Affine nilTemperley-Lieb Algebras and Generalized Weyl Algebras:

Combinatorics and Representation Theory

Dissertation

zur Erlangung des Doktorgrades (Dr. rer. nat.)

der

Mathematisch-Naturwissenschaftlichen Fakultät

der

Rheinischen Friedrich-Wilhelms-Universität Bonn

vorgelegt von

Joanna Meinel

aus

Bonn-Duisdorf

Bonn, März 2016

Angefertigt mit Genehmigung der Mathematisch-Naturwissenschaftlichen Fakultät der Rheinischen Friedrich-Wilhelms-Universität Bonn

1. Gutachter:	Prof. Dr. Catharina Stroppel
2. Gutachter:	Prof. Dr. Henning Haahr Andersen
Tag der Promotion:	29.07.2016
Erscheinungsjahr:	2016

Contents

Summary

This thesis lies at the crossroads of representation theory and combinatorics. It is subdivided into two parts, each of which is devoted to a particular combinatorial technique in the study of weight modules.

In the first part, we start out by a short review of crystal bases for finite-dimensional simple modules of the quantum group $\mathcal{U}_q(\mathfrak{sl}_n(\mathbb{C}))$ and for Kirillov–Reshetikhin modules of the quantum affine algebra $\mathcal{U}_q(\widetilde{\mathfrak{sl}}_n(\mathbb{C}))$. We identify crystal bases with combinatorially defined particle configurations on a lattice. Such particle configurations consist of a finite number of particles distributed along a line segment (the finite/classical case) or along a circle (the affine case). There are two versions present: Fermionic configurations where only one particle is allowed at each position, and bosonic configurations where arbitrarily many particles are admissible. Under this identification, Kashiwara crystal operators correspond to particle propagation operators, pushing particles from one position in the lattice to another. These operators satisfy the plactic relations, and we want to describe the algebras that act faithfully on the particle configurations.

It is known that the nilTemperley–Lieb algebra acts faithfully on fermionic particle configurations on a line segment. For bosonic particle configurations on line segments, we prove faithfulness of the action of the so-called partic algebra, which we define as a quotient of the plactic algebra. We construct a basis of the partic algebra, and we describe its center.

The question becomes substantially harder in the affine case. For fermionic particle configurations on a circle it is the affine nilTemperley–Lieb algebra that acts faithfully. This is an infinite dimensional algebra defined by generators and relations. Our main results for the affine nilTemperley–Lieb algebras include different bases of the algebra, an explicit description of its center, and a classification of its simple modules. Furthermore, we define embeddings of the affine nilTemperley–Lieb algebra on N generators into the affine nilTemperley–Lieb algebra on $N+1$ generators.

For bosonic particle configurations on a circle we find an interesting family of additional relations that are not obvious from the classical case.

The second part of the thesis exhibits a different combinatorial approach to weight modules, namely that of discrete geometry applied to the support of a module. This time we consider the representation theory of generalized Weyl algebras, a class of algebras that generalizes the definition of the Weyl algebra, the algebra of differential operators on a polynomial ring. Its weight modules allow a beautiful description in terms of lattice points and hyperplanes.

We apply a theorem by Musson and Van den Bergh [MB98] to a special class of generalized Weyl algebras, thereby proving a Duflo type theorem stating that the annihilator of any simple module is in fact given by the annihilator of a simple highest weight module.

Introduction

The interplay of representation theory and combinatorics builds on a long tradition. In particular the study of algebras that admit a notion of highest weight modules has turned out to be remarkably fruitful. Famous examples are provided by universal enveloping algebras of Lie algebras, quantum groups and Weyl algebras. Within the usually unfathomable category of all modules over such an algebra, it is the subcategory of weight modules that allows for neat combinatorial descriptions. Weight combinatorics have been studied extensively over the past decades, and they continue to be a source of beautiful results with many applications in algebra, geometry and mathematical physics.

In all of the examples above, the algebra is generated by a nice subalgebra whose representation theory is well understood – e.g. a commutative subalgebra – together with some additional generators that come in pairs (often called "positive" and "negative" generators) so that the product or the commutator of each such pair lies in the nice subalgebra. Weight modules are fully reducible modules over the nice subalgebra, the irreducible summands are called the weight spaces of the module. The labelling set of the isomorphism classes of simple modules over the nice subalgebra is called set of weights. The positive and negative generators take weight spaces to weight spaces (or 0) in a controlled way – ideally, each weight space is taken to one particular other weight space, so one gets an action of the positive and negative generators on the set of weights.

The classical example is the simple Lie algebra $\mathfrak{sl}_n(\mathbb{C})$ with its triangular decomposition into upper and lower triangular matrices and the commutative subalgebra of diagonal matrices $\mathfrak{h}$, together with its highest weight modules in category $\mathcal{O}$ with weights in $\mathfrak{h}^*$, see [BGG76], [Hum08]. This can be generalized to a theory of Lie algebras with a triangular decomposition as in [MP95, Sections 2.1, 2.2], [RCW82]. Also the notion of category $\mathcal{O}$ can be extended to Lie algebras with a triangular decomposition [Kha15]. Some characterizations of simple highest weight modules carry over from the complex semisimple Lie algebra case to more general Lie algebras with a triangular decomposition [MZ13].

An important result about highest weight modules for semisimple complex Lie algebras is Duflo's theorem [Duf77]. It states that inside the universal enveloping algebra, all the annihilators of simple modules are given by the annihilators of simple highest weight modules. In contrast, the simple modules themselves are far from being classified in general. This theorem underlines the significance of highest weight modules inside the category of all modules over a semisimple complex Lie algebra. There are some Duflo type theorems for other families of algebras known, see Section II.1.1. One result of this thesis is the proof of a Duflo type theorem for a class of generalized Weyl algebras.

The definition of weight modules opens many possibilities to apply combinatorics to representation theoretic questions. Some of the tools that also appear in this thesis include crystal bases, Young tableaux, and geometry of weight lattices. But also further combinatorial techniques like gradings, central characters, diagrammatical calculus, and (affine) cellular structures are present.

Certain crystal bases for highest weight modules of the quantum group $\mathcal{U}_q(\mathfrak{sl}_n(\mathbb{C}))$ and the quantum affine algebra $\mathcal{U}_q(\widetilde{\mathfrak{sl}}_n(\mathbb{C}))$ can be identified with particle configurations on a lattice, so that the Kashiwara operators correspond to particle propagation operators. Such particle configurations were used in [KS10, Theorem 1.3] to describe the $\widehat{\mathfrak{sl}}_n(\mathbb{C})$-Verlinde algebras, which in turn can be identified with a quotient of the quantum cohomology ring of the Grassmannian, see e.g. [Buc03], [Pos05] and see [ST97] for a presentation by generators and relations. An alternative combinatorial realisation in terms of vicious and osculating walkers is given e.g. in [Kor14].

Overview of the thesis

The thesis consists of two parts. The first part on "Particle configurations and crystals" is split into three chapters, the second part on "Generalized Weyl algebras" contains a single chapter. All chapters are independent from each other, although we include cross-references to indicate connections among them.

Conventions and notation

In both parts of the thesis we use the following conventions unless stated otherwise:

By a module, we always mean a left module. All rings and algebras are associative and unital.

We denote our ground field by $\Bbbk$. If the ground field should satisfy any additional properties (uncountable, algebraically closed and/or of characteristic 0) we indicate this in the beginning of the chapter or section where it applies. In Chapter I.1 we work over the complex numbers $\Bbbk = \mathbb{C}$. In most of Chapters I.2 and I.3 it suffices to assume that $\Bbbk$ be a (commutative) ring, for details see Remark I.2.1.3.

We use δ to denote the Kronecker symbol, i.e. $\delta_{xy} = 1$ if $x = y$ and $\delta_{xy} = 0$ if $x \neq y$. The symmetric group generated by $m - 1$ simple transpositions $(i, i + 1)$ is denoted by $\mathcal{S}_m$.

Part I: Particle configurations and crystals

The first part of the thesis deals with the (classical and affine) plactic algebra, and two interesting quotients: The affine nilTemperley–Lieb algebra, a quotient of the affine local plactic algebra that has been known before, and the partic algebra, a quotient of the classical local plactic algebra that we introduce in this thesis. These algebras are defined by generators and relations over the ground field (or ground ring) $\Bbbk$, and they appear in the study of representation theory and crystal combinatorics of $\mathcal{U}(\mathfrak{sl}_n(\mathbb{C}))$ and $\mathcal{U}(\widetilde{\mathfrak{sl}}_n(\mathbb{C}))$.

Let us briefly introduce these algebras and explain where they come from and why they are interesting. After that we give an overview of our results. For precise definitions and statements see the cross-references.

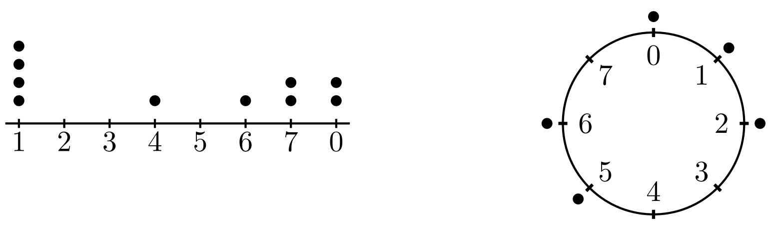
The classical (local) plactic algebra is generated by $a_1, \dots, a_{N-1}$ subject to the so-called plactic relations

$$\begin{aligned} a_i a_j &= a_j a_i && \text{for } |i-j| > 1, \\ a_i a_{i-1} a_i &= a_i a_i a_{i-1} && \text{for } 2 \le i \le N-1, \\ a_i a_{i+1} a_i &= a_{i+1} a_i a_i && \text{for } 1 \le i \le N-2. \end{aligned}$$

For the affine version of the plactic algebra, take generators $a_1, \dots, a_{N-1}, a_0$ with "the same" relations, except that the indices of the generators are now read modulo N. In particular we have additional relations $a_0 a_{N-1} a_0 = a_0 a_0 a_{N-1}$ and $a_0 a_{N-1} a_{N-1} = a_{N-1} a_0 a_{N-1}$, and the generators a_0 and a_{N-1} are neighbours that do not commute (Definition I.3.1.2).

The classical plactic algebra was studied in [FG98]. It is a quotient of the algebra over the "monoide plaxique" defined by Lascoux and Schützenberger [LS81]. These relations are also known as 0-Serre relations from a specialisation of the negative or positive half of $\mathcal{U}_q(\mathfrak{sl}_N(\mathbb{C}))$ to $q = 0$ (Remark I.1.1.9), and they are precisely the relations satisfied in the Hall monoid from [Rei01], [Rei02] (classical type A) and [DD05] (affine type $\widehat{\mathsf{A}}$). Moreover, the Kashiwara operators on certain crystals of type A and $\widehat{\mathsf{A}}$ satisfy the above relations (Section I.1.1). These are the crystals $\mathcal{B}(\omega_k)$ and $\mathcal{B}(k\omega_1)$ associated with the alternating representation $\Lambda^k(\mathbb{C}^N)$ and the symmetric representation $\mathrm{Sym}^k(\mathbb{C}^N)$ of $\mathfrak{sl}_N(\mathbb{C})$, and in the affine case the corresponding Kirillov-Reshetikhin crystals, as discussed in Chapter I.1.

In [KS10], the plactic algebra appears in the study of certain particle configurations. This is also our point of view in Chapter I.2 and I.3: Combinatorially, a particle configuration is defined as a tuple $(k_1, \dots, k_{N-1}, k_0)$ in $\mathbb{Z}_{\ge 0}^N$ (called bosonic) or in $\{0,1\}^N$ (fermionic). One can think of such a tuple as a finite number of particles distributed on a discrete lattice of N positions on a line segment (the finite/classical case) or along a circle (the affine case). In bosonic configurations, arbitrarily many particles are admissible, while in fermionic configurations at most one particle is allowed at each position.



Example for $N = 8$: A bosonic particle configuration on a line segment and a fermionic particle configuration on a circle.

The generators a_i act on the particle configurations (or their $\Bbbk$-span) by lowering k_i by 1 and increasing k_{i+1} by 1, if possible. If not possible, i.e. because $k_i = 0$ or, in the fermionic case, $k_{i+1} = 1$, the result is 0. In the picture this would correspond to (clockwise) propagation of a particle from position i to $i+1$ (Sections I.2.4 and I.3.4). This action can be identified with the action of Kashiwara operators $\tilde{f}_i$ on crystals $\mathcal{B}(\omega_k)$ and $\mathcal{B}(k\omega_1)$ (Section I.1.2).

On affine particle configurations, the additional generator a_0 takes a particle from position 0 and moves it to position 1. If we consider the $\Bbbk[q]$-span instead of the $\Bbbk$-span, we can keep track of the application of a_0 by multiplication with an additional factor q (bosonic) or $\pm q$ (fermionic).

In Chapter I.2 we describe a quotient of the affine plactic algebra that acts faithfully on the $\Bbbk[q]$-span of fermionic particle configurations on a circle. This is the affine nilTemperley–Lieb algebra $\mathrm{n}\widehat{\mathrm{TL}}_N$: It is defined by the additional nil relation $a_i^2 = 0$ for all i. Together with the plactic relations we obtain immediately that also $a_i a_{i\pm1} a_i = 0$ for all i, where we take the indices modulo N. The subalgebra of $\mathrm{n}\widehat{\mathrm{TL}}_N$ generated by $a_1, \dots, a_{N-1}$ is the (classical/finite) nilTemperley–Lieb algebra nTL_N.

Chapter I.3 is devoted to the quotient of the classical plactic algebra that acts faithfully on the $\Bbbk$-span of the bosonic particle configurations on a line segment. The additional defining relation is $a_i a_{i-1} a_{i+1} a_i = a_{i+1} a_i a_{i-1} a_i$ for all $2 \le i \le N-2$. We call this the partic algebra because of its faithful action on the particle configurations. The corresponding action of the affine plactic algebra on bosonic particle configurations on a circle is much harder to describe: We encounter an infinite family of additional relations of the form

$$\begin{aligned} &a_{i+1}^m a_{i+2}^m \dots a_{i-2}^m a_{i-1}^m a_i^{2m} a_{i+1}^m a_{i+2}^m \dots a_{i-2}^m a_{i-1}^m \\ &= a_{j+1}^m a_{j+2}^m \dots a_{j-2}^m a_{j-1}^m a_j^{2m} a_{j+1}^m a_{j+2}^m \dots a_{j-2}^m a_{j-1}^m \qquad \text{for all } i, j \in \mathbb{Z}/N\mathbb{Z},\ m \in \mathbb{Z}_{\ge 1}, \end{aligned}$$

and it is not yet clear whether these relations together with $a_i a_{i-1} a_{i+1} a_i = a_{i+1} a_i a_{i-1} a_i$ for all $i \in \mathbb{Z}/N\mathbb{Z}$ suffice to produce a faithful action.

The following picture recaps the relationship among the algebras studied in Part I:

(affine) plactic algebra

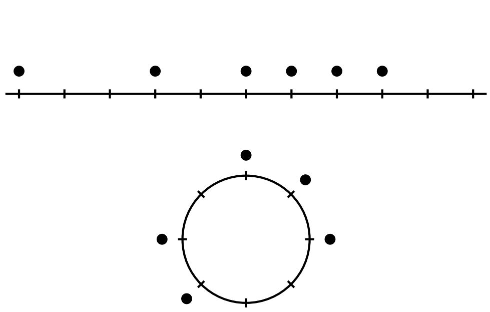
$$
\begin{aligned}
a_i a_j &= a_j a_i \qquad \text{if } |i-j|>1\\
a_i a_{i-1} a_i &= a_i^2 a_{i-1}\\
a_i a_{i+1} a_i &= a_{i+1} a_i^2
\end{aligned}
$$

$a_i^2 = 0$ (to the nilTemperley–Lieb algebra) and $a_{i+1}a_i^2a_{i-1} = a_i a_{i-1} a_{i+1} a_i$ (to the partic algebra)

(affine) nilTemperley–Lieb algebra

$$
\begin{aligned}
a_i a_j &= a_j a_i \qquad \text{if } |i-j|>1\\
a_i a_{i-1} a_i &= 0\\
a_i a_{i+1} a_i &= 0\\
a_i^2 &= 0
\end{aligned}
$$

acts on

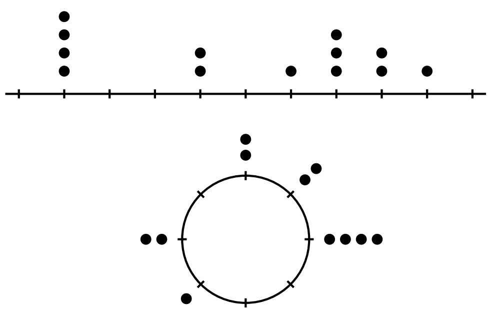


classical and affine versions of fermionic particle configurations/ crystal $\mathcal{B}(\omega_k)$

(affine) partic algebra

$$
\begin{aligned}
a_i a_j &= a_j a_i \qquad \text{if } |i-j|>1\\
a_i a_{i-1} a_i &= a_i^2 a_{i-1}\\
a_i a_{i+1} a_i &= a_{i+1} a_i^2\\
a_{i+1} a_i^2 a_{i-1} &= a_i a_{i-1} a_{i+1} a_i
\end{aligned}
$$

acts on



classical and affine versions of bosonic particle configurations/ crystal $\mathcal{B}(k\omega_1)$

Chapter I.1: Crystal bases and particle configurations

The first chapter is mainly devoted to a review of crystals of classical type A and affine type $\widehat{\mathsf{A}}$. We briefly recall the basic definitions of quantum groups and quantum affine algebras, their finite dimensional irreducible modules, and their crystal bases in Section I.1.1. We consider the action of Kashiwara operators on the crystals $\mathcal{B}(\omega_k)$ and $\mathcal{B}(k\omega_1)$ for the simple $\mathcal{U}_q(\mathfrak{sl}_N(\mathbb{C}))$-modules $L_q(\omega_k)$ and $L_q(k\omega_1)$, corresponding to the alternating representation $\Lambda^k(\mathbb{C}^N)$ and the symmetric representation $\mathrm{Sym}^k(\mathbb{C}^N)$ of $\mathcal{U}(\mathfrak{sl}_N(\mathbb{C}))$, respectively. In affine type $\widehat{\mathsf{A}}$ we study the crystals of the Kirillov-Reshetikhin modules $W^{k,1}$ and $W^{1,k}$ that are isomorphic to $L_q(\omega_k)$ and $L_q(k\omega_1)$ as $\mathcal{U}_q(\mathfrak{sl}_N(\mathbb{C}))$-modules, respectively. In this special case it is particularly easy to describe this operation. We make the following two observations for classical type A, as well as the analogous observations for Kirillov–Reshetikhin crystals in affine type $\widehat{\mathsf{A}}$:

- On $\mathcal{B}(\omega_k)$ and $\mathcal{B}(k\omega_1)$, the Kashiwara operators satisfy the plactic relations, i.e. the 0-Serre relations.
- The crystals $\mathcal{B}(\omega_k)$ and $\mathcal{B}(k\omega_1)$ can be identified with fermionic and bosonic particle configurations, so that the action of the Kashiwara operators is identified with particle propagation operators. These fermionic and bosonic particle configurations are defined purely combinatorially in Section I.1.2.

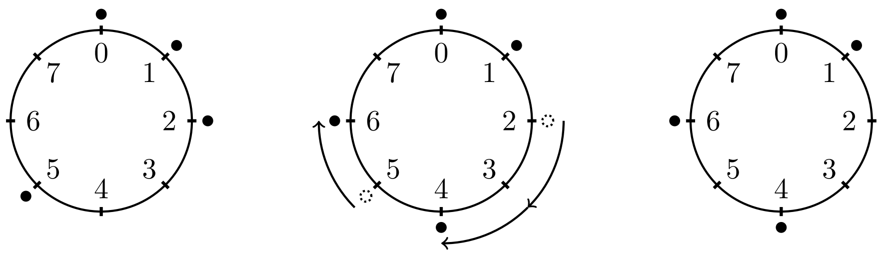
Chapter I.2: Affine nilTemperley–Lieb algebras

The main result of this chapter is a description of the center of the affine nilTemperley–Lieb algebra $\mathrm{n}\widehat{\mathrm{TL}}_N$ over any ground field. Only two tools are used: a fine grading on $\mathrm{n}\widehat{\mathrm{TL}}_N$ and a faithful representation of $\mathrm{n}\widehat{\mathrm{TL}}_N$ on fermionic particle configurations on a circle. We give another, more direct proof of the faithfulness result from [KS10, Proposition 9.1] by constructing a basis for $\mathrm{n}\widehat{\mathrm{TL}}_N$ that is especially adapted to the problem. This basis has further advantages: It can be used to prove that the affine nilTemperley–Lieb algebra is finitely generated over its center. Hence, central quotients are finite dimensional. Also, it can be used to exhibit an explicit embedding of $\mathrm{n}\widehat{\mathrm{TL}}_N$ into $\mathrm{n}\widehat{\mathrm{TL}}_{N+1}$ defined on basis elements that otherwise would not be apparent, since the defining relations of these algebras are affine, and there is no embedding of the corresponding Coxeter graphs.

As mentioned above, the affine nilTemperley–Lieb algebra $\mathrm{n}\widehat{\mathrm{TL}}_N$ acts faithfully on fermionic particle configurations on a circle. This is the graphical representation from [KS10] (see also [Pos05]), which we use in our description of the center of $\mathrm{n}\widehat{\mathrm{TL}}_N$. We consider circular particle configuration having N positions, where $k \leq N$ particles are distributed among the positions on the circle so that there is at most one particle at each position. On the space

$$\operatorname{span}_{\Bbbk[q]} \{\text{fermionic particle configurations of } k \text{ particles on a circle with } N \text{ positions}\},$$

the generators a_i of $\mathrm{n}\widehat{\mathrm{TL}}_N$ act by sending a particle lying at position i to position $i+1$. Additionally, the particle configuration is multiplied by $\pm q$ when applying a_0. The precise definition is given in Section I.2.4, here is a picture that illustrates the action:



Example for $N = 8$: Application of $a_3 a_2 a_5$ to the particle configuration $(0, 1, 2, 5)$ gives $(0, 1, 4, 6)$.

We proceed as follows: In Section I.2.1, we introduce our notation. In Section I.2.2 we explain the connection between affine nilTemperley–Lieb algebras and many other algebras, such as the affine plactic algebra and the affine Temperley–Lieb algebra, and we briefly recall the relationship with the small quantum cohomology ring of the Grassmannian. The $\mathbb{Z}^N$-grading of $\mathrm{n}\widehat{\mathrm{TL}}_N$ is given in Section I.2.3, and its importance for the description of the center is discussed. In Section I.2.4, we give a detailed definition of the $\mathrm{n}\widehat{\mathrm{TL}}_N$-action on fermionic particle configurations on a circle. Theorem I.2.4.5 of that section recalls [KS10, Proposition 9.1] stating that the representation is faithful. In [KS10], this fact is deduced from the finite nilTemperley–Lieb algebra case, as treated in [BJS93] and [BFZ96, Proposition 2.4.1]. We give a complete, self-contained proof in Section I.2.6. Our proof is elementary and relies on the construction of a basis in Section I.2.5. We use a normal form algorithm that reorders the factors of a nonzero monomial. Our basis is reminiscent of the Jones normal form for reduced expressions of monomials in the Temperley–Lieb algebra, as discussed in [RSA14], and is characterised in Theorem I.2.5.7 as follows (see also Theorem I.2.10.1 which gives a different description):

Theorem (Normal form). Every nonzero monomial in the generators a_j of $\mathrm{n}\widehat{\mathrm{TL}}_N$ can be rewritten uniquely in the form

$$(a_{i_1}^{(m)} \dots a_{i_k}^{(m)}) \dots (a_{i_1}^{(n+1)} \dots a_{i_k}^{(n+1)})(a_{i_1}^{(n)} \dots a_{i_k}^{(n)}) \dots (a_{i_1}^{(1)} \dots a_{i_k}^{(1)})(a_{i_1} \dots a_{i_k})$$

with $a_{i_\ell}^{(n)} \in \{1, a_0, a_1, \dots, a_{N-1}\}$ for all $1 \le n \le m,\ 1 \le \ell \le k$, such that

$$a_{i_\ell}^{(n+1)} \in \begin{cases} \{1\} & \text{if } a_{i_\ell}^{(n)} = 1, \\ \{1, a_{j+1}\} & \text{if } a_{i_\ell}^{(n)} = a_j. \end{cases}$$

The factors $a_{i_1}, \dots, a_{i_k}$ are determined by the property that the generator $a_{i_\ell - 1}$ does not appear to the right of a_{i_ℓ} in the original presentation of the monomial. Alternatively, every nonzero monomial is uniquely determined by the following data from its action on the graphical representation:

- the input particle configuration with the minimal number of particles on which it acts nontrivially,
- the corresponding output particle configuration,
- the power of q by which it acts.

For the proof of this result, we recall a characterisation of the nonzero monomials in $\mathrm{n}\widehat{\mathrm{TL}}_N$ from [Gre02]. Al Harbat [Alh13] has recently described a normal form for fully commutative elements of the affine Temperley–Lieb algebra, which differs from ours when passing to $\mathrm{n}\widehat{\mathrm{TL}}_N$.

In Section I.2.7 we define special monomials that serve as the projections onto a single particle configuration (up to multiplication by $\pm q$). Based on this, in Section I.2.8 we state the main result (Theorem I.2.8.5) of the chapter:

Theorem. The center of $\mathrm{n}\widehat{\mathrm{TL}}_N$ is the subalgebra

$$\mathsf{C}_N = \mathsf{Cent}(\mathrm{n}\widehat{\mathrm{TL}}_N) \ = \ \langle 1, \mathbf{t}_1, \dots, \mathbf{t}_{N-1} \rangle \ \cong \ \frac{\Bbbk[\mathbf{t}_1, \dots, \mathbf{t}_{N-1}]}{(\mathbf{t}_k \mathbf{t}_\ell \mid k \neq \ell)},$$

where the generator $\mathbf{t}_k = (-1)^{k-1} \sum_{|\mathrm{I}|=k} a(\hat{\imath})$ is the sum of monomials $a(\hat{\imath})$ corresponding to particle configurations given by increasing sequences $\mathrm{I} = \{1 \le i_1 < \ldots < i_k \le N\}$ of length k. The monomial $a(\hat{\imath})$ sends particle configurations with $n \neq k$ particles to 0 and acts on a particle configuration with k particles by projecting onto I and multiplying by $(-1)^{k-1}q$. Hence, $\mathbf{t}_k$ acts as multiplication by q on the configurations with k particles.

Our $N-1$ central generators $\mathbf{t}_k$ are essentially the $N-1$ central elements constructed by Postnikov. Lemma 9.4 of [Pos05] gives an alternative description of $\mathbf{t}_k$ as product of the k-th elementary symmetric polynomial (with factors cyclically ordered) with the $(N-k)$-th complete homogeneous symmetric polynomial (with factors reverse cyclically ordered) in the noncommuting generators of $\mathrm{n}\widehat{\mathrm{TL}}_N$. The above theorem shows that in fact these elements generate the entire center of $\mathrm{n}\widehat{\mathrm{TL}}_N$. In Section I.2.9, we establish that $\mathrm{n}\widehat{\mathrm{TL}}_N$ is finitely generated over its center. In Section I.2.10 we describe an alternative normal form for monomials in $\mathrm{n}\widehat{\mathrm{TL}}_N$ using the generators $\mathbf{t}_k$ of the center. Using the faithfulness of the graphical representation, we define monomials e_{IJ} that move particles from positions $\mathrm{J} = \{1 \leq j_1 < \ldots < j_k \leq N\}$ to $\mathrm{I} = \{1 \leq i_1 < \ldots < i_k \leq N\}$ so that the power of q in this action is minimal. Then the main result is Theorem I.2.10.1:

Theorem. The set of monomials

$$\{1\} \ \cup \ \{\mathbf{t}_k^{\ell} e_{\mathrm{IJ}} \mid \ell \in \mathbb{Z}_{\geq 0},\ 1 \leq |\mathrm{I}| = |\mathrm{J}| = k \leq N-1,\ 1 \leq k \leq N-1\}$$

defines a $\Bbbk$-basis of the affine nilTemperley–Lieb algebra $\mathrm{n}\widehat{\mathrm{TL}}_N$.

In Section I.2.11, we define yet another monomial basis for $\mathrm{n}\widehat{\mathrm{TL}}_N$ indexed by pairs of particle configurations together with a natural number indicating how often the particles have been moved around the circle. With this basis at hand, we obtain inclusions $\mathrm{n}\widehat{\mathrm{TL}}_N \subset \mathrm{n}\widehat{\mathrm{TL}}_{N+1}$. The inclusions are not as obvious as those for the nilCoxeter algebra nC_N having underlying Coxeter graph of type A_{N-1}, since one cannot deduce them from embeddings of the affine Coxeter graphs. Our result, Theorem I.2.11.1, reads as follows:

Theorem. For all $0 \leq m \leq N-1$, there are unital algebra embeddings $\varepsilon_m : \mathrm{n}\widehat{\mathrm{TL}}_N \to \mathrm{n}\widehat{\mathrm{TL}}_{N+1}$ given by

$$a_i \ \mapsto \ a_i \quad \text{for } 0 \leq i \leq m-1, \qquad a_m \ \mapsto \ a_{m+1}a_m, \qquad a_i \ \mapsto \ a_{i+1} \quad \text{for } m+1 \leq i \leq N-1.$$

In Section I.2.12 we turn towards the representation theory of $\mathrm{n}\widehat{\mathrm{TL}}_N$: In this section and the remainder of Chapter I.2 we have to assume that the ground field $\Bbbk$ of $\mathrm{n}\widehat{\mathrm{TL}}_N$ is restricted to be an uncountable algebraically closed field (of arbitrary characteristic). Let χ be an algebra homomorphism $\mathsf{C}_N \to \Bbbk$. Then with the help of localisations with respect to central elements, we classify the simple modules over $\mathrm{n}\widehat{\mathrm{TL}}_N$ with central character χ in Theorem I.2.12.3 as follows.

Theorem. Up to isomorphism, there is precisely one simple module of $\mathrm{n}\widehat{\mathrm{TL}}_N$ with central character χ. The simple modules of $\mathrm{n}\widehat{\mathrm{TL}}_N$ are given up to isomorphism by

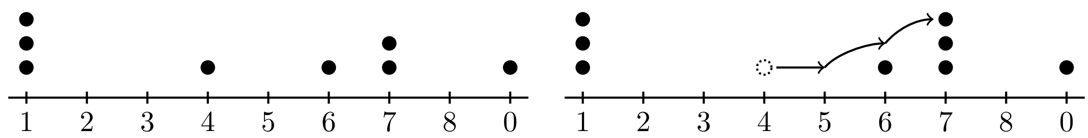
i) the trivial onedimensional module $\Bbbk$ with trivial central character,
ii) the $\binom{N}{k}$-dimensional module $\bigwedge^k \Bbbk^N$ with central character $\chi(\mathbf{t}_k) \in \Bbbk \smallsetminus \{0\}$, $\chi(\mathbf{t}_\ell) = 0$ for all $\ell \neq k$.

The localisation with respect to multiplicative subsets of the center can be considered as pseudo-commutative localisation since the Ore conditions are for free. In Section I.2.13 we use these localisations together with a rank argument to show that $\mathrm{n}\widehat{\mathrm{TL}}_N$ is not free over its center.

In analogy to the affine Temperley–Lieb algebra one would expect that also the affine nilTemperley–Lieb algebra can easily be equipped with the structure of an affine cellular algebra in the sense of [KX12]. Then the classification of simple modules for $\mathrm{n}\widehat{\mathrm{TL}}_N$ would follow from the general approach for affine cellular algebras. However, affine cellularity does not pass in an obvious way to the nil-case. In Section I.2.14 we discuss three approaches to identify $\mathrm{n}\widehat{\mathrm{TL}}_N$ as an affine cellular algebra.

Chapter I.3: The plactic and the partic algebra

Analogous to the results for the affine nilTemperley–Lieb algebra in Chapter I.2, our main results in this chapter are a description of the center of the partic algebra and the construction of a basis. Using this basis we prove that the action of the partic algebra on bosonic particle configurations is faithful. Again here is a picture illustrating this action:



Example for $N = 9$: The particle configuration $(3,0,0,1,0,1,2,0,1)$, and the element $a_6a_5a_4$ acting on it.

In Section I.3.1 we recall the definition of the classical and affine plactic algebra, and we put it into the context of the existing literature.

First we study the classical plactic algebra: In Section I.3.2 we discuss an action on bosonic particle configurations on line segments, and we define the quotient of the classical plactic algebra named partic algebra by the additional relation

$$a_ia_{i-1}a_{i+1}a_i = a_{i+1}a_ia_{i-1}a_i \qquad \text{for } 2 \le i \le N-2.$$

Since these relations only involve permutations of the generators we can define two gradings on the partic algebra, by the word length and by how often each generator occurs, similar to the affine nilTemperley–Lieb algebra before.

In Section I.3.3 we construct a normal form of the monomials in the partic algebra. Our main result of this section is Theorem I.3.3.1

Theorem. The partic algebra $\mathcal{P}_N^{\mathrm{part}}$ has a $\Bbbk$-basis given by monomials of the form

$$\{a_{N-1}^{d_{N-1}} \dots a_2^{d_2} a_1^{k_1} a_2^{k_2} \dots a_{N-1}^{k_{N-1}} \mid d_i \leq d_{i-1} + k_{i-1} \ \text{ for all } \ 3 \leq i \leq N-1,\ d_2 \leq k_1\}$$

where $d_i, k_i \in \mathbb{Z}_{\geq 0}$ for all $1 \leq i \leq N-1$.

In Section I.3.4 we consider the action of the classical plactic and the partic algebra on bosonic particle configurations, and we obtain the following faithfulness result in Theorem I.3.4.2

Theorem. The action of the partic algebra $\mathcal{P}_N^{\mathrm{part}}$ on bosonic particle configurations is faithful.

This allows us to define a labelling of the monomials in normal form. We get an alternative description of the basis from Theorem I.3.3.1 in Proposition I.3.4.5. If we write $a_{\underline{\mathrm{I}}\underline{\mathrm{J}}} = a_{N-1}^{d_{N-1}} \dots a_2^{d_2} a_1^{k_1} a_2^{k_2} \dots a_{N-1}^{k_{N-1}}$, it can be reformulated as follows:

Theorem. The set of monomials

$$\{1\} \ \cup \ \{a_{\underline{\mathrm{I}}\underline{\mathrm{J}}} \mid \underline{\mathrm{J}} = (k_1, k_2, k_3 \dots, k_{N-1}, 0),\ \underline{\mathrm{I}} = (0, k_1 - d_2, k_2 + d_2 - d_3, \dots, k_{N-1} + d_{N-1})\}$$

with $k_1, \dots, k_{N-1} \in \mathbb{Z}_{\geq 0}$ and $d_i \leq d_{i-1} + k_{i-1} \in \mathbb{Z}_{\geq 0}$ for all $3 \leq i \leq N-1$, $d_2 \leq k_1$, defines a $\Bbbk$-basis of the partic algebra.

In Section I.3.5 we describe the center of the partic algebra:

Theorem. The center of the partic algebra $\mathcal{P}_N^{\mathrm{part}}$ is given by the $\Bbbk$-span of the elements

$$\{a_{N-1}^r a_{N-2}^r \dots a_2^r a_1^r \mid r \geq 0\}.$$

Finally, in Section I.3.6 we turn to the affine case. We define the affine partic algebra and we consider its action on affine bosonic particle configurations. This is substantially harder to understand than the classical case, in particular we find a new type of relations of the form

$$\begin{aligned} & a_{i+1}^m a_{i+2}^m \dots a_{i-2}^m a_{i-1}^m a_i^{2m} a_{i+1}^m a_{i+2}^m \dots a_{i-2}^m a_{i-1}^m \\ = \ & a_{j+1}^m a_{j+2}^m \dots a_{j-2}^m a_{j-1}^m a_j^{2m} a_{j+1}^m a_{j+2}^m \dots a_{j-2}^m a_{j-1}^m \qquad \text{for all } i, j \in \mathbb{Z}/N\mathbb{Z},\ m \in \mathbb{Z}_{\geq 1}. \end{aligned}$$

We have not yet found a nice normal form for monomials for the affine partic algebra (and neither for its quotient with respect to the new type of relations). For the construction of the normal forms of the partic algebra and the affine nilTemperley–Lieb algebra, it was helpful to know the faithful representations on particle configurations. The particle configurations could be used for labelling sets of the basis elements. This approach fails for the affine partic algebra since it does not act faithfully on affine bosonic particle configurations. It is unclear whether faithfulness holds for the quotient with respect to the new type of relations.

Part II: Generalized Weyl algebras

Generalized Weyl algebras (GWA's) were introduced by Bavula in [Bav92]. A GWA is defined over a unital associative commutative $\Bbbk$-algebra R that is a noetherian domain, where $\Bbbk$ is an algebraically closed ground field of characteristic 0. For any choice of n nonzero elements $t = (t_1, \dots, t_n)$ in R and n pairwise commuting algebra automorphisms $\sigma = (\sigma_1, \dots, \sigma_n)$ in $\mathrm{Aut}(R)$ such that $\sigma_i(t_j) = t_j$ for all $i \neq j$ the corresponding GWA $A = R(\sigma, t)$ is the $\Bbbk$-algebra generated over R by $2n$ additional generators X_i, Y_i, $1 \leq i \leq n$, with relations

$$\begin{aligned} X_i r &= \sigma_i(r) X_i, & X_i Y_i &= \sigma_i(t_i), & [X_i, X_j] &= 0, \\ Y_i r &= \sigma_i^{-1}(r) Y_i, & Y_i X_i &= t_i, & [Y_i, Y_j] &= 0, \\ & & & & [X_i, Y_j] &= 0 \end{aligned}$$

for all $1 \leq i, j \leq n$ with $i \neq j$ and all $r \in R$. It is a $\mathbb{Z}^n$-graded algebra with $\deg(X_i) = e_i$ and $\deg(Y_i) = -e_i$ where we denote by e_i the i-th standard basis vector of $\mathbb{Z}^n$.

Chapter II.1: Duflo Theorem for a Class of Generalized Weyl Algebras

The main result of this chapter is a Duflo type theorem for a class of generalized Weyl algebras (GWA's).

For the universal enveloping algebra of a semisimple Lie algebra over $\Bbbk$, Duflo's Theorem [Duf77] states that all its primitive ideals (i.e. the annihilators of simple modules) are given by the annihilators of simple highest weight modules. In contrast, the simple modules themselves are far from being classified in general.

Now it is possible to define highest weight modules for GWA's and therefore natural to ask whether an analogous statement holds. We prove a Duflo type theorem for a special class of GWA's using a theorem by [MB98] that relates the annihilator of a simple weight module to its support.

This chapter is subdivided as follows: In Section II.1.1 we provide a quick overview of Duflo type theorems. In Section II.1.2 we review generalized Weyl algebras, and we introduce our special class of GWA's. In particular, our base ring is always a polynomial ring $R = \Bbbk[T_1, \dots, T_n]$ and the automorphisms are given by translations $\sigma_i(T_j) = T_j - \delta_{ij} b_i$ as considered already in [Bav92]. We discuss highest weight modules and graded modules over generalized Weyl algebras. We characterize moreover the highest weight modules as those modules with a locally nilpotent action of the X_i.

In Section II.1.3 we prepare to apply the result from [MB98] to our class of GWA's: We recall the description of weight modules by their support which is given in terms of lattice points and hyperplanes from [Bav92]. These hyperplanes "break" the weight lattice into regions, and a weight module can be characterised by these regions and its defining "breaks". This is made precise in Definition II.1.3.3. We give a careful description of the break conditions.

In Section II.1.4 we formulate and prove the main theorem of the chapter:

Theorem. Let $A = R(\sigma, t)$ be a GWA of rank n as defined in Section II.1.2 where we assume $R = \Bbbk[T_1, \dots, T_n]$, $\sigma_i(T_j) = T_j - \delta_{ij} b_i$ for $b_i \in \Bbbk \setminus \{0\}$ and $t_i \in \Bbbk[T_i] \subset \Bbbk[T_1, \dots, T_n]$, $t_i \notin \Bbbk$. Then all primitive ideals of A, i.e. the annihilator ideals of simple A-modules, are given by the annihilators of simple highest weight A-modules $L(\mathfrak{m})$ of highest weight $\mathfrak{m} \in \operatorname{mspec}(R)$.

The main tool is the Duflo type theorem from [MB98]. We show it applies to our situation and improve it by showing that it is enough to consider the much smaller class of *highest* weight modules (as in the classical Duflo theorem).

We provide a list of important examples of GWA's to which the main theorem applies, e.g. central quotients of the universal enveloping algebra $\mathcal{U}(\mathfrak{sl}_2(\mathbb{C}))$ and its generalisations by [Smi90] as discussed in [Bav92, Example 1.2.(4)]. We include a discussion why we require our assumptions on the special class of GWA's.

In Section II.1.5 we conclude the chapter by some examples that illustrate the relationship between the annihilator and the support of simple highest weight modules.

Publications and Coauthorships

Parts of this thesis have been published or accepted for publication during the PhD project: Most of Chapter I.2 as well as the corresponding parts of this introduction can be found in the paper [BM16] with Georgia Benkart. Except for Lemma II.1.2.2, all of Chapter II.1 is published in [Mei15].

Sections I.2.12 and I.2.13 grew out of discussions with Gwyn Bellamy and Uli Krähmer.

Acknowledgements

I am deeply thankful to my advisor Catharina Stroppel for sharing her deep insights and her enthusiasm, for patiently standing me by and supporting me. It has been a huge pleasure to work under her guidance, and I enjoyed every single hour during our many, many discussions. From the first time that I heard about Lie algebras to our most recent meeting she has been an inspiring teacher and a fantastic mentor, and I am grateful for all the time we spent together in great working atmosphere.

During my PhD project I spent five months at QGM Aarhus and five more months at the University of Uppsala. I would like to thank Henning Haahr Andersen and Volodymyr Mazorchuk for generously hosting me, for giving me their time and for the numerous interesting discussions that we had. I enjoyed my stays in Aarhus and in Uppsala a lot, and I would like to thank the members of QGM Aarhus and the people at the Department for Mathematics in Uppsala for their kind hospitality.

I am grateful to my coauthor Georgia Benkart for our collaboration, including long discussions and exciting example computations. I thank the MSRI Berkeley for giving us the opportunity to start this collaboration during the programme on "Noncommutative Algebraic Geometry and Representation Theory".

I would like to thank Gwyn Bellamy, Kenneth Brown, Christian Korff and Uli Krähmer for kind advice and discussions about the affine nilTemperley–Lieb algebra during a short visit to the University of Glasgow, and I am grateful to Jonas Hartwig for interesting conversations about GWA's.

I heartily thank the members of the representation theory working group in Bonn for sharing their knowledge, giving me support and advice. I am particularly indebted to Hanno Becker, Michael Ehrig, Deniz Kus, and Daniel Tubbenhauer for lots of feedback and discussions about this thesis, for proofreading, and for bearing with my terrible puns. I thank Viktoriya Ozornova for comments on the thesis, and further thanks are due to the referees for many improvements of the two articles underlying this thesis.

I gratefully acknowledge the support by the IMPRS programme of the Max Planck Institute for Mathematics, the Hausdorff stipend of the Bonn International Graduate School, and the scholarship of the Deutsche Telekom Stiftung. Their generous funding and the extremely helpful administrative staff enabled me to carry out my work, and I learned a lot from the exchange with the people I met thanks to these programmes.

My friends at the Mathematical Institutes in Bonn, Aarhus and Uppsala made my life as a PhD student very pleasant. The list of reasons is long, and it includes many mathematical discussions, feedback, and reciprocal encouragement as well as common coffee breaks, QGM lounge meetings, fika sessions, tea sessions, sushi dinners, Vietnamese dinners, cooking together in the tiny kitchen of the Mathematical Institute in Bonn, baking delicious cookies, processing tons of chestnuts, hiking in California, Catalonia, Corsica, Scotland and the Siebengebirge, cinema visits, shared rooms in strange hotels, shared offices, and many non-mathematical discussions.

I am lucky to have a wonderful family and patient friends who supported me unconditionally during the past years, even when I was fully absorbed by my thesis. It is a pleasure to thank them for giving me so much of their time and energy!

Part I.

Particle configurations and crystals

I.1. Crystal bases and particle configurations

In this chapter we discuss the relationship of particle configurations on a lattice with crystal combinatorics in type A and $\widehat{\mathsf{A}}$. It can be seen as a motivation for the definitions of the affine nilTemperley-Lieb algebra, the plactic and the partic algebra that we discuss in the following chapters. This chapter is otherwise independent of the following chapters.

In Section I.1.1 we review crystal bases for the quantum group $\mathcal{U}_q(\mathfrak{sl}_n(\mathbb{C}))$ and the quantum affine algebra $\mathcal{U}_q(\widetilde{\mathfrak{sl}}_n(\mathbb{C}))$, and we discuss relations among Kashiwara operators. In Section I.1.2 we describe particle configurations following [KS10] and we discuss identifications of crystal and particle combinatorics.

Throughout the chapter we work over the complex numbers $\Bbbk = \mathbb{C}$ for convenience. For tensor products over $\mathbb{C}$ we write $\otimes$ instead of $\otimes_{\mathbb{C}}$. We write $\mathbb{C}(q)$ for the field of rational functions in the variable q.

I.1.1. Quantum groups and crystal bases of type A_n and $\widehat{\mathsf{A}}_n$

In this section we review crystal bases for the quantum groups $\mathcal{U}_q(\mathfrak{sl}_n(\mathbb{C}))$ and $\mathcal{U}_q(\widetilde{\mathfrak{sl}}_n(\mathbb{C}))$ and fix our notation. We follow mainly [HK02] and [Jan96] unless otherwise stated. We focus on type A_n and $\widehat{\mathsf{A}}_n$, for more general statements see the references.

I.1.1.1. Finite case

Let $\mathfrak{sl}_n(\mathbb{C})$ be the Lie algebra of traceless complex $n \times n$-matrices with standard Cartan subalgebra $\mathfrak{h}$ consisting of the diagonal matrices generated by $h_i = e_{ii} - e_{(i+1)(i+1)}$ for $1 \le i \le n-1$. Here e_{ii} denotes the elementary matrix where the (i,i)th entry is one and all other entries are zero. The root decomposition of $\mathfrak{sl}_n(\mathbb{C})$ with respect to the adjoint $\mathfrak{h}$-action is given by $\mathfrak{sl}_n(\mathbb{C}) = \bigoplus_{\alpha \in \Phi} \mathfrak{sl}_n(\mathbb{C})_\alpha$ and simple roots $\alpha_i = \varepsilon_i - \varepsilon_{i+1} \in \mathfrak{h}^*$. Here ε_i denotes the function on $\mathfrak{h}$ that returns the ith diagonal entry, and $\Phi = \mathrm{span}_{\mathbb{Z}}\{\alpha_1,\ldots,\alpha_{n-1}\}$ is the root lattice of $\mathfrak{sl}_n(\mathbb{C})$. In our notation we do not distinguish between linear functions on $\mathfrak{h}$ and linear functions on the diagonal matrices. The fundamental weights are given by $\omega_i = \varepsilon_1 + \ldots + \varepsilon_i$. We denote the weight lattice by $P = \mathrm{span}_{\mathbb{Z}}\{\omega_1,\ldots,\omega_{n-1}\}$. It contains the dominant integral weights $P^+ = \mathrm{span}_{\mathbb{Z}_{\geq 0}}\{\omega_1,\ldots,\omega_{n-1}\}$.

The finite dimensional simple $\mathfrak{sl}_n(\mathbb{C})$-modules $L(\lambda)$ are labelled by their dominant integral highest weights $\lambda \in P^+ \subset \mathfrak{h}^* = \mathrm{span}_{\mathbb{C}}\{\varepsilon_i \mid 1 \leq i \leq n\}/\mathrm{span}_{\mathbb{C}}\{\varepsilon_1 + \ldots + \varepsilon_n\}$. Such a dominant integral highest weight can be represented by an element of the form $\lambda = \lambda_1\varepsilon_1 + \ldots + \lambda_{n-1}\varepsilon_{n-1}$ with coefficients $\lambda_1 \geq \ldots \geq \lambda_{n-1} \in \mathbb{Z}_{\geq 0}$. This in turn is identified with partitions $(\lambda_1,\ldots,\lambda_{n-1})$ with $n-1$ rows of length λ_i.

Now we turn to the quantum group:

I.1.1.1 Definition. The quantum group $\mathcal{U}_q(\mathfrak{sl}_n(\mathbb{C}))$ is the unital associative $\mathbb{C}(q)$-algebra generated by formal generators $E_i, F_i, K_i^{\pm 1}$ for $1 \leq i \leq n-1$ with relations

$$\begin{aligned}
K_iK_i^{-1} &= 1 = K_i^{-1}K_i && \text{for } 1 \leq i \leq n-1,\\
K_jE_i &= q^{\alpha_i(h_j)}E_iK_j && \text{for } 1 \leq i,j \leq n-1,\\
K_jF_i &= q^{-\alpha_i(h_j)}F_iK_j && \text{for } 1 \leq i,j \leq n-1,\\
[E_i,F_j] &= \delta_{ij}\frac{K_i - K_i^{-1}}{q-q^{-1}} && \text{for } 1 \leq i,j \leq n-1,\\
E_i^2E_{i\pm1} - [2]_qE_iE_{i\pm1}E_i + E_{i\pm1}E_i^2 &= 0, &&\\
[E_i,E_j] &= 0 && \text{for } |i-j|>1,\\
F_i^2F_{i\pm1} - [2]_qF_iF_{i\pm1}F_i + F_{i\pm1}F_i^2 &= 0, &&\\
[F_i,F_j] &= 0 && \text{for } |i-j|>1,
\end{aligned}$$

where $[n]_q = \frac{q^n-q^{-n}}{q-q^{-1}}$ is the usual notation for quantum integers, so $[2]_q = q+q^{-1}$. It can be equipped with a Hopf algebra structure where in particular the comultiplication Δ applied to F_i is given by $\Delta(F_i) = F_i \otimes 1 + K_i \otimes F_i$, the comultiplication applied to E_i is $\Delta(E_i) = E_i \otimes K_i^{-1} + 1 \otimes E_i$, and the elements $K_i^{\pm1}$ are grouplike, for $1 \leq i \leq n-1$.

I.1.1.2 Remark. This is the adjoint form of $\mathcal{U}_q(\mathfrak{sl}_n(\mathbb{C}))$ in the sense of [BG02], where the generators K_i correspond to the generators α_i of the root lattice Φ of $\mathfrak{sl}_n(\mathbb{C})$. Alternative forms of the quantum group $\mathcal{U}_q(\mathfrak{sl}_n(\mathbb{C}))$ can be defined for the (finer) weight lattice or any other lattice lying in between those two, see [BG02, Section 1.6.3], [CP95a, Section 9.1.A]. Furthermore, there is the Drinfeld-Jimbo quantum algebra whose elements are formal power series in e_i, f_i and h_i over the field $\mathbb{C}[[h]]$, see [CP95a, Definition 6.5.1], [Kas95]. There is a map of Hopf algebras from the quantum group defined above into the Drinfeld-Jimbo quantum group by $q \mapsto e^{\frac{h}{2}}$, $K_i^{\pm 1} \mapsto e^{\pm\frac{h}{2}h_i}$, $F_i \mapsto e^{-\frac{h}{4}} f_i$ and $E_i \mapsto e^{\frac{h}{4}} e_i$, see [Kas95, Proposition XVII.4.1] for $n = 2$. $\Diamond$

We are only interested in weight modules, i.e. $\mathcal{U}_q(\mathfrak{sl}_n(\mathbb{C}))$-modules with a weight space decomposition with respect to the action of K_i, $1 \leq i \leq n-1$, so that the K_i act by scalars in $\mathbb{C}(q)^\times$ on the weight spaces. In particular, we consider weight modules with weights of the form $\pm q^\mu$ for $\mu \in P \subset \mathfrak{h}^*$, meaning that K_i acts by $\pm q^{\mu(h_i)}$, for all $1 \leq i \leq n-1$.

All finite dimensional $\mathcal{U}_q(\mathfrak{sl}_n(\mathbb{C}))$-modules are completely reducible into simple highest weight modules of highest weight $\pm q^\lambda$ with $\lambda \in P^+$, see [CP95a, Propositions 10.1.1, 10.1.2]. In other words, the finite dimensional highest weight modules are labelled by partitions λ together with a choice of $(n-1)$ signs, so that K_i acts by $\pm q^{\lambda_i}$, for all $1 \leq i \leq n-1$.

One usually prefers the choice of all signs equal to $+1$ since the subcategory of these so-called type **1** modules is closed under tensor products. The abelian subcategory of finite dimensional $\mathcal{U}_q(\mathfrak{sl}_n(\mathbb{C}))$-modules with a fixed choice of signs is equivalent to the abelian category of finite dimensional $\mathfrak{sl}_n(\mathbb{C})$-modules. For type **1**, this is an equivalence of monoidal categories.

Under this equivalence, the finite dimensional simple $\mathfrak{sl}_n(\mathbb{C})$-module $L(\lambda)$ is mapped to the simple $\mathcal{U}_q(\mathfrak{sl}_n(\mathbb{C}))$-module $L_q(\lambda)$ of type **1** with the same character, see [BG02, Section I.6.12]. Here and in the following we adopt the shorthand notation of writing λ for $+q^\lambda$.

Let us now recall the combinatorics of some special crystals for $\mathfrak{sl}_n(\mathbb{C})$. We do not introduce Kashiwara operators and crystal bases in detail. We refer to [Kas91], but also e.g. to [HK02, Section 4] for the general statements and background material and to [HK02, Sections 7.4, 8.2] for details about type A_n.

Let $\tilde{f}_i$ denote the Kashiwara operator on a $\mathcal{U}_q(\mathfrak{sl}_n(\mathbb{C}))$-module M associated with the operator $F_i \in \mathcal{U}_q(\mathfrak{sl}_n(\mathbb{C}))$, i.e. $\tilde{f}_i u = \sum_k F_i^{(k+1)} u_k$ for a weight vector $u \in M_\mu$ written in the form $u = \sum_k F_i^{(k)} u_k$ with $u_k \in M_{\lambda+k\alpha_i} \cap \ker(E_i)$. Here $F_i^{(k)} = \frac{1}{[k]_q!} F_i^k$ is the notation for divided powers. The Kashiwara operator $\tilde{e}_i$ associated with E_i is defined analogously.

By [Kas91] there exists a crystal basis $(\mathcal{L}(\lambda), \mathcal{B}(\lambda))$ for the simple $\mathcal{U}_q(\mathfrak{sl}_n(\mathbb{C}))$-module $L_q(\lambda)$. Here $\mathcal{L}(\lambda)$ denotes the crystal lattice, the minimal lattice over the rational functions regular at 0 that contains a highest weight vector v_λ of $L_q(\lambda)$ and that is stable under the action of the Kashiwara operators $\tilde{f}_i$, $\tilde{e}_i$. The subset $\mathcal{B}(\lambda)$ of $\mathcal{L}(\lambda)/q\mathcal{L}(\lambda)$ is given by all nonzero elements of the form $\tilde{f}_{i_1} \dots \tilde{f}_{i_r}(v_\lambda) + q\mathcal{L}(\lambda)$.

One defines the crystal graph to be an oriented graph with vertices $\mathcal{B}(\lambda)$ and edges labelled by $1,\dots,n-1$, so that there is an i-labelled edge from b to $b' \in \mathcal{B}(\lambda)$ if and only if $\tilde{f}_i(b) = b'$ modulo $q\mathcal{L}(\lambda)$. This is the case if and only if $\tilde{e}_i(b') = b$ modulo $q\mathcal{L}(\lambda)$. By abuse of notation, the crystal graph is also denoted by $\mathcal{B}(\lambda)$.

Crystal bases are particularly suitable for the computation of tensor products. Given $L_q(\lambda)$ with crystal basis $(\mathcal{L}(\lambda),\mathcal{B}(\lambda))$ and $L_q(\lambda')$ with crystal basis $(\mathcal{L}(\lambda'),\mathcal{B}(\lambda'))$, one can easily determine a crystal graph for $L(\lambda) \otimes_{\mathbb{C}(q)} L(\lambda')$ on the set of vertices $\mathcal{B}(\lambda)\otimes\mathcal{B}(\lambda') := \mathcal{B}(\lambda)\times\mathcal{B}(\lambda')$. The tensor product rule prescribes on which tensor factor the Kashiwara operator $\tilde{f}_i$ acts, see [HK02, Theorem 4.4.1].

In type A_n, for any simple $\mathcal{U}_q(\mathfrak{sl}_n(\mathbb{C}))$-module $L_q(\lambda)$, the set $\mathcal{B}(\lambda)$ can be realized by semistandard Young tableaux of shape λ with entries $1,\dots,n$. The highest weight vector v_λ of $L_q(\lambda)$ is represented by the "standard" semistandard Young tableau of shape λ where all entries in the kth row are equal to k. In the crystal graph $\mathcal{B}(\lambda)$, if two semistandard Young tableaux are connected by an i-labelled edge, then their entries are the same except that in one box the entry i is replaced by $i{+}1$. Let us recall the details: In Figure I.1.1.1 we depict the crystal graph for the standard/vector representation $L_q(\omega_1)$ of $\mathcal{U}_q(\mathfrak{sl}_n(\mathbb{C}))$. As a $\mathbb{C}(q)$-vector space, $L_q(\omega_1) \cong \mathbb{C}(q)^n$.

$$\boxed{1} \xrightarrow{\quad 1 \quad} \boxed{2} \xrightarrow{\quad 2 \quad} \cdots \xrightarrow{\quad n-1 \quad} \boxed{n}$$

Figure I.1.1.1.: The crystal graph for the standard/vector representation of $\mathcal{U}_q(\mathfrak{sl}_n(\mathbb{C}))$.

Here, a crystal basis of $L_q(\omega_1)$ is given by $(\mathcal{L}(\omega_1),\mathcal{B}(\omega_1))$. The crystal lattice $\mathcal{L}(\omega_1)$ is spanned over the rational functions regular at 0 by the standard basis vectors $v_1,\dots,v_n$ on which F_i acts by $F_i v_i = v_{i+1}$, $F_i v_j = 0$ for $i \neq j$. The set $\mathcal{B}(\omega_1)$ is given by the residue classes of the standard basis vectors in $\mathcal{L}(\omega_1)/q\mathcal{L}(\omega_1)$. In Figure I.1.1.1 a box with entry i is identified with the residue class of the ith standard basis vector v_i of $\mathbb{C}(q)^n$.

The vertices of the crystal graph of $L_q(\lambda)$ can be identified with the set of semistandard Young tableaux of shape λ as follows: The tensor product rule allows to compute the crystal graph of $L_q(\omega_1)^{\otimes(\lambda_1+\ldots+\lambda_{n-1})}$. Then the crystal graph $\mathcal{B}(\lambda)$ is identified with a connected component in $\mathcal{B}(\omega_1)^{\otimes(\lambda_1+\ldots+\lambda_{n-1})}$ by an admissible reading. The tensor product rule prescribes on which tensor factor the Kashiwara operator $\tilde{f}_i$ acts, hence, in which box the entry i is turned into $i+1$. In general there are many possible choices of admissible readings, but the crystal structure on $\mathcal{B}(\lambda)$ does not depend on this choice, see [HK02, Theorem 7.3.6].

For a tensor product $\boxed{a_1} \otimes \boxed{a_2} \otimes \ldots \otimes \boxed{a_d}$ in $\mathcal{B}(\omega_1)^{\otimes d}$, the tensor product rule can be summarized as follows: We need to determine the box with entry i on which $\tilde{f}_i$ has to act. The boxes with entries $j \neq i, i+1$ are irrelevant and thus removed. Then all "increasing pair of boxes" are removed, that is, a box with entry $a_r = i$ which is followed immediately by a box with entry $a_{r+s} = i+1$ in the remaining tensor product (where all boxes with entries $a_{r+1}, \ldots, a_{r+s-1}$ have been previously removed). This process is repeated for the remaining tensor factors until no increasing pair of boxes remains. If the final result does not contain any box with entry i, then $\tilde{f}_i$ acts by zero. If there are some boxes with entry i left, then $\tilde{f}_i$ acts on the leftmost such box.

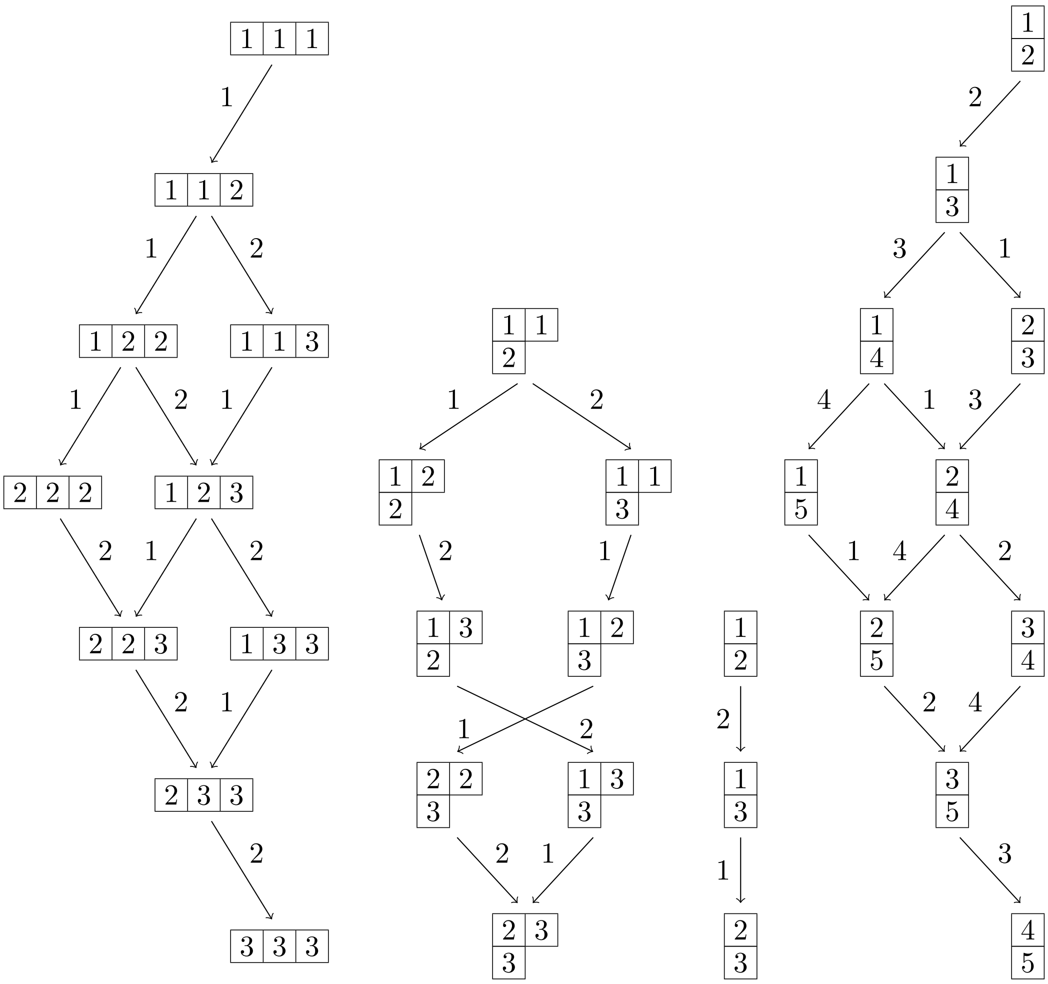


Figure I.1.1.2.: Examples of crystal graphs for $L_q(3\varepsilon_1)$, $L_q(2\varepsilon_1+\varepsilon_2)$, $L_q(\varepsilon_1+\varepsilon_2) \in \mathcal{U}_q(\mathfrak{sl}_3(\mathbb{C}))\text{-mod}$ and $L_q(\varepsilon_1+\varepsilon_2) \in \mathcal{U}_q(\mathfrak{sl}_5(\mathbb{C}))\text{-mod}$.

The examples in Figure I.1.1.2 illustrate the crystal graphs for the $\mathcal{U}_q(\mathfrak{sl}_3(\mathbb{C}))$-modules $L_q(3\varepsilon_1)$, $L_q(2\varepsilon_1+\varepsilon_2)$, $L_q(\varepsilon_1+\varepsilon_2)$, and for the $\mathcal{U}_q(\mathfrak{sl}_5(\mathbb{C}))$-module $L_q(\varepsilon_1+\varepsilon_2)$. For $L_q(2\varepsilon_1+\varepsilon_2)$ this is Example 7.4.3 from [HK02].

The crystal graphs $\mathcal{B}(\lambda)$ for $\lambda = k\omega_1 = \square\square\square\square$ and $\lambda = \omega_k = \begin{smallmatrix}\square\\\square\\\square\end{smallmatrix}$ are special. Recall that $\mathcal{B}(k\omega_1)$ is the crystal graph corresponding to the symmetric representation $\mathrm{Sym}^k(\mathbb{C}^n)$ of $\mathfrak{sl}_n(\mathbb{C})$, and $\mathcal{B}(\omega_k)$ is the crystal graph for the alternating representation $\Lambda^k(\mathbb{C}^n)$. The tensor product rule is particularly easy to formulate for $\mathcal{B}(k\omega_1)$ and $\mathcal{B}(\omega_k)$, and the action of the Kashiwara operators on semistandard Young tableaux of shape $k\omega_1$ or ω_k is independent of the relative positions of the boxes with entries i, $i+1$. Let us discuss this in detail:

Let us start with Young tableaux of shape $k\omega_1$. There is precisely one admissible reading. It is given by

$$\boxed{i_1}\boxed{i_2}\boxed{\dots}\boxed{i_k} \rightsquigarrow \boxed{i_k}\otimes\dots\otimes\boxed{i_2}\otimes\boxed{i_1}.$$

Since the sequence $i_1, i_2, \dots, i_k \in \{1, \dots, n\}$ is (weakly) increasing, the "reversed" sequence obtained from the admissible reading $i_k, \dots, i_2, i_1$ is decreasing. In particular, there are no "increasing pairs" of boxes. In this case, the tensor product rule for crystals simply amounts to the following rule:

I.1.1.3 Lemma. Let $1 \le i \le n-1$ and $k \in \mathbb{Z}_{\ge 0}$. On semistandard Young tableaux of shape $k\omega_1$ that contain a box with entry i the Kashiwara operator $\tilde{f}_i$ acts on the rightmost box with entry i, replacing it by $i+1$. On semistandard Young tableaux of shape $k\omega_1$ that do not contain any box with entry i the Kashiwara operator $\tilde{f}_i$ acts by zero.

For Young tableaux of shape ω_k, there is precisely one admissible reading given by

$$\begin{array}{|c|}\hline i_1\\\hline i_2\\\hline \dots\\\hline i_k\\\hline\end{array} \rightsquigarrow \boxed{i_1}\otimes\boxed{i_2}\otimes\dots\otimes\boxed{i_k}.$$

The sequence $i_1, i_2, \dots, i_k \in \{1, \dots, n\}$ is strictly increasing. In particular, no entry is repeated, and a quick case-by-case analysis gives the following rule equivalent to the tensor product rule:

I.1.1.4 Lemma. Let $1 \le i \le n-1$ and $1 \le k \le n-1$. On semistandard Young tableaux of shape ω_k that contain a box with entry i and that do not contain any box with entry $i+1$ the Kashiwara operator $\tilde{f}_i$ acts on (the only) box with entry i, replacing it by $i+1$.

On any other semistandard Young tableaux of shape ω_k the Kashiwara operator $\tilde{f}_i$ acts by zero.

I.1.1.5 Remark. The rules from Lemma I.1.1.3 and Lemma I.1.1.4 are formulated independently of the relative positions of the boxes with entries i, $i+1$. $\Diamond$

I.1.1.6 Remark. For hooks of the form $\begin{array}{|c|c|c|c|}\hline \, & \, & \, & \, \\ \hline \, \\ \cline{1-1}\end{array}$ there is only one admissible reading, too. But it is not guaranteed that the sequence we obtain from the admissible reading is decreasing or strictly increasing, and the result of the application of $\tilde{f}_i$ depends on the exact positions of the boxes with entries $i, i+1$ in the Young tableau. For example, in Figure I.1.1.2 we have seen that $\tilde{f}_2$ maps $\begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1}\end{array}$ to $\begin{array}{|c|c|}\hline 1 & 3 \\ \hline 3 \\ \cline{1-1}\end{array}$, whereas $\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array}$ is mapped to zero. $\Diamond$

Let us now investigate some of the relations among the Kashiwara operators $\tilde{f}_i$.

I.1.1.7 Lemma. i) Let $k \in \mathbb{Z}_{\geq 0}$. On $\mathcal{B}(k\omega_1) \cup \{0\}$ we have

$$\begin{aligned}
\tilde{f}_i\tilde{f}_j &= \tilde{f}_j\tilde{f}_i && \text{for all } 1 \leq i,j \leq n-1 \text{ so that } |i-j| > 1,\\
\tilde{f}_i\tilde{f}_{i-1}\tilde{f}_i &= \tilde{f}_i^2\tilde{f}_{i-1} && \text{for all } 2 \leq i \leq n-1,\\
\tilde{f}_i\tilde{f}_{i+1}\tilde{f}_i &= \tilde{f}_{i+1}\tilde{f}_i^2 && \text{for all } 1 \leq i \leq n-2,\\
\tilde{f}_i\tilde{f}_{i-1}\tilde{f}_{i+1}\tilde{f}_i &= \tilde{f}_{i+1}\tilde{f}_i^2\tilde{f}_{i-1} && \text{for all } 2 \leq i \leq n-2.
\end{aligned}$$

ii) Let $1 \leq k \leq n-1$. On $\mathcal{B}(\omega_k) \cup \{0\}$ we have

$$\begin{aligned}
\tilde{f}_i\tilde{f}_j &= \tilde{f}_j\tilde{f}_i && \text{for all } 1 \leq i,j \leq n-1 \text{ so that } |i-j| > 1,\\
\tilde{f}_i^2 &= 0 && \text{for all } 1 \leq i \leq n-1,\\
\tilde{f}_i\tilde{f}_{i-1}\tilde{f}_i &= 0 && \text{for all } 2 \leq i \leq n-1,\\
\tilde{f}_i\tilde{f}_{i+1}\tilde{f}_i &= 0 && \text{for all } 1 \leq i \leq n-2.
\end{aligned}$$

Proof. This follows from the explicit realisation of the Kashiwara operators in Lemma I.1.1.3 and Lemma I.1.1.4. $\square$

In particular, Lemma I.1.1.7 implies that the relations

$$\tilde{f}_i\tilde{f}_{i-1}\tilde{f}_i = \tilde{f}_i^2\tilde{f}_{i-1} \quad \text{and} \quad \tilde{f}_i\tilde{f}_{i+1}\tilde{f}_i = \tilde{f}_{i+1}\tilde{f}_i^2 \tag{I.1.1}$$

hold for all crystals $\mathcal{B}(k\omega_1)$ and $\mathcal{B}(\omega_k)$. In contrast, the relation $\tilde{f}_i\tilde{f}_{i-1}\tilde{f}_{i+1}\tilde{f}_i = \tilde{f}_{i+1}\tilde{f}_i^2\tilde{f}_{i-1}$ is special for $\mathcal{B}(k\omega_1)$ and does not hold for $\mathcal{B}(\omega_k)$. For example, for $n=5$ and ω_2 we have

$$\tilde{f}_2\tilde{f}_1\tilde{f}_3\tilde{f}_2\left(\begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline\end{array}\right) = \begin{array}{|c|}\hline 3 \\ \hline 4 \\ \hline\end{array} \neq 0.$$

The relations given in Lemma I.1.1.7 are not a complete list of relations, e.g. we have in adddition $\tilde{f}_i^{k+1} = 0$ on $\mathcal{B}(k\omega_1) \cup \{0\}$.

One can also define abstract crystals in a purely combinatorial way as a set $\mathcal{B}$ together with some maps, including operators $\mathcal{B} \to \mathcal{B} \cup \{0\}$, that satisfy a list of axioms, see [HK02, Definition 4.5.1]. These axioms are satisfied by crystal graphs and Kashiwara operators obtained from integrable highest weight modules of quantum symmetrizable Kac–Moody algebras, in particular from finite-dimensional $\mathcal{U}_q(\mathfrak{sl}_n(\mathbb{C}))$-modules.

For abstract crystals of simply laced finite and affine type, [Ste03] gives a list of relations that hold if and only if the abstract crystal graph can be realized as a crystal graph of an integrable highest weight representation. These relations are formulated using i-strings in the crystal graph. An i-string is defined at a node x to be the path of maximum (finite) length of the form

$$\tilde{e}_i^d x \longrightarrow \ldots \longrightarrow \tilde{e}_i x \longrightarrow x \longrightarrow \tilde{f}_i x \longrightarrow \ldots \longrightarrow \tilde{f}_i^r x.$$

In this case, write $\varepsilon(x, i) = r$, where we adopt the notation from [Ste03].

A subset of these relations is equivalent to the abstract crystal axioms. The additional relations are given by $\tilde{f}_i \tilde{f}_j x = \tilde{f}_j \tilde{f}_i x$ or $\tilde{f}_i \tilde{f}_j^2 \tilde{f}_i x = \tilde{f}_j \tilde{f}_i^2 \tilde{f}_j x$ at nodes x of the crystal graph where $\tilde{f}_i, \tilde{f}_j$ are both defined, i.e. $\tilde{f}_i x \neq 0$ and $\tilde{f}_i x \neq 0$ (analogously for $\tilde{e}_i$). Stembridge gives precise conditions in terms of the i-strings to determine which of the two relations must hold for a pair of Kashiwara operators $\tilde{f}_i$, $\tilde{f}_j$, see Relations (P5'), (P6') in [Ste03]. These additional relations can be considered as crystal versions of the Serre relations.

For crystals of type A and hence for all crystals of finite-dimensional $\mathcal{U}_q(\mathfrak{sl}_n(\mathbb{C}))$-modules this result implies

$$\begin{aligned} \tilde{f}_i \tilde{f}_j x &= \tilde{f}_j \tilde{f}_i x && \text{for all } i, j \text{ with } |i-j| > 1, \\ \tilde{f}_i \tilde{f}_{i+1}^2 \tilde{f}_i x &= \tilde{f}_{i+1} \tilde{f}_i^2 \tilde{f}_{i+1} x && \text{if } \varepsilon(x, i+1) - \varepsilon(\tilde{f}_i x, i+1) = -1 = \varepsilon(x, i) - \varepsilon(\tilde{f}_{i+1} x, i) \end{aligned}$$

at nodes x of the crystal graph where $\tilde{f}_i, \tilde{f}_j$ (respectively $\tilde{f}_i, \tilde{f}_{i+1}$) are both defined.

Notice that $\tilde{f}_i, \tilde{f}_{i+1}$ cannot both be defined at a node x of the crystal $\mathcal{B}(\omega_k)$: For $\tilde{f}_{i+1} x \neq 0$ we need a box labelled $i+1$ in the semistandard Young tableau corresponding to x, in which case $\tilde{f}_i x = 0$.

For the crystals $\mathcal{B}(k\omega_1)$, $\mathcal{B}(\omega_k)$ considered in Lemma I.1.1.7 one can deduce the relation $\tilde{f}_i \tilde{f}_{i+1}^2 \tilde{f}_i x = \tilde{f}_{i+1} \tilde{f}_i^2 \tilde{f}_{i+1} x$ for all nodes x of the crystal graph from the relations (I.1.1). One does not need the condition that $\tilde{f}_i, \tilde{f}_{i+1}$ have to be defined at x in this special case.

I.1.1.8 Remark. The relations from [Ste03] are necessary and sufficient to determine the crystals of integrable highest weight representations, but they do not form a complete list of relations. In particular, the relation

$$\tilde{f}_i\tilde{f}_{i-1}\tilde{f}_{i+1}\tilde{f}_i \;=\; \tilde{f}_{i+1}\tilde{f}_i^2\tilde{f}_{i-1}$$

that holds for Kashiwara operators on $\mathcal{B}(k\omega_1)$ does not appear in [Ste03]. Still it is surprisingly similar to the Stembridge relation

$$\tilde{f}_i\tilde{f}_{i+1}^2\tilde{f}_i \;=\; \tilde{f}_{i+1}\tilde{f}_i^2\tilde{f}_{i+1}.$$

I.1.1.9 Remark. The theory of crystal bases and Kashiwara operators is often understood as a theory of quantum groups "at $q = 0$", see e.g. [HK02, Chapter 4.2]. There are different approaches to define quantum groups "at $q = 0$", but these approaches do not necessarily give the same result.

In particular, one can only define specializations at $q = 0$ of the negative (or positive) half of the quantum groups after desymmetrizing the quantum Serre relations so that they can be rewritten without appearance of q^{-1}. For this one uses a twisted version of the quantum group $\mathcal{U}_q(\mathfrak{sl}_n(\mathbb{C}))$. The nonsymmetrised Euler form gives on simple roots $\langle\alpha_i,\alpha_i\rangle = 1$, $\langle\alpha_i,\alpha_{i-1}\rangle = -1$ and $\langle\alpha_i,\alpha_j\rangle = 0$ for all $j \neq i, i-1$. Then the twisted product in the negative half of the quantum group $\mathcal{U}_q(\mathfrak{sl}_n(\mathbb{C})^-)$ is defined by $F_i * F_j = q^{-\langle\alpha_i,\alpha_j\rangle}F_iF_j$. From the q-Serre relations one computes new assymmetric relations of the form

$$F_i * F_j \;-\; F_j * F_i \;=\; 0, \;|i-j| > 1, \tag{I.1.2}$$

$$F_i * F_i * F_{i-1} \;-\; (1+q^2)F_i * F_{i-1} * F_i \;+\; q^2 F_{i-1} * F_i * F_i \;=\; 0, \tag{I.1.3}$$

$$F_i * F_{i-1} * F_{i-1} \;-\; (1+q^2)F_{i-1} * F_i * F_{i-1} \;+\; q^2 F_{i-1} * F_{i-1} * F_i \;=\; 0. \tag{I.1.4}$$

The twisted negative half of the quantum group $\mathcal{U}_q(\mathfrak{sl}_n(\mathbb{C}))$ is defined to be the $\mathbb{Q}[[q^2]]$-subalgebra of $(\mathcal{U}_q(\mathfrak{sl}_n(\mathbb{C})), *)$ generated by $F_1, \dots, F_{n-1}$ with respect to the twisted multiplication $*$. In order to compare Hall algebra constructions and quantum groups one needs to twist the usual multiplication in one of the algebras in question. In [Rei02] it is proven that the twisted (positive) half of the quantum group specialized to $q = 0$ is isomorphic to the linearisation of the Hall monoid, see also Section I.3.1. The relations that are given by (I.1.2), (I.1.3) and (I.1.4) with $q = 0$ are known as (local) plactic relations.

By Lemma I.1.1.7, the Kashiwara operators $\tilde{f}_i$ satisfy the plactic relations on crystals $\mathcal{B}(k\omega_1)$ and $\mathcal{B}(\omega_k)$, see in particular equation (I.1.1).

In general, the Kashiwara operators $\tilde{f}_i$ cannot be identified with the operators F_i in the above specialisation of the quantum group at $q = 0$. For example, while $F_2 * F_1 * F_1 = F_1 * F_2 * F_1$ in the above specialisation, we can read off from the crystal graph of the $\mathfrak{U}_q(\mathfrak{sl}_3(\mathbb{C}))$-module $L_q(2\varepsilon_1 + \varepsilon_2)$ that $\tilde{f}_2\tilde{f}_1^2 \neq \tilde{f}_1\tilde{f}_2\tilde{f}_1$, see Figure I.1.1.2. $\Diamond$

I.1.1.2. Affine case

The extended loop algebra is a Lie algebra defined by

$$\widetilde{\mathfrak{sl}}_n(\mathbb{C}) \;=\; \mathfrak{sl}_n(\mathbb{C}) \otimes \mathbb{C}[T^{\pm 1}] \oplus \mathbb{C} \cdot c$$

with Lie bracket so that c is central and

$$[g \otimes T^m, g' \otimes T^{m'}] = [g, g'] \otimes T^{m+m'} + m\delta_{m,-m'}(g, g')c$$

for $g, g' \in \mathfrak{sl}_n(\mathbb{C})$, where $(\cdot,\cdot)$ denotes the Killing form on $\mathfrak{sl}_n(\mathbb{C})$. Its standard Cartan subalgebra is given by $\widetilde{\mathfrak{h}} = \mathfrak{h} \oplus \mathbb{C} \cdot c$. Denote $h_0 = c - \sum_i h_i$. By abuse of notation we write $\lambda \in \widetilde{\mathfrak{h}}^*$ for the linear function that restricts to $\lambda \in \mathfrak{h}^*$ and is extended by 0 to $\mathbb{C} \cdot c$. The extended loop algebra decomposes into $\widetilde{\mathfrak{h}}$-root spaces $\widetilde{\mathfrak{sl}}_n(\mathbb{C}) = \bigoplus_{\alpha\in\Phi} \widetilde{\mathfrak{sl}}_n(\mathbb{C})_\alpha$ with $\widetilde{\mathfrak{sl}}_n(\mathbb{C})_\alpha = \mathfrak{sl}_n(\mathbb{C})_\alpha \otimes \mathbb{C}[T^{\pm 1}]$, where the roots $\alpha \in \Phi$ are seen as elements of $\widetilde{\mathfrak{h}}^*$ by extension by 0 to $\mathbb{C} \cdot c$.

The loop algebra $\mathfrak{sl}_n(\mathbb{C}) \otimes \mathbb{C}[T^{\pm 1}]$ is a quotient of the extended loop algebra. Representations of the loop algebra can be lifted to representations of the extended loop algebra where c acts trivially.

The extended loop algebra is the derived Lie subalgebra of the affine Kac-Moody Lie algebra $\widehat{\mathfrak{sl}}_n(\mathbb{C})$ of type $\widehat{\mathsf{A}}_n = \mathsf{A}_n^{(1)}$, see [Kum02, Chapter 13.1]. The nontrivial simple modules for $\widehat{\mathfrak{sl}}_n(\mathbb{C})$ are all infinite dimensional, see [HK02, Section 10] or [Sen10]. In contrast, the (extended) loop algebra $\widetilde{\mathfrak{sl}}_n(\mathbb{C})$ has finite dimensional simple modules, see [Sen10]. A class of examples is provided by the evaluation modules: For any number $a \in \mathbb{C}^\times$ and $\lambda \in P^+$ one can lift the finite dimensional irreducible $\mathfrak{sl}_n(\mathbb{C})$-module $L(\lambda)$ along the evaluation map

$$\mathrm{ev}_a : \widetilde{\mathfrak{sl}}_n(\mathbb{C}) \;\to\; \mathfrak{sl}_n(\mathbb{C}), \quad T \;\mapsto\; a, \quad c \;\mapsto 0.$$

Since the evaluation map is an algebra homomorphism which restricts to the identity on $\mathfrak{sl}_n(\mathbb{C})$, the result is a finite dimensional $\widetilde{\mathfrak{sl}}_n(\mathbb{C})$-module which is indeed irreducible. See also [Sen10] for a classification of finite dimensional irreducible modules of the loop algebra in terms of tuples of so-called Drinfeld polynomials with constant term equal to 1.

Let us now turn to the quantum affine algebra:

I.1.1.10 Definition. The quantum affine algebra $\mathcal{U}_q(\widetilde{\mathfrak{sl}}_n(\mathbb{C}))$ is the unital associative $\mathbb{C}(q)$-algebra generated by formal generators $E_i, F_i, K_i^{\pm 1}$ for $0 \leq i \leq n-1$ with defining relations

$$
\begin{aligned}
K_i K_i^{-1} &= 1 = K_i^{-1} K_i && \text{for } i \in \mathbb{Z}/n\mathbb{Z},\\
K_j E_i &= q^{\alpha_i(h_j)} E_i K_j && \text{for } i,j \in \mathbb{Z}/n\mathbb{Z},\\
K_j F_i &= q^{-\alpha_i(h_j)} F_i K_j && \text{for } i,j \in \mathbb{Z}/n\mathbb{Z},\\
[E_i, F_j] &= \delta_{ij} \frac{K_i - K_i^{-1}}{q - q^{-1}} && \text{for } i,j \in \mathbb{Z}/n\mathbb{Z},\\
E_i^2 E_{i\pm1} - [2]_q E_i E_{i\pm1} E_i + E_{i\pm1} E_i^2 &= 0, &&\\
[E_i, E_j] &= 0 && \text{for } j \neq i \pm 1,\\
F_i^2 F_{i\pm1} - [2]_q F_i F_{i\pm1} F_i + F_{i\pm1} F_i^2 &= 0, &&\\
[F_i, F_j] &= 0 && \text{for } j \neq i \pm 1,
\end{aligned}
$$

where now all indices are understood modulo n.

I.1.1.11 Remark. i) There are many different definitions of the quantum affine algebra $\mathcal{U}_q(\widetilde{\mathfrak{sl}}_n(\mathbb{C}))$ available in the literature. We follow here the definition given in [CP95a, Theorem 12.2.1, Section 9.1]. Several presentations of $\mathcal{U}_q(\widetilde{\mathfrak{sl}}_n(\mathbb{C}))$ are available, see e.g. the overview in [CP95a], [Bec94], [CP95b]. The Drinfeld presentation is more complicated, but also makes it more obvious that $\mathcal{U}_q(\widetilde{\mathfrak{sl}}_n(\mathbb{C}))$ is a quantisation of the extended loop algebra. In particular, the central element $c \in \widetilde{\mathfrak{sl}}_n(\mathbb{C})$ corresponds to the central element $K_0 K_\theta \in \mathcal{U}_q(\widetilde{\mathfrak{sl}}_n(\mathbb{C}))$, where K_θ is a certain product of K_i, $1 \leq i \leq n-1$, see [Cha01, Section 2].

ii) Often the quantum affine algebra is denoted $\mathcal{U}_q'(\widehat{\mathfrak{sl}}_n(\mathbb{C}))$ or $\mathcal{U}_q'(\mathfrak{sl}_n(\mathbb{C}))$.

iii) The name quantum affine algebra is sometimes also used for quantisations of the (non-extended) loop algebra $\mathfrak{sl}_n(\mathbb{C}) \otimes \mathbb{C}[T^{\pm 1}]$ or the affine Kac-Moody Lie algebra $\widehat{\mathfrak{sl}}_n(\mathbb{C})$. See e.g. [Cha01] for a definition of the quantum group associated with the loop algebra as quotient of $\mathcal{U}_q(\widetilde{\mathfrak{sl}}_n(\mathbb{C}))$.

iv) As expected, there is an embedding of algebras $\mathcal{U}_q(\mathfrak{sl}_n(\mathbb{C})) \to \mathcal{U}_q(\widetilde{\mathfrak{sl}}_n(\mathbb{C}))$ given by $E_i \mapsto E_i$, $F_i \mapsto F_i$ and $K_i \mapsto K_i$. This is a nontrivial result, a proof can be found in [MP95, Proposition 2 of Section 3.4]. ◊

A $\mathcal{U}_q(\widetilde{\mathfrak{sl}}_n(\mathbb{C}))$-module is said to be a (classical) weight module if it decomposes into weight spaces with respect to the action of K_i, $1 \leq i \leq n-1$, just as in the non-affine case. Again we write μ for weights of the form $+q^\mu$, where $\mu \in P$ originally denotes an integral weight of $\mathfrak{sl}_n(\mathbb{C})$, see Section I.1.1.1. A $\mathcal{U}_q(\widetilde{\mathfrak{sl}}_n(\mathbb{C}))$-module is called highest weight module if it is highest weight as $\mathcal{U}_q(\mathfrak{sl}_n(\mathbb{C}))$-module, and the central element K_0K acts by 1.

The finite dimensional irreducible $\mathcal{U}_q(\widetilde{\mathfrak{sl}}_n(\mathbb{C}))$-modules are all highest weight up to some sign twist. By [CP95b, Theorem 3.3] the finite dimensional irreducible $\mathcal{U}_q(\widetilde{\mathfrak{sl}}_n(\mathbb{C}))$-modules (of type $\mathbf{1}$) are parametrized by $(n-1)$-tuples of polynomials in one variable with constant term 1, see also [CP91] (and note that the results from [CP91], [CP95b] have been obtained for $q = \epsilon \in \mathbb{C}^\times$ transcendental). In general it is difficult to describe these modules explicitly. In the quantum case it is only possible in type $\widehat{\mathsf{A}}$ to construct finite dimensional irreducible $\mathcal{U}_q(\widetilde{\mathfrak{sl}}_n(\mathbb{C}))$-modules from $\mathcal{U}_q(\mathfrak{sl}_n(\mathbb{C}))$-modules via evaluation homomorphisms, see [CP95b, Section 4.1] and [CP91, Proposition 4.1] for the definition in case $n = 2$.

In general, an important class of finite dimensional irreducible $\mathcal{U}_q(\widetilde{\mathfrak{sl}}_n(\mathbb{C}))$-modules is given by Kirillov–Reshetikhin modules $W^{i,r}$. The name originally refers to evaluation modules of the Yangian developed in [KR87]. They are labelled by a node i of the Dynkin diagram of classical type A_{n-1} and a positive integer $r \in \mathbb{Z}_{>0}$. In [Cha01] a definition of the Kirillov–Reshetikhin modules $W^{i,r}$ in terms of generators and relations is given. They are constructed for the quantum loop algebra which is a quotient of $\mathcal{U}_q(\widetilde{\mathfrak{sl}}_n(\mathbb{C}))$, so the central element K_0K_θ acts by zero on $W^{i,r}$. Chari proved a decomposition theorem for Kirillov–Reshetikhin modules as $\mathcal{U}_q(\mathfrak{sl}_n(\mathbb{C}))$-modules conjectured in [Hat+02, Conjecture 2.1]. The Kirillov–Reshetikhin modules are minimal affinizations in the sense of [CP95b, Section 6], see [CH10, Section 8]. In particular, for type $\widehat{\mathsf{A}}$ there is an isomorphism $W^{i,r} \cong L_q(r\omega_i)$ as $\mathcal{U}_q(\mathfrak{sl}_n(\mathbb{C}))$-modules [CP96, Theorem 3.1].

Abstract crystals can be defined similarly to the finite case situation from Section I.1.1.1, see e.g. [Kan+92]. It is proven in [Kan+92] that Kirillov–Reshetikhin modules admit crystal (pseudo)bases. Previously, results for type $\widehat{\mathsf{A}}$ have been obtained in [MM90] and [Jim+91], see furthermore [Shi02] and the overview in [Kus13], [Kus16]. In type $\widehat{\mathsf{A}}$ these Kirillov-Reshetikhin crystals are perfect [Kan+92, Theorem 1.2.2], see also [Par12].

The vertices of the crystal graph of a Kirillov–Reshetikhin module of type $\widehat{\mathsf{A}}$ can be realised by semistandard Young tableaux of rectangular shape, see [Shi02, Theorem 3.9]. The Kashiwara operators $\tilde{f}_i$, $\tilde{e}_i$ for $1 \le i \le n-1$ act as described in Section I.1.1.1 - this is the crystal version of the isomorphism of $W^{i,r} \cong L_q(r\omega_i)$ as $\mathcal{U}_q(\mathfrak{sl}_n(\mathbb{C}))$-modules. Then the Kashiwara operator $\tilde{f}_0$ (and similarly $\tilde{e}_0$) can be defined combinatorially as follows. Recall the rotation automorphism of the Dynkin diagram of type $\widehat{\mathsf{A}}$ given on the nodes by $i \mapsto i+1 \in \mathbb{Z}/n\mathbb{Z}$. It induces an isomorphism on crystals for Young tableaux of rectangular shape (this fails if the shape is not rectangular). This automorphism ψ is given by the

Schützenberger promotion operator realised in [Shi02, Proposition 3.15], according to which, ψ is applied to a semistandard Young tableau by the following steps: (i) remove all entries n, (ii) perform jeu-de-taquin to slide the remaining entries to the empty boxes, (iii) add 1 to all entries, (iv) fill the vacated boxes by 1.

For general Young tableaux, jeu-de-taquin is defined by a combinatorial rule e.g. in [Ful97, Section 1.2]. For Young tableaux of shape $k\omega_1$ or ω_k that consist of a single row or column, respectively, it is simply given by sliding all entries to the left or downwards, respectively.

I.1.1.12 Example. Let $n = 5$ and consider the following semistandard Young tableau of shape $6\omega_1$:

$$\begin{aligned}
\begin{array}{|c|c|c|c|c|c|}\hline 1&2&2&3&5&5\\ \hline\end{array} &\overset{\text{(i)}}{\rightsquigarrow} \begin{array}{|c|c|c|c|c|c|}\hline 1&2&2&3&\cdot&\cdot\\ \hline\end{array}\\
&\overset{\text{(ii)}}{\rightsquigarrow} \begin{array}{|c|c|c|c|c|c|}\hline \cdot&\cdot&1&2&2&3\\ \hline\end{array}\\
&\overset{\text{(iii)}}{\rightsquigarrow} \begin{array}{|c|c|c|c|c|c|}\hline \cdot&\cdot&2&3&3&4\\ \hline\end{array}\\
&\overset{\text{(iv)}}{\rightsquigarrow} \begin{array}{|c|c|c|c|c|c|}\hline 1&1&2&3&3&4\\ \hline\end{array}.
\end{aligned}$$

◊

Then ψ^{-1} is given by the reversed application of these steps: i) remove all entries 1, ii) subtract 1 from all remaining entries, iii) perform jeu-de-taquin to slide the remaining entries to the empty boxes, iv) fill the vacated boxes by n.

Finally, the Kashiwara operator $\tilde{f}_0$ applied to a rectangular semistandard Young tableau is given by $\tilde{f}_0 = \psi^{-1}\tilde{f}_1\psi$, see [Shi02, Equation 3.7].

I.1.1.13 Example. Let $n = 5$ and consider again the semistandard Young tableau of shape $6\omega_1$ from Example I.1.1.12:

$$\begin{array}{|c|c|c|c|c|c|}\hline 1&2&2&3&5&5\\ \hline\end{array} \overset{\psi}{\longmapsto} \begin{array}{|c|c|c|c|c|c|}\hline 1&1&2&3&3&4\\ \hline\end{array} \overset{\tilde{f}_1}{\longmapsto} \begin{array}{|c|c|c|c|c|c|}\hline 1&2&2&3&3&4\\ \hline\end{array} \overset{\psi^{-1}}{\longmapsto} \begin{array}{|c|c|c|c|c|c|}\hline 1&1&2&2&3&5\\ \hline\end{array}.$$

For other nonexceptional types, the Kirillov–Reshetikhin crystals were constructed explicitly in [FOS09].

$$\begin{tikzcd}
\boxed{1} \arrow[r, "1"] & \boxed{2} \arrow[r, "2"] & \cdots \arrow[r, "n-1"] & \boxed{n} \arrow[lll, bend left, "0"]
\end{tikzcd}$$

Figure I.1.1.3.: The Kirillov–Reshetikhin crystal graph for $W^{1,1} \cong_{\mathcal{U}_q(\mathfrak{sl}_n(\mathbb{C}))} L_q(\omega_1)$, analogue of the standard/vector representation for $\mathcal{U}_q(\widetilde{\mathfrak{sl}}_n(\mathbb{C}))$.

We focus on Kirillov–Reshetikhin modules $W^{1,k}$ ($k \in \mathbb{Z}_{>0}$) and $W^{k,1}$ ($1 \le k \le n-1$). In this case the action of the Kashiwara operators $\tilde{f}_i$ for $0 \le i \le n-1$ is particularly simple: For $1 \le i \le n-1$ the action has been described in Lemma I.1.1.3 and Lemma I.1.1.4 for the finite case. For $i = 0$ the jeu-de-taquin rule is simply given by sliding all entries downwards for $W^{k,1}$, respectively to the right for $W^{1,k}$. Therefore $\psi^{-1}\tilde{f}_1\psi$ applied to a semistandard Young tableau of shape $k\omega_1$ or ω_k is given by replacing the (unique or rightmost, respectively) box with entry n by a box in the top left corner with entry 1, if possible, otherwise the result is zero. We obtain the following two lemmata:

I.1.1.14 Lemma. Let $i \in \mathbb{Z}/n\mathbb{Z}$ and $k \in \mathbb{Z}_{\ge 0}$. On semistandard Young tableaux of shape $k\omega_1$ that contain a box with entry i the Kashiwara operator $\tilde{f}_i$ replaces a box with entry i by a box with entry $i+1 \bmod n$ so that the result is again a semistandard Young tableau of shape $k\omega_1$. On semistandard Young tableaux of shape $k\omega_1$ that do not contain any box with entry i the Kashiwara operator $\tilde{f}_i$ acts by zero.

Proof. For $\tilde{f}_i$, $1 \le i \le n-1$, this follows from Lemma I.1.1.3 together with the isomorphism $W^{1,k} \cong L_q(k\omega_1)$ of $\mathcal{U}_q(\mathfrak{sl}_n(\mathbb{C}))$-modules. For $i = 0$ this is an application of the identity $\tilde{f}_0 = \psi^{-1}\tilde{f}_1\psi$ [Shi02, Equation 3.7] together with the simplified jeu-de-taquin rule for Young tableaux of shape $k\omega_1$. □

I.1.1.15 Lemma. Let $i \in \mathbb{Z}/n\mathbb{Z}$ and $1 \le k \le n-1$. On semistandard Young tableaux of shape ω_k that contain a box with entry $i \bmod n$ and that do not contain a box with entry $i+1 \bmod n$ the Kashiwara operator $\tilde{f}_i$ replaces the box with entry $i \bmod n$ by a box with entry $i+1 \bmod n$ so that the result is again a semistandard Young tableau of shape ω_k. On semistandard Young tableaux of shape ω_k that do not contain a box with entry $i \bmod n$ or that do contain a box with entry $i+1 \bmod n$ the Kashiwara operator $\tilde{f}_i$ acts by zero.

Proof. For $\tilde{f}_i$, $1 \le i \le n-1$, this follows from Lemma I.1.1.4 together with the isomorphism $W^{k,1} \cong L_q(1\omega_k)$ of $\mathcal{U}_q(\mathfrak{sl}_n(\mathbb{C}))$-modules. For $i = 0$ this is an application of the identity $\tilde{f}_0 = \psi^{-1}\tilde{f}_1\psi$ [Shi02, Equation 3.7] together with the simplified jeu-de-taquin rule for Young tableaux of shape ω_k. □

I.1.1.16 Remark. The rules from Lemma I.1.1.14 and Lemma I.1.1.15 are formulated independently of the relative positions of the boxes with entries i, $i+1 \bmod n$. ◊

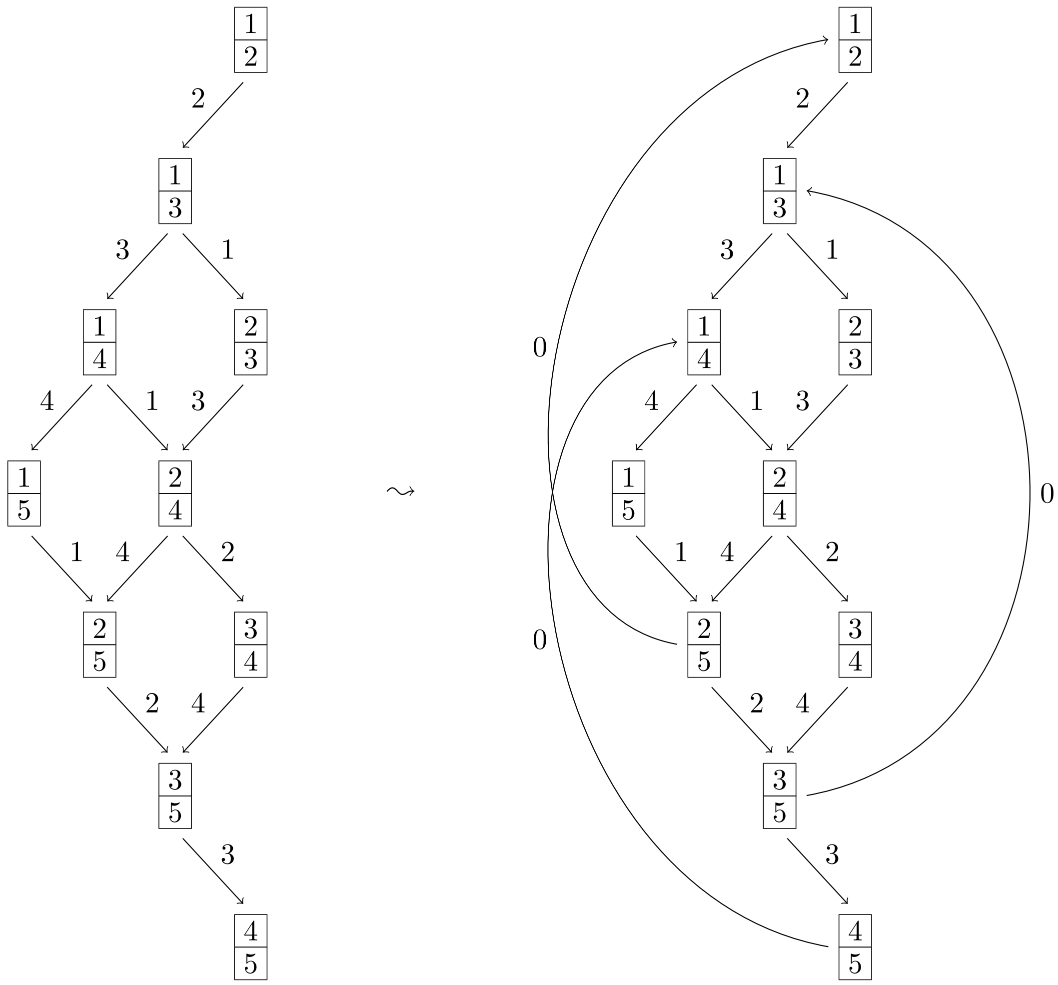


Figure I.1.1.4.: The crystal graph for $W^{2,1} \in \mathcal{U}_q(\widetilde{\mathfrak{sl}}_5(\mathbb{C}))\text{–}\mathsf{mod}$ obtained from the crystal graph for $L(\omega_2) \in \mathcal{U}_q(\mathfrak{sl}_5(\mathbb{C}))\text{–}\mathsf{mod}$.

I.1.2. Combinatorics of particle configurations

In this section we identify the crystal combinatorics described in Section I.1.1 with certain particle configurations following [Jim+91], [KS10]. Notice that the representations of the affine Kac-Moody Lie algebra considered in [KS10] factor indeed over the extended loop algebra $\widetilde{\mathfrak{sl}}_n(\mathbb{C})$.

The particle configurations are discussed in more detail in Chapters I.2 and I.3, where also graphical realisations are given. Combinatorially they can be defined as follows:

I.1.2.1 Definition. i) For $k \in \mathbb{Z}_{>0}$, $n \geq 2$, classical and affine bosonic particle configurations of k particles on n positions are defined to be partitions of k with n parts, i.e. tuples $(k_1, \ldots, k_n) \in \mathbb{Z}^n_{\geq 0}$ with $\sum_i k_i = k$. For classical bosonic particle configurations, particle propagation operators a_r are defined for $1 \leq r \leq n-1$ by

$$\begin{aligned} a_r(k_1, \ldots, k_r, k_{r+1}, \ldots, k_n) &= (k_1, \ldots, k_r - 1, k_{r+1} + 1, \ldots, k_n) && \text{if } k_r > 0, \\ a_r(k_1, \ldots, k_r, k_{r+1}, \ldots, k_n) &= 0 && \text{else.} \end{aligned}$$

For affine bosonic particle configurations, the particle propagation operators are defined for $1 \leq r \leq n-1$ in the same way. Additionally, there is a propagation operator a_0 given by

$$\begin{aligned} a_0(k_1, \ldots, k_n) &= (k_1 + 1, \ldots, k_n - 1) && \text{if } k_n > 0, \\ a_0(k_1, \ldots, k_n) &= 0 && \text{else.} \end{aligned}$$

Similarly, a reversed particle propagation operator a_r^* is defined for classical or affine bosonic particle configurations $(k_1, \ldots, k_n)$ with $1 \leq r \leq n-1$ or $0 \leq r \leq n-1$, respectively, by

$$\begin{aligned} a_r^*(k_1, \ldots, k_r, k_{r+1}, \ldots, k_n) &= (k_1, \ldots, k_r + 1, k_{r+1} - 1, \ldots, k_n) && \text{if } k_{r+1} > 0, \\ a_r^*(k_1, \ldots, k_r, k_{r+1}, \ldots, k_n) &= 0 && \text{else.} \end{aligned}$$

ii) For $1 \leq k \leq n-1$, $n \geq 2$, classical and affine fermionic particle configurations of k particles on n positions are defined to be tuples $(i_1, \ldots, i_n) \in \{0,1\}^n$ with $\sum_r i_r = k$. For classical fermionic particle configurations, particle propagation operators a_r are defined for $1 \leq r \leq n-1$ by

$$\begin{aligned} a_r(k_1, \ldots, k_r, k_{r+1}, \ldots, k_n) &= (k_1, \ldots, k_r - 1, k_{r+1} + 1, \ldots, k_n) && \text{if } k_r = 1,\ k_{r+1} = 0, \\ a_r(k_1, \ldots, k_r, k_{r+1}, \ldots, k_n) &= 0 && \text{else.} \end{aligned}$$

For affine fermionic particle configurations, the particle propagation operators are defined for $1 \leq r \leq n-1$ in the same way. Additionally, there is a propagation operator a_0 given by

$$\begin{aligned} a_0(k_1, \ldots, k_n) &= (k_1 + 1, \ldots, k_n - 1) && \text{if } k_n = 1,\ k_1 = 0, \\ a_0(k_1, \ldots, k_n) &= 0 && \text{else.} \end{aligned}$$

Similarly, define the reversed particle propagation operator a_r^* for a classical or affine fermionic particle configuration $(k_1, \ldots, k_n)$ with $1 \leq r \leq n-1$ or $0 \leq r \leq n-1$, respectively, by

$$\begin{aligned} a_r^*(k_1, \ldots, k_r, k_{r+1}, \ldots, k_n) &= (k_1, \ldots, k_r + 1, k_{r+1} - 1, \ldots, k_n) && \text{if } k_{r+1} = 1,\ k_r = 0, \\ a_r^*(k_1, \ldots, k_r, k_{r+1}, \ldots, k_n) &= 0 && \text{else.} \end{aligned}$$

Hence, if $a_i(k_1',\dots,k_n') = (k_1,\dots,k_n) \neq 0$, then we have $a_i^*(k_1,\dots,k_n) = (k_1',\dots,k_n')$.

One can identify particle configurations and crystals as follows, see [KS10, Remark 5.9]:

I.1.2.2 Proposition. i) Let $k \in \mathbb{Z}_{>0}$. There are bijections of sets

$$\begin{aligned} &\{\text{semistandard Young tableaux of shape } k\omega_1 \text{ with entries } 1,\dots,n\} \\ \leftrightarrow\ &\{\text{classical bosonic particle configurations of } k \text{ particles with } n \text{ positions}\}, \end{aligned}$$

$$\begin{aligned} &\{\text{semistandard Young tableaux of shape } k\omega_1 \text{ with entries in } \mathbb{Z}/n\mathbb{Z}\} \\ \leftrightarrow\ &\{\text{affine bosonic particle configurations of } k \text{ particles with } n \text{ positions}\} \end{aligned}$$

that identify the unique semistandard Young tableaux of shape $k\omega_1$ with entries $(i_1,\dots,i_k)$ so that $k_r := \#\{j \mid 1 \leq j \leq k,\ i_j = r\}$ with the particle configuration $(k_1,\dots,k_n)$. These bijections identify the Kashiwara operators $\tilde{f}_i$ with particle propagation operators a_i for $1 \leq i \leq n$ and $i \in \mathbb{Z}/n\mathbb{Z}$, respectively. The Kashiwara operators $\tilde{e}_i$ are identified with particle propagation operators a_i^* for $1 \leq i \leq n$ and $i \in \mathbb{Z}/n\mathbb{Z}$, respectively.

ii) Let $1 \leq k \leq n-1$. There are bijections of sets

$$\begin{aligned} &\{\text{semistandard Young tableaux of shape } \omega_k \text{ with entries } 1,\dots,n\} \\ \leftrightarrow\ &\{\text{classical fermionic particle configurations of } k \text{ particles with } n \text{ positions}\}, \end{aligned}$$

$$\begin{aligned} &\{\text{semistandard Young tableaux of shape } \omega_k \text{ with entries in } \mathbb{Z}/n\mathbb{Z}\} \\ \leftrightarrow\ &\{\text{affine fermionic particle configurations of } k \text{ particles with } n \text{ positions}\} \end{aligned}$$

that identify the unique semistandard Young tableaux of shape ω_k with entries $(i_1,\dots,i_k)$ so that $k_r := \#\{j \mid 1 \leq j \leq k,\ i_j = r\}$ with the particle configuration $(k_1,\dots,k_n)$. These bijections identify the Kashiwara operators $\tilde{f}_i$ with particle propagation operators a_i for $1 \leq i \leq n$ and $i \in \mathbb{Z}/n\mathbb{Z}$, respectively. The Kashiwara operators $\tilde{e}_i$ are identified with particle propagation operators a_i^* for $1 \leq i \leq n$ and $i \in \mathbb{Z}/n\mathbb{Z}$, respectively.

Proof. For the bijections between semistandard Young tableaux and particle configurations it suffices to observe that semistandard Young tableaux of shape ω_k or $k\omega_1$ are uniquely determined by their entries without recording the precise position of each entry. The identification of Kashiwara operators with particle propagation operators follows from Lemma I.1.1.3, Lemma I.1.1.14, Lemma I.1.1.4 and Lemma I.1.1.15 together with the property of crystals graphs that there is an edge from node x to node x' labelled $\tilde{f}_i$ if and only if there is an edge from node x' to x labelled $\tilde{e}_i$. □

I.1.2.3 Remark. By Proposition I.1.2.2 one can identify Kashiwara operators with particle propagation operators. In Chapters I.2 and Chapter I.3 we see that one can identify the particle propagation operators with generators of the (affine) plactic algebra. The (affine) plactic algebra is isomorphic to the specialisation of one half of the (affine) quantum group at $q = 0$ by desymmetrized Serre relations, see Remark I.1.1.9.

Thereby the two different approaches of specialisation of the quantum group at $q = 0$ – by crystals and by desymmetrized Serre relations – are related in the special case of Young diagrams of shape $k\omega_1$, ω_k. This fails for other crystals, see Remark I.1.1.9. It is special about $k\omega_1$ and ω_k that the action of Kashiwara operators, particle propagation operators and plactic operators is uniquely determined by the list of entries in the Young tableaux, independently of the position of the entries. In general, the Kashiwara operators are sensitive to the position of the entries, see Figure I.1.1.2 $\Diamond$

I.2. The affine nilTemperley–Lieb algebra

This chapter is structured as follows: In Section I.2.1 we define the affine nilTemperley–Lieb algebra $\mathrm{n}\widehat{\mathrm{TL}}_N$ and fix some notation. In Section I.2.2 we give an overview over algebras related to the affine nilTemperley–Lieb algebra. In Section I.2.3 we discuss two gradings of $\mathrm{n}\widehat{\mathrm{TL}}_N$ that are important tools when we compute the center of the algebra. The second main tool for the computation of the center is the faithfulness of the action on affine fermionic particle configurations: In Section I.2.4 we define this action, in Section I.2.5 we describe a normal form for monomials and hence a basis of $\mathrm{n}\widehat{\mathrm{TL}}_N$ that we use in Section I.2.6 to give an elementary proof of faithfulness.

In Section I.2.7 we define certain projectors inside $\mathrm{n}\widehat{\mathrm{TL}}_N$ that sum up to central elements. Then in Section I.2.8 we show that the center of $\mathrm{n}\widehat{\mathrm{TL}}_N$ is in fact generated by these special central elements. In Section I.2.9 we show that $\mathrm{n}\widehat{\mathrm{TL}}_N$ is finitely generated over the center. In Section I.2.10 we describe another normal form for the monomials that makes use of the center. At that moment we have collected enough information to construct inclusions $\mathrm{n}\widehat{\mathrm{TL}}_N \subset \mathrm{n}\widehat{\mathrm{TL}}_{N+1}$ in Section I.2.11.

While all results so far hold over an arbitrary ground field $\Bbbk$ (see Remark I.2.1.3), we have to assume that $\Bbbk$ is an uncountable algebraically closed field in the remaining sections. In Section I.2.12 we compute localisations with respect to central elements and we classify the simple modules over $\mathrm{n}\widehat{\mathrm{TL}}_N$. In Section I.2.13 we use these localisations together with a rank argument to show that $\mathrm{n}\widehat{\mathrm{TL}}_N$ is not free over its center. In Section I.2.14 we discuss possible approaches to equip $\mathrm{n}\widehat{\mathrm{TL}}_N$ with an affine cellular structure in the sense of [KX12] based on our knowledge of different normal forms for monomials in $\mathrm{n}\widehat{\mathrm{TL}}_N$.

I.2.1. Notation

Let $\Bbbk$ be any field, denote by $\Bbbk[q]$ the polynomial ring in an indeterminate q, and assume N is a positive integer. *Throughout we will assume $N \geq 3$.*

I.2.1.1 Definition. The affine nilTemperley–Lieb algebra $\mathrm{n}\widehat{\mathrm{TL}}_N$ of rank N is the unital associative $\Bbbk$-algebra generated by elements $a_0, \ldots, a_{N-1}$ subject to the relations

$$\begin{aligned} a_i^2 &= 0 && \text{for all } 0 \leq i \leq N-1, \\ a_i a_j &= a_j a_i && \text{for all } i-j \neq \pm 1 \bmod N, \\ a_i a_{i+1} a_i &= a_{i+1} a_i a_{i+1} = 0 && \text{for all } 0 \leq i \leq N-1, \end{aligned}$$

where all indices are taken modulo N, so in particular $a_{N-1}a_0a_{N-1} = a_0a_{N-1}a_0 = 0$. The finite nilTemperley–Lieb algebra nTL_N is the subalgebra of $\mathrm{n}\widehat{\mathrm{TL}}_N$ generated by $a_1, \ldots, a_{N-1}$. We adopt the convention that $\mathrm{nTL}_1 = \Bbbk 1$.

We fix the following notation for monomials in $\mathrm{n}\widehat{\mathrm{TL}}_N$ and nTL_N: For an ordered index sequence $\underline{j} = (j_1, \ldots, j_m)$ with $0 \leq j_1, \ldots, j_m \leq N-1$, we define the element

$$a(\underline{j}) = a_{j_1} \ldots a_{j_m}$$

and we call it (ordered) monomial. Unless otherwise specified, we use the letters i, j for indices from $\mathbb{Z}/N\mathbb{Z}$, in particular, we identify the indices 0 and N in this case.

I.2.1.2 Example. For $N = 5$, the monomial associated with the ordered index sequence $\underline{j} = (1,3,2,0,1,4,3)$ is given by

$$a(\underline{j}) = a(1,3,2,0,1,4,3) = a_1a_3a_2a_0a_1a_4a_3.$$

I.2.1.3 Remark. Except for Sections I.2.12, I.2.13 and I.2.14, all of our results hold over an arbitrary ground field $\Bbbk$, even one of characteristic 2, simply by ignoring signs in that case.

In fact, our arguments in Sections I.2.1 – I.2.7 work for any associative unital ground ring R if we adapt our notation: We need to replace $\Bbbk$-vector spaces and $\Bbbk$-algebras with free R-modules and R-algebras, respectively. In particular, the affine nilTemperley–Lieb algebra over $\Bbbk$ is replaced by the R-algebra with the same generators and relations, and the polynomial ring $\Bbbk[q]$ is replaced by $R[q]$. In addition, in Sections I.2.8 – I.2.11 we have to assume in addition that R is commutative unless we modify our statements slightly, e.g. replace R by its center in Proposition I.2.8.2 and Theorem I.2.8.5 and assume that R is finitely generated over its center in Theorem I.2.9.1.

This is possible because our arguments mainly rely on investigating monomials in the generators of $\mathrm{n}\widehat{\mathrm{TL}}_N$. However, for simplicity we have chosen to assume $\Bbbk$ is a field throughout the chapter. ◊

I.2.2. Related algebras

The affine nilTemperley–Lieb algebra appears in many different settings, some of which we describe in this section.

I.2.2.1. The affine nilCoxeter algebra

$\mathrm{n}\widehat{\mathrm{TL}}_N$ is a quotient of the affine nilCoxeter algebra of type $\widehat{\mathsf{A}}_{N-1}$:

The affine nilCoxeter algebra $\mathsf{n}\widehat{\mathsf{C}}_N$ of type $\widehat{\mathsf{A}}_{N-1}$ over a field $\Bbbk$ is the unital associative algebra generated by elements u_i, $0 \le i \le N-1$, satisfying the relations $u_i^2 = 0$; $u_i u_j = u_j u_i$ for $i - j \neq \pm 1 \bmod N$; and $u_i u_{i+1} u_i = u_{i+1} u_i u_{i+1}$ for $1 \le i \le N-1$, where the subscripts are read modulo N. The algebra $\mathrm{n}\widehat{\mathrm{TL}}_N$ is isomorphic to the quotient of $\mathsf{n}\widehat{\mathsf{C}}_N$ obtained by imposing the additional relations $u_i u_{i+1} u_i = u_{i+1} u_i u_{i+1} = 0$ for $1 \le i \le N-1$.

The nilCoxeter algebra nC_N has generators $u_i, 1 \le i \le N-1$, which satisfy the same relations as they do in $\mathsf{n}\widehat{\mathsf{C}}_N$. It first appeared in work on the cohomology of flag varieties [BGG73] and has played an essential role in studies on Schubert polynomials, Stanley symmetric functions, and the geometry of flag varieties (see for example [LS89], [Mac91], [KK86], [FS94]). The definition of nC_N was inspired by the divided difference operators ∂_i on polynomials in variables $\mathbf{x} = \{x_1, \dots, x_N\}$ defined by Demazure operators

$$\partial_i(f) = \frac{f(\mathbf{x}) - f(\sigma_i \mathbf{x})}{x_i - x_{i+1}},$$

where the transposition σ_i fixes all the variables except for x_i and x_{i+1}, which it interchanges. The operators ∂_i satisfy the nilCoxeter relations above, and applications of these relations enabled Fomin and Stanley [FS94] to recover known properties and establish new properties of Schubert polynomials.

The algebra nC_N belongs to a two-parameter family of algebras having generators u_i, $1 \le i \le N-1$, which satisfy the relations $u_i u_j = u_j u_i$ for $|i-j| > 1$ and $u_i u_{i+1} u_i = u_{i+1} u_i u_{i+1}$ for $1 \le i \le N-2$ from above, together with the relation $u_i^2 = \alpha u_i + \beta$ for all i, where α, β are fixed parameters. In particular, the specialization $\alpha = \beta = 0$ yields the nilCoxeter algebra; $\alpha = 0$, $\beta = 1$ gives the standard presentation of the group algebra of the symmetric group $\Bbbk \mathcal{S}_N$; and $\alpha = q - 1, \beta = q$ gives the Hecke algebra $\mathsf{H}_N(q)$ of type A.

Khovanov [Kho01] introduced restriction functors F_D and induction functors F_X corresponding to the natural inclusion of algebras $\mathsf{nC}_N \hookrightarrow \mathsf{nC}_{N+1}$ on the direct sum $\mathcal{C}$ of the categories $\mathcal{C}_N$ of finite-dimensional nC_N-modules. These functors categorify the Weyl algebra of differential operators with polynomial coefficients in one variable and correspond to the Weyl algebra generators ∂ and x (derivative and multiplication by x), which satisfy the relation $\partial x - x\partial = 1$.

Brichard [Bri11] used a diagram calculus on cylinders to determine the dimension of the center of nC_N and to describe a basis of the center for which the multiplication is trivial. In this diagram calculus on N strands, the generator u_i corresponds to a crossing of the strands i and $i+1$. The nil relation $u_i^2 = 0$ is represented by demanding that any two strands may cross at most once; otherwise the diagram is identified with zero.

For convenience let us include an overview of the various 0- and nil-versions of Hecke and Coxeter algebras of type A or $\widehat{\mathsf{A}}$ over the ground ring $\Bbbk$ (e.g. $\Bbbk = \mathbb{C}(q)$). Let $\nu \in \Bbbk^\times$ be a unit (e.g. $\nu = q$).

i) The *nilCoxeter algebra* nC_N of type A with its defining relations $u_i^2 = 0$; $u_i u_j = u_j u_i$ for $|i-j| > 1$; and $u_i u_{i+1} u_i = u_{i+1} u_i u_{i+1}$ for $1 \leq i \leq N-2$ is sometimes also called nilHecke algebra [GR04]. See [Kha16] for generalizations of the nilCoxeter algebra.
ii) The *(polynomial) affine nilHecke algebra* of type A is the algebra generated by u_i, $1 \leq i \leq N-1$, and X_i, $1 \leq i \leq N$, with relations $u_i^2 = 0$; $u_i u_j = u_j u_i$ for $|i-j| > 1$; $u_i u_{i+1} u_i = u_{i+1} u_i u_{i+1}$ for $1 \leq i \leq N-2$; $X_i X_j = X_j X_i$ for all i, j; and with mixed relations $X_j u_i = u_i X_j$ for all $j \neq i, i+1$ and

$$u_i X_{i+1} = X_i u_i + 1, \quad X_{i+1} u_i = u_i X_i + 1.$$

It contains the nilCoxeter algebra nC_N and the polynomial ring $\Bbbk[X_1, \ldots, X_n]$ as subalgebras. Many authors use the name nilHecke algebra for this algebra [KK86], [KL09]. Because of this ambiguity we avoid to use the terminology "nilHecke algebra". In [KL09, Examples 2.2 3)] a graphical realisation for the affine nilHecke algebra is given. Every monomial corresponds to a string diagram connecting N points in the bottom with N points in the top of the diagram. The generator X_i is given by the identity diagram with a dot on the i-th strand, while u_i is given by the crossing of the strands connecting i, $i+1$.
iii) The *(localised) affine nilHecke algebra* of type A is the localisation of the affine nilHecke algebra at all X_i, $1 \leq i \leq N$.
iv) The *Hecke algebra* or *Iwahori–Hecke algebra* $\mathsf{H}_N(\nu)$ of type A is "the" Hecke algebra defined by generators T_i, $1 \leq i \leq N-1$, with defining relations $(T_i - \nu)(T_i + 1) = 0$; $T_i T_j = T_j T_i$ for $|i-j| > 1$; and $T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1}$ for $1 \leq i \leq N-2$. Since $\nu \in \Bbbk^\times$ one can deduce frome these relations in addition that there exists $T_i^{-1} = \nu^{-1}(T_i + 1 - \nu)$ for all $1 \leq i \leq N-1$.

v) The *(polynomial) affine Hecke algebra* of type A is given by generators T_i, $1 \le i \le N-1$, and X_i, $1 \le i \le N$, with relations $(T_i - \nu)(T_i + 1) = 0$; $T_i T_j = T_j T_i$ for $|i-j| > 1$; and $T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1}$ for $1 \le i \le N-2$; $X_i X_j = X_j X_i$ for all i, j; and with mixed relations $T_i X_i T_i = \nu X_{i+1}$ for $1 \le i \le N-1$ and $T_i X_j = X_j T_i$ for $j \ne i, i+1$. One can deduce frome these relations that there exists $T_i^{-1} = \nu^{-1}(T_i + 1 - \nu)$ for all $1 \le i \le N-1$, and furthermore

$$T_i X_i \;=\; X_{i+1} T_i + X_{i+1} - \nu X_{i+1}, \quad X_i T_i \;=\; T_i X_{i+1} + X_{i+1} - \nu X_{i+1}.$$

Although these relations have been deduced using $\nu \in \Bbbk^\times$, they also make sense for $\nu = 0$, so they will appear again in the definition of the affine 0-Hecke algebra below.

vi) The *(localised) affine Hecke algebra* or *extended Iwahori–Matsumoto Hecke algebra* of type A is equal to the polynomial affine Hecke algebra localised at X_i for all $1 \le i \le N$ [IM65], [MS16].

vii) The 0-*Hecke algebra* of type A is defined by generators T_i, $1 \le i \le N-1$ and relations $T_i^2 = -T_i$; $T_i T_j = T_j T_i$ for $|i-j| > 1$; and $T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1}$ for $1 \le i \le N-2$. For alternative generators $\underline{H}_i = T_i + 1$ the relations read $\underline{H}_i^2 = \underline{H}_i$; $\underline{H}_i \underline{H}_j = \underline{H}_j \underline{H}_i$ for $|i-j| > 1$; and $\underline{H}_i \underline{H}_{i+1} \underline{H}_i = \underline{H}_{i+1} \underline{H}_i \underline{H}_{i+1}$ for $1 \le i \le N-2$. Notice that some authors call this algebra as well nilHecke algebra, see [Kha16] and references therein.

viii) Define the *(polynomial) affine* 0-*Hecke algebra* of type A as the algebra generated by T_i, $1 \le i \le N-1$, and X_i, $1 \le i \le N$, and relations $T_i^2 = -T_i$; $T_i T_j = T_j T_i$ for $|i-j| > 1$; $T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1}$ for $1 \le i \le N-2$; $X_i X_j = X_j X_i$ for all i, j; and with mixed relations $T_i X_j = X_j T_i$ for $j \ne i, i+1$; and furthermore

$$T_i X_i \;=\; X_{i+1} T_i + X_{i+1}, \quad X_i T_i \;=\; T_i X_{i+1} + X_{i+1}.$$

From this it follows that $T_i X_i T_i = 0$ for $1 \le i \le N-1$. For alternative generators $\underline{H}_i = T_i + 1$ the relations read $\underline{H}_i^2 = \underline{H}_i$; $\underline{H}_i \underline{H}_j = \underline{H}_j \underline{H}_i$ for $|i-j| > 1$; $\underline{H}_i \underline{H}_{i+1} \underline{H}_i = \underline{H}_{i+1} \underline{H}_i \underline{H}_{i+1}$ for $1 \le i \le N-2$; $X_i X_j = X_j X_i$ for all i, j; and with mixed relations $T_i X_i T_i = 0$ for $1 \le i \le N-1$ and $\underline{H}_i X_j = X_j \underline{H}_i$ for $j \ne i, i+1$; and furthermore

$$\underline{H}_i X_i \;=\; X_{i+1} \underline{H}_i + X_i, \quad X_i \underline{H}_i \;=\; \underline{H}_i X_{i+1} + X_i.$$

ix) The *(localised) affine* 0-*Hecke algebra* of type A is the localisation of the affine 0-Hecke algebra at all X_i, $1 \le i \le N$.

x) We use the name *cyclic affine Hecke algebra* when we refer to the Hecke algebra associated with the affine Coxeter group of type $\widehat{\mathsf{A}}$. It is defined by generators T_i, $0 \le i \le N-1$, with relations $(T_i - q)(T_i + 1) = 0$; $T_i T_j = T_j T_i$ for $|i-j| > 1$; and $T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1}$ for $0 \le i \le N-1$, where all indices are understood modulo N.

It is not immediate how these algebras defined by generators and relations are related. One can check that there is an isomorphism between the (localised) affine nilHecke algebra and the (localised) affine 0-Hecke algebra given by

$$\begin{aligned}\text{(Localised) affine nilHecke algebra} &\cong \text{(Localised) affine 0-Hecke algebra}\\ u_i &\mapsto -X_i^{-1}\underline{H}_i\\ X_i^{\pm 1} &\mapsto X_i^{\pm 1}.\end{aligned}$$

Although the affine nilCoxeter algebra $\mathsf{n}\widehat{\mathsf{C}}_N$ of type $\widehat{\mathsf{A}}$ and the localised affine nilHecke algebra of type A are defined in quite different ways, there is hope that they can be related similarly to the group algebras of "cyclic" affine symmetric group $\mathbb{C}[\widetilde{\mathcal{S}}_N]$ and the "extended" affine symmetric group $\mathbb{C}[\mathcal{S}_N] \ltimes \mathbb{C}[X_1^{\pm 1}, \dots, X_N^{\pm 1}]$. In fact, $\mathbb{C}[\mathcal{S}_N] \ltimes \mathbb{C}[X_1^{\pm 1}, \dots, X_N^{\pm 1}]$ is isomorphic to the $\mathbb{C}$-algebra defined by generators s_i, $1 \leq i \leq N$, and τ, so that $s_i^2 = 1$; $s_is_j = s_js_i$ for $|i-j| > 1$; $s_is_{i+1}s_i = s_{i+1}s_is_{i+1}$ for $1 \leq i \leq N-2$; $s_i\tau = \tau s_{i+1}$ for all i; and all indices are understood modulo N, see [GJ11, Section 6.2]. Indeed, for the (localised) affine Hecke algebra it is known that the "cyclic" presentation (x) and the "extended" presentation (vi) are equivalent, see [MS16, Lemma 3.2]. The Bernstein and the Iwahori-Matsumoto presentation are related by [HP02].

I.2.2.2. The universal enveloping algebra of the Lie algebra of affine type A

$\mathrm{n}\widehat{\mathrm{TL}}_N$ is a quotient of the negative part of the universal enveloping algebra of the affine Kac–Moody Lie algebra $\widehat{\mathfrak{sl}}_N$:

The negative part $\mathcal{U}^-(\widehat{\mathfrak{sl}}_N)$ of the universal enveloping algebra $\mathcal{U}(\widehat{\mathfrak{sl}}_N)$ of the affine Kac–Moody Lie algebra $\widehat{\mathfrak{sl}}_N$ has generators f_i, $0 \leq i \leq N-1$, which satisfy the Serre relations $f_i^2f_{i+1} - 2f_if_{i+1}f_i + f_{i+1}f_i^2 = 0 = f_{i+1}^2f_i - 2f_{i+1}f_if_{i+1} + f_if_{i+1}^2$ and $f_if_j = f_jf_i$ for $i - j \neq \pm 1 \mod N$ (all indices modulo N). Factoring $\mathcal{U}^-(\widehat{\mathfrak{sl}}_N)$ by the ideal generated by the elements f_i^2, $0 \leq i \leq N-1$, gives $\mathrm{n}\widehat{\mathrm{TL}}_N$ whenever the characteristic of $\Bbbk$ is different from 2.

I.2.2.3. The affine plactic algebra

The affine nilTemperley–Lieb algebra is a quotient of the affine plactic algebra and the local affine plactic algebra that we encounter again in Chapter I.3. The local affine plactic algebra $\widehat{\mathcal{P}}_N$ is the unital associative $\Bbbk$-algebra generated by $a_0, a_1, \dots, a_{N-1}$ with defining relations $a_ia_j = a_ja_i$ for $i - j \neq \pm 1 \mod N$; $a_ia_{i-1}a_i = a_ia_ia_{i-1}$ for $i, i-1 \in \mathbb{Z}/N\mathbb{Z}$;

and $a_i a_{i+1} a_i = a_{i+1} a_i a_i$ for $i, i+1 \in \mathbb{Z}/N\mathbb{Z}$. The name "local" refers to the commutativity relation $a_i a_j = a_j a_i$. See the discussion of the plactic algebra in Chapter I.3 for references and more details about this algebra.

After we quotient out the additional relation $a_i^2 = 0$ for all i, we obtain the affine nilTemperley–Lieb algebra.

I.2.2.4. Combinatorial actions

$\mathbf{n}\widehat{\mathbf{TL}}_N$ acts on the small quantum cohomology ring of the Grassmannian:

As in [Pos05, Section 2] (see also [KS10]), consider the cohomology ring $\mathrm{H}^\bullet(\mathrm{Gr}(k,N))$ with integer coefficients for the Grassmannian $\mathrm{Gr}(k,N)$ of k-dimensional subspaces of $\Bbbk^N$. It has a basis given by the Schubert classes $[\Omega_\lambda]$, where λ runs over all partitions with k parts, the largest part having size $N-k$. By recording the k vertical and $N-k$ horizontal steps that identify the Young diagram of λ inside the northwest corner of a $k \times (N-k)$ rectangle, such a partition corresponds to a $(0,1)$-sequence of length N with k ones (respectively and $N-k$ zeros) in the positions corresponding to the vertical (respectively horizontal) steps.

As a $\mathbb{Z}[q]$-module for an indeterminate q, the quantum cohomology ring of the Grassmannian is given by $\mathrm{qH}^\bullet(\mathrm{Gr}(k,N)) = \mathbb{Z}[q] \otimes_{\mathbb{Z}} \mathrm{H}^\bullet(\mathrm{Gr}(k,N))$ together with a q-multiplication, [Buc03], [FGP97, Theorem 1.3] (see also [FGP97, Theorem 1.2] for an identification of Schubert classes with quantum Schubert polynomials, analogously to [BGG73], and references therein).

The $\mathrm{n}\widehat{\mathrm{TL}}_N$-action can be defined combinatorially on

$$\mathrm{qH}^\bullet(\mathrm{Gr}(k,N)) \cong \mathrm{span}_{\mathbb{Z}[q]} \{(0,1)\text{-sequences of length } N \text{ with } k \text{ ones}\}$$

as described in the next item, and the multiplication of two Schubert classes $[\Omega_\lambda] \cdot [\Omega_\mu]$ is equal to $s_\lambda \cdot [\Omega_\mu]$ where s_λ is a certain Schur polynomial in the noncommutative generators of $\mathrm{n}\widehat{\mathrm{TL}}_N$ as defined in [Pos05, Corollary 8.3].

$\mathbf{n}\widehat{\mathbf{TL}}_N$ acts faithfully on fermionic particle configurations on a circle:

First, a $(0,1)$-sequence with k ones is identified with a circular particle configuration having N positions, where the k particles are distributed at the position on the circle that corresponds to their position in the sequence, so that there is at most one particle at each position. On the space

$$\mathrm{span}_{\Bbbk[q]}\,\{\text{fermionic particle configurations of } k \text{ particles on a circle with } N \text{ positions}\},$$

the generators a_i of $\mathrm{n}\widehat{\mathrm{TL}}_N$ act by sending a particle lying at position i to position $i+1$. Additionally, the particle configuration is multiplied by $\pm q$ when applying a_0. The generator a_i acts by zero if there is no particle at position i. This is the graphical representation from [KS10] (see also [Pos05]), which we use in our description of the center of $\mathrm{n}\widehat{\mathrm{TL}}_N$. The precise definition is given in Section I.2.4, here is a representative picture:

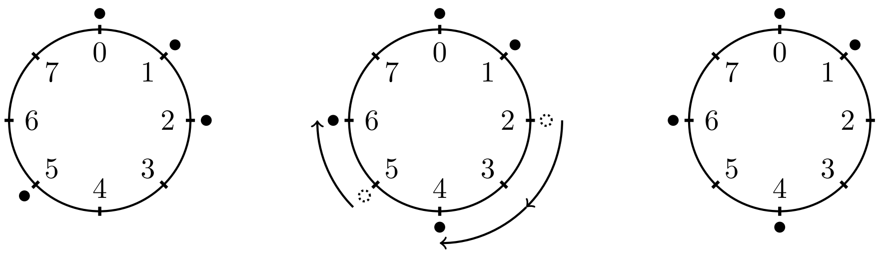


Figure I.2.2.1.: Example for $N=8$: Application of $a_3a_2a_5$ to the particle configuration $(0,1,2,5)$ gives $(0,1,4,6)$.

$\mathbf{n}\widehat{\mathbf{TL}}_N$ acts on Young diagrams:

This is another combinatorial description of the action described above. Here the generator a_i adds a box in the $(i-k)$-th diagonal of a Young diagram that is contained in a rectangle of size $k\times(N-k)$ whose diagonals are numbered from $-k$ to $N-k$. The generator a_0 removes a rim hook from the Young diagram. See [Pos05], [KS10, Remark 9.2], and for the finite case also [FG98, Example 2.4].

I.2.2.5. The creation/annihilation algebra

The finite nilTemperley–Lieb algebra is a subalgebra of the creation/annihilation algebra, a Clifford algebra having generators $\{\xi_i,\xi_i^* \mid 0\le i\le N-1\}$ and relations $\xi_i\xi_j+\xi_j\xi_i=0$, $\xi_i^*\xi_j^*+\xi_j^*\xi_i^*=0$, $\xi_i\xi_j^*+\xi_j^*\xi_i=\delta_{ij}$. The Clifford generators ξ_i (respectively ξ_i^*) act on the fermionic particle configurations by annihilation (respectively creation) of a particle at position i. The finite nilTemperley–Lieb algebra appears inside the Clifford algebra via $a_i\mapsto\xi_{i+1}^*\xi_i$. For the affine version one can take the scalar extension with $\Bbbk[q]$ of the Clifford algebra on $2(N+1)$ generators $\{\xi_i,\xi_i^*\mid 0\le i\le N\}$ and identify $\xi_N=q^{-1}\xi_0$, $\xi_N^*=q\xi_0$ as discussed in [KS10, Section 8].

I.2.2.6. The affine Temperley–Lieb algebra

$\mathrm{n}\widehat{\mathrm{TL}}_N$ is the associated graded algebra of the affine Temperley–Lieb algebra:

The affine Temperley–Lieb algebra $\widehat{\mathrm{TL}}_N(\delta)$ generated by $a_0, \ldots, a_{N-1}$ has the usual commuting relations and the relations $a_i a_{i\pm1} a_i = a_i$ and $a_i^2 = \delta a_i$ for some parameter $\delta \in \Bbbk$ instead of the nil relations (where again all indices are mod N). It contains the famous Temperley–Lieb algebra $\mathrm{TL}_N(\delta)$ as the subalgebra generated by $a_1, \ldots, a_{N-1}$, see [TL71], [Kau90]. Both are filtered algebras with ℓ-th filtration space generated by all monomials of length at most ℓ. Since the associated graded algebra of $\widehat{\mathrm{TL}}_N(\delta)$ is $\mathrm{n}\widehat{\mathrm{TL}}_N$ for any value of δ, elements of $\mathrm{n}\widehat{\mathrm{TL}}_N$ can be identified with reduced expressions in $\widehat{\mathrm{TL}}_N(\delta)$.

The Temperley–Lieb algebra $\mathrm{TL}_N(\delta)$ is known for its diagrammatical realisation by crossingless string diagrams in the plane that connect $2N$ points. Multiplication of diagrams is given by connecting and smoothing the strands. Whenever the strands form a circle, this is removed from the diagram at the expense of multiplying by the parameter δ. This is an example for a diagram algebra, in the sense that it is a quotient of an algebra with a presentation by generators and relations so that the generators can be identified with string diagrams (possibly with labelled strands and additional decoration, e.g. dots) inside $[0,1] \times \mathcal{M}$, where $\mathcal{M}$ is a compact manifold. The strands in such a diagram form closed loops or connect a finite number of end points in $\{0,1\} \times \mathcal{M}$, multiplication is given by stacking the diagrams on top of each other and connecting the strands at the end points, and isotopy relations hold. See [Koe08] for an overview of diagram algebras.

The diagram algebra structure of $\widehat{\mathrm{TL}}_N(\delta)$ is given by the same pictures as for the Temperley–Lieb algebra, but now the diagrams are wrapped around the cylinder (see e.g. [FG99], [KX12]). The top and bottom of the cylinder each have N nodes. Monomials in the affine Temperley–Lieb algebra are represented by diagrams of N non-crossing strands, each connecting a pair of those $2N$ nodes. Multiplication of two monomials is realized by stacking the cylinders one on top of the other, and then proceeding as for $\mathrm{TL}_N(\delta)$. The relation $a_i a_{i\pm1} a_i = a_i$ corresponds to the isotopy between a strand that changes direction and a strand that is pulled straight.

In contrast, the affine *nil*Temperley–Lieb algebra is *not* a diagram algebra in this sense. The relation $a_i a_{i\pm1} a_i = 0$ implies that isotopy would identify zero and nonzero elements. Nevertheless, the diagram of a reduced expression in $\widehat{\mathrm{TL}}_N$ may be considered as an element of $\mathrm{n}\widehat{\mathrm{TL}}_N$. Such a diagram consists of a number (possibly zero) of arcs that connect two nodes on the top of the cylinder, the same number of arcs connecting two nodes on the bottom, and arcs that connect a top node and a bottom one. The latter arcs wrap around the cylinder either all in a strictly clockwise direction or all in a strictly counterclockwise way. Since the multiplication of two such diagrams may give zero, we will not use this diagrammatic realization here.

In quantum Schur-Weyl duality the Temperley–Lieb algebra appears as the quotient of the Iwahori–Hecke algebra $\mathsf{H}_N(q)$ that acts faithfully on tensor powers of the two-dimensional simple module of $\mathcal{U}_q(\mathfrak{sl}_2)$ of type $\mathbf{1}$. More precisely, the Temperley–Lieb algebra $\mathrm{TL}_N(\delta)$ is the following quotient of the Iwahori–Hecke algebra $\mathsf{H}_N(q)$ over $\mathbb{Z}[q^{\pm\frac{1}{2}}]$ of type (A) [Jon87, Section 11]:

$$\begin{aligned} \mathsf{H}_N(q)/(T_iT_{i+1}T_i + T_iT_{i+1} + T_{i+1}T_i + T_i + T_{i+1} + 1 \mid 1 \le i \le N-2) \;&\cong\; \mathrm{TL}_N(q^{\frac{1}{2}} + q^{-\frac{1}{2}}) \\ T_i \;&\mapsto\; q^{\frac{1}{2}}a_i - 1. \end{aligned}$$

The standard basis of $\mathsf{H}_N(q)$ consists of monomials labelled by elements of $\mathcal{S}_N$, the Coxeter group of type A. According to [Fan96, Proposition 1], the subset of monomials labelled by 321-avoiding permutations is mapped under the quotient map to a basis of $\mathrm{TL}_N(q^{\frac{1}{2}} + q^{-\frac{1}{2}})$ (see also [GL01] and the reference [Gra95, Theorem 6.2] therein). The leading term in the image of a monomial $T_w \in \mathsf{H}_N(q)$ for a 321-avoiding permutation $w = s_{i_1}\dots s_{i_r}$ is the reduced monomial $a_{i_1}\dots a_{i_r}$ in the Temperley–Lieb algebra.

Likewise, the Kazhdan-Lusztig basis elements of $\mathsf{H}_N(q)$ labelled by 321-avoiding permutations are mapped to the canonical basis of $\mathrm{TL}_N(q^{\frac{1}{2}} + q^{-\frac{1}{2}})$, see [GL01, Lemma 2.2.1] and [GL99] for the canonical basis of the Temperley–Lieb algebra. There are many further bases known for TL_N, see e.g. [Jon83, Aside 4.1.4], [Mur95], [RSA14], [Här99], [Gob15] and references therein.

Also in the affine case, the affine Temperley–Lieb algebra $\widehat{\mathrm{TL}}_N(q^{\frac{1}{2}} + q^{-\frac{1}{2}})$ is a quotient of the affine Hecke algebra $\widetilde{\mathsf{H}}_N(q)$ of type $\widehat{\mathsf{A}}$, although by [GL98] the affine Temperley–Lieb algebra defined in terms of diagrams on a cylinder is slightly larger than the quotient of the affine Hecke algebra: The additional elements are given by so-called twists that allow strands to wrap around the cylinder, see the picture in [GL98, p. 182].

One can define generalized Temperley–Lieb algebras as a quotient of the Hecke algebra associated with a Coxeter group of any type. Then for any type, the subset of the standard basis or the Kazhdan-Lusztig basis of the Hecke algebra that is labelled by fully commutative elements induces a basis of the Temperley–Lieb quotient, see [GL01, Proposition 2.1.3 and Lemma 2.2.1]. Fully commutative elements of a Coxeter group are defined to be those elements for which any two words that represent this element differ only by a sequence of transpositions applied to adjacent pairs of commuting generators

[Ste96]. Then one can identify the fully commutative elements (considered as equivalence classes of words in the Coxeter group) with a basis of the corresponding generalized Temperley–Lieb algebra. For certain types (in particular type A), the fully commutative elements of the Coxeter group form a union of twosided Kazhdan–Lusztig cells. Equivalently, the Kazhdan–Lusztig basis elements not labelled by fully commutative elements are mapped to zero under the quotient map [GL01, Theorem 2.2.3].

In particular, the reduced expressions in the affine Temperley–Lieb algebra $\widehat{\mathrm{TL}}_N(\delta)$ can be identified with fully commutative elements in the affine Coxeter group of type $\widehat{\mathsf{A}}_{N-1}$ [Fan96], see also [FG99, Section 2.1]. This is similar to the finite case: The monomials in the nilTemperley–Lieb algebra nTL_N correspond to reduced monomials in the Temperley–Lieb algebra TL_N, which are also known to be labelled by 321-avoiding permutations in the symmetric group [Fan96]. By [BJS93, Theorem 2.1], being 321-avoiding is equivalent to being fully commutative for the words in the symmetric group.

In [HJ10] abacus diagrams are used in order to find a generating function for the number of fully commutative elements of a given Coxeter length in type $\widehat{\mathsf{A}}$. This can be interpreted as a graded dimension formula for the affine (nil)Temperley–Lieb algebra. Descriptions in terms of heaps and generating functions can be found for any affine type in [BJN15], together with an overview of the literature, see in particular [BJN15, Section 2.6] for type $\widehat{\mathsf{A}}$. More on the properties of the generating functions can be found e.g. in [Nad15] Al Harbat [Alh13] has recently described a normal form for the fully commutative elements of type $\widehat{\mathsf{A}}$.

For the Temperley–Lieb algebra $\mathrm{TL}_N(\delta)$ there are several elements of the center known, e.g. the Jones-Wenzl projectors, see [RSA14, Appendix A]. Some description of the center of the affine Temperley–Lieb algebra $\widehat{\mathrm{TL}}_N(\delta)$ is available in [Vla04]. In [HMR09] a commuting family of elements in the affine Temperley–Lieb algebra analogous to the Jucys-Murphy elements is defined.

The (diagrammatically defined) affine Temperley–Lieb algebra is known to be affine cellular in the sense of Koenig and Xi [KX12], see Proposition 2.5 therein. Affine cellularity generalizes the notion of cellularity for finite dimensional algebras from [GL96]. The Temperley–Lieb algebra $\mathrm{TL}_N(\delta)$ is known for being cellular, even graded cellular [PRH14] with grading induced from the grading on cyclotomic quiver Hecke algebras [BK09]. Therefore it is very tempting to ask whether the affine nilTemperley–Lieb algebra is affine cellular, too. This is far from being obvious, and we come back to this in Section I.2.14.

I.2.3. Gradings

One of the ingredients needed in Section I.2.8 to study the center of $\mathrm{n}\widehat{\mathrm{TL}}_N$ is a fine grading on the algebra. Gradings by abelian groups faciliate the computation of the center of an algebra, as the following standard result reduces the work to determining homogeneous central elements.

I.2.3.1 Lemma. Let $A = \bigoplus_{g \in G} A_g$ be an algebra graded by an abelian group G. The center of A is homogeneous, i.e. it inherits the grading.

Proof. Let $a = \sum_{g \in G} a_g$ be a central element of the graded algebra $A = \bigoplus_{g \in G} A_g$. We have for $b_h \in A_h$ that $\sum_{g \in G} a_g b_h = ab_h = b_h a = \sum_{g \in G} b_h a_g$. Since this equality must hold in every graded component, we get $a_g b_h = b_h a_g$ for all homogeneous elements b_h. Now take any element $b = \sum_{h \in G} b_h$ in A, then $a_g b = \sum_{h \in G} a_g b_h = \sum_{h \in G} b_h a_g = ba_g$, hence a_g is central. □

Since the defining relations are homogeneous, both $\mathrm{n}\widehat{\mathrm{TL}}_N$ and nTL_N have a $\mathbb{Z}$-grading by the length of a monomial, i.e. all generators a_i have $\mathbb{Z}$-degree 1. This can be refined to a $\mathbb{Z}^N$-grading by assigning to the generator a_i the degree e_i, the i-th standard basis vector in $\mathbb{Z}^N$. In either grading, we say that the degree 0 part of an element in $\mathrm{n}\widehat{\mathrm{TL}}_N$ or nTL_N is its constant term.

I.2.3.2 Remark. We exclude the case of $N \leq 2$ from our considerations since there are isomorphisms $\mathrm{n}\widehat{\mathrm{TL}}_N \cong \mathrm{nTL}_{N+1}$ for $N = 1, 2$, and in these cases the center is known (and uninteresting). The algebra $\mathrm{n}\widehat{\mathrm{TL}}_1$ is 2-dimensional and commutative; while $\mathrm{n}\widehat{\mathrm{TL}}_2$ has dimension 5, and its center can be computed by hand making use of Lemma I.2.3.1 and can be shown to be the $\Bbbk$-span of $1, a_0 a_1, a_1 a_0$. ◊

I.2.3.3 Remark. The affine (or finite) Temperley–Lieb algebra, which has relations $a_i a_j = a_j a_i$ for $i - j \neq \pm 1 \pmod N$, $a_i a_{i\pm 1} a_i = a_i$, and $a_i^2 = \delta a_i$ for some $\delta \in \Bbbk$, is a filtered algebra with respect to the length filtration. For this algebra, the ℓ-th filtration space is generated by all monomials of length $\leq \ell$. Its associated graded algebra is $\mathrm{n}\widehat{\mathrm{TL}}_N$ (or nTL_N). Thus, $\mathrm{n}\widehat{\mathrm{TL}}_N$ is infinite dimensional when $N \geq 3$, while nTL_N has dimension equal to the N-th Catalan number $\frac{1}{N+1}\binom{2N}{N}$. ◊

I.2.4. The graphical representation of the affine nilTemperley–Lieb algebra

The second ingredient we use to determine the center is a faithful representation of $\mathrm{n}\widehat{\mathrm{TL}}_N$. Here we recall the definition of the representation from [KS10] and describe its graphical realization, which is very convenient to work with.

Fix a basis $v_1, \ldots, v_N$ of $\Bbbk^N$. Consider the vector space $\mathsf{V} = \bigoplus_{k=0}^{N} \left(\Bbbk[q] \otimes \bigwedge^k \Bbbk^N\right)$. It has a standard $\Bbbk[q]$-basis consisting of wedge products

$$v(\mathrm{I}) \;:=\; v_{i_1} \wedge \ldots \wedge v_{i_k} \quad \text{for all (strictly) increasing sequences } \; \mathrm{I} = \{1 \le i_1 < \ldots < i_k \le N\} \tag{I.2.1}$$

for all $0 \le k \le N$, where the basis element of $\Bbbk = \bigwedge^0 \Bbbk^N$ is denoted $v(\varnothing)$. Recall that unless otherwise stated all tensor products are taken over $\Bbbk$, and we omit the tensor symbol in $\Bbbk[q]$-linear combinations of wedge products.

It is helpful to visualize the basis elements $v(\mathrm{I})$ as particle configurations having $0 \le k \le N$ particles arranged on a circle with N positions, where there is at most one particle at each site, as pictured below for $N = 8$ and $v(1,5,6) = v_1 \wedge v_5 \wedge v_6$. The vector $v(\varnothing)$ corresponds to the configuration with no particles. Then V is the $\Bbbk[q]$-span of such circular particle configurations.

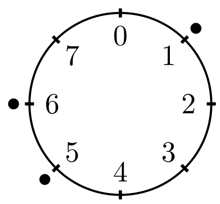


Figure I.2.4.1.: The element $v_1 \wedge v_5 \wedge v_6$ in the graphical realization.

There is an action of the affine nilTemperley–Lieb algebra $\mathrm{n}\widehat{\mathrm{TL}}_N$ defined on the basis vectors $v(\mathrm{I})$ of V as follows:

I.2.4.1 Definition. For $1 \le j \le N-1$,

$$a_j v(\mathrm{I}) \;=\; \begin{cases} v_{i_1} \wedge \ldots \wedge v_{i_{\ell-1}} \wedge v_{j+1} \wedge v_{i_{\ell+1}} \wedge \ldots \wedge v_{i_k}, & \text{if } i_\ell = j \text{ for some } \ell, \\ 0, & \text{otherwise.} \end{cases}$$

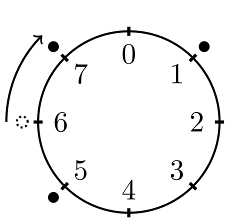
For the action of a_0, note that v_N appears in the basis element $v(\mathrm{I})$ if and only if it occurs in the last position, i.e. $v_{i_k} = v_N$, and define

$$
\begin{aligned}
a_0 v(\mathrm{I}) &= \begin{cases} q \cdot v_{i_1} \wedge \ldots \wedge v_{i_{k-1}} \wedge v_1, & \text{if } i_k = N, \\ 0, & \text{otherwise;} \end{cases} \\
&= \begin{cases} (-1)^{k-1} q \cdot v_1 \wedge v_{i_1} \wedge \ldots \wedge v_{i_{k-1}}, & \text{if } i_k = N, \\ 0, & \text{otherwise.} \end{cases}
\end{aligned}
$$

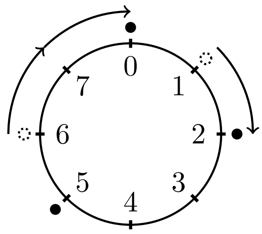
I.2.4.2 Remark. The indices of the vectors v_j can be interpreted modulo N, in the sense that we make no distinction between v_0 and v_N and often use the two interchangeably. This does not affect the order of factors in the wedge basis. We identify e.g. $v_1 \wedge v_3 \wedge v_4 \wedge v_6$ and $v_1 \wedge v_3 \wedge v_4 \wedge v_0$ for $N = 6$. $\lozenge$

I.2.4.3 Remark. It follows that $a_j v(\mathrm{I}) = 0$ if the sequence I contains $j + 1$ or if it does not contain j. In other words, a_j acts by replacing v_j by v_{j+1}. If this creates a wedge expression with two factors equal to v_{j+1}, the result is zero. Thus, for any monomial $a(\underline{j})$ there is a unique increasing sequence $\mathrm{J} = \{1 \leq j_1 < \ldots < j_k \leq N\}$ with k minimal on which the monomial acts nontrivially. Under the identification of basis elements $v(\mathrm{J})$ with particle configurations, we refer to J as the minimal particle configuration on which $a(\underline{j})$ acts nontrivially. $\lozenge$

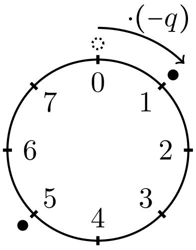
In the graphical description, a_j moves a particle clockwise from position j to position $j + 1$, and one records "passing position 0" by multiplying by $\pm q$ as illustrated by the particle configurations below.



(a) $a_6(v_1 \wedge v_5 \wedge v_6) = v_1 \wedge v_5 \wedge v_7$



(b) $a_7 a_1 a_6(v_1 \wedge v_5 \wedge v_6) = v_2 \wedge v_5 \wedge v_0$



(c) $a_0(v_5 \wedge v_0) = -q \cdot v_1 \wedge v_5$.

Figure I.2.4.2.: Examples for the action of $\mathrm{n}\widehat{\mathrm{TL}}_N$ on a particle configuration.

It is easy to verify that the defining relations for $\mathrm{n}\widehat{\mathrm{TL}}_N$ hold for this action, assuming that $N \geq 3$. Hence we obtain

I.2.4.4 Lemma. i) Definition I.2.4.1 gives a representation of $\mathrm{n}\widehat{\mathrm{TL}}_N$ on V.

ii) The number of wedge factors (i.e., the number of particles) remains constant under the action of the generators a_i, so that $\mathsf{V} = \bigoplus_{k=0}^{N} \left(\Bbbk[q] \otimes \bigwedge^k \Bbbk^N\right)$ is a direct sum decomposition of V as an $\mathrm{n}\widehat{\mathrm{TL}}_N$-module.

The following crucial statement is taken from [KS10, Proposition 9.1.(2)], see also [BFZ96, Proposition 2.4.1]. We will give a detailed proof adapted to our notation in Section I.2.6.

I.2.4.5 Theorem. The action from Definition I.2.4.1 gives a faithful representation of $\mathrm{n}\widehat{\mathrm{TL}}_N$ on V when $N \geq 3$.

The spaces $\Bbbk[q] \otimes \bigwedge^0 \Bbbk^N$ and $\Bbbk[q] \otimes \bigwedge^N \Bbbk^N$ are trivial summands in V on which every generator a_i acts as 0, and so they are not needed for the faithfulness of the graphical representation from Theorem I.2.4.5.

I.2.4.6 Remark. From now on we identify elements of $\mathrm{n}\widehat{\mathrm{TL}}_N$ with the corresponding operators on the particle configurations of the graphical representation. $\lozenge$

I.2.5. A normal form of monomials in the affine nilTemperley–Lieb algebra

In this section we present a normal form algorithm for nonzero monomials in the affine nilTemperley–Lieb algebra $\mathrm{n}\widehat{\mathrm{TL}}_N$.

I.2.5.1 Remark. Observe that the defining relations of $\mathrm{n}\widehat{\mathrm{TL}}_N$ only allow to replace a monomial expression by another monomial expression or 0. In other words, $\mathrm{n}\widehat{\mathrm{TL}}_N$ is the semigroup algebra of the semigroup defined by generators $a_0, a_1, \ldots, a_{n-1}, \varnothing$ and relations from Definition I.2.1.1 (where in addition the zero $\varnothing$ of the semigroup – also called the absorbing element – is identified with the zero of the semigroup algebra), see also [Hey00, Proposition 2.1]. This implies that

i) a nonzero monomial can only be manipulated using the commutativity relation, and
ii) any set of pairwise distinct nonzero monomials is linearly independent.

Hence, the construction of a normal form for nonzero monomials gives a basis for $\mathrm{n}\widehat{\mathrm{TL}}_N$. We automatically get a basis for the affine Temperley–Lieb algebra $\widehat{\mathrm{TL}}_N(\delta)$ for any δ since $\mathrm{n}\widehat{\mathrm{TL}}_N$ is the associated graded algebra of $\widehat{\mathrm{TL}}_N(\delta)$. $\lozenge$

The following lemma characterises nonzero monomials in $\mathrm{n}\widehat{\mathrm{TL}}_N$. They correspond to fully commutative elements in $\widehat{\mathrm{TL}}_N(\delta)$, see [Gre02] and the discussion in Section I.2.2.6.

I.2.5.2 Lemma. The monomial $a(\underline{j})$ is nonzero if and only if for any two neighbouring appearances of a_i in $a(\underline{j})$ there are exactly one a_{i+1} and one a_{i-1} in between, apart from possible factors a_ℓ for $\ell \neq i-1, i, i+1$ (indices to be understood modulo N).

Formulated differently, two consecutive factors a_i have to enclose a_{i+1} and a_{i-1}, i.e. $a_i \ldots a_{i\pm 1} \ldots a_{i\mp 1} \ldots a_i$, with the dots being possible products of a_ℓ's with $\ell \neq i \pm 1, i$. This lemma is a special case of [Gre02, Lemma 2.6]; here is a quick proof for the convenience of the reader.

Proof. The monomial $a(\underline{j})$ is zero if and only if we can bring two neighbouring factors a_i together so that we obtain either a_i^2 ("square") or $a_i a_{i\pm1} a_i$ ("braid"). But expressions of the form $a_i \dots a_{i\pm1} \dots a_{i\mp1} \dots a_i$ cannot be resolved this way by commutativity relations. On the other hand, if there are two neighbouring factors a_i with either none or only one of the terms $a_{i\pm1}$ in between, we immediately get either a_i^2 or $a_i a_{i\pm1} a_i$. If there are at least two factors a_{i+1} (or a_{i-1}) in between the two a_i, one can repeat the argument: Either we can create a square or a braid, or we have at least two factors of the same kind in between. In the case of a square or a braid we are done; otherwise we pick two neighbouring a_{i+k} in the k-th step of the argument. Since we always consider the space in between two neighbouring factors $a_i, a_{i+1}, \dots, a_{i+k}$, none of the previous $a_i, a_{i+1}, \dots, a_{i+k-1}$ occurs between the two neighbouring a_{i+k}. Unless we found a square or a braid in an earlier step, we end up in step $N-1$ with a subexpression of the form $a_r a_{r\pm1}^m a_r$ which is zero for any $m \geq 0$. □

I.2.5.3 Definition. For any $i \in \{0, 1, \dots, N-1\}$, we define a (clockwise) order $\overset{i}{<}$ on the set $\{0, 1, \dots, N-1\}$ starting at i by

$$i \overset{i}{<} i+1 \overset{i}{<} \dots \overset{i}{<} i+N-1.$$

A normal form algorithm

Now we are ready to describe an algorithm that reorders the factors of any given nonzero monomial in $\mathrm{n}\widehat{\mathrm{TL}}_N$ so that the resulting expression is independent of the choice of representative (as shown in Lemma I.2.5.6). Hence we describe a normal form for monomials in the affine nilTemperley–Lieb algebra.

The basic idea of the algorithm is as follows. Recall the defining relations of the affine nilTemperley–Lieb algebra: We have zero relations $a_i^2 = 0$, $a_i a_{i+1} a_i = 0 = a_{i+1} a_i a_{i+1}$ that we cannot apply because our monomial is assumed to be nonzero. According to Remark I.2.5.1 we can only apply the commutativity relations $a_i a_j = a_j a_i$ for $j \neq i \pm 1$, where all indices are taken modulo N. Inside a monomial we can freely move around a factor a_i as long as we never commute it with $a_{i\pm1}$. In particular, whenever we find a factor a_i without $a_{i\pm1}$ to the right of it, we may push it all the way to the right end of the monomial.

Now we give the algorithm in detail. Let $a(\underline{j})$ be an arbitrary nonzero monomial in $\mathrm{n}\widehat{\mathrm{TL}}_N$. As usual, the indices are considered modulo N. Reorder its factors according to the following algorithm:

i) Find all factors a_i in $a(\underline{j})$ with no a_{i-1} to their right. We denote them by $a_{i_1}, \dots, a_{i_k}$, ordered according to their appearance in $a(\underline{j})$; in other words, $a(\underline{j})$ is of the form
$$a(\underline{j}) = \dots a_{i_1} \dots a_{i_2} \dots \;\; \dots a_{i_k}.$$

ii) Move the $a_{i_1}, \dots, a_{i_k}$ to the far right, without changing their internal order,
$$a(\underline{j}) = a(\underline{j}') \cdot (a_{i_1} a_{i_2} \dots a_{i_k}) = a(\underline{j}') \cdot a(\underline{j}^{(0)})$$
for $\underline{j}^{(0)} = (i_1, \dots, i_k)$ and some sequence $\underline{j}' = (\underline{j}$ with $i_1, \dots, i_k$ removed). This is possible because

   a) by assumption, there is no a_{i-1} to the right of an a_i in this list;
   b) if for some i, a_{i+1} occurs to the right of some a_i, then either $a_i \dots a_{i+1} \dots a_i$ would occur as a subword without a_{i-1} in between, hence $a(\underline{j}) = 0$, or else a_{i+1} does not have a_i to its right, so it is one of the $a_{i_1}, \dots, a_{i_k}$ itself, and will be moved to the far right of $a(\underline{j})$, too;
   c) a_i commutes with all a_ℓ for $\ell \neq i-1, i+1$.

iii) Repeat for $a(\underline{j}')$ until we get
$$a(\underline{j}) = a(\underline{j}^{(m)}) \cdot a(\underline{j}^{(m-1)}) \cdot \ldots \cdot a(\underline{j}^{(1)}) \cdot a(\underline{j}^{(0)})$$
for sequences $\underline{j}^{(m)}, \dots, \underline{j}^{(1)}$ obtained successively the same way as described above. Notice:

   - Inside a sequence $\underline{j}^{(n)}$, every index occurs at most once. If two consecutive indices occur within $\underline{j}^{(n)}$, they are increasingly ordered using the order $\overset{i_k}{<}$ from Definition I.2.5.3.
   - For two consecutive sequences $\underline{j}^{(n+1)}$, $\underline{j}^{(n)}$ and for every index $i_r^{(n+1)}$ occurring in $\underline{j}^{(n+1)}$, we can find some index $i_s^{(n)}$ in $\underline{j}^{(n)}$ such that $i_r^{(n+1)} = i_s^{(n)} + 1$.
   - From that property, it also follows that the length of $\underline{j}^{(n+1)}$ is less or equal than the length of $\underline{j}^{(n)}$.

iv) Reorder the factors $a(\underline{j}^{(m)}), \dots, a(\underline{j}^{(1)}), a(\underline{j}^{(0)})$ internally:

   a) Start with $a(\underline{j}^{(0)})$. There is some $0 \leq \hat{\imath} \leq N-1$ which does not occur in $\underline{j}^{(0)}$, but $\hat{\imath}-1$ occurs. For example, this is satisfied by $\hat{\imath} = i_k + 1$, as i_k occurs in $\underline{j}^{(0)}$ and is to the right of every other factor of $a(\underline{j})$. Choose the largest such $\hat{\imath}$ (with respect to the usual order). Then we can move $\hat{\imath}-1$ to the very right of the sequence $\underline{j}^{(0)}$, because $\hat{\imath}$ is not present, and $\hat{\imath}-2$ may only occur to the left of $\hat{\imath}-1$ due to the construction of $\underline{j}^{(0)}$. We proceed in the same way with those indices $\hat{\imath}-2, \hat{\imath}-3, \dots, \hat{\imath}-(N-1)$ that appear in $\underline{j}^{(0)}$. The result is a reordering of the sequence $\underline{j}^{(0)}$ so that it is increasing from left to right with respect to $\overset{\hat\imath}{\prec}$.

b) Repeat with all other factors $a(\underline{j}^{(1)}), a(\underline{j}^{(2)}), \dots, a(\underline{j}^{(m)})$ taking as the initial right-hand index of the sequence $\hat\imath, \hat\imath{+}1, \dots, \hat\imath{+}m{-}1$ respectively, and reordering within each $a(\underline{j}^{(n)})$ so that the indices are increasing from left to right with respect to $\overset{\hat\imath+n}{\prec}$.

The resulting monomial is called the normal form of the monomial $a(\underline{j})$ that we started with.

I.2.5.4 Example. As an example for $\mathrm{n}\widehat{\mathrm{TL}}_7$, suppose $a(\underline{j}) = a(4\;3\;5\;4\;2\;0\;6\;1\;3\;2\;5)$. (We omit the commas to simplify the notation.)

Find all a_i without a_{i-1} to their right:	$a(4\;3\;5\;4\;2\;0\;6\;\underline{1}\;3\;\underline{2}\;\underline{5})$
Move them to the far right, and do not change their internal order:	$a(4\;3\;5\;4\;2\;0\;6\;3)\cdot a(1\;2\;5)$
Repeat:	$a(4\;3\;5\;4\;\underline{2}\;0\;\underline{6}\;\underline{3})\cdot a(1\;2\;5)$
	$a(4\;3\;5\;4\;0)\cdot a(2\;6\;3)\cdot a(1\;2\;5)$
	$a(4\;\underline{3}\;5\;\underline{4}\;\underline{0})\cdot a(2\;6\;3)\cdot a(1\;2\;5)$
	$a(4\;5)\cdot a(3\;4\;0)\cdot a(2\;6\;3)\cdot a(1\;2\;5)$
Reorder the factors in each $a(\underline{j}^{(n)})$, increasingly with respect to $\overset{\hat\imath+n}{\prec}$:	$a(4\;5)\cdot a(3\;4\;0)\cdot a(2\;3\;6)\cdot a(1\;2\;5)$.

Then the monomial $a(4\;5){\cdot}a(3\;4\;0){\cdot}a(2\;3\;6){\cdot}a(1\;2\;5)$ is the normal form of the monomial $a(4\;3\;5\;4\;2\;0\;6\;1\;3\;2\;5)$. $\lozenge$

As a shorthand notation, in the following we often identify the index sequence $\underline{j}$ with $a(\underline{j})$ (and manipulate $\underline{j}$ according to the same relations as $a(\underline{j})$) as demonstrated in the following example.

I.2.5.5 Example. Let $N = 6$. In our shorthand notation we identify the index sequences

$$\begin{aligned}
&(5\;1\;2\;3\;0\;4\;1\;5\;0\;2\;3\;1\;4\;5\;0\;2\;3\;1\;4\;2)\\
={}&(1)(5\;0\;2)(3\;4\;5\;1)(2\;3\;4\;0)(1\;2\;3\;5)(0\;1\;2\;4)\\
={}&(1\quad 5\;0\;2\quad 3\;4\;5\;1\quad 2\;3\;4\;0\quad 1\;2\;3\;5\quad 0\;1\;2\;4),
\end{aligned}$$

where we omit the commas to simplify the notation again. $\Diamond$

I.2.5.6 Lemma. Let $a(\underline{j})$ be an arbitrary nonzero monomial in $\mathrm{n}\widehat{\mathrm{TL}}_N$. We consider the monomials $a(\underline{j}^{(m)}), a(\underline{j}^{(m-1)}), \dots, a(\underline{j}^{(1)}), a(\underline{j}^{(0)})$ constructed from $a(\underline{j})$ by the algorithm above.

i) The equality $a(\underline{j}) = a(\underline{j}^{(m)})a(\underline{j}^{(m-1)}) \cdots a(\underline{j}^{(1)})a(\underline{j}^{(0)})$ holds in $\mathrm{n}\widehat{\mathrm{TL}}_N$.
ii) Given any two representatives $a(\underline{j})$, $a(\underline{j^{\#}})$ of the same element in $\mathrm{n}\widehat{\mathrm{TL}}_N$, the above algorithm creates the same representative $a(\underline{j}^{(m)})a(\underline{j}^{(m-1)}) \cdots a(\underline{j}^{(1)})a(\underline{j}^{(0)})$ for both $a(\underline{j})$ and $a(\underline{j^{\#}})$.

Proof. i) The algorithm only uses the defining relations of $\mathrm{n}\widehat{\mathrm{TL}}_N$: It never interchanges the order of two factors a_i, $a_{i\pm1}$ with consecutive indices within $a(\underline{j})$. Hence, the reordering of the factors of $a(\underline{j})$ uses only the commutativity relation $a_ia_j = a_ja_i$ for $i - j \neq \pm1 \mod N$ of $\mathrm{n}\widehat{\mathrm{TL}}_N$.

ii) Two nonzero monomials $a(\underline{j})$, $a(\underline{j^{\#}})$ in $\mathrm{n}\widehat{\mathrm{TL}}_N$ are equal if and only if they only differ by applications of commutativity relations $a_ia_j = a_ja_i$ for $i - j \neq \pm1 \mod N$, hence, if and only if they contain the same number of factors a_i for each i and the relative position of each a_i and $a_{i\pm1}$ is the same. Since the outcome of the algorithm depends only on the relative positions of consecutive indices, the resulting decomposition $a(\underline{j}^{(m)})a(\underline{j}^{(m-1)}) \cdots a(\underline{j}^{(1)})a(\underline{j}^{(0)})$ is the same. $\square$

Subsequently, whenever we refer to monomials in normal form, we assume the monomial is nonzero and nonconstant, in particular the sequence $\underline{j}$ is nonempty.

I.2.5.7 Theorem. Assume $N \geq 3$.

i) The algorithm above provides a normal form for nonzero monomials $a(\underline{j})$ in the generators a_i of $\mathrm{n}\widehat{\mathrm{TL}}_N$, or equivalently for nonzero fully commutative monomials in $\widehat{\mathrm{TL}}_N$, so that

$$a(\underline{j}) = (a_{i_1}^{(m)} \dots a_{i_k}^{(m)}) \dots (a_{i_1}^{(n+1)} \dots a_{i_k}^{(n+1)})(a_{i_1}^{(n)} \dots a_{i_k}^{(n)}) \dots (a_{i_1}^{(1)} \dots a_{i_k}^{(1)})(a_{i_1} \dots a_{i_k}),$$

where $a_{i_\ell}^{(n)} \in \{1, a_0, a_1, \dots, a_{N-1}\}$ for all $1 \leq n \leq m,\ 1 \leq \ell \leq k$, and

$$a_{i_\ell}^{(n+1)} \in \begin{cases} \{1\} & \text{if } a_{i_\ell}^{(n)} = 1, \\ \{1, a_{j+1}\} & \text{if } a_{i_\ell}^{(n)} = a_j. \end{cases}$$

The factors $a_{i_1}, \dots, a_{i_k}$ are determined by the property that the generator $a_{i_\ell - 1}$ does not appear to the right of a_{i_ℓ} in the original presentation of the monomial. The internal ordering of the factors is increasing with respect to the relation $\overset{\widehat{\imath}}{\prec}$ as in Definition I.2.5.3, where $\widehat{\imath}$ is the largest value in $\{0, 1, \dots, N-1\}$ such that $\widehat{\imath} - 1 \notin \{i_1, \dots, i_k\}$, but $\widehat{\imath} \in \{i_1, \dots, i_k\}$.

ii) The set $\{a(\underline{j}) \text{ in normal form}\} \cup \{1\}$ is a $\Bbbk$-basis of $\mathrm{n}\widehat{\mathrm{TL}}_N$.

Proof. We have shown in Lemma (I.2.5.6.i) that any nonzero nonconstant monomial can be rewritten by the algorithm as a monomial in normal form. By Lemma (I.2.5.6.ii) these monomials are in pairwise distinct equivalence classes with respect to the defining relations of $\mathrm{n}\widehat{\mathrm{TL}}_N$. Then by Remark I.2.5.1, the set of monomials in normal form together with 1 is linearly independent. □

I.2.5.8 Corollary. The set of monomials obtained by the normal form algorithm regarded as elements in $\widehat{\mathrm{TL}}_N$ forms a $\Bbbk$-basis of the positively graded part $(\widehat{\mathrm{TL}}_N)_{>0}$ in the filtration by length of monomials. In other words, $\{a(\underline{j}) \text{ in normal form}\} \cup \{1\}$ is a $\Bbbk$-basis of $\widehat{\mathrm{TL}}_N$.

Proof. Recall that the affine nilTemperley–Lieb algebra $\mathrm{n}\widehat{\mathrm{TL}}_N$ is the associated graded algebra for the affine Temperley–Lieb algebra $\widehat{\mathrm{TL}}_N$ with respect to the filtration by length of monomials. A $\Bbbk$-basis for $\mathrm{n}\widehat{\mathrm{TL}}_N$ is automatically a $\Bbbk$-basis for $\widehat{\mathrm{TL}}_N$ and vice versa. □

I.2.5.9 Remark. We obtain a different normal form for monomials in $\mathrm{n}\widehat{\mathrm{TL}}_N$ if we replace the steps

i) Find all factors a_i in $a(\underline{j})$ with no a_{i-1} to their right,
ii) Move them to the far right, without changing their internal order

in our algorithm by the alternative steps

i) Find all factors a_i in $a(\underline{j})$ with no a_{i+1} to their left,
ii) Move them to the far left, without changing their internal order.

After we fix some rule for the internal reordering of factors, we obtain the following two versions of normal forms in Example I.2.5.5:

$$\begin{aligned} &(5\ 1\ 2\ 3\ 0\ 4\ 1\ 5\ 0\ 2\ 3\ 1\ 4\ 5\ 0\ 2\ 3\ 1\ 4\ 2) \\ =\ &(1)(5\ 0\ 2)(3\ 4\ 5\ 1)(2\ 3\ 4\ 0)(1\ 2\ 3\ 5)(0\ 1\ 2\ 4) \\ =\ &(5\ 1\ 2\ 3)(4\ 0\ 1\ 2)(3\ 5\ 0\ 1)(2\ 4\ 5\ 0)(1\ 3\ 4)(2). \end{aligned}$$

Here, the first expression is the given monomial, the second expression is the normal form obtained by the original algorithm, and the third expression is the alternative normal form. However, we do not use the alternative normal form in the following. ◊

I.2.6. Faithfulness of the graphical representation

In this section, we prove Theorem I.2.4.5 which we recall here:

Theorem. For $N \geq 3$, V is a faithful $\mathrm{n}\widehat{\mathrm{TL}}_N$-module with respect to the action described in Definition I.2.4.1.

For the proof, we explicitly show the linear independence of the matrices representing the monomials in $\mathrm{n}\widehat{\mathrm{TL}}_N$. We proceed in two steps: First, we find a bijection between the monomials in normal form constructed in Section I.2.5 and certain pairs of particle configurations together with a power of q. In other words, we find a labelling for the basis of $\mathrm{n}\widehat{\mathrm{TL}}_N$ from Theorem I.2.5.7. The final step is the description of the action of a monomial on V using its matrix realization. The matrices representing the monomials have a distinguished nonzero entry that is given in terms of the particle configurations and the power of q from the bijection, and for most matrices, this is the only nonzero entry. From this description it will quickly follow that all these matrices are linearly independent.

I.2.6.1. Labelling of basis elements

In this section we use the following shorthand notation:

I.2.6.1 Definition. i) Given a tuple $(n_1, \ldots, n_r) \in \mathbb{Z}^r$ for some $r > 0$ we write $m \in (n_1, \ldots, n_r)$ if there exists some index $1 \leq \ell \leq r$ so that $m = n_\ell$. Similarly, for $(n_1, \ldots, n_r) \in (\mathbb{Z}/N\mathbb{Z})^r$ we write $m \in (n_1, \ldots, n_r)$ if there exists some index $1 \leq \ell \leq r$ so that $m = n_\ell \bmod N$.

ii) Given two tuples $(n_1, \ldots, n_r) \in \mathbb{Z}^r$ or $(\mathbb{Z}/N\mathbb{Z})^r$, and $(n'_1, \ldots, n'_{r'}) \in \mathbb{Z}^{r'}$ or $(\mathbb{Z}/N\mathbb{Z})^{r'}$, we write

$$(n_1, \ldots, n_r) \cdot (n'_1, \ldots, n'_{r'}) = (n_1, \ldots, n_r, n'_1, \ldots, n'_{r'}) \in \mathbb{Z}^{r+r'}$$

for their concatenation, where $r, r' > 0$.

iii) For two tuples $(n_1, \ldots, n_r) \in \mathbb{Z}^r$ or $(\mathbb{Z}/N\mathbb{Z})^r$, and $(n'_1, \ldots, n'_{r'}) \in \mathbb{Z}^{r'}$ or $(\mathbb{Z}/N\mathbb{Z})^{r'}$, with pairwise distinct elements $n_\ell \neq n_m$ and $n'_\ell \neq n'_m$ for $\ell \neq m$, we write

$$(n_1, \ldots, n_r) \subset (n'_1, \ldots, n'_{r'})$$

if there is an inclusion $\{n_1, \ldots, n_r\} \subset \{n'_1, \ldots, n'_{r'}\}$ of the underlying sets.

We need the following notion of blocks and strands of index sequences:

I.2.6.2 Definition. Let $a(\underline{j}) = a(\underline{j}^{(m)})a(\underline{j}^{(m-1)}) \cdots a(\underline{j}^{(1)})a(\underline{j}^{(0)})$ be a normal form monomial.

i) We call $\underline{j}^{(\ell)}$ the ℓ-th block of $\underline{j}$.
ii) A string of indices of maximal length of the form $i_s \in \underline{j}^{(0)}$, $i_s{+}1 \in \underline{j}^{(1)}$, $i_s{+}2 \in \underline{j}^{(2)}, \ldots$ (modulo N) is called the s-th strand of $\underline{j}$.

I.2.6.3 Example. Let $N = 6$, and consider Example I.2.5.5 once again, where

$$\underline{j} = (1 \quad 5\,0\,2 \quad 3\,4\,5\,1 \quad 2\,3\,4\,0 \quad 1\,2\,3\,5 \quad 0\,1\,2\,4).$$

The blocks are $\underline{j}^{(0)} = (0124)$, $\underline{j}^{(1)} = (1235)$, $\underline{j}^{(2)} = (2340)$, $\underline{j}^{(3)} = (3451)$, $\underline{j}^{(4)} = (502)$, and $\underline{j}^{(5)} = (1)$. The strands are [3210], [54321], [105432] and [21054]. In particular, strands (and blocks) can have different lengths, but the longest strand has length $m = 6$. $\Diamond$

Each monomial $a(\underline{j}) \in \mathrm{n}\widehat{\mathrm{TL}}_N$ determines two sets $\mathrm{I}^{\mathrm{in}}_{\underline{j}}, \mathrm{I}^{\mathrm{out}}_{\underline{j}}$ and an integer $\ell_{\underline{j}} \in \mathbb{Z}_{\geq 0}$ as follows:

$$\begin{aligned} \mathrm{I}^{\mathrm{in}}_{\underline{j}} &= \{i \in \{0,1,\ldots,N-1\} \mid \text{no } i-1 \text{ to the right of } i \text{ in } \underline{j}\} \qquad \text{(I.2.2)} \\ \mathrm{I}^{\mathrm{out}}_{\underline{j}} &= \{i \in \{0,1,\ldots,N-1\} \mid \text{no } i+1 \text{ to the left of } i \text{ in } \underline{j}\} \\ \ell_{\underline{j}} &= \text{the number of zeros in } \underline{j}. \end{aligned}$$

These are well defined because, as in the proof of Lemma I.2.5.6, any element of $\mathrm{n}\widehat{\mathrm{TL}}_N$ is uniquely determined by the number of factors a_i and the relative position of each a_i and $a_{i\pm1}$, for all i. The set $\mathrm{I}^{\mathrm{in}}_{\underline{j}}$ equals the underlying set of $\underline{j}^{(0)}$ in the normal form from the algorithm above. All strands of $\underline{j}$ begin with an element in $\mathrm{I}^{\mathrm{in}}_{\underline{j}}$ and end with an element from $\mathrm{I}^{\mathrm{out}}_{\underline{j}}$.

The goal of this subsection is to show

I.2.6.4 Proposition. The mapping

$$\begin{aligned} \psi:\ \{a(\underline{j}) \in \mathrm{n}\widehat{\mathrm{TL}}_N \text{ in normal form}\} &\to \mathcal{P}_N \times \mathcal{P}_N \times \mathbb{Z}_{\geq 0} \qquad \text{(I.2.3)} \\ a(\underline{j}) &\mapsto (\mathrm{I}^{\mathrm{in}}_{\underline{j}}, \mathrm{I}^{\mathrm{out}}_{\underline{j}}, \ell_{\underline{j}}), \end{aligned}$$

is injective, where $\mathcal{P}_N$ is the power set of $\{0,1,\ldots,N-1\}$.

I.2.6.5 Remark. i) The map ψ is defined so that in the graphical description of the representation V of $\mathrm{n}\widehat{\mathrm{TL}}_N$, the set $\mathrm{I}^{\mathrm{in}}_{\underline{j}}$ equals the set of positions where $a(\underline{j})$ expects a particle to be. The set $\mathrm{I}^{\mathrm{out}}_{\underline{j}}$ equals the set of positions where $a(\underline{j})$ moves the particles from $\mathrm{I}^{\mathrm{in}}_{\underline{j}}$, but each one is translated by 1, that is,

$$a(\underline{j}) \text{ applied to a particle at } i \in \mathrm{I}^{\mathrm{in}}_{\underline{j}} \text{ gives a particle at } j+1 \text{ for some } j \in \mathrm{I}^{\mathrm{out}}_{\underline{j}}.$$

ii) The map ψ is far from being surjective. An obvious constraint is that $|\mathrm{I}^{\mathrm{in}}_{\underline{j}}| = |\mathrm{I}^{\mathrm{out}}_{\underline{j}}|$, and furthermore, for some pairs $(\mathrm{I}^{\mathrm{in}}_{\underline{j}}, \mathrm{I}^{\mathrm{out}}_{\underline{j}})$, one can only obtain sufficiently large values $\ell_{\underline{j}}$. ◊

We start by proving injectivity of the restriction ψ_0 of ψ to those monomials $a(\underline{j})$ in normal form whose first element i_1 of $\underline{j}^{(0)}$ is 0. The proof itself will amount to counting indices.

I.2.6.6 Proposition. The map

$$\begin{aligned}\psi_0 : \{a(\underline{j}) \in \mathrm{n}\widehat{\mathrm{TL}}_N \text{ in normal form, with } i_1 = 0\} &\to \mathcal{P}_N \times \mathcal{P}_N \times \mathbb{Z}_{\geq 0},\\ a(\underline{j}) &\mapsto (\mathrm{I}^{\mathrm{in}}_{\underline{j}}, \mathrm{I}^{\mathrm{out}}_{\underline{j}}, \ell_{\underline{j}})\end{aligned}$$

that sends a basis element in normal form with rightmost index $i_1 = 0$ to the tuple $(\mathrm{I}^{\mathrm{in}}_{\underline{j}}, \mathrm{I}^{\mathrm{out}}_{\underline{j}}, \ell_{\underline{j}})$ defined in Equation I.2.2 is injective.

Before beginning the proof of this result, we note that for monomials $a(\underline{j})$ with $i_1 = 0$, the inequality $i_k < N-1$ must hold in $\mathrm{I}^{\mathrm{in}}_{\underline{j}}$, since $i_1 = 0$ implies that $i_1 - 1 = N-1$ is not an element of $\mathrm{I}^{\mathrm{in}}_{\underline{j}}$. Consequently, the ordering of the indices in $\mathrm{I}^{\mathrm{in}}_{\underline{j}}$ agrees with the natural ordering of $\mathbb{Z}$, so we can regard $(\mathrm{I}^{\mathrm{in}}_{\underline{j}}, <)$ as a subset of $(\mathbb{Z}, <)$ and replace the modular index sequence $\underline{j}$ by an integral index sequence $\underline{j}^{\mathbb{Z}}$ such that $\underline{j}^{\mathbb{Z}} (\bmod N) = \underline{j}$.

I.2.6.7 Definition. Assume $\underline{j} = \underline{j}^{(m)} \cdot \ldots \cdot \underline{j}^{(1)} \cdot \underline{j}^{(0)}$ is a normal form sequence with $\underline{j}^{(0)} = \{0 = i_1 < \ldots < i_k < N-1\}$ and $\underline{j}^{(n)} = (i_{h_1} + n, \ldots, i_{h_{k(n)}} + n) \subseteq (i_1 + n, \ldots, i_k + n)$, where indices in $\underline{j}^{(n)}$ are modulo N and $1 \leq k(n) \leq k$ for all $1 \leq n \leq m$. The integral normal form sequence for $\underline{j}$ is

$$\underline{j}^{\mathbb{Z}} = (\underline{j}^{(m)})^{\mathbb{Z}} \cdot \ldots \cdot (\underline{j}^{(1)})^{\mathbb{Z}} \cdot \underline{j}^{(0)} \quad \text{where} \quad (\underline{j}^{(n)})^{\mathbb{Z}} := (i_{h_1} + n, \ldots, i_{h_{k(n)}} + n) \in \mathbb{Z}^{k(n)}$$

for $n = 1, \ldots, m$.

I.2.6.8 Example. We illustrate our notation with our running Example I.2.5.5 for $N = 6$.

$$\begin{aligned}\text{If } \underline{j} &= (1 \quad 5\,0\,2 \quad 3\,4\,5\,1 \quad 2\,3\,4\,0 \quad 1\,2\,3\,5 \quad 0\,1\,2\,4),\\ \text{then } \underline{j}^{\mathbb{Z}} &= (7 \quad 5\,6\,8 \quad 3\,4\,5\,7 \quad 2\,3\,4\,6 \quad 1\,2\,3\,5 \quad 0\,1\,2\,4).\end{aligned}$$ ◊

Our proof of Proposition I.2.6.6 will hinge upon the following technical lemma.

I.2.6.9 Lemma. Let $\underline{j}^{\mathbb{Z}}$ be the integral normal form sequence for $\underline{j}$ and let $[i_s, \ldots, i_s + n_s]$ for $s = 1, \ldots, k$ be the strands of $\underline{j}^{\mathbb{Z}}$ in the sense of Definition I.2.6.2. Assume $i_1 = 0$. Then

i) $n_1 = i_1 + n_1 \ < \ i_2 + n_2 \ < \ \dots \ < \ i_k + n_k$,
ii) $i_k + n_k < i_1 + n_1 + N = n_1 + N$.

Assume this lemma for the moment. We postpone the proof of this result and proceed directly to proving the proposition.

Proof (Proposition I.2.6.6). Since we will fix the sequence $\underline{j}$ throughout the proof, we will drop the subscript $\underline{j}$ on $\mathrm{I}^{\mathrm{in}}_{\underline{j}}$, $\mathrm{I}^{\mathrm{out}}_{\underline{j}}$, $\ell_{\underline{j}}$. To show the injectivity of ψ_0, we consider the factorization $\psi_0 = \gamma \circ \beta \circ \alpha$ given by

$$\psi_0 : \ a(\underline{j}) \ \xmapsto{\alpha} \ a(\underline{j}^{\mathbb{Z}}) \ \xmapsto{\beta} \ ((\mathrm{I}^{\mathrm{in}})^{\mathbb{Z}}, (\mathrm{I}^{\mathrm{out}})^{\mathbb{Z}}) \ \xmapsto{\gamma} \ (\mathrm{I}^{\mathrm{in}}, \mathrm{I}^{\mathrm{out}}, \ell),$$

where $(\mathrm{I}^{\mathrm{in}})^{\mathbb{Z}} = \mathrm{I}^{\mathrm{in}}$ and $(\mathrm{I}^{\mathrm{out}})^{\mathbb{Z}} = \{i \in \underline{j}^{\mathbb{Z}} \mid \text{no } i+1 \text{ to the left of } i\}$ similar to the definition of $\mathrm{I}^{\mathrm{out}}$. The map α replaces indices in $\mathbb{Z}/N\mathbb{Z}$ by indices in $\mathbb{Z}$ as in Definition I.2.6.7 above. The map β is given by reading off $(\mathrm{I}^{\mathrm{out}})^{\mathbb{Z}}$ and $(\mathrm{I}^{\mathrm{in}})^{\mathbb{Z}}$ from $\underline{j}^{\mathbb{Z}}$. The map γ sends the pair $((\mathrm{I}^{\mathrm{in}})^{\mathbb{Z}}, (\mathrm{I}^{\mathrm{out}})^{\mathbb{Z}})$ to a triple consisting of the respective images $\mathrm{I}^{\mathrm{in}}, \mathrm{I}^{\mathrm{out}}$ modulo N of the pair and the integer $\ell = 1 + \sum \ell_r$ where $\ell_r = \lfloor \frac{j_r}{N} \rfloor$ for each $j_r \in (\mathrm{I}^{\mathrm{out}})^{\mathbb{Z}}$. The summand 1 corresponds to $0 = i_1$; all other occurrences of 0 are counted by $\sum \ell_r$.

Now we check injectivity of all factors of ψ_0 in the factorisation $\psi_0 = \gamma \circ \beta \circ \alpha$.

The map α is clearly injective since $\underline{j}^{\mathbb{Z}} \ \mapsto \ \underline{j}^{\mathbb{Z}}(\bmod N)$ is a left inverse map.

To see that β is injective, we need to know that $\underline{j}^{\mathbb{Z}}$ can be uniquely reconstructed from $((\mathrm{I}^{\mathrm{in}})^{\mathbb{Z}}, (\mathrm{I}^{\mathrm{out}})^{\mathbb{Z}})$. Observe that $\underline{j}^{\mathbb{Z}}$ is determined by knowing all the "strands" $i_s, i_s+1, i_s+2, \dots, i_s+n_s$ for $1 \le s \le k$, hence by assigning an element $i_s + n_s \in (\mathrm{I}^{\mathrm{out}})^{\mathbb{Z}}$ to each $i_s \in (\mathrm{I}^{\mathrm{in}})^{\mathbb{Z}}$. But it follows from Lemma (I.2.6.9.i) that $i_1 + n_1$ must be the smallest element of $(\mathrm{I}^{\mathrm{out}})^{\mathbb{Z}}$, i_2+n_2 the second smallest, etc., so that the element i_s+n_s is assigned to the s-th element in I^{in}, that is, to i_s.

Now to see that γ is injective, we need to recover $((\mathrm{I}^{\mathrm{in}})^{\mathbb{Z}}, (\mathrm{I}^{\mathrm{out}})^{\mathbb{Z}})$ in a unique way from $(\mathrm{I}^{\mathrm{in}}, \mathrm{I}^{\mathrm{out}}, \ell)$. Write $\mathrm{I}^{\mathrm{in}} = \{0 = i_1 < \dots < i_k < N-1\}$, and set $(\mathrm{I}^{\mathrm{in}})^{\mathbb{Z}} := \mathrm{I}^{\mathrm{in}}$. By Lemma (I.2.6.9.i), we know that $(\mathrm{I}^{\mathrm{out}})^{\mathbb{Z}}$ is of the form $(i_1+n_1 < \dots < i_k+n_k)$, and since the elements of $\mathrm{I}^{\mathrm{out}}$ have to be equal to the elements of $(\mathrm{I}^{\mathrm{out}})^{\mathbb{Z}}$ modulo N, we can write $i_r + n_r = N\ell_r + d_r$ for $\ell_r = \lfloor \frac{i_r+n_r}{N} \rfloor$ and some $d_r \in \mathrm{I}^{\mathrm{out}}$. Comparing ℓ_r and ℓ_s for $r < s$, we have

$$N\ell_r \ \le \ N\ell_r + d_r \ = \ i_r + n_r \ < \ i_s + n_s \ = \ N\ell_s + d_s \ \le \ N(\ell_s + 1).$$

So $\ell_r < \ell_s + 1$, i.e. $\ell_r \le \ell_s$. Similarly, we obtain from Lemma (I.2.6.9.ii) that $\ell_k \le \ell_1 + 1$.

As a result,

$$N\ell_k \ \le \ N\,\ell_k + d_k \ = \ i_k + n_k \ < \ i_1 + n_1 + N \ = \ N(\ell_1+1) + d_1 \ \le \ N(\ell_1+2),$$

i.e. $\ell_k < \ell_1 + 2$. Together we have $\ell_1 = \ldots = \ell_s < \ell_{s+1} = \ldots = \ell_1 + 1$ for some $1 < s \leq k$ (where we treat the case $s = k$ by $\ell_1 = \ldots = \ell_k$). Set $\widetilde{\ell} := \ell_1$. Then

$$\begin{aligned} i_r + n_r &= N\widetilde{\ell} + d_r && \text{for } 1 \leq r \leq s, \\ i_r + n_r &= N(\widetilde{\ell} + 1) + d_s && \text{for } s+1 \leq r \leq k. \end{aligned}$$

As a first consequence,

$$\ell = 1 + \sum_r \ell_r = 1 + k\widetilde{\ell} + (k - s),$$

which determines $\widetilde{\ell} = \lfloor \frac{\ell-1}{k} \rfloor$, and hence all ℓ_r, as well as the index s. Using Lemma I.2.6.9, we determine that

$$i_{s+1} + n_{s+1} < \ldots < i_k + n_k < i_1 + n_1 + N < \ldots < i_s + n_s + N,$$

and so

$$N(\widetilde{\ell} + 1) + d_{s+1} < \ldots < N(\widetilde{\ell} + 1) + d_k < N(\widetilde{\ell} + 1) + d_1 < \ldots < N(\widetilde{\ell} + 1) + d_s.$$

Therefore, $d_{s+1} < \ldots < d_k < d_1 < \ldots < d_s$, which fixes the choice of d_r for all r. We conclude that given $(\mathrm{I}^{\mathrm{in}}, \mathrm{I}^{\mathrm{out}}, \ell)$, we can reconstruct $(\mathrm{I}^{\mathrm{out}})^{\mathbb{Z}}$ by setting $i_r + n_r := N\ell_r + d_r$. This completes the proof of Proposition I.2.6.6. □

Proof (Lemma I.2.6.9). i) Let $\underline{j}^{\mathbb{Z}}$ be a nonempty integral normal form sequence with $0 = i_1 < \ldots < i_k \leq N - 1$ and strands $[i_r, \ldots, i_r + n_r]$ for $1 \leq r \leq k$ (recall Definitions I.2.6.2 and I.2.6.7). Assume that there is some index $1 \leq t \leq k - 1$ such that $i_t + n_t \geq i_{t+1} + n_{t+1}$. Since $i_t < i_{t+1}$, we have $n_t > n_{t+1}$. So

$$\underline{j}^{\mathbb{Z}} = \ldots \underbrace{(\ldots\ i_t + n_t\ \ldots)}_{\text{the } n_t\text{th bracket}} \ldots \underbrace{(\ldots\ i_t + n_{t+1}\quad i_{t+1} + n_{t+1}\ \ldots)}_{\text{the } n_{t+1}\text{th bracket}} \ldots .$$

From $i_t + n_{t+1} < i_{t+1} + n_{t+1} \leq i_t + n_t$ it follows that there is some integer $n_{t+1} < p \leq n_t$ such that $i_{t+1} + n_{t+1} = i_t + p$ appears in the strand $[i_t, \ldots, i_t + n_t]$, i.e.

$$\underline{j}^{\mathbb{Z}} = \ldots \underbrace{(\ldots\ i_t + n_t\ \ldots)}_{\text{the } n_t\text{th bracket}} \ldots \underbrace{(\ldots\ i_t + p\ \ldots)}_{\text{the } p\text{th bracket}} \ldots \underbrace{(\ldots\ i_t + n_{t+1}\quad i_{t+1} + n_{t+1}\ \ldots)}_{\text{the } n_{t+1}\text{th bracket}} \ldots$$

with $i_t + p = i_{t+1} + n_{t+1}$. But by the definition of the strands, there is no $i_{t+1} + n_{t+1} + 1$ appearing to the left of $i_{t+1} + n_{t+1}$. Due to Lemma I.2.5.2, we know that (even modulo N) there is no repetition of $i_{t+1} + n_{t+1}$ to the left. Thus $i_t + p = i_{t+1} + n_{t+1}$ is not possible, and we obtain $i_1 + n_1 < i_2 + n_2 < \ldots < i_k + n_k$.

ii) For the second statement of Lemma I.2.6.9, assume $i_k + n_k \geq i_1 + n_1 + N$. It is true generally that $N > i_k$, so we get $i_k + n_k \geq i_1 + n_1 + N > i_k + n_1$. Hence $i_1 + n_1 + N = i_k + b$ for some $n_1 < b \leq n_k$, i.e. $i_1 + n_1 + N$ appears in the strand $[i_k, \ldots, i_k + n_k]$ and we have

$$\underline{j}^{\mathbb{Z}} = \ldots \underbrace{(\ldots\ i_k + n_k)}_{\text{the } n_k\text{th bracket}} \ \ldots\ \underbrace{(\ldots\ i_k + b\ \ldots)}_{\text{the } b\text{th bracket}} \ \ldots\ \underbrace{(i_1 + n_1\ \ldots\ i_k + n_1)}_{\text{the } n_1\text{th bracket}} \ \ldots\ .$$

Here it may be that the n_k-th bracket and the b-th bracket coincide, but in any case, we find that $i_k + b = i_1 + n_1 + N = i_1 + n_1 \bmod N$, and so $i_k + b$ appears to the left of $i_1 + n_1$. By the definition of the strands, there is no $i_1 + n_1 + 1$ to the left of $i_1 + n_1$, and from Lemma I.2.5.2 we deduce that in $\underline{j} = \underline{j}^{\mathbb{Z}} \bmod N$ there is no $i_1 + n_1 \bmod N$ to the left of $i_1 + n_1$ allowed, which leads to a contradiction. Hence $i_k + n_k < i_1 + n_1 + N$ must hold. □

Having established that ψ is injective when restricted to sequences with $i_1 = 0$, we now show the injectivity of ψ in general.

Proof (Proposition I.2.6.4). We have the following disjoint decompositions according to the smallest value i_1 in $\underline{j}^{(0)}$ for $\underline{j}$:

$$\begin{aligned}
\{a(\underline{j}) \text{ in normal form}\} &= \coprod_i \{a(\underline{j}) \text{ in normal form},\ \ i_1 = i\} \\
\{(\mathrm{I}^{\mathrm{in}}_{\underline{j}}, \mathrm{I}^{\mathrm{out}}_{\underline{j}}, \ell_{\underline{j}})\} &= \coprod_i \{(\mathrm{I}^{\mathrm{in}}_{\underline{j}}, \mathrm{I}^{\mathrm{out}}_{\underline{j}}, \ell_{\underline{j}}) \mid i_1 = i \in \mathrm{I}^{\mathrm{in}}_{\underline{j}}\} \\
\psi &= \coprod_i \psi_i \\
\text{where } \psi_i : \{a(\underline{j}) &\text{ in normal form},\ \ i_1 = i\} \ \to\ \{(\mathrm{I}^{\mathrm{in}}_{\underline{j}}, \mathrm{I}^{\mathrm{out}}_{\underline{j}}, \ell_{\underline{j}}) \mid i_1 = i \in \mathrm{I}^{\mathrm{in}}_{\underline{j}}\}.
\end{aligned}$$

By Proposition I.2.6.6, the map $\psi_0 :\ a(\underline{j}) \mapsto (\mathrm{I}^{\mathrm{in}}_{\underline{j}}, \mathrm{I}^{\mathrm{out}}_{\underline{j}}, \ell_{\underline{j}})$ restricted to those $a(\underline{j})$ with $i_1 = 0$ is injective. We argue next that by an index shift this result is true for all other ψ_i. It follows from Proposition I.2.6.6 that the map $\widehat{\psi}_0$ defined by

$$\widehat{\psi}_0 :\ \{a(\underline{j}) \in \mathrm{n}\widehat{\mathrm{TL}}_N \text{ in normal form, with } i_1 = 0\} \ \to\ \{(\mathrm{I}^{\mathrm{in}}_{\underline{j}}, \mathrm{I}^{\mathrm{out}}_{\underline{j}}, \widehat{\ell_{\underline{j}}}) \mid i_1 = 0 \in \mathrm{I}_{\mathrm{in}}\}$$

is injective, where $\widehat{\ell_{\underline{j}}} = \widehat{\ell_{\underline{j}}}(i)$ counts the occurences of $N - i$ in $\underline{j}$. Recall that

$$\ell_{\underline{j}} = \sum_r \ell_r + 1 \ \text{ where } \ \ell_r \text{ is the number of 0 in the } r\text{th strand } \ [i_r, \ldots, i_r + n_r] \text{ of } \underline{j} \bmod N.$$

Now observe that we can obtain $\ell_{\underline{j}}$ from $\widehat{\ell_{\underline{j}}}$ as

$$\ell_{\underline{j}} = \widehat{\ell_{\underline{j}}} - \left|\{d_r \in \mathrm{I}^{\mathrm{out}}_{\underline{j}} \mid d_r \geq N - i\}\right| + \left|\{i_r \in \mathrm{I}^{\mathrm{in}}_{\underline{j}} \mid i_r > N - i\}\right| + 1,$$

which follows from a computation using $\widehat{\ell_{\underline{j}}} = \sum_r \widehat{\ell_r}$ and

$$\begin{aligned}
\widehat{\ell_r} &= \text{the number of } \ N-i \ \text{ in the } r\text{th strand } \ [i_r,\dots,i_r+n_r] \bmod N \\
&= \begin{cases} \lfloor \frac{i_r+n_r+i}{N} \rfloor & \text{if } i_r \le N-i \\ \lfloor \frac{i_r+n_r+i}{N} \rfloor - 1 & \text{if } i_r > N-i \end{cases} \\
&= \begin{cases} \lfloor \frac{N\ell_r+d_r+i}{N} \rfloor & \text{if } i_r \le N-i \\ \lfloor \frac{N\ell_r+d_r+i}{N} \rfloor - 1 & \text{if } i_r > N-i \end{cases} \\
&= \begin{cases} \ell_r+1 & \text{if } i_r \le N-i \text{ and } d_r+i \ge N \\ \ell_r & \text{if } i_r \le N-i \text{ and } d_r+i < N \\ \ell_r & \text{if } i_r > N-i \text{ and } d_r+i \ge N \\ \ell_r-1 & \text{if } i_r > N-i \text{ and } d_r+i < N. \end{cases}
\end{aligned}$$

We obtain ψ_i by first shifting the indices of $\underline{j}$ by subtracting i from each index, $\underline{j} - (i,\dots,i)$, then applying $\widehat{\psi}_0$, and finally shifting the indices from $\mathrm{I}^{\mathrm{in}}_{\underline{j}}$ and $\mathrm{I}^{\mathrm{out}}_{\underline{j}}$ by adding i to each. Hence, ψ_i is injective for each i, and ψ is injective because the unions are disjoint. □

I.2.6.2. Description and linear independence of the matrices

Recall that the standard $\Bbbk$-basis of the representation $\mathsf{V} = \bigoplus_{k=0}^{N} \left(\Bbbk[q] \otimes \bigwedge^k \Bbbk^N\right)$ is given by

$$\{q^\ell \cdot v_{i_1} \wedge \ldots \wedge v_{i_k} \mid \ell \in \mathbb{Z}_{\ge 0},\ 1 \le i_1 < \ldots < i_k \le N,\ 0 \le k \le N\}$$

where $(i_1,\dots,i_k)$ is identified with the particle configuration having particles in those positions in the graphical description. Now we describe, with respect to this basis, the matrix representing a nonzero nonconstant monomial $a(\underline{j}) \in \mathrm{n}\widehat{\mathrm{TL}}_N$ as a $2^N \times 2^N$-matrix with entries in $\Bbbk[q]$. Since V decomposes as a $\mathrm{n}\widehat{\mathrm{TL}}_N$-module into submodules $\Bbbk[q] \otimes \bigwedge^k \Bbbk^N$ for $k = 0,1,\dots,N$, the matrix of $a(\underline{j})$ is block diagonal with $N+1$ blocks $A_0, A_1, \dots, A_N$, where $A_0 = A_N = (0)$ corresponding to the trivial representation.

$$a(\underline{j}) = \begin{pmatrix} 0 & 0 & & \cdots & 0 \\ 0 & \boxed{A_1} & & & \vdots \\ & & \ddots & & \\ \vdots & & & \boxed{A_{N-1}} & 0 \\ 0 & \cdots & & 0 & 0 \end{pmatrix}$$

The block A_k is a $\binom{N}{k} \times \binom{N}{k}$-matrix, with entries from $\Bbbk[q]$ indexed by the standard basis of $\bigwedge^k \Bbbk^N$, corresponding to all possible particle configurations whose number of particles is equal to k.

Now fix a nonzero monomial $a(\underline{j})$ in normal form specified by the triple $(\mathrm{I}^{\mathrm{in}}_{\underline{j}}, \mathrm{I}^{\mathrm{out}}_{\underline{j}}, \ell_{\underline{j}})$ defined in the previous subsection. Let $k = |\mathrm{I}^{\mathrm{in}}_{\underline{j}}|$. All blocks $A_1, \ldots, A_{k-1}$ are zero since $a(\underline{j})$ expects at least k particles. For $r > k$, there might be nonzero blocks (unless the particles from $\mathrm{I}^{\mathrm{in}}_{\underline{j}}$ are moved around the whole circle with no position left out, in which case there are no surplus particles allowed. This occurs if $a(\underline{j})$ contains at least every other generator $a_i, a_{i+2}, \ldots$). More importantly, the block A_k has precisely one nonzero entry, and this is given by

$$(A_k)_{\mathrm{I}^{\mathrm{in}}_{\underline{j}}, \mathrm{I}^{\mathrm{out}}_{\underline{j}}} \;=\; \pm q^{\ell_{\underline{j}}}.$$

Now we can show the following lemma:

I.2.6.10 Lemma. The set of matrices representing monomials $a(\underline{j})$ in normal form is $\Bbbk$-linearly independent.

Proof. First of all we observe that the matrices representing monomials $a(\underline{j})$ in normal form with $|\mathrm{I}^{\mathrm{in}}_{\underline{j}}| = N-1$ are $\Bbbk$-linearly independent. There is only one nonzero entry which is equal to $\pm q^{\ell_{\underline{j}}}$ at position $(\mathrm{I}^{\mathrm{in}}_{\underline{j}}, \mathrm{I}^{\mathrm{out}}_{\underline{j}})$.

Furthermore, if all matrices representing monomials $a(\underline{j})$ in normal form with $|\mathrm{I}^{\mathrm{in}}_{\underline{j}}| \geq k$ are $\Bbbk$-linearly independent, then also all matrices representing monomials $a(\underline{j})$ in normal form with $|\mathrm{I}^{\mathrm{in}}_{\underline{j}}| \geq k-1$ are $\Bbbk$-linearly independent. This follows because the additional monomials $a(\underline{j})$ with $|\mathrm{I}^{\mathrm{in}}_{\underline{j}}| = k-1$ have nonzero entries $(A_{k-1})_{\mathrm{I}^{\mathrm{in}}_{\underline{j}}, \mathrm{I}^{\mathrm{out}}_{\underline{j}}} = \pm q^{\ell_{\underline{j}}}$ in the $(k-1)$-th block which is zero for all $a(\underline{j})$ with $|\mathrm{I}^{\mathrm{in}}_{\underline{j}}| \geq k$. So by induction, all matrices representing monomials $a(\underline{j})$ in normal form are $\Bbbk$-linearly independent. □

I.2.6.11 Corollary. The representation of $\mathrm{n}\widehat{\mathrm{TL}}_N$ on V is faithful.

Proof. According to Theorem I.2.5.7, $\{a(\underline{j}) \text{ in normal form}\} \cup \{1\}$ is a $\Bbbk$-basis of $\mathrm{n}\widehat{\mathrm{TL}}_N$. According to Lemma I.2.6.10, the set of matrices representing monomials $a(\underline{j})$ in normal form is $\Bbbk$-linearly independent. Since all of the matrices representing some $a(\underline{j})$ in normal form have a zero entry in the upper left (and lower right) corner, we may include the identity matrix into the linearly independent set of matrices, and it remains linearly independent. □

The submodule $\bigoplus_{k=1}^{N-1} \left(\Bbbk[q] \otimes \bigwedge^k \Bbbk^N\right)$ is already faithful. This can be seen directly from the following modification of the proof of Corollary I.2.6.11: Since the diagonal entries of the matrices representing some $a(\underline{j})$ in normal form are either 0 or $\pm q^{\ell_{\underline{j}}}$ for $\ell_{\underline{j}} \geq 1$, we may add the identity matrix to the linearly independent set of matrices, and it remains linearly independent over $\Bbbk$.

Section I.2.5 has given a normal form for each monomial. Section I.2.6 has provided an alternate proof of the faithfulness of the representation of $\mathrm{n}\widehat{\mathrm{TL}}_N$ by elementary arguments.

I.2.7. Projectors

This section describes certain projectors for the graphical representation. They turn out to be the main ingredients in the definition of central elements in Section I.2.8. Here and in the following, we use the name projector for a map that is a nonzero $\Bbbk[q]$-multiple of a proper projection.

For a standard basis element $v(\mathrm{I})$ of $1 \leq k \leq N-1$ wedge factors corresponding to an increasing sequence $\mathrm{I} = \{1 \leq i_1 < \ldots < i_k \leq N\}$, the next lemma defines a certain monomial $a(\hat{\mathrm{I}})$ that projects $v(\mathrm{I})$ onto $(-1)^{k-1} q\, v(\mathrm{I})$ and sends $v(\mathrm{I}')$ to zero for an increasing sequence $\mathrm{I}' \neq \mathrm{I}$ of any length. Before stating the result, we give an example to demonstrate in the graphical description how this projector will be defined.

I.2.7.1 Example. Let $N = 8$, and consider the particle configuration $v(\mathrm{I}) = v_1 \wedge v_5 \wedge v_6$. Up to some factor in $\Bbbk[q]$ we want to map $v_1 \wedge v_5 \wedge v_6$ to itself:

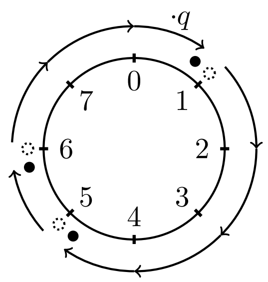


Figure I.2.7.1.: The action of $a(\widehat{1\ 5\ 6})$ on the particle configuration $v_1 \wedge v_5 \wedge v_6$

We read off from the picture that the monomial $a(\widehat{1\ 5\ 6}) = (a_0 a_7) \cdot (a_4 a_3 a_2) \cdot (a_1 a_5 a_6)$ sends $v_1 \wedge v_5 \wedge v_6$ to $(-1)^2 q \cdot v_1 \wedge v_5 \wedge v_6$:

The factor $a_1 a_5 a_6$ moves every particle one step forward clockwise. It is critical that we start by moving the particle at position 6 before moving the particle at position 5, as otherwise the result would be zero. But since there is a "gap" at position 7, we can move the particle from site 6 to 7, and afterwards the particle from site 5 to 6, without obtaining zero. The assumption that $k < N$ ensures such a gap always exists.

After applying $a_1a_5a_6$, the particles are at positions 2, 6, and 7. The particle previously at position 5 is now at position 6, which is where we want a particle to be. The particle currently at position 2 can be moved to position 5 by applying the product $a_4a_3a_2$. The particle now at position 7 can be moved by a_0a_7 to position 1. Hence, the result of applying $(a_0a_7)\cdot(a_4a_3a_2)\cdot(a_1a_5a_6)$ is the same particle configuration as the original one. However, the answer must be multiplied by $\pm q$, since applying a_0a_7 involves crossing the zero position once. To determine the sign, note from Definition I.2.4.1 that $(a_0a_7)\cdot(a_4a_3a_2)\cdot(a_1a_5a_6)(v_1\wedge v_5\wedge v_6) = q\cdot v_5\wedge v_6\wedge v_1 = (-1)^2 q\cdot v_1\wedge v_5\wedge v_6$, so the sign is $+$. $\Diamond$

Now we describe the general procedure that defines the projector $a(\hat{\imath})$:

I.2.7.2 Lemma. Assume $v(\mathrm{I})$ is a particle configuration, where $\mathrm{I} = \{1 \le i_1 < \ldots < i_k \le N\}$ is an increasing sequence and $1 \le k \le N-1$. Then there exists an index ℓ such that $i_\ell + 1 < i_{\ell+1}$ (or $i_k + 1 < i_1$), i.e. the sequence has a "gap" between i_ℓ and $i_{\ell+1}$. Split the sequence I into the two parts $\{i_1 < \ldots < i_\ell\}$ and $\{i_{\ell+1} < \ldots < i_k\}$. Set

$$\begin{aligned} a(\hat{\imath}) \;&:=\; (a_{i_1-1}a_{i_1-2}\ldots a_{i_k+2}a_{i_k+1})\cdot\prod_{s=1}^{k-1}(a_{i_{s+1}-1}a_{i_{s+1}-2}\ldots a_{i_s+2}a_{i_s+1}) \qquad (\star)\\ &\quad\cdot(a_{i_{\ell+1}}a_{i_{\ell+2}}\ldots a_{i_{k-1}}a_{i_k})\cdot(a_{i_1}a_{i_2}\ldots a_{i_{\ell-1}}a_{i_\ell}), \end{aligned}$$

where the indices are modulo N in the factor $(a_{i_1-1}a_{i_1-2}\ldots a_{i_k+2}a_{i_k+1})$. Then

$$a(\hat{\imath})v(\mathrm{I}') = \begin{cases} (-1)^{k-1}q\cdot v(\mathrm{I}) & \text{if } \mathrm{I}' = \mathrm{I},\\ 0 & \text{for all } \mathrm{I}' \neq \mathrm{I} \text{ (of any length)}, \end{cases}$$

and $a(\hat{\imath})$ has $\mathbb{Z}^N$-degree $(1,1,\ldots,1)$.

Proof. The assertions can be seen using the graphical realization of V. The terms in the second line of equation $(\star)$ move a particle at site $i_j \in \mathrm{I}$ one step forward to i_j+1 for each j, while the terms in the first line send the particle from i_j+1 to the original position of i_{j+1}.

Consider first $a(\hat{\imath})v(\mathrm{I})$. By applying $(a_{i_{\ell+1}}a_{i_{\ell+2}}\ldots a_{i_{k-1}}a_{i_k})\cdot(a_{i_1}a_{i_2}\ldots a_{i_{\ell-1}}a_{i_\ell})$, every particle is first moved clockwise by one position. By our choice of the index i_ℓ, we avoid mapping the whole particle configuration to zero. After that step, every particle is moved by one of the factors $(a_{i_{s+1}-1}a_{i_{s+1}-2}\ldots a_{i_s+2}a_{i_s+1})$ to the original position of its successor in the sequence I, so the particle configuration remains the same. One of the particles has passed the zero position, so we have to multiply by $\pm q$. Definition I.2.4.1 tells us the appropriate sign is $(-1)^{k-1}$.

Now consider $a(\hat{\imath})v(\mathrm{I}')$ for $\mathrm{I}' \neq \mathrm{I}$. The term $(a_{i_{\ell+1}} a_{i_{\ell+2}} \dots a_{i_{k-1}} a_{i_k}) \cdot (a_{i_1} a_{i_2} \dots a_{i_{\ell-1}} a_{i_\ell})$ expects a particle at each of the sites $i_1, \dots, i_k$, so if any of these positions is empty in $v(\mathrm{I}')$, the result of applying $a(\hat{\imath})$ is zero. If the positions $i_1, \dots, i_k$ are already filled, and there is an additional particle somewhere, multiplication by $(a_{i_{\ell+1}-1} a_{i_{\ell+1}-2} \dots a_{i_\ell+2} a_{i_\ell+1})$ will cause two particles to be at the same position, hence the result is again zero.

Since every a_j appears in $a(\hat{\imath})$ exactly once, the $\mathbb{Z}^N$-degree of $a(\hat{\imath})$ is $(1, 1, \dots, 1)$. □

I.2.7.3 Example. In the previous example, $N = 8$, $\mathrm{I} = (1, 5, 6)$, and we may assume the two subsequences are (1) and $(5, 6)$. Then the terms in the second line of $(\star)$ are $(a_5 a_6) \cdot (a_1) = a_1 a_5 a_6$. The term corresponding to $j = 1$ in the product on the first line of $(\star)$ is $a_4 a_3 a_2$, and the expression corresponding to $j = 2$ is empty, hence taken to be 1. The first factor on the first line is $a_0 a_7$. Thus, for $\mathrm{I} = (1, 5, 6)$, $a(\hat{\imath}) = (a_0 a_7) \cdot (a_4 a_3 a_2) \cdot (a_1 a_5 a_6)$, as in Example I.2.7.1. If the gap between 6 and 0 is used instead, the right-hand factor of the second line is $a_1 a_5 a_6$ and the left-hand factor is 1. The factors in the first line remain the same, and so one obtains the same expression for $a(\hat{\imath})$. ◊

I.2.7.4 Corollary. For $\mathrm{I} \neq \mathrm{J}$ the product $\{a(\hat{\imath}) \cdot \{a(\hat{\jmath})$ of two distinct projectors defined in Lemma I.2.7.2 is zero. In particular, the subalgebra generated by the set of projectors

$$\{a(\hat{\imath}) \in \mathrm{n}\widehat{\mathrm{TL}}_N \mid \mathrm{I} \text{ increasing sequence}\}$$

is commutative.

Proof. This follows from Lemma I.2.7.2 together with the faithfulness of the particle configuration module V, see Theorem I.2.4.5. □

I.2.7.5 Remark. Because V is a faithful module (Theorem I.2.4.5), $a(\hat{\imath})$ is, as an element in $\mathrm{n}\widehat{\mathrm{TL}}_N$ (i.e. up to reordering according to the defining relations), uniquely determined by the increasing sequence I. One can read off I from $a(\hat{\imath})$ as follows: In the defining equation $(\star)$ of $a(\hat{\imath})$, the factors in the first line are pairwise commuting. The underlying subsequence $(i_{s+1} - 1, i_{s+1} - 2, \dots, i_s + 2, i_s + 1)$ corresponding to the factor $a_{i_{s+1}-1} a_{i_{s+1}-2} \dots a_{i_s+2} a_{i_s+1}$ of $a(\hat{\imath})$ is a decreasing sequence. After all such decreasing sequences are removed from $a(\hat{\imath})$, what remains is a product of generators a_j with an increasing subsequence of indices or a product of two such subsequences corresponding to the factors in the second line. This is I. Given any monomial $a(\underline{r})$ of $\mathbb{Z}^N$-degree $(1, \dots, 1)$, one can rewrite it using the relations in $\mathrm{n}\widehat{\mathrm{TL}}_N$ so that it is of the form $a(\hat{\imath})$ for some increasing sequence I. Then $v(\text{I})$ is the unique standard basis element upon which $a(\underline{r}) = a(\hat{\text{I}})$ acts by multiplication by $\pm q$. $\Diamond$

I.2.8. Description of the center

In this section, we give an explicit description of the center C_N of $\text{n}\widehat{\text{TL}}_N$. We start with the following initial characterisation of the central elements:

I.2.8.1 Lemma. Any central element c in $\text{n}\widehat{\text{TL}}_N$ with zero constant term is a linear combination of monomials $a(\underline{j}) = a_{j_1} \cdot \ldots \cdot a_{j_m}$ where every generator a_i, $0 \leq i \leq N-1$, appears at least once. In particular, a homogeneous nonconstant central element c has $\mathbb{Z}$-degree at least N.

Proof. Assume $c = \sum_{\underline{j}} c_{\underline{j}} a(\underline{j})$, where $c_{\underline{j}} \in \Bbbk$ for all $\underline{j}$. By Lemma I.2.3.1, we can assume c is a homogeneous central element with respect to the $\mathbb{Z}^N$-grading. By our assumption, $c \notin \Bbbk$. For all i, we need to show that a_i occurs in each monomial $a(\underline{j})$ appearing in c. Without loss of generality, we show this for $i = 0$. Suppose some summand is missing a_0, then no summand contains a_0 because c is homogeneous. Hence $a_0 a(\underline{j}) \neq 0$ and $a(\underline{j}) a_0 \neq 0$ for all $\underline{j}$ with $c_{\underline{j}} \neq 0$, and since $a_0 c = c a_0$, none of the $a(\underline{j})$ can contain the factor a_1 either, as otherwise the factor a_0 cannot pass through c from left to right (so also a_{N-1} cannot be contained in the $a(\underline{j})$). Proceeding inductively, we see that all $a(\underline{j})$ must be a constant, contrary to our assumption. $\square$

The next proposition states that on the standard wedge basis vector $v(\text{I})$ of V defined in Equation I.2.1, any central element acts via multiplication by a polynomial $p_k \in \Bbbk[q]$ that only depends on the length $k = |\text{I}|$ of the increasing sequence $\text{I} = \{1 \leq i_1 < \ldots < i_k \leq N\}$. In other words, the decomposition of V into the summands $\Bbbk[q] \otimes \bigwedge^k \Bbbk^N$ is a decomposition with respect to different central characters (apart from the two trivial summands for $k \in \{0, N\}$).

I.2.8.2 Proposition. For any central element $c \in \text{n}\widehat{\text{TL}}_N$ and all increasing sequences I with fixed length $0 \leq k \leq N$, there is some element $p_k \in \Bbbk[q]$ depending only on c and the length k such that $cv(\text{I}) = p_k v(\text{I})$.

Proof. By Lemma I.2.3.1 we may assume c is a $\mathbb{Z}^N$-homogeneous central element of $\text{n}\widehat{\text{TL}}_N$. If c is constant, then $p_k = c$ for any $0 \leq k \leq N$, independently of I. Now assume that c is nonconstant.

For $k \in \{0, N\}$, the action of a generator a_i on a monomial of length k is 0, so $p_k = 0$ for such values of k. Now consider $1 \le k \le N-1$, and suppose that $\mathrm{I} = \{1 \le i_1 < \ldots < i_k \le N\}$ is an increasing sequence of length k. According to Lemma (I.2.4.4.ii), the number of wedges in a vector remains constant under the action of the a_i. Hence $cv(\mathrm{I}) = \sum_{|\mathrm{I}'|=k} c_{\mathrm{I}'}\, v(\mathrm{I}')$ for some polynomials $c_{\mathrm{I}'} \in \Bbbk[q]$. We want to prove that $c_{\mathrm{I}'} = 0$ for all $\mathrm{I}' \neq \mathrm{I}$.

We have shown in Lemma I.2.7.2 that to each increasing sequence $\mathrm{J} \subset \{1, \ldots, N\}$ there corresponds a monomial $a(\hat{\mathrm{J}}) \in \mathrm{n}\widehat{\mathrm{TL}}_N$ that allows us to select a single basis vector:

$$a(\hat{\mathrm{J}})v(\mathrm{I}) \;=\; \begin{cases} (-1)^{k-1} q v(\mathrm{J}) & \text{if } \;\mathrm{I} = \mathrm{J}, \\ 0 & \text{otherwise.} \end{cases}$$

Thus, for $\mathrm{J} \neq \mathrm{I}$, we see that

$$0 \;=\; c(a(\hat{\mathrm{J}})v(\mathrm{I})) \;=\; a(\hat{\mathrm{J}})(cv(\mathrm{I})) \;=\; a(\hat{\mathrm{J}})\left(\sum_{|\mathrm{I}'|=k} c_{\mathrm{I}'}\, v(\mathrm{I}')\right) = c_{\mathrm{J}}\,(-1)^{k-1} q v(\mathrm{J}),$$

implying $c_{\mathrm{J}} = 0$ for $\mathrm{J} \neq \mathrm{I}$. Hence, we may assume for each increasing sequence I that $cv(\mathrm{I}) = p_{\mathrm{I}}\, v(\mathrm{I})$ for some polynomial $p_{\mathrm{I}} \in \Bbbk[q]$. Now it is left to show that $p_{\mathrm{I}} = p_{\mathrm{I}'}$ for all I' with $|\mathrm{I}'| = |\mathrm{I}| = k$. It is enough to verify this for I, I' which differ in exactly one entry, i.e. $i_s = i$, $i'_s = i+1$, and $i_\ell = i'_\ell$ for all $\ell \neq s$, for some $1 \le s \le k$ and $i \in \mathbb{Z}/N\mathbb{Z}$. If $1 \le i \le N-1$, we have

$$p_{\mathrm{I}'}\, v(\mathrm{I}') \;=\; cv(\mathrm{I}') \;=\; c(a_i v(\mathrm{I})) \;=\; a_i(cv(\mathrm{I})) \;=\; a_i(p_{\mathrm{I}}\, v(\mathrm{I})) \;=\; p_{\mathrm{I}}\, v(\mathrm{I}'),$$

and if $i = 0$, we get

$$\begin{aligned}(-1)^{k-1} q p_{\mathrm{I}'}\, v(\mathrm{I}') \;&=\; (-1)^{k-1} q c v(\mathrm{I}') \;=\; c(a_0 v(\mathrm{I})) \;=\; a_0(cv(\mathrm{I})) \;=\; a_0(p_{\mathrm{I}}\, v(\mathrm{I})) \\ &=\; (-1)^{k-1} q p_{\mathrm{I}}\, v(\mathrm{I}').\end{aligned}$$

Hence, $p_{\mathrm{I}'} = p_{\mathrm{I}}$, and this common polynomial is the desired polynomial p_k. □

I.2.8.3 Corollary. Any central element in $\mathrm{n}\widehat{\mathrm{TL}}_N$ with constant term 0 acts on a standard basis vector $v(\mathrm{I}) \in \mathsf{V}$ as multiplication by an element of $q\Bbbk[q]$.

Proof. According to Lemma I.2.8.1, each summand of such a central element must contain the factor a_0, and a_0 acts on a wedge product by 0 or multiplication by $\pm q$. □

Now we are ready to introduce nontrivial central elements in $\mathrm{n}\widehat{\mathrm{TL}}_N$. For each $1 \le k \le N-1$, set

$$\mathbf{t}_k \;:=\; (-1)^{k-1} \sum_{|\mathrm{I}|=k} a(\hat{\mathrm{I}}), \tag{I.2.4}$$

where the monomials $a(\hat{\mathrm{I}})$ correspond to increasing sequences $\mathrm{I} = \{1 \le i_1 < \ldots < i_k \le N\}$ of length k as defined in Lemma I.2.7.2.

I.2.8.4 Example. In $\mathrm{n}\widehat{\mathrm{TL}}_3$ the elements $\mathbf{t}_1$, $\mathbf{t}_2$ look as follows:

$$\begin{aligned}\mathbf{t}_1 &= a_2a_1a_0 + a_0a_2a_1 + a_1a_0a_2,\\ \mathbf{t}_2 &= -a_0a_1a_2 - a_1a_2a_0 - a_2a_0a_1.\end{aligned}$$

In $\mathrm{n}\widehat{\mathrm{TL}}_4$ we have

$$\begin{aligned}\mathbf{t}_1 &= a_3a_2a_1a_0 + a_0a_3a_2a_1 + a_1a_0a_3a_2 + a_2a_1a_0a_3,\\ \mathbf{t}_2 &= -a_0a_3a_1a_2 - a_0a_2a_1a_3 - a_3a_2a_0a_1 - a_1a_0a_2a_3 - a_1a_3a_0a_2 - a_2a_1a_3a_0,\\ \mathbf{t}_3 &= a_0a_1a_2a_3 + a_1a_2a_3a_0 + a_2a_3a_0a_1 + a_3a_0a_1a_2. \qquad \lozenge\end{aligned}$$

In the graphical realization of V, $\mathbf{t}_k$ acts by annihilating all particle configurations whose number of particles is different from k. For particle configurations having k particles, every particle is moved clockwise to the original site of the next particle. Hence, the particle configuration itself remains fixed by the action of $\mathbf{t}_k$ (and it is multiplied with $(-1)^{2(k-1)}q = q$, since a particle has been moved through position 0). All the $\mathbf{t}_k$ have $\mathbb{Z}^N$-degree equal to $(1,\dots,1)$ and $\mathbb{Z}$-degree equal to N. Any monomial whose $\mathbb{Z}^N$-degree is $(1,\dots,1)$ occurs as a summand in some central element (after possibly reordering the factors), and the number of summands of $\mathbf{t}_k$ equals $\binom{N}{k} = \dim(\bigwedge^k \Bbbk^N)$, see Remark I.2.7.5.

Now we formulate our main theorem of this section:

I.2.8.5 Theorem. Let $\mathrm{n}\widehat{\mathrm{TL}}_N$ be the affine nilTemperley–Lieb algebra for $N \geq 3$.

i) The $\mathbf{t}_k$ defined in Equation I.2.4 are central for all $1 \leq k \leq N-1$, and the center of $\mathrm{n}\widehat{\mathrm{TL}}_N$ is generated by 1 and the $\mathbf{t}_k$, $1 \leq k \leq N-1$.

ii) The subalgebra generated by $\mathbf{t}_k$ is isomorphic to the polynomial ring $\Bbbk[q]$ for all $1 \leq k \leq N-1$. Moreover $\mathbf{t}_k\mathbf{t}_\ell = 0$ for all $k \neq \ell$. Hence the center of $\mathrm{n}\widehat{\mathrm{TL}}_N$ is the subalgebra

$$\mathsf{C}_N = \Bbbk \oplus (\mathbf{t}_1\Bbbk[\mathbf{t}_1] \oplus \ldots \oplus \mathbf{t}_{N-1}\Bbbk[\mathbf{t}_{N-1}]) \cong \frac{\Bbbk[\mathbf{t}_1,\dots,\mathbf{t}_{N-1}]}{(\mathbf{t}_k\mathbf{t}_\ell \mid k \neq \ell)}.$$

Proof. i) The action of $\mathbf{t}_k$ on V is the projection onto the $\mathrm{n}\widehat{\mathrm{TL}}_N$-submodule $\Bbbk[q] \otimes \bigwedge^k \Bbbk^N$ followed by multiplication by q. This commutes with the action of every other element of $\mathrm{n}\widehat{\mathrm{TL}}_N$. Since V is a faithful module, $\mathbf{t}_k$ commutes with any element of $\mathrm{n}\widehat{\mathrm{TL}}_N$. As we have seen in Proposition I.2.8.2, any central element c without constant term acts on the summand $\Bbbk[q] \otimes \bigwedge^k \Bbbk^N$ via multiplication by some polynomial $p_k^c \in q\Bbbk[q]$. Once again using the faithfulness of V, we get that $c = \sum_{k=1}^{N-1} p_k^c(\mathbf{t}_k)$.

ii) Note that $\Bbbk[q] \otimes \bigwedge^k \Bbbk^N$ is a free $\Bbbk[q]$-module of rank $\binom{N}{k}$. Since $\mathbf{t}_k$ acts by multiplication with q on that module, the subalgebra of $\mathrm{n}\widehat{\mathrm{TL}}_N$ generated by $\mathbf{t}_k$ must be isomorphic to the polynomial ring $\Bbbk[q]$. Since $a(\hat{\jmath})a(\hat{\imath}) = 0$ for all $\mathrm{J} \neq \mathrm{I}$, in particular for J, I of different length, we get $\mathbf{t}_k\mathbf{t}_\ell = 0$ for $k \neq \ell$, as they consist of pairwise different summands. □

Theorem I.2.8.5 enables us to describe the endomorphism algebra $\mathrm{End}_{\mathrm{n}\widehat{\mathrm{TL}}_N}(\mathsf{W})$ of the space of nontrivial particle configurations $\mathsf{W} := \bigoplus_{k=1}^{N-1} \left(\Bbbk[q] \otimes \bigwedge^k \Bbbk^N\right) \subset \mathsf{V}$. We first observe that on W multiplication by q is given by the action of a central element in C_N, therefore it is justified to speak about $\Bbbk[q]$-linearity of a $\mathrm{n}\widehat{\mathrm{TL}}_N$-endomorphism of W.

I.2.8.6 Lemma. Any $\mathrm{n}\widehat{\mathrm{TL}}_N$-module endomorphism φ of W is $\Bbbk[q]$-linear, that is, we have $\mathrm{End}_{\mathrm{n}\widehat{\mathrm{TL}}_N}(\mathsf{W}) \subset \mathrm{End}_{\Bbbk[q]}(\mathsf{W})$.

Proof. Observe that $\sum_{k=1}^{N-1} \mathbf{t}_k \in \mathrm{n}\widehat{\mathrm{TL}}_N$ acts by multiplication by q on every element in W. Therefore multiplication by q commutes with the application of every $\varphi \in \mathrm{End}_{\mathrm{n}\widehat{\mathrm{TL}}_N}(\mathsf{W})$. □

I.2.8.7 Proposition. The endomorphism algebra $\mathrm{End}_{\mathrm{n}\widehat{\mathrm{TL}}_N}(\mathsf{W})$ is isomorphic to a direct sum of $N-1$ polynomial algebras $\Bbbk[T_1] \oplus \ldots \oplus \Bbbk[T_{N-1}]$.

Proof. The proof is very similar to the one of Proposition I.2.8.2. First we show that $\varphi(v(\mathrm{I}))$ is a $\Bbbk[q]$-linear multiple of $v(\mathrm{I})$ for any $\varphi \in \mathrm{End}_{\mathrm{n}\widehat{\mathrm{TL}}_N}(\mathsf{W})$ and any increasing sequence I. This statement holds if and only if $\pm q\varphi(v(\mathrm{I})) \in \Bbbk[q]\, v(\mathrm{I})$. Indeed, by Lemma I.2.7.2 and Lemma I.2.8.6 we get

$$\pm q\varphi(v(\mathrm{I})) \;=\; \varphi(\pm q v(\mathrm{I})) \;=\; \varphi(a(\hat{\imath})v(\mathrm{I})) \;=\; a(\hat{\imath})\varphi(v(\mathrm{I})) \;\in \Bbbk[q]\, v(\mathrm{I}).$$

Therefore, we can write $\varphi(v(\mathrm{I})) = p_{\mathrm{I}} \cdot v(\mathrm{I})$ for some polynomial $p_{\mathrm{I}} \in \Bbbk[q]$. Note that this implies

$$\mathrm{End}_{\mathrm{n}\widehat{\mathrm{TL}}_N}\left(\bigoplus_{k=1}^{N-1}\left(\Bbbk[q] \otimes \textstyle\bigwedge^k \Bbbk^N\right)\right) \;=\; \bigoplus_{k=1}^{N-1}\left(\mathrm{End}_{\mathrm{n}\widehat{\mathrm{TL}}_N}\left(\Bbbk[q] \otimes \textstyle\bigwedge^k \Bbbk^N\right)\right).$$

What remains is to show that these polynomials only depend on the number of particles in I, in other words there exists $p_k \in \Bbbk[q]$ so that $p_{\mathrm{I}} = p_k$ for all I with $|\mathrm{I}| = k$. Again it suffices to show this for two sequences I, I' of length k which differ in exactly one entry. So say $i_s = i$, $i'_s = i+1$, and $i_\ell = i'_\ell$ for all $\ell \neq s$, for some $1 \leq s \leq k$ and $i \in \mathbb{Z}/N\mathbb{Z}$. When $1 \leq i \leq N-1$,

$$p_{\mathrm{I}'}\, v(\mathrm{I}') \;=\; \varphi(v(\mathrm{I}')) \;=\; \varphi(a_i v(\mathrm{I})) \;=\; a_i\varphi(v(\mathrm{I})) \;=\; a_i(p_{\mathrm{I}}\, v(\mathrm{I})) \;=\; p_{\mathrm{I}}\, v(\mathrm{I}'),$$

and when $i = 0$,

$$\begin{aligned}(-1)^{k-1} q p_{\mathrm{I}'} v(\mathrm{I}') &= (-1)^{k-1} q \varphi(v(\mathrm{I}')) = \varphi(a_0 v(\mathrm{I})) = a_0 \varphi(v(\mathrm{I})) = a_0 (p_{\mathrm{I}} v(\mathrm{I})) \\ &= (-1)^{k-1} q p_{\mathrm{I}} v(\mathrm{I}').\end{aligned}$$

Hence we can write $\varphi = \sum_{k=1}^{N-1} p_k \pi_k$ where π_k is the projection onto $\Bbbk[q] \otimes \bigwedge^k \Bbbk^N$, and we get that

$$\mathrm{End}_{\mathrm{n}\widehat{\mathrm{TL}}_N} \left(\Bbbk[q] \otimes \textstyle\bigwedge^k \Bbbk^N \right) = \Bbbk[T_k],$$

where T_k denotes the multiplication action of the central element $\mathbf{t}_k$, which is indeed a $\mathrm{n}\widehat{\mathrm{TL}}_N$-module endomorphism of W. Thus, $\mathrm{End}_{\mathrm{n}\widehat{\mathrm{TL}}_N}(\mathsf{W})$ is isomorphic to a direct sum of polynomial algebras as claimed. □

I.2.8.8 Remark. The arguments in the proof of Proposition I.2.8.7 remain valid even if we specialize the indeterminate q to some element in $\Bbbk \smallsetminus \{0\}$. In this case, we obtain that the summands $\bigwedge^k \Bbbk^N$ are simple modules and

$$\mathrm{End}_{\mathrm{n}\widehat{\mathrm{TL}}_N} \left(\bigoplus_{k=1}^{N-1} \bigwedge^k \Bbbk^N \right) \cong \Bbbk^{N-1}.$$

For $q = 0$, the situation is more complicated: If q is specialized to zero, the generator a_0 acts by zero on the module. The action of $\mathrm{n}\widehat{\mathrm{TL}}_N$ factorizes over nTL_N and the module $\bigwedge^k \Bbbk^N$ is no longer simple. Instead it has a one-dimensional head spanned by the particle configuration $v(1, \dots, k)$, and any endomorphism is given by choosing an image of this top configuration. It is always possible to map it to itself and to the one-dimensional socle spanned by $v(N - k, \dots, N)$, but in general there are more endomorphisms.

For example, in $\bigwedge^4 \Bbbk^8$, the image of $v(1, 2, 3, 4)$ may be any linear combination of $v(1, 2, 3, 4)$, $v(2, 3, 4, 8)$, $v(3, 4, 7, 8)$, $v(4, 6, 7, 8)$ and $v(5, 6, 7, 8)$, so that $\mathrm{End}_{\mathrm{n}\widehat{\mathrm{TL}}_8} \left(\bigwedge^4 \Bbbk^8 \right)$ is 5-dimensional.

In $\bigwedge^3 \Bbbk^8$, the image of $v(1, 2, 3)$ may be any linear combination of $v(1, 2, 3)$, $v(2, 3, 8)$, $v(3, 7, 8)$ and $v(6, 7, 8)$, so its endomorphism algebra is 4-dimensional. ◊

I.2.9. The affine nilTemperley–Lieb algebra is finitely generated over its center

In this section we prove that $\mathrm{n}\widehat{\mathrm{TL}}_N$ is finitely generated over its center (it is not free over its center, as we will see in Section I.2.13). We show this using a basis of $\mathrm{n}\widehat{\mathrm{TL}}_N$

that is adapted to the problem, but recall the normal form constructed in Section I.2.5 and the labelling thereof from Section I.2.6.

The affine nilTemperley–Lieb algebra is infinite dimensional when $N \geq 3$; however, the following finiteness result holds:

I.2.9.1 Theorem. The algebra $\mathrm{n}\widehat{\mathrm{TL}}_N$ is finitely generated over its center.

Proof. Given an arbitrary monomial $a(\underline{j}) \in \mathrm{n}\widehat{\mathrm{TL}}_N$, we first factor it as $a(\underline{j}') \cdot a(\underline{j}^{(0)})$ in the following way: Take the minimal particle configuration $\mathrm{J} = \{1 \leq j_1 < \ldots < j_k \leq N\}$ on which the monomial $a(\underline{j})$ acts nontrivially, in the sense of Remark I.2.4.3. The monomial $a(\underline{j})$ moves all of the particles by at least one step, because the particle configuration was assumed to be minimal. Using the faithfulness of the representation, we know that we may reorder the monomial $a(\underline{j})$ so that first each particle is moved one step clockwise, and afterwards the remaining particle moves are carried out. Hence, we may choose some factorization $a(\underline{j}) = a(\underline{j}') \cdot a(\underline{j}^{(0)})$, where $\underline{j}^{(0)}$ is a sequence obtained by permuting $j_1, \ldots, j_k$. The remaining particle moves are carried out by $a(\underline{j}')$. In Section I.2.5, this decomposition is explicitly constructed (not using the faithful representation). Next, we want to find an expression of the form

$$a(\underline{j}) \; = \; a_{\mathrm{fin}} \cdot \mathbf{t}_k^n \cdot a(\underline{j}^{(0)}),$$

where a_{fin} is a monomial of some subalgebra ${}^{i}\mathrm{nTL}_N$ of $\mathrm{n}\widehat{\mathrm{TL}}_N$, $\mathbf{t}_k^n$ is in the center of $\mathrm{n}\widehat{\mathrm{TL}}_N$, and $a(\underline{j}^{(0)})$ is the above factor. Here

$${}^{i}\mathrm{nTL}_N = \langle a_0, \ldots, a_{i-1}, a_{i+1}, \ldots, a_{N-1} \rangle \tag{I.2.5}$$

is a copy of the finite nilTemperley–Lieb algebra nTL_N sitting in $\mathrm{n}\widehat{\mathrm{TL}}_N$. To accomplish this, we have to subdivide the action of $a(\underline{j})$ on the particle configuration $\mathrm{J} = \{j_1 < \ldots < j_k\}$ one more time. There are two cases:

i) There is an index i not appearing in $\underline{j}'$: In this case, $a(\underline{j}')$ is an element of ${}^{i}\mathrm{nTL}_N$ and we are done.

ii) All indices appear at least $n \geq 1$ times in $\underline{j}'$: Let us investigate the action of $a(\underline{j}')$ on the particle configuration $v(\mathrm{I}) = a(\underline{j}^{(0)})v(\mathrm{J})$, where $\mathrm{I} = \{j_1 + 1, \ldots, j_k + 1\}$. Note that I is the minimal particle configuration for $a(\underline{j}')$ in the sense of Remark I.2.4.3. Each of the particles in I is moved by $a(\underline{j}')$ to the position of the next particle in the sequence I, because there is no index missing (a missing index is equivalent to a particle being stopped before reaching the position of its successor), before possibly continuing to move along the circle. Again invoking the faithfulness of the representation, we can rewrite $a(\underline{j}') = a(\underline{j}'') \cdot a(\hat{\imath})^n$, with the monomial $a(\hat{\imath})$ from

Lemma I.2.7.2. For maximal n, the remaining factor $a(\underline{j''})$ is an element of ${}^i\mathrm{nTL}_N$ for some i. Observe that $a(\hat{\imath})^n a(\underline{j}^{(0)}) = \mathbf{t}_k^n a(\underline{j}^{(0)})$, which follows immediately from the definition of $\mathbf{t}_k$ and Lemma I.2.7.2.

Therefore, we have shown that

$$a(\underline{j}) \;=\; a(\underline{j'}) \cdot a(\underline{j}^{(0)}) \;=\; a_{\mathrm{fin}} \cdot a(\hat{\imath})^n \cdot a(\underline{j}^{(0)}) \;=\; a_{\mathrm{fin}} \cdot \mathbf{t}_k^n \cdot a(\underline{j}^{(0)}),$$

where $n = 0$ in the first case. Since there is only a finite number of monomials in ${}^0\mathrm{nTL}_N, {}^1\mathrm{nTL}_N, \ldots, {}^{N-1}\mathrm{nTL}_N$ and only finitely many monomials $a(\underline{j}^{(0)})$ such that every index $0, 1, \ldots, N-1$ occurs at most once in the sequence $\underline{j}^{(0)}$, the affine nilTemperley–Lieb algebra is indeed finitely generated over its center. □

I.2.10. An alternative normal form using the center

Motivated by the proof of Theorem I.2.9.1 we introduce another basis of $\mathrm{n}\widehat{\mathrm{TL}}_N$. In this section we use faithfulness of the graphical representation V from Theorem I.2.4.5 and our knowledge of the center C_N of $\mathrm{n}\widehat{\mathrm{TL}}_N$ from Theorem I.2.8.5. Therefore the approach given here is much simpler that the elementary but lengthy computation of a normal form in Sections I.2.5 and I.2.6.

For any two particle configurations with the same number $1 \le k \le N-1$ of particles corresponding to the increasing sequences $\mathrm{I} = \{1 \le i_1 < \ldots < i_k \le N\}$ and $\mathrm{J} = \{1 \le j_1 < \ldots < j_k \le N\}$, there is a monomial in $\mathrm{n}\widehat{\mathrm{TL}}_N$ moving particles at the positions J to the positions I. We require that every particle from J is moved by at least one step, but we do not prescribe explicitly which of the j's is mapped to which of the i's. For $\mathrm{I} \neq \mathrm{J}$, take e_{IJ} to be the monomial such that the power of q in $e_{\mathrm{IJ}} v(\mathrm{J}) = \pm q^\ell v(\mathrm{I})$ is minimal (under the assumption that every particle from J must be moved). By faithfulness of the graphical representation, e_{IJ} is uniquely determined. For $\mathrm{I} = \mathrm{J}$, we have $e_{\mathrm{II}} = a(\hat{\imath})$, the special monomial defined in Section I.2.7, hence $e_{\mathrm{II}} v(\mathrm{I}) = \pm q v(\mathrm{I})$. Observe that one can write $\mathbf{t}_k = \sum_{|\mathrm{I}|=k} e_{\mathrm{II}}$, where the sum runs over all possible increasing sequences I of length k, and that $\mathbf{t}_k^\ell e_{\mathrm{IJ}}$ is a monomial, since all but one summand vanish for $k = |\mathrm{I}|$. The condition that e_{IJ} moves all particles from J by at least one step guarantees that it acts by zero on all particle configurations with fewer particles than $|\mathrm{I}| = |\mathrm{J}|$. For example, when $N = 7$,

$$e_{(2)(1)} = a_1, \quad e_{(0,2)(0,1)} = a_6 a_5 a_4 a_3 a_1 a_2 a_0 a_1.$$

(Note that a_1 moves $v(0,1)$ to $v(0,2)$, but this does not satisfy the requisite property that all the particles must be moved by at least one step.) If we apply the factorization of monomials from Theorem I.2.9.1 to e_{IJ}, the minimality condition implies that $e_{\mathrm{IJ}} = a_{\mathrm{fin}} \cdot 1 \cdot a(\underline{j}^{(0)})$, where if $\mathrm{J} = \{j_1 < \ldots < j_k\}$, then $\underline{j}^{(0)}$ is a sequence obtained by permuting the elements of J.

I.2.10.1 Theorem. The set of monomials

$$\{1\} \ \cup \ \{\mathbf{t}_k^\ell e_{\mathrm{IJ}} \mid \ell \in \mathbb{Z}_{\geq 0},\ 1 \leq |\mathrm{I}| = |\mathrm{J}| = k \leq N-1,\ 1 \leq k \leq N-1\}$$

defines a $\Bbbk$-basis of the affine nilTemperley–Lieb algebra $\mathrm{n}\widehat{\mathrm{TL}}_N$.

Proof. First, observe that $\mathbf{t}_k^\ell e_{\mathrm{IJ}}$ is indeed a monomial since $|\mathrm{I}| = k$. We show that the elements $\mathbf{t}_k^\ell e_{\mathrm{IJ}}$ act $\Bbbk$-linearly independently on the graphical representation $\mathsf{V} = \bigoplus_{k=0}^{N} \left(\Bbbk[q] \otimes \bigwedge^k \Bbbk^N\right)$. By definition, the monomial e_{IJ} acts by zero on summands $\Bbbk[q] \otimes \bigwedge^{k'} \Bbbk^N$ for $k' < |\mathrm{I}|$. On $\Bbbk[q] \otimes \bigwedge^{|\mathrm{I}|} \Bbbk^N$, the matrix representing the action of $\mathbf{t}_k^\ell e_{\mathrm{IJ}}$ relative to the standard basis has exactly one nonzero entry, and this one distinguishes all monomials with the same minimal number of particles $|\mathrm{I}| = |\mathrm{J}|$. From these two observations, the linear independence follows. On the other hand, given any nonzero monomial in $\mathrm{n}\widehat{\mathrm{TL}}_N$, there exists a minimal particle configuration J on which it acts nontrivially. Recording the image particle configuration I and the power of q, we conclude that there is some ℓ so that the element $\mathbf{t}_k^\ell e_{\mathrm{IJ}}$ acts on the minimal particle configuration J in the same way as the given monomial does. The action of a monomial on the minimal particle configuration determines its action on the module V. Due to the faithfulness of this representation (see Theorem I.2.4.5 or Section I.2.6), the proposition follows. □

The following proposition describes the multiplication of the monomials e_{IJ} from the $\Bbbk$-basis in Theorem I.2.10.1.

I.2.10.2 Proposition. The product of basis elements e_{IJ} and e_{KL} for any particle configurations given by I, J, K, L is given by

$$e_{\mathrm{IJ}} \cdot e_{\mathrm{KL}} \ = \ \begin{cases} 0 & \text{if } \exists k \in \mathrm{K} :\ k \notin \mathrm{J} \text{ but } k \in \mathrm{J}^{(0)} \cup \mathrm{J}^{(1)} \cup \ldots \cup \mathrm{J}^{(m-1)} \cup \mathrm{I} \\ 0 & \text{if } \exists j \in \mathrm{J} :\ j \notin \mathrm{K} \text{ but } j \in \mathrm{L} \cup \mathrm{L}^{(0)} \cup \ldots \cup \mathrm{L}^{(n-2)} \cup \mathrm{L}^{(n-1)} \\ \pm \mathbf{t}_{|\mathrm{I}^*|}^{r^*} e_{\mathrm{I}^*\mathrm{L}^*} & \text{else, where } \mathrm{L}^* = \mathrm{L} \cup (\mathrm{J} \smallsetminus (\mathrm{J} \cap \mathrm{K})),\ \mathrm{I}^* = \mathrm{I} \cup (\mathrm{K} \smallsetminus (\mathrm{J} \cap \mathrm{K})), \end{cases}$$

where

$$v(\mathrm{J}^{(s)}) \ = \ a(\underline{j}^{(s)}) \ \ldots \ a(\underline{j}^{(1)}) a(\underline{j}^{(0)}) v(\mathrm{J})$$

for $e_{\mathrm{IJ}} = a(\underline{j}^{(m)}) a(\underline{j}^{(m-1)}) \ \ldots \ a(\underline{j}^{(1)}) a(\underline{j}^{(0)})$ in the normal form from Theorem I.2.5.7. Similarly,

$$v(\mathrm{L}^{(s)}) = a(\underline{l}^{(s)}) \ \ldots \ a(\underline{l}^{(1)}) a(\underline{l}^{(0)}) v(\mathrm{L})$$

for $e_{\mathrm{KL}} = a(\underline{l}^{(n)}) a(\underline{l}^{(n-1)}) \ \ldots \ a(\underline{l}^{(1)}) a(\underline{l}^{(0)})$ in normal form. The power r^* depends on all particle configurations I, J, K and L.

Proof. This can be seen using the graphical representation: The product $e_{\text{IJ}} \cdot e_{\text{KL}}$ is zero if and only if it acts by zero on the particle configuration $\text{L}^* = \text{L} \cup (\text{J} \smallsetminus (\text{J} \cap \text{K}))$. This is the case if e_{KL} acts by zero on L^*, or if e_{IJ} acts by zero on

$$\text{K} \cup \text{J} \;=\; \text{K} \cup (\text{J} \smallsetminus (\text{J} \cap \text{K})) \;=\; e_{\text{KL}} \text{ applied to } \text{L}^*, \text{ if nonzero.}$$

Example I.2.10.3 illustrates the dependence of the power r^* in the product

$$e_{\text{IJ}} \cdot e_{\text{KL}} \;=\; \mathbf{t}_{|\text{I}^*|}^{r^*} e_{\text{I}^*\text{L}^*}$$

from Proposition I.2.10.2 on the particle configurations I, J, K and L.

I.2.10.3 Example. Let $N = 10$. Let $\text{I} = (8)$, $\text{J} = \text{K} = (5)$ and $\text{L} = (3)$. Then $e_{\text{IJ}} \cdot e_{\text{KL}} \;=\; e_{\text{IL}}$. If we replace I by $\text{I}' = (4)$ we obtain $e_{\text{I}'\text{J}} \cdot e_{\text{KL}} \;=\; \mathbf{t}_1^1 e_{\text{I}'\text{L}}$. For $\text{J}' = \text{K}' = (1)$ we have $e_{\text{IJ}'} \cdot e_{\text{K}'\text{L}} \;=\; \mathbf{t}_1^1 e_{\text{IL}}$. ◊

I.2.10.4 Remark. The bases from Section I.2.10 and Section I.2.5 are both labelled by pairs of particle configurations (pairs of increasing sequences) together with a natural number ℓ, see also Section I.2.6.1. The labelling sets agree up to an index shift in the output configuration I and a shift of the natural number ℓ. ◊

I.2.10.5 Remark. From a monomial e_{IJ}, we can read off the sequences

$$\begin{aligned} \text{J} &= \{i \mid \text{ no } a_{i-1} \text{ occurs to the right of } a_i \text{ in the monomial } \; e_{\text{IJ}}\}, \\ \text{I} &= \{i \mid \text{ no } a_i \text{ occurs to the left of } a_{i-1} \text{ in the monomial } \; e_{\text{IJ}}\} \end{aligned}$$ ◊

I.2.11. Embeddings of affine nilTemperley–Lieb algebras

In this section we use the basis constructed in Section I.2.10 and construct embeddings of affine nilTemperley–Lieb algebras $\mathrm{n}\widehat{\mathrm{TL}}_N \subset \mathrm{n}\widehat{\mathrm{TL}}_{N+1}$.

In the proof of Theorem I.2.9.1, we have used the N obvious embeddings of nTL_N into $\mathrm{n}\widehat{\mathrm{TL}}_N$ coming from the N different embeddings of the Coxeter graph A_{N-1} into $\widetilde{\mathsf{A}}_{N-1}$. For finite nilTemperley–Lieb algebras there are obvious embeddings of nTL_N into nTL_{N+1} coming from the embeddings of the Coxeter graphs A_{N-1} into A_N.

Embeddings for affine nilTemperley–Lieb algebras cannot be defined in an obvious way. There are no corresponding embeddings of Coxeter graphs. Instead our embeddings of $\mathrm{n}\widehat{\mathrm{TL}}_N$ into $\mathrm{n}\widehat{\mathrm{TL}}_{N+1}$ correspond to the subdivision of an edge of $\widetilde{\mathsf{A}}_{N-1}$ by inserting a vertex on the edge to obtain $\widetilde{\mathsf{A}}_N$.

I.2.11.1 Theorem. For any number $0 \le m \le N-1$, there is a unital embedding of algebras $\varepsilon_m : \mathrm{n}\widehat{\mathrm{TL}}_N \to \mathrm{n}\widehat{\mathrm{TL}}_{N+1}$ given by

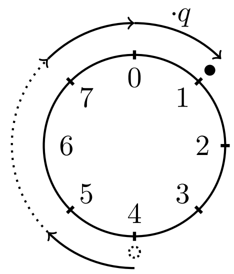
$$a_i \mapsto \begin{cases} a_i & \text{for } 0 \le i \le m-1, \\ a_{m+1}a_m & \text{for } i = m, \\ a_{i+1} & \text{for } m+1 \le i \le N-1. \end{cases} \tag{I.2.6}$$

Let us make some remarks before we prove the theorem.

I.2.11.2 Remark. It is not difficult to see that (I.2.6) defines an algebra homomorphism ε_m from $\mathrm{n}\widehat{\mathrm{TL}}_N$ to $\mathrm{n}\widehat{\mathrm{TL}}_{N+1}$ when $N \ge 3$. Due to the circular nature of the relations, it suffices to check this for ε_0. This amounts to showing the following, since all the other relations are readily apparent. To avoid confusion, we indicate generators of $\mathrm{n}\widehat{\mathrm{TL}}_{N+1}$ in these calculations by $\widetilde{a}_i$:

$$\begin{aligned} (\widetilde{a}_1\widetilde{a}_0)(\widetilde{a}_1\widetilde{a}_0) &= \widetilde{a}_1(\widetilde{a}_0\widetilde{a}_1\widetilde{a}_0) = 0, \\ \widetilde{a}_2(\widetilde{a}_1\widetilde{a}_0)\widetilde{a}_2 &= (\widetilde{a}_2\widetilde{a}_1\widetilde{a}_2)\widetilde{a}_0 = 0, & \widetilde{a}_N(\widetilde{a}_1\widetilde{a}_0)\widetilde{a}_N &= \widetilde{a}_1(\widetilde{a}_N\widetilde{a}_0\widetilde{a}_N) = 0, \\ (\widetilde{a}_1\widetilde{a}_0)\widetilde{a}_2(\widetilde{a}_1\widetilde{a}_0) &= (\widetilde{a}_1\widetilde{a}_2)(\widetilde{a}_0\widetilde{a}_1\widetilde{a}_0) = 0, & (\widetilde{a}_1\widetilde{a}_0)\widetilde{a}_N(\widetilde{a}_1\widetilde{a}_0) &= (\widetilde{a}_1\widetilde{a}_0\widetilde{a}_1)(\widetilde{a}_N\widetilde{a}_0) = 0. \end{aligned}$$ ◊

I.2.11.3 Remark. One can visualize the action of $\varepsilon_m(\mathrm{n}\widehat{\mathrm{TL}}_N) \subset \mathrm{n}\widehat{\mathrm{TL}}_{N+1}$ on the particle configurations on a circle with $N+1$ positions as follows: Except for a_m, all generators of $\mathrm{n}\widehat{\mathrm{TL}}_N$ are mapped to corresponding generators of $\mathrm{n}\widehat{\mathrm{TL}}_{N+1}$. They will act as before, by moving a particle one step clockwise around the circle. Since a_m is mapped by ε_m to the product $\widetilde{a}_{m+1}\widetilde{a}_m$ in $\mathrm{n}\widehat{\mathrm{TL}}_{N+1}$, it will move a particle from m to $m+2$, ignoring position $m+1$, as depicted below.



◊

Figure I.2.11.1.: $\varepsilon_5(\mathrm{n}\widehat{\mathrm{TL}}_7) \subset \mathrm{n}\widehat{\mathrm{TL}}_8$: The action of $\varepsilon_5(a_0a_6a_5a_4) = \widetilde{a}_0\widetilde{a}_7\widetilde{a}_6\widetilde{a}_5\widetilde{a}_4$ on the particle configuration $v(4)$.

I.2.11.4 Remark. Formulas similar to our embeddings appear in [Mak15, Section 3.5] in the construction of $\widetilde{\mathfrak{sl}}_N$-categorical actions from $\widetilde{\mathfrak{sl}}_{N+1}$-categorical actions in the sense of [Mak15, Section 3.4]. They come from embeddings of loop algebras $\widetilde{\mathfrak{sl}}_N \subset \widetilde{\mathfrak{sl}}_{N+1}$, see also [Mak15, Section 3.3] and [Kum02, Section 13.1]. ◊

Proof (Theorem I.2.11.1). Recall the basis given by $\{1\} \cup \{\mathbf{t}_k^{\ell} e_{\mathrm{IJ}} \mid \ell \in \mathbb{Z}_{\geq 0},\ 1 \leq |\mathrm{I}| = |\mathrm{J}| = k \leq N-1\}$ defined in Section I.2.10. We have already noted in Remark I.2.11.2 that ε_m is an algebra homomorphism. Using Remark I.2.11.3, observe that the monomial $e_{\mathrm{IJ}} \in \mathrm{n}\widehat{\mathrm{TL}}_N$ is mapped to a monomial $\tilde{e}_{\mathrm{I'J'}} \in \mathrm{n}\widehat{\mathrm{TL}}_{N+1}$ (tilde again indicates in $\mathrm{n}\widehat{\mathrm{TL}}_{N+1}$), where the new index sets are obtained by $i \mapsto i$ for $0 \leq i \leq m$ and $i \mapsto i+1$ for $m+1 \leq i \leq N-1$. The injectivity follows since distinct basis elements $\left(\sum_{|\mathrm{K}|=k} e_{\mathrm{KK}}\right)^{\ell} \cdot e_{\mathrm{IJ}}$ of $\mathrm{n}\widehat{\mathrm{TL}}_N$ are mapped to distinct basis elements $\left(\sum_{|\mathrm{K'}|=k} \tilde{e}_{\mathrm{K'K'}}\right)^{\ell} \cdot \tilde{e}_{\mathrm{I'J'}}$ of $\mathrm{n}\widehat{\mathrm{TL}}_{N+1}$. □

I.2.11.5 Remark. It is possible to verify this theorem on generators and relations in the language of Section I.2.5 without using the graphical description. The idea is that from a monomial e_{IJ}, we can read off the sequences

$$\begin{aligned} \mathrm{J} &= \{i \mid \text{ no } a_{i-1} \text{ occurs to the right of } a_i \text{ in the monomial } e_{\mathrm{IJ}}\}, \\ \mathrm{I} &= \{i \mid \text{ no } a_i \text{ occurs to the left of } a_{i-1} \text{ in the monomial } e_{\mathrm{IJ}}\} \end{aligned}$$

as in Remark I.2.10.5. Now using Lemma I.2.5.2 one checks that the image of e_{IJ} under ε_m is a nonzero monomial, which must be equal to the monomial $\tilde{e}_{\mathrm{I'J'}}$ determined by

$$\begin{aligned} \{i \mid \text{ no } \widetilde{a}_{i-1} \text{ occurs to the right of } \widetilde{a}_i \text{ in the monomial } \varepsilon_m(e_{\mathrm{IJ}})\} &= \mathrm{J}', \\ \{i \mid \text{ no } \widetilde{a}_i \text{ occurs to the left of } \widetilde{a}_{i-1} \text{ in the monomial } \varepsilon_m(e_{\mathrm{IJ}})\} &= \mathrm{I}'. \end{aligned}$$ ◊

I.2.11.6 Remark. These embeddings work specifically for the affine *nil*Temperley–Lieb algebras but fail for the ordinary Temperley–Lieb algebras. The relation that fails to hold is the braid relation for Temperley–Lieb algebras, i.e. $a_i a_{i\pm1} a_i = a_i$. Interestingly, the relation $a_i^2 = \delta a_i$ is respected for $\delta = 1$. ◊

I.2.12. Classification of simple modules

In this section we have to assume that the ground ring $\Bbbk$ of $n\widehat{TL}_N$ is restricted to be an uncountable algebraically closed field of arbitrary characteristic.

The classification of simple modules for $\mathrm{n}\widehat{\mathrm{TL}}_N$ uses central characters.

I.2.12.1 Definition. Let A be a k-algebra over some field k. An A-module M has central character $\chi : \mathsf{C}(A) \to k$ if χ is an algebra homomorphism such that $cv = \chi(c)v$ for all $c \in \mathsf{C}(A)$ and all $v \in M$. In case χ is the central character of M we denote it by χ_M.

Recall the following fact:

I.2.12.2 Proposition. Let k be an algebraically closed field and A be a k-algebra with $\dim(A) < |k|$. Then every simple A-module has central character.

This is proven e.g. in [CG97, Corollary 8.1.2] for the affine Hecke algebra, and in [Maz10, Theorem 4.7] for $A = \mathcal{U}(\mathfrak{sl}_2)$, compare Section II.1.4. The general statement can be proven analogously.

Let us now turn to the affine nilTemperley–Lieb algebra $\mathrm{n}\widehat{\mathrm{TL}}_N$. *From now on, the ground ring* $\Bbbk$ *of* $n\widehat{TL}_N$ *is an uncountable algebraically closed field.* Denote the category of left $\mathrm{n}\widehat{\mathrm{TL}}_N$-modules by $\mathrm{n}\widehat{\mathrm{TL}}_N\text{–}\mathsf{mod}$. We will use two main facts about $\mathrm{n}\widehat{\mathrm{TL}}_N$ to describe its simple modules: The finiteness of $\mathrm{n}\widehat{\mathrm{TL}}_N$ over its center, and the explicit description of the center $\mathsf{C}_N \cong \frac{\Bbbk[\mathbf{t}_1,\ldots,\mathbf{t}_{N-1}]}{(\mathbf{t}_k\mathbf{t}_\ell | k\neq\ell)}$. Observe that due to the relation $\mathbf{t}_k\mathbf{t}_\ell = 0$ in C_N the only nonzero algebra homomorphisms $\chi : \mathsf{C}_N \to \Bbbk$ are given by the choice of some $\chi(\mathbf{t}_k) = \zeta \in \Bbbk \smallsetminus \{0\}$, $\chi(\mathbf{t}_\ell) = 0$ for all $\ell \neq k$. The following theorem classifies all simple $\mathrm{n}\widehat{\mathrm{TL}}_N$-modules:

I.2.12.3 Theorem. Let $\Bbbk$ be an uncountable algebraically closed field of arbitrary characteristic. Let χ be an algebra homomorphism $\mathsf{C}_N \to \Bbbk$. Then up to isomorphism there is precisely one simple module of $\mathrm{n}\widehat{\mathrm{TL}}_N$ with central character χ.
The simple modules of $\mathrm{n}\widehat{\mathrm{TL}}_N$ are given up to isomorphism by

i) the trivial onedimensional module $\Bbbk$ with trivial central character,
ii) the $\binom{N}{k}$-dimensional module $\bigwedge^k \Bbbk^N$ with central character $\chi(\mathbf{t}_k) \in \Bbbk \smallsetminus \{0\}$, $\chi(\mathbf{t}_\ell) = 0$ for all $\ell \neq k$.

Proof. Thanks to our assumption that $\Bbbk$ is an uncountable algebraically closed field we know that every simple module of $\mathrm{n}\widehat{\mathrm{TL}}_N$ has central character.

Let us first consider the case $\chi = 0$. Given any simple $\mathrm{n}\widehat{\mathrm{TL}}_N$-module M with $\chi_M = 0$, we have that M is simple as $\mathrm{n}\widehat{\mathrm{TL}}_N/\langle\mathbf{t}_1,\ldots,\mathbf{t}_{N-1}\rangle$-module. This quotient algebra is graded by the length of monomials (since $\mathsf{C}_N \subset \mathrm{n}\widehat{\mathrm{TL}}_N$ is homogeneously generated) and furthermore finite dimensional by Theorem I.2.9.1. Its degree 0 component equals $\Bbbk$. Its Jacobson radical is given by all positively graded elements $\left(\mathrm{n}\widehat{\mathrm{TL}}_N/\langle\mathbf{t}_1,\ldots,\mathbf{t}_{N-1}\rangle\right)_{>0}$. Since the simple modules over an algebra can be identified with the simple modules over the quotient with respect to the Jacobson radical, we only need to determine all simple modules of the quotient $\left(\mathrm{n}\widehat{\mathrm{TL}}_N/\langle\mathbf{t}_1,\ldots,\mathbf{t}_{N-1}\rangle\right) / \left(\mathrm{n}\widehat{\mathrm{TL}}_N/\langle\mathbf{t}_1,\ldots,\mathbf{t}_{N-1}\rangle\right)_{>0}$, which is $\Bbbk$. By the Artin-Wedderburn Theorem [Lam01, Chapter 1.3] (or directly by the lack of nontrivial maximal ideals in $\Bbbk$), the only simple module of $\Bbbk$ and hence of $\mathrm{n}\widehat{\mathrm{TL}}_N/\langle\mathbf{t}_1,\ldots,\mathbf{t}_{N-1}\rangle$ is the one-dimensional (trivial) module $\Bbbk$.

Now we turn to the case $\chi \neq 0$, i.e. χ is given by $\chi(\mathbf{t}_k) = \zeta \in \Bbbk \setminus \{0\}$, $\chi(\mathbf{t}_\ell) = 0$ for all $\ell \neq k$. In this case we want to form the localisation $\mathrm{n}\widehat{\mathrm{TL}}_N[\mathbf{t}_k^{-1}]$. There are two possibilities to convince oneself that one can localise $\mathrm{n}\widehat{\mathrm{TL}}_N$ with respect to the multiplicative subset generated by $\mathbf{t}_k$: Either one checks that localisation with respect to a multiplicative subset for commutative rings as in [AM69, Chapter 3] can be easily imitated for multiplicative central subsets in an arbitrary ring. Or one applies Ore localisation as discussed in [Lam99, Chapter 4, Section 10] to the noncommutative ring $\mathrm{n}\widehat{\mathrm{TL}}_N$ and the (right and left) denominator set $\{1, \mathbf{t}_k, \mathbf{t}_k^2, \ldots\}$, where the Ore conditions are automatically satisfied since $\{1, \mathbf{t}_k, \mathbf{t}_k^2, \ldots\}$ is central (see [Lam99, (10.15)]). The resulting right and left ring of fractions is unique up to unique isomorphism. We denote the localisation of $\mathrm{n}\widehat{\mathrm{TL}}_N$ with respect to the set $\{1, \mathbf{t}_k, \mathbf{t}_k^2, \ldots\}$ by $\mathrm{n}\widehat{\mathrm{TL}}_N[\mathbf{t}_k^{-1}]$.

Now our goal is to show that the functor $\mathrm{n}\widehat{\mathrm{TL}}_N[\mathbf{t}_k^{-1}] \otimes_{\mathrm{n}\widehat{\mathrm{TL}}_N} -$ induces a bijection

$$\begin{aligned} &\{\text{simple } \mathrm{n}\widehat{\mathrm{TL}}_N\text{-modules with } \mathbf{t}_k \text{ acts by } \zeta \neq 0\}/\cong \\ \overset{1:1}{\longleftrightarrow}\ &\{\text{simple } \mathrm{n}\widehat{\mathrm{TL}}_N[\mathbf{t}_k^{-1}]\text{-modules with } \mathbf{t}_k \text{ acts by } \zeta \neq 0\}/\cong \end{aligned} \tag{I.2.7}$$

that preserves the $\Bbbk$-dimensions of the simple modules. First of all, localisation is exact, see [Lam99, Exercise 10.18] or [Mat89, Theorem 4.5] for the commutative version of the statement. Therefore $\mathrm{n}\widehat{\mathrm{TL}}_N[\mathbf{t}_k^{-1}] \otimes_{\mathrm{n}\widehat{\mathrm{TL}}_N} L$ is a simple $\mathrm{n}\widehat{\mathrm{TL}}_N[\mathbf{t}_k^{-1}]$-module for any simple $\mathrm{n}\widehat{\mathrm{TL}}_N$-module L.

Let L be a simple $\mathrm{n}\widehat{\mathrm{TL}}_N$-module so that $\mathbf{t}_k$ acts by $\zeta \neq 0$ on L. In this case the dimension of L is preserved under localisation since $L \cong \mathrm{n}\widehat{\mathrm{TL}}_N[\mathbf{t}_k^{-1}] \otimes_{\mathrm{n}\widehat{\mathrm{TL}}_N} L$ even as $\mathrm{n}\widehat{\mathrm{TL}}_N[\mathbf{t}_k^{-1}]$-modules. The $\mathrm{n}\widehat{\mathrm{TL}}_N[\mathbf{t}_k^{-1}]$-action on L is given by letting $\mathbf{t}_k^{-1}$ act by ζ^{-1}. More precisely, by the universal property of localisation ([Lam99, Proposition 9.2]), any ring homomorphism from $\mathrm{n}\widehat{\mathrm{TL}}_N$ that maps $\{1, \mathbf{t}_k, \mathbf{t}_k^2, \ldots\}$ into the units of the codomain ring factors uniquely over $\mathrm{n}\widehat{\mathrm{TL}}_N[\mathbf{t}_k^{-1}]$. In this way we can see a simple $\mathrm{n}\widehat{\mathrm{TL}}_N$-module L as $\mathrm{n}\widehat{\mathrm{TL}}_N[\mathbf{t}_k^{-1}]$-module. The natural map $L \to \mathrm{n}\widehat{\mathrm{TL}}_N[\mathbf{t}_k^{-1}] \otimes_{\mathrm{n}\widehat{\mathrm{TL}}_N} L$ is injective since its kernel is given by $\{m \in L \mid \mathbf{t}_k^n m = \zeta^n m = 0 \text{ for some } n\} = \{0\}$. It is also surjective since $\mathbf{t}_k^{-1} \otimes_{\mathrm{n}\widehat{\mathrm{TL}}_N} m = 1 \otimes_{\mathrm{n}\widehat{\mathrm{TL}}_N} \zeta^{-1} m$ for all $m \in L$.

Vice versa, any simple $\mathrm{n}\widehat{\mathrm{TL}}_N[\mathbf{t}_k^{-1}]$-module L' so that $\mathbf{t}_k$ acts by $\zeta (\neq 0)$ on L' is naturally an $\mathrm{n}\widehat{\mathrm{TL}}_N$-module. As such, L' is simple: Let $0 \to N' \to L' \to M' \to 0$ be a short exact sequence of $\mathrm{n}\widehat{\mathrm{TL}}_N$-modules. The central element $\mathbf{t}_k$ acts by ζ on N', L' and M'. Now apply the exact functor $\mathrm{n}\widehat{\mathrm{TL}}_N[\mathbf{t}_k^{-1}] \otimes_{\mathrm{n}\widehat{\mathrm{TL}}_N} -$. We have $0 = \mathrm{n}\widehat{\mathrm{TL}}_N[\mathbf{t}_k^{-1}] \otimes_{\mathrm{n}\widehat{\mathrm{TL}}_N} N' \cong N'$ or $0 = \mathrm{n}\widehat{\mathrm{TL}}_N[\mathbf{t}_k^{-1}] \otimes_{\mathrm{n}\widehat{\mathrm{TL}}_N} M' \cong M'$, where we use again that the natural map is an isomorphism.

These two maps are inverses of each other on isomorphism classes of simples, and we get the bijection from Equation (I.2.7).

Next we show that $\mathrm{n}\widehat{\mathrm{TL}}_N[\mathbf{t}_k^{-1}] \cong \mathrm{End}_{\Bbbk}\left(\Bbbk[q^{\pm 1}] \otimes \bigwedge^k \Bbbk^N\right)$:
Consider the composition of the $\Bbbk$-linear embedding of $\mathrm{n}\widehat{\mathrm{TL}}_N$ into the $\Bbbk[q]$-algebra of endomorphisms of its faithful module $\mathsf{V} = \bigoplus_{k=0}^{N} \left(\Bbbk[q] \otimes \bigwedge^k \Bbbk^N\right)$ with the $\Bbbk$-linear embedding of $\mathrm{End}_{\Bbbk[q]}(\mathsf{V}) \hookrightarrow \mathrm{End}_{\Bbbk[q^{\pm 1}]}(\Bbbk[q^{\pm}1] \otimes_{\Bbbk[q]} \mathsf{V})$. Then

$$\mathrm{n}\widehat{\mathrm{TL}}_N \hookrightarrow \mathrm{End}_{\Bbbk[q^{\pm 1}]}(\Bbbk[q^{\pm}1] \otimes_{\Bbbk[q]} \mathsf{V})$$

is a $\Bbbk[\mathbf{t}_k]$-linear map where $\mathbf{t}_k$ acts on the affine nilTemperley–Lieb algebra $\mathrm{n}\widehat{\mathrm{TL}}_N$ by (left) multiplication and on the endomorphism space $\mathrm{End}_{\Bbbk[q^{\pm 1}]}(\Bbbk[q^{\pm}1] \otimes_{\Bbbk[q]} \mathsf{V})$ by multiplication with q times the projection map onto the summand $\Bbbk[q^{\pm 1}] \otimes_{\Bbbk} \bigwedge^k \Bbbk^N$.

Localisation with respect to $\mathbf{t}_k$ gives an embedding of $\mathrm{n}\widehat{\mathrm{TL}}_N[\mathbf{t}_k^{-1}]$ into the $\Bbbk[q^{\pm 1}]$-algebra of endomorphisms of $\bigoplus_{k=0}^{N} \left(\Bbbk[q^{\pm 1}] \otimes \bigwedge^k \Bbbk^N\right)$. Since $\mathbf{t}_k$ acts by zero on $\Bbbk[q^{\pm 1}] \otimes \bigwedge^\ell \Bbbk^N$ for all $\ell \neq k$, we obtain an embedding

$$\mathrm{n}\widehat{\mathrm{TL}}_N[\mathbf{t}_k^{-1}] \;\hookrightarrow\; \mathrm{End}_{\Bbbk}\left(\Bbbk[q^{\pm 1}] \otimes \bigwedge^k \Bbbk^N\right).$$

This map is an isomorphism – surjectivity follows from the fact that the basis element e_{IJ} is mapped to q^m times the elementary matrix E_{IJ}, where I labels the basis element $v(\mathrm{I})$ of $\bigwedge^k \Bbbk^N$ and m equals the number of appearances of a_0 in e_{IJ}. In other words, $\mathbf{t}_k^{-m} e_{\mathrm{IJ}}$ is mapped to the elementary matrix E_{IJ}.

Finally it suffices to observe that the only simple module of the $\Bbbk[q^{\pm 1}]$-linear matrices of size $\binom{N}{k} \times \binom{N}{k}$ where $q\mathrm{Id}$ acts by multiplication with $\zeta \neq 0$ is the vector representation $\Bbbk^{\binom{N}{k}}$. □

I.2.13. The affine nilTemperley–Lieb algebra is not free over its center

In this section we have to assume that the ground ring $\Bbbk$ of $\mathrm{n}\widehat{TL}_N$ is restricted to be an uncountable algebraically closed field of arbitrary characteristic.

I.2.13.1 Theorem. The affine nilTemperley–Lieb algebra is not free (as a module) over its center.

Proof. Recall that the algebra $\mathrm{n}\widehat{\mathrm{TL}}_N$ is finitely generated over its center by Theorem I.2.9.1. If $\mathrm{n}\widehat{\mathrm{TL}}_N$ was freely generated over its center, its rank as a module over the center would agree with its rank as a module over any localisation of the center since localisation commutes with direct sums, see [Lam99, Exercise 10.18] or [Mat89, Theorem 4.4] for the commutative algebra version of this statement. In particular, its rank as a module over any localisation of the center would be constant. In the proof of Theorem I.2.12.3 we have seen that the localisation with respect to $\mathbf{t}_k$ is a matrix algebra over $\mathsf{C}\left(\mathrm{n}\widehat{\mathrm{TL}}_N\right)[\mathbf{t}_k^{-1}] = \Bbbk[\mathbf{t}_k^{\pm 1}]$ of size $\binom{N}{k}$ for $0 \leq k \leq N$. Hence the rank is not constant and $\mathrm{n}\widehat{\mathrm{TL}}_N$ is not freely generated over the center. □

I.2.14. Affine cellularity of the affine nilTemperley–Lieb algebra

In this section we assume that the ground ring $\Bbbk$ of $n\widehat{TL}_N$ is restricted to be an uncountable algebraically closed field of arbitrary characteristic.

In [KX12] Koenig and Xi introduce the notion of affine cellular algebras generalizing the concept of cellular algebras from [GL96]. One of the motivating examples is given by affine Temperley–Lieb algebras that can be equipped with an affine cellular structure [KX12, Proposition 2.5]. It would be interesting to know whether this is the case for the affine nilTemperley–Lieb algebra $\mathrm{n}\widehat{\mathrm{TL}}_N$ as well. In this section we first recall the basic properties of affine cellular algebras from [KX12]. Then we discuss possible approaches to the construction of an affine cellular structure for $\mathrm{n}\widehat{\mathrm{TL}}_N$ (and their limitations).

By definition, a $\Bbbk$-algebra A together with a $\Bbbk$-linear anti-algebra involution i is called affine cellular if

$$A \;\cong\; J_1' \oplus \ldots \oplus J_n' \quad \text{as } \Bbbk\text{-vector spaces} \tag{I.2.8}$$

for some $J_1', \ldots, J_n'$ stable under i so that $J_k := \bigoplus_{\ell=1}^{k} J_\ell'$ is a twosided ideal of A, and $n \geq 1$. The subquotient J_k/J_{k-1} is required to be an affine cell ideal in A/J_{k-1} for $1 \leq k \leq n$, where $J_0 := \{0\}$. This means that

$$J_k/J_{k-1} \;\cong\; V_k \otimes_{\Bbbk} B_k \otimes_{\Bbbk} V_k \quad \text{as } (A/J_{k-1} - A/J_{k-1})\text{-bimodules}, \tag{I.2.9}$$

where B_k is a quotient of a polynomial ring in finitely many variables and V_k is some finite-dimensional $\Bbbk$-vector space. The tensor product $V_k \otimes_{\Bbbk} B_k$ is equipped with a left A/J_{k-1}-action that commutes with the right multiplication with elements in B_k. This also induces a right A/J_{k-1}-action on $B_k \otimes_{\Bbbk} V_k$ by twisting with the anti-algebra involution i (to be precise, i is identified with the induced map on the quotient A/J_{k-1}), which commutes with the left multplication action of B_k. Under the identification (I.2.9), the anti-algebra involution i is required to be of the form

$$i:\; v \otimes b \otimes v' \;\mapsto\; v' \otimes \sigma_k(b) \otimes v \quad \text{for } v, v' \in V_k,\; b \in B_k, \tag{I.2.10}$$

where σ_k is some involution of B_k.

The monomial basis of $\mathrm{n}\widehat{\mathrm{TL}}_N$ from Section I.2.10 suggests that one can equip $\mathrm{n}\widehat{\mathrm{TL}}_N$ with an affine cellular structure in the following natural way. Namely, one can decompose $\mathrm{n}\widehat{\mathrm{TL}}_N \cong J_1' \oplus \ldots \oplus J_N'$ as a $\Bbbk$-vector space with

$$\begin{array}{rcll} J_k' &=& \{\mathbf{t}_{N-k}^{\ell} e_{\mathrm{IJ}} \mid |\mathrm{I}| = |\mathrm{J}| = N-k\} & \text{for } 1 \le k \le N, \\ J_k &=& \{\mathbf{t}_{N-k}^{\ell} e_{\mathrm{IJ}} \mid |\mathrm{I}| = |\mathrm{J}| \ge N-k\} & \text{for } 1 \le k \le N, \\ V_k &=& \mathrm{span}_{\Bbbk}\{\mathrm{I} \mid |\mathrm{I}| = N-k\} & \text{for } 1 \le k \le N, \\ B_k &=& \Bbbk[\mathbf{t}_{N-k}] & \text{for } 1 \le k \le N, \end{array}$$

so that there is an isomorphism of $\Bbbk$-vector spaces $J_k/J_{k-1} \cong V_k \otimes_{\Bbbk} B_k \otimes_{\Bbbk} V_k$ given by

$$\mathbf{t}_{N-k}^{\ell} e_{\mathrm{IJ}} \mapsto \mathrm{I} \otimes \mathbf{t}_{N-k}^{\ell} \otimes \mathrm{J}.$$

For $k = N$, we identify $e_{\varnothing\varnothing} = \mathbf{t}_0 = 1$, so we have $J_N' \cong \Bbbk$. An involution that matches these definitions is given by

$$i\colon\ \mathrm{n}\widehat{\mathrm{TL}}_N \to \mathrm{n}\widehat{\mathrm{TL}}_N \text{ defined by } \mathbf{t}_k^{\ell} e_{\mathrm{IJ}} \mapsto \mathbf{t}_k^{\ell} e_{\mathrm{JI}}.$$

A left $\mathrm{n}\widehat{\mathrm{TL}}_N/J_{k-1}$-action on $V_k \otimes_{\Bbbk} B_k$ can be given by $\mathbf{t}_{N-k}^{d} e_{\mathrm{JI}}(\mathrm{I} \otimes \mathbf{t}_{N-k}^{\ell}) = \mathrm{J} \otimes \mathbf{t}_{N-k}^{\ell+d+r^*}$ for $N-k = |\mathrm{J}| = |\mathrm{I}|$ and r^* is the power of $\mathbf{t}_{N-k}$ from Proposition I.2.10.2. It naturally commutes with the right B_k-action since $B_k = \Bbbk[\mathbf{t}_{N-k}]$ is central.

But there are several problems about this definition: The involution i fails to be an anti-algebra homomorphism, as can be seen from the following example for $N = 5$, where we have $i(e_{\{4\}\{3\}} e_{\{3\}\{2\}}) = i(e_{\{4\}\{2\}}) = e_{\{2\}\{4\}}$, but $i(e_{\{3\}\{2\}}) i(e_{\{4\}\{3\}}) = e_{\{2\}\{3\}} e_{\{3\}\{4\}} = \mathbf{t}_1^1 e_{\{2\}\{4\}}$. Likewise, products do not obey the multiplication rule for affine cellular algebras from [KX12, Proposition 2.2]: The power ℓ'' of $\mathbf{t}_{N-k}$ in the product

$$(\mathrm{I} \otimes \mathbf{t}_{N-k}^{\ell} \otimes \mathrm{J})(\mathrm{I}' \otimes \mathbf{t}_{N-k}^{\ell'} \otimes \mathrm{J}') = \mathrm{I} \otimes \delta_{\mathrm{JI}'} \mathbf{t}_{N-k}^{\ell+\ell'+\ell''} \otimes \mathrm{J}' \in V_k \otimes B_k \otimes V_k$$

depends on all of I, $\mathrm{J} = \mathrm{I}'$ and J, see Example I.2.10.3. According to the multiplication rule for affine cellular algebras, an additional factor in B_k (apart from $\mathbf{t}_{N-k}^{\ell+\ell'}$) may only depend on J and I'.

Let us discuss another approach to defining an affine cellular structure on $\mathrm{n}\widehat{\mathrm{TL}}_N$, based on the labelling set of monomials from Proposition I.2.6.4, see also Section I.2.5 and Remark I.2.10.4. There we have seen that a monomial $a(\underline{j})$ can be uniquely identified by the triple $(\mathrm{I}_{\underline{j}}^{\mathrm{in}}, \mathrm{I}_{\underline{j}}^{\mathrm{out}}, \ell_{\underline{j}})$ where

$$\begin{array}{rcl} \mathrm{I}_{\underline{j}}^{\mathrm{in}} &=& \{i \in \{0,1,\ldots,N-1\} \mid \text{no } i-1 \text{ to the right of } i \text{ in } \underline{j}\} \\ \mathrm{I}_{\underline{j}}^{\mathrm{out}} &=& \{i \in \{0,1,\ldots,N-1\} \mid \text{no } i+1 \text{ to the left of } i \text{ in } \underline{j}\} \\ \ell_{\underline{j}} &=& \text{the number of zeros in } \underline{j}. \end{array}$$

We define an affine cellular algebra by the following data (see [KX12, Proposition 2.3]):

$$\begin{aligned} V_k &= \mathrm{span}_{\Bbbk}\{\textsc{i} \mid |\textsc{i}| = N-k\} && \text{for } 1 \le k \le N-1, \\ B_k &= \Bbbk[q] && \text{for } 1 \le k \le N-1, \\ V_N &= \Bbbk, \\ B_N &= \Bbbk, \\ A_+ &= \bigoplus_{k=1}^{N} V_k \otimes B_k \otimes V_k, \\ i:\ \textsc{i} \otimes q^\ell \otimes \textsc{j} &\mapsto \textsc{j} \otimes q^\ell \otimes \textsc{i}. \end{aligned}$$

The multiplication on A_+ is defined for $\textsc{i}, \textsc{j} \in V_k$, $\textsc{i}', \textsc{j}' \in V_{k'}$ and $\ell, \ell' \in \mathbb{Z}_{\ge 0}$ by

$$(\textsc{i} \otimes q^\ell \otimes \textsc{j})(\textsc{i}' \otimes q^{\ell'} \otimes \textsc{j}') = \begin{cases} \textsc{i}^* \otimes q^{\ell+\ell'} \otimes \textsc{j}^* & \text{if } e_{\textsc{ij}} e_{\textsc{i}'\textsc{j}'} \neq 0, \\ 0 & \text{else,} \end{cases}$$

where $\textsc{i}^*$, $\textsc{j}^*$ are defined precisely like in Proposition I.2.10.2 by

$$\textsc{i}^* = \textsc{i} \cup (\textsc{i}' \smallsetminus (\textsc{j} \cap \textsc{i}')), \qquad \textsc{j}^* = \textsc{j}' \cup (\textsc{j} \smallsetminus (\textsc{j} \cap \textsc{i}')).$$

In particular, the induced multiplication on the subquotient $V_k \otimes B_k \otimes V_k$ of A_+ is given by

$$(\textsc{i} \otimes q^\ell \otimes \textsc{j})(\textsc{i}' \otimes q^{\ell'} \otimes \textsc{j}') = \textsc{i} \otimes \delta_{\textsc{ji}'} q^{\ell+\ell'} \otimes \textsc{j}'. \tag{I.2.11}$$

I.2.14.1 Lemma. There is a proper embedding of algebras

$$\mathrm{n}\widehat{\mathrm{TL}}_N \subset A_+, \qquad \mathbf{t}_k^\ell e_{\textsc{ij}} \mapsto \textsc{i} \otimes q^{\ell + r_{\textsc{ij}}} \otimes \textsc{j},$$

where $r_{\textsc{ij}}$ is the number of generators a_0 appearing in the basis monomial $e_{\textsc{ij}}$.

Proof. This map is a $\Bbbk$-linear embedding of vector spaces. By definition of the multiplication, it is an algebra homomorphism. An element of the form $\textsc{i} \otimes q^r \otimes \textsc{j} \in A_+$ with $r < r_{\textsc{ij}}$ does not have a preimage in $\mathrm{n}\widehat{\mathrm{TL}}_N$. □

In this approach, we find an affine cellular algebra A_+, but the problem is that the embedding $\mathrm{n}\widehat{\mathrm{TL}}_N \subset A_+$ is not surjective.

Let us now modify this approach slightly to obtain another affine cellular algebra A_-: As a $\Bbbk$-vector space, we define $A_- = A_+$, and also the involution i remains the same. We define a new multiplication on A_- by

$$(\textsc{i} \otimes q^\ell \otimes \textsc{j})(\textsc{i}' \otimes q^{\ell'} \otimes \textsc{j}') = \begin{cases} \textsc{i}^* \otimes q^{|\textsc{j} \cap \textsc{i}'| + \ell + \ell'} \otimes \textsc{j}^* & \text{if } e_{\textsc{ij}} e_{\textsc{i}'\textsc{j}'} \neq 0, \\ 0 & \text{else,} \end{cases}$$

where we use the notation from above. For the induced multiplication on the subquotient $V_k \otimes B_k \otimes V_k$ of A_- we obtain

$$(\mathrm{I} \otimes q^{\ell} \otimes \mathrm{J})(\mathrm{I}' \otimes q^{\ell'} \otimes \mathrm{J}') \;=\; \mathrm{I} \otimes (\delta_{\mathrm{JI}'} q^{N-k}) q^{\ell+\ell'} \otimes \mathrm{J}', \tag{I.2.12}$$

I.2.14.2 Lemma. There is a proper embedding of algebras

$$A_- \;\subset\; \mathrm{n}\widehat{\mathrm{TL}}_N, \qquad \mathrm{I} \otimes q^{\ell} \otimes \mathrm{J} \;\mapsto\; \mathbf{t}_k^{\ell+k-r_{\mathrm{IJ}}} e_{\mathrm{IJ}},$$

where r_{IJ} is the number of generators a_0 appearing in the basis monomial e_{IJ}.

Proof. This map is a $\Bbbk$-linear embedding of vector spaces. By definition of the multiplication, it is an algebra homomorphism. A basis element of the form $e_{\mathrm{IJ}} \in \mathrm{n}\widehat{\mathrm{TL}}_N$ with $r_{\mathrm{IJ}} < k$ does not have a preimage in A_-. □

We obtain that the affine nilTemperley–Lieb algebra is sandwiched between two affine cellular algebras:

I.2.14.3 Corollary. There are proper inclusions of algebras

$$A_- \;\subset\; \mathrm{n}\widehat{\mathrm{TL}}_N \;\subset\; A_+,$$

where A_-, A_+ are affine cellular algebras that are isomorphic as $\Bbbk$-vector spaces.

Proof. This is precisely the statement of Lemma I.2.14.1 and Lemma I.2.14.2. □

The reason why we work in this section over an uncountable algebraically closed field $\Bbbk$ (instead of an arbitrary field or commutative ring) is that we will now compare the classifications of simple modules for the algebras A_- and A_+ with the classification of simple modules of $\mathrm{n}\widehat{\mathrm{TL}}_N$ from Section I.2.12.

I.2.14.4 Proposition. The isomorphism classes of simple modules of the affine cellular algebra A_- are labelled by the set

$$\{(N,0)\} \;\cup\; \{(k,\mathfrak{m}) \mid 1 \le k \le N-1,\ \mathfrak{m} \subset \Bbbk[q] \text{ maximal ideal with } q \notin \mathfrak{m}\}.$$

The isomrphism classes of simple modules of the affine cellular algebra A_+ are labelled by the set

$$\{(N,0)\} \;\cup\; \{(k,\mathfrak{m}) \mid 1 \le k \le N-1,\ \mathfrak{m} \subset \Bbbk[q] \text{ maximal ideal}\}.$$

Proof. This is an immediate consequence of [KX12, Theorem 3.12] together with the multiplication laws (I.2.11), (I.2.12) for subquotients of A_+ and A_-. □

It is not clear yet whether the affine nilTemperley–Lieb algebra $\mathrm{n}\widehat{\mathrm{TL}}_N$ can be equipped with an affine cellular structure or not. Nevertheless, $\mathrm{n}\widehat{\mathrm{TL}}_N$ contains a slightly smaller affine cellular algebra, so that the classifications of simple modules for both algebras agree:

I.2.14.5 Corollary. The labelling sets of the isomorphism classes of simple modules agree for $\mathrm{n}\widehat{\mathrm{TL}}_N$ and the affine cellular algebra A_-.

Proof. This follows from Proposition I.2.14.4 and Theorem I.2.12.3. □

I.3. The plactic and the partic algebra

This chapter is devoted to the plactic algebra and its action on bosonic particle configurations. It turns out that this action factors over a quotient algebra that we call partic algebra, whose induced action on bosonic particle configurations is faithful.

The chapter is subdivided as follows: In Section I.3.1 we recall the definition of the classical and affine plactic algebra, and we give a short overview over the literature so far. We dedicate the following sections to the classical case: In Section I.3.2 we discuss the action of the classical plactic algebra on classical bosonic particle configurations, and we define a quotient of the classical plactic algebra named partic algebra. In Section I.3.3 we construct a normal form of the monomials in the partic algebra. In Section I.3.4 we discuss the action of the classical plactic and the partic algebra on bosonic particle configurations, and we prove faithfulness of the action of the partic algebra. In Section I.3.5 we describe the center of the partic algebra.

Finally, in Section I.3.6 we turn to the affine case. We define the affine partic algebra and we consider its action on affine bosonic particle configurations. This is substantially harder to understand than the classical case, in particular we find a new type of relations.

I.3.1. The classical and the affine plactic algebra

Let $\Bbbk$ be a field. Similar to Chapter I.2 our results hold over an arbitrary unitary associative ring, see the discussion in Remark I.2.1.3, but for simplicity we choose to work over a field.

I.3.1.1 Definition. Define the (local) plactic algebra $\mathcal{P}_N$ to be the unital associative $\Bbbk$-algebra generated by $a_1, \ldots, a_{N-1}$ subject to the plactic relations

$$a_i a_j = a_j a_i \qquad \text{for } |i-j| > 1, \tag{I.3.1}$$

$$a_i a_{i-1} a_i = a_i a_i a_{i-1} \qquad \text{for } 2 \le i \le N-1, \tag{I.3.2}$$

$$a_i a_{i+1} a_i = a_{i+1} a_i a_i \qquad \text{for } 1 \le i \le N-2. \tag{I.3.3}$$

In order to distinguish $\mathcal{P}_N$ from the affine plactic algebra defined in Definition I.3.1.2 we refer to it as classical plactic algebra.

An affine version of the plactic algebra can be obtained by a very similar construction, except that the indices of the generators are now read modulo N.

I.3.1.2 Definition. Define the affine plactic algebra $\widehat{\mathcal{P}}_N$ to be the unital associative $\Bbbk$-algebra generated by $a_0, a_1, \dots, a_{N-1}$ subject to the plactic relations

$$\begin{aligned} a_i a_j &= a_j a_i && \text{for } i - j \neq \pm 1 \mod N, && \text{(I.3.4)} \\ a_i a_{i-1} a_i &= a_i a_i a_{i-1} && \text{for } i, i-1 \in \mathbb{Z}/N\mathbb{Z}, && \text{(I.3.5)} \\ a_i a_{i+1} a_i &= a_{i+1} a_i a_i && \text{for } i, i+1 \in \mathbb{Z}/N\mathbb{Z}. && \text{(I.3.6)} \end{aligned}$$

The plactic relations go back to Lascoux and Schützenberger [LS81]. They study the monoid defined by the "plaxic relations" (in the original, "plaxique" or "a placche") (I.3.2), (I.3.3) and the non-local Knuth relation, a slightly weaker commutativity relation $(a_i a_j)a_k = (a_j a_i)a_k$, $a_k(a_i a_j) = a_k(a_j a_i)$ for $i < k < j$ (in particular for $|i - j| > 1$). This monoid is isomorphic to the monoid of semistandard Young tableaux with entries $1, \dots, N-1$ (and multiplication defined by row bumping) by reading off the entries of a tableau from left to right and bottom to top, see [Ful97, Section 2.1] for the details.

The name local plactic algebra for the algebra defined by the relations (I.3.1), (I.3.2) and (I.3.3) goes back to [FG98] due to the additional "local" commutativity relation (I.3.1). Fomin and Greene develop a theory of Schur functions in noncommutative variables that applies in particular to the (local) plactic algebra (and to the nilTemperley-Lieb algebra), see [FG98, Example 2.6], including a generalized Littlewood-Richardson rule for Schur functions defined over the plactic algebra. The plactic algebra acts on Young diagrams by Schur operators, i.e. a_i adds a box in the i-th column if possible, and otherwise maps the diagram to zero [Fom95].

In [KS10], the plactic algebra appears in the study of bosonic particle configurations in the finite as well as in the affine case that we discuss in Sections I.3.4 and I.3.6. Schur functions in the generators of the affine plactic algebra are defined, using Bethe Ansatz techniques to show that they are well-defined. One can identify bosonic particle configurations with Young diagrams, then the operator a_i acts by adding a box in the $(i+1)$st row of the Young diagram. Up to an index shift and switching rows and columns, in the finite case this is the same as the action on Young diagrams by Schur operators from [FG98].

The monoid defined by the plactic relations (I.3.1), (I.3.2), (I.3.3) appears as a Hall monoid or "quantic monoid" of type A_{N-1} in [Rei01], [Rei02]: Reineke defines the structure of a monoid on isomorphism classes of modules over $\Bbbk Q$ for an oriented Dynkin quiver Q. The product of two isomorphism classes $[M]$ and $[M']$ is defined by $[M * M']$, the isomorphism class of the generic extension of M by M' in $\Bbbk Q-\mathrm{mod}$. The generic extension is up to isomorphism uniquely determined to be the extension with $\dim\mathrm{End}_{\Bbbk Q}(M * M')$ minimal among all possible extensions. Equivalently, the orbit of the generic extension is dense in the subset of extensions of $[M]$ and $[M']$ inside the representation variety of Q. In particular, Reineke shows that for $Q = \mathsf{A}_{N-1}$ (with orientation given e.g. by $i \to (i-1)$ for the vertices $2 \leq i \leq N-1$ of Q), the $\Bbbk$-linearisation of the resulting monoid is isomorphic the plactic algebra as defined above, where the isomorphism classes of the one-dimensional simple modules $[S_i]$ are mapped to the generators a_i. This is furthermore identified with the positive half of the twisted quantum group at $q = 0$, see Remark I.1.1.9. By Ringel's theorem [Rin90] we know that the positive half of the twisted quantum group is isomorphic to the generic Hall algebra for any Dynkin quiver Q. Hence, the specialisation of the generic Hall algebra at $q = 0$ gives the Hall monoid. Different normal forms for monomials in the plactic algebra are given in terms of enumerations of the roots [Rei02, Theorem 2.10].

In [DD05] a similar approach is taken for affine type $\widehat{\mathsf{A}}_{N-1}$. A normal form for monomials in the affine plactic algebra is constructed using generic extensions of nilpotent representations of the quiver $\widehat{\mathsf{A}}_{N-1}$ with cyclic orientation. In [Wol07, Theorem 2.8] it is shown that also for affine type $\widehat{\mathsf{A}}_{N-1}$, the specialisation of the generic Hall algebra at $q = 0$ gives the Hall monoid of generic extensions. Schiffmann [Sch00] and Hubery [Hub05] describe the center of the Hall algebra of nilpotent modules of the cyclic quiver.

I.3.2. The partic algebra

In this section we introduce a quotient of the classical plactic algebra:

I.3.2.1 Definition. Define the partic algebra $\mathcal{P}_N^{\mathrm{part}}$ to be the quotient of $\mathcal{P}_N$ by the additional relation

$$a_i a_{i-1} a_{i+1} a_i = a_{i+1} a_i a_{i-1} a_i \qquad \text{for } 2 \leq i \leq N-2. \tag{I.3.7}$$

Note that one can interpret relations (I.3.2), (I.3.3) as commutativity of the product $(a_{i+1}a_i)$ with the generators a_{i+1} and a_i. Relation (I.3.7) together with (I.3.2) implies in particular that $(a_{i+1}a_i)$ and $(a_i a_{i-1})$ commute.

I.3.2.2 Remark. This relation appears naturally in the study of bosonic particle configurations, see Section I.3.4 and Section I.1.2. In contrast, in the Hall monoid of finite type A_{N-1} one cannot expect $[S_{i+1} * S_i]$ and $[S_i * S_{i-1}]$ to commute. This is because precisely one of $S_{i+1} * S_i * S_i * S_{i-1}$, $S_i * S_{i-1} * S_{i+1} * S_i$ is a nontrivial extension of $S_{i+1} * S_i$ and $S_i * S_{i-1}$ (it depends on the choice of orientation which one is nontrivial) – in much the same way as $[S_i]$ and $[S_{i\pm 1}]$ do not commute. ◊

I.3.2.3 Remark. We have two gradings on both the plactic and the partic algebra:

i) All relations preserve the length of monomials, hence $\mathcal{P}_N$ and $\mathcal{P}_N^{\mathrm{part}}$ can be equipped with a $\mathbb{Z}$-grading by the length of monomials.
ii) All relations preserve the number of different generators in a monomial, hence $\mathcal{P}_N$ and $\mathcal{P}_N^{\mathrm{part}}$ can be equipped with a $\mathbb{Z}^{N-1}$-grading that assigns to the generator a_i the degree e_i, the i-th standard basis vector in $\mathbb{Z}^{N-1}$. This is a refinement of the above length grading. ◊

I.3.2.4 Lemma. In the plactic (and hence also in the partic) algebra, the following relations hold:

i) For all generators a_i, a_{i-1}, $2 \le i \le N-1$ and all $m \ge 0$, we have

$$\begin{aligned} a_i^m a_{i-1}^m &= (a_i a_{i-1})^m, \\ a_i(a_i^m a_{i-1}^m) &= (a_i^m a_{i-1}^m) a_i. \end{aligned} \tag{I.3.8}$$

ii) For all $i \ge k \ge j$ we have

$$(a_i a_{i-1} \dots a_{j+1} a_j) a_k = a_k (a_i a_{i-1} \dots a_{j+1} a_j). \tag{I.3.9}$$

Proof. i) The second equation of Lemma I.3.2.4.(i) follows from the first by the plactic relation (I.3.2). By induction, $a_i^m a_{i-1}^m = a_i(a_i a_{i-1})^{m-1} a_{i-1} = (a_i a_{i-1})^{m-1} a_i a_{i-1} = (a_i a_{i-1})^m$.
ii) This equality follows from the calculation

$$\begin{aligned} (a_i a_{i-1} \dots a_{j+1} a_j) a_k &\overset{\text{(I.3.1)}}{=} a_i a_{i-1} \dots a_{k+1} a_k a_{k-1} a_k \dots a_{j+1} a_j \\ &\overset{\text{(I.3.2)}}{=} a_i a_{i-1} \dots a_{k+1} a_k a_k a_{k-1} \dots a_{j+1} a_j \\ &\overset{\text{(I.3.3)}}{=} a_i a_{i-1} \dots a_k a_{k+1} a_k a_{k-1} \dots a_{j+1} a_j \\ &\overset{\text{(I.3.1)}}{=} a_k (a_i a_{i-1} \dots a_{j+1} a_j). \end{aligned}$$ □

I.3.3. A basis of the partic algebra

In this section we formulate the following main theorem:

I.3.3.1 Theorem. The partic algebra $\mathcal{P}_N^{\text{part}}$ has a basis given by monomials of the form

$$\{a_{N-1}^{d_{N-1}} \dots a_2^{d_2} a_1^{k_1} a_2^{k_2} \dots a_{N-1}^{k_{N-1}} \mid d_i \leq d_{i-1} + k_{i-1} \text{ for all } 3 \leq i \leq N-1,\ d_2 \leq k_1\} \quad \text{(I.3.10)}$$

where $d_i, k_i \in \mathbb{Z}_{\geq 0}$ for all $1 \leq i \leq N-1$.

The proof consists of two steps: Since the partic algebra is defined by monomial relations, it suffices to construct a normal form for monomials to obtain a $\Bbbk$-basis for the algebra. This is similar to the affine nilTemperley–Lieb algebra, see Remark I.2.5.1. In this section we show that every monomial in the partic algebra is equivalent to a monomial of the form (I.3.10). In Section I.3.4 we observe that these monomials act pairwise differently on the particle configuration module, and we conclude that they must have been distinct.

I.3.3.2 Proposition. Every monomial in the partic algebra $\mathcal{P}_N^{\text{part}}$ is equivalent to a monomial of the form (I.3.10), i.e. $a_{N-1}^{d_{N-1}} \dots a_2^{d_2} a_1^{k_1} a_2^{k_2} \dots a_{N-1}^{k_{N-1}}$ with $d_i \leq d_{i-1} + k_{i-1}$ for all $3 \leq i \leq N-1$ and $d_2 \leq k_1$.

Proof. The proof works by induction on the length of monomials. If the length is equal to 1, we have $a_i = a_i^{k_i}$ for $k_i = 1$, and the condition from (I.3.10) is preserved. For the induction step our goal is to show that

$$a_i \cdot \left(a_{N-1}^{d_{N-1}} \dots a_2^{d_2} a_1^{k_1} a_2^{k_2} \dots a_{N-1}^{k_{N-1}}\right) = a_{N-1}^{d_{N-1}} \dots a_i^{d_i'} \dots a_2^{d_2} a_1^{k_1} a_2^{k_2} \dots a_i^{k_i'} \dots a_{N-1}^{k_{N-1}}, \quad \text{(I.3.11)}$$

where $d_i' = d_i$ and $k_i' = k_i + 1$, or $d_i' = d_i + 1$ and $k_i' = k_i$ are such that the inequality condition (I.3.10) is preserved. Since we can commute a_i with all a_j as long as $j \neq i \pm 1$, we only need to consider

$$a_i \cdot \left(a_{i+1}^{d_{i+1}} a_i^{d_i} a_{i-1}^{d_{i-1}} \dots a_{i-1}^{k_{i-1}} a_i^{k_i}\right).$$

In order to prove that this can be rewritten as in (I.3.11), we have to show that either we can pass a_i through to the right hand side, increasing the exponent k_i by one, or we leave it at the left hand side, increasing d_i by one.

i) Case $d_{i+1} = d_i = d_{i-1} = k_{i-1} = 0$: Set $k_i' = k_i + 1$. The inequality condition (I.3.10) is automatically satisfied if we increase one of the k's, so there is nothing to check. The equality (I.3.11) is obvious since we only apply the commutativity relation (I.3.1).

ii) Case $d_{i+1} = d_i = d_{i-1} = 0$, $k_{i-1} > 0$: Set $d_i' = 1$. The inequality condition (I.3.10) is preserved since $k_{i-1} \geq 1$, and again we only apply the commutativity relation (I.3.1).

iii) Case $d_{i+1} = d_i = 0$, $d_{i-1} > 0$, k_{i-1} arbitrary: Set $d_i' = 1$. The inequality condition (I.3.10) is preserved since $d_{i-1} \geq 1$, and as before we only apply the commutativity relation (I.3.1).

iv) Case $d_{i+1} = 0$, $d_i > 0$, d_{i-1} and k_{i-1} arbitrary so that $d_i \leq d_{i-1} + k_{i-1}$:

- $d_i < d_{i-1} + k_{i-1}$: Set $d_i' = d_i + 1$.
- $d_i = d_{i-1} + k_{i-1}$: We cannot increase d_i, hence we have to show that we can commute a_i past $a_{i-1}^{d_{i-1}}$ and $a_{i-1}^{k_{i-1}}$ to increase k_i. Indeed, we can apply equality (I.3.8) from Lemma I.3.2.4.(i) to obtain

$$\begin{aligned} a_i\left(a_i^{d_i} a_{i-1}^{d_{i-1}} \dots a_{i-1}^{k_{i-1}} a_i^{k_i}\right) &= a_i^{d_{i-1}+k_{i-1}+1} a_{i-1}^{d_{i-1}} \dots a_{i-1}^{k_{i-1}} a_i^{k_i} \\ &= a_i^{k_{i-1}+1} (a_i a_{i-1})^{d_{i-1}} \dots a_{i-1}^{k_{i-1}} a_i^{k_i} \\ &= (a_i a_{i-1})^{d_{i-1}} a_i^{k_{i-1}+1} \dots a_{i-1}^{k_{i-1}} a_i^{k_i} \\ &= (a_i a_{i-1})^{d_{i-1}} \dots a_i^{k_{i-1}+1} a_{i-1}^{k_{i-1}} a_i^{k_i} \\ &= (a_i a_{i-1})^{d_{i-1}} \dots a_i (a_i a_{i-1})^{k_{i-1}} a_i^{k_i} \\ &= (a_i a_{i-1})^{d_{i-1}} \dots (a_i a_{i-1})^{k_{i-1}} a_i^{k_i+1} \\ &= (a_i a_{i-1})^{d_{i-1}} a_i^{k_{i-1}} \dots a_{i-1}^{k_{i-1}} a_i^{k_i+1} \\ &= a_i^{d_{i-1}+k_{i-1}} a_{i-1}^{d_{i-1}} \dots a_{i-1}^{k_{i-1}} a_i^{k_i+1}. \end{aligned}$$

v) Case $d_{i+1} > 0$, d_i, k_i and d_{i-1}, k_{i-1} arbitrary so that $d_{i+1} \leq d_i + k_i$, $d_i \leq d_{i-1} + k_{i-1}$: We reduce to the previous cases by proving

$$a_i a_{i+1}^{d_{i+1}} \cdot \left(a_i^{d_i} a_{i-1}^{d_{i-1}} \dots a_{i-1}^{k_{i-1}} a_i^{k_i}\right) = a_{i+1}^{d_{i+1}} a_i \left(a_i^{d_i} a_{i-1}^{d_{i-1}} \dots a_{i-1}^{k_{i-1}} a_i^{k_i}\right).$$

- $d_{i+1} > d_i$: Here we can apply Lemma I.3.2.4.(i) to obtain

$$a_i(a_{i+1}^{d_{i+1}} a_i^{d_i}) = a_i (a_{i+1} a_i)^{d_{i+1}} a_i^{d_i - d_{i+1}} = (a_{i+1} a_i)^{d_{i+1}} a_i^{d_i - d_{i+1}} = a_{i+1}^{d_{i+1}} a_i a_i^{d_i}.$$

- $d_{i+1} \leq d_i$: In this case we have $k_i \geq d_{i+1} - d_i > 0$. It suffices to prove that

$$a_i a_{i+1}^m a_{i-1}^{d_{i-1}} \dots a_{i-1}^{k_{i-1}} a_i^m = a_{i+1}^m a_i a_{i-1}^{d_{i-1}} \dots a_{i-1}^{k_{i-1}} a_i^m$$

for any $m > 0$. Then the desired statement follows using equality (I.3.8) from Lemma I.3.2.4.(i):

$$\begin{aligned} a_i a_{i+1}^{d_{i+1}} a_i^{d_i} a_{i-1}^{d_{i-1}} \dots a_{i-1}^{k_{i-1}} a_i^{k_i} &= (a_{i+1} a_i)^{d_i} \left(a_i a_{i+1}^{d_{i+1}-d_i} a_{i-1}^{d_{i-1}} \dots a_{i-1}^{k_{i-1}} a_i^{d_{i+1}-d_i}\right) a_i^{k_i+d_i-d_{i+1}} \\ &= (a_{i+1} a_i)^{d_i} \left(a_{i+1}^{d_{i+1}-d_i} a_i a_{i-1}^{d_{i-1}} \dots a_{i-1}^{k_{i-1}} a_i^{d_{i+1}-d_i}\right) a_i^{k_i+d_i-d_{i+1}} \\ &= a_{i+1}^{d_{i+1}} a_i a_i^{d_i} a_{i-1}^{d_{i-1}} \dots a_{i-1}^{k_{i-1}} a_i^{k_i}. \end{aligned}$$

Now for $d_{i-1}, d_{i-2}, \dots, d_j > 0$ and $d_{j-1} = 0$ (possibly $j = i$, or $j = 1$), we apply equations (I.3.1), (I.3.7), (I.3.8) and (I.3.9) to pass the factor a_i (distinguished by bold print) through the whole expression, thereby proving the desired equality.

$$
\begin{aligned}
& \boldsymbol{a_i} a_{i+1}^{m} a_{i-1}^{d_{i-1}} a_{i-2}^{d_{i-2}} \dots a_j^{d_j} a_{j-2}^{d_{j-2}} \dots a_{j-2}^{k_{j-2}} a_{j-1}^{k_{j-1}} a_j^{k_j} \dots a_{i-1}^{k_{i-1}} a_i^{m} \\
\stackrel{(\mathrm{I.3.1})}{=}\ & \boldsymbol{a_i} a_{i-1}^{d_{i-1}} a_{i-2}^{d_{i-2}} \dots a_j^{d_j} a_{j-2}^{d_{j-2}} \dots a_{j-2}^{k_{j-2}} a_{j-1}^{k_{j-1}} a_j^{k_j} \dots a_{i-1}^{k_{i-1}} a_{i+1}^{m} a_i^{m} \\
\stackrel{(\mathrm{I.3.8})}{=}\ & \boldsymbol{a_i} a_{i-1}^{d_{i-1}} a_{i-2}^{d_{i-2}} \dots a_j^{d_j} a_{j-2}^{d_{j-2}} \dots a_{j-2}^{k_{j-2}} a_{j-1}^{k_{j-1}} a_j^{k_j} \dots a_{i-1}^{k_{i-1}} (a_{i+1} a_i)^{m} \\
=\ & (\boldsymbol{a_i} a_{i-1}) a_{i-1}^{d_{i-1}-1} a_{i-2}^{d_{i-2}} \dots a_j^{d_j} a_{j-2}^{d_{j-2}} \dots a_{j-2}^{k_{j-2}} a_{j-1}^{k_{j-1}} a_j^{k_j} \dots a_{i-1}^{k_{i-1}} (a_{i+1} a_i)^{m} \\
\stackrel{(\mathrm{I.3.9})}{=}\ & a_{i-1}^{d_{i-1}-1} a_{i-2}^{d_{i-2}-1} \dots (\boldsymbol{a_i} a_{i-1} a_{i-2} \dots a_j) a_j^{d_j-1} a_{j-2}^{d_{j-2}} \dots a_{j-2}^{k_{j-2}} a_{j-1}^{k_{j-1}} a_j^{k_j} \dots a_{i-1}^{k_{i-1}} (a_{i+1} a_i)^{m} \\
\stackrel{(\mathrm{I.3.9})}{=}\ & a_{i-1}^{d_{i-1}-1} a_{i-2}^{d_{i-2}-1} \dots a_j^{d_j-1} (\boldsymbol{a_i} a_{i-1} a_{i-2} \dots a_j) a_{j-2}^{d_{j-2}} \dots a_{j-2}^{k_{j-2}} a_{j-1}^{k_{j-1}} a_j^{k_j} \dots a_{i-1}^{k_{i-1}} (a_{i+1} a_i)^{m} \\
\stackrel{(\mathrm{I.3.1})}{=}\ & a_{i-1}^{d_{i-1}-1} a_{i-2}^{d_{i-2}-1} \dots a_j^{d_j-1} a_{j-2}^{d_{j-2}} \dots a_{j-2}^{k_{j-2}} (\boldsymbol{a_i} a_{i-1} a_{i-2} \dots a_j) a_{j-1}^{k_{j-1}} a_j^{k_j} \dots a_{i-1}^{k_{i-1}} (a_{i+1} a_i)^{m} \\
\stackrel{(\mathrm{I.3.9})}{=}\ & a_{i-1}^{d_{i-1}-1} a_{i-2}^{d_{i-2}-1} \dots a_j^{d_j-1} a_{j-2}^{d_{j-2}} \dots a_{j-2}^{k_{j-2}} a_{j-1}^{k_{j-1}-1} (\boldsymbol{a_i} a_{i-1} a_{i-2} \dots a_j a_{j-1}) a_j^{k_j} \dots a_{i-1}^{k_{i-1}} (a_{i+1} a_i)^{m} \\
\stackrel{(\mathrm{I.3.9})}{=}\ & a_{i-1}^{d_{i-1}-1} a_{i-2}^{d_{i-2}-1} \dots a_j^{d_j-1} a_{j-2}^{d_{j-2}} \dots a_{j-2}^{k_{j-2}} a_{j-1}^{k_{j-1}-1} a_j^{k_j} \dots a_{i-1}^{k_{i-1}} (\boldsymbol{a_i} a_{i-1} a_{i-2} \dots a_j a_{j-1}) (a_{i+1} a_i)^{m} \\
\stackrel{(\mathrm{I.3.1})}{=}\ & a_{i-1}^{d_{i-1}-1} a_{i-2}^{d_{i-2}-1} \dots a_j^{d_j-1} a_{j-2}^{d_{j-2}} \dots a_{j-2}^{k_{j-2}} a_{j-1}^{k_{j-1}-1} a_j^{k_j} \dots a_{i-1}^{k_{i-1}} (\boldsymbol{a_i} a_{i-1}) (a_{i+1} a_i)^{m} (a_{i-2} \dots a_j a_{j-1}) \\
\stackrel{(\mathrm{I.3.7})}{=}\ & a_{i-1}^{d_{i-1}-1} a_{i-2}^{d_{i-2}-1} \dots a_j^{d_j-1} a_{j-2}^{d_{j-2}} \dots a_{j-2}^{k_{j-2}} a_{j-1}^{k_{j-1}-1} a_j^{k_j} \dots a_{i-1}^{k_{i-1}} (a_{i+1} a_i)^{m} (\boldsymbol{a_i} a_{i-1}) (a_{i-2} \dots a_j a_{j-1}) \\
\stackrel{(\mathrm{I.3.8})}{=}\ & a_{i-1}^{d_{i-1}-1} a_{i-2}^{d_{i-2}-1} \dots a_j^{d_j-1} a_{j-2}^{d_{j-2}} \dots a_{j-2}^{k_{j-2}} a_{j-1}^{k_{j-1}-1} a_j^{k_j} \dots a_{i-1}^{k_{i-1}} a_{i+1}^{m} a_i^{m} (\boldsymbol{a_i} a_{i-1}) (a_{i-2} \dots a_j a_{j-1}) \\
\stackrel{(\mathrm{I.3.1})}{=}\ & a_{i+1}^{m} a_{i-1}^{d_{i-1}-1} a_{i-2}^{d_{i-2}-1} \dots a_j^{d_j-1} a_{j-2}^{d_{j-2}} \dots a_{j-2}^{k_{j-2}} a_{j-1}^{k_{j-1}-1} a_j^{k_j} \dots a_{i-1}^{k_{i-1}} a_i^{m} (\boldsymbol{a_i} a_{i-1} a_{i-2} \dots a_j a_{j-1}) \\
\stackrel{(\mathrm{I.3.9})}{=}\ & a_{i+1}^{m} a_{i-1}^{d_{i-1}-1} a_{i-2}^{d_{i-2}-1} \dots a_j^{d_j-1} a_{j-2}^{d_{j-2}} \dots a_{j-2}^{k_{j-2}} (\boldsymbol{a_i} a_{i-1} a_{i-2} \dots a_j a_{j-1}) a_{j-1}^{k_{j-1}-1} a_j^{k_j} \dots a_{i-1}^{k_{i-1}} a_i^{m} \\
=\ & a_{i+1}^{m} a_{i-1}^{d_{i-1}-1} a_{i-2}^{d_{i-2}-1} \dots a_j^{d_j-1} a_{j-2}^{d_{j-2}} \dots a_{j-2}^{k_{j-2}} (\boldsymbol{a_i} a_{i-1} a_{i-2} \dots a_j) a_{j-1}^{k_{j-1}} a_j^{k_j} \dots a_{i-1}^{k_{i-1}} a_i^{m} \\
\stackrel{(\mathrm{I.3.1})}{=}\ & a_{i+1}^{m} a_{i-1}^{d_{i-1}-1} a_{i-2}^{d_{i-2}-1} \dots a_j^{d_j-1} (\boldsymbol{a_i} a_{i-1} a_{i-2} \dots a_j) a_{j-2}^{d_{j-2}} \dots a_{j-2}^{k_{j-2}} a_{j-1}^{k_{j-1}} a_j^{k_j} \dots a_{i-1}^{k_{i-1}} a_i^{m} \\
\stackrel{(\mathrm{I.3.9})}{=}\ & a_{i+1}^{m} a_{i-1}^{d_{i-1}-1} a_{i-2}^{d_{i-2}-1} \dots (\boldsymbol{a_i} a_{i-1} a_{i-2} \dots a_j) a_j^{d_j-1} a_{j-2}^{d_{j-2}} \dots a_{j-2}^{k_{j-2}} a_{j-1}^{k_{j-1}} a_j^{k_j} \dots a_{i-1}^{k_{i-1}} a_i^{m} \\
\stackrel{(\mathrm{I.3.9})}{=}\ & a_{i+1}^{m} a_{i-1}^{d_{i-1}-1} (\boldsymbol{a_i} a_{i-1} a_{i-2}) a_{i-2}^{d_{i-2}-1} \dots a_j^{d_j} a_{j-2}^{d_{j-2}} \dots a_{j-2}^{k_{j-2}} a_{j-1}^{k_{j-1}} a_j^{k_j} \dots a_{i-1}^{k_{i-1}} a_i^{m} \\
\stackrel{(\mathrm{I.3.9})}{=}\ & a_{i+1}^{m} (\boldsymbol{a_i} a_{i-1}) a_{i-1}^{d_{i-1}-1} a_{i-2}^{d_{i-2}} \dots a_j^{d_j} a_{j-2}^{d_{j-2}} \dots a_{j-2}^{k_{j-2}} a_{j-1}^{k_{j-1}} a_j^{k_j} \dots a_{i-1}^{k_{i-1}} a_i^{m} \\
=\ & a_{i+1}^{m} \boldsymbol{a_i} a_{i-1}^{d_{i-1}} a_{i-2}^{d_{i-2}} \dots a_j^{d_j} a_{j-2}^{d_{j-2}} \dots a_{j-2}^{k_{j-2}} a_{j-1}^{k_{j-1}} a_j^{k_j} \dots a_{i-1}^{k_{i-1}} a_i^{m}.
\end{aligned}
$$

This concludes the proof of Proposition I.3.3.2. □

Note that the relation special for the partic algebra (I.3.7) was only used once in the proof of Proposition I.3.3.2, namely in the long computation at the end. All other steps have been carried out using only the commutativity relation (I.3.1) and the plactic relations (I.3.2) and (I.3.3). The following corollary recaps what we obtained for the multiplication in the partic algebra:

I.3.3.3 Corollary. Assume we are given a monomial $a_{N-1}^{d_{N-1}} \dots a_2^{d_2} a_1^{k_1} a_2^{k_2} \dots a_{N-1}^{k_{N-1}}$ of the form (I.3.10) in the partic algebra. Then left multiplication with a_i gives

$$a_i \cdot \left(a_{N-1}^{d_{N-1}} \dots a_2^{d_2} a_1^{k_1} a_2^{k_2} \dots a_{N-1}^{k_{N-1}}\right) \tag{I.3.12}$$
$$= \begin{cases} a_{N-1}^{d_{N-1}} \dots a_i^{d_i} \dots a_2^{d_2} a_1^{k_1} a_2^{k_2} \dots a_i^{k_i+1} \dots a_{N-1}^{k_{N-1}} & \text{if } d_i = d_{i-1} + k_{i-1}, \\ a_{N-1}^{d_{N-1}} \dots a_i^{d_i+1} \dots a_2^{d_2} a_1^{k_1} a_2^{k_2} \dots a_i^{k_i} \dots a_{N-1}^{k_{N-1}} & \text{if } d_i < d_{i-1} + k_{i-1}. \end{cases}$$

Right multiplication with a_i gives

$$\left(a_{N-1}^{d_{N-1}} \dots a_2^{d_2} a_1^{k_1} a_2^{k_2} \dots a_{N-1}^{k_{N-1}}\right) \cdot a_i \tag{I.3.13}$$
$$= \begin{cases} a_{N-1}^{d_{N-1}} \dots a_i^{d_i} a_{i+1}^{d_{i+1}+1} \dots a_2^{d_2} a_1^{k_1} a_2^{k_2} \dots a_i^{k_i+1} a_{i+1}^{k_{i+1}-1} \dots a_{N-1}^{k_{N-1}} & \text{if } k_{i+1} \geq 1 \\ a_{N-1}^{d_{N-1}} \dots a_i^{d_i} a_{i+1}^{d_{i+1}} \dots a_2^{d_2} a_1^{k_1} a_2^{k_2} \dots a_i^{k_i+1} a_{i+1}^{0} \dots a_{N-1}^{k_{N-1}} & \text{if } k_{i+1} = 0, \end{cases}$$

with the result written again in the normal form (I.3.10).

Proof. For the left multiplication, equation (I.3.12) is contained in the proof of Proposition I.3.3.2. For the right multiplication, equation (I.3.13) follows from the repeated application of the rule for left multiplication of $a_{N-1}^{d_{N-1}} \dots a_2^{d_2} a_1^{k_1} a_2^{k_2} \dots a_{N-1}^{k_{N-1}}$ to a_i. □

I.3.3.4 Example. The partic algebra has zero divisors, e.g. in $\mathcal{P}_3^{\text{part}}$,

$$a_2 \cdot \left(a_3^5 a_2^8 a_1^8 a_2^3 a_3^1 \; - \; a_3^5 a_2^7 a_1^8 a_2^4 a_3^1\right) \; = \; a_3^5 a_2^8 a_1^8 a_2^4 a_3^1 \; - \; a_3^5 a_2^8 a_1^8 a_2^4 a_3^1 \; = \; 0$$

(it follows from Theorem I.3.3.1 that $a_3^5 a_2^8 a_1^8 a_2^3 a_3^1 \; - \; a_3^5 a_2^7 a_1^8 a_2^4 a_3^1 \; \neq \; 0$). ◊

I.3.3.5 Remark. Let us compare our normal form with the monomial bases of the plactic algebra from [Rei02]: The plactic algebra $\mathcal{P}_N$ surjects onto the partic algebra $\mathcal{P}_N^{\text{part}}$, mapping generators to generators and hence monomials to monomials. Given a monomial of the normal form from Proposition I.3.3.2, finding the (finitely many) preimages of basis monomials in the plactic algebra amounts to solving a system of linear equations over the nonnegative integers, i.e. finding lattice points inside a polyhedron.

For example, consider the basis of the plactic algebra $\mathcal{P}_5$ from [Rei02, Theorem 2.10] given by monomials

$$(a_1)^{n_1}(a_2a_1)^{n_{21}}(a_2)^{n_2}(a_3a_2a_1)^{n_{321}}(a_3a_2)^{n_{32}}(a_3)^{n_3}(a_4a_3a_2a_1)^{n_{4321}}(a_4a_3a_2)^{n_{432}}(a_4a_3)^{n_{43}}(a_4)^{n_4}$$

where all $n_i \in \mathbb{Z}_{\geq 0}$ and compare it with the basis of the partic algebra $\mathcal{P}_5^{\text{part}}$ from Proposition I.3.3.2

$$\{a_4^{d_4} a_3^{d_3} a_2^{d_2} a_1^{k_1} a_2^{k_2} a_3^{k_3} a_4^{k_4} \mid \text{all } k_i, d_i \in \mathbb{Z}_{\geq 0}, \; d_i \leq d_{i-1} + k_{i-1}\}.$$

While $a_1a_2a_3a_4 \in \mathcal{P}_5^{\text{part}}$ has only one preimage, namely $(a_1)^1(a_2)^1(a_3)^1(a_4)^1 \in \mathcal{P}_5$, we find two preimages of $a_4a_3a_2a_1a_2 \in \mathcal{P}_5^{\text{part}}$, namely $(a_2)^1(a_4a_3a_2a_1)^1, (a_2a_1)^1(a_4a_3a_2)^1 \in \mathcal{P}_5$. This corresponds to the number of possible applications of the additional partic relation (I.3.7). ◊

I.3.4. The action on bosonic particle configurations

In this section we discuss an action of the plactic algebra $\mathcal{P}_N$ on the polynomial ring $\Bbbk[x_1, \ldots, x_{N-1}, x_0]$ in N variables. It was defined in [KS10, Proposition 5.8]. We recall the definition here: Let $x_1^{k_1} \ldots x_{N-1}^{k_{N-1}} x_0^{k_0}$ be a monomial in $\Bbbk[x_1, \ldots, x_{N-1}, x_0]$. Set

$$a_i \cdot x_1^{k_1} \ldots x_{N-1}^{k_{N-1}} x_0^{k_0} = \begin{cases} x_1^{k_1} \ldots x_i^{k_i - 1} x_{i+1}^{k_{i+1}+1} \ldots x_{N-1}^{k_{N-1}} x_0^{k_0} & \text{if } k_i > 0, \\ 0 & \text{else,} \end{cases} \tag{I.3.14}$$

$$a_{N-1} \cdot x_1^{k_1} \ldots x_{N-1}^{k_{N-1}} x_0^{k_0} = \begin{cases} x_1^{k_1} \ldots x_{N-1}^{k_{N-1}-1} x_0^{k_0+1} & \text{if } k_{N-1} > 0, \\ 0 & \text{else.} \end{cases} \tag{I.3.15}$$

This defines an action of the plactic algebra which factors over the partic algebra:

I.3.4.1 Lemma. Equations (I.3.14) and (I.3.15) define an action of the plactic algebra $\mathcal{P}_N$ on the polynomial ring $\Bbbk[x_1, \ldots, x_{N-1}, x_0]$. This action factors over an action of the partic algebra $\mathcal{P}_N^{\text{part}}$.

Proof. This can be verified by direct computation. □

In this section our goal is the proof of the following main theorem:

I.3.4.2 Theorem. The action of the partic algebra $\mathcal{P}_N^{\text{part}}$ on $\Bbbk[x_1, \ldots, x_{N-1}, x_0]$ defined by equations (I.3.14) and (I.3.15) is faithful.

I.3.4.3 Remark. In [KS10, Proposition 5.8] it is stated incorrectly that the action of the plactic algebra $\mathcal{P}_N$ on $\Bbbk[x_1, \ldots, x_{N-1}, x_0]$ is faithful. ◊

One can think of the monomials $x_1^{k_1} \ldots x_{N-1}^{k_{N-1}} x_0^{k_0}$ as configurations of particles on a line with N positions, with k_i particles at the i-th position. The 0-th position is regarded as the deposit for particles moved to the end of the line. Then a_i moves a particle from position i to position $i+1$. We call $\Bbbk[x_1, \ldots, x_{N-1}, x_0]$ with the above action the (classical bosonic) particle configuration module of $\mathcal{P}_N$ or $\mathcal{P}_N^{\text{part}}$, and we refer to the monomials inside $\Bbbk[x_1, \ldots, x_{N-1}, x_0]$ as (classical bosonic) particle configurations.

I.3.4.4 Definition. We use the shorthand notation $\underline{\mathrm{I}} := (k_1, \ldots, k_{N-1}, k_0) \in \mathbb{Z}_{\geq 0}^N$ for the monomial $v(\underline{\mathrm{I}}) := x_1^{k_1} \ldots x_{N-1}^{k_{N-1}} x_0^{k_0}$.

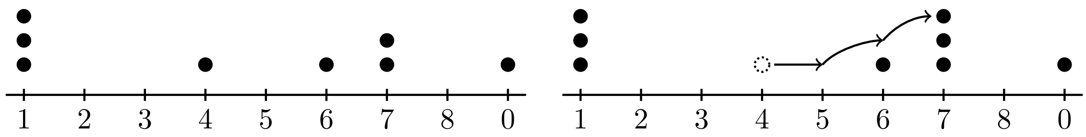


Figure I.3.4.1.: Example for $N = 9$: The particle configuration $(3,0,0,1,0,1,2,0,1)$ corresponding to the monomial $x_1^3 x_2^0 x_3^0 x_4^1 x_5^0 x_6^1 x_7^2 x_8^0 x_0^1$, and the element $a_6 a_5 a_4$ acting on it.

Now we investigate the action of the partic algebra on the particle configuration module.

I.3.4.5 Proposition. Fix a monomial $a_{N-1}^{d_{N-1}} \dots a_3^{d_3} a_2^{d_2} a_1^{k_1} a_2^{k_2} a_3^{k_3} \dots a_{N-1}^{k_{N-1}}$ in the partic algebra satisfying condition (I.3.10). There is a unique particle configuration with the number of particles minimal, i.e. a monomial in $\Bbbk[x_1, \dots, x_{N-1}, x_0]$ of minimal degree, so that the monomial acts nontrivially on it. This minimal particle configuration is given by

$$\underline{\mathrm{I}}_{\mathrm{in}} = (k_1, k_2, k_3 \dots, k_{N-1}, 0).$$

The image of $\underline{\mathrm{I}}_{\mathrm{in}}$ under the action of $a_{N-1}^{d_{N-1}} \dots a_2^{d_2} a_1^{k_1} a_2^{k_2} \dots a_{N-1}^{k_{N-1}}$ is given by

$$\underline{\mathrm{I}}_{\mathrm{out}} = (0, k_1 - d_2, k_2 + d_2 - d_3 \dots, k_{N-2} + d_{N-2} - d_{N-1}, k_{N-1} + d_{N-1}).$$

Proof. First we show that $a_1^{k_1} a_2^{k_2} a_3^{k_3} \dots a_{N-1}^{k_{N-1}}$, hence $a_{N-1}^{d_{N-1}} \dots a_3^{d_3} a_2^{d_2} a_1^{k_1} a_2^{k_2} a_3^{k_3} \dots a_{N-1}^{k_{N-1}}$ annihilates any particle configuration $(r_1, r_2, r_3, \dots, r_{N-1}, r_0)$ with $r_i < k_i$ for some i. We compute

$$\begin{aligned}
&a_1^{k_1} a_2^{k_2} \dots a_j^{k_j} a_{j+1}^{k_{j+1}} \dots a_{N-1}^{k_{N-1}} (x_1^{r_1} x_2^{r_2} \dots x_j^{r_j} x_{j+1}^{r_{j+1}} \dots x_{N-1}^{r_{N-1}} x_0^{r_0}) \\
&= \begin{cases} a_1^{k_1} a_2^{k_2} \dots a_j^{k_j} (x_1^{r_1} x_2^{r_2} \dots x_j^{r_j} x_{j+1}^{r_{j+1}-k_{j+1}} \dots x_{N-1}^{r_{N-1}-k_{N-1}+k_{N-2}} x_0^{r_0+k_{N-1}}), & r_i \geq k_i \text{ for } j < i \leq N-1 \\ 0 & \text{else} \end{cases} \\
&= \begin{cases} x_1^{r_1-k_1} x_2^{r_2-k_2+k_1} \dots x_i^{r_i-k_i+k_{i-1}} x_{i+1}^{r_{i+1}-k_{i+1}+k_i} \dots x_{N-1}^{r_{N-1}-k_{N-1}+k_{N-2}} x_0^{r_0+k_{N-1}}, & r_i \geq k_i \text{ for all } i \\ 0 & \text{else.} \end{cases}
\end{aligned}$$

Together with condition (I.3.10) it follows that the action of a monomial of the form $a_{N-1}^{d_{N-1}} \dots a_3^{d_3} a_2^{d_2} a_1^{k_1} a_2^{k_2} a_3^{k_3} \dots a_{N-1}^{k_{N-1}}$ on a particle configuration $(r_1, r_2, r_3, \dots, r_{N-1}, r_0)$ is nontrivial iff $r_i \geq k_i$ for all i (recall that $r_0 \geq 0 = k_0$ is automatically satisfied). This proves that $\underline{\mathrm{I}}_{\mathrm{in}}$ is indeed the minimal particle configuration on which the monomial acts nontrivially. Now compute the image of $\underline{\mathrm{I}}_{\mathrm{in}}$ under the action of the monomial: Plug in

$r_i = k_i$ for all i to see that

$$\begin{aligned} a_{N-1}^{d_{N-1}} \dots a_3^{d_3} a_2^{d_2} a_1^{k_1} a_2^{k_2} a_3^{k_3} \dots a_{N-1}^{k_{N-1}} (\underline{\mathrm{I}}_{\mathrm{in}}) &= a_{N-1}^{d_{N-1}} \dots a_3^{d_3} a_2^{d_2} (x_1^0 x_2^{k_1} \dots x_{N-1}^{k_{N-2}} x_0^{k_{N-1}}) \\ &= \underline{\mathrm{I}}_{\mathrm{out}}. \end{aligned}$$

This proves Proposition I.3.4.5. □

See also Remark I.2.4.3 for the notion of minimal (affine fermionic) particle configurations for the affine nilTemperley–Lieb algebra.

Proof (of Theorem I.3.3.1). By Proposition I.3.3.2 any monomial in the partic algebra is equivalent to one of the form (I.3.10). We have shown in Proposition I.3.4.5 that the action on the particle configuration module distinguishes any two monomials of the form (I.3.10), hence (I.3.10) describes a normal form for the monomials in the partic algebra $\mathcal{P}_N^{\mathrm{part}}$, hence a basis of $\mathcal{P}_N^{\mathrm{part}}$. □

Now Theorem I.3.4.2 follows as a corollary from Proposition I.3.4.5:

Proof (of Theorem I.3.4.2). We have seen in Proposition I.3.4.5 that the normal form monomials, hence the basis elements in $\mathcal{P}_N^{\mathrm{part}}$ act linearly independent on the particle configurations. In other words, the action of $\mathcal{P}_N^{\mathrm{part}}$ is faithful. □

I.3.4.6 Remark. The faithfulness of the action of the algebra $\mathcal{P}_N^{\mathrm{part}}$ on the particle configuration module motivates us to give $\mathcal{P}_N^{\mathrm{part}}$ the name “partic” algebra. ◊

By Proposition I.3.3.2 and Proposition I.3.4.5, we can identify each monomial in the partic algebra uniquely by the minimal particle configuration $\underline{\mathrm{J}} \in \mathbb{Z}_{\geq 0}^N$ on which it acts nontrivially and the output particle configuration $\underline{\mathrm{I}} \in \mathbb{Z}_{\geq 0}^N$ that one gets back from the action of the monomial on $\underline{\mathrm{J}}$. Hence the following is welldefined:

I.3.4.7 Definition. Given a monomial in normal form with $d_i \leq d_{i-1} + k_{i-1}$ for all $3 \leq i \leq N-1$ and $d_2 \leq k_1$, see Proposition I.3.4.5, we write

$$a_{\underline{\mathrm{IJ}}} = a_{N-1}^{d_{N-1}} \dots a_2^{d_2} a_1^{k_1} a_2^{k_2} \dots a_{N-1}^{k_{N-1}}$$

for bosonic particle configurations $\underline{\mathrm{I}} = (0, k_1 - d_2, k_2 + d_2 - d_3 \dots, k_{N-2} + d_{N-2} - d_{N-1}, k_{N-1} + d_{N-1})$ and $\underline{\mathrm{J}} = (k_1, k_2, k_3 \dots, k_{N-1}, 0)$. The number of particles $|\underline{\mathrm{I}}| = |\underline{\mathrm{J}}| = \sum_i k_i$ in $\underline{\mathrm{I}}$ and $\underline{\mathrm{J}}$ is the same.

This labelling is made so that $a_{\underline{\mathrm{IJ}}} \cdot v(\underline{\mathrm{J}}) = v(\underline{\mathrm{I}})$ in the notation of Definition I.3.4.4.

I.3.4.8 Definition. For $\underline{\mathrm{I}} = (r_1, \ldots, r_{N-1}, r_0) \in \mathbb{Z}^N_{\geq 0}$, we set

$$\underline{\mathrm{I}} \cup \{i\} = (r_1, \ldots, r_i + 1, \ldots, r_{N-1}, r_0), \qquad \underline{\mathrm{I}} \smallsetminus \{i\} = (r_1, \ldots, r_i - 1, \ldots, r_{N-1}, r_0),$$

where the latter is only defined for $r_i > 0$.

With this notation we can rewrite Corollary I.3.3.3 to obtain the following multiplication rule.

I.3.4.9 Corollary. Let $a_{\underline{\mathrm{I}}\underline{\mathrm{J}}}$ be a monomial in normal form as in Definition I.3.4.7. Then left and right multiplication by some generator $a_i \in \mathcal{P}^{\mathrm{part}}_N$ are given by

$$\begin{aligned} a_i a_{\underline{\mathrm{I}}\underline{\mathrm{J}}} &= \begin{cases} a_{\underline{\mathrm{I}}''\underline{\mathrm{J}}''} & \text{if } i \in \underline{\mathrm{I}} \\ a_{\underline{\mathrm{I}}'\underline{\mathrm{J}}'} & \text{if } i \notin \underline{\mathrm{I}}, \end{cases} \\ a_{\underline{\mathrm{I}}\underline{\mathrm{J}}} a_i &= \begin{cases} a_{\underline{\mathrm{I}}'''\underline{\mathrm{J}}'''} & \text{if } i+1 \in \underline{\mathrm{J}} \\ a_{\underline{\mathrm{I}}'\underline{\mathrm{J}}'} & \text{if } i+1 \notin \underline{\mathrm{J}}. \end{cases} \end{aligned}$$

Here we denote

$$\begin{array}{lll} \underline{\mathrm{I}}' = \underline{\mathrm{I}} \cup \{i+1\} & \underline{\mathrm{I}}'' = (\underline{\mathrm{I}} \smallsetminus \{i\}) \cup \{i+1\} & \underline{\mathrm{I}}''' = \underline{\mathrm{I}} \\ \underline{\mathrm{J}}' = \underline{\mathrm{J}} \cup \{i\} & \underline{\mathrm{J}}'' = \underline{\mathrm{J}} & \underline{\mathrm{J}}''' = (\underline{\mathrm{J}} \smallsetminus \{i+1\}) \cup \{i\}. \end{array}$$

I.3.4.10 Example. Let $N = 6$, and consider the monomial $a_{\underline{\mathrm{I}}\underline{\mathrm{J}}} = a_5^1 a_2^2 a_3^1 a_4^2$ with minimal input configurataion $\underline{\mathrm{J}} = (0, 2, 1, 2, 0, 0)$ and output configuration $\underline{\mathrm{I}} = (0, 0, 2, 1, 1, 1)$. Now consider the left and right multiplication with a_i for $i = 3$:

$$\begin{aligned} a_3 \cdot a_{(0,0,2,1,1,1)(0,2,1,2,0,0)} &= a_{(0,0,1,2,1,1)(0,2,1,2,0,0)}, \\ \text{with } \underline{\mathrm{I}}'' &= (0, 0, 1, 2, 1, 1), \\ \underline{\mathrm{J}}'' &= (0, 2, 1, 2, 0, 0), \\ a_{(0,0,2,1,1,1)(0,2,1,2,0,0)} a_3 &= a_{(0,0,2,1,1,1)(0,2,2,1,0,0)}, \\ \text{with } \underline{\mathrm{I}}''' &= (0, 0, 2, 1, 1, 1), \\ \underline{\mathrm{J}}''' &= (0, 2, 2, 1, 0, 0). \end{aligned}$$

In contrast, left and right multiplication with a_i for $i = 1$ gives

$$\begin{aligned} a_1 \cdot a_{(0,0,2,1,1,1)(0,2,1,2,0,0)} &= a_{(0,1,1,2,1,1)(1,2,1,2,0,0)}, \\ \text{with } \underline{\mathrm{I}}' &= (0, 1, 1, 2, 1, 1), \\ \underline{\mathrm{J}}' &= (1, 2, 1, 2, 0, 0), \\ a_{(0,0,2,1,1,1)(0,2,1,2,0,0)} a_1 &= a_{(0,0,2,1,1,1)(1,1,1,2,0,0)}, \\ \text{with } \underline{\mathrm{I}}''' &= (0, 0, 2, 1, 1, 1), \\ \underline{\mathrm{J}}''' &= (1, 1, 1, 2, 0, 0). \end{aligned}$$

We observe that the product $a_1 \cdot a_{(0,0,2,1,1,1)(0,2,1,2,0,0)}$ requires an additional particle at position 1, so that the cardinality of the minimal particle configuration of the product $a_1 \cdot a_{(0,0,2,1,1,1)(0,2,1,2,0,0)}$ is by one higher than that of $a_{(0,0,2,1,1,1)(0,2,1,2,0,0)}$. ◊

I.3.5. The center of the partic algebra

Now that we have a basis of the partic algebra with a convenient labelling at our disposal, the goal of this section is to describe the center of the partic algebra $\mathcal{P}_N^{\text{part}}$.

I.3.5.1 Theorem. The center of the partic algebra $\mathcal{P}_N^{\text{part}}$ is given by the $\Bbbk$-span of the elements

$$\{a_{N-1}^r a_{N-2}^r \dots a_2^r a_1^r \mid r \geq 0\}.$$

The monomial $a_{N-1}^r a_{N-2}^r \dots a_2^r a_1^r = (a_{N-1}a_{N-2}\dots a_2 a_1)^r = a_{(0,\dots,0,r)(r,0,\dots,0)}$ acts on the bosonic particle configurations by moving r particles from the first position 1 to the last position 0 if there are at least r particles at position 1, and it acts by zero if there are less than r particles at position 1. This action can be visualized as follows:

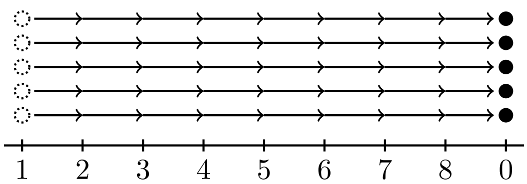


Figure I.3.5.1.: Example for $N = 9$: The action of the central element $(a_8a_7a_6a_5a_4a_3a_2a_1)^5$ on the particle configuration $(5,0,0,0,0,0,0,0,0)$.

Proof. Let $z := \sum_{\underline{I},\underline{J}} c_{\underline{I}\underline{J}} a_{\underline{I}\underline{J}}$ be an element in the center, where we label the monomial $a_{\underline{I}\underline{J}}$ by minimal input and output particle configurations as in Definition I.3.4.7, with coefficients $c_{\underline{I}\underline{J}} \in \Bbbk$. Notice that $a_{(0,\dots,0,r)(r,0,\dots,0)}$ commutes with all a_i by equation (I.3.9) from Lemma I.3.2.4.(ii). We show that $c_{\underline{I}\underline{J}} = 0$ for all $\underline{J}$ that contain some $i \neq 1$, and for all $\underline{I}$ that contain some $i \neq 0$.

Let $i \geq 2$. First we prove that $c_{\underline{I}\underline{J}} = 0$ for all $\underline{J}$ that contain a particle at position i. Since $z = \sum_{\underline{I},\underline{J}} c_{\underline{I}\underline{J}} a_{\underline{I}\underline{J}}$ is central, it commutes in particular with $a_{i-1}a_{i-2}\dots a_2 a_1$. Using Corollary

I.3.4.9 we calculate

$$\begin{aligned}(a_{i-1}a_{i-2}\dots a_2a_1)a_{\underline{\mathrm{I}}\underline{\mathrm{J}}} &= a_{(\underline{\mathrm{I}}\cup\{i\})(\underline{\mathrm{J}}\cup\{1\})},\\ a_{\underline{\mathrm{I}}\underline{\mathrm{J}}}(a_{i-1}a_{i-2}\dots a_2a_1) &= \begin{cases} a_{(\underline{\mathrm{I}}\cup\{i\})(\underline{\mathrm{J}}\cup\{1\})} & \text{if } i\notin\underline{\mathrm{J}},\\ a_{\underline{\mathrm{I}}((\underline{\mathrm{J}}\smallsetminus\{i\})\cup\{1\})} & \text{if } i\in\underline{\mathrm{J}}.\end{cases}\end{aligned}$$

Therefore $(a_{i-1}a_{i-2}\dots a_2a_1)a_{\underline{\mathrm{I}}\underline{\mathrm{J}}} = a_{\underline{\mathrm{I}}\underline{\mathrm{J}}}(a_{i-1}a_{i-2}\dots a_2a_1)$ for $i\notin\underline{\mathrm{J}}$. This we use to deduce that we have $(a_{i-1}a_{i-2}\dots a_2a_1)z = z(a_{i-1}a_{i-2}\dots a_2a_1)$ if and only if

$$(a_{i-1}a_{i-2}\dots a_2a_1)\left(\sum_{\substack{\underline{\mathrm{I}},\ \underline{\mathrm{J}}\\ i\notin\underline{\mathrm{J}}}} c_{\underline{\mathrm{I}}\underline{\mathrm{J}}}a_{\underline{\mathrm{I}}\underline{\mathrm{J}}} + \sum_{\substack{\underline{\mathrm{I}},\ \underline{\mathrm{J}}\\ i\in\underline{\mathrm{J}}}} c_{\underline{\mathrm{I}}\underline{\mathrm{J}}}a_{\underline{\mathrm{I}}\underline{\mathrm{J}}}\right) = \left(\sum_{\substack{\underline{\mathrm{I}},\ \underline{\mathrm{J}}\\ i\notin\underline{\mathrm{J}}}} c_{\underline{\mathrm{I}}\underline{\mathrm{J}}}a_{\underline{\mathrm{I}}\underline{\mathrm{J}}} + \sum_{\substack{\underline{\mathrm{I}},\ \underline{\mathrm{J}}\\ i\in\underline{\mathrm{J}}}} c_{\underline{\mathrm{I}}\underline{\mathrm{J}}}a_{\underline{\mathrm{I}}\underline{\mathrm{J}}}\right)(a_{i-1}a_{i-2}\dots a_2a_1),$$

which holds if and only if

$$(a_{i-1}a_{i-2}\dots a_2a_1)\left(\sum_{\substack{\underline{\mathrm{I}},\ \underline{\mathrm{J}}\\ i\in\underline{\mathrm{J}}}} c_{\underline{\mathrm{I}}\underline{\mathrm{J}}}a_{\underline{\mathrm{I}}\underline{\mathrm{J}}}\right) = \left(\sum_{\substack{\underline{\mathrm{I}},\ \underline{\mathrm{J}}\\ i\in\underline{\mathrm{J}}}} c_{\underline{\mathrm{I}}\underline{\mathrm{J}}}a_{\underline{\mathrm{I}}\underline{\mathrm{J}}}\right)(a_{i-1}a_{i-2}\dots a_2a_1).$$

The latter is precisely the equality

$$\sum_{\substack{\underline{\mathrm{I}},\ \underline{\mathrm{J}}\\ i\in\underline{\mathrm{J}}}} c_{\underline{\mathrm{I}}\underline{\mathrm{J}}}a_{(\underline{\mathrm{I}}\cup\{i\})(\underline{\mathrm{J}}\cup\{1\})} = \sum_{\substack{\underline{\mathrm{I}},\ \underline{\mathrm{J}}\\ i\in\underline{\mathrm{J}}}} c_{\underline{\mathrm{I}}\underline{\mathrm{J}}}a_{\underline{\mathrm{I}}((\underline{\mathrm{J}}\smallsetminus\{i\})\cup\{1\})}. \tag{I.3.16}$$

Observe on the other hand that for fixed i the set of monomials

$$\{a_{\underline{\mathrm{I}}((\underline{\mathrm{J}}\smallsetminus\{i\})\cup\{1\})} \mid \underline{\mathrm{I}},\ \underline{\mathrm{J}} \text{ such that } i\in\underline{\mathrm{J}}\}$$

is linearly independent since the sets $((\underline{\mathrm{J}}\smallsetminus\{i\})\cup\{1\})$ are all distinct for distinct $\underline{\mathrm{J}}$.

Next, we show by induction on the number k_i of particles at position i in $\underline{\mathrm{J}}$ that all coefficients $c_{\underline{\mathrm{I}}\underline{\mathrm{J}}}$ are zero for $k_i \geq 1$:
For $k_i = 1$, the set $(\underline{\mathrm{J}}\smallsetminus\{i\})\cup\{1\}$ does not contain any particle at position i any more. Hence the monomial $a_{\underline{\mathrm{I}}((\underline{\mathrm{J}}\smallsetminus\{i\})\cup\{1\})}$ cannot appear in the left sum in equation (I.3.16), and so its coefficient $c_{\underline{\mathrm{I}}\underline{\mathrm{J}}}$ must have been zero. For the induction step, assume that the coefficient $c_{\underline{\mathrm{I}}\underline{\mathrm{J}}}$ is zero for all $a_{\underline{\mathrm{I}}\underline{\mathrm{J}}}$ with at most k_i particles at position i in the minimal input particle configuration $\underline{\mathrm{J}}$. Consider some $a_{\underline{\mathrm{I}}\underline{\mathrm{J}}}$ with k_i+1 particles at position i in $\underline{\mathrm{J}}$. So the set $(\underline{\mathrm{J}}\smallsetminus\{i\})\cup\{1\}$ contains k_i particles at position i in $\underline{\mathrm{J}}$, and so the monomial $a_{\underline{\mathrm{I}}((\underline{\mathrm{J}}\smallsetminus\{i\})\cup\{1\})}$ cannot appear in the sum (I.3.16). Therefore we see that the coefficient $c_{\underline{\mathrm{I}}\underline{\mathrm{J}}}$ must have been zero.

We have shown that any central element in $\mathcal{P}_N^{\mathrm{part}}$ is of the form

$$z = \sum_{\underline{\mathrm{I}},\ \underline{\mathrm{J}}} c_{\underline{\mathrm{I}}\underline{\mathrm{J}}}a_{\underline{\mathrm{I}}\underline{\mathrm{J}}},$$

where the particle configurations $\underline{\mathrm{J}}$ are of the form $(r, 0, \dots, 0)$, $r \in \mathbb{Z}_{\geq 0}$. We use the convention that $i+1 = 0$ for $i = N-1$ which matches our definition of the action of the partic algebra $\mathcal{P}_N^{\mathrm{part}}$ on the bosonic particle configuration module. Notice that 0 is never contained in the minimal input particle configuration, so that for $1 \leq i \leq N-1$ we have that $i+1 \notin \underline{\mathrm{J}}$ for all $c_{\underline{\mathrm{I}}\underline{\mathrm{J}}} \neq 0$.

Now we use a similar induction argument to show that $c_{\underline{\mathrm{I}}\underline{\mathrm{J}}} = 0$ for all $\underline{\mathrm{I}}$ that contain a particle at position $i \neq 0$. So let $1 \leq i \leq N-1$. Using Corollary I.3.4.9 we calculate that

$$\begin{aligned} a_i a_{\underline{\mathrm{I}}\underline{\mathrm{J}}} &= \begin{cases} a_{(\underline{\mathrm{I}} \cup \{i+1\})(\underline{\mathrm{J}} \cup \{i\})} & \text{if } i \notin \underline{\mathrm{I}}, \\ a_{((\underline{\mathrm{I}} \setminus \{i\}) \cup \{i+1\})\underline{\mathrm{J}}} & \text{if } i \in \underline{\mathrm{I}}, \end{cases} \\ a_{\underline{\mathrm{I}}\underline{\mathrm{J}}} a_i &= \begin{cases} a_{(\underline{\mathrm{I}} \cup \{i+1\})(\underline{\mathrm{J}} \cup \{i\})} & \text{if } i+1 \notin \underline{\mathrm{J}}, \\ a_{\underline{\mathrm{I}}((\underline{\mathrm{J}} \setminus \{i+1\}) \cup \{i\})} & \text{if } i+1 \in \underline{\mathrm{J}}. \end{cases} \end{aligned}$$

Since we have shown already that $i+1 \notin \underline{\mathrm{J}}$, we know that $a_i z = z a_i$ is nothing but the equality

$$a_i \left(\sum_{\substack{\underline{\mathrm{I}},\ \underline{\mathrm{J}} \\ i+1 \notin \underline{\mathrm{J}} \\ i \notin \underline{\mathrm{I}}}} c_{\underline{\mathrm{I}}\underline{\mathrm{J}}} a_{\underline{\mathrm{I}}\underline{\mathrm{J}}} + \sum_{\substack{\underline{\mathrm{I}},\ \underline{\mathrm{J}} \\ i+1 \notin \underline{\mathrm{J}} \\ i \in \underline{\mathrm{I}}}} c_{\underline{\mathrm{I}}\underline{\mathrm{J}}} a_{\underline{\mathrm{I}}\underline{\mathrm{J}}} \right) = \left(\sum_{\substack{\underline{\mathrm{I}},\ \underline{\mathrm{J}} \\ i+1 \notin \underline{\mathrm{J}} \\ i \notin \underline{\mathrm{I}}}} c_{\underline{\mathrm{I}}\underline{\mathrm{J}}} a_{\underline{\mathrm{I}}\underline{\mathrm{J}}} + \sum_{\substack{\underline{\mathrm{I}},\ \underline{\mathrm{J}} \\ i+1 \notin \underline{\mathrm{J}} \\ i \in \underline{\mathrm{I}}}} c_{\underline{\mathrm{I}}\underline{\mathrm{J}}} a_{\underline{\mathrm{I}}\underline{\mathrm{J}}} \right) a_i.$$

This in turn is equivalent to the equality

$$a_i \left(\sum_{\substack{\underline{\mathrm{I}},\ \underline{\mathrm{J}} \\ i+1 \notin \underline{\mathrm{J}} \\ i \in \underline{\mathrm{I}}}} c_{\underline{\mathrm{I}}\underline{\mathrm{J}}} a_{\underline{\mathrm{I}}\underline{\mathrm{J}}} \right) = \left(\sum_{\substack{\underline{\mathrm{I}},\ \underline{\mathrm{J}} \\ i+1 \notin \underline{\mathrm{J}} \\ i \in \underline{\mathrm{I}}}} c_{\underline{\mathrm{I}}\underline{\mathrm{J}}} a_{\underline{\mathrm{I}}\underline{\mathrm{J}}} \right) a_i,$$

which can be rewritten as

$$\sum_{\substack{\underline{\mathrm{I}},\ \underline{\mathrm{J}} \\ i+1 \notin \underline{\mathrm{J}} \\ i \in \underline{\mathrm{I}}}} c_{\underline{\mathrm{I}}\underline{\mathrm{J}}} a_{((\underline{\mathrm{I}} \setminus \{i\}) \cup \{i+1\})\underline{\mathrm{J}}} = \sum_{\substack{\underline{\mathrm{I}},\ \underline{\mathrm{J}} \\ i+1 \notin \underline{\mathrm{J}} \\ i \in \underline{\mathrm{I}}}} c_{\underline{\mathrm{I}}\underline{\mathrm{J}}} a_{(\underline{\mathrm{I}} \cup \{i+1\})(\underline{\mathrm{J}} \cup \{i\})}. \tag{I.3.17}$$

Again, we observe that the set of monomials $\{a_{((\underline{\mathrm{I}} \setminus \{i\}) \cup \{i+1\})\underline{\mathrm{J}}} \mid i+1 \notin \underline{\mathrm{J}},\ i \in \underline{\mathrm{I}}\}$ is linearly independent for fixed i.

By induction on the number k_i' of particles at position i in $\underline{\mathrm{I}}$ we see that all coefficients $c_{\underline{\mathrm{I}}\underline{\mathrm{J}}}$ are zero for $k_i' \geq 1$:
For $k_i' = 1$, the set $(\underline{\mathrm{I}} \setminus \{i\}) \cup \{i+1\}$ does not contain any particle at position i any more. Hence the monomial $a_{((\underline{\mathrm{I}} \setminus \{i\}) \cup \{i+1\})\underline{\mathrm{J}}}$ cannot appear in the right sum in equation (I.3.17), and its coefficient $c_{\underline{\mathrm{I}}\underline{\mathrm{J}}}$ must have been zero. For the induction step we assume that

the coefficients for all $a_{\underline{\mathrm{IJ}}}$ with at most k_i' particles at position i in the output particle configuration $\underline{\mathrm{I}}$ are zero. Consider some $a_{\underline{\mathrm{IJ}}}$ with $k_i' + 1$ particles at position i in $\underline{\mathrm{I}}$. So the set $(\underline{\mathrm{I}} \setminus \{i\}) \cup \{i+1\}$ contains k_i' particles at position i in $\underline{\mathrm{J}}$, and the monomial $a_{((\underline{\mathrm{I}} \setminus \{i\}) \cup \{i+1\})\underline{\mathrm{J}}}$ cannot appear in the sum (I.3.16). Again we see that its coefficient $c_{\underline{\mathrm{IJ}}}$ must have been zero.

We have deduced now that only those monomials labelled by minimal input particle configurations $\underline{\mathrm{J}} = (r, 0, \dots, 0)$ and output particle configuration $\underline{\mathrm{I}} = (0, \dots, 0, s)$ may have nonzero coefficients. Since the number of particles has to be the same in $\underline{\mathrm{I}}$ and $\underline{\mathrm{J}}$, any central element is of the form

$$\sum_{r \in \mathbb{Z}_{\geq 0}} c_{(0,\dots,0,r)(r,0,\dots,0)} a_{(0,\dots,0,r)(r,0,\dots,0)}$$

as claimed. $\square$

I.3.5.2 Remark. In the proof of Theorem I.3.5.1 one has to be careful: One cannot simply compare the coefficients in equalities of the form

$$a_i \left(\sum c_{\underline{\mathrm{IJ}}} a_{\underline{\mathrm{IJ}}}\right) = \left(\sum c_{\underline{\mathrm{IJ}}} a_{\underline{\mathrm{IJ}}}\right) a_i$$

since the partic algebra $\mathcal{P}_N^{\mathrm{part}}$ has zero divisors, see Example I.3.3.4. Therefore, when we consider the coefficients $c_{\underline{\mathrm{IJ}}}$, we first have to determine linearly independent sets of monomials, e.g. of the form

$$\{a_{((\underline{\mathrm{I}} \setminus \{i\}) \cup \{i+1\})\underline{\mathrm{J}}} \mid i+1 \notin \underline{\mathrm{J}},\ i \in \underline{\mathrm{I}}\}.$$

This is in fact an application of the faithfulness result from Theorem I.3.4.2 combined with the normal form for monomials from Theorem I.3.3.1. $\lozenge$

I.3.5.3 Remark. The partic algebra is not finitely generated over its center: The center is concentrated in degree $\mathbb{Z}_{\geq 0} \cdot (1, \dots 1)$ with respect to the $\mathbb{Z}^{N-1}$-grading from Remark I.3.2.3. On the other hand one can see from the normal form in Proposition I.3.3.2 that all $\mathbb{Z}_{\geq 0}^{N-1}$-graded components of the partic algebra are nontrivial, hence the partic algebra cannot be finitely generated over its degree $\mathbb{Z}_{\geq 0} \cdot (1, \dots 1)$ component. $\lozenge$

I.3.6. The affine partic algebra

In this section we discuss the affine situation. Analogously to the classical case we introduce the following quotient of the affine plactic algebra $\widehat{\mathcal{P}}_N$ defined in Definition I.3.1.2:

I.3.6.1 Definition. Define the affine partic algebra $\widehat{\mathcal{P}}_N^{\mathrm{part}}$ to be the quotient of $\widehat{\mathcal{P}}_N$ by the additional relations

$$a_i a_{i-1} a_{i+1} a_i = a_{i+1} a_i a_{i-1} a_i \quad \text{for } i-1, i, i+1 \in \mathbb{Z}/N\mathbb{Z}. \tag{I.3.18}$$

The affine plactic algebra and the affine partic algebra both act on the polynomial ring $\Bbbk[x_1, \ldots, x_N, q]$ in $N+1$ variables as follows:

$$a_i \cdot x_1^{k_1} \ldots x_N^{k_N} q^t = \begin{cases} x_1^{k_1} \ldots x_i^{k_i - 1} x_{i+1}^{k_{i+1}+1} \ldots x_N^{k_N} q^t & \text{if } k_i > 0, \\ 0 & \text{else,} \end{cases} \tag{I.3.19}$$

$$a_0 \cdot x_1^{k_1} \ldots x_N^{k_N} q^t = \begin{cases} x_1^{k_1+1} \ldots x_N^{k_N - 1} q^{t+1} & \text{if } k_N > 0, \\ 0 & \text{else.} \end{cases} \tag{I.3.20}$$

Again one can verify by calculation that this defines indeed an action of the affine plactic algebra which factors over the affine partic algebra:

I.3.6.2 Lemma. Equations (I.3.19) and (I.3.20) define an action of the affine plactic algebra $\widehat{\mathcal{P}}_N$ on the polynomial ring $\Bbbk[x_1, \ldots, x_N, q]$. This action factors over an action of the affine partic algebra $\widehat{\mathcal{P}}_N^{\mathrm{part}}$.

We call this representation the affine bosonic particle representation of $\widehat{\mathcal{P}}_N$ or $\widehat{\mathcal{P}}_N^{\mathrm{part}}$, respectively. We use the shorthand notation $\underline{\mathrm{I}} = (k_1, \ldots, k_N) \in \mathbb{Z}_{\geq 0}^N$ for the monomial $x_1^{k_1} \ldots x_N^{k_N}$ and refer to it as (affine bosonic) particle configuration. Similar to the classical bosonic case discussed in Section I.3.4 we can identify a monomial $x_1^{k_1} \ldots x_N^{k_N}$ with a particle configuration on a circle with N positions, with k_i particles lying at position i. The indeterminate q protocols how often we apply a_0 to a particle configuration.

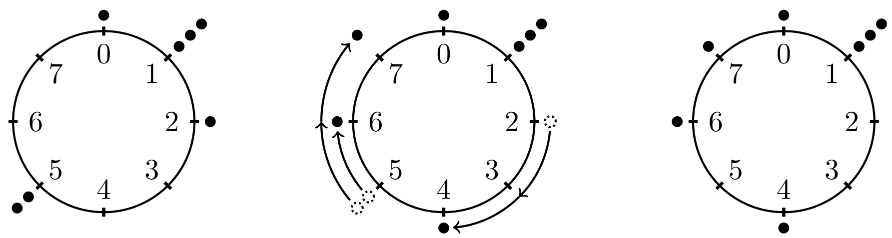


Figure I.3.6.1.: Example for $N = 8$: Application of $a_6 a_5 a_3 a_2 a_5$ to the particle configuration $(3, 1, 0, 0, 2, 0, 0, 1)$ gives $(3, 0, 0, 1, 0, 1, 1, 1)$.

Unlike in the case of the fermionic particle representation discussed in Chapter I.2, the *affine* bosonic particle representation is very different from the *classical* bosonic particle configuration. In the fermionic case, we have a faithful action of the affine/finite nilTemperley-Lieb algebra on affine/finite fermionic particle configurations, respectively. In the bosonic case, although we have a faithful action of the partic algebra on classical bosonic particle configurations, this is no longer true for the action of the affine partic algebra on affine bosonic particle configurations:

Additionally, we get an infinite family of relations of the form

$$\begin{aligned} & a_{i+1}^m a_{i+2}^m \dots a_{i-2}^m a_{i-1}^m a_i^{2m} a_{i+1}^m a_{i+2}^m \dots a_{i-2}^m a_{i-1}^m \\ = \; & a_{j+1}^m a_{j+2}^m \dots a_{j-2}^m a_{j-1}^m a_j^{2m} a_{j+1}^m a_{j+2}^m \dots a_{j-2}^m a_{j-1}^m \quad \text{for all } i, j \in \mathbb{Z}/N\mathbb{Z},\ m \in \mathbb{Z}_{\geq 1}, \end{aligned} \tag{I.3.21}$$

and in particular, neither the affine plactic nor the affine partic algebra act faithfully on the affine bosonic particle representation. Faithfulness of the affine plactic algebra action was claimed in [KS10, Proposition 5.8].

On the affine bosonic particle configuration the relation from equation (I.3.21) can be visualized as depicted in Figure I.3.6.2: The minimal particle configuration on which any such monomial $a_{i+1}^m a_{i+2}^m \dots a_{i-2}^m a_{i-1}^m a_i^{2m} a_{i+1}^m a_{i+2}^m \dots a_{i-2}^m a_{i-1}^m$ acts nontrivially is given by $(1, 1, \dots, 1, 1)$, i.e. one particle at each position. Each particle is moved by two steps in total. The output configuration is the same as the input configuration, namely $(1, 1, \dots, 1, 1)$, which we have to multiply by q^{2m}. We see immediately that i is not recorded by the minimal input configuration, the output configuration or the power of q. Therefore, these monomials cannot be distinguished by the affine bosonic particle representation.

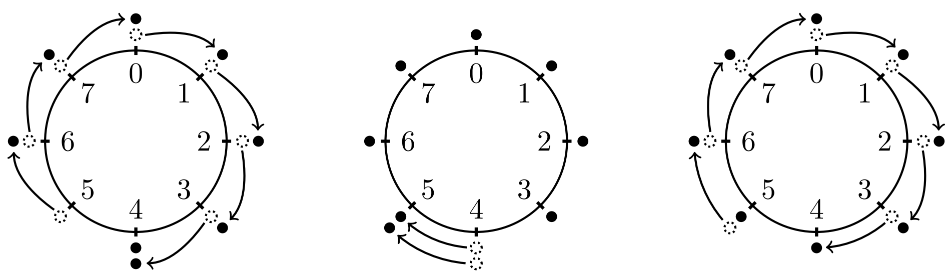


Figure I.3.6.2.: Example for $N = 8$, $i = 4$, $m = 1$: Application of $a_{i+1}a_{i+2}\dots a_{i-2}a_{i-1}$ followed by a_i^2 followed by $a_{i+1}a_{i+2}\dots a_{i-2}a_{i-1}$ to the particle configuration $(1, 1, 1, 1, 1, 1, 1, 1)$ gives $(1, 1, 1, 1, 1, 1, 1, 1)$ (multiplied by an additional factor q^2 that we omit in the picture).

It follows that it is much harder to find a normal form for the affine partic algebra: We cannot expect a labelling of monomials by input/output particle configurations together with a power of q as in the case of the (affine) nilTemperley-Lieb algebra and the partic algebra. This labelling would be equivalent to faithfulness of the particle representation. Such a labelling allowed us to reorder monomials so that the indices of the rightmost factors correspond to the the minimal particle configuration on which the monomial acts nontrivially, see Proposition I.3.4.5, Section I.2.6, and also compare with Theorem I.2.10.1. The whole approach fails for the affine partic algebra:

The minimal particle configuration does not indicate a natural reordering of factors. For example, consider the following monomials in $\widehat{\mathcal{P}}_5^{\mathrm{part}}$:

$$a_1^2 a_2 a_3 a_4 a_0,\ a_2^2 a_3 a_4 a_0 a_1,\ a_3^2 a_4 a_0 a_1 a_2,\ a_4^2 a_0 a_1 a_2 a_3,\ a_0^2 a_1 a_2 a_3 a_4.$$

All of these expect precisely one particle at each position $1, 2, 3, 4, 0$ in the minimal particle configuration, but none of them can be reordered in any way. Of course the output configurations are all different, so these five different monomials could even be distinguished by the affine particle representation. In fact one cannot expect a basis labelled by tuples of a minimal input configuration, the output configuration, and some power of q.

Part II.

Generalized Weyl algebras

II.1. A Duflo theorem for a class of generalized Weyl algebras

II.1.1. An overview of Duflo type theorems

Let $\Bbbk$ be an algebraically closed field of characteristic 0. For the universal enveloping algebra of a semisimple Lie algebra over $\Bbbk$, Duflo's Theorem [Duf77] states that all its primitive ideals (i.e. the annihilators of simple modules) are given by the annihilators of simple highest weight modules. In contrast, the simple modules themselves are far from being classified in general. Fortunately, for several other classes of algebras the notion of a highest weight module makes sense and the analogue of Duflo's theorem holds:

In [Smi90], Smith introduced a family of algebras similar to $\mathcal{U}(\mathfrak{sl}_2)$. These are $\mathbb{C}$-algebras generated by three elements E, F, H subject to the relations $[H, E] = E$, $[H, F] = -F$ and $[E, F] = f(H)$ where f can be any polynomial. They share many properties with $\mathcal{U}(\mathfrak{sl}_2)$ (which is of course included in this family for $f(H) = 2H$). In particular it is straightforward to generalize the notion of highest weight modules to these algebras and indeed all primitive ideals are given by annihilators of highest weight modules (see [Smi90, Theorem 3.3]).

For classical simple Lie superalgebras, Musson defines $\mathbb{Z}/2\mathbb{Z}$-graded highest weight modules depending on a choice of a triangular decomposition. Then all $\mathbb{Z}/2\mathbb{Z}$-graded primitive ideals in the universal enveloping algebra of a classical simple Lie superalgebra are given by the annihilators of $\mathbb{Z}/2\mathbb{Z}$-graded simple highest weight modules (see [Mus92, Theorem 2.2]).

In [MB98], Musson and Van den Bergh introduce algebras that, roughly speaking, allow a weight space decomposition with weight spaces cyclic over a commutative subalgebra. This class of algebras is closed under taking certain graded subalgebras, tensor products and central quotients. They show that (under some further assumptions, see Theorem II.1.4.4 for details) all prime, hence all primitive ideals are given by the annihilators of simple weight modules. In particular, this applies to localizations of Weyl algebras and their central subquotients (see [MB98, Chapter 6]). Note that for a classical Weyl algebra, given by differential operators on a polynomial ring in n variables, the primitive ideals are not very interesting: These algebras are simple, i.e. the only proper twosided ideal is the zero ideal. Since an annihilator is always twosided, the only primitive ideal of a classical Weyl algebra is the zero ideal.

Now it is natural to ask whether an analogous statement holds for generalized Weyl algebras, a class of algebras that includes many interesting examples, in particular Smith's generalizations of $\mathcal{U}(\mathfrak{sl}_2)$. These noncommutative algebras are generated by a commutative $\Bbbk$-algebra R together with $2n$ elements $X_1, \dots, X_n$, $Y_1, \dots, Y_n$. For the relations see Section II.1.2. They are $\mathbb{Z}^n$-graded by setting $\deg(X_i) = e_i$, $\deg(Y_i) = -e_i$ where e_i denotes the i-th standard basis vector in $\mathbb{Z}^n$. Each graded component is a cyclic R-module. In this situation, we can define highest weight modules and formulate a Duflo theorem. We prove it for a special class of generalized Weyl algebras using a theorem by [MB98] that relates the annihilator of a simple weight module to its support and obtain as main result (see Theorem II.1.4.1):

Theorem. Let $A = R(\sigma, t)$ be a GWA of rank n as defined in Section II.1.2 where we assume $R = \Bbbk[T_1, \dots, T_n]$, $\sigma_i(T_j) = T_j - \delta_{ij} b_i$ for $b_i \in \Bbbk \setminus \{0\}$ and $t_i \in \Bbbk[T_i] \subset \Bbbk[T_1, \dots, T_n]$, $t_i \notin \Bbbk$. Then all primitive ideals of A, i.e. the annihilator ideals of simple A-modules, are given by the annihilators of simple highest weight A-modules $L(\mathfrak{m})$ of highest weight $\mathfrak{m} \in \operatorname{mspec}(R)$.

In Section II.1.2 we recall the definition of generalized Weyl algebras, define highest weight modules and discuss graded modules over generalized Weyl algebras. We characterize moreover the highest weight modules as those modules with a locally nilpotent action of the X_i. In Section II.1.4 we formulate and prove the main theorem. The principal tool is the Duflo type theorem using *weight* modules from [MB98]. We show it applies to our situation and improve it by showing that it is enough to consider the much smaller class of *highest weight* modules (as in the classical Duflo theorem). In Section II.1.5 we finally give some examples to illustrate the relationship between the annihilator and the support of simple highest weight modules.

II.1.2. Generalized Weyl algebras and graded modules

II.1.2.1. Definition of a GWA and first observations

Fix a base field $\Bbbk = \overline{\Bbbk}$ of characteristic 0. Fix a unital associative commutative $\Bbbk$-algebra R that is a noetherian domain. Given n nonzero elements $t = (t_1, \ldots, t_n)$ in R and n pairwise commuting algebra automorphisms $\sigma = (\sigma_1, \ldots, \sigma_n)$ in $\mathrm{Aut}(R)$ such that $\sigma_i(t_j) = t_j$ for all $i \neq j$, define the corresponding ***generalized Weyl algebra (GWA)*** $A = R(\sigma, t)$ of rank n as follows: It is the $\Bbbk$-algebra generated over R by $2n$ generators X_i, Y_i, $1 \leq i \leq n$ with relations given by

$$\begin{aligned} X_i r &= \sigma_i(r) X_i, & X_i Y_i &= \sigma_i(t_i), & [X_i, X_j] &= 0, \\ Y_i r &= \sigma_i^{-1}(r) Y_i, & Y_i X_i &= t_i, & [Y_i, Y_j] &= 0, \\ & & & & [X_i, Y_j] &= 0 \end{aligned}$$

for all $1 \leq i, j \leq n$ with $i \neq j$ and all $r \in R$. It was introduced originally by Bavula in [Bav92].
The GWA $A = \bigoplus_{\alpha \in \mathbb{Z}^n} R \cdot a^\alpha$ is a left and right R-module with generators

$$a^\alpha = a_1^{\alpha_1} \cdot \ldots \cdot a_n^{\alpha_n}, \quad a_i^{\alpha_i} = \begin{cases} X_i^{\alpha_i} & \text{for } \alpha_i \geq 0 \\ Y_i^{|\alpha_i|} & \text{for } \alpha_i < 0. \end{cases}$$

Denote $A_\alpha = R \cdot a^\alpha$. Since $A_\alpha \cdot A_\beta \in A_{\alpha+\beta}$, any GWA A is a $\mathbb{Z}^n$-graded algebra with $\deg(X_i) = e_i$ and $\deg(Y_i) = -e_i$ where we denote by e_i the i-th standard basis vector of $\mathbb{Z}^n$, see eg. [Bav92, Section 1.1]. The degree 0 part of A is given by $A_0 = R$. Notice that the $\sigma_1, \ldots, \sigma_n$ from the defining data of a GWA $A = R(\sigma, t)$ give a $\mathbb{Z}^n$-action on R by $e_i \mapsto \sigma_i$. Write $\sigma^\alpha = \sigma_1^{\alpha_1} \cdot \ldots \cdot \sigma_n^{\alpha_n}$. The following lemma summarizes [Bav92, Proposition 1.3 (1)] and [Bav92, p. 1.1], [BO09].

II.1.2.1 Lemma. Let $A = R(\sigma, t)$ be a GWA of finite rank. Then A is left and right noetherian, and the tensor product over R or $\Bbbk$ of two GWA's is again a GWA.

We give a detailed proof of the following statement from [BB00, Proposition 1.3 (2)] or [BO09, Lemma 2.3]:

II.1.2.2 Lemma. Let $A = R(\sigma, t)$ be a GWA, in particular R is a domain, $t_i \neq 0$ for all i and $\sigma_i(t_j) = t_j$ for all $i \neq j$. Then $A = \bigoplus_{\alpha \in \mathbb{Z}^n} R \cdot a^\alpha$ is a free left and right R-module, and A is a domain.

Proof. First we show that $A \cong R \otimes \Bbbk[Z_1^{\pm1}, \dots, Z_n^{\pm1}]$ as $\Bbbk$-vector spaces. This implies $R{\cdot}a^\alpha \cong R$ and freeness of A as a left R-module. Freeness as a right R-module follows since σ^α is a ring automorphism of R. For the proof, construct the following representation of A: Consider the semidirect product $R \ltimes \Bbbk[Z_1^{\pm1}, \dots, Z_n^{\pm1}]$ with respect ot the action of $\Bbbk[Z_1^{\pm1}, \dots, Z_n^{\pm1}]$ on R given by $Z_i^{\pm1} \cdot r = \sigma_i^{\pm1}(r)$. This is well defined since σ_i, σ_j are pairwise commuting automorphisms of R for all i, j. Now define an A-module structure on $R \ltimes \Bbbk[Z_1^{\pm1}, \dots, Z_n^{\pm1}]$ by

$$r(s,v) = (rs, v), \quad X_i(s,v) = (\sigma_i(s), Z_i v), \quad Y_i(s,v) = (\sigma_i^{-1}(s) \cdot t_i, Z_i^{-1} v)$$

for $(s,v) \in R \ltimes \Bbbk[Z_1^{\pm1}, \dots, Z_n^{\pm1}]$. One quickly checks that this defines indeed an A-module structure because all defining GWA relations are satisfied. Here one needs again that $\sigma_i\sigma_j = \sigma_j\sigma_i$, and additionally $\sigma_i(t_j) = t_j$ for all $i \neq j$. Next, one observes that this is a faithful representation of A. Then $a \mapsto a(1,1)$ gives the desired isomorphism $A \cong R \otimes \Bbbk[Z_1^{\pm1}, \dots, Z_n^{\pm1}]$.
From $R \cdot a^\alpha \cong R$ it follows that A is a domain, i.e. for any $b, b' \in A \smallsetminus \{0\}$ their product is nonzero: Write $b = \sum_\alpha r_\alpha a^\alpha$ and $b' = \sum_{\alpha'} r'_\alpha a^{\alpha'}$. By lexicographic ordering we can compare the multiindices that appear, and we have $bb' = 0$ if and only if $r_\beta a^\beta \cdot r_{\beta'} a^{\beta'} = 0$ for $r_\beta, r_{\beta'} \neq 0$, where $\beta > \alpha$ for all $\alpha \neq \beta$ with $r_\alpha \neq 0$, $\alpha' \neq \beta'$ with $r_{\alpha'} \neq 0$. Using the defining relations of the GWA A we compute $r_\beta a^\beta \cdot r_{\beta'} a^{\beta'} = r_\beta \sigma^\beta(r_{\beta'}) s a^{\beta+\beta'}$, where $s = s_1 \dots s_n \in R$ is given by

$$s_i = \begin{cases} 1 & \text{if } \beta_i, \beta'_i \text{ have the same parity,} \\ \sigma_i^{\beta_i - 1}(t_i) \cdot \ldots \cdot \sigma_i^1(t_i) \cdot t_i & \text{if } |\beta_i| \leq |\beta'_i| \text{ and } \beta_i \geq 0, \beta'_i < 0, \\ \sigma_i^{\beta_i + 1}(t_i) \cdot \ldots \cdot \sigma_i^{-1}(t_i) & \text{if } |\beta_i| \leq |\beta'_i| \text{ and } \beta_i < 0, \beta'_i \geq 0, \\ \sigma_i^{\beta'_i - 1}(t_i) \cdot \ldots \cdot \sigma_i^1(t_i) \cdot t_i & \text{if } |\beta'_i| \leq |\beta_i| \text{ and } \beta'_i \geq 0, \beta_i < 0, \\ \sigma_i^{\beta'_i + 1}(t_i) \cdot \ldots \cdot \sigma_i^{-1}(t_i) & \text{if } |\beta'_i| \leq |\beta_i| \text{ and } \beta'_i < 0, \beta_i \geq 0. \end{cases}$$

as shown in the proof for [BO09, Lemma 2.3]. By $R \cdot a^\alpha \cong R$, injectivity of σ^β and since R is a domain and $t_i \neq 0$ for all i, we get that $r_\beta a^\beta \cdot r_{\beta'} a^{\beta'}$ is nonzero if and only if $r_\beta, r_{\beta'} \neq 0$. This concludes the proof. □

Observe that the condition $\sigma_i(t_j) = t_j$ for all $i \neq j$ is essential for A being a domain: Otherwise $(\sigma_i(t_j) - t_j)X_i = X_i Y_j X_j - Y_j X_j X_i = X_i Y_j X_j - X_i Y_j X_j = 0$ is a counterexample.

II.1.2.2. A special class of GWA's

We confine ourselves to the study of GWA's with base ring $R = \Bbbk[T_1, \dots, T_n]$, automorphisms $\sigma_i(T_j) = T_j - \delta_{ij} b_i$ for $b_i \in \Bbbk \smallsetminus \{0\}$ and $t_i \in \Bbbk[T_i] \subset \Bbbk[T_1, \dots, T_n]$, $t_i \notin \Bbbk$. This is the tensor product of n GWA's of rank 1 over the polynomial ring in one variable $\Bbbk[T]$, with $\sigma \in \mathrm{Aut}(\Bbbk[T])$ of the form $T \mapsto T - b$ for some $b \neq 0$ in $\Bbbk$ and a nonconstant element $t \in \Bbbk[T]$. As $\Bbbk$ is algebraically closed, we can factorize $t = (T - z_1) \cdot \ldots \cdot (T - z_s)$ for some $z_1, \ldots, z_s \in \Bbbk$ (multiplying t by some nonzero scalar would give an isomorphic GWA, so we can assume this scalar is 1).

II.1.2.3 Remark. With this choice of $\sigma_1, \ldots, \sigma_n$ the $\mathbb{Z}^n$-action on R is free on $R \smallsetminus \Bbbk$ (on $\Bbbk$ the action is trivial). Additionally, the $\mathbb{Z}^n$-action on $\mathrm{mspec}(R)$ given by $\alpha_\bullet \mathfrak{m} := \sigma^\alpha(\mathfrak{m})$ is free. As freeness is defined pointwise, every orbit $\{\sigma^\alpha(\mathfrak{m}) \mid \alpha \in \mathbb{Z}\}$ is infinite. So we only deal with pure translations, i.e. $a = 1$ in a general automorphism $\sigma : T \mapsto aT - b$, $a \neq 0$, of $\Bbbk[T]$. For the application of [MB98], we need to work with $\mathbb{Z}$-lattices, and we want to keep things easy. $\Diamond$

II.1.2.3. Weight modules

In this section, $A = R(\sigma, t)$ can be any GWA. By a module, we always mean a left module unless stated otherwise. Denote by $\mathrm{mspec}(R)$ the set of maximal ideals of R. For $\mathfrak{m} \in \mathrm{mspec}(R)$ define the $\mathfrak{m}$-***weight space*** of an A-module M to be

$$M_\mathfrak{m} \;=\; \{v \in M \mid \mathfrak{m} \cdot v = 0\}$$

and say that M is a ***weight module*** if M decomposes as vector space into its weight spaces $M = \sum_{\mathfrak{m} \in \mathrm{mspec}(R)} M_\mathfrak{m}$. Define the ***support*** of the weight module M to be

$$\mathrm{Supp}\,(M) \;=\; \{\mathfrak{m} \in \mathrm{mspec}(R) \mid M_\mathfrak{m} \neq 0\}.$$

Furthermore, for a weight A-module M we have $X_i(M_\mathfrak{m}) \subset M_{\sigma_i(\mathfrak{m})}$ and $Y_i(M_\mathfrak{m}) \subset M_{\sigma_i^{-1}(\mathfrak{m})}$. In other words, $A_\alpha \cdot M_\mathfrak{m} \subset M_{\sigma^\alpha(\mathfrak{m})}$ for $\alpha \in \mathbb{Z}^n$. M is called a ***highest weight*** A-module if it is generated as A-module by $M_\mathfrak{m}$ and $X_i \cdot M_\mathfrak{m} = 0$ for all $1 \leq i \leq n$. In particular, for the support of a highest weight module M we have $\mathrm{Supp}(M) \subset \{\sigma^\alpha(\mathfrak{m}) \mid \alpha \in \mathbb{Z}^n_{\leq 0}\}$.

II.1.2.4 Lemma. Let A be a GWA of finite rank.

i) Let M be a weight A-module. Then $M = \bigoplus_{\mathfrak{m} \in \mathrm{mspec}(R)} M_\mathfrak{m}$.
ii) Let M be a weight A-module, $U \subset M$ some A-submodule. Then U and hence M/U inherit the weight decomposition from M, i.e. U is a homogeneous submodule.
iii) Let M, N be weight A-modules and $f : M \to N$ be a homomorphism of A-modules. Then $f(M_\mathfrak{m}) \subset N_\mathfrak{m}$.

Proof. The proof uses standard arguments.

i) Let $v_1 + \ldots + v_n = 0$ with $v_i \in M_{\mathfrak{m}_i}$, i.e. $\mathfrak{m}_i \cdot v_i = 0$, and assume $\mathfrak{m}_i \neq \mathfrak{m}_j$ for all $i \neq j$. In particular, $\prod_{i \neq j} \mathfrak{m}_i \not\subset \mathfrak{m}_j$ because $\mathfrak{m}_j$ is maximal and hence prime. Each v_j is zero: As $\prod_{i \neq j} \mathfrak{m}_i \ni r$ annihilates all v_i, $i \neq j$, we get $0 = r \cdot (v_1 + \ldots + v_n) = r \cdot v_j$. So v_j is annihilated by $\prod_{i \neq j} \mathfrak{m}_i$ and $\mathfrak{m}_j$ which generate the whole R. In particular $1 \cdot v_j = v_j = 0$ and the sum is direct.

ii) We have to check that $U = \bigoplus_{\mathfrak{m} \in \operatorname{mspec}(R)} U_{\mathfrak{m}}$ with $U_{\mathfrak{m}} := U \cap M_{\mathfrak{m}}$. Decompose $v \in U$ as element of M into $v = v_1 + \ldots + v_n$ with nonzero $v_j \in M_{\mathfrak{m}_j}$. We show by a diagonal argument that $v_j \in U$ for all j. Take some element $r := \prod_{i \neq j} r_i$, where the r_i are some nonzero elements of the maximal ideals $\mathfrak{m}_i$. Hence r is nonzero, $r \cdot v_j \neq 0$ and $r \notin \mathfrak{m}_j$. Thus there is some $r' \in R$ with $r'r = 1 \in \Bbbk \cong R/\mathfrak{m}_j$. We get $r'r \cdot v = r'r \cdot v_j = v_j$ because $\mathfrak{m}_j$ annihilates v_j. Therefore $v_j \in U$. It follows that M/U is isomorphic to $\bigoplus_{\mathfrak{m} \in \operatorname{mspec}(R)} M_{\mathfrak{m}}/U_{\mathfrak{m}}$.

iii) Since f is an A-module homomorphism, $\mathfrak{m} \cdot f(v) = f(\mathfrak{m} \cdot v)$ for all $v \in M$. Hence $\mathfrak{m} \cdot f(M_{\mathfrak{m}}) = 0$, in other words, $f(M_{\mathfrak{m}}) \subset N_{\mathfrak{m}}$. □

From the lemma it follows that the weight A-modules together with A-module homomorphisms that preserve the weight spaces form a full abelian subcategory of the category of left A-modules.

II.1.2.4. A characterization of highest weight modules for special GWA's

Here, A is a special GWA as defined in Section II.1.2.2. The following lemma characterizes highest weight A-modules. A similar result for Lie algebras can be found in [MZ13].

II.1.2.5 Proposition. Let M be a simple left A-module. The following are equivalent:

i) M is a highest weight module.

ii) For all $1 \leq i \leq n$, the action of X_i on M is locally nilpotent, i.e. for every $v \in M$ there exists a natural number k_i such that $X_i^{k_i} \cdot v = 0$.

iii) There exists $v \in M$ such that X_i acts nilpotently on v for all $1 \leq i \leq n$.

Proof. (i)⇒(ii): Let M be a highest weight module with highest weight $\mathfrak{m}$ and weight space decomposition $M = \bigoplus_{\alpha \in \mathbb{Z}^n_{\leq 0}} M_{\sigma^\alpha(\mathfrak{m})}$. So any $v \in M$ decomposes as $v = v_{\alpha(1)} + \ldots + v_{\alpha(r)}$ for weight vectors $v_{\alpha(j)} \in M_{\sigma^{\alpha(j)}(\mathfrak{m})}$. In particular, $X^{-\alpha(j)} \cdot v_{\alpha(j)} = a^{-\alpha(j)} \cdot v_{\alpha(j)} \in M_{\mathfrak{m}}$, which is the highest weight space, hence $X_i^{-\alpha(j)_i+1} \cdot v_{\alpha(j)} = 0$. Now choose $k_i \in \mathbb{Z}$ such that $k_i \geq -\alpha(j)_i + 1$ for all j. Then $X_i^{k_i} \cdot v_{\alpha(j)} = 0$ for all j and therefore $X_i^{k_i} \cdot v = 0$.

(ii)⇒(iii): Clear.

(iii)⇒(i): Assume we have an element $v \in M$ such that $X_i^{k_i} \cdot v = 0$ for some natural numbers k_i. We construct a nonzero element v' in M that is annihilated by all X_i and a maximal ideal $\mathfrak{m}$. Since M is simple, this suffices to prove that

$$M = A \cdot v' = \bigoplus_{\alpha \in \mathbb{Z}^n_{\leq 0}} A_\alpha v' = \bigoplus_{\alpha \in \mathbb{Z}^n_{\leq 0}} M_{\sigma^\alpha(\mathfrak{m})}$$

is a highest weight module of highest weight $\mathfrak{m}$. Since the X_i commute, we can find for all i a natural number β_i such that $0 \leq \beta_i < k_i$ and $\widetilde{v} := X_1^{\beta_1} \dots X_n^{\beta_n} \cdot v \neq 0$ but $X_i \cdot \widetilde{v} = 0$ for all i. Hence

$$t_i \cdot \widetilde{v} = Y_i X_i \cdot \widetilde{v} = 0.$$

Now according to our assumption $t_i \in \Bbbk[T_i]$ is a polynomial, say $t_i = (T_i - a(i)_1) \dots (T_i - a(i)_{s(i)})$ for some $s(i) \in \mathbb{Z}_{>0}$ and $a(i)_r \in \mathbb{C}$. So there is a linear factor $(T_i - a(i)_{r(i)})$ such that

$$\begin{aligned} v(i) &:= (T_i - a(i)_{r(i)+1}) \dots (T_i - a(i)_{s(i)})\widetilde{v} \neq 0, \quad \text{and} \\ (T_i - a(i)_{r(i)})(T_i - a(i)_{r(i)+1}) \dots (T_i - a(i)_{s(i)})\widetilde{v} &= 0. \end{aligned}$$

In this way we construct successively nonzero elements $v(1), \dots, v(n)$ in M that are annihilated by all X_i since they differ from $\widetilde{v}$ only by multiplication with elements in the base ring R. Furthermore, $v' := v(n)$ is annihilated by the maximal ideal $\mathfrak{m} := \big(T_1 - a(1)_{r(1)}, \dots, T_n - a(n)_{r(n)}\big)$. □

II.1.2.5. Side remark on generalized gradings

Theorem II.1.4.4 describes primitive ideals of graded algebras in terms of annihilators of *graded* simple modules. Although GWA's are $\mathbb{Z}^n$-graded, their weight modules are not $\mathbb{Z}^n$-graded in general. Instead, weight modules M decompose into weight spaces $M_\mathfrak{m}$ indexed by $\mathrm{mspec}(R)$. It makes sense to think of a weight module as a graded module, but instead of the usual notion of graded modules over a graded algebra, where both objects are graded over the same additive group, one needs to generalize it as follows:

II.1.2.6 Definition. Let G be an abelian group and X be a set with G-action. Let $A = \bigoplus_{g \in G} A_g$ be a G-graded algebra. Then a $(G \circlearrowleft X)$-graded module (or a module with

X-grading respecting the G-action) is an A-module M with a decomposition $M = \bigoplus_{x \in X} M_x$ such that $A_g \cdot M_x \subset M_{gx}$.

This kind of graded modules was studied in [NRVO90], motivated by G-graded modules over the group algebra $\Bbbk[G]$ of a group G: Take a $\Bbbk[G]$-module graded by the group G itself, but consider it now as $\Bbbk[H]$-modules for a subgroup $H \subset G$. As a $\Bbbk[H]$-module, it is then naturally $(H \circlearrowright G)$-graded. In [BD96] an equivalence of the category of $(G \circlearrowright X)$-graded modules with the module category over a smash product ring is given.

Weight modules over a GWA A are naturally $(\mathbb{Z}^n \circlearrowright \operatorname{mspec}(R))$-graded because $A_\alpha \cdot M_\mathfrak{m} \subset M_{\sigma^\alpha(\mathfrak{m})}$. Nevertheless, for our special GWA's it is enough to change the indexing set of both the GWA A and the module M to find a common index set with group structure, with respect to which M is a classically graded A-module, see Section II.1.3.1. So we will work with the classical grading.

II.1.3. Description of weight modules in terms of breaks

II.1.3.1. Grading of weight modules

Let A be again a special GWA as introduced in Section II.1.2.2. Consider the left A-module $M(\mathfrak{m}) = A/A\mathfrak{m}$. As R-module it decomposes into

$$M(\mathfrak{m}) = \bigoplus_{\alpha \in \mathbb{Z}^n} A_\alpha / A_\alpha \mathfrak{m},$$

and this decomposition is already a weight space decomposition because

$$A_\alpha / A_\alpha \mathfrak{m} \;\cong\; \{m \in A \mid \sigma^\alpha(\mathfrak{m}) \cdot m \in A\mathfrak{m}\} \;\cong\; \{m \in M(\mathfrak{m}) \mid \sigma^\alpha(\mathfrak{m}) \cdot m = 0\} \;=\; M(\mathfrak{m})_{\sigma^\alpha(\mathfrak{m})}.$$

Here we use $\sigma^\alpha(\mathfrak{m}) \cdot A_\alpha = A_\alpha \cdot \mathfrak{m}$ and that A is graded, so that one can study whether $\sigma^\alpha(\mathfrak{m}) \cdot m$ is an element of $A\mathfrak{m}$ for homogeneous m. For $\mathfrak{m} = \mathfrak{m}_a$ and the shorthand notation $M(\mathfrak{m}_a)_{a'} = M(\mathfrak{m}_a)_{\mathfrak{m}_{a'}}$ and $\alpha \cdot \beta$ defined componentwise by $(\alpha \cdot b)_i = \alpha_i \cdot b_i$, we obtain

$$M(\mathfrak{m}_a) \;=\; \bigoplus_{\alpha \in \mathbb{Z}^n} M(\mathfrak{m}_a)_{\sigma^\alpha(\mathfrak{m}_a)} \;=\; \bigoplus_{\alpha \in \mathbb{Z}^n} M(\mathfrak{m}_a)_{a + \alpha \cdot b}$$

(notice that indeed $\sigma_i(\mathfrak{m}_a) = \mathfrak{m}_{a+b_i}$). This weight space decomposition turns $M(\mathfrak{m}_a)$ into a graded A-module, but only after reindexing the grading of A: The decomposition of $M(\mathfrak{m}_a)$ does not respect the usual $\mathbb{Z}^n$-grading of $A = \bigoplus_{\alpha \in \mathbb{Z}^n} A_\alpha$ because $A_\alpha \cdot M(\mathfrak{m}_a)_{a'}$ is a subset of $M(\mathfrak{m}_a)_{a'+\alpha\cdot b}$ instead of $M(\mathfrak{m}_a)_{a'+\alpha}$. We have to interpret the abstract $\mathbb{Z}^n$-grading of the GWA A as a $\mathbb{Z}^n \cdot b$-grading coming from the adjoint R-action as in (A1), where we write $\mathbb{Z}^n \cdot b = \{\alpha \cdot b \mid \alpha \in \mathbb{Z}^n\}$. Observe that

$$\begin{aligned} A_\alpha &= R \cdot a^\alpha \\ &= \{a \in A \mid r \cdot a = a \cdot \sigma^{-\alpha}(r) \quad \forall\ r \in R\} \\ &= \{a \in A \mid T_i \cdot a = a \cdot (T_i + \alpha_i b_i) \quad \forall\ 1 \leq i \leq n\} \\ &= \{a \in A \mid [T_i, a] = \alpha_i b_i \cdot a \quad \forall\ 1 \leq i \leq n\} \\ &= A_{\alpha\cdot b} \qquad \text{in the sense of (A1).} \end{aligned}$$

Then thanks to $A_{\alpha\cdot b} \cdot A_{\alpha'\cdot b} \subset A_{(\alpha+\alpha')\cdot b}$,

$$A = \bigoplus_{\tau \in \Bbbk^n} A_\tau \quad \text{with} \quad A_\tau = 0 \text{ for } \tau \neq \alpha \cdot b$$

is a $\Bbbk^n$-grading of A. Of course we do not change the decomposition of A, we only choose a concrete realization for the abstract $\mathbb{Z}^n$-grading and added some 0-summands to A. With respect to this new grading $M(\mathfrak{m})$ is a $\Bbbk^n$-graded A-module.

Let us recall some further properties of $M(\mathfrak{m})$:

- Since $\overline{1} \in A_0/A_0\mathfrak{m} = M(\mathfrak{m})_\mathfrak{m}$, it follows that $A_\alpha \cdot M(\mathfrak{m})_\mathfrak{m} = M(\mathfrak{m})_{\sigma^\alpha(\mathfrak{m})}$ for all $\alpha \in \operatorname{Supp}(A)$, therefore the support of $M(\mathfrak{m})$ is given by

  $$\operatorname{Supp}(M(\mathfrak{m}_a)) = a + \operatorname{Supp}(A) = a + \mathbb{Z}^n \cdot b = \{a + \alpha \cdot b \mid \alpha \in \mathbb{Z}^n\},$$

  i.e. as subset of $\operatorname{mspec}(R)$, the support equals the whole orbit $\{\sigma^\alpha(\mathfrak{m}) \mid \alpha \in \mathbb{Z}\}$.
- Every weight space of $M(\mathfrak{m})$ is one-dimensional since

  $$M(\mathfrak{m})_{\sigma^\alpha(\mathfrak{m})} = A_\alpha / A_\alpha \mathfrak{m} \cong R/\sigma^{-\alpha}(\mathfrak{m}) \cong \Bbbk$$

  with $A_\alpha\mathfrak{m} = R \cdot a_\alpha \cdot \mathfrak{m} = \sigma^{-\alpha}(\mathfrak{m}) A_\alpha$.
- Every submodule of $M(\mathfrak{m})$ is homogeneous (see Lemma II.1.2.4).
- $M(\mathfrak{m})$ has a unique simple top, denoted by $L(\mathfrak{m})$. It inherits the grading of $M(\mathfrak{m})$. Its support is denoted by $\langle\mathfrak{m}\rangle := \operatorname{Supp}(L(\mathfrak{m}))$. We usually consider $\langle\mathfrak{m}\rangle$ as subset of $\Bbbk^n$.
- Notice that although the modules $M(\mathfrak{m})$ seem to be very similar (as $\Bbbk$-vector spaces they are all isomorphic to $\bigoplus_{\alpha \in \mathbb{Z}^n} \Bbbk$), two modules $M(\mathfrak{m})$, $M(\mathfrak{m}')$ are only isomorphic iff their simple tops $L(\mathfrak{m})$ and $L(\mathfrak{m}')$ are isomorphic, too, and the latter are isomorphic iff they have the same support.

The weight space structure of the module $M(\mathfrak{m}) = A/A\mathfrak{m}$ and the existence of its simple top were discussed in [Bav92].

II.1.3.2. Breaks and the submodule lemma

Now recall how the submodules of $M(\mathfrak{m}_a)$ can be described in terms of its support and the breaks therein, see [Bav92] and [DGO96]. Later on we will see how this carries over to the primitive ideals.

II.1.3.1 Definition. A maximal ideal $\mathfrak{m} \in \operatorname{mspec}(R)$ is called a break ideal in direction i if $t_i \in \mathfrak{m}$.

It deserves this name since the module $M(\mathfrak{m}_a)$ 'breaks' into submodules precisely between its weight spaces $M(\mathfrak{m}_a)_{\mathfrak{m}}$ and $M(\mathfrak{m}_a)_{\sigma_i(\mathfrak{m})}$ for break ideals $\mathfrak{m}$:

II.1.3.2 Lemma. Let $M = M(\mathfrak{m}_a)$ for some $\mathfrak{m}_a \in \operatorname{mspec}(R)$. Let $\mathfrak{m}$ be in the support of M. If $t_i \in \mathfrak{m}$ then X_i or Y_i act by 0 between $M_{\mathfrak{m}}$ and $M_{\sigma_i(\mathfrak{m})}$. Otherwise, X_i and Y_i act up to nonzero scalars as mutually inverse bijections between the weight spaces.

Proof. Every weight space of $M(\mathfrak{m}_a)$ is of the form $M(\mathfrak{m}_a)_{a+\alpha\cdot b}$ of $M(\mathfrak{m}_a)$ and in particular one-dimensional. Therefore,

$$X_i \cdot M(\mathfrak{m}_a)_{a+\alpha\cdot b} = \begin{cases} 0, & \text{iff} \quad X_i \cdot a^{\alpha} \in A_{\alpha+e_i}\mathfrak{m}_a; \\ M(\mathfrak{m}_a)_{a+(\alpha+e_i)\cdot b}, & \text{else.} \end{cases}$$

For $\alpha_i \geq 0$, we have $X_i a^{\alpha} = a^{\alpha+e_i} \notin A_{\alpha+e_i}\mathfrak{m}_a$. For $\alpha_i < 0$, the defining relations of a GWA give $X_i a^{\alpha} = \sigma_i(t_i) a^{\alpha+e_i}$. So $X_i a^{\alpha} \in A_{\alpha+e_i}\mathfrak{m}_a = \sigma^{\alpha+e_i}(\mathfrak{m}_a)A_{\alpha+e_i}$ iff $\sigma_i(t_i) \in \sigma^{\alpha+e_i}(\mathfrak{m}_a)$ (use that A_α is a free R-module), iff $t_i \in \sigma^{\alpha}(\mathfrak{m}_a)$. Similarly, we obtain for Y_i that $Y_i \cdot M(\mathfrak{m}_a)_{a+\alpha\cdot b} = 0$ iff $\alpha_i > 0$ and $t_i \in \sigma^{\alpha-e_i}(\mathfrak{m}_a)$. In other words:

$$\begin{aligned} X_i \text{ acts by zero on } M(\mathfrak{m})_{\sigma^{\alpha}(\mathfrak{m})} &\text{ iff } \alpha_i < 0 \text{ and } t_i \in \sigma^{\alpha}(\mathfrak{m}), \\ Y_i \text{ acts by zero on } M(\mathfrak{m})_{\sigma^{\alpha}(\mathfrak{m})} &\text{ iff } \alpha_i > 0 \text{ and } t_i \in \sigma^{\alpha-e_i}(\mathfrak{m}). \end{aligned}$$

Together this proves the claim. □

Since $\sigma_j(t_i) = t_i$ for $i \neq j$, a maximal ideal $\mathfrak{m}$ is a break ideal in direction i iff so is $\sigma_j(\mathfrak{m})$. The break ideals that are in the same σ_j-orbits for $j \neq i$ lie on a common hyperplane.

II.1.3.3 Definition. We call a hyperplane in $\Bbbk^n$ containing all σ_j-orbits of $\mathfrak{m}$ for $j \neq i$ a break in direction i.

Notice that every point inside a break is indeed a break ideal. If we identify once more $\Bbbk^n$ with $\operatorname{mspec}(R)$, the breaks correspond to hyperplanes parallel to the coordinate hyperplanes. From Lemma II.1.3.2 we know that breaks should be interpreted as 'forward breaks'. Examples will be given in Section II.1.5.

II.1.3.4 Lemma. The module $M(\mathfrak{m})$ has at most $2^{\prod_{i=1}^{n}(1+\text{number of zeroes of } t_i)}$ submodules. The subquotients occur with multiplicity 1. In particular, $M(\mathfrak{m})$ has finite length bounded by $\prod_{i=1}^{n}(1+\text{number of zeroes of } t_i)$, independent of $\mathfrak{m}$.

Proof. Every submodule N inherits the weight space decomposition from $M(\mathfrak{m})$, and because every weight space of $M(\mathfrak{m})$ is at most one-dimensional, we have

$$N = \bigoplus_{\mathfrak{m}' \in \operatorname{Supp}(N)} M(\mathfrak{m})_{\mathfrak{m}'}.$$

The submodules are therefore completely determined by their supports, in the sense that $N = N'$ iff $\operatorname{Supp}(N) = \operatorname{Supp}(N')$. From the discussion of the breaks we know that X_i and Y_i act as mutually inverse (up to multiplication by elements in R) bijections between the weight spaces, unless we encounter a weight space that belongs to a break. If a weight between two successive breaks belongs to the support of N, all the other weights between these two breaks do as well. The choice of a submodule is thus equivalent to the choice of the breaks (or no breaks at all) for each coordinate direction i. The polynomial t_i is contained in the maximal ideal $\mathfrak{m}_a = (T_1 - a_1, \ldots, T_n - a_n)$ iff a_i is a zero of t_i. In particular, t_i can only be contained in finitely many maximal ideals in the orbit $\{\sigma_i^{\alpha_i}(\mathfrak{m}) \mid i \in \mathbb{Z}\}$. So there are only finitely many breaks in each direction i, and they all occur at zeros of t_i. Since there are $\#(\text{zeroes of } t_i)$ breaks in the i-th coordinate direction, the statement of the lemma follows. $\square$

The breaks provide in particular a description of the support of the simple modules $L(\mathfrak{m}_a)$: We have $\operatorname{Supp}(L(\mathfrak{m}_a)) \subset \operatorname{Supp}(M(\mathfrak{m}_a))$. In other words, $\langle \mathfrak{m}_a \rangle \subset a + \mathbb{Z}^n \cdot b$, i.e. the support consists of lattice points. Since $\overline{1} \in L(\mathfrak{m}_a)_a$, we know that $a \in \langle \mathfrak{m}_a \rangle$. Again, X_i and Y_i act (up to multiplication by elements in R) as mutually inverse bijections between the weight spaces, unless we encounter a weight space that belongs to a break ideal. So informally speaking $\langle \mathfrak{m}_a \rangle$ is given by those weights that can be reached from a without crossing a break.

More precisely: For every i, pick the largest $\gamma_i^{\text{LOW}} < 0$ with $t_i \in \sigma_i^{\gamma_i^{\text{LOW}}}(\mathfrak{m}_a)$ and the smallest $\gamma_i^{\text{UP}} > 0$ with $t_i \in \sigma_i^{\gamma_i^{\text{UP}} - e_i}(\mathfrak{m}_a)$ (if they exist). Under the isomorphism $\operatorname{mspec}(R) \cong \mathbb{k}^n$, denote the i-th coordinate of the image of $\sigma_i^{\gamma_i^{\text{LOW}}}(\mathfrak{m}_a)$ by g_i^{LOW} and the image of $\sigma_i^{\gamma_i^{\text{UP}} - e_i}(\mathfrak{m}_a)$ by g_i^{UP} (and set $g_i^{\text{LOW}} = -\infty$ resp. $g_i^{\text{UP}} = \infty$ in case this does not exist). Then as a subset of $\mathbb{k}^n$,

$$\begin{aligned} \langle \mathfrak{m}_a \rangle = \operatorname{Supp}(L(\mathfrak{m}_a)) &= (a + \mathbb{Z}^n \cdot b) \cap \{x \in \mathbb{k}^n \mid g_i^{\text{LOW}} < x_i \le g_i^{\text{UP}} \text{ for all } i\} \\ &= (a + \mathbb{Z}^n \cdot b) \cap \{x \in \mathbb{k}^n \mid g_i^{\text{LOW}} + b_i \le x_i \le g_i^{\text{UP}} \text{ for all } i\}. \end{aligned}$$

As these inequalities involve only one coordinate each, the support has the shape of a rectangle with sides consisting of hyperplanes parallel to the coordinate hyperplanes, in case there exist g_i^{UP} and g_i^{LOW} (otherwise drop the corresponding hyperplane from the picture). Of course g_i^{LOW}, g_i^{UP} are just two zeroes of t_i chosen such that $g_i^{\text{LOW}} < a_i \leq g_i^{\text{UP}}$ and there is no other zero of the polynomial t_i between them in the lattice $a_i + \mathbb{Z} \cdot b_i$. The choice of these zeroes depends on a (so we should really write ${}^a g_i^{\text{UP}}$ if it wasn't too much index notation).

II.1.4. Primitive ideals of generalized Weyl algebras

II.1.4.1. The main result

Let A be the special GWA described in Section II.1.2.2. Denote by $\mathrm{Ann}_A(M) := \{a \in A \mid a \cdot M = 0\}$ the annihilator of M. It is a twosided ideal of A. For a simple A-module L, the annihilator $\mathrm{Ann}_A(L)$ is called a primitive ideal. Then our main result reads as follows:

II.1.4.1 Theorem. Let A be the GWA of rank n given by $R = \Bbbk[T_1, \dots, T_n]$, $\sigma_i(T_j) = T_j - \delta_{ij} b_i$ for some $b_i \in \Bbbk \setminus \{0\}$ and some $t_i \in \Bbbk[T_i] \subset \Bbbk[T_1, \dots, T_n]$, $t_i \notin \Bbbk$. Then all primitive ideals of A are of the form $\mathrm{Ann}_A(L(\mathfrak{m}))$ for some simple highest weight A-module $L(\mathfrak{m})$ of highest weight $\mathfrak{m} \in \mathrm{mspec}(R)$. In other words, there is a bijection

$$\begin{aligned}&\{\mathrm{Ann}_A(L(\mathfrak{m})) \mid \mathfrak{m} \in \mathrm{mspec}(R) \text{ such that } L(\mathfrak{m}) \text{ is a highest weight module}\}\\ \leftrightarrow\ &\{\text{primitive ideals of } A\}.\end{aligned}$$

This theorem is analogous to the classical Duflo theorem from [Duf77] for the universal enveloping algebra $\mathcal{U}(\mathfrak{g})$ of a semisimple Lie algebra $\mathfrak{g}$, stating that its primitive ideals are given by the annihilators of highest weight modules $L(\lambda)$ where $\lambda \in \mathfrak{h}^*$ for a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$. The proof is an application of Theorem II.1.4.4 from [MB98], which we recall in Section II.1.4.2. In Section II.1.3.1 we will give more details about the simple highest weight module $L(\mathfrak{m})$. The proof itself follows in Sections II.1.4.3 and II.1.4.4. From the proof it follows that

II.1.4.2 Corollary. *A as above has only finitely many different primitive ideals.*

We give some important examples of the class of GWA's to which Theorem II.1.4.1 applies:

II.1.4.3 Example. i) The classical Weyl algebras $A_n = \Bbbk[x_1, \dots, x_n, \partial_1, \dots, \partial_n]$ (see [Bav92, Example 1.2.(1)]). Since these algebras are simple, every primitive ideal is zero.

ii) The universal enveloping algebra

$$\mathcal{U}(\mathfrak{sl}_2) = \mathbb{C}\langle e, f, h\rangle / \ ([h, e] = 2e,\ [h, f] = -2f,\ [e, f] = h)$$

is not included in this class of algebras: It is isomorphic to the GWA $\mathbb{C}[C, H](\sigma, t)$ with $\sigma(H) = H - 2$, $\sigma(C) = C$ and $t = \frac{1}{4}(C - H(H + 2))$. The isomorphism is given by $X \mapsto e$, $Y \mapsto f$, $H \mapsto h$ and $C \mapsto c$ where $c = h(h + 2) + 4fe$ denotes the Casimir element in the universal enveloping algebra. Hence t is mapped to fe. However, every simple $\mathfrak{sl}_2$-module L has central character, so for every simple module L there is some $\chi \in \mathbb{C}$ such that $c \cdot v = \chi \cdot v$ for all $v \in L$. Hence we have

$$\begin{aligned}\{\text{primitive ideals of } \mathcal{U}(\mathfrak{sl}_2)\} &= \bigcup_{\chi\in\mathbb{C}} \{\text{primitive ideals of } \mathcal{U}(\mathfrak{sl}_2) \text{ that contain } (c - \chi)\} \\ &\leftrightarrow \bigcup_{\chi\in\mathbb{C}} \{\text{primitive ideals of } \mathcal{U}(\mathfrak{sl}_2)/(c - \chi)\}.\end{aligned}$$

But the central quotient $\mathcal{U}(\mathfrak{sl}_2)/(c - \chi)$ is isomorphic to the GWA $\mathbb{C}[H](\sigma : H \mapsto H + 2,\ t = \frac{1}{4}(\chi - H(H + 2))$ (see [Bav92, Example 1.2.(3)]), to which our theorem applies. Hence we recover the Duflo theorem in this case.

iii) More generally, for all $\Bbbk$-algebras A with $\dim(A) < |\Bbbk|$ it is true that every simple module has central character, see the argument in [CG97, Corollary 8.1.2] or [Maz10, Theorem 4.7] (it is shown in [Maz10] that the Casimir element C of $\mathcal{U}(\mathfrak{sl}_2)$ acts by a scalar on any simple $\mathcal{U}(\mathfrak{sl}_2)$-module, but one can apply exactly the same argument for a central element $C \in A$ of any algebra with $\dim(A) < |\Bbbk|$, eg. A with countable dimension and $\Bbbk$ uncountable (and still algebraically closed!)). To obtain a Duflo statement for A, it is enough to establish a Duflo theorem for all central quotients $A/(Z - \chi(Z))$, where Z denotes the center of A and $\chi \in Z^*$ is a central character - similarly to the $\mathcal{U}(\mathfrak{sl}_2)$-example. The primitive ideals in $A/(Z - \chi(Z))$ can then be lifted to ideals in A, which are indeed primitive and exactly those primitive ideals of A that contain $(Z - \chi(Z))$ (all simple A-modules with central character χ are lifts of the simple $A/(Z - \chi(Z))$-modules). But notice that if some X_i is central, a simple highest weight $A/(Z - \chi(Z))$-module need not be *highest* weight as A-module in the sense of the defintion given in Section II.1.2.3. However, it seems to be adequate to adapt the notion of a highest weight module so that a central X_i is not supposed to act by 0 on the 'highest weight space'.

iv) More generally, Smith's generalizations of $\mathcal{U}(\mathfrak{sl}_2)$, defined in [Smi90], have central quotients that are GWA's in the special class we consider here. The realization as GWA is given in [Bav92, Example 1.2.(4)]. The primitive ideals were already described in [Smi90, Section 3].

v) The class of GWA's and all examples discussed in [Bav92, Section 1.2]: They agree with our special GWA's, except that the automorphism σ is given by translation by 1 instead of any nonzero b. In [Bav92, Theorem 3.2, 3.8], a classification of simple modules for these algebras is given. $\Diamond$

We confine ourselves to the special class of GWA's because we want the following properties to hold, mainly for the application of Theorem II.1.4.4. Some of them could be weakened slightly, but without greater insight and to the cost of additional technical considerations (as illustrated in the enveloping algebra example).

- The base ring R is in particular noetherian, hence by Lemma II.1.2.1 the GWA A is noetherian, too. This is a requirement of Theorem II.1.4.4.
- The base ring is the polynomial ring and not just a quotient of such since otherwise we cannot guarantee that there are only finitely many 'breaks', see Section II.1.3.2. But such a finiteness condition is needed in Theorem II.1.4.4.
- To satisfy $\sigma_i(t_j) = t_j$ for $i \neq j$, it is convenient to consider only tensor products of rank 1 GWA's.
- The application of Theorem II.1.4.4 is only possible for a GWA where $\mathbb{Z}^n$ acts freely on R, i.e. $\sigma^\alpha = \sigma^\beta$ iff $\alpha = \beta$: This ensures that the graded components A_α are cyclic over R, see (A2) below.
- The grading should come from a weight space decomposition with respect to the adjoint action of R on A. In this case, any twosided ideal inherits the grading of A, and this is fundamental for Theorem II.1.4.4. Therefore in the rank 1 case, some automorphism of the polynomial ring $\sigma : T \mapsto aT - b$ must be of the form $\sigma : T \mapsto T - b$.
- Furthermore, $b_i \neq 0$ because otherwise σ_i would be trivial. This contradicts the free $\mathbb{Z}^n$-action on R.

II.1.4.2. The result of [MB98]

We would like to apply the following result of [MB98, Theorem 3.2.4], slightly reformulated:

II.1.4.4 Theorem. Let $\Bbbk$ be an algebraically closed field of characteristic 0. Let A be any unital associative $\Bbbk$-algebra satisfying the following assumptions:

(A1) A carries a grading $\bigoplus_{\tau \in \Bbbk^n} A_\tau$ with $A_0 = R := \Bbbk[T_1, \dots, T_n]$ commutative, where the grading comes from the weight space decomposition of A with respect to the adjoint action of $\operatorname{span}_\Bbbk \{T_1, \dots, T_n\}$,

$$A_\tau = \{a \in A \mid [T_i, a] = \tau_i a\}.$$

(A2) $R \twoheadrightarrow A_\tau = R \cdot a_\tau$ for all τ, i.e. each nonzero A_τ is generated by one element over R.

(A3) A is graded left noetherian.

(A4) For a maximal ideal $\mathfrak{m} \subset R$, the A-module $M(\mathfrak{m}) := A/A\mathfrak{m}$ has uniformly bounded length, independent of $\mathfrak{m}$.

(A5) The number of different Zariski closed sets $\overline{\langle \mathfrak{m} \rangle} \subset \Bbbk^n$ is finite.
Here, the set $\langle \mathfrak{m} \rangle$ is defined as follows: For an algebra A satisfying (A1) and (A2), the A-module $M(\mathfrak{m})$ has a weight space decomposition which turns it into a $\Bbbk^n$-graded module with $M(\mathfrak{m})_a := M(\mathfrak{m})_{\mathfrak{m}_a}$ and $\mathfrak{m}_a = (T_1 - a_1, \dots, T_n - a_n)$ is the maximal ideal corresponding to $a = (a_1 \dots, a_n) \in \Bbbk^n$: Indeed $A_\tau \cdot M(\mathfrak{m})_\alpha \subset M(\mathfrak{m})_{\alpha+\tau}$. It is easy to see that $M(\mathfrak{m})$ has a unique maximal submodule, because a submodule is proper iff it does not contain $\overline{1} \in A/A\mathfrak{m}$. Hence $M(\mathfrak{m})$ has simple top, denoted $L(\mathfrak{m})$. It inherits the grading of $M(\mathfrak{m})$. Its support is denoted by $\langle \mathfrak{m} \rangle := \operatorname{Supp}(L(\mathfrak{m}))$. We usually consider $\langle \mathfrak{m} \rangle$ as subset of $\Bbbk^n$.

(A6) For all $\mathfrak{m}_\alpha \in \operatorname{mspec}(R)$ and all $\tau \in \operatorname{Supp}(A)$ we have

$$\overline{(\tau + \langle \mathfrak{m} \rangle)} \cap \overline{\langle \mathfrak{m} \rangle} = \overline{(\tau + \langle \mathfrak{m} \rangle) \cap \langle \mathfrak{m} \rangle}.$$

Then all prime ideals, hence all primitive ideals of A are of the form $\operatorname{Ann}_A(L(\mathfrak{m})) =: J(\mathfrak{m})$ for some $\mathfrak{m} \in \operatorname{mspec}(R)$, and

$$\{\overline{\langle \mathfrak{m} \rangle} \mid \mathfrak{m} \in \operatorname{mspec}(R)\} \leftrightarrow \{J(\mathfrak{m}) \mid \mathfrak{m} \in \operatorname{mspec}(R)\} \leftrightarrow \{\text{primitive ideals of } A\}.$$

The first bijection is given by $J(\mathfrak{m}) = A \cdot I(\overline{\langle \mathfrak{m} \rangle}) \cdot A$ where $I(\overline{\langle \mathfrak{m} \rangle}) = \bigcap_{\mathfrak{m}' \in \overline{\langle \mathfrak{m} \rangle}} \mathfrak{m}'$.

The formulation of the theorem is slightly modified: In [MB98] the subalgebra R can be any finitely generated commutative subalgebra. We will obtain a slight refinement, by finding the above correspondence for *highest* weight modules $L(\mathfrak{m})$.

As mentioned in Section II.1.3.1, the weight space structure of the module $M(\mathfrak{m}) = A/A\mathfrak{m}$ and the existence of its simple top were treated for GWA's already in [Bav92]. But in fact they are a general consequence of conditions (A1) and (A2) (see [MB98, Proposition 3.1.7]).

II.1.4.3. The proof of Theorem II.1.4.1: Reduction to weight modules

We now check the conditions of Theorem II.1.4.4.

Condition (A1) is valid for any GWA (here we have to use the unusual grading as described in Section II.1.3.1).

Condition (A2) holds for any GWA with free $\mathbb{Z}^n$-action on $\mathrm{Aut}(R)$. For σ_i given by translations in coordinate direction i, it follows from $\sigma^\alpha = \sigma^\beta$ that $\alpha = \beta$, so the $\mathbb{Z}^n$-action on $\mathrm{Aut}(R)$ is indeed free.

Condition (A3) holds for any GWA whose ground ring R is noetherian (Lemma II.1.2.1), in particular in our case where $R = \Bbbk[T_1, \dots, T_n]$ is the polynomial ring.

Condition (A4) is satisfied according to Lemma II.1.3.4, and the length is uniformly bounded by $\prod_{i=1}^{n}(1 + \text{number of zeroes of } t_i)$.

For the verification of (A5) and (A6), we first notice that there are only finitely many breaks (i.e. hyperplanes consisting of those points in $\Bbbk^n$ that correspond to maximal ideals $\mathfrak{m} \subset \Bbbk[T_1, \dots, T_n]$ containing one of the t_i).

II.1.4.5 Remark. In case $\mathfrak{m}$ is contained in an orbit without breaks, the support of $L(\mathfrak{m})$ is the whole orbit $\langle\mathfrak{m}\rangle = \mathrm{Supp}\,(A) \cdot \mathfrak{m}$. For our special choice of GWA's A we have $\mathrm{Supp}\,(A) \cdot \mathfrak{m} = \mathbb{Z}^n \cdot \mathfrak{m}$ which is dense in $\mathrm{mspec}(R)$, and therefore $\overline{\langle\mathfrak{m}\rangle} = \mathrm{mspec}(R)$. So these closures give all the same contribution when we count the different closures to verify (A5). Also, $\sigma^\alpha(\langle\mathfrak{m}\rangle) = \langle\mathfrak{m}\rangle$ for any $\sigma^\alpha \in \mathrm{Supp}\,(A)$ and so (A6) is satisfied for those $\mathfrak{m}$. ◊

For $\mathfrak{m}_a$ inside an orbit $\mathbb{Z}^n \cdot \mathfrak{m}_a$ containing a break, we can first translate the whole orbit by $-a$ to the origin. Then rescale in every coordinate direction by b_i^{-1}, so that the orbit becomes the standard $\mathbb{Z}$-lattice in $\Bbbk^n$. In particular, the breaks $g_i, d_i \in (a_i + \mathbb{Z} \cdot b_i)$ become points in $\mathbb{Z}$ (to be precise, $g_i^{\text{UP}} \mapsto \widehat{g_i}^{\text{UP}} = b_i^{-1}(g_i^{\text{UP}} - a_i)$, $g_i^{\text{LOW}} \mapsto \widehat{g_i}^{\text{LOW}} = b_i^{-1}(g_i^{\text{LOW}} - a_i)$). Rescaling and translation are isomorphisms of varieties, so these manipulations are allowed when computing the closure. Furthermore, we can compute the closure of $\langle\mathfrak{m}\rangle$ over $\mathbb{Q}$ since $\overline{\langle\mathfrak{m}\rangle_{\Bbbk}} = k \otimes_{\mathbb{Q}} \overline{\langle\mathfrak{m}\rangle_{\mathbb{Q}}}$. Use the following results from [MB98, Section 7.1]:

II.1.4.6 Proposition. Consider $\mathbb{Z}^n \subset \mathbb{Q}^n$.

i) Given any $\lambda_1, \dots, \lambda_m \in (\mathbb{Q}^n)^*$, there is a unique decomposition of the index set $T = \{1, \dots, m\}$ into two disjoint parts $I \dot\cup J$, such that there are $e \in \mathbb{Q}^n$, $z = (z_1, \dots, z_m) \in \mathbb{Q}^m$ with

$$\sum_{i=1}^m z_i \lambda_i = 0, \text{ and } \langle \lambda_i, e\rangle = \lambda_i(e) = \begin{cases} > 0, & \text{for } i \in I \\ = 0, & \text{for } i \in J \end{cases} \text{ and } z_i = \begin{cases} = 0, & \text{for } i \in I \\ > 0, & \text{for } i \in J. \end{cases}$$

ii) Given furthermore $q_1, \dots, q_m \in \mathbb{Q}$, define $E = \bigcap_{j \in J} \ker(\lambda_j)$ and

$$\begin{aligned} C &= \{x \in \mathbb{Q}^n \mid \langle \lambda_i, x\rangle = \lambda_i(x) \leq q_i, \ \forall i \in T\}, \\ C' &= \{x \in \mathbb{Q}^n \mid \langle \lambda_j, x\rangle = \lambda_j(x) \leq q_j \ \forall j \in J\}, \end{aligned}$$

then the Zariski closure of $C \cap \mathbb{Z}^n$ equals $\overline{C \cap \mathbb{Z}^n} = C' \cap (\mathbb{Z}^n + E)$ and $C' \cap (\mathbb{Z}^n + E)$ is a finite union of translates of E.

iii) For $x \in \mathbb{Z}^n$, one has $\overline{(x + C \cap \mathbb{Z}^n) \cap (C \cap \mathbb{Z}^n)} = \overline{(x + C \cap \mathbb{Z}^n)} \cap \overline{(C \cap \mathbb{Z}^n)}$.

This proposition can be applied to the translated, rescaled support of $L(\mathfrak{m})$ given by $\mathbb{Z}^n \cap C$ with

$$\begin{aligned} C &= \{x \in \mathbb{Q}^n \mid \widehat{g}_i^{\text{LOW}} + 1 \leq x_i \leq \widehat{g}_i^{\text{UP}} \text{ for all } i\} \\ &= \{x \in \mathbb{Q}^n \mid -\varepsilon_i(x) \leq -\widehat{g}_i^{\text{LOW}} - 1, \quad \varepsilon_i(x) \leq \widehat{g}_i^{\text{UP}}, \quad 1 \leq i \leq n\} \\ &= \{x \in \mathbb{Q}^n \mid \lambda_k(x) \leq q_k, \ 1 \leq k \leq 2n\} \end{aligned}$$

where ε_i denotes the i-th coordinate function, $\lambda_k = \varepsilon_k$, $q_k = \widehat{g}_k^{\text{UP}}$ for $1 \leq k \leq n$ and $\lambda_k = -\varepsilon_{k-n}$, $q_k = -\widehat{g}_{k-n}^{\text{LOW}} - 1$ for $n+1 \leq k \leq 2n$. Inequalities where $\widehat{g}_i^{\text{UP}}$ or $\widehat{g}_i^{\text{LOW}}$ are $\pm\infty$ are dropped. In our easy situation, we can make the index set $J \subset \{1, \dots, 2n\}$ concrete:

$$J = \{i \mid \text{ neither } \widehat{g}_i^{\text{UP}} \text{ nor } \widehat{g}_i^{\text{LOW}} = \pm\infty\}$$

(choose eg. $e = (e_k)_k$ with $e_k = e_{k+n} = 0$ for those $1 \leq k \leq n$ with neither $\widehat{g}_k^{\text{UP}}$ nor $\widehat{g}_k^{\text{LOW}}$ are $\pm\infty$, and $e_k = 1$ resp. $e_{n+k} = -1$ otherwise. Similarly, $z = (z_k)_k$ with $z_k = z_{k+n} = 1$ for those $1 \leq k \leq n$ with neither $\widehat{g}_k^{\text{UP}}$ nor $\widehat{g}_k^{\text{LOW}}$ are $\pm\infty$, and $z_k = 0$ otherwise). We get

$$\overline{C \cap \mathbb{Z}^n} = \{x \in \mathbb{Q}^n \mid \widehat{g}_i^{\text{LOW}} + 1 \leq x_i \leq \widehat{g}_i^{\text{UP}} \text{ for all } i \text{ st. } \widehat{g}_i^{\text{LOW}} \text{ and } \widehat{g}_i^{\text{UP}} \neq \pm\infty\} \cap (\mathbb{Z}^n + \mathbb{Q}^{n-J})$$

where we denote $\mathbb{Q}^{n-J} = \operatorname{span}_{\mathbb{Q}} \{e_i \mid 1 \leq i \leq n \text{ and } i \notin J\}$. Tensor with $\Bbbk$ and undo the rescaling and translating, then we get

$$\begin{aligned} \overline{\langle \mathfrak{m} \rangle} &= \{x \in \Bbbk^n \mid g_i^{\text{LOW}} + b_i \leq x_i \leq g_i^{\text{UP}} \text{ for all } i \in J, \text{ i.e. } g_i^{\text{LOW}} \text{ and } g_i^{\text{UP}} \neq \pm\infty\} \\ &\cap \ (\mathbb{Z}^n \cdot b + a + \operatorname{span}_{\Bbbk} \{b_i \mid 1 \leq i \leq n \text{ and } i \notin J\}) \end{aligned}$$

(note here that the inequalities still make sense over an arbitrary field $\Bbbk$ because in the i-th coordinate for $i \in J$, we work in a lattice). But because there are only finitely many breaks, there are only finitely many possibilities to choose g_i^{UP} and g_i^{LOW} corresponding to a break, as well as for $J \subset \{1,\dots,n\}$. Therefore there are only finitely many different Zariski closed sets $\overline{\langle\mathfrak{m}\rangle}$, so (A5) holds. Finally, (A6) is the consequence of Proposition (II.1.4.6.iii).

II.1.4.7 Remark. Of course in this easy case the closures can be computed by hands. But this proposition indicates how to deal with (twisted) GWA's where the breaks need no longer be parallel to the coordinate hyperplanes (for twisted GWA's, see [MT99]). ◊

II.1.4.4. The proof: The refinement

Given any primitive ideal $\mathfrak{a}$, Theorem II.1.4.4 assigns a simple weight module $L(\mathfrak{m})$ such that $\mathrm{Ann}_A(L(\mathfrak{m})) = \mathfrak{a}$. Now we show that it is possible to choose $\mathfrak{m}'$ to be *highest* weight with $\mathrm{Ann}_A(L(\mathfrak{m}')) = \mathfrak{a}$, under the assumption that none of the t_i is a unit. In that case the tensor factor A_i of $A = A_1 \otimes \ldots \otimes A_n$ would be a commutative algebra and not of interest. Once the theorem gave us $\mathfrak{m}$, there are two possibilities:

- Either there are breaks $\sigma_i^{\gamma_i^{\text{UP}}}(\mathfrak{m})$ for $\gamma_i^{\text{UP}} > 0$ in all coordinate directions i. This means that $\sigma^{\gamma^{\text{UP}}-\underline{1}}(\mathfrak{m}) =: \mathfrak{m}'$ is a highest weight (where $\underline{1} = (1,\dots,1)$), and since $\mathfrak{m}'$ lies in the support of $L(\mathfrak{m})$, we have $L(\mathfrak{m}) \cong L(\mathfrak{m}')$. Hence $J(\mathfrak{m}) = J(\mathfrak{m}')$.
- Or we have some coordinate i for which $g_i^{\text{UP}} = \infty$, so in

$$\begin{aligned}\overline{\langle\mathfrak{m}\rangle} &= \{x \in \Bbbk^n \mid g_i^{\text{UP}} \geq x_i \geq g_i^{\text{LOW}} + b_i \text{ for all } i \in J, \text{ i.e. } g_i^{\text{UP}} \text{ and } g_i^{\text{LOW}} \neq \pm\infty\} \\ &\cap\ (\mathbb{Z}^n \cdot b + a + \mathrm{span}_{\Bbbk}\{b_i \mid 1 \leq i \leq n \text{ and } i \notin J\}),\end{aligned}$$

  there is no inequality restricting the coordinate x_i of any element $x \in \overline{\langle\mathfrak{m}\rangle}$. In other words, $\overline{\langle\mathfrak{m}\rangle} + \Bbbk \cdot e_i = \overline{\langle\mathfrak{m}\rangle}$. We want to replace $\mathfrak{m}$ by some other maximal ideal $\mathfrak{m}'$ so that their closures are the same, but $L(\mathfrak{m}')$ is a highest weight module. All we need to do is to keep the inequalities and the index set J in the description of $\overline{\langle\mathfrak{m}\rangle}$ unchanged. Replace for this purpose $\mathfrak{m} = \mathfrak{m}_a = (T_1 - a_1, \dots, T_n - a_n)$ by any other maximal ideal of the form $(T_1 - a_1, \dots, T_i - z, \dots, T_n - a_n)$ such that $(T_i - z)$ is a root of t_i (recall that we assumed $t_i \notin \Bbbk$). Assume that it is the smallest break in the orbit $\sigma_i^{\mathbb{Z}}(T_1 - a_1, \dots, T_i - z, \dots, T_n - a_n)$. This is possible because t_i has only finitely many roots. Then $\sigma_i(T_1 - a_1, \dots, T_i - z, \dots, T_n - a_n) =: \mathfrak{m}'$ is a highest weight in the i-th coordinate direction. Let us check that we preserved the closure $\overline{\langle\mathfrak{m}\rangle} = \overline{\langle\mathfrak{m}'\rangle}$: Because we chose the break to be smallest possible, we have $g_i^{\text{UP}} = z$ and $g_i^{\text{LOW}} = -\infty$, and in the computation of the closure the corresponding i-th inequality will be dropped. The other coordinate directions are not concerned. Repeating this for all coordinates with $g_i^{\text{UP}} = \infty$, we end up with a maximal ideal that is highest weight.

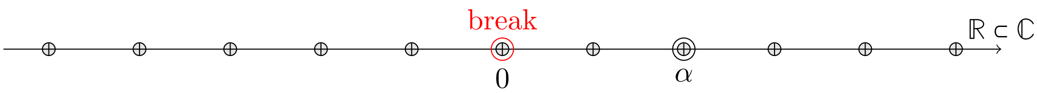
Notice that in the last case the two simple modules $L(\mathfrak{m})$ and $L(\mathfrak{m}')$ are no longer isomorphic (we even changed the weight lattice), but their annihilators satisfy $J(\mathfrak{m}) = A \cdot I(\overline{\langle \mathfrak{m} \rangle}) = A \cdot I(\overline{\langle \mathfrak{m}' \rangle}) = J(\mathfrak{m}')$, hence the result is the same primitive ideal we started with.

II.1.5. Examples

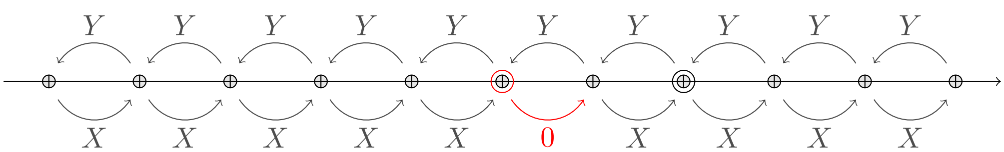
In this section, our ground field $\Bbbk = \mathbb{C}$ are the complex numbers.

II.1.5.1. The first Weyl algebra

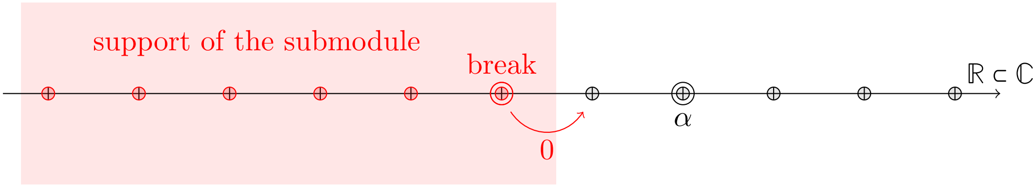
The first Weyl algebra $\mathbb{C}[x, \partial] = \mathbb{C}\langle x, \partial \rangle / [\partial, x] = 1$ of differential operators on a polynomial ring in one variable can be described as a GWA A of rank one with base ring $R = \mathbb{C}[T]$, defining element $t = T$ and automorphism $\sigma(T) = T - 1$, see [Bav92, Example 1.2 (1)]. In particular, since σ is a translation with $b := -1$, it is a GWA of the special form we discuss here. The defining element t has only one zero, namely $z = 0$, hence only one orbit inside $\mathbb{C} \cong \operatorname{mspec}(\mathbb{C}[T])$ contains a break, and this is $0 + \mathbb{Z} \cdot (-1) = \mathbb{Z}$. All modules $M(\mathfrak{m}_a)$ with $a \notin \mathbb{Z}$ are already simple, i.e. $L(\mathfrak{m}_a) = M(\mathfrak{m}_a)$, and $\langle \mathfrak{m}_a \rangle = a + \mathbb{Z}$ is dense in $\mathbb{C}$, therefore $\operatorname{Ann}_A(L(\mathfrak{m}_a)) = A \cdot I(\overline{\langle \mathfrak{m}_a \rangle}) \cdot A = (0)$. Instead, concentrate on those $L(\mathfrak{m}_a)$ with $a \in \mathbb{Z}$, eg. $a = 2$. The following picture shows the weight lattice of $M(\mathfrak{m}_2)$:



The action of X and Y on the weight spaces are bijective (gray arrows) except for $M(\mathfrak{m}_s)_{\mathfrak{m}_0}$, where the break is: Here $X \cdot M(\mathfrak{m}_2)_{\mathfrak{m}_0} = 0$.



Thus, $M(\mathfrak{m}_2)$ has one submodule generated by $M(\mathfrak{m}_2)_{\mathfrak{m}_0}$:



So we see that also for the two simple weight modules with support $\mathbb{Z}_{\leq 0}$ resp. $\mathbb{Z}_{>0}$, the closure of the support is $\overline{\langle \mathfrak{m}_0 \rangle} = \overline{\langle \mathfrak{m}_1 \rangle} = \mathbb{C}$ and $\mathrm{Ann}_{\mathbb{C}[x,\partial]}(L(\mathfrak{m}_0)) = \mathrm{Ann}_{\mathbb{C}[x,\partial]}(L(\mathfrak{m}_1)) = (0)$. In other words, the only primitive ideal in $\mathbb{C}[x,\partial]$ is (0), which matches the fact that the Weyl algebra is simple, so there are no nontrivial twosided ideals.

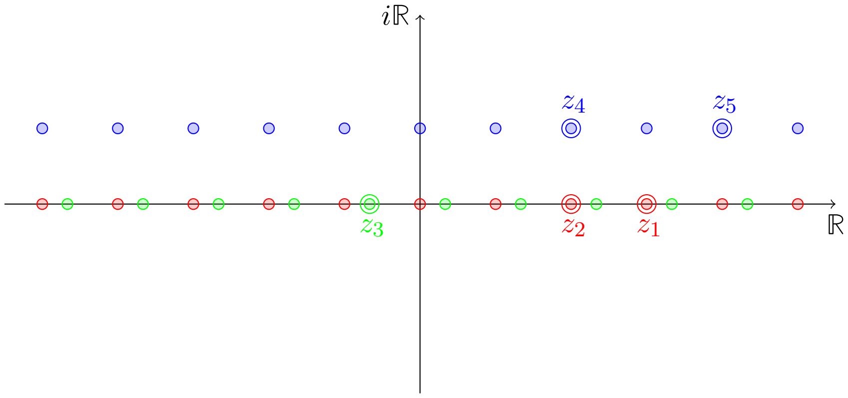
II.1.5.1 Remark. Notice that we get a break at 0, while the computations in [MB98, Section 6] correspond to a break at −1. This difference can be explained by the choice of R. We follow the convention in [Bav92], where $R = \Bbbk[T] = \Bbbk[YX]$, while in [MB98] $R = \Bbbk[T] = \Bbbk[XY]$. Since $YX - XY = 1$, it follows that

$$\mathfrak{m}_0^{\text{Bavula}} = (YX) = (XY + 1) = \mathfrak{m}_{-1}^{\text{MvdB}},$$

which explains the 'shift by 1'. The same has to be kept in mind for the n-th Weyl algebra. ◊

II.1.5.2. A rank 1 example with two breaks

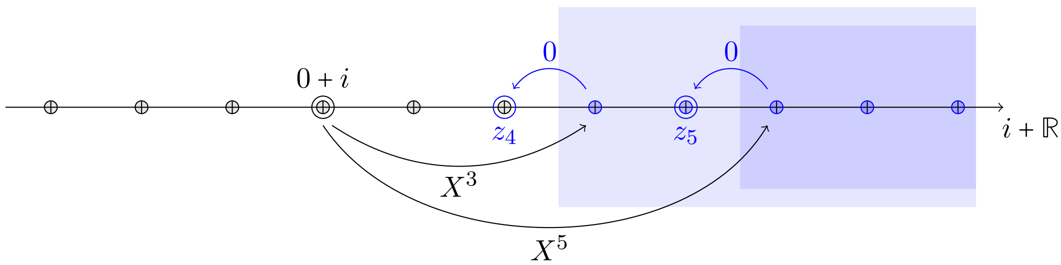
We stay in the rank 1 case, we keep the translation $\sigma(T) = T - 1$, but we change t to be some other polynomial (these are the 'main objects' considered in [Bav92]). For example, choose $t = (T-3)(T-2)(T+\frac{2}{3})(T-(2+i))(T-(4+i))$. Then we have three orbits with breaks: $\mathbb{Z}$, $-\frac{2}{3} + \mathbb{Z}$ and $i + \mathbb{Z}$. First we depict how these orbits lie inside the complex plane $\mathrm{mspec}(\mathbb{C}[T]) \cong \mathbb{C}$ (not to be confused with the following discussion of the rank 2 case!):



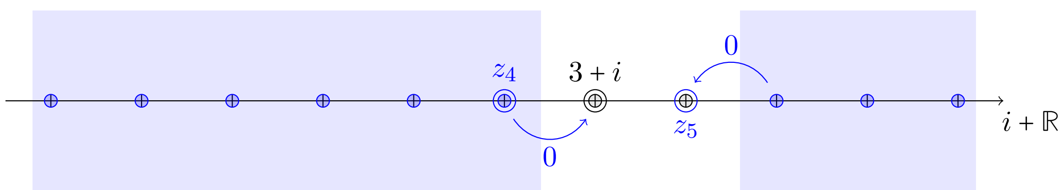
Pick the blue orbit, it is the support of eg. $M(\mathfrak{m}_{0+i})$. Determine its submodules: We have two breaks in the orbit of $0+i$, namely $z_4 = 2+i$ and $z_5 = 4+i$. We have observed earlier that for $\alpha > 0$,

$$\begin{aligned} YX^\alpha = 0 \ \text{ iff } & \ \sigma^{\alpha-1}(t) \in \mathfrak{m}_{0+i}, \\ \text{iff } & \ (T-(2+i))(T-(4+i)) \in \mathfrak{m}_{\alpha-1+i}, \\ \text{iff } & \ \alpha = 3 \text{ or } \alpha = 5. \end{aligned}$$

X^3 and X^5 are bijective, while YX^3 and YX^5 are zero. So there are two submodules, one generated by X^3 and the other by X^5. This is depicted below, where we shade the support of the two submodules blue.



Notice that the support of $M(\mathfrak{m}_{i+3})$ is the same, but the submodule structure is different (still, the subquotients are of course isomorphic):



In our notation from above, the lower and upper break for $3+i$ are $g^{\text{LOW}} = z_4$ and $g^{\text{UP}} = z_5$, so the support of $L(\mathfrak{m}_{3+i})$ is

$$\langle \mathfrak{m}_{3+i} \rangle \;=\; \operatorname{Supp}\left(L(\mathfrak{m}_{3+i})\right) \;=\; (i+\mathbb{Z}) \;\cap\; \{x \in \mathbb{C} \mid z_5 \geq x > z_4\} \;=\; \{3+i,\; 4+i\}.$$

Since it consists only of two points, it agrees with its closure and hence

$$\operatorname{Ann}_A(L(\mathfrak{m}_{3+i})) = A \cdot (\mathfrak{m}_{3+i} \cap \mathfrak{m}_{4+i}) \cdot A = A \cdot (\mathfrak{m}_{3+i}\mathfrak{m}_{4+i}) \cdot A.$$

There are up to isomorphism two more simple modules with support in the orbit $i+\mathbb{Z}$, namely $L(\mathfrak{m}_{2+i})$ and $L(\mathfrak{m}_{5+i})$, both of which have infinite support $i+\mathbb{Z}_{\leq 2}$ and $i+\mathbb{Z}_{>4}$, resp. The closure of the support is in both cases equal to $\mathbb{C}$, so the annihilators of both simple modules are (0). The two other orbits containing breaks can be treated similarly. We find only one more nonzero annihilator, namely $\operatorname{Ann}_A(L(\mathfrak{m}_3)) = A\mathfrak{m}_3 A$, since an orbit needs to contain at least two breaks to allow finite support.

II.1.5.3. A rank 2 example

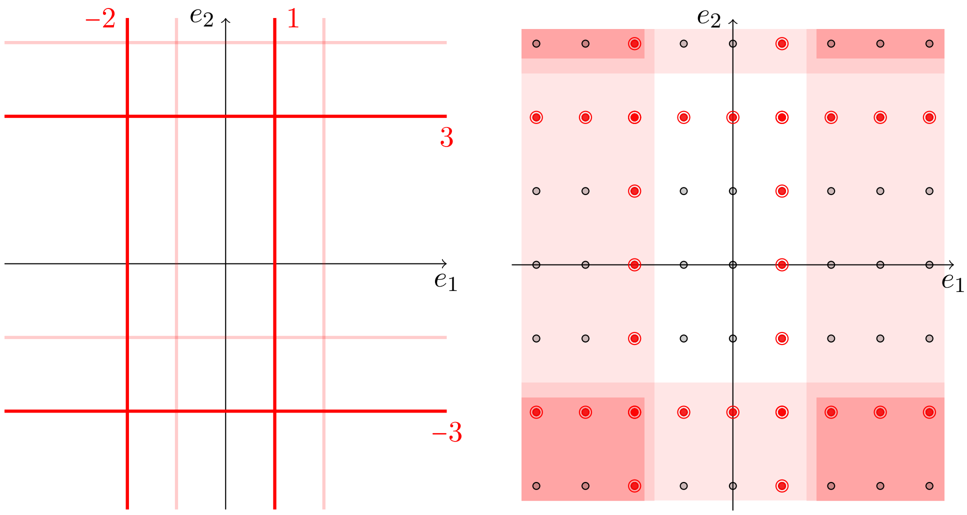
Consider the GWA A with base ring $R = \Bbbk[T_1, T_2]$, with automorphisms $\sigma_1(T_1) = T_1 - 1$, $\sigma_2(T_2) = T_2 - \frac{3}{2}$ and with defining elements $t_1 = (T_1+2)(T_1-1)$ and $t_2 = (T_2+3)(T_2-3)$. Now choose $\mathfrak{m} = \mathfrak{m}_{(0,0)}$. The support of $M(\mathfrak{m}_{(0,0)})$ is given by

$$\operatorname{Supp}\left(M(\mathfrak{m}_{(0,0)})\right) \;=\; (0,0) \;+\; \mathbb{Z}\cdot e_1 \;+\; \frac{3}{2}\mathbb{Z}\cdot e_2,$$

so it contains both breaks -2 and 1 for the first coordinate (corresponding to the maximal ideals $\mathfrak{m}_{(-2,\alpha_2)}$ and $\mathfrak{m}_{(1,\alpha_2)}$ for arbitrary $\alpha_2 \in \frac{3}{2}\mathbb{Z}$) and both breaks -3 and 3 for the second coordinate (corresponding to the maximal ideals $\mathfrak{m}_{(\alpha_1,-3)}$ and $\mathfrak{m}_{(\alpha_1,3)}$ for arbitrary $\alpha_1 \in \mathbb{Z}$). The left picture shows the breaks as (red) hyperplanes in $\Bbbk^2$. Since break ideals are those ideals $\mathfrak{m}$ for which

$$M_{\mathfrak{m}} \xrightarrow{X_i=0} M_{\sigma_i(\mathfrak{m})} \quad \text{or} \quad M_{\mathfrak{m}} \xleftarrow{Y_i=0} M_{\sigma_i(\mathfrak{m})},$$

we furthermore depict $\sigma_i(\text{break in direction } i)$ (light red). The right picture shows the resulting submodule structure of $M(\mathfrak{m}_{(0,0)})$:



From the break structure, read off the annihilators of the simple modules:

$$
\begin{aligned}
\mathrm{Ann}_A(L(\mathfrak{m}_{(2,\frac{9}{2})})) &= \mathrm{Ann}_A(L(\mathfrak{m}_{(-2,-3)})) = \mathrm{Ann}_A(L(\mathfrak{m}_{(2,-3)})) = \mathrm{Ann}_A(L(\mathfrak{m}_{(-2,\frac{9}{2})})) \\
&= (0) \\
\mathrm{Ann}_A(L(\mathfrak{m}_{(0,\frac{9}{2})})) &= \mathrm{Ann}_A(L(\mathfrak{m}_{(0,-3)})) = A \cdot ((T_1+1) \cap (T_1) \cap (T_1-1)) \cdot A \\
\mathrm{Ann}_A(L(\mathfrak{m}_{(-2,0)})) &= \mathrm{Ann}_A(L(\mathfrak{m}_{(2,0)})) = A \cdot ((T_2+1) \cap (T_2) \cap (T_2-1) \cap (T_2-2)) \cdot A \\
\mathrm{Ann}_A(L(\mathfrak{m}_{(0,0)})) &= A \cdot \Big(\mathfrak{m}_{(-1,-\frac{3}{2})} \cap \mathfrak{m}_{(0,-\frac{3}{2})} \cap \mathfrak{m}_{(1,-\frac{3}{2})} \\
&\qquad \cap \mathfrak{m}_{(-1,0)} \cap \mathfrak{m}_{(0,0)} \cap \mathfrak{m}_{(1,0)} \\
&\qquad \cap \mathfrak{m}_{(-1,\frac{3}{2})} \cap \mathfrak{m}_{(0,\frac{3}{2})} \cap \mathfrak{m}_{(1,\frac{3}{2})} \\
&\qquad \cap \mathfrak{m}_{(-1,3)} \cap \mathfrak{m}_{(0,3)} \cap \mathfrak{m}_{(1,3)} \Big) \cdot A
\end{aligned}
$$

There is no further annihilator ideal in A since we considered already all the breaks.

Bibliography

[Alh13] S. Alharbat. "A classification of affine fully commutative elements". In: *arXiv* (2013). eprint: `arXiv/1311.7089`.

[AM69] M. F. Atiyah and I. G. Macdonald. *Introduction to commutative algebra*. Addison-Wesley Publishing Co., Reading, Mass.-London-Don Mills, Ont., 1969, pp. ix+128.

[Bav92] V. V. Bavula. "Generalized Weyl algebras and their representations". In: *Algebra i Analiz* 4.1 (1992), pp. 75–97.

[BB00] V. V. Bavula and V. Bekkert. "Indecomposable representations of generalized Weyl algebras". In: *Comm. Algebra* 28.11 (2000), pp. 5067–5100.

[BD96] M. Beattie and S. Dăscălescu. "Categories of modules graded by G-sets. Applications". In: *J. Pure Appl. Algebra* 107.2-3 (1996). Contact Franco-Belge en Algèbre (Diepenbeek, 1993), pp. 129–139.

[Bec94] J. Beck. "Braid group action and quantum affine algebras". In: *Comm. Math. Phys.* 165.3 (1994), pp. 555–568.

[BM16] G. Benkart and J. Meinel. "The center of the affine nilTemperley–Lieb algebra". In: *Math. Zeit.* (2016). To appear. eprint: `arXiv/math/1505.02544`.

[BO09] G. Benkart and M. Ondrus. "Whittaker modules for generalized Weyl algebras". In: *Represent. Theory* 13 (2009), pp. 141–164.

[BFZ96] A. Berenstein, S. Fomin, and A. Zelevinsky. "Parametrizations of canonical bases and totally positive matrices". In: *Adv. Math.* 122.1 (1996), pp. 49–149.

[BGG73] I. N. Bernšteĭn, I. M. Gel′fand, and S. I. Gel′fand. "Schubert cells, and the cohomology of the spaces G/P". In: *Uspehi Mat. Nauk* 28.3(171) (1973), pp. 3–26.

[BGG76] I. N. Bernšteĭn, I. M. Gel′fand, and S. I. Gel′fand. "A certain category of $\mathfrak{g}$-modules". In: *Funkcional. Anal. i Priložen.* 10.2 (1976), pp. 1–8.

[BJN15] R. Biagioli, F. Jouhet, and P. Nadeau. "Fully commutative elements in finite and affine Coxeter groups". In: *Monatsh. Math.* 178.1 (2015), pp. 1–37.

[BJS93] S. C. Billey, W. Jockusch, and R. P. Stanley. "Some combinatorial properties of Schubert polynomials". In: *J. Algebraic Combin.* 2.4 (1993), pp. 345–374.

[Bri11] J. Brichard. *On Using Graphical Calculi: Centers, Zeroth Hochschild Homology and Possible Compositions of Induction and Restriction Functors in Various Diagrammatical Algebras.* Ph.D. Thesis – Columbia University. ProQuest LLC, Ann Arbor, MI, 2011, p. 107.

[BG02] K. A. Brown and K. R. Goodearl. *Lectures on algebraic quantum groups.* Advanced Courses in Mathematics. CRM Barcelona. Birkhäuser Verlag, Basel, 2002, pp. x+348.

[BK09] J. Brundan and A. Kleshchev. "Blocks of cyclotomic Hecke algebras and Khovanov-Lauda algebras". In: *Invent. Math.* 178.3 (2009), pp. 451–484.

[Buc03] A. S. Buch. "Quantum cohomology of Grassmannians". In: *Compositio Math.* 137.2 (2003), pp. 227–235.

[Cha01] V. Chari. "On the fermionic formula and the Kirillov-Reshetikhin conjecture". In: *Internat. Math. Res. Notices* 12 (2001), pp. 629–654.

[CH10] V. Chari and D. Hernandez. "Beyond Kirillov-Reshetikhin modules". In: *Quantum affine algebras, extended affine Lie algebras, and their applications.* Vol. 506. Contemp. Math. Amer. Math. Soc., Providence, RI, 2010, pp. 49–81.

[CP91] V. Chari and A. Pressley. "Quantum affine algebras". In: *Comm. Math. Phys.* 142.2 (1991), pp. 261–283.

[CP95a] V. Chari and A. Pressley. *A guide to quantum groups.* Corrected reprint of the 1994 original. Cambridge: Cambridge University Press, 1995, pp. xvi+651.

[CP95b] V. Chari and A. Pressley. "Quantum affine algebras and their representations". In: *Representations of groups (Banff, AB, 1994).* Vol. 16. CMS Conf. Proc. Amer. Math. Soc., Providence, RI, 1995, pp. 59–78.

[CP96] V. Chari and A. Pressley. "Minimal affinizations of representations of quantum groups: the simply laced case". In: *J. Algebra* 184.1 (1996), pp. 1–30.

[CG97] N. Chriss and V. Ginzburg. *Representation theory and complex geometry.* Birkhäuser Boston, Inc., Boston, MA, 1997, pp. x+495.

[DD05] B. Deng and J. Du. "Monomial bases for quantum affine $\mathfrak{sl}_n$". In: *Adv. Math.* 191.2 (2005), pp. 276–304.

[DGO96] Yu. A. Drozd, B. L. Guzner, and S. A. Ovsienko. “Weight modules over generalized Weyl algebras”. In: *J. Algebra* 184.2 (1996), pp. 491–504.

[Duf77] M. Duflo. “Sur la classification des idéaux primitifs dans l’algèbre enveloppante d’une algèbre de Lie semi-simple”. In: *Ann. of Math. (2)* 105.1 (1977), pp. 107–120.

[Fan96] C. K. Fan. “A Hecke algebra quotient and some combinatorial applications”. In: *J. Algebraic Combin.* 5.3 (1996), pp. 175–189.

[FG99] C. K. Fan and R. M. Green. “On the affine Temperley-Lieb algebras”. In: *J. London Math. Soc. (2)* 60.2 (1999), pp. 366–380.

[Fom95] S. Fomin. “Schur operators and Knuth correspondences”. In: *J. Combin. Theory Ser. A* 72.2 (1995), pp. 277–292.

[FGP97] S. Fomin, S. Gelfand, and A. Postnikov. “Quantum Schubert polynomials”. In: *J. Amer. Math. Soc.* 10.3 (1997), pp. 565–596.

[FG98] S. Fomin and C. Greene. “Noncommutative Schur functions and their applications”. In: *Discrete Math.* 193.1-3 (1998). Selected papers in honor of Adriano Garsia (Taormina, 1994), pp. 179–200.

[FS94] S. Fomin and R. P. Stanley. “Schubert polynomials and the nil-Coxeter algebra”. In: *Adv. Math.* 103.2 (1994), pp. 196–207.

[FOS09] G. Fourier, M. Okado, and A. Schilling. “Kirillov-Reshetikhin crystals for nonexceptional types”. In: *Adv. Math.* 222.3 (2009), pp. 1080–1116.

[Ful97] W. Fulton. *Young tableaux.* Vol. 35. London Mathematical Society Student Texts. With applications to representation theory and geometry. Cambridge University Press, Cambridge, 1997, pp. x+260.

[GJ11] M. Geck and N. Jacon. *Representations of Hecke algebras at roots of unity.* Vol. 15. Algebra and Applications. Springer-Verlag London, Ltd., London, 2011, pp. xii+401.

[Gob15] T. Gobet. “Noncrossing partitions, fully commutative elements and bases of the Temperley-Lieb algebra”. In: *arXiv* (2015). eprint: `arXiv/arXiv:1409.6500v2`.

[Gra95] J. J. Graham. *Modular representations of Hecke algebras and related algebras.* Ph.D. Thesis – University of Sydney. 1995.

[GL96] J. J. Graham and G. I. Lehrer. “Cellular algebras”. In: *Invent. Math.* 123.1 (1996), pp. 1–34.

[GL98] J. J. Graham and G. I. Lehrer. “The representation theory of affine Temperley-Lieb algebras”. In: *Enseign. Math. (2)* 44.3-4 (1998), pp. 173–218.

[Gre02] R. M. Green. “On 321-avoiding permutations in affine Weyl groups”. In: *J. Algebraic Combin.* 15.3 (2002), pp. 241–252.

[GL99] R. M. Green and J. Losonczy. “Canonical bases for Hecke algebra quotients”. In: *Math. Res. Lett.* 6.2 (1999), pp. 213–222.

[GL01] R. M. Green and J. Losonczy. “Fully commutative Kazhdan-Lusztig cells”. In: *Ann. Inst. Fourier (Grenoble)* 51.4 (2001), pp. 1025–1045.

[GR04] S. Griffeth and A. Ram. “Affine Hecke algebras and the Schubert calculus”. In: *European J. Combin.* 25.8 (2004), pp. 1263–1283.

[HP02] T. J. Haines and A. Pettet. “Formulae relating the Bernstein and Iwahori-Matsumoto presentations of an affine Hecke algebra”. In: *J. Algebra* 252.1 (2002), pp. 127–149.

[HMR09] T. Halverson, M. Mazzocco, and A. Ram. “Commuting families in Hecke and Temperley-Lieb algebras”. In: *Nagoya Math. J.* 195 (2009), pp. 125–152.

[HJ10] C. R. H. Hanusa and B. C. Jones. “The enumeration of fully commutative affine permutations”. In: *European J. Combin.* 31.5 (2010), pp. 1342–1359.

[Här99] M. Härterich. “Murphy bases of generalized Temperley-Lieb algebras”. In: *Arch. Math. (Basel)* 72.5 (1999), pp. 337–345.

[Hat+02] G. Hatayama et al. “Paths, crystals and fermionic formulae”. In: *MathPhys odyssey, 2001.* Vol. 23. Prog. Math. Phys. Birkhäuser Boston, Boston, MA, 2002, pp. 205–272.

[Hey00] A. Heyworth. “Rewriting as a special case of non-commutative Gröbner basis theory”. In: *Computational and geometric aspects of modern algebra (Edinburgh, 1998).* Vol. 275. London Math. Soc. Lecture Note Ser. Cambridge Univ. Press, Cambridge, 2000, pp. 101–105.

[HK02] J. Hong and S.-J. Kang. *Introduction to quantum groups and crystal bases.* Vol. 42. Graduate Studies in Mathematics. Providence, RI: American Mathematical Society, 2002, pp. xviii+307.

[Hub05] A. Hubery. “Symmetric functions and the centre of the Ringel-Hall algebra of a cyclic quiver”. In: *Math. Z.* 251.3 (2005), pp. 705–719.

[Hum08] J. E. Humphreys. *Representations of semisimple Lie algebras in the BGG category* $\mathcal{O}$. Vol. 94. Graduate Studies in Mathematics. American Mathematical Society, Providence, RI, 2008, pp. xvi+289.

[IM65] N. Iwahori and H. Matsumoto. "On some Bruhat decomposition and the structure of the Hecke rings of $\mathfrak{p}$-adic Chevalley groups". In: *Inst. Hautes Études Sci. Publ. Math.* 25 (1965), pp. 5–48.

[Jan96] J. C. Jantzen. *Lectures on quantum groups*. Vol. 6. Graduate Studies in Mathematics. American Mathematical Society, Providence, RI, 1996, pp. viii+266.

[Jim+91] M. Jimbo et al. "Combinatorics of representations of $U_q(\widehat{\mathfrak{gl}}(n))$ at $q = 0$". In: *Comm. Math. Phys.* 136.3 (1991), pp. 543–566.

[Jon83] V. F. R. Jones. "Index for subfactors". In: *Invent. Math.* 72.1 (1983), pp. 1–25.

[Jon87] V. F. R. Jones. "Hecke algebra representations of braid groups and link polynomials". In: *Ann. of Math. (2)* 126.2 (1987), pp. 335–388.

[Kan+92] S.-J. Kang et al. "Perfect crystals of quantum affine Lie algebras". In: *Duke Math. J.* 68.3 (1992), pp. 499–607.

[Kas91] M. Kashiwara. "On crystal bases of the q-analogue of universal enveloping algebras". In: *Duke Math. J.* 63.2 (1991), pp. 465–516.

[Kas95] C. Kassel. *Quantum groups*. Vol. 155. Graduate Texts in Mathematics. New York: Springer-Verlag, 1995, pp. xii+531.

[Kau90] L. H. Kauffman. "An invariant of regular isotopy". In: *Trans. Amer. Math. Soc.* 318.2 (1990), pp. 417–471.

[Kha15] A. Khare. "Axiomatic framework for the BGG category $\mathcal{O}$". In: *arXiv* (2015). eprint: `arXiv/math/1502.06706`.

[Kha16] A. Khare. "On the classification of finite-dimensional nil-Coxeter algebras over complex reflection groups". In: *arXiv* (2016). eprint: `arXiv:1601.08231`.

[Kho01] M. Khovanov. "Nilcoxeter algebras categorify the Weyl algebra". In: *Comm. Algebra* 29.11 (2001), pp. 5033–5052.

[KL09] M. Khovanov and A. D. Lauda. "A diagrammatic approach to categorification of quantum groups. I". In: *Represent. Theory* 13 (2009), pp. 309–347.

[KR87] A. N. Kirillov and N. Yu. Reshetikhin. "Representations of Yangians and multiplicities of the inclusion of the irreducible components of the tensor product of representations of simple Lie algebras". In: *Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI)* 160.Anal. Teor. Chisel i Teor. Funktsii. 8 (1987), pp. 211–221, 301.

[Koe08] S. Koenig. “A panorama of diagram algebras”. In: *Trends in representation theory of algebras and related topics*. EMS Ser. Congr. Rep. Eur. Math. Soc., Zürich, 2008, pp. 491–540.

[KX12] S. Koenig and C. Xi. “Affine cellular algebras”. In: *Adv. Math.* 229.1 (2012), pp. 139–182.

[Kor14] C. Korff. “Quantum cohomology via vicious and osculating walkers”. In: *Lett. Math. Phys.* 104.7 (2014), pp. 771–810.

[KS10] C. Korff and C. Stroppel. “The $\widehat{\mathfrak{sl}}(n)_k$-WZNW fusion ring: a combinatorial construction and a realisation as quotient of quantum cohomology”. In: *Adv. Math.* 225.1 (2010), pp. 200–268.

[KK86] B. Kostant and S. Kumar. “The nil Hecke ring and cohomology of G/P for a Kac-Moody group G”. In: *Adv. in Math.* 62.3 (1986), pp. 187–237.

[Kum02] S. Kumar. *Kac-Moody groups, their flag varieties and representation theory.* Vol. 204. Progress in Mathematics. Birkhäuser Boston, Inc., Boston, MA, 2002, pp. xvi+606.

[Kus13] D. Kus. “Realization of affine type A Kirillov-Reshetikhin crystals via polytopes”. In: *J. Combin. Theory Ser. A* 120.8 (2013), pp. 2093–2117.

[Kus16] D. Kus. “Kirillov–Reshetikhin crystals, energy function and the combinatorial R-matrix”. In: *J. Algebraic Combin.* 43.1 (2016), pp. 45–74.

[Lam99] T. Y. Lam. *Lectures on modules and rings.* Vol. 189. Graduate Texts in Mathematics. Springer-Verlag, New York, 1999, pp. xxiv+557.

[Lam01] T. Y. Lam. *A first course in noncommutative rings.* Second. Vol. 131. Graduate Texts in Mathematics. Springer-Verlag, New York, 2001, pp. xx+385.

[LS81] A. Lascoux and M.-P. Schützenberger. “Le mono$\ddot{}^{-1}$de plaxique”. In: *Noncommutative structures in algebra and geometric combinatorics (Naples, 1978).* Vol. 109. Quad. “Ricerca Sci.” CNR, Rome, 1981, pp. 129–156.

[LS89] A. Lascoux and M.-P. Schützenberger. “Fonctorialité des polynômes de Schubert”. In: *Invariant theory (Denton, TX, 1986).* Vol. 88. Contemp. Math. Amer. Math. Soc., Providence, RI, 1989, pp. 585–598.

[Mac91] I. G. Macdonald. “Schubert polynomials”. In: *Surveys in combinatorics, 1991 (Guildford, 1991).* Vol. 166. London Math. Soc. Lecture Note Ser. Cambridge Univ. Press, Cambridge, 1991, pp. 73–99.

[Mak15] R. Maksimau. “Categorical representations, KLR algebras and Koszul duality”. In: *arXiv* (2015). eprint: `arXiv/math/1512.04878`.

[Mat89] H. Matsumura. *Commutative ring theory*. Second. Vol. 8. Cambridge Studies in Advanced Mathematics. Translated from the Japanese by M. Reid. Cambridge University Press, Cambridge, 1989, pp. xiv+320.

[Maz10] V. Mazorchuk. *Lectures on $\mathfrak{sl}_2(\mathbb{C})$-modules*. London: Imperial College Press, 2010, pp. x+263.

[MT99] V. Mazorchuk and L. Turowska. "Simple weight modules over twisted generalized Weyl algebras". In: *Comm. Algebra* 27.6 (1999), pp. 2613–2625.

[MZ13] V. Mazorchuk and K. Zhao. "Characterization of simple highest weight modules". In: *Canad. Math. Bull.* 56.3 (2013), pp. 606–614.

[Mei15] J. Meinel. "Duflo theorem for a class of generalized Weyl algebras". In: *J. Algebra Appl.* 14.10 (2015), pp. 1550147, 21.

[MS16] V. Miemietz and C. Stroppel. "Affine quiver Schur algebras and p-adic GL_n". In: *arXiv* (2016). eprint: `arXiv:1601.07323`.

[MM90] K. Misra and T. Miwa. "Crystal base for the basic representation of $U_q(\widehat{\mathfrak{sl}}(n))$". In: *Comm. Math. Phys.* 134.1 (1990), pp. 79–88.

[MP95] R. V. Moody and A. Pianzola. *Lie algebras with triangular decompositions*. Canadian Mathematical Society Series of Monographs and Advanced Texts. A Wiley-Interscience Publication. John Wiley & Sons, Inc., New York, 1995, pp. xxii+685.

[Mur95] G. E. Murphy. "The representations of Hecke algebras of type A_n". In: *J. Algebra* 173.1 (1995), pp. 97–121.

[Mus92] I. M. Musson. "A classification of primitive ideals in the enveloping algebra of a classical simple Lie superalgebra". In: *Adv. Math.* 91.2 (1992), pp. 252–268.

[MB98] I. M. Musson and M. Van den Bergh. "Invariants under tori of rings of differential operators and related topics". In: *Mem. Amer. Math. Soc.* 136.650 (1998).

[Nad15] P. Nadeau. "On the length of fully commutative elements". In: *arXiv* (2015). eprint: `arXiv:1511.08788v1`.

[NRVO90] C. Năstăsescu, Š. Raianu, and F. Van Oystaeyen. "Modules graded by G-sets". In: *Math. Z.* 203.4 (1990), pp. 605–627.

[Par12] E. Park. "Quiver varieties and adjoint crystals of level ℓ for type $A_n^{(1)}$". In: *Comm. Algebra* 40.12 (2012), pp. 4513–4539.

[PRH14] D. Plaza and S. Ryom-Hansen. “Graded cellular bases for Temperley-Lieb algebras of type A and B”. In: *J. Algebraic Combin.* 40.1 (2014), pp. 137–177.

[Pos05] A. Postnikov. “Affine approach to quantum Schubert calculus”. In: *Duke Math. J.* 128.3 (2005), pp. 473–509.

[Rei01] M. Reineke. “Generic extensions and multiplicative bases of quantum groups at $q = 0$”. In: *Represent. Theory* 5 (2001), 147–163 (electronic).

[Rei02] M. Reineke. “The quantic monoid and degenerate quantized enveloping algebras”. In: *arXiv* (2002). eprint: `arXiv/math/0206095`.

[RSA14] D. Ridout and Y. Saint-Aubin. “Standard Modules, Induction and the Temperley-Lieb Algebra”. In: *Adv. Theor. Math. Phys.* 18.5 (2014), pp. 957–1041.

[Rin90] C. M. Ringel. “Hall algebras and quantum groups”. In: *Invent. Math.* 101.3 (1990), pp. 583–591.

[RCW82] A. Rocha-Caridi and N. R. Wallach. “Projective modules over graded Lie algebras. I”. In: *Math. Z.* 180.2 (1982), pp. 151–177.

[Sch00] O. Schiffmann. “The Hall algebra of a cyclic quiver and canonical bases of Fock spaces”. In: *Internat. Math. Res. Notices* 8 (2000), pp. 413–440.

[Sen10] P. Senesi. “Finite-dimensional representation theory of loop algebras: a survey”. In: *Quantum affine algebras, extended affine Lie algebras, and their applications.* Vol. 506. Contemp. Math. Amer. Math. Soc., Providence, RI, 2010, pp. 263–283.

[Shi02] M. Shimozono. “Affine type A crystal structure on tensor products of rectangles, Demazure characters, and nilpotent varieties”. In: *J. Algebraic Combin.* 15.2 (2002), pp. 151–187.

[ST97] B. Siebert and G. Tian. “On quantum cohomology rings of Fano manifolds and a formula of Vafa and Intriligator”. In: *Asian J. Math.* 1.4 (1997), pp. 679–695.

[Smi90] S. P. Smith. “A class of algebras similar to the enveloping algebra of $\mathfrak{sl}_2$”. In: *Trans. Amer. Math. Soc.* 322.1 (1990), pp. 285–314.

[Ste96] J. R. Stembridge. “On the fully commutative elements of Coxeter groups”. In: *J. Algebraic Combin.* 5.4 (1996), pp. 353–385.

[Ste03] J. R. Stembridge. “A local characterization of simply-laced crystals”. In: *Trans. Amer. Math. Soc.* 355.12 (2003), 4807–4823 (electronic).

[TL71] H. N. V. Temperley and E. H. Lieb. "Relations between the "percolation" and "colouring" problem and other graph-theoretical problems associated with regular planar lattices: some exact results for the "percolation" problem". In: *Proc. Roy. Soc. London Ser. A* 322.1549 (1971), pp. 251–280.

[Vla04] M. Vlasenko. "Description of the center of the affine Temperley-Lieb algebra of type $\widetilde{\mathsf{A}}_N$". In: *arXiv* (2004). eprint: `arXiv/math/0408392`.

[Wol07] S. Wolf. "The Hall algebra of a cyclic quiver at $q = 0$". In: *arXiv* (2007). eprint: `arXiv:0709.4348`.