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Contents

Introduction

This thesis comprises four essays that belong to different strands of the theoretical economic literature. Chapter 1 and Chapter 2 study competitions among heterogeneous agents. Chapter 1 investigates the effects of head starts in innovation contests in which firms invest over time to find innovations. Chapter 2 studies how firms and managers pair up with each other when firms play an imperfect competition in the same goods market. Chapter 3 and Chapter 4 are on optimal disclosure policies in contests. Chapter 3 models a contest as an all-pay auction and studies whether a contest organizer should disclose the actual number of participating bidders if each bidder's participation is random. Chapter 4 considers whether an innovation contest organizer, upon receiving a submission of an innovation, should reveal the information of the submitted innovation in time. Chapter 1 and Chapter 3 are based on joint work with Xiandeng Jiang and Dmitriy Knyazev. For this reason, I use the pronoun "we" in these chapters, whereas I use "I" in Chapter 2 and Chapter 4.

Chapter 1 considers innovation contests among firms with heterogeneous starting positions. This topic derives from some phenomena observed in the market. On one hand, many companies, such as Nokia, that once "ruled the roost" of their respective industries got knocked off by more innovative competitors and came crashing down. On the other hand, some companies, such as Tesla, gave up patents for their technologies in early stages of their business. What are the effects of head starts in innovation contests

Attempting to answer the this question, this chapter analyzes a two-firm winner-takes-all contest in which there is a deadline and each firm decides when to stop a privately observed search for innovations (with recall). The firm with a superior innovation at the outset has a head start, and the firm with the most successful innovation at a common deadline wins. It is found that a large head start guarantees a firm victory without incurring cost. However, In contrast to what a naive person would expect, a medium-sized head start ensures defeat for the firm if the deadline is sufficiently long. In the latter case, the competitor wins the entire rent of the contest. A key insight is that when having a large head start a firm loses incentive to innovate. Hence, a head start firm might want to get rid of its head start and commit to innovate.

Chapter 2 considers matchings between firms and managers. Firms compete in the goods market. A firm's production cost of the good depends on both this firm's technology and the human capital of the manager hired. However, if a manager is hired by one firm, this manager cannot be hired by the other firms. Hence, a firm's hiring decision has externalities on the other firms' profits and managers' salaries.

In this chapter, I model the problem as a two-stage game: A first stage of simultaneous 1-to-1 matching and a second stage of Cournot competition among matched pairs. If firm technology and human capital are complementary, it is rational for each firm-manager pair to expect that the remaining agents will form a positive assortative matching (PAM), and PAM on the whole market is a stable matching under rational expectations. Even if they are substitutable yet the substitutive effect is relatively small, PAM is still stable. However, if the goods market is sufficiently large, a negative assortative matching is stable. Social welfare induced by matchings is discussed.

Chapter 3 considers information disclosure problem of a contest organizer. In many contests, such as job promotions and patent races, each contestant has private information about her valuation on the prize for winning and faces an uncertain number of competitors. The organizer, although having no information on the values of the prize to the contestants, may know the actual number of participating contestants. In such a case, to maximize the total expected revenue, should the organizer commit to revealing or concealing the number of active contestants?

This chapter models such a contest as an all-pay auction with stochastic entry. The contest entails one prize and n potential bidders. Each bidder receives a signal about the value of the prize and has a signal-dependent participation probability. All bidders bear a cost of bidding that is an increasing function of their bids. It is shown that the contest organizer prefers fully concealing (disclosing) the information about the number of participating bidders when the cost functions are convex (concave). If bidders have independent and private values, this result holds even if the probability of participating is endogenous. However, this result does not hold when firms have heterogeneous participation probabilities.

Chapter 4 considers an information disclosure problem of an organizer when she is organizing a winner-takes-all dynamic contest to procure a valuable innovation. Contestants innovate and submit innovations. Generally, submissions are made in the course of time but not at the same time. Should the principal choose (1) to publish the information on a submission upon receiving it or (2) to conceal the information until all contestants have made submissions?

I answer this question in a two-agent winner-takes-all contest in which each agent decides when to stop a privately observed search for innovations. Each agent makes one-time submission of her best discovery to the organizer. The contest ends when both submissions are make, and the agent that submitted the best innovation wins. I find that the organizer prefers revealing information on a submitted innovation upon receiving it to concealing it until the end not only because the former policy elicits a higher expected revenue but also because it is more efficient. Any hidden equal sharing rule is outcome equivalent to the winner-takes-all contest with information concealment.

Chapter 1

Head Starts and Doomed Losers: Contest via Search

Introduction

> "[U]nfortunately, for every Apple out there, there are a thousand other companies . . . like Woolworth, Montgomery Ward, Borders Books, Blockbuster Video, American Motors and Pan Am Airlines, that once 'ruled the roost' of their respective industries, to only get knocked off by more innovative competitors and come crashing down." (*Forbes*, January 8, 2014)

This chapter studies innovation contests, which are widely observed in a variety of industries. In many innovation contests, some firms have head starts: One firm has a more advanced existing technology than its rivals at the outset of a competition. The opening excerpt addresses a prominent phenomenon that is often observed in innovation contests: Companies with a head start ultimately lose a competition in the long run. It seems that having a head start sometimes results in being trapped. The failure of Nokia, the former global mobile communications giant, to compete with the rise of Apple's iPhone is one example. James Surowiecki (2013) pointed out that Nokia's focus on (improving) hardware, its existing technology, and neglect of (innovating) software contributed to the company's downfall. In his point of view, this was "a classic case of a company being enthralled (and, in a way, imprisoned) by its past success" (*New Yorker Times*, September 3, 2013).

Motivated by these observations, we investigate the effects of head starts on firms' competition strategies and payoffs in innovation contests. Previous work on innovation contests focuses on reduced form games and symmetric players, and previous work on contests with head starts considers all-pay auctions with either sequential bidding or simultaneous bidding. By contrast, we consider a stochastic contest model in which one firm has a superior existing innovation at the outset of the contest and firms' decisions are dynamic. The main contribution of our study is the identification of the long-run effects of a head start. In particular, in a certain range of the head start value, the *head start firm* becomes the ultimate loser in the long run and its competitor (or competitors) benefits greatly from its initial apparent "disadvantage". The key insight to the above phenomenon is that a large head start (e.g., a patent) indicates a firm's demise as an innovator.

Specifically, the model we develop in Section 1.1 entails two firms and one fixed prize. At the beginning of the game, each firm may or may not have an initial innovation. Whether a firm has an initial innovation, as well as the value of the initial innovation if this firm has one, is common knowledge. If a firm conducts a search for innovations, it incurs a search cost. As long as a firm continues searching, innovations arrive according to a Poisson process. The value of each innovation is drawn independently from a fixed distribution. The search activity and innovation process of each firm are privately observed. At any time point before a common deadline, each firm decides whether to stop its search process. At the deadline, each firm releases its most effective innovation to the public, and the one whose released innovation is deemed superior wins the prize.

First, we consider equilibrium behavior in the benchmark case, in which no firm has any innovation initially, in Section 1.2. We divide the deadline-cost space into three regions (as in figure 1.1). For a given deadline, (1) if the search cost is relatively high, there are two equilibria, in each of which one firm searches until it discovers an innovation and the other firm does not search; (2) if the search cost is in the middle range, each firm searches until it discovers an innovation; (3) if the search cost is relatively low, each firm searches until it discovers an innovation with a value above a certain positive cut-off value. In the third case, the equilibrium cut-off value strictly increases as the deadline extends and the arrival rate of innovations increases, and it strictly decreases as the search cost increases.

We then extend the benchmark case to include a head start: The head start firm is assigned a better initial innovation than its competitor, called the latecomer. Section 1.3 considers equilibrium behavior in the case with a head start and compares equilibrium payoffs across firms, and Section 1.4 analyzes the effects of a head start on each firm's equilibrium payoff.

Firms' equilibrium strategies depend on the value of the head starter's initial innovation (head start). Our main findings concern the case in which the head start lies in the middle range. In this range, the head starter loses its incentive to search because of its high initial position. The latecomer takes advantage of that and searches more actively, compared to when there is no head start.

An immediate question is: who does the head start favor? When the deadline is short, the latecomer does not have enough time to catch up, and thus the head starter obtains a higher expected payoff than the latecomer does. When the deadline is long, the latecomer is highly likely to obtain a superior innovation than the head starter, and thus the latecomer obtains a higher expected payoff. In the latter case, the latecomer's initial apparent "disadvantage", in fact, puts it in a more favorable position than the head starter. When the deadline is sufficiently long, the head starter is doomed to lose the competition with a payoff of zero because of its unwillingness to search, and all benefits of the head start goes to the latecomer.

Then, does the result that the latecomer is in a more favorable position than the head starter when the deadline is long imply that the head start hurts the head starter and benefits the latecomer in the long run? Focus on the case in which the latecomer does not have an initial innovation. When the search cost is relatively low, the head start, in fact, always benefits the head starter, but the benefit ceases as the deadline extends. It also benefits the latecomer when the deadline is long. When the search cost is relatively high, the head start could potentially hurt the head starter.

If the head start is large, neither firm will conduct a search, because the latecomer is deterred from competition. In this scenario, no innovation or technological progress is created, and the head starter wins the contest directly. If the head start is small, both firms play the same equilibrium strategy as they do when neither firm has an initial innovation. In both cases, the head start benefits the head starter and hurts the latecomer.

Section 1.5.1 extends our model to include stages at which the firms sequentially have an option to discard their initial innovation before the contest starts. Suppose that both firms' initial innovations are of values in the middle range and that the deadline is long. If the head starter can take the first move in the game, it can increase its expected payoff by discarding its initial innovation and committing to search. When search cost is low, by sacrificing the initial innovation, the original head starter actually makes the competitor the new head starter; this new head starter has no incentive to discard its initial innovation or to search any more. It is possible that by discarding the head start, the original head starter may benefit both firms. When search cost is high, discarding the initial innovation is a credible threat to the latecomer, who will find the apparent leveling of the playing field discouraging to conducting a high-cost search. As a result, the head starter suppresses the innovation progress.

In markets, some firms indeed give up head starts (Ulhøi, 2004), and our result provides a partial explanation of this phenomenon. For example, Tesla gave up its patents for its advanced technologies on electric vehicles at an early stage of its business.[1] While there may be many reasons for doing so, one significant reason is to maintain Tesla's position as a leading innovator in the electronic vehicle market.[2] As Elon Musk (2014), the CEO of Tesla, wrote,

[1]Toyota also gave up patents for its hydrogen fuel cell vehicles at an early stage.

[2]Another reason is to help the market grow faster by the diffusion of its technologies. A larger market

> technology leadership is not defined by patents, which history has repeatedly shown to be small protection indeed against a determined competitor, but rather by the ability of a company to attract and motivate the world's most talented engineers.[3]

Whilst Tesla keeps innovating to win a large share of the future market, its smaller competitors have less incentive to innovate since they can directly adopt Tesla's technologies. One conjecture which coincides with our result is that "Tesla might be planning to distinguish itself from the competitors it helps . . . by inventing and patenting better electric cars than are available today" (*Discovery Newsletter*, June 13, 2014).

Section 1.5.2 considers intermediate information disclosure. Suppose the firms are required to reveal their discoveries at an early time point after the starting of the contest, how would firms compete against each other? If the head start is in the middle range, before the revelation point, the head starter will conduct a search, whereas the latecomer will not. If the head starter obtains a very good innovation before that point, the latecomer will be deterred from competition. Otherwise, the head starter is still almost certain to lose the competition. Hence, such an information revelation at an early time point increases both the expected payoff to the head starter and the expected value of the winning innovation.

Section 1.6 compares the effects of a head start to those of a cost advantage and points out a significant difference. A cost advantage reliably encourages a firm to search more actively for innovations, whereas it discourages the firm's competitor.

Section 1.7 concludes this chapter. The overarching message this chapter conveys is that a market regulator who cares about long-run competitions in markets may not need to worry too much about the power of the current market dominating firms if these firms are not in excessively high positions. In the long run, these firms are to be defeated by latecomers. On the other hand, if the dominating firms are in excessively high positions, which deters entry, a regulator can intervene the market.

Literature

There is a large literature on innovation contests. Most work considers reduced form models (Fullerton and McAfee, 1999; Moldovanu and Sela, 2001; Baye and Hoppe, 2003; Che and Gale, 2003).[4] Head starts are studied in various forms of all-pay auctions. Leininger (1991), Konrad (2002), and Konrad and Leininger (2007) model a head start as a first-mover advantage in a

increases demand and lowers cost.

[3]See "All Our Patent Are Belong To You," June 12, 2014, on http://www.teslamotors.com/blog/all-our-patent-are-belong-you.

[4]Also see, for example, Hillman and Riley (1989), Baye et al. (1996), Krishna and Morgan (1998), Che and Gale (1998), Cohen and Sela (2007), Schöttner (2008), Bos (2012), Siegel (2009, 2010, 2014), Kaplan et al. (2003), and Erkal and Xiao (2015).

sequential all-pay auction and study the first-mover's performance. Casas-Arce and Martinez-Jerez (2011), Siegel (2014), and Seel (2014) model a head start as a handicap in a simultaneous all-pay auction and study the effect on the head starter. Kirkegaard (2012) and Seel and Wasser (2014) also model a head start as a handicap in a simultaneous all-pay auction but study the effect on the auctioneer's expected revenue. Segev and Sela (2014) analyzes the effect a handicap on the first mover in a sequential all-pay auction. Unlike these papers, we consider a framework in which players' decisions are dynamic.

The literature considering settings with dynamic decisions is scarce, and most studies focus on symmetric players. The study by Taylor (1995) is the most prominent.[5] In his symmetric T-period private search model, there is a unique equilibrium in which players continue searching for innovations until they discover one with a value above a certain cut-off. We extend Taylor's model to analyze the effects of a head start and find the long-run effects of the head start, which is our main contribution.

Seel and Strack (2013, forthcoming) and Lang et al. (2014) also consider models with dynamic decisions. Same as in our model, in these models each player also solves an optimal stopping problem. However, the objectives and the results of these papers are different from ours. In the models of Seel and Strack (2013, forthcoming), each player decides when to stop a privately observed Brownian motion with a drift. In their earlier model, there is no deadline and no search cost and a process is forced to stop when it hits zero. They find that players do not stop their processes immediately even if the drift is negative. In their more resent model, each search incurs a cost that depends on the stopping time. This more recent study finds that when noise vanishes the equilibrium outcome converges to the symmetric equilibrium outcome of an all-pay auction. Lang et al. (2014) consider a multi-period model in which each player decides when to stop a privately observed stochastic points-accumulation process. They find that in equilibrium the distribution over successes converges to the symmetric equilibrium distribution of an all-pay auction when the deadline is long.

Our study also contributes to the literature on information disclosure in innovation contests. Aoyagi (2010), Ederer (2010), Goltsman and Mukherjee (2011), and Wirtz (2013) study how much information on intermediate performances a contest designer should disclose to the contestants. Unlike what we do, these papers consider two-stage games in which the value of a contestant's innovation is its total outputs from the two stages. Bimpikis et al. (2014) and Halac et al. (forthcoming) study the problem of designing innovation contests, which includes both the award structures and the information disclosure policies. Halac et al. (forthcoming) consider a model in which each contestant searches for innovations, but search outcomes

[5]Innovation contests were modeled as a race in which the first to reach a defined finishing line gains a prize, e.g., Loury (1979), Lee and Wilde (1980), and Reinganum (1981, 1982).

are binary. A contest ends after the occurrence of a single breakthrough, and a contestant becomes more and more pessimistic over time if there has been no breakthrough. Bimpikis et al. (2014) consider a model which shares some features with Halac et al. (forthcoming). In the model, an innovation happens only if two breakthroughs are achieved by the contestants, the designer decides whether to disclose the information on whether the first breakthrough has been achieved by a contestant, and intermediate awards can be used. In both models, contestants are symmetric. In contrast, the contestants in our model are always asymmetric. Rieck (2010) studies information disclosure in the two-period case of Taylor's (1995) model. In contrast to our finding, he shows that the contest designer prefers concealing the outcome in the first stage. Unlike all the above papers, Gill (2008), Yildirim (2005), and Akcigit and Liu (2015) address the incentives for contestants, rather than the designer, to disclose intermediate outcomes.

Last but most importantly, our study contributes to the literature on the relationship between market structure and incentive for R&D investment. The debate over the effect of market structure on R&D investment dates back to Schumpeter (1934, 1942).[6] Due to the complexity of the R&D process, earlier theoretical studies tend to focus on one facet of the process. Gilbert and Newbery (1982), Fudenberg et al. (1983), Harris and Vickers (1985a,b, 1987), Judd (2003), Grossman and Shapiro (1987), and Lippman and McCardle (1987) study preemption games. In these models, an incumbent monopolist has more incentive to invest in R&D than a potential entrant. In fact, a potential entrant sees little chance to win the competition, because of a lag at the starting point of the competition, and is deterred from competition. In our model, the intuition for the result in the case of a large head start is similar to this "preemption effect", except that no firm invests in our case.

By contrast, Arrow (1962) and Reinganum (1983, 1985) show, in their respective models, that an incumbent monopolist has less incentive to innovate than a new entrant.[7] The cause for this is what is called the "replacement effect" by Tirole (1997). While an incumbent monopolist can increase its profit by innovating, it has to lose the profit from the old technology once it adopts a new technology. This effectively reduces the net value of the new technology to the incumbent. It is then natural that a firm who has a lower value of an innovation has less incentive to innovate, which is exactly what happens in our model with asymmetric costs. On the other hand, our main result, on medium-sized head start, has an intuition very similar to the "replacement effect". Rather than a reduction in the value of an innovation to the head starter, a head start decreases the increase in the probability of winning from innovating. In

[6]See Gilbert (2006) for a comprehensive survey.

[7]Doraszelski (2003) generalizes the models of Reinganum (1981, 1982) to a history-dependent innovation process model and shows, in some circumstances, the catching-up behavior in equilibrium.

both Reinganum's models and our model, an incumbent could have a lower probability of winning than a new entrant. However, different from her models, in our model an incumbent (head starter) can also have a lower expected payoff than a new entrant (latecomer).

1.1 The Model

Firms and Tasks

There are two risk neutral firms, Firm 1 and Firm 2, competing for a prespecified prize, normalized to 1, in the contest. Time is continuous, and each firm searches for innovations before a deadline T. At the deadline T, each firm releases to the public the best innovation it has discovered, and the firm who releases a superior innovation wins the prize. If no firm has discovered any innovation, the prize is retained. If there is a tie between the two firms, the prize is randomly allocated to them with equal probability.

At any time point $t \in [0, T)$ before the deadline, each firm decides whether to continue searching for innovations. If a firm continues searching, the arrival of innovations in this firm follows a Poisson process with an arrival rate of λ. That is, the probability of discovering m innovations in an interval of length δ is $\frac{e^{-\lambda\delta}(\lambda\delta)^m}{m!}$. The values of innovations are drawn independently from a distribution F, defined on $(0, 1]$ with $F(0) := \lim_{a\to 0} F(a) = 0$. F is continuous and strictly increasing over the domain.

Each firm's search cost is $c > 0$ per unit of time. We assume that $c < \lambda$, because if $c > \lambda$ the cost is so high that no firm is going to conduct a search. To illustrate this claim, suppose Firm 2 does not search, Firm 1 will not continue searching if it has an innovation with a value above 0, whereas Firm 1's instantaneous gain from searching at any moment when it has no innovation is

$$\lim_{\delta\to 0} \frac{\sum_{m=1}^{+\infty} \frac{e^{-\lambda\delta}(\lambda\delta)^m}{m!} - c\delta}{\delta} = \lambda - c,$$

which is negative if $c > \lambda$.

Information

The search processes of the two firms are independent and with recall. Whether the opponent firm is actively searching is unobservable; whether a firm has discovered any innovation, as well as the values of discovered innovations, is private information until the deadline T.

For convenience, we say a firm is in a **state** $a \in [0, 1]$ at time t if the value of the best innovation it has discovered by time t is a, where $a = 0$ means that the firm has no innovation. The **initial states** of Firm 1 and Firm 2 are denoted by a_1^I and a_2^I, respectively. Firms' initial states are commonly known.

Strategies

In our model, each firm's information on its opponent is not updated. Hence, the game is static, although the firms' decisions are dynamic. Then, the solution concept we use is Nash equilibrium. In accordance with the standard result from search theory that each firm's optimal strategy is a constant cut-off rule, we make the following assumption.[8]

Assumption 1.1. *We focus on equilibria that consist of* ***constant cut-off rules****: Denote $\hat{a}_i^*$ as an equilibrium strategy.* $\hat{a}_i^* \in \mathcal{S}_i := \{-1\} \cup [a_i^I,\ 1]$.

If a constant cut-off strategy $\hat{a}_i^* \in [a_i^I,\ 1]$ is played, at any time point $t \in [0, T)$, Firm i stops searching if it is in a state above $\hat{a}_i^*$ and continues searching if in a state below or at $\hat{a}_i^*$.[9] The strategy $\hat{a}_i = -1$ represents that Firm i does not conduct a search.

Suppose both firms have no initial innovation. Without this assumption, for any given strategy played by a firm's opponent, there is a constant cut-off rule being the firm's best response. Such a cut-off value being above zero is the unique best response strategy, ignoring elements associated with zero probability events. However, in the cases in which a firm is indifferent between continuing searching and not if it is in state 0, this firm has (uncountably) many best response strategies. The above assumption helps us to focus on the two most natural strategies: not to search at all and to search with 0 as the cut-off.[10] A full justification for this assumption is provided in the appendix.

Let $\tilde{P}[a|\hat{a}_i, a_i^I]$ denote the probability of Firm i ending up in a state below a if it adopts a strategy $\hat{a}_i$ and its initial state is a_i^I; let $E[cost|\hat{a}_i]$ denote Firm i's expected cost on search if it adopts a strategy $\hat{a}_i$. Firm i's ex ante expected utility is

$$U_i = \int_0^1 P[a|\hat{a}_{-i}, a_{-i}^I] dP[a|\hat{a}_i, a_i^I] - E[cost|\hat{a}_i].$$

Now, we are ready to study equilibrium behavior. Before solving the head start case, we first look at the case with no initial innovation.

[8] See Lippman and McCall (1976) for the discussion on optimal stopping strategies for searching with finite horizon and recall.

[9] Once Firm i stops searching at some time point it shall not search again later.

[10] Without this assumption, there can be additional best response strategies of the following type: a firm randomizes between searching and not searching until a time $T' < T$ with cutoff 0 and stops at T' even if no discovery was made.

1.2 The Symmetric-Firms Benchmark ($a_1^I = a_2^I = 0$)

In this section, we look at the benchmark case, in which both firms start with no innovation. It is in the spirit of Taylor's (1995), except that it is in continuous time. The equilibrium strategies are presented below.[11]

Theorem 1.1. *Suppose* $a_1^I = a_2^I = 0$.

i. If $c \in [\frac{1}{2}\lambda(1+e^{-\lambda T}), \lambda)$, *there are two equilibria, in each of which one firm searches with* 0 *as the cut-off and the other firm does not search.*

ii. If $c \in [\frac{1}{2}\lambda(1-e^{-\lambda T}), \frac{1}{2}\lambda(1+e^{-\lambda T}))$, *there is a unique equilibrium, in which both firms search with* 0 *as the cut-off.*

iii. If $c \in (0, \frac{1}{2}\lambda(1-e^{-\lambda T}))$, *there is a unique equilibrium, in which both firms search with* a^* *as the cut-off, where* $a^* > 0$ *is the unique value that satisfies*

$$\frac{1}{2}\lambda[1-F(a^*)]\left[1-e^{-\lambda T[1-F(a^*)]}\right] = c. \tag{1.1}$$

Proof. See Appendix 1.A.2. □

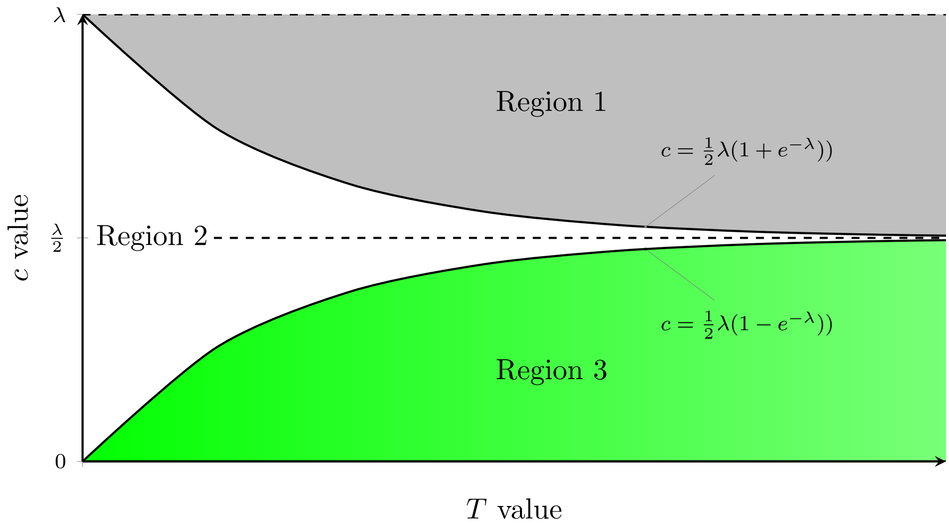


Figure 1.1

The result is illustrated in figure 1.1. The deadline-cost space is divided into three regions.[12] In Region 1, the search cost is so high that it is not profitable for both firms to innovate. In Region 2, both firms would like to conduct a search in order to discover an innovation with

[11] When search cost is low, the equilibrium is unique even without Assumption 2.1. When search is high, without Assumption 2.1, there are additional symmetric equilibria of the following type: each firm randomizes between not participating and participating until a time $T' < T$ with 0 as the cutoff.

[12] An " area" we say is the interior of the corresponding area.

any value, but none has the incentive to spend additional effort to find an innovation with a high value. In Region 3, both firms exert efforts to find an innovation with a value above a certain level. In this case, a firm in the cut-off state is indifferent between continuing and stopping searching. This is represented by equation (1.1), in which $1 - e^{-\lambda T[1-F(a^*)]}$ is the probability of a firm's opponent ending up in a state above a^* and $\frac{1}{2}[1 - F(a^*)]$ is the increase in the probability of winning if the firm, in state a^*, obtains a new innovation. Hence, this equation represents that, in the cut-off state, the instantaneous increase in the probability of winning from continuing searching equals the instantaneous cost of searching. As T goes to infinity, $c = \frac{\lambda}{2}$ becomes the separation line for Case [i] and Case [iii].

Generally, there is no closed form solution for the cut-off value in Case [iii]. However, if the search cost is very low, we have a simple approximation for it.

Corollary 1.1. *Suppose $a_1^I = a_2^I = 0$. When c is small,* $a^* \approx F^{-1}\left(1 - \sqrt{\frac{2c}{\lambda^2 T}}\right)$.

Proof. First, we assume that $\lambda T[1 - F(a^*)]$ is small, and we come back to check that it is implied by that c is small. Applying equation (1.1), we have

$$\begin{aligned}
&\frac{c}{\lambda} = \frac{1}{2}[1 - F(a^*)]\left[1 - e^{-\lambda T[1-F(a^*)]}\right] \approx \frac{1}{2}\lambda T[1 - F(a^*)]^2 \\
\Leftrightarrow &[1 - F(a^*)]^2 \approx \frac{2c}{\lambda^2 T} \\
\Leftrightarrow &a^* \approx F^{-1}\left(1 - \sqrt{\frac{2c}{\lambda^2 T}}\right) \quad \text{and} \quad \lambda T[1 - F(a^*)] \approx \sqrt{2cT}.
\end{aligned}$$

□

For later reference we, based on the previous theorem, define a function $a^* : (0, \lambda) \times [0, +\infty) \to [0, 1]$ where

$$a^*(c, T) = \begin{cases} 0 & \text{for } c \in [\frac{1}{2}\lambda(1 - e^{-\lambda T}), \lambda) \\ \text{the } a^* \text{ that solves (1.1)} & \text{for } c \in (0, \frac{1}{2}\lambda(1 - e^{-\lambda T})). \end{cases}$$

A simple property which will be used in later sections is stated below.

Lemma 1.1. *In Region 3, $a^*(c, T)$ is strictly increasing in T (and λ) and strictly decreasing in c.*

There are two observations. One is that $a^*(c, T) = 0$ if $c \geq \frac{\lambda}{2}$. The other is that $a^*(c, T)$ converges to $F^{-1}(1 - \frac{2c}{\lambda})$ as T goes to infinity if $c < \frac{\lambda}{2}$, which derives from taking the limit of equation (1.1) w.r.t. T. Let us denote a^*_L as the limit of $a^*(c,T)$ w.r.t. T:

$$a^*_L := \lim_{T\to+\infty} a^*(c,T) = \begin{cases} 0 & \text{for } c \geq \frac{\lambda}{2}, \\ F^{-1}(1-\frac{2c}{\lambda}) & \text{for } c < \frac{\lambda}{2}. \end{cases}$$

We end this section by presenting a full rent dissipation property of the contest when the deadline approaches infinity.

Lemma 1.2. *Suppose $a^I_1 = a^I_2 = 0$. If $c < \frac{\lambda}{2}$, each firm's expected payoff in equilibrium goes to 0 as the deadline T goes to infinity.*

Proof. See Appendix 1.A.2. □

The intuition is as follows. The instantaneous increase in the expected payoff from searching for a firm who is in state $a^*(c,T)$, the value of the equilibrium cut-off, is 0 (it is indifferent between continuing searching and not). If the deadline is finite, a firm in a state below $a^*(c,T)$ has a positive probability of winning even if it stops searching. Hence, the firms have positive rents in the contest. As the deadline approaches infinity, there is no difference between being in a state below $a^*(c,T)$ and at $a^*(c,T)$, because the firm will lose the contest for sure if it does not search. In either case the instantaneous increase in the expected payoff from searching is 0. Hence, in the limit the firms' rents in the contest are fully dissipated.

Though the equilibrium expected payoff goes to 0 in the limit, it is not monotonically decreasing to 0 as the deadline approaches infinity, because each firm's expected payoff converges to 0 as the deadline approaches 0 as well.[13]

1.3 Main Results: Exogenous Head Starts ($a^I_1 > a^I_2$)

In this section, we add head starts into the study. Without loss of generality, we assume that Firm 1 has a better initial innovation than does Firm 2 before competition begins, i.e., $a^I_1 > a^I_2$. We first derive the equilibrium strategies, and then we explore equilibrium properties.

1.3.1 Equilibrium Strategies

Theorem 1.2. *Suppose $a^I_1 > a^I_2$.*

1. *For $a^I_1 > F^{-1}(1-\frac{c}{\lambda})$, there is a unique equilibrium, in which no firm searches, and thus Firm 1 wins the prize.*

[13] In fact, by taking the derivative of (1.12) (as in the appendix) w.r.t. T, one can show that the derivative at $T = 0$ is $\lambda - c > 0$ and that, if $c < \frac{\lambda}{2}$, (1.12) is increasing in T for $T < \min\{\frac{1}{\lambda}\ln\frac{\lambda}{c}, \frac{1}{\lambda}\ln\frac{\lambda}{\lambda-2c}\}$ and decreasing in T for $T > \max\{\frac{1}{\lambda}\ln\frac{\lambda}{c}, \frac{1}{\lambda}\ln\frac{\lambda}{\lambda-2c}\}$.

2. *For $a_1^I = F^{-1}(1-\frac{c}{\lambda})$, there are many equilibria. In one equilibrium, both firms do not search. In the other equilibria, Firm 1 does not search and Firm 2 searches with a value $\hat{a}_2 \in [a_2^I, a_1^I]$ as the cut-off.*

3. *For $a_1^I \in (a^*(c,T), F^{-1}(1-\frac{c}{\lambda}))$, there is a unique equilibrium, in which Firm 1 does not search and Firm 2 searches with a_1^I as the cut-off .*

4. *For $a_1^I = a^*(c,T)$, there are two equilibria. In one equilibrium, both firms search with a_1^I as the cut-off. In the other equilibrium, Firm 1 does not search and Firm 2 searches with a_1^I as the cut-off.*

5. *For $a_1^I \in (0, a^*(c,T))$, there is a unique equilibrium, in which both firms search with $a^*(c,T)$ as the cut-off.*

Proof. See Appendix 1.A.3. □

Remark. *Case [4] and [5] exist only when $c \leq \frac{1}{2}\lambda[1-e^{-\lambda T}]$ (Region 3).*

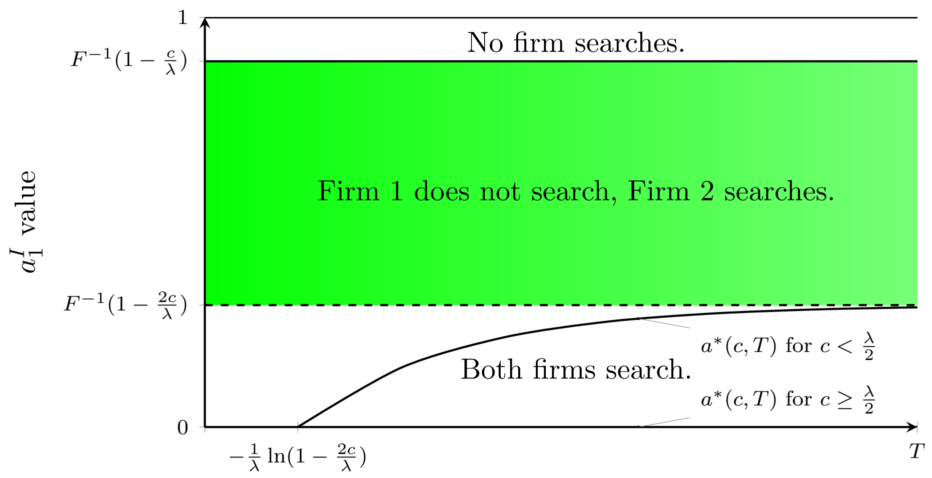


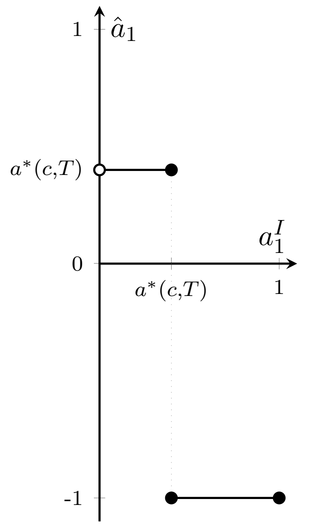
Figure 1.2: Thresholds (when $c < \frac{\lambda}{2}(1-e^{-\lambda T})$).

The thresholds in the theorem are depicted in figure 1.2. The leading case is Case [3], when the head start is in the middle range. A head start reduces the return of a search, in terms of the increase in the probability of winning. Having a sufficiently large initial innovation, Firm 1 loses incentive to search because the marginal increase in the probability of winning from searching for Firm 1 is too small compared to the marginal cost of searching, whether Firm 2 searches or not. Firm 2 takes advantage of that and commits to search until it discovers an innovation better than Firm 1's initial innovation. Hence, compared to its equilibrium behavior in the benchmark case, Firm 2 is more active in searching (in terms of a higher cut-off value) when Firm 1 has a medium-sized head start, and a larger value of head start forces Firm 2 to search more actively.

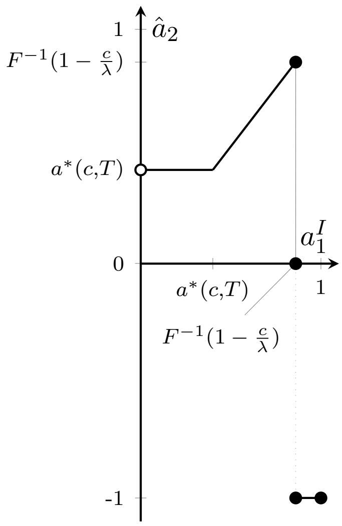
In Case [1], Firm 1's head start is so large that Firm 2 is deterred from competition because Firm 2 has little chance to win if it searches. Firm 1 wins the prize without incurring any cost. Moreover, it is independent of the deadline T.

In Case [5], in which Firm 1's head start is small, the head start has no effect on either firm's equilibrium strategy, and both firms search with $a^*(c,T)$ as the cut-off, same as in the benchmark case. The only effect of the head start is an increase in Firm 1's probability of winning (and a decrease in Firm 2's).

In brief, a comparison of Theorem 1.2 and Theorem 1.1 shows that a head start of Firm 1 does not alter its own equilibrium behavior but Firm 2's. The effect on Firm 2's equilibrium strategy is not monotone in the head start of Firm 1. The initial state of Firm 2, the latecomer, is irrelevant to the equilibrium strategies. Figure 1.3 illustrates how each firm's equilibrium strategy changes as the value of the initial innovation of Firm 1, the head starter, varies.



(a) Firm 1



(b) Firm 2

Figure 1.3: Firms' equilibrium cut-off values as the value of Firm 1's initial innovation, a_1^I, varies.

Figure 1.4 illustrates Firm 2'the best responses (when it has no initial innovation) to Firm 1's strategies for various values of Firm 1's initial states. The case in which Firm 1 has a high-value initial innovation is significantly different from the case in which Firm 1 has no initial innovation.

Turning back to Case [3] in the previous result, we notice that the lower bound for this case to happen does not converge to the upper bound as the deadline approaches infinity, i.e., $a_L^* < F^{-1}(1-\frac{c}{\lambda})$. The simplest but most interesting result of this study, the case of "head starts and doomed losers", derives.

Corollary 1.2. *Suppose* $a_1^I \in (a_L^*, F^{-1}(1-\frac{c}{\lambda}))$.

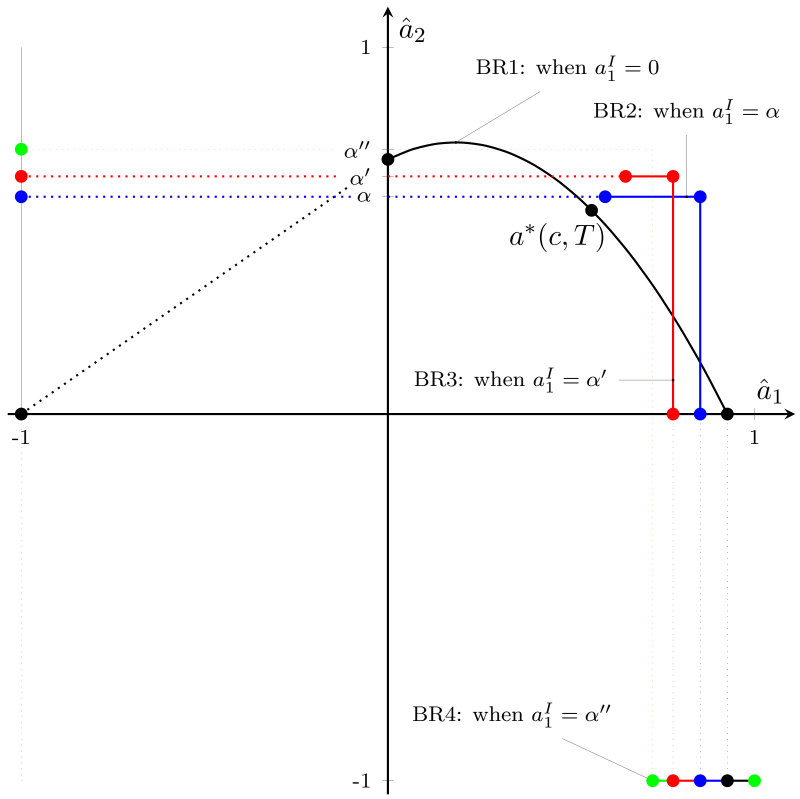


Figure 1.4: Best response projections for Firm 2 as a_1^I takes values in $\{-1, \alpha, \alpha', \alpha''\}$ ($c < \frac{\lambda}{2}(1 - e^{-\lambda T})$ and $a_2^I = 0$).

BR1 represents Firm 2's best responses when no firm has an initial innovation. If Firm 1 does not search, Firm 2 would search with 0 as the cut-off. If Firm 1 searches with 0 as the cut-off, Firm 2 would search with a cut-off higher than the equilibrium cut-off. As Firm 1 further rises its cut-off, Firm 2 would first rise its cut-off and then lower its cut-off. When the deadline is long, Firm 2 would not search if Firm 1's cut-off is high. BR2 and BR3 represent Firm 2's best responses when Firm 1 has an initial innovation with a value slightly above $a^*(c,T)$, the equilibrium cut-off when there is no initial innovation. In this case, if Firm 1 does not search, Firm 2's best response is to search with the value Firm 1's initial innovation as the cut-off. If Firm 1 searches with a cut-off slightly above the value of its initial innovation, Firm 2's best response is still to search with the value of Firm 1's initial innovation as the cut-off. Once Firm 1's cut-off is greater than a certain value, Firm 2 would not search. BR4 represents Firm 2's best responses when Firm 1 has a high-value initial innovation. In this case, if Firm 1 does not search, Firm 2 would still search with the value of Firm 1's initial innovation as the cut-off; if Firm 1 searches, Firm 2 would have no incentive to search. On the other hand, when the value of Firm 1's initial innovation is above $a^*(c,T)$, Firm 1's best response to any strategy of Firm 2 is not to search.

1. *Firm 2's (Firm 1's) probability of winning increases (decreases) in the deadline.*

2. *As T goes to infinity, Firm 2's probability of winning goes to* 1, *and Firm 1's goes to* 0.

Proof. In equilibrium Firm 1 does not search and Firm 2 searches with a_1^I as the cut-off. Firm 2's probability of winning is thus

$$1 - e^{-\lambda T[1-F(a_1^I)]},$$

which is increasing in T, and it converges to 1 as T goes infinity. Firm 1's probability of winning is $e^{-\lambda T[1-F(a_1^I)]}$, which is, in the contrast, decreasing in T, and it converges to 0 as T approaches infinity. □

This property results from our assumption that search processes are with recall. The larger the head start is, the smaller the marginal increase in the probability of winning from searching is, given any strategy played by the latecomer. Hence, even if the head starter knows that in the long run the latecomer will almost surely obtain an innovation with a value higher than its initial innovation, the head starter is not going to conduct a search as long as the instantaneous increase in the probability of winning is smaller than the instantaneous cost of searching.

1.3.2 Payoff Comparison across Firms

A natural question arises: which firm does a head start favor? Will Firm 1 or Firm 2 achieve a higher expected payoff? To determine that, we need a direct comparison of the two firms' expected payoffs. When $a_1^I \in (a^*(c,T), F^{-1}(1-\frac{c}{\lambda}))$, the difference between the payoffs of Firm 1 and Firm 2 is[14]

$$D^F(T, a_1^I) := e^{-\lambda T[1-F(a_1^I)]} - (1 - e^{-\lambda T[1-F(a_1^I)]})(1 - \frac{c}{\lambda[1-F(a_1^I)]}). \tag{1.2}$$

The head start of Firm 1 favors Firm 1 (Firm 2) if $D^F(T, a_1^I) > (<) 0$.

$D^F(T, a_1^I)$ is increasing in a_1^I and decreasing in T. Hence, a longer deadline tends toward to favor Firm 2 when the head start is in the middle range. Since

$$D^F(0, a_1^I) = 1 > 0$$

[14] $1 - e^{-\lambda T[1-F(a_1^I)]}$ is Firm 2's probability of obtaining an innovation better than Firm 1's initial innovation, a_1^I, and $\frac{1}{\lambda[1-F(a_1^I)]}$ is the unconditional expected interarrival time of innovations with a value higher than a_1^I. The second term in $D^F(T, a_1^I)$ thus represents the expected payoff of Firm 2.

and

$$\lim_{T\to\infty} D^F(T, a_1^I) = -(1 - \frac{c}{\lambda[1-F(a_1^I)]}) < 0 \text{ for any } a^I < F^{-1}(1-\frac{c}{\lambda}),$$

there must be a unique $\hat{T}(a_1^I) > 0$ such that $DE(\hat{T}(a_1^I), a_1^I) = 0$. The following result derives.

Proposition 1.1. *For $a_1^I \in (a_L^*, F^{-1}(1-\frac{c}{\lambda}))$, there is a unique $\hat{T}(a_1^I) > 0$ such that Firm 1 (Firm 2) obtains a higher expected payoff if $T < (>)\hat{T}(a_1^I)$.*

That is, for any given value of the head start in the middle range $(a_L^*, F^{-1}(1-\frac{c}{\lambda}))$, the head start favors the latecomer (head starter) if the deadline is long (short). The effects of a head start do not vanish as the deadline approaches infinity. In fact, as the deadline approaches infinity, the head start eventually pushes the whole share of the surplus to Firm 2.

Lemma 1.3. *As the deadline increases to infinity,*

1. *for $a_1^I \in (0, a_L^*)$, both Firms' equilibrium payoffs converge to 0;*
2. *for $a_1^I \in (a_L^*, F^{-1}(1-\frac{c}{\lambda}))$, Firm 1's equilibrium payoff converges to 0, whereas Firm 2's equilibrium payoff converges to $1 - \frac{c}{\lambda[1-F(a_1^I)]} \in (0, \frac{1}{2})$.*

Proof. [1] follows from Lemma 1.2. [2] follows from Corollary 1.2 and the limit of Firm 2's expected payoff

$$(1 - e^{-\lambda T[1-F(a^I)]})\left(1 - \frac{c}{\lambda[1-F(a_1^I)]}\right) \tag{1.3}$$

w.r.t. T. □

A comparison to Lemma 1.2 shows that, just as having no initial innovation, when Firm 1 has an innovation whose value is not of very high, its expected payoff still converges to 0 as the deadline becomes excessively long. When there is no initial innovation, the expected total surplus for the firms (i.e., the sum of the two firms' expected payoff) converges to 0. In contrast, when there is a head start with a value above a_L^*, the expected total surplus is strictly positive even when the deadline approaches infinity. However, as it approaches infinity, this total surplus created by the head start of Firm 1 goes entirely to Firm 2, the latecomer, if the head start is in the middle range. The intuition is as follows. For Firm 1, it is clear that its probability of winning converges to 0 as the deadline goes to infinity. For Firm 2, we first look at the case that $a_1^I = F^{-1}(1-\frac{c}{\lambda})$. In this case Firm 2 is indifferent between searching and not searching, and thus the expected payoff is 0. As the deadline approaches infinity, both expected cost of searching and the probability of winning converges to 1, if Firm 2 conducts a search. Then, if a_1^I is below $F^{-1}(1-\frac{c}{\lambda})$ (but above a_L^*), as the deadline approaches infinity, Firm 2's probability of winning still goes to 1, but the expected cost of searching drops to a value below 1 because it adopts a lower cut-off for stopping. Hence, Firm 2's expected payoff converges to a positive value.

The relationship between the rank order of the two Firms' payoffs and the deadline is illustrated in figure 1.5, in each of which Firm 2 obtains a higher expected payoff at each point in the colored area. (a) is for the cases in which $c \leq \frac{\lambda}{2}$. In these cases a longer deadline tends to favor the latecomer. (b) and (c) are for the cases in which $c > \frac{\lambda}{2}$. In these cases, the rank order is not generally monotone in the deadline and the head start.

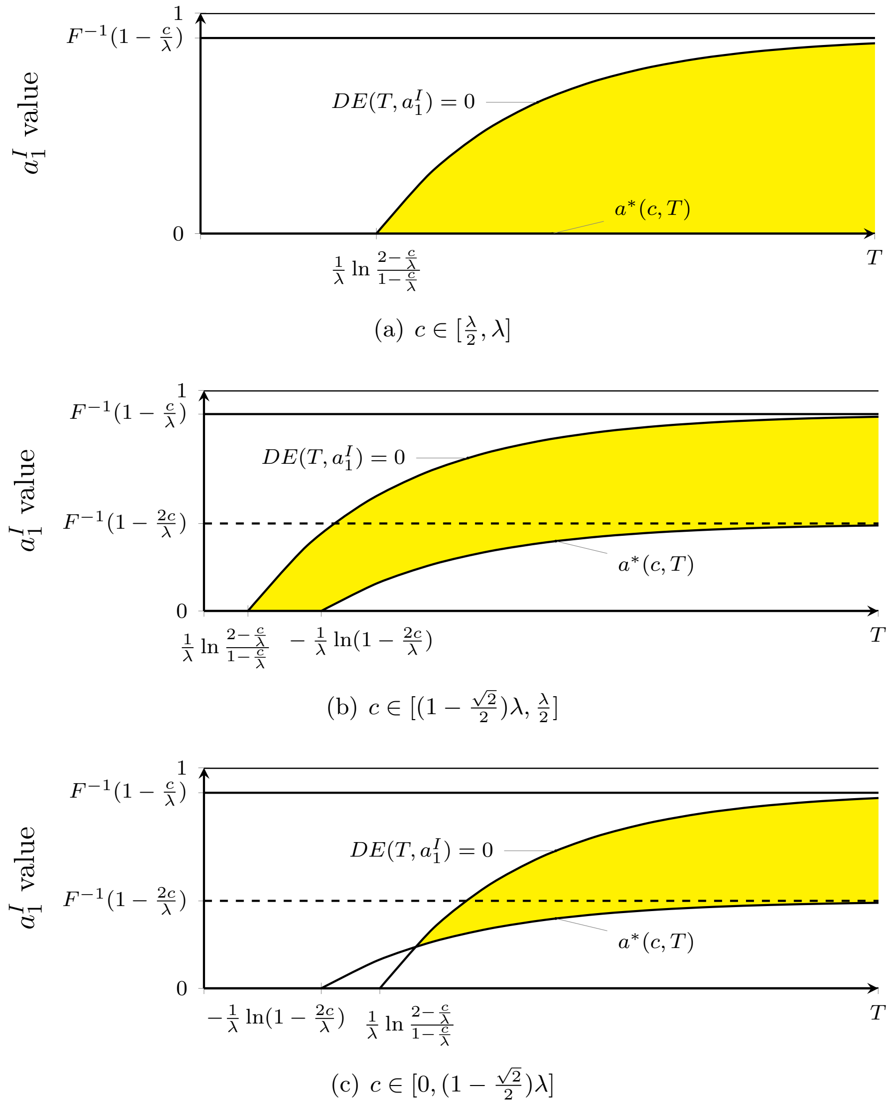


Figure 1.5: When (a_1^I, T) lies in the colored area, the head start favors the latecomer.

We notice in all the figures that if the deadline is sufficiently short, a head start is ensured to bias toward Firm 1, whereas if it is long, only a relatively large head start biases toward Firm 1.

1.4 Effects of Head Starts on Payoffs

In this section, we study the effects of a head start on both firms' payoffs. Suppose Firm 2 has no initial innovation, who does a head start of Firm 1 benefit or hurt? The previous comparison between Theorem 1.1 and Theorem 1.2 already shows that a head start a_1^I benefits Firm 1 and hurts Firm 2 if $a_1^I < a^*(c,T)$ or $a_1^I > F^{-1}(1-\frac{c}{\lambda})$. In the former case, which happens only when $a^*(c,T) > 0$ (Region 3 of figure 1.1), both firms search with $a^*(c,T)$ as the cut-off, the same as when there is no head start, and the head start increases Firm 1's probability of winning and decreases Firm 2's. As the deadline goes to infinity, the expected payoffs to both firms converge to 0, with the effect of the head start disappearing. In the latter case, Firm 1 always obtains a payoff of 1, and Firm 2 always 0.

The interesting case occurs then when the head start is in the middle range, $a_1^I \in (a^*(c,T), F^{-1}(1-\frac{c}{\lambda}))$, which will be the focus in the remaining parts of this study. To answer the above question regarding the head start being in the middle range, we first analyze the case that point (c,T) lies in Regions 2 and 3 (in figure 1.1), and then we turn to analyze the case of Region 1.

1.4.1 Regions 2 and 3

In the previous section, we showed that for T being sufficiently long, Firm 1 is almost surely going to lose the competition if a_1^I is in the middle range. Although it seems reasonable that in this case a head start may make Firm 1 worse off, the following proposition shows that this conjecture is not true.

Proposition 1.2. *Suppose $a_2^I = 0$. In Regions 2 and 3, in which $c < \frac{1}{2}\lambda(1+e^{-\lambda T})$, a head start $a_1^I > 0$ always benefits Firm 1, compared to the equilibrium payoff it gets in the benchmark case.*

To give the intuition, we consider the case of $a^*(c,T) > 0$. Suppose Firm 1 has a head start $a_1^I = a^*(c,T)$. As shown in Case [4] of Theorem 1.2, we have the following two equilibria: in one equilibrium both firms search with $a^*(c,T)$ as the cut-off; in another equilibrium Firm 1 does not search and Firm 2 searches with $a^*(c,T)$ as the cut-off. Firm 1 is indifferent between these two equilibria, hence its expected payoffs from both equilibria are $e^{-\lambda T[1-F(a^*(c,T))]}$, the probability of Firm 2 finding no innovation with a value higher than $a^*(c,T)$. However, Firm 1's probability of winning increases in its head start, hence a larger head start gives Firm 1 a higher expected payoff.

The above result itself corresponds to expectation. What unexpected is the mechanism through which Firm 1 gets better off. As a head start gives Firm 1 a higher position, we would expect that it is better off by (1) having a better chance to win and (2) spending less on searching. Together with Theorem 1.2, the above proposition shows that Firm 1 is better off purely from an increase in the probability of winning when $a_1^I < a^*(c,T)$; purely from spending nothing on searching when $a_1^I \in (a^*(c,T), F^{-1}(1-\frac{c}{\lambda}))$ (though there could be a loss from a decrease in the probability of winning); from an increase in the probability of winning and a reduction in the cost of searching when $a_1^I > F^{-1}(1-\frac{c}{\lambda})$.

In contrast to the effect of a head start of Firm 1 on Firm 1's own expected payoff, the effect on Firm 2's expected payoff is not clear-cut. Instead of giving a general picture of the effect, we present some properties in the following.

Proposition 1.3. *Suppose* $a_2^I = 0$.

1. *A head start* $a_1^I \in (0, F^{-1}(1-\frac{c}{\lambda}))$ *hurts Firm 2 if the deadline* T *is sufficiently small.*
2. *If* $c < \frac{\lambda}{2}$, *a head start* $a_1^I \in (a_L^*, F^{-1}(1-\frac{c}{\lambda}))$ *benefits Firm 2 if the deadline is sufficiently long.*

Proof. See Appendix 1.A.3. □

Case [1] occurs because a head start of Firm 1 reduces Firm 2's probability of winning and may increase its expected cost of searching. Case [2] follows from Propositions 1.2 and 1.1. Because a head start of Firm 1 always benefits Firm 1 and a long deadline favors Firm 2, a head start must also benefit Firm 2 if the deadline is long.[15]

Figure 1.6 illustrates how Firm 2's equilibrium payoff changes as Firm 1's head start increases. In particular, a head start of Firm 1 slightly above $a^*(c,T)$, the equilibrium cut-off when there is no initial innovation, benefits Firm 2 if $a^*(c,T)$ is low. Some more conditions under which a head start benefits or hurts the latecomer are given below.

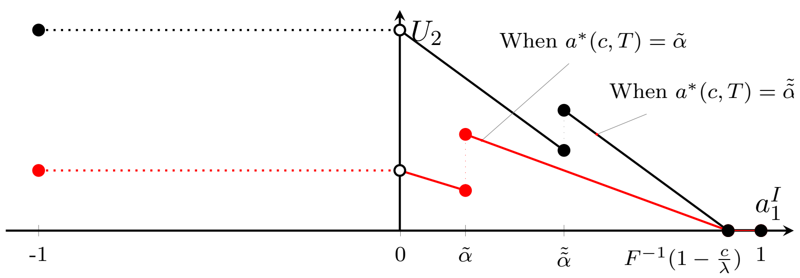


Figure 1.6: Firm 2's equilibrium payoffs as $a^*(c,T)$ varies.

Proposition 1.4. *In Region* 2 *and* 3, *in which* $c < \frac{1}{2}\lambda(1+e^{-\lambda T})$,

[15] Alternatively, it also follows from Lemmas 1.2 and 1.3. If the deadline is very long and the head start of Firm 1 is in the middle range, Firm 2's payoff converges to 0 same as in the benchmark case and some positive value in head start case.

1. if

$$(1-e^{-\lambda T[1-F(a^*(c,T))]})-\frac{1}{2}(1-e^{-2\lambda T})>0, \tag{1.4}$$

there exists a $\tilde{a}_1^I \in (a^*(c,T), F^{-1}(1-\frac{c}{\lambda}))$ such that the head start a_1^I hurts Firm 2 if $a_1^I \in (\tilde{a}_1^I, F^{-1}(1-\frac{c}{\lambda}))$ and benefits Firm 2 if $a_1^I \in (a^*(c,T), \tilde{a}_1^I)$;

2. if (1.4) holds in the opposite direction, any head start $a_1^I \in (a^*(c,T), F^{-1}(1-\frac{c}{\lambda}))$ hurts Firm 2.

Proof. See Appendix 1.A.3. □

The first term on the left side of inequality (1.4) is Firm 2's probability of winning in the equilibrium in which Firm 2 searches and Firm 1 does not search in the limiting case that Firm 1 has a head start of $a^*(c,T)$. The second term, excluding the minus sign, is Firm 1's probability of winning when there is no head start. The expected searching costs are the same in both cases. The following corollary shows some scenarios in which inequality (1.4) holds.

Corollary 1.3. *In Region 2, when $a^*(c,T)=0$, inequality (1.4) holds.*

This shows that for search cost lying in the middle range, a head start of Firm 1 must benefit Firm 2, if it is slightly above 0. The simple intuition is as follows. When Firm 1 has such a small head start, Firm 2's cut-off value of searching increases by only a little bit, and thus the expected cost of searching also increases slightly. However, the increase in Firm 2's probability of winning is very large, because Firm 1, when having a head start, does not search any more. Thus, in this case Firm 2 is strictly better off.

Lastly, even though Firm 1 does not search when the head start $a_1^I > a^*(c,T)$, it seems that a low search cost may benefit Firm 2. On the contrary, a head start of Firm 1 would always hurt Firm 2 when the search cost is sufficiently small.

Corollary 1.4. *For any fixed deadline T, if the search cost is sufficiently small, inequality (1.4) holds in the opposite direction.*

Proof. As c being close to 0, $a^*(c,T)$ is close to 1, and thus the term on left side of inequality (1.4) is close to $-\frac{1}{2}(1-e^{-2\lambda T})<0$. □

That is because when c is close to 0, $a^*(c,T)$ is close to 1, and the interval in which Firm 1 does not search while Firm 2 searches is very small, and thus the chance for Firm 2 to win is too low when $a_1^I > a^*(c,T)$, even though the expected cost of searching is low as well.

1.4.2 Region 1

Since there are multiple equilibria in the benchmark case when (c, T) lies in Region 1, whether a head start hurts or benefits a firm depends on which equilibrium we compare to. If we compare the two equilibria in each of which Firm 1 does not search and Firm 2 searches, then the head start benefits Firm 1 and hurts Firm 2. If we compare to the other equilibrium in the benchmark case, the outcome is not clear-cut.

Proposition 1.5. *Suppose $a_2^I = 0$. In Region 1, in which $c > \frac{1}{2}\lambda(1 + e^{-\lambda T})$ and $a^*(c, T) = 0$, for $a_1^I \in (0, F^{-1}(1 - \frac{c}{\lambda}))$, if*

$$(1 - e^{-\lambda T})(1 - \frac{c}{\lambda}) - e^{-\lambda T[1-F(a_1^I)]} < 0, \tag{1.5}$$

Firm 1's equilibrium payoff is higher than its expected payoff in any equilibrium in the benchmark case. If the inequality holds in the opposite direction, Firm 1's equilibrium payoff is lower than its payoff in the equilibrium in which Firm 1 searches and Firm 2 does not search in the benchmark case.

This result is straightforward. The first term on the left side of inequality (1.5) is Firm 1's expected payoff in the equilibrium in which Firm 1 searches and Firm 2 does not in the benchmark case and the second term, excluding the minus sign, is its expected payoff when there is no head start.

Moreover, the left hand side of inequality (1.5) strictly increases in T, and it reaches -1 when T approaches 0 and $1 - \frac{c}{\lambda}$ when T approaches infinity. The intermediate value theorem insures that inequality (1.5) holds in the opposite direction for the deadline T being large.

As a result of the above property, when the head start is small and the deadline is long, in an extended game in which Firm 1 can publicly discard its head start before the contest starts, there are two subgame perfect equilibria: in one equilibrium, Firm 1 does not discard its head start and Firm 2 searches with the Firm 1's initial innovation value as the cut-off; in the other equilibrium, Firm 1 discards the head start and searches with 0 as the cut-off and Firm 2 does not search. Hence, there is the possibility that Firm 1 can improve its expected payoff if it discards its head start.

Last, we discuss Firm 2's expected payoff. The result is also straightforward.

Proposition 1.6. *Suppose $a_2^I = 0$. In Region 1, in which $c > \frac{1}{2}\lambda(1 + e^{-\lambda T})$, for $a_1^I \in (0, F^{-1}(1 - \frac{c}{\lambda}))$, Firm 2's equilibrium payoff is*

- *less than its expected payoff in the equilibrium in which Firm 1 does not search and Firm 2 searches in the benchmark case, and*

- *higher than the payoff in the equilibrium in which Firm* 1 *searches and Firm* 2 *does not search in the benchmark case.*

Proof. Compared to the equilibrium in which Firm 2 searches in the benchmark case, in the equilibrium when Firm 1 has a head start, Firm 2 has a lower expected probability of winning and a higher expected cost because of a higher cut-off, and and thus a lower expected payoff. But this payoff is positive. □

1.5 Extended Dynamic Models ($a_1^I > a_2^I$)

In this section, we study two extended models.

1.5.1 Endogenous Head Starts

We first study how the firms would play if each firm has the option to discard its initial innovation before the contest starts. Formally, a game proceeds as below.

Model*:

- Stage 1: Firm i decides whether to discard its initial innovation.
- Stage 2: Firm i's opponent decides whether to discard its initial innovation.
- Stage 3: Upon observing the outcomes in the previous stages, both firms simultaneously start playing the contest as described before.

The incentive for a head starter to discard its initial innovation when the latecomer has no initial innovation has been studied in the previous section. The focus of the section is on the case in which both firms have an initial innovation in the middle range.[16] The main result of this section is as follows.

Proposition 1.7. *Suppose $a_1^I, a_2^I \in (a_L^*, F^{-1}(1-\frac{c}{\lambda}))$. In Model* with Firm 1 having the first move, there is a $\breve{T}(a_1^I, a_2^I) > 0$ such that*

- *if $T > \breve{T}(a_1^I, a_2^I)$, there is a unique subgame perfect equilibrium (SPE), in which Firm* 1 *discards its initial innovation and searches with a_2^I as the cut-off and Firm* 2 *keeps its initial innovation and does not search;*
- *if $T < \breve{T}(a_1^I, a_2^I)$, subgame perfect equilibria exist, and in each equilibrium Firm* 2 *searches with a_1^I as the cut-off and Firm* 1 *keeps its initial innovation and does not search.*

[16] As shown in Proposition 1.2, when (c, T) lies in Regions 2 and 3, the head starter with a medium-sized initial innovation has no incentive to discard its head start if the latecomer has a no innovation. The head starter would also have no incentive to discard its initial innovation when the latecomer has a low-value initial innovation.

Proof. See Appendix 1.A.4. □

This proposition shows that in the prescribed scenario Firm 1 is better off giving up its initial innovation if the deadline is long.[17] The intuition is simple. For the deadline being long, Firm 1's expected payoff is low, because its probability of winning is low. By giving up its initial innovation, it makes Firm 2 the head starter, and thus Firm 1 obtains a higher expected payoff than before by committing to searching whereas Firm 2 has no incentive to search. Yet the reasoning for the case in which $c \leq \frac{\lambda}{2}$ differs from that for the case in which $c > \frac{\lambda}{2}$. After Firm 1 discards its initial innovation, Firm 2 turns to the new head start firm. In the former case, Firm 2 would then have no incentive to discard its initial innovation any more as shown in Proposition 1.2, and its dominant strategy in the subgame is not to search whether Firm 1 is to search or not. In the latter case, Firm 2 may have the incentive to discard its initial innovation and search if the deadline is long. However, discarding the initial innovation is a credible threat for Firm 1 to deter Firm 2 from doing that.

Remark. *When the deadline is sufficiently long, by giving up the initial innovation, Firm* 1 *makes itself better off but Firm* 2 *worse off. However, if the deadline is not too long, by doing so, Firm* 1 *can benefit both firms. This is because the total expected cost of searching after Firm 1 discards its initial innovation is lower than before and hence there is an increase in the total surplus for the two firms. It is then possible that both firms get a share of the increase in the surplus. We illustrate that in the following example.*

[17] Discarding a head start is one way to give up one's initial leading position. In reality, a more practical and credible way is to give away the head start innovation. A head start firm could give away it patent for its technology. By doing so, any firm can use this technology for free. That is, every firm's initial state becomes a_1^I. If firms can enter the competition freely, the value of the head start technology is approximately zero to any single firm, because everyone has approximately zero probability to win with this freely obtained innovation. For a head start being in the middle range, the market is not large enough to accommodate two firms to compete. Hence, to model giving away head starts with free entry to the competition, we can study a competition between two firms but with some modified prize allocation rules. Formally, the game proceeds as below.

Model**:

- Stage 1: Firm 1 decides whether to give away its initial innovation.
- Stage 2: Upon observing the stage 1 outcome, Firm 1 and Firm 2 simultaneously start playing the contest as described before.
- If Firm 1 gives away its initial innovation:
  - Both firms' states at time 0 become a_1^I.
  - The prize is retained if no firm is in a state above a_1^I at the deadline T.
  - The firm with a higher state, which is higher than a_1^I, at the deadline wins.
- If Firm 1 retains its initial innovation, the firm with a higher state at the deadline wins.

Suppose $a_1^I \in (a_L^*, F^{-1}(1 - \frac{c}{\lambda}))$. If the deadline is long, in Model** there are two subgame perfect equilibria. In one equilibrium, Firm 1 gives away its initial innovation and searches with a_1^I as the cut-off and Firm 2 does not search. In the other equilibrium, Firm 1 retains its initial innovation and does not search and Firm 2 searches with a_1^I as the cut-off. However, forward induction selects the first equilibrium as the refined equilibrium, because giving away a head start is a credible signal of Firm 1 to commit to search.

Example 1.1. *Suppose F is the uniform distribution, $c = \frac{1}{3}$, $\lambda = 1$, $a_1^I = \frac{1}{2}$, and $a_2^I = \frac{1}{3}$. If Firm 1 discards its initial innovation, then its expected payoff would be $\frac{1}{2}(1 - e^{-\frac{T}{3}})$, and Firm 2's expected payoff would be $e^{-\frac{T}{3}}$; if Firm 1 does not discard its initial innovation, then its expected payoff would be $e^{-\frac{T}{2}}$, and Firm 2's expected payoff would be $\frac{1}{3}(1 - e^{-\frac{T}{2}})$.*

Firm 1 would be better off by discarding its initial innovation if $T > 2.52$. If $T \in (2.52, 3.78)$, by discarding the initial innovation, Firm 1 makes both firms better off. If T is larger, then doing so would only make Firm 2 worse off.

The previous result is conditional on Firm 1 having the first move. If Firm 2 has the first move, it may, by discarding its initial innovation, be able to prevent Firm 1 from discarding its own head start and committing to searching. However, if the deadline is not sufficiently long, Firm 1 would still have the incentive to discard its initial innovation.

Proposition 1.8. *Suppose $a_1^I, a_2^I \in (a_L^*, F^{-1}(1 - \frac{c}{\lambda}))$. In Model* with Firm 2 having the first move, there is a $\hat{T}(a_1^I, a_2^I) > 0$ such that*

- *for $T > \hat{T}(a_1^I, a_2^I)$, there is a unique SPE, in which Firm 2 discards its initial innovation and searches with a_2^I as the cut-off and Firm 1 keeps the initial innovation and does not search;*
- *for $T \in (\check{T}(a_1^I, a_2^I), \hat{T}(a_1^I, a_2^I))$, there is a unique SPE, in which Firm 2 keeps its initial innovation and does not search and Firm 1 discards its initial innovation and searches with a_1^I as cut-off;*
- *for $T < \check{T}(a_1^I, a_2^I)$, subgame perfect equilibria exist, and in each equilibrium Firm 2 searches with cut-off a_1^I and Firm 1 keeps its initial innovation and does not search.*

Proof. See Appendix 1.A.4. □

In the middle range of the deadline, even though Firm 2 can credibly commit to searching and scare Firm 1 away from competition by discarding its initial innovation, it is not willing to do so, yet Firm 1 would like to discard its initial innovation and commit to searching. This is because Firm 1's initial innovation is of a higher value than Firm 2's. The cut-off value of the deadline at which Firm 2 is indifferent between discarding the initial innovation to commit to searching, and keeping the initial innovation, is higher than that of Firm 1.

1.5.2 Intermediate Information Disclosure

In the software industry, it is common to preannounce with a long lag to launch (Bayus et al., 2001). Many firms do that by describing the expected features or demonstrating prototypes at trade shows. Many other firms publish their findings in a commercial disclosure service, such as Research Disclosure, or in research journals.[18] Suppose there is a regulator who would like to impose an intermediate information disclosure requirement on innovation contests. What are the effects of the requirement on firms' competition strategies and the expected value of the winning innovation. Specifically, suppose there is a time point $t_0 \in (0, T)$ at which both firms have to reveal everything they have, how would firms compete against each other?

When the head start a_1^I is larger the threshold $F^{-1}(1 - \frac{c}{\lambda})$, it is clear that no firm has an incentive to conduct any search. When the head start is below this threshold, if t_0 is very close to T, information revelation has little effect on the firms' strategies. Both firms will play approximately the same actions before time t_0 as they do when there is no revelation requirement. After time t_0, the firm in a higher state at time t_0 stops searching. The other firm searches with this higher state as the cut-off if this higher state is below $F^{-1}(1 - \frac{c}{\lambda})$, and stops searching as well if it is higher than $F^{-1}(1 - \frac{c}{\lambda})$.

Our main finding in this part regards the cases in which the head start is in the middle range and the deadline is sufficiently far from the information revelation point.[19] That is, firms have to reveal their progress at an early stage of a competition.

Proposition 1.9. *Suppose at a time point $t_0 \in (0, T)$ both firms have to reveal their discoveries. For $a_1^I \in (a_L^*, F^{-1}(1 - \frac{c}{\lambda}))$ and $a_2^I < a_1^I$, if $T - t_0$ is sufficiently large, there is a unique subgame perfect equilibrium, in which*

- *Firm 1 searches with $F^{-1}(1 - \frac{c}{\lambda})$ as the cut-off before time t_0 and stops searching from time t_0;*
- *Firm 2 does not search before time t_0 and searches with a_1^I as the cut-off from time t_0 if $a_1^I < F^{-1}(1 - \frac{c}{\lambda})$.*

Proof. See Appendix 1.A.4. □

In the proof, we show that between time 0 and time t_0, the dominant action of Firm 2 is not to search, given that the equilibria in the subgames from time t_0 are described as in Theorem 1.2. If Firm 1 is in a state higher than the threshold $F^{-1}(1 - \frac{c}{\lambda})$ at time t_0, Firm 2's effort will be futile if it searches before time t_0. If Firm 1 is in a state in between its initial state a_1^I and and the threshold $F^{-1}(1 - \frac{c}{\lambda})$ at time t_0, Firm 2 has the chance to get into a state above the threshold $F^{-1}(1 - \frac{c}{\lambda})$ and thus a continuation payoff of 1, but this instantaneous benefit only compensates the instantaneous cost of searching. Firm 2 also has the chance to get into a state above that of Firm 1 but below the threshold $F^{-1}(1 - \frac{c}{\lambda})$ at time t_0, which

[18]Over 90% of the world's leading companies have published disclosures in Research Disclosure's pages (see www.researchdisclosure.com).

[19]Generally, for the cases in which $a_1^I < a_L^*$, there are many subgame perfect equilibria, including two equilibria in each of which one firm searches with $F^{-1}(1 - \frac{c}{\lambda})$ as the cut-off between time 0 and time t_0 and the other firm does not.

results in a continuation payoff of approximately 0 if the deadline is sufficiently long, whereas it obtains a strictly positive payoff if it does not search before time t_0. It is thus not worthwhile for Firm 2 to conduct a search before time t_0. If Firm 2 does not search before time t_0, Firm 1 then has the incentive to conduct a search if the deadline is far from t_0. If it does not search, it obtains a payoff of approximately 0 when $T - t_0$ is sufficiently large. If it conducts a search before time t_0, the benefit from getting into a higher state can compensate the cost.

An early stage revelation requirement therefore hurts the latecomer and benefits the head starter. It gives the head starter a chance to get a high-value innovation so as to deter the latecomer from competition. It also increases the expected value of the winning innovation.

1.6 Discussion: Asymmetric Costs ($a_1^I = a_2^I = 0$, $c_1 < c_2$)

In this section, we show that, compared to the effects of head starts, the effects of cost advantages are simpler. A head start probably discourages a firm from conducting searching and can either discourage its competitor from searching or encourage its competitor to search more actively. In contrast, a cost advantage encourages a firm to search more actively and discourages its opponent.

We now assume that the value of pre-specified prize to Firm i, $i = 1, 2$, is V_i and that the search cost is for Firm i is C_i per unit of time. However, at each time point Firm i only makes a binary decision on whether to stop searching or to continue searching. Whether it is profitable to continue searching depends on the ratio of $\frac{C_i}{V_i}$ rather than the scale of V_i and C_i. Therefore, we can normalize the valuation of each player to be 1 and the search cost to be $\frac{C_i}{V_i} =: c_i$. W.l.o.g, we assume $c_1 < c_2$. For convenience, we define a function

$$I(a_i|a_j, c_i) := \lambda \int_{a_i}^{\bar{a}} [Z(a|a_j) - Z(a_i|a_j)] dF(a) - c_i,$$

where $Z(a|a_j)$ is defined, in Lemma 4.2 in the appendix, as Firm j's probability of ending up in a state **below** a if it searches with a_j as the cut-off. We emphasize on the most important case, in which both firms' search costs are low.

Proposition 1.10. *For $0 < c_1 < c_2 < \frac{1}{2}\lambda(1 - e^{-\lambda T})$, there must exist a unique equilibrium (a_1^*, a_2^*), in which $a_1^* > a_2^* \geq 0$. Specifically,*

1. *if $I\left(0|F^{-1}\left(1 - \sqrt{\frac{2c_1}{\lambda(1-e^{-\lambda T})}}\right), c_2\right) > 0$, the unique equilibrium is a pair of cut-off rules (a_1^*, a_2^*), $a_1^* > a_2^* > 0$, that satisfy*

$$\lambda \int_{a_i^*}^{\bar{a}} \left[Z(a|a_j^*, T) - Z(a_i^*|a_j^*, T)\right] dF(a) = c_i;$$

2. *if* $I\left(0|F^{-1}\left(1-\sqrt{\frac{2c_1}{\lambda(1-e^{-\lambda T})}}\right), c_2\right) \leq 0$, *the unique equilibrium is a pair of cut-off rules* $\left(F^{-1}\left(1-\sqrt{\frac{2c_1}{\lambda(1-e^{-\lambda T})}}\right), 0\right)$.

Proof. See Appendix 1.A.5. □

The existence of equilibrium is proved by using Brouwer's fixed point theorem. As expected, a cost (valuation) advantage would drive a firm to search more actively than its opponent. The following statement shows that while an increase in cost advantage of the firm in advantage would make the firm more active in searching and its opponent less active, a further cost disadvantage of the firm in disadvantage would make both firms less active in searching.

Proposition 1.11. *For* $0 < c_1 < c_2 < \frac{1}{2}\lambda(1-e^{-\lambda T})$ *and* $I\left(0|F^{-1}\left(1-\sqrt{\frac{2c}{\lambda(1-e^{-\lambda T})}}\right), c_2\right) > 0$, *in which case there is a unique equilibrium* (a_1^*, a_2^*), $a_1^*, a_2^* > 0$,

1. *for fixed* c_2, $\frac{\partial a_1^*}{\partial c_1} < 0$ *and* $\frac{\partial a_2^*}{\partial c_1} > 0$;
2. *for fixed* c_1, $\frac{\partial a_1^*}{\partial c_2} < 0$ *and* $\frac{\partial a_2^*}{\partial c_2} < 0$.

Proof. See Appendix 1.A.5. □

The intuition is simple. When the cost of the firm in advantage decreases, this firm would be more willing to search, while the opponent firm would be discouraged because the marginal increase in the probability of winning from continuing searching in any state is reduced, and therefore the opponent firm would lower its cut-off. When the cost of the firm at a disadvantage increases, the firm would be less willing to search, and the opponent firm would consider it less necessary to search actively because the probability of winning in any state has increased.

A comparison between the equilibrium strategies in this model and that of the benchmark model can be made.

Corollary 1.5. *For* $0 < c_1 < c_2 < \frac{1}{2}\lambda(1-e^{-\lambda T})$ *and* $I\left(0|F^{-1}\left(1-\sqrt{\frac{2c}{\lambda(1-e^{-\lambda T})}}\right), c_2\right) > 0$, *in which case there is a unique equilibrium* (a_1^*, a_2^*), $a_1^* > a_2^* > 0$, a_1^* *and* a_2^* *satisfy*

1. $a_1^* < a^*(c, T)$ *for the corresponding* $c = c_2 > c_1$;
2. $a_2^* < a^*(c, T)$ *for the corresponding* $c = c_1 < c_2$.

Based on the benchmark model, a cost reduction for Firm 1 will result in both firms searching with cut-offs below the original one; a cost increase for Firm 2 will certainly result in Form 2 searching with a cut-off below the original one.

The equilibrium for the other non-marginal cases (conditional on $c_1 < c_2$), together with the above case, are stated in Table 1.1 without proof. The regions in Table 1.1 are the same as in Figure 1.1. The row (column) number indicates in which region c_1 (c_2) lies, and each element

$c_2 \ \backslash \ c_1$	Region 1	Region 2	Region 3
Region 1	(a_1^*, a_2^*)	$\left(F^{-1}\left(1-\sqrt{\frac{2c_1}{\lambda(1-e^{-\lambda T})}}\right), 0\right)$	$(0, -1)$
Region 2	/	$(0, 0)$	$(0, -1)$
Region 3	/	/	$(0, -1), (-1, 0)$

Table 1.1: Equilibria in all non-marginal cases.

in each cell represents a corresponding equilibrium. For example, the element in the cell at the second row and the second column means that for $c_1, c_2 \in (\frac{1}{2}\lambda(1-e^{-\lambda T}), \frac{1}{2}\lambda(1+e^{-\lambda T}))$, there is a unique equilibrium, in which both firms search with cut-off 0. This shows that Firm 1 is more active in searching than is Firm 2.

Remark. *Similar results can be found in a model with the same search cost but with different arrival rates of innovations for the two firms.*

1.7 Concluding Remarks

In this chapter, we studied the long-run effects of head starts in innovation contests in which each firm decides when to stop a privately observed repeated sampling process before a preset deadline. Unlike an advantage in innovation cost or innovation ability, which encourages a firm to search more actively for innovations and discourages its opponent, a head start has non-monotone effects. The head starter is discouraged from searching if the head start is large, and its strategy remains the same if the head start is small. The latecomer is discouraged from searching if the head start is large but is encouraged to search more actively if it is in the middle range. Our main finding is that, if the head start is in the middle range, in the long run, the head starter is doomed to lose the competition with a payoff of zero and the latecomer will take the entire surplus for the competing firms. As a consequence, our model can exhibit either the "preemption effect" or the "replacement effect", depending on the value of the head start.

Our results have implications on antitrust problems. Market regulators have concerns that the existence of market dominating firms, such as Google, may hinder competitions, and they take measures to curb the monopoly power of these companies. For instance, the European Union voted to split Google into smaller companies.[20] Our results imply that in many cases the positions of the dominant firms are precarious. In the long run, they will be knocked off their perch. These firms' current high positions, in fact, may promote competitions in

[20] "Google break-up plan emerges from Brussels," Financial Times, November 21, 2014.

the long run because they encourage their rivals to exert efforts to innovate and reach high targets. Curbing the power of the current dominating firms may benefit the society and these firms' rivals in the short run, but in the long run it hurts the society because it discourages innovation. However, the the dominating firms' positions are excessively high, which deters new firms from entering the the market, a market regulator could take some actions.

The results have also implications on R&D policies. When selecting an R&D policy, policy makers have to consider both the nature of the R&D projects and the market structure. If the projects are on radical innovations, subsidizing innovation costs effectively increases competition when the market is blank (no advanced substitutive technology exists in the market). However, when there is a current market dominating firm with an existing advanced technology, a subsidy may not be effective. The dominating firm has no incentive to innovate, and the latecomer, even if it is subsidized, will not innovate more actively.

In our model we have only one head starter and one latecomer. The model can be extended to include more than two firms, and similar results still hold. One extension is to study the designing problem in our framework. For example, one question is how to set the deadline. If the designer is impatient, she may want to directly take the head starter's initial innovation without holding a contest; if she is patient, she may set a long deadline in order to obtain a better innovation. Some other extensions include: to consider a model with a stochastic number of firms; to consider a model with cumulative scores with or without regret instead of a model with repeated sampling.

1.A Appendix

1.A.1 Preliminaries

To justify Assumption 2.1, we show in the following that, for any given strategy played by a firm's opponent, there is a constant cut-off rule being the firm's best response. If the cut-off value is above zero, it is actually the unique best response strategy, ignoring elements associated with zero probability events. We argue only for the case that both firms' initial states are 0. The arguments for the other cases are similar and thus are omitted.

Suppose $a_1^I = a_2^I = 0$. For a given strategy played by Firm j, we say at time t

$$\underline{a}_i^t := \inf\{\tilde{a} \in A | \text{Firm } i \text{ weakly prefers stopping to continuing searching in state } \tilde{a}\}$$

is Firm i's **lower optimal cut-off** and

$$\bar{a}_i^t := \inf\{\tilde{a} \in A | \text{Firm } i \text{ strictly prefers stopping to continuing searching in state } \tilde{a}\}$$

is Firm i's **upper optimal cut-off** .

Lemma 1.4. *Suppose $a_1^I = a_2^I = 0$. For any fixed strategy played by Firm j, Firm i's best response belongs to one of the three cases.*

i. Not to search: $\bar{a}_i^t = \underline{a}_i^t = -1$ for all $t \in [0, T]$,

ii. Search with a constant cut-off rule $\hat{a}_i \geq 0$: $\bar{a}_i^t = \underline{a}_i^t = \hat{a}_i \geq 0$ for all $t \in [0, T]$.

iii. Both not to search and search until being in a state above 0: $\bar{a}_i^t = 0$ and $\underline{a}_i^t = -1$ for all $t \in [0, T)$.

Proof of Lemma 4.1. Fix a strategy of Firm j. Let $P(a)$ denote the probability of Firm j ending up in a state **below** a at time T. $P(a)$ is either constant in a or strictly increasing in a. It is a constant if and only if Firm j does not search.[21] If this is the case, Firm i's best response is to continue searching with a fixed cut-off $\hat{a}_i^t = \bar{a}^{t_i} = \underline{a}^{t_i} = 0$ for all t. In the following, we study the case in which $P(a)$ is strictly increasing in a.

Step 1. We argue that, given a fixed strategy played by Firm j, Firm i's best response is a (potentially history-dependent) cut-off rule. Suppose at time t Firm i is in a state $\tilde{a} \in [0, 1]$. If it is strictly marginally profitable to stop (continue) searching at t, then it is also strictly marginally profitable to continue searching if it is in a state higher (lower) than $\tilde{a}$. Let the upper and lower optimal cut-offs at time t be $\bar{a}_i^t$ and $\underline{a}_i^t$, respectively, as defined previously.

Step 2. We show that $\{\bar{a}_i^t\}_{t=0}^T$ and $\{\underline{a}_i^t\}_{t=0}^T$ should be history-independent. We use a discrete version to approximate the continuous version. Take any $\tilde{t} \in [0, T)$. Let $\{t_l\}_{l=0}^k$, where $t_l - t_{l-1} = \frac{T-\tilde{t}}{k} =: \delta$ for $l = 1, ..., k$, be a partition of the interval $[\tilde{t}, T]$. Suppose Firm i can only make decisions at $\{t_l\}_{l=0}^k$ in the interval $[\tilde{t}, T]$. Let $\{\bar{a}^{t_l}\}_{l=0}^{k-1}$ and $\{\underline{a}^{t_l}\}_{l=0}^{k-1}$ be the corresponding upper and lower optimal cut-offs, respectively, and $G^\delta(a)$ be Firm i's probability of discovering **no** innovation with a value **above** a in an interval δ.

At t_{k-1}, for Firm i in a state a, if it stops searching, the expected payoff is $P(a)$; if it continues searching, the expected payoff is

$$
\begin{aligned}
&G^\delta(a)P(a) + \int_a^1 P(\tilde{a}) dG^\delta(\tilde{a}) - \delta c_i \\
=&P(a) + \int_a^1 [P(\tilde{a}) - P(a)] dG^\delta(\tilde{a}) - \delta c_i.
\end{aligned}
$$

The firm strictly prefers continuing searching if and only if searching in the last period strictly increases its expected payoff,

$$
e^\delta(a) := \int_a^1 [P(\tilde{a}) - P(a)] dG^\delta(\tilde{a}) - \delta c_i > 0.
$$

$e^\delta(a)$ strictly decreases in a and $e^\delta(1) \leq 0$. Because $e^\delta(0)$ can be either negative or positive, we have to discuss several cases.

[21] More generally, it is constant if and only if the opponent firm conducts search with a measure 0 over $[0, T]$.

Case 1. If $e^\delta(0) < 0$, Firm i is strictly better off stopping searching in any state $a \in [0,1]$. Thus, $\bar{a}^{t_{k-1}} = \underline{a}^{t_{k-1}} = -1$.
Case 2. If $e^\delta(0) = 0$, Firm i is indifferent between stopping searching and continuing searching with 0 as the cut-off, if it is in state 0; strictly prefers stopping searching, if it is in any state above 0. Then $\bar{a}^{t_{k-1}} = 0$ and $\underline{a}^{t_{k-1}} = -1$.
Case 3. If $e^\delta(0) > 0 \geq \lim_{a\to 0} e^\delta(a)$, Firm i is strictly better off continuing searching in state 0, but stopping searching once it is in a state above 0. Thus, $\bar{a}^{t_{k-1}} = \underline{a}^{t_{k-1}} = 0$.
Case 4. If $\lim_{a\to 0} e^\delta(a) > 0$, then Firm i's is strictly better off stopping searching if it is in a state above $\hat{a}^{t_{k-1}}$ and continuing searching if it is in a state below $\hat{a}^{t_{k-1}}$, where the optimal cut-off $\hat{a}^{t_{k-1}} > 0$ is the unique value of a that satisfies,

$$\int_a^1 [P(\tilde{a}) - P(a)]dG^\delta(\tilde{a}) - \delta c = 0.$$

Thus, in this case $\bar{a}^{t_{k-1}} = \underline{a}^{t_{k-1}} = \hat{a}^{t_{k-1}}$.

Hence, the continuation payoff at $t_{k-1} \geq 0$ for Firm i in a state $a \in [0,1]$ is

$$\omega(a) = \begin{cases} P(a) + \int_a^1 [P(\tilde{a}) - P(a)]dG^\delta(\tilde{a}) - \delta c & for\ a < \underline{a}^{t_{k-1}} \\ P(a) & for\ a \geq \underline{a}^{t_{k-1}}. \end{cases}$$

Then, we look at the time point t_{k-2}. In the following, we argue that $\bar{a}^{t_{k-2}} = \bar{a}^{t_{k-1}}$. The argument for $\underline{a}^{t_{k-2}} = \underline{a}^{t_{k-1}}$ is very similar and thus is omitted.

First, we show that $\bar{a}^{t_{k-2}} \leq \bar{a}^{t_{k-1}}$. Suppose $\bar{a}^{t_{k-2}} > \bar{a}^{t_{k-1}}$. Suppose Firm i is in state $\bar{a}^{t_{k-2}}$ at time t_{k-2}. Suppose Firm i searches between t_{k-2} and t_{k-1}. If it does not discover any innovation with a value higher than $\bar{a}^{t_{k-2}}$, then at the end of this period it stops searching and takes $\bar{a}^{t_{k-2}}$. However, $\bar{a}^{t_{k-2}} > \bar{a}^{t_{k-1}}$ implies

$$\begin{aligned} 0 &= \int_{\bar{a}^{t_{k-2}}}^1 [P(\tilde{a}) - P(\bar{a}^{t_{k-2}})]dG^\delta(\tilde{a}) - \delta c_i \\ &< \int_{\bar{a}^{t_{k-1}}}^1 [P(\tilde{a}) - P(\bar{a}^{t_{k-1}})]dG^\delta(\tilde{a}) - \delta c \leq 0. \end{aligned}$$

The search cost is not compensated by the increase in the probability of winning from searching between t_{k-2} and t_{k-1}, and thus the firm strictly prefers stopping searching to continuing searching at time t_{k-2}, which contradicts the assumption that $\bar{a}^{t_{k-2}}$ is the upper optimal cut-off. Hence, it must be the case that $\bar{a}^{t_{k-2}} \leq \bar{a}^{t_{k-1}}$.

Next, we show that $\bar{a}^{t_{k-2}} = \bar{a}^{t_{k-1}}$.

In *Case 1*, it is straightforward that Firm i strictly prefers stopping searching at t_{k-2}, since it is for sure not going to search between t_{k-1} and t_k. Hence, Firm i stops searching before t_{k-1}, and $\bar{a}^{t_{k-2}} = \bar{a}^{t_{k-1}} = \underline{a}^{t_{k-2}} = \underline{a}^{t_{k-1}} = -1$.

For $\bar{a}^{t_{k-1}} \geq 0$, we prove by contradiction that $\bar{a}^{t_{k-2}} < \bar{a}^{t_{k-1}}$ is not possible. Suppose the inequality holds. If Firm i stops searching at t_{k-2}, it would choose to continue searching at

t_{k-1}, and its expected continuation payoff at t_{k-2} is $\omega(\bar{a}^{t_{k-2}})$. If the firm continues searching, its expected continuation payoff is

$$\omega(\bar{a}^{t_{k-2}}) + \int_{\bar{a}^{t_{k-2}}}^{1} [\omega(a) - \omega(\bar{a}^{t_{k-2}})] dG^{\delta}(a) - \delta c. \tag{1.6}$$

In *Cases 2 and 3*, $\bar{a}^{t_{k-1}} = 0$ implies $\bar{a}^{t_{k-2}} = -1$. Then,

$$\begin{aligned}
&\int_{\bar{a}^{t_{k-2}}}^{1} [\omega(a) - \omega(\bar{a}^{t_{k-2}})] dG^{\delta}(a) - \delta c_i \\
=&\int_{-1}^{1} [P(a)] dG^{\delta}(a) - \delta c_i \\
=&e^{\delta}(-1) \\
\geq&0
\end{aligned}$$

which means that Firm i in state 0 is weakly better off continuing searching between t_{k-2} and t_{k-1}, which implies that $\bar{a}^{t_{k-2}} \geq 0$, resulting in a contradiction.

For *Case 4*, in which $\bar{a}^{t_{k-1}} > 0$, we have in (1.6)

$$\begin{aligned}
&\int_{\bar{a}^{t_{k-2}}}^{1} [\omega(a) - \omega(\bar{a}^{t_{k-2}})] dG^{\delta}(a) \\
=&\int_{\bar{a}^{t_{k-2}}}^{\bar{a}^{t_{k-1}}} \left[\left(P(a) + \int_{a}^{1} [P(\tilde{a}) - P(a)] dG^{\delta}(\tilde{a}) \right) - \left(P(\bar{a}^{t_{k-2}}) + \int_{\bar{a}^{t_{k-2}}}^{1} [P(\tilde{a}) - P(\bar{a}^{t_{k-2}})] dG^{\delta}(\tilde{a}) \right) \right] dG^{\delta}(a) \\
&\quad + \int_{\bar{a}^{t_{k-1}}}^{1} \left[P(a) - \left(P(\bar{a}^{t_{k-2}}) + \int_{\bar{a}^{t_{k-2}}}^{1} [P(\tilde{a}) - P(\bar{a}^{t_{k-2}})] dG^{\delta}(\tilde{a}) \right) \right] dG^{\delta}(a) \\
=&\int_{\bar{a}^{t_{k-2}}}^{1} \left[P(a) - P(\bar{a}^{t_{k-2}}) \right] dG^{\delta}(a) + \int_{\bar{a}^{t_{k-2}}}^{\bar{a}^{t_{k-1}}} \left[\int_{a}^{1} [P(\tilde{a}) - P(a)] dG^{\delta}(\tilde{a}) \right] dG^{\delta}(a) \\
&\quad - \int_{\bar{a}^{t_{k-2}}}^{1} \left[\int_{\bar{a}^{t_{k-2}}}^{1} [P(\tilde{a}) - P(\bar{a}^{t_{k-2}})] dG^{\delta}(\tilde{a}) \right] dG^{\delta}(a) \\
=&G^{\delta}(\bar{a}^{t_{k-2}}) \int_{\bar{a}^{t_{k-2}}}^{1} \left[P(a) - P(\bar{a}^{t_{k-2}}) \right] dG^{\delta}(a) + \int_{\bar{a}^{t_{k-2}}}^{\bar{a}^{t_{k-1}}} \left[\int_{a}^{1} [P(\tilde{a}) - P(a)] dG^{\delta}(\tilde{a}) \right] dG^{\delta}(a) \\
>&0.
\end{aligned}$$

Hence, at t_{k-2} Firm i would strictly prefer continuing searching, which again contradicts the assumption that $\bar{a}^{t_{k-2}}$ is the upper optimal cut-off. Consequently, $\bar{a}^{t_{k-2}} = \bar{a}^{t_{k-1}}$.

By backward induction from t_{k-1} to t_0, we have $\bar{a}^{t_0} = \bar{a}^{t_{k-1}}$. Taking the limit we get

$$\bar{a}^{t} = \lim_{\delta \to 0} \bar{a}^{T-\delta} =: \bar{a} \quad \text{for all } t \in [0, T).$$

Similarly,

$$\underline{a}^{t} = \lim_{\delta \to 0} \underline{a}^{T-\delta} =: \underline{a} \quad \text{for all } t \in [0, T).$$

In addition, $\bar{a} \neq \underline{a}$ when and only when $\bar{a} = 0$ and $\underline{a} = -1$.

As a consequence, Firm i's best response is not to search, if $\bar{a} = \underline{a} = -1$; to continue searching if it is in a state below $\bar{a}$ and to stop searching once the firm in a state above $\bar{a}$, if $\bar{a} = \underline{a} \geq 0$. $\square$

In brief, the above property is proved by backward induction. Take Case [ii] for example. If at the last moment a firm is indifferent between continuing and stopping searching when it is in a certain state, which means the increase in the probability of winning from continuing searching equals the cost of searching, and therefore there is no gain from searching. Immediately before the last moment the firm should also be indifferent between continuing searching and not given the same state. This is because, if the firm reaches a higher state from continuing searching, it weakly prefers not to search at the last moment, and thus the increase in the probability of winning from continuing searching at this moment equals the cost of searching as well. By induction, the firm should be indifferent between continuing and stopping searching in the same state from the very beginning.

In Case [iii], Firm i generally has uncountably many best response strategies. By Assumption 2.1, we rule out most strategies and consider only two natural strategies: not to search at all and to search with 0 as the cut-off.

Lemma 1.5. *Suppose a firm's initial state is* 0*, and she searches with a cut-off* $\hat{a} \geq 0$*. Then, the firm's probability of ending up in a state **below** $a \in [0, 1]$ at time T is*

$$Z(a|\hat{a}, T) = \begin{cases} 0 & if \ a = 0 \\ e^{-\lambda T[1-F(a)]} & if \ 0 < a \leq \hat{a} \\ e^{-\lambda T[1-F(\hat{a})]} + \left[1 - e^{-\lambda T[1-F(\hat{a})]}\right] \frac{F(a)-F(\hat{a})}{1-F(\hat{a})} & if \ a > \hat{a}. \end{cases}$$

$1 - e^{-\lambda T[1-F(\hat{a})]}$ is the probability that the firm stops searching before time T, and $\frac{F(a)-F(\hat{a})}{1-F(\hat{a})}$ is the conditional probability that the innovation above the threshold the firm discovers is in between $\hat{a}$ and a.

Proof of Lemma 4.2. For $a = 0$, it is clear that $Z(a|\hat{a}, T) = 0$.

For $0 < a \leq \hat{a}$,

$$Z(a|\hat{a}, T) = \sum_{l=0}^{\infty} \frac{e^{-\lambda T}(\lambda T)^l}{l!} F^l(a) = e^{-\lambda T[1-F(a)]}.$$

For $a > \hat{a}$, we approximate it by a discrete time model. Let $\{t_l\}_{l=0}^{k}$, where $0 = t_0 < t_1 < ... < t_k = T$, be a partition of the interval $[0, T]$, and let $\delta_l = t_l - t_{l-1}$ for $l = 1, 2, ..., k$. Define

π as

$$\| \pi \| = \max_{1 \leq l \leq k} |\delta_l|.$$

Then,

$$\begin{aligned}
Z(a|\hat{a}, T) =& Z(\hat{a}|\hat{a}, T) + \lim_{\|\pi\| \to 0} \sum_{l=1}^{k} Z(a|\hat{a}, t_{l-1}) \left[\sum_{n=1}^{\infty} \frac{e^{-\lambda\delta_l}(\lambda\delta_l)^n}{n!} [F^n(a) - F^n(\hat{a})] \right] \\
=& Z(\hat{a}|\hat{a}, T) + \lim_{\|\pi\| \to 0} \sum_{l=1}^{k} e^{-\lambda t_{l-1}[1-F(\hat{a})]} \lambda e^{-\lambda\delta_l} \big([F(a) - F(\hat{a})] + O(\delta_l) \big) \delta_l \\
=& Z(\hat{a}|\hat{a}, T) + \int_0^T \lambda e^{-\lambda t[1-F(\hat{a})]} [F(a) - F(\hat{a})] dt \\
=& Z(\hat{a}|\hat{a}, T) + \left[1 - e^{-\lambda T[1-F(\hat{a})]} \right] \frac{F(a) - F(\hat{a})}{1 - F(\hat{a})},
\end{aligned}$$

where the second term on the right hand side of each equality is the firm's probability of ending up in a state between $\hat{a}$ and a. The term $Z(\hat{a}|\hat{a}, t_n)$ used here is a convenient approximation when δ_l is small. The second equality is derived from the fact that

$$\sum_{n=2}^{\infty} \frac{(\lambda\delta_l)^n}{n!} [F^n(a) - F^n(a^*)] < \frac{\lambda^2\delta_l^2}{2(1 - \lambda\delta_l)} = o(\delta_l).$$

$\square$

Lemma 1.6. *Given $a > a'$, $Z(a|\tilde{a}, T) - Z(a'|\tilde{a}, T)$*

1. *is constant in $\tilde{a}$ for $\tilde{a} \geq a$;*
2. *strictly decreases in $\tilde{a}$ for $\tilde{a} \in (a', a)$;*
3. *strictly increases in $\tilde{a}$ for $\tilde{a} \leq a'$.*

This single-peaked property says that the probability of ending up in a state between a' and a is maximized if a firm chooses strategy a'.

Proof of Lemma 1.6. First, we show that $\frac{1-e^{-\lambda T x}}{x}$ strictly decreases in x over $(0, 1]$ as follows. Define $s := \lambda T$ and take x_1, x_2, $0 < x_1 < x_2 \leq 1$, we have

$$\frac{1 - e^{-sx_1}}{x_1} > \frac{1 - e^{-sx_2}}{x_2},$$

implied by

$$\begin{aligned}
&\frac{\partial (1 - e^{-sx_1})x_2 - (1 - e^{-sx_2})x_1}{\partial s} = x_1 x_2 (e^{-sx_1} - e^{-sx_2}) \geq 0 \quad (= 0 \text{ iff } s = 0) \quad \text{and} \\
&(1 - e^{-sx_1})x_2 - (1 - e^{-sx_2})x_1 = 0 \quad \text{for} \quad s = 0.
\end{aligned}$$

Next, define $x := 1 - F(a)$, $x' := 1 - F(a')$, and $\tilde{x} := 1 - F(\tilde{a})$. We have

$$Z(a|\tilde{a}, T) - Z(a'|\tilde{a}, T) = \begin{cases} e^{-\lambda T x} - e^{-\lambda T x'} & \text{for } \tilde{a} \geq a \\ (1 - e^{-\lambda T x'}) - (1 - e^{-\lambda T \tilde{x}})\frac{x}{\tilde{x}} & \text{for } \tilde{a} \in (a', a) \\ (1 - e^{-\lambda T \tilde{x}})\frac{x' - x}{\tilde{x}} & \text{for } \tilde{a} \leq a'. \end{cases}$$

It is independent of $\tilde{a}$ for $\tilde{a} \geq a$, strictly increasing in $\tilde{x}$ and thus strictly decreasing in $\tilde{a}$ for $\tilde{a} \leq a'$, and strictly decreasing in $\tilde{x}$ and thus strictly increasing in $\tilde{a}$ for $\tilde{a} \leq a'$. □

Lemma 1.7. *Suppose Firm j with initial state* 0 *plays a strategy $\hat{a}_j$. Then, the* ***instantaneous gain*** *on payoff from searching for Firm i in a state a is*

$$\lambda \int_{a_i}^{1} [Z(a|\hat{a}_j, T) - Z(a_i|\hat{a}_j, T)] dF(a) - c.$$

Proof of Lemma 2.2. For convenience, denote $H(a)$ as $Z(a|\hat{a}_j, T)$ for short. The instantaneous gain from searching for Firm i in a state a_i is

$$\begin{aligned}
&\lim_{\delta \to 0} \frac{\left(e^{-\lambda\delta} H(a_i) + \lambda\delta e^{-\lambda\delta}\left[\int_{a_i}^{1} H(a)dF(a) + F(a_i)H(a_i)\right] + o(\delta) - \delta c\right) - H(a_i)}{\delta} \\
=&\lim_{\delta \to 0} \frac{-(1 - e^{-\lambda\delta})H(a_i) + \lambda\delta e^{-\lambda\delta}\left[\int_{a_i}^{1} H(a)dF(a) + F(a_i)H(a_i)\right] + o(\delta) - \delta c}{\delta} \\
=&\lim_{\delta \to 0} \frac{-\lambda\delta e^{-\lambda\delta} H(a_i) + \lambda\delta e^{-\lambda\delta}\left[\int_{a_i}^{1} H(a)dF(a) + F(a_i)H(a_i)\right] + o(\delta) - \delta c}{\delta} \\
=& - \lambda H(a_i) + \lambda\left[\int_{a_i}^{1} H(a)dF(a) + F(a_i)H(a_i)\right] - c \\
=&\lambda \int_{a_i}^{1} \left[Z(a|\hat{a}_j, T) - Z(a_i|\hat{a}_j, T)\right] dF(a) - c.
\end{aligned}$$

□

1.A.2 Proofs for the Benchmark Case

Proof of Theorem 1.1. We prove the theorem case by case.

Case[i]. When Firm i does not search, Firm j's best response is to search with cut-off 0. For $\frac{1}{2}\lambda(1 + e^{-\lambda T}) \leq c$, when Firm j searches with any cut-off $a_j \geq 0$, Firm i's best response is not to search, since the instantaneous gain from searching for Firm i in state 0 is

$$\begin{aligned}
&\lambda \int_0^1 Z(a|a_j, T)dF(a) - c \\
\leq &\lambda \int_0^1 Z(a|0, T)dF(a) - c \\
= &\lambda \int_0^1 [e^{-\lambda T} + (1 - e^{-\lambda T})F(a)]dF(a) - c \\
= &\lambda \left[e^{-\lambda T} + \frac{1}{2}(1 - e^{-\lambda T}) \right] - c \\
= &\frac{1}{2}\lambda(1 + e^{-\lambda T}) - c \\
\leq &0 \ \ (= 0 \text{ iff } \frac{1}{2}\lambda(1 + e^{-\lambda T}) = c),
\end{aligned}$$

where the first inequality follows from Lemma 1.6. Hence, there are two pure strategy equilibria, in each of which one firm does not search and the other firm searches with 0 as the cut-off, and if $\frac{1}{2}\lambda(1 + e^{-\lambda T}) = c$ there is also an equilibrium in which both firms search with 0 as the cut-off.

Case [ii]. First, we show that any strategy with a cut-off value higher than zero is a dominated strategy. When Firm j does not search, Firm i prefers searching with 0 as the cut-off to any other strategy. Suppose Firm j searches with $\hat{a}_j \geq 0$ as the cut-off. The instantaneous gain from searching for Firm i in a state $a_i > 0$ is

$$\begin{aligned}
&\lambda \int_{a_i}^1 [Z(a|\hat{a}_j, T) - Z(a_i|\hat{a}_j, T)]dF(a) - c \\
\leq &\lambda \int_{a_i}^1 [Z(a|a_i, T) - Z(a_i|a_i, T)]dF(a) - c \\
= &\frac{1}{2}\lambda(1 - e^{-\lambda T})[1 - F(a_i)]^2 - c \\
< &0,
\end{aligned}$$

where the first inequality follows from Lemma 1.6. Hence, once Firm i is in a state above 0, it has no incentive to continue searching any more. In this case, Firm i prefers either not to conduct any search or to search with 0 as the cut-off to any strategy with a cut-off value higher than zero.

Second, we show that the prescribed strategy profile is the unique equilibrium. It is sufficient to show that searching with 0 as the cut-off is the best response to searching with 0 as the cut-off. Suppose Firm j searches with 0 as the cut-off, the instantaneous gain from searching for Firm i in state $a = 0$ is

$$\begin{aligned}&\lambda \int_0^1 Z(a|0,T)dF(a) - c\\ =&\frac{1}{2}\lambda(1+e^{-\lambda T}) - c\\ >&0.\end{aligned}$$

That is, Firm i is strictly better off continuing searching if it is in state 0, and strictly better off stopping searching once it is in a state above 0. Hence, the prescribed strategy profile is the unique equilibrium.

Case [iii]. First, we prove that among the strategy profiles in which each firm searches with a cut-off higher than 0, the prescribed symmetric strategy profile is the unique equilibrium. Suppose a pair of cut-off rules (a_1^*, a_2^*), in which $a_1^*, a_2^* > 0$, is an equilibrium, then Firm i in state a_i^* is indifferent between continuing searching and not. That is, by Lemma 2.2, we have

$$\lambda \int_{a_i^*}^1 \left[Z(a|a_j^*, T) - Z(a_i^*|a_j^*, T)\right] dF(a) - c = 0. \tag{1.7}$$

Suppose $a_1^* \neq a_2^*$. W.l.o.g., we assume $a_1^* < a_2^*$. Then,

$$\begin{aligned}c =&\lambda \int_{a_1^*}^1 [Z(a|a_2^*, T) - Z(a_1^*|a_2^*, T)]dF(a)\\ >&\lambda \int_{a_2^*}^1 [Z(a|a_2^*, T) - Z(a_2^*|a_2^*, T)]dF(a)\\ >&\lambda \int_{a_2^*}^1 [Z(a|a_1^*, T) - Z(a_2^*|a_1^*, T)]dF(a) = c\end{aligned}$$

resulting in a contradiction. Hence, it must be the case that $a_1^* = a_2^*$.

Next, we show the existence of equilibrium by deriving the unique equilibrium cut-off value $a^* := a_1^* = a_2^*$ explicitly. Applying Lemma 4.2 to (1.7), we have

$$\begin{aligned}&\lambda \int_{a^*}^1 \left[1 - e^{-\lambda T[1-F(a^*)]}\right] \frac{F(a) - F(a^*)}{1 - F(a^*)} dF(a) = c\\ \Leftrightarrow&\frac{1}{2}[1 - F(a^*)]\left[1 - e^{-\lambda T[1-F(a^*)]}\right] = \frac{c}{\lambda}.\end{aligned} \tag{1.8}$$

The existence of a solution is ensured by the intermediate value theorem: when $F(a^*) = 1$, the term on the left hand side of (1.8) equals to 0, smaller than $\frac{c}{\lambda}$; when $F(a^*) = 0$, it equals to $\frac{1-e^{-\lambda T}}{2}$, larger than or equals to $\frac{c}{\lambda}$. The uniqueness of the solution is insured by that the term on the left hand side of the above equality is strictly decreasing in a^*.

Second, we show that there is no equilibrium in which one firm searches with 0 as the cut-off. Suppose Firm j searches with 0 as the cut-off. The instantaneous gain from searching for Firm i in a state $a_i > 0$ is

$$\begin{aligned}
&\lambda \int_{a_i}^{1} [Z(a|0,T) - Z(a_i|0,T)]dF(a) - c \\
=&\lambda \int_{a_i}^{1} (1 - e^{-\lambda T})[F(a) - F(a_i)]dF(a) - c \\
=&\frac{1}{2}\lambda(1 - e^{-\lambda T})[1 - F(a_i)]^2 - c,
\end{aligned} \tag{1.9}$$

which is positive when $a_i = 0$ and negative when $a_i = 1$. By the intermediate value theorem, there must be a value $\hat{a}_i > 0$ such that (1.9) equals 0 when $a_i = \hat{a}_i$. Hence, Firm i's best response is to search with $\hat{a}_i$ as the cut-off. However, if Firm i searches with $\hat{a}_i$ as the cut-off, it is not Firm j's best response to search with 0 as the cut-off, because

$$\begin{aligned}
0 =& \int_{\hat{a}_j}^{1} [Z(a|0,T) - Z(\hat{a}_j|0,T)]dF(a) - c \\
<& \int_{\hat{a}_j}^{1} [Z(a|\hat{a}_j,T) - Z(\hat{a}_j|\hat{a}_j,T)]dF(a) - c \\
<& \int_{0}^{1} [Z(a|\hat{a}_j,T) - Z(0|\hat{a}_j,T)]dF(a) - c,
\end{aligned}$$

which means that Firm j strictly prefers continuing searching when it is in a state slightly above 0.

Last, we show that there is no equilibrium in which one firm does not search. Suppose Firm j does not search. Firm i's best response is to search with 0 as the cut-off. However, Firm j then strictly prefers searching when it is in state 0, since the instantaneous gain from searching for the firm in state 0 is again

$$\lambda \int_{0}^{1} Z(a|a_i,T)dF(a) - c > 0.$$

□

Proof of Lemma 1.2. The expected total cost of a firm who searches with a cut-off $a \geq 0$ is

$$\begin{aligned}
&c\left[\int_{0}^{T} \frac{\partial(1 - Z(a|a,t))}{\partial t} t dt + TZ(a|a,T)\right] \\
=&(1 - e^{-\lambda T[1-F(a)]})\frac{c}{\lambda[1 - F(a)]},
\end{aligned} \tag{1.10}$$

which strictly increases in a. In Regions 2 and 3, in equilibrium, the probability of winning for each firm is

$$\begin{aligned}&\frac{1}{2}[1 - Z^2(0|a^*(c,T),T)]\\=&\frac{1}{2}(1 - e^{-2\lambda T}),\end{aligned} \tag{1.11}$$

and thus the expected payoff to each firm is the difference between the expected probability of winning (1.11) and the expected search cost (1.10), setting a to be $a^*(a,T)$:

$$\frac{1}{2}(1 - e^{-2\lambda T}) - (1 - e^{-\lambda T[1-F(a^*(c,T))]})\frac{c}{\lambda[1 - F(a^*(c,T))]}. \tag{1.12}$$

The limit of (1.12) as T approaches infinity is 0.

□

1.A.3 Proofs for the Head Start Case

First, we state two crucial lemmas for the whole section.

Lemma 1.8.

1. *For $a_1^I > a^*(c,T)$, not to search is Firm 1's strictly dominant strategy.*
2. *For $a_1^I = a^*(c,T)$, not to search is Firm 1's weakly dominant strategy. If Firm 2 searches with cut-off a_1^I, Firm 1 is indifferent between not to search and search with a_1^I as the cut-off; Otherwise, Firm 1 strictly prefers not to search.*

Proof of Lemma 1.8. Suppose Firm 2 does not search, Firm 1's best response is not to search. Suppose Firm 2 searches with cut-off $a_2 \geq a_1^I$. If Firm 1 searches with cut-off $a_1 \geq a_1^I$, following from Lemma 1.6, the instantaneous gain from searching for Firm 1 in any state $a_1 \geq a_1^I \geq a^*(c,T)$ is

$$\begin{aligned}&\lambda\int_{a_1}^{1}[Z(a|a_2,T) - Z(a_1|a_2,T)]dF(a) - c\\\leq&\lambda\int_{a^*(c,T)}^{1}[Z(a|a^*(c,T),T) - Z(a^*(c,T)|a^*(c,T),T)]dF(a) - c,\end{aligned} \tag{1.13}$$

where equality holds if and only if $a_1 = a_2 = a^*(c,T)$. The right hand side of inequality (1.13) is less than or equal to 0 (it equals to 0 iff $c \geq \frac{1}{2}\lambda[1 - e^{-\lambda T}]$). Hence, the desired results follow. □

Lemma 1.9.

1. *For $a_1^I > F^{-1}(1 - \frac{c}{\lambda})$, not to search is Firm 2's strictly dominant strategy.*
2. *For $a_1^I = F^{-1}(1 - \frac{c}{\lambda})$, not to search is Firm 2's weakly dominant strategy. If Firm 1 does not search, Firm 2 is indifferent between not to search and search with any $\hat{a}_2 \in [a_2^I, a_1^I]$ as the cut-off. If Firm 1 searches, Firm 2's strictly prefers not to search.*

Proof of Lemma 1.9. If Firm 1 does not search, the instantaneous gain from searching for Firm 2 in a state $a_2 \leq a_1^I$ is

$$\lim_{\delta \to 0} \frac{\lambda\delta e^{-\lambda\delta}[1 - F(a_1^I)] + o(\delta) - c\delta}{\delta} = \lambda[1 - F(a_1^I)] - c \begin{cases} < 0 & \text{in Case [1]} \\ = 0 & \text{in Case [2].} \end{cases}$$

If Firm 1 searches, Firm 2's instantaneous gain is even lower. Hence, the desired results follow. □

Proof of Theorem 1.2. [1],[2], and [3] directly follow from Lemmas 1.8 and 1.9. We only need to prove [4] and [5] in the following.

[4]. Following Lemma 1.8, if Firm 2 searches with cut-off a_1^I, Firm 1 has two best responses: not to search and search with cut-off a_1^I. If Firm 1 searches with cut-off a_1^I, the instantaneous gain from searching for Firm 2 is

$$\begin{aligned}
&\lambda \int_{a_1^I}^1 Z(a|a_1^I, T)dF(a) - c \\
=&\lambda \int_{a_1^I}^1 \left[e^{-\lambda T[1-F(a_1^I)]} + (1 - e^{-\lambda T[1-F(a_1^I)]}) \frac{F(a) - F(a_1^I)}{1 - F(a_1^I)} \right] dF(a) - c \\
=&\frac{1}{2}\lambda(1 + e^{-\lambda T[1-F(a_1^I)]})[1 - F(a_1^I)] - c \\
>&\lambda[1 - F(a_1^I)] - c \\
>&0
\end{aligned}$$

if it is in a state $a_2 < a_1^I = a^*(c, T)$; it is

$$\lambda \int_{a_2}^1 [Z(a|a_1^I, T) - Z(a_2|a_1^I, T)]dF(a) - c < 0$$

if it is in a state $a_2 > a_1^I = a^*(c, T)$. Hence, the two prescribed strategy profiles are equilibria.

[5]. First, there is no equilibrium in which either firm does not search. If Firm 2 does not search, Firm 1's best response is not to search. However, if Firm 1 does not search, Firm 2's best response is to search with cut-off a_1^I rather than not to search. If Firm 2 searches with cut-off a_1^I, then not to search is not Firm 1's best response, because the instantaneous gain from searching for Firm 1 in state a_1^I is

$$\begin{aligned}
&\lambda \int_{a_1^I}^1 [Z(a|a_1^I, T) - Z(a_1^I|a_1^I, T)]dF(a) - c \\
=&\frac{1}{2}\lambda(1 - e^{-\lambda T[1-F(a_1^I)]})[1 - F(a_1^I)] - c \\
>&0,
\end{aligned}$$

where inequality holds because $a_1^I < a^*(c, T)$.

Next, we argue that there is no equilibrium in which either firm searches with cut-off a_1^I. Suppose Firm i searches with cut-off a_1^I. Firm j's best response is to search with a cut-off $\hat{a}_j \in [a_1^I, a^*(c,T))$. This is because the instantaneous gain from searching for Firm j in a state $a' \geq a_1^I$ is

$$\lambda \int_{a'}^{1} p_2[Z(a|a_1^I, T) - Z(a'|a_1^I, T)]dF(a) - c. \tag{1.14}$$

(1.14) is larger than 0 when $a' = a_1^I$. It is less than 0 if $a' = a^*(c,T)$, because by Lemma 1.6 we have

$$\begin{aligned}
&\lambda \int_{a^*}^{1} p_2[Z(a|a_1^I, T) - Z(a^*|a_1^I, T)]dF(a) - c \\
<&\lambda \int_{a^*(c,T)}^{1} [Z(a|a^*(c,T), T) - Z(a^*|a^*(c,T), T)]dF(a) - c \\
=&0.
\end{aligned}$$

Then, the intermediate value theorem and the strict monotonicity yield the unique cut-off value of $\hat{a}_j \in (a_1^I, a^*(c,T))$.

However, if Firm j searches with cut-off $\hat{a}_j \in [a_1^I, a^*(c,T))$, Firm i's best response is to search with a cut-off value $\hat{a}_i \in (\hat{a}_j, a^*(c,T))$ rather than a_1^I, because the instantaneous gain from searching for Firm i in a state $\tilde{a}$ is

$$\lambda \int_{\tilde{a}}^{1} [Z(a|\hat{a}_1, T) - Z(\tilde{a}|\hat{a}_1, T)]dF(a) - c \begin{cases} < 0 & \text{for } \tilde{a} = a^*(c,T) \\ > 0 & \text{for } \tilde{a} = \hat{a}_1 \end{cases}$$

and it is monotone w.r.t. $\tilde{a}$. This results in contradiction. Hence, there is no equilibrium in which either firm searches with a_1^I as the cut-off.

Last, we only need to consider the case in which each firm searches with a cut-off higher than a_1^I. Following the same argument as in the proof of Theorem 1.1, we have $(a^*(c,T), a^*(c,T))$ being the unique equilibrium.

□

Proof of Proposition 1.2. We apply Theorem 1.2 here for the analysis. We only need to show the case for $a_1^I \in (a^*(c,T), F^{-1}(1-\frac{c}{\lambda}))$. In this case, Firm 1 does not to search and Firm 2 searches with a_1^I as the cut-off. Now, take the limit $a_1^I \to a^*(c,T)$ from the right hand side of $a^*(c,T)$. In the limit, where Firm 2 searches with $a^*(c,T)$ as the cut-off, Firm 1 weakly prefers not to search. If $a_1^I = a^*(c,T)$, Firm 1 is actually indifferent between searching and not. Hence, a head start in the limit makes Firm 1 weakly better off. Firm 1's payoff when it does not search is $e^{-\lambda T[1-F(a_1^I)]}$, the probability of Firm 2 ending up in a state below a_1^I, is strictly increasing in a_1^I. Hence, a higher value of the head start makes Firm 1 even better off. □

Proof of Proposition 1.3. [1]. For T being small, $a^*(c, T) = 0$. $D^M(0, a_1^I) = 0$, and the partial derivative of $D^M(T, a_1^I)$ w.r.t. T when T is small is

$$\frac{\partial D^M(T, a_1^I)}{\partial T} = \lambda(1 - a_1^I)e^{-\lambda T(1-a_1^I)}[1 - \frac{c}{\lambda(1 - a_1^I)}] - \lambda e^{-2\lambda T} + ce^{-\lambda T},$$

which equals to $-\lambda a_1^I < 0$ at the limit of $T = 0$.

[2]. Follows from Propositions 1.1 and 1.2. □

Proof of Proposition 1.4. $D^M(T, a_1^I)$ is strictly decreasing in a_1^I, and it goes to the opposite of (1.12), which is less than 0, as a_1^I goes to $F^{-1}(1 - \frac{c}{\lambda})$, and

$$(1 - e^{-\lambda T[1-F(a^*(c,T))]}) - \frac{1}{2}(1 - e^{-2\lambda T}) \tag{1.15}$$

as a_1^I goes to $a^*(c, T)$. Hence, if (1.15) is positive, Case 1 yields from the intermediate value theorem; Case 2 holds if (1.15) is negative. □

1.A.4 Proofs for the Extended Models

Proof of Proposition 1.7. We argue that, to determine a subgame perfect equilibrium, we only need to consider two kinds of strategies profiles:

a Firm 1 retains its initial innovation and does not search, and Firm 2 searches with a_1^I as the cut-off;

b Firm 1 discards its initial innovation and searches with a_2^I as the cut-off, and Firm 2 retains its initial innovation.

First, suppose $c < \frac{\lambda}{2}(1 + e^{-\lambda T})$. If Firm 1 retains the initial innovation, it will have no incentive to search, and Firm 2 is indifferent between discarding the initial innovation and not. In either case, Firm 2 searches with a_1^I as the cut-off. Given that Firm 1 has discarded its initial innovation, Firm 2 has no incentive to discard its initial innovation as shown in Proposition 1.2.

Second, suppose $c > \frac{\lambda}{2}(1 + e^{-\lambda T})$. In the subgame in which both firms discard their initial innovation, there are two equilibria, in each of which one firm searches with 0 as the cut-off and the other firm does not search. Hence, to determine a subgame perfect equilibrium, we have to consider another two strategy profiles, in addition to [a] and [b]:

c Firm 1 discards its initial innovation and searches with 0 as the cut-off, and Firm 2 discards its initial innovation and does not search.

d Firm 1 discards its initial innovation and does not search, and Firm 2 discards its initial innovation and searches with 0 as the cut-off.

However, we can easily rule out [c] and [d] from the candidates for equilibria. In [c], Firm 2 obtains a payoff of 0. It can deviate by retaining its initial innovation so as to obtain a positive payoff. Similarly, in [d], Firm 1 can deviate by retaining its initial innovation to obtain a positive payoff rather than 0.

Last, it remains to compare Firm 1's payoff in [a] and [b]. In [a], Firm 1's payoff is

$$e^{-\lambda T[1-F(a_1^I)]}. \tag{1.16}$$

In [b], it is

$$(1-e^{-\lambda T[1-F(a_2^I)]})(1-\frac{c}{\lambda[1-F(a_2^I)]}). \tag{1.17}$$

The difference between these two payoffs, (1.17) and (1.16), is increasing in T, and it equals -1 when $T=0$ and goes to $1-\frac{c}{\lambda[1-F(a_2^I)]}>0$ as T approaches infinity. Hence, the desired result is implied by the intermediate value theorem. □

Proof of Proposition 1.8. The backward induction is similar to the proof of Proposition 1.7, and thus is omitted. □

Proof of Proposition 1.9. The equilibrium for the subgame starting from time t_0 derives from Theorem 1.2. Suppose at time t_0, Firm i is in a state a_i^0, where $\max\{a_1^0, a_2^0\} \geq a_1^I$. Assume $a_i^0 > a_j^0$. If $a_i^0 > F^{-1}(1-\frac{c}{\lambda})$, then Firm i obtains a continuation payoff of 1, and Firm j obtains 0. If $a_i^0 \in (a^*(c, T-t_0), F^{-1}(1-\frac{c}{\lambda}))$, then Firm i obtains a continuation payoff of $e^{-\lambda(T-t_0)[1-F(a_i^0)]}$, and Firm j obtains $(1-e^{-\lambda(T-t_0)[1-F(a_i^0)]})(1-\frac{c}{\lambda[1-F^{-1}(a_i^0)]})$.

To prove this result, we first show that not to search before t_0 is Firm 2's best response regardless of Firm 1's action before time t_0. It is equivalent to showing that not to search before t_0 is Firm 2's best response if Firm 2 knows that Firm 1 is definitely going to be in any state $a_1^0 \geq a_L^*$ at time t_0.

As we have shown before, for any $a_1^0 \geq (>) F^{-1}(1-\frac{c}{\lambda})$, Firm 2 (strictly) prefers not to conduct searching before time t_0.

If $a_1^0 \in [a_L^*, F^{-1}(1-\frac{c}{\lambda}))$, Firm 2's unique best response before time t_0 is not to search. The instantaneous gain from searching at any time point before t_0 for Firm 2 in a state below a_1^0 is

$$\begin{aligned}
&\lambda\big[[1-F(F^{-1}(1-\frac{c}{\lambda}))]+\int_{a_1^0}^{F^{-1}(1-\frac{c}{\lambda})} e^{-\lambda(T-t_0)[1-F(a)]}dF(a) \\
&\qquad -[1-F(a_1^0)](1-e^{-\lambda(T-t_0)[1-F(a_i^0)]})(1-\frac{c}{\lambda[1-F^{-1}(a_i^0)]})\big]-c \\
=&\lambda\big[\int_{a_1^0}^{F^{-1}(1-\frac{c}{\lambda})} e^{-\lambda(T-t_0)[1-F(a)]}dF(a) \\
&\qquad -[1-F(a_1^0)](1-e^{-\lambda(T-t_0)[1-F(a_i^0)]})(1-\frac{c}{\lambda[1-F^{-1}(a_i^0)]})\big],
\end{aligned}$$

which is strictly negative when $T - t_0$ is sufficiently large, and thus conducting a search before time t_0 actually makes Firm 2 strictly worse off in this case.

Next, we show that Firm 1's best response before time t_0 is to search with $F^{-1}(1-\frac{c}{\lambda})$ as the cut-off, if Firm 2 does not search before t_0. To see this, look at the instantaneous gain from searching for Firm 1 in a state below $F^{-1}(1-\frac{c}{\lambda})$:

$$
\begin{aligned}
&\lambda\Big[[1-F(F^{-1}(1-\frac{c}{\lambda}))] \\
&\qquad + \int_{a_1^I}^{F^{-1}(1-\frac{c}{\lambda})} e^{-\lambda(T-t_0)[1-F(a)]}dF(a) - [1-F(a_1^I)]e^{-\lambda(T-t_0)[1-F(a_1^I)]}\Big] - c \\
=&\int_{a_1^I}^{F^{-1}(1-\frac{c}{\lambda})} e^{-\lambda(T-t_0)[1-F(a)]}dF(a) - [1-F(a_1^I)]e^{-\lambda(T-t_0)[1-F(a_1^I)]} \\
>&\int_{\tilde{a}}^{F^{-1}(1-\frac{c}{\lambda})} e^{-\lambda(T-t_0)[1-F(a)]}dF(a) - [1-F(\tilde{a})]e^{-\lambda(T-t_0)[1-F(a_1^I)]} \\
>&(1-\frac{c}{\lambda}-F(\tilde{a}))e^{-\lambda(T-t_0)[1-F(\tilde{a})]} - [1-F(\tilde{a})]e^{-\lambda(T-t_0)[1-F(a_1^I)]} \\
=&e^{-\lambda(T-t_0)[1-F(a_1^I)]}[1-\frac{c}{\lambda}-F(\tilde{a})]\left(e^{\lambda(T-t_0)[F(\tilde{a})-F(a_1^I)]} - \frac{1-F(\tilde{a})}{1-\frac{c}{\lambda}-F(\tilde{a})}\right),
\end{aligned}
$$

where $\tilde{a}$ is any value in $(a_1^I, F^{-1}(1-\frac{c}{\lambda}))$. The term on the right hand side of the last equality is strictly positive if $T-t_0$ is sufficiently large. Hence, the desired result yields.

□

1.A.5 Proofs for the Case with Asymmetric Costs

Proposition 1.12. *If $0 < c_1 < c_2 < \frac{1}{2}\lambda(1-e^{-\lambda T})$ there exists a pure strategy equilibrium (a_1^*, a_2^*) with $a_1^*, a_2^* \geq 0$.*

Proof of Proposition 1.12. We prove the existence of equilibrium by applying Brouwer's fixed point theorem. First, same as in the previous proofs, if Firm j searches with a cut-off $\hat{a}_j \geq 0$, the instantaneous gain from searching for Firm i in state 0 is

$$\lambda \int_0^1 Z(a|1,T) - c_i > 0,$$

and thus Firm i is better off continuing searching if it is in state 0.

Next, let us define for each Firm j a critical value

$$\alpha_j = \sup\{a_j \in [0,1] \mid I(0|\alpha_j, c_i) = \lambda \int_0^1 [Z(a|\alpha_j) - Z(0|\alpha_j)]dF(a) - c_i > 0\}.$$

Suppose there is a $\alpha_j \in (0,1)$ such that

$$I(0|\alpha_j, c_i) = \lambda \int_0^1 [Z(a|\alpha_j) - Z(0|\alpha_j)]dF(a) - c_i = 0.$$

For any $\hat{a}_j \in [0, \alpha_j]$,

$$I(0|\hat{a}_j, c_i) \geq 0 \text{ and}$$
$$I(1|\hat{a}_j, c_i) < 0.$$

By the intermediate value theorem and the strict monotonicity of $Q(a|\hat{a}_j, c_i)$ in a, there must exist a unique $\tilde{a}_i \in [0, 1)$ such that

$$I(\tilde{a}_i|\hat{a}_j, c_i) = 0.$$

That is, if Firm j searches with cut-off $\hat{a}_j$, Firm i's best response is to search with cut-off $\tilde{a}_i$.

For any $\hat{a}_j \in (\alpha_j, 1]$, if the set is not empty,

$$I(0|\hat{a}_j, c_i) < 0.$$

That is, Firm i's best response is to search with cut-off 0.

Then, we could define two best response functions $BR_i : [0, 1] \to [0, 1]$ where

$$BR_i(\hat{a}_j) := \begin{cases} 0 & \text{for } \hat{a}_j \in (\alpha_j, 1] \text{ if it is not empty} \\ \tilde{a}_i & \text{where } I(\tilde{a}_i|\hat{a}_j, c_i) = 0 \text{ for } \hat{a}_j \in [0, \alpha_j]. \end{cases}$$

It is also easy to verify that BR_i is a continuous function over $[0, 1]$. Hence, we have a continuous self map $BR : [0, 1]^2 \to [0, 1]^2$ where

$$BR = (BR_1, BR_2)$$

on a compact set, and by Brouwer's fixed point theorem, there must exist of a pure strategy equilibrium in which each Firm searches with a cut-off higher than or equal to 0. $\square$

Proof of Proposition 3.3. First, using the same arguments as in the proof of Proposition 1.1, we claim that if there exists an equilibrium it must be the case that each firm searches with a cut-off higher than or equal to 0 with one strictly positive value for one firm.

Next, we show that there can be no equilibrium in which Firm 2 searches with a cut-off $\hat{a}_2 > 0$ and Firm 1 searches with cut-off 0. Such a strategy profile $(0, \hat{a}_2)$ is an equilibrium if and only if

$$\lambda \int_0^1 [Z(a|\hat{a}_2, T) - Z(0|\hat{a}_2, T)] dF(a) - c_1 \leq 0, \quad \text{and}$$
$$\lambda \int_{\hat{a}_2}^1 [Z(a|0, T) - Z(\hat{a}_2|0, T)] dF(a) - c_2 = 0.$$

However,

$$
\begin{aligned}
0 &= \lambda \int_{\hat{a}_2}^{1} [Z(a|0,T) - Z(\hat{a}_2|0,T)]dF(a) - c_2 \\
&< \lambda \int_{\hat{a}_2}^{1} [Z(a|\hat{a}_2,T) - Z(\hat{a}_2|\hat{a}_2,T)]dF(a) - c_2 \\
&< \lambda \int_{0}^{1} [Z(a|\hat{a}_2,T) - Z(0|\hat{a}_2,T)]dF(a) - c_1 \leq 0,
\end{aligned}
$$

resulting in a contradiction.

Next, we derive the necessary and sufficient conditions for the existence of an equilirbium in which Firm 2 searches with a cut-off 0 and Firm 1 searches with a cut-off strictly higher than 0. A pair of cut-off rules $(\hat{a}_1, 0)$, $\hat{a}_1 > 0$, is an equilibrium if and only if

$$
\lambda \int_{\hat{a}_1}^{1} [Z(a|0,T) - Z(\hat{a}_1|0,T)]dF(a) - c_1 = 0 \ \text{ and} \tag{1.18}
$$

$$
\lambda \int_{0}^{1} [Z(a|\hat{a}_1,T) - Z(0|\hat{a}_1,T)]dF(a) - c_2 \leq 0, \tag{1.19}
$$

where

$$
(1.18) \Leftrightarrow \frac{1}{2}\lambda(1-e^{-\lambda T})[1-F(\hat{a}_1)]^2 - c = 0 \Leftrightarrow \hat{a}_j = F^{-1}\left(1 - \sqrt{\frac{2c_1}{\lambda(1-e^{-\lambda T})}}\right). \tag{1.20}
$$

Then, (1.20) and (1.19) together imply that $(\hat{a}_i, 0)$ is an equilibrium if and only if

$$
I\left(0|F^{-1}\left(1 - \sqrt{\frac{2c_1}{\lambda(1-e^{-\lambda T})}}\right), c_2\right) \leq 0. \tag{1.21}
$$

We will see that if (1.21) holds there is no other equilibrium.

When (1.21) does not hold, there is a unique equilibrium, in which each firm searches with a cut-off strictly higher than 0. Because by Proposition 1.12 an equilibrium must exists. Let (a_1^*, a_2^*) be such an equilibrium. We first show that $a_1^* > a_2^*$ must hold by proof by contradiction, and then we show that it must be a unique equilibrium. Such a pair (a_1^*, a_2^*) is an equilibrium if and only if

$$
\lambda \int_{a_i^*}^{1} [Z(a|a_j^*,T) - Z(a_i^*|a_j^*,T)]dF(a) = c_i \ \text{ for } i = 1,2 \text{ and } j \neq i. \tag{1.22}
$$

Suppose $a_1^* \leq a_2^*$. Applying Lemma 1.6, we have

$$\begin{aligned} c_1 =& \lambda \int_{a_1^*}^1 [Z(a|a_2^*, T) - Z(a_1^*|a_2^*, T)] dF(a) \\ \geq& \lambda \int_{a_2^*}^1 [Z(a|a_2^*, T) - Z(a_2^*|a_2^*, T)] dF(a) \\ \geq& \lambda \int_{a_2^*}^1 [Z(a|a_1^*, T) - Z(a_2^*|a_1^*, T)] dF(a) = c_2, \end{aligned}$$

resulting in a contradiction.

Then, we show the uniqueness of the equilibrium for Cases [1] – [3] by contradiction. For Case [1] we show that the solution to (1.22) is unique, and for Cases [2] and [3] we show that there can be no equilibrium in which each firm searches with a cut-off higher than 0 coexisting with equilibrium $\left(F^{-1}\left(1 - \sqrt{\frac{2c_1}{\lambda(1-e^{-\lambda T})}}\right), 0\right)$. We can prove all of them together. Suppose there are two equilibria (a_1^*, a_2^*) and $(\tilde{a}_1^*, \tilde{a}_2^*)$, where (a_1^*, a_2^*) is a solution to (1.22) and $(\tilde{a}_1^*, \tilde{a}_2^*)$ is either $\left(F^{-1}\left(1 - \sqrt{\frac{2c_1}{\lambda(1-e^{-\lambda T})}}\right), 0\right)$ or a solution to (1.22). It is sufficient to show that the following two cases are not possible:

1. $\tilde{a}_1^* > a_1^* > a_2^* > \tilde{a}_2^* \geq 0$ and

2. $a_1^* > \tilde{a}_1^* > a_2^* > \tilde{a}_2^* \geq 0$.

Suppose $\tilde{a}_1^* > a_1^* > a_2^* > \tilde{a}_2^* \geq 0$. Applying Lemma 1.6 we have

$$\begin{aligned} 0 =& \lambda \int_{a_1^*}^1 [Z(a|a_2^*, T) - Z(a_1^*|a_2^*, T)] dF(a) - c_1 \\ <& \lambda \int_{\tilde{a}_1^*}^1 [Z(a|a_2^*, T) - Z(\tilde{a}_1^*|a_2^*, T)] dF(a) - c_1 \\ <& \lambda \int_{\tilde{a}_1^*}^1 [Z(a|\tilde{a}_2^*, T) - Z(\tilde{a}_1^*|\tilde{a}_2^*, T)] dF(a) - c_1 = 0, \end{aligned}$$

resulting in a contradiction.

Suppose $a_1^* > \tilde{a}_1^* > a_2^* > \tilde{a}_2^* \geq 0$. Applying Lemma 1.6 again, we have

$$\begin{aligned} 0 \geq& \lambda \int_{\tilde{a}_2^*}^1 [Z(a|\tilde{a}_1^*, T) - Z(\tilde{a}_2^*|\tilde{a}_1^*, T)] dF(a) - c_2 \\ >& \lambda \int_{a_2^*}^1 [Z(a|\tilde{a}_1^*, T) - Z(a_2^*|\tilde{a}_1^*, T)] dF(a) - c_2 \\ >& \lambda \int_{a_2^*}^1 [Z(a|a_1^*, T) - Z(a_2^*|a_1^*, T)] dF(a) - c_2 = 0, \end{aligned}$$

resulting in another contradiction.

□

Proof of Proposition 1.11. For fixed c_2 we have

$$\frac{\partial a_2^*}{\partial a_1^*} = -\frac{\frac{\partial \int_{a_2^*}^1 [Z(a|a_1^*,T)-Z(a_2^*|a_1^*,T)]dF(a)}{\partial a_1^*}}{\frac{\partial \int_{a_2^*}^1 [Z(a|a_1^*,T)-Z(a_2^*|a_1^*,T)]dF(a)}{\partial a_2^*}} = \frac{\int_{a_2^*}^1 \frac{\partial [Z(a|a_1^*,T)-Z(a_2^*|a_1^*,T)]}{\partial a_1^*} dF(a)}{\frac{\partial Z(a_2^*|a_1^*,T)}{\partial a_2^*}} < 0.$$

Then,

$$\frac{\partial a_1^*}{\partial c_1} = -\frac{-1}{\lambda \int_{a_1^*}^1 \left[\frac{\partial [Z(a|a_2^*,T)-Z(a_1^*|a_2^*,T)]}{\partial a_2^*}\frac{\partial a_2^*}{\partial a_1^*} - \frac{\partial Z(a_1^*|a_2^*,T)}{a_1^*}\right] dF(a)} < 0 \text{ and}$$
$$\frac{\partial a_2^*}{\partial c_1} = \frac{\partial a_2^*}{\partial a_1^*}\frac{\partial a_1^*}{\partial c_1} > 0.$$

For fixed c_1 we have

$$\frac{\partial a_1^*}{\partial a_2^*} = -\frac{\frac{\partial \int_{a_1^*}^1 [Z(a|a_2^*,T)-Z(a_1^*|a_2^*,T)]dF(a)}{\partial a_2^*}}{\frac{\partial \int_{a_1^*}^1 [Z(a|a_2^*,T)-Z(a_1^*|a_2^*,T)]dF(a)}{\partial a_1^*}} = \frac{\int_{a_1^*}^1 \frac{\partial [Z(a|a_2^*,T)-Z(a_1^*|a_2^*,T)]}{\partial a_2^*} dF(a)}{\frac{\partial Z(a_1^*|a_2^*,T)}{\partial a_1^*}} > 0.$$

Then,

$$\frac{\partial a_2^*}{\partial c_2} = -\frac{-1}{\lambda \int_{a_2^*}^1 \left[\frac{\partial [Z(a|a_1^*,T)-Z(a_2^*|a_1^*,T)]}{\partial a_1^*}\frac{\partial a_1^*}{\partial a_2^*} - \frac{\partial Z(a_2^*|a_1^*,T)}{a_2^*}\right] dF(a)} < 0 \text{ and}$$
$$\frac{\partial a_1^*}{\partial c_1} = \frac{\partial a_1^*}{\partial a_2^*}\frac{\partial a_2^*}{\partial c_2} < 0.$$

□

Chapter 2

Matching Prior to Cournot Competition: An Assignment Game with Externalities

Introduction

Recently, an observation was addressed in the following piece of news:

> *"Apple is reported to be working on an electric car . . . The . . . company has been on a hiring spree attracting key members of rival car manufacturers' electric and new vehicle system teams. . . . A new team has reportedly been . . . under the control of the former Ford executive Steve Zadesky and former chief executive of Mercedes-Benz's research and development Johann Jungwirth." (The Guardian February 20, 2015)*

When making hiring decisions, a large firm does not only care about the ability of the workers to be hired by itself but also the ability of the workers to be hired by its rival firms. A worker does not only affect her employer's competing strategy but also her employer's rival firms' strategies in the goods market. That is, each firm's hiring decision has externalities on the other firms and workers. If such externalities are taken into account, what would be a stable matching outcome between firms and workers? And what would be the consequent goods market outcome? Such a labor market matching problem could not be solved by the standard models introduced by Gale and Shapley (1962) and Shapley and Shubik (1971) since externalities are not considered in these models.[1]

This chapter extends the literature on matching with externalities. It is the first attempt to link the labor market matching problem to the imperfect competition problem in a goods market. Specifically, the problem is modeled as a two-stage game in which firms and managers simultaneously for a 1-to-1 matching before a Cournot competitions starts among all firm-manager pairs. I provide sufficient conditions for the outcomes that good firms hire good

[1] See Roth and Sotomayor (1992) for a detailed discussion of literature.

managers and that good firms hire lesser managers, respectively.

To analyze the above problem, one needs a proper concept of stable matching. I start by a review of the general model for 1-to-1 assignment games with externalities in Section 2.1. The solution concept I use is an adaption of Sasaki and Toda's (1996) conjectural equilibrium (Hahn, 1987) for pessimistic (Aumann and Peleg, 1960) agents. A matching is stable if there is an associated feasible payoff profile such that: (1) the matching is consistent with agents' conjectures (expectations about the resulting matchings), i.e., the matching is in the expectation of each pair of agents; (2) no pair of agents can block the outcome. A result parallel to the finding of Sasaki and Toda in matching with externalities and ordinal preferences is that a stable matching is ensured to exist if and only if each agent considers all matchings possible.

Section 2.2 puts the genera framework of 1-to-1 assignment games into use in the environment of a Cournot competition. Firms produce a homogeneous good and sell it to the same market, and the market price of the good is decided by the aggregate output. Hence, each firm's hiring outcome affects all other firm's production decisions and profits, which in turn affects these firms' hiring decisions. I assume that (i) each firm has to hire one manager to produce the good; (ii) the market demand function is linear; (iii) the unit production cost of a firm depends only on the technology the firm and the human capital of the manager hired by the firm. Under these assumptions, it is shown in Section 2.3 that if the unit cost function is submodular, it is rational for each firm-manager pair to believe that, if they are to pair up, the remaining agents will form a positive assortative matching (PAM) among themselves, and the PAM on the whole market is a stable matching under these rational conjectures. That is, if firm technology and human capital are complementary, it will be a stable matching that a good manager works for a good firm and a lesser manager works for a lesser firm. Even if the the unit cost function is supermodular, as long as the cross derivative is sufficiently smaller than the product of the partial derivatives, the PAM is still stable under rational conjectures. That is, if firm technology and human capital are substitutable, yet the substitutive effect is dominated by the marginal effects of technology and human capital, it would still be a stable matching that good firms hire good managers. However, if the market demand level is sufficiently high, the negative assortative matching is a stable matching. Because of multi-dimensionality of externalities, it is not easy to provide a short-cut to prove the above results. I prove them in a recursive method, using Sharpley and Shubik's linear programming technology. On the other hand, an example shows that a stable matching does not generally exist.

Afterwards, I analyze social welfare induced by matchings. A surprising result is that when technology and human capital are complementary, the social welfare induced by a PAM, which is stable, could be lower than that induced by some other matching. Hence, when firm technology and human capital are substitutable, a government may improve social welfare by subsidizing production cost to induce a more desirable matching outcome in the labor market.

Literature

Ever since Gale and Shapley's (1971) and Shapley and Shubik's (1971) pioneer work, there has been a huge literature on matching and assignment games. Two-sided matching with externalities was first studied by Sasaki and Toda (1996), but the literature had been very scarce until the past few years. In Sasaki and Toda's one-to-one matching model, externalities are captured by preferences over the set of complete matchings rather than the set of agents on the other side of the market. Each pair of deviators have exogenously given conjectures about the possible matchings resulting from their block. They provide a concept of conjectural equilibrium for their model and show that a stable matching is guaranteed to exist if and only if each conjecture consists of all possible matchings.[2] My model applies Sasaki and Toda's abstract framework into the concrete environment of a Cournot competition.[3]

Most existing work on matching with externalities focuses on the cases with non-trasferable utilities. Hafalir (2008) perfects Sasaki and Toda's model of one-to-one matching with ordinal preferences by proposing endogenized "sophisticated conjectures" and shows the existence of stable matchings under these conjectures. In contrast to Sasaki and Toda's far-sighted notions, Mumcu and Saglam (2010) consider a setting of one-to-one matching in which each pair of deviators hold the actions of all other players constant when considering their block. Fisher and Hafalir (2016) study stable matchings in a setting of one-to-one matching in which each agent's utility is affected by the aggregate level of externality and each agent has little influence on the externality level. Similar to Mumcu and Saglam (2010), Bando (2012, 2014) studies stable matchings and analyzes deferred acceptance algorithm in a setting of many-to-one matching allowing externalities in the choice behavior of firms but not of workers. Pycia and Yenmez (2015) consider a general many-to-many matching market with contracts and externalities. They extend the classical substitutes condition to allow for externalities and establish existence of stable matchings under the condition. In another recent study, Teytelboym (2012) looks at multilateral matching markets in networks with contacts and externalities and shows that

[2] The proof is for the case of two-sided externalities. The same negative result holds under one-sided externalities, and the existence of stable matching is not guaranteed under rational conjectures as well.

[3] There is an independent study by Gudmundsson and Habis (2013). Their model is similar to the one here for general assignment games. The major difference in the settings is on specifying the agents pairs' values of deviation. In their setting, a value of deviation is directly given. By contrast, in my setting a pair's value of deviation subjects to their conjecture and a stable matching has to be consistent with conjectures. Their study complements this study by showing properties of equilibria of general two-sided one-to-one assignment games with externalities. I emphasize on the application of the framework.

a stable matching exists if and only if agents' preferences are pairwise aligned in the sense of Pycia (2012). There is also a literature that examines externalities between couples matched to the same firm or among peers matched to the same college (Dutta and Massó, 1997; Klaus and Klijn, 2005; Echenique and Yenmez, 2007; Pycia, 2012; Kojima et al., 2013; Ashlagi et al., 2014; Inal, 2015). In contrast to all the above papers, my study considers transferable utilities and focuses on sorting patterns of stable matchings.

A recent study by Chade and Eeckhout (2016) is closely related to this chapter. They also consider a two-stage game: A first stage of one-to-one matching and a second stage pairwise competition among agent pairs (teams). Their model has three distinguishing features: (1) Each agent has only two possible types; (2) there is a mass of agents, so each agent pair's actions have no effect on the overall market; (3) each agent pair (team) competes only against another pair and thus has no effect on the other teams. Due to these features, a complete solution to the game can be derived. They also study sorting patterns of stable matchings and efficiency. My model is also, in some sense, an extension to that of Jehiel and Moldovanu (1996). Unlike my model in which there are multiple agents on each side of the matching market, their auction model entails many buyers but only one seller with one indivisible good. They emphasize on individual strategic play and show that agents might be better off not participating.

Besides the above theoretical work, there has also been some work emphasizing on applications and methodologies for quantifying externality effects in matching problems. Baccara et al. (2012) study an office assignment problem with transferable utilities and quantify the effects of network externalities on choices and outcomes. Uetake and Watanabe (2012) conduct a comprehensive analysis of an industry game in which firms enter a market by merging an incumbent firm, using a matching model with externalities.

Apart from the studies of matching, I would also like to mention a study by Brander and Spencer (1985). The Stackelberg-Cournot version of their model is a two-stage game: At the first stage, a government decides whether to subsidize the production of a domestic firm; at the second stage firms play a Cournot competition. Under some conditions, subsidization may increase domestic social welfare. The two-stage game of this chapter shares some features with Brander and Spencer's model. Comparing to a lesser manager, a good manager can relatively reduce the unit production cost of a firm. Thus, my model can be regarded as a model in which the good managers choose firms to subsidize before the firms start a Cournot competition.

2.1 The Model

General Matching and Assignment

Let I be the set of agents on one side (firms) of the market and J be the set of agents on the other side (managers). I and J are finite and disjoint. The market is denoted by $M = I \cup J$. To simplify the exposition, I assume (i) that I and J have the same cardinality (i.e., $n = |I| = |J| \geq 2$) and (ii) that each of the agents must be matched to one agent on the other side of the market. Let i and j denote a generic firm and manager, respectively.

A **matching** is a bijection function $\mu: I \cup J \to I \cup J$ such that: (i) for each $m \in I \cup J$, $\mu \circ \mu(m) = m$; (ii) $\mu(i) \in J$, $\mu(j) \in I$ for all $i \in I$ and for all $j \in J$. $\mu(i) = j$ (or equivalently, $\mu(j) = i$) is written as $(i, j) \in \mu$. Let $A(I, J)$ be the set of all matchings. For each $(i, j) \in I \times J$, $A(i, j) = \{\mu \in A(I, J) | (i, j) \in \mu\}$.

Let $\pi : I \times J \times A(I, J) \to R_+$ denote a **surplus function**. $\pi(i, j, \mu)$ is interpreted as the surplus agent i and j create (for themselves) in matching μ. I denote $\pi(\mu) := \sum_{(i,j)\in\mu} \pi(i, j, \mu)$ as the **total surplus** of the agents on a matching μ.

Let $(u, v) \in R_+^{|I|} \times R_+^{|J|}$ denote a payoff profile.[4] A payoff profile is a redistribution of the surpluses to the agents. For simplicity, I assume both π and (u, v) to be nonnegative.[5]

The triplet $< I, J, \pi >$ is called an **assignment game (matching with transferable utilities) with externalities**.

Estimation Function

The notion of stability in this chapter is pair-wise stability. When a firm and a manager are trying to leave their current respective partner and form a new partnership between themselves, they need to take into account how the other agents will behave. I here adapt the non-Bayesian setting proposed by Sasaki and Toda (1996). Let $\varphi(i, j) \subset A(i, j)$ be the set of matchings that agent i and j consider *possible* if they form a partnership. φ is called an **estimation function** by Sasaki and Toda. Agents are assumed to be "pessimistic". That is, a pair of agents not matched would like to pair up if they believe they can gain by pairing to each other. Let us denote the value of deviation by

$$V(ij|\varphi) := \min_{\mu\in\varphi(i,j)} \pi(i, j, \mu).$$

[4]The lower bound of the payoffs is normalized to be 0. Typically, in some market games, if an agent quits from the market, this agent gets 0 profit, and the resulting matching among the remaining agents do not influence this agent's payoff.

[5]So that I can directly apply the linear programming method without modification.

Equilibrium Concept

The equilibrium concept I propose is defined as below.

Definition 2.1. *Given an estimation function φ, a triplet (μ, u, v) is a φ-equilibrium if*

1. *φ-admissibility: for any pair $(i, j) \in \mu$, $\mu \in \varphi(i, j)$;*
2. *φ-stability: there is no pair of agents (i, j) for whom $V(ij|\varphi) > u_i + u_j$;*
3. *feasibility: for all $(i, j) \in \mu$, $u_i + v_j = \pi(i, j, \mu)$.*

This matching μ is called a φ-stable matching.

φ-admissibility requires a matching to be consistent with agents' conjectures (i.e., if a matching is going to be an equilibrium matching, it must be in the expectations of the agent pairs). φ-stability requires that the outcome cannot be blocked by any pair of agents who are not paired to each other. Feasibility requires that there is no cross-pair monetary transfer and that no money is left on the table.[6]

Let $E_\varphi(I, J, \pi)$ denote the set of all φ-equilibria. A negative result, which parallel to that of Sasaki and Toda (1996) for the case with ordinal preferences, can be shown.

Theorem 2.1. *The set $E_\varphi(I, J, \pi)$ of φ-equilibria is non-empty for any surplus function π if and only if $\varphi(i, j) = A(i, j)$ for all $i \in I$ and $j \in J$.*

Proof. See Appendix 2.A.1. □

2.2 Cournot Competition among Firm-Manager Pairs

Suppose that firms produce a homogeneous consumer good and sell in the same goods market. Firms differ in production technologies. Each firm must employ one manager so can it produce the good. Managers differ in human capital. The unit production cost of a firm depends on both the production technology of the firm and the human capital of the manager hired by the firm. Each firm wants to maximize its profit, and each manager goes to the firm that is willing to pay her the highest salary.

The question is whether high-type (with good production technology) firms would hire high-type managers (with high human capital). It seems ambiguous because there are two effects. On the one hand, a high-type manager can help a firm produce more output at a lower production cost and occupy a larger share of the market. On the other hand, if high-type managers work for high-type firms, the total output would be high, which results in a

[6] This requirement can be relaxed.

low market price, and thus a high-type firm may not be willing to pay a high salary to hire a high-type manager.

To study this problem, I model it as a two stage game and put the previous framework into use. The timeline of the two-stage game is as below.

- Stage 1: Firms pair up with managers.
- Stage 2: Firms play a Cournot competition.

At the first stage, firms and managers simultaneously form partnerships, and at the second stage firms simultaneously decide their respective output quantities after observing the matching formed at the first stage. Since each firm's output quantity has some influence on the market price of the good, the surplus of a firm-manager pair depends on the full matching realized. The game is very complicated because of the multi-dimensionality brought by externalities. However, some clear results can be derived under some mild assumptions. Before solving problem, I need to specify the surplus function π and the estimation function φ.

2.2.1 Specification of π

Firms and managers are assigned to types via maps $f : I \to F$ and $s : J \to S$, where $F := [\underline{f}, \overline{f}]$ and $S := [\underline{s}, \overline{s}]$ are compact subsets of R. Their types are publicly known. f_i is firm i's technology type and s_j is manager j's human capital type. Then, I make the following assumptions.

Assumption 2.1. *The inverse demand function is linear, i.e.,*

$$p(\mu, q) := \max\{\ A - \alpha \sum_{(i,j)\in\mu} q^{ij},\quad 0\}.$$

A is a constant, μ is the matching on the labor market, $q^{ij} \geq 0$ is the amount of the good produced and sold by firm i who employs manager j, and q is a vector of q^{ij}'s.

Assumption 2.2. *The unit production cost of each firm-manager pair is constant.*

Let us denote the unit cost function by $c : F \times S \to [\underline{c}, \overline{c}]$, where $\underline{c} > 0$. $c(f, s)$ represents the unit production cost of a type-f firm who hires a type-s manager.

Assumption 2.3. *The unit production cost function $c(\cdot, \cdot)$ is twice continuously differentiable and decreasing in both types (i.e., $c_1 < 0$ and $c_2 < 0$).*

Then, the maximum value of c is $\overline{c} = c(\underline{f}, \underline{s})$, and the minimum is $\underline{c} = c(\overline{f}, \overline{s})$.

Assumption 2.4. *The value of c is bounded from above by $\frac{A}{n}$, i.e., $\frac{A}{n} > \overline{c}$.*

By imposing Assumption 2.4, I restrict attention to the cases in which no manager or firm will leave the market or produce 0 output. While this saves us some tedious computational work, this is without loss of generality for the analysis.[7] Further more, I say firm technology and human capital are substitutable if $c_{12} \geq 0$; complementary if $c_{12} \leq 0$.

Hence, for a given matching μ, the surplus[8] created by $(i,j) \in \mu$, given the other firm-manager pairs produce $q^{-(ij)}$, is

$$\pi(i,j,\mu|\, q^{-(ij)}) = \max_{q^{ij} \geq 0} q^{ij} \cdot [p(\mu, q) - c(f_i, s_j)].$$

For a realized matching μ, the solution to the Cournot competition in the second stage is as follows.

Lemma 2.1. *Suppose Assumptions 2.1, 2.2, and 2.4 hold. For a given matching μ, in a Cournot game among the matched pairs in μ, a pair $(i,j) \in \mu$ produce*

$$q^{ij}(\mu) = \frac{1}{(n+1)\alpha}[A - (n+1)c(f_i, s_j) + \sum_{(i',j') \in \mu} c(f_{i'}, s_{j'})]$$

and get a surplus of

$$\pi(i,j,\mu) = \frac{1}{(n+1)^2\alpha}[A - (n+1)c(f_i, s_j) + \sum_{(i',j') \in \mu} c(f_{i'}, s_{j'})]^2. \tag{2.1}$$

Proof. See Appendix 2.A.2. □

In the two-stage game, when (i,j) believe that if they pair to each other, μ is going to be the resulting matching and firms will play a Cournot competition in the second stage, $q^{ij}(\mu)$ is the amount of output that (i,j) are going to produce. That is, $q^{ij}(\mu)$ is (i,j)'s best response to their conjecture, and $\pi(i,j,\mu)$ is the corresponding surplus for (i,j).

2.2.2 Specification of φ

According to Theorem 2.1, if each agent's conjectures consist of all possible matchings, a stable matching must exist. However, it is not a good idea to assume that agents have universal conjectures, because that would include many unreasonable matchings. In the following, I will focus on rational conjectures: conjectures based on rational deductions. Although Theorem 2.1 does not apply any more in this case, the same techniques are used in the proofs.

[7] If I do not impose the upper bound on the c value, there are cases in which the high-type firms and managers would produce a large amount of the good, and then the market price is too low for the low-type firms and managers to survive. For these cases, I can simply exclude these low-type agents from the analysis.

[8] Notice that although other firms' output quantities are also involved in the surplus function of a pair of agents (i,j), this does not mean that there are indeed externalities. A simple example is that if $c(f,s) = f + s$, then the output and the surplus of a pair of agents (i,j) do not depend on how the other agents are going to match since the value of $\sum_{(i,j)\in\mu} f_i + s_j$ does not depend on μ.

Rational Conjectures

First, rational conjectures are formally described below. Let $M_c := I_c \cup J_c$ be a submarket in which $|I_c| = |J_c|$, and let $M_s := I_s \cup J_s = M \backslash M_c$ be the complement of set M_c. Let μ_c denote a generic matching on M_c, and μ_s a generic matching on M_s. For a given **conditional reduced market**

$$M_r := \{M_s \mid \mu_c\},$$

denote firm i and manager j's conjecture, when considering to pair to each other, on this conditional reduced market by

$$\varphi^{M_r}(i,j) \subset \{\mu_s \in A(I_s, J_s) \mid (i,j) \in \mu_s\}.$$

Denote a further conditional reduced market by

$$M_r^{ij} := \{M_s \backslash \{i,j\} \mid (i,j) \cup \mu_c\}$$

and the corresponding conditional conjecture function by $\varphi^{M_r^{ij}}$. Then, I say $\varphi^{M_r}(i,j)$ is **rational** (on market M_r) if: (1) for any matching $\mu_s \in \varphi^{M_r}(i,j)$, $\mu_s \backslash \{(i,j)\}$ is a $\varphi^{M_r^{ij}}$-stable matching on the reduced market M_r^{ij}; (2) for any $i' \in I_s \backslash \{i\}$ and $j' \in J_s \backslash \{j\}$, $\varphi^{M_r^{ij}}(i',j')$ is **rational** (on market M_r^{ij}).[9] When M_s consists of only 4 agents, every firm-manager pair's conjecture is rational since each conjecture contains exactly one sub-matching.[10] φ^{M_r} is **rational** if $\varphi^{M_r}(i,j)$ is **rational** for any $i, j \in M_s$. I illustrate this concept by the following example.

Example 2.1. *Consider a matching market with 5 agents on each side of the market. I show an example of forming rational conjectures by backward deduction.*

First, suppose conditional on (i_1, j_1), (i_2, j_2), and (i_3, j_3) being paired up, respectively, it is an equilibrium[11] that i_4 hires j_4 and i_5 hires j_5 on the reduced market $\{i_4, i_5, j_4, j_5\}$. Then, consider the reduced market $\{i_3, i_4, i_5, j_3, j_4, j_5\}$ conditional on (i_1, j_1) and (i_2, j_2) being paired up, respectively. With a little abuse of notation, I denote this conditional reduced market by

$$M'' := \{i_3, i_4, i_5, j_3, j_4, j_5 \mid (i_1, j_1), (i_2, j_2)\}.$$

[9]Notice that I can not directly kick i and j out, since they still have an role at the Cournot competition stage of the game.

[10]Li (1993) applies the idea of rational expectations to his model of a one-to-one matching with externalities and shows that the existence of the equilibrium is ensured when externalities are very weak.

[11]Since there are only 4 agents, a stable matching must exist.

If conjectures are rational, it must be the case that

$$\varphi^{M''}(i_3, j_3) = \{\{(i_3, j_3), (i_4, j_4), (i_5, j_5)\}\}.$$

Next, suppose $\{(i_3, j_3), (i_4, j_4), (i_5, j_5)\}$ *is indeed a* $\varphi^{M''}$*-stable matching on market* M'' *conditional on* (i_1, j_1) *and* (i_2, j_2) *being paired up, respectively. Consider the conditional reduced market*

$$M' := \{i_2, i_3, i_4, i_5, j_2, j_3, j_4, j_5 \mid (i_1, j_1)\}$$

in which (i_1, j_1) *are paired up. If conjectures are rational, it must be the case that*

$$\varphi^{M'}(i_2, j_2) = \{\{(i_2, j_2), (i_3, j_3), (i_4, j_4), (i_5, j_5)\}\}.$$

Finally, suppose $\{(i_2, j_2), (i_3, j_3), (i_4, j_4), (i_5, j_5)\}$ *is indeed a* $\varphi^{M'}$*-stable matching on market* M' *conditional on* (i_1, j_1) *being paired up. Consider the full market*

$$M := \{i_1, i_2, i_3, i_4, i_5, j_1, j_2, j_3, j_4, j_5\}.$$

If conjectures are rational, it must be the case that

$$\varphi(i_1, j_1) = \{\{(i_1, j_1), (i_2, j_2)), (i_3, j_3), (i_4, j_4), (i_5, j_5)\}\}.$$

That is, when firm i_1 *and manager* j_1 *are considering to pair up, they believe that this matching will be the resulting matching.*

In brief, a rational conjecture, e.g., $\varphi(i_1, j_1)$*, requires agents* i_1 *and* j_1 *to have a correct belief about what* i_2 *and* j_2*, when considering to pair up conditional on* (i_1, j_1) *being paired up, believe about what* i_3 *and* j_3*, when considering to pair up conditional on* (i_1, j_1) *and* (i_2, j_2) *being paired up, respectively, believe about what matching will be formed among* $\{i_4, i_5, j_4, j_5\}$ *conditional on*(i_1, j_1)*,* (i_2, j_2)*, and* (i_3, j_3) *being paired up, respectively.*

Assortative Matchings and Assortative Conjectures

It is complicated to derive rational conjectures, and I now consider assortative matchings as natural candidates. It will be shown that in many cases conjectures consisting of assortative matchings on the reduced markets are rational. Let's review the following definition adapted to the model in which no two agents share the same type.

Definition 2.2. *A matching* μ *on market* M *is a positive assortative matching (PAM) if for any two pairs* $(i, j), (i', j') \in \mu$*,*

$$f_i > f_{i'} \Leftrightarrow s_j > s_{j'}.$$

A negative assortative matching (NAM) is defined similarly.[12]

Let us denote μ_+^M as the PAM on M and $\mu_+^{M(ij)} = (i,j) \cup \mu_+^{M\backslash\{i,j\}}$ as the matching in which the submatching $\mu^{M\backslash\{i,j\}}$ is the PAM on $M\backslash\{i,j\}$. Similarly, denote μ_-^M as the NAM on M and $\mu_-^{M(ij)} = (i,j) \cup \mu_-^{M\backslash\{i,j\}}$ as the matching in which the submatching $\mu^{M\backslash\{i,j\}}$ is the NAM on $M\backslash\{i,j\}$. For a given conditional reduced market M^r, denote

$$\varphi_+^{M_r}(i,j) := \{\mu_+^{M_s(ij}\}, \text{ for } i,j \in M_s$$

as the i and j's conjecture in which it is a PAM on the corresponding further reduced market. Similarly, denote

$$\varphi_-^{M_r}(i,j) := \{\mu_-^{M_s(ij}\}, \text{ for } i,j \in M_s$$

as the i and j's conjecture in which it is a NAM on the corresponding further reduced market.

2.3 Main Results

First, unfortunately, a stable matching in this two-stage game may not necessarily exist under rational conjectures.

Proposition 2.1. *In the two-stage game with Assumptions 2.1-2.4, a stable matching may not exist under rational conjectures.*

It is proved by the following example.

Example 2.2. *Consider a game with 3 firms and 3 managers. Let the parameters be as follows: Agents' types are* $(f_1, f_2, f_3) = (0.2, 0.25, 0.8)$ *and* $(s_1, s_2, s_3) = (0.2, 0.5, 0.8)$*; the unit production cost function is* $c(f,s) = \frac{1}{f \cdot s}$*;* $A = 80$ *and* $\alpha = 1$*. The surplus function assigns (approximately) values to the pairs of agents in the six matchings as follows.*

$$\mu_1 \begin{array}{ccc} 13 & 426 & 733 \\ f_1 & f_2 & f_3 \\ s_1 & s_2 & s_3 \end{array} \qquad \mu_2 \begin{array}{ccc} 10 & 535 & 657 \\ f_1 & f_2 & f_3 \\ s_1 & s_3 & s_2 \end{array} \qquad \mu_3 \begin{array}{ccc} 320 & 62 & 693 \\ f_1 & f_2 & f_3 \\ s_2 & s_1 & s_3 \end{array}$$

$$\mu_4 \begin{array}{ccc} 234 & 412 & 363 \\ f_1 & f_2 & f_3 \\ s_2 & s_3 & s_1 \end{array} \qquad \mu_5 \begin{array}{ccc} 438 & 52 & 609 \\ f_1 & f_2 & f_3 \\ s_3 & s_1 & s_2 \end{array} \qquad \mu_6 \begin{array}{ccc} 356 & 293 & 356 \\ f_1 & f_2 & f_3 \\ s_3 & s_2 & s_1 \end{array}$$

Let $r(ij)$ *denote the stable matchings in the conditional reduced market* $\{M \setminus \{i,j\} | (i,j)\}$*. For* $n = 2$*, there are no externalities, and then rational conjectures can be easily obtained by*

[12] See Legros and Newman (2007) for more on general assortative matchings.

looking at the maximal total surplus of the four agents. Thus,

$$
\begin{array}{lll}
r(11) = \{\mu_2\} & r(12) = \{\mu_4\} & r(13) = \{\mu_5\} \\
r(21) = \{\mu_5\} & r(22) = \{\mu_1\} & r(23) = \{\mu_2\} \\
r(31) = \{\mu_6\} & r(32) = \{\mu_2\} & r(33) = \{\mu_1\}.
\end{array}
$$

These are also agents' estimations. Consequently, the values for the characteristic function are obtained as follows.

$$
\begin{array}{lll}
V(11) = 10 & V(12) = 234 & V(13) = 438 \\
V(21) = 52 & V(22) = 426 & V(23) = 535 \\
V(31) = 356 & V(32) = 656 & V(33) = 733.
\end{array}
$$

Putting these values into the primal problem, I obtain the maximum value $z_{max} = V(13) + V(22) + V(31) = 1221$. *The only admissible matching here is* μ_2. *However,* μ_2 *can only achieve a total surplus of 1201 which is smaller than* z_{max}. *Hence, there exists no stable matching under rational conjectures in this game.*

Remark. *Taxing the unit production cost or subsidizing the production cost can change the existence of stable matching. In the above example, if the government either taxes on production cost by 10 percent (unit production cost function is then* $\frac{1.1}{f \cdot s}$*) or subsidizes production cost by 10 percent (unit production cost function is effectively* $\frac{0.9}{f \cdot s}$*),* $\mu_2 = \{(i_1, j_1), (i_2, j_3), (i_3, j_2)\}$ *would be stable. If the government subsidizes production cost by 30 percent (unit production cost function is then* $\frac{0.7}{f \cdot s}$*), the NAM* $\mu_6 = \{(i_1, j_3), (i_2, j_2), (i_3, j_1)\}$ *would be stable.*

Nevertheless, existence of stable matching under rational conjectures could be ensured in some restricted scenarios. Denote $(c^2)_{12}$ as the cross derivative of c^2. The following theorem provides sufficient conditions to ensure $\varphi_+^{M_r}$ and $\varphi_-^{M_r}$ to be stable under rational conjectures in the corresponding scenarios.

Theorem 2.2. *In the two-stage game with Assumptions 2.1-2.4,*

1. *if* $c_{12} \leq 0$, *then* $\varphi_+^{M_r}$ *is rational for* M_r, *and* μ_+^M *is a stable matching under rational conjectures;*
2. *if* $c_{12} > 0$ *and* $\frac{(c^2)_{12}}{c_{12}} \geq 4 \cdot \frac{n^2-n}{n^2+1} \cdot \frac{A}{n}$, *then* $\varphi_+^{M_r}$ *is rational for* M_r, *and* μ_+^M *is a stable matching under rational conjectures;*
3. *if* $c_{12} > 0$ *and* $\frac{(c^2)_{12}}{c_{12}} \leq 2 \cdot \frac{n^2-n}{n^2+1} \cdot \frac{A}{n}$, *then* $\varphi_-^{M_r}$ *is rational for* M_r, *and* μ_-^M *is a stable matching under rational conjectures.*

Proof. See Appendix 2.A.3. □

These sufficient conditions are based on (*i*) the complementary and substitutive effects between technology and human capital (captured by c_{12}), (*ii*) the relationship between marginal effects (captured by c_1 and c_2) and the substitutive effect, and (*iii*) the market size. In Scenario 1, in which firm technology and human capital are complementary, a PAM is ensured to be a stable matching under rational conjectures.[13] In Scenario 2, in which firm technology and human capital are substitutable, if the ratio of the cross derivative of the square of the cost function and the cross derivative of the unit cost function is sufficiently large, the PAM is still stable. In Scenario 3, in which the market demand is very high, a change in the resulting matching has little influence on the final total output and the market price. In this case a NAM is stable under rational conjectures. The proof is a top-down recursive method. The intuition is provided as below.

We first analyze Scenario 1. Increase the human capital of the manager hired by the best firm (i.e., firm 1) increases by $\Delta > 0$ and keep the other variables constant. Firm 1's best response to this is to produce more, and the other firms' best responses to firm 1's action are to produce less. In the matching framework, there must be some other manager (hired by some firm i other than firm 1) whose human capital decreases by Δ. Firm i would then produce even less, and the other firms response to this by producing more. However, when firm technology and human capital are complementary, for firms other than firm 1 and firm i, firm 1's effect dominates. Hence, by hiring the best manager (i.e., manager 1) firm 1 is the only firm that gains and all other firms suffer. This logic can be applied recursively to firm 2 and manager 2, firm 3 and manager 3, and so on. As a consequence, in this scenario the game ends up with a PAM.

In Scenario 2, the ratio, $\frac{(c^2)_{12}}{c_{12}}$, can be written as $\frac{c_1 c_2}{c_{12}} + c$. The condition is based on the marginal effects to the substitutive effect. The marginal effects and the substitutive effect counter each other. To give the intuition, I start from an assignment game with no externalities. Suppose each firm can only produce one unit and the market price of the good is fixed, but the unit production cost still depends on firm type and manager type. The separation point is $c_{12} = 0$: the PAM is stable if and only if $c_{12} \leq 0$. Then, take into account the effects of c_1, c_2. $c_2 < 0$ implies that by hiring a marginally better manager, a firm can produce marginally more at a lower cost, and thus get a higher profit. Hence, even when $c_{12} > 0$, as long as c_{12} is not too large, a good firm is still willing to spend a little more money to hire a better manager, and the higher the $|c_2|$ value is, the more the firm is willing to pay. Similar logic applies to c_1. Therefore, the separation point, in terms of the value of c_{12}, for PAM being stable and unstable moves upward. As a consequence, when the marginal effects

[13] Noticed that although $\mu_+^{M(ij)}$ is the worse case for agents i and j when they are paired to each other, I cannot directly apply Theorem 2.1 to prove the result for this case.

dominate the substitutive effect, good firms should hire good managers; otherwise, good firms may hire lesser managers and lesser firms may hire good managers.

Theorem 2.2 does not provide a complete solution to the problem, and there are cases not covered. A natural question follows is, could a non-assotative matching be stable when technology and human capital are substitutive but neither condition 3 nor condition 2 is satisfied? The answer is yes, and I prove it by the the following example, which is a modification of Example 2.2.

Example 2.3 (A non-assortative matching being stable.). *Consider a game with 3 firms and 3 managers. Let the parameters be as follows: agents' types are* $(f_1, f_2, f_3) = (0.2, 0.5, 0.8)$ *and* $(s_1, s_2, s_3) = (0.2, 0.5, 0.8)$*; the unit production cost function is* $c(f, s) = \frac{1}{f \cdot s}$*;* $A = 80$ *and* $\alpha = 1$*. The surplus function assigns (approximately) values to the pairs of agents in the six matchings as follows.*

$$\mu_1 \begin{array}{ccc} 7 & 559 & 680 \\ f_1 & f_2 & f_3 \\ s_1 & s_2 & s_3 \end{array} \qquad \mu_2 \begin{array}{ccc} 6 & 625 & 625 \\ f_1 & f_2 & f_3 \\ s_1 & s_3 & s_2 \end{array} \qquad \mu_3 \begin{array}{ccc} 237 & 237 & 568 \\ f_1 & f_2 & f_3 \\ s_2 & s_1 & s_3 \end{array}$$

$$\mu_4 \begin{array}{ccc} 216 & 492 & 340 \\ f_1 & f_2 & f_3 \\ s_2 & s_3 & s_1 \end{array} \qquad \mu_5 \begin{array}{ccc} 340 & 216 & 493 \\ f_1 & f_2 & f_3 \\ s_3 & s_1 & s_2 \end{array} \qquad \mu_6 \begin{array}{ccc} 320 & 405 & 320 \\ f_1 & f_2 & f_3 \\ s_3 & s_2 & s_1 \end{array}$$

The stable matchings in the conditional reduced market $\{M \setminus \{i, j\} | (i, j)\}$*,* $r(ij)$*, can be easily obtained by looking at the maximal total surplus of the four agents in this market.*

$$\begin{array}{lll} r(11) = \{\mu_2\} & r(12) = \{\mu_4\} & r(13) = \{\mu_6\} \\ r(21) = \{\mu_5\} & r(22) = \{\mu_1\} & r(23) = \{\mu_2\} \\ r(31) = \{\mu_6\} & r(32) = \{\mu_2\} & r(33) = \{\mu_1\}. \end{array}$$

These are also agents' estimations. Consequently, the values for the characteristic function are obtained:

$$\begin{array}{lll} V(11) = 7 & V(12) = 237 & V(13) = 340 \\ V(21) = 237 & V(22) = 559 & V(23) = 625 \\ V(31) = 340 & V(32) = 625 & V(33) = 680. \end{array}$$

Putting these values into the primal problem, I obtain the maximum value $z_{max} = V(11) + V(23) + V(32) = 1257$*. The only admissible matching here is* μ_2*, and* μ_2 *can achieve a total surplus of 1257. Therefore, this matching* μ_2*, neither positive assortative nor negative assortative, is stable under rational conjectures.*

2.4 Social Welfare

In this section, I conduct an analysis of social welfare induced by matchings. I say a matching μ **dominates** a matching μ' if the social surplus (i.e., total consumer benefit minus total production cost) generated in the Cournot game conditional on μ being formed is higher than that induced by matching μ', and μ is a **dominant** matching if no other matching induces a higher social surplus than μ does. The result is describes as follows.

Proposition 2.2. *In the two-stage game with Assumptions 2.1-2.4,*

(0). if $c_{12} < 0$ and $n \leq 3$, then μ_+^M is ensured to be a dominant matching;

(1). if $c_{12} < 0$ and $n > 3$, then μ_+^M is not ensured to be a dominant matching;

(2). if $c_{12} < 0$ and $\frac{c_1 c_2}{c_{12}} \leq -\frac{A}{n\alpha}$, then μ_+^M is a dominant matching;

(3). if $c_{12} > 0$ and $\frac{c_1 c_2}{c_{12}} \geq 3 \cdot \frac{A}{n\alpha}$, then μ_+^M is a dominant matching;

(4). if $c_{12} > 0$ and $\frac{A}{n\alpha}$ is sufficiently large, then μ_-^M is a dominant matching;

(5). if $c_{12} = 0$, then μ_+^M is a dominant matching.

Proof. See Appendix 2.A.4. □

A naive guess is that when firm technology and human capital are complementary, the PAM should induce the highest social welfare since it induces the highest total output and low cost agents produces more; when substitutable, the NAM is more likely to be dominant. The above result shows that this conjecture is not correct. In the proof for case (1), I constructe an example to show that when (i) the gap between of the average unit cost of the best $n-2$ firms, who positive assortatively hire the best $n-2$ managers, respectively, and the unit cost of the second worst firm, who hires the second worst manager, is sufficiently large and (ii) it makes little difference on the unit cost for the worst firm to hire the worst manager or the second worst manager, matching $\{(i_n, j_{n-1}), (i_{n-1}, j_n)\} \cup \mu_+^{M \setminus \{i_{n-1}, j_{n-1}, i_n, j_n\}}$ dominates the PAM. (2) and (3) say that the PAM is dominant if the marginal effects are sufficiently larger than the complementary or substitutive effect, respectively. (4) says that the NAM is dominant if the market is sufficiently large. (5) is about the cases in which no externalities present. Similar results can be found for producer surpluses.

2.5 Concluding Remarks

This chapter studies a matching problem in the environment of a Cournot competition. A PAM can be a stable matching under rational expectations when firm technology and human capital are either complimentary or substitutable, while a NAM can be stable under rational expectations only when the two production factors are substitutable and the goods market is sufficiently large. An example was given to show that a stable matching under rational beliefs may not exist when technology and human capital are substitutable. Although I discussed about the existence of stable matchings among assortative matchings, I did not give a general pattern about stable matchings. Example 2.3 and some other simulations lead to the guess that, when firm technology and human capital are substitutable, if a stable matching exists under rational beliefs, this stable matching satisfies a partition pattern: Firms and managers are sorted from the highest type to the lowest type and are partitioned into to several groups, and firms and managers in each group match assortatively. Come back to the existence of equilibrium, some other meaningful sufficient conditions may also be found. Moreover, the core and the equilibrium payoffs are not characterized, and hence further work might be done.[14]

2.A Appendix

2.A.1 Review of Linear Programming (LP)

Given a conditional reduced market $M_r := \{M_s|\mu_c\}$ and a conjecture function φ^{M_r}, we can use Shapley and Shubik's (1971) Linear Programming method to find a φ^{M_r}-stable matching.

(Primal)

$$\max_x z = \sum_{i\in I_s}\sum_{j\in J_s} V(ij|\varphi^{M_r})x_{ij}$$
$$s.t. \ \textstyle\sum_{i\in I_s} x_{ij} \leq 1, \quad \forall j \in J_s$$
$$\textstyle\sum_{j\in J_s} x_{ij} \leq 1, \quad \forall i \in I_s$$
$$x_{ij} \geq 0$$

(Dual)

$$\min_{u,v} w = \sum_{i\in I_s} u_i + \sum_{j\in J_s} v_j$$
$$s.t. \ u_i + v_j \geq V(ij|\varphi^{M_r}), \quad \forall i \in I_s, \forall j \in J_s$$
$$u_i \geq 0$$
$$v_j \geq 0$$

The maximum value z_{max} of the *primal* problem is attained with all $x_{ij} = 0$ or 1. By strong duality both the primal and the dual have the same optimal value, i.e., $z_{max} = w_{min}$. By complementary slackness, if $x^*_{ij} = 1$ then the corresponding dual constraint must be tight which implies $u_i + v_j \geq V(ij|\varphi^{M_r})$.

Take an extreme solution x^* to the PP, I construct a matching μ^*_s such that it satisfies

[14] Generally, it is complicated to define a core. First, for example, in my to-stage game members of a coalition must not only have beliefs about what kind of matchings the agents outside of the coalition will form but also what kind of actions they will play at the second stage. A coalition at the matching stage can play either non-cooperatively or cooperatively at the Cournot competition stage. If they play cooperative, they can ask the best firm (to hire the best manager) to produce at the lowest unit cost for all the firms in the coalition. Hence, one has to be careful about specifying values to the characteristic function.

that

$$(i,j) \in \mu_s^* \textit{ if and only if } x_{ij}^* = 1.$$

Take an solution (u^*, v^*) to the *dual* problem.

Lemma 2.2. *μ_s^* is a φ^{M_r}-stable matching if and only if μ_s^* is admissible.*

Proof. Define $u'_i := \pi(i,j,\mu_s^*) - v_j^*$ for $i, j \in \mu_s^*$. Then, the triplet (μ^*, u', v^*) is a φ-equilibrium. □

The estimation function φ plays the key role in this problem, since problems arise if there is no *admissible* matching with a total surplus of the agents being larger than or equal to z_{max}.

Remark. *For a matching μ to be a φ^{M_r}-stable matching it is necessary but not sufficient that*[15,16]

i. μ is φ^{M_r}-admissible and

ii. $z_{max} \leq \pi(\mu)$.

However, following from Lemma 2.2, the following conditions are sufficient.

Lemma 2.2'. *A matching μ_s is a φ^{M_r}-stable matching if*

i. $\mu_s = \arg\max_{\mu'_s \in A(I_s,J_s)} V(\mu'_s|\varphi^{M_r})$;

ii. μ_s is φ^{M_r}-admissible.

Example. *(More efficient matching may not be stable)*: Let $n = 3$. Then, there are six possible matchings. Consider a surplus function which assigns values to the pairs of agents in the six matchings as follows.

Let the universal conjectures hold: $\varphi_i(j) = \varphi_j(i) = A(i,j)$ for all $i \in I, j \in J$. Then, all the matchings are admissible, and the values for the characteristic function are as following:

$$V(i_2 j_2) = V(i_3 j_3) = 0$$
$$V(ij) = 1 \; for \; all \; (i,j) \notin \{(i_2,j_2),(i_3,j_3)\}.$$

[15] In assignment games with no externalities, these two conditions are sufficient. Moreover, it is also possible that between two admissible matchings, the less efficient (with lower total surplus) matching is a stable matching while the more efficient matching is not. It is illustrated by the following example that follows.

[16] This result differs from Proposition 3.4 of Gudmundsson and Habis (2013). In their proof, the tricky case that when a pair of agents want to increase their payoffs they must take (back) some money from the other agents was overlooked.

$$\begin{array}{cccc} & 100 & 0 & 0 \\ \mu_1 & i_1 & i_2 & i_3 \\ & j_1 & j_2 & j_3 \end{array} \qquad \begin{array}{cccc} & 1 & 1 & 1 \\ \mu_2 & i_1 & i_2 & i_3 \\ & j_2 & j_3 & j_1 \end{array} \qquad \begin{array}{cccc} & 1 & 1 & 1 \\ \mu_3 & i_1 & i_2 & i_3 \\ & j_3 & j_1 & j_2 \end{array}$$

$$\begin{array}{cccc} & 1 & 1 & 1 \\ \mu_4 & i_1 & i_2 & i_3 \\ & j_1 & j_3 & j_2 \end{array} \qquad \begin{array}{cccc} & 1 & 1 & 1 \\ \mu_5 & i_1 & i_2 & i_3 \\ & j_3 & j_2 & j_1 \end{array} \qquad \begin{array}{cccc} & 1 & 1 & 1 \\ \mu_6 & i_1 & i_2 & i_3 \\ & j_2 & j_1 & j_3 \end{array}$$

Hence, the maximum value to the primal problem is $z_{max} = 1$, and $\mu_2, \mu_3, \mu_4, \mu_5,$ *and* μ_6 are all stable matchings. But matching μ_1, the most efficient matching, is not a stable matching since it is blocked by (i_2, j_3) and (i_3, j_2).

Proof of Theorem 2.1. First, for any given problem (I, J, π), if $\varphi(i, j) = A(i, j)$ for all $i \in I$ and all $j \in J$, the set $E_\varphi(I, J, \pi)$ of φ-equilibria is non-empty. This follows from Lemma 2.2, because a μ^* constructed in the above way is admissible.

Second, I show that, for a problem (I, J, π), if $\varphi(i, j) \neq A(i, j)$ for some pair of $i \in I$ and for $j \in J$, then there exists a surplus function π such that $E_\varphi(I, J, \pi) = \emptyset$. Notice that this can not happen if there are only two agents on each side of the market, I only need to look at the cases in which $n = |I| = |J| \geq 3$. The proof is divided into several steps.

Step 1. Let us first consider a sub-LP problem of a pair of agents (i', j'), by ignoring the characteristic value related them. Let $z_{max}(i', j')$ denote the maximum value to the following PP:[17]

$$\begin{aligned} \max_x z = &\sum_{i \in I'} \sum_{j \in J'} V(ij|A(i,j)) \\ s.t. \quad &\sum_{i \in I'} x_{ij} \leq 1 && \forall j \in J', \\ &\sum_{j \in J'} x_{ij} \leq 1 && \forall i \in I', \\ &x_{ij} \geq 0 && \forall i \in I' \textit{ and } j \in J', \end{aligned}$$

where $I' = I \setminus \{i'\}$ and $J' = J \setminus \{j'\}$.

Let $w_{min}(i', j')$ denote the minimum value to the following DP:

$$\begin{aligned} \min_{u,v} w = &\sum_{i \in I'} u_i + \sum_{j \in J'} v_j \\ st. \quad &u_i + v_j \geq V(ij|A(i,j)) && \forall i \in I' \textit{ and } \forall j \in J', \\ &u_i \geq 0 \textit{ and } v_j \geq 0 && \forall i \in I' \textit{ and } \forall j \in J'. \end{aligned}$$

[17] $V(ij|A(i,j)) = min\{\pi(i,j,\mu)|\ (i,j) \in \mu, \mu \in A(i,j)\}$. That is, although I ignore i' and j', the characteristic value of a pair (i,j) is still taken from all the full matchings in $A(i,j)$, but not necessarily in $A(i,j) \cap A(i',j')$.

Again, the fundamental duality theorem tells us that $w_{min}(i', j') = z_{max}(i', j')$, where $w_{min}(i', j')$ is the minimum value of the sum of the payoffs that the other agents demand when (i', j') are going to pair up.

Step 2. W.l.o.g., assume that $\varphi(i_1, j_2) \neq A(i_1, j_2)$. Then there is a $\mu \in A(i_1, j_2) \setminus \varphi(i_1, j_2)$.Let us denote $j'_k = \mu(i_k)$ for all $k \geq 2$. For all $k \geq 3$, let π satisfy

$$\pi(i_k, j'_k, \tilde{\mu}) + z_{max}(i_k, j'_k) > \pi(\tilde{\mu}')$$

for all $\tilde{\mu} \in A(i_k, j'_k)$ and for all $\tilde{\mu}' \notin A(i_k, j'_k)$. This is equivalent to

$$\pi(i_k, j'_k, \tilde{\mu}) > \pi(\tilde{\mu}') - w_{min}(i_k, j'_k)$$

for all $\tilde{\mu} \in A(i_k, j'_k)$ and for all $\tilde{\mu}' \notin A(i_k, j'_k)$. Then any matchings other than μ and $\mu' = \{(i_1, j'_2), (i_2, j_2), (i_3, j'_3), ..., (i_n, j'_n)\}$ are blocked by some pair (i_k, j'_k) for $k \geq 3$. Then, similarly, let π satisfy

$$\pi(i_1, j'_2, \bar{\mu}) > \pi(\mu) - w_{min}(i_1, j'_2)$$

for all $\bar{\mu} \in A(i_1, j'_2)$. Then μ is blocked by (i_1, j'_2). Lastly, let π satisfy

$$\pi(i_1, j_2, \hat{\mu}) > \pi(\mu') - w_{min}(i_1, j_2)$$

for all $\hat{\mu} \in \varphi(i_1, j_2)$. Then, I have that μ' is blocked by (i_1, j_2). Therefore, $E_\varphi(I, J, \pi) = \emptyset$ for the problem (I, J, π) constructed in this way. □

2.A.2 Cournot Competition

Proof of Lemma 2.1. It is a quadratic problem and thus can be solved by a linear system. In the Cournot game each pair $(i, j) \in \mu$, (i, j) chooses $q^{ij}(\mu)$ (also written as $q(i, j, \mu)$) such that

$$q^{ij}(\mu) = \arg\max_{q \geq 0} q \cdot [A - \alpha(q + \sum_{(i',j') \in \mu, i' \neq i, j' \neq j} q^{i'j'}) - c(f_i, s_j)]$$

The first order condition for each pair $(i, j) \in \mu$ gives us the following linear system:

$$\begin{bmatrix} A - c(f_{i_1}, s_{\mu(i_1)}) \\ A - c(f_{i_2}, s_{\mu(i_2)}) \\ \vdots \\ A - c(f_{i_n}, s_{\mu(i_n)}) \end{bmatrix} = \alpha \begin{bmatrix} 2 & 1 & \dots & 1 \\ 1 & 2 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots \\ 1 & \dots & 1 & 2 \end{bmatrix} \begin{bmatrix} q(i_1, \mu(i_1), \mu) \\ q(i_2, \mu(i_2), \mu) \\ \vdots \\ q(i_n, \mu(i_n), \mu) \end{bmatrix}.$$

Thus,

$$\begin{bmatrix} q(i_1,\mu(i_1),\mu) \\ q(i_2,\mu(i_2),\mu) \\ \vdots \\ q(i_n,\mu(i_n),\mu) \end{bmatrix} = \frac{1}{\alpha} \begin{bmatrix} \frac{n}{n+1} & -\frac{1}{n+1} & \dots & -\frac{1}{n+1} \\ -\frac{1}{n+1} & \frac{n}{n+1} & -\frac{1}{n+1} & \dots \\ \vdots & \vdots & \vdots & \vdots \\ -\frac{1}{n+1} & \dots & -\frac{1}{n+1} & \frac{n}{n+1} \end{bmatrix} \begin{bmatrix} A - c(f_{i_1}, s_{\mu(i_1)}) \\ A - c(f_{i_2}, s_{\mu(i_2)}) \\ \vdots \\ A - c(f_{i_n}, s_{\mu(i_n)}) \end{bmatrix}.$$

Therefore, for a pair $(i,j) \in \mu$, their optimal output quantity for a realized matching μ is

$$q^{ij}(\mu) = \frac{1}{(n+1)\alpha}[A - (n+1)c(f_i, s_j) + \sum_{(i',j')\in\mu} c(f_{i'}, s_{j'})] > 0.$$

Substituting back the above solution to the surplus function, I get

$$\pi(i,j,\mu) = \frac{1}{(n+1)^2\alpha}[A - (n+1)c(f_i, s_j) + \sum_{(i',j')\in\mu} c(f_{i'}, s_{j'})]^2 = \alpha\left[q^{ij}(\mu)\right]^2.$$

□

2.A.3 Stable Matchings under Rational Conjectures

For all the following proofs, I do the following computations with the surplus function derived from above. Let $[M_1, M_2]$ be a partition of market M, in which

$$\begin{aligned} M_1 =& I_1 \cup J_1 := \{i_1, ..., i_{n-2}\} \cup \{j_1, ..., j_{n-2}\} \ \textit{and} \\ M_2 =& I_2 \cup J_2 := M \backslash M_1 = \{l,p\} \cup \{l,p\} \text{ where } f_l > f_p \text{ and } s_l > s_p. \end{aligned}$$

We write $c(f_i, s_j)$ by c^{ij} for short sometimes. For a given $\mu^{M_1} \in A(I_1, J_1)$, I have the following equations:

$$\begin{aligned} &\left[\pi^2(i_l, j_l, \mu^{M_1} \cup \mu_+^{M_2}) + \pi^2(i_p, j_p, \mu^{M_1} \cup \mu_+^{M_2})\right] - \left[\pi^2(i_l, j_p, \mu^{M_1} \cup \mu_-^{M_2}) + \pi^2(i_p, j_l, \mu^{M_1} \cup \mu_-^{M_2})\right] \\ =& \left[\pi^2(i_l, j_l, \mu^{M_1} \cup \mu_+^{M_2}) - \pi^2(i_l, j_p, \mu^{M_1} \cup \mu_-^{M_2})\right] - \left[\pi^2(i_p, j_l, \mu^{M_1} \cup \mu_-^{M_2}) - \pi^2(i_p, j_p, \mu^{M_1} \cup \mu_+^{M_2})\right] \\ =& \frac{1}{(n+1)^2\alpha} \cdot \left[-n(c^{ll} - c^{lp}) + (c^{pp} - c^{pl})\right] \cdot \left(2A - n(c^{ll} + c^{lp}) + (c^{pp} + c^{pl}) + 2\sum_{(i,j)\in\mu^{M_1}} c^{ij}\right) \\ &- \frac{1}{(n+1)^2\alpha} \cdot \left[-n(c^{pl} - c^{pp}) + (c^{lp} - c^{ll})\right] \cdot \left(2A - n(c^{pp} + c^{pl}) + (c^{lp} + c^{ll}) + 2\sum_{(i,j)\in\mu^{M_1}} c^{ij}\right). \end{aligned}$$

For convenience, denote T^1, T^2, T^3, T^4 as the terms in the 4 big brackets on the right hand side of the last equality in order.

Lemma 2.3. *In the two-stage game with Assumptions 2.2-2.4, for a given* $[M_1, M_2]$ *and a*

$\mu^{M_1} \in A(I_1, J_1)$,

1. *if* $c_{12} \leq 0$, *then* $T_1T_2 - T_3T_4 > 0$;
2. *if* $c_{12} > 0$ *and* $\frac{(c^2)_{12}}{c_{12}} \geq 2 \cdot \frac{n^2-n}{n^2+1} \cdot \frac{A+\left(n-2\right)\bar{c}}{n}$, *then* $T_1T_2 - T_3T_4 > 0$;
3. *if* $c_{12} > 0$ *and* $\frac{(c^2)_{12}}{c_{12}} \leq 2 \cdot \frac{n^2-n}{n^2+1} \cdot \frac{A}{n}$, *then* $T_1T_2 - T_3T_4 < 0$.

Proof. First, I have

$$\begin{aligned} c_2 < 0 \ &\Rightarrow T^1, T^3 > 0; \\ \frac{A}{n} - c > 0 \ &\Rightarrow T^2, T^4 > 0. \end{aligned}$$

For [1], it is sufficient to show $T^1 \geq T^3$ and $T^2 > T^4$.

$$\begin{aligned} T^1 - T^3 =& -(n-1)[(c^{ll} - c^{lp}) - (c^{pl} - c^{pp})] = -(n-1)\int_{f_p}^{f_l}\int_{s_p}^{s_l} c_{12}(\hat{f}, \hat{s})d\hat{s}d\hat{f} \geq 0. \\ T^2 - T^4 =& -(n+1)\left[(c^{ll} + c^{lp}) - (c^{pp} + c^{pl})\right] = -(n+1)\int_{f_p}^{f_l} [c_1(\hat{f}, s_l) + c_1(\hat{f}, s_p)]d\hat{f} > 0. \end{aligned}$$

For [2] and [3],

$$\begin{aligned} T_1T_2 - T_3T_4 =& -(n-1)\left[(c^{ll} - c^{lp}) - (c^{pl} - c^{pp})\right]\left(2A + 2\sum_{(i,j)\in\mu^{M_1}} c^{ij}\right) \\ &+ (n^2+1)\left[\left((c^{ll})^2 - (c^{lp})^2\right) - \left((c^{pl})^2 - (c^{pp})^2\right)\right] \\ =& -(n-1)\int_{f_p}^{f_l}\int_{s_p}^{s_l} c_{12}(\tilde{f,\tilde{s}})d\tilde{s}d\tilde{f}\left(2A + 2\sum_{(i,j)\in\mu^{M_1}} c^{ij}\right) \\ &+ (n^2+1)\int_{f_p}^{f_l}\int_{s_p}^{s_l}\left(c_{12}(\tilde{f,\tilde{s}})\right)^2 d\tilde{s}d\tilde{f}, \end{aligned}$$

greater than 0 in [2] and less than 0 in [3].

□

Proof of Theorem 2.2. I will prove by generalization. Take a conditional reduced market $M_r^t := \{M_s^t|\mu_c^t\}$ where $M_s^t := I_s^t \cup J_s^t = \{i_1, i_2, ..., i_t\} \cup \{j_1, j_2, ..., j_t\}$ for a t in $\{2, ..., n\}$. For [1] and [2], want to show that if $\varphi_+^{M_r^t}$ is rational on M_r^t then $\varphi_+^{M_r^{t+1}}$ is rational on M_r^{t+1}. When $t = 2$ it holds, following from Lemma 2.3. When $t > 2$, it is sufficient to show

$$(i). \quad \mu_+^{M_s^t} := \arg\max_{\mu\in A(I_s^t, J_s^t)} V(\mu|\varphi_+^{M_r^t})$$

where $V(\mu|\varphi_+^{M_r^t}) = \sum_{(i,j)\in\mu} \pi^2(i, j, \mu_+^{M_s^t(ij)} \cup \mu_c^t)$. I prove it in a recursive method. That is, $\mu_+^{M_s^t}$ induced by a solution to the PP of the sub-game (see Lemma 2.2' in Appendix 2.A.1).

W.o.l.g., assume $f_{i_1} > f_{i_2} > ... > f_{i_t}$ and $s_{i_1} > s_{i_2} > ... > s_{i_t}$. Take a matching $\mu \in A(I^t_s, J^t_s)$. If $(i_1, j_1) \notin \mu$, replace μ by another matching $\mu' := \{(i_1, j_1), (\mu(j_1), \mu(i_1))\} \cup [\mu \backslash \{(i_1, \mu(i_1)), (\mu(j_1), j_1)\}]$. By Lemma 2.3, I have

$$V(\mu'|\varphi_+^{M^t_r}) > V(\mu|\varphi_+^{M^t_r}).$$

V is maximized only if (i_1, j_1) are paired to each other.

Next, repeat the above procedure, I can get that, conditional on (i_1, j_1) being paired to each other, V is maximized only if (i_2, j_2) are paired to each other. By repeating these steps recursively, I have the V value maximized at $\mu_+^{M^t_s}$.

[3] is proved in a similar way. □

2.A.4 Social Welfare

Proof of Proposition 2.2. W.l.o.g., assume $\alpha = 1$ since α is a scaler, and suppose $f_1 > f_2 > ... > f_n$ and $s_1 > s_2 > ... > s_n$. For a given matching μ, the social benefit is

$$\begin{aligned}
B(\mu) :=& \int_0^{Q(\mu)} A - q \; dq \\
=& AQ(\mu) - \frac{Q^2(\mu)}{2} \\
=& \frac{A}{n+1}[nA - c(\mu)] - \frac{1}{2(n+1)^2}[nA - c(\mu)]^2 \\
=& \frac{1}{n+1}\left(\left[n - \frac{n^2}{2(n+1)}\right] A^2 - \frac{1}{n+1} \cdot Ac(\mu) - \frac{1}{2(n+1)} \cdot [c(\mu)]^2\right),
\end{aligned}$$

and the total cost of production is

$$\begin{aligned}
C(\mu) :=& \frac{1}{n+1} \sum_{(i,j)\in\mu} c^{ij}[A - (n+1)c^{ij} + c(\mu)] \\
=& \frac{1}{n+1}\left[Ac(\mu) - (n+1)\sum_{(i,j)\in\mu} \left(c^{ij}\right)^2 + [c(\mu)]^2\right].
\end{aligned}$$

The social surplus is then

$$\begin{aligned}
&B(\mu) - C(\mu) \\
=& \frac{1}{n+1}\left(\left[n - \frac{n^2}{2(n+1)}\right] A^2 - \frac{n+2}{n+1} Ac(\mu) - \frac{2n+3}{2(n+1)}[c(\mu)]^2 + (n+1)\sum_{(i,j)\in\mu}\left(c^{ij}\right)^2\right).
\end{aligned}$$

For the following part, I take a matching $\mu' \neq \mu^M_+$ and a matching $\mu \neq \mu^M_-$. $(i, j'), (i', j) \in \mu'$, in which $i > i'$ and $j > j'$, and $\mu := \{(i, j), (i', j')\} \cup [\mu' \backslash \{(i, j'), (i', j)\}]$. Then, I derive the following equations:

$$
\begin{aligned}
&[B(\mu) - C(\mu)] - [B(\mu') - C(\mu')] \\
= &- \frac{n+2}{n+1} A[c(\mu) - c(\mu')] - \frac{2n+3}{2(n+1)} \cdot [c(\mu) - c(\mu')][c(\mu) + c(\mu')] \\
&+ (n+1)\left(\left[\left(c^{ij}\right)^2 + \left(c^{i'j'}\right)^2\right] - \left[\left(c^{ij'}\right)^2 + \left(c^{i'j}\right)^2\right]\right) \\
= &- \frac{n+2}{n+1} A[(c^{ij} - c^{ij'}) - (c^{i'j} - c^{i'j'})] - \frac{2n+3}{2(n+1)} \cdot [(c^{ij} - c^{ij'}) - (c^{i'j} - c^{i'j'})][c(\mu) + c(\mu')] \\
&+ (n+1)\left[(c^{ij} - c^{ij'})[(c^{ij} + c^{ij'}) - (c^{i'j'} + c^{i'j})] + [(c^{ij} - c^{ij'}) - (c^{i'j} - c^{i'j'})](c^{i'j'} + c^{i'j})\right]. \\
= &- \frac{n+2}{n+1} A[(c^{ij} - c^{ij'}) - (c^{i'j} - c^{i'j'})] - \frac{2n+3}{2(n+1)} \cdot [(c^{ij} - c^{ij'}) - (c^{i'j} - c^{i'j'})][c(\mu) + c(\mu')] \\
&+ (n+1)\left[[(c^{ij} - c^{ij'}) - (c^{i'j} - c^{i'j'})](c^{ij} + c^{ij'}) + (c^{i'j} - c^{i'j'})[(c^{ij} + c^{ij'}) - (c^{i'j'} + c^{i'j})]\right].
\end{aligned}
$$

For Case 0, I have

$$
\begin{aligned}
&[B(\mu) - C(\mu)] - [B(\mu') - C(\mu')] \\
\geq &- \frac{n+2}{n+1} A[(c^{ij} + c^{i'j'}) - (c^{ij'} + c^{i'j})] \\
&- \frac{2n+3}{2(n+1)} \cdot [(c^{ij} + c^{i'j'}) - (c^{ij'} + c^{i'j})][(c^{ij} + c^{i'j'}) + (c^{ij'} + c^{i'j})] \\
&+ (n+1)[(c_{ij}^2 + c_{i'j'}^2) - (c_{ij'}^2 + c_{i'j}^2)] \\
= &- \frac{n+2}{n+1} A[(c^{ij} + c^{i'j'}) - (c^{ij'} + c^{i'j})] + \frac{n^2-2}{n+1} \cdot \left(\left[\left(c^{ij}\right)^2 + \left(c^{i'j'}\right)^2\right] - \left[\left(c^{ij'}\right)^2 + \left(c^{i'j}\right)^2\right]\right) \\
&+ \frac{2n+3}{2(n+1)}[(c^{ij} - c^{i'j'})^2 - (c^{ij'} - c^{i'j})^2] \\
> &- \frac{n+2}{n+1} A[(c^{ij} + c^{i'j'}) - (c^{ij'} + c^{i'j})] + \frac{n^2-2}{n+1} \cdot [(c^{ij} - c^{ij'}) - (c^{i'j} - c^{i'j'})](c^{i'j} + c^{i'j'}) \\
> &- \frac{1}{n+1}\left[\int_{f_{i'}}^{f_i}\int_{f_{i'}}^{f_i} c_{12}(\hat{f}, \hat{s}) d\hat{f} d\hat{s}\right][(4 + 2n - n^2) \cdot \frac{A}{n}] \\
> &0 \;\; when\ n \leq 3.
\end{aligned}
$$

Hence, when $n \leq 3$, for any matching $\mu' \neq \mu_+^M$, there is another matching that dominates μ', and therefore μ_+^M is a dominant matching.

Case 1. I construct the following example. Let us take two values

$$
\hat{c} := \frac{1}{n-2}\sum_{k=1}^{n-2} c^{kk} \; and \; \tilde{c} := \frac{c^{n-1n-1} + c^{n-1n}}{2},
$$

where $\hat{c} < \tilde{c}$. Take matching $\mu'' := \{(i_n, j_{n-1}), (i_{n-1}, j_n)\} \cup [\mu_+^M \backslash \{(i_{n-1}, j_{n-1}), (i_n, j_n)\}]$. I show in the following that there are some scenarios in which μ'' dominates μ_+^M.

$$
\begin{aligned}
&[B(\mu_+^M) - C(\mu_+^M)] - [B(\mu'') - C(\mu'')] \\
=&(c^{n-1n-1} - c^{n-1n})\big[-\frac{n+2}{n+1}A - \frac{2n+3}{2(n+1)} \cdot [(c^{n-1n-1} + c^{n-1n}) + (c^{nn-1} + c^{nn})] \\
&\qquad + (n+1)(c^{n-1n-1} + c^{n-1n})\big] \\
&- \frac{2n+3}{2(n+1)} \cdot (c^{n-1n-1} - c^{n-1n}) \cdot 2\sum_{k=1}^{n-2} c^{kk} \\
&+ \frac{2n+3}{2(n+1)} \cdot (c^{nn-1} - c^{nn})[c(\mu) + c(\mu')] \\
&- (n+1)(c^{nn-1} - c^{nn})(c^{nn} + c^{nn-1}) \\
<&(c^{n-1n-1} - c^{n-1n}) \left(-\frac{n+2}{n+1}A + \left[(n+1) - \frac{2n+3}{2(n+1)}\right] \cdot 2\hat{c} - \frac{2n+3}{2(n+1)} \cdot 2 \cdot \frac{A}{n}\right) \\
&- \frac{(2n+3)(n-2)}{n+1} \cdot (c^{n-1n-1} - c^{n-1n}) \cdot \tilde{c} \\
&+ (c^{nn-1} - c^{nn}) \left[\frac{2n+3}{2(n+1)} \cdot [c(\mu) + c(\mu')] - (n+1)(c^{nn} + c^{nn-1})\right] \\
=&(c^{n-1n-1} - c^{n-1n}) \cdot \frac{2n^2+2n-1}{n+1} \cdot \left[\hat{c} - \frac{n^2+4n+3}{2n^2+2n-1} \cdot \frac{A}{n} - \frac{2n^2-n-6}{2n^2+2n-1} \cdot \tilde{c}\right] \\
&+ (c^{nn-1} - c^{nn}) \left[\frac{2n+3}{2(n+1)} \cdot [c(\mu) + c(\mu')] - (n+1)(c^{nn} + c^{nn-1})\right] \\
<&(c^{n-1n-1} - c^{n-1n}) \cdot \frac{2n^2+2n-1}{n+1} \cdot \left[(\hat{c} - \tilde{c}) - \frac{n^2+4n+3}{2n^2+2n-1} \cdot \frac{A}{n}\right] \\
&+ (c^{nn-1} - c^{nn}) \left[\frac{2n+3}{2(n+1)} \cdot [c(\mu) + c(\mu')] - (n+1)(c^{nn} + c^{nn-1})\right],
\end{aligned}
$$

in which $\frac{n^2+4n+3}{2n^2+2n-1} < 1$ when $n > 3$. Thus, the term on the right hand side of the last inequality is less than 0 when: $c^{nn-1} - c^{nn} \to 0$, $n > 3$, and $\hat{c} - \tilde{c} > \frac{n^2+4n+3}{2n^2+2n-1} \cdot \frac{A}{n}$. Hence, μ_+^M is dominated by μ'.

Case 2.

$$
\begin{aligned}
&[B(\mu) - C(\mu)] - [B(\mu') - C(\mu')] \\
>&[(c^{ij} - c^{ij'}) - (c^{i'j} - c^{i'j'})]\big(-\frac{n+2}{n+1}A + 2n\frac{A}{n} + (n+1)\frac{(c^{ij} - c^{ij'})[(c^{ij} - c^{i'j}) + (c^{ij'} - c^{i'j'})]}{(c^{ij} - c^{ij'}) - (c^{i'j} - c^{i'j'})}\big) \\
>&\left[\int_{f_{i'}}^{f_i}\int_{s_{j'}}^{s_j} c_{12}(\hat{f},\hat{s})d\hat{s}d\hat{f}\right] \big(\frac{n}{n+1}A + (n+1) \cdot \frac{\left[\int_{s_{j'}}^{s_j} c_2(f_i,\hat{s})d\hat{s}\right] \cdot \left[\int_{f_{i'}}^{f_i}[c_1(\hat{f},s_j) + c_1(\hat{f},s_{j'})]d\hat{f}\right]}{\int_{f_{i'}}^{f_i}\int_{s_{j'}}^{s_j} c_{12}(\hat{f},\hat{s})d\hat{s}d\hat{f}}\big) \\
>&\left[\int_{f_{i'}}^{f_i}\int_{f_{i'}}^{f_i} c_{12}(\hat{f},\hat{s})d\hat{f}d\hat{s}\right] \big(\frac{n}{n+1}A + (n+1) \cdot \frac{\int_{f_{i'}}^{f_i}\int_{s_{j'}}^{s_j} c_1(\hat{f},\hat{s}) \cdot c_2(\hat{f},\hat{s})d\hat{s}d\hat{f}}{\int_{f_{i'}}^{f_i}\int_{s_{j'}}^{s_j} c_{12}(\hat{f},\hat{s})d\hat{s}d\hat{f}}\big) \\
\geq&0.
\end{aligned}
$$

Hence, μ_+^M is a dominant matching.

Case 3.

$$
\begin{aligned}
&[B(\mu) - C(\mu)] - [B(\mu') - C(\mu')] \\
> &\left[\int_{f_{i'}}^{f_i}\int_{s_{j'}}^{s_j} c_{12}(\hat{f},\hat{s})d\hat{s}d\hat{f}\right]\Big(-\frac{n+2}{n+1}A - \frac{2n+3}{2(n+1)}\cdot 2A \\
&\qquad + (n+1)\frac{\left[\int_{s_{j'}}^{s_j} c_2(f_{i'},\hat{s})d\hat{s}\right]\cdot\left[\int_{f_{i'}}^{f_i}[c_1(\hat{f},s_j)+c_1(\hat{f},s_{j'})]d\hat{f}\right]}{\int_{f_{i'}}^{f_i}\int_{s_{j'}}^{s_j} c_{12}(\hat{f},\hat{s})d\hat{s}d\hat{f}}\Big) \\
> &\left[\int_{f_{i'}}^{f_i}\int_{s_{j'}}^{s_j} c_{12}(\hat{f},\hat{s})d\hat{s}d\hat{f}\right]\Big(-\frac{3n+5}{n+1}A + (n+1)\cdot\frac{\int_{f_{i'}}^{f_i}\int_{s_{j'}}^{s_j} c_1(\hat{f},\hat{s})\cdot c_2(\hat{f},\hat{s})d\hat{s}d\hat{f}}{\int_{f_{i'}}^{f_i}\int_{s_{j'}}^{s_j} c_{12}(\hat{f},\hat{s})d\hat{s}d\hat{f}}\Big) \\
\geq &0.
\end{aligned}
$$

Hence, μ_+^M must be the most efficient matching.

Case 4. $[B(\mu) - C(\mu)] - [B(\mu') - C(\mu')] < 0$ when $\frac{A}{n}$ is sufficiently large.

Case 5.

$$
[B(\mu) - C(\mu)] - [B(\mu') - C(\mu')] = \left[\int_{s_{j'}}^{s_j} c_2(f_{i'},\hat{s})d\hat{s}\right]\cdot\left[\int_{f_{i'}}^{f_i}[c_1(\hat{f},s_j)+c_1(\hat{f},s_{j'})]d\hat{f}\right] > 0.
$$

Hence, μ_+^M is a dominant matching. □

Chapter 3

On Disclosure Policies in All-pay Auctions with Stochastic Entry

Introduction

Many real-world competitions, such as rent seeking, political campaigns, and job promotions, are commonly viewed as contests. In these contests, participants spend resources in order to win some prizes. Classic studies of contest theory focused on setting in which there is fixed and known number of contestants and contestants have no private information. However, in many competitions, an individual has no information about the actual number of contestants she has to compete with nor the values of the prizes to the other competitors. For instance, when an individual seeks a job promotion, she has to compete not only with colleagues whom she knows but also anonymous candidates from outside, and she does not know the values of the position to the others. Similarly, in a patent race, a firm may not be aware of the number of other firms who are developing similar products nor the values of the patent to those firms. With some government-contract contests, the government invites selected contractors to submit proposals. In this type of contests, although the contest organizer does not know the values of prizes to the competitors, he may know the exact number of contestants who are participating. In order to maximize his own expected revenue, should he reveal or conceal the actual number of participants?

We address the above issue in a setting of all-pay auctions with a stochastic number of bidders. There is one prize and a fixed number of potential risk neutral bidders, each of whom can receive a signal on the value of the prize to her and an the signal and has a signal-dependent probability of participating in the auction. The realized number of participants is known privately to the organizer. The organizer chooses between two policies: (1) fully revealing the actual number of participants, or (2) fully concealing the number. He publicly announces the chosen policy before the contest starts and commits to implement it during the contest (before bidders start bidding). Each bidder upon participating receives a private signal about her valuation of the prize. The function governing the joint distribution of the signals is common knowledge. Each participating bidder makes a private bid for the prize. The prize is assigned to the bidder with the highest bid, and it is retained if there is no bid. All bidders incur a cost that is a strictly increasing function of their bid. As in the model of Gavious et al. (2002), we differentiate three cases where the cost function is, respectively, convex, concave, or linear.[1] Each bidder chooses her bid to maximize her expected utility, and the organizer chooses a disclosure policy to maximize the expected total bids.

Our first main finding is that the optimal disclosure policy depends on the curvature of bidders' cost function. Specifically, if the cost function is concave (convex), concealing the actual number of participating bidders leads to a lower (higher) expected total bids. However, the ex ante expected utilities of each potential bidder are the same under these two disclosure policies. The key reason is that a participating bidder's interim expected utility, at the time when she learns her signal but not the number of bidders, is invariant to both the curvature of the cost function and the disclosure policy. The cost a participating bidder incurs under full concealment is simply the average value over the costs he incurs under full disclosure. Hence, if the cost function is concave (and the inverse function is convex), under full disclosure the organizer extracts a risk premium from each bidder. Because bidders are indifferent between the two disclosure policies, the optimal disclosure policy for the organizer is thus also Pareto dominant.

We further consider disclosure under endogenous entry: A bidder incurs a cost to enter the contest. Our second and third main results apply the first main result and show that under endogenous entry, concealing the actual number of participating bidders still leads to a lower (higher) expected total bids if the cost function is concave (convex). The reason is that, implied the payoff equivalence result for the potential bidders with exogenous entry, disclosure policies are irrelevant to the potential bidders' entry decisions.

However, the main result cannot be extended to the cases with asymmetric participation probabilities. To illustrate that, we analyze a two-player contest with linear bid cost functions and uniformly distributed independent and private values. The bidder with a lower probability of participation tends to bid more aggressively than her opponent. The contest organizer prefers fully revealing the number of participating bidders to fully concealing it.

This chapter contributes to the scarce literature on information revelation and stochastic entry in contests.[2] Lim and Matros (2009) are the first to study information disclosure policies in Tullock contests with a stochastic number of contestants. In their paper, each potential

[1] See Gavious et al. (2002) for interpretations.

[2] See Morath and Münster (2008), Fu et al. (2014), and Serena (2015) for studies on disclosure policies on contestants' abilities.

contestant has an exogenous probability of participation, and they find that the disclosure policies are irrelevant to the expected total effort. Based on their work, Fu et al. (2011) investigate the same problem in a similar setting with a more general contest success function. They find that the disclosure irrelevance principle does not hold in non-Tullock contests and the optimal disclosure policy depends on the curvature of the characteristic function they define.[3] This is similar to our first main result. Fu et al. (forthcoming) study disclosure policies in a two-player Tullock contest with asymmetric valuations and asymmetric probabilities of participation. In these papers, contestants' valuations of the prize are commonly known, costs are linear, and the probabilities of participation is exogenous. In contrast, our model's major innovation is private information on valuations.[4] We link the optimal disclosure policy with the curvature of the cost function under both exogenous and endogenous probability of participation.

This chapter is also closely related to McAfee and McMillan's (1987) work. They study disclosure policies in first price sealed bid auctions with a stochastic number of participants and independent and private values. Their setting is with a more general exogenous and correlated participation probability structure, and they relate the optimal disclosure policy to bidders' risk altitudes.[5] More specifically, the seller prefers concealing the number of bidders if the bidders have constant absolute risk aversion. They also find that a (potential) bidder's interim expected utilities are the same across the two disclosure policies. We go beyond their paper by considering signal-dependent participation probabilities and endogenous entry.

3.1 The Model

Consider a contest with a set $N = \{1, 2, ..., n\}$ potential risk neutral bidders and one indivisible prize. Each potential bidder participates in the contest with an independent probability. The number of participants is only observable to the contest organizer, and the organizer has to announce publicly and commit to his disclosure policy — either to disclose or conceal the information about the number of participating bidders.

Each bidder i can receive a private signal, $X_i \in [0, 1]$ about her valuation of the prize. The

[3]In Tullock contests, contestant i wins with a probability $\frac{x_i^r}{\sum_j x_j^R}$. Fu et al. (2011) generalize the winning probability function to $\frac{f(x_i)}{\sum_j f(x_j)}$ and define the characteristic function as $\frac{f(x_i)}{f'(x_i)}$.

[4]See Matthews (1987), Levin and Ozdenoren (2004), Levin and Smith (1994), and Ye (2004) for studies on auctions with a stochastic number of bidders.

[5]Take a participating bidder and fixed her interim probability of winning. He is effectively risk averse if the curvature of the cost function is convex, and our result shows that the organizer is better off concealing the information.

value of the prize to the bidder i is

$$V_i = u(X_i, \{X_j\}_{j \in N \setminus i})$$

where $u : [0,1]^n \to [0,1]$ is continuous, strictly increasing in the first variable, and weakly increasing and symmetric in the other variables.

The random variables $X_1, ..., X_n$ are governed by the joint distribution function F, with the density function f. f is strictly positive and symmetric in the signals. Let F_X denote the marginal distribution of a signal.

Stochastic participation and believe updating. A bidder with a signal X_i has a conditional independent probability (CIP) $p(X_i)$ of participating the contest.[6] Because of the dependency of bidders' probabilities of participation on their signals, a bidder updates her belief about the distribution of the other potential bidders' signals according to the Bayesian rule, based on the actual number of participants. Suppose there is a subset $M \subset N$ of participating bidders, besides bidder $i \notin M$. With a little abuse of notation, denote random variable $Y^m = \max\{X_j\}_{j \in M}$, where $m = |M|$, as the highest signal among the participating bidders other than i, and let $\tilde{F}_{Y^m}(\cdot|x)$ denote the corresponding updated cumulative distribution function of Y^m given $X_i = x$, and $\tilde{f}_{Y^m}(\cdot|x)$ be the density function. The updated distribution $\tilde{F}_{Y^m}(y|x)$ is

$$\frac{\int_0^y \cdots \int_0^y \left[\int_0^1 \cdots \int_0^1 \prod_{k=2}^{m+1} p(x_k) \prod_{l=m+2}^{n} (1 - p(x_l)) f_{X_{-1}}(x_2, ..., x_n|x) dx_2 ... dx_{m+1}\right] dx_{m+2} ... dx_n}{\int_0^1 \cdots \int_0^1 \prod_{k=2}^{m+1} p(x_k) \prod_{l=m+2}^{n} (1 - p(x_l)) f_{X_{-1}}(x_2, ..., x_n|x) dx_2 ... dx_n}.$$

Define $v_m(x, y) = E[V_i | X_i = x, Y^m = y]$ as the participating bidder i's expected value of the prize conditional on when there are m participating opponent bidders, i's signal is x, and the highest signal among the m other bidders is y.

Each participating bidder i submits a bid b_i, and bids are submitted simultaneously and independently of each other. The bidder with the highest bid wins the prize, but all participating bidders pay their bids. Ties are resolved by random allocation. A bid b causes a cost $g(b)$, where function $g : R_+ \to R_+$ is strictly increasing, twice continuously differentiable with $g(0) = 0$. When there is a subset M of participating bidders, each bidder bids b_i and the

[6] At this moment, we assume that potential bidder's probability of participation is exogenous but depends on her signal.

payoffs are:

$$W_i = \begin{cases} V_i - g(b_i) & \text{if } b_i > \max_{j\in M\setminus\{i\}} b_j \\ -g(b_i) & \text{if } b_i < \max_{j\in M\setminus\{i\}} b_j \\ \frac{1}{\#\{k\in M: b_k=b_i\}} V_i - g(b_i) & \text{if } b_i = \max_{j\in M\setminus\{i\}} b_j. \end{cases}$$

In detail, the model has the following timing:

1. The contest organizer commits to reveal or conceal her private information before the contest starts.
2. Bidders receive signals, and nature chooses the number of participating bidders.
3. The organizer implements his commitment.[7]
4. Bidders submit their bids privately.
5. The one with the highest bid wins the prize.

3.2 Exogenous Participation Probabilities

We will give symmetric equilibrium strategies under the two disclosure policies and compare the expected revenues for the organizer. It is beyond the scope of this chapter to discuss the existence of symmetric equilibrium, and we directly maintain the following assumption throughout the chapter:

Assumption 3.1. *The environment is such that a symmetric pure strategy equilibrium bidding function exists, and it is increasing.*[8] *In particular, this assumption is true if signals are* ***independent****.*

3.2.1 Contests with Full Disclosure

We first consider the subgame in which the organizer reveals the actual number of bidders before they make their bids.

Lemma 3.1. *Suppose the organizer commits to disclose the number of participating bidders. [i.] In a subgame with $m+1$ participating bidders, in a symmetric equilibrium a bidder with a signal x bids*

$$\beta_m(x) = g^{-1}\Big(\int_0^x v_m(s,s)\tilde{f}_{Y^m}(s|s)ds\Big)$$

[7] A thorough analysis on the issue of commitment is beyond the scope of this chapter.

[8] As shown by Krishna and Morgan (1997), a sufficient condition for the existence of a symmetric equilibrium in each case of our model is $v_m(x,s)\tilde{f}_{Y^m}(s|x)$ increasing in x for $m = 1, 2, ..., n-1$.

and the associated expected payoff is

$$\pi_m(x) = \int_0^x v_m(x,s)\tilde{f}_{Y^m}(s|x)ds - \int_0^x v_m(s,s)\tilde{f}_{Y^m}(s|s)ds.$$

[ii.] The organizer's ex ante expected revenue is

$$ER^D = n\sum_{m=0}^{n-1}\int_0^1 p(x)P_m(x)\beta_m(x)dF_X(x),$$

where $P_m(x) := \binom{n-1}{m}\int_0^1 ... \int_0^1 \prod_{k=2}^{m+1} p(x_k)\prod_{l=m+2}^{n}(1-p(x_l))f_{X_{-1}}(x_2,...,x_n|x_1)dx_2...dx_n$ *is the probability of a participating bidder with a signal* x *facing* m *opponents.*

Proof. [i.] Krishna and Morgan (1997) have shown, in their heuristic derivation, that each bidder's equilibrium strategy in terms of spending is

$$c_m(x) = \int_0^x v_m(s,s)\tilde{f}_{Y^m}(s|s)ds.$$

Hence, the equilibrium strategy in terms of bid is $\beta_m(x) = g^{-1}\big(c_m(x)\big)$. The associated expected payoff is $\pi_m(x) = \int_0^x v_m(x,y)f_{Y^m}(y|x)dx - g(\beta_m(x))$.
[ii.] The organizer's ex ante expected revenue is

$$\begin{aligned}
ER^D =& \sum_{m=0}^{n-1}\binom{n}{m+1}\int_0^1 ... \int_0^1 \prod_{k=1}^{m+1} p(x_k)\prod_{l=m+2}^{n}(1-p(x_l))\sum_{i=1}^{m+1}\beta_m(x_i)f(x_1,...,x_n)dx_1....dx_n \\
=& \sum_{m=0}^{n-1}\binom{n}{m+1}\int_0^1 ... \int_0^1 \prod_{k=1}^{m+1} p(x_k)\prod_{l=m+2}^{n}(1-p(x_l))(m+1)\beta_m(x_1)f(x_1,...,x_n)dx_1....dx_n \\
=& \sum_{m=0}^{n-1}\int_0^1 p(x_1)\beta_m(x_1)f_{X_1}(x_1) \\
& \cdot n\binom{n-1}{m}\int_0^1 ... \int_0^1 \prod_{k=2}^{m+1} p(x_k)\prod_{l=m+2}^{n}(1-p(x_l))f(x_2,...,x_n|x_1)dx_2....dx_n dx_1 \\
=& n\sum_{m=0}^{n-1}\int_0^1 p(x_1)P_m(x_1)\beta_m(x_1)f_{X_1}(x_1)dx_1.
\end{aligned}$$

□

In this case a participating bidder with a signal x and facing m opponents has a expected payoff of

$$\pi_m(x) = \int_0^x v_m(x|y)\tilde{f}_{Y^m}(y|x)dy - g(\beta_m(x)).$$

3.2.2 Contests with Full Concealment

Next, we analyze the subgame in which the organizer conceals the information about the actual number of bidders.

Lemma 3.2. *Suppose the organizer commits to conceal the number of participating bidders. [i.] In a symmetric equilibrium, a bidder with a signal x bids*

$$\beta(x|p) = g^{-1}\big(\sum_{m=0}^{n} P_m(x) \int_0^x v_m(s,s)\tilde{f}_{Y^m}(s|s)ds\big)$$

and the associated expected payoff is

$$\pi(x|p) = \sum_{m=0}^{n-1} P_m(x) \left[\int_0^x v_m(x,y)\tilde{f}_{Y^m}(y|x)dy - \int_0^x v_m(s,s)\tilde{f}_{Y^m}(s|s)ds\right].$$

[ii.] The organizer's ex ante expected revenue is

$$ER^C = n\int_0^1 p(x)\beta(x|p)dF_X(x).$$

Proof. [i.] Suppose each of the other participating bidders plays the symmetric strategy $\beta(\cdot|p)$. Suppose bidder i receives a signal x. She bids b to maximize her expected payoff

$$\pi(b,x|p) = \sum_{m=0}^{n-1} P_m(x) \int_0^{\beta^{-1}(b|p)} v_m(x,y)\tilde{f}_{Y^m}(y|x)dy - g(b). \tag{3.1}$$

The first order condition with respect to b yields

$$\sum_{m=0}^{n-1} P_m(x)\tilde{f}_{Y^m}(\beta^{-1}(b|p)|x)v_m(x,\beta^{-1}(b|p))\frac{1}{\beta'(\beta^{-1}(b|p)|p)} - g'(b) = 0. \tag{3.2}$$

At a symmetric equilibrium, $\beta(x|p) = b$ and thus (3.2) becomes

$$\sum_{m=0}^{n-1} P_m(x)\tilde{f}_{Y^m}(x|x)v_m(x,x) - g'(\beta(x|p))\beta'(x|p) = 0$$

and thus

$$\beta(x|p) = g^{-1}\big(\sum_{m=0}^{n-1} P_m(x) \int_0^x v(s,s)\tilde{f}_{Y^m}(s|s)ds\big).$$

The associated expected payoff is $\pi(x|p) = \pi(x, x|p)$. [ii.] The organizer's expected revenue is

$$
\begin{aligned}
ER^C =& \sum_{m=0}^{n-1} \int_0^1 \dots \int_0^1 \binom{n}{m+1} \prod_{k=1}^{m+1} p(x_k) \prod_{l=m+2}^{n} (1 - p(x_l)) \sum_{i=1}^{m+1} \beta(x_i|p) f(x_1, ..., x_n) dx_1 ... dx_n \\
=& n \sum_{m=0}^{n-1} \int_0^1 p(x_1) P_m(x_1) \beta(x_1|p) f_{X_1}(x_1) dx_1 \\
=& n \int_0^1 p(x_1) \beta(x_1|p) f_{X_1}(x_1) dx_1.
\end{aligned}
$$

□

3.2.3 Comparison across Disclosure Policies

Before we compare the ex ante expected revenues to the organizer across the two disclosure policies, we first consider the ex ante expected payoffs to each potential bidder. It is found that each potential bidder is actually indifferent between these two policies no matter what the cost function is.

Proposition 3.1. *With exogenous and CIP's $\{p(x)\}_{x\in[0,1]}$, for whatever g is, under either disclosure policy a participating bidder's interim expected payoff is*

$$
\pi(p) = \int_0^1 \sum_{m=0}^{n-1} P_m(x) \left[\int_0^x v_m(x, y) \tilde{f}_{Y^m}(y|x) dy - \int_0^x v_m(s, s) \tilde{f}_{Y^m}(s|s) ds \right] dF_X(x).
$$

Proof. From Lemmas 3.1 and 3.2, $\pi(p) = \int_0^1 \sum_{m=0}^{n-1} P_m(x)\pi_m(x) dF_X(x) = \int_0^1 \pi(x|p) dF_X(x)$. □

This result is similar to that of McAfee and McMillan (1987). They find payoff equivalence in their setting with independent and private values and constant absolute risk aversion utilities. The intuition is that, under full concealment, a bidder expected value of prize conditional on winning and the cost she exerts, respectively, simply average those under full disclosure. In particular, under each disclosure policy the curvature of the cost function irrelevant to each bidder's interim expected utility and cost, because the allocation rule and bidders' information remain exactly the same and a similar logic of the revenue equivalence theorem applies.[9]

The curvature of the cost function has no effect on a bidder's interim cost, and it translates into the curvature of the bidding function. Hence, it has an effect on the rank of the expected revenues to the organizer between the two disclosure policies. The first main result is stated as below.

Theorem 3.1. *With exogenous and CIP's $\{p(x)\}_{x\in[0,1]}$,*

[9]The incentive compatibility constraints are the same under different cost functions, and the bidder with the lowest signal receives a payoff of 0.

1. *if g is convex, the organizer prefers to conceal the number of participating bidders;*

2. *if g is concave, the organizer prefers to disclose the number of participating bidders;*

3. *if g is linear, the organizer is indifferent between concealing and revealing.*

Proof. We only show the case of g being convex. g^{-1} is then concave. Applying Jensen's inequality to $\beta(x|p)$, we have

$$\begin{aligned}
ER^C &= n\int_0^1 p(x)\beta(x|p)dF_X(x) \\
&= n\int_0^1 p(x)g^{-1}\big(\sum_{m=0}^{n-1} P_m(x)\int_0^x v_m(s,s)\tilde{f}_{Y^m}(s|s)ds)dF_X(x)\big) \\
&\geq n\int_0^1 \sum_{m=0}^{n} p(x)P_m(x)g^{-1}\big(\int_0^x v_m(s,s)f_{Y^m}(s|s)ds)dF_X(x)\big) \\
&= n\int_0^1 \sum_{m=0}^{n} p(x)P_m(x)\beta_m(x)dF_X(x) \\
&= ER^D.
\end{aligned}$$

□

The above theorem states that the organizer prefers concealing the number of participating bidders if the marginal cost of bidding is increasing, whereas he prefers disclosing the information if the marginal cost is decreasing. It is a result similar to that of Fu et al. (2011).[10] They link the disclosure policies to the curvature of their characteristic function in a public information setting. In contrast, we link disclosure policies to the curvature of the cost function in a private information setting. Both findings result from the fact that uncertainty in the number of participating bidders smooths the bidding strategy of each participating bidder. The organizer extracts a risk premium from each participating bidder under full disclosure in our model with a concave cost function, same as in their model with a concave impact function. It can be further shown that full concealment (disclosure) is an optimal disclosure policy among all disclosure policies if the cost function is convex (concave).[11] Together with Proposition 3.1, this property also implies that the optimal disclosure policy for the organizer is also Pareto optimal.

[10]Unlike our finding, in Fu et al. (2011) the ex ante expected payoffs to each potential participant under the two disclosure policies are not equivalent.

[11]The organizer can choose a set of signals, each signal corresponds to a set of numbers of bidders. He can then design a consistent rule to release signals upon observing the actual number of bidders.

3.3 Endogenous Entry Decisions

Previously, we assumed that each agent has an exogenous probability of participation. In this section we consider endogenous participation decisions. Entry is costly, and thus potential bidders decide whether to incur a cost to enter. We make two case distinctions. In one case, a potential bidder receives her signal on the value of the prize only after entering the auction (henceforth, ex ante participation decision). In the other case, a potential bidder receives her signal before making her entry decision (henceforth, interim participation decision). The main result can be applied to these cases.

3.3.1 Ex ante Participation Decisions

We start from ex ante participation decisions and discuss two types of participation costs, fixed and publicly known participation cost and private and independent participation cost. Before proceeding to solve the problems, we make a technical assumption for this section.

Assumption 3.2. *Under disclosure policy D, the interim expected payoff of each bidder is strictly decreasing in the number of opponents, i.e.,* $\pi_m(x) > \pi_{m+1}(x)$.[12]

This assumption is not unnatural. A bidder wins only if her signal is higher than all the other participating bidders' signals. Hence, an additional opponent brings a bad news to the winner. We derive a crucial property which leads to our second main result.

Lemma 3.3. *Suppose* $p(x) = q$ *for all* $x \in [0, 1]$*. Under Assumption 3.2,* $\pi(q)$ *is strictly decreasing in* q*.*

Proof.

$$\pi(q) = \sum_{m=0}^{n-1} P_m(x) \int_0^1 \pi_m(x) dF_X(x) = \int_0^1 \sum_{m=0}^{n-1} \binom{n-1}{m} q^m (1-q)^{n-1-m} \pi_m(x) dF_X(x).$$

By Assumption 3.2, $\pi_m(x)$ is strictly decreasing in m. A binomial distribution with a higher probability of success first order stochastic dominates a binomial distribution with a lower probability of success (see Lemma 8.3 of Wolfstetter, 1999). Then, the desired result follows from the property of first order stochastic dominance. □

The strict monotonicity of $\pi(q)$ helps us to rule out the case of multiple symmetric equilibria under each disclosure policy. In the following, we study two types of endogenous entry, and show that the ex ante probabilities of participation of each potential bidder under the two disclosure policies are identical.

[12]For example, this assumption is satisfied if bidders have independent and private values (IPV), because under IPV $\pi_m(x) = \int_0^x F^m(y) dF_X(y)$.

Fixed Participation Cost. First, we consider fixed and publicly known participation cost. Suppose it costs a potential bidder an amount c to participate. Under disclosure policy K, $K = C, D$, an equilibrium participation probability q_K satisfies

$$\begin{aligned} q_K = 1 \quad &\text{if} \quad \pi^K(1) \geq c \\ q_K \in (0,1) \quad &\text{if} \quad \pi^K(q_K) = c, \; \textit{and} \\ q_K = 0 \quad &\text{if} \quad \pi^K(0) \leq c. \end{aligned}$$

That is, a potential bidder will definitely participate in bidding if the expected benefit from bidding is higher than the cost even if all the other potential bidders participate; she is indifferent between participating or not participating if the other players also play a mixed strategy; she prefers staying outside of the auction if the participation cost is higher than the expected value of the prize ($\pi(0) = E[V_i]$).

Since $\pi^C(q) = \pi^D(q) = \pi(q)$ and $\pi(q)$ is strictly decreasing in q by Lemma 3.3, there must be a unique equilibrium participation probability q_K under disclosure policy K, and the equilibrium probabilities must be of the same value, i.e., $q_C = q_D$.

Proposition 3.2. *Suppose the participation cost for each agent is c. Each agent participates in bidding with the same probability in the symmetric equilibria under the two disclosure policies.*

Private Participation Costs. Second, we consider private and independent participation costs. Each potential bidder i's cost $c_i \in [0, 1]$ is governed by distribution G, with a probability density function $G'(c) > 0$ for all $c \in [0, 1]$. Each potential bidder learns her participation cost before she participates (and her private signal after joining). In a symmetric equilibrium, each agent's participation decision must be a cut-off rule. That is, low cost types participate and high cost types do not. Under disclosure policy K, $K = C, D$, an equilibrium cut-off type c_K is indifferent between participating and not, and thus c_K satisfies[13]

$$\pi^K(q(c_K)) = c_K, \quad \text{where} \;\; q(c) = G(c).$$

As $\pi^K(q)$ is strictly decreasing in q and $q(c)$ is strictly increasing in c, the solution, c_K, of the above equation must be unique. Because $\pi^C(q) = \pi^D(q) = \pi(q)$, the equilibrium cut-off types under the disclosure policies must be of the same value, i.e., $c_C = c_D$. A bidder with any signal x has a probability $q(c_C)$ of participation.

Proposition 3.3. *Suppose each potential bidder has an independent and private participation cost c_i. Each potential bidder has the same probability of participation in the symmetric equilibria under the two disclosure policies.*

[13]Notice that $\pi^K(1) < 1$ and $\pi^K(0) > 0$.

As a consequence of Theorem 4.1 and Propositions 3.2 and 3.3, we have the following result.

Theorem 3.2. *Under ex ante participation decisions,*

1. *if g is convex, the organizer prefers to conceal the number of participating bidders;*
2. *if g is concave, the organizer prefers to disclose the number of participating bidders;*
3. *if g is linear, the organizer is indifferent between concealing and revealing.*

3.3.2 Interim Participation Decisions

Next, we discuss interim participation decisions, and we consider fixed and publicly known participation cost. Suppose it costs a potential bidder c to participate. Each potential bidder independently decides whether to participate after learning her own valuation of the prize. In a symmetric equilibrium, each potential bidder's participation decision must be a cut-off rule. That is, only those who receive high-value signals will participate. Denote x_K as the equilibrium cut-off type under disclosure policy K, $K = C, D$. A potential bidder who receives a signal above x_K strictly prefers participating the contest, and a potential bidder who receives a signal below x_K strictly prefers not to participate. We ignore all zero probability events.

Claim 3.1. *Under disclosure policy K, $K = C, D$, in a symmetric equilibrium with monotone bidding strategies, the cut-off value $x_K = \hat{x}$, where $\hat{x}$ solves*

$$\begin{cases} F_{Y^{n-1}}(\hat{x})E[v_{n-1}(\hat{x}, y)|y \leq \hat{x}] = c & \text{if } c < E[v_{n-1}(1, y)], \\ \hat{x} = 1 & \text{if } c \geq E[v_{n-1}(1, y)]. \end{cases}$$

If $c \geq E[v_{n-1}(1, y)]$, the participation cost is so high that no one is to participate. If $c < E[v_{n-1}(1, y)]$, a potential bidder who receives a signal x_K is in different between participating and not. She obtains a payoff of 0 if she does not participate. If she participates, she will bid 0, because a participating bidder with the lowest type has no incentive to bid more than 0. She loses for sure if there is another participant. She wins only when there is no other participant, which happens only when all other potential bidders' signals are below $\hat{x}$. Her expected payoff is thus the expected value of the prize minus the entry cost, $F_{Y^{n-1}}(\hat{x})E[v_{n-1}(\hat{x}, y)|y \leq \hat{x}] - c$. Equating this term to zero yields the value of $\hat{x}$.

The cut-off type for participation, $\hat{x}$, is invariant to the cost function g and the disclosure policy. Hence, regardless of the disclosure policy, each potential bidder with a signal x has a conditional independent probability of participation

$$p(x) = \begin{cases} 1 & \text{if } x > \hat{x} \\ 0 & \text{if } x < \hat{x}. \end{cases}$$

Applying Theorem 4.1, we have the third main result of this chapter.

Theorem 3.3. *Under interim participation decisions,*

1. *if g is convex, the organizer prefers to conceal the number of participating bidders;*
2. *if g is concave, the organizer prefers to disclose the number of participating bidders;*
3. *if g is linear, the organizer is indifferent between concealing and revealing.*

3.4 Asymmetric Entry with IPV

In the previous sections, agents are symmetric ex ante. However, in many cases agents have different probabilities of participation. For instance, an agent who faces a higher entry cost has a lower probability of participation. A question is, does the main result hold if agents have different probabilities of participation? The answer is no.

To show that, we turn back to exogenous and independent probabilities of participation:p_1 for bidder 1 and p_2 for bidder 2. Without loss of generality, assuming that $p_1 > p_2 > 0$, we conduct an analysis of a case of two agents with independent and private values, i.e., $V_i = X_i$ and $f(x_1, ..., x_n) = \prod_{i=1}^{n} f(x_i)$, and linear bid cost functions, i.e., $g(b) = b$. The distribution functions of the two agents' valuations are identical. However, the result for a general distribution function of valuations is not traceable. We thus consider only the uniform distribution, $F(x) = x$, under which the equilibrium bidding strategies can be derived explicitly.

Contests with Full Disclosure. First, suppose the organizer commits to reveal the number of participants. Because a participating bidder bids 0 if she is the only participant, the organizer earns revenue only when both agents participate. In the case that both agents participate, in the symmetric equilibrium (of this subgame) the bid function of each participating bidder is

$$\beta_1(v) = \int_0^x s dF(s) = \int_0^x s ds = \frac{x^2}{2}.$$

The organizer's ex ante expected revenue can be computed:

$$ER^D = 2p_1p_2 \int_0^1 \beta_1(x)dF(x) = 2p_1p_2 \int_0^1 \frac{x^2}{2}dx = \frac{p_1p_2}{3}.$$

Lemma 3.4. *Suppose $F(x) = x$ and $g(b) = b$. If the organizer commits to disclose the number of participating bidders, the ex ante expected revenue is $ER^D = \frac{p_1p_2}{3}$.*

Contests with Full Concealment. Next, suppose the organizer commits to conceal the number of participants. The equilibrium in our auction with asymmetric entry probabilities is then solved below.

Lemma 3.5. *Suppose $F(x) = x$ and $g(b) = b$. If the organizer commits to conceal the number of participating bidders, there is a unique increasing bidding equilibrium, in which*

$$\beta_1(x|p_1, p_2) = \frac{x^{\frac{p_1}{p_2}+1}}{\frac{1}{p_1} + \frac{1}{p_2}} \quad \text{and} \quad \beta_2(x|p_1, p_2) = \frac{x^{\frac{p_2}{p_1}+1}}{\frac{1}{p_1} + \frac{1}{p_2}}.$$

Proof. See the appendix. □

In equilibrium bidder 2, who has a higher probability of entry, bids more aggressively than bidder 1. This is due to the special assumption we made on the distribution. In general, this is not true, although it seems to be very intuitive.[14]

Comparison across Disclosure Policies. Now, we are ready to compare the expected revenues across the disclosure policies. A direct comparison shows that the organizer prefers to reveal the number of participating bidders. The result is formally stated below.

Proposition 3.4. *Suppose $F(x) = x$ and $g(b) = b$. Full disclosure generates a higher expected revenue than full concealment.*

Proof. See the appendix. □

Hence, our main result can not be extended to the case with asymmetric participation probabilities. Due to the technical difficulty, we are only able to compare the expected payoffs under uniform the distribution. However, we suspect under linear cost it is always true that full disclosure induces a higher expected revenue than full concealment, because full disclosure symmetrizes the game and the bidders in a symmetric game should compete more vigorously.

[14]Intuitively, one may guess that bidder 2 tends to bid more aggressively than bidder 1, because bidder 1 has a higher probability of entry. This is not true. Suppose the distribution function takes the form of $F(x) = x^k$, $k > 0$. Then, bidder 1 bids less aggressively than bidder 2 if $k < 1$; more aggressively if $k \geq 1$

3.5 Concluding Remarks

We examine the impact of disclosure on expected revenue in all-pay auctions with a stochastic number of bidders, and give important insights into the designing problem. We show that whenever the cost function is convex (concave), concealing the number of bidders yields a higher (lower) expected revenue to the organizer. However, any bidder is indifferent between the two policies. The results hold even when participation is endogenous. Although the proof of our main result is similar to that of Fu et al. (2011), the logic behind it is closer to that of McAfee and McMillan (1987).

We end this chapter with two remarks. First, the better policy between full disclosure and full concealment is actually optimal among all disclosure policies, including partial disclosure policies. Second, the question on general revenue comparison under asymmetric participation probabilities is still open. To solve the problem, one has to find a proper methodology.

3.A Appendix

Proof of Lemma 3.5. Firstly, It is a standard result that in equilibrium the maximal bids of the potential bidders are the same. Denote $[0, \bar{b}]$ as each potential agent's biding range. A bidder who has a value of 0 will bid 0, and a bidder who has a value of 1 will bid $\bar{b}$. In an increasing strategy equilibrium, it must be the case that

$$\max\{\beta_1(x|p_1, p_2),\ \beta_2(x|p_1, p_2)\} > 0 \text{ for all } x > 0. \tag{3.3}$$

Otherwise, an bidder with a valuation close to 0 can increase her expected payoff significantly by bidding slightly above 0 to secure her winning.

Secondly, let ϕ_i be bidder i's inverse equilibrium bidding function over the domain $[0, \bar{b})$ and define $\phi_i(\bar{b}) := \min\{x|\beta_i(x|p_1, p_2) = \bar{b}\}$. Suppose bidder i receives a signal $x \in (0, \phi_i(\bar{b}))$. She bids b to maximize her expected payoff

$$\pi(b, x) = (1 - p_{-i})x + p_{-i}F(\phi_{-i}(b))x - b \tag{3.4}$$

The first order condition with respect to b yields

$$p_{-i}F'(\phi_{-i}(b))\phi'_{-i}(b)x = 1 \tag{3.5}$$

In equilibrium, $x = \phi_i(b)$ and thus (3.5) becomes

$$p_{-i}F'(\phi_{-i}(b))\phi'_{-i}(b)\phi_i(b) = 1. \tag{3.6}$$

Substituting $F(x) = x$ back to (3.6), we have

$$\begin{aligned}
p_{-i}\phi'_{-i}(b)\phi_i(b) &= 1. \qquad (3.7)\\
\Leftrightarrow p_{-i} \cdot \frac{\phi'_{-i}(b)}{\phi_{-i}(b)} &= p_i \cdot \frac{\phi'_i(b)}{\phi_i(b)}\\
\Leftrightarrow \int_y^{\bar{b}} p_{-i} \cdot \frac{\phi'_{-i}(b)}{\phi_{-i}(b)} &= \int_y^{\bar{b}} p_i \cdot \frac{\phi'_i(b)}{\phi_i(b)}\\
\Leftrightarrow p_{-i}\big[\ln \phi_{-i}(\bar{b}) - \ln \phi_{-i}(y)\big] &= p_i\big[\ln \phi_i(\bar{b}) - \ln \phi_i(y)\big]. \qquad (3.8)
\end{aligned}$$

Thirdly, incentive compatibility also requires $\phi_i(\bar{b}) = 1$. If this is not true, then bidder i bids $\bar{b}$ if her valuation is in between $\phi_i(\bar{b})$ and 1. Then, if her opponent has a type very close to 1, the opponent can obtain a substantial increase in payoff by bidding slightly above $\bar{b}$ to secure her winning rather than bid below $\bar{b}$. To see that, suppose $\phi_i(\bar{b}) < 1$, then

$$\begin{gathered}
(1 - p_{-i})x + p_{-i}F(1)x - (\bar{b} + \epsilon) > (1 - p_{-i})x + p_{-i}F(\phi_{-i}(b))x - b \qquad (3.9)\\
\text{when } b < \bar{b},\ x \to 1_-,\ \text{and } \epsilon \to 0_+,
\end{gathered}$$

where the term on the left hand side is bidder $-i$'s expected payoff if she bids $\bar{b} + \epsilon$, and the term on the right hand side is the expected payoff if she bids $\phi_{-i}(b)$.

Substituting $\phi_i(\bar{b}) = 1$ back to equation (3.8), we have

$$\begin{aligned}
p_{-i} \ln \phi_{-i}(y) &= p_i \ln \phi_i(y)\\
\Leftrightarrow \big(\phi_{-i}(y)\big)^{p_{-i}} &= \big(\phi_i(y)\big)^{p_i}\\
\Leftrightarrow \phi_i(y) &= \big(\phi_{-i}(y)\big)^{\frac{p_{-i}}{p_i}}. \qquad (3.10)
\end{aligned}$$

Together with 3.3, equation (3.10) implies that $\phi_1(0) = \phi_2(0) = 0$. Substituting equation (3.10) back to equation (3.7), we have

$$\begin{aligned}
\big(\phi_{-i}(y)\big)^{\frac{p_{-i}}{p_i}} \phi'_{-i}(b) &= 1\\
\Leftrightarrow \int_0^y \big(\phi_{-i}(b)\big)^{\frac{p_{-i}}{p_i}} \phi'_{-i}(b)db &= y\\
\Leftrightarrow \frac{p_{-i}\big(\phi_{-i}(y)\big)^{\frac{p_{-i}}{p_i}+1}}{\frac{p_{-i}}{p_i} + 1} &= y\\
\Leftrightarrow \frac{\big(\phi_{-i}(y)\big)^{\frac{p_{-i}}{p_i}+1}}{\frac{1}{p_i} + \frac{1}{p_{-i}}} &= y. \qquad (3.11)
\end{aligned}$$

Hence, in equilibrium bidders' bidding strategies are, for $x \in [0, 1]$,

$$\beta_1(x|p_1, p_2) = \frac{x^{\frac{p_1}{p_2}+1}}{\frac{1}{p_1} + \frac{1}{p_2}} \quad \text{and} \quad \beta_2(x|p_1, p_2) = \frac{x^{\frac{p_2}{p_1}+1}}{\frac{1}{p_1} + \frac{1}{p_2}}.$$

□

Proof of Proposition 3.4. Using the equilibrium bidding functions in Lemma 3.5, we have the expected revenue under full concealment being

$$\begin{aligned}
ER^C &= p_1 \int_0^1 \beta_1(x|p_1,p_2)dx + p_2 \int_0^1 \beta_2(x|p_1,p_2)dx \\
&= p_1 \int_0^1 \frac{x^{\frac{p_1}{p_2}+1}}{\frac{1}{p_1}+\frac{1}{p_2}} dx + p_2 \int_0^1 \frac{x^{\frac{p_2}{p_1}+1}}{\frac{1}{p_1}+\frac{1}{p_2}} dx \\
&= \frac{p_1}{\left(\frac{1}{p_1}+\frac{1}{p_2}\right)\left(\frac{p_1}{p_2}+2\right)} + \frac{p_2}{\left(\frac{1}{p_1}+\frac{1}{p_2}\right)\left(\frac{p_2}{p_1}+2\right)} \\
&= p_1p_2 \cdot \frac{\left(\frac{p_1}{p_2}+1\right)\left(\frac{p_2}{p_1}+2\right)+\left(\frac{p_2}{p_1}+1\right)\left(\frac{p_1}{p_2}+2\right)}{\left(\frac{p_2}{p_1}+1\right)\left(\frac{p_1}{p_2}+2\right)\left(\frac{p_1}{p_2}+1\right)\left(\frac{p_2}{p_1}+2\right)} \\
&= p_1p_2 \cdot \frac{3+2\cdot\frac{p_1}{p_2}+\frac{p_2}{p_1}+3+2\cdot\frac{p_2}{p_1}+\frac{p_1}{p_2}}{\left(2+\frac{p_2}{p_1}+\frac{p_1}{p_2}\right)\left(5+2\cdot\frac{p_1}{p_2}+2\cdot\frac{p_2}{p_1}\right)} \\
&= p_1p_2 \cdot \frac{6+3\cdot\frac{p_1}{p_2}+3\cdot\frac{p_2}{p_1}}{14+9\cdot\frac{p_1}{p_2}+9\cdot\frac{p_2}{p_1}+2\left(\frac{p_2}{p_1}\right)^2+2\left(\frac{p_1}{p_2}\right)^2} \\
&< p_1p_2 \cdot \frac{6+3\cdot\frac{p_1}{p_2}+3\cdot\frac{p_2}{p_1}}{14+9\cdot\frac{p_1}{p_2}+9\cdot\frac{p_2}{p_1}+4} = \frac{p_1p_2}{3} = ER^D,
\end{aligned}$$

where the strict inequality holds because $p_1 \neq p_2$. □

Chapter 4

On Disclosure Policies in Innovation Contests

Introduction

Contests have been practical and prevalent mechanisms for procuring innovations. Prior to World War II, the U.S. Army Air Corps regularly sponsored prototype tournaments to award production contracts to the winning manufacturers. The Federal Communication Commission (FCC) determined the American broadcast standard for high-definition television by a tournament. IBM sponsors annual tournaments to grant winning contestants to develop their projects for commercial use. In 2006 the Internet television company Netflix announced a $1 million prize to induce a 10 percent improvement in the accuracy of its movie recommendation algorithm.

This chapter studies information disclosure in innovation contests. Most existing studies on information disclosure in innovation contests are input oriented. They focus on how to optimally disclosing intermediate performances so as to elicit a higher deterministic total input (effort). However, in many cases, two ways that elicits the same total input may induce two different outcomes with one outcome strictly dominates the other. In contrast to those studies, my study is output oriented. It is on how to optimally choose disclosure policies to induce a more desirable outcome. In my model, agents search for innovations privately over time — exerting costly effort and observing their own outcomes. At each moment an agent can decide whether to continue searching and whether to submit her best discovery. The game ends once both agents have made a submission, and the organizer only values the best submitted innovation. In such a contest, in order to maximize his own benefit, should the organizer choose (1) to reveal the information on a submission upon receiving it or (2) to conceal the information until all agents have made submissions.

The timing of the game is as follows. First, the organizer announces publicly and commits to his disclosure policy. Second, the organizer implements his commitment throughout the game, and each agent searches for innovations and makes a one-time irrevisable submission of the best innovation she has discovered. Finally the agents who submitted a better innovation wins a prize.

I show that the organizer prefers revealing the information on a submission upon receiving it. On the one hand, it elicits a higher expected revenue to the organizer. On the other hand, the expected length the contest lasts is shorter under this policy. However, each agent is indifferent between these two disclosure policies if the marginal cost of search is low and prefers information concealing if it is high. Moreover, under information concealment, equally share the prize to agents who submit an innovation with a type above a certain cut-off is outcome equivalent to let the agent who submits the best innovation to take the whole prize.

When the organizer chooses to reveal the information on a submission immediately, the contest turns out to be a war of attrition. When the organizer chooses concealing the information until the end, the game effectively turns out to be a special case of Taylor's (1995) model.

There is a rich literature on information disclosure in innovation contests.[1] Aoyagi (2010), Ederer (2010), Goltsman and Mukherjee (2011), and Wirtz (2013) consider two-stage games and study how much information on intermediate performances a contest designer should disclose to elicit the optimal total input. Bimpikis et al. (2014) and Halac et al. (forthcoming) study innovation contest design, which includes both award structures and information disclosure policies. Their models are also output oriented. However, contrasting to my setting, they model an innovation process as a repeated costly sampling process with binary outcomes: failure and success. Therefore, the organizers in their models care about the probability of success rather than the value of the winning innovation. Another common trait in their models but not in my model is believe updating: There is uncertainty about the environment and agents update their beliefs about the environment over time through their past sampling outcomes. If an organizer chooses to make intermediate outcomes public, each agent also updates her belief through her opponents' past sampling outcomes. They show that, among all winner-takes-all contests, the public winner-takes-all contest elicits the highest probability of obtaining an innovation to the organizer. However, the hidden equal sharing rule may be even better for the organizer. On the one hand, I also find, in my winner-takes-all model, that full information disclosure induces a strictly higher expected revenue than full information concealment. On the other hand, any hidden equal sharing rule, however, is outcome equivalent to the winner-takes-all contest with full information concealment and thus elicits a strictly lower expected revenue than the winner-takes-all contest with full information dis-

[1] The incentives for contestant to voluntarily reveal information is also investigated by Yildirim (2005), Gill (2008), and Akcigit and Liu (2015).

closure. Rieck (2010) investigates the organizer's incentive to reveal each agent's innovation progress in the two-period case of Taylor's (1995) multi-period repeated sampling model. He shows that revealing intermediate outcomes would lower the organizer's expected revenue but increase the agents' expected payoff, totally contrasting to my results. This is due to the finiteness and discreteness of his model.

4.1 The Model

In order to obtain an innovation, a risk neutral principal organizes a contest between two risk neutral agents, Agent 1 and Agent 2. Time is continuous and runs from 0 to infinity. At every instance $t \geq 0$, each agent makes three decisions in the following order:

$D1$. Whether to continue searching for innovations.

$D2$. Whether to submit the best innovation discovery to the organizer.

When an agent searches actively, innovations privately arrive to this agent according to a Poisson process with an arrival rate of λ. λ is publicly known, and I assume that $\lambda > c$ to ensure there is at least one participant. The types of innovations are drawn independently from a distribution F, defined on $(0, 1]$. F is continuous and has a strictly positive density f over the domain. The cost of conducting a search is $c > 0$ per unit of time for each agent. Whether an agent is actively searching is not observable to her rival and the organizer.

For convenience, I say an agent is in a **state** $a \in [0, 1]$ at time t if the type of the best innovation she has discovered by time t is a, where $a = 0$ means that the agent has discovered no innovation. Both agents start from state 0.

Each agent can make at most one submission.[2] In particular, I say an agent chooses to quit the contest if she submits a "type-0 innovation". In this case, this previous one-time submission assumption means that once she quits she can not re-enter the contest. The contest ends when both agents have made a submission. When the contest ends, the agent who submitted a better innovation wins a prize normalized to 1. If both agents quit, the prize is reserved; otherwise, a tie between the two submissions is solved by a random allocation of the prize with equal probabilities.

An innovation with a type a is worth $v(a)$ to the organizer, and the organizer only needs one innovation. $v(a)$ is strictly increasing in a. Before the contest starts, the organizer has to announce publicly and commit to one of the two disclosure policies: fully revealing (R) or fully concealing (C) the information about the submissions of the agents. Under full revelation, once the organizer receives a submission he reveals the information on the submission to the

[2]This assumption is without loss of generality and simplifies the analysis.

public, and thus the public knows when and who has submitted what type of innovation or has quit; under full concealment, the organizer reveals the information on the submissions only when the game ends.

In detail, the model has the following timing:

1. The organizer publicly announces and commits to Policy R or Policy C.
2. Each agent decides on $D1 - D2$ as described above, based on the information about the opponent's submission and her own state.
3. Once an agent has made submission, the organizer implements his commitment.
4. Once both agents have made a submission, the game ends and the agent who submitted a better innovation wins the prize.

4.2 Results

In this part I solve the contest under the two disclosure policies and then compare the expected revenues to the organizer across the two disclosure policies.

4.2.1 Contests with Full Disclosure

We first consider the case in which the organizer reveals the information a submission upon receiving it. The solution concept I use here is the subgame perfect equilibrium. I focus on symmetric (strategy) equilibrium.

Proposition 4.1. *Under disclosure policy R, there is a symmetric equilibrium described as below.*[3]

i. On the equilibrium path:

- *a. Both agents continue searching if no one has submitted an innovation with a type above $F^{-1}(1 - \frac{c}{\lambda})$.*
- *b. Once an agent discovers an innovation with a type above $F^{-1}(1 - \frac{c}{\lambda})$, this agent stops searching and submits this innovation immediately.*
- *c. An agent submits (or equivalently quits) immediately once the opponent has submitted an innovation with a type above $F^{-1}(1 - \frac{c}{\lambda})$.*

ii. Off the equilibrium path:

- *If an agent's opponent has submitted something with a type a below $F^{-1}(1-\frac{c}{\lambda})$, this agent continues searching until finding an innovation with a type above a. Once she finds such an innovation, she submits it immediately.*

[3]Elements which are associated with zero probability events are ignored throughout the chapter.

Each agent's ex ante expected payoff in equilibrium is 0*.*

Proof. We prove this proposition by backward induction. I first argue for the off the equilibrium path. Suppose an agent j's opponent, agent i, submitted something with a type $a_i < F^{-1}(1 - \frac{c}{\lambda})$. If agent j is in a state a_j above a_i, stopping searching and submitting the innovation a_j is a best response. If agent j is in a state below a_i, her instantaneous gain from continuing searching is

$$\lambda[1 - F(a_i)] - c > 0,$$

where $\lambda[1 - F(a_i)]$ is the instantaneous rate of obtaining an innovation with a type above a_i, or say the instantaneous rate of winning. Therefore, agent j would continue searching until reaching a state above a_i.

Next, I argue for the on the equilibrium path. Suppose an agent j's opponent, agent i, submitted something with a type $a_i > F^{-1}(1 - \frac{c}{\lambda})$. For agent j, in a state $a_j < a_i$, stop searching and submit a_j is a best response, because the instantaneous gain from continuing searching is

$$\lambda[1 - F(a_i)] - c < 0.$$

Then, given [i(c)], each agent who has discovered an innovation with a type above $F^{-1}(1 - \frac{c}{\lambda})$ should submit it immediately so as to deter her opponent and win the contest.

Last, it is easy to verify that given [ii] and [i(c)]when no when no submission has been made, it is optimal to continue searching until one agent reaches a state above $F^{-1}(1 - \frac{c}{\lambda})$. Given that agent j plays the prescribed strategy, agent i can win the contest only if she is the first to submit an innovation with a type above $F^{-1}(1 - \frac{c}{\lambda})$. However, the instantaneous rate of obtaining such innovation from searching is $\lambda[1 - F(F^{-1}(1 - \frac{c}{\lambda}))]$, which equals to c. She is indifferent between continuing search and not. Hence, continuing searching until one agent reaches a state above $F^{-1}(1 - \frac{c}{\lambda})$ is a best response. To this point, I have shown that the prescribed strategy profile is a subgame perfect equilibrium.

Additionally, I can also show that, given [ii] and [i(c)], in a symmetric equilibrium, no agent would like to submit an innovation with a type below $F^{-1}(1 - \frac{c}{\lambda})$. Suppose there is a none-zero probability event that an agent submits an innovation with a type below $F^{-1}(1 - \frac{c}{\lambda})$. I argue that when no submission has been made and the event has not happened, an agent in a state below $F^{-1}(1 - \frac{c}{\lambda})$ strictly prefers continuing searching to stopping searching. Suppose agent i is in a state below $F^{-1}(1 - \frac{c}{\lambda})$. If she makes a submission, by [ii], she will lose the contest for sure, obtaining an expected continuation payoff of 0. However, if she continues searching, she can obtain a strictly positive continuation payoff: At any instance, she has a rate of c to obtain an innovation with a type above $F^{-1}(1 - \frac{c}{\lambda})$ to win the contest; there is also a positive probability that agent j submits an innovation with a type below $F^{-1}(1 - \frac{c}{\lambda})$, in which case, by [*ii*], agent i can obtain a strictly positive continuation payoff.

In equilibrium, an agent's instantaneous payoff is always 0, and thus her ex ante expected payoff is also 0.

□

In fact, under policy R the contest turns to be a war of attrition. Both agents continue investing until eventually one agent gets into a high state and the other agent gives up. It then is natural that each agent obtains an ex ante expected payoff of zero.

Remark. *1. There are many other outcome equivalent equilibria.*[4] *I choose a most fast-ended equilibrium, in which there is no lag between stopping searching and making a submission, because this is robust to small perturbations on the setting. Adding a discount rate and a small reward for quiting the contest, this equilibrium would be the unique equilibrium.*

2. The setting with full disclosure is equivalent to award the prize to the agent who is the first to submit an innovation with a type above $F^{-1}(1-\frac{c}{\lambda})$.[5]

4.2.2 Contests with Full Concealment

Next, I analyze the case in which the organizer conceals the information about the first submission. This is, however, effectively the same as the model of Taylor (1995) in the case that the deadline approaches infinity. In both models each agent's searching and submission decisions and searching outcomes are privately known. If the deadline in the Taylor's model goes to infinity, the contest effectively ends when both agents stop searching, which is exactly the case here. Hence, the solution my model should coincide with the solution to the continuous time version of Taylor's model (see Chapter 1). Same as these two mentioned papers, I only consider constant cut-off equilibrium.[6,7]

Proposition 4.2. *Suppose the organizer commits to policy C.*

i. If $c < \frac{\lambda}{2}$, there is an equilibrium, in which each agent continue searching with $F^{-1}(1-\frac{2c}{\lambda})$ as the cut-off and submits the innovation once she stops searching.

ii. If $c > \frac{\lambda}{2}$, there are two equilibria, in each of which one agent does not search and the other agent searches with $F^{-1}(1-\frac{2c}{\lambda})$ as the cut-off and submits the innovation once she stops searching.

Proof. See Theorem 1.1 of Chapter 1 and take the limit of the deadline in the benchmark model to be infinity. □

[4] For example, another equilibrium would be the above equilibrium strategy profile with a modification on i(c): An agent quits some time after the opponent has submitted an innovation with a type above $F^{-1}(1-\frac{c}{\lambda})$.

[5] There are many outcome equivalent designs for the contest. My focus is on information disclosure policies.

[6] A cut-off is a value of an agent's state, above which this agent stops searching and at or below which she continues searching.

[7] Same as in before, I select the most fast-ended equilibrium (or equilibria).

Lemma 4.1. *Suppose the organizer commits to policy C.*

i If $c < \frac{\lambda}{2}$, each agent's ex ante expected payoff is 0.

ii If $c > \frac{\lambda}{2}$, in an equilibrium, the agents who searches has an ex ante expected payoff of $1 - \frac{c}{\lambda}$ and the other agent has an ex ante expected payoff of 0.

Proof. See Lemma 1.2 of Chapter 1 for [i]. For [ii], the firm who searches wins the contest with a probability of 1, and thus her expected duration of search is $\frac{1}{\lambda}$. Therefore, her expected payoff is $1 - \frac{c}{\lambda}$. The other agent does not search, and thus obtains a payoff of 0. □

Remark. *Another realistic disclosure policy is partial revealing (P), in which it is publicly known whether an agent has submitted an innovation or not, but type of any submitted innovation is not. One can easily verify that all equilirbia under policy R and policy C are equilibria under policy P.*

4.2.3 Comparison across Disclosure Policies

Under policy R, there is only one effective submission in equilibrium, because the other agent quits. The effective submission is of an innovation with a type above $F^{-1}(1-\frac{c}{\lambda})$. Under policy C, if $c > \frac{\lambda}{2}$ there is also only one submission, but the submission could be an innovation with a type below $F^{-1}(1 - \frac{c}{\lambda})$. Hence, for $c > \frac{\lambda}{2}$, policy R elicits a higher expected revenue to the organizer.

However, if $c < \frac{\lambda}{2}$ there are two effective submissions in equilibrium. Although both submitted innovations are of types only above $F^{-1}(1-\frac{2c}{\lambda})$, only the better submission matters to the organizer. It is thus not straightforward which disclosure policy is better for the organizer in this case, one submission above a higher cut-off or two submissions above a lower cut-off. A comparison of the ex ante expected revenues for the organizer across the two disclosure policies shows that policy R elicits a strictly higher expected revenue than policy C does.

Theorem 4.1. *The contest organizer strictly prefers policy R to policy C.*

Proof. The case that $c > \frac{\lambda}{2}$ is argues as above, I only need to show for the case that $c < \frac{\lambda}{2}$.

Under policy R, the expected revenue for the organizer is the expected revenue a submission can generate conditional on the type of the submitted innovation is higher than $F^{-1}(1 - \frac{c}{\lambda})$,

$$\int_{F^{-1}(1-\frac{c}{\lambda})}^{1} \frac{f(a)}{1 - F(F^{-1}(1 - \frac{c}{\lambda}))} \cdot v(a)da.$$

Under policy C, in the case when there are two submissions, the expected revenue is the expected value of the better submission conditional on the types of both submitted innovations are higher than $F^{-1}(1-\frac{2c}{\lambda})$,

$$\int_{F^{-1}(1-\frac{2c}{\lambda})}^{1} \frac{2f(a)F(a)}{1-[F(F^{-1}(1-\frac{2c}{\lambda}))]^2} \cdot v(a)da.$$

To prove that policy R elicits a higher revenue than C does to the organizer, it is sufficient to show that, the distribution of the winning innovation type under policy R first order stochastic dominates the one under policy C. Let us define $G^D : [F^{-1}(1-\frac{2c}{\lambda}), 1] \to [0,1]$ as the distribution under policy $D \in \{R, C\}$. For $a \in [F^{-1}(1-\frac{2c}{\lambda}), F^{-1}(1-\frac{c}{\lambda})]$,

$$G^R(a) = 0 \leq \frac{F^2(a) - [F(F^{-1}(1-\frac{2c}{\lambda}))]^2}{1-[F(F^{-1}(1-\frac{2c}{\lambda}))]^2} = G^C(a).$$

For $a \in (F^{-1}(1-\frac{c}{\lambda}), 1]$,

$$\begin{aligned}
\frac{G^R(a)}{G^C(a)} &= \frac{F(a) - F(F^{-1}(1-\frac{c}{\lambda}))}{1-F(F^{-1}(1-\frac{c}{\lambda}))} \Big/ \frac{F^2(a) - [F(F^{-1}(1-\frac{2c}{\lambda}))]^2}{1-[F(F^{-1}(1-\frac{2c}{\lambda}))]^2} \\
&= \frac{F(a) - (1-\frac{c}{\lambda})}{\frac{c}{\lambda}} \Big/ \frac{F^2(a) - (1-\frac{2c}{\lambda})^2}{1-(1-\frac{2c}{\lambda})^2} \\
&= \frac{F(a) - (1-\frac{c}{\lambda})}{\frac{c}{\lambda}} \Big/ \frac{F^2(a) - (1-\frac{2c}{\lambda})^2}{\frac{2c}{\lambda}(2-\frac{2c}{\lambda})} \\
&= \frac{4[F(a) - (1-\frac{c}{\lambda})](1-\frac{c}{\lambda})}{F^2(a) - (1-\frac{2c}{\lambda})^2} \\
&= \frac{F^2(a) - [2(1-\frac{c}{\lambda}) - F(a)]^2}{F^2(a) - (1-\frac{2c}{\lambda})^2} \\
&= \frac{F^2(a) - [(2-F(a)-\frac{2c}{\lambda})]^2}{F^2(a) - (1-\frac{2c}{\lambda})^2} \\
&\leq 1 \ (= 1 \text{ iff } a = 1).
\end{aligned}$$

Hence, the desired result derives. □

As what is shown in the proof, although there could be two submissions in the equilibrium under policy C, the probability of obtaining a good submission is lower than that under policy R, whatever the distribution of innovation types is.

Though the organizer prefers policy R to policy C, the agents' preferences have two possibilities. If search cost is below $\frac{\lambda}{2}$, agents are indifferent between the two policies. As shown previously, each agent obtains an ex ante expected payoff of 0 under either disclosure policy in this case. If search cost is above $\frac{\lambda}{2}$, then the agents prefer policy C to policy R, as under policy C each of them has some probability to be the single player in the contest and thus can obtain a positive expected payoff, whereas under policy R both of them obtain an expected payoff of 0. Formally, I state the result as below.

Corollary 4.1.

i. For $c < \frac{\lambda}{2}$, agents are indifferent between policy R and policy C.

ii. For $c > \frac{\lambda}{2}$, agents prefers policy C to policy R.

The next question is, which game is more efficient, or say, under which policy does the contest ends quicker in expectation? The answer is unambiguous.

For search cost being below $\frac{\lambda}{2}$, under either policy, an agent obtains the prize with a probability of $\frac{1}{2}$ and an expected payoff of 0. Hence, each agent's expected cost on search is $\frac{1}{2}$, and the expected during of search is $\frac{1}{2c}$ under either disclosure policy. However, under policy R agents stop searching at the same time, whereas under policy C agents stop searching independently. The game ends when both agents stop searching. Therefore, expectedly, the contest ends quicker under policy R in this case. For search cost being above $\frac{\lambda}{2}$, under policy R the expected duration is still $\frac{1}{2c}$, whereas under policy C it is $\frac{1}{\lambda}$, a standard result from the probability theory. Because $\frac{1}{2c} < \frac{1}{\lambda}$, the contest ends quicker under policy R in this case as well. I formally state this result as below.

Corollary 4.2. *The expected duration of the contest is shorter under disclosure policy R than under policy C.*

Hence, the organizer prefers policy R not only for that it elicits a higher expected revenue but also that it is more efficient.

In contrast to my results, Rieck (2010) finds, in a multi-period model of Taylor (1995), that revealing intermediate outcomes would, however, lower the organizer's expected revenue but increases the agents' expected payoffs. This is because of the finiteness of the deadline in his model. In a continuous time version of his model, taking the limit of the deadline to be infinity, the equilibrium under information revealing policy actually coincides with the equilibrium in my model.[8]

Halac et al. (forthcoming) builds a continuous time model in which sampling outcomes of each agent are binary and beliefs about the state of the world are updated over time based on the past sampling outcomes. They also find in their model that the public winner-takes-all contest is optimal among all winner-takes-all contests. However, the hidden equal-sharing contest may induce a higher probability of success than the public winner-takes-all contest. In my setting, there is no belief updating. There could also be hidden equal-sharing rules:

ER: *the prize is shared equally to both agents if both submitted innovations are above certain*

[8] In the multi-period model of Rieck (2010), in the limit of the deadline being infinity, it is not an equilibrium that both agents continue searching until one obtains an innovation with a type above a certain cut-off.

cut-off and the prize is given to the agent who submitted a better innovation otherwise.[9]

However, unlike the finding of Halac et al. (forthcoming), in my setting any hidden equal-sharing rule is strictly dominated by the winner-takes-all contest with policy R. In fact, I can make the following claim.

Claim 4.1. *Any hidden equal-sharing rule is outcome equivalent to the winner-takes-all contest with policy C.*

Proof. We first analyze the case that $c < \frac{\lambda}{2}$. If the cut-off set by the organizer is below or equal to $F^{-1}(1 - \frac{2c}{\lambda})$, it is not effective, because no agent is going to search with that value as the cut-off. The game is exactly the same as the winner-takes-all contest with policy C. If the cut-off set by the organizer is above $F^{-1}(1 - \frac{2c}{\lambda})$, the expected share of the prize an agent can win is still $\frac{1}{2}$, same as that in the winner-takes-all contest with policy C, and thus both agents will still search with $F^{-1}(1 - \frac{2c}{\lambda})$ as the cut-off in equilibrium.

The analysis for the cases that $c > \frac{\lambda}{2}$ is similar and thus is omitted. □

In short, a hidden equal-sharing rule is just a modification of the winner-takes-all contest with policy C. It merely changes the realized prize allocation but not the expected share of the prize each agent will win, and thus it does not alter agents' incentives for searching. Hence, the competition strategy of each agent remains the same.

4.3 Concluding Remarks

In this study, I found that in innovation contests revealing information on contestants' submitted innovations yields a higher expected revenue to the organizer and is more efficient than concealing it until the end. However, contestants are indifferent between the two policies if the search cost is low and prefer information concealment if the search cost is high. The results can be extended to the cases with more than two agents (see the appendix). There are two other possible extensions. One is on contests with asymmetric agents and the other is on contests with a deadline.[10] Unfortunately, due to lack of closed form equilibrium solutions, one cannot make direct comparisons of the expected revenues across disclosure policies. Proper methodologies have to be found. However, I conjecture that the main result holds in these two extensions. Last, like in many studies on information disclosure, the organizer's ability of commitment is also an issue in my model. When announced to reveal information, the organizer may want to hide the information when he receives a submission so that the remaining agent(s) may submit an even better innovation.

[9] One can also look at another rule, in which the prize is shared equally to agents who submitted innovations above a certain cut-off and the prize is reserved otherwise.

[10] One can refer to Section 1.6 of Chapter 1 for the case with heterogeneous search costs.

4.A Appendix

I discuss the extension to include $N > 2$ agents. In this extension, the main result still holds.

Under disclosure policy R, when there are more than 2 agents, there are uncountably many symmetric equilibria. Some properties a (most fast-ended) symmetric strategy equilibrium has to satisfy are describe below.

i. On the equilibrium path:

   a. Before some agent submit an innovation with a type above or at $F^{-1}(1-\frac{c}{\lambda})$, no agent stops searching before she discovers an innovation.

   b. Once an agent discovers an innovation with a type above $F^{-1}(1-\frac{c}{\lambda})$, this agent stops searching and submits this innovation immediately.

   c. An agent stops searching and submits immediately once one of the opponent has submitted an innovation with a type above $F^{-1}(1-\frac{c}{\lambda})$.

ii. Off the equilibrium path:

   a. If $N-1$ agents have submitted but no submission is of an innovation with a type above $F^{-1}(1-\frac{c}{\lambda})$, the last agent continues searching until finding an innovation with a type above a. Once she finds such an innovation, she submits it immediately.

   b. If $N-2$ agents have submitted but no submission is of an innovation with a type above $F^{-1}(1-\frac{c}{\lambda})$, the remaining 2 agents continue searching until one of them finds an innovation with a type above a. Once an agent finds such an innovation, she submits it immediately.

For example, suppose there are three agents. One equilibrium path can be like that: All agents continue searching until some agent reaches a state above or at $F^{-1}(1-\frac{c}{\lambda})$ and makes a submission, and then both other agents quit.

Some other symmetric equilibria paths can be like follows. At the beginning, each agent continues searching with $\tilde{a} \in [0, F^{-1}(1-\frac{c}{\lambda}))$ as the cut-off. The first agent who discovers an innovation with a type above this value submits it immediately. If the first submitted innovation is with a type above $F^{-1}(1-\frac{c}{\lambda})$, all other agents quit and the contest ends. If the first submitted innovation is with a type below $F^{-1}(1-\frac{c}{\lambda})$, the remaining two agents continue searching until one agent reaches a state above or at $F^{-1}(1-\frac{c}{\lambda})$.

In brief, on an equilibrium path there must be at least two agents playing the war of attrition. The distribution of the winning innovation type is the distribution of an innovation type conditional on it being above $F^{-1}(1-\frac{c}{\lambda})$, i.e., $G^R(\cdot)$, same as when there are only two agents.

Under disclosure policy C, equilibrium strategies are also similar to the case when there are two agents.

i. If $c \in (0, \frac{\lambda}{N})$, there is a unique equilibrium, in which all agents continue searching with $F^{-1}(1-\frac{Nc}{\lambda})$ as the cut-off.

ii. If $c \in (\frac{\lambda}{m}, \frac{\lambda}{m-1})$, $m = 2, 3, ..., N$, there are $\binom{N}{N-m+1}$ equilibria, in each of which $m-1$ agents search with 0 as the cut-off and the other agents do not search.

Hence, if $c \in (0, \frac{\lambda}{N})$, the distribution of the winning innovation type is the distribution of the first order statistic for a sample of N from the distribution of an innovation type conditional on it being above $F^{-1}(1-\frac{Nc}{\lambda})$; if $c \in (\frac{\lambda}{m}, \frac{\lambda}{m-1})$, $m = 2, 3, ..., N$, the distribution of the winning innovation type is the distribution of the first order statistic for a sample of $m-1$ from the distribution F.

I can show that the main result still holds when there are N agents.

Claim 4.2. *Given $N > 2$ agents, the contest organizer strictly prefers policy R to policy C.*

Proof. It is sufficient to prove for the case that $c \in (0, \frac{\lambda}{N})$. As what I do in the proof of Theorem 4.1, I prove it by showing that the distribution of the winning innovation type under policy R first order stochastic dominates that under policy C.

Let us define the the distribution of the winning innovation type under policy C by $\tilde{G}^C$: $[F^{-1}(1-\frac{Nc}{\lambda}), 1] \to [0, 1]$. For $a \in [F^{-1}(1-\frac{Nc}{\lambda}), F^{-1}(1-\frac{c}{\lambda})]$,

$$G^R(a) = 0 \leq \frac{F^N(a) - [F(F^{-1}(1-\frac{2c}{\lambda}))]^N}{1-[F(F^{-1}(1-\frac{2c}{\lambda}))]^N} = \tilde{G}^C(a).$$

For $a \in (F^{-1}(1-\frac{c}{\lambda}), 1]$,

$$\begin{aligned}
G^R(a) - \tilde{G}^C(a) &= \frac{F(a) - F(F^{-1}(1-\frac{c}{\lambda}))}{1 - F(F^{-1}(1-\frac{c}{\lambda}))} - \frac{F^N(a) - [F(F^{-1}(1-\frac{2c}{\lambda}))]^N}{1 - [F(F^{-1}(1-\frac{2c}{\lambda}))]^N} \\
&= \frac{[F(a) - (1-\frac{c}{\lambda})][1-(1-\frac{2c}{\lambda})^N] - \frac{c}{\lambda}[F^N(a) - (1-\frac{2c}{\lambda})^N]}{\frac{c}{\lambda}[1-(1-\frac{2c}{\lambda})^N]} \\
&= \frac{[F(a) - (1-\frac{c}{\lambda}) - F(a)(1-\frac{2c}{\lambda})^N + (1-\frac{2c}{\lambda})^{N+1}] - \frac{c}{\lambda}F^N(a) - [\frac{c}{\lambda}(1-\frac{2c}{\lambda})^N]}{\frac{c}{\lambda}[1-(1-\frac{2c}{\lambda})^N]} \\
&= \frac{\frac{c}{\lambda}[1-F^N(a)] - [1-(1-\frac{2c}{\lambda})^N][1-F(a)]]}{\frac{c}{\lambda}[1-(1-\frac{2c}{\lambda})^N]} \\
&= \frac{[1-F(a)]\left[\frac{c}{\lambda}\sum_{k=0}^{N-1} F^k(a) - \frac{Nc}{\lambda}\sum_{k=0}^{N-1}(1-\frac{Nc}{\lambda})^k\right]}{\frac{c}{\lambda}[1-(1-\frac{2c}{\lambda})^N]} \\
&= \frac{[1-F(a)]\left[\left(\sum_{k=0}^{N-1} F^k(a) - N\right) - \sum_{k=1}^{N-1}(1-\frac{Nc}{\lambda})^k\right]}{1-(1-\frac{2c}{\lambda})^N} \\
&\leq 0 \ (= 0 \text{ iff } a = 1).
\end{aligned}$$

Hence, the desired result derives.

$\square$

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