Scalar curvature rigidity on locally conformally flat manifolds with boundary

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"Always pass on what you have learned."

Yoda[1]

[1]from "Star Wars: Episode III – Revenge of the Sith"

Abstract

Inspired by the work of F. Hang and X. Wang and partial results by S. Raulot, we prove a scalar curvature rigitidy result for locally conformally flat manifolds with boundary in the spirit of the well-known Min-Oo conjecture. Our results imply that Min-Oo's conjecture is true provided the considered manifold is locally conformally flat. In exchange, we require less knowledge on the geometry of the boundary than in the original statement of Min-Oo's conjecture. Furthermore, our result can be extended to yield a similar rigidity result for geodesic balls in a hemisphere.

Applications of our techniques include rigidity results for more general domains in a hemisphere and geodesic balls in Euclidean space as well as an extension of our result to locally conformally symmetric manifolds. To that end, we additionally establish that our results are valid for manifolds with parallel Ricci tensor, under slightly stronger assumptions on the geometry of the boundary.

Contents

Chapter 1

Introduction and overview of the results

The famous Min-Oo conjecture has fascinated mathematicians for over a decade: The story began in 1995, when Maung Min-Oo claimed to have proven the following ([MO98, Theorem 4]):

Conjecture 1.1. *Let M be a compact connected spin manifold with simply-connected boundary and g be a Riemannian metric on M with the following properties:*

i) ∂M is totally geodesic in M,
ii) the metric induced on ∂M has constant sectional curvature 1,
iii) the scalar curvature of g satisfies $\mathrm{Scal}(g) \geq n(n-1)$ on M.

Then (M, g) is isometric to the round hemisphere $S^n_+ = \{x \in S^n \mid x_{n+1} \geq 0\}$ equipped with the standard metric.

Min-Oo intended to give a proof in an upcoming paper, but it turned out that his argument was flawed – Conjecture 1.1 became known as the *Min-Oo conjecture*.

Due to its analogies with the positive mass theorem (see our exposition in Section 2.4), Min-Oo's conjecture is very natural and was widely believed to be true in the mathematical community but proven wrong in 2011 when Brendle, Marques and Neves [BMN11] were able to construct a counterexample valid in dimensions $n \geq 3$. Min-Oo's conjecture is true in dimension two by an old result due to Topogonov [Top59], compare also Corollary 4.1.1, i). However, several partial results have been obtained and modified versions of Min-Oo's conjecture hold in many special cases. For more information on the topic, the reader is referred to Section 2.5 where we give a more detailed overview and discuss several positive results obtained until today as well as the construction of a counterexample given by Brendle, Marques and Neves.

The main result of this thesis, Theorem I below, is a scalar curvature rigidity theorem for locally conformally flat manifolds in the spirit of the Min-Oo conjecture. It implies that the conjecture is true provided the manifold in consideration is *locally conformally flat*, see below for an explanation. Moreover, under these circumstances, the condition on the boundary to be totally geodesic can be weakened and our statement can be extended to geodesic balls in a hemisphere.

To motivate our results let us consider a special case where Min-Oo's conjecture has been proven to hold: Metrics conformally equivalent to the standard metric on the hemisphere. In [HW06], Hang and Wang have proven the following:

THEOREM 1.2. *Let $g = e^{2f} g_{S^n_+}$ be a C^2-metric on S^n_+. Assume that*

i) $\mathrm{Scal}(g) \geq n(n-1)$ *everywhere,*
ii) The boundary is isometric to S^{n-1}.

Then g is isometric to the standard metric $g_{S^n_+}$.

In [HW09], they were able to prove a similar result for domains in S^n_+, see Proposition 2.5.2. The proofs rely on the analysis of the equations for conformal scalar and mean curvature: If $n \geq 3$ and g as well as $\tilde{g} = u^{\frac{4}{n-2}} g$ are conformally equivalent metrics, then

$$\frac{n-2}{4(n-1)} \mathrm{Scal}(\tilde{g})\, u^{\frac{n+2}{n-2}} = \frac{n-2}{4(n-1)} \mathrm{Scal}(g) u - \Delta u, \tag{1.1}$$

$$\frac{n-2}{2} H(\tilde{g}) u^{\frac{n}{n-2}} = \frac{n-2}{2} H(g) u + \frac{\partial u}{\partial \eta}, \tag{1.2}$$

where H denotes the mean curvature computed with respect to the inner unit normal $\nu = -\eta$.

A disadvantage of Theorem 1.2 is that one needs to fix the differentiable structure of the manifold in consideration in order to be able to assume that g is conformally equivalent to the standard metric on S^n_+. This makes it impossible to see any influence of the curvature assumptions and geometry of the boundary on the topology or differentiable structure of M.

Motivated by this, Raulot [RAU12] was able to extend Theorem 1.2 to a class of locally conformally flat manifolds, that is, manifolds which are not globally conformally equivalent to the upper hemisphere but locally look like a conformal deformation of the sphere (for a precise definition see Definition 2.2.1). Using the Chern-Gauß-Bonnet formula, he proved:

PROPOSITION 1.3 ([RAU12, Corollaire 1]). *Let (M^n, g) be a compact connected Riemannian manifold with boundary of dimension $n = 4$ or $n = 6$ with $\chi(M) = 1$. Suppose that the boundary ∂M is umbilic with nonnegative mean curvature and isometric to the round sphere S^{n-1}. If (M, g) is locally conformally flat with scalar curvature* $\mathrm{Scal} \geq n(n-1)$*, then (M, g) is isometric to the standard hemisphere.*

Here, one already sees some influence of our assumptions on the differentiable structure of the manifold, but, in order to employ the Chern-Gauß-Bonnet formula, Raulot needs the additional topological assumption $\chi(M) = 1$ and his proof is restricted to dimensions 4 and 6.

The main result of this thesis is that Raulot's result is in fact valid in all dimensions without any additional assumptions on the Euler characteristic. Moreover, we are able to extend it to spherical caps of radius $0 < \rho \leq \frac{\pi}{2}$.

To state it, let us fix the following definitions and conventions which we will use throughout this thesis: Let $p \in S^n$ be arbitrary, $0 < \rho \leq \frac{\pi}{2}$ and $D_\rho := D_\rho(p) := \{x \in S^n \mid d^{S^n}(x,p) < \rho\}$ be the geodesic ball of radius ρ around p in S^n. Let $H_\rho := \cot(\rho)$ be the mean curvature of the boundary $\Sigma_\rho := \Sigma_\rho(p) := \partial D_\rho(p)$. Note that Σ_ρ is isometric to a sphere of radius $\sin(\rho)$.

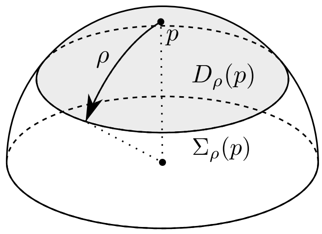

FIGURE 1. D_ρ and Σ_ρ

Then we have:

THEOREM I. *Let (M^n, g), $n \geq 3$, be a compact connected locally conformally flat Riemannian manifold with boundary. Assume that*

i) $\mathrm{Scal}(g) \geq n(n-1)$ *everywhere,*
ii) The boundary ∂M is umbilic with mean curvature $H(g) \geq H_\rho$ and every connected component is isometric to Σ_ρ.

Then (M, g) is isometric to $\overline{D_\rho}$ with the standard metric.

REMARK 1.4. i) We stated Theorem I in this way as we wanted the statement to be short and clear. Nevertheless, it also holds in slightly more general settings, see Corollary 4.1.1.

ii) The bound $\rho \leq \frac{\pi}{2}$ above is optimal. In fact, if the convexity of the boundary Σ_ρ fails (i.e. when $\frac{\pi}{2} < \rho < \pi$), Hang and Wang constructed metrics of the form $g = e^{2f} g_{S^n}$ on $\overline{D_\rho}$ with $\mathrm{Scal}(g) \geq n(n-1)$, $f \neq 0$ and $\mathrm{supp}(f) \subseteq D_\rho$, see [HW06, Theorem 2.1].

Rather than the Chern-Gauß-Bonnet formula, our proof relies on results by Schoen and Yau [SY88], [SY94] concerning the injectivity of the so-called *developing map*, a conformal immersion from the universal covering of M to S^n obtained from the locally conformally flat structure. A key observation in the proof is that – under the assumptions of Theorem I – this conformal immersion is injective (Proposition 3.2.1) which allows us to model the universal covering of M on the image of the developing map in S^n. We will then employ analytical techniques similar to those used by Hang and Wang to prove Theorem 1.2 and Proposition 2.5.2 to show that M is isometric to a geodesic ball. We will summarise our strategy in greater detail at the beginning of Chapter 3 on page 27.

Apart from geodesic balls we consider arbitrary domains in the hemisphere. A consequence of our main Theorem I is the following:

THEOREM (Theorem II on page 44). *Let* $\Omega \subseteq S^n_+$, $n \geq 3$, *be an* n*-dimensional manifold with boundary such that* $S^n \setminus \Omega$ *is a smooth domain. Let* (M^n, g) *be a compact, connected locally conformally flat Riemannian manifold with boundary. Assume that*

i) $\mathrm{Scal}(g) \geq n(n-1)$ *or* $\mathrm{Scal}(g)$ *attains its minimum at the boundary,*

ii) There exists an isometry $\phi\colon \partial M \to \partial\Omega$ *with the property that* $\phi^* \mathrm{II}_{\partial\Omega} = \mathrm{II}_{\partial M}$ *and* $\phi^*\left(R^{S^n}(\cdot, \eta^{\partial\Omega}, \eta^{\partial\Omega}, \cdot)\right) = R^M(\cdot, \eta^{\partial M}, \eta^{\partial M}, \cdot)$.

Then (M, g) *is isometric to* Ω *with the standard metric.*

The stronger conditions on the geometry of the boundary are needed in order to ensure that Theorem I can be applied to the manifold $M \cup_\phi (S^n_+ \setminus \Omega)$ obtained by gluing M to $S^n_+ \setminus \Omega$ along ϕ. We think that this should not be necessary but a complete proof is subject to further research.

Another application of the techniques employed to prove our main result are geodesic balls in Euclidean space. Let B_r denote an open ball of radius r in $\mathbb{R}^n$ with boundary sphere $S_r := \partial B_r$. Analogous to Theorem I, we prove:

THEOREM (Theorem III on page 47). *Let* (M^n, g), $n \geq 3$, *be a compact connected locally conformally flat Riemannian manifold with boundary. Assume that*

i) $\mathrm{Scal}(g) \geq 0$ *everywhere,*

ii) The boundary ∂M *is umbilic and every connected component is isometric to* S_r, *with mean curvature* $H(g) \geq H^r := r^{-1}$.

Then (M, g) *is isometric to* $\overline{B_r}$ *with the standard metric.*

The argumentation here is substantially easier as the conformal scalar curvature equation (1.1) greatly simplifies if the background metric has vanishing scalar curvature: For example, if $\tilde{g} = u^{\frac{4}{n-2}} g_{\mathbb{R}^n}$ is a metric on a subset of $\mathbb{R}^n$ with nonnegative scalar curvature, this simply means that u is superharmonic, i.e. $-\Delta u \geq 0$. Then the maximum principle and the Hopf lemma imply the following rigidity result: A metric $\tilde{g} = u^{\frac{4}{n-2}} g_{\mathbb{R}^n}$ on a bounded domain Ω with smooth boundary with nonnegative scalar curvature and $u = 1$ on $\partial\Omega$ must satisfy $u \geq 1$ and $H(\tilde{g}) \leq H(g_{\mathbb{R}^n})$; with equality at a point if and only if $u = 1$. In contrast, the proof of the corresponding statement for the upper hemisphere (Proposition 2.5.2) is highly nontrivial while the statement is not even true for domains not contained in a hemisphere when the boundary is not convex (cf. Remark 1.4, ii)). This fact also gives an explanation why we do not need any restrictions on r in Theorem III whereas we always assume $0 < \rho \leq \frac{\pi}{2}$ in Theorems I and IV.

Comparable results under different conditions have been obtained by, for example, Miao [MIA02] (for metrics on the unit ball), Shi and Tam [ST02] and

Raulot [Rau08] (for spin manifolds). We refer to the introduction in Section 4.3 for a short survey.

As another possible extension to Theorem I, one can try to weaken the condition on local conformal flatness. As Min-Oo's conjecture is incorrect, we will need other conditions on the manifold in order to ensure that the statement still holds. However, in the proof of Theorem I, the assumption that M is locally conformally flat is crucial as it allows us to model the universal covering on a domain of the sphere using the developing map.

Our starting point is the *Weyl-Schouten theorem* (Theorem 2.2.3) which characterises local conformal flatness by vanishing conditions of certain conformally invariant tensors: In dimension $n \geq 4$, a manifold is locally conformally flat if and only if its Weyl tensor W vanishes. This inspired us to investigate manifolds where the Weyl tensor not necessarily vanishes but is merely parallel ($\nabla W = 0$). We call such manifolds *locally conformally symmetric*. It turns out that such manifolds are either locally conformally flat or locally symmetric, see the paper [DR77, Theorem 2] by Derdziński and Roter.

As locally conformally flat manifolds are already covered by Theorem I, we turned our attention to manifolds with parallel Ricci tensor, which form a larger class than locally symmetric manifolds. Our result is:

Theorem (Theorem IV on page 50). *Let (M^n, g), $n \geq 3$, be a compact connected Riemannian manifold with boundary, $0 < \rho \leq \frac{\pi}{2}$. Assume that the Ricci tensor is parallel and*

i) $\mathrm{Scal}(g) \geq n(n-1)$ *everywhere,*
ii) The boundary ∂M is umbilic with mean curvature $H(g) = H_\rho$ and every connected component is isometric to Σ_ρ.

Then (M, g) is isometric to $\overline{D_\rho}$ with the standard metric.

The main step in the proof is to show that $\mathrm{Ric} \geq n-1$ at the boundary, then $\nabla \mathrm{Ric} = 0$ implies $\mathrm{Ric} \geq n-1$ everywhere. It follows that the boundary is connected and the statement is a consequence of Theorem 2.5.3 by Hang and Wang.

From Theorem IV it follows that Theorem I is valid for locally conformally symmetric manifolds (Corollary 4.4.4).

This thesis is structured as follows: For the reader's convenience we recall all background material necessary to understand the proof of Theorem I in Chapter 2, Sections 2.1–2.3. We then discuss basics on scalar curvature rigidity and give a detailed overview on the Min-Oo conjecture including many positive results as well as a sketch of the construction of a counterexample by Brendle, Marques and Neves. We conclude the chapter with a short discussion of the Yamabe problem on manifolds with boundary.

In Chapter 3, we give a complete proof of Theorem I.

We then turn our attention to possible applications and extensions and prove Theorems II–IV in Chapter 4.

For these results and a delicate issue concerning the positive mass theorem, it is necessary to review the proofs of the Weyl-Schouten theorem and the injectivity of the developing map proven by Schoen and Yau. For a clearer arrangement, we decided to collect them in a separate chapter, Chapter 5, where we also discuss regularity properties of the canonical Riemannian metric on a manifold obtained by gluing Riemannian manifolds along their boundaries.

Parts of this thesis have already been published in the author's preprint [Spi15].

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Chapter 2

Background material

In this chapter we present all background material necessary to understand the proof of Theorem I including a short discussion on the Yamabe problem on manifolds with boundary. In addition, we provide some foundational knowledge on scalar curvature geometry and rigidity as well as an overview on the Min-Oo conjecture. We begin by recalling the basic concepts of conformal geometry (Section 2.1), locally conformally flat manifolds (Section 2.2) and Möbius transformations (Section 2.3).

Note our convention that geodesic balls in spheres are denoted by $D_\rho(p)$ with boundary sphere $\Sigma_\rho(p) = \partial D_\rho(p)$, while geodesic balls in Euclidean space are denoted by $B_r(p)$ with boundary sphere $S_r(p) = \partial B_r(p)$. Provided that the center of a ball or sphere is irrelevant, it will be omitted in our notation and we simply write Σ_ρ, D_ρ, B_r or S_r. We denote the mean curvature of Σ_ρ by $H_\rho = \cot(\rho)$ and the mean curvature of S_r by $H^r = r^{-1}$. For a list of all frequently used symbols and expressions see page 67.

2.1. Basic conformal geometry and umbilic hypersurfaces

Let (M, g) be an n-dimensional Riemannian manifold. Another Riemannian metric $\bar{g}$ is called *conformally equivalent* to g if, at each point, $\bar{g}$ is a multiple of g, that is, $\bar{g} = e^{2f}g$ for some function f. The *conformal class* $[g]$ of g is the set of all metrics conformal to g.

An immersion $\Phi\colon (M, g) \to (N, h)$ is called *conformal* if it is angle-preserving or, equivalently, if Φ^*h is conformal to g. Two Riemannian manifolds (M, g) and (N, h) are called *conformally equivalent* if there is a conformal diffeomorphism $\Phi\colon (M, g) \to (N, h)$.

Let $\Sigma \subset M$ be a hypersurface with trivial normal bundle; in our later discussions, Σ will be the boundary ∂M of M. Let ν be the inner unit normal with associated scalar second fundamental form $\mathrm{II}(X, Y) = \langle \nabla_X Y, \nu \rangle$ and mean curvature $H = \frac{1}{n-1} \operatorname{trace} \mathrm{II}$. The outer unit normal is denoted by $\eta = -\nu$. We say that a point $x \in \Sigma$ is *umbilic* if, at x, the second fundamental form is diagonal with respect to the first fundamental form, i.e.

$$\mathrm{II}_x = H_x g_x|_{T_x\Sigma\times T_x\Sigma}.$$

We say that Σ is umbilic if all points $x \in \Sigma$ are umbilic points. Σ is called *minimal* if $H = 0$. It follows that Σ is totally geodesic (i.e. $\mathrm{II} = 0$) if and only if it is both umbilic and minimal. Note that – in general – umbilic hypersurfaces do not have constant mean curvature.

From the conformal transformation law for the second fundamental form (cf. e.g. [Esc92b, Equation (1.3)]),

$$\mathrm{II}(e^{2f}g) = e^f\,\mathrm{II}(g) + \frac{\partial f}{\partial\eta}e^f g,$$

it follows that being umbilic is a conformal invariant: $\Sigma \subset (M, g)$ is umbilic if and only if it is umbilic with respect to all $\bar{g} \in [g]$.

One can check that connected umbilic hypersurfaces in $\mathbb{R}^n$ are either contained in a hyperplane (hence $H = 0$) or a sphere ($H \neq 0$), see e.g. [Spi75, Lemma 7.1]. As, for $p \in S^n$, $(S^n \setminus \{p\}, g_{S^n})$ and $(\mathbb{R}^n, g_{\mathbb{R}^n})$ are conformally equivalent via a stereographic projection π_p from p (see Definition 2.3.1 below), it follows that connected umbilic hypersurfaces in S^n are contained in geodesic spheres $\Sigma_\rho(q)$.

2.2. Locally conformally flat manifolds

We now present some basics and examples of locally conformally flat manifolds as well as the Weyl-Schouten theorem which characterises locally conformally flat manifolds using conformally invariant tensors. We define:

Definition 2.2.1. A C^k-Riemannian manifold (M, g) is called *locally conformally flat* if for every point $p \in M$, there exists a neighbourhood U of p and $f \in C^k(U)$ such that the metric $e^{2f}g$ is flat on U.

Here we say that a C^k-metric h is *flat* if it is locally isometric to the Euclidean metric, i.e. for all $p \in M$, there exists a neighbourhood U of p and an isometry $f\colon U \to f(U) \subset \mathbb{R}^n$ (necessarily of class C^{k+1}).

Note that locally conformally flat manifolds are sometimes just called *conformally flat* in the literature. We reserve the term conformally flat for manifolds (M, g) which are globally conformally flat in the sense that there is $f \in C^k(M)$ such that $e^{2f}g$ is flat.

For the rest of this section we assume that M is a smooth manifold with C^3-Riemannian metric, so that the Cotton tensor defined below is well-defined and continuous.

Before coming to more theoretical investigations let us discuss some examples:

EXAMPLE 2.2.2. Locally conformally flat manifolds include:

i) Two-dimensional Riemannian manifolds,
ii) All spaces of constant sectional curvature,
iii) Open submanifolds and umbilic hypersurfaces[1] of locally conformally flat spaces are again locally conformally flat.
iv) The products $S^1 \times S^{n-1}$ and $S^m \times H^m$, although products of locally conformally flat manidolds are not necessarily locally conformally flat when equipped with the product metric. To see this consider the product metric on $S^m \times S^m$, $m \geq 2$, for instance: For $2m$-dimensional locally conformally flat manifolds we have a Bochner-Weitzenböck formula on m-forms
$$\Delta = \nabla^* \nabla + \frac{m}{4(m-1)} \operatorname{Scal},$$
cf. [LIS14, Lemma 3], key observation due to Bourguignon [BOU81]. This implies that closed locally conformally flat manifolds with positive scalar curvature must have $H^m_{\mathrm{dR}}(M) = 0$. For more topological obstructions on locally conformally flat manifolds with positive scalar curvature see e.g. [SY94, Chapter VI].
v) [BES87, Example 1.167], [LAF88, Proposition D.2]: More generally, one can check that a Riemannian product is locally conformally flat if and only if one factor is one-dimensional and the other one is of constant sectional curvature or if both factors are of constant sectional curvature with sectional curvatures κ and $-\kappa$, respectively.
vi) [LAF88, Proposition D.1 ii)]: If (M, g) is of constant sectional curvature, then warped products of the form $(M \times I, e^{2f(t)} g + \mathrm{d}t^2)$, $f \in C^\infty(I)$ are locally conformally flat.

In dimensions $n \geq 4$, the condition on a metric to be locally conformally flat is reflected in a vanishing condition on a certain conformally invariant tensor, called *Weyl tensor*. As a $(4,0)$-tensor, it is given by (cf. e.g. [BES87, Section 1G]):
$$W := R - \frac{\operatorname{Scal}}{2n(n-1)} g \owedge g - \frac{1}{n-2} \mathring{\operatorname{Ric}} \owedge g. \tag{2.2.1}$$
Here, $\mathring{\operatorname{Ric}} = \left(\operatorname{Ric} - \frac{\operatorname{Scal}}{n} g\right)$ denotes the traceless Ricci tensor and $\owedge$ is the Kulkarni-Nomizu product of two symmetric $(2,0)$-tensors h, k defined by
$$\begin{aligned}(h \owedge k)(v_1, v_2, v_3, v_4) := {} & h(v_1, v_3) k(v_2, v_4) + h(v_2, v_4) k(v_1, v_3) \\ & - h(v_1, v_4) k(v_2, v_3) - h(v_2, v_3) k(v_1, v_4).\end{aligned}$$

[1]In fact, if $\Sigma \subseteq (M, g)$ is umbilic, then it is also umbilic with respect to a locally defined flat metric $e^{2f} g$. Recall from Section 2.1 that umbilic hypersurfaces in Euclidean space are contained in either a hyperplane or a hypersphere, both of which are locally conformally flat.

Using the *Schouten tensor* (we adapt the sign convention of [BJ10], see Equation (1.2.13) therein)

$$S := \frac{1}{n-2}\left(\mathrm{Ric} - \frac{\mathrm{Scal}}{2(n-1)}\right), \tag{2.2.2}$$

one can write (2.2.1) as

$$W = R - S \owedge g. \tag{2.2.3}$$

In dimension three, the Weyl tensor automatically vanishes and the condition to be locally conformally flat is equivalent to the vanishing of the *Cotton tensor* given by

$$C(X, Y, Z) := (\nabla_X S)(Y, Z) - (\nabla_Y S)(X, Z), \tag{2.2.4}$$

see e.g. [BJ10, Section 2.2.3]. This can be summarized to the *Weyl-Schouten theorem* (first proven by Cotton [Cot97] in dimension three and by Weyl [Wey18] and Schouten [Sch21] in dimensions $n \geq 4$, see also [Laf88]):

THEOREM 2.2.3 (Weyl-Schouten). *Let (M, g) be an n-dimensional Riemannian manifold. Then*

i) If $n = 2$, then (M, g) is locally conformally flat.
ii) If $n = 3$, then (M, g) is locally conformally flat if and only if the Cotton tensor vanishes.
iii) If $n \geq 4$, then (M, g) is locally conformally flat if and only if the Weyl tensor vanishes.

As we need to refer to it later on – in a situation where the metric in consideration is not C^3 – we included a proof in Section 5.2.

We conclude the investigation of locally conformally flat manifolds for now and will return to them in Section 3.1

2.3. Conformal transformation groups and the Poincaré extension

When dealing with locally conformally flat manifolds, conformal transformations will arise naturally. For a better understanding it is thus worthwhile having a look at the conformal transformation groups of S^n and $\mathbb{R}^n$, which we will discuss below. For a more elaborate discussion see e.g. [Rat06].

To relate the conformal transformation groups of S^n and $\mathbb{R}^n$, respectively, we make use of the well-known stereographic projection:

DEFINITION 2.3.1. A stereographic projection $\pi\colon S^n \setminus \{e_{n+1}\} \to \mathbb{R}^n$ from e_{n+1} is given by $\pi(x) := (1-x_{n+1})^{-1}(x_1, \dots, x_n)$. For $p \in S^n$, a *stereographic projection from p* is a map of the form $\pi_p = \pi \circ \Phi\colon S^n \setminus \{p\} \to \mathbb{R}^n$, where Φ is an isometry of S^n mapping p to e_{n+1}.

Stereographic projections are conformal diffeomorphisms $S^n \setminus \{p\} \to \mathbb{R}^n$ as $(\pi^{-1})^* g_{S^n} = 4\left(1+|x|^2\right)^{-2} g_{\mathbb{R}^n}$.

Writing $\overline{\mathbb{R}^n} = \mathbb{R}^n \cup \{\infty\}$ for the one-point compactification of $\mathbb{R}^n$, we can extend any stereographic projection π_p to a homeomorphism $\pi_p \colon S^n \to \overline{\mathbb{R}^n}$ by setting $\pi_p(p) := \infty$. Equip $\overline{\mathbb{R}^n}$ with the metric, differentiable and conformal structure of S^n induced by such an extension.

We now define the *Möbius transformation* groups $M(\overline{\mathbb{R}^n})$ and $M(S^n)$ to be the subgroups of the respective diffeomorphism group generated by reflections in hyperspheres, where a hyperplane in $\mathbb{R}^n$ is seen as a hypersphere in $\overline{\mathbb{R}^n}$ containing infinity. As S^n and $\overline{\mathbb{R}^n}$ are conformally equivalent via a stereographic projection, their Möbius transformation groups $M(\overline{\mathbb{R}^n})$ and $M(S^n)$ are isomorphic via conjugation with a stereographic projection.

When considering locally defined conformal maps of $\mathbb{R}^n$, it turns out that these are exactly given by Möbius transformations as stated by Liouville's theorem on conformal transformations:

THEOREM 2.3.2 (Liouville, 1850). *Let $\Omega \subseteq \mathbb{R}^n$, $n \geq 3$, be a domain and $\phi \colon \Omega \to \overline{\mathbb{R}^n}$ be a conformal map of class C^2. Then ϕ is a composition of translations, rotations, reflections, scalings and inversions. In particular, ϕ is the restriction of a globally defined Möbius transformation $\Phi \colon \overline{\mathbb{R}^n} \to \overline{\mathbb{R}^n}$.*

Note that a "standard" proof (as presented in, e.g. [SY94]) requires ϕ to be C^3. However, there are proofs with less regularity assumptions, see e.g. [HAR47] and [HAR58].

Liouville's theorem implies that the conformal transformation groups of $\overline{\mathbb{R}^n}$ and S^n are $M(\overline{\mathbb{R}^n})$ and $M(S^n)$, respectively.

We set $M(B^n)$ to be the subgroup of $M(\overline{\mathbb{R}^n})$ containing all Möbius transformations preserving the unit ball B^n. One can show that the homomorphism $M(B^{n+1}) \to M(S^n)$ induced by restriction is an isomorphism, that is:

PROPOSITION 2.3.3. *Let $\varphi \in M(S^n)$. Then there is a unique $\Phi \in M(B^{n+1})$ with $\Phi|_{S^n} = \varphi$, called* Poincaré extension *of φ.*

PROOF. As $M(S^n)$ is generated by reflections in hyperspheres, it is enough to extend these. Let a reflection σ in a hypersphere $\Sigma \subseteq S^n$ be given. Let $\tilde{\Sigma}$ be the generalized hypersphere ($\tilde{\Sigma}$ may be a hyperplane) orthogonal to S^n which intersects S^n in Σ. Then the reflection of $\overline{\mathbb{R}^{n+1}}$ in $\tilde{\Sigma}$ extends σ and leaves B^n invariant. For more details, see [RAT06, Section 4.4] □

REMARK 2.3.4. The construction above is not restricted to the unit ball: More generally, if $S_r(p)$ or $\Sigma_\rho(p)$ is a sphere in $\mathbb{R}^{n+1}$ or S^{n+1}, respectively, and if $\varphi \in M(S_r)$ or $\varphi \in M(\Sigma_\rho)$ is a Möbius transformation of that sphere, then we can extend it to a Möbius transformation of the whole space ($\overline{\mathbb{R}^{n+1}}$ or S^{n+1}, respectively) preserving $B_r(p)$ or $D_\rho(p)$. This can be seen by adjusting the proof of Proposition 2.3.3 to this case or by conjugation with a conformal diffeomorphism mapping $B_r(p)$ or $D_\rho(p)$ to B_1 and making use of Proposition 2.3.3 directly.

One finds that the conformal transformation group $M(S^n)$ of the sphere is rather large. In fact, among compact manifolds, the conformal class of the standard metric on S^n is the only one with a noncompact automorphism group (first proven by Obata [OBA71] and Lelong-Ferrand [LF71]). Noncompactness of $M(S^n)$ is also reflected in the following result due to Obata [OBA71, Proposition 6.1]:

THEOREM 2.3.5 (Obata). *Let $\bar{g} \in [g_{S^n}]$ be a metric on S^n conformally equivalent to the standard one. Then $\bar{g}$ has constant scalar curvature if and only if it has constant sectional curvature.*

An important consequence is that any metric $\bar{g} \in [g_{S^n}]$ with constant scalar curvature $n(n-1)$ is of the form $\bar{g} = \Phi^* g_{S^n}$ for some $\Phi \in M(S^n)$.

Also, it is worth noting that $M(S^n)$ acts transitively on the set of hyperspheres in S^n, i.e.

LEMMA 2.3.6. *Let $\Sigma^i \subseteq S^n$, $i = 1, 2$, be hyperspheres. Then there exists $\Phi \in M(S^n)$ with $\Phi(\Sigma^1) = \Sigma^2$.*

PROOF. Let $p \in S^n \setminus (\Sigma^1 \cup \Sigma^2)$ and let $\pi_p \colon S^n \to \overline{\mathbb{R}^n}$ be a stereographic projection from p. Then $\pi_p(\Sigma^i)$ are spheres in $\mathbb{R}^n$, say $S_{r_i}(q_i)$. Let $M_r(x) := rx$ and $T_q(x) := x + q$ denote dilation by r and translation by q, respectively. Then

$$\Phi := \pi_p^{-1} \circ T_{q_2} \circ M_{r_2 r_1^{-1}} \circ T_{-q_1} \circ \pi_p$$

maps Σ^1 to Σ^2. □

Combining this with the Poincaré extension, we obtain a very useful result for our investigations in Chapter 3:

PROPOSITION 2.3.7. *Let $D \subseteq S^n$ be a geodesic ball in S^n with boundary sphere $\Sigma = \partial D$ and let $h \in [g_{S^n}|_\Sigma]$ be a metric on Σ conformal to the restriction of the standard metric. Assume that (Σ, h) is isometric to some hypersphere Σ_ρ of radius $0 < \rho < \pi$ equipped with the standard metric. Then, for $p \in S^n$, there exists $\Phi \in M(S^n)$ with $\Phi(D_\rho(p)) = D$, $\Phi(\Sigma_\rho(p)) = \Sigma$ and $\Phi^* h = g_{S^n}|_{\Sigma_\rho(p)}$.*

PROOF. As shown in Lemma 2.3.6 we can find $\phi \in M(S^n)$ with $\phi(\Sigma_\rho(p)) = \Sigma$. By composing with a reflection in Σ if necessary, we can furthermore ensure that $\phi(D_\rho(p)) = D$; call this composition ϕ again. As ϕ is conformal, $\phi^* h$ is conformal to $g_{S^n}|_{\Sigma_\rho(p)}$ because h is conformal to $g_{S^n}|_\Sigma$.

By assumption, there exists an isometry $\psi\colon (\Sigma_\rho(p), g_{S^n}|_{\Sigma_\rho(p)}) \to (\Sigma_\rho(p), \phi^*h)$. Since ϕ^*h is conformal to $g_{S^n}|_{\Sigma_\rho(p)}$, the map ψ is a conformal transformation of $\Sigma_\rho(p)$, hence ψ is the restriction of its Poincaré extension $\Psi \in M(S^n)$ preserving $D_\rho(p)$ (see Proposition 2.3.3 and Remark 2.3.4 thereafter). Set $\Phi \coloneqq \phi \circ \Psi$, then

$$\Phi(D_\rho(p)) = \phi(\Psi(D_\rho(p))) = \phi(D_\rho(p)) = D,$$

therefore $\Phi(\Sigma_\rho(p)) = \Sigma$ and additionally

$$\Phi^*h = \Psi^*(\phi^*h) = \psi^*(\phi^*h) = g_{S^n}|_{\Sigma_\rho(p)}. \qquad \square$$

2.4. Scalar curvature rigidity

Let (M, g) be some Riemannian manifold. In many mathematical contexts it is interesting to know whether the combination of certain known properties or invariants of M bears additional information on the geometry, topology or differentiable structure of the manifold. In this fashion, a *rigidity theorem* completely recovers the geometry (topology, differentiable structure) of a manifold from some of its invariants which correspondingly determine the manifold up to isometry (homeomorphism, diffeomorphism). For example, the probably best-known rigidity theorem in geometry is:

THEOREM 2.4.1. *A closed, simply-connected complete Riemannian manifold of constant sectional curvature is isometric to Euclidean space, a sphere or hyperbolic space (of adequate radius).*

Compared to sectional curvature as utilised in the theorem above, scalar curvature is a much weaker invariant and consequently bears less information; the case $n = 2$, where both are equivalent, being exceptional. For instance, a metric of constant scalar curvature can be found on any manifold and even in every conformal class, see the discussion on the Yamabe problem in Section 2.6. However, the sign of the scalar curvature may tell us something about the underlying manifold as was observed by Kazdan and Warner [KW75] and Bérard-Bergery [BB81], see also [BES87, Theorem 4.35]:

THEOREM 2.4.2. *Compact manifolds M of dimension $n \geq 3$ can be divided in three classes, each determined by one of the following properties:*

i) Any function on M is the scalar curvature of some Riemannian metric;

ii) A function on M is the scalar curvature of some Riemannian metric if and only if it is either identically zero or strictly negative somewhere; furthermore, any metric with vanishing scalar curvature is Ricci-flat;

iii) A function on M is the scalar curvature of some Riemannian metric if and only if it is strictly negative somewhere.

However, since all these classes are quite large, the result is unsatisfying from the standpoint of rigidity. Nevertheless it indicates that – to obtain a scalar curvature rigidity theorem in the spirit of Theorem 2.4.1 – the conditions on the geometry and topology have to be chosen very carefully.

An example for such a condition is the notion of *asymptotic flatness*, which is substantial for one of the best-known scalar curvature rigidity theorems: The positive mass theorem.

DEFINITION 2.4.3. A Riemannian three-manifold (M, g) is called *asymptotically flat* if $\mathrm{Scal} \in L^1(M)$ and there exists a compact set $K \subseteq M$ such that $M \setminus K$ is diffeomorphic to $\mathbb{R}^3 \setminus \overline{B_1}$ such that – with respect to the coordinates provided by this – the metric satisfies

$$g_{ij} = \delta_{ij} + \mathcal{O}(|x|^{-1}), \qquad \partial_k g_{ij} = \mathcal{O}(|x|^{-2}), \qquad \partial_k \partial_l g_{ij} = \mathcal{O}(|x|^{-3}).$$

The positive mass theorem, first proved by Schoen and Yau [SY79], [SY81] states:

THEOREM 2.4.4 (Positive mass theorem). *The* ADM[2]*-mass* [SY81, page 232]

$$m_{\mathrm{ADM}} := \lim_{r\to\infty} \frac{1}{16\pi} \sum_{i,j=1}^{3} \int_{|x|=r} \big(\partial_j g_{ij}(x) - \partial_i g_{jj}(x)\big) \frac{x^i}{r} \, \mathrm{d}S(x)$$

of an asymptotically flat three-manifold with nonnegative scalar curvature is nonnegative. Moreover, $m_{\mathrm{ADM}} = 0$ *if and only if* (M, g) *is isometric to* $\mathbb{R}^3$ *with the standard metric.*

Until today, various generalizations and extensions of the original positive mass theorem have been established. For example, Schoen and Yau's proof can be extended up to dimension 7 while Witten was able to prove a generalization valid for spin manifolds of all dimensions [WIT81].

From the positive mass theorem, we conclude the following rigidity result: A Riemannian metric on $\mathbb{R}^n$ with nonnegative scalar curvature agreeing with the standard metric outside a compact set must be flat. Hence the Euclidean metric is rigid in the sense that it is not possible to deform it locally and increase scalar curvature without decreasing it somewhere.

In contrast, it is always possible to decrease scalar curvature by locally deforming a metric as shown by Lohkamp:

THEOREM 2.4.5 ([LOH99, Theorem 1]). *Let* (M, g) *be a Riemannian manifold and* $U \subseteq M$ *open. Then, for all* $f \in C^\infty(M)$ *with* $f < \mathrm{Scal}(g)$ *on* U *and* $f = \mathrm{Scal}(g)$ *on* $M \setminus U$ *and* $\varepsilon > 0$*, there exists a smooth metric* g_ε *on* M *with* $f - \varepsilon \leq \mathrm{Scal}(g_\varepsilon) \leq f$ on $U_\varepsilon := \{x \in M \mid d(x, U) < \varepsilon\}$ *and* $g = g_\varepsilon$ *on* $M \setminus U_\varepsilon$.

[2] named after Richard Arnowitt, Stanley Deser and Charles W. Misner.

Summing up, we have seen that decreasing scalar curvature by locally changing the metric is always possible while increasing it may be problematic. Nevertheless, it can be achieved for nonstatic manifolds due to a result by Corvino, [Cor00]:

Definition 2.4.6. We say that a Riemannian manifold (M, g) is *static* if the linearization L_g of the scalar curvature map $g \mapsto \mathrm{Scal}(g)$ has a formal L^2-adjoint $L_g^*\colon H^2_{\mathrm{loc}}(M) \to L^2_{\mathrm{loc}}(M)$ with nontrivial kernel.

By computing L_g^* explicitely, one sees that a closed Riemannian manifold is static if and only if there is $f \in C^\infty(M) \setminus \{0\}$ with

$$\mathrm{Hess}\, f = f\,\mathrm{Ric}(g) + \Delta f \cdot g. \tag{2.4.1}$$

Corvino's result [Cor00, Theorem 4] now states that, given a compactly contained nonstatic domain $\Omega \subseteq M$ with smooth boundary and $f \in C^\infty(M)$ with $\mathrm{supp}(f - \mathrm{Scal}(g)) \subseteq \Omega$ and $f - \mathrm{Scal}(g)$ sufficiently small, there exists a local deformation of g with scalar curvature f.

Note that the standard metric on both the sphere S^n and the hemisphere S^n_+ are static: In fact, the restriction of the coordinate function x_{n+1} to S^n or S^n_+, respectively, lies in the kernel of L_g^*. Hence the above results do not give us insight whether or not these metrics can be deformed locally to increase scalar curvature. This is the starting point for the Min-Oo conjecture which will be discussed in the next section.

2.5. The Min-Oo conjecture

In 1995, Maung Min-Oo claimed to have proven the following scalar curvature rigidity theorem [MO98, Theorem 4]:

Conjecture 2.5.1. *Let M be a compact connected spin manifold with simply-connected boundary and g be a Riemannian metric on M with the following properties:*

i) ∂M is totally geodesic in M,
ii) the metric induced on ∂M has constant sectional curvature 1*,*
iii) the scalar curvature of g satisfies $\mathrm{Scal}(g) \geq n(n-1)$ *on M.*

Then (M, g) is isometric to the round hemisphere with the standard metric.

Min-Oo announced a proof in an upcoming paper, but he realized that his argument was incorrect. Conjecture 2.5.1 became known as the *Min-Oo conjecture.*

Min-Oo's conjecture can be seen as an analogue of the (rigidity part of the) positive mass theorem in positive curvature, where the asymptotic conditions on the manifold are replaced by boundary conditions. Due to these analogies and validity of similar results for zero and negative sectional curvature (see e.g. the rigidity result on the unit ball obtained by Miao [Mia02, Corollary 1.1]), Min-Oo's conjecture was long believed to be true in the mathematical community until Brendle, Marques and Neves [BMN11] were able to construct a counterexample in 2011 valid in all dimensions $n \geq 3$. Min-Oo's conjecture is true in dimension two due to a result by Topogonov [Top59]. However, several partial results have been obtained and modified versions of Min-Oo's conjecture hold in many special cases. In this section, we survey some of those results and discuss the construction of a counterexample by Brendle, Marques and Neves. For an excellent survey on the topic see the survey article [Bre12].

Positive results. One of the first positive results on the Min-Oo conjecture, and a very important one when considering the focus of this thesis, was obtained by Hang and Wang in 2006. In [HW06], they showed that Min-Oo's conjecture is true provided the metric in consideration is conformally equivalent to the standard one, see Theorem 1.2. It is quite remarkable that no condition on the second fundamental form of the boundary is needed in order to obtain the result. In the same paper, they were able to verify Min-Oo's conjecture for Einstein metrics ([HW06, Theorem 4.1]).

In their later work [HW09], Hang and Wang were able to extend both results to more general settings. With help of the already-established Theorem 1.2, they used the conformal scalar and mean curvature equation (1.1), (1.2) to prove:

Proposition 2.5.2 ([HW09, Proposition 1]). *Let $\Omega \subseteq S^n_+$ be a smooth domain and $\tilde{g} = u^{\frac{4}{n-2}} g_{S^n}$ be a metric on Ω in the conformal class of the standard metric. Assume that*

i) $\mathrm{Scal}(\tilde{g}) \geq n(n-1)$,
ii) the metric induced on $\partial\Omega$ agrees with the standard metric.

Then $u \geq 1$ and $H(\tilde{g}) \leq H(g_{S^n})$. Moreover, if equality holds somewhere, then $u = 1$.

Moreover, they extended their result on Einstein manifolds to obtain a "Ricci-version" of Min-Oo's conjecture. Their most general result in this direction is:

Theorem 2.5.3 ([HW09, Theorem 3]). *Let (M, g) be a compact Riemannian manifold with boundary $\partial M = \Sigma$ and $\overline{\Omega} \subseteq S^n_+$ be a compact domain with smooth boundary in the open hemisphere. Suppose that*

i) $\mathrm{Ric}(g) \geq (n-1)$,
ii) there is an isometric embedding $\iota \colon (\Sigma, g|_\Sigma) \to \partial\overline{\Omega}$ with the property that the second fundamental form II^Σ of Σ in M and the second fundamental form $\mathrm{II}^{\partial\overline{\Omega}}$ of $\partial\overline{\Omega}$ in S^n satisfy $\mathrm{II}^\Sigma \geq \mathrm{II}^{\partial\overline{\Omega}} \circ \iota$.

Then (M, g) is isometric with $(\overline{\Omega}, g_{S^n}|_\Omega)$.

The same conclusion is valid for the whole hemisphere, i.e. $\overline{\Omega} = S^n_+$, see [HW09, Theorem 2].

Hang and Wang's proof uses a Lipschitz version of the following result due to Reilly: If a Riemannian manifold (M, g) satisfies $\mathrm{Ric}(g) \geq n-1$ then the first Dirichlet eigenvalue λ_1 of $-\Delta$ satisfies $\lambda_1 \geq n$, with equality if and only if (M, g) is isometric to S^n_+. The statement of Theorem 2.5.3 now follows by proving that the manifold $M \cup_\iota (S^n_+ \setminus \Omega)$ has $\lambda_1 = n$.

Another remarkable result has been obtained by Eichmair in [Eic09]. He was able to prove that a three-dimensional Riemannian manifold with totally geodesic boundary such that $\mathrm{Scal}(g) \geq 6$ and $\mathrm{Area}(\partial M) \geq 4\pi$ is isometric to a hemisphere provided that $\mathrm{Ric}(g) > 0$ and that the boundary is an isoparametric surface for the double manifold.

A different approach was taken by Huang and Wu. In their papers [HW10] and [HW11], they investigated under which conditions a version of Min-Oo's conjecture is valid for hypersurfaces in spaces of constant sectional curvature equipped with the induced metric. For surfaces in Euclidean or hyperbolic space, they obtain:

Theorem 2.5.4 ([HW10, Theorems 1 and 3]). *Let $M \subseteq \mathbb{R}^{n+1}$ or $M \subseteq H^{n+1}$ be a compact connected hypersurface with boundary satisfying the (hyperbolic) incorporation condition. Suppose that the scalar curvature of M is at least $n(n-1)$. Then M is isometric to the hemisphere S^n_+.*

Here, the *(hyperbolic) incorporation condition* demands that, among other technical conditions, the boundary is diffeomorphic to a sphere and contained in a hyperplane $\mathbb{R}^n \times \{1\}$ such that $B^n_1 \times \{1\}$ is contained in the region enclosed by ∂M.

The main advantage when working with hypersurfaces is that the scalar curvature condition can be reformulated to a mean curvature condition using the Gauß equation. Huang and Wu's proof then relies on adequate maximum and comparison principles applied to the mean curvature operator, which is elliptic in nonpositive constant sectional curvature.

For spheres, the situation is more complicated as the latter ellipticity fails. Nevertheless, they were able to prove:

Theorem 2.5.5 ([HW11, Theorem 1]). *Let $M \subseteq S^{n+1}$ be a compact connected hypersurface with boundary satisfying*

i) $\mathrm{Scal} \geq n(n-1)$,

ii) M is tangent to a great n-sphere at ∂M and ∂M is a great $(n-1)$-sphere.

Then M is a hemisphere S^n_+.

Inspired by the Min-Oo conjecture, several authors obtained "local" rigidity results for geodesic balls in the hemisphere. For example, Brendle and Marques proved the following:

THEOREM 2.5.6 ([BM11, Theorem 3]). *Let* $\Omega = D_\rho$ *be a geodesic ball in* S^n *with radius* ρ *such that* $\cos(\rho) \geq \frac{2}{\sqrt{n+3}}$. *Let* g *be a Riemannian metric on* Ω *with*

i) $\mathrm{Scal}(g) \geq n(n-1)$,
ii) $H(g) \geq H(g_{S^n})$,
iii) the metrics g *and* g_{S^n} *induce the same metric on* $\partial\Omega$.

If $||g - g_{S^n}||_{C^2(\Omega)}$ *is sufficiently small, then* $g = \varphi^*(g_{S^n})$ *for some diffeomorphism* φ *of* Ω *with* $\varphi|_{\partial\Omega} = \mathrm{id}$.

In [CMT13], Cox, Miao and Tam were able to decrease the lower bound $\frac{2}{\sqrt{n+3}}$ in Theorem 2.5.6 by carefully investigating and improving the technique of the proof in two different ways. The result is that Theorem 2.5.6 holds for larger balls, namely provided $\cos(\rho) > \min\{\zeta_1, \zeta_2\}$, where

$$\zeta_1 = \left(\frac{4(n+4) - 4\sqrt{2n-1}}{n^2+6n+17}\right)^{\frac{1}{2}} \qquad \text{and} \qquad \zeta_2 = \left(\frac{7n-1}{2n^2+5n-1}\right)^{\frac{1}{2}}.$$

Still, the result is not valid for the whole hemisphere. However, in [MT12], Miao and Tam proved that one can extend Theorem 2.5.6 provided the metric in consideration satisfies an additional volume constraint. They obtain:

THEOREM 2.5.7 ([MT12, Theorem 1.2]). *Let* g *be a Riemannian metric on the hemisphere* S^n_+ *with*

i) $\mathrm{Scal}(g) \geq n(n-1)$,
ii) $H(g) \geq 0$,
iii) the metrics g *and* $g_{S^n_+}$ *induce the same metric on* ∂S^n_+,
iv) $\mathrm{Vol}(g) \geq \mathrm{Vol}(g_{S^n_+})$.

If $||g - g_{S^n_+}||_{C^2(S^n_+)}$ *is sufficiently small, then* $g = \varphi^*(g_{S^n_+})$ *for some diffeomorphism* φ *of* S^n_+ *with* $\varphi|_{\partial S^n_+} = \mathrm{id}$.

Theorems 2.5.6 and 2.5.7 are remarkable taking into account that the counter-example to Min-Oo's conjecture constructed by Brendle, Marques and Neves can be chosen arbitrarily close to $g_{S^n_+}$ in the C^∞-topology.

Construction of a counterexample. In their paper [BMN11], Brendle, Marques and Neves proved Min-Oo's conjecture to be false in all dimensions $n \geq 3$. They construct a metric on S^n_+ which agrees with the standard metric on the boundary such that the boundary is totally geodesic, but with scalar curvature strictly larger than $n(n-1)$ [BMN11, Corollary 6]. Additionally, they show that there exists a metric $\hat{g}$ on S^n_+ which not only satisfies $\mathrm{Scal}(\hat{g}) \geq n(n-1)$ and

$\mathrm{Scal}(\hat{g}) > n(n-1)$ somewhere, but which even agrees with the standard metric in a neighbourhood of the boundary [BMN11, Theorem 7]. In particular, Min-Oo's conjecture is actually false under stronger (local) boundary conditions.

We give a short overview of the arguments used to obtain these results. The main steps are the following theorems:

THEOREM 2.5.8 ([BMN11, Theorem 4]). *For any $n \geq 3$, there exists a smooth metric g on the hemisphere S^n_+ with the following properties:*

i) $\mathrm{Scal}(g) > n(n-1)$,
ii) $g = g_{S^n_+}$ *along* ∂S^n_+,
iii) $H(g) > 0$.

THEOREM 2.5.9 ([BMN11, Theorem 5]). *Let M be a compact manifold with boundary and g_1, g_2 smooth Riemannian metrics on M with $g_1 = g_2$ along ∂M and mean curvatures $H(g_1) > H(g_2)$. Given any $\varepsilon > 0$ and a neighbourhood U of ∂M, there exists a smooth metric $\hat{g}$ on M with the following properties:*

i) $\mathrm{Scal}(\hat{g}) \geq \min\{\mathrm{Scal}(g_i), i = 1,2\} - \varepsilon$ *pointwise on* M ,
ii) $g = g_1$ *outside* U,
iii) $g = g_2$ *in a neighbourhood of* ∂M.

We can then construct the counterexample given in [BMN11, Corollary 6] as follows: Let g_1 be a metric as in Theorem 2.5.8 and g_2 be any metric on S^n_+ with totally geodesic boundary, $g_2 = g_{S^n_+}$ along ∂S^n_+ and $\mathrm{Scal}(g_2) > n(n-1)$ in a neighbourhood U of ∂S^n_+. Then apply Theorem 2.5.9.

To prove [BMN11, Theorem 7], one picks $\delta > 0$ and a smooth metric g_2^δ on S^n_+ with

$$g_2^\delta = \begin{cases} g_{S^n_+} & \text{if } x_{n+1} \leq \delta, \\ \left(1 - \exp\left(-\frac{1}{x_{n+1}-\delta}\right)\right)^{\frac{4}{n-2}} g_{S^n_+} & \text{if } \delta < x_{n+1} < 3\delta. \end{cases}$$

For δ sufficiently small, one has $\mathrm{Scal}(g_2^\delta) > n(n-1)$ on $\{\delta < x_{n+1} < 3\delta\}$.

Let g be a metric as in Theorem 2.5.8. By pulling back g with an appropriate conformal transformation and after scaling, one obtains a metric g_1^δ on the set $\{x_{n+1} \geq 2\delta\}$ with $\mathrm{Scal}(g_1^\delta) > n(n-1)$ and which agrees with g_2^δ on $\{x_{n+1} = 2\delta\}$. For δ sufficiently small, one can apply Theorem 2.5.9 to g_1^δ and g_2^δ to obtain a metric $\hat{g}$. Then $\hat{g}$ can be extended by the standard metric to a metric on S^n_+ which satisfies the desired properties.

We now present the ideas to prove Theorems 2.5.8 and 2.5.9.

PROOF (THEOREM 2.5.8). The proof relies on perturbation analysis and is inspired by the construction of counterexamples to Schoen's compactness conjecture to the Yamabe problem. The condition $n \geq 3$ is crucially used as it implies that there exist deformations of the equator ∂S^n_+ which increase area and have positive mean curvature. It follows that there exists a function $\eta\colon \partial S^n_+ \to \mathbb{R}$ with

$$\Delta_{\partial S^n_+}\eta + (n-1)\eta < 0.$$

Let X be a vector field on S^n_+ with $X = \eta\nu$ and $\mathscr{L}_X g_{S^n_+} = 0$ along ∂S^n_+. One considers the families of metrics

$$g_0(t) := g_{S^n_+} + t\mathscr{L}_X g_{S^n_+} \qquad \text{and} \qquad g_1(t) := (\phi^X_t)^* g_{S^n_+},$$

where ϕ^X_t is the flow of X. From the choice of X, it follows that the metrics $g_0(t)$ agree with the standard metric on the boundary while the metrics $g_1(t)$ do not. As g_0 and g_1 agree up to terms of second order, the mean curvature of the boundary with respect to g_0 is

$$H(g_0(t)) = H(g_1(t)) + \mathcal{O}(t^2) = -t\big(\Delta_{\partial S^n_+}\eta + (n-1)\eta\big) + \mathcal{O}(t^2),$$

which is positive for t sufficiently small. In order to satisfy the assertion on the scalar curvature, one adds a second order correction term to g_0 and defines

$$g(t) := g_{S^n_+} + t\mathscr{L}_X g_{S^n_+} + \frac{1}{2(n-1)} t^2 u g_{S^n_+},$$

where u is a solution to a certain elliptic equation on S^n_+ with Dirichlet boundary condition which ensures that $\operatorname{Scal}(g(t)) > n(n-1)$ for t sufficiently small. Then, for $t > 0$ small, $g(t)$ satisfies all properties claimed. □

We now give a sketch of the proof of Theorem 2.5.9:

Proof (Theorem 2.5.9). The idea of the proof is to perturb the metrics g_1 and g_2 using appropriately chosen cut-off functions.

First, let $\rho \in C^\infty(M)$ be a defining function for the boundary, that is $\rho \geq 0$, $\partial M = \rho^{-1}(0)$ and $|\nabla\rho| = 1$ on ∂M. Using that $g_1 = g_2$ along ∂M, we find a symmetric two-tensor T with $g_2 = g_1 + \rho T$ in a neighbourhood of ∂M and $T = 0$ outside U.

Let $\chi\colon [0,\infty) \to [0,1]$ be a smooth cut-off function with $\chi(s) = s - \frac{s^2}{2}$ for $0 \leq s \leq \frac{1}{2}$, $\chi(s)$ constant for $s \geq 1$ and $\chi''(s) < 0$ for $0 \leq s < 1$. Furthermore, let $\xi\colon (-\infty, 0] \to [0,1]$ be a smooth cut-off function with $\xi(s) = \frac{1}{2}$ for $-1 \leq s \leq 0$ and $\xi(s) = 0$ for $s \leq -2$. Then, for λ sufficiently large, we may define a smooth metric by

$$g^\lambda := \begin{cases} g_1 + \lambda^{-1}\chi(\lambda\rho)T & \text{for } \rho \geq e^{-\lambda^2}, \\ g_2 - \lambda\rho^2\xi(\lambda^{-2}\log(\rho))T & \text{for } \rho < e^{-\lambda^2}. \end{cases}$$

Using this and the condition on the mean curvatures (which translates into $\operatorname{trace} T|_{\partial M} > 0$) one can then check that, for λ sufficiently large, the metric g^λ satisfies all desired properties. □

2.6. The Yamabe problem on manifolds with boundary

This section is devoted to the Yamabe problem, with focus on manifolds with boundary. We will only give a rough sketch of the problem; for a more detailed overview as well as a complete and self-contained solution to the Yamabe problem on closed manifolds see the expository article [LP87] as well as [SY94, Chapter V].

Given a closed Riemannian manifold (M, g) of dimension[3] $n \geq 3$, the *Yamabe problem* asks for a conformal metric $\bar{g} \in [g]$ with constant scalar curvature. The problem is named after Hidehiko Yamabe who claimed to have solved it in [YAM60]. However, Trudinger [TRU68] found a serious flaw in Yamabe's proof and the problem remained open until 1984.

In view of the conformal scalar curvature equation (1.1), the Yamabe problem is equivalent to finding a positive $u \in C^\infty(M)$ satisfying

$$\frac{n-2}{4(n-1)} C u^{\frac{n+2}{n-2}} = \frac{n-2}{4(n-1)} \operatorname{Scal}(g) u - \Delta u,$$

where C is some constant which can – after normalization – be chosen to be -1, 0 or $+1$, respectively. One finds that this is the Euler-Lagrange equation of the functional

$$Q(u) := \frac{\displaystyle\int_M \left(|\nabla u|^2 + \frac{n-2}{4(n-1)} \operatorname{Scal}(g) u^2 \right) \mathrm{d}V_g}{\left(\displaystyle\int_M |u|^{\frac{2n}{n-2}} \, \mathrm{d}V_g \right)^{\frac{n-2}{n}}} = \frac{\displaystyle\frac{n-2}{4(n-1)} \int_M \operatorname{Scal}(\bar{g}) \, \mathrm{d}V_{\bar{g}}}{\left(\displaystyle\int_M \mathrm{d}V_{\bar{g}} \right)^{\frac{n-2}{n}}},$$

where $\bar{g} = u^{\frac{4}{n-2}} g \in [g]$. Based on the works by Trudinger [TRU68] and Yamabe [YAM60], Aubin [AUB76] was able to show that the Yamabe problem possesses a solution provided $Y(M, [g]) < Y(S^n, [g_{S^n}])$, where Y is the *Yamabe invariant*

$$Y(M, [g]) := \inf_{u \in C^\infty(M),\, u > 0} Q(u). \tag{2.6.1}$$

Furthermore, it holds that $Y(M, [g]) \leq Y(S^n, [g_{S^n}])$ and the works of Aubin [AUB76] and Schoen [SCH84] show that equality holds if and only if (M, g) is conformally equivalent to the sphere. This implies that the Yamabe problem on a closed manifold is always solvable.

The sign of the scalar curvature of a constant scalar curvature metric is given by the sign of the first eigenvalue of the *conformal Laplacian* $L = -\frac{4(n-1)}{n-2}\Delta + \operatorname{Scal}$, which is the same as the sign of the Yamabe invariant.

[3]Considerations are slightly different in dimension two. Since we only work in higher dimensions, we chose to simplify the presentation and only discuss the case $n \geq 3$.

Considering compact Riemannian manifolds with boundary, one can similarly ask the following questions:

1. Is there a conformally equivalent metric with constant scalar curvature and vanishing mean curvature?
2. Is there a conformally equivalent metric with vanishing scalar curvature and constant mean curvature?

We will only discuss the first question which is of greater interest for our considerations as an umbilic hypersurface will become totally geodesic when the manifold is equipped with a conformally equivalent metric with minimal boundary. We will refer to this as the *relative Yamabe problem.*

The problem was first studied by Escobar in [Esc92b]: As in the case of a closed manifold one can reformulate the problem into finding $u > 0$ satisfying

$$\frac{n-2}{4(n-1)} C u^{\frac{n+2}{n-2}} = \frac{n-2}{4(n-1)} \operatorname{Scal}(g) u - \Delta u,$$
$$0 = \frac{n-2}{2} H(g) u + \frac{\partial u}{\partial \eta},$$

for a constant $C \in \mathbb{R}$.

Analogously, one defines the functional

$$Q(u) := \frac{\displaystyle\int_M \left(|\nabla u|^2 + \frac{n-2}{4(n-1)} \operatorname{Scal}(g) u^2 \right) \mathrm{d}V_g + \frac{n-2}{2} \int_{\partial M} H(g) u^2 \, \mathrm{d}S_g}{\left(\displaystyle\int_M |u|^{\frac{2n}{n-2}} \, \mathrm{d}V_g \right)^{\frac{n-2}{n}}}$$
$$= \frac{\displaystyle\frac{n-2}{4(n-1)} \int_M \operatorname{Scal}(\bar{g}) \, \mathrm{d}V_{\bar{g}} + \frac{n-2}{2} \int_{\partial M} H(\bar{g}) \, \mathrm{d}S_{\bar{g}}}{\left(\displaystyle\int_M \mathrm{d}V_{\bar{g}} \right)^{\frac{n-2}{n}}},$$

where again $\bar{g} = u^{\frac{4}{n-2}} g \in [g]$, and the *relative Yamabe invariant*

$$Y(M, \partial M, [g]) := \inf_{u \in C^\infty(M),\, u > 0} Q(u). \tag{2.6.2}$$

The solution to the relative Yamabe problem is obtained similar to the closed case: One can show that $Y(M, \partial M, [g]) \leq Y(S^n_+, \partial S^n_+, [g_{S^n_+}])$, the relative Yamabe problem admits a solution provided $Y(M, \partial M, [g]) < Y(S^n_+, \partial S^n_+, [g_{S^n_+}])$ and $Y(M, \partial M, [g]) = Y(S^n_+, \partial S^n_+, [g_{S^n_+}])$ if and only if (M, g) is conformally equivalent to S^n_+. The crucial fact that $Y(M, \partial M, [g]) < Y(S^n_+, \partial S^n_+, [g_{S^n_+}])$ if M is not conformally equivalent to S^n_+ is the only point where the relative Yamabe problem substantially differs from the Yamabe problem on compact manifolds. It was first shown by Escobar in the following cases: If $3 \leq n \leq 5$; $n \geq 6$ and there exists a nonumbilic point at ∂M; $n \geq 6$, ∂M is umbilic and M is locally conformally flat or $n \geq 6$, ∂M is umbilic and the Weyl tensor does not vanish identically on ∂M (see [Esc92b]). However, when trying to treat the remaining cases similarly to the closed case, one needs a version of the positive mass theorem which has not been proven yet. Using a different approach, Mayer and Ndiaye [MN15] were recently able to solve the remaining cases based on earlier work by Brendle and Chen [BC14].

Similar to the closed case, the sign of the scalar curvature of a constant scalar curvature metric is given by the sign of the smallest eigenvalue of the conformal Laplacian L, but with boundary condition $Bu := \frac{\partial u}{\partial \eta} + \frac{n-2}{2} H u = 0$. This is the same as the sign of the relative Yamabe invariant of M.

Chapter 3

Proof of Theorem I

This chapter is devoted to the proof of Theorem I. Before getting started with the complete proof, we shortly give an overview over the central arguments:

In Section 3.1, we introduce the *developing map*: Given a simply-connected locally conformally flat manifold M, one can patch together the locally defined isometries $\phi_U\colon (U, e^{2f}g) \to (\phi(U), g_{\mathbb{R}^n}) \subseteq \mathbb{R}^n$ to obtain a conformal immersion $\Phi\colon M \to \overline{\mathbb{R}^n} \cong S^n$, called the developing map, which is unique up to conformal transformations of S^n. As a consequence, we obtain Theorem I under the additional assumption that M is simply-connected and ∂M is connected.

If M is not simply-connected, one can apply the arguments to the universal covering $\tilde{M}$. In Section 3.2 we will show that, if M is a locally conformally flat manifold with boundary and positive relative Yamabe invariant, the developing map is injective. This is basically an adaption of a deep result by Schoen and Yau on closed manifolds (see [SY88] and [SY94]), which will be discussed in greater detail in Section 5.3.

We will identify the image with a subset of the sphere of the form

$$C = C(\varepsilon_i, p_i, \Lambda) \coloneqq S^n \setminus \left(\bigcup_{i \in \pi_0(\partial \tilde{M})} D_{\varepsilon_i}(p_i) \cup \Lambda \right),$$

where the $D_{\varepsilon_i}(p_i)$ are geodesic balls in S^n with disjoint closures and Λ is the so-called *limit set*, a closed subset of Hausdorff dimension at most $\frac{n-2}{2}$, which we will comment on in Remark 3.2.3.

Using this, we will be able to extend the metric to a locally Lipschitz metric on $S^n \setminus \Lambda$ in Section 3.3. In fact, we will only extend the metric to $S^n \setminus (D_{\varepsilon_i}(p_i) \cup \Lambda)$ for one particular i and then use analytical methods similar to those used by Hang and Wang to prove Theorem 1.2 to conclude that $S^n \setminus (D_{\varepsilon_i}(p_i) \cup \Lambda)$ equipped with the pull-back metric is isometric to a closed geodesic ball in S^n, see Section 3.4.

Recall our convention that geodesic balls in the sphere are denoted by $D_\rho(p)$, while geodesic balls in $\mathbb{R}^n$ are denoted by $B_r(p)$; with respective boundaries $\Sigma_\rho(p) = \partial D_\rho(p)$ and $S_r(p) = \partial B_r(p)$.

Unless stated otherwise, M is a manifold of dimension $n \geq 3$.

3.1. The developing map

In this section we show how to use Liouville's theorem on conformal mappings to obtain the *developing map* of a locally conformally flat manifold. We have:

PROPOSITION 3.1.1. *Let (M, g) be a simply-connected locally conformally flat manifold. Then there exists a conformal immersion $\Phi\colon M \to S^n$ which is unique up to conformal transformations of S^n.*

PROOF. If M is locally conformally flat, then for every $p \in M$ there is a neighbourhood U of p and $f \in C^\infty(U)$ such that $e^{2f} g|_U$ is flat. Hence $(U, e^{2f} g|_U)$ is locally isometric to Euclidean space, that is, $(U, g|_U)$ is locally conformally equivalent to Euclidean space. It follows that we can cover M with charts $(U_\alpha, \phi_\alpha)_{\alpha\in A}$, where $\phi_\alpha\colon (U_\alpha, g|_{U\alpha}) \to (\phi_\alpha(U_\alpha), g_{\mathbb{R}^n})$ are conformal. By shrinking the U_α's if necessary, we may assume that all of them are convex and thus arbitrary intersections will be connected.

For two indices $\alpha, \beta \in A$ such that $U_\alpha \cap U_\beta \neq \emptyset$, we define

$$\psi_{\alpha\beta} \coloneqq \phi_\alpha \circ \phi_\beta^{-1}\colon \phi_\beta(U_\alpha \cap U_\beta) \to \phi_\alpha(U_\alpha \cap U_\beta).$$

Then $\psi_{\alpha\beta}$ is a locally defined conformal transformation of $\overline{\mathbb{R}^n}$. By Liouville's theorem (Theorem 2.3.2), there is $\Psi_{\alpha\beta} \in M(\overline{\mathbb{R}^n})$ with $\Psi_{\alpha\beta}|_{\phi_\beta(U_\alpha\cap U_\beta)} = \psi_{\alpha\beta}$. On $\phi_\beta(U_\alpha \cap U_\beta)$, we have $\Psi_{\alpha\beta} \circ \Psi_{\beta\alpha} = \phi_\alpha \circ \phi_\beta^{-1} \circ \phi_\beta \circ \phi_\alpha^{-1} = \mathrm{id}$, therefore

$$\Psi_{\alpha\beta} \circ \Psi_{\beta\alpha} = \mathrm{id}_{\overline{\mathbb{R}^n}}. \tag{3.1.1}$$

Similarly, one sees that

$$\Psi_{\alpha\beta} \circ \Psi_{\beta\gamma} = \Psi_{\alpha\gamma} \tag{3.1.2}$$

whenever $U_\alpha \cap U_\beta \cap U_\gamma \neq \emptyset$.

We now construct Φ as follows: Pick any $p \in M$ and $\alpha_1 \in A$ with $p \in U_{\alpha_1}$ and define

$$\Phi \coloneqq \phi_{\alpha_1} \text{ on } U_{\alpha_1}$$

For $q \notin U_{\alpha_1}$, we pick a chain $\alpha_2, \ldots, \alpha_k$ with $q \in U_{\alpha_k}$ and $U_{\alpha_i} \cap U_{\alpha_{i+1}} \neq \emptyset$ for $i = 1, \ldots, k-1$. Then set

$$\Phi \coloneqq \Psi_{\alpha_1\alpha_2} \circ \Psi_{\alpha_2\alpha_3} \circ \ldots \circ \Psi_{\alpha_{k-1}\alpha_k} \circ \phi_k \text{ on } U_{\alpha_k}.$$

Using (3.1.1), the cocycle condition (3.1.2) and the simply-connectedness of M, one checks that this is independent of the choices, cf. e.g. [SY94, Theorem VI.1.6]. Hence Φ is well-defined.

Uniqueness up to conformal transformations again follows from Liouville's theorem. □

We now present an easy topological lemma which will be used several times:

LEMMA 3.1.2. *Let $\Phi\colon M \to N$ be a local homeo-(diffeo-)morphism with M being compact and N being simply-connected. Then Φ is bijective, i.e. a homeo-(diffeo-)morphism.*

PROOF. As Φ is a local homeomorphism, Φ is open, so $\Phi(M)$ is open. Since M is compact, $\Phi(M)$ is compact, hence closed. Thus, Φ is surjective and therefore a covering. As N is simply-connected, Φ is bijective. □

As a corollary, we obtain a well-known result due to Kuiper (see [KUI49]):

COROLLARY 3.1.3. *Any n-dimensional closed simply-connected locally conformally flat manifold is conformally equivalent to S^n.*

PROOF. Apply Lemma 3.1.2 to the developing map. □

Together with Theorem 2.3.5, this implies:

COROLLARY 3.1.4. *Let (M, g) be a closed simply-connected locally conformally flat manifold with constant scalar curvature $r^{-2}n(n-1)$. Then (M, g) is isometric with S^n_r.*

The simply-connected case. We are now in the position to prove Theorem I under the condition that M is simply-connected and ∂M is connected. Although the proof is similar to the non-simply-connected case, we present it separately to illustrate the technique and to argue that the assumption on the mean curvature can be dropped in this case provided $\rho = \frac{\pi}{2}$, compare Corollary 4.1.1.

We proceed in three steps: We will first show that the developing map of M is injective, then compose with a stereographic projection and a Möbius transformation to obtain $\overline{D_\rho}$ as image. In the end, we use the results of Hang and Wang discussed in Section 2.5. Note that we will show the injectivity of the developing map without the simply-connectedness of M using a deep result by Schoen and Yau in Section 3.2.

PROOF (THEOREM I; M SIMPLY-CONNECTED, ∂M CONNECTED).

Step 1: The developing map is injective. As M is simply-connected, there exists a developing map $\Phi\colon M \to S^n$. Since ∂M is umbilic and being umbilic is a conformal invariant, the image of ∂M must be umbilic in S^n, that is, it is contained in a hypersphere $\Sigma \subseteq S^n$. Applying Lemma 3.1.2 to $\Phi|_{\partial M}$, we see that $\Phi|_{\partial M}\colon \partial M \to \Sigma$ is a diffeomorphism. Composing with a Möbius transformation of S^n, if necessary, we may assume that Σ is the equator $\partial S^n_+ = \{x \in S^n \mid x_{n+1} = 0\}$, see Lemma 2.3.6.

Consider the double manifold $\hat{M} = M \underset{\partial M}{\cup} (-M)$, see Section 5.1. We extend Φ to a map $\hat{\Phi}\colon \hat{M} \to S^n$ in the following way: Write $\Phi = (\Phi_1, \ldots, \Phi_{n+1})$ and set

$$\hat{\Phi}(x) := \begin{cases} \Phi(x) & \text{if } x \in M, \\ (\Phi_1(x), \ldots, \Phi_n(x), -\Phi_{n+1}(x)) & \text{if } x \in -M. \end{cases}$$

Then $\hat{\Phi}$ is well-defined and continuous because $\Phi_{n+1}(x) = 0$ for $x \in \partial M$. Moreover, it is a local homeomorphism. Lemma 3.1.2 implies that $\hat{\Phi}$ is a homeomorphism and hence Φ is injective. Furthermore, the image is either S^n_+ or S^n_-.

Step 2: The image has a nice form. As we want to show that (M, g) is isometric to $\overline{D_\rho}$, we would prefer to have that the image of Φ is $\overline{D_\rho(N)}$ for some point N. We can achieve this by composing with a Möbius transformation of S^n; we may take the inverse of the one constructed in Proposition 2.3.7. For simplicity, we call this composition Φ again. Therefore we may assume that the image of Φ is $\overline{D_\rho(N)}$ and that the pulled-back metric $(\Phi^{-1})^* g$ agrees with the standard metric on the boundary $\Sigma_\rho(N)$.

Step 3: Conclusion. By construction, we have obtained a metric $(\Phi^{-1})^* g$ on $\overline{D_\rho(N)}$ which is conformal to the standard metric and agrees with it on the boundary. We can now apply the results by Hang and Wang:

If $\rho = \frac{\pi}{2}$, Theorem 1.2 implies that $(\Phi^{-1})^* g$ is the standard metric, hence M is isometric to S^n_+ (without any assumptions on the mean curvature). In all other cases, $H(g) \geq H_\rho$ and Proposition 2.5.2 imply that $(\Phi^{-1})^* g$ is the standard metric. □

3.2. Injectivity of the developing map

If M is not simply-connected, we do not know whether there is a conformal map from M to S^n. However, we can pass to the universal covering $\tilde{M}$ to obtain a developing map $\Phi\colon \tilde{M} \to S^n$. In this section we establish that, under the assumptions of Theorem I, this developing map is injective (see also [LN14, Theorem 1.4]):

PROPOSITION 3.2.1. *Let (M, g) be a compact locally conformally flat manifold with boundary and positive relative Yamabe invariant. Assume that ∂M is umbilic and every connected component of ∂M is simply-connected. Let $\tilde{M} \to M$ be the universal covering. There exists an injective conformal map $\Phi\colon \tilde{M} \to S^n$ which is* *a conformal diffeomorphism onto its image. The image is of the form*

$$C = C(\varepsilon_i, p_i, \Lambda) := S^n \setminus \left(\bigcup_{i \in \pi_0(\partial \tilde{M})} D_{\varepsilon_i}(p_i) \cup \Lambda \right),$$

where the $D_{\varepsilon_i}(p_i)$ are geodesic balls in S^n with disjoint closures and every index i corresponds to a connected component of $\partial \tilde{M}$. Λ is the so-called limit set*, a closed subset of Hausdorff dimension at most $\frac{n-2}{2}$.*

We will present the proof at the end of this section. Prior to that, we explain how to see that our conditions on the scalar and mean curvature in Theorem I imply that the relative Yamabe invariant is positive, so Proposition 3.2.1 can be applied in this situation. Then we comment on the limit set Λ occurring in the statement as well as on results by Schoen and Yau utilised in the proof.

Lemma 3.2.2. *Let (M, g) be a compact connected Riemannian manifold with boundary with* Scal ≥ 0 *and $H \geq 0$. If either* Scal *or H is strictly positive at some point, then the relative Yamabe invariant of M is positive.*

Proof. As explained in Section 2.6, the sign of the relative Yamabe invariant is the same as the sign of the smallest eigenvalue of the conformal Laplacian $L = -\frac{4(n-1)}{n-2}\Delta + \mathrm{Scal}$ with boundary condition $Bu = \frac{\partial u}{\partial \eta} + \frac{n-2}{2} H u = 0$.

Let f_1 be a first eigenfunction of L with boundary condition B to the eigenvalue λ_1 with $||f_1||_{L^2(M)} = 1$. We can choose f_1 such that $f_1 > 0$ on M: In fact, from the variational characterisation it follows that $|f_1|$ is an eigenfunction as well, so we may assume $f_1 \geq 0$. Then the maximum principle and the Hopf lemma imply $f_1 > 0$, see [Esc92a, Proposition 1.3].

Hence we estimate

$$\begin{aligned}
\lambda_1 &= \langle L f_1, f_1 \rangle_{L^2(M)} \\
&= \int_M \left(|\nabla f_1|^2 + \frac{n-2}{4(n-1)} \mathrm{Scal}(g) f_1^2 \right) \mathrm{d}V_g + \frac{n-2}{2} \int_{\partial M} H(g) f_1^2 \, \mathrm{d}S_g \\
&\geq \frac{n-2}{4(n-1)} \int_M \mathrm{Scal}(g) f_1^2 \, \mathrm{d}V_g + \frac{n-2}{2} \int_{\partial M} H(g) f_1^2 \, \mathrm{d}S_g \\
&> 0
\end{aligned}$$

as $f_1 > 0$ and Scal ≥ 0, $H \geq 0$ with either Scal > 0 or $H > 0$ at some point. This implies $\mathrm{sign}\,(Y(M, \partial M, [g])) = \mathrm{sign}(\lambda_1) = 1$. ☐

Remark 3.2.3. Let us shortly comment on the limit set Λ occurring in Proposition 3.2.1: Let $\Gamma \subseteq M(S^n) \cong M(B^{n+1})$ be a discrete subgroup of the conformal transformation group of S^n. Then the *limit set* of Γ is defined as

[Rat06, § 12.1]:

$$\Lambda(\Gamma) := \left\{ x \in S^n \mid \text{there exist } x' \in B^{n+1} \text{ and } \gamma_i \in \Gamma \text{ such that } \gamma_i x' \to x \right\}.$$

One can show that $|\Lambda(\Gamma)| \in \{0, 1, 2, \infty\}$ and if $\Lambda(\Gamma)$ is infinite, then it is a minimal closed nonempty Γ-invariant subset of S^n (cf. [Rat06, Theorems 12.2.1 and 12.1.3]).

Now if (M, g) is a closed locally conformally flat manifold there is a developing map $\Phi\colon \tilde{M} \to S^n$, hence $\pi_1(M)$ (viewed as the group of deck transformations) acts on S^n by Möbius transformations. The homomorphism $\rho\colon \pi_1(M) \to M(S^n)$ obtained is called *holonomy representation* of $\pi_1(M)$ in $M(S^n)$. If the scalar curvature of M is positive, one can show that the developing map is injective, the holonomy representation is one-to-one and $\Lambda(\rho(\pi_1(M))) = \partial(\Phi(\tilde{M})) = S^n \setminus \Phi(\tilde{M})$. For more background see e.g. [SY94, Chapter VI] or [Rat06, Chapter 12].

The main idea of the proof of Proposition 3.2.1 is to apply the following result on the injectivity of developing maps by Schoen and Yau (see [SY88] and [SY94]) to the double manifold $\hat{M} = M \underset{\partial M}{\cup} (-M)$:

Theorem 3.2.4. *Let (M, g) be a closed locally conformally flat Riemannian manifold of positive scalar curvature. Then the developing map $\Phi\colon \tilde{M} \to S^n$ is injective.*

Remark 3.2.5. Theorem 3.2.4 would immediately follow from [SY88, Theorem 4.5] or [SY94, Theorem 3.5], which state that the developing map of a closed locally conformally flat manifold is injective provided that the scalar curvature is nonnegative. However, to prove [SY88, Theorem 4.5] and [SY94, Theorem 3.5], Schoen and Yau use a version of the positive mass theorem which, to the author's knowledge, is widely believed to be true but has not been proven yet. Still, validity of Theorem 3.2.4 which only covers metrics of positive scalar curvature, follows from the results of Schoen and Yau. We will address this issue in greater detail in Section 5.3, where we also prove a $C^{2,1}$-version of Theorem 3.2.4 which will be needed at a later point.

Proof of Proposition 3.2.1. We proceed in three steps: First we show how to obtain a smooth metric on the double manifold conformal to the canonical one, then apply Schoen and Yau's results to its universal cover. In the last step, we show that the image has the claimed form.

Step 1: Finding a smooth metric on the double manifold. If the Yamabe invariant satisfies $Y(M, \partial M, [g]) = Y(S^n_+, \partial S^n_+, [g_{S^n_+}])$, then M is conformally equivalent to the hemisphere and the assertion is clear. Else, by Escobar's solution to the Yamabe problem on manifolds with boundary for locally conformally flat manifolds (see [Esc92b] and Section 2.6), there exists a metric g' in the conformal class of g with (constant) positive scalar curvature and totally geodesic boundary, see our discussion in Section 2.6.

Consider the double manifold $\hat{M} = M \underset{\partial M}{\cup} (-M)$. Since ∂M is totally geodesic with respect to g', the canonical metric $\hat{g}'$ extending g' is $C^{2,1}$ as can be seen in Fermi coordinates, see Section 5.1 for a proof. One can now check that Theorem 3.2.4 remains valid for $C^{2,1}$-metrics, which we will do in Section 5.3. However, in this special case we can easily find a smooth metric with positive scalar curvature conformal to $\hat{g}$:

In fact, for every connected component Z of ∂M, we pick $\varepsilon > 0$ small enough and a neighbourhood U of Z in M diffeomorphic to $Z \times [0, \varepsilon)$. As the latter is simply-connected, we obtain a conformal map $\Phi_U : U \to S^n$. Since Z is umbilic, $\Phi(Z)$ is also umbilic and thus contained in a hypersphere. By Lemma 3.1.2, $\Phi_U|_Z$ is a diffeomorphism onto that hypersphere. Hence, as Φ_U is an immersion, we may shrink U to obtain an injective conformal map. This proves that there exists a neighbourhood of Z in $\hat{M}$ which is isometric to a tubular neighbourhood of the equator in S^n equipped with a metric of the form $\mu^2 g_{S^n}$, where μ is C^2 and smooth away from the equator. We may approximate μ with a suitable smooth function $\tilde{\mu} > 0$ such that $\tilde{\mu}^2 g_{S^n}$ has positive scalar curvature. Proceeding like this for every connected component of the boundary, we obtain a smooth metric g^* on $\hat{M}$ conformal to $\hat{g}'$ with positive scalar curvature.

Step 2: Obtaining an injective developing map. Let $\pi\colon N \to \hat{M}$ be the universal covering of $\hat{M}$. Equipped with the Riemannian metric induced by g^*, N is a simply-connected complete locally conformally flat manifold with positive scalar curvature. From Theorem 3.2.4, it follows that the developing map $\phi\colon N \to S^n$ is injective and $\Lambda' := S^n \setminus \phi(N)$ is the limit set of $\pi_1(\hat{M})$ as explained in Remark 3.2.3. As g^* has positive scalar curvature, the results of Schoen and Yau imply that the Hausdorff dimension of Λ' is at most $\frac{n-2}{2}$.

Let $\tilde{M} \subseteq N$ be a connected component of $\pi^{-1}(M)$. Then $\pi\colon \tilde{M} \to M$ is a covering and since $\phi\colon N \to S^n$ was injective, $\Phi := \phi|_{\tilde{M}}$ is also injective. Furthermore, the metrics $\tilde{g}$ and $\tilde{g}^*$ on $\tilde{M}$ induced by g and g^*, respectively, are conformally equivalent (since g and g^* are) and so Φ is also conformal with respect to $\tilde{g}$. We conclude that $\Phi\colon \operatorname{int}(\tilde{M}) \to \Phi(\operatorname{int}(\tilde{M}))$ is a conformal diffeomorphism. To show that it is actually a diffeomorphism of $\tilde{M}$, we need to verify that Φ is also a local diffeomorphism near the boundary, i.e. $\Phi(\partial\tilde{M}) \subseteq \partial\Phi(\tilde{M})$.

To see this, first note that $\partial\tilde{M}$ is diffeomorphic to disjoint copies of S^{n-1}: Since the covering $\tilde{M} \to M$ induces a covering $\partial\tilde{M} \to \partial M$ and the latter has simply-connected connected components, we know that all connected components of $\partial\tilde{M}$ are simply-connected. As any connected component Σ of $\partial\tilde{M}$ is umbilic, it is locally conformally flat itself and hence conformally equivalent to a sphere, see Corollary 3.1.3. Arguing as above, we see that for any such Σ, $\Phi|_\Sigma \colon \Sigma \to \Phi(\Sigma)$ is a diffeomorphism, where $\Phi(\Sigma)$ is some geodesic hypersphere in S^n. Thus $S^n \setminus \Phi(\Sigma)$ has two connected components. As $\tilde{M} \setminus \partial\tilde{M}$ is connected and Φ is injective, it follows that $\Phi(\tilde{M} \setminus \partial\tilde{M})$ lies in exactly one of these connected components, therefore $\Phi(\partial\tilde{M}) \subseteq \partial\Phi(\tilde{M})$.

Step 3: Identification of the image. We have seen that $\Phi\colon \tilde{M} \to \Phi(\tilde{M})$ is a conformal diffeomorphism and that the boundary $\partial\Phi(\tilde{M})$ of the image can be written as

$$\partial\Phi(\tilde{M}) = \Phi(\partial\tilde{M}) \cup \left(\overline{\Phi(\tilde{M})} \cap \Lambda'\right)$$

We will denote the closed subset $\overline{\Phi(\tilde{M})} \cap \Lambda'$ by Λ. Note that Λ is empty if $\tilde{M}$ is compact, i.e. $|\pi_1(M)| < \infty$.

As shown above, the image of the boundary of $\tilde{M}$ consists of disjoint geodesic hyperspheres. The image $\Phi(\tilde{M})$ is hence of the form

$$C = C(\varepsilon_i, p_i, \Lambda) := S^n \setminus \left(\bigcup_{i \in \pi_0(\partial\tilde{M})} D_{\varepsilon_i}(p_i) \cup \Lambda \right),$$

where, by injectivity, the $D_{\varepsilon_i}(p_i)$ are geodesic balls in S^n with disjoint closures. Then $\Phi\colon \tilde{M} \to C \subseteq S^n$ is a conformal diffeomorphism as claimed. □

3.3. Extension of the metric

As we have seen, any M satisfying the hypotheses of Theorem I can be conformally covered by a subset $C \subseteq S^n$ as above. We want to argue similarly to the simply-connected case, but certain problems arise: First of all, if M is not simply-connected, then the image of $\tilde{M}$ under the developing map has more than one "hole" $D_{\varepsilon_i}(p_i)$ and if $\pi_1(M)$ is infinite, then the limit set Λ is nonempty.

This makes it impossible to apply Proposition 2.5.2 to $(\Phi^{-1})^*\tilde{g}$ directly as we did in the simply-connected case: First of all, Proposition 2.5.2 does not apply when $\Lambda \neq \emptyset$ and secondly, the metric $(\Phi^{-1})^*\tilde{g}$ does not agree with the standard metric on the boundary ∂C as required. Utilising Proposition 2.3.7 as in the simply-connected case, we could ensure this for *one* boundary component $\partial D_{\varepsilon_i}(p_i)$, but then $(\Phi^{-1})^*\tilde{g} = g_{S^n}$ on the other boundary components would immediately contradict the condition that they are isometric with Σ_ρ. We thus have to argue differently; let us shortly sketch the arguments:

By Proposition 3.2.1, the universal covering of M is isometric to a manifold of the form (C, h), where $h := (\Phi^{-1})^*\tilde{g}$ and C is as in the statement of the proposition. Up to a conformal transformation we will be able to assume $D_{\varepsilon_i}(p_i) = D_{\pi-\rho}(N)$ for one particular i and $h = g_{S^n}$ on its boundary $\Sigma_{\pi-\rho}(N) = \Sigma_\rho(S)$, where $S = -N$. As h is conformally equivalent to g_{S^n}, we may write $h = u^{\frac{4}{n-2}} g_{S^n}$, where u satisfies

$$\begin{cases} u = 1 & \text{on } \Sigma_\rho(S), \\ \frac{\partial u}{\partial \eta} \geq 0 & \text{on } \Sigma_\rho(S), \\ -\Delta u \geq \frac{n(n-2)}{4}\left(u^{\frac{n+2}{n-2}} - u\right) & \text{in } \operatorname{int}(C), \end{cases}$$

by the conformal scalar and mean curvature equations (1.1) and (1.2).

Our aim is to show $u = 1$ on C which will imply Theorem I. This will follow from the Hopf lemma once we know that $u \geq 1$ because the latter implies that u is superharmonic by the last equation above. To show that $u \geq 1$, we first extend the metric h to the discs $D_{\varepsilon_i}(p_i)$, then use a subsolution-supersolution method and Proposition 2.5.2 by Hang and Wang to work out a maximum principle-type result for positive supersolutions of

$$-\Delta u = \frac{n(n-2)}{4}\left(u^{\frac{n+2}{n-2}} - u\right)$$

which will be used to deduce $u \geq 1$.

Interestingly, no problems arise from the occurrence of the limit set Λ: In fact, completeness of the metric h implies that $u(x) \to \infty$ for $x \to \Lambda$ (see below), hence $u \geq 1$ near Λ is always guaranteed.

In this section we discuss how to extend the metric $h = (\Phi^{-1})^* \tilde{g}$ to $S^n \setminus (D_{\varepsilon_i}(p_i) \cup \Lambda)$ for some fixed i. The basic idea is to glue in spherical caps $\overline{D_{\pi-\rho}}$ along the boundaries; we make this construction more explicit below. The important part is that the resulting metric is continuous (in fact locally Lipschitz) and satisfies $\mathrm{Scal} \geq n(n-1)$ in a weak sense, see Proposition 3.3.1 below.

In Section 3.4, we will then show that the extended metric is isometric with the standard one contradicting the existence of those caps.

Pick any i. For every $j \neq i$, we extend the metric h to $D_{\varepsilon_j}(p_j)$ in the following way:

Let $\{S, N\}$ be a pair of antipodal points. By assumption, $(\Sigma_{\varepsilon_j}(p_j), h|_{\Sigma_{\varepsilon_j}(p_j)})$ is isometric to $\Sigma_\rho(S)$ with the standard metric, therefore Proposition 2.3.7 guarantees the existence of a Möbius transformation ϕ_j with $\phi_j(D_{\pi-\rho}(N)) = D_{\varepsilon_j}(p_j)$ such that $\phi_j^* h$ agrees with the standard metric on $\Sigma_{\pi-\rho}(N) = \Sigma_\rho(S)$.

Then $h_j := \phi_j^* h$ is a metric on the set $C_j := \phi_j^{-1}(C)$ which is of a similar form as C (with different parameters) and where one of the balls – corresponding to the ball $D_{\varepsilon_j}(p_j)$ for our fixed j – is $D_{\pi-\rho}(N)$. As h_j is conformal to the standard metric, we may write

$$h_j = u_j^{\frac{4}{n-2}} g_{S^n},$$

for some function u_j. By construction and assumption, we have that h_j coincides with the standard metric on $\Sigma_\rho(S)$, $\mathrm{Scal}(h_j) \geq n(n-1)$ and the mean curvature of $\Sigma_\rho(S)$ is at least H_ρ. In terms of u_j, this becomes:

$$\begin{cases} u_j = 1 & \text{on } \Sigma_\rho(S), \\ \frac{\partial u_j}{\partial \eta} \geq 0 & \text{on } \Sigma_\rho(S), \\ -\Delta u_j \geq \frac{n(n-2)}{4}\left(u_j^{\frac{n+2}{n-2}} - u_j\right) & \text{in } \operatorname{int}(C_j), \end{cases}$$

Hence the function $\bar{u}_j$ defined by

$$\bar{u}_j(x) := \begin{cases} 1 & \text{if } x \in \overline{D_{\pi-\rho}(N)}, \\ u_j(x) & \text{if } x \in C_j \end{cases}$$

extends u_j to $C_j \cup D_{\pi-\rho}(N)$ and is locally Lipschitz. Now extend h_j by setting

$$\bar{h}_j := \bar{u}_j^{\frac{4}{n-2}} g_{S^n}.$$

Pulling back $\bar{h}_j$ with ϕ_j^{-1}, we obtain a metric $\tilde{h}_j$ on $C \cup D_{\varepsilon_j}(p_j)$ extending h. Repeating the construction for all $j \neq i$, we combine all the extensions by setting

$$\tilde{h}(x) := \tilde{h}_j(x) \quad \text{if } x \in C \cup D_{\varepsilon_j}(p_j).$$

Using this extension, we show:

PROPOSITION 3.3.1. *The metric $h := (\Phi^{-1})^*\tilde{g}$ can be extended to a metric $\tilde{h}$ on $S^n \setminus (D_{\varepsilon_i}(p_i) \cup \Lambda)$ which is locally Lipschitz and satisfies $\mathrm{Scal}(\tilde{h}) \geq n(n-1)$ weakly in the sense that $\tilde{h} = \tilde{u}^{\frac{4}{n-2}} g_{S^n}$ with*

$$-\Delta \tilde{u} \geq \frac{n(n-2)}{4}\left(\tilde{u}^{\frac{n+2}{n-2}} - \tilde{u}\right)$$

weakly. With respect to $\tilde{h}$, the boundary $\partial D_{\varepsilon_i}(p_i)$ has mean curvature at least H_ρ and is isometric to Σ_ρ.

PROOF. Writing $\tilde{h}$ constructed above as

$$\tilde{h} = \tilde{u}^{\frac{4}{n-2}} g_{S^n},$$

it remains to show that the extension $\tilde{u}$ satisfies

$$-\Delta \tilde{u} \geq \frac{n(n-2)}{4}\left(\tilde{u}^{\frac{n+2}{n-2}} - \tilde{u}\right) \tag{3.3.1}$$

weakly. Note that, by construction, $\tilde{u}$ satisfies (3.3.1) in $\operatorname{int}(C)$ and on every $D_{\varepsilon_j}(p_j)$, so we only need to take care of the boundary values.

Let $H^j = \cot(\varepsilon_j) > 0$ be the mean curvature of $\partial D_{\varepsilon_j}(p_j)$ with respect to the spherical metric and computed with respect to the outer unit normal $\eta = -\nu$ (pointing into $D_{\varepsilon_j}(p_j)$).

Since, by construction, $(D_{\varepsilon_j}(p_j), \tilde{h})$ is isometric to $D_{\pi-\rho}$ with the standard metric, the mean curvature of the boundary with respect to $\tilde{h}$ viewed from the inside is $-H_\rho$. Writing

$$\frac{\partial \tilde{u}}{\partial \eta^\pm}(x) := \lim_{\varepsilon\to 0^+} \frac{\tilde{u}(\cos(\pm\varepsilon)x + \sin(\pm\varepsilon)\eta) - \tilde{u}(x)}{\pm\varepsilon} = -\frac{\partial \tilde{u}}{\partial \nu^\mp}(x)$$

for the one-sided derivatives, this implies

$$\frac{n-2}{2}(-H_\rho)\tilde{u}^{\frac{n}{n-2}} = \frac{n-2}{2}H^j\tilde{u} + \frac{\partial \tilde{u}}{\partial \nu^-}$$

on $\partial D_{\varepsilon_j}(p_j)$. By hypothesis on $\tilde{h}$, we also have

$$\frac{n-2}{2}H_\rho\tilde{u}^{\frac{n}{n-2}} \leq \frac{n-2}{2}H(\tilde{h})\tilde{u}^{\frac{n}{n-2}} = -\frac{n-2}{2}H^j\tilde{u} + \frac{\partial \tilde{u}}{\partial \eta^-}.$$

Adding these inequalities, we obtain

$$\frac{\partial \tilde{u}}{\partial \nu^-} + \frac{\partial \tilde{u}}{\partial \eta^-} \geq 0, \text{ i.e. } \frac{\partial \tilde{u}}{\partial \nu^-} \geq -\frac{\partial \tilde{u}}{\partial \eta^-} = \frac{\partial \tilde{u}}{\partial \nu^+}.$$

Let $\varphi \in \mathcal{D}^+(S^n \setminus (\overline{D_{\varepsilon_i}(p_i)} \cup \Lambda)) = \{\phi \in C_c^\infty(S^n \setminus (\overline{D_{\varepsilon_i}(p_i)} \cup \Lambda)) \mid \phi \geq 0\}$ be a smooth test function. Using Green's formula, we have:

$$\begin{aligned}
&\int_{S^n\setminus(D_{\varepsilon_i}(p_i)\cup\Lambda)} \tilde{u}(-\Delta\varphi) \\
&\quad= \int_C \tilde{u}(-\Delta\varphi) + \sum_{j\neq i}\int_{D_{\varepsilon_j}(p_j)} \tilde{u}(-\Delta\varphi) \\
&\quad= \int_C \varphi(-\Delta\tilde{u}) + \sum_{j\neq i}\int_{D_{\varepsilon_j}(p_j)} \varphi(-\Delta\tilde{u}) \\
&\qquad - \int_{\partial C}\left(\tilde{u}\frac{\partial\varphi}{\partial\eta} - \varphi\frac{\partial\tilde{u}}{\partial\eta^-}\right) - \sum_{j\neq i}\int_{\partial D_{\varepsilon_j}(p_j)}\left(\tilde{u}\frac{\partial\varphi}{\partial\nu} - \varphi\frac{\partial\tilde{u}}{\partial\nu^-}\right) \\
&\quad= \int_C \varphi(-\Delta\tilde{u}) + \sum_{j\neq i}\int_{D_{\varepsilon_j}(p_j)} \varphi(-\Delta\tilde{u}) \\
&\qquad + \int_{\partial C}\varphi\left(\frac{\partial\tilde{u}}{\partial\nu^-} - \frac{\partial\tilde{u}}{\partial\nu^+}\right) \\
&\quad\geq \frac{n(n-2)}{4}\left[\int_C \varphi\left(\tilde{u}^{\frac{n+2}{n-2}} - \tilde{u}\right) + \sum_{j\neq i}\int_{D_{\varepsilon_j}(p_j)} \varphi\left(\tilde{u}^{\frac{n+2}{n-2}} - \tilde{u}\right)\right] \\
&\quad= \frac{n(n-2)}{4}\int_{S^n\setminus(D_{\varepsilon_i}(p_i)\cup\Lambda)}\left(\tilde{u}^{\frac{n+2}{n-2}} - \tilde{u}\right)\varphi
\end{aligned}$$

as claimed. □

3.4. Conclusion

Using the extension of the metric constructed above, we will now conclude the proof of Theorem I. This will be done at the end of this section, where we also briefly recall all previous steps for the reader's convenience. Prior to that, we work out some consequences and extensions to Proposition 2.5.2 which will be used in the proof.

Note that we can interchangeably work on either the sphere or Euclidean space, as both are conformally equivalent via a stereographic projection π. Working in $\mathbb{R}^n$ has the advantage that the conformal scalar curvature equation is simpler when the background metric has vanishing scalar curvature, but we find that the argumentation is clearer when presented on the sphere.

However, to demonstrate how both settings may be interchanged, we begin by presenting an analogue of Proposition 2.5.2 for spherical metrics on the unit ball. Observe that $(\pi^{-1})^* g_{S^n} = w^{\frac{4}{n-2}} g_{\mathbb{R}^n}$, where

$$w(x) := \left(\frac{2}{1+|x|^2}\right)^{\frac{n-2}{2}}.$$

Our first result is:

Corollary 3.4.1. *Let $\Omega \subseteq B_1$ be open with smooth boundary and $\tilde{g} = u^{\frac{4}{n-2}} g_{\mathbb{R}^n}$ be a metric on Ω in the conformal class of the standard metric. Suppose that*

i) $\operatorname{Scal}(\tilde{g}) \geq n(n-1)$,

ii) along the boundary $\partial\Omega$, we have $u = w$.

Then $u \geq w$ and $H(\tilde{g}) \leq H(w^{\frac{4}{n-2}} g_{\mathbb{R}^n})$.

Note that we did not assume Ω to be connected. In fact, connectedness in Proposition 2.5.2 is only needed to ensure $u = 1$ on the whole of Ω (and not just a connected component) provided $H(\bar{g}) = H(g_{S^n})$ at a point.

Proof. We have

$$\tilde{g} = u^{\frac{4}{n-2}} g_{\mathbb{R}^n} = \left(uw^{-1}\right)^{\frac{4}{n-2}} w^{\frac{4}{n-2}} g_{\mathbb{R}^n} =: \left(uw^{-1}\right)^{\frac{4}{n-2}} \bar{g}.$$

Thus

$$\pi^*\tilde{g} = \left(\left(uw^{-1}\right)^{\frac{4}{n-2}} \circ \pi\right) \pi^*\bar{g} = \left(\left(uw^{-1}\right) \circ \pi\right)^{\frac{4}{n-2}} g_{S^n}$$

and $\pi^*\tilde{g}$ satisfies the hypotheses of Proposition 2.5.2 by our assumptions on $\tilde{g}$. Applying it to all connected components of $\pi^{-1}(\Omega)$, we obtain $\left(uw^{-1}\right) \circ \pi \geq 1$ and $H(\pi^*\tilde{g}) \leq H(g_{S^n})$, thus $u \geq w$ and $H(\tilde{g}) \leq H(w^{\frac{4}{n-2}} g_{\mathbb{R}^n})$. □

We furthermore obtain that condition (ii) can be weakened to $u \geq w$ on $\partial\Omega$. For our purposes, it is important to note that this result is valid even for *continuous* conformal deformations of the standard metric:

Proposition 3.4.2. *Let $\Omega \subseteq B_1$ be open with smooth boundary and $u \in C^0(\overline{\Omega})$ be a positive, continuous function on Ω. Suppose*

i) u is a weak solution to

$$-\Delta u \geq \frac{n(n-2)}{4} u^{\frac{n+2}{n-2}},$$

ii) $u \geq w$ along the boundary $\partial\Omega$.

Then $u \geq w$ on Ω.

Note that condition (i) corresponds to $\mathrm{Scal}(u^{\frac{4}{n-2}} g_{\mathbb{R}^n}) \geq n(n-1)$.

Proof. We define $\underline{u} := \inf\{u(x) \mid x \in \Omega\} > 0$ and furthermore denote by $m := \min\{\underline{u}, 1\} \leq \min\{w(x) \mid x \in \Omega\}$. Consider the boundary value problem

$$\begin{cases} -\Delta f = \frac{n(n-2)}{4} f^{\frac{n+2}{n-2}} & \text{in } \Omega, \\ f(x) = w(x) & \text{on } \partial\Omega. \end{cases} \tag{3.4.1}$$

Then u is a supersolution while the constant function m is a subsolution with $u \geq m$. Hence there exists a solution v of (3.4.1) with

$$u \geq v \geq m > 0,$$

see e.g. the paper [CS87] by Clément and Sweers. Their proof is based on the Schauder fixed point theorem (rather than the method of monotone iterations) which makes it possible to obtain the result even for continuous u.

The solution v obtained by the above method is continuous and bounded. Using that $v > 0$ solves (3.4.1), it thus follows that v is smooth by standard regularity theory. Consider the metric $\tilde{g} := v^{\frac{4}{n-2}} g_{\mathbb{R}^n}$: As v is a solution to (3.4.1), we have $\mathrm{Scal}(\tilde{g}) = n(n-1)$ and $v = w$ on $\partial\Omega$, hence Corollary 3.4.1 is applicable and implies

$$u \geq v \geq w$$

on Ω as claimed. □

Using a stereographic projection as in the proof of Corollary 3.4.1, we obtain an analogous result for domains in the hemisphere which generalises a part of Proposition 2.5.2:

Corollary 3.4.3. *Let $\Omega \subseteq S^n_+$ be open with smooth boundary, $u \in C^0(\overline{\Omega})$ be positive. Assume*

i) u is a weak solution to

$$-\Delta u \geq \frac{n(n-2)}{4} \left(u^{\frac{n+2}{n-2}} - u \right),$$

ii) $u \geq 1$ along the boundary $\partial\Omega$.

Then $u \geq 1$ on Ω.

As announced in the introduction to Section 3.3, we will now argue why a possible limit set Λ does not give rise to problems in our further discussion: The following proposition implies that Corollary 3.4.3 still holds if there is a nonsmooth part of the boundary on which u is not necessarily defined or continuous, provided that we a priori know $u > 1$ near that part anyway.

In the statement of the proposition below, Σ and Λ are not assumed to be sphere and limit set, respectively, but the result will be applied to that case in our later discussion, hence our choice of symbols. We will make use of the case $\lambda = \infty$ in which u can be seen as a continuous function with values in $\mathbb{R}_+ \cup \{\infty\}$.

PROPOSITION 3.4.4. *Let $\Omega \subseteq S^n_+$ be open with boundary $\partial\Omega = \Sigma \cup \Lambda$, where Σ and Λ are closed and disjoint and Σ is smooth. Let $u \in C^0(\Omega \cup \Sigma)$ be positive. Assume*

i) u is a weak solution to

$$-\Delta u \geq \frac{n(n-2)}{4}\left(u^{\frac{n+2}{n-2}} - u\right),$$

ii) $u \geq 1$ along Σ and $\lambda := \liminf_{x\to\Lambda} u(x) - 1 > 0$.

Then $u \geq 1$ on Ω.

PROOF. To apply Corollary 3.4.3, we construct an open set Ω_ϵ with smooth boundary containing $u^{-1}((0, 1 + \frac{\lambda}{2}))$.

Let $d^\Lambda(x) := \inf\{d^{S^n}(x, y) \mid y \in \Lambda\}$ denote the distance function to Λ and set $D_\epsilon(\Lambda) = \{x \in \Omega \mid d^\Lambda(x) < \epsilon\}$. As Λ and Σ are closed, thus compact, and disjoint, they have positive distance. Hence there is $\epsilon > 0$ so that $D_{5\epsilon}(\Lambda) \cap \Sigma = \emptyset$. Since $\lambda > 0$, we may further decrease ϵ to ensure $u > 1 + \frac{\lambda}{2}$ on $D_{5\epsilon}(\Lambda)$.

Let $\delta \in C^0(D_{5\epsilon}(\Lambda))$ be a continuous function which is smooth on $D_{4\epsilon}(\Lambda)\backslash\overline{D_\epsilon(\Lambda)}$ and approximates d^Λ in the sense that

$$|\delta(x) - d^\Lambda(x)| < \epsilon \quad \text{for } x \in D_{5\epsilon}(\Lambda).$$

In particular, δ is smooth at all x with $\delta(x) \in (2\epsilon, 3\epsilon)$, therefore Sard's theorem implies that we may pick a regular value ξ of δ in that interval. Set

$$\Lambda_\epsilon := \delta^{-1}((-\epsilon, \xi)) \quad \text{and} \quad \Omega_\epsilon := \Omega \setminus \overline{\Lambda_\epsilon}.$$

Then $\partial\Omega_\epsilon = \Sigma \cup \partial\Lambda_\epsilon$ which is smooth because ξ is a regular value and Σ does not intersect $\partial\Lambda_\epsilon$. Furthermore, we have $\Lambda_\epsilon \subseteq D_{5\epsilon}(\Lambda)$ and thus $u > 1 + \frac{\lambda}{2}$ on Λ_ϵ, so $u \geq 1$ on $\partial\Omega_\epsilon$.

Corollary 3.4.3 implies that $u \geq 1$ on Ω_ϵ, so $u \geq 1$ on Ω. □

We can now finish the proof of Theorem I.

Proof (Theorem I). Let us quickly recall the previous steps:

Let (M, g) be a manifold as in the statement of Theorem I. In Proposition 3.2.1 we have seen that the developing map $\Phi\colon \tilde{M} \to S^n$ from the universal covering is injective, hence $(\tilde{M}, \tilde{g})$ is isometric to a set $C \subseteq S^n$ of the form

$$C = S^n \setminus \left(\bigcup_{i \in \pi_0(\partial \tilde{M})} D_{\varepsilon_i}(p_i) \cup \Lambda \right),$$

equipped with the pulled-back metric $h \coloneqq \left(\Phi^{-1}\right)^* \tilde{g}$. In Proposition 3.3.1, we have shown how to extend this metric to the geodesic balls $D_{\varepsilon_i}(p_i)$ for all but one (arbitrary) i to obtain a continuous metric $\tilde{h}$ on $S^n \setminus (D_{\varepsilon_i}(p_i) \cup \Lambda)$.

Let $\{S, N\}$ be a pair of antipodal points. By pulling back $\tilde{h}$ with a Möbius transformation as in Proposition 2.3.7, we may assume that $D_{\varepsilon_i}(p_i) = D_{\pi-\rho}(N)$ and $\tilde{h}$ restricted to the boundary $\partial D_{\pi-\rho}(N) = \Sigma_\rho(S)$ coincides with the restriction of the standard metric g_{S^n}.

Then $\tilde{h}$ is a metric on the set $S^n \setminus (D_{\pi-\rho}(N) \cup \Lambda) = \overline{D_\rho(S)} \setminus \Lambda$. We will henceforth omit the center S in our notation and simply write D_ρ and Σ_ρ.

As $\tilde{h}$ is conformally equivalent to g_{S^n}, there is a function u such that $\tilde{h}$ can be written as $\tilde{h} = u^{\frac{4}{n-2}} g_{S^n}$. From Proposition 3.3.1 and the conformal scalar and mean curvature equation (Equations 1.1 and 1.2), our assumptions on $\tilde{h}$ imply:

$$\begin{cases} u = 1 & \text{on } \Sigma_\rho, \\ \frac{\partial u}{\partial \eta} \geq 0 & \text{on } \Sigma_\rho, \\ -\Delta u \geq \frac{n(n-2)}{4}\left(u^{\frac{n+2}{n-2}} - u\right) & \text{in } D_\rho \setminus \Lambda. \end{cases}$$

Remark 3.4.5. The latter equation actually holds on the whole of D_ρ, see [SY88, Theorem 5.1].

As the metric $\tilde{h}$ is complete, it follows that $u(x) \to \infty$ uniformly for $x \to \Lambda$, compare [SY88, Proposition 2.6]. Thus Proposition 3.4.4 is applicable and shows $u \geq 1$. It follows that

$$-\Delta u \geq \frac{n(n-2)}{4}\left(u^{\frac{n+2}{n-2}} - u\right) \geq 0,$$

so u is superharmonic. On the one hand, as $u \geq 1$ and $u = 1$ on Σ_ρ, we have $\frac{\partial u}{\partial \eta} \leq 0$ there. On the other hand, we have $\frac{\partial u}{\partial \eta} \geq 0$ by assumption and thus $\frac{\partial u}{\partial \eta} = 0$ along Σ_ρ. The Hopf lemma implies that u is constant on C (recall that $C = \overline{D_\rho} \setminus \left(\bigcup_{j \neq i} D_{\varepsilon_j}(p_j) \cup \Lambda\right)$) and hence $\tilde{h}$ is the standard metric there.

As $u(x) \to \infty$ for $x \to \Lambda$, it follows that Λ is empty. Moreover, if $\partial \tilde{M}$ were not connected (that is, ∂M were not connected or M were not simply-connected), then there is an index $j \neq i$ and a corresponding ball $D_{\varepsilon_j}(p_j)$. However, we have just seen that $\tilde{h}$ agrees with the standard metric on the boundary sphere $\Sigma_{\varepsilon_j}(p_j)$.

As $\Sigma_{\varepsilon_j}(p_j) \subseteq D_\rho$, we have $\varepsilon_j < \rho$. This would contradict the fact that – with respect to $\tilde{h}$ – $\Sigma_{\varepsilon_j}(p_j)$ is isometric to Σ_ρ.

It follows that $\partial\tilde{M}$ is connected, therefore ∂M is connected and M is simply-connected. Moreover, $C = \overline{D_\rho}$ and $h = \tilde{h}$ is the standard metric. Hence M is isometric with $\overline{D_\rho}$. □

Chapter 4

Applications and related results

In this chapter, we discuss applications and extensions of Theorem I as well as related results for domains in Euclidean space (Section 4.3). Our applications include more general spherical domains in Section 4.2 and manifolds which are not necessarily locally conformally flat, but locally conformally symmetric, see Definition 4.4.1 below. In addition, we prove a result similar to Theorem I for manifolds with parallel Ricci tensor: Theorem IV in Section 4.4.

4.1. Immediate extensions

In the following corollary, we collect some small extensions to Theorem I which we did not include in the statement for the sake of clarity and simplicity.

Corollary 4.1.1. *Theorem I remains valid in the following situations:*

i) If $\dim(M) = 2$. *Note that the condition to be locally conformally flat is void in this case.*

ii) The assumption on the connected components of the boundary being isometric to Σ_ρ *can be relaxed to being simply-connected and of constant scalar curvature* $\csc(\rho)^2(n-1)(n-2)$.

iii) The assumption on the mean curvature can be dropped provided ∂M *is connected,* M *is simply-connected and* $\rho = \frac{\pi}{2}$. *This is impossible if* $\rho \neq \frac{\pi}{2}$ *as we cannot distinguish* $\overline{D_\rho}$ *from* $\overline{D_{\pi-\rho}}$ *or from geodesic balls in smaller spheres.*

Proof. i) follows from the results of Hang and Wang, see [HW09, Theorem 4].

ii) This is not actually an extension as the assumptions imply that every connected component of the boundary is isometric to Σ_ρ: In fact, as umbilic hypersurfaces in locally conformally flat manifolds are again locally conformally flat (see Example 2.2.2, iii)), this is a consequence of Corollary 3.1.4.

iii) was proved in Section 3.1 □

4.2. Other spherical domains

The generalization of Theorem I to arbitrary domains in a hemisphere turns out to be quite hard. If a connected component of the boundary is not simply-connected, then the image of the developing map will have a more complicated boundary. This makes it difficult to extend the metric as shown in Section 3.3 and to use the assumptions on the geometry of the boundary which led to the convenient boundary condition $u = 1$ on Σ_ρ in Section 3.4.

However, making use of Theorem I directly, one obtains the following:

THEOREM II. *Let $\Omega \subseteq S^n_+$, $n \geq 3$, be an n-dimensional manifold with boundary such that $S^n \setminus \Omega$ is a smooth domain. Let (M^n, g) be a compact, connected locally conformally flat Riemannian manifold with boundary. Assume that*

i) $\mathrm{Scal}(g) \geq n(n-1)$ *or* $\mathrm{Scal}(g)$ *attains its minimum at the boundary,*

ii) There exists an isometry $\phi\colon \partial M \to \partial\Omega$ with the property that $\phi^* \mathrm{II}_{\partial\Omega} = \mathrm{II}_{\partial M}$ and $\phi^*\left(R^{S^n}(\cdot, \eta^{\partial\Omega}, \eta^{\partial\Omega}, \cdot)\right) = R^M(\cdot, \eta^{\partial M}, \eta^{\partial M}, \cdot)$.

Then (M, g) is isometric to Ω with the standard metric.

The main idea of the proof is to apply Theorem I to $N_+ := M \cup_\phi (S^n_+ \setminus \Omega)$. This technique makes it necessary to strengthen the assumption on the geometry of the boundary of M (compared to Theorem I) to obtain a metric on N_+ which is regular enough. We think that this should not be necessary, but lack a proof at the moment. However, we suspect:

CONJECTURE 4.2.1. *Theorem II holds provided*

i) $\mathrm{Scal}(g) \geq n(n-1)$,

ii') There exists an isometry $\phi\colon \partial M \to \partial\Omega$ with $\mathrm{II}_{\partial M} \geq \phi^* \mathrm{II}_{\partial\Omega}$.

The rest of this section is devoted to the proof of Theorem II.

PROOF (THEOREM II). Consider the manifold $N := M \cup_\phi (S^n \setminus \Omega)$. By Lemma 5.1.1, the canonical metric h is $C^{2,1}$ on N and smooth up to the image of the boundary ∂M. Furthermore, N is connected. As the metric is C^2, the scalar curvature is continuous, so $\mathrm{Scal}(g) = n(n-1)$ at the boundary. Hence $\mathrm{Scal}(g) \geq n(n-1)$ if $\mathrm{Scal}(g)$ attains its minimum at the boundary.

To apply Thorem I to N_+, we have to check the following: First of all, we need to ensure that N is locally conformally flat; then we need to show that Thorem I holds for manifolds with $C^{2,1}$-metrics.

LEMMA 4.2.2. *N is locally conformally flat.*

As the proof basically repeats the arguments from the proof of the Wey-Schouten theorem (Theorem 5.2.1 and Proposition 5.2.2), we recommend the reader to consult Section 5.2 at this point. The key observation is the fact that $g \in C^2$ suffices to construct $f \in C^2$ as in the proof of Proposition 5.2.2 and $g \in C^3$ is used only at the point where we check that $e^{2f}g$ is flat. Here, this can be circumvented by using the fact that the metrics on M and $S^n \setminus \Omega$ are (C^∞) locally conformally flat.

PROOF. We let Σ be the image of the boundaries of M and Ω, respectively. Since both M and S^n are locally conformally flat, every point not contained in Σ has a neighbourhood conformally equivalent to an open set of $\mathbb{R}^n$. It remains to show this for points $p \in \Sigma$.

Again, since both M and S^n are locally conformally flat, there exist f_1 locally defined in a neighbourhood of p in M and f_2 locally defined in a neighbourhood of p in $S^n \setminus \Omega$ such that the metrics $e^{2f_1}g$ and $e^{2f_2}g_{S^n}$ are flat. We fix $c \in \mathbb{R}$ and $\omega_0 \in T_p^*M$. In view of Corollary 5.2.3, f_1 and f_2 are unique (in a neighbourhood of p in M and $S^n \setminus \Omega$, respectively) if we assume that $f_1(p) = f_2(p) = c$ and $\mathrm{d}f_1(p) = \mathrm{d}f_2(p) = \omega_0$.

We check that such f_i agree along Σ and that they can be glued together to a C^2-function. To do so, we construct a locally defined function $f \in C^2$ which agrees with f_i on the respective domains of definition.

Recall the proof of Theorem 5.2.1: To find a function f such that $e^{2f}g$ is flat, we need to solve

$$\nabla\omega = S + \omega \otimes \omega - \frac{1}{2}|\omega|^2 g. \tag{4.2.1}$$

for a one-form ω, which will then be $\mathrm{d}f$. Using the vanishing of the Weyl- and Cotton tensor, this is done using a version of the Frobenius theorem (Proposition 5.2.2). We will repeat this procedure in Fermi coordinates: Pick a chart (U, x) of N around p with $x(p) = 0$, $M \cap U = \{x^n \leq 0\}$, $(S^n \setminus \Omega) \cap U = \{x^n \geq 0\}$ so that $\Sigma \cap U = \{x^n = 0\}$. In local coordinates, (4.2.1) is equivalent to

$$\frac{\partial \omega_j}{\partial x^i} = S_{ij} + \omega_i\omega_j - \frac{1}{2}|\omega| g_{ij} + \sum_{k=1}^{n} \Gamma_{ij}{}^k \omega_k =: (X_i)_j. \tag{4.2.2}$$

The solution is obtained inductively for $k = 1, \ldots, n$ where in each step, we define $\omega(t^1, \ldots, t^{k-1}, t^k, 0, \ldots, 0)$ to be the unique solution $\beta_k(t^k)$ of

$$\begin{cases} \beta_k(0) &= \omega(t^1, \ldots, t^{k-1}, 0, 0, \ldots, 0), \\ \beta_k'(\tau) &= X_k(t^1, \ldots, t^{k-1}, \tau, 0, \ldots, 0, \beta_k(\tau)), \end{cases} \tag{4.2.3}$$

with $\omega(0) = \omega_0$. A unique solution exists because X_k is Lipschitz and hence ω is differentiable. As X is smooth outside Σ, the same holds for ω. Let f be a local primitive of ω defined on a neighbourhood V of p with $f(p) = c$. Then $f \in C^2(V)$

and $f \in C^\infty(V \setminus (V \cap \Sigma))$ which by construction and uniqueness implies that $e^{2f}h$ is flat and $f = f_i$ $i = 1, 2$ wherever both sides are defined. □

REMARK 4.2.3. From the proof, we see that a manifold with C^1-metric obtained by gluing locally conformally flat manifolds is not necessarily locally conformally flat again: In fact, unique solvability of (4.2.3) was crucial in order to be able to glue the functions f_i together. If the metric is only C^1, then X as in equation (4.2.2) is not well-defined on Σ as it depends on the metric to second order. It follows that the unique solutions ω^1, ω^2 to (4.2.1) computed on both sides of Σ will in general not agree on Σ.

As N is locally conformally flat, so is $N_+ := M \cup_\phi (S^n_+ \setminus \Omega)$. If we can apply Theorem I to N_+, it will imply that N_+ is isometric with S^n_+. However, the metric on N_+ is merely $C^{2,1}$, so we need:

LEMMA 4.2.4. *Theorem I is valid for N_+.*

PROOF. The crucial point is the injectivity of the developing map; all other arguments are certainly valid for $C^{2,1}$-metrics. To show that the developing map of N_+ is injective, we apply Theorem 3.2.4 to N. We will take this for granted for now and provide a proof in Section 5.3 in order to point out that it still applies in a $C^{2,1}$-setting. □

Thus N_+ is isometric with S^n_+. Hence N is isometric with S^n. The result now follows from:

LEMMA 4.2.5. *Let $U \subseteq S^n$ be a smooth domain and M a manifold with boundary. Let $\phi\colon \partial M \to \partial U$ be an isometry and suppose that $M \cup_\phi U$ is isometric to S^n. Then M is isometric to $S^n \setminus U$.*

PROOF. Let $\Psi\colon M \cup_\phi U \to S^n$ be an isometry. By restricting, we obtain a locally defined isometry $\psi := \Psi|_U\colon U \to \Psi(U)$ of S^n. As all locally defined isometries of S^n with connected domain of definition come from global isometries, we can extend ψ to an isometry $\tilde{\psi}$ of S^n. Then $\tilde{\psi}^{-1} \circ \Psi|_M$ is an isometry between M and $S^n \setminus U$. □

This completes the proof of Theorem II. □

4.3. Domains in Euclidean space

Another possible application of the technique employed to prove Theorem I are locally conformally flat manifolds with nonnegative scalar curvature. Here, our aim is to show that – under the correct assumptions on the boundary – these manifolds are isometric to domains in Euclidean space. As we did for spherical domains, we first focus on geodesic balls.

Results of this type have been obtained by, for example, Miao [Mia02], Shi and Tam [ST02] and Raulot [Rau08]: Using a version of the positive mass theorem for metrics which fail to be differentiable along a hypersurface, Miao was able to prove that a metric on the unit ball with nonnegative scalar curvature agreeing with the standard metric on the boundary with mean curvature at least 1 must be isometric to the standard metric ([Mia02, Corollary 1.1]). Using a similar technique, Shi and Tam were able to extend this result to more general domains and proved:

Theorem 4.3.1 ([ST02, Theorems 4.1 and 4.2]). *Let (M, g) be a Riemannian spin manifold with nonnegative scalar curvature and with boundary components Σ_i, $1 \leq i \leq k$, each with positive mean curvature H. Assume there are isometric embeddings $\iota_i \colon \Sigma_i \to \mathbb{R}^n$ such that the image of Σ_i is a strictly convex hypersurface with mean curvature $H^{(i)}$ (with respect to the Euclidean metric). Then, for all $1 \leq i \leq k$,*

$$\int_{\Sigma_i} H \, \mathrm{d}S \leq \int_{\iota_i(\Sigma_i)} H^{(i)} \, \mathrm{d}S.$$

If equality holds for some i then ∂M is connected and M is isometric to a domain in $\mathbb{R}^n$.

In 2008, Raulot [Rau08, Corollary 5] gave an alternative proof of Miao's result for spin manifolds without using the positive mass theorem.

We are going to give an independent proof of this result for locally conformally flat manifolds with umbilic boundary, while we do not assume the manifold to be spin. Recall that we write B_r for a ball of radius r in $\mathbb{R}^n$ and $S_r = \partial B_r$. Precisely, we show:

Theorem III. *Let (M^n, g), $n \geq 3$, be a compact connected locally conformally flat Riemannian manifold with boundary. Assume that*

i) $\mathrm{Scal}(g) \geq 0$ *everywhere,*

ii) The boundary ∂M is umbilic and every connected component is isometric to S_r, with mean curvature $H(g) \geq H^r = r^{-1}$.

Then (M, g) is isometric to $\overline{B_r}$ with the standard metric.

Proof. We follow the lines of the proof of Theorem I, but the argument turns out to be easier in the end. This is mainly due to the fact that – in contrast to the spherical metric – the Euclidean metric has vanishing scalar curvature which simplifies the conformal scalar curvature equation (1.1) and consequently the application of the Hopf lemma.

By Lemma 3.2.2, the Yamabe invariant of (M, g) is positive so Proposition 3.2.1 shows that the developing map $\Phi \colon \tilde{M} \to S^n$ is injective.

Composing with a stereographic projection π and a conformal transformation, we may assume that the image is

$$A = A(\varepsilon_i, p_i, \Lambda) \coloneqq \overline{B_r} \setminus \left(\bigcup_{i \in \pi_0(\partial \tilde{M})} B_{\varepsilon_i}(p_i) \cup \Lambda \right).$$

Write $\Psi \coloneqq \pi \circ \Phi$ and $h \coloneqq (\Psi^{-1})^* \tilde{g}$. Reasoning as in the proof of Proposition 2.3.7 we can pull back h with a Möbius transformation so that we may assume that h agrees with the Euclidean metric on ∂B_r. Writing

$$h = v^{\frac{4}{n-2}} g_{\mathbb{R}^n},$$

we obtain $v \in C^\infty(A)$ satisfying

$$\begin{cases} v = 1 & \text{on } \partial B_r, \\ \frac{\partial v}{\partial \eta} \geq 0 & \text{on } \partial B_r \\ -\Delta v \geq 0 & \text{on } A. \end{cases} \tag{4.3.1}$$

We claim that v attains its minimum at the boundary ∂B_r: In fact, since (A, h) is complete (as a metric space), we have $v(x) \to \infty$ for $x \to \Lambda$ (see [SY88, Proposition 2.6]). Hence v attains its minimum; this can only occur at a point $x \in \partial B_r$ or $x \in \partial B_{\varepsilon_i}(p_i)$ for some i because v is superharmonic.

For the sake of contradiction assume that v attains its minimum at a point $x \in \partial B_{\varepsilon_i}(p_i)$. Let η be the interior unit normal of $B_{\varepsilon_i}(p_i)$ with respect to the euclidean metric (pointing inwards $B_{\varepsilon_i}(p_i)$). By assumption, the mean curvature $H(h)$ of $S_{\varepsilon_i}(p_i)$ with respect to h and η is $\leq -r^{-1}$. Moreover, its mean curvature with respect to the Euclidean metric is $H^{\varepsilon_i} = \varepsilon_i^{-1}$. From the conformal mean curvature equation (Equation (1.2)), we obtain

$$\frac{\partial v}{\partial \eta} = -\frac{\partial v}{\partial \nu} = \frac{n-2}{2} \left(H^{\varepsilon_i} v - H(h) v^{\frac{n}{n-2}} \right) > 0$$

as $v > 0$. This is a contradiction to the fact that x is a minimum and we have proven that $v \geq 1$.

As $v = 1$ on the boundary ∂B_r, it follows that v attains its minimum there, therefore $\frac{\partial v}{\partial \eta} \leq 0$. On the other hand, we have $\frac{\partial v}{\partial \eta} \geq 0$ by Equation (4.3.1). The Hopf lemma now implies that $v = 1$ on A and thus h is the standard metric $g_{\mathbb{R}^n}$ there. This contradicts the fact that, with respect to h, the $S_{\varepsilon_i}(p_i)$ are isometric to S_r. Hence there are no $B_{\varepsilon_i}(p_i)$, ∂M is connected, Λ is empty and M is simply-connected and isometric to $\overline{B_r}$. □

By employing the same techniques as in the proof of Theorem II, we can extend Theorem III to more general subsets of Euclidean space. The result is:

Corollary 4.3.2. *Let $\Omega \subseteq \mathbb{R}^n$, $n \geq 3$, be an n-dimensional manifold with nonempty boundary such that $\mathbb{R}^n \setminus \Omega$ is a smooth domain. Let (M^n, g) be a compact, connected locally conformally flat Riemannian manifold with boundary. Assume that*

i) $\mathrm{Scal}(g) \geq 0$ *or* $\mathrm{Scal}(g)$ *attains its minimum at the boundary,*
ii) There exists an isometry $\phi\colon \partial M \to \partial\Omega$ *with the property that* $\phi^* \mathrm{II}_{\partial\Omega} = \mathrm{II}_{\partial M}$*,*
iii) $R^M(\cdot, \eta, \eta, \cdot) = 0$.

Then (M, g) is isometric to Ω with the standard metric.

4.4. Locally conformally symmetric manifolds and manifolds with parallel Ricci tensor

Another attempt to generalize Theorem I is to weaken the assumption that the manifold in consideration is locally conformally flat. As Min-Oo's conjecture is incorrect (see Section 2.5), we need other requirements to make a statement like Theorem I hold. Also, local conformal flatness was used in a crucial way to prove it using the developing map. This makes it difficult to replace the condition of local conformal flatness with something sensible and still preserve correctness of the statement of Theorem I.

In this section, we discuss two approaches using locally conformally symmetric manifolds, see Definition 4.4.1 below, and manifolds with parallel Ricci tensor.

Assume that $n \geq 4$. Recall from Theorem 2.2.3 that a Riemannian manifold is locally conformally flat if and only if its Weyl tensor W vanishes. Hence a natural class of manifolds to consider is the class of *locally conformally symmetric manifolds*:

Definition 4.4.1. We say that a Riemannian manifold (M, g), $\dim(M) \geq 4$, is *locally conformally symmetric*[1] if its Weyl tensor is parallel.

That is, instead of $W = 0$, we only assume $\nabla W = 0$. Of course, locally conformally flat spaces are locally conformally symmetric. From the decomposition $W = R - S \owedge g$, cf. Equation 2.2.3, we see that locally symmetric spaces (i.e. $\nabla R = 0$) are also locally conformally symmetric. One can show that the converse also holds: By a result of Derdziński and Roter, any locally conformally symmetric manifold is locally conformally flat or locally symmetric, see [DR77, Theorem 2]. The case that M is locally conformally flat is handled in Theorem I.

If M is locally symmetric and $n \geq 4$, it admits a non-totally geodesic umbilical hypersurface if and only if M is locally conformally flat by a result of Chen

[1] In accordance with Definition 2.2.1, we prefer the term *locally conformally symmetric* over *conformally symmetric*.

[CHE80, Theorem 4.1]. Thus, locally symmetric spaces are not of special interest for us. Instead, we study manifolds with parallel Ricci tensor, that is, instead of $\nabla R = 0$, we merely assume $\nabla \operatorname{Ric} = 0$.

We will prove:

THEOREM IV. *Let (M^n, g), $n \geq 3$, be a compact connected Riemannian manifold with boundary, $0 < \rho \leq \frac{\pi}{2}$. Assume that the Ricci tensor is parallel and*

i) $\operatorname{Scal}(g) \geq n(n-1)$ *everywhere,*

ii) The boundary ∂M is umbilic with mean curvature $H(g) = H_\rho$ and every connected component is isometric to Σ_ρ.

Then (M, g) is isometric to $\overline{D_\rho}$ with the standard metric.

PROOF. We proceed in three steps: We first show that $\operatorname{Ric} \geq n-1$ at all points $x \in \partial M$, then use $\nabla \operatorname{Ric} = 0$ to conclude $\operatorname{Ric} \geq n-1$ everywhere. In the last step, we show that the boundary is connected, therefore Theorem 2.5.3 by Hang and Wang (or [HW09, Theorem 2] in the case of a whole hemisphere) implies that (M, g) is isometric to $\overline{D_\rho}$.

Step 1: $\operatorname{Ric} \geq n-1$ *on the boundary.* Let $x \in \partial M$, $X \in T_x\partial M$ with $|X| = 1$, η a unit normal at x. Recall that every connected component of the boundary is a sphere of radius $\sin(\rho)$, i.e. with constant sectional curvature $\csc^2(\rho)$, and mean curvature $H = \cot(\rho)$. Using that the boundary is umbilic, the Gauß equation [BES87, Theorem 1.72 c)] implies

$$\operatorname{Ric}^M(X, X) = \operatorname{Ric}^{\partial M}(X, X) + R^M(X, \eta, \eta, X) - (n-2)H^2, \tag{4.4.1a}$$

$$\operatorname{Scal}^M = \operatorname{Scal}^{\partial M} + 2\operatorname{Ric}^M(\eta, \eta) - (n-1)(n-2)H^2. \tag{4.4.1b}$$

Therefore

$$\begin{aligned} 2\operatorname{Ric}^M(\eta, \eta) &= \operatorname{Scal}^M - \operatorname{Scal}^{\partial M} + (n-1)(n-2)H^2 \\ &\geq n(n-1) + (n-1)(n-2)\left(\cot^2(\rho) - \csc^2(\rho)\right) \\ &= 2(n-1). \end{aligned}$$

Since ∂M is umbilic with constant mean curvature, the Codazzi-Mainardi equation [BES87, Theorem 1.72 d)] reads

$$\left\langle R^M(X, Y)Z, \eta \right\rangle = \langle (\nabla_X \mathrm{II})(Y, Z), \eta \rangle - \langle (\nabla_Y \mathrm{II})(X, Z), \eta \rangle = 0$$

for all vector fields $X, Y, Z \in \Gamma(T\partial M)$. Hence also $\operatorname{Ric}(X, \eta) = 0$ along ∂M.

Next, we prove that $\operatorname{Ric}(X,X) = c|X|^2$ for some $c \in \mathbb{R}$ and all $X \in T_x\partial M$: Let $X, Y \in T_x\partial M$ with $|X| = |Y| = 1$. As every connected component of ∂M is isometric to a sphere, we know that the holonomy group (with respect to the induced connection) of the connected component containing x is isomorphic to $\mathrm{SO}(n-1)$. Hence there exists a curve $\gamma\colon [0,1] \to \partial M$ with $\gamma(0) = \gamma(1) = x$ and $P(\gamma)_0^1 X = Y$, where $P(\gamma)$ denotes parallel transport in ∂M along γ. We write $V(t) \coloneqq P(\gamma)_0^t X$ for short.

Since V is parallel with respect to the induced connection of ∂M, we have $\nabla_{\dot\gamma} V = \mathrm{II}(\dot\gamma, V)\nu$. Using moreover that $\nabla \operatorname{Ric} = 0$ and $\mathrm{II} = Hg$, we compute:

$$\begin{aligned}
\operatorname{Ric}(Y,Y) &= \int_0^1 \frac{\mathrm{d}}{\mathrm{d}t} \operatorname{Ric}(V(t),V(t))\,\mathrm{d}t + \operatorname{Ric}(X,X)\\
&= 2\int_0^1 \operatorname{Ric}\left(\nabla_{\dot\gamma} V(t), V(t)\right)\,\mathrm{d}t + \operatorname{Ric}(X,X)\\
&= 2\int_0^1 \operatorname{Ric}\left(\mathrm{II}(\dot\gamma(t),V(t))\cdot \nu(\gamma(t)), V(t)\right)\,\mathrm{d}t + \operatorname{Ric}(X,X)\\
&= 2H\int_0^1 \langle \dot\gamma(t), V(t)\rangle \operatorname{Ric}\left(\nu(\gamma(t)), V(t)\right)\,\mathrm{d}t + \operatorname{Ric}(X,X)\\
&= \operatorname{Ric}(X,X)
\end{aligned}$$

because $\operatorname{Ric}(\nu, V) = 0$. This implies $\operatorname{Ric}(X,X) = c|X|^2$ for some constant c for all $X \in T_x\partial M$. We will now show that $c \geq n-1$:

Let $\{E_1, \dots, E_{n-1}\}$ be an orthonormal basis of $T_x\partial M$. Using equation (4.4.1b), we compute:

$$\begin{aligned}
\mathrm{Scal}^M &= \sum_{i=1}^{n-1} \operatorname{Ric}(E_i,E_i) + \operatorname{Ric}(\eta,\eta)\\
&= c(n-1) + \frac{1}{2}\Big(\mathrm{Scal}^M - \mathrm{Scal}^{\partial M} + H^2(n-1)(n-2)\Big)\\
&= c(n-1) + \frac{1}{2}\left(\mathrm{Scal}^M + (n-1)(n-2)\left(\cot^2(\rho) - \csc^2(\rho)\right)\right)\\
&= c(n-1) + \frac{1}{2}\left(\mathrm{Scal}^M - (n-1)(n-2)\right).
\end{aligned}$$

Thus,

$$c = \frac{\mathrm{Scal}^M}{2(n-1)} + \frac{n-2}{2} \geq n-1.$$

This implies $\operatorname{Ric}(X,X) \geq (n-1)|X|^2$ for all $X \in T_x\partial M$. For a general $Y \in T_xM$, decompose $Y = Y^\perp + Y^\top$ into its normal and tangential component.

Then

$$\begin{aligned}\operatorname{Ric}(Y,Y) &= \operatorname{Ric}(Y^\perp,Y^\perp) + 2\operatorname{Ric}(Y^\perp,Y^\top) + \operatorname{Ric}(Y^\top,Y^\top)\\ &= \operatorname{Ric}(\eta,\eta)|Y^\perp|^2 + \operatorname{Ric}(Y^\top,Y^\top)\\ &\geq (n-1)|Y^\perp|^2 + (n-1)|Y^\top|^2\\ &= (n-1)|Y|^2.\end{aligned}$$

Step 2: $\operatorname{Ric} \geq n-1$ *everywhere.* We have shown that $\operatorname{Ric} \geq (n-1)$ at $x \in \partial M$. Now, for $y \in M$ and $v \in T_yM$ with $|v| = 1$, let $\gamma\colon [0,1] \to M$ be a curve with $\gamma(0) = x$, $\gamma(1) = y$. Let V be the parallel vector field along γ with $V(1) = v$. Then

$$\begin{aligned}\operatorname{Ric}(v,v) &= \int_0^1 \frac{\mathrm{d}}{\mathrm{d}t}\operatorname{Ric}(V(t),V(t))\,\mathrm{d}t + \operatorname{Ric}(V(0),V(0))\\ &\geq \int_0^1 (\nabla_{\dot\gamma}\operatorname{Ric})(V(t),V(t))\,\mathrm{d}t + (n-1)|V(0)|^2\\ &= n-1.\end{aligned}$$

Thus, $\operatorname{Ric} \geq n-1$ at y.

Step 3: The boundary is connected. To show that ∂M is connected we assume the contrary. Let C_1 be a connected component of ∂M and set $C_2 := \partial M \setminus C_1$. As M is compact, there exists $l > 0$ and a shortest unit speed geodesic $\gamma\colon [0,l] \to M$ joining C_1 and C_2. As γ is shortest, the image of γ lies in the interior of M except at its endpoints, i.e. $\gamma([0,l]) \cap \partial M = \{\gamma(0), \gamma(l)\}$.

Let $\{E_1, \ldots, E_{n-1}\}$ be an orthonormal basis of $T_{\gamma(0)}\partial M$. Again because γ is shortest, we know that $\dot\gamma(0)$ is perpendicular to $T_{\gamma(0)}\partial M$ and therefore $\{\dot\gamma(0), E_1, \ldots, E_{n-1}\}$ is an orthonormal basis of $T_{\gamma(0)}M$. Extend the E_i by letting V_i be the parallel vector field along γ with $V_i(0) = E_i$. This way, we obtain an orthonormal frame $\{\dot\gamma(t), V_1(t), \ldots, V_{n-1}(t)\}$ along γ.

Let $\gamma_i(s)$ be a variation of γ with variational vector field V_i such that the endpoints of $\gamma_i(s)$ lie on ∂M for all s. From the second variation formula of energy, we obtain (cf. Remark 2.10 in Chapter 9 of [dC92]; note that do Carmo's sign convention for the Riemann curvature tensor R differs from ours):

$$0 \leq \frac{1}{2}\,\frac{\mathrm{d}^2}{\mathrm{d}s^2}E\left(\gamma_i(s)\right)\bigg|_{s=0} = \int_0^l \left(\left|\nabla_{\frac{\partial}{\partial t}}V_i\right|^2 - R(V_i,\dot\gamma,\dot\gamma,V_i)\right)\mathrm{d}t + \left\langle \nabla_{V_i}V_i, \dot\gamma\right\rangle\Big|_0^l$$

Taking into account that the V_i are parallel and that $\dot\gamma(0)$ and $\dot\gamma(l)$ are inner and outer unit normals, we obtain

$$0 \leq -\int_0^l R(V_i,\dot\gamma,\dot\gamma,V_i)\,\mathrm{d}t - \mathrm{II}_{\gamma(l)}(V_i,V_i) - \mathrm{II}_{\gamma(0)}(V_i,V_i).$$

Summing over all $1 \le i \le n-1$, we arrive at

$$\begin{aligned} 0 &\le -\int_0^l \mathrm{Ric}(\dot\gamma,\dot\gamma)\,\mathrm{d}t - 2(n-1)H \\ &\le -(n-1)(2H+l), \end{aligned}$$

which contradicts our assumptions $H = H_\rho = \cot(\rho) \ge 0$ and $l > 0$. Hence ∂M is connected.

From Theorem 2.5.3 (for $\rho < \frac{\pi}{2}$) and [HW09, Theorem 2] ($\rho = \frac{\pi}{2}$) it follows that (M, g) is isometric to $\overline{D_\rho}$. □

Remark 4.4.2. Step 3 of the proof can of course be applied to a more general setting: An analogous calculation implies that the distance of two connected components of a disconnected boundary ∂M in a compact Riemannian manifold can be estimated in terms of lower bounds on the Ricci and mean curvature. To be precise, assume that $H \ge h \in \mathbb{R}$ and $\mathrm{Ric} \ge \kappa(n-1)$, where $\kappa \in \mathbb{R}$, then the distance l of two connected components satisfies

$$l\kappa \le -2h. \tag{4.4.2}$$

In particular, if κ and h can be chosen in a way so that (4.4.2) cannot be satisfied with $l > 0$, then ∂M is connected.

A similar computation as in the first step of the proof gives the following:

Corollary 4.4.3. *The second condition in Theorem IV can be replaced by*

ii') The boundary ∂M is umbilic with mean curvature $H(g) \ge H_\rho$ and every connected component is isometric to Σ_ρ. There exists a point $x \in \partial M$ where $H(g) = H_\rho$ and $\sec(X,\eta) \ge 1$ for all $X \in T_x\partial M$.

Proof. Again, the main argument is to show $\mathrm{Ric} \ge n-1$ at $x \in \partial M$; then the statement will follow as above.

Let $X \in T_x\partial M$ with $|X| = 1$. From Equation (4.4.1a), we obtain

$$\mathrm{Ric}^M(X,X) = (n-2)(\csc(\rho)^2 - \cot(\rho)^2) + \sec(X,\eta) \ge n-1.$$

As in the proof of Theorem IV, it follows that $\mathrm{Ric}(\eta,\eta) \ge n-1$. Also, since the boundary is umbilic and its mean curvature attains a minimum at x, we have $\mathrm{Ric}(X,\eta) = 0$ for all $X \in T_x\partial M$ by the Codazzi-Mainardi equation. As above, it follows that $\mathrm{Ric} \ge n-1$ at x. □

Summarizing our discussion of locally conformally symmetric manifolds, we state:

Corollary 4.4.4. *Let* (M^n, g)*,* $n \geq 4$*, be a compact connected locally conformally symmetric Riemannian manifold with boundary. Assume that*

i) $\mathrm{Scal}(g) \geq n(n-1)$ *everywhere,*

ii) The boundary ∂M *is umbilic with mean curvature* $H(g) \geq H_\rho$ *and every connected component is isometric to* Σ_ρ*.*

Then (M, g) *is isometric to* $\overline{D_\rho}$ *with the standard metric.*

Proof. If M is locally conformally flat, this is Theorem I. As we said above, by the results of Derdziński and Roter [DR77, Theorem 2] and Chen [Che80, Theorem 4.1], the only other possibility is that M is locally symmetric and ∂M is totally geodesic, i.e. $\rho = \frac{\pi}{2}$ and $H(g) = 0$, which is covered by Theorem IV. □

Chapter 5

Proofs of selected results

In this chapter we present the proofs of some results which we referred to in the preceding chapters but did not find substantial enough to be presented in the main part of this thesis. We begin by discussing regularity properties of the canonical Riemannian metric on manifolds obtained by gluing Riemannian manifolds, then present a proof of the Weyl-Schouten theorem using a version of the Frobenius theorem from [Spi70]. In Section 5.3, we discuss the results by Schoen and Yau ([SY88] and [SY94]) on the injectivity of the developing map for compact manifolds with positive scalar curvature.

5.1. Gluing Riemannian manifolds along their boundaries

In this section, we investigate the regularity of Riemannian metrics on manifolds obtained by gluing two Riemannian manifolds along their boundaries via a diffeomorphism. The result is as follows:

Lemma 5.1.1. *Let M, N be compact manifolds with boundaries ∂M, ∂N and Riemannian metrics g, h, respectively. Let $f\colon \partial M \to \partial N$ be a diffeomorphism. Then*

i) The topological space $M \cup_f N = {}^{M \sqcup N}\!/_{x \sim f(x)}$ admits the structure of a smooth manifold such that the inclusion maps $N, M \to M \cup_f N$ are embeddings. The image of the boundary has a neighbourhood diffeomorphic to $\partial M \times (-1, 1)$, where ∂M is identified with $\partial M \times \{0\}$.

ii) If $f^*(h|_{\partial N}) = g|_{\partial M}$ then the canonical metric $g \cup_f h$ on $M \cup_f N$ defined by $g \cup_f h = g$ on M and $g \cup_f h = h$ on N is well-defined and Lipschitz continuous.

iii) If additionally $f^*\mathrm{II}_{\partial N} = -\mathrm{II}_{\partial M}$ then $g \cup_f h$ is of class $C^{1,1}$.

iv) If additionally $f^*\left(R^N(\cdot, \eta^{\partial N}, \eta^{\partial N}, \cdot)\right) = R^M(\cdot, \eta^{\partial M}, \eta^{\partial M}, \cdot)$ then $g \cup_f h$ is $C^{2,1}$.

Proof. To define a smooth structure on $M \cup_f N$, we pick an atlas as follows: Take all charts of the interiors $\mathrm{int}(M)$ and $\mathrm{int}(N)$, these cover $M \cup_f N$ except for the points in the boundaries. For those, apply the tubular neighbourhood theorem to obtain neighbourhoods U, V of ∂M and ∂N, respectively and diffeomorphisms $\phi\colon \partial M \times (-\varepsilon, 0] \to U$, $\psi\colon \partial N \times [0, \varepsilon) \to V$ with $\phi(\cdot, 0) = \mathrm{id}_{\partial M}$ and $\psi(\cdot, 0) = \mathrm{id}_{\partial N}$.

Then define a chart by

$$\begin{aligned} \partial M \times (-\varepsilon, \varepsilon) &\to U \cup_f V \\ (x,t) &\mapsto \begin{cases} \phi(x,t) & \text{if } t \leq 0, \\ \psi(f(x),t) & \text{if } t \geq 0. \end{cases} \end{aligned}$$

Taking the union of this chart with the charts of the interiors gives an atlas of $M \cup_f N$. This implies i). ii) follows since g and h are smooth on the compact manifolds M and N, respectively, and hence Lipschitz, so $g \cup_f h$ is also Lipschitz.

For iii), we pick Fermi coordinates $(x^1, \dots, x^{n-1}, t)$ adapted to the boundary on M: Given coordinates $x = (x^1, \dots, x^{n-1})$ on ∂M and t sufficiently small, they are defined by

$$(x,t) \mapsto \exp_x(t\nu).$$

The metric takes the form

$$g = \mathrm{d}t^2 + \sum_{i,j=1}^{n-1} g_{ij}(x,t)\, \mathrm{d}x^i \otimes \mathrm{d}x^j.$$

On N, we pick Fermi coordinates $(x^1 \circ f, \dots, x^{n-1} \circ f, t)$.

In view of ii), we only have to check that the derivatives $\frac{\partial g_{ij}}{\partial t}$ and $-\frac{\partial h_{ij}}{\partial t}$ agree on corresponding points of the boundaries. Note that $\nu = \frac{\partial}{\partial t}$ along ∂M. Writing $\partial_t \coloneqq \frac{\partial}{\partial t}$ and $\partial_i \coloneqq \frac{\partial}{\partial x^i}$ for short, we have

$$\begin{aligned} \left.\frac{\partial g_{ij}}{\partial t}\right|_{t=0} &= g(\nabla_{\partial_t}\partial_i, \partial_j)|_{t=0} + g(\partial_i, \nabla_{\partial_t}\partial_j)|_{t=0} \\ &= g(\nabla_{\partial_i}\partial_t, \partial_j)|_{t=0} + g(\partial_i, \nabla_{\partial_j}\partial_t)|_{t=0} \\ &= -g(\nu, \nabla_{\partial_i}\partial_j) - g(\nabla_{\partial_j}\partial_i, \nu) \\ &= -2\mathrm{II}_{ij}, \end{aligned}$$

the same calculation for h shows that $g \cup_f h$ is differentiable. The assertion on Lipschitz derivatives follows from smoothness of g and h as above, so iii) follows.

For iv), we are left so show that the derivatives $\frac{\partial^2 g_{ij}}{\partial t^2}$ and $\frac{\partial^2 h_{ij}}{\partial t^2}$ agree along the boundary. Note that $\nabla_{\frac{\partial}{\partial t}} \frac{\partial}{\partial t} = 0$ because $t \mapsto \exp_x(t\nu)$ is a geodesic. It follows that

$$\begin{aligned} \left.\frac{\partial^2 g_{ij}}{\partial t^2}\right|_{t=0} &= g(\nabla_{\partial_t}\nabla_{\partial_t}\partial_i, \partial_j)\big|_{t=0} + 2g(\nabla_\eta\partial_i, \nabla_\eta\partial_j) + g(\partial_i, \nabla_{\partial_t}\nabla_{\partial_t}\partial_j)\big|_{t=0} \\ &= g(\nabla_{\partial_t}\nabla_{\partial_i}\partial_t, \partial_j)\big|_{t=0} + g(\partial_i, \nabla_{\partial_t}\nabla_{\partial_j}\partial_t)\big|_{t=0} + 2g(\nabla_\eta\partial_i, \nabla_\eta\partial_j) \\ &= g(R^M(\eta, \partial_i)\eta, \partial_j) + g(\partial_i, R^M(\eta, \partial_j)\eta) + 2g(\nabla_\eta\partial_i, \nabla_\eta\partial_j) \\ &= 2\left(g(R^M(\eta, \partial_i)\eta, \partial_j) + g(\nabla_\eta\partial_i, \nabla_\eta\partial_j)\right). \end{aligned}$$

The first terms agree by assumption while the second terms are of first order and agree by iii). This proves iv). $\square$

A special (and very important) case to consider is the *double manifold* which is obtained from a compact manifold M with boundary by gluing it with itself, that is $\hat{M} = M \cup_{\mathrm{id}_{\partial M}} M$. This is often written as $M \underset{\partial M}{\cup} (-M)$, where we write $-M$ for the second copy of M in $\hat{M}$.

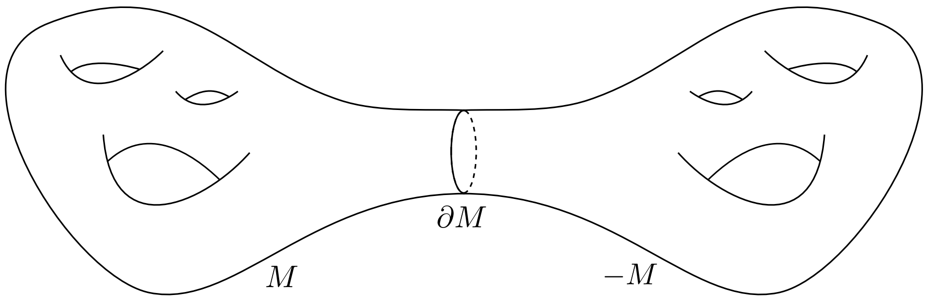

FIGURE 2. The double manifold $\hat{M}$

From Lemma 5.1.1, we obtain:

COROLLARY 5.1.2. *Let* (M, g) *be a compact Riemannian manifold with boundary and* $\hat{M} = M \underset{\partial M}{\cup} (-M)$ *be the double manifold. Then* $\hat{M}$ *admits the structure of a smooth manifold and the canonical Riemannian metric* $\hat{g}$ *is Lipschitz continuous. Furthermore, if* ∂M *is totally geodesic,* $\hat{g}$ *is* $C^{2,1}$.

5.2. The Weyl-Schouten theorem

In this section, we show how one can prove the Weyl-Schouten theorem stating equivalent characterisations of locally conformally flat manifolds as presented in Section 2.2, but omit most of the tedious calculations. For the reader's convenience we recall:

THEOREM 5.2.1 (Weyl-Schouten). *Let* (M, g) *be an* n*-dimensional Riemannian manifold. Then*

i) If $n = 2$*, then* (M, g) *is locally conformally flat.*
ii) If $n = 3$*, then* (M, g) *is locally conformally flat if and only if the Cotton tensor vanishes.*
iii) If $n \geq 4$*, then* (M, g) *is locally conformally flat if and only if the Weyl tensor vanishes.*

PROOF (FOLLOWING [LAF88]). In dimension two, every Riemannian manifold is locally conformally flat due to the existence of isothermal coordinates.

For one direction in (ii) and (iii), note that the Weyl tensor (in any dimension) and the Cotton tensor (in dimension three) are conformally invariant in the sense that

$$W(e^{2f}g) = e^{2f}W(g), \qquad C(e^{2f}g) = C(g).$$

It follows that, if (M, g) is conformally flat, then $W(g) = 0$ and $C(g) = 0$ in dimension three, as $W(g) = C(g) = 0$ for flat metrics.

For the other direction first note that in dimension three, the Weyl tensor automatically vanishes due to its symmetries. In dimension $n \geq 3$, Weyl- and Cotton-tensor are related by the formula $\sum_a \nabla^a W_{aijk} = \frac{n-3}{n-2} C_{ijk}$. Hence in dimension $n \geq 4$, the Cotton tensor vanishes if the Weyl tensor does and we may assume $W(g) = C(g) = 0$. Locally, we solve for f satisfying $R(e^{2f}) = 0$:

Defining

$$A(f) := \operatorname{Hess} f - \mathrm{d}f \otimes \mathrm{d}f + \frac{1}{2}|\nabla f|^2 g$$

and using $W = R - S \owedge g$ with $W(g) = 0$, we can write the transformation law for the curvature tensor as a $(4, 0)$-tensor as

$$R(e^{2f}g) = e^{2f} \left(R(g) - A(f) \owedge g\right) = e^{2f} \left(S(g) - A(f)\right) \owedge g.$$

Hence we search for f with $A(f) = S$.

Claim: To solve $A(f) = S$ locally it is necessary and sufficient to find a one-form ω with

$$\nabla\omega = S + \omega \otimes \omega - \frac{1}{2}|\omega|^2 g. \tag{5.2.1}$$

Indeed, if f solves $A(f) = S$, then $\omega = \mathrm{d}f$ solves (5.2.1). On the other hand, if ω solves (5.2.1), then $\nabla\omega$ is symmetric. This implies that ω is closed, thus locally exact.

Rewriting (5.2.1) in local coordinates, we obtain

$$\frac{\partial \omega_j}{\partial x^i} = S_{ij} + \omega_i\omega_j - \frac{1}{2}|\omega| g_{ij} + \sum_{k=1}^{n} \Gamma_{ij}{}^{k}\omega_k. \tag{5.2.2}$$

The statement now follows from a version of Frobenius' theorem (see Proposition 5.2.2 below). The integrability condition for existence of a solution of equation (5.2.1) arises from

$$\nabla^2_{X,Y}\omega - \nabla^2_{Y,X}\omega = R^{T^*M}(X, Y)\omega \quad \text{for all } X, Y$$

and is equivalent to

$$(\nabla_X S)(Y, Z) = (\nabla_Y S)(X, Z) \quad \text{for all } X, Y, Z,$$

that is, to $C = 0$. □

We will now comment on the version of the Frobenius theorem used in the last step of the proof. The following is found in, for instance, [Spi70, Theorem 6.1] and can be applied to Equation (5.2.2) directly:

Proposition 5.2.2. *Let $U \times V \subseteq \mathbb{R}^m \times \mathbb{R}^n$ be open with $0 \in U$ and $X_i \colon U \times V \to \mathbb{R}^n$, $i = 1, \dots, m$, be differentiable functions. Then, for $x \in V$, there exists a unique*[1] *function $\alpha \colon W \to V$ defined in a neighbourhood W of 0 in $\mathbb{R}^m$ satisfying*

$$\begin{cases} \alpha(0) &= x, \\ \frac{\partial \alpha}{\partial t^j}(t) &= X_j(t, \alpha(t)) \qquad \text{for } t \in W \end{cases} \tag{5.2.3}$$

if and only if there is a neighbourhood of $(0, x)$ in $U \times V$ on which

$$\frac{\partial X_i}{\partial t^j} + \sum_{k=1}^{n} \frac{\partial X_i}{\partial x^k} X_j^k = \frac{\partial X_j}{\partial t^i} + \sum_{k=1}^{n} \frac{\partial X_j}{\partial x^k} X_i^k \tag{5.2.4}$$

for all i, j.

We again give a sketch of the proof:

Proof. Necessity of (5.2.4) follows from $\frac{\partial^2 \alpha}{\partial t^j \partial t^i}(t) = \frac{\partial^2 \alpha}{\partial t^i \partial t^j}(t)$, Equation (5.2.3) and the chain rule.

Assuming (5.2.4), we construct α satisfying (5.2.3) inductively as follows: We first define $\alpha(t^1, 0, \dots, 0) := \beta_1(t^1)$, where β_1 is the solution to the ODE

$$\begin{cases} \beta_1(0) &= x, \\ \beta_1'(\tau) &= X_1(\tau, 0, \dots, 0, \beta_1(\tau)), \end{cases}$$

which has a unique solution for $|\tau| < \varepsilon_1$. Then

$$\frac{\partial \alpha}{\partial t^1}(t^1, 0, \dots, 0) = X_1(t^1, 0, \dots, 0, \alpha(t^1, 0, \dots, 0)).$$

Assuming that we have constructed $\alpha(t^1, \dots, t^{k-1}, 0, \dots, 0)$ satisfying (5.2.3) for $1 \leq j \leq k-1 < m$, we set $\alpha(t^1, \dots, t^{k-1}, t^k, 0, \dots, 0) := \beta_k(t^k)$, where β_k solves

$$\begin{cases} \beta_k(0) &= \alpha(t^1, \dots, t^{k-1}, 0, 0, \dots, 0), \\ \beta_k'(\tau) &= X_k(t^1, \dots, t^{k-1}, \tau, 0, \dots, 0, \beta_k(\tau)), \end{cases}$$

which again has a solution for $|\tau| < \varepsilon_k$ (one may have to choose $\varepsilon_1, \dots, \varepsilon_{k-1}$ sufficiently small). Then

$$\frac{\partial \alpha}{\partial t^k}(t^1, \dots t^k, 0 \dots, 0) = X_k(t^1, \dots, t^k, 0, \dots, 0, \alpha(t^1, \dots, t^k, 0, \dots, 0)) \tag{5.2.5}$$

and we need to check that (5.2.3) is still satisfied for $1 \leq j \leq k-1$.

[1] More precisely: Two such functions α_1 and α_2 defined on W_1 and W_2 agree on the connected component of $W_1 \cap W_2$ containing 0.

To do so pick any j and define

$$\begin{aligned} h(\tau) := {} & \frac{\partial \alpha}{\partial t^j}(t^1, \ldots, t^{k-1}, \tau, 0, \ldots, 0) \\ & - X_j(t^1, \ldots, t^{k-1}, \tau, 0, \ldots, 0, \alpha(t^1, \ldots, t^{k-1}, \tau, 0, \ldots, 0)). \end{aligned}$$

Then $h(0) = 0$ by the induction hypothesis.

Using condition (5.2.4), we will derive a differential equation for h. To simplify notation, we introduce the convention that all expressions of α are to be evaluated at $(t^1, \ldots, t^{k-1}, \tau, 0, \ldots, 0)$ while all expressions of X_j are to be evaluated at $(t^1, \ldots, t^{k-1}, \tau, 0, \ldots, 0, \alpha(t^1, \ldots, t^{k-1}, \tau, 0, \ldots, 0))$.

We have

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}\tau} h(\tau) &= \frac{\partial^2 \alpha}{\partial t^k \partial t^j} - \frac{\partial X_j}{\partial t^k} - \sum_{l=1}^{n} \frac{\partial X_j}{\partial x^l} \frac{\partial \alpha^l}{\partial t^k} \\ &= \frac{\partial}{\partial t^j}(X_k) - \frac{\partial X_j}{\partial t^k} - \sum_{l=1}^{n} \frac{\partial X_j}{\partial x^l} X_k^l && \text{by (5.2.5)} \\ &= \frac{\partial X_k}{\partial t^j} + \sum_{l=1}^{n} \frac{\partial X_k}{\partial x^l} \frac{\partial \alpha^l}{\partial t^j} - \frac{\partial X_j}{\partial t^k} - \sum_{l=1}^{n} \frac{\partial X_j}{\partial x^l} X_k^l && \text{chain rule} \\ &= \frac{\partial X_k}{\partial t^j} + \sum_{l=1}^{n} \frac{\partial X_k}{\partial x^l} \left(h^l(\tau) + X_j^l \right) - \frac{\partial X_j}{\partial t^k} - \sum_{l=1}^{n} \frac{\partial X_j}{\partial x^l} X_k^l && \text{definition of } h \\ &= \sum_{l=1}^{n} \frac{\partial X_k}{\partial x^l} h^l(\tau) && \text{by (5.2.4)} \end{aligned}$$

By uniqueness of solutions to ordinary differential equations, it follows that the only solution with initial condition $h(0) = 0$ is $h(\tau) = 0$. Hence (5.2.3) is true for all $1 \leq j \leq k$. The proof is complete. □

From the proof of Theorem 5.2.1, we obtain:

Corollary 5.2.3. *Let (M, g) be a locally conformally flat manifold and $p \in M$. Let $c \in \mathbb{R}$ and $\omega \in T_p^* M$. Then there exists a function f defined on a neighbourhood U of p such that $\mathrm{e}^{2f} g$ is flat and $f(p) = c$, $\mathrm{d}f(p) = \omega$. Moreover f is unique in the sense that if f_0 is another function defined on a neighbourhood V with $\mathrm{e}^{2f_0} g$ flat and $f_0(p) = c$, $\mathrm{d}f_0(p) = \omega$, then $f = f_0$ on the connected component of $V \cap U$ containing p.*

5.3. Injectivity of the developing map

In this section, we prove Theorem 3.2.4 and also argue why it still applies to $C^{2,1}$-metrics, that is:

PROPOSITION 5.3.1. *Let (M,g) be a closed locally conformally flat Riemannian manifold with $g \in C^{2,1}(M)$ of positive scalar curvature. Then the developing map $\Phi\colon \tilde{M} \to S^n$ is injective.*

As noted in Section 3.2, we cannot use [SY88, Theorem 4.5] or [SY94, Theorem 3.5], which state that the developing map of a complete locally conformally flat manifold without boundary is injective provided $\mathrm{Scal} \geq 0$, due to ambiguity with the version of the positive mass theorem used to prove these results: Schoen and Yau remark that "for this application it is necessary to extend the positive energy theorems to the case of complete manifolds; that is, assuming that the manifold has an asymptotically flat end and other ends which are merely complete. This extension will be carried out in a future work" [SY88, p. 65]. To the author's knowledge, such a generalization of the positive mass theorem is widely believed to be true but no such extension has yet been published.

Since we do not want to rely on this extension to be true, we build our argumentation on [SY88, Proposition 3.3] in dimension $n \geq 4$ and Witten's version of the positive mass theorem in dimension three. See also Appendix A of [CH06], where the same problem occured, as well as the works of Lohkamp [LOH06], [LOH15] who claims to have proven the positive mass theorem in every dimension and without additional topological assumptions, but has not published a complete proof yet. However, his results seem quite promising.

To prove Proposition 5.3.1 for $n \geq 4$ we argue along the lines of the proof of Theorem VI.3.1 of [SY94] (or Theorem 3.1 of [SY88], resp.). Their statement is as follows:

THEOREM 5.3.2. *Let (M,g) be a complete n-dimensional Riemannian manifold and $\Phi\colon M \to S^n$ be a conformal map. Assume that for $n \geq 5$, $\mathrm{Scal}(g) \geq -C$ and for $n = 3, 4$, $|\mathrm{Scal}(g)| \leq C$. If*

$$d(M) < \frac{(n-2)^2}{n},$$

then Φ is a conformal diffeomorphism of M onto $\Phi(M)$.

Here, $d(M)$ is a constant which – for compact M – only depends on the conformal structure [SY94, Proposition VI.2.6], see Definition 5.3.7 below.

REMARK 5.3.3. The condition $d(M) < \frac{(n-2)^2}{n}$ is fulfilled provided $n \geq 4$ and $\mathrm{Scal} > 0$, see [SY88, Proposition 3.3 (i)]. If $n \geq 7$ then $\mathrm{Scal} \geq 0$ is enough to ensure $d(M) < \frac{(n-2)^2}{n}$ [SY88, Proposition 3.3 (iii)]; in this case Proposition 5.3.1 holds for manifolds with nonnegative scalar curvature and Proposition 3.2.1 is valid for manifolds with nonnegative relative Yamabe invariant.

PROOF (PROPOSITION 5.3.1). We first consider the case $n = 3$: In this case every orientable manifold is spin, hence we may replace M by its orientation covering, if necessary, and assume that M is spin. Therefore Witten's proof of the positive mass theorem [WIT81] applies to M and implies that [SY88, Theorem 4.5] holds. In fact, Witten's version of the positive mass theorem extends to the framework needed in the proof of [SY88, Theorem 4.5] and is valid for C^2-metrics, see the appendix of [CH06], the references therein and, for example, [GT14] for a version of the positive mass theorem with low regularity assumptions.

As remarked above, if $n \geq 4$, we have $d(M) < \frac{(n-2)^2}{n}$ by [SY88, Proposition 3.3 (i)] because $\mathrm{Scal} > 0$. Hence Proposition 5.3.1 follows from Theorem 5.3.2 provided the latter is applicable to $C^{2,1}$-metrics which we will discuss now. □

PROOF (THEOREM 5.3.2, FOLLOWING [SY94]). We have to show that Φ is injective. Let $P \in M$. Up to scaling, we may assume $|\Phi'|(P) = 1$, where $|\Phi'|^2$ is the conformal factor of Φ, i.e. $\Phi^* g_{S^n} = |\Phi'|^2 g$.

Let G be the minimal positive Green's function for the conformal Laplacian

$$L := -\Delta + \frac{n-2}{4(n-1)} \mathrm{Scal}$$

of (M, g) with pole P. Existence of such follows from the existence of the conformal map Φ (see [SY94, Proposition VI.2.4]).

Note that – as g is only $C^{2,1}$ – G is not smooth on $M \setminus \{P\}$. However, from $LG = 0$ on $M \setminus \{P\}$ and elliptic regularity, it follows that $G \in C^{2,\alpha}_{\mathrm{loc}}(M \setminus \{P\})$.

REMARK 5.3.4. In the presence of a smooth conformally equivalent background metric g', one can also see this from the transformation formula

$$G(u^{\frac{4}{n-2}} g') = \frac{1}{u(P) \cdot u} G(g'),$$

see e.g. [HAB00, Lemma 2.2.7].

Let G_0 be the Green's function for the conformal Laplacian L_0 of S^n with pole $y = \Phi(P)$ and define

$$\overline{G} := |\Phi'|^{\frac{n-2}{2}} G_0 \circ \Phi.$$

From the transformation law of the conformal Laplacian L, it follows

$$L\overline{G} = \sum_{Q \in \Phi^{-1}(\{y\})} |\Phi'(Q)|^{\frac{n+2}{n}} \delta_Q,$$

thus it is enough to show $G = \overline{G}$.

Our strategy is to consider $v := G/\overline{G}$. In order to show that $v = 1$, we use various estimates on v and its derivatives. As we wish to use Bochner's formula, we first check that v is at least C^3.

LEMMA 5.3.5 (Lemma VI.3.2 of [SY94]). *The function v is a positive harmonic function with respect to the flat metric $\overline{g} := \overline{G}^{p-2}g$, where $p = 2n/(n-2)$. Writing $v(x) = 1 + h(x)$, then $h \in C^3$ and we have in normal coordinates centred at P:*

$$h(x) = \mathcal{O}(|x|^{n-2}) \quad \textit{and} \quad |\nabla h|_g = \mathcal{O}(|x|^{n-3}).$$

PROOF. Flatness of $\overline{g}$ follows as we can write

$$\overline{g} = \Phi^* \left(\pi^* g_{\mathbb{R}^n} \right),$$

where π is a stereographic projection from y. From the conformal invariance of the conformal Laplacian, we have

$$-\Delta_{\overline{g}} v = L_{\overline{g}} v = L_g G = 0$$

on $M \setminus \{P\}$. As G was minimal, by construction (see [SY94, Proposition VI.2.4]), $G \leq \bar{G}$, so $0 < v \leq 1$. Therefore v is harmonic with respect to $\overline{g}$ on the whole of M. As $\overline{g}$ is a $C^{2,\alpha}$-metric, we observe $v \in C^3(M)$ by elliptic regularity.

By changing the metric locally to a conformally equivalent one, we may assume $\mathrm{Scal}(g) \geq 0$ (this will not be necessary in our applications as we always have $\mathrm{Scal} \geq 0$ by assumption). Then the maximum principle is applicable to L and implies $\overline{G}(x) \leq (1+\varepsilon)G(x)$ for any $\varepsilon > 0$ and x sufficiently close to P. Hence

$$v(P) = \lim_{x \to 0} \frac{G(x)}{\overline{G}(x)} = 1.$$

We may thus write $v(x) = 1 + h(x)$, where $h = \mathcal{O}(|x|)$ is C^3.

In a small punctured neighbourhood of P, we have

$$L(hG) = L(\overline{G} - G) = L\overline{G} - LG = 0.$$

Since $G = \mathcal{O}(|x|^{2-n})$, we have $hG = \mathcal{O}(|x|^{3-n})$, so P is a removable singularity of hG. We conclude $h = \mathcal{O}(1/|x|^{2-n}) = \mathcal{O}(|x|^{n-2})$ and $|\nabla h|_g = \mathcal{O}(|x|^{n-3})$. □

REMARK 5.3.6. We have seen that v is at least C^3. By going over to harmonic coordinates with respect to $\bar{g}$, we could even assume that v is smooth.

The rest of the proof of Theorem 5.3.2 follows as presented in [SY94]. For convenience of the reader, we give a sketch of the main steps, but do not provide the computations. Let us first define the invariant $d(M)$ occurring in Theorem 5.3.2:

DEFINITION 5.3.7. Let M be a locally conformally flat manifold, $\tilde{M}$ its universal covering and $\Phi\colon \tilde{M} \to S^n$ be its developing map. As Φ is unique up to a conformal transformation of S^n, we obtain the *holonomy representation* $\rho\colon \pi_1(M) \to M(S^n)$

and the *holonomy covering* $\check{M} := \tilde{M}/\ker(\rho)$. Let G be a minimal positive Green's function on $\check{M}$ with arbitrary pole P (existence of such again follows from [SY94, Proposition VI.2.4]). Then we define

$$d(M) := \frac{n-2}{2} \inf \Big\{ q > 0 \,\Big|\, \forall U \ni P\colon \int_{\check{M}\setminus U} G^q \,\mathrm{d}V_g < \infty \Big\}. \tag{5.3.1}$$

The proof of Theorem 5.3.2 can now be completed as follows. We write C to denote a finite constant whose exact value may change from line to line.

Using the Bochner formula for v, one sees that for $q = \frac{2(n-2)}{n}$,

$$\overline{\Delta}|\overline{\nabla}v|^q_{\bar{g}} \geq C|\overline{\nabla}v|^{q-2}_{\bar{g}} \left|\overline{\nabla}|\overline{\nabla}v|_{\bar{g}}\right|^2_{\bar{g}},$$

where $\overline{\nabla}$ and $\overline{\Delta}$ denote the gradient and Laplacian with respect to $\bar{g}$. By integration by parts and the Schwarz inequality, this implies

$$\int_M \phi^2 |\overline{\nabla}v|^{q-2}_{\bar{g}} \left|\overline{\nabla}|\overline{\nabla}v|_{\bar{g}}\right|^2_{\bar{g}} \,\mathrm{d}V_{\bar{g}} \leq C \int_M |\nabla\phi|^2_g \bar{G}^q |\nabla v|^q_g \,\mathrm{d}V_g \tag{5.3.2}$$

for any $\phi \in C^\infty_c(M \setminus \{P\})$. By multiplying with a cut-off function and using the expansion $|\nabla v|_g = \mathcal{O}(|x|^{n-3})$ from Lemma 5.3.5, one can then obtain (5.3.2) for all $\phi \in C^\infty_c(M)$.

For ρ sufficiently large, we apply this to a function $\phi \in C^\infty_c(M)$ with $\phi = 1$ on $B_\rho(p)$ and $\phi = 0$ on $M \setminus B_{2\rho}(p)$, $0 \leq \phi \leq 1$ and $|\nabla\phi|_g \leq 2\rho^{-1}$ (all balls are with respect to the metric g). A computation involving [SY88, Lemma VI.3.4] shows that (5.3.2) applied to such ϕ implies

$$\int_{B_\rho(P)} |\overline{\nabla}v|^{q-2}_{\bar{g}} \left|\overline{\nabla}|\overline{\nabla}v|_{\bar{g}}\right|^2_{\bar{g}} \,\mathrm{d}V_{\bar{g}} \leq C\rho^{-2} \int_{B_{4\rho(P)}\setminus B_{\rho/2}(P)} G^q \left(1 + |\nabla \log \overline{G}|^2_g\right) \mathrm{d}V_g,$$

and the right hand side can be bounded by

$$C\rho^{-2} \int_{M\setminus B_{\rho/4}} G^{q_1} \,\mathrm{d}V_g, \tag{5.3.3}$$

where $q_1 = q$ for $n \neq 4$ and $q_1 \in (1/3, 1)$ for $n = 4$.

Now, by assumption, $\frac{2}{n-2} d(M) < \frac{2(n-2)}{n} = q$, hence there exists a sequence $q_i \searrow \frac{2}{n-2} d(M)$ with $q_i < q$. By definition of $d(M)$ in (5.3.1), we have

$$\int_{M\setminus B_1(P)} G^{q_i} \,\mathrm{d}V_g < \infty.$$

Using [SY88, Lemma VI.3.3], we arrive at

$$\int_{M\setminus B_1(P)} G^q \,\mathrm{d}V_g \leq C \int_{M\setminus B_1(P)} G^{q_i} \,\mathrm{d}V_g < \infty.$$

Combining this with the estimate (5.3.3) and letting $\rho \to \infty$, we see that

$$\int_M |\overline{\nabla} v|_{\overline{g}}^{q-2} \left|\overline{\nabla}|\overline{\nabla} v|_{\overline{g}}\right|_{\overline{g}}^2 \, \mathrm{d}V_{\overline{g}} = 0,$$

thus $|\overline{\nabla} v|_{\overline{g}}$ is constant. As $|\overline{\nabla} v|_{\overline{g}}(P) = 0$, v is constant and thus $v = v(P) = 1$. This proves the statement. □

List of frequently used symbols

Balls and spheres

Symbol	**Definition**	**Description**
$D_\rho(p)$	$\{x \in S^n \mid d^{S^n}(x,p) < \rho\}$	geodesic ball of radius ρ around p in S^n
D_ρ	$D_\rho(p)$	geodesic ball in S^n when center is irrelevant or understood
$\Sigma_\rho(p)$	$\partial D_\rho(p)$	geodesic sphere of radius ρ around p in S^n
Σ_ρ	$\Sigma_\rho(p)$	geodesic sphere in S^n when center is irrelevant or understood
S^n_+	$\Sigma_{\frac{\pi}{2}}$	hemisphere
$B_r(p)$	$\{x \in \mathbb{R}^n \mid \lvert x-p\rvert < r\}$	ball of radius r around p in $\mathbb{R}^n$
B_r	$B_r(p)$	ball of radius r in $\mathbb{R}^n$ when center is irrelevant or understood
B^n	B_1	unit ball in $\mathbb{R}^n$
$S_r(p)$	$\partial B_r(p)$	sphere of radius r around p in $\mathbb{R}^n$
S_r	$S_r(p)$	sphere of radius r in $\mathbb{R}^n$ when center is irrelevant or understood
S^{n-1}	S_1	standard unit sphere

Curvature tensors

Symbol	**Definition**	**Description**
R	$R(X,Y) = \nabla^2_{X,Y} - \nabla^2_{Y,X}$	Riemann curvature tensor
Ric	$\operatorname{trace}(\xi \mapsto R(\xi,\cdot)\cdot)$	Ricci tensor
Scal	$\operatorname{trace}\operatorname{Ric}$	scalar curvature
sec	–	sectional curvature
W	see Equation (2.2.1)	Weyl curvature tensor
S	see Equation (2.2.2)	Schouten tensor
C	see Equation (2.2.4)	Cotton tensor

Geometry of the boundary

Symbol	**Definition**	**Description**
ν and η	–	inner and outer unit normal
II	$\mathrm{II}(X,Y)=\langle\nabla_X Y,\nu\rangle$	scalar second fundamental form
H	$\frac{1}{n-1}\operatorname{trace}\mathrm{II}$	mean curvature
H_ρ	$\cot(\rho)$	mean curvature of Σ_ρ
H^r	r^{-1}	mean curvature of S_r

Conformal geometry

Symbol	**Definition**	**Description**
π_p	see Definition 2.3.1	stereographic projection from p
$M(\Xi)$	–	Möbius transformations of $\Xi\in\{S^n,\mathbb{R}^n,B^n\}$
$C=C(\varepsilon_i,p_i,\Lambda)$	$\Phi(\tilde{M})$	image of the developing map
Λ	$\partial\Phi(\tilde{M})\setminus\Phi(\partial\tilde{M})$	limit set
$Y(M,[g])$	see Equation (2.6.1)	Yamabe invariant
$Y(M,\partial M,[g])$	see Equation (2.6.2)	relative Yamabe invariant
L	$-\frac{4(n-1)}{n-2}\Delta+\mathrm{Scal}$	conformal Laplacian

Miscellaneous

Symbol	**Definition**	**Description**
$g_{\mathbb{R}^n}, g_{S^n}, g_{S^n_+}$	–	standard metric on $\mathbb{R}^n$, S^n or S^n_+
G	–	minimal positive Green's function
$\mathrm{d}V_g$	$\sqrt{\det(g)}\lvert\mathrm{d}x^{\{1,\dots,n\}}\rvert$	Riemannian volume element or density
$\mathrm{int}(M)$	$M\setminus\partial M$	interior of M
$\hat{M}$	$M\underset{\partial M}{\cup}(-M)$	double of of M
$\tilde{M}$	–	universal covering of M
$\mathscr{L}$	–	Lie derivative
$\pi_0(M)$	–	set of path components
$\pi_1(M)$	–	fundamental group
$H^m_{\mathrm{dR}}(M)$	–	m-th de Rham cohomology group

List of figures

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Summary

Given a compact, connected Riemannian spin manifold M with boundary, the Min-Oo conjecture states that M is isometric to a hemisphere provided that the scalar curvature is bounded below by $n(n-1)$ and that the boundary is totally geodesic and isometric to a round sphere S^{n-1}. It can be seen as a spherical analogue of the rigidity statement of the positive mass theorem in which conditions on the asymptotic geometry of the manifold are replaced by boundary conditions.

Min-Oo claimed to have proven the result in 1995, but he realised that his argument was incorrect. The conjecture was unresolved until 2011 when Brendle, Marques and Neves were able to give a counterexample in all dimensions $n \geq 3$; while the conjecture is true in dimension two. They constructed a metric on the hemisphere which not only satisfies all conditions of Min-Oo's conjecture, but also agrees with the standard metric near the boundary, hence the conjecture is false even under stronger (local) boundary conditions and it requires additional assumptions on the geometry on the interior to make a statement like Min-Oo's conjecture true.

Our starting point is a rigidity result obtained by F. Hang and X. Wang, who showed that the Min-Oo conjecture is true for metrics on the hemisphere which are conformally equivalent to the standard metric. The drawback here is that we already have to assume that the manifold in consideration is diffeomorphic to a hemisphere in order to make sense of this prerequisite. Hence we cannot see any influence of the geometry on the topology or differentiable structure of M. Motivated by partial results in lower dimensions obtained by S. Raulot, we turned our attention to *locally conformally flat manifolds*, that is, manifolds which are not globally conformally equivalent to a hemisphere but locally look like a conformal deformation of the latter.

Our main result is a rigidity result for geodesic balls in a hemisphere which generalises the results by Hang and Wang and Raulot. It implies that Min-Oo's conjecture is true provided the manifold is locally conformally flat. Furthermore, we find that weaker conditions on the geometry of the boundary than in the original statement of the conjecture are sufficient for our result:

THEOREM. *Let (M^n, g), $n \geq 3$, be a compact connected locally conformally flat Riemannian manifold with boundary, $0 < \rho \leq \frac{\pi}{2}$. Assume that*

(i) $\mathrm{Scal}(g) \geq n(n-1)$ *everywhere,*

(ii) ∂M *is umbilic with mean curvature* $H(g) \geq \cot(\rho)$ *and every connected component is isometric to a round sphere of radius* $\sin(\rho)$.

Then (M, g) is isometric to a closed geodesic ball of radius ρ in S^n equipped with the standard metric.

Our result is proved using results by Schoen and Yau concerning the injectivity of the so-called *developing map*, a conformal immersion from the universal covering of M to S^n obtained from the locally conformally flat structure. This allows us to model the universal covering of M on the sphere. Then, analytical considerations of the conformal scalar and mean curvature equation together with results by Hang and Wang, which we extend to hold in a setting suitable for our investigations, imply the result.

From our main result above, we conclude a rigidity result for more general domains in a hemisphere by applying it to a manifold obtained by gluing a part of the sphere to the manifold in consideration. Also, the technique of the proof can be used with slight modifications to obtain an analogous rigidity result for balls in Euclidean space.

Last but not least, we try to weaken the assumption on the local conformal flatness of M. As Min-Oo's conjecture is incorrect, this is rather delicate. Nevertheless, we are able to extend the result above to manifolds of dimension $n \geq 4$ which are not necessarily locally conformally flat, but *locally conformally symmetric.* To that end, we establish a result similar to the theorem above for manifolds with parallel Ricci tensor, but with slightly stronger boundary conditions.