves $<\kappa$-presentable objects. Indeed, let $X \in \mathscr{A}^{<\kappa}$ is $<\kappa$-presentable. Then the canonical morphism $G^{\coprod \operatorname{Hom}_{\mathscr{A}}(G,X)} \to X$ is an epimorphism [KS06, Proposition 5.2.3(iv)], and hence so is $F(G)^{\coprod \operatorname{Hom}_{\mathscr{A}}(G,X)} \to F(X)$ since F commutes with colimits by assumption. As $F(G) \in \mathscr{B}^{<\kappa}$ and $|\operatorname{Hom}_{\mathscr{A}}(G,X)| < \kappa$ by assumption, Lemma II.B.8 implies $F(X) \in \mathscr{B}^{<\kappa}$ as claimed. □

Theorem II.B.11. *Let $U : \mathscr{B} \to \mathscr{A}$ be a cocontinuous, monadic functor between Grothendieck categories, and let $\mathcal{F} \subset \mathscr{A}$ be a deconstructible class. Then $U^*(\mathcal{F}) := \{X \in \mathscr{B} \mid U(X) \in \mathcal{F}\}$ is again deconstructible.*

Proof. By definition of monadic functors, we may assume that U is the forgetful functor $\perp$-Alg $\to \mathscr{A}$ for a cocontinuous monad $\perp$ on $\mathscr{A}$, and then $U^*(\mathcal{F}) = \perp\text{-Alg}_{\mathcal{F}}$. Since $\mathcal{F} = \text{filt-}\mathcal{F}$ by [Što13, Lemma 1.6], Lemma II.B.8 implies that $\mathcal{F} = \text{filt-}(\mathcal{F} \cap \mathscr{A}^{<\kappa})$ for all sufficiently large cardinals κ. Here, by slight abuse of notation $\mathcal{F} \cap \mathscr{A}^{<\kappa}$ means a representative set of isomorphism classes of objects in $\mathcal{F} \cap \mathscr{A}^{<\kappa}$ (it is a set by Lemma II.B.9((ii))). Moreover, by Lemma II.B.10 we may also assume that $\perp$ preserves κ-presentable objects, and hence the claim follows from Proposition II.B.7. □

Meanwhile, Theorem II.B.11 has been generalized to the non-abelian context of combinatorial categories by Makkai and Rosický: Reversing the passage from ordinary model categories to abelian model categories, instead of looking at Grothendieck abelian categories equipped with a deconstructible class of objects the authors of loc.cit. consider locally presentable categories $\mathcal{K}$ equipped with a subclass $\operatorname{Cof}\mathcal{K} \subset \operatorname{Mor}\mathcal{K}$ of morphisms that is closed under retracts, pushouts and transfinite compositions, and which is already generated by a set of morphisms through these properties. The pair $(\mathcal{K}, \operatorname{Cof}\mathcal{K})$ is then called a *combinatorial category*, and the following generalizes our Theorem II.B.11:

Theorem II.B.12 [MR14, Corollary 3.6]**.** *Let $\perp\colon \mathcal{K} \to \mathcal{K}$ be a cocontinuous monad on a combinatorial category $(\mathcal{K}, \operatorname{Cof}\mathcal{K})$. Then $(\perp\text{-Alg}, \operatorname{Cof}\mathcal{K})$ is combinatorial.*

II.C. The homotopy category of an abelian model category

In this appendix we study the relationship between the homotopy category of a hereditary abelian model structure $\mathcal{M}$ on an abelian category $\mathscr{A}$ and the derived category $\mathbf{D}(\mathscr{A})$ of the latter. To begin, in Section II.C.1 we show that any short exact sequence in $\mathscr{A}$ gives rise to a canonical distinguished triangle $\mathrm{Ho}(\mathcal{M})$ (Corollary II.C.1.2), a principle well-known from the ordinary derived category of an abelian category. In Section II.C.2, we show how this gives rise to homomorphisms $\mathbf{D}(\mathscr{A})(X, \Sigma^k Y) \cong \mathrm{Ext}^k_{\mathscr{A}}(X, Y) \to \mathrm{Ho}(\mathcal{M})(X, \Sigma^k Y)$ (Proposition II.C.2.2). Finally, in the main Section II.C.3 we show that these homomorphisms are induced by a triangulated functor $\mathbf{D}(\mathscr{A}) \to \mathrm{Ho}(\mathcal{M})$, at least for hereditary, cofibrantly generated abelian model structures on Grothendieck categories (Proposition II.C.3.26). The crucial steps in the construction are Theorems II.C.3.15 and II.C.3.23 which are also of independent interest: they show that $\mathcal{M}$ induces several Quillen equivalent model structures on $\mathrm{Ch}(\mathscr{A})$ that are connected to suitable models for the derived category $\mathbf{D}(\mathscr{A})$ via a butterfly of model structures – the stabilization functor associated to the induced recollement is the desired functor $\mathbf{D}(\mathscr{A}) \to \mathrm{Ho}(\mathcal{M})$.

II.C.1. Distinguished triangles from short exact sequences

Recall the description of the shift functor and the distinguished triangles in the stable category $\underline{\mathscr{F}} = \mathscr{F}/\omega$ of a Frobenius category $\mathscr{F}$ with class ω of projective-injective objects: Given some $X \in \mathscr{F}$, ΣX can be computed through any admissible short exact sequence $0 \to X \to \mathrm{C}\,X \to \Sigma X \to 0$ with $\mathrm{C}\,X \in \omega$, and any two choices of such short exact sequences lead to canonically isomorphic objects in $\underline{\mathscr{F}}$. Further, given an admissible short exact sequence $0 \to X \to Y \to Z \to 0$, one constructs a *connecting homomorphism* $\delta : Z \to \Sigma X$ in $\underline{\mathscr{F}}$ leading to the *standard distinguished triangle* $X \to Y \to Z \to \Sigma X$ attached to $0 \to X \to Y \to Z \to 0$ as follows: firstly, one picks an admissible short exact sequence $0 \to X \to \mathrm{C}\,X \to \Sigma X \to 0$ with $\mathrm{C}\,X \in \omega$ computing the shift ΣX of X. Then, forming the pushout diagram

$$\begin{array}{ccccccccc} & & 0 & & 0 & & & & \\ & & \downarrow & & \downarrow & & & & \\ 0 & \xrightarrow{f} & X & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \| & & \\ 0 & \longrightarrow & \mathrm{C}\,X & \longrightarrow & \mathrm{C}\,f & \longrightarrow & Z & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & & & \\ & & \Sigma X & = & \Sigma X & & & & \\ & & \downarrow & & \downarrow & & & & \\ & & 0 & & 0 & & & & \end{array} \tag{II.C.1.1}$$

the exact sequence $0 \to \mathrm{C}\,X \to \mathrm{C}\,f \to Z \to 0$ splits by the injectivity of $\mathrm{C}\,X$, and hence $\mathrm{C}\,f \to Z$ is an isomorphism in $\underline{\mathscr{F}}$. The connecting homomorphism of $0 \to X \to Y \to Z \to 0$ is then defined as the composition $Z \to \mathrm{C}\,f \to \Sigma X$ in $\underline{\mathscr{F}}$. The distinguished triangles in $\underline{\mathscr{F}}$ are by definition those which are isomorphic to the standard triangles constructed above.

Back in our original situation of an hereditary abelian model structure $\mathcal{M} = (\mathcal{C}, \mathcal{W}, \mathcal{F})$ on an abelian category $\mathscr{A}$, we now use the Resolution Lemma II.3.1.5 to transfer the construction of the connecting homomorphism of the Frobenius category $\mathcal{C} \cap \mathcal{F}$ to $\mathrm{Ho}(\mathcal{M})$:

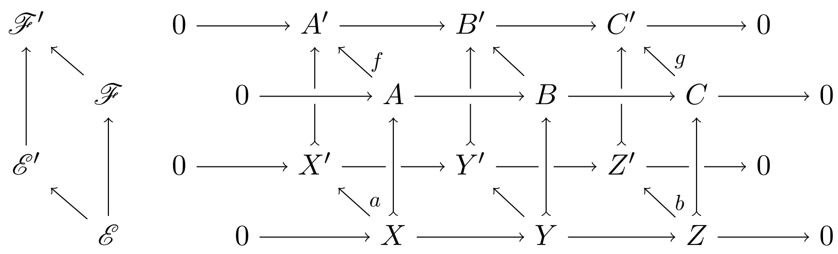
Proposition II.C.1.1. *Let $\mathcal{M} = (\mathcal{C}, \mathcal{W}, \mathcal{F})$ be an hereditary abelian model structure on an abelian category $\mathscr{A}$. Then there is a unique functor δ assigning to each short exact sequence $\mathscr{E} := 0 \to X \to Y \to Z \to 0$ in $\mathscr{A}$ a morphism $\delta(\mathscr{E}) : Z \to \Sigma X$ in $\mathrm{Ho}(\mathcal{M})$, equal to the connecting morphism in $\mathcal{C} \cap \mathcal{F} / \mathcal{C} \cap \mathcal{W} \cap \mathcal{F}$ in case $X, Y, Z \in \mathcal{C} \cap \mathcal{F}$.*

Proof. Assume first that $X, Y, Z \in \mathcal{C}$ and pick some commutative diagram

$$\begin{array}{ccccccccccc} \mathscr{E} & & 0 & \longrightarrow & X & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & 0 \\ \downarrow & & & & \rightarrowtail & & \rightarrowtail & & \rightarrowtail & & \\ \mathscr{F} & & 0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0 \end{array} \tag{II.C.1.2}$$

where the vertical arrows are trivial cofibrations and $A, B, C \in \mathcal{C} \cap \mathcal{F}$; this is possible by Lemma II.3.1.5 applied to the cotorsion pair $(\mathcal{C} \cap \mathcal{W}, \mathcal{F})$. The connecting homomorphism $C \to \Sigma A$ in $\mathcal{C} \cap \mathcal{F} / \mathcal{C} \cap \mathcal{W} \cap \mathcal{F} \cong \mathrm{Ho}(\mathcal{M})$ together with the fact that the vertical maps are isomorphisms in $\mathrm{Ho}(\mathcal{M})$ then yields the composition $Z \cong C \to \Sigma A \cong \Sigma X$ as a candidate for $\delta(\mathscr{E})$. In fact, if the functor δ is to exist, this is the only possible choice for $\delta(\mathscr{E})$; however, it remains to be proved that the our definition of $\delta(\mathscr{E})$ is independent on the choice of $\mathcal{C} \cap \mathcal{F}$-resolution (II.C.1.2).

Suppose next that we are given a commutative diagram



(II.C.1.3)

where all vertical arrows are trivial cofibrations, the bottom layer belonging to $\mathcal{F}$ and the upper layer belonging to $\mathcal{C}\cap\mathcal{F}$. Then, by the naturality of the connecting homomorphism,

$$\begin{array}{ccc} C & \longrightarrow & \Sigma A \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle \Sigma f} \\ C' & \longrightarrow & \Sigma A', \end{array}$$

is commutative in $\mathcal{C}\cap\mathcal{F}/\mathcal{C}\cap\mathcal{W}\cap\mathcal{F} \cong \mathrm{Ho}(\mathcal{M})$, so we deduce that also the square

$$\begin{array}{ccc} Y & \longrightarrow & \Sigma X \\ {\scriptstyle \Sigma a}\downarrow & & \downarrow{\scriptstyle b} \\ Y' & \longrightarrow & \Sigma X' \end{array}$$

commutes in $\mathrm{Ho}(\mathcal{M})$, which is a special case of the naturality we require; however, we had to assume the existence of the map $\mathscr{F} \to \mathscr{F}'$ making (II.C.1.3) commutative. While it's not clear if such maps always exist, one can always extend a diagram (II.C.1.3) with the map $\mathscr{F} \to \mathscr{F}'$ missing by some map $\mathscr{F}' \to \mathscr{F}''$ of short exact sequences, which firstly is termwise a trivial cofibration, and for which secondly there is a map $\mathscr{F} \to \mathscr{F}''$ making everything commutative: for this, first form the pushout of $\mathscr{E} \to \mathscr{E}' \to \mathcal{F}'$ along $\mathscr{E} \to \mathscr{F}$, giving a pair of maps of short exact sequences $\mathcal{F}' \to \mathscr{E}'' \leftarrow \mathscr{F}$ in which $\mathcal{F}' \to \mathscr{E}''$ is a termwise trivial cofibration. Then, applying Lemma II.3.1.5 to $\mathscr{E}''$ gives a termwise trivial cofibration $\mathscr{E}'' \to \mathscr{F}''$ with $\mathscr{F}''$ having terms in $\mathcal{C} \cap \mathcal{F}$, and the composition $\mathscr{F}' \to \mathscr{E}'' \to \mathscr{F}''$ has the required properties. In this situation, we may then apply the restricted naturality we have just proved twice to prove naturality in general. In particular, the choice of resolving sequence $0 \to A \to B \to C \to 0$ does not affect the morphism $Z \to \Sigma X$ we obtain.

So far we restricted to the case $X, Y, Z \in \mathcal{C}$. In the general case, we repeat the above arguments, this time for resolutions of the form

$$\begin{array}{ccccccccc} 0 & \longrightarrow & X & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & 0 \\ & & \uparrow & & \uparrow & & \uparrow & & \\ 0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0 \end{array}$$

with $A, B, C \in \mathcal{C}$ and the vertical maps being trivial fibrations; the existence of those is guaranteed by the dual of Resolution Lemma II.3.1.5 applied to the cotorsion pair $(\mathcal{C}, \mathcal{W} \cap \mathcal{F})$. In this way, we may reduce the construction of the required morphism $Z \to \Sigma X$ and its naturality to the previously treated case $X, Y, Z \in \mathcal{C}$. □

In particular, the value of the shift functor $\Sigma : \mathrm{Ho}(\mathcal{M}) \to \mathrm{Ho}(\mathcal{M})$ on some $X \in \mathscr{A}$ can be computed through any short exact sequence $0 \to X \to W \to X' \to 0$ with $X \in \mathscr{A}$: for any such sequence, its connecting homomorphism is a canonical isomorphism $X' \to \Sigma X$.

Corollary II.C.1.2. *In the situation of Proposition II.C.1.1, the triangle $X \to Y \to Z \to \Sigma X$ is distinguished in* $\mathrm{Ho}(\mathcal{M})$*, and any distinguished triangle is isomorphic to such a triangle. In particular, any short exact sequence in $\mathscr{A}$ gives rise to a distinguished triangle in* $\mathrm{Ho}(\mathcal{M})$*.*

Corollary II.C.1.3. *In any hereditary abelian model structure $\mathcal{M} = (\mathcal{C}, \mathcal{W}, \mathcal{F})$ on an abelian category $\mathscr{A}$, the weak equivalences of $\mathcal{M}$ are precisely the morphisms $f : X \to Y$ that can be factored as $X \overset{\alpha}{\rightarrowtail} Z \overset{\beta}{\twoheadrightarrow} Y$ with α a monomorphism with* $\mathrm{coker}(\alpha) \in \mathcal{W}$ *and β an epimorphism with* $\ker(\beta) \in \mathcal{W}$

Proof. The implication "⇒" follows from the factorization axiom, while the non-trivial direction "⇐" follows from the stability of weak equivalences under composition and Corollary II.C.1.2. □

Remark II.C.1.4. In fact, Corollary II.C.1.3 is true for *any* abelian model structure [Hov02, Lemma 5.8] – we nonetheless kept the Corollary here as a simple application of Corollary II.C.1.2. ◊

For a general abelian model category, the naturality of δ as well as the possibility of successively resolving short exact sequences by such in $\mathcal{C}$ and then by such in $\mathcal{C} \cap \mathcal{F}$, we deduce the following recipe for actually computing the connecting homomorphism of some short exact sequence:

Corollary II.C.1.5. *Assume the setting of Proposition II.C.1.1 and let $0 \to X \to Y \to Z \to 0$, $0 \to X \to W \to X' \to 0$ be short exact sequences in $\mathscr{A}$ with $W \in \mathcal{W}$. Then the* *connecting homomorphism* $Z \to \Sigma X \cong X'$ *is equal to the composition* $Z \leftarrow T \to X'$ *in the pushout diagram*

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & X & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & W & \longrightarrow & T & \longrightarrow & Z & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & \overset{\delta}{\dashleftarrow} & & & \\
 & & X' & = & X' & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array} \tag{II.C.1.4}$$

Proof. First, the dual of Lemma II.3.1.5 applied to the cotorsion pair $(\mathcal{C}, \mathcal{W} \cap \mathcal{F})$ provides a termwise trivial fibration from some short exact sequence in $\mathcal{C}$ to the given short exact sequence $0 \to W \to T \to Z \to 0$. Taking the pullback of this resolution along the morphism between the first and second row, we therefore reduce to the case $W, T, Z \in \mathcal{C}$. Similarly, we may reduce to also $Y, X' \in \mathcal{C}$ by resolving the short exact sequence $0 \to Y \to T \to X' \to 0$. Then, it follows that $X \in \mathcal{C}$ since $\mathcal{C}$ is closed under kernels of epimorphisms by assumption, so we may assume all terms of (II.C.1.4) to belong to $\mathcal{C}$.

In this situation, we may apply Lemma II.3.1.5 twice again, this time applied to the cotorsion pair $(\mathcal{C} \cap \mathcal{W}, \mathcal{F})$, to resolve the exact sequences $0 \to X \to Y \to Z \to 0$ and $0 \to X \to W \to X' \to 0$ by sequences in $\mathcal{F}$ and to therefore achieve $X, Y, Z, W, X' \in \mathcal{C} \cap \mathcal{F}$. Then $T \in \mathcal{C} \cap \mathcal{F}$ is automatic since $\mathcal{C}$ and $\mathcal{F}$ are closed under extensions.

We have therefore reduced to the case where all terms of (II.C.1.4) belong to $\mathcal{C} \cap \mathcal{F}$, and in this case the claim is precisely the definition of the connecting morphism, as we recalled in (II.C.1.1) above. □

Finally, we include a formulation of Corollary II.C.1.2 in the language of derivators, which we quickly introduce first; for precise definitions and a thorough treatment of derivators, we refer to [Gro13]. Informally, a *derivator* $\mathbb{D}$ based on category $\mathcal{C}$ is a family of categories $\mathbb{D}(I)$, indexed over small diagram categories I, in which one thinks of $\mathbb{D}(I)$ as some suitable category of I-diagrams in $\mathcal{C}$, which, however, need not equal $\mathcal{C}^I$. For $I = *$ one requires $\mathbb{D}(*) = \mathcal{C}$, and further part of $\mathbb{D}$ are abstractions $\alpha^* : \mathbb{D}(I) \to \mathbb{D}(J)$ of the restriction functors $\mathcal{C}^I \to \mathcal{C}^J$ associated to functors $\alpha : J \to I$. Among other things, it is required that these functors have left and right adjoints $\alpha_!, \alpha^* : \mathbb{D}(J) \to \mathbb{D}(I)$, to be thought of abstractions of the left and right Kan extension functors $\mathcal{C}^J \to \mathcal{C}^I$.

Two sources of derivators are the following: Firstly, if $\mathcal{C}$ is a bicomplete category, then $\mathbb{D}(I) := \mathcal{C}^I$ defines a derivator based on $\mathcal{C}$, the *derivator represented by* $\mathcal{C}$; see [Gro13, Example 1.2, Example 1.11]. Secondly, if $\mathcal{M}$ is a model structure on some category $\mathcal{C}$ with weak equivalences W, then $\mathbb{D}_{\mathcal{M}}(I) := (\mathcal{C}^I)[\mathrm{W}_I^{-1}]$ defines a derivator based on $\mathrm{Ho}(\mathcal{M}) = \mathcal{C}[\mathcal{W}^{-1}]$, where W_I is the class of termwise weak equivalences; see [Gro13, Theorem 1.38]. In this situation, the left and right adjoints $\alpha_!$ and α_* of $\alpha^* : \mathbb{D}_{\mathcal{M}}(I) \to \mathbb{D}_{\mathcal{M}}(J)$ are called *homotopy left/right Kan extension*, respectively, and objects of $\mathbb{D}_{\mathcal{M}}(I)$ are called *coherent I-diagrams* to distinguish them from the diagrams in $\mathcal{C}[\mathcal{W}^{-1}]^I = \mathrm{Ho}(\mathcal{M})^I$ which can only be represented by diagrams that are commutative up to homotopy. Note also that $\mathbb{D}_{\mathcal{M}}(I)$ is not defined as the homotopy category of a particular model structure on $\mathcal{C}^I$, but that suitable model structures on $\mathcal{C}^I$ may, if present, be used to calculate homotopy left/right Kan extensions; for example, if $\mathcal{M}$ is combinatorial and I is arbitrary, or if $\mathcal{M}$ is arbitrary and I is directed [Hov99, Definition 5.1.1], then $\mathbb{D}_{\mathcal{M}}(I)$ is modelled by the projective model structure on $\mathcal{C}^I$ [Hov99, Theorem 5.1.3] with respect to which the left homotopy Kan extension may be computed; this will be used and illustrated in the proof of Proposition II.C.1.6 below.

In any derivator $\mathbb{D}$ there is a notion of *cartesian and cocartesian squares* [Gro13, Section 3.2, in particular Definition 3.9]: The inclusions $\mathrm{i}_{\ulcorner} : \ulcorner \hookrightarrow \Box$ and $\mathrm{i}_{\lrcorner} : \lrcorner \hookrightarrow \Box$ induce fully faithful homotopy Kan extension functors $\mathrm{i}_{\lrcorner *} : \mathbb{D}(\lrcorner) \to \mathbb{D}(\Box)$ and $\mathrm{i}_{\ulcorner !} : \mathbb{D}(\ulcorner) \to \mathbb{D}(\Box)$, and a coherent square in $\mathbb{D}(\Box)$ is called cartesian resp. cocartesian if it belongs to the essential image of $\mathrm{i}_{\lrcorner *}$ resp. $\mathrm{i}_{\ulcorner !}$ (this reduces to the ordinary notion of cartesian and cocartesian squares for the derivator represented by a category). Finally, there's the notion of *stable derivators* [Gro13, Section 4, in particular Definition 4.1] for which cocartesian and cartesian squares coincide, and the derivator $\mathbb{D}_{\mathcal{M}}$ associated to a combinatorial stable model structure $\mathcal{M}$ is stable [Gro13, Example 4.2]. In particular, the derivators associated to cofibrantly generated, hereditary abelian model structures are stable.

We have the following version of Proposition II.C.1.1 in the language of derivators.

Proposition II.C.1.6. *Let $\mathcal{M} = (\mathcal{C}, \mathcal{W}, \mathcal{F})$ be a hereditary abelian model structure on an abelian category $\mathscr{A}$ and let $\mathbf{D}_{\mathcal{M}}$ be its stable derivator. Further, assume that*

$$\begin{array}{ccc} X & \rightarrowtail & Y \\ \downarrow & & \downarrow \\ A & \rightarrowtail & Z \end{array}$$

is a commutative, bicartesian square in $\mathscr{A}$, with the vertical arrows being epimorphisms and the horizontal arrows being monomorphisms. Then this square is also bicartesian considered as a coherent square in $\mathbf{D}_{\mathcal{M}}(\Box)$.

Proof. Let $U \hookrightarrow X$ be the kernel of $X \twoheadrightarrow A$. We then have a commutative diagram

$$\begin{array}{ccccc} U & \rightarrowtail & X & \rightarrowtail & Y \\ \big\downarrow & & \big\downarrow & & \big\downarrow \\ 0 & \rightarrowtail & A & \rightarrowtail & Z \end{array}$$

in which both small inner squares and hence also the large outer square are bicartesian. By [Gro13, Proposition 3.14(1)] it is therefore sufficient to prove the proposition in case $A = 0$. In other words, we have to show that for any short exact sequence $0 \to X \to Y \to Z \to 0$ in $\mathscr{A}$ the coherent square

$$\begin{array}{ccc} X & \overset{f}{\rightarrowtail} & Y \\ \big\downarrow & & \big\downarrow \\ 0 & \longrightarrow & Z \end{array} \tag{II.C.1.5}$$

is cartesian, i.e. that it belongs to the essential image of the homotopy left Kan extension functor $\mathrm{i}_{\ulcorner !} : \mathbb{D}_{\mathcal{M}}(\ulcorner) \to \mathbb{D}_{\mathcal{M}}(\square)$ associated to the inclusion $\mathrm{i}_{\ulcorner} : \ulcorner \hookrightarrow \square$. Further, by Lemma II.3.1.5, it is sufficient to treat the case where $X, Y, Z \in \mathcal{C}$.

Since $\ulcorner$ and $\square$ are directed, $\mathscr{A}^{\ulcorner}$ and $\mathscr{A}^{\square}$ carry the projective model structures, with respect to which the adjunction $\mathrm{i}_{\ulcorner !} \dashv \mathrm{i}_{\ulcorner}{}^* : \mathscr{A}^{\ulcorner} \rightleftarrows \mathscr{A}^{\square}$ is a Quillen adjunction. The homotopy left Kan extension functor $\mathrm{i}_{\ulcorner !} : \mathbb{D}_{\mathcal{M}}(\ulcorner) \to \mathbb{D}_{\mathcal{M}}(\square)$ may therefore be computed naively on projectively cofibrant objects in $\mathscr{A}^{\ulcorner}$, which by [Hov99, Theorem 5.1.3] are those diagrams $V \overset{i}{\leftarrow} U \overset{j}{\rightarrow} W$ in which V, U, W are cofibrant and i and j are cofibrations. Assuming $X, Y, Z \in \mathcal{C}$ we already have that $f : X \rightarrowtail Y$ is a cofibration, so that choosing a cofibration $X \rightarrowtail I$ with $I \in \mathcal{C} \cap \mathcal{W} \cap \mathcal{F}$ we get

$$\mathrm{i}_{\ulcorner !}\left[\begin{array}{ccc} X & \overset{f}{\rightarrowtail} & Y \\ \big\downarrow & & \\ 0 & & \end{array}\right] \cong \mathrm{i}_{\ulcorner !}\left[\begin{array}{ccc} X & \overset{f}{\rightarrowtail} & Y \\ \big\downarrow & & \\ I & & \end{array}\right] \cong \left[\begin{array}{ccc} X & \overset{f}{\rightarrowtail} & Y \\ \big\downarrow & & \big\downarrow \\ I & \overset{g}{\rightarrowtail} & C \end{array}\right]$$

Since $\operatorname{coker}(g) \cong \operatorname{coker}(f) \cong Z$, this cocartesian square is isomorphic in $\mathbb{D}_{\mathcal{M}}(\square)$ to (II.C.1.5), and so the latter is cocartesian as well, as claimed. □

II.C.2. Higher Extensions

Proposition II.C.1.1 defines a morphism $\delta : \operatorname{Ext}^1_{\mathscr{A}}(A, B) \to \operatorname{Ho}(\mathcal{M})(A, \Sigma B)$, inducing the structure of an unbounded cohomological δ-functor on the functors $\operatorname{Ho}(\mathcal{M})(A, \Sigma^*(-))$

for fixed $A \in \mathscr{A}$. Indeed, any short exact sequence $\mathscr{E} : 0 \to X \to Y \to Z \to 0$ in $\mathscr{A}$ gives rise to the distinguished triangle $X \to Y \to Z \xrightarrow{\delta(\mathscr{E})} \Sigma X$ in $\mathrm{Ho}(\mathcal{M})$, hence to a long exact sequence

$$\ldots \to \mathrm{Ho}(\mathcal{M})(A, \Sigma^k Y) \to \mathrm{Ho}(\mathcal{M})(A, \Sigma^k Z) \xrightarrow{\Sigma^k \delta(\mathscr{E})} \mathrm{Ho}(\mathcal{M})(A, \Sigma^{k+1} X) \to \ldots$$

Similarly, for any fixed $X \in \mathscr{A}$ the family of functors $\mathrm{Ho}(\mathcal{M})(-, \Sigma^* X)$ admits a canonical structure of an unbounded contravariant cohomological δ-functor, with the connecting homomorphism $\mathrm{Ho}(\mathcal{M})(A, \Sigma^k X) \to \mathrm{Ho}(\mathcal{M})(C, \Sigma^{k+1} X)$ associated to a short exact sequence $\mathscr{E} : 0 \to A \to B \to C \to 0$ being given by the composition

$$\mathrm{Ho}(\mathcal{M})(A, \Sigma^k X) \xrightarrow{\Sigma} \mathrm{Ho}(\mathcal{M})(\Sigma A, \Sigma^{k+1} X) \xrightarrow{-\circ\delta(\mathscr{E})} \mathrm{Ho}(\mathcal{M})(C, \Sigma^{k+1} X).$$

Now, recall that for any abelian category $\mathscr{A}$ there is the distinguished cohomological δ-functor of *Yoneda extensions* $\mathrm{Ext}^k_{\mathscr{A}}(-,-)$, defined as follows: For any two $X, Y \in \mathscr{A}$, $\mathrm{Ext}^k_{\mathscr{A}}(X, Y)$ is the set of connected components of the category $\mathscr{E}^k_{\mathscr{A}}(X, Y)$ of $(k+2)$-term exact sequences $0 \to Y \to Z_1 \to \ldots \to Z_k \to X \to 0$, with morphisms being commutative diagrams

$$\begin{array}{ccccccccccccc}
0 & \longrightarrow & Y & \longrightarrow & Z_1 & \longrightarrow & \cdots & \longrightarrow & Z_k & \longrightarrow & X & \longrightarrow & 0 \\
 & & \| & & \downarrow & & & & \downarrow & & \| & & \\
0 & \longrightarrow & Y & \longrightarrow & Z_1' & \longrightarrow & \cdots & \longrightarrow & Z_k' & \longrightarrow & X & \longrightarrow & 0.
\end{array}$$

Morphisms $X' \to X$ and $Y \to Y'$ induce functors $\mathscr{E}^*_{\mathscr{A}}(X', Y) \leftarrow \mathscr{E}^*_{\mathscr{A}}(X, Y) \to \mathscr{E}^*_{\mathscr{A}}(X, Y')$ by pushout and pullback, hence maps $\mathrm{Ext}^*_{\mathscr{A}}(X', Y) \leftarrow \mathrm{Ext}^*_{\mathscr{A}}(X, Y) \to \mathrm{Ext}^*_{\mathscr{A}}(X, Y')$ on passage to connected components. Similarly, splicing together exact sequences defines functors $\mathscr{E}^k_{\mathscr{A}}(X, Y) \times \mathscr{E}^l_{\mathscr{A}}(Y, Z) \to \mathscr{E}^{k+l}_{\mathscr{A}}(X, Z)$, hence maps $\star : \mathrm{Ext}^k_{\mathscr{A}}(X, Y) \times \mathrm{Ext}^l_{\mathscr{A}}(Y, Z) \to \mathrm{Ext}^{k+l}_{\mathscr{A}}(X, Z)$ on connected components – this is called the *Yoneda product*. Finally, given a short exact sequence $\mathscr{F} : 0 \to X \to Y \to Z \to 0$, the Yoneda product with $\mathscr{F}$ defines connecting maps

$$[\mathscr{F}] \star - : \ \mathrm{Ext}^*_{\mathscr{A}}(-, Z) \to \mathrm{Ext}^{*+1}_{\mathscr{A}}(-, X) \text{ and } - \star [\mathscr{F}] : \ \mathrm{Ext}^*_{\mathscr{A}}(X, -) \to \mathrm{Ext}^{*+1}_{\mathscr{A}}(Z, -)$$

that turn $\mathrm{Ext}^*_{\mathscr{A}}(X, -)$ and $\mathrm{Ext}^*_{\mathscr{A}}(-, X)$ into δ-functors. These δ-functors are universal in the following sense: for any another covariant, $\mathbb{Z}$-indexed (possibly unbounded) δ-functor $T^* : \mathscr{A} \to \mathrm{Ab}$ and any $\alpha \in T^0 X$ there is a unique morphism of δ-functors $\psi^* : \mathrm{Ext}^*_{\mathscr{A}}(X, -) \to T^*$ such that $\psi^0(X)(\mathrm{id}_X) = \alpha$, and similar for morphisms of contravariant δ-functors $\mathrm{Ext}^*_{\mathscr{A}}(-, X) \to T^*$ in case T^* is contravariant. For details, see e.g. [Buc59, Theorem 3.1] and [Buc60, Theorem 4.3].

Remark II.C.2.1. The previous paragraph raises some set-theoretic issues, since it is known that the Yoneda extension group $\mathrm{Ext}^k_{\mathscr{A}}(X,Y)$ need not be realizable as a set. To be precise, the entirety of objects in $\mathscr{E}^k_{\mathscr{A}}(X,Y)$ is definable as a class, and the relation of belonging to the same connected component of $\mathscr{E}^k_{\mathscr{A}}(X,Y)$ is definable as a class equivalence relation – however, there need not be a set-quotient of this relation. For example, one can consider the category $\mathscr{A}$ of triples (A,λ,ρ) where A is an abelian group, $\lambda\in\mathrm{Ord}$ is an ordinal and $\rho:\lambda\to\mathrm{End}_{\mathbb{Z}}(A)$ is a map of sets. Thinking of the action of remaining ordinals $\mu\notin\Lambda$ on A as 0, such a triple is like a "locally small" module over the large polynomial ring $\mathbb{Z}[X_\lambda\mid\lambda\in\mathrm{Ord}]$. With this in mind one has a natural notion of morphism and henceforth gets a category $\mathscr{A}$ well-defined in ZFC. This category $\mathscr{A}$ is AB5, but it does not admit a generator, since for any (A,λ,ρ) one has $\mathrm{Hom}_{\mathscr{A}}((A,\lambda,\rho),\mathbb{Z}_\lambda)=0$, where $\mathbb{Z}_\lambda$ is the triple $(\mathbb{Z},\lambda+1,\rho)$ with $\rho(\mu):=0$ for $\mu<\lambda$ and $\rho(\lambda)=\mathrm{id}$. Moreover, for the module $\mathbb{Z}:=(\mathbb{Z},\emptyset,\emptyset)$ we have a class embedding $\mathrm{Ord}\hookrightarrow\mathrm{Obj}(\mathscr{E}^1_{\mathscr{A}}(\mathbb{Z},\mathbb{Z}))$ sending $\lambda\in\mathrm{Ord}$ to $0\to\mathbb{Z}\to\mathbb{Z}[X_\lambda]/(X_\lambda^2)\to\mathbb{Z}\to 0$, where the module in the middle is understood to have trivial μ-action for all ordinals $\mu\neq\lambda$. Since $\mathbb{Z}[X_\lambda]/(X_\lambda^2)\not\cong\mathbb{Z}[X_\mu]/(X_\mu^2)$ for $\lambda\neq\mu$, this map hits different connected components of $\mathscr{E}^1_{\mathscr{A}}(X,Y)$, and hence the hypothetical $\mathrm{Ext}^1_{\mathscr{A}}(\mathbb{Z},\mathbb{Z})=\pi_0\mathscr{E}^1_{\mathscr{A}}(X,Y)$ is not defined in ZFC.

In the end, however, these issues are not serious, as we can work with $\mathrm{Ext}^*_{\mathscr{A}}(-,-)$ as a class equipped with a class equivalence relation, and the morphism of δ-functors $\mathrm{Ext}^*_{\mathscr{A}}(X,-)\to T^*$ associated to some $\alpha\in T^0(X)$ can, in principle, be explicitly described as a class function $\mathscr{E}^*_{\mathscr{A}}(X,Y)\to T^*(Y)$ compatible with the class equivalence relation on $\mathscr{E}^*_{\mathscr{A}}(X,Y)$ (see the proof of Proposition II.C.2.2 below).

Alternatively, one might abandon all definability issues by working in ZFC with universes, but the question of preservation of universe also remains in that case. ◊

Proposition II.C.2.2. *Let $\mathcal{M}=(\mathcal{C},\mathcal{W},\mathcal{F})$ be a hereditary abelian model structure on the abelian category $\mathscr{A}$. We equip $\mathrm{Ho}(\mathcal{M})(-,-)$ with the structure of a cohomological δ-functor in both variables as described in the previous paragraph.*

(i) *For any $A\in\mathscr{A}$ there is a unique "realization" morphism of cohomological δ-functors $\mathrm{real}^*_{A,-}:\mathrm{Ext}^*_{\mathscr{A}}(A,-)\to\mathrm{Ho}(\mathcal{M})(A,\Sigma^*(-))$ which coincides with the canonical morphism $\mathscr{A}(A,-)\to\mathrm{Ho}(\mathcal{M})(A,-)$ for $*=0$.*

(ii) *The morphisms $\mathrm{real}^*_{A,X}:\mathrm{Ext}^*_{\mathscr{A}}(A,X)\to\mathrm{Ho}(\mathcal{M})(A,\Sigma^*X)$ also constitute the unique morphism of δ-functors $\mathrm{Ext}^*_{\mathscr{A}}(-,X)\to\mathrm{Ho}(\mathcal{M})(-,\Sigma^*X)$ determined by its degree 0 being given by $\mathscr{A}(-,X)\to\mathrm{Ho}(\mathcal{M})(-,X)$.*

Proof. Part (a) follows from the universality of $\mathrm{Ext}^*_{\mathscr{A}}(A,-)$. Explicitly, the morphism $\mathrm{real}^k_{A,X}$ is given as follows: By definition, an extension class $\alpha\in\mathrm{Ext}^k_{\mathscr{A}}(A,X)$ is represented by a $(k+2)$-term exact sequence $0\to X\to Z_1\to\ldots\to Z_k\to A\to 0$ in $\mathscr{A}$

splitting into short exact sequences $0 \to X \to Z_1 \to S_1 \to 0$, $0 \to S_1 \to Z_2 \to S_2 \to 0$, $\ldots$, $0 \to S_{k-1} \to Z_k \to A \to 0$ and giving rise to a sequence of connecting homomorphisms $\delta_1 : S_1 \to \Sigma X$, $S_2 \to \Sigma S_1$, $\ldots$, $\delta_k : A \to \Sigma S_k$ in $\mathrm{Ho}(\mathcal{M})$. The image of α under $\mathrm{Ext}^k_{\mathscr{A}}(A, X) \to \mathrm{Ho}(\mathcal{M})(A, \Sigma^k X)$ is then given by the composition $\Sigma^{k-1}\delta_1 \circ \Sigma^{k-2}\delta_2 \circ \ldots \circ \delta_k : A \to \Sigma^k X$.

For part (b) we have to check that for any short exact sequence $\mathscr{F} := 0 \to A \to B \to C \to 0$ in $\mathscr{A}$ and any $X \in \mathscr{A}$ the diagram

$$\begin{array}{ccccc}
\mathrm{Ext}^k_{\mathscr{A}}(A,X) & & \xrightarrow{\;-\,\star\,[\mathscr{F}]\;} & & \mathrm{Ext}^{k+1}_{\mathscr{A}}(C,X) \\
\big\downarrow{\scriptstyle \mathrm{real}^k_{A,X}} & & & & \big\downarrow{\scriptstyle \mathrm{real}^{k+1}_{C,X}} \\
\mathrm{Ho}(\mathcal{M})(A,\Sigma^k X) & \xrightarrow[\Sigma]{} & \mathrm{Ho}(\mathcal{M})(\Sigma A,\Sigma^{k+1} X) & \xrightarrow[-\,\circ\,\delta(\mathscr{F})]{} & \mathrm{Ho}(\mathcal{M})(C,\Sigma^{k+1} X)
\end{array}$$

is commutative, and this follows directly from the explicit description of the comparison morphism given in (a). ☐

II.C.3. A realization functor

Interpreting $\mathrm{Ext}^*_{\mathscr{A}}(-,-)$ as the morphisms in $\mathbf{D}(\mathscr{A})$, we have constructed assignments

$$\mathrm{real}^*_{-,-} : \mathbf{D}(\mathscr{A})(-, \Sigma^*(-)) \longrightarrow \mathrm{Ho}(\mathcal{M})(-, \Sigma^*(-)),$$

and it is tempting to ask whether these maps are induced by a "realization functor"

$$\mathrm{real} : \mathbf{D}(\mathscr{A}) \longrightarrow \mathrm{Ho}(\mathcal{M}) \qquad \text{(II.C.3.6)}$$

making the diagram

$$\begin{array}{ccc}
\mathrm{Ho}(\mathcal{M}) & \xleftarrow{\;\mathrm{real}\;} & \mathbf{D}(\mathscr{A}) \\
\uparrow & & \uparrow \\
 & \mathscr{A} &
\end{array} \qquad \text{(II.C.3.7)}$$

commute. This question is studied in detail in this section, and the affirmative answer is given in Proposition II.C.3.26. We begin by studying some examples:

Example II.C.3.1. Let R be Gorenstein and $\mathcal{M}^{\text{G-proj}}(R) = (\text{G-proj}(R), \mathcal{P}^{<\infty}, R\text{-Mod})$ be the Gorenstein projective model structure on $\mathscr{A} := R\text{-Mod}$ (see Proposition II.2.1.7). Then, firstly we know that there is a Quillen equivalence $Q^0 : \mathcal{M}^{\text{G-proj}}(R) \rightleftarrows {}^p\mathcal{M}^{\text{ctr}}_{\text{sing}}(R) : \iota^0$ (Theorem II.5.1.5) and in particular a commutative diagram

$$\begin{array}{ccc}
\mathrm{Ho}(\mathcal{M}^{\text{G-proj}}(R)) & \xrightarrow[\mathbf{R}\iota^0 = \iota^0]{\cong} & \mathrm{Ho}({}^p\mathcal{M}^{\text{ctr}}_{\text{sing}}(R)) \\
\uparrow & & \uparrow \\
 & R\text{-Mod}\,. &
\end{array} \qquad \text{(II.C.3.8)}$$

Secondly, $\mathbf{K}_{\mathrm{ac}}(\mathrm{Proj}(R))$ belongs to Krause's recollement (see Corollary II.4.2.7),

$$\mathbf{K}_{\mathrm{ac}}(\mathrm{Proj}(R)) \rightleftarrows \mathbf{K}(\mathrm{Proj}(R)) \rightleftarrows \mathbf{D}(R\text{-Mod}),$$

so that we have a stabilization functor [Kra05, §5]

$$\mathbf{D}(R\text{-Mod}) \xrightarrow{Q_\lambda} \mathbf{K}(\mathrm{Proj}(R)) \xrightarrow{I_\rho} \mathbf{K}_{\mathrm{ac}}(\mathrm{Proj}(R)). \tag{II.C.3.9}$$

Composing the horizontal equivalence in (II.C.3.8) with (II.C.3.9), we obtain a functor $\mathrm{real} : \mathbf{D}(R\text{-Mod}) \longrightarrow \mathrm{Ho}(\mathcal{M}^{\text{G-proj}}(R))$, of which we claim that is makes the diagram

$$\begin{array}{ccc} \mathrm{Ho}(\mathcal{M}^{\text{G-proj}}(R)) & \xleftarrow{\quad\mathrm{real}\quad} & \mathbf{D}(R\text{-Mod}) \\ \uparrow & & \uparrow \\ & R\text{-Mod} & \end{array} \tag{II.C.3.10}$$

commutative. In view of (II.C.3.8), this follows once we prove commutativity of

$$\begin{array}{ccccc} \mathbf{D}^{\mathrm{ctr}}_{\mathrm{sing}}(R\text{-Mod}) & \xleftarrow{I_\rho} & \mathbf{D}^{\mathrm{ctr}}(R\text{-Mod}) & \xleftarrow{Q_\lambda} & \mathbf{D}(R\text{-Mod}). \\ \uparrow & & \uparrow & & \uparrow \\ & & R\text{-Mod} & & \end{array} \tag{II.C.3.11}$$

This follows from the explicit description of $Q_\lambda : \mathbf{D}(R\text{-Mod}) \to \mathbf{D}^{\mathrm{ctr}}(R\text{-Mod})$ and $I_\rho : \mathbf{D}^{\mathrm{ctr}}(R\text{-Mod}) \to \mathbf{D}^{\mathrm{ctr}}_{\mathrm{sing}}(R\text{-Mod})$ that can be read off from the projective analogue of the butterfly ($\bowtie$): The left adjoint $Q_\lambda : \mathbf{D}(R\text{-Mod}) \to \mathbf{D}^{\mathrm{ctr}}(R\text{-Mod})$ is computed by choice of dg projective resolution; since any bounded above acyclic complex is trivial in $\mathbf{D}^{\mathrm{ctr}}(R\text{-Mod})$, this shows that the right wing of (II.C.3.11) is commutative. The right adjoint $I_\rho : \mathbf{D}^{\mathrm{ctr}}(R\text{-Mod}) \to \mathbf{D}^{\mathrm{ctr}}_{\mathrm{sing}}(R\text{-Mod})$ can be computed naively, hence the left wing of (II.C.3.11) commutes, too.

From the commutativity of (II.C.3.10) we see that the morphism of δ-functors

$$\mathrm{Ext}^*_{R\text{-Mod}}(-,-) = \mathbf{D}(R\text{-Mod})(-,\Sigma^*(-)) \to \mathrm{Ho}(\mathcal{M}^{\text{G-proj}}(R))(-,\Sigma^*(-))$$

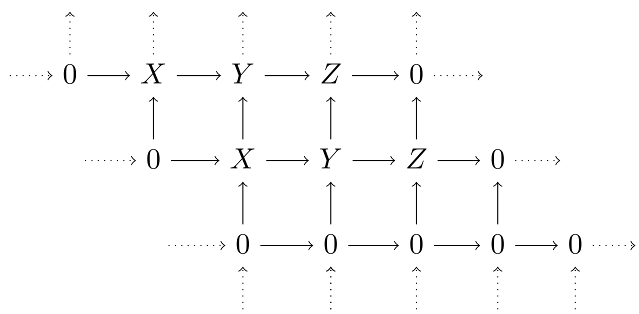
induced by real coincides with the one constructed in Proposition II.C.2.2. ◊

Example II.C.3.2. Suppose $\mathscr{C}$ is a Grothendieck abelian category. By Remark II.2.3.19, the category $\mathscr{A} := \mathrm{Ch}(\mathscr{C})$ carries the injective model structure $\mathcal{M}^{\mathrm{inj}}(\mathscr{C})$, and the realization functor (II.C.3.6) we are looking for would be a triangulated functor $\mathbf{D}(\mathscr{A}) \to \mathbf{D}(\mathscr{C})$ from the localization $\mathbf{D}(\mathscr{A})$ of the category of bicomplexes over $\mathscr{C}$ at the class of horizontal quasi-isomorphisms to the derived category $\mathbf{D}(\mathscr{C})$ of $\mathscr{C}$.

In case $\mathscr{C} = R\text{-Mod}$ for some ring, one might take as a functor $\mathbf{D}(\mathscr{A}) \to \mathbf{D}(\mathscr{C})$ the following "Laurent totalization": Given a bicomplex $(X^{**}, \mathrm{d}_{\mathrm{ver}}, \mathrm{d}_{\mathrm{hor}})$ over $\mathscr{C}$ (where the first index denotes the horizontal grading), denote $\mathrm{Tot}^{\sqsupset} X \in \mathrm{Ch}(\mathscr{C})$ the subcomplex of the totalization $\mathrm{Tot}^{\Pi} X$ by products given by the semi-infinite product

$$(\mathrm{Tot}^{\sqsupset} X)^n := \varinjlim_{a\to\infty} \prod_{\substack{p+q=n \\ p\leq a}} X^{p,q} \subset \prod_{p+q=n} X^{p,q} = (\mathrm{Tot}^{\Pi} X)^n$$

A small diagram-chase (see [Wei94, Acyclic Assembly Lemma 2.7.3]) shows that $\mathrm{Tot}^{\sqsupset} X$ is acyclic if each $(X^{*,q}, \mathrm{d}_{\mathrm{hor}})$ is acyclic, i.e. if X vanishes in $\mathbf{D}(\mathscr{A})$, so that $\mathrm{Tot}^{\sqsupset}$ descends naively to a triangulated functor $\mathbf{D}(\mathscr{A}) \to \mathbf{D}(\mathscr{C})$ making (II.C.3.7) commute. Note again, however, that real does not arise as a left or right Quillen functor. Also, the Laurent totalization does not descend naively to $\mathbf{D}(\mathscr{A})$ unless products in $\mathscr{C}$ are exact, as can be seen by considering the bicomplexes of the form



with $0 \to X \to Y \to Z \to 0$ a short exact sequence in $\mathscr{C}$: the Laurent totalization of this bicomplex is a shift of $0 \to X^{\mathbb{N}} \to Y^{\mathbb{N}} \to Z^{\mathbb{N}} \to 0$, and this sequence need not be exact for a general Grothendieck abelian category $\mathscr{C}$.

Alternatively, one can use the equivalence $\mathbf{D}(\mathscr{A}) \cong \mathbf{K}(\text{dg-Inj}(\mathscr{A}))$ first and apply any of the variants $\mathrm{Tot}^{\oplus}, \mathrm{Tot}^{\Pi}, \mathrm{Tot}^{\sqsupset}, \mathrm{Tot}^{\sqsubset}$ of totalization afterwards to get a functor to $\mathbf{D}(\mathscr{C})$. When restricted to $\mathrm{Ch}(\mathscr{C}) = \mathscr{A} \subset \mathrm{Ch}(\mathscr{A})$, the variants using $\mathrm{Tot}^{\oplus}$ and $\mathrm{Tot}^{\sqsupset}$ agree, as do the variants using Tot^{Π} and $\mathrm{Tot}^{\sqsubset}$. In particular, the composition

$$\mathbf{D}(\mathscr{A}) \ \cong\ \mathbf{K}(\text{dg-Inj}(\mathscr{A})) \xrightarrow{\mathrm{Tot}^{\oplus}} \mathbf{D}(\mathscr{C}) \tag{II.C.3.12}$$

makes (II.C.3.7) commute as well, hence gives another solution for our problem of constructing a realization functor. On the contrary, considering the variants using Tot^{Π} and $\mathrm{Tot}^{\sqsubset}$, their restrictions to $\mathscr{A}$ vanish: Recall first that any injective object in $\mathscr{A} = \mathrm{Ch}(\mathscr{C})$ is acyclic (even contractible) as a complex by Lemma II.2.3.3. It follows that any bounded below complex of injectives in $\mathrm{Ch}(\mathscr{A})$ is a left-bounded bicomplex with acyclic columns, and the totalization-by-products of any such bicomplex is acyclic. ◊

General idea. I am thankful to Jan Stovicek for an interesting discussion that brought to life the following idea: Instead of trying to directly construct a functor from a fixed localization of $\mathrm{Ch}(\mathscr{A})$ to $\mathrm{Ho}(\mathscr{A})$, we propose to proceed in two steps: Firstly, try to construct a model structure on $\mathrm{Ch}(\mathscr{A})$ which is Quillen equivalent to the given model structure on $\mathscr{A}$, and secondly, understand how this model structure relates to (suitable models for) the derived category of $\mathscr{A}$. We have just seen this working in the example of the Gorenstein projective and injective model structures on R-Mod for R Gorenstein, and as noted in Remark II.5.1.7, Bravo, Gillespie and Hovey succeeded in doing so for the more general Gorenstein AC-projective and Gorenstein AC-injective model structures. Finally, a strong evidence to believe in the possibility of realizing the first step is the following classical statement:

Proposition II.C.3.3. *Let $\mathscr{F}$ be a Frobenius category and $\omega := \mathrm{ProjInj}(\mathscr{F})$ its class of projective-injective objects. Then there are equivalences of triangulated categories*

$$Q^0, Z^0 : \mathbf{K}_{\mathrm{ac}}(\omega) \longrightarrow \underline{\mathscr{F}} = \mathscr{F}/\omega$$

which coincide up to shift, $Q^0 \cong \Sigma \circ Z^0$.

Since for any hereditary abelian model structure $\mathcal{M} = (\mathcal{C}, \mathcal{W}, \mathcal{F})$ on $\mathscr{A}$ its homotopy category $\mathrm{Ho}(\mathcal{M})$ is equivalent to the stable category $\underline{\mathcal{C} \cap \mathcal{F}} = \mathcal{C} \cap \mathcal{F}/\mathcal{C} \cap \mathcal{W} \cap \mathcal{F}$ of the Frobenius category $\mathcal{C} \cap \mathcal{W}$ (Proposition II.2.1.21), this suggests that there should be a model structure on $\mathrm{Ch}(\mathscr{A})$ having the class of acyclic complexes in $\mathcal{C} \cap \mathcal{W} \cap \mathcal{F}$ with syzygies in $\mathcal{C} \cap \mathcal{F}$ as its bifibrant objects. We begin by recalling some classical notation concerning classes of complexes (see e.g. [Gil04, Definition 3.3], [Šťo13, Notation 4.1]):

Definition II.C.3.4. *Let $(\mathcal{C}, \mathcal{D})$ be a cotorsion pair in the abelian category $\mathscr{A}$.*

(i) $\text{dw-}\,\mathcal{C} := \mathrm{Ch}(\mathcal{C})$ *and* $\text{dw-}\,\mathcal{D} := \mathrm{Ch}(\mathcal{D})$.

(ii) $\widetilde{\mathcal{C}}$ *resp.* $\widetilde{\mathcal{D}}$ *denote the classes of acyclic complexes in $\mathscr{A}$ with syzygies in $\mathcal{C}$ resp. $\mathcal{D}$.*

(iii) $\text{dg-}\,\widetilde{\mathcal{C}} := {}^{\perp}\widetilde{\mathcal{D}}$ *and* $\text{dg-}\,\widetilde{\mathcal{D}} := \widetilde{\mathcal{C}}^{\perp}$.

Example II.C.3.5. Considering the cotorsion pairs $(\mathcal{P}, \mathscr{A})$ and $(\mathscr{A}, \mathcal{I})$ one recovers the classes of dg projective and dg injective complexes as $\text{dg-}\,\mathcal{P}$ and $\text{dg-}\,\mathcal{I}$, respectively. ◊

Proposition II.C.3.6. *Let $(\mathcal{C}, \mathcal{D})$ be a cotorsion pair over $\mathscr{A}$ and $X \in \mathrm{Ch}(\mathscr{A})$.*

(i) *$X \in \text{dg-}\,\mathcal{C}$ if and only if $X \in \text{dw-}\,\mathcal{C}$ and $[X, D] = 0$ for all $D \in \widetilde{\mathcal{D}}$.*

(ii) *$X \in \text{dg-}\,\mathcal{D}$ if and only if $X \in \text{dw-}\,\mathcal{D}$ and $[C, X] = 0$ for all $C \in \widetilde{\mathcal{C}}$.*

Further, we have the following inclusions:

(iii) $\mathrm{Ch}^-(\mathcal{C}) \subset \text{dg-}\widetilde{\mathcal{C}}$ *and* $\mathrm{Ch}^+(\mathcal{D}) \subset \text{dg-}\widetilde{\mathcal{D}}$.

(iv) $\mathrm{Ch}^+(\mathscr{A}) \cap \widetilde{\mathcal{C}} \subset {}^\perp[\text{dw-}\mathcal{D}]$ *and* $\mathrm{Ch}^-(\mathscr{A}) \cap \widetilde{\mathcal{D}} \subset [\text{dw-}\mathcal{C}]^\perp$.

(v) $\widetilde{\mathcal{C}} \subset \text{dg-}\widetilde{\mathcal{C}}$ *and* $\widetilde{\mathcal{D}} \subset \text{dg-}\widetilde{\mathcal{D}}$.

Proof. This is mostly contained in [Gil04, §3], but for convenience we include an argument here. First, the (exact) adjoints $G^\pm : \mathscr{A}^\mathbb{Z} \to \mathrm{Ch}(\mathscr{A})$ to the (exact) forgetful functor $(-)^\sharp : \mathrm{Ch}(\mathscr{A}) \to \mathscr{A}^\mathbb{Z}$ (see Proposition II.2.3.2) map $\mathcal{C}^\mathbb{Z}$ resp. $\mathcal{D}^\mathbb{Z}$ to $\widetilde{\mathcal{C}}$ resp. $\widetilde{\mathcal{D}}$. Hence, given $X \in \text{dg-}\widetilde{\mathcal{C}}$ and $D \in \mathcal{D}^\mathbb{Z}$, we have $0 = \mathrm{Ext}^1_{\mathrm{Ch}(\mathscr{A})}(X, G^-(D)) \cong \mathrm{Ext}^1_{\mathscr{A}^\mathbb{Z}}(X^\sharp, D)$, so $X^\sharp \in {}^\perp(\mathcal{D}^\mathbb{Z}) = \mathcal{C}^\mathbb{Z}$, i.e. $X \in \text{dw-}\mathcal{C}$. Similarly, we have $\text{dg-}\widetilde{\mathcal{D}} \subset \text{dw-}\mathcal{D}$.

Next, if $X \in \text{dw-}\mathcal{D}$ and $Z \in \text{dw-}\mathcal{C}$, then any short exact sequence $0 \to X \to Y \to Z \to 0$ in $\mathrm{Ch}(\mathscr{A})$ is degree-wise split, so that $\mathrm{Ext}^1_{\mathrm{Ch}(\mathscr{A})}(Z, X)$ is canonically isomorphic to the extension group $\mathrm{Ext}^1_{\text{dw-}\mathrm{Ch}(\mathscr{A})}(Z, X) \cong [Z, \Sigma X]$ with respect to the degree-wise split exact structure on $\mathrm{Ch}(\mathscr{A})$. This proves the first two claims (i) and (ii).

The proof of the inclusions in (iii) and (iv) is analogous to the proof of the classical fact that chain maps from bounded above complexes of projectives to acyclic complexes are nullhomotopic, as are chain maps from acyclic complexes to bounded below complexes of injectives. Finally, the inclusions $\widetilde{\mathcal{C}} \subset \text{dg-}\widetilde{\mathcal{C}}$ and $\widetilde{\mathcal{D}} \subset \text{dg-}\widetilde{\mathcal{D}}$ from part (v) are proved in [Gil04, Lemma 3.9]. □

The following beautiful theorem is the result of long work by Gillespie [Gil04; Gil06] in his studies of the flat model structures on $\mathrm{Ch}(R\text{-Mod})$ and $\mathrm{Ch}(\mathcal{O}_X)$ (for a survey, see [Hov07, §7]) and new results of Stovicek [Šťo13] on deconstructible classes in Grothendieck categories. To be precise, [Gil06, Proposition 3.6, Corollary 3.7] essentially prove parts (i) and (iii), while the completeness of $(\widetilde{\mathcal{C}}, \text{dg-}\widetilde{\mathcal{D}})$ crucial for part (ii) is guaranteed by the deconstructibility of $\widetilde{\mathcal{C}}$ established in [Šťo13, Theorem 4.2]. We collect these arguments and give a proof for convenience of the reader.

Theorem II.C.3.7 [Gil06; Šťo13]**.** *Let $\mathscr{A}$ be a Grothendieck category and $(\mathcal{C}, \mathcal{D})$ be a small and hereditary cotorsion pair in $\mathscr{A}$. Then the following hold:*

(i) $(\text{dg-}\widetilde{\mathcal{C}}, \widetilde{\mathcal{D}})$ *is a small, hereditary cotorsion pairs in* $\mathrm{Ch}(\mathscr{A})$.

(ii) $(\widetilde{\mathcal{C}}, \text{dg-}\widetilde{\mathcal{D}})$ *is a small, hereditary cotorsion pairs in* $\mathrm{Ch}(\mathscr{A})$.

(iii) $\widetilde{\mathcal{C}} = \text{dg-}\widetilde{\mathcal{C}} \cap \mathrm{Acyc}(\mathscr{A})$ *and* $\widetilde{\mathcal{D}} = \text{dg-}\widetilde{\mathcal{D}} \cap \mathrm{Acyc}(\mathscr{A})$.

Corollary II.C.3.8. *In the situation of Theorem II.C.3.7,* $(\text{dg-}\widetilde{\mathcal{C}}, \mathrm{Acyc}(\mathscr{A}), \text{dg-}\widetilde{\mathcal{D}})$ *is a cofibrantly generated abelian model structure on* $\mathrm{Ch}(\mathscr{A})$ *with homotopy category* $\mathbf{D}(\mathscr{A})$.

Example II.C.3.9. Applying Corollary II.C.3.8 to the cotorsion pair $(\mathscr{A}, \mathcal{I})$ gives rise to the injective model $(\mathrm{Ch}(\mathscr{A}), \mathrm{Acyc}(\mathscr{A}), \text{dg-}\mathcal{I})$ for $\mathbf{D}(\mathscr{A})$. Similarly, if $\mathscr{A}$ has enough projectives the cotorsion pair $(\mathcal{P}, \mathscr{A})$ yields the projective model $(\text{dg-}\mathcal{P}, \mathrm{Acyc}(\mathscr{A}), \mathrm{Ch}(\mathscr{A}))$ for $\mathbf{D}(\mathscr{A})$. A nontrivial example – and in fact *the* example that started the theory – is obtained from the flat cotorsion pair $(\mathrm{flat}(R), \mathrm{flat}(R)^{\perp})$ on R-Mod, with R any ring (see Example II.2.2.8): in this case, one obtains Gillespie's flat model structure. ◊

Proof of Theorem II.C.3.7. We first prove (i). To begin, [Gil04, Proposition 3.6] shows that $(\text{dg-}\widetilde{\mathcal{C}}, \widetilde{\mathcal{D}})$ and $(\widetilde{\mathcal{C}}, \text{dg-}\widetilde{\mathcal{D}})$ are cotorsion pairs; in addition to $(\mathcal{C}, \mathcal{D})$ being a cotorsion pair, this only needs the assumption that $\mathcal{C}$ is generating and $\mathcal{D}$ is cogenerating in $\mathscr{A}$.

Concerning the smallness of $(\text{dg-}\widetilde{\mathcal{C}}, \widetilde{\mathcal{D}})$ and $(\widetilde{\mathcal{C}}, \text{dg-}\widetilde{\mathcal{D}})$, applying [Što13, Theorem 4.2] (generalizing the ideas used by Gillespie [Gil04] in the case of the flat model structure on $\mathrm{Ch}(R\text{-Mod})$) shows that $\text{dg-}\widetilde{\mathcal{C}}$ and $\widetilde{\mathcal{C}}$ are deconstructible, so it remains to check that $\widetilde{\mathcal{C}}$ is generating. For this, note that since $\mathcal{C}$ is generating in $\mathscr{A}$, $\mathcal{C}^{\mathbb{Z}}$ is generating in $\mathscr{A}^{\mathbb{Z}}$; the counit $G^{+}(X^{\sharp}) \to X$ being an epimorphism for $X \in \mathrm{Ch}(\mathscr{A})$, it follows that $G^{+}(\mathcal{C}^{\mathbb{Z}})$ is generating in $\mathrm{Ch}(\mathscr{A})$. We have $G^{+}(\mathcal{C}^{\mathbb{Z}}) \subset \widetilde{\mathcal{C}}$, so $\widetilde{\mathcal{C}}$ is generating, too.

To check that $(\text{dg-}\widetilde{\mathcal{C}}, \widetilde{\mathcal{D}})$ and $(\widetilde{\mathcal{C}}, \text{dg-}\widetilde{\mathcal{D}})$ are hereditary, it suffices (by Corollary II.2.1.19) to show that $\text{dg-}\widetilde{\mathcal{C}}$ is resolving while $\text{dg-}\widetilde{\mathcal{D}}$ is coresolving. We only check that $\text{dg-}\widetilde{\mathcal{C}}$ is resolving, the proof of $\text{dg-}\widetilde{\mathcal{D}}$ being analogous. For that, recall from Proposition II.C.3.6 that $\text{dg-}\widetilde{\mathcal{C}}$ consists of those $X \in \text{dw-}\mathcal{C}$ for which $\mathrm{Hom}^{*}_{\mathscr{A}}(X, D) \in \mathrm{Acyc}(\mathbb{Z})$ for all $D \in \widetilde{\mathcal{D}}$, and suppose $0 \to X \to Y \to Z \to 0$ is a short exact sequence in $\mathrm{Ch}(\mathscr{A})$ with $Y, Z \in \text{dg-}\widetilde{\mathcal{C}}$. Then, firstly, $X \in \text{dw-}\mathcal{C}$ since $\mathcal{C}$ is resolving and $Y, Z \in \text{dw-}\mathcal{C}$. Further, for any $D \in \widetilde{\mathcal{D}}$ (even any $D \in \text{dw-}\mathcal{D}$), the sequence of complexes of abelian groups

$$0 \to \mathrm{Hom}^{*}_{\mathscr{A}}(Z, D) \to \mathrm{Hom}^{*}_{\mathscr{A}}(Y, D) \to \mathrm{Hom}^{*}_{\mathscr{A}}(X, D) \to 0$$

is exact, and since $\mathrm{Hom}^{*}_{\mathscr{A}}(Z, D)$ and $\mathrm{Hom}^{*}_{\mathscr{A}}(Y, D)$ are exact by our assumption that $Y, Z \in \text{dg-}\widetilde{\mathcal{C}}$, it follows that $\mathrm{Hom}^{*}_{\mathscr{A}}(X, D)$ is exact, too.

Concerning (ii), [Gil04, Theorem 3.12] shows that $\text{dg-}\widetilde{\mathcal{C}} \cap \mathrm{Acyc} = \widetilde{\mathcal{C}}$, and in view of [Gil04, Lemma 3.14(1)] and the completeness of $(\text{dg-}\widetilde{\mathcal{C}}, \widetilde{\mathcal{D}})$ we already proved, this also shows $\text{dg-}\widetilde{\mathcal{D}} \cap \mathrm{Acyc} = \widetilde{\mathcal{D}}$. □

Suppose now that $\mathcal{M} = (\mathcal{C}, \mathcal{W}, \mathcal{F})$ is a cofibrantly generated and hereditary abelian model structure on the Grothendieck category $\mathscr{A}$, and put $\omega := \mathcal{C} \cap \mathcal{W} \cap \mathcal{F}$, the *core* of $\mathcal{M}$. We will be concerned with quite a number of induced cotorsion pairs all of which will have $\widetilde{\mathcal{C}} \cap \widetilde{\mathcal{W} \cap \mathcal{F}} = \widetilde{\mathcal{C} \cap \mathcal{W}} \cap \widetilde{\mathcal{F}} =: \widetilde{\omega}$ as their core, the class of acyclic complexes with syzygies in ω. In view of the following lemma, these are precisely the contractible complexes with entries in ω:

Lemma II.C.3.10. *For $\mathscr{A}$ abelian and $X \in \mathrm{Acyc}(\mathscr{A})$, the following are equivalent:*

(i) *X is contractible.*

(ii) *The exact sequences* $0 \to Z^n X \to X \to Z^{n+1} X \to 0$ *split.*

The following is roughly analogous to Lemma II.2.3.4:

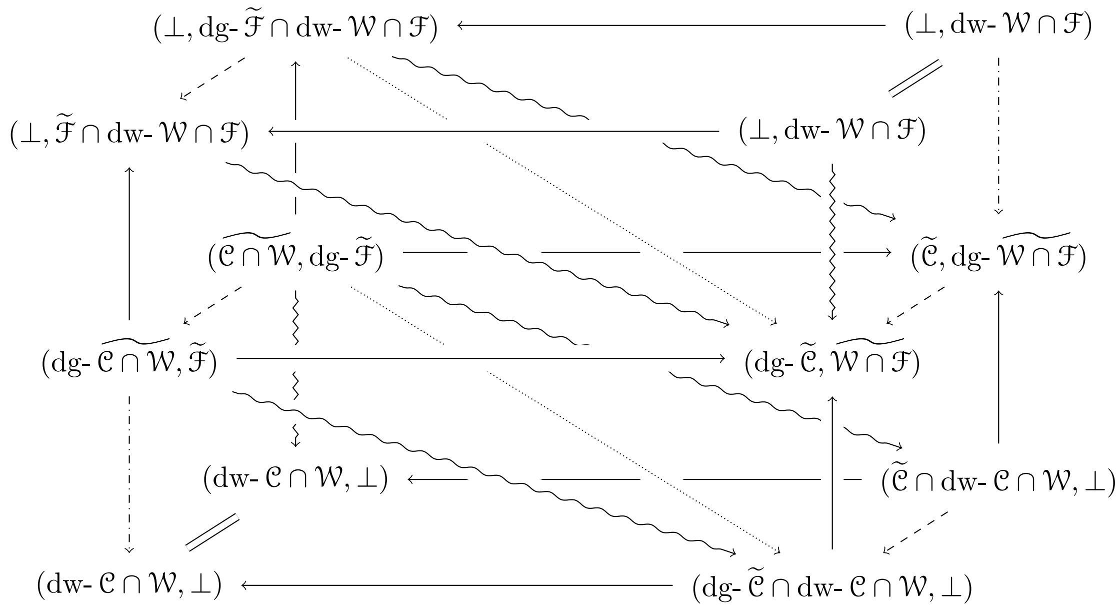
Lemma II.C.3.11. *Let* $\mathscr{A}$ *be an abelian category and* ω *be a self-orthogonal class of objects in* $\mathscr{A}$*, i.e.* $\omega \subset {}^{\perp}\omega$*. Then* $\widetilde{\omega}$*, the class of contractible complexes with entries in* ω*, is the largest self-orthogonal,* Σ*-stable class in* $\mathrm{Ch}(\mathscr{A})$ *contained in* dw-ω*.*

Proof. By self-orthogonality of ω, any short exact sequence $0 \to X \to Y \to Z \to 0$ with $X, Y \in$ dw-ω is degree-wise split, and hence determined by a homotopy class in $[Z, \Sigma X]$. This shows that $\widetilde{\omega}$ is self-orthogonal in $\mathrm{Ch}(\mathscr{A})$. Conversely, suppose $\mathcal{E} \subset$ dw-ω is self-orthogonal and Σ-stable, i.e. $\Sigma\mathcal{E} \subset \mathcal{E}$. Then, given any $X \in \mathcal{E}$ we have $0 = \mathrm{Ext}^1_{\mathrm{Ch}(\mathscr{A})}(\Sigma X, X) = \mathrm{Ext}^1_{\text{dw-}\mathrm{Ch}(\mathscr{A})}(\Sigma X, X) \cong [\Sigma X, \Sigma X]$, so X is contractible. □

Theorem II.C.3.12. *Let* $\mathcal{M} = (\mathcal{C}, \mathcal{W}, \mathcal{F})$ *be a cofibrantly generated, hereditary abelian model structure on the Grothendieck category* $\mathscr{A}$ *with core* $\omega := \mathcal{C} \cap \mathcal{W} \cap \mathcal{F}$*.*

Then Figure II.C.3.1 shows a diagram of small, hereditary cotorsion pairs on $\mathrm{Ch}(\mathscr{A})$ *all of whose cores are equal to* $\widetilde{\omega}$*, the class of contractible complexes in* ω*. An arrow* $(\mathcal{D}', \mathcal{E}') \to (\mathcal{D}, \mathcal{E})$ *signifies (independent of its style) that* $\mathcal{D}' \subseteq \mathcal{D}$*, and* $\perp$ *indicates that the corresponding entry is to be taken as the left/right orthogonal of the other entry.*

Proof. We begin by showing that all cotorsion pairs are small. For the top square, it suffices to check that the right hand sides of the cotorsion pairs listed in it are of the form $\mathcal{S}^{\perp}$ for a generating set $\mathcal{S} \subset \mathrm{Ch}(\mathscr{A})$. This property is preserved under intersection, so we need to check it for $\widetilde{\mathcal{F}}$, dg-$\widetilde{\mathcal{F}}$ and dw-$\mathcal{W} \cap \mathcal{F}$ only. Concerning the first two, we know from Theorem II.C.3.7 that $\widetilde{\mathcal{F}} = \left[\text{dg-}\widetilde{\mathcal{C} \cap \mathcal{W}}\right]^{\perp}$ and dg-$\widetilde{\mathcal{F}} = \left[\widetilde{\mathcal{C} \cap \mathcal{W}}\right]^{\perp}$, with $\widetilde{\mathcal{C} \cap \mathcal{W}}$ and dg-$\widetilde{\mathcal{C} \cap \mathcal{W}}$ both deconstructible and generating. For dw-$\mathcal{W} \cap \mathcal{F}$, note that $X \in$ dw-$\mathcal{W} \cap \mathcal{F}$ if and only if for all $C \in \mathcal{C}^{\mathbb{Z}}$ we have $0 = \mathrm{Ext}^1_{\mathscr{A}^{\mathbb{Z}}}(C, X^{\sharp}) \cong \mathrm{Ext}^1_{\mathrm{Ch}(\mathscr{A})}(G^+(C), X)$, so that by cocontinuity and exactness of G^+ we conclude that dw-$\mathcal{W} \cap \mathcal{F} = G^+(\mathcal{S})^{\perp}$ for $\mathcal{S} \subset \mathscr{A}$ some set chosen such that $\mathcal{C} =$ filt-$\mathcal{S}$; as $\mathcal{C}$ is generating, we may assume $\mathcal{S}$ generating, too, and then $G^+(\mathcal{S})$ is generating in $\mathrm{Ch}(\mathscr{A})$ since the counit $G^+ X^{\sharp} \to X$ is an epimorphism for all $X \in \mathrm{Ch}(\mathscr{A})$. This concludes the proof that all cotorsion pairs in the upper square are small. For the middle square, all cotorsion pairs contained in it are of the form studied in Theorem II.C.3.7, hence small. Finally, to prove that the cotorsion pairs in the lower square are small it suffices to show that their left hand sides are generating and deconstructible. They are generating as they all contain the generating class $G^+(\mathcal{C} \cap \mathcal{W})$, and deconstructibility follows from the stability of deconstructible classes under intersection [Šťo13, Proposition 2.9] as well as the deconstructibility of $\widetilde{\mathcal{C}}$, dg-$\widetilde{\mathcal{C}}$ and dw-$\widetilde{\mathcal{C} \cap \mathcal{W}}$ [Šťo13, Theorem 4.2].



(II.C.3.13)

Figure II.C.3.1. Cotorsion pairs on $\mathrm{Ch}(\mathscr{A})$ induced by a model structure on $\mathscr{A}$

Next we check that all cotorsion pairs are hereditary. For the ones in middle square, this follows from Theorem II.C.3.7 above. Concerning the ones in the upper square, their right hand sides are coresolving as intersections of the classes $\widetilde{\mathcal{F}}$, $\text{dg-}\widetilde{\mathcal{F}}$ and $\text{dw-}\mathcal{W} \cap \mathcal{F}$, each of which is coresolving: the first two are again treated as part of Theorem II.C.3.7, while $\text{dw-}\mathcal{W} \cap \mathcal{F}$ is coresolving since $\mathcal{W} \cap \mathcal{F}$ is. Applying Corollary II.2.1.19 then shows that all cotorsion pairs in the upper square are hereditary, and the reasoning for the lower square is analogous.

Finally we check that all cotorsion pairs have core equal to $\widetilde{\omega}$, the class of contractible complexes with values in $\mathcal{C} \cap \mathcal{W} \cap \mathcal{F}$. First, using the fact $\mathrm{Ext}^1_{\text{dw-}\mathrm{Ch}\,\mathscr{A}}(-,-) \cong [\Sigma(-),-]$ it is a quick check that $\widetilde{\omega}$ is contained in all the cores. For the reverse inclusion, Lemma II.C.3.11 and the stability under shift of all the classes involved show that it suffices to check that all cores are contained in $\text{dw-}\omega$. For the middle square, this is clear. For the upper square, all the cotorsion pairs in it have their right hand sides contained in $\text{dw-}\mathcal{W} \cap \mathcal{F}$, and the fact that they are all connected to a cotorsion pair in the middle row by a chain of arrows shows that their left hand sides are all contained in $\text{dw-}\mathcal{C}$. Similarly, the left hand sides of the cotorsion pairs in the lower square are all contained in $\text{dw-}\mathcal{C} \cap \mathcal{W}$, while all of them receiving an arrow from the middle square shows that their right hand sides are all contained in $\text{dw-}\mathcal{F}$. □

Since the cores of all cotorsion pairs from Theorem II.C.3.12 all agree, each arrow in it gives rise to an abelian model structure by virtue of the following recent generalization, due to Gillespie, of our Localization Theorem II.3.1.2:

Theorem II.C.3.13 [Gil14a, Theorem 1.1]**.** *Let $\mathscr{A}$ be an abelian category and suppose $(\mathcal{Q}, \widetilde{\mathcal{R}})$ and $(\widetilde{\mathcal{Q}}, \mathcal{R})$ are complete (small), hereditary cotorsion pairs over $\mathscr{A}$ with $\widetilde{\mathcal{Q}} \subseteq \mathcal{Q}$ and $\mathcal{Q} \cap \widetilde{\mathcal{R}} = \widetilde{\mathcal{Q}} \cap \mathcal{R}$. Then there exists a unique (cofibrantly generated) abelian model structure $(\mathcal{Q}, \mathcal{W}, \mathcal{R})$, and its class $\mathcal{W}$ of weakly trivial objects is given by*

$$\mathcal{W} = \{X \in \mathscr{A} \mid \exists\ 0 \to X \to \widetilde{\mathcal{R}} \to \widetilde{\mathcal{Q}} \to 0 \quad \textit{and} \quad 0 \to \widetilde{\mathcal{R}} \to \widetilde{\mathcal{Q}} \to X \to 0\}. \qquad \text{(II.C.3.14)}$$

Notation II.C.3.14. We write $(\mathcal{D}', \mathcal{E}') \to (\mathcal{D}, \mathcal{E})$, or $(\mathcal{D}', \mathcal{E}') \xrightarrow{\alpha} (\mathcal{D}, \mathcal{E})$ to give it the name α, as an abbreviation for $(\mathcal{D}', \mathcal{E}')$ and $(\mathcal{D}, \mathcal{E})$ being complete and hereditary cotorsion pairs having the same core $\mathcal{D}' \cap \mathcal{E}' = \mathcal{D} \cap \mathcal{E}$ and satisfying $\mathcal{D}' \subset \mathcal{D}$. Such a situation will be called a *localization context*. Its induced model structure is denoted $\mathrm{Loc}(\alpha) := (\mathcal{D}, ?, \mathcal{E}')$ on $\mathscr{A}$ and called the *localization* of the localization context. ◊

In this terminology, all arrows in (II.C.3.13) from Theorem II.C.3.12 are localization contexts, and we are about to describe their localizations next.

Overview. In the rest of this section, we elaborate on the following statements:

(i) Each triangle  in (II.C.3.13) gives rise to a localization sequence between the three model structures induced by its edges (Proposition II.C.3.19).

(ii) Each square  in (II.C.3.13) yields a butterfly-shaped diagram of adjunctions between the two localization sequences associated via (i) to its triangle faces (Fact II.C.3.22). However, these are not necessarily butterflies in the sense of II.4.2.11.

(iii) The dashed arrows in (II.C.3.13) yield four models for $\mathbf{D}(\mathscr{A})$ (Proposition II.C.3.18).

(iv) The snaked arrows yield four models structures on $\mathrm{Ch}(\mathscr{A})$ Quillen equivalent to our given model structure $(\mathcal{C}, \mathcal{W}, \mathcal{F})$ on $\mathscr{A}$ (Theorem II.C.3.15).

(v) The zigzag arrows induce model structures analogous to the co- and contraderived models, and the upper right and lower left triangles induce localization sequences connecting them to the models for $\mathbf{D}(\mathscr{A})$ from (iii) and the model structures associated to the dashdotted arrows. This generalizes Corollary II.3.1.6 and is discussed in Example II.C.3.20. In contrast to the Example from Section II.5.1, however, the dashdotted arrows are in general not Quillen equivalent to the given model structure on $\mathscr{A}$; instead, they are connected to the model structures associated to the snaked arrows through localization sequences associated to the upper back and lower front triangles.

(vi) Each of the two tilted squares in (II.C.3.13) composed out of two dashed, two snaked and one dotted arrow, gives rise to a butterfly between the model structures associated to the dotted arrows and the models from (iii) and (iv).

We begin by studying the model structures induced by the snaked arrows in (II.C.3.13):

Theorem II.C.3.15. *Let $\mathcal{M} = (\mathcal{C}, \mathcal{W}, \mathcal{F})$ be a cofibrantly generated, hereditary abelian model structure on the Grothendieck category $\mathscr{A}$. Then there is a square of cofibrantly generated abelian model structures on* $\mathrm{Ch}(\mathscr{A})$ *and identity Quillen equivalences:*

$$\begin{array}{ccc} (\widetilde{\mathcal{C}}, ?, \text{dg-}\widetilde{\mathcal{F}} \cap \text{dw-}\mathcal{W} \cap \mathcal{F}) & \underset{\mathrm{R}}{\overset{\mathrm{L}}{\rightleftarrows}} & (\text{dg-}\widetilde{\mathcal{C}}, ?, \widetilde{\mathcal{F}} \cap \text{dw-}\mathcal{W} \cap \mathcal{F}) \\ \mathrm{R}\downarrow\uparrow\mathrm{L} & & \mathrm{R}\downarrow\uparrow\mathrm{L} \\ (\widetilde{\mathcal{C}} \cap \text{dw-}\mathcal{C} \cap \mathcal{W}, ?, \text{dg-}\widetilde{\mathcal{F}}) & \underset{\mathrm{R}}{\overset{\mathrm{L}}{\rightleftarrows}} & (\text{dg-}\widetilde{\mathcal{C}} \cap \text{dw-}\mathcal{C} \cap \mathcal{W}, ?, \widetilde{\mathcal{F}}) \end{array} \tag{II.C.3.15}$$

Their homotopy categories are equivalent to the homotopy category of acyclic complexes with entries in $\mathcal{C}\cap\mathcal{W}\cap\mathcal{F}$ and syzygies in $\mathcal{C}\cap\mathcal{F}$. Moreover, there are Quillen equivalences

$$Q^0 : (\widetilde{\mathcal{C}} \cap \text{dw-}\mathcal{C} \cap \mathcal{W}, ?, \text{dg-}\widetilde{\mathcal{F}}) \rightleftarrows (\mathcal{C}, \mathcal{W}, \mathcal{F}) : \iota^0 \tag{II.C.3.16}$$

$$\iota^0 : (\mathcal{C}, \mathcal{W}, \mathcal{F}) \rightleftarrows (\text{dg-}\widetilde{\mathcal{C}}, ?, \widetilde{\mathcal{F}} \cap \text{dw-}\mathcal{W} \cap \mathcal{F}) : Z^0 \tag{II.C.3.17}$$

which on the homotopy categories yield the classical equivalences from Proposition II.C.3.3.

In particular, one has to beware that the derived adjoint equivalences of (II.C.3.16) and (II.C.3.17) are *not* isomorphic, but are shifts of one another.

Example II.C.3.16. Suppose R is a Gorenstein ring and consider Hovey's Gorenstein projective model structure $\mathcal{M}^{\text{G-proj}}(R) = (\text{G-proj}(R), \mathcal{P}^{<\infty}(R), R\text{-Mod})$ on R-Mod. Then $(\widetilde{\mathcal{C}}\cap\text{dw-}\mathcal{C}\cap\mathcal{W}, ?, \text{dg-}\widetilde{\mathcal{F}})$ coincides with ${}^{p}\mathcal{M}^{\text{ctr}}_{\text{sing}}(R) = (\mathrm{Ch}(\mathrm{Proj}(R)), ?, \mathrm{Ch}(R))$ since the syzygies of any acyclic complex of projectives are automatically Gorenstein projective. The equivalence (II.C.3.17) therefore agrees with the Quillen equivalence from Theorem II.5.1.5. Also, we claim that $(\text{dg-}\widetilde{\mathcal{C}} \cap \text{dw-}\mathcal{C} \cap \mathcal{W}, ?, \widetilde{\mathcal{F}})$ coincides with the classical singular contraderived model $\mathcal{M}^{\text{ctr}}_{\text{sing}}(R) = (\mathrm{Ch}(\mathrm{Proj}(R)), ?, \mathrm{Acyc}(R))$, i.e. that $\mathrm{Ch}(\mathrm{Proj}(R)) \subseteq {}^{\perp}\widetilde{\mathcal{I}^{<\infty}}$: For this, note first that for $P \in \mathrm{Ch}(\mathrm{Proj}(R))$ the short exact sequence $0 \to P \to G^{+}(\Sigma P^{\sharp}) \to \Sigma P \to 0$ exhibits P as the syzygy of ΣP in the abelian category $\mathrm{Ch}(R)$, since $G^{+}(\Sigma P^{\sharp})$ is projective in $\mathrm{Ch}(R)$ by Lemma II.2.3.3. Iterating this procedure, we see that any $P \in \mathrm{Ch}(\mathrm{Proj}(R))$ is an arbitrarily high syzygy in $\mathrm{Ch}(R)$. On the other hand, any complex in $\widetilde{\mathcal{I}^{<\infty}}$ admits a finite resolution by contractible complexes of injectives, i.e. has finite injective dimension in the abelian category $\mathrm{Ch}(R)$. The claim $\mathrm{Ch}(\mathrm{Proj}(R)) \subseteq {}^{\perp}\widetilde{\mathcal{I}^{<\infty}}$ follows. ◊

Proof of Theorem II.C.3.15. It only remains to prove that (II.C.3.16) and (II.C.3.17) are Quillen equivalences. We start by checking that (II.C.3.16) is a Quillen adjunction. Suppose $\iota : X \to Y$ is a cofibration in $(\widetilde{\mathcal{C}} \cap \text{dw-}\,\mathcal{C} \cap \mathcal{W}, ?, \text{dg-}\,\widetilde{\mathcal{F}})$, i.e. ι is a monomorphism in $\mathrm{Ch}(\mathscr{A})$ with cokernel $Z := \mathrm{coker}(\iota)$ belonging to $\widetilde{\mathcal{C}} \cap \text{dw-}\,\mathcal{C} \cap \mathcal{W}$. Since $\widetilde{\mathcal{C}} \subset \mathrm{Acyc}(\mathscr{A})$, it follows as in Lemma II.5.1.1 that $0 \to Q^0X \to Q^0Y \to Q^0Z \to 0$ is exact in $\mathscr{A}$, and as $Q^0Z \in \mathcal{C}$ by definition, it follows that $Q^0\iota$ is a cofibration in $(\mathcal{C}, \mathcal{W}, \mathcal{F})$. If ι is a trivial cofibration, then $Z \in {}^{\perp}\text{dg-}\,\widetilde{\mathcal{F}} = \widetilde{\mathcal{C} \cap \mathcal{W}}$, hence exact, and we deduce an exact sequence $0 \to Q^0X \to Q^0Y \to Q^0Z \to 0$ with $Q^0Z \in \mathcal{C} \cap \mathcal{W}$. This shows that (II.C.3.16) is a Quillen adjunction, and for (II.C.3.17) the proof is analogous.

Next we prove that (II.C.3.16) is a Quillen equivalence. In the one direction, consider a bifibrant X in $(\widetilde{\mathcal{C}} \cap \text{dw-}\,\mathcal{C} \cap \mathcal{W}, ?, \text{dg-}\,\widetilde{\mathcal{F}})$, that is, $X \in \widetilde{\mathcal{C}} \cap \text{dg-}\,\widetilde{\mathcal{F}} \cap \text{dw-}\,\mathcal{C} \cap \mathcal{W}$. Since $\text{dg-}\,\widetilde{\mathcal{F}} \cap \mathrm{Acyc}(\mathscr{A}) = \widetilde{\mathcal{F}}$, we have $X \in \widetilde{\mathcal{C}} \cap \widetilde{\mathcal{F}} \cap \text{dw-}\,\mathcal{C} \cap \mathcal{W}$, and in particular $Q^0X \in \mathcal{F}$ is fibrant. Hence, to show the derived unit is an equivalence, it suffices to show that such an X, the underived unit $\varepsilon : X \to \iota^0 Q^0(X)$ is a weak equivalence in $(\widetilde{\mathcal{C}} \cap \text{dw-}\,\mathcal{C} \cap \mathcal{W}, ?, \text{dg-}\,\widetilde{\mathcal{F}})$. We have that ε is an epimorphism and $\ker(\varepsilon) \cong \tau_{\leq 0}X \oplus \sigma_{>0}X$, and we consider the two summands separately. The first summand $\tau_{\leq 0}X$ belongs to $\mathrm{Ch}^-(\mathscr{A}) \cap \widetilde{\mathcal{F}}$ which is contained in $[\text{dw-}\,\mathcal{C} \cap \mathcal{W}]^{\perp}$ by Proposition II.C.3.6(iv), hence trivially fibrant in $(\widetilde{\mathcal{C}} \cap \text{dw-}\,\mathcal{C} \cap \mathcal{W}, ?, \text{dg-}\,\widetilde{\mathcal{F}})$. The second summand $\sigma_{>0}(X)$ belongs to $\mathrm{Ch}^+(\mathcal{W} \cap \mathcal{F})$ which is contained in $\widetilde{\mathcal{C}}^{\perp}$ by Proposition II.C.3.6(iii), hence trivially fibrant, too. It follows that $\varepsilon : X \to \iota^0 Q^0(X)$ is indeed weak equivalence in $(\widetilde{\mathcal{C}} \cap \text{dw-}\,\mathcal{C} \cap \mathcal{W}, ?, \text{dg-}\,\widetilde{\mathcal{F}})$.

We have just proved that the derived unit $\mathrm{id} \Rightarrow \mathbf{R}\iota^0 \circ \mathbf{L}Q^0$ is an equivalence, which means that $\mathbf{L}Q^0$ is fully faithful. To prove that $\mathbf{L}Q^0 \dashv \mathbf{R}\iota^0$ is an equivalence, it is therefore enough to show that $\mathbf{L}Q^0$ is also essentially surjective. For this, it suffices to check that any bifibrant $M \in \mathcal{C} \cap \mathcal{F}$ occurs as the 0-th syzygy Q^0X of some bifibrant "complete resolution" $X \in \widetilde{\mathcal{C}} \cap \widetilde{\mathcal{F}} \cap \text{dw-}\,\mathcal{C} \cap \mathcal{W}$. Such a resolution can be built inductively using the completeness of the cotorsion pairs $(\mathcal{C} \cap \mathcal{W}, \mathcal{F})$ and $(\mathcal{C}, \mathcal{W} \cap \mathcal{F})$.

The proof that (II.C.3.17) is a Quillen equivalence is analogous. □

Corollary II.C.3.17. *Any hereditary, cofibrantly generated abelian model structure on a Grothendieck category is Quillen equivalent to an abelian model structure on* $\mathrm{Ch}(\mathscr{A})$.

Next we study the model structures induced by the dashed arrows in (II.C.3.13).

Proposition II.C.3.18. *Let* $\mathcal{M} = (\mathcal{C}, \mathcal{W}, \mathcal{F})$ *be a cofibrantly generated, hereditary abelian model structure on the Grothendieck category* $\mathscr{A}$*. Then there is a square of cofibrantly* *generated abelian model structures on* $\mathrm{Ch}(\mathscr{A})$ *and identity Quillen equivalences:*

$$\begin{array}{ccc}
(\mathrm{dg}\text{-}\widetilde{\mathcal{C}\cap\mathcal{W}}, \mathrm{Acyc}(\mathscr{A}), \mathrm{dg}\text{-}\widetilde{\mathcal{F}}) & \underset{\mathrm{R}}{\overset{\mathrm{L}}{\rightleftarrows}} & (\mathrm{dg}\text{-}\widetilde{\mathcal{C}}\cap\mathrm{dw}\text{-}\mathcal{C}\cap\mathcal{W}, \mathrm{Acyc}(\mathscr{A}), (\widetilde{\mathcal{C}}\cap\mathrm{dw}\text{-}\mathcal{C}\cap\mathcal{W})^{\perp}) \\
{\scriptstyle\mathrm{L}}\downarrow\uparrow{\scriptstyle\mathrm{R}} & & {\scriptstyle\mathrm{R}}\uparrow\downarrow{\scriptstyle\mathrm{L}} \\
({}^{\perp}(\widetilde{\mathcal{F}}\cap\mathrm{dw}\text{-}\mathcal{W}\cap\mathcal{F}), \mathrm{Acyc}(\mathscr{A}), \mathrm{dg}\text{-}\widetilde{\mathcal{F}}\cap\mathrm{dw}\text{-}\mathcal{W}\cap\mathcal{F}) & \underset{\mathrm{R}}{\overset{\mathrm{L}}{\rightleftarrows}} & (\mathrm{dg}\text{-}\widetilde{\mathcal{C}}, \mathrm{Acyc}(\mathscr{A}), \mathrm{dg}\text{-}\widetilde{\mathcal{W}\cap\mathcal{F}})
\end{array} \tag{II.C.3.18}$$

Their homotopy categories are equivalent to the ordinary derived category $\mathbf{D}(\mathscr{A})$.

Proof. Applying Gillespie's Theorem II.C.3.13 to the dashed arrows in (II.C.3.13) gives four model structures matching the triples listed in (II.C.3.18) in the left and right hand parts; it therefore suffices to check that their classes of weakly trivial objects all coincide with the class $\mathrm{Acyc}(\mathscr{A})$ of acyclic complexes.

For the model structures associated to the arrows $(\widetilde{\mathcal{C}\cap\mathcal{W}}, \mathrm{dg}\text{-}\widetilde{\mathcal{F}}) \to (\mathrm{dg}\text{-}\widetilde{\mathcal{C}\cap\mathcal{W}}, \widetilde{\mathcal{F}})$ and $(\widetilde{\mathcal{C}}, \mathrm{dg}\text{-}\widetilde{\mathcal{W}\cap\mathcal{F}}) \to (\mathrm{dg}\text{-}\widetilde{\mathcal{C}}, \mathrm{dg}\text{-}\widetilde{\mathcal{W}\cap\mathcal{F}})$ we already know this from Corollary II.C.3.8 above. Next, consider model structure associated to $(\widetilde{\mathcal{C}}\cap\mathrm{dw}\text{-}\mathcal{C}\cap\mathcal{W}, \perp) \to (\mathrm{dg}\text{-}\widetilde{\mathcal{C}}\cap\mathrm{dw}\text{-}\mathcal{C}\cap\mathcal{W}, \perp)$: By (II.C.3.14) the weakly trivial objects in the associated model structure are those $X \in \mathrm{Ch}(\mathscr{A})$ which admit a short exact sequence of the form $0 \to F \to C \to X \to 0$ with $F \in \left[\mathrm{dg}\text{-}\widetilde{\mathcal{C}}\cap\mathrm{dw}\text{-}\mathcal{C}\cap\mathcal{W}\right]^{\perp} \subset \widetilde{\mathcal{F}}$ and $C \in \widetilde{\mathcal{C}}\cap\mathrm{dw}\text{-}\mathcal{C}\cap\mathcal{W}$. Note that $F, C \in \mathrm{Acyc}(\mathscr{A})$, so the existence of such a sequence implies that $X \in \mathrm{Acyc}(\mathscr{A})$. Conversely, suppose $X \in \mathrm{Acyc}(\mathscr{A})$ and pick an approximation sequence $0 \to F \to C \to X \to 0$ for the cotorsion pair $(\mathrm{dg}\text{-}\widetilde{\mathcal{C}}\cap\mathrm{dw}\text{-}\mathcal{C}\cap\mathcal{W}, \left[\mathrm{dg}\text{-}\widetilde{\mathcal{C}}\cap\mathrm{dw}\text{-}\mathcal{C}\cap\mathcal{W}\right]^{\perp})$. Then again $F \in \mathrm{Acyc}(\mathscr{A})$, and also $X \in \mathrm{Acyc}(\mathscr{A})$ by assumption, so $C \in \mathrm{dg}\text{-}\widetilde{\mathcal{C}}\cap\mathrm{Acyc}(\mathscr{A})\cap\mathrm{dw}\text{-}\mathcal{C}\cap\mathcal{W} = \widetilde{\mathcal{C}}\cap\mathrm{dw}\text{-}\mathcal{C}\cap\mathcal{W}$.

The proof that the weak equivalences in the model structure associated to the arrow $(\perp, \mathrm{dg}\text{-}\widetilde{\mathcal{F}}\cap\mathrm{dw}\text{-}\mathcal{W}\cap\mathcal{F}) \to (\perp, \widetilde{\mathcal{F}}\cap\mathrm{dw}\text{-}\mathcal{W}\cap\mathcal{F})$ in (II.C.3.13) is analogous. □

Finally, we study the relation between the model structures induced by the tilted squares in (II.C.3.13), beginning with some observations that hold in general:

Proposition II.C.3.19. *For localization contexts* $(\mathcal{D}'', \mathcal{E}'') \xrightarrow{\alpha} (\mathcal{D}', \mathcal{E}') \xrightarrow{\beta} (\mathcal{D}, \mathcal{E})$ *their induced model structures are related via a localization sequence of triangulated categories:*

$$\begin{array}{ccccc}
\mathrm{Loc}(\alpha) & \underset{\mathbf{R}\,\mathrm{id}}{\overset{\mathbf{L}\,\mathrm{id}}{\rightleftarrows}} & \mathrm{Loc}(\beta\circ\alpha) & \underset{\mathbf{R}\,\mathrm{id}}{\overset{\mathbf{L}\,\mathrm{id}}{\leftrightarrows}} & \mathrm{Loc}(\beta) \\
\| & & \| & & \| \\
(\mathcal{D}', ?, \mathcal{E}'') & & (\mathcal{D}, ?, \mathcal{E}'') & & (\mathcal{D}, ?, \mathcal{E}')
\end{array} \tag{II.C.3.19}$$

Proof. For ω the common core of the given cotorsion pairs, we have $\mathrm{Ho}\,\mathrm{Loc}(\alpha) \cong \mathcal{C}'\cap\mathcal{E}''/\omega$ and $\mathrm{Ho}\,\mathrm{Loc}(\beta\circ\alpha) \cong \mathcal{C}\cap\mathcal{E}''/\omega$, and $\mathbf{L}\,\mathrm{id} : \mathrm{Ho}\,\mathrm{Loc}(\alpha) \to \mathrm{Ho}\,\mathrm{Loc}(\beta\circ\alpha)$ is the canonical functor $\mathcal{C}'\cap\mathcal{E}''/\omega \to \mathcal{C}\cap\mathcal{E}''/\omega$, hence fully faithful. Similarly, $\mathbf{R}\,\mathrm{id} : \mathrm{Ho}\,\mathrm{Loc}(\beta) \to \mathrm{Ho}\,\mathrm{Loc}(\beta\circ\alpha)$ is given by the embedding of $\mathcal{D}\cap\mathcal{E}'/\omega$ into $\mathcal{D}\cap\mathcal{E}''/\omega$, hence fully faithful.

It remains to prove the exactness of (II.C.3.19). Up to isomorphism in $\operatorname{Ho}\operatorname{Loc}(\beta\circ\alpha)$, $\ker[\operatorname{Ho}\operatorname{Loc}(\beta\circ\alpha) \xrightarrow{\mathbf{L}\,\mathrm{id}} \operatorname{Ho}\operatorname{Loc}(\beta)]$ consists of those $D \in \mathcal{D}$ admitting a short exact sequence $0 \to E \to D' \to D \to 0$ with $E \in \mathcal{E}$, $D' \in \mathcal{D}'$ (recall that these characterize the weakly trivial objects in $\operatorname{Loc}(\beta)$). As $\mathcal{E}$ is the class of trivially fibrant objects of $\operatorname{Loc}(\beta\circ\alpha)$, such a sequence already implies $D \cong D'$ in $\operatorname{Ho}\operatorname{Loc}(\beta\circ\alpha)$. Hence, up to isomorphism in $\operatorname{Ho}\operatorname{Loc}(\beta\circ\alpha)$, $\ker[\operatorname{Ho}\operatorname{Loc}(\beta\circ\alpha) \xrightarrow{\mathbf{L}\,\mathrm{id}} \operatorname{Ho}\operatorname{Loc}(\beta)]$ consists of the objects of $\mathcal{D}'$, and the same is true for $\operatorname{im}[\operatorname{Ho}\operatorname{Loc}(\alpha) \xrightarrow{\mathbf{L}\,\mathrm{id}} \operatorname{Ho}\operatorname{Loc}(\beta\circ\alpha)]$ by definition of $\mathbf{L}\,\mathrm{id}$. □

Example II.C.3.20. Proposition II.C.3.19 applies to the chain of localization contexts

$$(\perp, \text{dw-}\,\mathcal{W}\cap\mathcal{F}) \to (\widetilde{\mathcal{C}}, \text{dg-}\,\widetilde{\mathcal{W}\cap\mathcal{F}}) \to (\text{dg-}\,\widetilde{\mathcal{C}}, \widetilde{\mathcal{W}\cap\mathcal{F}}), \tag{II.C.3.20}$$

$$(\widetilde{\mathcal{C}\cap\mathcal{W}}, \text{dg-}\,\widetilde{\mathcal{F}}) \to (\text{dg-}\,\widetilde{\mathcal{C}\cap\mathcal{W}}, \widetilde{\mathcal{F}}) \to (\text{dw-}\,\mathcal{C}\cap\mathcal{W}, \perp) \tag{II.C.3.21}$$

in the upper right resp. lower left corner of (II.C.3.13). The model structures

$$(\text{dg-}\,\widetilde{\mathcal{C}}, ?, \text{dw-}\,\mathcal{W}\cap\mathcal{F}) \quad \text{and} \quad (\text{dw-}\,\mathcal{C}\cap\mathcal{W}, ?, \text{dg-}\,\widetilde{\mathcal{F}}) \tag{II.C.3.22}$$

associated with the composed localization contexts are generalizations of both the injective/projective models for $\mathbf{D}(\mathscr{A})$ and the contraderived/coderived model structures:

- Choosing $(\mathcal{C},\mathcal{W},\mathcal{F})$ as $(\mathscr{A},\mathscr{A},\mathcal{I}(\mathscr{A}))$, we have $(\text{dg-}\,\widetilde{\mathcal{C}}, ?, \text{dw-}\,\mathcal{W}\cap\mathcal{F}) = \mathcal{M}^{\mathrm{co}}(\mathscr{A})$ and $(\text{dw-}\,\mathcal{C}\cap\mathcal{W}, ?, \text{dg-}\,\widetilde{\mathcal{F}}) = \mathcal{M}^{\mathrm{inj}}(\mathscr{A})$.
- Choosing $(\mathcal{C},\mathcal{W},\mathcal{F})$ as $(\mathcal{P}(\mathscr{A}),\mathscr{A},\mathscr{A})$ in case $\mathscr{A}$ has enough projectives, we have $(\text{dg-}\,\widetilde{\mathcal{C}}, ?, \text{dw-}\,\mathcal{W}\cap\mathcal{F}) = \mathcal{M}^{\mathrm{proj}}(\mathscr{A})$ and $(\text{dw-}\,\mathcal{C}\cap\mathcal{W}, \text{dg-}\,\widetilde{\mathcal{F}}) = \mathcal{M}^{\mathrm{ctr}}(\mathscr{A})$.

Further, in the case of $(\mathscr{A},\mathscr{A},\mathcal{I}(\mathscr{A}))$ the localization sequences induced by (II.C.3.20) is the known one $\mathbf{D}(\mathscr{A}) \rightleftarrows \mathbf{K}(\mathcal{I}(\mathscr{A})) \rightleftarrows \mathbf{K}_{\mathrm{ac}}(\mathcal{I}(\mathscr{A}))$, while the one induced by (II.C.3.21) is the trivial localization sequence $0 \rightleftarrows \mathbf{D}(\mathscr{A}) \rightleftarrows \mathbf{D}(\mathscr{A})$. Similarly, in the case of $(\mathcal{P}(\mathscr{A}),\mathscr{A},\mathscr{A})$, the localization sequence associated with (II.C.3.20) is trivial, while the one associated with (II.C.3.21) is the classical one $\mathbf{K}_{\mathrm{ac}}(\mathcal{P}(\mathscr{A})) \rightleftarrows \mathbf{K}(\mathcal{P}(\mathscr{A})) \rightleftarrows \mathbf{D}(\mathscr{A})$. ◊

Generalizing the inclusions $\mathrm{Acyc}^- \subset \mathcal{W}^{\mathrm{ctr}} \subset \mathrm{Acyc} \supset \mathcal{W}^{\mathrm{co}} \supset \mathrm{Acyc}^+$, we have:

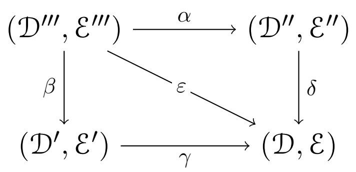
Fact II.C.3.21. *The model structures* (II.C.3.22) *have the following properties:*

(i) *The class $\mathcal{W}$ of weakly trivial complexes in the model structure* $(\text{dg-}\,\widetilde{\mathcal{C}}, ?, \text{dw-}\,\mathcal{W}\cap\mathcal{F})$ *satisfies* $\mathrm{Acyc}^+(\mathscr{A}) \subseteq \mathcal{W} \subseteq \mathrm{Acyc}(\mathscr{A})$.

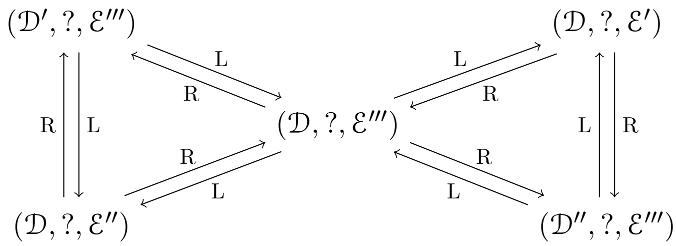
(ii) *The class $\mathcal{W}$ of weakly trivial complexes in the model structure* $(\text{dw-}\,\mathcal{C}\cap\mathcal{W}, ?, \text{dg-}\,\widetilde{\mathcal{F}})$ *satisfies* $\mathrm{Acyc}^-(\mathscr{A}) \subseteq \mathcal{W} \subseteq \mathrm{Acyc}(\mathscr{A})$.

Proof. Recall that the model structure (dg-$\widetilde{\mathcal{C}}$, ?, dw-$\mathcal{W} \cap \mathcal{F}$) arises as the localization of the composed localization context in (II.C.3.20). Therefore, by Gillespie's Theorem II.C.3.13, its class $\mathcal{W}$ of weakly trivial complexes consists of those $X \in \mathrm{Ch}(\mathscr{A})$ which admit a short exact sequence $0 \to X \to F \to C \to 0$ with $F \in \widetilde{\mathcal{W} \cap \mathcal{F}}$ and $C \in {}^{\perp}[\text{dw-}\mathcal{W} \cap \mathcal{F}]$. Now $\widetilde{\mathcal{W} \cap \mathcal{F}} \subseteq \mathrm{Acyc}(\mathscr{A})$ by definition, and ${}^{\perp}[\text{dw-}\mathcal{W} \cap \mathcal{F}] \subseteq \mathrm{Acyc}(\mathscr{A})$ as witnessed by the upper right dashdotted arrow in (II.C.3.13), so $\mathcal{W} \subseteq \mathrm{Acyc}(\mathscr{A})$ by the 2-out-of-3 property of $\mathrm{Acyc}(\mathscr{A})$. Conversely, suppose $X \in \mathrm{Acyc}^{+}(\mathscr{A})$. Applying the Resolution Lemma II.3.1.5 to the cotorsion pair $(\mathcal{C}, \mathcal{W} \cap \mathcal{F})$ and all short exact sequences $0 \to Z^n X \to X^n \to Z^{n+1} X \to 0$ we can construct a short exact sequence $0 \to X \to F \to C \to 0$ in $\mathrm{Ch}(\mathscr{A})$ with $F \in \widetilde{\mathcal{W} \cap \mathcal{F}} \cap \mathrm{Ch}^{+}(\mathscr{A})$ and $C \in \widetilde{\mathcal{C}} \cap \mathrm{Ch}^{+}(\mathscr{A})$. Since $\widetilde{\mathcal{C}} \cap \mathrm{Ch}^{+}(\mathscr{A}) \subseteq {}^{\perp}[\text{dw-}\mathcal{W} \cap \mathcal{F}]$, it follows that $X \in \mathcal{W}$. This finishes the proof of statement (i), and (ii) is analogous. □

Fact II.C.3.22. *Suppose given a square of localization contexts*



Then their localizations fit into a diagram of identity Quillen adjunctions



(II.C.3.23)

According to Fact II.C.3.22, each oriented square in (II.C.3.13) gives rise to a diagram of the form (II.C.3.23) in which the diagonals are exact. In case of the tilted squares in (II.C.3.13) these diagrams are even butterflies in the sense of Definition II.4.2.11:

Theorem II.C.3.23. *For a hereditary, cofibrantly generated abelian model structure* $\mathcal{M} = (\mathcal{C}, \mathcal{W}, \mathcal{F})$ *over a Grothendieck category* $\mathscr{A}$*, consider Figure II.C.3.2. It shows a diagram of identity Quillen adjunctions between hereditary, cofibrantly generated abelian model structures on* $\mathrm{Ch}(\mathscr{A})$ *with the following properties:*

(i) *The horizontal layers are butterflies in the sense of Definition II.4.2.11, and all vertical adjunctions are Quillen equivalences.*

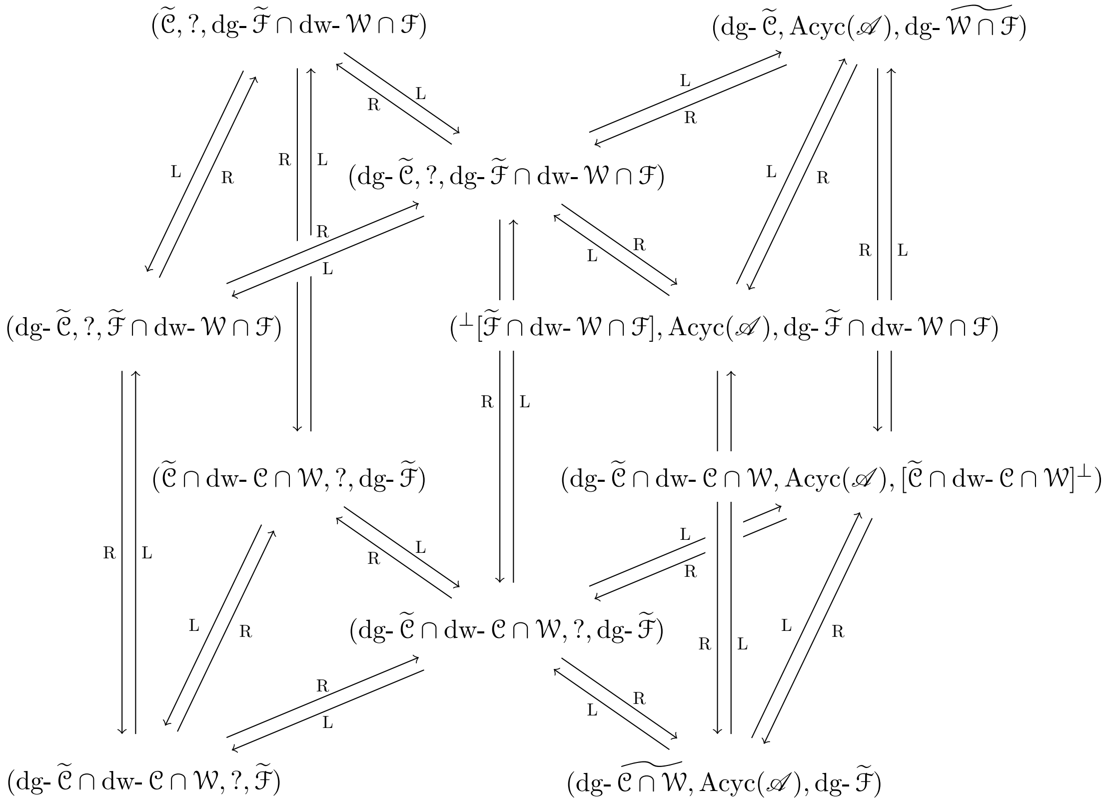


Figure II.C.3.2. Model structures on $\text{Ch}(\mathscr{A})$ induced by a model structure on $\mathscr{A}$

(ii) *The four model structures on the left side have their homotopy category canonically equivalent to* $\text{Ho}(\mathcal{M})$ *as explained in Theorem II.C.3.15.*

(iii) *The four model structures on the right side hand have their homotopy category canonically equivalent to* $\mathbf{D}(\mathscr{A})$ *as explained in Proposition II.C.3.18.*

Denote $\mathcal{T}$ the common homotopy category of the two middle model structures in Figure II.C.3.2; explicitly, this is the homotopy category of complexes with components in $\omega = \mathcal{C} \cap \mathcal{W} \cap \mathcal{F}$ which belong to both dg-$\widetilde{\mathcal{C}}$ and dg-$\widetilde{\mathcal{F}}$. Passing to homotopy categories in Figure II.C.3.2 now yields a recollement

$$\text{Ho}(\mathcal{M}) \; \substack{\longleftarrow \\ \longrightarrow \\ \longleftarrow} \; \mathcal{T} \; \substack{\longleftarrow \\ \longrightarrow \\ \longleftarrow} \; \mathbf{D}(\mathscr{A})$$

and we claim that the induced stabilization functor real : $\mathbf{D}(\mathscr{A}) \to \text{Ho}(\mathcal{M})$ makes the following diagram commutative:

$$\begin{array}{ccc} \mathrm{Ho}(\mathcal{M}) & \xleftarrow{\quad\text{real}\quad} & \mathbf{D}(\mathscr{A}) \\ \uparrow & & \uparrow \\ & \mathscr{A} & \end{array}$$

We begin with some generalities on stabilization functors associated to recollements:

Definition II.C.3.24 (see [Kra05, §5])**.** *Given a recollement* $\mathcal{T}' \leftrightarrows \mathcal{T} \leftrightarrows \mathcal{T}''$ *of triangulated categories, the functors* $I_\rho Q_\lambda : \mathcal{T}'' \to \mathcal{T}'$ *resp.* $I_\lambda Q_\rho : \mathcal{T}'' \to \mathcal{T}'$ *are called the* left resp. right stabilization functors *associated to the recollement.*

Fact II.C.3.25. *For any* $X'' \in \mathcal{T}''$ *there is a non-canonical isomorphism*

$$I_\lambda Q_\rho X'' \;\cong\; \Sigma I_\rho Q_\lambda X''.$$

Proof. For $X \in \mathcal{T}$ the localization sequence $\mathcal{T}' \rightleftarrows \mathcal{T} \rightleftarrows \mathcal{T}''$ induces a non-canonical distinguished triangle $Q_\lambda Q X \to X \to I I_\lambda X \to \Sigma Q_\lambda Q X$, which for $X = Q_\rho X''$ with $X'' \in \mathcal{T}''$ transforms into $Q_\lambda X'' \to Q_\rho X'' \to I I_\lambda Q_\rho X'' \to \Sigma Q_\lambda X''$. Applying I_ρ from the left annihilates $Q_\rho X$ and hence yields an isomorphism $I_\lambda Q_\rho X'' \cong \Sigma I_\rho Q_\lambda X''$. □

Proposition II.C.3.26. *In the situation of Theorem II.C.3.23, the composition*

$$\mathbf{D}(\mathscr{A}) \cong \mathrm{Ho}(\text{dg-}\widetilde{\mathcal{C}\cap\mathcal{W}}, \mathrm{Acyc}(\mathscr{A}), \text{dg-}\widetilde{\mathcal{F}}) \longrightarrow \mathrm{Ho}(\widetilde{\mathcal{C}} \cap \text{dw-}\,\mathcal{C}\cap\mathcal{W}, ?, \text{dg-}\widetilde{\mathcal{F}}) \xrightarrow[\cong]{\mathbf{L}Q^0} \mathrm{Ho}(\mathcal{M})$$

of the left stabilization functor (Definition II.C.3.24) associated to the lower butterfly in Figure II.C.3.2 and the equivalence $\mathbf{L}Q^0$ *from* (II.C.3.16) *in Theorem II.C.3.15 makes the following diagram commutative:*

$$\begin{array}{ccc} \mathrm{Ho}(\mathcal{M}) & \longleftarrow & \mathbf{D}(\mathscr{A}) \\ \uparrow & & \uparrow \\ & \mathscr{A} & \end{array} \qquad \text{(II.C.3.24)}$$

Dually, the composition of the right stabilization functor associated to the upper butterfly in Figure II.C.3.2 and the equivalence $\mathbf{R}Z^0$ from II.C.3.17 in Theorem II.C.3.15 gives another functor $\mathbf{D}(\mathscr{A}) \to \mathrm{Ho}(\mathcal{M})$ making (II.C.3.24) commutative. Comparing the two functors $\mathbf{D}(\mathscr{A}) \to \mathrm{Ho}(\mathcal{M})$ obtained this way, Fact II.C.3.25 and $\mathbf{L}Q^0 \cong \Sigma\mathbf{R}Z^0$ show that they are pointwise non-canonically isomorphic; to show that they are even naturally isomorphic we'd need to find an enhancement of Fact II.C.3.25.

Proof of Proposition II.C.3.26. Step 1: Firstly, we note that the derived functor

$$Q_\lambda : \mathrm{Ho}(\text{dg-}\widetilde{\mathcal{C}\cap\mathcal{W}}, \mathrm{Acyc}(\mathscr{A}), \text{dg-}\widetilde{\mathcal{F}}) \to \mathrm{Ho}(\text{dg-}\widetilde{\mathcal{C}} \cap \text{dw-}\,\mathcal{C}\cap\mathcal{W}, ?, \text{dg-}\widetilde{\mathcal{F}}) \qquad \text{(II.C.3.25)}$$

may be computed naively on $\mathrm{Ch}^-(\mathscr{A})$: Namely, it can be computed through any resolution by a quasi-isomorphic bounded above complex with entries in $\mathcal{C}\cap\mathcal{W}$, and by Fact II.C.3.21 such a resolution is still a weak equivalence in $(\text{dw-}\,\mathcal{C}\cap\mathcal{W}, ?, \text{dg-}\,\widetilde{\mathcal{F}})$, hence a fortiori also in $(\text{dg-}\,\widetilde{\mathcal{C}}\cap\text{dw-}\,\mathcal{C}\cap\mathcal{W}, ?, \text{dg-}\,\widetilde{\mathcal{F}})$.

Step 2: We claim that the stabilization functor in question annihilates all $X\in\mathcal{W}$. For that, step 1 and the exactness (Proposition II.C.3.19) of the sequence of functors

$$\begin{array}{c}\mathrm{Ho}(\text{dg-}\,\widetilde{\mathcal{C}}\cap\text{dw-}\,\mathcal{C}\cap\mathcal{W}, \mathrm{Acyc}(\mathscr{A}), [\widetilde{\mathcal{C}}\cap\text{dw-}\,\mathcal{C}\cap\mathcal{W}]^{\perp})\\ \big\downarrow \mathbf{R}\,\mathrm{id}\\ \mathrm{Ho}(\text{dg-}\,\widetilde{\mathcal{C}}\cap\text{dw-}\,\mathcal{C}\cap\mathcal{W}, ?, \text{dg-}\,\widetilde{\mathcal{F}})\\ \big\downarrow \mathbf{R}\,\mathrm{id}\\ \mathrm{Ho}(\widetilde{\mathcal{C}}\cap\text{dw-}\,\mathcal{C}\cap\mathcal{W}, ?, \widetilde{\mathcal{F}})\end{array}$$

show that it suffices to prove that any $X\in\mathcal{W}\subset\mathrm{Ch}(\mathscr{A})$ belongs to $[\widetilde{\mathcal{C}}\cap\text{dw-}\,\mathcal{C}\cap\mathcal{W}]^{\perp}$ up to weak equivalence in $(\text{dg-}\,\widetilde{\mathcal{C}}\cap\text{dw-}\,\mathcal{C}\cap\mathcal{W}, ?, \text{dg-}\,\widetilde{\mathcal{F}})$. Now, the presence of enough injectives with respect to $(\mathcal{C}\cap\mathcal{W},\mathcal{F})$ shows that X admits a resolution $\iota: X\rightarrowtail F$ with $F\in\mathrm{Ch}^{\geq 0}(\mathcal{F})$ such that $Z^kF\in\mathcal{C}\cap\mathcal{W}$ for $k>0$. The thickness of $\mathcal{W}$ then implies that even $F\in\mathrm{Ch}^{\geq 0}(\mathcal{W}\cap\mathcal{F})\subset\text{dg-}\,\widetilde{\mathcal{W}\cap\mathcal{F}}$, and moreover $Z:=\mathrm{coker}(\iota)\in\widetilde{\mathcal{C}\cap\mathcal{W}}$. Since $\widetilde{\mathcal{C}\cap\mathcal{W}}$ are the trivially cofibrant objects in $(\text{dg-}\,\widetilde{\mathcal{C}}\cap\text{dw-}\,\mathcal{C}\cap\mathcal{W}, ?, \text{dg-}\,\widetilde{\mathcal{F}})$ and $\text{dg-}\,\widetilde{\mathcal{W}\cap\mathcal{F}}\subseteq[\widetilde{\mathcal{C}}\cap\text{dw-}\,\mathcal{C}\cap\mathcal{W}]^{\perp}$ as witnessed by the right vertical arrows in Figure II.C.3.2, it follows that $X\cong F\in[\widetilde{\mathcal{C}}\cap\text{dw-}\,\mathcal{C}\cap\mathcal{W}]^{\perp}$ in $\mathrm{Ho}(\text{dg-}\,\widetilde{\mathcal{C}}\cap\text{dw-}\,\mathcal{C}\cap\mathcal{W}, ?, \text{dg-}\,\widetilde{\mathcal{F}})$ as claimed.

Step 3: Since any $X\in\mathscr{A}$ admits a functorial resolution of the form $0\to X\to F\to C\to 0$ with $F\in\mathcal{F}$ and $C\in\mathcal{C}\cap\mathcal{W}$, the second step shows that it suffices to prove the commutativity of (II.C.3.24) when restricted to the fibrant objects $\mathcal{F}\subseteq\mathscr{A}$. In this case, by definition as well as step 1, both the left stabilization

$$\mathrm{Ho}(\text{dg-}\,\widetilde{\mathcal{C}\cap\mathcal{W}}, \mathrm{Acyc}(\mathscr{A}), \text{dg-}\,\widetilde{\mathcal{F}}) \longrightarrow \mathrm{Ho}(\widetilde{\mathcal{C}}\cap\text{dw-}\,\mathcal{C}\cap\mathcal{W}, ?, \text{dg-}\,\widetilde{\mathcal{F}})$$

and the functor

$$\mathbf{R}\iota^0 : \mathrm{Ho}(\mathcal{M}) \to \mathrm{Ho}(\widetilde{\mathcal{C}}\cap\text{dw-}\,\mathcal{C}\cap\mathcal{W}, ?, \text{dg-}\,\widetilde{\mathcal{F}})$$

can be computed naively, and the commutativity of (II.C.3.24) follows. □

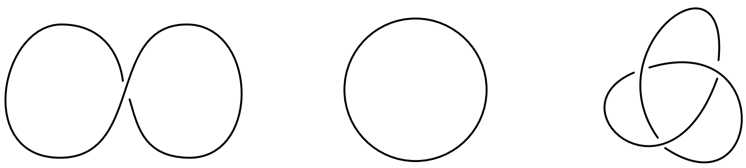
Index of notation for Part I

$\mathrm{Ho}^*(R)$	Homotopy category of chain complexes over a ring R with boundedness condition $* \in \{\emptyset, +, -, b\}$, page 93
$\mathcal{H}_n(q)$	Generic type A (Iwahori-)Hecke algebra, page 166
H^t	Total cohomology of a linear factorization, page 62
k	Fixed number indicating which $\mathfrak{sl}(?)$ homology we consider, page 67
$K(A, w)$	Koszul algebra, page 63
$\Bbbk$	Base ring in the construction of Khovanov-Rozansky homology, page 67
$\Bbbk_{\mathrm{eq}}$	Base ring for equivariant Khovanov-Rozansky homology, page 160
$\widetilde{\Bbbk}_{\mathrm{eq}}$	Base ring for extended equivariant Khovanov-Rozansky homology, page 160
$\mathcal{KR}^k$	Khovanov-Rozansky homology, the Poincaré polynomial of $\mathcal{CKR}^k$, page 67
$\mathrm{LF}(A, w)$	Category of linear factorizations of type (A, w), page 61
$\widetilde{M}$	2-periodic curved complex associated to a linear factorization, page 62
$\mathrm{MC}(A, w)$	Category of curved mixed complexes of type (A, w), page 63
$\underline{\mathrm{MF}}(A, w)$	Homotopy category of matrix factorizations of type (A, w), page 61
$\mathrm{MF}(A, w)$	Category of matrix factorizations of type (A, w), page 61
$\mathcal{RC}_{\Bbbk}(\beta)$	Rouquier complex associated to braid word β, page 91
$\mathbf{R}\,\mathrm{fold}^{\Pi}$	Derived folding via products, page 78
$\mathcal{SBM}_{\Bbbk}(n)$	Category of Soergel bimodules over $\Bbbk[x_1, \ldots, x_n]$, page 166
$\mathrm{SCh}(A)$	Category of spectral complexes over A, page 86
${}^w\,\mathrm{sHH}_*^{A/\Bbbk}(M)$	w-stable Hochschild homology of the $\widehat{A}/(\widehat{w})$-module M, page 81
${}^w\,\mathrm{SHH}^{A/\Bbbk}(M)$	Spectral complex connecting ordinary and w-stable Hochschild homology of M, page 86
${}^w\,\mathrm{sHH}_t^{A/\Bbbk}(M)$	Total w-stable Hochschild homology of the $\widehat{A}/(\widehat{w})$-module M, page 81
$\tau_{\geq n}$	Truncation of (curved mixed) complexes, page 63
$\mathrm{Tr}_{\mathbb{Z}}^{n+1}$	Partial trace functor for curved mixed complexes, page 100
$\mathrm{Tr}_{\mathbb{Z}_2}^{n+1}$	Partial trace functor for linear factorizations, page 102
$\mathbb{V}_{\mathbb{Z}_2}^{n+1}$	Forgetful functor for linear factorizations over $\widehat{\mathbb{A}}_{\Bbbk}^{n+1}$, page 102
$\mathbb{V}_{\mathbb{Z}}^{n+1}$	Forgetful functor for mixed complexes over $\widehat{\mathbb{A}}_{\Bbbk}^{n+1}$, page 100
w_n	The $\mathfrak{sl}(k)$ Khovanov-Rozansky potential $\sum_{i=1}^{n} x_i^{k+1} \in \mathbb{A}_{\Bbbk}^n$, page 67
$\mathcal{W}^{\mathrm{abs}}$	Class of absolutely acyclic modules, page 70
$\mathcal{W}^{\mathrm{ctr}}$	Class of contraacyclic modules, page 70
$\widehat{w}_n$	The doubled $\mathfrak{sl}(k)$ Khovanov-Rozansky potential $\sum_{i=1}^{n} x_i^{k+1} - y_i^{k+1} \in \widehat{\mathbb{A}}_{\Bbbk}^n$, page 67
$\{\underline{x}, \underline{y}\}$	Koszul factorization associated to sequences $\underline{x}$ and $\underline{y}$, page 64
$\{x, y\}$	Elementary Koszul factorization, page 63
$X_{\Bbbk}^{n,i}$	i-th twisted diagonal $\mathbb{A}_{\Bbbk}^n$-bimodule, page 67

Index of notation for Part II

$\mathcal{I}^{<\infty}$	Objects of finite injective dimension, page 181
$\iota^k(M)$	Stalk complex with M in degree k, page 225
$K_{S,w}$	Koszul algebra, page 230
$\mathrm{LF}(S,w)$	Category of linear factorizations of type (S,w), page 230
$\mathcal{M}$	Symbol used for model structures, page 179
$\mathcal{M}_1/\mathcal{M}_2$	Right localization of $\mathcal{M}_1$ w.r.t. $\mathcal{M}_2$, page 205
$\mathcal{M}_2\backslash\mathcal{M}_1$	Left localization of $\mathcal{M}_1$ w.r.t. $\mathcal{M}_2$, page 210
$\mathcal{M}^{\mathrm{co}}$	Coderived model structure, page 197
$\mathcal{M}^{\mathrm{co}}_{\mathrm{sing}}(A)$	Absolute singular coderived model structure, page 215
$\mathcal{M}^{\mathrm{co}}_{\mathrm{sing}}(A/R)$	Relative singular coderived model structure, page 215
${}^{i}\mathcal{M}^{\mathrm{co}}_{\mathrm{sing}}$	Injective variant of $\mathcal{M}^{\mathrm{co}}_{\mathrm{sing}}$, page 216
$\mathcal{M}^{\mathrm{ctr}}$	Contraderived model structure, page 197
$\mathcal{M}^{\mathrm{ctr}}_{\mathrm{sing}}(A)$	Absolute singular contraderived model structure, page 215
$\mathcal{M}^{\mathrm{ctr}}_{\mathrm{sing}}(A/R)$	Relative singular contraderived model structure, page 215
${}^{p}\mathcal{M}^{\mathrm{ctr}}_{\mathrm{sing}}$	Projective variant of $\mathcal{M}^{\mathrm{ctr}}_{\mathrm{sing}}$, page 216
$\mathrm{MF}(S,w)$	Category of matrix factorizations of type (S,w), page 230
$\mathcal{M}^{\mathrm{flat}}(R)$	Flat model structure on $\mathrm{Ch}(R\text{-Mod})$, page 181
$\mathcal{M}^{\text{G-inj}}(R)$	Gorenstein injective model structure on R-Mod, page 181
$\mathcal{M}^{\text{G-proj}}(R)$	Gorenstein projective model structure on R-Mod, page 181
$\mathcal{M}^{\mathrm{inj}}$	Standard injective model structure, page 181
${}^{m}\mathcal{M}^{\mathrm{inj}}$	Mixed variant of $\mathcal{M}^{\mathrm{inj}}$, page 218
$\mathcal{M}^{\mathrm{proj}}$	Standard projective model structure, page 181
$\mathcal{P}^{<\infty}$	Objects of finite projective dimension, page 181
$Q^k(X,\delta)$	k-th cosyzygy module of (X,δ), i.e. $Q^k(X,\delta) := \mathrm{coker}(\delta^{k-1})$, page 225
S_w	$\mathbb{Z}/2\mathbb{Z}$-graded curved dg ring s.t. $S_w\text{-Mod} \cong \mathrm{LF}(S,w)$., page 230
U	Forgetful functor, page 201
$Z^k(X,\delta)$	k-th syzygy module of (X,δ), i.e. $Z^k(X) := \ker(\delta^k)$, page 225

Eine Einführung in die Knotentheorie



a b c

Abbildung II.5.3.3. Offenbar sind b und c nicht äquivalent – doch *warum*?

Das Grundproblem der *Knotentheorie* ist das folgende: Gegeben zwei Knoten a, b im Raum, entscheide ob a und b ineinander verformbar sind oder nicht. Sind sie es, so nennen wir a und b äquivalent und schreiben $a \sim b$, anderenfalls $a \not\sim b$. Beispielsweise ist der Knoten a aus Abbildung II.5.3.3 in den *Unknoten* b überführbar, d.h. $a \sim b$, der *Kleeblattknoten* c jedoch nicht, d.h. $b \not\sim c$. Beide Aussagen scheinen plausibel, doch während $a \sim b$ durch die Angabe einer expliziten Deformation zwischen a und b auch in der Tat leicht zu beweisen ist, reicht für den Beweis von $b \not\sim c$ die bloße Feststellung, dass es keine „offensichtliche" Deformation gibt, nicht aus: auch nach vielen gescheiterten Versuchen einer Überführung besteht die Möglichkeit, dass es lediglich unser mangelndes Geschick ist, das uns den richtigen „Trick" noch nicht hat entdecken lassen. Ferner zeigt das folgende Beispiel, dass die Anschauung bisweilen täuscht: Wir betrachten die beiden in Abbildung II.5.3.4 dargestellten Verschlingungen und fragen jeweils, ob die mit a beschriftete Schlaufe ohne Zerschneiden der Schlaufen b und c aus dem Gesamtgefüge entfernt werden kann oder nicht. Dem ersten Eindruck nach mag die linke Verschlingung komplizierter erscheinen, doch tatsächlich stellt sich heraus, dass Schlaufe a aus dem linken Gefüge zu entfernen ist, im rechten hingegen eine unauflösbare Verknotung mit b und c besteht. Empirie und Intuition bilden demnach keine Grundlage für einen Nachweis der Nicht-Äquivalenz zweier Knoten, stattdessen muss nach Strategien gesucht werden, durch die die Existenz einer Überführung der betrachteten Knoten grundsätzlich ausgeschlossen werden kann – eine solche ist die Betrachtung von *Knoteninvarianten*. Eine Knoteninvariante ist ein Verfahren, das einem Knoten ein mathematisches Objekt – beispielsweise eine Zahl oder ein Polynom – in einer Weise zuordnet, dass Verformen des

Knotens das ihm zugeordnete Objekt nicht verändert. Wendet man ein solches Verfahren auf zwei gegebene Knoten an, und sind die ihnen zugeordneten Objekte verschieden, so folgt, dass beide Knoten nicht äquivalent sind.

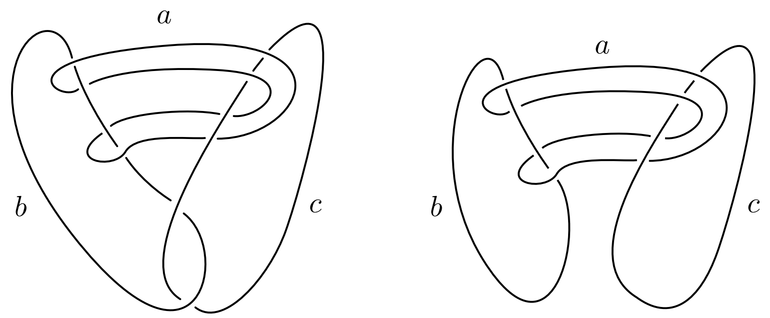
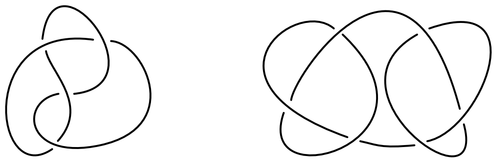


Abbildung II.5.3.4. Ist a mit b und c verschlungen?

Ein einfaches Beispiel einer Knoteninvariante ist die *Dreifärbbarkeit*: Ist ein Knoten in Form einer planaren Zeichnung wie denen in den Abbildungen II.5.3.3 und II.5.3.4 gegeben, so versuche man, seine Segmente mit drei Farben derart einzufärben, dass an jeder Überkreuzung die drei involvierten Segmente entweder sämtlich gleicher oder paarweise verschiedener Farbe sind. Ein Knoten, für den es dafür neben den drei uninteressanten Lösungen, bei denen jeweils für alle Segmente die gleiche Farbe verwendet wird, noch weitere Lösungen gibt, heißt *dreifärbbar*. Die Knoten a und b in Abbildung II.5.3.3 sind in diesem Sinne nicht dreifärbbar, der Kleeblattknoten hingegen schon. Nehmen wir hin, dass die Frage nach der Dreifärbbarkeit eines Knoten nicht davon abhängt, wie wir ihn gezeichnet haben, so ist damit bewiesen, dass der Kleeblattknoten tatsächlich ein „echter“ Knoten, d.h. vom Unknoten verschieden ist. Der Nutzen der Dreifärbbarkeit zur Unterscheidung von Knoten ist jedoch sehr gering: da der einem Knoten zugeordnete Wert entweder „ja“ oder „nein“ ist, unterteilt die Frage nach der Dreifärbbarkeit die Gesamtheit aller Knoten lediglich in zwei Lager: jene Knoten, die dreifärbbar sind, und jene, die es nicht sind. Sind aber zwei Knoten gegeben, die beide dreifärbbar oder beide nicht dreifärbbar sind, so können wir ohne weitere Hilfsmittel abermals nichts über ihre Äquivalenz aussagen. Beispielsweise ist der *Kreuzknoten b* in Abbildung II.5.3.5 dreifärbbar und somit mit unseren bisherigen Mitteln nicht vom Kleeblattknoten zu unterscheiden, und der *Achtknoten a* in Abbildung II.5.3.5 nicht dreifärbbar und somit bisher nicht vom Unknoten zu unterscheiden.

Es gibt nun zwei Möglichkeiten, unser Wissen um die Verschiedenheit von Knoten zu erweitern: Wir können erstens versuchen, mit gänzlich neuen Ideen weitere Knoteninvarianten zu konstruieren, oder zweitens bestehende Knoteninvarianten verfeinern.



a *b*

Abbildung II.5.3.5. Der Achtknoten und der Kreuzknoten

Unter einer Verfeinerung einer Knoteninvariante $\mathcal{I}$ verstehen wir hierbei eine weitere Knoteninvariante $\mathcal{J}$, die die Eigenschaft hat, dass je zwei über $\mathcal{J}$ nicht zu unterscheidende Knoten auch durch $\mathcal{I}$ nicht unterschieden werden. Ein Beispiel für eine Verfeinerung der Dreifärbbarkeit erhalten wir, indem wir nicht nur fragen, *ob* es eine gültige Dreifärbung neben den Uninteressanten gibt, sondern *wie viele*. Es stellt sich heraus, dass diese Zahl immer noch eine Knoteninvariante ist, und zwar eine Verfeinerung der Dreifärbbarkeit: Fragte letztere nur, ob der Knoten dreifärbbar ist, d.h. ob die Anzahl der gültigen Dreifärbungen gleich 3 (für die drei uninteressanten Lösungen) oder größer 3 ist, so betrachten wir nun stattdessen die genaue Zahl der gültigen Dreifärbungen. Diese feinere Invariante kann nun tatsächlich den Kleeblattknoten c aus Abbildung II.5.3.3 vom Kreuzknoten b aus Abbildung II.5.3.5 unterscheiden: der Kreuzknoten hat 27 gültige Färbungen, der Kleeblattknoten hingegen nur 9. Für die Unterscheidung des Achtknotens a aus Abbildung II.5.3.5 vom Unknoten fehlen uns jedoch weiter die Mittel.

Die hier besprochene Dreifärbbarkeit ist die einfachste von mittlerweile sehr vielen bekannten Knoteninvarianten von immer größerer Komplexität. Zu den jüngsten von ihnen gehört auch die in dieser Arbeit untersuchte Khovanov-Rozansky Homologie.

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