Quiver Modulations and Potentials

Dissertation

zur
Erlangung des Doktorgrades (Dr. rer. nat.)
der
Mathematisch-Naturwissenschaftlichen Fakultät
der
Rheinischen Friedrich-Wilhelms-Universität Bonn

vorgelegt von
Jan Geuenich
aus Köln

Bonn 2016

Angefertigt mit Genehmigung der Mathematisch-Naturwissenschaftlichen Fakultät
der Rheinischen Friedrich-Wilhelms-Universität Bonn

Erstgutachter:	Prof. Dr. Jan Schröer
Zweitgutachter:	Prof. Dr. Igor Burban
Tag der Promotion:	16.03.2017
Erscheinungsjahr:	2017

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Acknowledgments

I thank Jan Schröer for many helpful discussions. Moreover, I am grateful for the hospitality of Christof Geiß and Daniel Labardini Fragoso during a stay in Mexico City. Special thanks go to Daniel for the collaboration whose outcome is presented in the last two chapters.

1 Introduction

1.1 Motivation

Cluster algebras are a recent and active field of research. They were introduced by Fomin and Zelevinsky [FZ02] in 2002 as a new approach to study the duals of Lusztig's canonical bases for quantum groups from a combinatorial point of view.

Cluster theory was soon revealed to have intimate connections to numerous mathematical disciplines: among others, Kac-Moody algebras, root systems, representations of quivers (with potential), Teichmüller theory, preprojective algebras, and Calabi-Yau categories.

A groundbreaking discovery was that acyclic skew-symmetric cluster algebras admit a categorification by *cluster categories* [BMR+06]. Another more general categorical model was developed in [DWZ08; DWZ10] and uses representations of quivers with potential.

One important feature shared by Kac-Moody Lie algebras and cluster algebras is that both can be defined in terms of generalized Cartan matrices. In this relationship, cluster algebras with skew-symmetric exchange matrices have symmetric Cartan counterparts.

The symmetric case has been closely investigated in the recent past and is comparatively well-understood. In contrast, there are many open problems in the symmetrizable situation. Answering these problems and lifting constructions from the symmetric to the symmetrizable setting is an area of current research (see e.g. [GLS16a; GLS16b; GLS16c]).

In the representation theory of finite-dimensional algebras symmetric generalized Cartan matrices are linked to quiver representations, while the symmetrizable case is covered by representations of *modulations* for weighted quivers.

Gabriel initiated the investigation of modulations (or *species*) in [Gab73]. In a sequence of articles [DR74; DR75; DR76] Dlab and Ringel developed the theory further. Their most striking result is the finite-type classification of modulations, which parallels the classification of semi-simple Lie algebras.

Finite-Dimensional Algebras

Fix a field K. We briefly revise how finite-dimensional algebras over K lead to modulations.

The module category $\mathrm{mod}(\Lambda)$ of a finite-dimensional algebra Λ over K is equivalent to the module category $\mathrm{mod}(\Lambda_{\mathrm{basic}})$ of the basic algebra $\Lambda_{\mathrm{basic}} = \mathrm{End}_\Lambda(P)^{\mathrm{op}}$ where P is any multiplicity-free projective generator of $\mathrm{mod}(\Lambda)$. This is why it is common practice to focus in representation theory on finite-dimensional basic algebras.

For every finite-dimensional basic algebra Λ over K its reduction $\Lambda_{\mathrm{red}} = \Lambda/\mathrm{rad}(\Lambda)$ is semi-simple and by the Artin-Wedderburn theorem even a product of division algebras.

Let $A = \mathrm{Ext}^1_\Lambda(\Lambda_{\mathrm{red}}, \Lambda_{\mathrm{red}})$ considered as a bimodule over $R = \mathrm{End}_\Lambda(\Lambda_{\mathrm{red}})^{\mathrm{op}}$. Naturally associated with Λ we therefore have the tensor algebra $R\langle A\rangle$ of A.

If K is a perfect field, Benson [Ben98] motivated by [Gab73; Gab80] proves $\Lambda \cong R\langle A\rangle/J$ for an admissible ideal J of $R\langle A\rangle$. The algebra Λ is hereditary if and only if $J = 0$.

Writing $1 = e_1 + \cdots + e_n$ for a complete set of primitive orthogonal idempotents $e_1, \ldots, e_n$ in Λ, the modules $S_i = \Lambda e_i/\mathrm{rad}(\Lambda e_i)$ with $1 \leq i \leq n$ form an up to isomorphism complete set of simple Λ-modules. There is an induced factorization $R = R_1 \times \cdots \times R_n$ of rings and a decomposition $A = \bigoplus_{i,j} {}_jA_i$ of bimodules where $R_i = \mathrm{End}_\Lambda(S_i)^{\mathrm{op}}$ and ${}_jA_i = \mathrm{Ext}^1_\Lambda(S_i, S_j)$.

If K is an algebraically closed field, then $R_i \cong K$ for all i and the tensor algebra $R\langle A\rangle$ is the path algebra KQ of a quiver Q with $\dim_K\big({}_jA_i\big)$ arrows $j \leftarrow i$.

Modulations

Let Q be a *weighted quiver*, i.e. a finite quiver Q equipped with a function $Q_0 \to \mathbb{N}_+, i \mapsto d_i$.

A *modulation* for Q is a family $(R_i, A_a)_{i,a}$ of connected K-algebras R_i with $\dim_K(R_i) = d_i$ and non-zero $R_j \otimes_K R_i^{\mathrm{op}}$-modules A_a indexed by the vertices $i \in Q_0$ and arrows $j \xleftarrow{a} i \in Q_1$ such that both ${}_{R_j}(A_a)$ and $(A_a)_{R_i}$ are free of finite rank. Let ${}_jA_i = \bigoplus_{j \xleftarrow{a} i} A_a$.

The modulation is *minimal* if $\dim_K(A_a) = \mathrm{lcm}(d_j, d_i)$ for all arrows $j \xleftarrow{a} i$.

The situation where all R_i are division algebras corresponds to what Gabriel called a species in [Gab73] and what we shall call a *division modulation*.

The previous paragraph pointed out that every finite-dimensional K-algebra gives rise to a division modulation.

Path Algebras of Modulations

Every modulation $\mathcal{H} = (R_i, A_a)_{i,a}$ for a weighted quiver Q defines a *path algebra*: the tensor algebra $R\langle A\rangle$ of the bimodule $A = \bigoplus_a A_a$ over the *ground ring* $R = \prod_i R_i$.

Work of Roganov [Rog75] and Iwanaga [Iwa80] shows that the projectivity of ${}_RA$ and A_R implies that the path algebra $R\langle A\rangle$ is $(n+1)$-Gorenstein if the ground ring R is n-Gorenstein. In this case, a module over $R\langle A\rangle$ has finite homological dimension if and only if it has finite homological dimension when considered as a module over R.

Geiß, Leclerc, and Schröer exploited this fact recently in [GLS16a] for self-injective ground rings R where $n = 0$. Another manifestation of this result is the famous slogan "Path algebras of quivers are hereditary."

Representations of Modulations

A *representation* of the modulation $\mathcal{H}$ is a module M over the path algebra $R\langle A\rangle$. It can be identified with a pair $({}_RM, {}_AM)$ consisting of an R-module ${}_RM = M$ and ${}_AM \in \mathrm{Rep}(A, M)$

for the space of A-representations

$$\operatorname{Rep}(A, M) \;=\; \operatorname{Hom}_R(A \otimes_R M, M) \;=\; \bigoplus_{i,j} \operatorname{Hom}_{R_j}({}_jA_i \otimes_{R_i} e_iM, e_jM)\,.$$

Following [GLS16a] an R-module M is *locally free* if each e_iM is a free module over R_i. Finite-dimensional locally free R-modules M have a rank vector $\underline{\operatorname{rank}}(M) = (\operatorname{rank}_{R_i}(e_iM))_i$.

Preprojective Algebras

Assume that R carries the structure $R \xrightarrow{\varphi} K$ of a symmetric algebra. Let $A^* = \operatorname{Hom}_K(A, K)$. Denote by ${}_AM \mapsto {}_AM^\vee$ the isomorphism $\operatorname{Rep}(A, M) \to \operatorname{Hom}_R(M, A^* \otimes_R M)$ induced by φ.

A *compatible A-double representation* $(M, {}_AM, {}_{A^*}M)$ consists of an R-module M and representations ${}_AM \in \operatorname{Rep}(A, M)$ and ${}_{A^*}M \in \operatorname{Rep}(A^*, M)$ with ${}_{A^*}M \circ {}_AM^\vee = {}_AM \circ {}_{A^*}M^\vee$.

Compatible A-double representations are the same as modules over the *preprojective algebra* $\Pi = R\langle A \oplus A^*\rangle/\langle\rho\rangle$ where $\rho \in (A \otimes A^*) \oplus (A^* \otimes A)$ is the *preprojective relation*.

More generally, the *deformed preprojective algebra* $\Pi^\lambda = R\langle A \oplus A^*\rangle/\langle\rho - \lambda\rangle$ is defined for every λ in the center of R. All this builds on [GP79; Rie80; DR80; CH98; GLS16a].

If Q is acyclic, the path algebra $H = R\langle A\rangle$ is finite-dimensional and Baer, Geigle, Lenzing's [BGL87] alternative definition of the preprojective algebra Π as the tensor algebra of the H-bimodule $\operatorname{Ext}^1_H(H^*, H) \cong \operatorname{Hom}_H(H, \tau^-(H))$ gives a conceptual explanation for the significance of Π in the representation theory of finite-dimensional algebras.

Gabriel's Theorem

Gabriel's Theorem [Gab72] classifies all representation-finite acyclic quivers. Namely, he shows that an acyclic quiver is representation-finite if and only if it is a finite union of Dynkin quivers. Right after proving this, Gabriel [Gab73] introduced division modulations for weighted quivers (or *valued graphs*) to provide the framework for a generalization of his celebrated classification to the non-simply laced situation.

Dlab and Ringel [DR75] were first in giving a proof of Gabriel's Theorem in this more general setting: A division modulation $\mathcal{H}$ for a weighted acyclic quiver Q is representation-finite if and only if $\mathcal{H}$ is minimal and Q a union of Dynkin quivers. In this case, $M \mapsto \underline{\operatorname{rank}}(M)$ establishes a bijection between the isomorphism classes of indecomposable representations in $\operatorname{rep}(\mathcal{H})$ and the set of positive roots of the quadratic form q_Q defined by Q.

Coxeter Functors

To give a satisfactory explanation for the appearance of finite simply-laced root systems in Gabriel's Theorem, Bernstein, Gelfand, and Ponomarev [BGP73] described for acyclic Q an adjoint pair (C^-, C^+) of endofunctors of $\operatorname{rep}(Q)$. The *Coxeter functor* C^+ (resp. C^-) acts on rank vectors of indecomposable non-projective (resp. non-injective) representations as the Coxeter transformation (resp. its inverse) of the root lattice defined by Q.

Again, it were Dlab and Ringel [DR74; DR76] who constructed similar endofunctors of $\operatorname{rep}(\mathcal{H})$ for division modulations $\mathcal{H}$ of weighted acyclic quivers.

Inspired by an idea of Riedtmann, Gabriel [Gab80] makes the remarkable observation that there are isomorphisms $C^+T \cong \tau^+$ and $C^-T \cong \tau^-$ where $\tau^+ = (-)^* \circ \mathrm{Tr}$ and $\tau^- = \mathrm{Tr} \circ (-)^*$ are the Auslander-Reiten translations of $\operatorname{rep}(Q)$ and T is a "twist" functor.

GLS Modulations

Let Q be a weighted acyclic quiver and let $f_{ij} = d_j/\gcd(d_i, d_j)$ for $i, j \in Q_0$.

Geiß, Leclerc, and Schröer [GLS16a] thoroughly examined the modulation $\mathcal{H} = (R_i, A_a)_{i,a}$ for Q where for $j \xleftarrow{a} i$

$$R_i \;=\; K[\varepsilon_i] \Big/ \Big(\varepsilon_i^{d_i}\Big), \qquad\qquad A_a \;=\; K[\varepsilon_j, \varepsilon_i] \Big/ \Big(\varepsilon_j^{f_{ij}} - \varepsilon_i^{f_{ji}},\, \varepsilon_j^{d_j},\, \varepsilon_i^{d_i}\Big).$$

For constant weights $d_i = c$ the path algebra of $\mathcal{H}$ is the path algebra of the quiver Q over the truncated polynomial ring $K[\varepsilon]/(\varepsilon^c)$. In particular, $\operatorname{rep}(\mathcal{H}) = \operatorname{rep}(Q)$ in case $c = 1$.

The arguments in [GLS16a] implicitly but consistently use that the ground ring R carries the structure $R \xrightarrow{\varphi} K$ of a symmetric algebra in the sense of [Nak39] where φ is the linear form dual to $\sum_i \varepsilon_i^{d_i - 1}$ with respect to the basis formed by the ε_i^r.

Geiß, Leclerc, and Schröer's notable insight is that, for a suitable generalization of the Coxeter functors $C^\pm$, the Brenner-Butler-Gabriel isomorphisms $C^\pm T \cong \tau^\pm$ are valid on the full subcategory of $\operatorname{rep}(\mathcal{H})$ consisting of the locally free representations.

Moreover, the functors $\tau^\pm$ leave the subcategory of locally free representations M that are *rigid* (i.e. $\operatorname{Ext}^1_{\mathcal{H}}(M, M) = 0$) invariant.

Using this, they were able to prove a much more general version of Gabriel's Theorem: The number of isomorphism classes of indecomposable locally free rigid representations in $\operatorname{rep}(\mathcal{H})$ is finite if and only if Q is a union of Dynkin quivers. In this case, $M \mapsto \underline{\operatorname{rank}}(M)$ yields a bijection between these isomorphism classes and the positive roots of q_Q.

Cluster Algebras

Cluster algebras $\mathcal{A}_Q$ are subalgebras of the field $\mathbb{Q}(x_1, \ldots, x_n)$ of rational functions. Their generators, the *cluster variables*, are grouped into overlapping *clusters*. All clusters are obtained from the initial cluster $(x_1, \ldots, x_n)$ by an iterative process called *mutation*. Cluster mutation is governed by a weighted quiver Q associated with the initial cluster.

It was proved in [FZ03] that a cluster algebra $\mathcal{A}_Q$ has only a finite number of cluster variables if and only if Q is mutation-equivalent to a union of Dynkin quivers.

Potentials and Caldero-Chapoton Formula

String theorists [Sei95; DM96] associated with certain supersymmetric gauge theories quivers with *superpotential*, which often describe the endomorphism algebra $\operatorname{End}(\bigoplus_i E_i)$ of an exceptional collection $E_1, \ldots, E_n$ in a triangulated category [BP01; AF06; Bri05; BP06].

Loosely based on these ideas, Derksen, Weyman, and Zelevinsky [DWZ08] developed a mutation theory for quivers Q with *potential* (QPs) and their representations.

In doing so, they provided a categorical model for cluster mutation [DWZ10]. Namely, the k-th variable in a non-initial cluster of $\mathcal{A}_Q$ obtained via mutation at a sequence $\mathbf{i}$ can be computed as

$$x_{\mathbf{i},k} \;=\; \prod_{j=1}^{n} x_j^{-\langle S_j,\, M_{\mathbf{i},k}\rangle} \sum_{\alpha\in\mathbb{N}^n} \chi(\mathrm{Gr}_\alpha(M_{\mathbf{i},k}))\,\mathbf{x}^{B\alpha}\,.$$

In the formula, $M_{\mathbf{i},k}$ is a representation of a *non-degenerate* QP (Q, W) over the complex numbers $\mathbb{C}$ that is obtained from the negative simple representation S_{-k} by mutation at $\mathbf{i}$. Furthermore, $\langle -, -\rangle$ is the 1-truncated Euler form, B the skew-symmetric matrix, and Gr_α the quiver Grassmannian of α-dimensional subrepresentations associated with (Q, W).

Originally, Caldero and Chapoton [CC06] discovered this formula for Dynkin quivers.

Representations of (Q, W) are by definition modules over the *Jacobian algebra* $\mathcal{J}(W)$, the quotient of the completed path algebra of Q by the cyclic derivatives $\partial_\xi(W)$.

Cluster-Tilting Subcategories

Let Π be the preprojective algebra and $\mathcal{W}_Q$ the Weyl group of a finite acyclic quiver Q.

For every $w \in \mathcal{W}_Q$ Buan, Iyama, Reiten, and Scott [BIRS09] described a subcategory $\mathcal{C}_w$ of the category of finite-dimensional nilpotent Π-modules. This category $\mathcal{C}_w$ is Frobenius and its stable category is 2-Calabi-Yau. Moreover, $\mathcal{C}_w$ can be regarded as a categorification of a cluster algebra constructed by Geiß, Leclerc, and Schröer in [GLS11].

Each reduced expression $\underline{s} = (s_1, \ldots, s_\ell)$ for w determines a maximal rigid Π-module $T(\underline{s})$, which induces a *cluster-tilting subcategory* of $\mathcal{C}_w$. The 2-*CY tilted* algebra $\mathrm{End}_\Pi(T(\underline{s}))$ is a strongly quasi-hereditary Jacobian algebra according to [BIRS11; GLS11].

Surface-Type Cluster Algebras

A cluster algebra $\mathcal{A}_Q$ has *finite mutation type* if the number of (isomorphism classes of) weighted quivers that are mutation-equivalent to Q is finite. It has *surface type* if Q is the weighted adjacency quiver of a triangulation of a weighted orbifold. Fomin, Shapiro, and Thurston [FST08] began with the investigation of surface-type cluster algebras in the skew-symmetric case and Felikson, Shapiro, and Tumarkin [FST12a] extended their results to the skew-symmetrizable setting.

The importance of surface-type cluster algebras is revealed by Felikson, Shapiro, and Tumarkin's [FST12a; FST12b] mutation-type classification: If Q is a connected weighted quiver with at least three vertices and is not mutation-equivalent to one of 18 exceptions, the cluster algebra $\mathcal{A}_Q$ is of finite mutation type if and only if it is of surface type.

In theoretical particle physics, Cecotti and Vafa [CV13; Cec13] applied this result to classify all “complete” 4d $\mathcal{N} = 2$ supersymmetric gauge theories.

Let $Q(\tau)$ be the adjacency quiver of a triangulation τ of a surface without orbifold points. Assem et al. [ABCP10] (for unpunctured surfaces) and Labardini-Fragoso [Lab09a] (in the general situation) described a potential $W(\tau)$ on $Q(\tau)$. For almost all surfaces it is shown in [Lab09a; CL12; Lab16] that the QPs $(Q(\tau), W(\tau))$ and $(Q(\varsigma), W(\varsigma))$ correspond to each other under mutation if the triangulations τ and ς are related by flipping an arc.

1.2 Results

We sketch the main results chapter by chapter. More detailed introductions and summaries can be found at the beginning of each chapter and its major sections.

As before, let K be a field. Fix a K-modulation $\mathcal{H} = (R_i, A_a)_{i,a}$ for a weighted quiver Q. Let $R = \prod_i R_i$ and $A = \bigoplus_a A_a$.

Chapter 2: Background

For all elements W in the completed path algebra $R\langle\langle A\rangle\rangle$ we define Jacobian algebras $\mathcal{J}(W)$. Generalizing a result of [DWZ08] we have:

Proposition. *Assume that the enveloping algebra $R^{\mathrm{e}} = R \otimes_K R^{\mathrm{op}}$ is basic semi-simple. Every K^I-algebra automorphism f of $R\langle\langle A\rangle\rangle$ induces an isomorphism $\mathcal{J}(W) \to \mathcal{J}(f(W))$.*

An *SP over R* is a pair (A, W) consisting of a finite-dimensional R-bimodule A over K and a *potential* W, i.e. an element in the trace space of $R\langle\langle A\rangle\rangle$. In the joint work [GL16a] the mutation theory for quivers with potential (QPs) was lifted to a special class of SPs:

Generalization. *Assume that the field K contains a d_i-th primitive root of unity and the algebras R_i are intermediate fields of a cyclic Galois extension L/K. Then for every vertex j there is an involution μ_j of the set of reduced-equivalence classes of SPs over R that are 2-acyclic at j, which specializes to QP mutation if all weights d_i are equal to one.*

Parts of the mutation theory of QPs have previously been extended to several special classes of modulations in [Dem10; Ngu12; LZ16; BL16].

Chapter 3: Symmetric Modulations

Let us assume that the ground ring R carries the structure $R \xrightarrow{\varphi} K$ of a symmetric algebra. In this situation, an idea of [CH98] allows to identify the space of A^*-representations on a finite-dimensional locally free R-module M with the dual of the space of A-representations. More precisely, there is a trace pairing that induces an isomorphism of K-vector spaces

$$\operatorname{Rep}(A^*, M) \xrightarrow{\cong} \operatorname{Rep}(A, M)^* .$$

Along this line, it is possible to prove the central result of [GLS16a] for arbitrary *symmetric local modulations*, i.e. whenever all R_i are symmetric local algebras:

Theorem. *Let H be the path algebra of an acyclic symmetric local modulation. Moreover, let $C^{\pm}$ be the Coxeter functors of* $\mathrm{mod}(H)$ *constructed as in [BGP73; BK12; GLS16a].*

There is an H-bimodule Π_1 such that $C^{+}T \cong \mathrm{Hom}_H(\Pi_1, -)$ and $C^{-}T \cong \Pi_1 \otimes_H -$ as endofunctors of $\mathrm{mod}(H)$. *Restricted to the category of locally free H-modules, there are isomorphisms $C^{+}T \cong \tau^{+}$ and $C^{-}T \cong \tau^{-}$ where $\tau^{\pm}$ are the Auslander-Reiten translations.*

As a consequence, Gabriel's Theorem as stated in [GLS16a] for locally free rigid modules also holds in the setting of acyclic symmetric local modulations.

Finally, the following extensions of results of [CH98; BK12; GLS16a] could be of interest:

Proposition. *Let H be the path algebra and Π^{λ} with $\lambda \in \mathrm{Z}(R)$ a deformed preprojective algebra of a symmetric local modulation $(R_i, A_a)_{i,a}$ for a weighted acyclic quiver Q.*

- *For each vertex j in Q one can construct two* reflection functors $\Sigma_j^{\pm}$ *on* $\mathrm{Mod}(\Pi^{\lambda})$. *If $\lambda_j \in R_j^{\times}$, there is $r_j(\lambda) \in \mathrm{Z}(R)$ and $\Sigma_j^{+} \cong \Sigma_j^{-}$ induces an equivalence*

$$\mathrm{Mod}\big(\Pi^{\lambda}\big) \xrightarrow{\quad\simeq\quad} \mathrm{Mod}\big(\Pi^{r_j(\lambda)}\big)\,.$$

  If $\lambda_j = 0$, then $(\Sigma_j^{-}, \Sigma_j^{+})$ is a pair of adjoint endofunctors of $\mathrm{Mod}(\Pi^{\lambda})$.
- *The preprojective algebra $\Pi = \Pi^{0}$ is self-injective if Q is a Dynkin quiver.*
- *The module ${}_H\Pi$ is the direct sum of the "preprojective" modules $\tau^{-p}({}_H H)$ with $p \in \mathbb{N}$.*
- *The "symmetry" $\mathrm{Ext}^1_{\Pi}(M, N) \cong \mathrm{Ext}^1_{\Pi}(N, M)^{*}$ holds for locally free $M, N \in \mathrm{mod}(\Pi)$.*

Comparable extensions of [GLS16a] were independently proposed in [LY15; Kül16].

Chapter 4: Potentials for Cluster-Tilting Subcategories

Let w be an element in the Weyl group $\mathcal{W}_Q$ of a finite acyclic quiver Q. We will show:

Theorem. *For every reduced expression $\underline{s}$ for w the quiver of* $\mathrm{End}_{\Pi}(T(\underline{s}))$ *admits an up to right-equivalence unique non-degenerate potential $W(\underline{s})$.*

Chapter 5: Potentials for Tagged Triangulations

This chapter presents results of [GL16a].

We introduce and study potentials for (tagged) triangulations of *weighted orbifolds* in the sense of [FST12a]. Roughly speaking, a weighted orbifold $\mathbf{\Sigma}$ is a connected compact oriented Riemann surface of genus g with b boundary components, m marked points, and o weighted orbifold points. In this chapter we treat the case where all orbifold points have weight two. Depending on coefficient functions u and z, we associate with each (tagged) triangulation τ of $\mathbf{\Sigma}$ an SP $\mathcal{S}_{u,z}(\tau) = (A(\tau), W_{u,z}(\tau))$.

Generalizing [Lab16] to the weighted situation, the main result is that flipping arcs in tagged triangulations is compatible with the mutation of SPs:

Theorem. *Assume $g > 0$ or $b+m+o \geq 7$. Let ς be obtained from a tagged triangulation τ of $\mathbf{\Sigma}$ by flipping an arc i. Then the SP mutation of $\mathcal{S}_{u,z}(\tau)$ at i is equivalent to $\mathcal{S}_{u,z}(\varsigma)$.*

The important consequence is the non-degeneracy of the SPs $\mathcal{S}_{u,z}(\tau)$ for all tagged triangulations τ of weighted orbifolds $\mathbf{\Sigma}$ that satisfy the condition in the theorem.

For non-closed orbifolds the choice of coefficients will be shown to be not essential:

Proposition. *Assume $b > 0$. Let τ be a (tagged) triangulation of $\mathbf{\Sigma}$. The SPs $\mathcal{S}_{u,z}(\tau)$ and $\mathcal{S}_{u',z'}(\tau)$ are equivalent for all valid coefficient functions u, u' and z, z'.*

Chapter 6: Potentials for Colored Triangulations

The contents of this chapter are drawn from [GL16b].

We continue the investigation of weighted orbifolds $\mathbf{\Sigma}$. Now we impose no restriction on the weights of the orbifold points, but assume that all marked points lie on the boundary.

Following a construction of [AG16], we define chain complexes $C_\bullet(\tau)$ for triangulations τ. A *colored triangulation* is a pair (τ, ξ) where τ is a triangulation of $\mathbf{\Sigma}$ and ξ is a 1-cocycle in the cochain complex $C^\bullet(\tau)$ that is $\mathbb{F}_2$-dual to $C_\bullet(\tau)$.

After describing an SP $\mathcal{S}(\tau,\xi) = (A(\tau,\xi), W(\tau,\xi))$ for each colored triangulation (τ,ξ), we prove the compatibility of flip and mutation:

Theorem. *Assume that (ς, ξ') is obtained from another colored triangulation (τ,ξ) of $\mathbf{\Sigma}$ by flipping an arc i. Then the SP mutation of $\mathcal{S}(\tau,\xi)$ at i is equivalent to $\mathcal{S}(\varsigma,\xi')$.*

A corollary of this result is the non-degeneracy of the potential $W(\tau,\xi)$. On the other hand, we will argue that the property to be non-degenerate essentially determines $W(\tau,\xi)$:

Theorem. *Assume $\mathbf{\Sigma}$ is not a monogon in which all orbifold points have the same weight. Then every non-degenerate potential for $A(\tau,\xi)$ is equivalent to $W(\tau,\xi)$.*

It is often useful to know that Jacobian algebras are finite-dimensional. Thus we prove:

Proposition. *The Jacobian algebra $\mathcal{J}(\tau,\xi)$ defined by $\mathcal{S}(\tau,\xi)$ is finite-dimensional.*

It is also interesting to observe that the isomorphism classes of Jacobian algebras $\mathcal{J}(\tau,\xi)$ corresponding to a fixed triangulation τ are parametrized by a cohomology group:

Theorem. *Let τ be a triangulation of $\mathbf{\Sigma}$. There exists an isomorphism $\mathcal{J}(\tau,\xi) \cong \mathcal{J}(\tau,\xi')$ of K-algebras fixing the vertices pointwise if and only if $[\xi] = [\xi']$ in cohomology.*

Finally, we define a *flip graph* whose vertices are pairs $(\tau, [\xi])$ of triangulations τ of $\mathbf{\Sigma}$ and cohomology classes $[\xi] \in H^1(C^\bullet(\tau))$. This graph is disconnected unless $\mathbf{\Sigma}$ is a disk:

Theorem. *The flip graph of $\mathbf{\Sigma}$ has at least 2^{2g+b-1} connected components.*

1.3 Terminology

Unless it is otherwise specified, all rings are unital, all algebras associative and unital, all ideals two-sided, and all modules left modules.

1.4 Notations

K	an arbitrary field, fixed for this thesis
$(-)^*$	the functor $\mathrm{Hom}_K(-, K)$
$\mathbb{N}$	the set of natural numbers $0, 1, 2, \ldots$
$\mathbb{N}_+$	the set of positive integers $1, 2, 3, \ldots$
$\mathbb{Z}$	the ring of integers
$\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$	the field of rational, real, and complex numbers, respectively
$\mathbb{F}_q$	the finite field with q elements
$\mathrm{Mod}(R)$	the category of modules over the ring R
$\mathrm{mod}(R)$	the category of finite-dimensional modules over the K-algebra R
$\dim(V)$	the dimension of the vector space V
$\mathrm{rank}\,(M)$	the (well-defined) rank of a module M
$\mathrm{rad}(M)$	the radical of a module M
${}_RM$	used to stress the fact that M should be considered as a (left) R-module
M_R	used to stress the fact that M should be considered as a right R-module
$\langle x_1, \ldots, x_n \rangle$	the ideal generated by $x_1, \ldots, x_n$ in a ring that is clear from the context
$\mathrm{Z}(R)$	the center of the ring R
$\mathrm{char}(R)$	the characteristic of the ring R
$\mathrm{Gal}(L/E)$	the Galois group of the Galois extension L/E
$[L : E]$	the degree of the field extension L/E
$\mathrm{Tr}_{L/E}$	the trace in E relative to L of the E-algebra L as in [Bou70, III. §9 no. 3]
$\mathrm{Tr}_R(f)$	the trace of the R-linear endomorphism f in the commutative ring R
δ_P	the Kronecker delta; equal to 1, if P is true, and equal to 0, if P is false

2 Background

In this chapter we lay the foundation for the later parts of this thesis. § 2.1 briefly revises weighted quivers and their relationship to Cartan matrices. Operations like reflection and mutation are presented for modular quivers. § 2.2 introduces tensor algebras and § 2.3 tries to justify the concept of K-modulations. In § 2.4 modulations are treated in detail. We discuss natural constructions like pullback, dual, and double modulations. § 2.5 focuses on cyclic Galois modulations. The concluding § 2.6 is concerned with Jacobian algebras and species with potential (SPs). We describe SP mutation for a special class of modulations.

2.1 Weighted and Modular Quivers

2.1.1 Quivers

Most of the terminology introduced in this subsection is standard.

Definition 2.1.1. Let $\mathcal{K}$ be the *Kronecker category*, i.e. the category with two objects, 0 and 1, and two non-identity morphisms, s and t, both of the form $1 \to 0$. The *category of quivers* $\mathcal{Q}$ is the category of functors from $\mathcal{K}$ to the category of finite sets.

The objects Q of $\mathcal{Q}$ are called *quivers*. We write Q_x for the image of $x \in \{0,1\}$ under Q. The images under Q of the morphisms s and t are again denoted by s and t. In plain words, a quiver Q consists of two finite sets, Q_0 and Q_1, and two functions $s, t : Q_1 \to Q_0$. The elements in Q_0 are called the *vertices* and the elements in Q_1 the *arrows* of Q. An arrow $a \in Q_1$ with $i = s(a)$ and $j = t(a)$ *starts* in i and *ends* in j. This is visualized as

$$j \xleftarrow{a} i \ \in \ Q_1 \,.$$

To state the fact that Q is a quiver whose vertex set is I, we say that Q is an I*-quiver*. A *morphism of* I*-quivers* is a morphism of quivers $Q \xrightarrow{f} Q'$ with $f_0 = \mathrm{id}_I$.

A quiver Q' is a *subquiver* of another quiver Q if there is a monomorphism $Q' \xhookrightarrow{f} Q$ where f_x are the inclusions of subsets $Q'_x \subseteq Q_x$ for $x \in \{0,1\}$. We then write $Q' \subseteq Q$.

A *path* $a_n \cdots a_1$ in Q is a tuple $\big(i_n \xleftarrow{a_n} i_{n-1}, \cdots, i_2 \xleftarrow{a_2} i_1, i_1 \xleftarrow{a_1} i_0\big) \in Q_1^n$ with $n \in \mathbb{N}$ and the convention $Q_1^0 = \{i \xleftarrow{e_i} i \,|\, i \in Q_0\}$. We say that the path $p = a_n \cdots a_1$ *starts* in $s(p) := i_0$ and *ends* in $t(p) := i_n$. Paths $p \in Q_1^n$ have *length* $\ell(p) := n$.

Set $\mathcal{P}_Q^n(i,j) := \mathcal{P}_Q(i,j) \cap \mathcal{P}_Q^n$ where $\mathcal{P}_Q(i,j)$ is the set of paths in Q starting in i and ending in j and $\mathcal{P}_Q^n$ is the set of length-n paths in Q.

Cyclic paths are paths in $\mathcal{P}_Q^{\circlearrowleft} := \bigcup_{i \in Q_0} \mathcal{P}_Q(i,i)$. If $\mathcal{P}_Q^{\circlearrowleft} \cap \mathcal{P}_Q^n = \varnothing$, the quiver Q is said to be *n-acyclic*. It is *loop-free* if it is 1-acyclic, and *acyclic* if it is n-acyclic for all $n \in \mathbb{N}_+$.

The quiver Q is *n-acyclic at* $i \in Q_0$ and i is an *n-acyclic vertex in* Q if $\mathcal{P}_Q(i,i) \cap \mathcal{P}_Q^n = \varnothing$.

The *path category* $\mathcal{P}_Q$ of Q is the category with object set Q_0 and set of morphisms $i \to j$ given by $\mathcal{P}_Q(i,j)$. The trivial paths e_i are the identities. Composition is defined in the obvious way as concatenation of paths. The assignment $Q \mapsto \mathcal{P}_Q$ canonically extends to a functor from $\mathcal{Q}$ to the category of strict small categories,

The quiver Q is *connected* if $\mathcal{P}_Q$ is a connected category. A subquiver $Q' \subseteq Q$ is *full* if the functor $\mathcal{P}_f$ is full for the canonical inclusion $Q' \overset{f}{\hookrightarrow} Q$.

Notation 2.1.2. For $X \subseteq Q_1$ write $Q - X$ for the Q_0-subquiver of Q with arrow set $Q_1 \setminus X$.

For $X \subseteq Q_0$ write $Q|_X$ for the full subquiver of Q with vertex set X.

For subquivers $Q', Q'' \subseteq Q$ write $Q = Q' \oplus Q''$ if $Q_0 = Q'_0 \cup Q''_0$ and $Q_1 = Q'_1 \,\dot{\cup}\, Q''_1$.

2.1.2 Weighted Quivers

We introduce weighted quivers and describe two operations, *reflection* and *premutation*, modifying a weighted quiver locally and producing another one. The first operation plays a significant role in Chapter 3, whilst the second one is prominently used in Chapters 5 and 6.

Definition 2.1.3. A *weighted quiver* Q is a pair (Q, d^Q) consisting of a quiver Q and a function $Q_0 \xrightarrow{d^Q} \mathbb{N}_+$, $i \mapsto d_i^Q$. The integer d_i^Q is called the *weight* of the vertex i.

A *morphism of weighted quivers* is a morphism $Q \xrightarrow{f} Q'$ of quivers satisfying $d^{Q'} \circ f_0 = d^Q$.

We simply write d instead of d^Q where it does not cause confusion.

Notation 2.1.4. Whenever a weighted quiver Q with weights $d = d^Q$ has been fixed, we use the following notations for $i, j \in Q_0$:

$$d_{ij} := \gcd(d_i, d_j)\,, \qquad d^{ij} := \operatorname{lcm}(d_i, d_j)\,, \qquad f_{ij} := \frac{d_j}{d_{ij}} = \frac{d^{ij}}{d_i}\,,$$

$$c_{ij} := 2\delta_{i=j} - f_{ij} m_{ij}\,, \qquad m_{ij} := |\{j \leftarrow i \in Q_1\}| + |\{j \to i \in Q_1\}|$$

Occasionally, we also use the abbreviation $d_{kji} := \gcd(d_k, d_j, d_i)$ for $i, j, k \in Q_0$.

Remark 2.1.5. Fix a finite set I. A *Cartan matrix* C with *symmetrizer* D is a pair (C, D) of integer $(I \times I)$-matrices with the following properties:

(a) The diagonal entries of C are 2 and the remaining entries non-positive.

(b) D is a diagonal matrix with positive entries.

(c) DC is symmetric.

For every loop-free weighted I-quiver Q the matrix $C_Q := (c_{ij})_{i,j\in I}$ is a Cartan matrix with symmetrizer $D_Q := \operatorname{diag}(d_i \,|\, i \in I)$.

Vice versa, every Cartan matrix C with symmetrizer D arises from a weighted quiver. More specifically, there is a 2-acyclic weighted I-quiver Q such that $C_Q = C$ and $D_Q = D$. The underlying graph G_Q of Q is determined by (C, D) up to isomorphism. The number of edges joining two vertices $i \neq j$ in G_Q is equal to

$$m_{ij} \;=\; \begin{cases} \gcd(|c_{ij}|, |c_{ji}|) & \text{if } c_{ij}c_{ji} \neq 0, \\ 0 & \text{otherwise.} \end{cases}$$

Remark 2.1.6. Let I be a finite set. According to [FZ02, Definition 4.4] a pair (B, D) of integer $(I \times I)$-matrices is called a *skew-symmetrizable* matrix B with *symmetrizer* D if D is a diagonal matrix with positive entries and DB is skew-symmetric. For the relation between skew-symmetrizable matrices and Cartan matrices see [FZ02, Remark 4.6].

Let $\mathcal{M}_I$ be the set of skew-symmetrizable integer $(I \times I)$-matrices with symmetrizer and let $\mathcal{Q}_I$ be the set of isomorphism classes of 2-acyclic weighted I-quivers. We have a bijection

$$\begin{aligned} \mathcal{M}_I &\longrightarrow \mathcal{Q}_I \\ (B, \operatorname{diag}(d_i)) &\longmapsto (Q_B, i \mapsto d_i) \end{aligned}$$

where Q_B has $\gcd(|b_{ij}|, |b_{ji}|)$ arrows $j \leftarrow i$ if $b_{ji} > 0$ and no arrows $j \leftarrow i$ otherwise.

Definition 2.1.7. The *dual* of a weighted quiver Q is the weighted Q_0-quiver Q^* with

$$Q_1^* \;=\; \left\{ j \xrightarrow{a^*} i \,|\, j \xleftarrow{a} i \in Q_1 \right\}$$

and $d_i^{Q^*} = d_i^Q$ for all $i \in Q_0$. For $b = a^* \in Q_1^*$ set $b^* := a \in Q_1$.

The *double* of Q is the weighted Q_0-quiver $\overline{Q}$ such that $\overline{Q} = Q \oplus Q^*$ as weighted quivers.

The *reflection* of a weighted quiver Q at $j \in Q_0$ is the subquiver Q^{*j} of $\overline{Q}$ with arrow set

$$Q_1^{*j} \;=\; \left\{ k \leftarrow i \in Q_1 \,|\, j \notin \{i, k\} \right\} \cup \left\{ k \rightarrow i \in Q_1^* \,|\, j \in \{i, k\} \right\}.$$

The following definition is inspired by [LZ16, Definition 2.5].

Definition 2.1.8. The *premutation* of a weighted quiver Q at a 2-acyclic vertex $j \in Q_0$ is the weighted Q_0-quiver $Q^{\sim j}$ such that $Q^{\sim j} = Q^{*j} \oplus Q^{-j-}$ where

$$Q_1^{-j-} \;=\; \left\{ k \xleftarrow{[ba]_r^q} i \,|\, k \xleftarrow{b} j, j \xleftarrow{a} i \in Q_1, r \in \mathbb{Z}/r_{kji}\mathbb{Z},\, q \in \mathbb{Z}/q_{kji}\mathbb{Z} \right\}.$$

Here, r_{kji} and q_{kji} are the positive integers $r_{kji} = d_{ki}/d_{kji}$ and $q_{kji} = d_{kji}\, d_j/(d_{kj}d_{ji})$.

Remark 2.1.9. For each $k \xleftarrow{b} j \xleftarrow{a} i$ in Q the premutation $Q^{\sim j}$ has $q_{kji}r_{kji} = d_{ki}\, d_j/(d_{kj}d_{ji})$ *composite* arrows $[ba]_r^q$. The reason for this number and the labeling of the arrows will become clear in § 2.5.4. Note that $q_{kji} = r_{kji} = 1$ if $d_k = d_j = d_i$.

Example 2.1.10. Let Q be the weighted quiver $k \xleftarrow{b} j \xleftarrow{a} i$ with $d_k = 12$, $d_j = 40$, $d_i = 30$. Then $d_{kj} = 4$, $d_{ji} = 10$, $d_{ki} = 6$, $d_{kji} = 2$, $r_{kji} = 3$, $q_{kji} = 2$. The premutation $Q^{\sim j}$ has six composite arrows $k \leftarrow i$ and looks as follows:

$$\begin{tikzcd} & j \arrow[dr, "a^*"] & \\ k \arrow[ur, "b^*"] & & i \arrow[ll, shift right, "{[ba]^1_0, [ba]^1_1, [ba]^1_2}"'] \arrow[ll, shift left, "{[ba]^0_0, [ba]^0_1, [ba]^0_2}"] \end{tikzcd}$$

2.1.3 Modular Quivers

Modular quivers will capture combinatorial aspects of *cyclic Galois modulations.*

Definition 2.1.11. A weighted quiver Q equipped with a function $Q_1 \xrightarrow{\sigma^Q} \bigcup_{i,j \in Q_0} \mathbb{Z}/d_{ji}\mathbb{Z}$ such that $a \mapsto \sigma^Q_a \in \mathbb{Z}/d_{ji}\mathbb{Z}$ for all $j \xleftarrow{a} i \in Q_1$ is called a *modular quiver.*

A *morphism of modular quivers* is a morphism $Q \xrightarrow{f} Q'$ of weighted quivers that satisfies in addition $\sigma^{Q'} \circ f_1 = \sigma^Q$.

We write σ instead of σ^Q where it does not lead to confusion.

Definition 2.1.12. Let Q be a modular quiver.

The double $\overline{Q}$ of the weighted quiver Q is a modular quiver with $\sigma^{\overline{Q}}_a = \sigma^Q_a$ and $\sigma^{\overline{Q}}_{a^*} = -\sigma^Q_a$.

The dual Q^* and the reflections Q^{*j} are modular subquivers of $\overline{Q}$.

The premutation $Q^{\sim j} = Q^{*j} \oplus Q^{-j-}$ is a modular quiver where for $k \xleftarrow{b} j \xleftarrow{a} i$ in Q

$$\sigma_{[ba]^q_r} \;=\; (\sigma_b + \sigma_a)_* + d_{kji} r \;\in\; \mathbb{Z}/d_{ki}\mathbb{Z}\,.$$

Here, $x \mapsto x_*$ denotes the map $\mathbb{Z}/d_{kji}\mathbb{Z} \hookrightarrow \mathbb{Z}/d_{ki}\mathbb{Z}$, $n + d_{kji}\mathbb{Z} \mapsto n + d_{ki}\mathbb{Z}$, for $0 \le n < d_{kji}$.

Let $[ba]^q_{!r}$ be the arrow $[ba]^q_{r'}$ in $Q^{\sim j}$ with $r' \in \mathbb{Z}/r_{kji}\mathbb{Z}$ and $\sigma_{[ba]^q_{r'}} = -(-\sigma_a - \sigma_b)_* - d_{kji} r$.

Example 2.1.13. Let Q be the weighted quiver from Example 2.1.10 considered as a modular quiver with $\sigma_b = 1 \in \mathbb{Z}/4\mathbb{Z}$ and $\sigma_a = 4 \in \mathbb{Z}/10\mathbb{Z}$. Then $\sigma_b + \sigma_a = 1 \in \mathbb{Z}/2\mathbb{Z}$ such that $\sigma_{[ba]^q_r} = 1 + 2r \in \mathbb{Z}/6\mathbb{Z}$ and $\sigma_{b^*} = -\sigma_b = 3 \in \mathbb{Z}/4\mathbb{Z}$ and $\sigma_{a^*} = -\sigma_a = 6 \in \mathbb{Z}/10\mathbb{Z}$.

Definition 2.1.14. A *canceling* 2*-cycle* in a modular quiver Q is a subquiver of Q spanned by two arrows $j \xleftarrow{a} i$ and $j \xrightarrow{b} i$ with $\sigma_b + \sigma_a = 0 \in \mathbb{Z}/d_{ji}\mathbb{Z}$.

A subquiver $T \subseteq Q$ is *trivial* if $T = T^1 \oplus \cdots \oplus T^n$ for canceling 2-cycles $T^1, \ldots, T^n$ in Q.

A modular quiver Q is *reduced at* $j \in Q_0$ if it has no canceling 2-cycles that contain j. It is *reduced* if it is reduced at all vertices.

A *reduction of* Q is a modular subquiver Q' of Q that is reduced and satisfies $Q = Q' \oplus T$ for some trivial T. Similarly, $Q' \subseteq Q$ is a *reduction at* j if $Q = Q' \oplus T$ and $Q(j) - T_1$ is a reduction of the subquiver $Q(j)$ of Q spanned by all arrows incident with j and $T \subseteq Q(j)$.

We say that a modular quiver is 2*-acyclic (at* j*) after reduction* if it has a reduction that is 2-acyclic (at j). Two modular I-quivers are *reduced-equivalent* if they have reductions that are isomorphic as modular I-quivers.

Remark 2.1.15. All reductions of Q are isomorphic as modular Q_0-quivers.

Example 2.1.16. The modular quiver $i \underset{b_0}{\overset{a_0}{\rightleftarrows}} j \underset{b_1}{\overset{a_1}{\rightleftarrows}} k$ with $d_i = d_j = d_k = 2$, $\sigma_{a_0} = \sigma_{b_0} = 1$, $\sigma_{a_1} = 0$, $\sigma_{b_1} = 1$ is 2-acyclic at i after reduction, but is not 2-acyclic at j after reduction.

Lemma 2.1.17. *Let I be a finite set. For $j \in I$ let $\mathcal{Q}(j)$ be the set of reduced-equivalence classes of modular I-quivers that are 2-acyclic at j after reduction. There is an involution*

$$\mathcal{Q}(j) \xrightarrow{\mu_j} \mathcal{Q}(j)\,,$$

called mutation, *given by $Q \mapsto Q^{\sim j}$ for modular quivers Q that are 2-acyclic at j.*

Proof. For each $k \xleftarrow{b} j \xleftarrow{a} i$ in Q the arrows $[ba]^q_r$ and $[a^*b^*]^{-q}_{!r}$ span a canceling 2-cycle $T^{r,q}_{b,a}$ in $(Q^{\sim j})^{\sim j} = Q \oplus \bigoplus_{b,a,r,q} T^{r,q}_{b,a}$. ☐

Example 2.1.18. Let Q be the modular quiver $\begin{tikzcd} & j \arrow[dl, "b"'] & \\ k \arrow[rr, "c"'] & & i \arrow[ul, "a"'] \end{tikzcd}$ with $d_k = d_i = 2$, $d_j = 1$. Note that $\sigma_b = \sigma_a = 0$ and $\sigma_c = r \in \mathbb{Z}/2\mathbb{Z}$. The two arrows c and $[ba]^0_r$ span a canceling 2-cycle in the premutation $Q^{\sim j}$. The (in this example only) reduction of $Q^{\sim j}$ is the following modular quiver with $\sigma_{[ba]^0_{r+1}} = r + 1 \neq r = \sigma_c$:

$$\begin{tikzcd} & j \arrow[dr, "a^*"] & \\ k \arrow[ur, "b^*"] & & i \arrow[ll, "{[ba]^0_{r+1}}"] \end{tikzcd}$$

Definition 2.1.19. A modular I-quiver Q is said to be X*-admissible* for $X \subseteq I$ if it is 2-acyclic after reduction and if, recursively, the elements of $\mu_j(Q)$ are X-admissible for all $j \in X$. It is called admissible if it is I-admissible.

Example 2.1.20. The modular quiver in Example 2.1.18 is easily checked to be admissible.

Example 2.1.21. Let Q be the quiver from Example 2.1.18 regarded as a modular quiver with $d_k = d_i = 4$, $d_j = 2$ and $\sigma_b = \sigma_a = 0 \in \mathbb{Z}/2\mathbb{Z}$, $\sigma_c = 1 \in \mathbb{Z}/4\mathbb{Z}$. The premutation $Q^{\sim j}$ is not 2-acyclic after reduction, because its arrows $k \leftarrow i$ are $\underline{c}_0 := [ba]^0_0$ and $\underline{c}_1 := [ba]^0_1$ and $\sigma_c + \sigma_{\underline{c}_r} = 1 + 2r \neq 0 \in \mathbb{Z}/4\mathbb{Z}$ for all $r \in \{0, 1\}$. In particular, Q is not admissible.

Example 2.1.22. The adjacency quivers investigated in Chapters 5 and 6 form a large and interesting class of admissible modular quivers.

2.1.4 Weyl Groups and Root Systems

Convention 2.1.23. Fix a weighted quiver Q.

Notation 2.1.24. Let $\mathbb{Z}^{Q_0}$ be the free abelian group of integer-valued functions on Q_0. The *standard basis* of $\mathbb{Z}^{Q_0}$ is $\{e_i \mid i \in Q_0\}$ where e_i is the function $Q_0 \to \mathbb{Z}$ with $e_i(j) = \delta_{i=j}$.

Definition 2.1.25. For $i \in Q_0$ the *simple reflection* s_i is the automorphism of $\mathbb{Z}^{Q_0}$ given on the standard basis as $s_i(e_j) = e_j - c_{ij}e_i$.

The *Weyl group* is the subgroup $\mathcal{W}$ of $\mathrm{Aut}\big(\mathbb{Z}^{Q_0}\big)$ generated by all s_i with $i \in Q_0$.

Denote by $\langle -,- \rangle$ and $(-,-)$ the integer-valued bilinear forms on $\mathbb{Z}^{Q_0}$ given on the standard basis as $\langle e_i, e_j \rangle = c_{ji}$ and $(e_i, e_j) = d_j c_{ji} = d_i c_{ij}$.

The set Δ^+_{re} of *positive real roots* is $\mathcal{W}\big(\{e_i \mid i \in Q_0\}\big) \cap \{\alpha \in \mathbb{Z}^{Q_0} \mid \alpha(i) \geq 0 \text{ for all } i \in Q_0\}$.

We might write $(-,-)_Q$ for $(-,-)$ and $\Delta^+_{\mathrm{re}}(Q)$ for Δ^+_{re} to stress the dependence on Q.

Remark 2.1.26. The form $(-,-)$ is always symmetric, while $\langle -,- \rangle$ is symmetric if and only if C_Q is symmetric. It is $\langle -,- \rangle = (-,-)$ if and only if $d_i = 1$ for all i.

Remark 2.1.27. We can express the simple reflections as $s_i = \mathrm{id} - \langle -, e_i \rangle\, e_i$.

2.2 Bimodules

We give an overview of bimodules and tensor algebras. The different notions of dual for bimodules are recalled. Bimodule representations are seen to manifest themselves in up to three different forms. Moreover, we observe that the category of bimodule representations and the category of tensor-algebra modules are equivalent. Finally, the different notions of dual bimodule are shown to coincide when R carries the structure of a symmetric algebra.

Convention 2.2.1. Fix a ring R.

For $M \in \mathrm{Mod}(R)$ and $x \in \mathrm{Z}(R)$ let x^M be the element in $\mathrm{End}_R(M)$ given by $m \mapsto xm$.

2.2.1 R-Algebras

Definition 2.2.2. An *R-algebra* is a ring H carrying an R-bimodule structure subject to the conditions $((rxs)y)t = r(x(syt))$ and $rz = zr$ for all $r, s, t \in R$, $x, y \in H$, $z \in \mathrm{Z}(H)$.

A *morphism of R-algebras* is a map $H \to H'$ that is a morphism of rings and R-bimodules.

Remark 2.2.3. Let H be an R-algebra and $r, s, t \in R$, $x, y \in H$, and $1_H = 1 \in H$. We can unambiguously write $rxsyt$ for $((rxs)y)t = r(x(syt)) \in H$ and $r \in H$ for $r1_H = 1_H r \in H$.

Remark 2.2.4. An R-algebra is the same as a monoid in the category of R-bimodules.

2.2.2 Tensor Algebras

Notation 2.2.5. We denote the tensor algebra of an R-bimodule A by $R\langle A \rangle$, which has as an R-bimodule the form

$$R\langle A \rangle \;=\; \bigoplus_{n \in \mathbb{N}} A^{\otimes n}$$

where $A^{\otimes 0} = R$ and $A^{\otimes n} = A \otimes_R A^{\otimes(n-1)}$ is the n-fold tensor product of A with itself.

Let ι_A be the canonical inclusion $A \hookrightarrow R\langle A\rangle$.

The assignment $A \mapsto R\langle A\rangle$ extends in the obvious way to a functor from the category of R-bimodules to the category of R-algebras. The tensor algebra is the free R-algebra on A in the sense that $R\langle -\rangle$ is left adjoint to the forgetful functor:

Lemma 2.2.6. *Let H be an R-algebra and $A \xrightarrow{f} H$ a morphism of R-bimodules. Then there exists a unique morphism $\hat{f}$ of R-algebras making the following diagram commute:*

$$\begin{tikzcd} A \arrow[rr, hook, "\iota_A"] \arrow[dr, "f"'] & & R\langle A\rangle \arrow[dl, dashed, "\hat{f}"] \\ & H & \end{tikzcd}$$

Proof. This is straightforward. □

Remark 2.2.7. Let S be a subring of $\mathrm{Z}(R)$. Call an R-bimodule A an R-bimodule *over* S if S acts centrally on A, i.e. $sx = xs$ for all $s \in S$ and $x \in A$. An R-algebra *over* S is an R-algebra H that is an R-bimodule over S. The functor $R\langle -\rangle$ restricts to a functor from the full subcategory spanned by R-bimodules over S to the full subcategory spanned by R-algebras over S. This restriction is still left adjoint to the forgetful functor, i.e. the obvious generalization "over S" of Lemma 2.2.6 is true.

Remark 2.2.8. An R-bimodule is the same as an R-bimodule over $\mathbb{Z}$.

Remark 2.2.9. The category of R-bimodules over S can be identified with $\mathrm{Mod}(R \otimes_S R^{\mathrm{op}})$ and the category of R-algebras over S with the category of monoids in $\mathrm{Mod}(R \otimes_S R^{\mathrm{op}})$.

2.2.3 Dual Bimodules

Let A be an R-bimodule over K. In other words, A is a (left) module over the enveloping algebra $R^{\mathrm{e}} = R \otimes_K R^{\mathrm{op}}$. The left R-dual ${}^R\!A$, the right R-dual A^R, the K-dual A^*, and the bimodule dual $A^\dagger$, carry natural R^{e}-module structures. Namely, for $r, s \in R$, and $x \in A$,

$$(sfr)(x) \;=\; \begin{cases} f(xs)r & \text{for } f \in {}^R\!A \;:= \mathrm{Hom}_R({}_RA, {}_RR)\,, \\ sf(rx) & \text{for } f \in A^R \;:= \mathrm{Hom}_R(A_R, R_R)\,, \\ f(rxs) & \text{for } f \in A^* \;:= \mathrm{Hom}_K(A, K)\,, \\ rf(x)s & \text{for } f \in A^\dagger \;:= \mathrm{Hom}_{R^{\mathrm{e}}}(A, R^{\mathrm{e}})\,. \end{cases}$$

Remark 2.2.10. There is an isomorphism $(-)^* \cong \mathrm{Hom}_{R^{\mathrm{e}}}(-, (R^{\mathrm{e}})^*)$ induced by the adjoint pair $(R^{\mathrm{e}} \otimes_{R^{\mathrm{e}}} -, \mathrm{Hom}_K(R^{\mathrm{e}}, -))$. Hence, $(-)^* \cong (-)^\dagger$ if and only if ${}_{R^{\mathrm{e}}}(R^{\mathrm{e}})^* \cong {}_{R^{\mathrm{e}}}R^{\mathrm{e}}$.

2.2.4 Bimodule Representations

Definition 2.2.11. Let A be an R-bimodule. An *A-representation* is a pair (M, r_M) consisting of an R-module M and an R-module morphism $A \otimes_R M \xrightarrow{r_M} M$.

A *morphism of A-representations* $(M, r_M) \xrightarrow{f} (N, r_N)$ is a morphism $M \xrightarrow{f} N$ of R-modules making the following square commute:

$$\begin{tikzcd} A \otimes_R M \arrow[r, "r_M"] \arrow[d, "\mathrm{id}\otimes f"] & M \arrow[d, "f"] \\ A \otimes_R N \arrow[r, "r_N"] & N \end{tikzcd}$$

Denote by $\mathrm{Rep}(A)$ the category of A-representations.

For $M \in \mathrm{Mod}(R)$ the *space of A-representations* is $\mathrm{Rep}(A, M) := \mathrm{Hom}_R(A \otimes_R M, M)$.

Remark 2.2.12. If A is an R-bimodule over S, the space $\mathrm{Rep}(A, M)$ of A-representations on M carries a natural S-module structure.

Remark 2.2.13. Under the adjunction $(A \otimes_R -, \mathrm{Hom}_R(A, -))$ an A-representation (M, r_M) corresponds to a pair $(M, \tilde{r}_M)$ where $\tilde{r}_M$ is an R-module morphism $M \to \mathrm{Hom}_R(A, M)$.

An R-module morphism $M \xrightarrow{f} N$ defines a morphism $(M, \tilde{r}_M) \xrightarrow{f} (N, \tilde{r}_N)$ whenever the following square is commutative:

$$\begin{tikzcd} M \arrow[r, "\tilde{r}_M"] \arrow[d, "f"] & \mathrm{Hom}_R(A, M) \arrow[d, "\mathrm{Hom}_R(A,f)"] \\ N \arrow[r, "\tilde{r}_N"] & \mathrm{Hom}_R(A, N) \end{tikzcd}$$

To state a variation of the last remark, we recall the following classical result:

Lemma 2.2.14. *Let A be an R-bimodule such that ${}_RA$ is finitely generated projective. For every $M \in \mathrm{Mod}(R)$ there is an isomorphism, natural in M, of left R-modules:*

$$\begin{aligned} {}^{R}\!A \otimes_R M &\xrightarrow{\eta_M} \mathrm{Hom}_R(A, M) \\ f \otimes m &\longmapsto (x \mapsto f(x)m) \end{aligned}$$

Proof. See [Bou70, II. §4 no. 2]. □

Definition 2.2.15. Let A be an R-bimodule such that ${}_RA$ is finitely generated projective.

For $M, N \in \mathrm{Mod}(R)$ the isomorphism $\mathrm{Hom}_R(A \otimes_R M, N) \to \mathrm{Hom}_R(M, \mathrm{Hom}_R(A, N))$ of the tensor-hom adjunction given by $f \mapsto (m \mapsto x \mapsto f(x \otimes m))$ composed with postcomposition with η_N^{-1} yields an isomorphism

$$\mathrm{Hom}_R(A \otimes_R M, N) \xrightarrow{\mathrm{ad}} \mathrm{Hom}_R(M, {}^{R}\!A \otimes_R N)\,,$$

which we call the *adjunction correspondence.*

We write $f^\vee$ for $\mathrm{ad}(f)$ and ${}^\vee g$ for $\mathrm{ad}^{-1}(g)$.

Remark 2.2.16. Let A be an R-bimodule such that ${}_RA$ is finitely generated projective. Then ${}^R\!A \otimes_R - \cong \mathrm{Hom}_R(A, -)$ canonically by Lemma 2.2.14.

With the adjunction correspondence an A-representation (M, r_M) corresponds to a pair $(M, r_M^\vee)$ where $r_M^\vee$ is an R-module morphism $M \to {}^R\!A \otimes_R M$.

An R-module morphism $M \xrightarrow{f} N$ defines a morphism $(M, r_M^\vee) \xrightarrow{f} (N, r_N^\vee)$ whenever the following square is commutative:

$$\begin{tikzcd} M \arrow[r, "r_M^\vee"] \arrow[d, "f"] & {}^R\!A \otimes_R M \arrow[d, "\mathrm{id}\otimes f"] \\ N \arrow[r, "r_N^\vee"] & {}^R\!A \otimes_R N \end{tikzcd}$$

Let H be an R-algebra and $M \in \mathrm{Mod}(R)$. Recall that an H-module structure on M is an R-module morphism $H \otimes_R M \to M$, $x \otimes m \mapsto x \cdot m$, satisfying for all $x, y \in H$ and $m \in M$ the relations $1 \cdot m = m$ and $(xy) \cdot m = x \cdot (y \cdot m)$.

Lemma 2.2.17. *Let A be an R-bimodule and let $H = R\langle A\rangle$ be its tensor algebra. For every A-representation (M, r_M) there is a unique H-module structure $\hat{r}_M$ on M such that*

$$\begin{tikzcd} A \otimes_R M \arrow[rr, "\iota_A\otimes \mathrm{id}"] \arrow[dr, "r_M"'] & & H \otimes_R M \arrow[dl, dashed, "\hat{r}_M"] \\ & M & \end{tikzcd}$$

commutes. Furthermore, every morphism $(M, r_M) \xrightarrow{f} (N, r_N)$ of A-representations is a morphism $M \xrightarrow{f} N$ of H-modules when the module structures are given by $\hat{r}_M$ and $\hat{r}_N$.

Proof. This is straightforward. □

Notation 2.2.18. For $H = R\langle A\rangle$ and $M \in \mathrm{Mod}(H)$ denote by $A \otimes_R M \xrightarrow{{}_AM} M$ the map obtained by postcomposing $\iota_A \otimes \mathrm{id}$ with the map $H \otimes_R M \to M$ given by multiplication.

The rule $M \mapsto ({}_RM, {}_AM)$ canonically extends to a functor $\mathrm{Mod}(H) \to \mathrm{Rep}(A)$.

Corollary 2.2.19. *Let A be an R-bimodule and let $H = R\langle A\rangle$ be its tensor algebra. Then the functor $M \mapsto ({}_RM, {}_AM)$ defines an equivalence* $\mathrm{Mod}(H) \to \mathrm{Rep}(A)$.

Proof. This is a direct consequence of Lemma 2.2.17. □

2.2.5 Symmetric Ground Rings

The following is one of the several common definitions of symmetric algebras:

Definition 2.2.20. A K-algebra Λ is said to carry a *Frobenius structure* if it is equipped with a K-linear form $\Lambda \xrightarrow{\varphi} K$ satisfying the property:

(a) The zero ideal is the only left ideal of Λ contained in the kernel of φ.

The form φ is called a *symmetric structure* on Λ if it has additionally to (a) the property:

(b) $\varphi(sr) = \varphi(rs)$ for all $s, r \in \Lambda$.

A *symmetric algebra* is a finite-dimensional K-algebra that admits a symmetric structure.

Remark 2.2.21. It is well-known (see [Lam99, §3B]) that (a) in Definition 2.2.20 is left-right symmetric. It can be substituted with the following equivalent condition:

(a') The zero ideal is the only right ideal of Λ contained in the kernel of φ.

Remark 2.2.22. Assume $R = \prod_i R_i$. Then the ground ring R is a symmetric algebra if and only if R_i is a symmetric algebra for each i. More precisely, every symmetric structure φ on R corresponds to symmetric structures φ_i on R_i such that $\varphi = \sum_i \varphi_i$.

Remark 2.2.23. The enveloping algebra $R^{\mathrm{e}} = R \otimes_K R^{\mathrm{op}}$ is symmetric if R is symmetric. More precisely, symmetric structures φ on R yield symmetric structures $\varphi^{\mathrm{e}} = \varphi \otimes \varphi$ on R^{e}.

Remark 2.2.24. Let φ be a Frobenius structure on R. It is clear (see again [Lam99, §3B]) that φ^{e} induces an isomorphism ${}_{R^{\mathrm{e}}}R^{\mathrm{e}} \xrightarrow{\cong} {}_{R^{\mathrm{e}}}(R^{\mathrm{e}})^*$ that is defined by $(r \mapsto (s \mapsto \varphi^{\mathrm{e}}(sr))$. With Remark 2.2.10 we obtain a canonical isomorphism $A^\dagger \xrightarrow{\varphi_\dagger} A^*$ given by $f \mapsto \varphi^{\mathrm{e}} \circ f$.

Lemma 2.2.25. *For every symmetric structure $R \xrightarrow{\varphi} K$ on R we have the following isomorphisms of R-bimodules over K:*

$$\begin{array}{ccccccc} {}^{R}A & \xrightarrow{\;\;{}_*\varphi\;\;} & A^* & \qquad\qquad & A^* & \xleftarrow{\;\;\varphi_*\;\;} & A^R \\ f & \longmapsto & \varphi \circ f & & \varphi \circ f & \longleftarrow\!\shortmid & f \end{array}$$

Proof. Property (a) in Definition 2.2.20 implies that ${}_*\varphi$ is bijective, while (b) shows that ${}_*\varphi$ is a morphism of R-bimodules. The existence of φ_* follows similarly from (a') and (b). $\square$

Notation 2.2.26. We write ${}^*\varphi$ and φ^* for the inverses of ${}^{R}A \xrightarrow{{}_*\varphi} A^*$ and $A^R \xrightarrow{\varphi_*} A^*$.

2.3 From Bimodules to Quivers

Starting from an R-bimodule A over K with the only assumption that both R and A are finite-dimensional over K, we discuss how decompositions $R = \prod_i R_i$ and $A = \bigoplus_a A_a$ give rise to a weighted quiver Q with vertices i and arrows a. We then observe how bimodule representations of A can be regarded as quiver representations of Q.

2.3.1 Decomposing Bimodules

Convention 2.3.1. Let R be a finite-dimensional algebra and let A be a finite-dimensional R-bimodule over K. Factorize R as a product of connected algebras R_i, indexed by the elements i of some set I. Formally,

$$R \;=\; \prod_{i\in I} R_i\,.$$

Denote by $e_i \in R$ the identity element of R_i. The bimodule A decomposes into the direct sum of the $R_j \otimes_K R_i^{\mathrm{op}}$-modules ${}_jA_i := e_jAe_i$. We refine this decomposition as

$${}_jA_i \;=\; \bigoplus_{a\in I_{ji}} A_a\,.$$

where I_{ji} are pairwise disjoint sets. In this way, we obtain an I-quiver Q with arrow set

$$Q_1 \;=\; \left\{ j \xleftarrow{a} i \,|\, i,j\in I,\, a\in I_{ji}\right\}.$$

The quiver Q can be regarded as a weighted quiver with weights $d_i = \dim_K(R_i)$.

Remark 2.3.2. The summands occurring in the decomposition ${}_jA_i = \bigoplus_{a\in I_{ji}} A_a$ are by the Krull-Remak-Schmidt Theorem up to permutation and isomorphism uniquely determined if we demand that each A_a is an indecomposable $R_j \otimes_K R_i^{\mathrm{op}}$-module.

The K-dual, the left dual, and the right dual of the bimodule A decompose as

$$A^* \;=\; \bigoplus_{a\in Q_1} A_a^*\,, \qquad {}^R\!A \;=\; \bigoplus_{a\in Q_1} A_{{}_*a}\,, \qquad A^R \;=\; \bigoplus_{a\in Q_1} A_{a_*}\,.$$

For $j \xleftarrow{a} i$ the summands in these decompositions are the K-duals, R_j-duals, and R_i-duals of A_a, which are defined as

$$A_a^* \;:=\; \mathrm{Hom}_K(A_a, K)\,, \qquad A_{{}_*a} \;:=\; \mathrm{Hom}_{R_j}(A_a, R_j)\,, \qquad A_{a_*} \;:=\; \mathrm{Hom}_{R_i}(A_a, R_i)\,.$$

All these duals carry natural $R_i \otimes_K R_j^{\mathrm{op}}$-module structures.

2.3.2 Quiver Representations

Notation 2.3.3. Every $M \in \mathrm{Mod}(R)$ is the direct sum of the R_i-modules $M_i := e_iM$:

$$M \;=\; \bigoplus_{i\in Q_0} M_i$$

With this decomposition we have for all $M, N \in \mathrm{Mod}(R)$ canonically

$$\mathrm{Hom}_R(M,N) \;=\; \bigoplus_{i\in Q_0} \mathrm{Hom}_{R_i}(M_i, N_i)\,.$$

By f_i denote the component of $f \in \mathrm{Hom}_R(M,N)$ belonging to $\mathrm{Hom}_{R_i}(M_i, N_i)$.

Similarly, there is a decomposition $\mathrm{Hom}_R(A \otimes_R M, N) = \bigoplus_a \mathrm{Hom}_{R_j}(A_a \otimes_{R_i} M_i, N_j)$ where $j \xleftarrow{a} i$ runs through all arrows of Q. Again, we will write f_a for the component of a map $f \in \mathrm{Hom}_R(A \otimes_R M, N)$ that belongs to $\mathrm{Hom}_{R_j}(A_a \otimes_{R_i} M_i, N_j)$. In particular, the space of A-representations of $M \in \mathrm{Mod}(R)$ decomposes as

$$\mathrm{Rep}(A, M) = \bigoplus_{j \xleftarrow{a} i} \mathrm{Hom}_{R_j}(A_a \otimes_{R_i} M_i, M_j)\,.$$

We simply write M_a for $({}_A M)_a$ if $M \in \mathrm{Mod}(R\langle A \rangle)$ (compare Notation 2.2.18).

We summarize the discussion by stating the analog of [GLS16a, Proposition 5.1].

Lemma 2.3.4. *Let $H = R\langle A \rangle$ be the tensor algebra of A. The category $\mathrm{Mod}(H)$ can be canonically identified with the category whose objects are families $(M_i, M_a)_{i,a}$ with*

$$M_i \in \mathrm{Mod}(R_i) \qquad \textit{indexed by } i \in Q_0,$$

$$M_a \in \mathrm{Hom}_{R_j}(A_a \otimes_{R_i} M_i, M_j) \qquad \textit{indexed by } j \xleftarrow{a} i \in Q_1,$$

and whose morphisms $(M_i, M_a)_{i,a} \to (N_i, N_a)_{i,a}$ are tuples $(f_i)_i$ with

$$f_i \in \mathrm{Hom}_{R_i}(M_i, N_i) \qquad \textit{indexed by } i \in Q_0,$$

making the following diagram commute for all arrows $j \xleftarrow{a} i$ in Q:

$$\begin{tikzcd}
A_a \otimes_{R_i} M_i \arrow[r, "M_a"] \arrow[d, "\mathrm{id}\otimes f_i"] & M_j \arrow[d, "f_j"] \\
A_a \otimes_{R_i} N_i \arrow[r, "N_a"] & N_j
\end{tikzcd}$$

Proof. This follows from Corollary 2.2.19 and the decompositions presented above. □

Remark 2.3.5. Clearly, M is finitely generated if and only if all M_i are finitely generated.

We also have a decomposition $\mathrm{Hom}_R(M, {}^R A \otimes_R N) = \bigoplus_{j \xleftarrow{a} i} \mathrm{Hom}_{R_i}(M_i, A_{*a} \otimes_{R_j} N_j)$ such that $f = \sum_a f_a$ with $f_a \in \mathrm{Hom}_{R_i}(M_i, A_{*a} \otimes_{R_j} N_j)$ for every $f \in \mathrm{Hom}_R(M, {}^R A \otimes_R N)$.

Remark 2.3.6. If ${}_R A$ is projective, choosing an H-module M amounts with Remark 2.2.16 also to the same as to specifying a family $(M_i, M_a^\vee)_{i,a}$ with

$$M_i \in \mathrm{Mod}(R_i) \qquad \text{indexed by } i \in Q_0,$$

$$M_a^\vee \in \mathrm{Hom}_{R_i}(M_i, A_{*a} \otimes_{R_j} M_j) \qquad \text{indexed by } j \xleftarrow{a} i \in Q_1.$$

2.3.3 Locally Free Modules

We extend the terminology of locally freeness introduced in [GLS16a] to arbitrary R-algebras and observe that the "adjunction formulas" from ibid. § 5.1 remain valid for locally free ${}_R A$.

Definition 2.3.7. Let H be an R-algebra. A module $M \in \operatorname{Mod}(H)$ is called *locally free* if M_i is a free R_i-module for all $i \in Q_0$.

Analogously, one defines locally freeness for right H-modules.

Remark 2.3.8. Let H be an R-algebra and assume that R_i is local for all $i \in Q_0$. Then an H-module M is locally free if and only if ${}_R M$ is projective.

If ${}_R A$ is locally free, the inverse of the isomorphism $\eta = \eta_M$ from Lemma 2.2.14 can be explicitly described in terms of bases of the finitely generated free modules ${}_{R_j}(A_a)$.

Lemma 2.3.9. *Assume ${}_{R_j}(A_a)$ is free for some $j \xleftarrow{a} i \in Q_1$ and let B_a be a basis of ${}_{R_j}(A_a)$ and $\{b^* \mid b \in B_a\}$ its R_j-dual basis. For each $M \in \operatorname{Mod}(R)$ the inverse of η_a acts as follows:*

$$\begin{array}{ccc} \operatorname{Hom}_{R_j}(A_a, M_j) & \xrightarrow{\eta_a^{-1}} & A_{*a} \otimes_{R_j} M_j \\ g & \longmapsto & \sum_{b \in B_a} b^* \otimes g(b) \end{array}$$

Proof. This is straightforward. □

Corollary 2.3.10. *Let $f \in \operatorname{Hom}_R(A \otimes_R M, N)$ and $g \in \operatorname{Hom}_R(M, {}^R\!A \otimes_R N)$ for two modules $M, N \in \operatorname{Mod}(R)$. If ${}_{R_j}(A_a)$ has a basis B_a, the formulas*

$$f_a^\vee(m) \;=\; \sum_{b \in B_a} b^* \otimes f(b \otimes m)\,, \qquad {}^\vee g_a(x \otimes m) \;=\; \sum_{b \in B_a} x_b\, g_{m,b}$$

hold for all elements $m \in M_i$ and $x = \sum_b x_b b \in A_a$ where $g(m) = \sum_b b^* \otimes g_{m,b} \in A_{*a} \otimes_{R_j} N_j$ with $x_b \in R_j$ and $g_{m,b} \in N_j$.

Proof. This follows by direct calculations using Lemma 2.3.9. □

We can restate Lemma 2.2.25 in the current context as follows:

Corollary 2.3.11. *For every symmetric structure $R \xrightarrow{\varphi} K$ on R and every arrow $j \xleftarrow{a} i$ in Q we have the following isomorphisms of $R_i \otimes_K R_j^{\mathrm{op}}$-modules:*

$$\begin{array}{ccc} A_{*a} & \xrightarrow{\;{}_*\varphi\;} & A_a^* \\ f & \longmapsto & \varphi_j \circ f \end{array} \qquad\qquad \begin{array}{ccc} A_a^* & \xleftarrow{\;\varphi_*\;} & A_{a*} \\ \varphi_i \circ f & \longleftarrow\!\shortmid & f \end{array}$$

Proof. Use Lemma 2.2.25 and the decomposition $A = \bigoplus_a A_a$. □

2.4 Path Algebras for Weighted Quivers

Convention 2.4.1. Fix a weighted quiver Q and recall Notation 2.1.4.

2.4.1 Modulations

Definition 2.4.2. A *K-modulation* for the weighted quiver Q is a family

$$\mathcal{H} \;=\; \big(R_i, A_a\big)_{i\in Q_0, a\in Q_1}$$

of connected K-algebras R_i, indexed by the vertices $i \in Q_0$, and non-zero $R_j \otimes_K R_i^{\mathrm{op}}$-modules A_a, indexed by the arrows $j \xleftarrow{a} i \in Q_1$, satisfying the following properties:

(a) $\dim_K(R_i) = d_i$ for each $i \in Q_0$.

(b) ${}_{R_j}(A_a)$ and $(A_a)_{R_i}$ are free of finite rank for each $j \xleftarrow{a} i \in Q_1$.

Clearly, properties (a) and (b) imply the existence of positive integers d_a such that

$$\dim_K(A_a) \;=\; d_a d^{ji}\,, \qquad \operatorname{rank}_{R_j}(A_a) \;=\; d_a f_{ji}\,, \qquad \operatorname{rank}(A_a)_{R_i} \;=\; d_a f_{ij}\,.$$

The modulation $\mathcal{H}$ is *minimal* if $d_a = 1$ for all $a \in Q_1$.

We call $\mathcal{H}$ *local* (resp. *symmetric*) if R_i is a local (resp. symmetric) algebra for all $i \in Q_0$.

We say that $\mathcal{H}$ is *decomposed* if for all $j \xleftarrow{a} i$ there is no decomposition $A_a = M_1 \oplus M_2$ with non-zero $R_j \otimes_K R_i^{\mathrm{op}}$-modules M_s such that ${}_{R_j}(M_s)$ and $(M_s)_{R_i}$ are free.

Let $\mathcal{H} = (R_i, A_a)_{i,a}$ and $\mathcal{H}' = (R_i', A_a')_{i,a}$ be K-modulations for Q. A morphism $\mathcal{H} \xrightarrow{f} \mathcal{H}'$ is a family $(f_i, f_a)_{i,a}$ of K-algebra homomorphisms $R_i \xrightarrow{f_i} R_i'$ and linear maps $A_a \xrightarrow{f_a} A_a'$ such that $f_a(sxr) = f_j(s) f_a(x) f_i(r)$ for all $j \xleftarrow{a} i \in Q_1$, $s \in R_j$, $x \in A_a$, $r \in R_i$.

An *$(R_i)_i$-modulation* for a weighted I-quiver Q is a K-modulation $(R_i, A_a)_{i,a}$ for Q.

Remark 2.4.3. If $\mathcal{H}$ is local, it is decomposed if and only if all A_a are indecomposable.

Remark 2.4.4. Every minimal K-modulation is decomposed.

Remark 2.4.5. Let $\mathcal{H} = (R_i, A_a)_{i,a}$ be a K-modulation where all R_i are division algebras. Then $(R_i, {}_jA_i)_{i,j\in Q_0}$ with ${}_jA_i \;:=\; \bigoplus_{j \xleftarrow{a} i \in Q_1} A_a$ is a modulation in the sense of [DR76].

Conversely, for families $(R_i)_{i\in Q_0}$ of connected K-algebras and $({}_jA_i)_{i,j\in Q_0}$ of $R_j \otimes_K R_i^{\mathrm{op}}$-modules ${}_jA_i$ with $\dim_K(R_i) = d_i$ and ${}_{R_j}({}_jA_i)$, $({}_jA_i)_{R_i}$ free of finite rank, we can make the following two observations:

(i) Suppose there is exactly one arrow $j \xleftarrow{a} i$ in Q whenever $\dim_K({}_jA_i) \neq 0$ and no arrow $j \leftarrow i$ otherwise. Set $A_a := {}_jA_i$. Then $(R_i, A_a)_{i,a}$ is a K-modulation for Q.

(ii) The Krull-Remak-Schmidt Theorem yields a decomposition ${}_jA_i = \bigoplus_{s=1}^m M_s$ whose summands are indecomposable and determined up to reordering and isomorphism. Let us assume that all R_i are local rings. Then ${}_{R_j}(M_s)$ and $(M_s)_{R_i}$ are free modules. Supposing that Q has precisely m arrows $a_1, \ldots, a_m$ from i to j, we set $A_{a_s} := M_s$. Then $(R_i, A_a)_{i,a}$ is a decomposed (local) K-modulation for Q.

Remark 2.4.6. Dlab and Ringel require in [DR76] that $\operatorname{Hom}_{R_i}(A_a, R_i) \cong \operatorname{Hom}_{R_j}(A_a, R_j)$

as $R_i \otimes_K R_j^{\mathrm{op}}$-modules. For every K-modulation $\mathcal{H} = (R_i, A_a)_{i,a}$ where all R_i are division algebras such isomorphisms always exist. See Corollary 2.3.11 and Example 3.2.5.

Example 2.4.7. Let Q be the weighted quiver $1 \xleftarrow{a} 2$ of type A_2 with $d_1 = d_2 = 2$. For $\rho \in \mathrm{Aut}_{\mathbb{R}}(\mathbb{C})$ we have an $\mathbb{R}$-modulation $\mathcal{H}^\rho = (R_i, A_a^\rho)_{i,a}$ for Q with $R_1 = R_2 = \mathbb{C}$ and $A_a^\rho = \mathbb{C}^\rho$. Here $\mathbb{C}^\rho$ is the $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}$-module with $\mathbb{C}^\rho = \mathbb{C}$ as $\mathbb{R}$-vector space and bimodule structure given by $wzv := w \cdot z \cdot \rho(v)$ for $w, v \in \mathbb{C}$ and $z \in \mathbb{C}^\rho$.

Let $\mathrm{Aut}_{\mathbb{R}}(\mathbb{C}) = \{\mathrm{id}, \rho\}$ where ρ is complex conjugation. Then the maps $f_1 = \mathrm{id}$, $f_2 = \rho^{-1}$ and $f_a = \mathrm{id}_{\mathbb{C}}$ define an isomorphism $\mathcal{H}^{\mathrm{id}} \xrightarrow{f} \mathcal{H}^\rho$ of K-modulations for Q.

Example 2.4.8. Taking $R_1 = R_2 = \mathbb{C}$, $A_a = \mathbb{C} \otimes_{\mathbb{R}} \mathbb{C} \cong \mathbb{C} \oplus \mathbb{C}^\rho$ defines an $\mathbb{R}$-modulation $\mathcal{H}$ for the weighted quiver from Example 2.4.7. This modulation is local but not decomposed. The K-modulation $\mathcal{H}'$ for Q given by $R_1 = R_2 = \mathbb{C}$, $A_a = \mathbb{C} \oplus \mathbb{C}$ is not isomorphic to $\mathcal{H}$.

Example 2.4.9. Let Q be the weighted quiver $1 \xleftarrow{a} 2$ with $d_1 = 2$, $d_2 = 3$. Then $R_1 = \mathbb{C}$, $R_2 = \left(\begin{smallmatrix} \mathbb{R} & \mathbb{R} \\ 0 & \mathbb{R} \end{smallmatrix}\right)$, $A_a = R_1 \otimes_{\mathbb{R}} R_2 \cong \left(\begin{smallmatrix} \mathbb{C} & \mathbb{C} \\ 0 & \mathbb{C} \end{smallmatrix}\right)$ defines a minimal non-local $\mathbb{R}$-modulation $\mathcal{H}$ for Q. In particular, $\mathcal{H}$ is decomposed. However, $A_a \cong \left(\begin{smallmatrix} \mathbb{C} & \mathbb{C} \\ 0 & 0 \end{smallmatrix}\right) \oplus \left(\begin{smallmatrix} 0 & 0 \\ 0 & \mathbb{C} \end{smallmatrix}\right)$ is not indecomposable.

Example 2.4.10. Let Q be as in Example 2.4.9 and let ζ be a primitive cube root of unity. Then $R_1 = \mathbb{Q}(\zeta)$, $R_2 = \mathbb{Q}(\sqrt[3]{2})$, $A_a = R_1 \otimes_{\mathbb{Q}} R_2$ yields a minimal local $\mathbb{Q}$-modulation for Q.

Example 2.4.11. § 5 in [GLS16a] describes the minimal local K-modulation $\mathcal{H} = (R_i, A_a)_{i,a}$ with $R_i = K[\varepsilon_i]/(\varepsilon_i^{d_i})$ for $i \in Q_0$ and $A_a = K[\varepsilon_j, \varepsilon_i]/(\varepsilon_j^{f_{ij}} - \varepsilon_i^{f_{ji}}, \varepsilon_j^{d_j}, \varepsilon_i^{d_i})$ for $j \xleftarrow{a} i \in Q_1$.

Example 2.4.12. With $R_i = K[\varepsilon_i]/(\varepsilon_i^2)$ and $A_a = R_1 \otimes_K R_2 \cong K[\varepsilon_1, \varepsilon_2]/(\varepsilon_1^2, \varepsilon_2^2)$ we get a K-modulation for the weighted quiver $1 \xleftarrow{a} 2$ with $d_1 = d_2 = 2$ from Example 2.4.7. This modulation is local and decomposed but not minimal.

2.4.2 Pullback Modulations

Definition 2.4.13. Let $\mathcal{H}$ be a K-modulation of Q. For a morphism $Q' \xrightarrow{f} Q$ of weighted quivers the *pullback of* $\mathcal{H}$ *along* f is the K-modulation

$$f^*\mathcal{H} \;=\; \big(R_{f(i')}, A_{f(a')}\big)_{i' \in Q_0', a' \in Q_1'} .$$

If f is the inclusion of a subquiver, we call $f^*\mathcal{H}$ the *submodulation* of $\mathcal{H}$ induced by f.

Remark 2.4.14. More generally, the *pullback* $f^*\mathcal{H}$ *of* $\mathcal{H}$ *along* f can be defined in a similar way for every functor $\mathcal{P}_{Q'} \xrightarrow{f} \mathcal{P}_Q$ where $A_{f(a')} := A_{a_n} \otimes_R \cdots \otimes_R A_{a_1}$ if $f(a') = a_n \cdots a_1$. Here, the quiver Q' has to be considered as a weighted quiver with weights $d_{i'}^{Q'} = d_{f(i')}^{Q}$.

2.4.3 Dual and Double Modulations

Convention 2.4.15. Fix a symmetric K-modulation $\mathcal{H} = (R_i, A_a)_{i,a}$ for Q.

Definition 2.4.16. The *double of* $\mathcal{H}$ is the K-modulation $\overline{\mathcal{H}} = (R_i, A_a)_{i\in\overline{Q}_0, a\in\overline{Q}_1}$ for $\overline{Q}$ with $A_{a^*} = A_a^*$. The *dual of* $\mathcal{H}$ is the submodulation $\mathcal{H}^*$ of $\mathcal{H}$ induced by $Q^* \subseteq \overline{Q}$.

Remark 2.4.17. Corollary 2.3.11 ensures that $\overline{\mathcal{H}}$ satisfies (b) in Definition 2.4.2.

Lemma 2.4.18. *If $\mathcal{H}$ is a minimal/decomposed/local/symmetric modulation, $\overline{\mathcal{H}}$ and $\mathcal{H}^*$ are minimal/decomposed/local/symmetric modulations.*

Proof. This is straightforward. □

Example 2.4.19. Let $\mathcal{H}$ be the *GLS modulation* from Example 2.4.11. Then $\mathcal{H}$ is symmetric. More precisely, the map $\varphi_i = t_i^{\max}$ in [GLS16a, § 8.1] defines a symmetric structure on R_i.

Fix $j \xleftarrow{a} i \in Q_1$. In [GLS16a, § 5.1] two maps λ and ρ are described, which yield an isomorphism $\lambda\circ\rho^{-1}$ between $A_{{}_*a}$ and A_{a_*}. Inspection shows that $\lambda\circ\rho^{-1}$ is the map $\varphi_i^* \circ {}_*\varphi_j$.

If one prefers to identify the K-dual A_a^* with ${}_i\Lambda_j = K[\varepsilon_i, \varepsilon_j]/\big(\varepsilon_i^{f_{ji}} - \varepsilon_j^{f_{ij}}, \varepsilon_i^{d_i}, \varepsilon_j^{d_j}\big)$, as Geiß, Leclerc, and Schröer do, it is possible to use the isomorphism ${}_i\Lambda_j \to A_a^*$ that is given, for $0 \le q < d_{ij}$, $0 \le r < f_{ji}$, $0 \le s < f_{ij}$, $\varepsilon_{ij} := \varepsilon_{ji} := \varepsilon_i^{f_{ji}} = \varepsilon_j^{f_{ij}}$, by

$$\varepsilon_i^r \varepsilon_{ij}^q \varepsilon_j^s \;\mapsto\; \left(\varepsilon_j^{(f_{ij}-1)-s} \varepsilon_{ji}^{(d_{ij}-1)-q} \varepsilon_i^{(f_{ji}-1)-r}\right)^* .$$

This shows how $(R_i, A_a)_{i\in\overline{Q}_0, a\in\overline{Q}_1}$ with $A_{a^*} = {}_i\Lambda_j$ can be regarded as the double of $\mathcal{H}$.

2.4.4 Path Algebras

Convention 2.4.20. Fix a K-modulation $\mathcal{H} = (R_i, A_a)_{i,a}$ for Q.

Definition 2.4.21. We will call the K-algebra $R_{\mathcal{H}} = \prod_{i\in Q_0} R_i$ the *ground ring* and the R-bimodule $A_{\mathcal{H}} = \bigoplus_{a\in Q_1} A_a$ over K the *species* of the modulation $\mathcal{H}$.

The tensor algebra $H_{\mathcal{H}} = R\langle A\rangle$ is the *path algebra* of Q defined by $\mathcal{H}$.

We write R, A, H instead of $R_{\mathcal{H}}$, $A_{\mathcal{H}}$, $H_{\mathcal{H}}$ when confusion seems unlikely.

Notation 2.4.22. We use the notation introduced in § 2.3. In particular, e_i stands for the identity of R_i considered as an element of R. So $\sum_{i\in Q_0} e_i = 1 \in R \subseteq H$.

Remark 2.4.23. If $d_i = 1$ for all i, then $d_{ij} = d^{ij} = f_{ij} = 1$ for all i, j. In this case, up to isomorphism, $\mathcal{H} = (R_i, A_a)_{i,a}$ with $R_i = K$ for all i and $A_a = K$ for all a is the only minimal K-modulation for Q and $H_{\mathcal{H}} \cong KQ$.

Remark 2.4.24. As an R-bimodule H decomposes as $\bigoplus_{j,i\in Q_0} e_j H e_i$. In this way, we can think of elements in H as $(Q_0 \times Q_0)$-matrices. Multiplication in H is then nothing else but multiplication of these matrices. See Examples 2.4.25 and 2.4.26 for an illustration.

Example 2.4.25. Let Q be the weighted quiver $1 \xleftarrow{a} 2$ with $d_1 = 1$, $d_2 = 2$ of type B_2 and let $\mathcal{H}$ be the $\mathbb{R}$-modulation for Q given by $R_1 = \mathbb{R}$, $R_2 = \mathbb{C}$ and $A_a = \mathbb{C}$. Then the path algebra H of Q defined by $\mathcal{H}$ is the matrix algebra $\left(\begin{smallmatrix}\mathbb{R} & \mathbb{C}\\ 0 & \mathbb{C}\end{smallmatrix}\right)$.

Example 2.4.26. Let $\mathcal{H}^\rho$ be one of the K-modulations for the weighted quiver Q of type A_2 described in Example 2.4.7. The path algebra H^ρ of Q defined by $\mathcal{H}^\rho$ is $\left(\begin{smallmatrix} \mathbb{C} & \mathbb{C}^\rho \\ 0 & \mathbb{C} \end{smallmatrix}\right)$.

Example 2.4.27. Let $\mathcal{H}$ be the *GLS modulation* for Q from Example 2.4.11. As discussed in detail in [GLS16a], the path algebra H of Q defined by $\mathcal{H}$ can be identified with $KQ^{\circlearrowleft}/I(d)$, where $Q^{\circlearrowleft}$ is the quiver obtained from Q by adding a loop ε_i at each vertex i, and $I(d)$ is the ideal generated by the relations $\varepsilon_i^{d_i} = 0$ for $i \in Q_0$ and $a\varepsilon_i^{f_{ji}} = \varepsilon_j^{f_{ij}} a$ for $j \xleftarrow{a} i \in Q_1$.

Example 2.4.28. As pointed out in [GLS16a], in case $d_i = 2$ for all vertices $i \in Q_0$, the path algebra in Example 2.4.27 is the path algebra of Q over the dual numbers $K[\varepsilon]/(\varepsilon^2)$, which was investigated in [RZ13].

We end this section with a few elementary but useful observations.

Notation 2.4.29. Let $R \xrightarrow{f} R'$ be a ring homomorphism and let A' be an R'-bimodule. Write f_*A' for A' regarded as an R-bimodule with $sxr := f(s)xf(r)$ for $s, r \in R$, $x \in f_*A'$.

Lemma 2.4.30. *Every morphism $\mathcal{H} \xrightarrow{f} \mathcal{H}'$ of K-modulations for Q induces a K-algebra morphism $R_{\mathcal{H}} \xrightarrow{f} R_{\mathcal{H}'}$ between ground rings and an R-bimodule morphism $A_{\mathcal{H}} \xrightarrow{f} f_*A_{\mathcal{H}'}$.*

Proof. This is straightforward. □

Corollary 2.4.31. *Every morphism $\mathcal{H} \xrightarrow{f} \mathcal{H}'$ of K-modulations for Q induces a morphism of R-algebras $H_{\mathcal{H}} \xrightarrow{f} f_*H_{\mathcal{H}'}$.*

Proof. Use Lemmas 2.2.6 and 2.4.30. □

Lemma 2.4.32. *Let $Q' \xrightarrow{f} Q$ be a morphism of weighted quivers with $Q'_0 \xhookrightarrow{f_0} Q_0$ injective. Then f induces a non-unital K-algebra morphism $H_{f^*\mathcal{H}} \xrightarrow{f_*} H_{\mathcal{H}}$. In particular, if f is a morphism of I-quivers, the induced morphism f_* is a morphism of (unital) K-algebras.*

Proof. Since $Q'_0 \xhookrightarrow{f_0} Q_0$ is injective, $R_{f^*\mathcal{H}} = \prod_{i'}(R_{\mathcal{H}})_{f(i')}$ can be canonically regarded as a non-unital subalgebra of $R_{\mathcal{H}} = \prod_i (R_{\mathcal{H}})_i$. Therefore the identities $(A_{f^*\mathcal{H}})_{a'} \to (A_{\mathcal{H}})_{f(a')}$ induce an $R_{f^*\mathcal{H}}$-bimodule morphism $A_{f^*\mathcal{H}} \to A_{\mathcal{H}}$. Now use Lemma 2.2.6. □

2.5 Path Algebras for Modular Quivers

This section is concerned with cyclic Galois modulations, a class of modulations suitable for explicit computations. Introduced by Labardini and Zelevinsky in [LZ16] in the strongly primitive setting, [GL16a] considers cyclic Galois modulations $(R_i, A_a)_{i,a}$ in the general situation. The idea is to take for the R_i intermediate fields of a cyclic Galois extension L/K. The valid choices for each bimodule A_a are then parametrized by a Galois group.

Convention 2.5.1. Fix a weighted quiver Q.

2.5.1 Cyclic Galois Modulations

Definition 2.5.2. A cyclic Galois extension L/K is *Q-admissible* if $d_i \mid [L : K]$ for all i. In this case, denote by L_i the intermediate field of L/K with $[L_i : K] = d_i$.

Example 2.5.3. An extension $\mathbb{F}_{q^m}/\mathbb{F}_q$ of finite fields is Q-admissible if and only if m is divisible by all of the weights d_i.

Definition 2.5.4. Let L/K be a Q-admissible cyclic Galois extension. A *cyclic Galois modulation* over L/K for Q is a minimal K-modulation $(R_i, A_a)_{i,a}$ for Q with $R_i \cong L_i$.

Remark 2.5.5. Cyclic Galois modulations are local, symmetric, and decomposed.

Example 2.5.6. Examples 2.4.7 and 2.4.25 presented cyclic Galois modulations over $\mathbb{C}/\mathbb{R}$.

2.5.2 Modulation of a Modular Quiver

The data σ^Q stored in a modular quiver Q is devised to determine a K-modulation over any Q-admissible cyclic Galois extension L/K with fixed isomorphism $\mathbb{Z}/m\mathbb{Z} \xrightarrow{\cong} \mathrm{Gal}(L/K)$.

Convention 2.5.7. For §§ 2.5.2 to 2.5.4 fix a Q-admissible cyclic Galois extension L/K.

Notation 2.5.8. We abbreviate $L_{ji} = L_j \cap L_i$ and $L^{ji} = L_j L_i$.

For any intermediate field F of L_{ji}/K and $\rho \in \mathrm{Gal}(F/K)$ denote by ${}_jL_i^\rho$ the $L_j \otimes_K L_i$-module $L_j \otimes_\rho L_i$ where the tensor product is taken with respect to $L_j \xleftarrow{\rho} F \xrightarrow{\mathrm{id}} L_i$.

Remark 2.5.9. It is $[L_{ji} : K] = d_{ji}$ and $[L^{ji} : K] = d^{ji}$. If $F = L_{ji}$, then $\dim_K({}_jL_i^\rho) = d^{ji}$.

Remark 2.5.10. Let $\rho \in \mathrm{Gal}(F/K)$ for some intermediate field F of the extension L_{ji}/K. Then $1 \otimes x = \rho(x) \otimes 1 \in {}_jL_i^\rho$ for all $x \in F$. In particular, ${}_jL_i^{\mathrm{id}_{L_{ji}}} \cong L^{ji}$ as $L_j \otimes_K L_i$-module.

Convention 2.5.11. Let $\mathbb{Z}/m\mathbb{Z} \xrightarrow{\alpha} \mathrm{Gal}(L/K)$ be an isomorphism and set $m_F = [F : K]$ for every intermediate field F of L/K. We have an isomorphism $\mathbb{Z}/m_F\mathbb{Z} \xrightarrow{\alpha_F} \mathrm{Gal}(F/K)$ making the following square commute, where the horizontal arrows are the canonical maps:

$$\begin{tikzcd}
\mathbb{Z}/m\mathbb{Z} \arrow[r, two heads] \arrow[d, "\alpha", "\cong"'] & \mathbb{Z}/m_F\mathbb{Z} \arrow[d, "\alpha_F", "\cong"'] \\
\mathrm{Gal}(L/K) \arrow[r, two heads] & \mathrm{Gal}(F/K)
\end{tikzcd}$$

We use the notation α_i for α_{L_i} and α_{ji} for $\alpha_{L_{ji}}$ with $i, j \in Q_0$.

Definition 2.5.12. Let Q be a modular quiver. The *modulation of Q over* $(L/K, \alpha)$ is the K-modulation $\mathcal{H} = (R_i, A_a)_{i,a}$ with $R_i = L_i$ for all i and $A_a = {}_jL_i^{\alpha_{ji}(\sigma_a)}$ for all $j \xleftarrow{a} i$.

We write a for $1 \otimes 1 \in A_a$ when considered as an element in $A_{\mathcal{H}} \subseteq H_{\mathcal{H}}$.

Example 2.5.13. Let Q be the weighted quiver of type A_2 and $\mathcal{H}^\rho$ the modulation for Q from Example 2.4.7, where $\rho \in \mathrm{Gal}(\mathbb{C}/\mathbb{R})$ is complex conjugation. If we turn Q into a modular quiver with $\sigma_a = 1$, the modulation of Q over $(\mathbb{C}/\mathbb{R}, \alpha)$ is $\mathcal{H}^\rho$.

The following classical result or, more precisely, its corollary below justifies the slogan "Every cyclic Galois modulation is the modulation of a modular quiver."

Lemma 2.5.14. *Let $i, j \in Q_0$. For every intermediate field F of the extension L_{ji}/K, the algebra $L_j \otimes_F L_i$ is a basic semi-simple algebra. The rule $\rho \mapsto {}_jL_i^\rho$ establishes a bijection between $\mathrm{Gal}(L_{ji}/F)$ and the set of isomorphism classes of simple $L_j \otimes_F L_i$-modules.*

Proof. The separability of L/K guarantees that the field extension L_j/F is generated by a primitive element whose minimal polynomial f over F is separable. It is a classical fact that $\Lambda = L_j \otimes_F L_i$ is semi-simple (see [Kna07, Proposition 2.29]). More precisely, the number n of irreducible factors of f when considered as a polynomial over L_i is the number of simple summands of ${}_\Lambda\Lambda$. It only remains to observe that $n = [L_{ji} : F]$ and that ${}_jL_i^{\rho_1} \cong {}_jL_i^{\rho_2}$ if and only if $\rho_1 = \rho_2$. □

Remark 2.5.15. The semi-simplicity of $\Lambda = L_j \otimes_F L_i$ is also a consequence of the following two facts: On the one hand, Λ is finite-dimensional, so its reduction $\Lambda/\mathrm{rad}(\Lambda)$ is semi-simple. On the other hand, Λ is reduced because of the separability of L_j/F.

Example 2.5.16. For the cyclic Galois extension $\mathbb{C}/\mathbb{R}$ Lemma 2.5.14 yields the decomposition $\mathbb{C} \otimes_\mathbb{R} \mathbb{C} \cong \mathbb{C} \oplus \mathbb{C}^\rho$ (compare Example 2.4.7).

Corollary 2.5.17. *Let $\mathcal{H} = (L_i, A_a)_{i,a}$ be a cyclic Galois modulation. For all $j \xleftarrow{a} i \in Q_1$ there exists $\rho_a \in \mathrm{Gal}(L_{ji}/K)$ such that $A_a \cong {}_jL_i^{\rho_a}$ as $L_j \otimes_K L_i$-modules.*

Proof. Apply Lemma 2.5.14 with $F = K$. □

2.5.3 Isotypical Components

Let $R = \prod_{i \in Q_0} L_i$. From Lemma 2.5.14 we know that the category of R-bimodules over K is semi-simple with simple objects ${}_jL_i^\rho$ parametrized by $i, j \in Q_0$ and $\rho \in \mathrm{Gal}(L_{ji}/K)$. This subsection introduces notation for isotypical components and investigates how the tensor product of R-bimodules over K decomposes into simples.

Notation 2.5.18. For $L_j \otimes_K L_i$-modules M denote by M^ρ their ${}_jL_i^\rho$-isotypical component and by $\pi_\rho : M \to M$ the idempotent corresponding to the canonical projection onto M^ρ. We have a decomposition

$$M = \bigoplus_{\rho \in \mathrm{Gal}(L_{ji}/K)} M^\rho .$$

We refer to M^ρ as the *ρ-isotypical component* of M. If the module M has finite length, let $[M : {}_jL_i^\rho]$ denote the Jordan-Hölder multiplicity of ${}_jL_i^\rho$ in M.

More generally, for intermediate fields F of L_{ji}/K and $\gamma \in \mathrm{Gal}(F/K)$ let G_{ji}^{γ} be the coset of $\mathrm{Gal}(L_{ji}/K)$ consisting of all ρ such that $\rho|_F = \gamma$. The submodule

$$M^{\gamma} \;=\; \bigoplus_{\rho \in G_{ji}^{\gamma}} M^{\rho}$$

is called the *γ-isotypical component* of M. Set $\pi_{\gamma} := \sum_{\rho \in G_{ji}^{\gamma}} \pi_{\rho}$.

Remark 2.5.19. The notation $\left[L^{ji} : L^{ji}\right]$ is unambiguous. It represents the integer one, no matter if it is interpreted as Jordan-Hölder multiplicity or as degree of a field extension.

Remark 2.5.20. The γ-isotypical component M^{γ} can be characterized as

$$M^{\gamma} \;=\; \{x \in M \mid xu = \gamma(u)x \text{ for all } u \in F\}\,. \tag{$\star$}$$

We can now formulate the following corollary of Lemma 2.5.14. See [GL16a, Lemma 2.14].

Corollary 2.5.21. *Let $i, j \in Q_0$. For every intermediate field F of the extension L_{ji}/K and every $\gamma \in \mathrm{Gal}(F/K)$ it is $\left[{}_jL_i^{\gamma} : {}_jL_i^{\rho}\right] = \delta_{\rho \in G_{ji}^{\gamma}}$.*

Proof. Equation ($\star$) shows that there is a non-zero $L_j \otimes_K L_i$-module morphism ${}_jL_i^{\gamma} \to {}_jL_i^{\rho}$ defined by $1 \otimes 1 \mapsto 1 \otimes 1$ for all $\rho \in G_{ji}^{\gamma}$, so $\left[{}_jL_i^{\gamma} : {}_jL_i^{\rho}\right] > 0$. The corollary now follows from the fact that $\bigoplus_{\rho \in G_{ji}^{\gamma}} {}_jL_i^{\rho}$ and ${}_jL_i^{\gamma}$ have the same dimension. □

For elements $\nu \in \mathrm{Gal}(L_{kj}/K)$ and $\mu \in \mathrm{Gal}(L_{ji}/K)$ the composition $\nu\mu$ is well-defined on the intersection $L_{kj} \cap L_{ji}$ and will be considered as an element in $\mathrm{Gal}(L_{kj} \cap L_{ji}/K)$.

The following result is standard. It can be found as Proposition 2.12 in [GL16a].

Lemma 2.5.22. *Let $i, j, k \in Q_0$. For every $\nu \in \mathrm{Gal}(L_{kj}/K)$ and $\mu \in \mathrm{Gal}(L_{ji}/K)$, it is*

$$\left[{}_kL_j^{\nu} \otimes_{L_j} {}_jL_i^{\mu} : {}_kL_i^{\rho}\right] \;=\; \begin{cases} \left[L_j : L_{kj}L_{ji}\right] & \text{if } \rho \in G_{ki}^{\nu\mu}, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Let $M = {}_kL_i^{\nu\mu} = L_k \otimes_{\nu\mu} L_i$ and $N = L_k \otimes_{\nu} L_j \otimes_{\mu} L_i$. Then $N \cong {}_kL_j^{\nu} \otimes_{L_j} {}_jL_i^{\mu}$. Choose a basis B of L_j over $L_{kj}L_{ji}$. Let $f = (f_b)_{b \in B}$ be the morphism $\bigoplus_{b \in B} M \to N$ of $L_k \otimes_K L_i$-modules where f_b is defined by $1 \otimes 1 \mapsto 1 \otimes b \otimes 1$. Then f is an isomorphism, since it is surjective by the choice of B and $|B| \cdot \dim_K M = \frac{d_{kji}d_j}{d_{kj}d_{ji}} \cdot \frac{d_k d_i}{d_{kji}} = \frac{d^{kj}d^{ji}}{d_j} = \dim_K N$. Now use $|B| = [L_j : L_{kj}L_{ji}]$ and $[M : {}_kL_i^{\rho}] = \delta_{\rho \in G_{ki}^{\nu\mu}}$ by Corollary 2.5.21. □

Remark 2.5.23. The integers $q_{kji} = d_{kji}d_j/(d_{kj}d_{ji})$ and $r_{kji} = d_{ki}/d_{kji}$ from Definition 2.1.8 are the degree $[L_j : L_{kj}L_{ji}]$ and the cardinality $|G_{ki}^{\nu\mu}| = [L_{ki} : L_{kj} \cap L_{ji}]$.

Remark 2.5.24. The length of ${}_kL_j^{\nu} \otimes_{L_j} {}_jL_i^{\mu}$ is $|G_{ki}^{\nu\mu}| \cdot [L_j : L_{kj}L_{ji}] = d_{ki}d_j/(d_{kj}d_{ji})$.

2.5.4 Dual, Double, and Premutation

Lemma 2.5.25. *Let $\mathcal{H}$ be the modulation of a modular quiver Q over $(L/K, \alpha)$. Then $\overline{\mathcal{H}}$ and $\mathcal{H}^*$ are isomorphic to the modulations of $\overline{Q}$ and Q^*, respectively, over $(L/K, \alpha)$.*

Proof. Given Definitions 2.1.12, 2.4.16 and 2.5.12, all we have to show is $\big({}_jL_i^{\rho}\big)^* \cong {}_jL_i^{\rho^{-1}}$. This readily follows from $(\star)$. □

Notation 2.5.26. Given an R-bimodule $A = \bigoplus_{i,k \in Q_0} {}_kA_i$, the *premutation of* A at $j \in Q_0$ is defined in [DWZ08, (5.3)] as the R-bimodule $A^{\sim j} = \bigoplus_{i,k \in Q_0} {}_k\big(A^{\sim j}\big)_i$ with

$$
{}_k\big(A^{\sim j}\big)_i \;=\; \begin{cases} \big({}_kA_i\big)^* & \text{if } j \in \{i,k\}, \\ {}_kA_i \oplus \big({}_kA_j \otimes_{R_j} {}_jA_i\big) & \text{otherwise.} \end{cases}
$$

Lemma 2.5.27. *Let j be a 2-acyclic vertex in a modular quiver Q. Denote by $\mathcal{H}$ and $\mathcal{H}^{\sim j}$ the modulations of Q and $Q^{\sim j}$ over $(L/K, \alpha)$, respectively. Then $A_{\mathcal{H}^{\sim j}} \cong A_{\mathcal{H}}^{\sim j}$.*

Proof. Inspecting how $Q^{\sim j}$ and $\mathcal{H}^{\sim j}$ are defined (see Definitions 2.1.12 and 2.5.12), this is an immediate consequence of $\big({}_kL_i^{\rho}\big)^* \cong {}_kL_i^{\rho^{-1}}$ and Lemma 2.5.22. □

2.5.5 Comfy Modulations

The modulations considered in Chapters 5 and 6 are slightly less general than cyclic Galois modulations. Below we define the kind of modulations that will be used in those chapters. We also give explicit formulas for the projections π_ρ for such modulations.

Convention 2.5.28. Fix a cyclic Galois extension L/K.

Remark 2.5.29. Assume that K contains a primitive m-th root ζ of unity for $m = [L : K]$. As a classical consequence of Hilbert's Satz 90 (see [Bou81, V. §11 no. 6]) the extension L/K is generated by a primitive element v with minimal polynomial $\varepsilon^m - w \in K[\varepsilon]$ over K. Then we have an isomorphism $\mathbb{Z}/m\mathbb{Z} \xrightarrow{\alpha_{\zeta,v}} \mathrm{Gal}(L/K)$ defined by $1 \mapsto (v \mapsto \zeta v)$.

Definition 2.5.30. A *com(putation)f(riendl)y extension* is a triple $(L/K, \zeta, v)$ consisting of a cyclic Galois extension L/K, a primitive $[L : K]$-th root of unity $\zeta \in K$, and a primitive element $v \in L$ for the extension L/K such that $v^{[L:K]} \in K$.

Remark 2.5.31. Every cyclic Galois extension L/K where K contains a primitive $[L : K]$-th root of unity is part of a comfy extension due to Remark 2.5.29.

Remark 2.5.32. The existence of a primitive m-th root of unity in K implies $\mathrm{char}(K) \nmid m$.

Example 2.5.33. The triples $(\mathbb{C}/\mathbb{R}, -1, v)$ where $v \in \mathbb{C}$ is any non-zero imaginary number are comfy extensions.

Example 2.5.34. Let $q = p^s$ be a prime power and $m \in \mathbb{N}_+$ divisible by all of the weights d_i. Then $\mathbb{F}_{q^m}/\mathbb{F}_q$ is a Q-admissible cyclic Galois extension (see Example 2.5.3). By Dirichlet's theorem on arithmetic progressions there are, for each fixed m, infinitely many choices for p such that $m \mid p-1$. For such a choice, the field $\mathbb{F}_p$ contains a primitive m-th root of unity and Remark 2.5.29 allows us to find a comfy extension $(\mathbb{F}_{q^m}/\mathbb{F}_q, \zeta, v)$.

Example 2.5.35. Let m be as in Example 2.5.34 and let ζ be a primitive m-th root of unity in the algebraic closure of K. Denote by $F(t^m)$ the function field in t^m over $F = K(\zeta)$. Then $(F(t)/F(t^m), \zeta, t)$ is a comfy extension.

Convention 2.5.36. Fix a comfy extension $(L/K, \zeta, v)$ such that L/K is Q-admissible.

Definition 2.5.37. Let Q be a modular quiver. The *modulation of Q over* $(L/K, \zeta, v)$ is the modulation of Q over $(L/K, \alpha_{\zeta,v})$.

Notation 2.5.38. For intermediate fields F of L/K let $m_F := [F : K]$ and $m^F := [L : F]$. Set $\zeta^F := \zeta^{m_F}$ and $\zeta_F := \zeta^{m^F}$ and $v_F := v^{m^F}$.

We abbreviate ζ_{L_i} as ζ_i and v_{L_i} as v_i for $i \in Q_0$ and $\zeta_{L_{ji}}$ as ζ_a for $j \xleftarrow{a} i \in Q_1$.

Remark 2.5.39. The triples $(F/K, \zeta_F, v_F)$ and $(L/F, \zeta^F, v)$ are again comfy extensions.

The statement of the next lemma partially appears in [GL16a, Proposition 2.15].

Lemma 2.5.40. *Let M be an $L_j \otimes_K L_i$-module for some $i, j \in Q_0$. For every intermediate field F of L_{ji}/K and every $\rho \in \mathrm{Gal}(F/K)$ we can write $\rho = \alpha_{\zeta_F, v_F}(r)$ for some $r \in \mathbb{Z}$. Then we have $\rho(v_F^s) = \zeta_F^{rs} v_F^s$ for all $s \in \mathbb{Z}$ and the projection π_ρ acts on $x \in M$ as*

$$\pi_\rho(x) \;=\; \frac{1}{m_F} \sum_{s=0}^{m_F-1} \rho(v_F^s) x v_F^{-s} \,.$$

Proof. Clearly, $\rho = \alpha_{\zeta_F, v_F}(r)$ for some $r \in \mathbb{Z}$ and $\rho(v_F^s) = \zeta_F^{rs} v_F^s$ for all $s \in \mathbb{Z}$, since α_{ζ_F, v_F} is the isomorphism $\mathbb{Z}/m_F\mathbb{Z} \xrightarrow{\cong} \mathrm{Gal}(F/K)$ given by $1 \mapsto (v_F \mapsto \zeta_F v_F)$. Now let $\pi'_\rho(x)$ be the right-hand side of the equation in the lemma. It is a straightforward exercise to verify $\pi'_\rho(x) v_F = \rho(v_F)\pi'_\rho(x)$. Equation ($\star$) then implies $\pi'_\rho(M) \subseteq M^\rho$ because v_F is a primitive element for F/K. The identities $\sum_{r=0}^{m_F-1} \zeta_F^{rs} = \delta_{s=0} m_F$ for $0 \le s < m_F$ easily yield $\sum_{\rho \in \mathrm{Gal}(F/K)} \pi'_\rho(x) = x$. All in all, we can conclude $\pi'_\rho(x) = \pi_\rho(x)$. ☐

Example 2.5.41. Write $\mathrm{Gal}(\mathbb{C}/\mathbb{R}) = \{\mathrm{id}, \rho\}$. Then we have $\pi_{\mathrm{id}}(x) = \frac{1}{2}(x + vxv^{-1})$ and $\pi_\rho(x) = \frac{1}{2}(x - vxv^{-1})$ for every comfy extension $(\mathbb{C}/\mathbb{R}, -1, v)$.

Example 2.5.42. Set $w := v^{m_L}$ such that $\varepsilon^{m_L} - w \in K[\varepsilon]$ is the minimal polynomial of v over K. Let $\mathcal{H} = (R_i, A_a)_{i,a}$ be the modulation of a modular quiver Q over $(L/K, \zeta, v)$.

Then $R_i \cong K[\varepsilon_i]/\big(\varepsilon_i^{d_i} - w\big)$ and $A_a \cong K[\varepsilon_j, \varepsilon_i]/\big(\zeta_a^{\sigma_a}\, \varepsilon_j^{f_{ij}} - \varepsilon_i^{f_{ji}},\, \varepsilon_j^{d_j} - w,\, \varepsilon_i^{d_i} - w\big)$ for $j \xleftarrow{a} i$.

Similarly as for the GLS modulation in Example 2.4.27, the path algebra $H_\mathcal{H}$ can be identified with the path algebra $KQ^{\circlearrowleft}$ modulo an ideal $I(d, \sigma)$, which is in this case generated by the relations $\varepsilon_i^{d_i} = w$ for $i \in Q_0$ and $a\varepsilon_i^{f_{ji}} = \zeta_a^{\sigma_a}\varepsilon_j^{f_{ij}}a$ for $j \xleftarrow{a} i \in Q_1$.

2.5.6 Comfy vs. GLS Modulations

Examples 2.4.27 and 2.5.42 revealed the striking similarities between comfy modulations and the modulations considered in [GLS16a]. In this subsection we sketch how this resemblance could be formalized and give a small example.

Convention 2.5.43. Fix a field K and a quiver Q and denote by $\widetilde{K}$ the quotient of the polynomial ring $K[w, \lambda_a : a \in Q_1]$ by the ideal $\big(\lambda_a^{d_{ji}} - 1 : j \xleftarrow{a} i \in Q_1\big)$.

Definition 2.5.44. Let Q be a weighted quiver. Then the family $\widetilde{\mathcal{H}} = \big(\widetilde{R}_i, \widetilde{A}_a\big)_{i,a}$ is called the *universal comfy modulation* of Q where $\widetilde{R}_i = \widetilde{K}[\varepsilon_i]/\big(\varepsilon_i^{d_i} - w\big)$ are indexed by $i \in Q_0$ and $\widetilde{A}_a = \widetilde{K}[\varepsilon_j, \varepsilon_i]/\big(\lambda_a\varepsilon_j^{f_{ij}} - \varepsilon_i^{f_{ji}},\, \varepsilon_j^{d_j} - w,\, \varepsilon_i^{d_i} - w\big)$ are indexed by $j \xleftarrow{a} i \in Q_1$.

Remark 2.5.45. The universal comfy modulation $\widetilde{\mathcal{H}}$ is a "$\widetilde{K}$-modulation for Q" in the sense that (a) $\widetilde{R}_i$ is free of rank d_i over $\widetilde{K}$ for all $i \in Q_0$, and (b) ${}_{\widetilde{R}_j}(\widetilde{A}_a)$ and $(\widetilde{A}_a)_{\widetilde{R}_i}$ are free of rank f_{ji} and f_{ij}, respectively, for all $j \xleftarrow{a} i \in Q_1$. Compare Definition 2.4.2.

Notation 2.5.46. The *path algebra* $\widetilde{H}$ of $\widetilde{\mathcal{H}}$ is the tensor algebra $\widetilde{R}\langle\widetilde{A}\rangle$ of $\widetilde{A} = \bigoplus_{a\in Q_1}\widetilde{A}_a$ over $\widetilde{R} = \prod_{i\in Q_0}\widetilde{R}_i$. We write a for $1 \in \widetilde{A}_a$ when considered as an element in $\widetilde{A} \subseteq \widetilde{H}$.

Remark 2.5.47. The path algebra $\widetilde{H} = \widetilde{R}\langle\widetilde{A}\rangle$ can be identified with the path-algebra quotient $\widetilde{K}Q^{\circlearrowleft}/\widetilde{I}(d)$ where $Q^{\circlearrowleft}$ is the quiver defined in Example 2.4.27 and $\widetilde{I}(d)$ is the ideal generated by the relations $\varepsilon_i^{d_i} = w$ for $i \in Q_0$ and $a\varepsilon_i^{f_{ji}} = \lambda_a\varepsilon_j^{f_{ij}}a$ for $j \xleftarrow{a} i \in Q_1$.

Remark 2.5.48. For the GLS modulation $\mathcal{H}$ for Q described in Examples 2.4.11 and 2.4.27 we have as K-algebras $H_{\mathcal{H}} \cong \widetilde{K}_{0,1} \otimes_{\widetilde{K}} \widetilde{H}$ with $\widetilde{K}_{0,1} = \widetilde{K}/(w, \lambda_a - 1 : a \in Q_1) \cong K$.

Here $\mathcal{H}$ arises from $\widetilde{\mathcal{H}}$ by specializing $w \to 0$ and $\lambda_a \to 1$.

Remark 2.5.49. For the modulation $\mathcal{H}$ of a modular quiver Q over $(L/K, \zeta, v)$ we have as K-algebras $H_{\mathcal{H}} \cong \widetilde{K}_{v^{[L:K]},\sigma} \otimes_{\widetilde{K}} \widetilde{H}$ with $\widetilde{K}_{v^{[L:K]},\sigma} = \widetilde{K}/(w - v^{[L:K]}, \lambda_a - \zeta_a^{\sigma_a} : a \in Q_1) \cong K$.

Now $\mathcal{H}$ arises from $\widetilde{\mathcal{H}}$ by specializing $w \to v^{[L:K]}$ and $\lambda_a \to \zeta_a^{\sigma_a}$.

For the next example we formalize the definition of the (unweighted) quiver $Q^{\circlearrowleft}$:

Definition 2.5.50. The *loop extension* of a weighted quiver Q is the Q_0-quiver $Q^{\circlearrowleft}$ with

$$Q_1^{\circlearrowleft} \;=\; Q_1 \,\dot{\cup}\, \big\{ i \xleftarrow{\varepsilon_i} i \,\big|\, i \in Q_0 \big\}\,.$$

Example 2.5.51. Let Q be the modular quiver $1 \xleftarrow{a} 2$ of type A_2 with $d_1 = d_2 = 2$ and $\sigma_a = 1 \in \mathbb{Z}/2\mathbb{Z}$, which already appeared in Example 2.5.13. Then $Q^{\circlearrowleft}$ is the quiver

$$\begin{tikzcd} 1 \arrow[loop above, "\varepsilon_1"] & 2 \arrow[l, "a"'] \arrow[loop above, "\varepsilon_2"] \end{tikzcd} .$$

Let I be the ideal of $\mathbb{R}Q^{\circlearrowleft}$ generated by the relations $\varepsilon_1^2 = w$, $\varepsilon_2^2 = w$, $a\varepsilon_2 = \lambda_a \varepsilon_1 a$ for some fixed $w, \lambda_a \in K$. For the choice $w = -1$ and $\lambda_a = (-1)^{\sigma_a} = -1$ the quotient $\mathbb{R}Q^{\circlearrowleft}/I$ is the path algebra of the cyclic Galois modulation $\mathcal{H}^\rho$ from Example 2.5.13, whereas the choice $w = 0$ and $\lambda_a = 1$ gives rise to the path algebra of the GLS modulation.

2.5.7 Path-Algebra Bases

Below we describe a $\widetilde{K}$-basis for the path algebra of the universal comfy modulation and related K-bases for the path algebras of GLS and comfy modulations. These bases are useful for explicit computations with path-algebra elements.

Definition 2.5.52. A path $\varepsilon_{i_\ell}^{q_\ell} a_\ell \cdots \varepsilon_{i_1}^{q_1} a_1 \cdot \varepsilon_{i_0}^{q_0}$ in $Q^{\circlearrowleft}$ with $a_1, \ldots, a_\ell \in Q_1$ and $q_0, \ldots, q_\ell \in \mathbb{N}$ is said to be *d-reduced* if $q_s < d_{i_s}$ for all $0 \le s \le \ell$. By convention $\varepsilon_i^0 = e_i$ for all $i \in Q_0$.

Let $\sim$ be the equivalence relation on the set of paths in $Q^{\circlearrowleft}$ generated by

$$\varepsilon_{i_\ell}^{q_\ell} a_\ell \cdots \varepsilon_{i_1}^{q_1} a_1 \cdot \varepsilon_{i_0}^{q_0} \;\sim\; \varepsilon_{i_\ell}^{q'_\ell} a_\ell \cdots \varepsilon_{i_1}^{q'_1} a_1 \cdot \varepsilon_{i_0}^{q'_0}$$

whenever $i_\ell \xleftarrow{a_\ell} \cdots \xleftarrow{a_1} i_0$ is a path in Q and there exist $0 < t \le \ell$ and $q_0, q'_0, \ldots, q_\ell, q'_\ell \in \mathbb{N}$ with $q_s = q'_s$ for all $s \in \{0, \ldots, \ell\} \setminus \{t-1, t\}$ and $q_t - q'_t = f_{i_{t-1} i_t}$ and $q'_{t-1} - q_{t-1} = f_{i_t i_{t-1}}$.

A *Q-tensor class* is an equivalence class of $\sim$ that contains only d-reduced paths.

The representative $\varepsilon_{i_\ell}^{q_\ell} a_\ell \cdots \varepsilon_{i_1}^{q_1} a_1 \cdot \varepsilon_{i_0}^{q_0}$ of a Q-tensor class that minimizes $(q_\ell, \ldots, q_0)$ with respect to the lexicographical order on $\mathbb{N}^{\ell+1}$ is called a Q-tensor path of type $a_\ell \cdots a_1$. For each vertex $i \in Q_0$ we have Q-tensor paths $e_i = \varepsilon_i^0, \varepsilon_i^1, \ldots, \varepsilon_i^{d_i - 1}$ of type e_i.

Denote by $\mathcal{T}_Q(p)$ the set of Q-tensor paths of type p. Set $\mathcal{T}_Q = \dot{\bigcup}_{p \text{ path in } Q} \mathcal{T}_Q(p)$.

Example 2.5.53. For the weighted quiver Q from Example 2.5.51 we have $\mathcal{T}_Q(a) = \{a, a\varepsilon_2\}$. The element $\varepsilon_1 a$ represents the same Q-tensor class as $a\varepsilon_2$ but is itself not a Q-tensor path. The path $\varepsilon_2 a \varepsilon_1 \sim a\varepsilon_1^2$ in $Q^{\circlearrowleft}$ does not define a Q-tensor class.

Lemma 2.5.54. $\bigl|\mathcal{T}_Q\bigl(i_\ell \xleftarrow{a_\ell} \cdots \xleftarrow{a_1} i_0\bigr)\bigr| = d_{i_0} \cdot \prod_{s=1}^{\ell} f_{i_{s-1} i_s} = \prod_{s=0}^{\ell} d_{i_s} \Big/ \prod_{s=1}^{\ell} d_{i_{s-1} i_s}.$

Proof. Clearly, $|\mathcal{T}_Q(e_i)| = d_i$. Assume $|\mathcal{T}_Q(a_{\ell-1} \cdots a_1)| = f_{i_{\ell-2} i_{\ell-1}} \cdots f_{i_0 i_1} \cdot d_{i_0}$ by induction and note $\mathcal{T}_Q(a_{\ell-1} \cdots a_1) \times \bigl\{0, \ldots, f_{i_{\ell-1} i_\ell} - 1\bigr\} \xrightarrow{\cong} \mathcal{T}_Q(a_\ell \cdots a_1)$ via $(p, q) \mapsto \varepsilon_{i_\ell}^q a_\ell p$. $\square$

Lemma 2.5.55. *The path algebra $\widetilde{H}$ of the universal comfy modulation of Q is free over $\widetilde{K}$. A $\widetilde{K}$-basis is given by the image of the map $\mathcal{T}_Q \to \widetilde{H}$ defined as*

$$\varepsilon_{i_\ell}^{q_\ell} a_\ell \cdots \varepsilon_{i_1}^{q_1} a_1 \cdot \varepsilon_{i_0}^{q_0} \;\mapsto\; \varepsilon_{i_\ell}^{q_\ell} a_\ell \cdots \varepsilon_{i_1}^{q_1} a_1 \cdot \varepsilon_{i_0}^{q_0}\,.$$

Proof. This is a consequence of the definitions of $\widetilde{H} = \widetilde{R}\langle\widetilde{A}\rangle$ and $\mathcal{T}_Q$. $\square$

Remark 2.5.56. The $\widetilde{K}$-basis of $\widetilde{H}$ described in Lemma 2.5.55 "specializes" under each of the isomorphisms in Remarks 2.5.48 and 2.5.49 to a K-basis of $H_{\mathcal{H}}$. The elements of these K-bases have the form $\varepsilon_{i_\ell}^{q_\ell} a_\ell \cdots \varepsilon_{i_1}^{q_1} a_1 \cdot \varepsilon_{i_0}^{q_0}$ and $v_{i_\ell}^{q_\ell} a_\ell \cdots v_{i_1}^{q_1} a_1 \cdot v_{i_0}^{q_0}$, respectively.

2.6 Jacobian Algebras and Potentials

We recollect some basics about topological rings equipped with the $\mathfrak{m}$-adic topology and discuss completed tensor algebras. After that, Jacobian algebras $\mathcal{J}(W)$ are introduced for arbitrary completed tensor algebras by a minor modification of [DWZ08, Definition 3.1]. For potentials we define R-equivalence and the weaker notion of I-equivalence, which both specialize to Derksen, Weyman, and Zelevinsky's right-equivalence.

The last two subsections contain the results from [GL16a, §§ 3 and 10.1–2]. Namely, we prove that the so-called Splitting Theorem remains true for cyclic Galois modulations. Finally, premutation of potentials will be described for comfy modulations.

2.6.1 Topological Rings

Definition 2.6.1. An *adic ring* is a topological ring H together with an ideal $\mathfrak{m}_H$ of H where H carries the $\mathfrak{m}_H$-adic topology, i.e. each point $z \in H$ has $\{z + \mathfrak{m}_H^n \mid n \in \mathbb{N}\}$ as a fundamental system of open neighborhoods.

The *order* of an element z in an adic ring H is $\operatorname{ord}_H(z) := \min\{n \in \mathbb{N} \mid z \notin \mathfrak{m}_H^{n+1}\}$ with the convention $\min \varnothing = \infty$. If confusion seems unlikely, we write ord instead of ord_H.

A *morphism* $H \xrightarrow{f} H'$ of adic rings is a continuous ring homomorphism $H \xrightarrow{f} H'$.

The *completion* of an adic ring H is the adic ring $\widehat{H} = \varprojlim H/\mathfrak{m}_H^n$ with $\mathfrak{m}_{\widehat{H}} = \widehat{H}\mathfrak{m}_H\widehat{H}$. A *complete ring* is an adic ring H for which the canonical map $H \xrightarrow{\iota} \widehat{H}$ is an isomorphism.

We collect some elementary facts about adic and complete rings in the next lemma.

Lemma 2.6.2. *Let H be an adic ring.*

(a) For every $X \subseteq H$ its closure is $\bigcap_{n\in\mathbb{N}}(X + \mathfrak{m}_H^n)$. In particular, $\mathfrak{m}_H^n$ is closed for all n.

(b) The map $H \times H \xrightarrow{\operatorname{dist}_H} \mathbb{R}$, $(z, z') \mapsto \exp(-\operatorname{ord}_H(z - z'))$, where $\exp(-\infty) := 0$, defines a pseudometric on H. The topology induced by dist_H is the topology of H.

(c) A sequence $(x_m)_{m\in\mathbb{N}}$ in H is a Cauchy sequence with respect to dist_H if and only for all $n \in \mathbb{N}$ there is $k \in \mathbb{N}$ such that $x_{m+1} - x_m \in \mathfrak{m}_H^n$ for all $m \geq k$.

(d) H is Hausdorff if and only if dist_H is a metric if and only if $\bigcap_{n\in\mathbb{N}} \mathfrak{m}_H^n = 0$.

(e) H is a complete ring if and only if dist_H is a complete metric.

(f) A ring homomorphism $H \xrightarrow{f} H'$ between two adic rings is (sequentially) continuous if and only if there exists $n \in \mathbb{N}$ such that $f(\mathfrak{m}_H^n) \subseteq \mathfrak{m}_{H'}$.

(g) If H is a complete ring, then $\mathfrak{m}_H \subseteq \operatorname{rad}(H)$ or, equivalently, $1 + \mathfrak{m}_H \subseteq H^\times$.

Proof. Statements (a)–(f) are well-known and straightforwardly verified. For a proof of (g) see [Lam91, (21.30)]. □

Example 2.6.3. Let $H = K[x]$ be the polynomial ring viewed as adic ring with $\mathfrak{m}_H = (x)$. The order of $p = \sum_n p_n x^n \in H$ with $p_n \in K$ is $\mathrm{ord}_H(p) = \min\{n \in \mathbb{N} \mid p_n \neq 0\}$.

Example 2.6.4. The adic ring H in Example 2.6.3 is not complete. The sequence $(x^n)_{n\in\mathbb{N}}$ is a Cauchy sequence with respect to dist_H that does not converge in H. However, in the completion $\widehat{H}$ one has $\lim_{n\to\infty} x^n = 0$. There is a canonical isomorphism $K[[x]] \xrightarrow{\cong} \widehat{H}$.

Lemma 2.6.5. *Let H be an adic ring. If H is Hausdorff, the canonical map $H \xrightarrow{\iota} \widehat{H}$ is a topological embedding with dense image.*

Proof. See [War93, Corollary 5.22]. □

2.6.2 Completed Path Algebras

Definition 2.6.6. Let A be an R-bimodule over K. The *completed tensor algebra* of A is the R-algebra $R\langle\langle A\rangle\rangle$ over K whose underlying R-bimodule is

$$R\langle\langle A\rangle\rangle \;=\; \prod_{n\in\mathbb{N}} A^{\otimes n}\,.$$

The product $xy = \sum_n (xy)_n \in R\langle\langle A\rangle\rangle$ of $x = \sum_n x_n, y = \sum_n y_n \in R\langle\langle A\rangle\rangle$ is defined by

$$(xy)_n \;=\; \sum_{k=0}^{n} x_k \otimes y_{n-k}\,.$$

If $R = R_{\mathcal{H}}$, $A = A_{\mathcal{H}}$ for a K-modulation $\mathcal{H}$ of a weighted quiver Q, we call $\widehat{H}_{\mathcal{H}} = R\langle\langle A\rangle\rangle$ the *completed path algebra* of Q defined by $\mathcal{H}$.

Notation 2.6.7. For maps $M \xrightarrow{f} R\langle\langle A\rangle\rangle$ set $f_{(n)} := \mathrm{pr}_n \circ f$ and $f_{\geq n} := \mathrm{pr}_{\geq n} \circ f$ where pr_n and $\mathrm{pr}_{\geq n}$ are the projections $R\langle\langle A\rangle\rangle \twoheadrightarrow A^{\otimes n}$ and $R\langle\langle A\rangle\rangle \twoheadrightarrow \prod_{k\geq n} A^{\otimes k}$ of R-bimodules.

Convention 2.6.8. Fix a finite-dimensional R-bimodule A over K.

Abbreviate $H = R\langle A\rangle$ and $\widehat{H} = R\langle\langle A\rangle\rangle$. The tensor algebra H and the completed tensor algebra $\widehat{H}$ are regarded as adic rings, where $\mathfrak{m}_H$ and $\mathfrak{m}_{\widehat{H}}$ are the ideals generated by A.

Remark 2.6.9. Explicitly, $\mathfrak{m}_H = \bigoplus_{n\in\mathbb{N}_+} A^{\otimes n}$ and $\mathfrak{m}_{\widehat{H}} = \prod_{n\in\mathbb{N}_+} A^{\otimes n}$. For each $n \in \mathbb{N}$ there is a commuting square of canonical maps:

$$\begin{tikzcd} H \arrow[r, hook, "\iota"] \arrow[d, two heads] & \widehat{H} \arrow[d, two heads] \\ H/\mathfrak{m}_H^n \arrow[r, "\cong"] & \widehat{H}/\mathfrak{m}_{\widehat{H}}^n \end{tikzcd}$$

We get an induced map $H \xhookrightarrow{\iota} \widehat{H} \xrightarrow{\cong} \varprojlim H/\mathfrak{m}_H^n$ and realize that $\widehat{H}$ is the completion of H. It is clear that $\widehat{H}$ is a complete ring. Since H is Hausdorff, Lemma 2.6.5 (a) applies.

Universal property of tensor algebra (see Lemma 2.2.6) and completion (inverse limit) combine to the following universal property of the completed tensor algebra.

Lemma 2.6.10. *Let Λ be a complete R-algebra and $A \xrightarrow{f} \Lambda$ a map of R-bimodules such that the induced map $R\langle A\rangle \xrightarrow{\hat{f}} \Lambda$ is continuous, i.e. $\hat{f}(A^{\otimes n}) \subseteq \mathfrak{m}_\Lambda$ for some n. Then there exists a unique morphism $\tilde{f}$ of adic R-algebras making the following diagram commute:*

$$\begin{tikzcd} A \arrow[rr, hook, "\kappa_A"] \arrow[dr, "f"'] & & R\langle\langle A\rangle\rangle \arrow[dl, dashed, "\tilde{f}"] \\ & \Lambda & \end{tikzcd}$$

Here κ_A denotes the canonical inclusion $A \hookrightarrow R\langle\langle A\rangle\rangle$.

Proof. The property $\hat{f}(A^{\otimes n}) \subseteq \mathfrak{m}_\Lambda$ ensures that f induces a map $R\langle\langle A\rangle\rangle \to \varprojlim \Lambda/\mathfrak{m}_\Lambda^n = \widehat{\Lambda}$. Since Λ is complete, we get a map as claimed in the lemma. $\square$

Definition 2.6.11. Assume a factorization $R = \prod_{i\in I} R_i$ is fixed.

Let $\Lambda = R\langle A\rangle/J$ and $\Lambda' = R\langle A'\rangle/J'$ with $J \subseteq \mathfrak{m}_\Lambda$ and $J' \subseteq \mathfrak{m}_{\Lambda'}$. A morphism $\Lambda \xrightarrow{f} \Lambda'$ of K-algebras is a K^I*-algebra morphism* if f induces automorphisms $R_i \xrightarrow{\cong} R_i$ for all $i \in I$.

Similarly, one defines K^I-algebra morphisms for quotients of completed tensor algebras.

For loop-free weighted I-quivers every K^I-algebra morphism between their (completed) path algebras is automatically a morphism of topological algebras:

Lemma 2.6.12. *Every K^I-algebra morphism $\Lambda \xrightarrow{f} \Lambda'$ between (completed) path algebras defined by $(R_i)_i$-modulations for loop-free weighted I-quivers Q and Q' maps $\mathfrak{m}_\Lambda$ into $\mathfrak{m}_{\Lambda'}$.*

Proof. Say $\Lambda = H_{\mathcal{H}}$ and $\Lambda' = H_{\mathcal{H}'}$ for K-modulations $\mathcal{H} = (R_i, A_a)_{i,a}$ and $\mathcal{H}' = (R_i, A'_a)_{i,a}$. If $f(\mathfrak{m}_\Lambda) \not\subseteq \mathfrak{m}_{\Lambda'}$, then there are $i, j \in I$ and $x \in {}_jA_i \neq 0$ such that $f(x) = e_j f(x) e_i = y_0 + y_+$ for some $y_0 \in {}_jR_i$ and $y_+ \in {}_j(\mathfrak{m}_{\Lambda'})_i$ with $y_0 \neq 0$. Hence ${}_jR_i \neq 0$, so $i = j$. But then ${}_iA_i \neq 0$ and Q is not loop-free. The proof for completed path algebras is the same. $\square$

In the completed setting, Lemma 2.6.12 has a generalization for quivers with loops:

Lemma 2.6.13. *Let $\widehat{H}$ and $\widehat{H}'$ be the completed path algebras defined by $(R_i)_i$-modulations for weighted I-quivers Q and Q' such that for all $i \in I$ either Q is loop-free at the vertex i or the algebra R_i is local with nilpotent maximal ideal $\mathfrak{p}_i$.*

Then every K^I-algebra morphism $\widehat{H} \xrightarrow{f} \widehat{H}'$ is continuous and $f\big(\mathfrak{m}_{\widehat{H}}^n\big) \subseteq \mathfrak{m}_{\widehat{H}'}$ where

$$n \;=\; \max\big(\{1\} \cup \{n_i \,|\, i \in I \text{ not loop-free in } Q\}\big)$$

and n_i is the nilpotency degree of $\mathfrak{p}_i$, i.e. the smallest positive integer k with $\mathfrak{p}_i^k = 0$.

Proof. Say $\widehat{H} = \widehat{H}_{\mathcal{H}}$ for a K-modulation $\mathcal{H} = (R_i, A_a)_{i,a}$. With the same argument given in the proof of Lemma 2.6.12 one shows that $f(x) \in \mathfrak{m}_{\widehat{H}'}$ for all $x \in {}_jA_i$ with $j \neq i$. For non-zero $x \in {}_iA_i$ the quiver Q has a loop at i and so R_i is local with nilpotent maximal ideal $\mathfrak{p}_i$. We can write $f(x) = e_i f(x) e_i = y_0 + y_+$ with $0 \neq y_0 \in R_i$ and $y_+ \in {}_i\big(\mathfrak{m}_{\widehat{H}'}\big)_i$. To prove the lemma, it suffices to show $y_0 \in \mathfrak{p}_i$. If this were not the case, then $y_0 \in R_i \backslash \mathfrak{p}_i = R_i^\times$. Let z be the inverse of $f^{-1}(y_0)$ in R_i. Lemma 2.6.2 (g) implies that the element $y_+ = -y_0 f(1 - zx)$ is a unit in ${}_i\widehat{H}'_i$. But this contradicts the fact that $\mathrm{pr}_0(y_+) = 0$ is not invertible in R_i. $\square$

Under the assumption that the ground ring is a product of division algebras, Lemma 2.6.12 is hence also true for quivers with loops, at least in the completed world:

Corollary 2.6.14. *Every K^I-algebra morphism $\widehat{H} \xrightarrow{f} \widehat{H}'$ between completed path algebras defined by $(R_i)_i$-modulations with division algebras R_i satisfies $f\big(\mathfrak{m}_{\widehat{H}}\big) \subseteq \mathfrak{m}_{\widehat{H}'}$.*

Proof. This is a direct consequence of Lemma 2.6.13. $\square$

Example 2.6.15. The GLS modulations (see Example 2.4.11) satisfy the assumption in Lemma 2.6.13. The nilpotency degree of the maximal ideal of $R_i = K[\varepsilon_i]/\big(\varepsilon_i^{d_i}\big)$ is $n_i = d_i$.

Example 2.6.16. Let Q be the weighted quiver $a \circlearrowleft 1$ with $d_1 = 2$. The completed path algebra $\widehat{H}$ of Q defined by the GLS modulation (see Example 2.4.27) is the ring $\widehat{H} = R[[a]]$ of formal power series over the dual numbers $R = K[\varepsilon]/(\varepsilon^2)$ with $\mathfrak{m}_{\widehat{H}} = (a)$. The rule $a \mapsto \varepsilon$ induces an R-algebra morphism $\widehat{H} \xrightarrow{f} \widehat{H}$. As claimed in Lemma 2.6.13, f is continuous, since $f\big(\mathfrak{m}_{\widehat{H}}^2\big) = 0 \subseteq \mathfrak{m}_{\widehat{H}}$. However, $f\big(\mathfrak{m}_{\widehat{H}}\big) \not\subseteq \mathfrak{m}_{\widehat{H}}$.

Example 2.6.17. The path algebra H of $a \circlearrowleft 1$ over K is the polynomial ring $K[a]$ with distinguished ideal $\mathfrak{m}_H = (a)$. The K-algebra endomorphism of H given by the rule $a \mapsto 1$ is not continuous. This shows that the statements of Lemma 2.6.13 and Corollary 2.6.14 are no longer true when replacing "completed path algebras" by "path algebras".

Notation 2.6.18. For adic R-algebras H and H' we use the notation $\mathrm{Hom}_R^n(H, H')$ for the set of R-algebra morphisms $H \xrightarrow{f} H'$ satisfying $f(\mathfrak{m}_H) \subseteq \mathfrak{m}_{H'}^n$.

Remark 2.6.19. Let $\mathcal{H}$ and $\mathcal{H}'$ be $(R_i)_i$-modulations for weighted I-quivers Q and Q'. If Q is loop-free, we have seen that $\mathrm{Hom}_R^1(H_{\mathcal{H}}, H_{\mathcal{H}'}) = \mathrm{Hom}_R(H_{\mathcal{H}}, H_{\mathcal{H}'})$. Furthermore, if Q is loop-free or all R_i are division rings, $\mathrm{Hom}_R^1(\widehat{H}_{\mathcal{H}}, \widehat{H}_{\mathcal{H}'}) = \mathrm{Hom}_R(\widehat{H}_{\mathcal{H}}, \widehat{H}_{\mathcal{H}'})$.

The next lemma is the analog of [DWZ08, Proposition 2.4].

Lemma 2.6.20. *Let A and A' be R-bimodules. For every $n \in \mathbb{N}_+$ there is a bijection:*

$$\begin{array}{ccc} \mathrm{Hom}_R^n\Big(R\langle\langle A\rangle\rangle, R\langle\langle A'\rangle\rangle\Big) & \xrightarrow{\ \cong\ } & \mathrm{Hom}_{R^e}\Big(A, \mathfrak{m}_{R\langle\langle A'\rangle\rangle}^n\Big) \\ f & \longmapsto & f|_A \end{array}$$

Moreover, f is an isomorphism if and only if the component $f_{(1)}|_A$ is an isomorphism.

Proof. The map in the proposition is well-defined, since $f(A) \subseteq \mathfrak{m}_{R\langle\langle A'\rangle\rangle}^n$. It is a bijection because of Lemma 2.6.10. It remains to verify the last claim. For $n > 1$ it is trivially true, since in this case neither f nor $f_{(1)}|_A$ can be an isomorphism. So let us assume $n = 1$.

For every element $x = \sum_m x_m \in R\langle\langle A\rangle\rangle$ we have $f_{(1)}(x_m) = 0$ for all $m > 1$ because of the continuity of f and $f_{(1)}(x_0) = 0$ since f is a map of R-algebras. Thus $f_{(1)} = f_{(1)}|_A \circ \mathrm{pr}_{(1)}$ such that for all $g \in \mathrm{Hom}^1_R(R\langle\langle A'\rangle\rangle, R\langle\langle A\rangle\rangle)$ one has $(fg)_{(1)}|_{A'} = f_{(1)}|_A \circ g_{(1)}|_{A'}$ and, similarly, $(gf)_{(1)}|_A = g_{(1)}|_{A'} \circ f_{(1)}|_A$. In particular, if f is an isomorphism, so is $f_{(1)}|_A$.

On the other hand, assume that $f_{(1)}|_A$ has an inverse $A' \xrightarrow{g} A$. Let $R\langle\langle A'\rangle\rangle \xrightarrow{\widetilde{g}} R\langle\langle A\rangle\rangle$ be the induced map such that $\widetilde{g}|_{A'} = \widetilde{g} \circ \kappa_{A'} = \kappa_A \circ g$. The universal property for $R\langle\langle A\rangle\rangle$ (Lemma 2.6.10) then implies $\widetilde{g} \circ f = \mathrm{id}$, since $\widetilde{g} \circ f \circ \kappa_A = \widetilde{g} \circ \kappa_{A'} \circ f_{(1)}|_A = \kappa_A \circ g \circ f_{(1)}|_A = \kappa_A$. Similarly, one can prove $f \circ \widetilde{g} = \mathrm{id}$ to conclude that f is an isomorphism. □

We adopt the terminology introduced in [DWZ08, Definition 2.5].

Definition 2.6.21. Let $f \in \mathrm{Aut}_R(R\langle\langle A\rangle\rangle)$ and $\mathfrak{m} = \mathfrak{m}_{R\langle\langle A\rangle\rangle}$.

Call f a *change of arrows*, if $f(A) = A$. We say f is *unitriangular*, if $(f - \mathrm{id})(A) \subseteq \mathfrak{m}^{n+1}$ for some $n \in \mathbb{N}_+$. For unitriangular f one defines the *depth* of f as

$$\mathrm{depth}(f) \ := \ \sup\left\{n \in \mathbb{N}_+ \mid (f - \mathrm{id})(A) \subseteq \mathfrak{m}^{n+1}\right\}.$$

Remark 2.6.22. An automorphism $f \in \mathrm{Aut}_R(R\langle\langle A\rangle\rangle)$ is unitriangular of depth $\geq n$ if and only if for all $a \in A$ there is $\nu_a \in \mathfrak{m}^{n+1}$ such that $f(a) = a + \nu_a$.

We repeatedly use the following statement implicitly. Compare [DWZ08, § 4].

Lemma 2.6.23. *Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of unitriangular R-algebra automorphisms of $R\langle\langle A\rangle\rangle$ with $\lim_{n\to\infty} \mathrm{depth}(f_n) = \infty$. Then the pointwise-defined limit $\lim_{n\to\infty} f_n \cdots f_0$ exists and is an R-algebra automorphism of $R\langle\langle A\rangle\rangle$.*

Proof. For all z the sequence $(f_n \cdots f_0(z))_{n\in\mathbb{N}}$ is Cauchy because $\lim_{n\to\infty} \mathrm{depth}(f_n) = \infty$. The completeness of $R\langle\langle A\rangle\rangle$ implies that the limit $f = \lim_{n\to\infty} f_n \cdots f_0$ exists. Clearly, f is an R-algebra morphism. It is an automorphism because $f_{(1)}|_A$ is an automorphism. □

Recall that $H = R\langle A\rangle$ and $\widehat{H} = R\langle\langle A\rangle\rangle$. Via restriction of scalars every $\widehat{H}$-module can also be viewed as an H-module. In the finite-dimensional world, $\widehat{H}$-modules are precisely the nilpotent H-modules. This observation generalizes the discussion in [DWZ08, § 10].

Lemma 2.6.24. *Restriction of scalars induces an equivalence* $\mathrm{mod}(\widehat{H}) \xrightarrow{\simeq} \mathrm{mod}^{\mathrm{nil}}(H)$ *where* $\mathrm{mod}^{\mathrm{nil}}(H)$ *is the full subcategory of* $\mathrm{mod}(H)$ *consisting of modules, called* nilpotent, *that are annihilated by $A^{\otimes n}$ for some large enough n.*

Proof. Nilpotent H-modules can be naturally regarded as $\widehat{H}$-modules and a map between nilpotent H-modules is an $\widehat{H}$-module homomorphism if and only if it is an H-module homomorphism. Therefore it suffices to check that every finite-dimensional $\widehat{H}$-module M is nilpotent. By Lemma 2.6.2 (g) it is $\mathfrak{m} = \mathfrak{m}_{\widehat{H}} \subseteq \mathrm{rad}(\widehat{H})$. Now $\mathfrak{m}^{n+1}M = \mathfrak{m}^n M$ for some n, if M is finite-dimensional. So $\mathfrak{m}^n M = 0$ by Nakayama's lemma (see [Lam91, (4.22)]). □

We close this subsection with the following easy observation.

Lemma 2.6.25. *Let $(X_n)_{n\in\mathbb{N}}$ be a family of additive subgroups $X_n \subseteq A^{\otimes n} \subseteq \widehat{H}$. Then the topological closure of $\bigoplus_{n\in\mathbb{N}} X_n$ in $\widehat{H}$ is $\prod_{n\in\mathbb{N}} X_n$.*

Proof. Let $X = \bigoplus_{n\in\mathbb{N}} X_n$ and $Y = \prod_{n\in\mathbb{N}} X_n$. On the one hand, $X \subseteq Y \subseteq \overline{X}$ because every $y \in Y$ is the limit of the sequence $\big(\sum_{n\le m} y_n\big)_{m\in\mathbb{N}}$ in X. On the other hand, $Y = \overline{Y}$, since a sequence in Y is a Cauchy sequence if and only if for all $n \in \mathbb{N}$ its image in X_n is an eventually constant sequence, i.e. its limit lies in Y. $\square$

2.6.3 Jacobian Algebras

Convention 2.6.26. As before, we denote by R^{e} the enveloping algebra $R \otimes_K R^{\mathrm{op}}$ of R. We regard R-bimodules M over K as left R^{e}-modules via $(s \otimes r) \cdot m = smr$ and as right R^{e}-modules via $m \cdot (s \otimes r) = rms$ for $s \otimes r \in R^{\mathrm{e}}$ and $m \in M$.

Recall the notations $A^\dagger = \mathrm{Hom}_{R^{\mathrm{e}}}(A, R^{\mathrm{e}})$ and $H = R\langle A\rangle$ and $\widehat{H} = R\langle\langle A\rangle\rangle$.

Definition 2.6.27. For $\xi \in A^\dagger$ denote by $A^{\otimes n} = A^{\otimes s-1} \otimes_R A \otimes_R A^{\otimes n-s} \xrightarrow{\partial_\xi^{n,s}} A^{\otimes n-1}$ the K-linear map induced, for $x \in A^{\otimes s-1}$, $a \in A$, $y \in A^{\otimes n-s}$, by the rule

$$\partial_\xi^{n,s}(xay) \;=\; yx \cdot \xi(a)\,.$$

The *cyclic derivative with respect to* ξ is the K-linear map $\widehat{H} \xrightarrow{\partial_\xi} \widehat{H}$ defined as

$$\partial_\xi \;:=\; \sum_{n=1}^{\infty}\sum_{s=1}^{n} \partial_\xi^{n,s}\,.$$

The *Jacobian ideal* ∂W of an element $W \in \widehat{H}$ is the closed ideal of $\widehat{H}$ generated by all cyclic derivatives $\partial_\xi(W)$ with $\xi \in A^\dagger$. The *Jacobian algebra* $\mathcal{J}(W)$ of W is $\mathcal{J}(W) := \widehat{H}/\partial W$.

Remark 2.6.28. Let $\mathfrak{m} = \mathfrak{m}_H$. The map ∂_ξ is well-defined as a map $H \to H$ and sends $\mathfrak{m}^n$ into $\mathfrak{m}^{n-1}$. The universal property of $\widehat{H} = \varprojlim H/\mathfrak{m}^n$ applied to $\widehat{H} \twoheadrightarrow H/\mathfrak{m}^n \xrightarrow{\partial_\xi} H/\mathfrak{m}^{n-1}$ yields the extension of ∂_ξ to a map $\widehat{H} \to \widehat{H}$.

Remark 2.6.29. Almost resembling [DWZ08, (3.1)], we can express ∂_ξ as

$$\partial_\xi(a_1 \cdots a_n) \;=\; \sum_{s=1}^{n} a_{s+1}\cdots a_n a_1 \cdots a_{s-1} \cdot \xi(a_s)$$

for $a_1, \dots, a_n \in A$. In general, it is however not possible to "move" $\xi(a_s)$ to the left.

Example 2.6.30. Let Q be the weighted quiver $\begin{tikzcd} & j \arrow[dl, "b"'] & \\ k \arrow[rr, shift left, "c_1"] \arrow[rr, shift right, "c_0"'] & & i \arrow[ul, "a"'] \end{tikzcd}$ with $d_k = d_i = 1$ and $d_j = 2$. We consider the GLS modulation $\mathcal{H} = (R_i, A_a)_{i,a}$ for Q (see Example 2.4.11). Set $\varepsilon := \varepsilon_j$. The bimodules $A_{c_r}^\dagger = \langle c_r^\dagger : c_r \mapsto e_i \otimes e_k\rangle$, $A_b^\dagger = \langle b^\dagger : b \mapsto e_k \otimes e_j\rangle$, $A_a^\dagger = \langle a^\dagger : a \mapsto e_j \otimes e_i\rangle$

are cyclically generated. For $W = (c_0 b\varepsilon + c_1 b)a$ one computes $\partial_{c_0^\dagger}(W) = b\varepsilon a$, $\partial_{c_1^\dagger}(W) = ba$, $\partial_{b^\dagger}(W) = \varepsilon a c_0 + a c_1$, and $\partial_{a^\dagger}(W) = c_0 b\varepsilon + c_1 b$. So the Jacobian algebra of W is

$$\mathcal{J}(W) \;=\; \widehat{H}_{\mathcal{H}} / \overline{\langle b\varepsilon a, ba, \varepsilon a c_0 + a c_1, c_0 b \varepsilon + c_1 b\rangle} \;\cong\; H_{\mathcal{H}} / \langle b\varepsilon a, ba, \varepsilon a c_0 + a c_1, c_0 b\varepsilon + c_1 b\rangle\,.$$

2.6.4 Semi-Simple Structures

Definition 2.6.31. A *basic semi-simple structure* on a K-modulation $\mathcal{H} = (R_i, A_a)_{i,a}$ for Q consists of a decomposition ${}_{R_j \otimes_K R_i^{\mathrm{op}}}(R_j \otimes_K R_i^{\mathrm{op}}) = \bigoplus_\rho {}_j L_i^\rho$ into pairwise non-isomorphic simple modules ${}_j L_i^\rho$ for all i, j and a cyclic generator a of A_a for all a.

Given such a basic semi-simple structure on $\mathcal{H}$, we can write $\sum_{i,j,\rho} 1_{ji}^\rho = 1 \in R^{\mathrm{e}}$ for uniquely determined $1_{ji}^\rho \in {}_j L_i^\rho$ and have $A_a \cong {}_j L_i^{\rho_a}$ for a unique index element ρ_a.

In this case, we use the notation $A_a^\dagger := \mathrm{Hom}_{R^{\mathrm{e}}}(A_a, {}_j L_i^{\rho_a})$ and denote by $a^\dagger$ the generator of $A_a^\dagger$ defined by $a \mapsto 1_a := 1_{ji}^{\rho_a}$.

Remark 2.6.32. A minimal K-modulation $\mathcal{H}$ admits a basic semi-simple structure if and only if the enveloping algebra R^{e} of its ground ring is basic semi-simple.

Example 2.6.33. Every modulation $\mathcal{H} = (R_i, A_a)_{i,a}$ of a modular quiver Q over $(L/K, \alpha)$ carries a canonical semi-simple structure. Namely, $1_{ji}^\rho = \pi_\rho(1 \otimes 1) \in L_j \otimes_K L_i$ indexed by $\rho \in \mathrm{Gal}(L_{ji}/K)$ and $a = 1 \otimes 1 \in A_a$. In this situation it is $\rho_a = \alpha_{ji}(\sigma_a)$ for $j \xleftarrow{a} i$.

Lemma 2.6.34. *Assume $\mathcal{H}$ is a K-modulation for Q with basic semi-simple structure. For $W \in \widehat{H}_{\mathcal{H}}$ the Jacobian ideal ∂W is generated as a closed ideal by all $\partial_{a^\dagger}(W)$ with $a \in Q_1$.*

Proof. This is obvious because $A^\dagger = \bigoplus_a A_a^\dagger$ and $a^\dagger$ generates $A_a^\dagger$. ☐

Lemma 2.6.35. *Assume $\mathcal{H} = (R_i, A_a)_{i,a}$ is a K-modulation for Q with basic semi-simple structure. For every R-bimodule A' and $\mathfrak{m}' = \mathfrak{m}_{R\langle\langle A'\rangle\rangle}$ there is a bijection:*

$$\begin{array}{ccc} \mathrm{Hom}_R^n\big(R\langle\langle A\rangle\rangle, R\langle\langle A'\rangle\rangle\big) & \xrightarrow{\cong} & \big\{Q_1 \xrightarrow{\nu} \mathfrak{m}'^n \mid \nu(a) \in ({}_j\mathfrak{m}'_i)^{\rho_a} \text{ for } j \xleftarrow{a} i \in Q_1\big\} \\ f & \longmapsto & f|_{Q_1} \end{array}$$

Proof. Every map $Q_1 \xrightarrow{\nu} \mathfrak{m}'^n$ with $\nu(a) \in ({}_j\mathfrak{m}'_i)^{\rho_a}$ for all $j \xleftarrow{a} i \in Q_1$ uniquely extends to an R-bimodule morphism $A \to \mathfrak{m}'^n$. Now use Lemma 2.6.20. ☐

Notation 2.6.36. For $X = \{a_1, \dots, a_\ell\} \subseteq Q_1$ we say that $f \in \mathrm{Hom}_R(R\langle\langle A\rangle\rangle, R\langle\langle A'\rangle\rangle)$ is given by the *substitution rules* $a_1 \mapsto f(a_1), \dots, a_\ell \mapsto f(a_\ell)$ if $f(a) = a$ for all $a \in Q_1 \setminus X$.

Notation 2.6.37. For a subquiver Q' of Q we use the notation $\mathrm{Aut}_{Q'}(R\langle\langle A\rangle\rangle)$ for the subset of $\mathrm{Aut}_R(R\langle\langle A\rangle\rangle)$ consisting of all f with $f(a) = a$ for all $a \in Q'_1$.

Example 2.6.38. Let Q be the modular quiver $a \circlearrowright i \circlearrowleft b$ with $d_i = 2$ and $\sigma_a = 0$, $\sigma_b = 1$ and let $\widehat{H}$ be the completed path algebra defined by the modulation of Q over $(\mathbb{C}/\mathbb{R}, \alpha)$.

The substitution rule $a \mapsto a+b^2$ determines an element $f \in \operatorname{Aut}_R(\widehat{H})$ with $f(b)=b$. There is no element in $\operatorname{End}_R(\widehat{H})$ determined by the rule $a \mapsto b$ because of $\sigma_a \neq \sigma_b$. Similarly, the rule $a \mapsto a+ba$ does not define an R-algebra endomorphism of $\widehat{H}$ because $\pi_{\rho_a}(ba) \neq ba$.

2.6.5 Cyclic Chain Rule

This subsection contains straightforward generalizations of the cyclic Leibniz and cyclic chain rule found in [DWZ08]. As pointed out there, cyclic derivatives and a version of the cyclic chain rule were considered for arbitrary non-commutative K-algebras in [RSS80].

Notation 2.6.39. Following [DWZ08, § 3] we set $H \mathbin{\widehat{\otimes}} H := \prod_{q,r\in\mathbb{N}} (A^{\otimes q} \otimes_K A^{\otimes r})$ and regard it as a topological R-bimodule with the sets $\prod_{q+r\geq n} (A^{\otimes q} \otimes_K A^{\otimes r})$ indexed by $n \in \mathbb{N}$ as a fundamental system of open neighborhoods of zero. Actually, $H \mathbin{\widehat{\otimes}} H$ is the completion of the adic ring $\Lambda = \Gamma \otimes_K \Gamma$ with $\mathfrak{m}_\Lambda = \mathfrak{m}_\Gamma \otimes_K \Gamma + \Gamma \otimes_K \mathfrak{m}_\Gamma$ where Γ is either of H or $\widehat{H}$.

We use the symbol $\star$ for the right action of R^{e} on $H \mathbin{\widehat{\otimes}} H$ given by $(u \otimes w) \star (x \otimes y) = ux \otimes yw$ for $x \otimes y \in R^{\mathrm{e}}$ and $u \otimes w \in A^{\otimes q} \otimes_K A^{\otimes r}$.

For every $\xi \in A^\dagger$ let Δ_ξ be the continuous R-bimodule morphism $\widehat{H} \to H \mathbin{\widehat{\otimes}} H$ defined by $\Delta_\xi(xay) = (x \otimes y) \star \xi(a)$ for $x \in A^{\otimes s-1}$, $a \in A$, $y \in A^{\otimes n-s}$. Furthermore, we write $t \,\square\, z$ for the image of $t \otimes z$ under the continuous K-linear map $(H \mathbin{\widehat{\otimes}} H) \otimes_K \widehat{H} \to \widehat{H}$ that is defined by $(u \otimes w) \,\square\, z = wzu$ for $u \otimes w \in A^{\otimes q} \otimes_K A^{\otimes r}$ and $z \in \widehat{H}$. This map is a morphism of right R^{e}-modules for the action $(t \otimes z) \star r = (t \star r) \otimes z$ of $r \in R^{\mathrm{e}}$ on $t \otimes z \in (H \mathbin{\widehat{\otimes}} H) \otimes_K \widehat{H}$.

Remark 2.6.40. For $a_1, \ldots, a_n \in A$ we can explicitly express Δ_ξ as

$$\Delta_\xi(a_1 \cdots a_n) \;=\; \sum_{s=1}^{n} (a_1 \cdots a_{s-1} \otimes a_{s+1} \cdots a_n) \star \xi(a_s)\,.$$

The following lemma is completely analogous to [DWZ08, Lemma 3.8].

Lemma 2.6.41 (Cyclic Leibniz rule)**.** *Assume $R = \prod_{i\in I} R_i$. Let $\xi \in A^\dagger$ and $(i_1, \ldots, i_\ell)$ a finite sequence in I. For all $z_s \in {}_{i_s}\widehat{H}_{i_{s+1}}$ with $s \in \mathbb{Z}/\ell Z$ we have the identity*

$$\partial_\xi(z_1 \cdots z_\ell) \;=\; \sum_{s=1}^{\ell} \Delta_\xi(z_s) \,\square\, (z_{s+1} \cdots z_\ell z_1 \cdots z_{s-1})\,.$$

Proof. Because of the K-linearity and continuity of the maps ∂_ξ, Δ_ξ, and $\square$ it suffices to prove the identity for all z_s of the form $a_{s1} \cdots a_{sn_s}$ with $a_{sr} \in A$. In this case it is clear that $\Delta_\xi(z_1 \cdots z_\ell) = \sum_s z_1 \cdots z_{s-1} \cdot \Delta_\xi(z_s) \cdot z_{s+1} \cdots z_\ell$. Thus

$$\begin{aligned} \partial_\xi(z_1 \cdots z_\ell) \;=\; \Delta_\xi(z_1 \cdots z_\ell) \,\square\, 1 \;&=\; \textstyle\sum_s (z_1 \cdots z_{s-1} \cdot \Delta_\xi(z_s) \cdot z_{s+1} \cdots z_\ell) \,\square\, 1 \\ &=\; \textstyle\sum_s \Delta_\xi(z_s) \,\square\, (z_{s+1} \cdots z_\ell z_1 \cdots z_{s-1})\,. \end{aligned}$$

$\square$

Convention 2.6.42. For the rest of this subsection fix $(R_i)_i$-modulations $\mathcal{H}$ and $\mathcal{H}'$ with basic semi-simple structure for two weighted I-quivers Q and Q'.

The cyclic chain rule below is a consequence of Lemma 2.6.41. Its proof in the current more general setting uses almost the same calculations as [DWZ08, Lemma 3.9].

Lemma 2.6.43 (Cyclic chain rule). *For every K^I-algebra homomorphism $\widehat{H}_{\mathcal{H}'} \xrightarrow{f} \widehat{H}_{\mathcal{H}}$, every $W \in \widehat{H}_{\mathcal{H}'}$, and every $\xi \in A^\dagger$ we have*

$$\partial_\xi(f(W)) \;=\; \sum_{a \in Q_1'} \Delta_\xi(f(a)) \,\square\, f(\partial_{a^\dagger}(W))\,.$$

Proof. By K-linearity and continuity of ∂_ξ, Δ_ξ, $\square$, $\partial_{a^\dagger}$, and f, we can assume $W = z_1 \cdots z_\ell$ with $z_s \in A'_{a_s}$ and $a_s \in Q_1'$ for all $1 \leq s \leq \ell$. Then $a_s^\dagger(z_s) \cdot a_s = z_s$ and one computes

$$\Delta_\xi(f(z_s)) \;=\; \Delta_\xi\big(f\big(a_s^\dagger(z_s)\big) \cdot f\big(a_s\big)\big) \;=\; f\big(a_s^\dagger(z_s)\big) \cdot \Delta_\xi\big(f(a_s)\big)\,.$$

With $W_{\widehat{s}} = z_{s+1} \cdots z_\ell z_1 \cdots z_{s-1}$ it is $\partial_{a^\dagger}(W) = \sum_s W_{\widehat{s}} \cdot a^\dagger(z_s)$. Now, using Lemma 2.6.41 and $a^\dagger(z_s) = 0$ for $a \neq a_s$, one obtains

$$\begin{aligned}
\partial_\xi(f(W)) &= \textstyle\sum_s \Delta_\xi(f(z_s)) \,\square\, f\big(W_{\widehat{s}}\big) \\
&= \textstyle\sum_a \Delta_\xi(f(a)) \,\square\, f\big(\sum_s W_{\widehat{s}} \cdot a^\dagger(z_s)\big) \;=\; \sum_a \Delta_\xi(f(a)) \,\square\, f(\partial_{a^\dagger}(W))\,.
\end{aligned}$$

□

The outcome of the cyclic chain rule is that K^I-algebra isomorphisms between completed path algebras induce isomorphisms between Jacobian algebras.

Proposition 2.6.44. *For every K^I-algebra morphism $\widehat{H}_{\mathcal{H}'} \xrightarrow{f} \widehat{H}_{\mathcal{H}}$ and every $W \in \widehat{H}_{\mathcal{H}'}$ one has $\partial f(W) \subseteq f\big(\partial W\big)$. In particular, if f is an isomorphism, $\partial f(W) = f\big(\partial W\big)$ such that f induces an isomorphism of Jacobian algebras $\mathcal{J}(W) \xrightarrow{\cong} \mathcal{J}(f(W))$.*

Proof. With Lemma 2.6.43 the proof of [DWZ08, Proposition 3.7] can be used as is. □

2.6.6 Potentials

Convention 2.6.45. Fix a factorization $R = \prod_i R_i$ and a finite-dimensional R-bimodule A over K. We continue to use the notation $\widehat{H} = R\langle\langle A\rangle\rangle$.

Definition 2.6.46. Recall that [DWZ08, Definition 3.4] defines the *trace space* $\operatorname{Tr}(H)$ of a topological R-algebra H over K as the K-vector space $H/\{H,H\}$, where $\{H,H\}$ stands for the closed K-vector subspace generated by the commutators $xy - yx$ with $x, y \in H$.

Remark 2.6.47. For $\widehat{H} = \prod_n A^{\otimes n}$ the closed K-vector subspace $\big\{\widehat{H}, \widehat{H}\big\}$ is generated by homogeneous elements. With Lemma 2.6.25 we see that $\big\{\widehat{H}, \widehat{H}\big\} = \prod_n \big\{\widehat{H}, \widehat{H}\big\}_n$ where

$$\big\{\widehat{H}, \widehat{H}\big\}_n \;=\; \operatorname{span}_K \Big\{ xy - yx \;\Big|\; x \in A^{\otimes k},\, y \in A^{\otimes n-k},\, 0 \leq k \leq n \Big\}\,.$$

Therefore we have a decomposition $\operatorname{Tr}\big(\widehat{H}\big) = \prod_n \operatorname{Tr}\big(\widehat{H}\big)_n$ with $\operatorname{Tr}\big(\widehat{H}\big)_n = A^{\otimes n}/\big\{\widehat{H}, \widehat{H}\big\}_n$.

Remark 2.6.48. The image of ${}_j\widehat{H}_i$ in $\operatorname{Tr}(\widehat{H})$ vanishes unless $i = j$. More precisely, it is $a_1a_2\cdots a_{n-1}a_n = a_2a_3\cdots a_na_1 = \cdots = a_na_1\cdots a_{n-2}a_{n-1}$ in $\operatorname{Tr}(\widehat{H})$ for all $a_1,\ldots,a_n \in A$.

Example 2.6.49. Assume $\widehat{H}$ is the completed path algebra of a weighted quiver Q defined by a K-modulation with basic semi-simple structure. Then $\{\widehat{H},\widehat{H}\}$ is generated as a closed K-vector space by all elements of the form $(\omega_\ell a_\ell\cdots\omega_2a_2)\omega_1a_1\omega_0 - a_1\omega_0(\omega_\ell a_\ell\cdots\omega_2a_2)\omega_1$ where $i_\ell \xleftarrow{a_\ell} \cdots \xleftarrow{a_1} i_0$ is a cyclic path in Q and $\omega_r \in R_{i_r}$. This characterization of $\{\widehat{H},\widehat{H}\}$ is used in [GL16a, Definition 3.11] to define (cyclic equivalence of) potentials.

Example 2.6.50. Assume that $\widehat{H}$ is the completed path algebra of the double-loop quiver from Example 2.6.38. Then $b^2a = bab = ab^2$ in $\operatorname{Tr}(\widehat{H})$.

Definition 2.6.51. The *order* of $z = \sum_n z_n \in \operatorname{Tr}(\widehat{H})$ with $z_n \in \operatorname{Tr}(\widehat{H})_n$ is defined as

$$\operatorname{ord}(z) \;:=\; \min\left\{n \in \mathbb{N} \,\middle|\, z_n \neq 0 \in \operatorname{Tr}(\widehat{H})\right\}.$$

Remark 2.6.52. For the image of an element $z \in \widehat{H}$ in the trace space $\operatorname{Tr}(\widehat{H})$ we usually write again z. We then have $\operatorname{ord}(z) \geq \operatorname{ord}_{\widehat{H}}(z)$.

Definition 2.6.53. A *potential* for A is any element z in $\operatorname{Tr}(\widehat{H})$ with $\operatorname{ord}(z) > 0$.

A *species with potential* (*SP*) over R is a pair (A, W) consisting of a finite-dimensional R-bimodule A over K and a potential W for A.

Remark 2.6.54. As already explained in the introduction of [DWZ08], it is natural to think of potentials as elements in the trace space. However, ibid. the term potential is used to refer to elements in $\widehat{H}_{\text{cyc}} = \bigoplus_i {}_i(\mathfrak{m}_{\widehat{H}})_i$. Elements in $\widehat{H}_{\text{cyc}}$ are then said to be *cyclical equivalent* if they have the same image in the trace space. In a nutshell, what we call a potential here is the cyclic-equivalence class of a potential in the terminology of [DWZ08].

The next two facts are discussed in [GL16a, § 10.2] for comfy modulations.

Lemma 2.6.55. *Assume A is the species of a cyclic Galois modulation for Q over L/K. For all $i, j \in Q_0$ and $m \in {}_j\widehat{H}_i$, $n \in {}_i\widehat{H}_j$ and $\mathrm{id} = \mathrm{id}_{L_j}$ the id-isotypical component of mn is*

$$\pi_{\mathrm{id}}(mn) \quad=\quad \sum_{\rho\in\operatorname{Gal}(L_{ji}/K)} \pi_{\mathrm{id}}(\pi_{\rho^{-1}}(m)\pi_\rho(n)).$$

Proof. Use $\pi_{\mathrm{id}}(mn) = \sum_{\rho,\rho'} \pi_{\mathrm{id}}(\pi_{\rho'}(m)\pi_\rho(n))$ and $(\star)$ in § 2.5.3. □

Corollary 2.6.56. *Assume A is the species of a cyclic Galois modulation for Q over L/K. For all $i, j \in Q_0$ and $m \in {}_j\widehat{H}_i$, $n \in {}_i\widehat{H}_j$ we have in $\operatorname{Tr}(\widehat{H})$ the identity*

$$mn \quad=\quad \sum_{\rho\in\operatorname{Gal}(L_{ji}/K)} \pi_{\rho^{-1}}(m)\pi_\rho(n).$$

Proof. Because of Lemma 2.6.55 it suffices to show that $\pi_\gamma(z) = 0$ in $\operatorname{Tr}(\widehat{H})$ for all $z \in {}_j\widehat{H}_j$ and non-identity $\gamma \in \operatorname{Gal}(L_j/K)$. Now $x = \gamma(u) - u \neq 0$ for some u, if $\gamma \neq \mathrm{id}$. By $(\star)$ it is $\gamma(u)\pi_\gamma(z) = \pi_\gamma(z)u = u\pi_\gamma(z)$ in $\operatorname{Tr}(\widehat{H})$. So $\pi_\gamma(z) = x^{-1}(\gamma(u)-u)\pi_\gamma(z) = 0$ in $\operatorname{Tr}(\widehat{H})$. □

Corollary 2.6.57. *Assume A is the species of a cyclic Galois modulation for Q over L/K. For every potential W for A there are elements $\nu_a \in {}_i\big(\mathfrak{m}_{\widehat{H}}\big)_j$ for all $j \xleftarrow{a} i \in Q_1$ such that*

$$W \;=\; \sum_{a\in Q_1} \nu_a a \;=\; \sum_{a\in Q_1} \pi_{\rho_a^{-1}}(\nu_a)a\,.$$

Proof. The existence of elements ν_a such that the first equality holds is clear. For the last equality use Corollary 2.6.56. □

Example 2.6.58. Let $\widehat{H}$ be the completed path algebra of $\begin{tikzcd} i \arrow[loop left, "a"] \arrow[loop right, "b"] \end{tikzcd}$ as in Example 2.6.38. Recall that $\sigma_a = 0$, $\sigma_b = 1 \in \mathbb{Z}/2\mathbb{Z}$. Then $ba = 0$ but $b^2a \neq 0$ in $\mathrm{Tr}\big(\widehat{H}\big)$.

It is clear that the cyclic derivatives ∂_ξ annihilate $\big\{\widehat{H},\widehat{H}\big\}$ (compare Definition 2.6.27 and Remark 2.6.48). Therefore the following definition makes sense.

Definition 2.6.59. The K-linear map $\mathrm{Tr}\big(\widehat{H}\big) \to \widehat{H}$ induced by ∂_ξ is once again called the *cyclic derivative with respect to $\xi \in A^\dagger$* and is also denoted by ∂_ξ.

2.6.7 Equivalence of Potentials

The main purpose of potentials W is to encode defining relations of Jacobian ideals ∂W. It is thus natural to introduce an equivalence relation on the space of potentials in such a way that equivalent potentials define isomorphic Jacobian algebras. Having this in mind, Proposition 2.6.44 motivates the next definition.

Convention 2.6.60. As before, assume $R = \prod_{i\in I} R_i$.

Let A, A' be R-bimodules over K and $\widehat{H} = R\langle\langle A\rangle\rangle$, $\widehat{H}' = R\langle\langle A'\rangle\rangle$.

Definition 2.6.61. Two SPs (A, W) and (A', W') are *R-equivalent* if there is an R-algebra isomorphism $\widehat{H} \xrightarrow{f} \widehat{H}'$ with $f(W) = W'$. In this case, we write $(A, W) \sim_R (A', W')$.

The SPs (A, W) and (A', W') are *I-equivalent* if there exists an isomorphism $\widehat{H} \xrightarrow{f} \widehat{H}'$ of K^I-algebras such that $f(W) = W'$. Then we write $(A, W) \sim_I (A', W')$.

In case $A = A'$ and $X \in \{R, I\}$, we say that the potentials W and W' are *X-equivalent*, formally $W \sim_X W'$, whenever the SPs (A, W) and (A, W') are *X-equivalent.*

Remark 2.6.62. In the unweighted situation, i.e. if $R = \prod_i K$, both R-equivalence and I-equivalence coincide with what is called right-equivalence in [DWZ08]. We leave it to the judgment of the reader to decide which equivalence is the "right" one in general.

Example 2.6.63. Let Q be the modular quiver $\begin{tikzcd} & j \arrow[dl, "b"'] & \\ k \arrow[rr, shift left, "c_1"] \arrow[rr, shift right, "c_0"'] & & i \arrow[ul, "a"'] \arrow[loop right, "e"] \end{tikzcd}$ with $d_k = d_i = 2$, $d_j = 1$ and $\sigma_{c_0} = 0$, $\sigma_{c_1} = 1$. Let A be the species of the modulation of Q over $(\mathbb{C}/\mathbb{R}, -1, v)$. Then the potentials $W = v(c_0 + c_1)ba + e(e + 1)$ and $W' = -v(c_0 + c_1)ba + e(e + 1)$ for A are I-equivalent. Indeed, it is $W' = f(W)$ for the element $f \in \mathrm{Aut}_{K^I}\big(\widehat{H}\big)$ that acts on the ground ring $R = \mathbb{C}e_k \times \mathbb{R}e_j \times \mathbb{C}e_i$ as $\mathrm{id}_{\mathbb{C}} \times \mathrm{id}_{\mathbb{R}} \times \rho$, where ρ is complex conjugation, and on the arrows as $f(c_0) = c_1$, $f(c_1) = c_0$, $f(b) = b$, $f(a) = a$, $f(e) = e$.

2.6.8 Subpotentials and Restricted Potentials

The suggestive notations for restricted species $A|_S$, restricted potential $W|_S$, subpotential W^S, induced subquiver Q^W, and induced subspecies A^W are made precise below.

Definition 2.6.64. Let $S \subseteq Q_1$ and $T = Q_1 \setminus S$. We define $A|_T := \bigoplus_{a \in T} A_a$.

Let $\mathcal{H} = (R_i, A_a)_{i,a}$ be a K-modulation for the weighted quiver Q and let $\mathcal{H}'$ be the submodulation of $\mathcal{H}$ induced by the inclusion of the Q_0-subquiver of Q spanned by S. Moreover, let J be the ideal of $\widehat{H}_{\mathcal{H}}$ generated by $A|_T$.

Note that $\widehat{H}_{\mathcal{H}} = \widehat{H}_{\mathcal{H}'} \oplus J$ and denote by $\widehat{H}_{\mathcal{H}} \overset{\pi_S}{\twoheadrightarrow} \widehat{H}_{\mathcal{H}'} \overset{\iota_S}{\hookrightarrow} \widehat{H}_{\mathcal{H}}$ and $\widehat{H}_{\mathcal{H}} \overset{\pi_J}{\twoheadrightarrow} J \overset{\iota_J}{\hookrightarrow} \widehat{H}_{\mathcal{H}}$ the canonical projections and inclusions. Finally, let W be a potential for $A_{\mathcal{H}}$ and $p_{\mathcal{H}}^{-1}(W)$ a preimage of W under the canonical projection $\widehat{H}_{\mathcal{H}} \overset{p_{\mathcal{H}}}{\twoheadrightarrow} \mathrm{Tr}(\widehat{H}_{\mathcal{H}})$.

The *restriction of* W *to* S is the potential $W|_S := p_{\mathcal{H}'}\big(\pi_S\big(p_{\mathcal{H}}^{-1}(W)\big)\big)$ for $A_{\mathcal{H}'}$. We will often regard $W|_S$ as the potential $p_{\mathcal{H}}\big(\iota_S\pi_S\big(p_{\mathcal{H}}^{-1}(W)\big)\big)$ for $A_{\mathcal{H}}$.

The *subpotential of* W *spanned by* T is the potential $W^T := p_{\mathcal{H}}\big(\iota_J\pi_J\big(p_{\mathcal{H}}^{-1}(W)\big)\big)$.

Remark 2.6.65. The potentials $W|_S$ and W^T do not depend on the choice of $p_{\mathcal{H}}^{-1}(W)$.

Remark 2.6.66. We have $W = W|_S + W^{Q_1 \setminus S}$.

Definition 2.6.67. We sometimes say that $a \in Q_1$ *occurs* in the potential W if $W^{\{a\}} \neq 0$. We use the notation Q^W for the Q_0-subquiver of Q spanned by all arrows that occur in W. We call Q^W the *subquiver* and $A^W := A|_{Q_1^W}$ the *subspecies induced by* W.

Remark 2.6.68. It is $W^{Q_1^W} = W|_{Q_1^W} = W$.

Example 2.6.69. For the potential $W = (c_0 b\varepsilon + c_1 b)a$ in Example 2.6.30 it is $W^{\{c_0\}} = c_0 b\varepsilon a$.

Example 2.6.70. Consider Example 2.6.63. Let Q' be the subquiver $k \xrightarrow{c_1} i \circlearrowleft e$ of Q. The restriction of $W = v(c_0 + c_1)ba + e(e+1)$ to Q_1' is $W|_{Q_1'} = e(e+1)$.

2.6.9 The Splitting Theorem

For modular quivers Q we have already seen that premutation requires “local” 2-acyclicity. However, even if a modular quiver was not 2-acyclic itself, it could have a 2-acyclic reduction Q_{red}. The mutation of Q was then defined as the reduced-equivalence class of the premutation of Q_{red} (see Lemma 2.1.17). In order to make a similar story work for quivers with potential, Derksen, Weyman, and Zelevinsky came up with the Splitting Theorem. We generalize it to species with potential defined by cyclic Galois modulations.

Convention 2.6.71. All species A, A' etc. are assumed to be defined by $(L_i)_i$-modulations

$\mathcal{H} = (L_i, A_a)_{i,a}$, $\mathcal{H}' = (L_i, A'_a)_{i,a}$ of modular loop-free I-quivers Q, Q' etc. over $(L/K, \alpha)$.

As usual, we use the notation $R = \prod_{i \in I} L_i$ and $\widehat{H} = \widehat{H}_{\mathcal{H}}$.

Definition 2.6.72. An SP (A, W) or the potential W is said to be *reduced* if $\operatorname{ord}(W) > 2$. It is called *trivial* if $W = \operatorname{pr}_2(W)$ and $\partial W = \mathfrak{m}_{\widehat{H}}$.

For two SPs (A, W) and (A', W') their *sum* is $(A, W) \oplus (A', W') := (A \oplus A', W + W')$.

Remark 2.6.73. The sum of two trivial (resp. reduced) SPs is trivial (resp. reduced).

Remark 2.6.74. In view of Lemma 2.6.12 being reduced is invariant under I-equivalence and R-equivalence. Proposition 2.6.44 shows that for every change of arrows $f \in \operatorname{Aut}_R(\widehat{H})$ a potential W for A is trivial if and only if the potential $f(W)$ is trivial.

Remark 2.6.75. Proposition 4.5 in [DWZ08] explains the terminology "trivial SP": For all SPs (A, W) and all trivial SPs (A', W') the canonical map $R\langle\langle A \rangle\rangle \to R\langle\langle A \oplus A' \rangle\rangle$ induces an isomorphism $\mathcal{J}(W) \to \mathcal{J}(W + W')$. In particular, $\mathcal{J}(W') \cong R$.

Given an SP (A, W) one is usually interested in the module category of its Jacobian algebra $\mathcal{J}(W)$. In view of the last remark it seems reasonable to "split off" the trivial part. Proposition 2.6.44 gives a clue how this can be done: Find a dimension-maximal trivial SP (A', W') with $(A, W) \sim_R (A_{\text{red}}, W_{\text{red}}) \oplus (A', W')$, then replace (A, W) by $(A_{\text{red}}, W_{\text{red}})$. It turns out that $(A_{\text{red}}, W_{\text{red}})$ is reduced and up to R-equivalence uniquely determined. The proof of this fact is almost identical to [DWZ08, § 4] as soon as we have the following result on "normal forms" of potentials, which generalizes (4.6) ibid.

Recall that a canceling 2-cycle in Q is a subquiver $i \underset{b}{\overset{a}{\rightleftarrows}} j$ with $\sigma_b + \sigma_a = 0 \in \mathbb{Z}/d_{ji}\mathbb{Z}$.

Lemma 2.6.76. *Every potential W for A is R-equivalent to a potential of the form*

$$\sum_{s=1}^{r} b_s a_s + \sum_{a \in T_1} \nu_a a + W' \tag{✠}$$

for canceling 2-cycles $T^s = \underset{b_s}{\overset{a_s}{\rightleftarrows}}$ in Q such that $T := \bigoplus_{s=1}^{r} T^s$ and $(W')^{T_1} = 0$, $\operatorname{ord}(W') > 2$, and $\nu_a \in {}_j\widehat{H}_i^{\rho_a^{-1}}$, $\operatorname{ord}(\nu_a a) > 2$ for all $j \overset{a}{\longleftarrow} i \in T_1$.

More precisely, there is a change of arrows $\varphi \in \operatorname{Aut}_R(\widehat{H})$ with $\varphi(W)$ of the form (✠).

Proof. Choose a total order $\leq$ on Q_0. Set $S_{ji}^{\rho} := \{j \overset{a}{\longleftarrow} i \in Q_1 \mid \rho_a = \rho\}$ for $i, j \in Q_0$ and $\rho \in \operatorname{Gal}(L_{ji}/K)$. Using Corollary 2.6.57 it is not hard to see that

$$W = \sum_{\substack{i<j \\ \gamma\rho = \text{id}}} \sum_{a \in S_{ji}^{\rho}} \sum_{b \in S_{ij}^{\gamma}} b \cdot (\mu_{ba} \cdot a) + W_{>2}$$

with $\operatorname{ord}(W_{>2}) > 2$ and $\mu_{ba} \in L_j \otimes_K L_i$. Pick a basis U_{ji} of L_j over L_{ji} for all $i, j \in Q_0$. For $a \in S_{ji}^{\rho}$ with $i < j$ we will regard A_a as an L^{ji}-vector space via $A_a = L_j \otimes_\rho L_i \overset{\cong}{\longrightarrow} L^{ji}$ induced by $xv \otimes u \mapsto xvu$ for $x \in L_{ji}$, $v \in U_{ji}$, $u \in U_{ij}$. Similarly, for $b \in S_{ij}^{\rho^{-1}}$ with $i < j$

we will view A_b as an L^{ji}-vector space via the isomorphism $A_b = L_i \otimes_{\rho^{-1}} L_j \xrightarrow{\cong} L^{ji}$ induced by $u \otimes vx \mapsto uvx$ for $u \in U_{ij}$, $v \in U_{ji}$, $x \in L_{ji}$. Note that ${}_jA_i^\rho = A|_{S_{ji}^\rho}$.

With this preparation the rest of the argument is similar to [Geu13, Remark 3.2.5 (c)]. Note that $a \mapsto \partial_{a^\dagger}(\mathrm{pr}_2(W))$ induces an L^{ji}-linear map ${}_jA_i^\rho \xrightarrow{g} {}_iA_j^{\rho^{-1}}$. Pick $f \in \mathrm{Aut}_{L^{ji}}\big({}_jA_i^\rho\big)$ and $h \in \mathrm{Aut}_{L^{ji}}\big({}_iA_j^{\rho^{-1}}\big)$ such that the matrix of $h \circ g \circ f$ has block form $\left(\begin{smallmatrix} \mathbf{1}_r & 0 \\ 0 & 0 \end{smallmatrix}\right)$ with respect to the (arbitrary) ordered bases $S_{ji}^\rho = \{a_1, \dots, a_p\}$ and $S_{ij}^{\rho^{-1}} = \{b_1, \dots, b_q\}$, where $\mathbf{1}_r$ is the $(r \times r)$-identity matrix. Let $f^{\mathbf{T}}$ be the transpose of f. A straightforward calculation shows that the element $\varphi \in \mathrm{Aut}_R(\widehat{H})$ defined for all $i < j$ by the substitutions $a \mapsto f^{\mathbf{T}}(a)$ for $a \in S_{ji}^\rho$ and $b \mapsto h(b)$ for $b \in S_{ij}^{\rho^{-1}}$ satisfies $\mathrm{pr}_2(\varphi(W)) = \sum_{s=1}^r b_s a_s$.

Using Corollary 2.6.57 it is now easy to see that $\varphi(W)$ is a potential of the form (✠). $\square$

Corollary 2.6.77. *For every trivial SP (A, W) it is $A \cong B \oplus B^*$ for some R-bimodule B.*

Proof. With Remark 2.6.74 and Lemma 2.6.76 we can assume that $W = \sum_{s=1}^r b_s a_s$ for cyclic paths $b_s a_s$ with $\rho_{b_s} = \rho_{a_s}^{-1}$ and $|\{a_s, b_s \,|\, 1 \le s \le r\}| = 2r$. Since $\partial W = \mathfrak{m}_{\widehat{H}}$, we must have $\{a_s, b_s \,|\, 1 \le s \le r\} = Q_1$. Take $B = A|_{\{a_s \,|\, 1 \le s \le r\}}$. $\square$

For convenience we reproduce a simplified version of the proof of the existence statement in the Splitting Theorem, since we use similar arguments in Chapters 5 and 6.

Theorem 2.6.78. *For every SP (A, W) there exists a reduced SP $(A_{\mathrm{red}}, W_{\mathrm{red}})$ and a trivial SP $(A_{\mathrm{triv}}, W_{\mathrm{triv}})$ such that $(A, W) \sim_R (A_{\mathrm{red}}, W_{\mathrm{red}}) \oplus (A_{\mathrm{triv}}, W_{\mathrm{triv}})$.*

Moreover, $(A_{\mathrm{red}}, W_{\mathrm{red}})$ and $(A_{\mathrm{triv}}, W_{\mathrm{triv}})$ are uniquely determined up to R-equivalence.

Definition 2.6.79. Every SP that is R-equivalent to $(A_{\mathrm{red}}, W_{\mathrm{red}})$ is called a *reduced part* and every trivial SP R-equivalent to $(A_{\mathrm{triv}}, W_{\mathrm{triv}})$ a *trivial part* of (A, W).

Proof of Theorem 2.6.78. Replacing W with an R-equivalent potential, we can assume by Lemma 2.6.76 that there are $r \in \mathbb{N}$ and cyclic paths $a_s^* a_s$ in Q with $\sigma_{a_s^*} + \sigma_{a_s} = 0$ such that $S = \{a_s, a_s^* \,|\, 1 \le s \le r\} \subseteq Q_1$ has $2r$ elements and, for $W_{\mathrm{triv}} = \sum_{s=1}^r a_s^* a_s$ and $W'_{-1} = 0$, it is $W_0 := W = W_{\mathrm{triv}} + \sum_{a \in S} \nu_{0,a} a + W'_0$ where $(W'_0)^S = 0$ and $\mathrm{ord}(W'_0 - W'_{-1}), \mathrm{ord}(\nu_{0,a} a) > 2$.

Now assume that for some $n \in \mathbb{N}$ we have

$$W_n \;=\; W_{\mathrm{triv}} + \sum_{a \in S} \nu_{n,a} a + W'_n$$

where $(W'_n)^S = 0$ and $\mathrm{ord}(W'_n - W'_{n-1}), \mathrm{ord}(\nu_{n,a} a) > n + 2$. The element $\varphi_{n+1} \in \mathrm{Aut}_R(\widehat{H})$ given by the substitution rules $a \mapsto a - \pi_{\rho_a}(e_j \nu_{n,a^*} e_i)$ for $j \xleftarrow{a} i \in S$ (with $a^{**} := a$) is unitriangular of depth $\ge n$. A straightforward computation shows

$$W_{n+1} \;:=\; \varphi_{n+1}(W_n) \;=\; W_{\mathrm{triv}} + \sum_{a \in S} \nu_{n+1,a} a + W'_{n+1}$$

where $(W'_{n+1})^S = 0$ and $\mathrm{ord}(W'_{n+1} - W'_n), \mathrm{ord}(\nu_{n+1,a} a) > (n+1) + 2$.

We get a sequence $(\varphi_n)_{n\in\mathbb{N}_+}$ of unitriangular automorphisms with $\lim_n \operatorname{depth}(\varphi_n) = \infty$ such that the potentials $W_n = \widetilde{\varphi}_n(W)$ with $\widetilde{\varphi}_n = \varphi_n \cdots \varphi_1$ define a convergent sequence with limit $\lim_n W_n = W_{\mathrm{triv}} + W_{\mathrm{red}}$ for $W_{\mathrm{red}} = \lim_n W'_n$. Note that $(W_{\mathrm{red}})^S = 0$.

The R-algebra automorphism $\widetilde{\varphi} = \lim_n \widetilde{\varphi}_n$ maps the potential W to $W_{\mathrm{red}} + W_{\mathrm{triv}}$ such that $(A, W) \sim_R (A_{\mathrm{red}}, W_{\mathrm{red}}) \oplus (A_{\mathrm{triv}}, W_{\mathrm{triv}})$ where $(A_{\mathrm{triv}}, W_{\mathrm{triv}})$ is trivial and $(A_{\mathrm{red}}, W_{\mathrm{red}})$ reduced, for $A_{\mathrm{triv}} = A|_S$ and $A_{\mathrm{red}} = A|_{Q_1\setminus S}$.

The arguments given in [DWZ08, § 4] to show that the SPs $(A_{\mathrm{red}}, W_{\mathrm{red}})$ and $(A_{\mathrm{triv}}, W_{\mathrm{triv}})$ are up to R-equivalence uniquely determined work without modification in our setting. □

Example 2.6.80. Let Q be the modular quiver
$$\begin{tikzcd} & k \arrow[dr, "f"] & \\ j \arrow[ur, "e"] \arrow[r, shift right, "a_0"'] & i \arrow[l, shift right, "b_0"'] \arrow[r, shift right, "a_1"'] & h \arrow[l, shift right, "b_1"'] \end{tikzcd}$$
with $d_k = 1$ and $d_h = d_i = d_j = 2$ and $\sigma_{a_0} = \sigma_{a_1} = \sigma_{b_1} = 0$ and $\sigma_{b_0} = 1$. Consider the species A defined by the modulation of Q over $(\mathbb{C}/\mathbb{R}, -1, v)$. The potential $W = b_1a_1 + feb_0(1 + b_1a_1)$ for A is R-equivalent to $W' = b_1a_1 + feb_0$ (indeed, $W = \varphi(W')$ for $\varphi \in \operatorname{Aut}_R(\widehat{H})$ defined by the substitution $b_1 \mapsto b_1 + \pi_{\mathrm{id}}(feb_0b_1)$). Hence, we have $(A, W) \sim_R (A_{\mathrm{red}}, W_{\mathrm{red}}) \oplus (A_{\mathrm{triv}}, W_{\mathrm{triv}})$ with $(A_{\mathrm{red}}, W_{\mathrm{red}}) = \big(A|_{\{a_0,b_0,e,f\}}, feb_0\big)$ and $(A_{\mathrm{triv}}, W_{\mathrm{triv}}) = \big(A_{\{a_1,b_1\}}, b_1a_1\big)$.

We conclude this subsection with a few convenient definitions.

Definition 2.6.81. Let $X \in \{R, I\}$ and $j \in Q_0$.

We write $(A, W) \approx_X (A', W')$ if SPs (A, W) and (A', W') are *reduced-X-equivalent*, i.e. there are trivial SPs (B, S) and (B', S') such that $(A, W) \oplus (B, S) \sim_X (A', W') \oplus (B', S')$.

An SP (A, W) is 2-*acyclic (at j)* if the corresponding quiver Q is 2-acyclic (at j). It is 2-*acyclic (at j) after reduction* if it has a reduced part $(A_{\mathrm{red}}, W_{\mathrm{red}})$ that is 2-acyclic (at j).

Remark 2.6.82. An SP (A, W) can fail to be 2-acyclic after reduction even if the modular quiver Q is 2-acyclic after reduction. However, if Q is 2-acyclic after reduction, there always exists a potential W for A such that (A, W) is 2-acyclic after reduction.

Example 2.6.83. Let Q be the modular quiver $\begin{tikzcd} i \arrow[r, shift left, "a"] & j \arrow[l, shift left, "b"] \end{tikzcd}$ with $d_i = d_j = 2$ and $\sigma_a = \sigma_b = 1$. Consider the species A defined by the modulation of Q over $(\mathbb{C}/\mathbb{R}, -1, v)$. The SP (A, ba) is 2-acyclic after reduction, whereas the SP $(A, 0)$ is not.

Definition 2.6.84. Let $S \subseteq Q_1$. An SP (A, W) or the potential W is in *S-split form* if the potential W^S has the form $(\maltese)$ with $\nu_a = 0$ for all $k \xleftarrow{a} i \in T_1$. In this case, we define

$$\begin{aligned} \operatorname{red}^W_S(Q) &:= Q - T_1\,, & \operatorname{triv}^W_S(Q) &:= T\,, \\ \operatorname{red}_S(W) &:= W|_{Q_1\setminus T_1}\,, & \operatorname{triv}_S(W) &:= W|_{T_1}\,, \\ \operatorname{red}_S(A, W) &:= \big(A|_{Q_1\setminus T_1}, W|_{Q_1\setminus T_1}\big)\,, & \operatorname{triv}_S(A, W) &:= \big(A|_{T_1}, W|_{T_1}\big)\,. \end{aligned}$$

Let $S^j = \{k \leftarrow i \in Q_1 \,|\, j \in \{i, k\}\}$. The SP (A, W) or the potential W is in *j-split form* if it is in S^j-split form. In this case, we simply write red^W_j for $\operatorname{red}^W_{S^j}$ and red_j for red_{S^j}.

Remark 2.6.85. If (A, W) is in S-split form, then $W = \mathrm{red}_S(W) + \mathrm{triv}_S(W)$ and $\mathrm{triv}_S(A, W)$ is trivial. In particular, $\mathrm{red}_S(A, W) \approx_R (A, W)$.

2.6.10 Mutation of Potentials

This final subsection defines (pre)mutation for SPs. These operations play a key role in Chapters 4 to 6. The concept was introduced in [DWZ08, § 5], generalized in [LZ16, § 8] to the strongly primitive setting, and appears in the form presented below in [GL16a, § 3].

Convention 2.6.86. Fix a comfy extension $(L/K, \zeta, v)$ and a 2-acyclic vertex j in Q.

We still assume all quivers to be loop-free.

Recall $Q^{\sim j} = Q^{*j} \oplus Q^{-j-}$ and $v_j = v^{[L:L_j]}$ and $r_{kji} = d_{ki}/d_{kji}$ and $q_{kji} = d_{kji}d_j/(d_{kj}d_{ji})$.

Denote by $\widehat{H} = R\langle\langle A\rangle\rangle$, $\widehat{\overline{H}} = R\langle\langle \overline{A}\rangle\rangle$, $\widehat{H}^{\sim j} = R\langle\langle A^{\sim j}\rangle\rangle$ the completed path algebras defined by the modulations of the modular quivers Q, $\overline{Q}$, $Q^{\sim j}$ over $(L/K, \zeta, v)$.

We begin with the construction of a K-linear map

$$\mathrm{Tr}\big(\widehat{H}\big) \xhookrightarrow{[-]} \mathrm{Tr}\big(\widehat{H}^{\sim j}\big)$$

making it possible to regard potentials W for A as potentials $[W]$ for $A^{\sim j}$.

There are embeddings of R-algebras $\widehat{H} \xhookrightarrow{\iota} \widehat{\overline{H}} \xhookleftarrow{\iota^{\sim j}} \widehat{H}^{\sim j}$ where ι is given by $\iota(a) = a$ for all $a \in Q_1$ and $\iota^{\sim j}$ by

$$\iota^{\sim j}(c) \;=\; \begin{cases} \pi_{\rho_c}\big(bv_j^q a\big) & \text{for } c = [ba]_r^q \in Q_1^{-j-} \text{ with } k \xleftarrow{b} j \xleftarrow{a} i \text{ in } Q \text{ and } 0 \le q < q_{kji}, \\ c & \text{for } c \in Q_1^{*j}. \end{cases}$$

Remark 2.6.87. We have $A^{\sim j} \xrightarrow{\cong} \overline{A}|_{Q_1^{*j}} \oplus \bigoplus_{k \xleftarrow{b} j \xleftarrow{a} i \text{ in } Q} A_b \otimes_{L_j} A_a \subseteq \widehat{\overline{H}}$ induced by the map $\iota^{\sim j}$ according to Lemma 2.5.27.

Lemma 2.6.88. *There is a K-linear map $\mathrm{Tr}\big(\widehat{H}\big) \xhookrightarrow{[-]} \mathrm{Tr}\big(\widehat{H}^{\sim j}\big)$ induced by ι and $\iota^{\sim j}$.*

Proof. Since Q is loop-free at j, for every $i_\ell \xleftarrow{a_\ell} \cdots \xleftarrow{a_1} i_0$ in Q with $\ell > 0$ there is $0 \le s \le \ell$ with $i_s \neq j$. Hence, $\iota(A_{a_\ell} \cdots A_{a_1}) = \iota((A_{a_s} \cdots A_{a_1})(A_{a_\ell} \cdots A_{a_{s+1}})e_{i_s}) \subseteq \mathrm{im}(\iota^{\sim j})$ in $\mathrm{Tr}\big(\widehat{\overline{H}}\big)$. This shows that $\mathrm{im}(\iota) \subseteq \mathrm{im}(\iota^{\sim j})$ in $\mathrm{Tr}\big(\widehat{\overline{H}}\big)$. The observation that ι and $\iota^{\sim j}$ induce injective maps $\mathrm{Tr}\big(\widehat{H}\big) \hookrightarrow \mathrm{Tr}\big(\widehat{\overline{H}}\big) \hookleftarrow \mathrm{Tr}\big(\widehat{H}^{\sim j}\big)$ yields a map $\mathrm{Tr}\big(\widehat{H}\big) \hookrightarrow \mathrm{Tr}\big(\widehat{H}^{\sim j}\big)$, $W \mapsto [W]$, with the property that $\iota(W) = \iota^{\sim j}([W])$ in $\mathrm{Tr}\big(\widehat{\overline{H}}\big)$. $\square$

For the sake of readability, we introduce the following two abbreviations:

Notation 2.6.89. Set $[ba]^q := \sum_{0 \le r < r_{kji}} [ba]_r^q$ for $k \xleftarrow{b} j \xleftarrow{a} i$ in Q.

Notation 2.6.90. Set $\Delta_j := \sum_{k \xleftarrow{b} j \xleftarrow{a} i \text{ in } Q} \Delta_{ba}$ where $\Delta_{ba} := \frac{1}{q_{kji}} \sum_{0 \le q < q_{kji}} v_j^{-q} b^* [ba]^q a^*$.

Remark 2.6.91. We have $\iota^{\sim j}([ba]^q) = bv_j^q a$ for $k \xleftarrow{b} j \xleftarrow{a} i$ in Q and $0 \le q < q_{kji}$.

Remark 2.6.92. It is $\iota^{\sim j}(\Delta_{ba}) = \frac{1}{q_{kji}} \sum_q v_j^{-q} b^* b v_j^q a a^*$. Hence, $\iota^{\sim j}(\Delta_{ba}) = \pi_{\mathrm{id}}(b^*b)\pi_{\mathrm{id}}(aa^*)$ in $\mathrm{Tr}\big(\widehat{\widehat{H}}\big)$ for $\mathrm{id} = \mathrm{id}_{L_j}$ by Lemmas 2.5.22 and 2.5.40 and Corollary 2.6.56.

Now we are ready for the central definition of this subsection.

Definition 2.6.93. Let $\mathrm{Tr}\big(\widehat{H}\big) \hookrightarrow \mathrm{Tr}\big(\widehat{H}^{\sim j}\big)$ be the map $W \mapsto W^{\sim j} := [W] + \Delta_j$.

For every potential W for A the *premutation* of W at j is the potential $W^{\sim j}$ for $A^{\sim j}$.

The *premutation* of the SP (A, W) at j is defined as the SP $\widetilde{\mu}_j(A, W) := (A^{\sim j}, W^{\sim j})$.

More generally, for every $k \in Q_0$ and SP (A, W) in k-split form with $\mathrm{red}_k^W(Q)$ 2-acyclic at k, the *premutation* at k is $\widetilde{\mu}_k(A, W) := \widetilde{\mu}_k(\mathrm{red}_k(A, W))$ and $\widetilde{\mu}_k(W) := (\mathrm{red}_k(W))^{\sim k}$.

Lemma 2.6.94. *Let $\big\{k \leftarrow i \in Q_1 \,|\, j \in \{i, k\}\big\} \subseteq S \subseteq Q_1$. For every potential W for A*

$$\widetilde{\mu}_j(A, W) \;=\; \Big(A|_{Q_1 \setminus S} \oplus \big(A|_S\big)^{\sim j},\, W|_{Q_1 \setminus S} + \big(W^S\big)^{\sim j}\Big)\,.$$

Proof. This is obvious. ☐

Example 2.6.95. Let Q be the modular quiver

$$\begin{tikzcd} & j \arrow[dl, "b"'] & \\ k \arrow[rr, "c"'] & & i \arrow[ul, "a"'] \end{tikzcd}$$

with $d_k = d_i = 2$, $d_j = 1$ and $\sigma_c = r \in \mathbb{Z}/2\mathbb{Z}$, which was examined in Example 2.1.18. Consider the species A defined by the modulation of Q over $(\mathbb{C}/\mathbb{R}, -1, v)$. The premutation of the potential $W = cba$ for A is $W^{\sim j} = [W] + \Delta_j$ where

$$[W] = c[ba]^0 = c\big([ba]_0^0 + [ba]_1^0\big) = c[ba]_r^0\,, \qquad \Delta_j = \Delta_{ba} = b^*[ba]^0 a^* = b^*\big([ba]_0^0 + [ba]_1^0\big)a^*\,.$$

One easily checks that $W \sim_R W_{\mathrm{red}} + W_{\mathrm{triv}}$ where $W_{\mathrm{red}} = b^*[ba]_{r+1}^0 a^*$ is a reduced potential and $\big(A|_{\{c, [ba]_r^0\}}, W_{\mathrm{triv}} = c[ba]_r^0\big)$ a trivial SP.

Example 2.6.96. Let Q be the modular quiver

$$\begin{tikzcd} & j \arrow[dl, "b"'] & \\ k \arrow[rr, "c"'] & & i \arrow[ul, "a"'] \end{tikzcd}$$

with $d_k = d_i = 1$, $d_j = 2$. Consider the species A defined by the modulation of Q over $(\mathbb{C}/\mathbb{R}, -1, v)$. We have $v_j = v$. The premutation of the potential $W = cb(x + yv)a$ for A with $x, y \in \mathbb{R}$ is $W^{\sim j} = [W] + \Delta_j$ where

$$[W] = xc[ba]^0 + yc[ba]^1\,, \qquad \Delta_j = \Delta_{ba} = \frac{1}{2}\big(b^*[ba]^0 a^* + v^{-1} b^*[ba]^1 a^*\big)\,.$$

If $x + yv \neq 0$, it is $W \sim_R W_{\mathrm{red}} + W_{\mathrm{triv}}$ for the reduced potential $W_{\mathrm{red}} = b^*[ba]^0 a^*$ and the trivial SP $\big(A|_{\{c, [ba]^1\}}, W_{\mathrm{triv}} = c[ba]^1\big)$.

The next lemma is the key to the verification that premutation preserves R-equivalence. It is Lemma 10.4 in [GL16a] and builds on [LZ16, Lemma 8.4] and [DWZ08, Lemma 5.3].

Notation 2.6.97. Set $\Delta_{j,\mathrm{out}} := \sum_{k \xleftarrow{b} j \in Q_1} \pi_{\mathrm{id}}(b^*b)$ and $\Delta_{j,\mathrm{in}} := \sum_{j \xleftarrow{a} i \in Q_1} \pi_{\mathrm{id}}(aa^*)$.

Remark 2.6.98. We have $\iota^{\sim j}(\Delta_j) = \Delta_{j,\mathrm{out}}\Delta_{j,\mathrm{in}}$ in $\mathrm{Tr}\big(\widehat{\widehat{H}}\big)$.

Lemma 2.6.99. *For every $\varphi \in \mathrm{Aut}_R\big(\widehat{H}\big)$ there exists $\overline{\varphi} \in \mathrm{Aut}_R\big(\widehat{\widehat{H}}\big)$ satisfying $\overline{\varphi} \circ \iota = \iota \circ \varphi$ and $\mathrm{im}\big(\overline{\varphi} \circ \iota^{\sim j}\big) = \mathrm{im}\big(\iota^{\sim j}\big)$ and $\overline{\varphi}\big(\Delta_{j,\mathrm{out}}\big) = \Delta_{j,\mathrm{out}}$ and $\overline{\varphi}\big(\Delta_{j,\mathrm{in}}\big) = \Delta_{j,\mathrm{in}}$.*

Proof. Abbreviate $\mathfrak{m} = \mathfrak{m}_{\widehat{\widehat{H}}}$. Set $\overline{\varphi}(c) := \varphi(c)$ for all $c \in Q_1$. This will ensure $\overline{\varphi} \circ \iota = \iota \circ \varphi$.

It remains to define $\overline{\varphi}(a^*)$ in ${}_i\mathfrak{m}_j^{\rho_{a^*}}$ for $j \xleftarrow{a} i \in Q_1$ and $\overline{\varphi}(b^*)$ in ${}_j\mathfrak{m}_k^{\rho_{b^*}}$ for $k \xleftarrow{b} j \in Q_1$ such that the induced $\overline{\varphi}$ is invertible, maps $\iota^{\sim j}\big(\widehat{H}^{\sim j}\big)$ to itself, and fixes $\Delta_{j,\mathrm{out}}$ and $\Delta_{j,\mathrm{in}}$.

Let $\widehat{\varphi}(a^*)$ in ${}_i\mathfrak{m}_j$ for $j \xleftarrow{a} i \in Q_1$ and $\widehat{\varphi}(b^*)$ in ${}_j\mathfrak{m}_k$ for $k \xleftarrow{b} j \in Q_1$ be defined just as in the proof of [LZ16, Lemma 8.4]. Using that $d_j\pi_{\mathrm{id}}(z) = \sum_{q=0}^{d_j-1} v_j z v_j^{-1} = \sum_{q=0}^{d_j-1} v_j^{-1} z v_j$ for $\mathrm{id} = \mathrm{id}_{L_j}$ by Lemma 2.5.40, the identities (8.20) and (8.15) in [LZ16] assume the form

$$\sum_{k \xleftarrow{b} j \in Q_1} \pi_{\mathrm{id}}\big(\widehat{\varphi}(b^*)b\big) \;=\; \Delta_{j,\mathrm{out}}\,, \qquad\qquad \sum_{j \xleftarrow{a} i \in Q_1} \pi_{\mathrm{id}}\big(a\widehat{\varphi}(a^*)\big) \;=\; \Delta_{j,\mathrm{in}}\,.$$

The invertibility of the matrices C_0 and D_0 in [LZ16, proof of Lemma 8.4] shows that $\widehat{\varphi}$ induces an R-bimodule automorphism of $e_j A^{*j} \oplus A^{*j} e_j \subseteq \overline{A}$.

Setting $\overline{\varphi}(c^*) := \pi_{\rho_{c^*}}(\widehat{\varphi}(c^*))$ for all $k \xleftarrow{c} i \in Q_1$ with $j \in \{i, k\}$, we get by Lemma 2.6.35 an induced endomorphism $\overline{\varphi} \in \mathrm{End}_R\big(\widehat{\widehat{H}}\big)$. The map $\overline{\varphi}$ is an automorphism according to Lemma 2.6.20 because $\overline{\varphi}_1|_{\overline{A}} = \widehat{\varphi}_1|_{\overline{A}}$ is an automorphism. Clearly, $\mathrm{im}\big(\overline{\varphi} \circ \iota^{\sim j}\big) = \mathrm{im}\big(\iota^{\sim j}\big)$.

Finally, $\overline{\varphi}(\Delta_{j,\mathrm{in}}) = \sum_{j \xleftarrow{a} i \in Q_1} \pi_{\mathrm{id}}\big(a\overline{\varphi}(a^*)\big) = \sum_{j \xleftarrow{a} i \in Q_1} \pi_{\mathrm{id}}\big(a\widehat{\varphi}(a^*)\big) = \Delta_{j,\mathrm{in}}$, where the identity in the middle uses Lemma 2.6.55. Similarly, one can prove $\overline{\varphi}(\Delta_{j,\mathrm{out}}) = \Delta_{j,\mathrm{out}}$. $\square$

The last lemma leads to the following important result.

Theorem 2.6.100. *If $(A, W) \sim_R (A', W')$, then $\widetilde{\mu}_j(A, W) \sim_R \widetilde{\mu}_j(A', W')$.*

Proof. The proof of [DWZ08, Theorem 5.2] works as is. $\square$

The next theorem is the SP analog of Lemma 2.1.17. It records the crucial fact that, up to reduced-R-equivalence, premutation at j is an involutive operation for SPs that are 2-acyclic at j after reduction. The consequence is that the rule $\mathcal{J}(W) \mapsto \mathcal{J}(W^{\sim j})$ defines an involution for isomorphism classes of Jacobian algebras by Proposition 2.6.44.

Theorem 2.6.101. *Let $(d_i)_{i \in I}$ be a tuple of positive integers and let $R = \prod_{i \in I} L_i$ where L_i is the intermediate field of L/K of degree d_i over K for a comfy extension $(L/K, \zeta, v)$.*

Denote by $\mathcal{A}(j)$ the set of reduced-R-equivalence classes of SPs over R that are 2-acyclic at $j \in I$ after reduction. We have an involution

$$\mathcal{A}(j) \xrightarrow{\mu_j} \mathcal{A}(j)\,,$$

called mutation*, given by $(A, W) \mapsto \widetilde{\mu}_j(A, W)$ for SPs (A, W) that are 2-acyclic at j.*

Proof. For the well-definedness of μ_j use Theorems 2.6.78 and 2.6.100. The proof that μ_j is involutive is almost the same as the one of [DWZ08, Theorem 5.7] and [LZ16, Theorem 8.10]. We sketch it briefly. A straightforward calculation shows

$$W_0 \;=\; (W^{\sim j})^{\sim j} \;=\; [W] + \sum_{k \xleftarrow[b]{} j \xleftarrow[a]{} i} q_{kji}^{-1} \sum_q v_j^{-\delta_{q\neq 0} q_{kji}} [a^* b^*]^{-q} \big([ba]^q + b v_j^q a\big)\,.$$

The R-algebra automorphism of $(\widehat{H}^{\sim j})^{\sim j} = R\langle\langle (A^{\sim j})^{\sim j}\rangle\rangle$ determined by the substitution rules $b \mapsto -b$ for $k \xleftarrow{b} j \in Q_1$ and $[a^* b^*]^q \mapsto q_{kji} v_j^{\delta_{q\neq 0} q_{kji}} [a^* b^*]^q$ maps W_0 to

$$W_1 \;=\; [W] + \sum_{k \xleftarrow[b]{} j \xleftarrow[a]{} i} \sum_q [a^* b^*]^{-q} \big([ba]^q - b v_j^q a\big)\,.$$

The element in $\mathrm{Aut}_R\big((\widehat{H}^{\sim j})^{\sim j}\big)$ given by the rules $[ba]^q \mapsto [ba]^q + b v_j^q a$ maps W_1 to

$$W_2 \;=\; W + \sum_{k \xleftarrow[b]{} j \xleftarrow[a]{} i} \sum_q [ba]^{-q} \big([a^* b^*]^q + \nu_{b,a,q}\big)$$

where $\mathrm{ord}(\nu_{b,a,q}) > 1$ and none of the arrows $[a^* b^*]^q_r$ occurs in any of the elements $\nu_{b',a',q'}$. Finally, the rules $[a^* b^*]^q \mapsto [a^* b^*]^q - \nu_{b,a,q}$ send W_2 to the potential $W + W_{\mathrm{triv}}$ where

$$W_{\mathrm{triv}} \;=\; \sum_{k \xleftarrow[b]{} j \xleftarrow[a]{} i} \sum_q [ba]^{-q} [a^* b^*]^q\,.$$

It merely remains to observe that the subquiver induced by W_{triv} is the trivial modular quiver $T = \bigoplus_{b,a,r,q} T^{r,q}_{b,a}$ described in the proof of Lemma 2.1.17. Thus $(Q^{\sim j})^{\sim j} = Q \oplus T$.

All in all, this proves $((A^{\sim j})^{\sim j}, (W^{\sim j})^{\sim j}) \sim_R ((A^{\sim j})^{\sim j}, W + W_{\mathrm{triv}}) \approx_R (A, W)$. □

Finally, we recall the important notion of non-degeneracy from [DWZ08, Definition 7.2].

Definition 2.6.102. An SP (A, W) or the potential W is *non-degenerate* if for all $\ell \in \mathbb{N}$ and every finite sequence $(i_1, \ldots, i_\ell)$ of vertices in Q each of the reduced-R-equivalence classes (A, W), $\mu_{i_1}(A, W)$, …, $\mu_{i_\ell} \cdots \mu_{i_1}(A, W)$ contains a 2-acyclic SP.

Remark 2.6.103. If the modular quiver Q is not admissible in the sense of Definition 2.1.19, every potential W for A is degenerate.

3 Symmetric Modulations

This chapter is concerned with a generalization of many of the results from [GLS16a] to a larger class of finite-dimensional algebras. In our terminology, the algebras considered ibid. are path algebras $R\langle A\rangle$ of weighted acyclic quivers $Q = (Q, d)$ defined by the K-modulation $\big(R_i, A_a\big)_{i\in Q_0,\, j\xleftarrow{a} i\in Q_1}$ where $R_i = K[\varepsilon_i]/\big(\varepsilon_i^{d_i}\big)$ are truncated polynomial rings and

$$A_a \;=\; K[\varepsilon_j, \varepsilon_i] \Big/ \Big(\varepsilon_j^{f_{ij}} - \varepsilon_i^{f_{ji}},\, \varepsilon_j^{d_j},\, \varepsilon_i^{d_i}\Big)$$

are R_j-R_i-bimodules, free on the left of rank $f_{ji} = d_i/\mathrm{gcd}(d_j, d_i)$ and free on the right of rank $f_{ij} = d_j/\mathrm{gcd}(d_i, d_j)$. The majority of the arguments given in [GLS16a] are independent of this explicit modulation. They work equally well for all K-modulations $(R_i, A_a)_{i,a}$ of weighted acyclic quivers Q where the R_i are symmetric local K-algebras.

This note was written while working on a version of Crawley-Boevey and Holland's [CH98] deformed preprojective algebras for symmetric modulations. Since then a very similar approach was proposed in [LY15] using the language of matrix algebras.

Convention 3.0.1. Fix a weighted quiver Q and a K-modulation $\mathcal{H} = (R_i, A_a)_{i,a}$ for Q.

We will assume that $\mathcal{H}$ is minimal, i.e. $\dim_K(A_a) = d^{ji} = \mathrm{lcm}(d_j, d_i)$ for all $j \xleftarrow{a} i \in Q_1$.

Denote by $H = R\langle A\rangle$ the path algebra defined by $\mathcal{H}$.

Remark 3.0.2. It is possible to drop the assumption that $\mathcal{H}$ is minimal by defining Cartan matrix, Weyl group, and bilinear forms in terms of $\mathcal{H}$ instead of Q.

3.1 Gorenstein Tensor Algebras

The algebras investigated in [GLS16a] belong to a special class of tensor algebras $H = R\langle A\rangle$, which are 1-Gorenstein due to the self-injectivity of R and the projectivity of ${}_RA$ and A_R.

Let us be more precise. For a ring Λ denote by $\mathrm{Proj}^n(\Lambda)$ the full subcategory of $\mathrm{Mod}(\Lambda)$ consisting of all modules of projective dimension less than $n+1$ where $n \in \mathbb{N} \cup \{\infty\}$ and the convention $\infty + 1 = \infty$. Dually, $\mathrm{Inj}^n(\Lambda)$ is defined with projective replaced by injective. Let R be an n-Gorenstein ring. Then, by [Iwa80] it is

$$\mathcal{F}(R) \;:=\; \mathrm{Proj}^\infty(R) \;=\; \mathrm{Inj}^\infty(R) \;=\; \mathrm{Proj}^n(R) \;=\; \mathrm{Inj}^n(R)\,.$$

Form the tensor algebra $H = R\langle A\rangle$ of an R-bimodule A that is projective on the left and projective on the right. An application of [Iwa80; Rog75] yields that H is $(n+1)$-Gorenstein with

$$\mathcal{F}(H) \;=\; \mathrm{Mod}(H) \cap \mathcal{F}(R) \;=:\; \mathcal{F}_R(H)\,.$$

This section revises and generalizes [GLS16a, §§ 3 and 4].

Remark 3.1.1. Given that $\dim_K R < \infty$, R is 0-Gorenstein if and only if R is self-injective.

3.1.1 Projectivity and Injectivity over the Ground Ring

Notation 3.1.2. We use the abbreviations

$$\mathrm{Proj}^n_R(H) \;:=\; \mathrm{Mod}(H) \cap \mathrm{Proj}^n(R)\,, \qquad \mathrm{Inj}^n_R(H) \;:=\; \mathrm{Mod}(H) \cap \mathrm{Inj}^n(R)\,.$$

Denote by $\mathrm{proj}^n(H)$, $\mathrm{inj}^n(H)$, $\mathrm{proj}^n_R(H)$, $\mathrm{inj}^n_R(H)$ the full subcategories of $\mathrm{Mod}(H)$ obtained by intersecting $\mathrm{Proj}^n(H)$, $\mathrm{Inj}^n(H)$, $\mathrm{Proj}^n_R(H)$, $\mathrm{Inj}^n_R(H)$, respectively, with $\mathrm{mod}(H)$.

If R is n-Gorenstein, define $\mathfrak{f}(R) := \mathcal{F}(R) \cap \mathrm{mod}(R)$ and $\mathfrak{f}_R(H) := \mathcal{F}_R(H) \cap \mathrm{mod}(H)$.

Remark 3.1.3. $M \in \mathrm{Mod}(H)$ lies in $\mathrm{Proj}^n_R(H)$ if and only if $M_i \in \mathrm{Proj}^n(R_i)$ for all $i \in Q_0$. Dually, $M \in \mathrm{Mod}(H)$ belongs to $\mathrm{Inj}^n_R(H)$ if and only if $M_i \in \mathrm{Inj}^n(R_i)$ for all $i \in Q_0$.

Notation 3.1.4. Denote by $\mathrm{Mod}_{\mathrm{l.f.}}(H)$ the full subcategory of $\mathrm{Mod}(H)$ consisting of the locally free modules (see Definition 2.3.7). Set $\mathrm{mod}_{\mathrm{l.f.}}(H) := \mathrm{Mod}_{\mathrm{l.f.}}(H) \cap \mathrm{mod}(H)$.

Definition 3.1.5. A locally free module is called *locally-free simple* if it has no non-zero proper locally free submodules. For $M \in \mathrm{mod}_{\mathrm{l.f.}}(H)$ define its *rank vector* as

$$\underline{\mathrm{rank}}(M) \;:=\; \sum_{i \in Q_0} \mathrm{rank}\,(M_i) \cdot e_i \;\in\; \mathbb{Z}^{Q_0}\,.$$

Remark 3.1.6. $\mathrm{mod}_{\mathrm{l.f.}}(H) \subseteq \mathrm{mod}(H) \cap \mathrm{proj}(R) = \mathrm{proj}^0_R(H)$. If $\mathcal{H}$ is local, equality holds.

Notation 3.1.7. For $i \in Q_0$ denote by E_i the locally-free simple H-module that is given by $(E_i)_i = e_i E_i = {}_{R_i}R_i$ and $(E_i)_j = e_j E_i = 0$ for $j \neq i$. Note that $\underline{\mathrm{rank}}(E_i) = e_i$.

Remark 3.1.8. If Q is acyclic and R_i is local, the projective H-module $P_i = He_i$ and the injective H-module $I_i = (e_i H)^*$ are indecomposable.

We begin with an elementary observation corresponding to [GLS16a, Proposition 3.1]. Recall that the acyclicity of Q is equivalent to $\dim_K H < \infty$.

Lemma 3.1.9. *Every projective H-module is a projective R-module. More generally,*

$$\mathrm{Proj}^n(H) \;\subseteq\; \mathrm{Proj}^n_R(H)\,.$$

If Q is acyclic, the projective H-modules P_i are locally free with

$$\underline{\mathrm{rank}}(P_i) \;=\; e_i + \sum_{j \xleftarrow{a} i \in Q_1} f_{ji} \cdot \underline{\mathrm{rank}}(P_j)\,.$$

Proof. By definition $H = R\langle A\rangle$ and ${}_RA$ is projective such that [Rog75, Corollary 3] implies the first statement. The second one is a straightforward consequence.

The last statement can be proved with a discussion similar to the one in [GLS16a, § 3.1]. We present a slightly less explicit version: The cokernel of the injective multiplication map $H \otimes_R A \to H$ is as an H-module isomorphic to $\bigoplus_{i\in Q_0} E_i$. With this observation we get an exact sequence of H-modules

$$0 \longrightarrow H \otimes_R A = \bigoplus_{a\in Q_1} H \otimes_R A_a \longrightarrow H = \bigoplus_{i\in Q_0} P_i \longrightarrow \bigoplus_{i\in Q_0} E_i \longrightarrow 0\,,$$

which is the sum over $i \in Q_0$ of the short exact sequences of H-modules

$$0 \longrightarrow \bigoplus_{j \xleftarrow{a} i\in Q_1} P_j \otimes_{R_j} A_a \longrightarrow P_i \longrightarrow E_i \longrightarrow 0\,.$$

If P_j is locally free for all j, then $\underline{\mathrm{rank}}(P_j \otimes_{R_j} A_a) = \mathrm{rank}_{R_j}(A_a)\cdot \underline{\mathrm{rank}}(P_j) = f_{ji}\cdot \underline{\mathrm{rank}}(P_j)$. So to prove the formula in the lemma it only remains to show that all P_j are locally free.

Now $P_j \cong \bigoplus_\ell \bigoplus_{a_1,\dots,a_\ell} A_{a_\ell} \otimes_R \cdots \otimes_R A_{a_1} \otimes_R E_j$ where the sum is taken over all $\ell \in \mathbb{N}$ and all arrows $a_1,\dots,a_\ell \in Q_1$ satisfying $s(a_1) = j$ and $i_p := s(a_{p+1}) = t(a_p)$ for $0 < p < \ell$. Set $i_\ell := t(a_\ell)$. The R_{i_ℓ}-module $A_{a_\ell} \otimes_R \cdots \otimes_R A_{a_1} \otimes_R E_j \cong A_{a_\ell} \otimes_{R_{i_{\ell-1}}} \cdots \otimes_{R_{i_1}} A_{a_1} \otimes_{R_j} E_j$ is free, since for $0 < p \le \ell$, each A_{a_p} is a free R_{i_p}-module. Thus P_j is locally free. □

We have the following dual version of Lemma 3.1.9.

Lemma 3.1.10. *Every injective H-module is an injective R-module. More generally,*

$$\mathrm{Inj}^n(H) \;\subseteq\; \mathrm{Inj}^n_R(H)\,.$$

If Q is acyclic and R self-injective, the injective H-modules I_i are locally free with

$$\underline{\mathrm{rank}}(I_i) \;=\; e_i + \sum_{i \xleftarrow{a} j\in Q_1} f_{ji}\cdot \underline{\mathrm{rank}}(I_j)\,.$$

Proof. Since $H = R\langle A\rangle$ and A_R is projective, [Rog75, Corollary 2] proves the first statement. The second one is a straightforward consequence.

There is an obvious dual version of Lemma 3.1.9 for right modules (since A_R is projective and $(A_a)_{R_i}$ is free for all $j \xleftarrow{a} i \in Q_1$). Hence, using ${}_{R_i}(R_i)^* \cong {}_{R_i}R_i$ for all i, the last claim follows by applying the duality $(-)^*$ to the projective right H-modules e_iH. □

Corollary 3.1.11. *If R is Gorenstein,* $\mathrm{Proj}^\infty(H) \cup \mathrm{Inj}^\infty(H) \subseteq \mathcal{F}_R(H)$.

Proof. Use Lemmas 3.1.9 and 3.1.10 and $\mathcal{F}_R(H) = \mathrm{Proj}^\infty_R(H) = \mathrm{Inj}^\infty_R(H)$. □

3.1.2 Rank Vectors of Projectives and Injectives

We state [GLS16a, Lemmas 3.2, 3.3 and Proposition 3.4] in our context.

Corollary 3.1.12. *Assume Q is acyclic. Choose a total order $i_1 < \cdots < i_n$ on Q_0 such that there are no arrows $i_k \leftarrow i_\ell$ in Q with $i_k > i_\ell$. Then*

$$\underline{\mathrm{rank}}(P_{i_k}) \;=\; s_{i_1} \cdots s_{i_{k-1}}(e_{i_k}) \;\in\; \Delta^+_{\mathrm{re}}(Q)\,,$$

and, if R is self-injective, also

$$\underline{\mathrm{rank}}(I_{i_k}) \;=\; s_{i_n} \cdots s_{i_{k+1}}(e_{i_k}) \;\in\; \Delta^+_{\mathrm{re}}(Q)\,.$$

Proof. Let $x_k = s_{i_1} \cdots s_{i_{k-1}}(e_{i_k})$. It is well-known and easily verified by induction that

$$x_k \;=\; e_{i_k} \;-\; \sum_{j<i_k} c_{ji_k} x_j \;=\; e_{i_k} \;+\; \sum_{j\leftarrow i_k} f_{ji_k} x_j\,.$$

In combination with Lemma 3.1.9 we get the first identity, since the e_j form a basis of $\mathbb{Z}^{Q_0}$. The second identity can be proved analogously using Lemma 3.1.10. □

3.1.3 Canonical Short Exact Sequences

The next result is standard. It can be found as a lemma in [Rog75].

Lemma 3.1.13. *There is a short exact sequence of H-bimodules*

$$0 \longrightarrow H \otimes_R A \otimes_R H \xrightarrow{\;\partial\;} H \otimes_R H \xrightarrow{\;\nu\;} H \longrightarrow 0$$

with $\partial(1 \otimes x \otimes 1) = x \otimes 1 - 1 \otimes x$ and $\nu(1 \otimes 1) = 1$. □

Corollary 3.1.14. *For all $M \in \mathrm{Mod}(H)$ there is a short exact sequence*

$$0 \longrightarrow H \otimes_R A \otimes_R M \xrightarrow{\;\partial\;} H \otimes_R M \xrightarrow{\;\nu\;} M \longrightarrow 0$$

of H-modules with $\partial(1 \otimes x \otimes m) = x \otimes m - 1 \otimes xm$ and $\nu(1 \otimes m) = m$.

This is a projective resolution for $M \in \mathrm{Proj}^0_R(H)$. More generally, for $M \in \mathrm{Proj}^n_R(H)$,

$$H \otimes_R M, \quad H \otimes_R A \otimes_R M \;\in\; \mathrm{Proj}^n(H)\,.$$

Proof. Apply $- \otimes_H M$ to the sequence in Lemma 3.1.13 to obtain a short exact sequence isomorphic to the one in the statement. Since ${}_RA$ is projective, ${}_R(A \otimes_R P)$ is projective for every projective R-module P. The functor $A \otimes_R -$ is exact, because A_R is projective. Applying it to a projective resolution for ${}_RM$ of minimal length shows $A \otimes_R M \in \mathrm{Proj}^n(R)$. Now $H \otimes_R M, H \otimes_R (A \otimes_R M) \in \mathrm{Proj}^n(H)$ by [Rog75, Corollary 4]. □

Corollary 3.1.14 has a dual version, which is also treated in [Rog75].

Corollary 3.1.15. *For all $M \in \operatorname{Mod}(H)$ there is a short exact sequence*

$$0 \longrightarrow M \longrightarrow \operatorname{Hom}_R(H, M) \longrightarrow \operatorname{Hom}_R(H, \operatorname{Hom}_R(A, M)) \longrightarrow 0$$

of H-modules.

This is an injective resolution for $M \in \operatorname{Inj}^0_R(H)$. More generally, for $M \in \operatorname{Inj}^n_R(H)$,

$$\operatorname{Hom}_R(H, M), \quad \operatorname{Hom}_R(H, \operatorname{Hom}_R(A, M)) \in \operatorname{Inj}^n(H)\,.$$

Proof. Apply $\operatorname{Hom}_H(-, M)$ to the sequence in Lemma 3.1.13 to obtain the short exact sequence in the statement. Because A_R is projective, ${}_R(\operatorname{Hom}_R({}_RA, I))$ is injective for every injective R-module I. The functor $\operatorname{Hom}_R({}_RA, -)$ is exact, since ${}_RA$ is projective. Applying it to an injective resolution for ${}_RM$ of minimal length shows that $\operatorname{Hom}_R(A, M) \in \operatorname{Inj}^n(R)$. Now $\operatorname{Hom}_R(H, M), \operatorname{Hom}_R(H, \operatorname{Hom}_R(A, M)) \in \operatorname{Inj}^n(H)$ by [Rog75, Corollary 4]. □

Remark 3.1.16. Let $M \in \operatorname{mod}_{\text{l.f.}}(H)$. A discussion similar to the one at the end of the proof of Lemma 3.1.9 shows that as H-modules

$$H \otimes_R M \;\cong\; \bigoplus_{i \in Q_0} P_i^{\operatorname{rank}(M_i)}\,, \qquad H \otimes_R A \otimes_R M \;\cong\; \bigoplus_{j \xleftarrow{a} i \in Q_1} P_j^{f_{ji} \cdot \operatorname{rank}(M_i)}\,.$$

Moreover, if Q is acyclic and R self-injective, similarly

$$\operatorname{Hom}_R(H, M) \;\cong\; \bigoplus_{i \in Q_0} I_i^{\operatorname{rank}(M_i)}\,, \quad \operatorname{Hom}_R(H, \operatorname{Hom}_R(A, M)) \;\cong\; \bigoplus_{i \xleftarrow{a} j \in Q_1} I_j^{f_{ji} \cdot \operatorname{rank}(M_i)}\,.$$

3.1.4 Gorenstein Ground Rings

We generalize [GLS16a, § 3.5] combining results from [Rog75; Iwa80; AS81; AR91].

Proposition 3.1.17. *There are inclusions:*

$$\begin{aligned} \operatorname{Proj}^n(H) \;&\subseteq\; \operatorname{Proj}^n_R(H) \;\subseteq\; \operatorname{Proj}^{n+1}(H) \\ \operatorname{Inj}^n(H) \;&\subseteq\; \operatorname{Inj}^n_R(H) \;\subseteq\; \operatorname{Inj}^{n+1}(H) \end{aligned}$$

Proof. This follows from Lemmas 3.1.9 and 3.1.10 and Corollaries 3.1.14 and 3.1.15. □

Proposition 3.1.18. *If R is n-Gorenstein, H is $(n+1)$-Gorenstein with $\mathcal{F}(H) = \mathcal{F}_R(H)$.*

Proof. On the one hand, Proposition 3.1.17 implies that $\mathcal{F}_R(H) \subseteq \operatorname{Proj}^{n+1}(H) \cap \operatorname{Inj}^{n+1}(H)$. On the other hand, $\operatorname{Proj}^\infty(H) \cup \operatorname{Inj}^\infty(H) \subseteq \mathcal{F}_R(H)$ by Corollary 3.1.11. Hence,

$$\mathcal{F}_R(H) \;=\; \operatorname{Proj}^\infty(H) \;=\; \operatorname{Inj}^\infty(H) \;=\; \operatorname{Proj}^{n+1}(H) \;=\; \operatorname{Inj}^{n+1}(H)\,.$$

In particular, we have $\operatorname{idim}_H H \leq n+1$. With right-module versions of Proposition 3.1.17 and Corollary 3.1.11 one can similarly deduce $\operatorname{idim} H_H \leq n+1$. □

The main theorem of [Rog75] for path algebras reads:

Corollary 3.1.19. $\mathrm{gldim}(R) \leq \mathrm{lgldim}(H) \leq \mathrm{gldim}(R) + 1$.

Proof. Use Proposition 3.1.17 and the fact that by Corollary 2.2.19 every module in $\mathrm{Mod}(R)$ can be extended to a module in $\mathrm{Mod}(H)$. □

We get the generalization of [GLS16a, Proposition 3.5, Corollary 3.7, and Theorem 3.9]:

Corollary 3.1.20. *If $\mathcal{H}$ is local and R self-injective, the path algebra H is 1-Gorenstein with $\mathrm{mod}_{\mathrm{l.f.}}(H) = \mathfrak{f}(H)$.*

Proof. This follows from Remarks 3.1.1 and 3.1.6 and Proposition 3.1.18. □

Corollary 3.1.21. *Assume that Q is acyclic and R self-injective. Then $\mathfrak{f}(H) = \mathfrak{f}_R(H)$ is functorially finite in $\mathrm{mod}(H)$. In particular, $\mathfrak{f}(H)$ has Auslander-Reiten sequences.*

Proof. By Proposition 3.1.18, we have $\mathfrak{f}(H) = \mathrm{proj}^1(H) = \mathrm{inj}^1(H)$. As pointed out in the proof of [GLS16a, Theorem 3.9] the category $\mathfrak{f}(H)$ is functorially finite in $\mathrm{mod}(H)$ by [AR91, Proposition 4.2] and thus has Auslander-Reiten sequences by [AS81, Theorem 2.4]. □

3.1.5 Filtered Modules

Definition 3.1.22. Let $\mathrm{mod}_{\mathrm{filtered}}(H)$ be the full subcategory of $\mathrm{mod}(H)$ consisting of all modules M admitting a filtration of H-modules

$$0 = M_0 \subseteq M_1 \subseteq \cdots \subseteq M_\ell = M \tag{\#}$$

such that for each $1 \leq p \leq \ell$ the factor M_p/M_{p-1} is isomorphic to E_{i_p} for some $i_p \in Q_0$.

It is clear that the integer ℓ is independent of the choice of the filtration. We will denote it by $\ell(M)$ and call it the *filtration length* of M.

Lemma 3.1.23. $\mathrm{mod}_{\mathrm{filtered}}(H) \subseteq \mathrm{proj}^0_R(H)$.

Proof. This is clear because of $\{E_i \mid i \in Q_0\} \subseteq \mathrm{proj}^0_R(H)$. □

Lemma 3.1.24. *Assume Q is acyclic. Then $\mathrm{mod}_{\mathrm{l.f.}}(H) \subseteq \mathrm{mod}_{\mathrm{filtered}}(H)$. More precisely, for every $M \in \mathrm{mod}_{\mathrm{l.f.}}(H)$ there is a filtration like in (#) where all M_p are locally free.*

Proof. Let $M \neq 0$ be in $\mathrm{mod}_{\mathrm{l.f.}}(H)$ and let $\sigma(M) = \{i \in Q_0 \mid M_i \neq 0\}$. We use induction on the cardinality of $\sigma(M)$. Choose a sink i in the full subquiver of Q with vertex set $\sigma(M)$. Then M_i is a submodule of M isomorphic to E_i^r for some $r > 0$ and there is a short exact sequence $0 \to E_i^r \to M \to N \to 0$. Clearly, N belongs to $\mathrm{mod}_{\mathrm{l.f.}}(H)$ and $|\sigma(N)| < |\sigma(M)|$. By induction $N \in \mathrm{mod}_{\mathrm{filtered}}(H)$. Hence, $M \in \mathrm{mod}_{\mathrm{filtered}}(H)$. □

Corollary 3.1.25. *If $\mathcal{H}$ is local and Q acyclic,* $\mathrm{mod}_{\mathrm{filtered}}(H) = \mathrm{mod}_{\mathrm{l.f.}}(H)$.

Proof. This follows from Lemmas 3.1.23 and 3.1.24 and Remark 3.1.6. $\square$

3.1.6 Euler Form

The *Euler form* $\langle -,- \rangle_H$ is defined for H-modules M and N as

$$\langle M,N \rangle_H \;=\; \sum_{n=0}^{\infty} \dim_K \operatorname{Ext}_H^n(M,N)$$

given that, either $M \in \operatorname{proj}^\infty(H)$ and $N \in \operatorname{mod}(H)$, or $M \in \operatorname{mod}(H)$ and $N \in \operatorname{inj}^\infty(H)$. In particular, it is well-defined on $\operatorname{mod}_{\text{l.f.}}(H) \times \operatorname{mod}_{\text{l.f.}}(H)$.

The *symmetrized Euler form* $(-,\,\cdot\,)_H := \langle -,\,\cdot\,\rangle_H + \langle\,\cdot\,,- \rangle_H$ agrees with the symmetric form defined by Q with similar reasoning as in [GLS16a, § 4]:

Lemma 3.1.26. *If Q is acyclic, $(M,N)_H = (\underline{\operatorname{rank}}(M), \underline{\operatorname{rank}}(N))_Q$ for $M,N \in \operatorname{mod}_{\text{l.f.}}(H)$.*

Proof. For $Y \in \{M,N\}$ there exist short exact sequences $0 \to X_Y \to Y \to Z_Y \to 0$ in $\operatorname{mod}_{\text{l.f.}}(H)$ by Lemma 3.1.24. Both $\omega = (-,-)_H$ and $\omega = (\underline{\operatorname{rank}}(-), \underline{\operatorname{rank}}(-))_Q$ satisfy

$$\omega(M,N) \;=\; \sum_{U,V \in \{X,Z\}} \omega(U_M, V_N)\,.$$

Therefore, by induction on the filtration length, we can assume $M = E_i$ and $N = E_k$. From the proof of Lemma 3.1.9 we have a short exact sequence of H-modules

$$0 \longrightarrow \bigoplus_{j \xleftarrow{a} i} P_j^{f_{ji}} \longrightarrow P_i \longrightarrow E_i \longrightarrow 0\,.$$

Applying $\operatorname{Hom}_H(-,E_k)$ and abbreviating $\dim_K \operatorname{Hom}_H(-,-)$ as $[-,-]$ we get

$$\langle E_i, E_k \rangle_H \;=\; [P_i, E_k] - \sum_{j \xleftarrow{a} i} f_{ji} \cdot [P_j, E_k] \;=\; d_k \cdot \left(\delta_{i=k} - \sum_{k \xleftarrow{a} i} f_{ki} \right),$$

where the last equality used that $[P_m, E_k] = \delta_{m=k} \cdot d_k$ because of $\operatorname{Hom}_H(P_m, E_k) \cong (E_k)_m$.

We conclude $(E_i, E_k)_H = \langle E_i, E_k \rangle_H + \langle E_k, E_i \rangle_H = (e_i, e_k)_Q$. $\square$

3.1.7 Auslander-Reiten Translation

Convention 3.1.27. Assume now that Q is acyclic and R self-injective.

In particular, the path algebra H is finite-dimensional and 1-Gorenstein.

Notation 3.1.28. Denote by τ^+ and τ^- the Auslander-Reiten translations in $\operatorname{mod}(H)$.

Lemma 3.1.29. *Let $M \in \mathfrak{f}(H)$. For every projective resolution $0 \to P^1 \to P^0 \to M \to 0$ and injective resolution $0 \to M \to I^0 \to I^1 \to 0$ there are exact sequences in* $\operatorname{mod}(H)$

$$0 \longrightarrow \tau^+(M) \longrightarrow \nu^+(P^1) \longrightarrow \nu^+(P^0) \longrightarrow \nu^+(M) \longrightarrow 0$$

$$0 \longrightarrow \nu^-(M) \longrightarrow \nu^-(I^0) \longrightarrow \nu^-(I^1) \longrightarrow \tau^-(M) \longrightarrow 0$$

that are induced by the Nakayama functors $\nu^+ := \operatorname{Hom}_H(-,H)^*$ *and* $\nu^- := \operatorname{Hom}_H(H^*,-)$.

Moreover, for all $L \in \operatorname{ind}(H)$ *and* $N \in \operatorname{mod}(H)$*:*

(a) $\tau^+(M) \cong \operatorname{Ext}^1_H(M,H)^*$

(b) $\tau^-(M) \cong \operatorname{Ext}^1_H(H^*,M)$

(c) $\operatorname{Ext}^1_H(M,N) \cong \operatorname{Hom}_H(N,\tau^+(M))^*$

(d) $\operatorname{Ext}^1_H(N,M) \cong \operatorname{Hom}_H(\tau^-(M),N)^*$

(e) $\tau^+(L) \in \mathfrak{f}(H)$ *non-zero* $\Leftrightarrow \nu^+(L) = 0$

(f) $\tau^-(L) \in \mathfrak{f}(H)$ *non-zero* $\Leftrightarrow \nu^-(L) = 0$

(g) $N \in \mathfrak{f}(H) \Leftrightarrow \nu^-\tau^+(N) = 0 \Leftrightarrow \nu^+\tau^-(N) = 0$

(h) $\operatorname{Ext}^1_H(M,M) = 0 \Leftrightarrow \operatorname{Hom}_H(M,\tau^+(M)) = 0 \Leftrightarrow \operatorname{Hom}_H(\tau^-(M),M) = 0$

Proof. These are well-known consequences of the identities $\mathfrak{f}(H) = \operatorname{proj}^1(H) = \operatorname{inj}^1(H)$, which hold by Proposition 3.1.18. See also [GLS16a, §§ 3.5 and 11.1]. □

Corollary 3.1.30. *Let* $M \in \operatorname{mod}_{\text{l.f.}}(H)$. *Then:*

(a) $\tau^+(M) \in \operatorname{mod}_{\text{l.f.}}(H)$ *non-zero* $\Leftrightarrow \nu^+(M) = 0$

(b) $\tau^-(M) \in \operatorname{mod}_{\text{l.f.}}(H)$ *non-zero* $\Leftrightarrow \nu^-(M) = 0$

Proof. The implications $\Rightarrow$ in (a) and (b) follow from Lemma 3.1.29 (e) and (f).

For $\Leftarrow$ in (a) apply Lemma 3.1.29 to the projective resolution from Corollary 3.1.14, recall Remark 3.1.16, and note that $\nu^+(P_i) = I_i$ are locally free by Lemma 3.1.10.

Corollary 3.1.15 and Lemma 3.1.9 can be used in a similar way to verify (b). □

3.1.8 Coxeter Transformation

This subsection generalizes results from [GLS16a, §§ 3.4 and 11.1].

Definition 3.1.31. The *Coxeter transformation* $\Phi = \Phi_H \in \operatorname{Aut}(\mathbb{Z}^{Q_0})$ of H is defined by

$$\Phi(\underline{\operatorname{rank}}(P_i)) \;=\; -\underline{\operatorname{rank}}(I_i)\,.$$

Remark 3.1.32. By Lemmas 3.1.9 and 3.1.10 the vectors $\underline{\operatorname{rank}}(P_i)$ and the vectors $\underline{\operatorname{rank}}(I_i)$ form two bases of $\mathbb{Z}^{Q_0}$. This ensures that Φ is well-defined.

Remark 3.1.33. The Coxeter transformation Φ can be computed as the product

$$-C_{\text{inj}} \circ C_{\text{proj}}^{-1}$$

where C_{proj} and C_{inj} act on $e_i = \underline{\operatorname{rank}}(E_i)$ as $C_{\text{proj}}(e_i) = \underline{\operatorname{rank}}(P_i)$ and $C_{\text{inj}}(e_i) = \underline{\operatorname{rank}}(I_i)$.

Lemma 3.1.34. *Let $M \in \operatorname{mod}_{\text{l.f.}}(H)$ be indecomposable and $p \in \mathbb{Z}$ such that $\tau^q(M)$ is locally free and non-zero for all $1 \leq q \leq p$ or for all $p \leq q \leq -1$. Then*

$$\underline{\operatorname{rank}}(\tau^p(M)) \;=\; \Phi^p(\underline{\operatorname{rank}}(M))\,.$$

Proof. This is standard. By induction it suffices to consider the case $p = +1$ and $p = -1$. For $p = +1$, just as in the proof of Corollary 3.1.30, apply Lemma 3.1.29 to the projective resolution from Corollary 3.1.14. Then use $\nu^+(P_i) = I_i$ and the fact that $\underline{\operatorname{rank}}$ is additive on short exact sequences of locally free modules. The case $p = -1$ is similar.

It is also possible to adapt the proof of [GLS16a, Proposition 10.6] to our situation. □

3.2 Symmetric Modulations

By the abstract nature of their proofs many of the results in [GLS16a] appear to be true in a broader context. Even though, some essential insights depend on explicit features of the modulations considered there. First and foremost, this concerns the possibility to identify the left dual ${}^{R}\!A$, the right dual A^R, and the K-dual bimodule A^* with one another in a canonical way (compare Lemma 2.2.25). This section provides the necessary tools to prove that Geiß, Leclerc, and Schröer's results are valid for symmetric (local) modulations.

3.2.1 Symmetric Structures

Recall that a modulation $(R_i, A_a)_{i,a}$ is called symmetric if all R_i are symmetric algebras.

Definition 3.2.1. A *strongly separable modulation* is a symmetric modulation $(R_i, A_a)_{i,a}$ with fixed symmetric structures on all R_i given by $\operatorname{Tr}_{R_i/K}$.

Remark 3.2.2. A K-modulation $(R_i, A_a)_{i,a}$ is strongly separable if and only if all R_i are *strongly separable* algebras, i.e. the trace pairings $(x, y) \mapsto \operatorname{Tr}_{R_i/K}(xy)$ are non-degenerate.

Remark 3.2.3. If $(R_i, A_a)_{i,a}$ is a strongly separable modulation, the ground ring $R = \prod_i R_i$ is a strongly separable algebra.

Example 3.2.4. Let L/K be a separable field extension and assume that $\mathcal{H} = (L_i, A_a)_{i,a}$ is a modulation where all L_i are intermediate fields of L/K. Then $\mathcal{H}$ is strongly separable. In particular, this is the case for cyclic Galois modulations $\mathcal{H}$.

Example 3.2.5. Every finite-dimensional semi-simple K-algebra is a symmetric algebra. This includes all finite-dimensional division algebras over K.

Example 3.2.6. The GLS modulations are symmetric (see Example 2.4.19) but, in general, not strongly separable.

Convention 3.2.7. Assume now that all R_i carry a symmetric structure φ_i.

Denote by φ the induced symmetric structure $\sum_{i\in Q_0}\varphi_i$ on R. Lemma 2.4.18 guarantees the well-definedness of the dual modulation $\mathcal{H}^*$.

By Lemma 2.2.25 and Corollary 2.3.11 the left and right dual bimodules $A_{{}_*a}$ and A_{a_*} can be identified with $A_{a^*} = A_a^*$ and, similarly, ${}^R\!A = \bigoplus_a A_{{}_*a}$ and $A^R = \bigoplus_a A_{a_*}$ with A^*. More precisely, we have a diagram

$$\begin{tikzcd}
{}^R\!A \arrow[r, "\cong"] \arrow[d, shift left, two heads] & A^* \arrow[d, shift left, two heads] & A^R \arrow[l, "\cong"'] \arrow[d, shift left, two heads] \\
A_{{}_*a} \arrow[r, "\cong"'] \arrow[u, shift left, hook] & A_a^* \arrow[u, shift left, hook] & A_{a_*} \arrow[l, "\cong"] \arrow[u, shift left, hook]
\end{tikzcd}$$

where the horizontal maps are the isomorphisms given by postcomposition with φ and the vertical maps are the canonical inclusions and projections.

As usual, we will also identify the double K-dual bimodules $A_a^{**} = \mathrm{Hom}_K(A_a^*, K)$ with A_a and, similarly, $A^{**} = \mathrm{Hom}_K(A^*, K)$ with A via evaluation. Thus we have a diagram

$$\begin{tikzcd}
A \arrow[r, "\cong"] \arrow[d, shift left, two heads] & A^{**} \arrow[d, shift left, two heads] \\
A_a \arrow[r, "\cong"'] \arrow[u, shift left, hook] & A_a^{**} \arrow[u, shift left, hook]
\end{tikzcd}$$

where the horizontal maps are the evaluation maps ev that send x to the function ev_x given by evaluation at x, i.e. $\mathrm{ev}_x(f) = f(x)$.

With these identifications the adjunction correspondence (see Definition 2.2.15) yields canonical isomorphisms of K-vector spaces for all $M, N \in \mathrm{Mod}(R)$:

$$\mathrm{Hom}_R(A \otimes_R M, N) \xrightarrow{\ \mathrm{ad}\ } \mathrm{Hom}_R(M, A^* \otimes_R N)$$

$$\mathrm{Hom}_R(A^* \otimes_R N, M) \xrightarrow{\ \mathrm{ad}_*\ } \mathrm{Hom}_R(N, A \otimes_R M)$$

Notation 3.2.8. Extending the notation introduced in Definition 2.2.15, we will write $f^\vee$ both for $\mathrm{ad}(f)$ and for $\mathrm{ad}_*(f)$ and, similarly, ${}^\vee g$ for $\mathrm{ad}^{-1}(g)$ and for $\mathrm{ad}_*^{-1}(g)$.

3.2.2 Non-Degenerate Trace Maps

The construction and statements about the trace maps presented below are variations of classical results. We use similar notation as [Bou70, III. §9 no. 1–4].

The symmetric structure φ on R can be used to define a K-valued *trace map*

$$\mathrm{End}_R(M) \xrightarrow{\mathrm{tr}} K$$

for every finitely generated projective R-module M.

The procedure to do this is standard: Because $R \xrightarrow{\varphi} K$ is symmetric, precomposing φ with the bilinear map $\mathrm{Hom}_R(M, R) \times M \longrightarrow R$, $(f, m) \mapsto f(m)$, induces a K-linear form

$$\mathrm{Hom}_R(M, R) \otimes_R M \xrightarrow{\ \nu\ } K\,.$$

Using that M is finitely generated projective, we get tr by precomposing ν with the inverse of the isomorphism $\mathrm{Hom}_R(M,R) \otimes_R M \longrightarrow \mathrm{End}_R(M)$ given by $f \otimes m \mapsto (n \mapsto f(n)m)$.

Analogously, using the symmetric structure φ_i on R_i, one gets a trace map

$$\mathrm{End}_{R_i}(M_i) \xrightarrow{\mathrm{tr}_i} K$$

for every finitely generated projective R_i-module M_i. By construction we have $\mathrm{tr} = \sum_i \mathrm{tr}_i$ as maps $\mathrm{End}_R(M) \to K$ for every finitely generated projective R-module $M = \bigoplus_i M_i$.

Lemma 3.2.9. *If R_i is commutative,* $\mathrm{tr}_i = \varphi_i \circ \mathrm{Tr}_{R_i}$. *If $\varphi_i = \mathrm{Tr}_{R_i/K}$, then* $\mathrm{tr}_i = \mathrm{Tr}_K$.

Proof. The first statement is clear by the construction of tr_i. For commutative R_i the second statement follows from the first one and $\mathrm{Tr}_{R_i/K} \circ \mathrm{Tr}_{R_i} = \mathrm{Tr}_K$. It is a straightforward, classical exercise to verify it also in the general situation. □

Lemma 3.2.10. *For $M, N \in \mathrm{proj}(R)$ and $f \in \mathrm{Hom}_R(M,N)$, $g \in \mathrm{Hom}_R(N,M)$ it is*

$$\mathrm{tr}(fg) = \mathrm{tr}(gf)\,.$$

For $M \in \mathrm{mod}_{\mathrm{l.f.}}(R)$ and $x \in \mathrm{Z}(R)$ it is $\mathrm{tr}\big(x^M\big) = \varphi(x) \cdot \underline{\mathrm{rank}}(M) = \sum_i \varphi_i(x_i)\,\mathrm{rank}\,(M_i)$.

Proof. The verification of the identity $\mathrm{tr}(fg) = \mathrm{tr}(gf)$ is a straightforward, classical exercise. For example, this can be done by computation with basis elements after reducing to the case that M and N are free. Here, the symmetry of φ plays a role. The rest is clear. □

Lemma 3.2.11. *For $M \in \mathrm{Mod}_{\mathrm{l.f.}}(R)$ and $N \in \mathrm{mod}_{\mathrm{l.f.}}(R)$ the* trace pairing $(f,g) \mapsto \mathrm{tr}(fg)$ *on* $\mathrm{Hom}_R(M,N) \times \mathrm{Hom}_R(N,M)$ *induces an isomorphism:*

$$\begin{array}{ccc} \mathrm{Hom}_R(M,N) & \xrightarrow{\quad\vartheta^{M,N}\quad} & \mathrm{Hom}_R(N,M)^* \\ f & \longmapsto & (g \mapsto \mathrm{tr}(fg)) \end{array}$$

Proof. We can reduce to the case $M = N = R$ and then have to prove the non-degeneracy of the bilinear form $(f,g) \overset{\omega}{\longmapsto} \mathrm{tr}(fg) = \varphi(f(1)g(1))$. The non-degeneracy of ω follows from the fact that the bilinear form $R \times R \to K$, $(x,y) \mapsto \varphi(xy)$, is non-degenerate, a reformulation of condition (a) in Definition 2.2.20 (see [Lam99, Theorem 3.15 and the ensuing Remark]). □

Let $M \in \mathrm{Mod}(R)$. Recall that H-module structures on M are parametrized by

$$\mathrm{Rep}(A,M) \;=\; \mathrm{Hom}_R(A \otimes_R M, M)\,.$$

Corollary 3.2.12. *For each $M \in \mathrm{mod}_{\mathrm{l.f.}}(R)$ there is an isomorphism of K-vector spaces:*

$$\begin{array}{ccc} \mathrm{Rep}(A^*,M) & \xrightarrow{\quad\theta^{M}\quad} & \mathrm{Rep}(A,M)^* \\ {}_{A^*}M & \longmapsto & \big({}_AM \mapsto \mathrm{tr}\big({}_{A^*}M \circ {}_AM^\vee\big)\big) \end{array}$$

Proof. Use Definition 2.2.15 and Lemma 3.2.11. $\square$

Inspired by [GLS16a, Proposition 8.3] one discovers the following relation.

Lemma 3.2.13. *Let $M \in \operatorname{Proj}(R)$ and $N \in \operatorname{proj}(R)$. Then we have $\operatorname{tr}\big(f^\vee g\big) = \operatorname{tr}\big(fg^\vee\big)$ for all $f \in \operatorname{Hom}_R(A \otimes_R M, N)$ and $g \in \operatorname{Hom}_R(A^* \otimes_R N, M)$.*

Proof. We can again reduce to the situation $M = N = R$. With a slight abuse of notation, identifying $A \otimes_R R$ and $A^* \otimes_R R$ in the canonical way with A and A^*, we have

$$f \in \operatorname{Hom}_R(A, R)\,, \quad g \in \operatorname{Hom}_R(A^*, R)\,, \quad f^\vee \in \operatorname{Hom}_R(R, A^*)\,, \quad g^\vee \in \operatorname{Hom}_R(R, A)\,.$$

Using $\operatorname{tr}(f^\vee g) = \operatorname{tr}(gf^\vee)$ it is enough to show $\operatorname{tr}(gf^\vee) = \operatorname{tr}(fg^\vee)$. This is easy to verify:

$$\operatorname{tr}(gf^\vee) = \varphi(gf^\vee(1)) = (\varphi g)(\varphi f) = (\varphi f)(\operatorname{ev}^{-1}(\varphi g)) = \varphi(fg^\vee(1)) = \operatorname{tr}(fg^\vee) \qquad \square$$

3.2.3 Characteristic Elements

Certain constructions for symmetric modulations mirror intrinsic properties of the fixed symmetric structure φ. Some information is encoded in the *characteristic elements.*

The isomorphisms $A^* \xrightarrow{{}^*\varphi} {}^R\!A$ and $A^* \xrightarrow{\varphi^*} A^R$ of R-bimodules give rise to non-degenerate R-bilinear maps ${}^\varphi\langle -, - \rangle$ and $\langle -, - \rangle^\varphi$ defined, for $x \in A$ and $f \in A^*$, as

$${}^\varphi\langle x, f\rangle \;=\; {}^*\varphi(f)(x)\,, \qquad\qquad \langle f, x\rangle^\varphi \;=\; \varphi^*(f)(x)\,.$$

The R-bimodule map $A^* \otimes_R A \to R$ given by $f \otimes x \mapsto \langle f, x\rangle^\varphi$ postcomposed with the inverse of $A^* \otimes_R A \to \operatorname{End}_R({}_RA)$ given by $f \otimes x \mapsto \big(y \mapsto {}^\varphi\langle x, f\rangle y\big)$ yields a map $\vec{\Phi} : \operatorname{End}_R({}_RA) \to R$.

Precomposing $\vec{\Phi}$ with the map $\mathrm{Z}(R) \to \operatorname{End}_R({}_RA)$, which sends elements $r \in \mathrm{Z}(R)$ to left multiplication with r, gives a map $\mathrm{Z}(R) \to R$. It is not hard to see that its image is contained in $\mathrm{Z}(R)$. In this way, we obtain a K-linear map $\vec{\Phi} : \mathrm{Z}(R) \to \mathrm{Z}(R)$.

The dual construction for $A \otimes_R A^* \to R$ given by $x \otimes f \mapsto {}^\varphi\langle x, f\rangle$ and the isomorphism $A \otimes_R A^* \to \operatorname{End}_R(A_R)$ given by $x \otimes f \mapsto \big(y \mapsto y\langle f, x\rangle^\varphi\big)$ yields $\overleftarrow{\Phi} : \mathrm{Z}(R) \to \mathrm{Z}(R)$.

For $j \xleftarrow{a} i \in Q_1$ the restriction of $\vec{\Phi}$ to $\mathrm{Z}(R_j)$ and of $\overleftarrow{\Phi}$ to $\mathrm{Z}(R_i)$ induce K-linear maps

$$\mathrm{Z}(R_j) \xrightarrow{\;\;\vec{\Phi}_a\;\;} \mathrm{Z}(R_i)\,, \qquad\qquad \mathrm{Z}(R_i) \xrightarrow{\;\;\overleftarrow{\Phi}_a\;\;} \mathrm{Z}(R_j)\,.$$

Definition 3.2.14. We call $\vec{\varphi}_a = \vec{\Phi}_a(1)$ and $\overleftarrow{\varphi}_a = \overleftarrow{\Phi}_a(1)$ the *characteristic elements.*

For $i, j \in Q_0$ define $\vec{\varphi}_{ji} = \vec{\Phi}_{ji}(1)$ and $\overleftarrow{\varphi}_{ji} = \overleftarrow{\Phi}_{ji}(1)$ where, for $y \in \mathrm{Z}(R_j)$ and $x \in \mathrm{Z}(R_i)$,

$$\vec{\Phi}_{ji}(y) \;=\; \sum_a \vec{\Phi}_a(y)\,, \qquad\qquad \overleftarrow{\Phi}_{ji}(x) \;=\; \sum_a \overleftarrow{\Phi}_a(x)$$

and the sums are taken over all arrows $j \xleftarrow{a} i \in \overline{Q}_1$.

Remark 3.2.15. Let C be a basis of ${}_{R_j}(A_a)$ and B a basis of $(A_a)_{R_i}$. For elements $y \in \mathrm{Z}(R_j)$ and $x \in \mathrm{Z}(R_i)$ we have the formulas

$$\vec{\Phi}_a(y) \;=\; \sum_{c \in C} \varphi_i^*(\varphi_j c^*)(yc)\,, \qquad\qquad \overleftarrow{\Phi}_a(x) \;=\; \sum_{b \in B} {}^*\varphi_j(\varphi_i b^*)(bx)\,.$$

3.2.4 Calculation Rules for Adjoints

In Corollary 2.3.10 we gave formulas for $f^\vee$ and ${}^\vee f$. We add the following calculation rules:

Lemma 3.2.16. *Let $M \in \mathrm{Mod}(R)$, $f \in \mathrm{Hom}_R(A \otimes_R M, M)$, $g \in \mathrm{Hom}_R(A^* \otimes_R M, M)$, and $x \in \mathrm{Z}(R)$. We have the following identities:*

(a) ${}^\vee\big(\mathrm{id}_{A \otimes_R M}\big) \circ \big(x^{A \otimes_R M}\big)^\vee \;=\; (\vec{\Phi}(x))^M$

(b) $f^\vee \circ g \;=\; (f \circ (\mathrm{id}_A \otimes g))^\vee$

(c) $f \circ g^\vee \;=\; {}^\vee(f^\vee \circ g) \circ \big(\mathrm{id}_{A^* \otimes_R M}\big)^\vee \;=\; {}^\vee\big(\mathrm{id}_{A^* \otimes_R M}\big) \circ (f^\vee \circ g)^\vee$

Proof. Using $R = \prod_i R_i$ and $A = \bigoplus_a A_a$, we can reduce to the case $A = A_a$, $M = M_j \oplus M_i$, $x = x_j \in \mathrm{Z}(R_j)$ for some $j \xleftarrow{a} i \in Q_1$. After choosing bases for ${}_{R_j}(A_a)$ and $(A_a)_{R_i}$, the identities can now be checked by straightforward, explicit calculations with the help of the formulas given in Corollary 2.3.10 and Remark 3.2.15. □

3.2.5 Rank-Aware Structures

Lemma 3.2.17. *$\varphi_i(\vec{\varphi}_a) = f_{ji} \cdot \varphi_j(1)$ and $\varphi_j(\overleftarrow{\varphi}_a) = f_{ij} \cdot \varphi_i(1)$ for all $j \xleftarrow{a} i$.*

Proof. Use Remark 3.2.15 and $\mathrm{rank}\,{}_{R_j}(A_a) = f_{ji}$ and $\mathrm{rank}\,(A_a)_{R_i} = f_{ij}$. □

Definition 3.2.18. The symmetric structure φ for $\mathcal{H}$ is *rank-aware* if for all $j \xleftarrow{a} i$

$$\vec{\varphi}_a \;=\; \mathrm{rank}\,(A_a)_{R_i} \;=\; f_{ij}\,, \qquad\qquad \overleftarrow{\varphi}_a \;=\; \mathrm{rank}\,{}_{R_j}(A_a) \;=\; f_{ji}\,.$$

Remark 3.2.19. Assume Q is connected and $\mathrm{char}(K) \nmid d_i$ for all i. Let φ be a rank-aware structure for $\mathcal{H}$. By Lemma 3.2.17 we have $d_j\varphi_i(1) = d_i\varphi_j(1)$ for all $i, j \in Q_0$. If $\varphi_i(1) \neq 0$, rescaling φ by $d_i/\varphi_i(1) = d_j/\varphi_j(1)$ yields a rank-aware structure $\tilde{\varphi} = \sum_i \tilde{\varphi}_i$ with $\tilde{\varphi}_i(1) = d_i$.

Example 3.2.20. Taking in Example 2.4.25 $\varphi_1 = \mathrm{Tr}_{\mathbb{R}/\mathbb{R}} = \mathrm{id}_{\mathbb{R}}$ and $\varphi_2 = \mathrm{Tr}_{\mathbb{C}/\mathbb{R}} = 2\,\mathrm{Re}$ to define the symmetric structure φ, one computes $\overleftarrow{\varphi}_a = 1 = f_{21}$ and $\vec{\varphi}_a = 2 = f_{12}$. Therefore, this choice of φ yields a rank-aware structure.

To illustrate that symmetric structures φ are not necessarily rank-aware, we consider Example 2.4.25 in the general situation, where $\mathbb{R} \xrightarrow{\varphi_1} \mathbb{R}$ and $\mathbb{C} \xrightarrow{\varphi_2} \mathbb{R}$ are arbitrary non-zero $\mathbb{R}$-linear maps. Then, $x = \varphi_1(1) \in \mathbb{R}^\times$ and $w = \varphi_2(1) - i\,\varphi_2(i) \in \mathbb{C}^\times$. It is not hard to check that ${}^*\varphi_1(f)(z) = x^{-1}f(z)$ and $\varphi_2^*(f)(z) = w^{-1}(f(z) - if(iz))$. Calculating, we see $\overleftarrow{\varphi}_a = 2\,(xw)^{-1}$ and $\vec{\varphi}_a = x^{-1}\varphi_2(1)$. In particular, always $\overleftarrow{\varphi}_a \neq 0$, while $\vec{\varphi}_a = 0$ is possible.

Example 3.2.21. Assume that $\mathcal{H}$ is the strongly separable modulation of a modular quiver Q over a comfy extension $(L/K, \zeta, v)$. We show that $\varphi = \mathrm{Tr}_{R/K} = \sum_i \mathrm{Tr}_{L_i/K}$ is rank-aware:

For $j \xleftarrow{a} i \in Q_1$ it is $A_a = {}_jL_i^{\rho_a} = L_j \otimes_{\rho_a} L_i$ for some $\rho_a \in \mathrm{Gal}(L_{ji}/K)$. Let $m = [L : K]$, $v_j = v^{m/d_j}$, $v_i = v^{m/d_i}$, $v_{ji} = v^{m/d_{ji}}$. The set $\{b_s = v_j^s \otimes 1 \,|\, 0 \le s < f_{ij}\}$ is a basis of $(A_a)_{R_i}$.

There are elements $h_{rs} \in K$ such that

$$\gamma_s \;=\; {}^*\!\varphi_j(\varphi_i b_s^*)(b_s) \;=\; \sum_{r=0}^{d_j-1} h_{rs} v_j^{-r}\,.$$

We have $d_j h_{rs} = \operatorname{Tr}_{L_j/K}(v_j^r \gamma_s) = \varphi_i b_s^*(v_j^{r+s} \otimes 1)$ because ${}^*\!\varphi_j(\varphi_i b_s^*)$ is L_j-linear. Let us write $r+s = p + q f_{ij}$ for some q and $0 \le p < f_{ij}$. Define $\zeta_a = \rho_a(v_{ji})/v_{ji}$. Then

$$d_j h_{rs} \;=\; \operatorname{Tr}_{L_i/K}\left(b_s^*(c_p)\cdot \zeta_a^{-q}\, v_i^{q f_{ji}}\right) \;=\; \delta_{p=s}\,\delta_{q\equiv 0 \bmod d_{ji}} \cdot d_i \cdot \zeta_a^{-q}\,\lambda^{q/d_{ji}} \;=\; \delta_{r=0}\cdot d_i\,.$$

This shows $\gamma_s = d_i/d_j$. Hence, $\overleftarrow{\varphi}_a = f_{ij}\cdot d_i/d_j = f_{ji}$ by Remark 3.2.15. Similarly, $\vec{\varphi}_a = f_{ij}$.

3.3 Deformed Preprojective Algebras

Extending [CH98] we define deformed preprojective algebras Π^λ for symmetric modulations. The choice $\lambda = 0$ recovers the ordinary preprojective algebra.

Convention 3.3.1. We still assume that a symmetric structure φ for $\mathcal{H}$ is fixed.

As before, we use φ to identify the duals $A_{{}_*a}$ and A_{a_*} with A_a^* as well as the duals ${}^R\!A$ and A^R with A^*. In the canonical way, A_a^{**} is identified with A_a and A^{**} with A.

3.3.1 Compatible Double Representations

Instead of working with modules over deformed preprojective algebras Π^λ we use the equivalent notion of λ-compatible A-double representations.

Recall that the double $\overline{\mathcal{H}}$ of $\mathcal{H}$ is the K-modulation $\overline{\mathcal{H}} = (R_i, A_a)_{i\in\overline{Q}_0, a\in\overline{Q}_1}$ with $A_{a^*} = A_a^*$. As usual, we do not distinguish $\overline{H}$-modules from $\overline{A}$-representations.

Because of $\overline{A} = A \oplus A^*$ an $\overline{A}$-representation can be regarded as a triple $(M, {}_AM, {}_{A^*}M)$ where $(M, {}_AM)$ is an A-representation and $(M, {}_{A^*}M)$ an A^*-representation.

Definition 3.3.2. Let $\lambda \in \mathrm{Z}(R)$. A *$\lambda$-compatible A-double representation* $(M, {}_AM, {}_{A^*}M)$ consists of an A-representation $(M, {}_AM)$ and an A^*-representation $(M, {}_{A^*}M)$ making the following square commute "up to adding λ":

$$\begin{tikzcd} M \arrow[r, "{}_AM^\vee"] \arrow[d, "{}_{A^*}M^\vee"'] & A^*\otimes_R M \arrow[d, "{}_{A^*}M"] \\ A\otimes_R M \arrow[r, "{}_AM"'] & M \end{tikzcd}$$

More precisely, we require ${}_{A^*}M \circ {}_AM^\vee - {}_AM \circ {}_{A^*}M^\vee = \lambda^M$.

Remark 3.3.3. By Lemma 2.2.14 and its right-module version we have canonical isomorphisms

$$\operatorname{End}_R({}_RA) \xrightarrow{\;\Delta\;} A^*\otimes_R A\,, \qquad \operatorname{End}_R(A_R) \xrightarrow{\;\Delta_*\;} A\otimes_R A^*\,.$$

Definition 3.3.4. Viewing $A^* \otimes_R A$ and $A \otimes_R A^*$ as summands of the degree-2 component of the path algebra $\overline{H}$, the *preprojective relation* $\rho \in \overline{H}$ is defined as

$$\rho \;=\; \Delta(\mathrm{id}) - \Delta_*(\mathrm{id})\,.$$

Definition 3.3.5. The *sign* of an arrow $a \in \overline{Q}_1$ is

$$\varepsilon_a \;:=\; \begin{cases} 1 & \text{if } a \in Q_1, \\ -1 & \text{if } a \in Q_1^*. \end{cases}$$

Remark 3.3.6. Choose for each $j \xleftarrow{a} i \in \overline{Q}_1$ a basis B_a of ${}_{R_j}(A_a)$ and let $\{b^* \,|\, b \in B_a\}$ be the R_j-dual basis of B_a. With respect to these bases and with Convention 3.3.1 the preprojective relation assumes the form $\rho = \sum_{a \in \overline{Q}_1,\, b \in B_a} \varepsilon_a \cdot b^* \cdot b$.

Remark 3.3.7. Clearly, $\rho = \sum_{i \in Q_0} \rho_i$ where $\rho_i = e_i \rho e_i$.

Remark 3.3.8. The preprojective relation ρ commutes with every element in the ground ring R, i.e. $\rho\, r = r \rho$ for all $r \in R$. As a consequence, for every module $M \in \mathrm{Mod}(\overline{H})$ left multiplication with ρ defines an endomorphism $\rho^M \in \mathrm{End}_R(M)$.

Definition 3.3.9. Let $\lambda \in \mathrm{Z}(R)$. The *deformed preprojective algebra* $\Pi^\lambda = \Pi^\lambda_{\mathcal{H}}$ is

$$\Pi^\lambda \;=\; \overline{H}/\langle \rho - \lambda \rangle\,.$$

The *preprojective algebra* $\Pi = \Pi_{\mathcal{H}}$ is the deformed preprojective algebra Π^0.

Lemma 3.3.10. $\rho^M = \sum_{a \in \overline{Q}_1} \varepsilon_a \cdot M_{a^*} M_a^\vee = {}_{A^*}M \circ {}_A M^\vee - {}_A M \circ {}_{A^*} M^\vee$ *for* $M \in \mathrm{Mod}(\overline{H})$.

Proof. Note that $\Delta = \sum_{a \in Q_1} \Delta_a$ and $\Delta_* = \sum_{a \in Q_1^*} \Delta_a$, where, for $j \xleftarrow{a} i \in Q_1$, the maps

$$\mathrm{End}_{R_j}({}_{R_j}(A_a)) \xrightarrow{\Delta_a} {}_{*}A_a \otimes_{R_j} A_a\,, \qquad\qquad \mathrm{End}_{R_i}((A_a)_{R_i}) \xrightarrow{\Delta_{a^*}} A_a \otimes_{R_i} A_{a_*}$$

are the canonical isomorphisms. For $j \xleftarrow{a} i \in \overline{Q}_1$ the formula for $M_a^\vee$ in Corollary 2.3.10, and the description of Δ_a given in Lemma 2.2.14 show that $M_{a^*} M_a^\vee \in \mathrm{End}_{R_i}(M_i)$ is multiplication with $\Delta_a(\mathrm{id})$. Now use the definition of ρ. □

Corollary 3.3.11. $M \mapsto ({}_R M, {}_A M, {}_{A^*} M)$ *defines an equivalence* $\mathrm{Mod}(\overline{H}) \to \mathrm{Rep}(\overline{A})$ *and an equivalence between* $\mathrm{Mod}(\Pi^\lambda)$ *and the category of* λ*-compatible* A*-double representations.*

Proof. Combine Corollary 2.2.19, Definition 3.3.2, and Lemma 3.3.10. □

Notation 3.3.12. Denote by ${}_{j\leftarrow}\overline{Q}_1$ the set of all arrows in $\overline{Q}$ ending in the vertex $j \in Q_0$.

For $M \in \mathrm{Mod}(\overline{H})$ abbreviate $A_a^M = A_a \otimes_{R_i} M_i$ for $j \xleftarrow{a} i \in \overline{Q}_1$ and

$${}_{j\leftarrow}M \;=\; e_j \overline{A} \otimes_R M \;=\; \bigoplus_{a \in {}_{j\leftarrow}\overline{Q}_1} A_a^M\,.$$

Let $A_a^M \xrightarrow{\mu_a} {}_{j\leftarrow}M \xrightarrow{\pi_a} A_a^M$ be the canonical inclusion and projection and

$$\pi_j \;=\; \sum_{a\in{}_{j\leftarrow}\overline{Q}_1} M_a\pi_a\,, \qquad\qquad \mu_j \;=\; \sum_{a\in{}_{j\leftarrow}\overline{Q}_1} \varepsilon_{a^*}\cdot\mu_a M_{a^*}^\vee\,.$$

Finally, define $\pi = \sum_{j\in Q_0}\pi_j$ and $\mu = \sum_{j\in Q_0}\mu_j$.

If it seems necessary to stress the dependence of π, μ etc. on M, we write π^M, μ^M etc.

Remark 3.3.13. Let $j \xleftarrow{a} i \in Q_1$. If M_i is free, then A_a^M is free. Furthermore, if M_i is free of finite rank m_i, we have $\operatorname{rank}_{R_j}(A_a^M) = \operatorname{rank}_{R_j}(A_a)\cdot m_i = f_{ji}\,m_i$.

The composition $\pi^M\mu^M$ can be viewed as an element in $\operatorname{End}_R(M)$. The following corollary corresponds to the discussion in [CH98, § 4] and [GLS16a, Proposition 5.2].

Corollary 3.3.14. *$\rho^M = \pi^M\mu^M$ for $M \in \operatorname{Mod}(\overline{H})$, so $M \in \operatorname{Mod}(\Pi^\lambda) \Leftrightarrow \pi^M\mu^M = \lambda^M$.*

Proof. Use Lemma 3.3.10. □

3.3.2 Lifting Representations

The next lemma generalizes [CH98, Lemma 4.2]. Our proof is conceptually the same.

Proposition 3.3.15. *For every $M \in \operatorname{mod}_{\text{l.f.}}(H)$ there is a short exact sequence*

$$0 \to \operatorname{Ext}^1_H(M,M)^* \longrightarrow \operatorname{Rep}(A^*,M) \xrightarrow{\mathfrak{d}} \operatorname{End}_R(M) \xrightarrow{\mathfrak{t}} \operatorname{End}_H(M)^* \to 0$$

where $\mathfrak{d}$ and $\mathfrak{t}$ act on ${}_{A^*}M \in \operatorname{Rep}(A^*,M)$ and $f \in \operatorname{End}_R(M)$ as

$$\mathfrak{d}\big({}_{A^*}M\big) = {}_{A^*}M \circ {}_{A}M^\vee - {}_{A}M \circ {}_{A^*}M^\vee\,, \qquad \mathfrak{t}(f) \;=\; \big(g \mapsto \operatorname{tr}(fg)\big)\,.$$

Proof. Applying $\operatorname{Hom}_H(-,M)$ to the short exact sequence from Corollary 3.1.14 yields an exact sequence

$$0 \to \operatorname{End}_H(M) \xrightarrow{\nu^\sharp} \operatorname{Hom}_H(H\otimes_R M, M) \xrightarrow{\partial^\sharp} \operatorname{Hom}_H(H\otimes_R A\otimes_R M, M) \longrightarrow \operatorname{Ext}^1_H(M,M) \to 0\,.$$

Using the tensor-hom adjunction we get a commutative square

$$\begin{tikzcd}
\operatorname{Hom}_H(H\otimes_R M,M) \arrow[r, "\partial^\sharp"] \arrow[d, "\alpha"'] & \operatorname{Hom}_H(H\otimes_R A\otimes_R M,M) \arrow[d] \\
\operatorname{End}_R(M) \arrow[r, "\partial^\flat"] & \operatorname{Rep}(A,M)
\end{tikzcd}$$

where the vertical maps are the canonical isomorphisms and $\partial^\flat$ is given, for all $f \in \operatorname{End}_R(M)$ and $x\otimes m \in A\otimes_R M$, by $\partial^\flat(f)(x\otimes m) = xf(m) - f(xm)$.

The map $\nu^\flat = \alpha\circ\nu^\sharp$ is the canonical inclusion. We get an exact sequence

$$0 \to \operatorname{End}_H(M) \xrightarrow{\nu^\flat} \operatorname{End}_R(M) \xrightarrow{\partial^\flat} \operatorname{Rep}(A,M) \longrightarrow \operatorname{Ext}^1_H(M,M) \to 0\,.$$

Apply $(-)^* = \mathrm{Hom}_K(-, K)$ to obtain an exact sequence

$$0 \to \mathrm{Ext}^1_H(M,M)^* \longrightarrow \mathrm{Rep}(A,M)^* \xrightarrow{(\partial^\flat)^*} \mathrm{End}_R(M)^* \xrightarrow{(\nu^\flat)^*} \mathrm{End}_H(M)^* \to 0\,.$$

The isomorphisms $\theta = \theta^M$ and $\vartheta = \vartheta^{M,M}$ from Lemma 3.2.11 and Corollary 3.2.12 make the following square commute:

$$\begin{tikzcd} \mathrm{Rep}(A,M)^* \arrow[r, "(\partial^\flat)^*"] & \mathrm{End}_R(M)^* \\ \mathrm{Rep}(A^*,M) \arrow[u, "\theta"] \arrow[r, "\mathfrak{d}"] & \mathrm{End}_R(M) \arrow[u, "\vartheta"'] \end{tikzcd}$$

It remains to verify that $\mathfrak{d}$ and $\mathfrak{t} = (\nu^\flat)^* \circ \vartheta$ act as claimed. For $\mathfrak{t}$ this is evident and for $\mathfrak{d}$ it suffices to check

$$((\partial^\flat)^* \circ \theta)\big({}_{A^*}M\big) \;=\; \vartheta\big({}_{A^*}M \circ {}_AM^\vee - {}_AM \circ {}_{A^*}M^\vee\big)\,.$$

Evaluating at f and applying Lemmas 3.2.10 and 3.2.13, this means

$$\mathrm{tr}\big({}_{A^*}M \circ (\partial^\flat(f))^\vee\big) \;=\; \mathrm{tr}\big({}_{A^*}M \circ \big({}_AM^\vee \circ f - (f \circ {}_AM)^\vee\big)\big)\,.$$

So we have to show $(\partial^\flat(f))^\vee = {}_AM^\vee \circ f - (f \circ {}_AM)^\vee$. According to Lemma 3.2.16 (b) this is equivalent to $\partial^\flat(f) = {}_AM \circ (\mathrm{id}_A \otimes f) - f \circ {}_AM$, so we are done. □

We have the following two corollaries. Compare [Cra01, Lemma 3.2].

Corollary 3.3.16. *Let $M \in \mathrm{mod}_{\mathrm{l.f.}}(\Pi^\lambda)$. Then $\mathrm{tr}(\lambda^M \circ g) = 0$ for all $g \in \mathrm{End}_H(M)$.*

Proof. It is $\lambda^M = \rho^M = {}_{A^*}M \circ {}_AM^\vee - {}_AM \circ {}_{A^*}M^\vee = \mathfrak{d}({}_{A^*}M) \in \ker(\mathfrak{t})$ by Lemma 3.3.10, Corollary 3.3.14, and Proposition 3.3.15. □

Corollary 3.3.17. *Let $M \in \mathrm{mod}_{\mathrm{l.f.}}(H)$ be indecomposable.*

(a) If M lifts to a Π^λ-module, then necessarily $\varphi(\lambda) \cdot \underline{\mathrm{rank}}(M) = 0$.

(b) If M satisfies $\mathrm{Ext}^1_H(M,M) = 0$, there is at most one way to lift M to a Π^λ-module.

Proof. For (a) combine Lemma 3.2.10 and Corollary 3.3.16 applied to $g = \mathrm{id}_M$. For (b) use Proposition 3.3.15. □

3.3.3 Reflection Functors

In this subsection we introduce reflection functors for deformed preprojective algebras of symmetric modulations. The construction we present is a painless adaptation of Crawley-Boevey and Holland's [CH98]. Furthermore, Baumann and Kamnitzer's [BK12] variant of these functors for non-deformed preprojective algebras are covered as a special case.

While Crawley-Boevey and Holland defined the reflection functors only for vertices with invertible deformation parameter – and obtained equivalences –, Baumann and Kamnitzer considered the non-deformed situation – and obtained a pair of adjoint functors.

Let $M \in \mathrm{Mod}(\Pi^\lambda)$ and fix $j \in Q_0$. For notational simplicity we write π, μ, and λ instead of π^M, μ^M, and λ^M. Recall that $\pi_j\mu_j = \lambda_j$ by Corollary 3.3.14. We define

$$M_j^+ \;=\; \ker(\pi_j)\,, \qquad\qquad M_j^- \;=\; {}_{j\leftarrow}M/\mathrm{im}(\mu_j)\,.$$

Inspired by the diagram depicted in [BK12, Remark 2.4 (ii)] we have schematically

$$\begin{tikzcd}
 & & M_j^- \arrow[d, dotted] \arrow[drr, "\mu_j^-"] & & \\
{}_{j\leftarrow}M \arrow[urr, two heads, "\pi_j^-"] \arrow[rr, "\pi_j"] \arrow[drr, "\pi_j^+"'] & & M_j \arrow[rr, "\mu_j"] \arrow[d, dotted] & & {}_{j\leftarrow}M \\
 & & M_j^+ \arrow[urr, hook, "\mu_j^+"'] & &
\end{tikzcd}$$

where μ_j^+ and π_j^- are the canonical inclusion and projection and the maps π_j^+ and μ_j^- are both induced by the endomorphism $\chi_j = \mu_j\pi_j - \lambda_j$.

Remark 3.3.18. The endomorphisms χ_j and $\gamma_j = \mu_j\pi_j$ form a pair of generalized orthogonal projections in the sense that $\chi_j^2 = \lambda_j\chi_j$, $\gamma_j^2 = \lambda_j\gamma_j$, and $\chi_j\gamma_j = \gamma_j\chi_j = 0$.

Obviously, $\pi_j^+\mu_j^+ = -\lambda_j$ and $\pi_j^-\mu_j^- = -\lambda_j$. If $\lambda_j \in R_j^\times$, then by Remark 3.3.18

$${}_{j\leftarrow}M \;=\; \ker(\pi_j) \oplus \mathrm{im}(\mu_j)\,, \qquad \ker(\pi_j) \;=\; M_j^+ \;\cong\; M_j^-\,, \qquad \mathrm{im}(\mu_j) \;\cong\; M_j\,.$$

On the other hand, if $\lambda_j = 0$, it is possible to fill in unique morphisms in the above diagram, indicated by dotted arrows, such that every subdiagram is commutative.

This construction yields modules $M^+, M^- \in \mathrm{Mod}(\overline{H})$, where for $\pm \in \{+,-\}$, $a \in {}_{j\leftarrow}\overline{Q}_1$,

$$M_a^\pm \;=\; \pi_j^\pm\mu_a\,, \qquad\qquad M_{a^*}^\pm \;=\; {}^\vee\!\left(\varepsilon_{a^*}\pi_a\mu_j^\pm\right),$$

and $M_i^\pm = M_i$ for $i \neq j$ and $M_a^\pm = M_a$ for $a \notin {}_{j\leftarrow}\overline{Q}_1$.

The assignment $M \mapsto M^\pm$ extends in a canonical way to a functor $\mathrm{Mod}(\Pi^\lambda) \to \mathrm{Mod}(\overline{H})$.

Definition 3.3.19. Let $r_j \in \mathrm{Aut}_K(\mathrm{Z}(R))$ be the map defined, for $\lambda \in \mathrm{Z}(R)$, $i \in Q_0$, by

$$(r_j(\lambda))_i \;=\; \begin{cases} -\lambda_j & \text{if } i = j,\\ \lambda_i + \vec{\Phi}_{ij}(\lambda_j) & \text{otherwise.}\end{cases}$$

Remark 3.3.20. It is $r_j^2 = \mathrm{id}_{\mathrm{Z}(R)}$, and $(r_j(\lambda))_j$ is invertible (resp. zero) if and only if λ_j is invertible (resp. zero). Moreover, $r_j = \mathrm{id}_{\mathrm{Z}(R)}$ if $\lambda_j = 0$.

Remark 3.3.21. Assume that φ is a rank-aware structure for $\overline{\mathcal{H}}$. This implies $\vec{\varphi}_{ij} = -c_{ij}$. Thus $(r_j(\lambda))_i = \lambda_i - c_{ij}\lambda_j$ for $i \neq j$ and $\lambda_j \in K$. So r_j restricts to a map $K^{Q_0} \to K^{Q_0}$ that is (induced by) the transpose of the simple reflection $s_j \in \operatorname{Aut}(\mathbb{Z}^{Q_0})$. Compare [CH98, § 5].

The next lemma shows that $M \mapsto M^{\pm}$ induces a functor $\operatorname{Mod}(\Pi^{\lambda}) \xrightarrow{\Sigma_j^{\pm}} \operatorname{Mod}(\Pi^{r_j(\lambda)})$.

Lemma 3.3.22. *The element $\rho - r_j(\lambda)$ annihilates $M^{\pm}$.*

Proof. We use Corollary 3.3.14. We already know $\pi_j^{\pm}\mu_j^{\pm} = -\lambda_j = (r_j(\lambda))_j$. For $i \neq j$ and arrows $j \xleftarrow{a} i \in \overline{Q}_1$ by definition

$$(M_{a^*}^{\pm})^{\vee} M_a^{\pm} \;=\; \varepsilon_{a^*}\pi_a\mu_j^{\pm}\pi_j^{\pm}\mu_a \;=\; \varepsilon_{a^*}\pi_a(\mu_j\pi_j - \lambda_j)\mu_a \;=\; M_{a^*}^{\vee}M_a - \varepsilon_{a^*}\lambda_j^{A_a \otimes_{R_i} M_i} .$$

Applying first $(-)^{\vee}$ and then ${}^{\vee}\big(\mathrm{id}_{A_a \otimes_{R_i} M_i}\big) \circ -$, Lemma 3.2.16 (c) and (a) show

$$M_{a^*}^{\pm}(M_a^{\pm})^{\vee} \;=\; M_{a^*}M_a^{\vee} - \varepsilon_{a^*}\,{}^{\vee}\Big(\mathrm{id}_{A_a \otimes_{R_i} M_i}\Big) \circ \Big(\lambda_j^{A_a \otimes_{R_i} M_i}\Big)^{\vee} \;=\; M_{a^*}M_a^{\vee} - \varepsilon_{a^*}\vec{\Phi}_a(\lambda_j) .$$

With the help of Lemma 3.3.10 and Corollary 3.3.14 we can conclude as desired

$$\pi_i^{\pm}\mu_i^{\pm} \;=\; \sum_{j \xleftarrow{a} i \in \overline{Q}_1} \varepsilon_a M_{a^*}^{\pm}(M_a^{\pm})^{\vee} \;=\; \pi_i\mu_i + \sum_{j \xleftarrow{a} i \in \overline{Q}_1} \vec{\Phi}_a(\lambda_j) \;=\; \lambda_i + \vec{\Phi}_{ij}(\lambda_j) \;=\; (r_j(\lambda))_i . \quad \square$$

If $\lambda_j \in R_j$ is invertible, the functors Σ_j^{+} and Σ_j^{-} are isomorphic and we set $\Sigma_j := \Sigma_j^{+}$. The next proposition is proved in complete analogy to [CH98, Theorem 5.1].

Proposition 3.3.23. *Let $j \in Q_0$ such that $\lambda_j \in R_j^{\times}$. We have quasi-inverse equivalences:*

$$\operatorname{Mod}(\Pi^{\lambda}) \underset{\Sigma_j}{\overset{\Sigma_j}{\rightleftarrows}} \operatorname{Mod}(\Pi^{r_j(\lambda)})$$

For $M \in \operatorname{mod}_{\text{l.f.}}(\Pi^{\lambda})$ the module $\Sigma_j(M)$ is locally free with $\underline{\operatorname{rank}}(\Sigma_j(M)) \;=\; s_j(\underline{\operatorname{rank}}(M))$.

Proof. It is an easy consequence of the construction and the invertibility of λ_j that the endofunctor $\Sigma_j\Sigma_j$ of $\operatorname{Mod}(\Pi^{\lambda})$ is isomorphic to the identity. Analogously, this is the case for the endofunctor $\Sigma_j\Sigma_j$ of $\operatorname{Mod}(\Pi^{r_j(\lambda)})$. The decomposition ${}_{j\leftarrow}M = \ker(\pi_j) \oplus \operatorname{im}(\mu_j)$, where $(\Sigma_j(M))_j \cong \ker(\pi_j)$ and $M_j \cong \operatorname{im}(\mu_j)$, shows that $\Sigma_j(M)$ is locally free because ${}_{j\leftarrow}M$ and M_j are free. It also proves the formula for the rank vectors. $\square$

Every module $M \in \operatorname{Mod}(\Pi^{\lambda})$ has a submodule $\operatorname{sub}_j(M)$ and a factor module $\operatorname{fac}_j(M)$ given as $\operatorname{sub}_j(M) = \ker(\mu_j)$ and $\operatorname{fac}_j(M) = M/(\operatorname{im}(\pi_j) + \sum_{i \neq j} M_i)$. We have the following version of [BK12, Proposition 2.5] and [GLS16a, Proposition 9.1].

Proposition 3.3.24. *Let $j \in Q_0$ such that $\lambda_j = 0$. Then $(\Sigma_j^{-}, \Sigma_j^{+})$ is a pair of adjoint endofunctors of* $\operatorname{Mod}(\Pi^{\lambda})$ *and there are short exact sequences*

$$0 \longrightarrow \operatorname{sub}_j \longrightarrow \mathrm{id} \longrightarrow \Sigma_j^{+}\Sigma_j^{-} \longrightarrow 0 , \qquad\qquad 0 \longrightarrow \Sigma_j^{-}\Sigma_j^{+} \longrightarrow \mathrm{id} \longrightarrow \operatorname{fac}_j \longrightarrow 0 .$$

Proof. The adjunction follows from Lemma 2.3.4, $\lambda_j = 0$, and the universal properties of kernels, cokernels, and direct sums. See [BK12, Proof of Proposition 2.5] and [GLS16a, Proof of Proposition 9.1], where the argument is given in detail. For the existence of the exact sequences note $M_j^{-+} = \mathrm{im}(\mu_j)$ and $M_j^{+-} = {}_{j\leftarrow}M/\ker(\pi_j) \cong \mathrm{im}(\pi_j)$. □

3.4 Auslander-Reiten Translation via Coxeter Functors

This section builds on [GLS16a, §§ 9–12]. We point out the necessary adaptations to make Geiß, Leclerc, and Schröer's arguments work for symmetric (local) modulations.

Gabriel proves in [Gab80] for path algebras $H = KQ$ of unweighted acyclic quivers that Bernstein, Gelfand, and Ponomarev's [BGP73] Coxeter functor C^+ coincides with the Auslander-Reiten translation τ^+ essentially up to a sign change. More precisely, the subbimodule Π_1 of the preprojective algebra Π generated by the dual arrows a^* represents both, the twisted Coxeter functor C^+T and the Auslander-Reiten translation τ^+, i.e.

$$C^+T \;\cong\; \mathrm{Hom}_H(\Pi_1, -) \;\cong\; \tau^+ .$$

Geiß, Leclerc, and Schröer showed that these isomorphisms are valid in a broader setting, not on the whole module category, but after restriction to locally free modules.

Convention 3.4.1. Assume that Q is acyclic and $\mathcal{H}$ local and symmetric.

Fix a symmetric structure φ on R.

3.4.1 BGP-Reflection and Coxeter Functors

Notation 3.4.2. For $j \in Q_0$ let $\mathcal{H}^{*j}$ be the submodulation of $\overline{\mathcal{H}}$ induced by $Q^{*j} \subseteq \overline{Q}$.

Let $H^{*j} = H_{\mathcal{H}^{*j}}$ and denote by ι_j the R-algebra map $H^{*j} \hookrightarrow \overline{H}$ given by Lemma 2.4.32.

There is a functor $\mathrm{Mod}(H) \xrightarrow{\mathcal{L}_0} \mathrm{Mod}(\Pi)$ that extends each H-module M to a Π-module with $M_{a^*} = 0$ for all $a \in Q_1$ (and maps morphisms to themselves).

Definition 3.4.3. For $j \in Q_0$ let $\mathrm{Mod}(\Pi) \xrightarrow{\mathcal{R}_j} \mathrm{Mod}(H^{*j})$ be the restriction functor induced by the composition of ι_j with the canonical projection $\overline{H} \twoheadrightarrow \Pi$. Define

$$F_j^+ \;=\; \mathcal{R}_j \circ \Sigma_j^+ \circ \mathcal{L}_0 , \qquad\qquad F_j^- \;=\; \mathcal{R}_j \circ \Sigma_j^- \circ \mathcal{L}_0 .$$

In the case that j is a sink or a source in Q we define $\mathrm{Mod}(H) \xrightarrow{F_j} \mathrm{Mod}(H^{*j})$ as

$$F_j \;=\; \begin{cases} F_j^+ & \text{if } j \text{ is a sink,} \\ F_j^- & \text{if } j \text{ is a source.} \end{cases}$$

Remark 3.4.4. Let j be a sink or a source in Q. The functor F_j is a generalization of the “image functor” in [BGP73, §1]. For example, if j is a sink in Q, let $M \in \mathrm{Mod}(H)$ and

$${}_{j\leftarrow}M \;=\; e_j A \otimes_R M \xrightarrow{\pi_j} M_j$$

given by multiplication (compare Notation 3.3.12). Then $(F_j^+(M))_j = M_j^+ = \ker(\pi_j)$.

Recall that a module $M \in \mathrm{Mod}(H)$ is called *rigid* if $\mathrm{Ext}^1_H(M, M) = 0$.

Proposition 3.4.5. *Let j be a sink or a source in the quiver Q. For every locally free and rigid module $M \in \mathrm{mod}(H)$ the module $F_j(M)$ is locally free and rigid, too.*

Proof. With Proposition 3.3.24 at hand, we can use [GLS16a, Proof of Proposition 9.6]. □

Definition 3.4.6. Let $\pm \in \{+, -\}$ and let $(i_1, \ldots, i_n)$ be any $\pm$-admissible sequence for Q. The *Coxeter functor* $C^\pm$ for $\mathcal{H}$ is the endofunctor $F^\pm_{i_n} \cdots F^\pm_{i_1}$ of $\mathrm{Mod}(H)$.

Remark 3.4.7. As indicated in [GLS16a, § 9.4], the argument given in the proof of [BGP73, Lemma 1.2] shows that $C^\pm$ does not depend on the choice of the $\pm$-admissible sequence.

Remark 3.4.8. With Proposition 3.3.24 it is easy to see that (C^-, C^+) is an adjoint pair.

Definition 3.4.9. The *twist* $T \in \mathrm{Aut}_R(H)$ of $H = R\langle A\rangle$ is induced by $A \xrightarrow{-\mathrm{id}} A \subseteq H$.

The automorphism of $\mathrm{Mod}(H)$ induced by $T \in \mathrm{Aut}_R(H)$ is again denoted by T.

The *twisted Coxeter bimodule* Π_1 is the sub H-bimodule of $\Pi = \Pi_{\mathcal{H}}$ generated by A^*.

Remark 3.4.10. It is $T^2 = \mathrm{id}$ and $C^\pm T = TC^\pm$ for $\pm \in \{+, -\}$.

The following is the key result [GLS16a, Theorem 10.1] in our setting:

Theorem 3.4.11. *We have $C^+T \cong \mathrm{Hom}_H(\Pi_1, -)$ and $C^-T \cong \Pi_1 \otimes_H -$ as endofunctors of* $\mathrm{Mod}(H)$*. Moreover, $C^+T \cong \tau^+$ and $C^-T \cong \tau^-$ as functors* $\mathrm{mod}_{\mathrm{l.f.}}(H) \to \mathrm{mod}(H)$.

3.4.2 Gabriel-Riedtmann Construction

Before giving a proof of Theorem 3.4.11, we discuss what form the Gabriel-Riedtmann construction takes in our context. Compare [Gab73, § 5] and [GLS16a, § 10.2].

Definition 3.4.12. The *Riedtmann quiver* defined by Q is the weighted quiver $\widetilde{Q}$ with vertex set $\widetilde{Q}_0 = \{i_\pm \,|\, i \in Q_0\,, \pm \in \{+,-\}\}$, weights $d_{i_\pm} = d_i$, and arrow set

$$\widetilde{Q}_1 \;=\; \{j_\pm \xleftarrow{a_\pm} i_\pm \,|\, j \xleftarrow{a} i \in Q_1,\, \pm \in \{+,-\}\} \;\dot\cup\; \{j_+ \xrightarrow{a^*} i_- \,|\, j \xleftarrow{a} i \in Q_1\}\,.$$

The *Riedtmann modulation* defined by $\mathcal{H}$ is the K-modulation $\widetilde{\mathcal{H}} = (R_{\tilde{\imath}}, A_{\tilde{a}})_{\tilde{\imath},\tilde{a}}$ for $\widetilde{Q}$ defined by $R_{i_\pm} = R_i$, $A_{a_\pm} = A_a$, $A_{a^*} = A_a^*$. Let $\widetilde{R} = R_{\widetilde{\mathcal{H}}}$ and $\widetilde{A} = A_{\widetilde{\mathcal{H}}}$ and $\widetilde{H} = H_{\widetilde{\mathcal{H}}}$.

Warning. In [GLS16a] the letter $\widetilde{H}$ is not used for $H_{\widetilde{\mathcal{H}}}$ but for its quotient $\widetilde{\Gamma}$ defined below.

Example 3.4.13. We reproduce [GLS16a, Example 10.2.2] to account for our slightly different notation. Let Q be the weighted quiver $j \xrightarrow{a} i \xleftarrow{b} k$ with $d_j = 1$, $d_i = d_k = 2$ of type B_3. Then the Riedtmann quiver $\widetilde{Q}$ looks as follows:

$$\begin{tikzcd}
& j_- \arrow[dl, "a_-"'] & & j_+ \arrow[dl, "a_+"'] \\
i_- & & i_+ \arrow[ul, "a^*"'] \arrow[dl, "b^*"] & \\
& k_- \arrow[ul, "b_-"] & & k_+ \arrow[ul, "b_+"]
\end{tikzcd}$$

Notation 3.4.14. For $\pm \in \{+,-\}$ there are injective non-unital K-algebra morphisms

$$H \xhookrightarrow{\eta_\pm} \widetilde{H}$$

induced by the identities $R_i \xrightarrow{\mathrm{id}} R_{i_\pm}$ and $A_a \xrightarrow{\mathrm{id}} A_{a_\pm}$.

Furthermore, there is an injective K-algebra morphism

$$H^* \xhookrightarrow{\eta_\bullet} \widetilde{H}$$

induced by $R_i \xrightarrow{(\mathrm{id}\,\mathrm{id})^{\mathbf{T}}} R_{i_-} \oplus R_{i_+}$ and $A_{a^*} \xrightarrow{\mathrm{id}} A_{a^*}$.

Inspired by [Gab73] (slightly deviating from [GLS16a]), we use the following notation for the restriction functors induced by $\eta_\pm$ and $\eta_\bullet$:

$$\begin{tikzcd}
\mathrm{Mod}(\widetilde{H}) \arrow[rr, "\mathcal{R}_\pm"] & & \mathrm{Mod}(H) \\
M \arrow[rr, mapsto] & & \mathcal{R}_\pm(M)
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathrm{Mod}(\widetilde{H}) \arrow[rr, "\mathcal{R}_\bullet"] & & \mathrm{Mod}(H^*) \\
M \arrow[rr, mapsto] & & \mathcal{R}_\bullet(M)
\end{tikzcd}$$

Remark 3.4.15. The ground ring of $\widetilde{\mathcal{H}}$ factorizes as $\widetilde{R} = R^- \times R^+$ with $R^\pm = \eta_\pm(R)$ and the species decomposes as $\widetilde{A} = A^- \oplus A^\bullet \oplus A^+$ with $A^\pm = \eta_\pm(A)$ and $A^\bullet = \eta_\bullet(A^*)$.

Remark 3.4.16. For $M \in \mathrm{Mod}(\widetilde{H})$ as R-modules $\mathcal{R}_\bullet(M) \cong \mathcal{R}_-(M) \oplus \mathcal{R}_+(M)$ canonically. Moreover, note that $xm_+ \in \mathcal{R}_-(M)$ and $xm_- = 0$ for all $x \in A^* \subseteq H^*$ and $m_\pm \in \mathcal{R}_+(M)$. Multiplication in the H^*-module $\mathcal{R}_\bullet(M)$ induces an R-module homomorphism

$$A^* \otimes_R \mathcal{R}_+(M) \xrightarrow{\ {}_\bullet M\ } \mathcal{R}_-(M)\,.$$

Definition 3.4.17. A *Riedtmann A-representation* is a triple (M_-, M_+, κ_M) consisting of two H-modules $M_\pm$ and an R-module morphism $A^* \otimes_R M_+ \xrightarrow{\kappa_M} M_-$.

A *morphism of Riedtmann A-representations* $(M_-, M_+, \kappa_M) \xrightarrow{(f_-,f_+)} (N_-, N_+, \kappa_N)$ is a pair of morphisms $M_\pm \xrightarrow{f_\pm} N_\pm$ of H-modules making the following square commute:

$$\begin{tikzcd}
A^* \otimes_R M_+ \arrow[r, "\kappa_M"] \arrow[d, "\mathrm{id}\otimes f_+"] & M_- \arrow[d, "f_-"] \\
A^* \otimes_R N_+ \arrow[r, "\kappa_N"] & N_-
\end{tikzcd}$$

Lemma 3.4.18. *The functor $M \mapsto (\mathcal{R}_-(M), \mathcal{R}_+(M), {}_\bullet M)$ defines an equivalence between the category $\mathrm{Mod}(\widetilde{H})$ and the category of Riedtmann A-representations.*

More precisely, for Riedtmann A-representations (M_-, M_+, κ_M) *let* M *be the* $\widetilde{R}$*-module with underlying set* $M_- \times M_+$ *and multiplication given as*

$$rm \;=\; (r_-m_-, r_+m_+)\,,$$

for all $r = (\eta_-(r_-), \eta_+(r_+)) \in R^- \times R^+ = \widetilde{R}$ *with* $r_-, r_+ \in R$ *and* $m = (m_-, m_+) \in M$*. There is a unique* $\widetilde{H}$*-module structure* ${}_{\widetilde{A}}M$ *on* M *making the following diagrams commute:*

$$\begin{tikzcd} A \otimes_R M_\pm \arrow[rr, "\eta_\pm \otimes \mathrm{id}"] \arrow[dr, "{}_AM_\pm"'] & & A^\pm \otimes_{\widetilde{R}} M_\pm \arrow[dl, "{}_{\widetilde{A}}M"] \\ & M_\pm & \end{tikzcd} \qquad \begin{tikzcd} A^* \otimes_R M_+ \arrow[rr, "\eta_\bullet \otimes \mathrm{id}"] \arrow[dr, "\kappa_M"'] & & A^\bullet \otimes_{\widetilde{R}} M_+ \arrow[dl, "{}_{\widetilde{A}}M"] \\ & M_- & \end{tikzcd}$$

Proof. This is similar to the proof of Corollaries 2.2.19 and 3.3.11. $\square$

As mentioned during the definition of the preprojective algebra (at the beginning of § 3.3), there are isomorphisms

$$\Delta^+ : \mathrm{End}_R({}_RA) \xrightarrow{\quad\Delta\quad} A^* \otimes_R A \xrightarrow{\;\eta_\bullet \otimes \eta_+\;} A^\bullet \otimes_{\widetilde{R}} A^+ \subseteq \widetilde{H}\,,$$

$$\Delta^- : \mathrm{End}_R(A_R) \xrightarrow{\quad\Delta_*\quad} A \otimes_R A^* \xrightarrow{\;\eta_- \otimes \eta_\bullet\;} A^- \otimes_{\widetilde{R}} A^\bullet \subseteq \widetilde{H}\,.$$

Definition 3.4.19. The *Riedtmann algebra* for $\mathcal{H}$ is $\widetilde{\Gamma} = \widetilde{H}/\langle\widetilde{\rho}\rangle$ where

$$\widetilde{\rho} \;=\; \Delta^+(\mathrm{id}) + \Delta^-(\mathrm{id}) \;\in\; \widetilde{H}\,.$$

Notation 3.4.20. Let $M \in \mathrm{Mod}(\widetilde{H})$. For arrows $j \xleftarrow{a} i \in \widetilde{Q}_1$ we abbreviate, similarly as before, $A_a^M = A_a \otimes_{R_i} M_i$ and $A_{a^*}^M = A_{a^*} \otimes_{R_j} M_j$. Moreover, define for each $j \in Q_0$

$${}_{j_-\leftarrow}M \;=\; \bigoplus_{j_- \xleftarrow{a} i} A_a^M\,, \qquad\qquad {}_{j_+\to}M \;=\; \bigoplus_{j_+ \xrightarrow{a} i} A_{a^*}^M\,.$$

where the sums are taken over arrows in $\widetilde{Q}$.

Let ${}_{j_-\leftarrow}M \xrightarrow{\widetilde{\pi}_a} A_a^M$ and $A_{a^*}^M \xrightarrow{\widetilde{\mu}_a} {}_{j_+\to}M$ be the canonical projection and inclusion. Set

$$\widetilde{\pi}_{j_-} \;=\; \sum_{j_- \xleftarrow{a} i} M_a \widetilde{\pi}_a\,, \qquad\qquad \widetilde{\mu}_{j_+} \;=\; \sum_{j_+ \xrightarrow{a} i} \widetilde{\mu}_a M_a^\vee\,.$$

Finally, define $\widetilde{\pi} = \sum_{j \in Q_0} \widetilde{\pi}_{j_-}$ and $\widetilde{\mu} = \sum_{j \in Q_0} \widetilde{\mu}_{j_+}$.

When we want to stress the dependence of $\widetilde{\pi}$, $\widetilde{\mu}$ etc. on M we write $\widetilde{\pi}^M$, $\widetilde{\mu}^M$ etc.

Using the correspondence given by $j_- \xleftarrow{a_-} i_- \leftrightarrow j_+ \xrightarrow{a^*} i_-$ and $j_- \xleftarrow{a^*} k_+ \leftrightarrow j_+ \xrightarrow{a_+} k_+$, we identify ${}_{j_+\to}M$ with ${}_{j_-\leftarrow}M$ so that $\widetilde{\pi}^M\widetilde{\mu}^M$ is a well-defined element in

$$\mathrm{End}_{R_+\to R_-}(M) \;:=\; \bigoplus_{j \in Q_0} \mathrm{Hom}_{R_j}(M_{j_+}, M_{j_-})\,.$$

We denote by $\widetilde{\rho}^M$ the element in $\mathrm{End}_{R_+\to R_-}(M)$ given by left multiplication with $\widetilde{\rho}$.

Lemma 3.4.21. *Let $M \in \operatorname{Mod}(\widetilde{H})$. Then $\widetilde{\rho}^M = \sum_{a \in Q_1} \left(M_{a^*} M_{a_+}^\vee + M_{a_-} M_{a^*}^\vee \right) = \widetilde{\pi}^M \widetilde{\mu}^M$ In particular, $M \in \operatorname{Mod}(\widetilde{\Gamma})$ if and only if $\widetilde{\pi}^M \widetilde{\mu}^M = 0$.*

Proof. This can be proved analogously to Lemma 3.3.10 and Corollary 3.3.14. □

Convention 3.4.22. Just as [Gab73, § 5.5] and [GLS16a, § 10] we will assume for notational simplicity that $Q_0 = \{1, \dots, n\}$ and $(1, \dots, n)$ is a +-admissible sequence for Q.

Remark 3.4.23. Let $0 \le \ell \le n$. The full subquiver of $\widetilde{Q}$ on the vertices i_+ with $1 \le i \le \ell$ and i_- with $\ell < i \le n$ can be identified with the quiver

$$Q^\ell \;:=\; (\cdots (Q^{*1})^{*2} \cdots)^{*\ell}$$

via $Q^\ell \xrightarrow{\iota_\ell} \widetilde{Q}$ given by $i \mapsto i_+$ for $1 \le i \le \ell$ and $i \mapsto i_-$ for $\ell < i \le n$ sending $j \xleftarrow{a} i \in Q_1^\ell$ to

$$\iota_\ell(a) \;=\; \begin{cases} a_- & \text{if } i > \ell \text{ and } j > \ell, \\ a_+ & \text{if } i \le \ell \text{ and } j \le \ell, \\ a & \text{otherwise.} \end{cases}$$

Definition 3.4.24. For $I \subseteq \widetilde{Q}_0$ we denote by $\widetilde{\Gamma}(I)$ the algebra $e_I \widetilde{\Gamma} e_I$ with $e_I = \sum_{i \in I} e_i$. For $0 \le \ell \le n$ set

$$\Gamma^\ell \;:=\; \widetilde{\Gamma}\big(\{\, i_+ \,|\, 1 \le i \le \ell \,\} \cup \{\, i_- \,|\, \ell < i \le n \,\}\big)\,,$$

$$\widetilde{\Gamma}^\ell \;:=\; \widetilde{\Gamma}\big(\{\, i_+ \,|\, 1 \le i \le \ell \,\} \cup \{\, i_- \,|\, 1 \le i \le n \,\}\big)\,.$$

Let H^ℓ be the path algebra of $(\cdots (\mathcal{H}^{*1})^{*2} \cdots)^{*\ell}$ and $H^\ell \xrightarrow{\widetilde{\eta}_\ell} \widetilde{\Gamma}^\ell$ the non-unital K-algebra homomorphism induced by the identities $R_i \to R_{\iota_\ell(i)}$ and $A_a \to A_{\iota_\ell(a)}$, whose image is Γ^ℓ.

For the restriction functors corresponding to $\widetilde{\eta}_\ell$ and the inclusion $\widetilde{\Gamma}^{\ell-1} \subseteq \widetilde{\Gamma}^\ell$ we write

$$\operatorname{Mod}(\widetilde{\Gamma}^\ell) \xrightarrow{\widetilde{\mathcal{R}}_\ell} \operatorname{Mod}(H^\ell)\,, \qquad \operatorname{Mod}(\widetilde{\Gamma}^\ell) \xrightarrow{\operatorname{Res}_\ell} \operatorname{Mod}(\widetilde{\Gamma}^{\ell-1})\,.$$

Remark 3.4.25. As was pointed out in [Gab73, § 5.5], the right adjoint Res^ℓ of Res_ℓ can be explicitly described as follows: It sends each $\widetilde{\Gamma}^{\ell-1}$-module M to the $\widetilde{\Gamma}^\ell$-module M^+ with

$$M^+_{\ell_+} \;=\; \ker\left(\widetilde{\pi}^M_{\ell_-}\right), \qquad M^+_i \;=\; M_i \quad \text{for } i \ne \ell_+\,,$$

such that for all $j \xleftarrow{a} i$ the map $(M_a^+)^\vee$ is induced by the canonical projection, if $i = \ell_+$, and is equal to M_a otherwise. We summarize this remark in the next lemma.

Lemma 3.4.26. $\widetilde{\mathcal{R}}_i \operatorname{Res}^i \cong F_i^+ \widetilde{\mathcal{R}}_{i-1}$.

Proof. This follows from Remark 3.4.25 and the definition of F_i^+. For a more detailed argument see [GLS16a, proof of Lemma 10.2]. □

Notation 3.4.27. Let $\pm \in \{+,-\}$. Denote by $\widetilde{\eta}_\pm$ the composition of $H \xhookrightarrow{\eta_\pm} \widetilde{H}$ with the projection $\widetilde{H} \twoheadrightarrow \widetilde{\Gamma}$ and by $\widetilde{\mathcal{R}}_\pm$ the corresponding restriction functor $\mathrm{Mod}(\widetilde{\Gamma}) \to \mathrm{Mod}(H)$.

As usual, $H \xrightarrow{\widetilde{\eta}_\pm} \widetilde{\Gamma}$ defines a coinduction functor:

$$\begin{array}{ccc} \mathrm{Mod}(H) & \xrightarrow{\qquad\widetilde{\mathcal{R}}^\pm\qquad} & \mathrm{Mod}(\widetilde{\Gamma}) \\ M & \longmapsto & \mathrm{Hom}_H(\widetilde{\mathcal{R}}_\pm(\widetilde{\Gamma}), M) \end{array}$$

Remark 3.4.28. It is a standard fact that $\big(\widetilde{\mathcal{R}}_\pm, \widetilde{\mathcal{R}}^\pm\big)$ is an adjoint pair.

Remark 3.4.29. It is $\widetilde{\eta}_+ = \widetilde{\eta}_n$ and $\widetilde{\eta}_- = \iota \circ \widetilde{\eta}_0$ for the canonical inclusion $\Gamma^0 = \widetilde{\Gamma}^0 \xhookrightarrow{\iota} \widetilde{\Gamma}$.

Therefore $\widetilde{\mathcal{R}}_+ = \widetilde{\mathcal{R}}_n$ and $\widetilde{\mathcal{R}}_- = \widetilde{\mathcal{R}}_0 \,\mathrm{Res}_1 \cdots \mathrm{Res}_n$, so $\widetilde{\mathcal{R}}^- \cong \mathrm{Res}^n \cdots \mathrm{Res}^1 \widetilde{\mathcal{R}}^0$ for every quasi-inverse of the equivalence $\widetilde{\mathcal{R}}^0$ of $\widetilde{\mathcal{R}}_0$.

Corollary 3.4.30. $C^+ \cong \widetilde{\mathcal{R}}_+ \widetilde{\mathcal{R}}^-$.

Proof. By Lemma 3.4.26 and Remark 3.4.29

$$C^+ \;\cong\; (F_n^+ \cdots F_1^+)\, \widetilde{\mathcal{R}}_0 \,\widetilde{\mathcal{R}}^0 \;\cong\; \widetilde{\mathcal{R}}_n \,(\mathrm{Res}^n \cdots \mathrm{Res}^1)\, \widetilde{\mathcal{R}}^0 \;\cong\; \widetilde{\mathcal{R}}_+ \widetilde{\mathcal{R}}^- \,. \qquad \square$$

Lemma 3.4.31. $\widetilde{\mathcal{R}}_+ \widetilde{\mathcal{R}}^- \cong \mathrm{Hom}_H(\Pi_1, T(-))$.

Proof. Let $I_\pm = \{i_\pm \,|\, i \in Q_0\}$ and ${}_-\widetilde{\Gamma}_+$ be the H-bimodule with underlying set $e_{I_-}\widetilde{\Gamma} e_{I_+}$ and multiplication given, for $x, y \in H$ and $z \in {}_-\widetilde{\Gamma}_+$, by

$$xzy \;=\; \eta_-(T(x)) \cdot z \cdot \eta_+(y)\,.$$

Then $\widetilde{\mathcal{R}}_+ \widetilde{\mathcal{R}}^- = \widetilde{\mathcal{R}}_+ \mathrm{Hom}_H(\widetilde{\mathcal{R}}_-(\widetilde{\Gamma}), -) \cong \mathrm{Hom}_H({}_-\widetilde{\Gamma}_+, T(-))$.

It only remains to observe that ${}_-\widetilde{\Gamma}_+ \cong \Pi_1$ as H-bimodule. $\square$

The next two definitions generalize [Gab73, § 5.4] and correspond to [GLS16a, § 10.4]. The aim is to provide an explicit description of $\widetilde{\mathcal{R}}^-$.

Definition 3.4.32. Let $\mathrm{Mod}(H) \xrightarrow{\overline{\mathcal{R}}^-} \mathrm{Mod}(\widetilde{H})$ be the functor that sends H-modules M to $\widetilde{H}$-modules $\widetilde{M}$ with

$$\mathcal{R}_-(\widetilde{M}) \;=\; M\,, \qquad \mathcal{R}_+(\widetilde{M}) \;=\; \mathrm{Hom}_R(A^* \otimes_R H, M)\,,$$

$$_\bullet\widetilde{M}(x \otimes f) \;=\; f(x \otimes 1) \quad \text{ for } x \otimes f \in A^* \otimes_R \mathcal{R}_+(\widetilde{M}).$$

Take for the action of $\overline{\mathcal{R}}^-$ on morphisms the obvious one.

Definition 3.4.33. For $f \in \mathcal{R}_+(\widetilde{M})$ we denote by $f^\sharp = f^\sharp_M$ the map $A \otimes_R A^* \otimes_R H \to M$ obtained by postcomposing $\mathrm{id}_A \otimes f$ with the multiplication map ${}_AM$. Define

$$\widetilde{\rho}_+ \;=\; \Delta^+(\mathrm{id})\,, \qquad \widetilde{\rho}_- \;=\; \Delta^-(\mathrm{id})\,.$$

For every $z \in H$ the element $\widetilde{\rho}_+ z$ will be regarded as an element in $A^* \otimes_R H$ and $\widetilde{\rho}_- z$ as an element in $A \otimes_R A^* \otimes_R H$.

By the next lemma, $M \mapsto \widehat{M}$ induces a subfunctor $\operatorname{Mod}(H) \xrightarrow{\widehat{\mathcal{R}}^-} \operatorname{Mod}(\widetilde{\Gamma})$ of $\overline{\mathcal{R}}^-$ where

$$\mathcal{R}_-(\widehat{M}) \;=\; M\,, \qquad \mathcal{R}_+(\widehat{M}) \;=\; \big\{ f \in \mathcal{R}_+(\widetilde{M}) \,|\, f(\widetilde{\rho}_+ z) + f^\sharp(\widetilde{\rho}_- z) = 0 \text{ for all } z \in H \big\}\,.$$

Remark 3.4.34. Obviously, $\widetilde{\rho}_+ + \widetilde{\rho}_- = \widetilde{\rho}$. In $\operatorname{Hom}_R(H, M)$ we have for each $f \in \mathcal{R}_+(\widetilde{M})$

$$f(\widetilde{\rho}_+ \cdot -) \;=\; f \circ {}_A H^\vee\,, \qquad f^\sharp(\widetilde{\rho}_- \cdot -) \;=\; {}_A M \circ f^\vee\,.$$

Lemma 3.4.35. *$\widehat{M}$ is an $\widetilde{H}$-submodule of $\widetilde{M}$ and is annihilated by $\widetilde{\rho}$.*

Proof. To prove that $\widehat{M}$ is an $\widetilde{H}$-submodule we have to verify $g = M_{a_+}(x \otimes f) \in \widehat{M}_{j_+}$ for every $j \xleftarrow{a} i \in Q_1$ and $x \otimes f \in A_a \otimes_{R_i} \widehat{M}_{i_+}$. This is clear, since for all $z \in He_j$

$$g(\widetilde{\rho}_+ z) \;=\; f(\widetilde{\rho}_+ zx)\,, \qquad g^\sharp(\widetilde{\rho}_- z) \;=\; f^\sharp(\widetilde{\rho}_- zx)\,.$$

A straightforward calculation yields for each $f \in \widetilde{M}_{i_+}$ the identities

$$\left(\sum_{a \in Q_1} \widetilde{M}_{a^*} \widetilde{M}_{a_+}^\vee\right)(f) \;=\; f(\widetilde{\rho}_+)\,, \qquad \left(\sum_{a \in Q_1} \widetilde{M}_{a_-} \widetilde{M}_{a^*}^\vee\right)(f) \;=\; f^\sharp(\widetilde{\rho}_-)\,.$$

Lemma 3.4.21 now implies that $\widetilde{\rho} = \widetilde{\rho}_+ + \widetilde{\rho}_-$ annihilates $\widehat{M}$. □

Lemma 3.4.36. *$\widehat{\mathcal{R}}^- \cong \widetilde{\mathcal{R}}^-$.*

Proof. It suffices to check that $\widehat{\mathcal{R}}^-$ is another right adjoint of $\widetilde{\mathcal{R}}_-$. The proof of this fact is similar to [Gab73, § 5.4] and [GLS16a, proof of Lemma 10.3].

Let $N \in \operatorname{Mod}(\widetilde{\Gamma})$ and $M \in \operatorname{Mod}(H)$. Then we have as R-module $\widetilde{\mathcal{R}}_-(N) \cong \bigoplus_{i \in Q_0} N_{i_-}$ canonically. It is clear that the rule

$$g \;=\; (g_i)_{i \in \widetilde{Q}_0} \;\mapsto\; (g_{i_-})_{i \in Q_0}$$

induces a map $\operatorname{Hom}_{\widetilde{\Gamma}}\big(N, \widehat{M}\big) \xrightarrow{r} \operatorname{Hom}_H\big(\widetilde{\mathcal{R}}_-(N), M\big)$, which is natural in N and M.

Using the definition of $\widetilde{M}$, the fact that g is a morphism of $\widetilde{H}$-modules can be reformulated as follows: The family $(g_{i_-})_{i \in Q_0}$ is a morphism $\widetilde{\mathcal{R}}_-(N) \to M$ of H-modules and $(\star)$ holds:

$(\star)$ For all $i \in Q_0$ and $n \in N_{i_+}$ and $\ell \xleftarrow{b} k \in Q_1$ and $y \in A_b^* \subseteq A^*$ and $z \in He_i$,

$$g_{i_+}(n)(y \otimes z) \;=\; \begin{cases} g_{j_+}\big(N_{a_+}(x \otimes n)(y \otimes z')\big) & \text{if } z = z'x \text{ with } j \xleftarrow{a} i \in Q_1 \text{ and } x \in A_a, \\ g_{k_-}\big(N_{b^*}(y \otimes n)\big) & \text{if } z = e_i \text{ and } \ell = i, \\ 0 & \text{if } z = e_i \text{ and } \ell \neq i. \end{cases}$$

Note that $N_{a_+}(x \otimes n) = \eta_+(x)n$ and $N_{b^*}(y \otimes n) = \eta_\bullet(y)n$. Since Q is acyclic, condition $(\star)$ is easily seen to be equivalent to the following:

($\hat{\star}$) For all $i \in Q_0$ and $n \in N_{i_+}$ and $\ell \xleftarrow{b} k \in Q_1$ and $y \in A_b^* \subseteq A^*$ and $z \in He_i$,

$$g_{i_+}(n)(y \otimes z) \;=\; g_{k_-}(\eta_\bullet(y)\eta_+(z)n)\,.$$

Clearly, ($\hat{\star}$) implies the injectivity of r.

Vice versa, given a morphism $g = (g_{i_-})_{i\in Q_0}$ from $\widetilde{\mathcal{R}}_-(N)$ to M, we can use ($\hat{\star}$) to extend it to a morphism $N \xrightarrow{g} \widetilde{M}$. Indeed, property ($\hat{\star}$) describes the well-defined R-linear map

$$g_{i_+}(n) \;=\; g \circ m_n\,,$$

where $A^* \otimes_R He_i \xrightarrow{m_n} N$ is given by $y \otimes z \mapsto \eta_\bullet(y)\eta_+(z)n$. Evidently, g_{i_+} is R_i-linear.

To conclude that r is surjective, it merely remains to check that g maps N into $\widehat{M}$. This means we have to show $f(\widetilde{\rho}_+ z) + f^\sharp(\widetilde{\rho}_- z) = 0$ for $n \in N_{i_+}$, $f = g_{i_+}(n)$, and $z \in He_i$. Straightforward calculations yield $f(\widetilde{\rho}_+ z) = g(\widetilde{\rho}_+\eta_+(z)n)$ and $f^\sharp(\widetilde{\rho}_- z) = g(\widetilde{\rho}_-\eta_+(z)n)$. This completes the proof, because the $\widetilde{\Gamma}$-module N is annihilated by $\widetilde{\rho}_+ + \widetilde{\rho}_- = \widetilde{\rho}$. $\square$

Lemma 3.4.37. *For all $M \in \operatorname{mod}_{\mathrm{l.f.}}(H)$ there is an isomorphism $\tau^+(M) \cong \widetilde{\mathcal{R}}_+\widehat{\mathcal{R}}^- T(M)$ that is natural in M.*

Proof. We have a commutative diagram

$$\begin{array}{ccc}
\mathcal{R}_+\overline{\mathcal{R}}^- T(M) \;=\; \operatorname{Hom}_R(A^* \otimes_R H, M) & \xrightarrow{\;\;\widetilde{\partial}\;\;} & \operatorname{Hom}_R(H, M) \\
\downarrow \theta & & \downarrow \vartheta \\
\operatorname{Hom}_R(A \otimes_R M, H)^* & \xrightarrow{\;\;(\partial^\flat)^*\;\;} & \operatorname{Hom}_R(M, H)^* \\
\downarrow & & \downarrow \\
\operatorname{Hom}_H(H \otimes_R A \otimes_R M, H)^* & \xrightarrow{\;\;(\partial^\sharp)^*\;\;} & \operatorname{Hom}_H(H \otimes_R M, H)^*
\end{array}$$

where the vertical arrows are isomorphisms. Namely, $\vartheta = \vartheta^{H,M}$ is the isomorphism induced by the trace pairing (see Lemma 3.2.11). The map θ is the isomorphism given by trace and adjunction (see Definition 2.2.15 and Lemma 3.2.11), i.e. $\theta(f)(g) = \operatorname{tr}(fg^\vee)$. The two unnamed arrows are the canonical isomorphisms. Finally, $\partial^\sharp = \operatorname{Hom}_H(\partial, H)$ for the map ∂ from Corollary 3.1.14, $\widetilde{\partial}(f)(z) = f(\widetilde{\rho}_+ z) - f_M^\sharp(\widetilde{\rho}_- z)$, and $\partial^\flat(g)(x \otimes m) = xg(m) - g(xm)$.

Compare $\partial^\flat$ to the map with the same name appearing in the proof of Proposition 3.3.15. Similarly as there, but using additionally Remark 3.4.34, the commutativity of the lower square boils down to the formula

$$\operatorname{tr}\big(f \circ (\partial^\flat(g))^\vee\big) \;=\; \operatorname{tr}\big((f \circ {}_AH^\vee - {}_AM \circ f^\vee) \circ g\big)\,.$$

for all $f \in \operatorname{Hom}_R(A^* \otimes_R H, M)$ and $g \in \operatorname{Hom}_R(M, H)$. Lemmas 3.2.10, 3.2.13 and 3.2.16 together with the fact $\partial^\flat(g) = {}_AH \circ (\mathrm{id} \otimes g) - g \circ {}_AM$ show that this is true.

Note that $\widetilde{\mathcal{R}}_+\widehat{\mathcal{R}}^- T(M) = \ker(\widetilde{\partial})$ because of $f^\sharp_{T(M)} = -f^\sharp_M$.

Hence $\widetilde{\mathcal{R}}_+\widehat{\mathcal{R}}^- T(M) = \ker(\widetilde{\partial}) \cong \ker((\partial^\sharp)^*) \cong \operatorname{Ext}^1_H(M, H)^* \cong \tau^+(M)$ by Corollary 3.1.14 and Lemma 3.1.29 (a) and the commutativity of the diagram, which is natural in M. $\square$

Proof of Theorem 3.4.11. By Corollary 3.4.30 and Lemma 3.4.31 $C^+T \cong \operatorname{Hom}_H(\Pi_1, -)$. We conclude $C^-T \cong \Pi_1 \otimes_H -$ with Remark 3.4.8.

Let $M \in \operatorname{mod}_{\mathrm{l.f.}}(H)$. Then $\tau^+(M) \cong \widetilde{\mathcal{R}}_+\widetilde{\mathcal{R}}^-T(M)$ due to Lemmas 3.4.36 and 3.4.37. Together with Corollary 3.4.30 this shows $\tau^+(M) \cong C^+T(M)$, natural in M.

To prove $\tau^-(M) \cong C^-T(M)$, recall that H^* is locally free by Lemma 3.1.10. Therefore

$$\tau^-(M) \;\cong\; \operatorname{Ext}^1_H(H^*, M) \;\cong\; \operatorname{Hom}_H(M, \tau^+(H^*))^*$$

by Lemma 3.1.29 (b) and (c). Furthermore, $\tau^+(H^*) \cong C^+T(H^*)$. So with Remark 3.4.8

$$\tau^-(M) \;\cong\; \operatorname{Hom}_H(M, C^+T(H^*))^* \;\cong\; \operatorname{Hom}_H(C^-T(M), H^*)^* \;\cong\; C^-T(M)\,.$$

It remains to observe that this isomorphism is also natural in M. □

3.4.3 τ-Locally Free Modules

This subsection generalizes the finite-type classification for $\operatorname{mod}_{\tau-\mathrm{l.f.}}(H)$ in [GLS16a, § 11].

Definition 3.4.38. Let $\operatorname{mod}_{\tau-\mathrm{l.f.}}(H)$ be the full subcategory of $\operatorname{mod}_{\mathrm{l.f.}}(H)$ consisting of all modules M such that $\tau^p(M) \in \operatorname{mod}_{\mathrm{l.f.}}(H)$ for all $p \in \mathbb{Z}$.

We state the analogs of [GLS16a, Proposition 11.4, Theorems 11.10 and 11.11].

Proposition 3.4.39. *Let $M \in \operatorname{mod}_{\mathrm{l.f.}}(H)$ be rigid. Then $M \in \operatorname{mod}_{\tau-\mathrm{l.f.}}(H)$.*

In particular, $P_i, I_i, E_i \in \operatorname{mod}_{\tau-\mathrm{l.f.}}(H)$ for all $i \in Q_0$.

Proof. The proof is identical to [GLS16a, proof of Proposition 11.4]. More precisely, combine Proposition 3.4.5, Theorem 3.4.11, Definition 3.4.6, and Lemmas 3.1.9 and 3.1.10. □

Theorem 3.4.40. *Let $H = H_{\mathcal{H}}$ be the path algebra of a symmetric local K-modulation $\mathcal{H}$ for a weighted acyclic quiver Q. Then:*

(a) $\underline{\operatorname{rank}}\left(\left\{\tau^{-p}(P_i),\, \tau^{+p}(I_i) \,|\, p \in \mathbb{N},\, i \in Q_0\right\} \setminus \{0\}\right) \subseteq \Delta^+_{\mathrm{re}}(Q)$.

(b) There are only finitely many isomorphism classes of indecomposables in $\operatorname{mod}_{\tau-\mathrm{l.f.}}(H)$ *if and only if Q is a finite union of Dynkin quivers.*

(c) If Q is a Dynkin quiver, $M \mapsto \underline{\operatorname{rank}}(M)$ yields a bijection between the set of isomorphism classes of indecomposables in $\operatorname{mod}_{\tau-\mathrm{l.f.}}(H)$ and the set of positive roots $\Delta^+_{\mathrm{re}}(Q)$.

(d) If Q is a Dynkin quiver, for every indecomposable $M \in \operatorname{mod}_{\mathrm{l.f.}}(H)$:

$$\begin{aligned} M \in \operatorname{mod}_{\tau-\mathrm{l.f.}}(H) \;&\Leftrightarrow\; M \textit{ is rigid} \\ &\Leftrightarrow\; M \cong \tau^{-p}(P_i) \textit{ for some } p \in \mathbb{N},\, i \in Q_0 \\ &\Leftrightarrow\; M \cong \tau^{+p}(I_i) \textit{ for some } p \in \mathbb{N},\, i \in Q_0 \end{aligned}$$

Proof. The proofs of [GLS16a, Theorem 11.10 and 11.11] can be used verbatim. □

3.4.4 Preprojective Algebras Revisited

This subsection generalizes results from [GLS16a, §§ 10.6, 11.3, and 12].

Recall that $H\langle\Pi_1\rangle$ stands for the tensor algebra of the H-bimodule Π_1.

Lemma 3.4.41. $\Pi \cong H\langle\Pi_1\rangle$ *as H-algebras.*

Proof. We use the same trick as the proof of [GLS16a, Proposition 6.5]. Consider $\overline{A} = A \oplus A^*$ as a graded bimodule where elements in A have degree 0 and elements in A^* degree 1. The tensor algebra $\overline{H} = R\langle\overline{A}\rangle$ inherits an $\mathbb{N}$-grading from $\overline{A}$. Namely, the homogeneous component $\overline{H}_s$ of degree s is generated by all $A^{\otimes r_1} \otimes_R (A^*)^{\otimes s_1} \otimes_R \cdots \otimes_R A^{\otimes r_n} \otimes_R (A^*)^{\otimes s_n}$ as an R-bimodule where $n \in \mathbb{N}$ and $r_1, s_1, \dots, r_n, s_n \in \mathbb{N}$ such that $s_1 + \cdots + s_n = s$.

We have $\Pi = \overline{H}/J$ where $J = \langle\rho\rangle$ is the ideal generated by the homogeneous element ρ of degree 1. Therefore the grading of $\overline{H}$ induces decompositions $J = \bigoplus_s J_s$ and $\Pi = \bigoplus_s \Pi_s$ where $\Pi_s = \overline{H}_s/J_s$. Note that $J_0 = 0$ and $\Pi_0 = \overline{H}_0 = \bigoplus_n A^{\otimes n} = H$.

There is a morphism $\overline{A} \hookrightarrow H\langle\Pi_1\rangle$ of R-bimodules induced by the inclusions $A \hookrightarrow H$ and $A^* \hookrightarrow \Pi_1$. We have the following commutative diagram of canonical maps:

$$\begin{tikzcd}
& \overline{A} \arrow[d, hook] \arrow[dr, hook] & \\
\Pi_1 \arrow[r, hook] \arrow[dr, hook] & H\langle\Pi_1\rangle \arrow[d, dotted, shift right, "g"'] & \overline{H} \arrow[l, dotted, "f"'] \arrow[dl, two heads] \\
& \Pi \arrow[u, dashed, shift right, "\bar{f}"'] &
\end{tikzcd}$$

The dotted morphisms are given by the universal property of the tensor algebras $\overline{H} = R\langle\overline{A}\rangle$ and $H\langle\Pi_1\rangle$ (see Lemma 2.2.6). The dashed map $\bar{f}$ is induced by the universal property of $\Pi = \overline{H}/J$. Indeed, $f(J) = 0$ because J is generated in degree 1 and, clearly, $f(J_1) = 0$. It is easy to see that $\bar{f}$ and g are inverse H-algebra isomorphisms. $\square$

Proposition 3.4.42. ${}_H\Pi \cong \bigoplus_{p\in\mathbb{N}} \tau^{-p}({}_H H)$. *In particular,* ${}_\Pi\Pi \in \operatorname{Mod}_{\text{l.f.}}(\Pi)$.

Proof. This is identical to [GLS16a, proof of Theorem 11.12]. Namely, $\Pi_1 \cong \operatorname{Ext}^1_H(H^*, H)$ as H-bimodule with the same argument as in [GLS16a, proof of Theorem 10.5]. Moreover, by induction and Lemma 3.1.29 (b) $\operatorname{Ext}^1_H(H^*, H)^{\otimes p} \cong \tau^{-p}(H)$. Now use Lemma 3.4.41 to obtain the first statement. For the last statement apply Proposition 3.4.39. $\square$

The next two facts are our version of [GLS16a, Proposition 12.1 and Corollary 12.2].

Lemma 3.4.43. *There is an exact sequence of Π^λ-bimodules*

$$\Pi^\lambda \otimes_R \Pi^\lambda \xrightarrow{\partial_1} \Pi^\lambda \otimes_R \overline{A} \otimes_R \Pi^\lambda \xrightarrow{\partial_0} \Pi^\lambda \otimes_R \Pi^\lambda \xrightarrow{\nu} \Pi^\lambda \longrightarrow 0$$

defined by $\partial_1(1 \otimes 1) = \rho \otimes 1 + 1 \otimes \rho$, $\partial_0(1 \otimes x \otimes 1) = x \otimes 1 - 1 \otimes x$, *and* $\nu(1 \otimes 1) = 1$.

Proof. The proof is standard. Let $\Lambda = \Pi^\lambda$ and J the ideal of $\overline{H}$ generated by $r = \rho - \lambda$.

By definition $\Lambda = \overline{H}/J$ and the Λ-bimodule morphism $\Lambda \otimes_R \Lambda \xrightarrow{\pi} J/J^2$ with $1 \otimes 1 \mapsto r$ is surjective. Recall that $\mathrm{Tor}_1^{\overline{H}}(\Lambda, \Lambda) \cong J/J^2$ (see [CE56, VI. Exercise 19]) and apply $\Lambda \otimes_{\overline{H}} -$ to the sequence from Corollary 3.1.14 for $M = \Lambda$ to get an exact sequence of Λ-bimodules

$$J/J^2 \xrightarrow{\hat{\partial}_1} \Lambda \otimes_R \overline{A} \otimes_R \Lambda \xrightarrow{\partial_0} \Lambda \otimes_R \Lambda \xrightarrow{\nu} \Lambda \longrightarrow 0\,.$$

Let $\overline{H} \xrightarrow{\delta} \overline{H} \otimes_R \overline{A} \otimes_R \overline{H}$ be the R-derivation defined by $\delta(x) = 1 \otimes x \otimes 1$ for $x \in \overline{A}$. According to [Sch85, Theorems 10.1 and 10.3] the map $\hat{\partial}_1$ can be taken to be induced by δ. Then $\hat{\partial}_1 \circ \pi = \partial_1$, which proves the lemma. □

Corollary 3.4.44. *For all $M \in \mathrm{Mod}(\Pi)$ there is an exact sequence*

$$\Pi \otimes_R M \xrightarrow{\partial_1} \Pi \otimes_R \overline{A} \otimes_R M \xrightarrow{\partial_0} \Pi \otimes_R M \xrightarrow{\nu} M \longrightarrow 0$$

of Π-modules with $\partial_1(1 \otimes m) = \rho \otimes m + 1 \otimes \sum_b (\rho'_b \otimes \rho''_b m)$ if $\rho = \sum_b \rho'_b \rho''_b$ with $\rho'_b, \rho''_b \in \overline{A}$, and $\partial_0(1 \otimes x \otimes m) = x \otimes m - 1 \otimes xm$, and $\nu(1 \otimes m) = m$.

This is the end of a projective resolution for all $M \in \mathrm{Proj}^0_R(\Pi)$.

Proof. Apply $- \otimes_\Pi M$ to the sequence in Lemma 3.4.43 to obtain a sequence isomorphic to the one in the statement. For the exactness note that the sequence in Lemma 3.4.43 for $\lambda = 0$ splits as a sequence of right R-modules, since Π_R is projective by the right-module version of Proposition 3.4.42. For the last claim we can argue as in the proof of Corollary 3.1.14, using that ${}_R\Pi$ is projective according to Proposition 3.4.42. □

Proposition 3.4.45. $\mathrm{Ext}^1_\Pi(M, N) \cong \mathrm{Ext}^1_\Pi(N, M)^*$ *for $M \in \mathrm{Mod}_{\mathrm{l.f.}}(\Pi)$, $N \in \mathrm{mod}_{\mathrm{l.f.}}(\Pi)$. This isomorphism is natural in M and N.*

Proof. Applying $\mathrm{Hom}_\Pi(-, N)$ to the exact sequence from Corollary 3.4.44 we get a commutative diagram where the vertical maps are the canonical isomorphisms:

$$\begin{array}{ccccc}
\mathrm{Hom}_\Pi(\Pi \otimes_R M, N) & \xrightarrow{\partial_0^\sharp} & \mathrm{Hom}_\Pi(\Pi \otimes_R \overline{A} \otimes_R M, N) & \xrightarrow{\partial_1^\sharp} & \mathrm{Hom}_\Pi(\Pi \otimes_R M, N) \\
\downarrow & & \downarrow & & \downarrow \\
\mathrm{Hom}_R(M, N) & \xrightarrow{\partial_0^{M,N}} & \mathrm{Hom}_R(\overline{A} \otimes_R M, N) & \xrightarrow{\partial_1^{M,N}} & \mathrm{Hom}_R(M, N)
\end{array}$$

Moreover, $\partial_0^{M,N}$ acts on $f \in \mathrm{Hom}_R(M, N)$ and $\partial_1^{M,N}$ on $g \in \mathrm{Hom}_R(\overline{A} \otimes_R M, N)$ as

$$\partial_0^{M,N}(f) \;=\; \sum_{a \in \overline{Q}_1} \big(N_a \circ (\mathrm{id} \otimes f) - f M_a\big)\,, \qquad \partial_1^{M,N}(g) \;=\; \sum_{a \in \overline{Q}_1} \varepsilon_a \cdot \big(N_{a^*} g_a^\vee - g_a M_{a^*}^\vee\big)\;.$$

By Corollary 3.4.44 and the commutative diagram above we know that

$$\mathrm{Hom}^1_\Pi(M, N) \;\cong\; \ker\big(\partial_0^{M,N}\big)\,, \qquad \mathrm{Ext}^1_\Pi(M, N) \;\cong\; \ker\big(\partial_1^{M,N}\big)/\mathrm{im}\big(\partial_0^{M,N}\big)\,.$$

Hence, it suffices to show that there is a commutative diagram, natural in M and N, where the vertical maps are isomorphisms:

$$\begin{tikzcd}
\mathrm{Hom}_R(M,N) \arrow[r, "\partial_0^{M,N}"] \arrow[d, "\vartheta"] & \mathrm{Hom}_R(\bar{A}\otimes_R M,N) \arrow[r, "\partial_1^{M,N}"] \arrow[d, "\theta"] & \mathrm{Hom}_R(M,N) \arrow[d, "-\vartheta"] \\
\mathrm{Hom}_R(N,M)^* \arrow[r, "\left(\partial_1^{N,M}\right)^*"] & \mathrm{Hom}_R(\bar{A}\otimes_R N,M)^* \arrow[r, "\left(\partial_0^{N,M}\right)^*"] & \mathrm{Hom}_R(N,M)^*
\end{tikzcd}$$

It is easy to check (similarly as for the last diagram in the proof of Proposition 3.3.15) that the isomorphisms ϑ and θ given by $\vartheta(f)(g) = \mathrm{tr}(fg)$ and $\theta(f)(g) = \sum_{a\in\bar{Q}_1} \varepsilon_a \,\mathrm{tr}(f_a g_{a^*}^\vee)$ make the diagram commute. □

Corollary 3.4.46. *For $M, N \in \mathrm{mod}_{\mathrm{l.f.}}(\Pi)$ one has*

$$(M,N)_H \;=\; \dim_K \mathrm{Hom}^1_\Pi(M,N) + \dim_K \mathrm{Hom}^1_\Pi(N,M) - \dim_K \mathrm{Ext}^1_\Pi(M,N)\,.$$

Proof. Use (part of) the proof of [GLS16a, Theorem 12.6]. □

The concluding corollary is the analog of [GLS16a, Corollary 12.7].

Corollary 3.4.47. *If Q is a Dynkin quiver,* Π *is self-injective.*

Proof. The algebra Π is finite-dimensional because of Theorem 3.4.40 and Proposition 3.4.42. Right-module versions of Corollary 3.4.44 and Proposition 3.4.42 yield an exact sequence $0 \to Z \to P \to \Pi^* \to 0$ of finite-dimensional locally free right Π-modules with P_Π projective. Applying $(-)^*$ leads to an exact sequence $0 \to \Pi \to P^* \to Z^* \to 0$ of locally free Π-modules. This sequence splits thanks to $\mathrm{Ext}^1_\Pi(Z^*,\Pi) \cong \mathrm{Ext}^1_\Pi(\Pi, Z^*)^* = 0$. Hence, ${}_\Pi\Pi$ is injective as a summand of the injective module ${}_\Pi P^*$. □

4 Potentials for Cluster-Tilting Subcategories

Let $\Delta = (V, E)$ be an undirected multigraph without loops and with $V = \{1, \dots, n\}$.

Buan, Iyama, Reiten, and Scott associated in [BIRS09, § II.4] a quiver $Q(\underline{s}) = Q(\Delta, \underline{s})$ with every finite sequence $\underline{s} = (s_1, \dots, s_\ell)$ in V. We show in this chapter that any full subquiver of $Q(\underline{s})$ admits an up to right-equivalence unique non-degenerate potential.

If $\underline{s}$ corresponds to a reduced expression for an element of the Weyl group $\mathcal{W}_\Delta$ of Δ, the quiver $Q'(\underline{s})$ of the cluster-tilting subcategory associated with $\underline{s}$ in [BIRS09] is a full subquiver of $Q(\underline{s})$. Furthermore, $Q'(\underline{s})$ is the quiver $\Gamma(\underline{s})^{\mathrm{op}} = \Gamma(-\underline{s})$ from [BFZ05, § 2.2] when $\underline{s}$ and $-\underline{s}$ are considered as reduced expressions for elements of $\mathcal{W}_\Delta \times \mathcal{W}_\Delta$.

4.1 Quiver of a Cluster-Tilting Subcategory

Let us recall and rephrase for our convenience the definition of $Q = Q(\Delta, \underline{s})$.

The vertices of Q are $Q_0 = \{1, \dots, \ell\}$. To define the arrows we need some notation:

(a) For $v \in V$ denote by $I^v = \left\{i_1^v < \cdots < i_{t_v}^v\right\}$ the subset of Q_0 such that $(s_i)_{i \in I^v}$ is the subsequence of $\underline{s}$ consisting of the members equal to v. We set $\tau(i_j^v) := i_{j-1}^v$.

(b) For $v \neq w$ in V let $I_1^{v,w} \,\dot\cup \cdots \dot\cup\, I_{t_{v,w}}^{v,w}$ be the partition of $I^v \cup I^w$ such that

- $I_j^{v,w} > I_{j-1}^{v,w}$ for each $j > 1$, i.e. $i > i'$ for all $i \in I_j^{v,w}$, $i' \in I_{j-1}^{v,w}$, and
- $(s_i)_{i \in I_j^{v,w}}$ is constant for each j, say with value $c_j^{v,w}$, where $c_j^{v,w} \neq c_{j-1}^{v,w}$ for $j > 1$.

Denote by $i_j^{v,w}$ the greatest element of $I_j^{v,w}$ and call it the *last vertex in the j-th v/w-group.* If $v \overset{e}{\frac{\quad}{}} w$ is an edge in Δ we write t_e, i_j^e, I_j^e etc. for $t_{v,w}$, $i_j^{v,w}$, $I_j^{v,w}$ etc.

Now we can describe the arrows of Q. For each $v \in V$ and $1 < j \le t_v$ there is an arrow

$$i_{j-1}^v \xleftarrow{\;b_j^v\;} i_j^v \,.$$

in Q. We set $b(i_j^v) := b_j^v$. Furthermore, for each $v \overset{e}{\frac{\quad}{}} w \in E$ and $1 \le j < t_e$ the quiver Q has an arrow

$$i_j^{v,w} \xrightarrow{\;a_j^e\;} i_{j+1}^{v,w} \,.$$

Hence, Q has as many parallel arrows from the last vertex in the j-th v/w-group to the last vertex in the $(j+1)$-st v/w-group as Δ has edges between v and w.

This concludes the definition of Q.

We continue with the construction of a rigid potential on Q.

4.2 Rigid Potentials

Convention 4.2.1. Let $\widehat{KQ}$ be the completed path algebra of Q over K.

Two potentials W and W' on Q are *right-equivalent* if there is $f \in \operatorname{Aut}_{KQ_0}\big(\widehat{KQ}\big)$ such that $f(W) = W'$. Compare Definition 2.6.61 and Remark 2.6.62.

For non-zero elements $x \in \widehat{KQ}$ (or for potentials x on Q) denote by $x_{\min} := \operatorname{pr}_{\operatorname{ord}(x)}(x)$ (see Notation 2.6.7) the non-zero homogeneous component of x of lowest degree with respect to the length grading. Set $0_{\min} := 0$.

We write $x \overset{\min}{\approx} y$ for elements $x, y \in \widehat{KQ}$ (or for potentials x, y on Q) if $x_{\min} = y_{\min}$. Potentials W and W' are *minimal-degree equivalent* if $W_{\min}$ and $W'_{\min}$ are right-equivalent.

The set of paths in the quiver Q is denoted by $\mathcal{P}_Q$. A *cycle* in Q is a subquiver of Q spanned by a cyclic path. Every cyclic path can be naturally regarded as a potential on Q.

Notation 4.2.2. For $v \neq w$ in V and $1 < j < t_{v,w}$ there exists a (unique) $o_j^{v,w} \in \mathbb{N}_+$ with

$$\tau^{o_j^{v,w}}\big(i_{j+1}^{v,w}\big) \;=\; i_{j-1}^{v,w}\,.$$

Set $b_j^{v,w}(t) := b\big(\tau^{t-1}\big(i_{j+1}^{v,w}\big)\big)$. Then for $v \overset{e}{\text{———}} w \in E$ we have the following cycle c_j^e in Q:

$$\begin{tikzcd}
 & & i_j^e \arrow[drr, "a_j^e"] & & \\
i_{j-1}^e \arrow[urr, "a_{j-1}^e"] & \cdot \arrow[l, "b_j^{v,w}(o_j^{v,w})"] & \cdots & \cdot & i_{j+1}^e \arrow[l, "b_j^{v,w}(1)"]
\end{tikzcd}$$

With the abbreviation $b_j^e := b_j^{v,w} := b_j^{v,w}(o_j^{v,w}) \cdots b_j^{v,w}(1)$ we can write $c_j^e = b_j^e a_j^e a_{j-1}^e$.

Definition 4.2.3. Taking the sum of the cycles c_j^e over all $e \in E$ and $1 < j < t_e$ yields a potential on $Q(\Delta, \underline{s})$:

$$W(\Delta, \underline{s}) \;:=\; \sum_{v,w} b_j^{v,w} \sum_e a_j^e a_{j-1}^e \;=\; \sum_{e,j} c_j^e$$

As was already observed in [BIRS11, Theorem 6.5], the potential $W(\Delta, \underline{s})$ is rigid.

We prove that it is up to right-equivalence the only non-degenerate potential on $Q(\Delta, \underline{s})$. These facts rely on the following observation.

Lemma 4.2.4. *Every cycle in $Q(\Delta, \underline{s})$ is of the form $p b_j^e a_j^e$ for some $e \in E$, $1 < j < t_e$, and some path p in $Q(\Delta, \underline{s})$.*

Proof. Let c be a cycle in $Q = Q(\Delta, \underline{s})$. Choose $i \in \{1, \ldots, \ell\} = Q_0$ minimal with the property that i belongs to c. Then by the definition of Q there exist $v \in V$, $1 < j' \leq t_v$, and $0 \leq r \leq t_v - j'$ such that $i = i_{j'-1}^v$ and $b = b_{j'}^v \cdots b_{j'+r}^v$ occurs in c. Let us assume that r is maximal with this property. Then there is $e \in E$ and $1 < j < t_e$ such that $b a_j^e$ occurs in c. By the choice of i we must have $b = b' b_j^e$ for some path b' in Q. $\square$

Before we continue with the central statements we introduce some additional notation.

Notation 4.2.5. For each path p in $Q = Q(\Delta, \underline{s})$ we denote by $\ell(p)$ its length and by $\vec{\ell}(p)$ the degree of p with respect to the $\rightarrow$-*grading*, i.e. the grading of KQ that assigns to each arrow a_j^e degree 1 and to each arrow b_j^v degree 0.

In plain words, $\vec{\ell}(p)$ counts the occurrences of arrows of type a_j^e in the path p.

Let x be an element in $\widehat{KQ}$ (or a potential on Q). We write $\ell(x)$ or $\vec{\ell}(x)$ for the degree and $\rightarrow$-degree of x, if x is length-homogeneous or $\rightarrow$-homogeneous, respectively.

Denoting by $\vec{\mathrm{pr}}_\ell(x)$ the projection of x onto its ℓ-th $\rightarrow$-homogeneous component, define

$$\overrightarrow{\mathrm{ord}}(x) \;:=\; \min\left\{\ell \in \mathbb{N} \mid \vec{\mathrm{pr}}_\ell(x) \neq 0\right\}.$$

A unitriangular automorphism $f \in \mathrm{Aut}_{KQ_0}\big(\widehat{KQ}\big)$ is said to be of $\rightarrow$-*depth* $n \in \mathbb{N}$ if

$$\overrightarrow{\mathrm{ord}}(f(x) - x) \;\geq\; \overrightarrow{\mathrm{ord}}(x) + n \;=\; \vec{\ell}(x) + n$$

for all $\rightarrow$-homogeneous $x \in \widehat{KQ}$.

Proposition 4.2.6. *The potential $W = W(\Delta, \underline{s})$ is rigid. More precisely, for every cycle c in $Q = Q(\Delta, \underline{s})$ there are $\rightarrow$-homogeneous p_j^e in KQ with $\vec{\ell}(p_j^e) = \vec{\ell}(c) - 1$ such that*

$$c \;=\; \sum_{e,j} p_j^e \partial_{a_j^e}(W)\,.$$

Proof. By Lemma 4.2.4 there exist $e' \in E$, $1 < j' < t_{e'}$, and $p \in \mathcal{P}_Q$ such that $c = pb_{j'}^{e'}a_{j'}^{e'}$. Clearly, $\vec{\ell}(p) = \vec{\ell}(c) - 1$.

If $j' = 2$, we have

$$c \;=\; p\partial_{a_{j'-1}^{e'}}(c_{j'}^{e'}) \;=\; p\partial_{a_{j'-1}^{e'}}(W)\,.$$

If $j' > 2$, the identity $\partial_{a_{j'-1}^{e'}}(W) = \partial_{a_{j'-1}^{e'}}(c_{j'-1}^{e'}) + \partial_{a_{j'-1}^{e'}}(c_{j'}^{e'}) = a_{j'-2}^{e'}b_{j'-1}^{e'} + b_{j'}^{e'}a_{j'}^{e'}$ shows

$$c \;=\; p\partial_{a_{j'-1}^{e'}}(W) \;-\; pa_{j'-2}^{e'}b_{j'-1}^{e'}\,.$$

By induction we can write $\tilde{c} := pa_{j'-2}^{e'}b_{j'-1}^{e'} = \sum_{e,j} \tilde{p}_j^e \partial_{a_j^e}(W)$ with $\vec{\ell}(\tilde{p}_j^e) = \vec{\ell}(\tilde{c}) - 1 = \vec{\ell}(c) - 1$.

Thus $c = \sum_{e,j} p_j^e \partial_{a_j^e}(W)$ where $p_j^e = -\tilde{p}_j^e$ for all $(e, j) \neq (e', j' - 1)$ and $p_{j'-1}^{e'} = p - \tilde{p}_{j'-1}^{e'}$.

Moreover, it is $\vec{\ell}(p_j^e) = \vec{\ell}(c) - 1$ as desired. □

Notation 4.2.7. Let Q' be a full subquiver of Q and let W be a potential on Q.

Following Definition 2.6.64 we write $W|_{Q'} := W|_{Q'_1}$ for the potential obtained by restricting W to Q'.

For elements $x = \sum_{p \in \mathcal{P}_Q} x_p\, p$ in $\widehat{KQ}$ with $x_p \in K$ we set $x|_{Q'} := \sum_{p \in \mathcal{P}_{Q'}} x_p\, p$.

Corollary 4.2.8. *Let Q' be a full subquiver of $Q = Q(\Delta, \underline{s})$ and let $W = W(\Delta, \underline{s})$. Then for every cycle c in Q' there are $\to$-homogeneous $p^e_j \in \widehat{KQ'}$ with $\vec{\ell}(p^e_j) = \vec{\ell}(c) - 1$ such that*

$$c \;=\; \sum_{e,j} p^e_j \partial_{a^e_j}(W|_{Q'})\,.$$

Proof. This follows immediately from Proposition 4.2.6 and the fact that for $a \in Q_1$ we have $\partial_a(W)|_{Q'} = \partial_a(W|_{Q'})$ if $a \in Q'_1$ and $\partial_a(W)|_{Q'} = 0$ otherwise. $\square$

4.3 Uniqueness of Non-Degenerate Potentials

Theorem 4.3.1. *Let Q' be a full subquiver of $Q = Q(\Delta, \underline{s})$ and let $W = W(\Delta, \underline{s})$. Then every non-degenerate potential on Q' is right-equivalent to $W|_{Q'}$.*

Proof. As a first step, we will show that for every cycle c^e_j completely contained in Q' every non-degenerate potential on Q' restricts to a potential on the full subquiver spanned by c^e_j that is minimal-degree equivalent to the restriction of W.

For adjacent vertices v, w in Δ and each j the subquiver $Q^{v,w}_j$ of Q' spanned by all cycles c^e_j contained in Q' with $v \overset{e}{-\!\!-} w \in E$ is empty or the following full subquiver of Q':

$$\begin{tikzcd}
 & i^e_j \arrow[dr, "a^e_j"] & \\
i^e_{j-1} \arrow[ur, "a^e_{j-1}"] & & i^e_{j+1} \arrow[ll, dotted, "b^{v,w}_j"]
\end{tikzcd}$$

Let $E_{v,w}$ be the subset of E consisting of the edges between v and w. If $Q^{v,w}_j$ is non-empty, every potential $W^{v,w}_j$ on $Q^{v,w}_j$ has the form

$$b^{v,w}_j \sum_{e,f\,\in\,E_{v,w}} \alpha_{e,f}\, a^f_j a^e_{j-1} \;+\; \tilde{W}$$

with $\alpha_{e,f} \in K$ and $\operatorname{ord}\big(\tilde{W}\big) > \ell\big(b^{v,w}_j\big) + 2$.

It is not hard to see that $W^{v,w}_j$ is non-degenerate if and only if the matrix $C = (\alpha_{e,f})_{e,f}$ is invertible. Assume this is the case. Write $C^{-1} = (\beta_{f,e})_{f,e}$ and let $\varphi^{v,w}_j$ be the automorphism of $\widehat{KQ'}$ given by the substitutions $a^f_j \mapsto \sum_e \beta_{f,e} a^e_j$. Then $W^{v,w}_j$ is mapped by $\varphi^{v,w}_j$ to

$$\sum_{e\,\in\,E_{v,w}} c^e_j \;+\; \varphi^{v,w}_j\big(\tilde{W}\big) \;\overset{\min}{\approx}\; \sum_{e\,\in\,E_{v,w}} c^e_j \;=\; W|_{Q^{v,w}_j}\,.$$

Now let W' be any non-degenerate potential on Q'. To prove the theorem, we must show that W' is right-equivalent to $W|_{Q'}$.

Assume there is some $j \geq 1$ such that $W'|_{Q^{v,w}_{j'}} \overset{\min}{\approx} W|_{Q^{v,w}_{j'}}$ for all $Q^{v,w}_{j'} \subseteq Q'$ with $j' < j$.

Set $\varphi_j = \prod_{v,w} \varphi_j^{v,w}$ where the product is taken over all $Q_j^{v,w} \subseteq Q'$ and $\varphi_j^{v,w}$ is defined as in the previous paragraph with $W_j^{v,w} = W'|_{Q_j^{v,w}}$. Note that $W'|_{Q_j^{v,w}}$ is indeed non-degenerate as the restriction of a non-degenerate potential to a full subquiver and φ_j does not depend on the order in which the $\varphi_j^{v,w}$ are composed.

Without loss of generality, we can replace W' by the right-equivalent potential $\varphi_j(W')$. Then we have $W'|_{Q_{j'}^{v,w}} \overset{\min}{\approx} W|_{Q_{j'}^{v,w}}$ for all $Q_{j'}^{v,w} \subseteq Q'$ with $j' < j+1$. Using induction we can assume that the non-degenerate potential W' satisfies $W'|_{Q_j^{v,w}} \overset{\min}{\approx} W|_{Q_j^{v,w}}$ for all $Q_j^{v,w} \subseteq Q'$.

By this assumption $W' = W|_{Q'} + \tilde{W}'$ for some potential $\tilde{W}'$ on Q' with $\overrightarrow{\operatorname{ord}}(\tilde{W}') \geq 3$. Using Corollary 4.2.8 there exist elements $p_0^{e,j} \in \widehat{KQ}'$ with $\overrightarrow{\operatorname{ord}}\big(p_0^{e,j}\big) \geq 2$ such that

$$W' \;=\; W|_{Q'} \;+\; \sum_{e,j} p_0^{e,j} \partial_{a_j^e}(W|_{Q'})\,.$$

Set $\varphi_0 := \mathrm{id}$. Assume that for some $n \in \mathbb{N}$ a unitriangular automorphism φ_n of $\widehat{KQ}'$ and a potential $\tilde{W}_n' = \sum_{e,j} p_n^{e,j} \partial_{a_j^e}(W|_{Q'})$ with $\overrightarrow{\operatorname{ord}}\big(p_n^{e,j}\big) \geq 2^n + 1$ are given such that

$$W_n' \;:=\; \varphi_n(W') \;=\; W|_{Q'} \;+\; \tilde{W}_n'\,. \tag{$*$}$$

The automorphism $\widetilde{\varphi}_{n+1}$ of $\widehat{KQ}'$ defined by the rules $a_j^e \mapsto a_j^e - p_n^{e,j}$ is unitriangular and has $\rightarrow$-depth 2^n. A straightforward calculation yields

$$\widetilde{\varphi}_{n+1}(W|_{Q'}) \;=\; W|_{Q'} \;-\; \tilde{W}_n' \;+\; \sum_{e,j} b_j^e p_n^{e,j} p_n^{e,j-1}\,.$$

By Corollary 4.2.8 there are $p_{n+1}^{e,j} \in \widehat{KQ}$ with $\overrightarrow{\operatorname{ord}}\big(p_{n+1}^{e,j}\big) \geq 2 \cdot (2^n+1) - 1 = 2^{n+1} + 1$ and

$$\sum_{e,j} b_j^e p_n^{e,j} p_n^{e,j-1} \;+\; \big(\widetilde{\varphi}_{n+1}(\tilde{W}_n') - \tilde{W}_n'\big) \;\;=\;\; \sum_{e,j} p_{n+1}^{e,j} \partial_{a_j^e}(W|_{Q'}) \;\;=:\;\; \tilde{W}_{n+1}'\,.$$

Setting $\varphi_{n+1} := \widetilde{\varphi}_{n+1} \circ \varphi_n$, Equation $(*)$ holds with n replaced by $n+1$.

Since $\lim_{n\to\infty} \overrightarrow{\operatorname{ord}}(x_n) = \infty$ is equivalent to $\lim_{n\to\infty} \operatorname{ord}(x_n) = \infty$, the sequence of potentials $(W_n')_{n\in\mathbb{N}}$ converges to $W|_{Q'}$ and the sequence $(\varphi_n)_{n\in\mathbb{N}}$ determines a unitriangular automorphism $\varphi := \lim_{n\to\infty} \varphi_n$ with $\varphi(W') = W|_{Q'}$ (compare Lemma 2.6.23). □

Corollary 4.3.2. *Let $\underline{s}$ be a reduced expression for an element in the Weyl group $\mathcal{W}_\Delta$ of Δ. Then $\Gamma(\underline{s})$ and $\Gamma(-\underline{s})$ admit an up to right-equivalence unique non-degenerate potential.* □

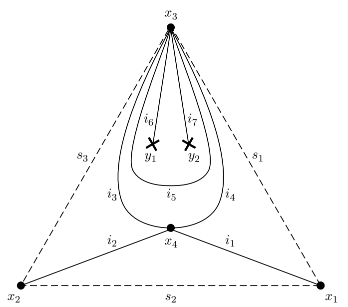
5 Potentials for Tagged Triangulations

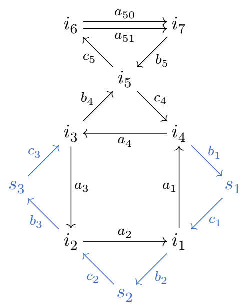
Motivation

The objects motivating this chapter are most practically introduced with an example.

Roughly speaking, an *orbifold* is a compact oriented surface Σ with two distinguished finite sets $\mathbb{M}, \mathbb{O} \subseteq \Sigma$ of points. These points are used as the vertices of *triangulations*.

For instance, we can see below the triangle with set of *marked points* $\mathbb{M} = \{x_1, x_2, x_3, x_4\}$ and *orbifold points* $\mathbb{O} = \{y_1, y_2\}$. It has been triangulated by *arcs* i_1, i_2, i_3, i_4, i_5, i_6, i_7. The sides s_1, s_2, s_3 are *boundary segments*.





Every triangulation τ gives rise to a quiver $Q(\tau)$. The vertices are the sides of the triangles in the triangulation and the arrows keep track of adjacencies. The quiver $Q(\tau)$ is weighted. Sides containing an orbifold point have weight 2, all other sides have weight 1.

For our example the quiver $Q(\tau)$ can be seen on the right; its weights are $d_{i_6} = d_{i_7} = 2$ and $d_{s_1} = d_{s_2} = d_{s_3} = d_{i_1} = d_{i_2} = d_{i_3} = d_{i_4} = d_{i_5} = 1$.

There is a path algebra $R\langle A\rangle$ for $Q(\tau)$ over $\mathbb{R}$ where $R = R(\tau) = \prod_{i \in Q_0(\tau)} R_i$ with

$$R_i \;=\; \begin{cases} \mathbb{R} & \text{if } d_i = 1, \\ \mathbb{C} & \text{if } d_i = 2. \end{cases}$$

The R-bimodule $A = A(\tau) = \bigoplus_{a \in Q_1(\tau)} A_a$ is chosen in such a way that

$$R_j \otimes_{\mathbb{R}} R_i \;=\; \begin{cases} \qquad A_a & \text{for all } j \xleftarrow{\;a\;} i \text{ with } d_i \neq 2 \text{ or } d_j \neq 2, \\ A_{a_0} \oplus A_{a_1} & \text{for all } j \overset{a_0}{\underset{a_1}{\Longleftarrow}} i \text{ with } d_i = 2 \text{ and } d_j = 2. \end{cases}$$

In our example, $A_{a_{50}} \oplus A_{a_{51}} = \mathbb{C} \otimes_{\mathbb{R}} \mathbb{C} \ni 1 \otimes 1 =: a_5$ and we have a potential

$$W(\tau) \;=\; W^{\Delta}(\tau) + W^{\bullet}(\tau)$$

where $W^{\Delta}(\tau) = c_1b_1a_1 + c_2b_2a_2 + c_3b_3a_3 + c_4b_4a_4 + c_5b_5a_5$ and $W^{\bullet}(\tau) = a_1a_2a_3a_4$.

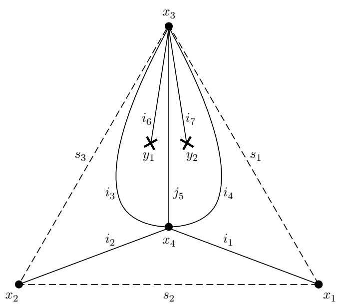
Such a potential can be constructed for every triangulated orbifold and will depend on the choice of two coefficient functions $u : \mathbb{M} \setminus \partial\Sigma \to \mathbb{R}^\times$ and $z : \mathbb{O} \to \mathbb{C} \setminus \mathbb{R}$. Just as in the example, the potential of a triangulation τ has the form

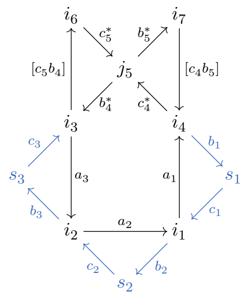
$$W_{u,z}(\tau) \;=\; \sum_{\maltese} W_{u,z}^{\maltese} \;+\; \sum_{\bullet} W_u^{\bullet}(\tau)$$

where the first sum is taken over all *puzzle pieces*, the building blocks of the triangulation τ, and the second sum over all (non-enclosed) *punctures*, the interior marked points.

What makes triangulations interesting, is that (some of) their arcs can be *flipped*, yielding other triangulations of the same orbifold.

Flipping the arc i_5 in our example, leads to another triangulation $\mu_{i_5}(\tau)$ with the same arcs as τ except that the "diagonal" i_5 of the "quadrilateral" with vertices y_1, y_2, x_3, x_4 is replaced by the other "diagonal" j_5:





On the level of weighted quivers, this change of triangulation is reflected by mutation.

The potential $W(\mu_{i_5}(\tau)) = W^\Delta(\mu_{i_5}(\tau)) + W^\bullet(\mu_{i_5}(\tau))$ of the flipped triangulation in the running example, defined by $W^\Delta(\mu_{i_5}(\tau)) = c_1b_1a_1 + c_2b_2a_2 + c_3b_3a_3 + c_5^*[c_5b_4]b_4^* + c_4^*[c_4b_5]b_5^*$ and $W^\bullet(\mu_{i_5}(\tau)) = a_1a_2a_3b_4^*c_4^*$, is easily seen to be obtained from $W(\tau)$ via mutation at i_5.

Results

The first main result of this chapter is the construction of an SP $\mathcal{S}_{u,z}(\tau) = (A(\tau), W_{u,z}(\tau))$ for every triangulation τ such that, whenever $\mu_i(\tau)$ is obtained from τ by flipping an arc, the SP $\mathcal{S}_{u,z}(\mu_i(\tau))$ coincides with the SPs in $\mu_i(\mathcal{S}_{u,z}(\tau))$ up to reduced-$Q_0(\tau)$-equivalence.

Ordinary triangulations can possess non-flippable arcs. For applications to cluster-algebra theory it is however desirable that every arc can be flipped. As a remedy, one can replace ordinary triangulations by an "enriched" version of so-called *tagged triangulations*.

We extend the result already mentioned to the tagged situation. That is, we will define an SP $\mathcal{S}_{u,z}(\boldsymbol{\tau}) = (A(\boldsymbol{\tau}), W_{u,z}(\boldsymbol{\tau}))$ for every tagged triangulation $\boldsymbol{\tau}$. Assuming that the orbifold under consideration is not a sphere whose total number of boundary components, marked points, and orbifold points is less than seven, we will again prove that the SP $\mathcal{S}_{u,z}(\mu_{\boldsymbol{i}}(\boldsymbol{\tau}))$ is reduced-$Q_0(\boldsymbol{\tau})$-equivalent to the SPs in the mutation $\mu_{\boldsymbol{i}}(\mathcal{S}_{u,z}(\boldsymbol{\tau}))$ for all tagged arcs $\boldsymbol{i}$. The important consequence is the non-degeneracy of $\mathcal{S}_{u,z}(\boldsymbol{\tau})$.

For orbifolds with non-empty boundary the $Q_0(\boldsymbol{\tau})$-equivalence class of the SP $\mathcal{S}_{u,z}(\boldsymbol{\tau})$ will be shown to be independent of the particular choice of the coefficient functions u and z.

5.1 Triangulated Orbifolds

Fomin, Shapiro, and Thurston [FST08] introduced *tagged triangulations* of *bordered surfaces with marked points* to model the mutation combinatorics of an interesting class of skew-symmetric cluster algebras: Cluster mutation corresponds to flipping arcs in triangulations.

Felikson, Shapiro, and Tumarkin [FST12a] defined tagged triangulations in a more general setting for *orbifolds*. In this way, they cover all non-exceptional skew-symmetrizable cluster algebras of finite mutation type.

This section is a short introduction to triangulated orbifolds.

5.1.1 Orbifolds

The following notion of orbifolds is due to [FST12a].

Definition 5.1.1. An *orbifold* is a triple $\mathbf{\Sigma} = (\Sigma, \mathbb{M}, \mathbb{O})$ satisfying the following properties:

(a) Σ is a connected compact oriented smooth surface with boundary $\partial\Sigma$.

(b) $\mathbb{M}$ is a non-empty finite subset of Σ.

(c) $\mathbb{O}$ is a finite subset of $\Sigma \setminus (\mathbb{M} \cup \partial\Sigma)$.

(d) $\mathbb{M}$ intersects each connected component of $\partial\Sigma$ at least once.

(e) $\mathbf{\Sigma}$ is neither a sphere with $|\mathbb{M} \cup \mathbb{O}| < 4$ nor a monogon with $|\mathbb{M} \cup \mathbb{O}| < 3$.

(f) $\mathbf{\Sigma}$ is neither a digon nor a triangle with $(\mathbb{M} \setminus \partial\Sigma) \cup \mathbb{O} = \varnothing$.

As in [FST08], $\mathbf{\Sigma}$ is called a *c-gon* if Σ is a disk and $|\mathbb{M} \cap \partial\Sigma| = c$. A *monogon*, *digon*, *triangle*, *quadrilateral* etc. is a 1-gon, 2-gon, 3-gon, 4-gon etc.

The points in $\mathbb{M}$ are called *marked points*, those in $\mathbb{P} = \mathbb{M} \setminus \partial\Sigma$ *punctures*, those in $\mathbb{M} \cap \partial\Sigma$ *boundary marked points*, and those in $\mathbb{O}$ *orbifold points*. We say that $\mathbf{\Sigma}$ is unpunctured, once-punctured, twice-punctured etc. if the cardinality of $\mathbb{P}$ is 0, 1, 2 etc.

A connected component of $\partial\Sigma \setminus \mathbb{M}$ is called a *boundary segment* of $\mathbf{\Sigma}$.

Convention 5.1.2. For the rest of the whole chapter we fix an orbifold $\mathbf{\Sigma} = (\Sigma, \mathbb{M}, \mathbb{O})$.

The set of punctures $\mathbb{M} \setminus \partial\Sigma$ will be denoted by $\mathbb{P}$, the genus of Σ by g, and the number of connected components of the boundary $\partial\Sigma$ by b. We define $m = |\mathbb{M}|$, $p = |\mathbb{P}|$, $o = |\mathbb{O}|$.

Moreover, we write $\mathfrak{s}$ for the set of *boundary segments* of $\mathbf{\Sigma}$.

Finally, let $\lambda = (\lambda_1, \dots, \lambda_b)$ be the integer partition of $m - p$ defined by $\lambda_s = |\mathbb{M} \cap C_s|$ where $\partial\Sigma = C_1 \dot\cup \cdots \dot\cup C_b$ is a partition of the boundary of Σ into connected components such that the cardinality of $\mathbb{M} \cap C_s$ decreases in s.

In illustrations we draw marked points as $\bullet$, orbifold points as $\times$, and boundary segments as $\bullet$- - - -$\bullet$. The orientation of drawn surfaces is always assumed to be clockwise.

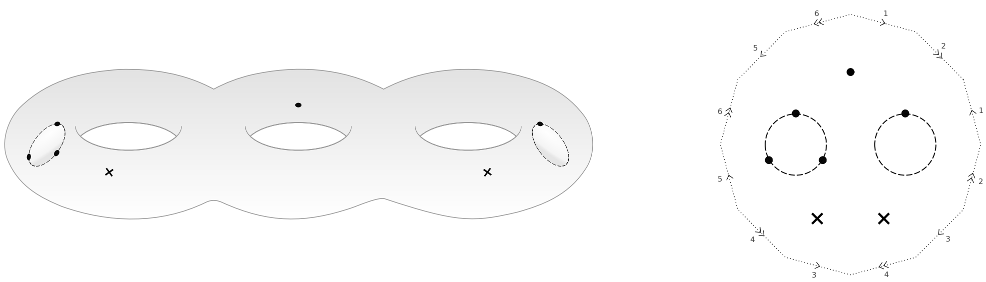
Remark 5.1.3. By the classification of compact orientable smooth surfaces with boundary, the invariant (g, λ, p, o) determines $\boldsymbol{\Sigma}$ up to diffeomorphism. The surface Σ can be obtained from the g-fold torus by cutting out b pairwise disjoint open disks. Compare Example 5.1.6.

Remark 5.1.4. Note that, contrary to the terminology, punctures belong to the surface and orbifold points have neighborhoods diffeomorphic to an open set in Euclidean space.

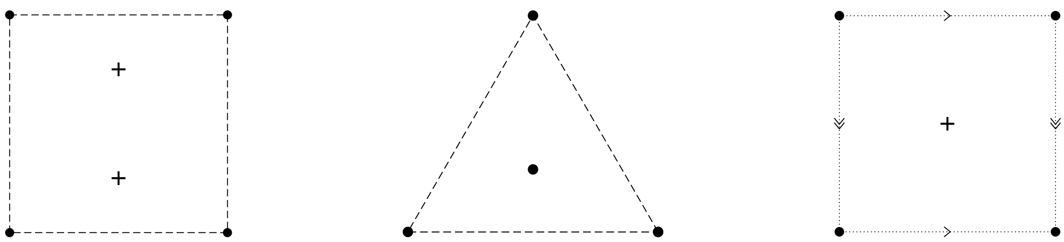
In [GL16a] orbifolds are called "surfaces with marked points and orbifold points."

Remark 5.1.5. An orbifold is a triple $(\Sigma, \mathbb{M}, \mathbb{O})$ where $(\Sigma, \mathbb{M} \cup \mathbb{O})$ is a bordered surface with marked points in the sense of [FST08, Definition 2.1] where $\mathbb{M} \neq \varnothing$ and $\mathbb{O} \subseteq \Sigma \setminus (\mathbb{M} \cup \partial\Sigma)$.

Example 5.1.6. The surface with invariant $g = 3$, $\lambda = (3, 1)$, $p = 1$, $o = 2$ is a 3-fold torus with two boundary components. It is the quotient of the $4g$-sided figure drawn below in which sides carrying the same label are identified as indicated.



Example 5.1.7. The quadrilateral with two orbifold points, the once-punctured triangle, and the once-punctured torus with one orbifold point:



Example 5.1.8. We regard the point at infinity as a point of the drawing plane. Both of the following two pictures represent a twice-punctured sphere with two orbifold points.

5.1.2 Triangulations

The definitions of arcs and triangulations given in this section are based on [FST12a]. In case of doubt, it could be helpful to consult [FST08, Definition 2.2]. However, a rigorous understanding of these definitions is not essential for this chapter and they should rather be considered a motivation. Namely, we will always work with *puzzle-piece decompositions*, which allow us to use purely combinatorial arguments. No topology or geometry is needed.

Definition 5.1.9. A *curve* γ in $\mathbf{\Sigma}$ is the image of a simple curve $[0,1] \xrightarrow{\iota} \Sigma$ such that:

(a) $\iota(\{0,1\}) \subseteq \mathbb{M} \cup \mathbb{O}$ and $\iota(\{0,1\}) \cap \mathbb{M} \neq \varnothing$ and $\iota((0,1)) \cap (\mathbb{M} \cup \mathbb{O} \cup \partial\Sigma) = \varnothing$.

(b) γ does not cut out an unpunctured monogon with less than two orbifold points.

(c) γ does not cut out an unpunctured digon without orbifold points.

The points in $\iota(\{0,1\})$ are the *endpoints* of γ; those in $\iota((0,1))$ are the *inner points* of γ. The curve γ is called *ordinary*, if its endpoints belong to $\mathbb{M}$, and *pending*, if one of its endpoints lies in $\mathbb{O}$ and the other in $\mathbb{M}$. Every curve in $\mathbf{\Sigma}$ is either ordinary or pending.

Two curves γ and γ' in $\mathbf{\Sigma}$ are *ambient-isotopic* if there exists a homotopy $\Sigma \times [0,1] \to \Sigma$ relative $\mathbb{M} \cup \mathbb{O} \cup \partial\Sigma$ given by $(s,t) \mapsto H_t(s)$ such that $H_0 = \mathrm{id}_\Sigma$, $H_1 \circ \gamma = \gamma'$, and H_t is a homeomorphism for all $t \in [0,1]$.

An *arc* i in $\mathbf{\Sigma}$ is an ambient-isotopy class of a curve in $\mathbf{\Sigma}$. The *endpoints* of an arc are the endpoints of a curve representing the arc. An arc is said to be *ordinary* resp. *pending* if it can be represented by an ordinary resp. pending curve.

Two arcs i and i' in $\mathbf{\Sigma}$ are *compatible* if they can be represented by curves γ and γ' that do not intersect each other except possibly in their endpoints belonging to $\mathbb{M}$.

A set of arcs in $\mathbf{\Sigma}$ is *compatible* if its elements are pairwise compatible.

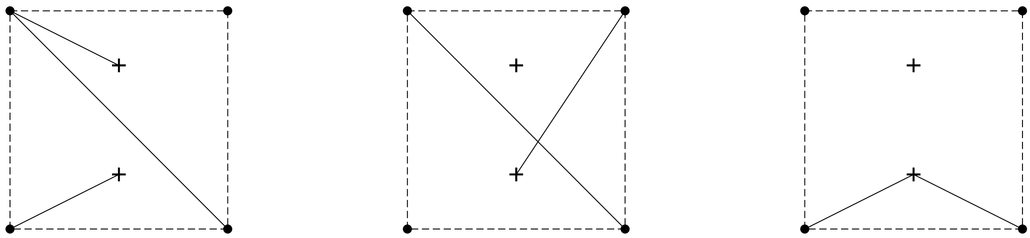
Remark 5.1.10. It is known that every compatible set $\{i_1, \dots, i_\ell\}$ of arcs in $\mathbf{\Sigma}$ can be represented by curves $\gamma_1, \dots, \gamma_\ell$ in $\mathbf{\Sigma}$ that do not intersect each other in $\Sigma \setminus \mathbb{M}$.

Furthermore, as a consequence of Whitney's Approximation Theorem, $\gamma_1, \dots, \gamma_\ell$ can be assumed to be images of smooth simple curves in Σ.

Remark 5.1.11. Let γ_0 and γ_1 be curves in $\mathbf{\Sigma}$ given as the images of simple curves ι_0 and ι_1 in Σ. It is known that γ_0 and γ_1 are ambient-isotopic if ι_0 and ι_1 are *isotopic*, i.e. there is a family of curves γ_t in $\mathbf{\Sigma}$ given as the images of simple curves ι_t in Σ such that $(s,t) \mapsto \iota_t(s)$ defines a homotopy $[0,1] \times [0,1] \to \Sigma$ from ι_0 to ι_1.

Example 5.1.12. Let $\mathbf{\Sigma}$ be the quadrilateral with two orbifold points. The left picture below displays images of simple curves $[0,1] \to \Sigma$ that are not curves in $\mathbf{\Sigma}$. On the right one can see three curves in $\mathbf{\Sigma}$ that all represent the same arc.

Example 5.1.13. The leftmost picture below illustrates three (representing curves of) compatible arcs, whereas each of the other two pictures shows a pair of incompatible arcs.



Definition 5.1.14. A *triangulation* τ of $\mathbf{\Sigma}$ is an inclusion-wise maximal set of pairwise compatible arcs in $\mathbf{\Sigma}$. We call the pair $(\mathbf{\Sigma}, \tau)$ a *triangulated orbifold*.

Remark 5.1.15. Triangulations are called ideal triangulations in [FST08] and [FST12a].

Proposition 5.1.16. *Every compatible set of arcs in* $\mathbf{\Sigma}$ *is contained in a triangulation. All triangulations of* $\mathbf{\Sigma}$ *have the same number* n *of arcs. Explicitly,*

$$n \;=\; 6(g-1) + 3b + m + 2(p+o)\,.$$

Every compatible set of $n-1$ *arcs in* $\mathbf{\Sigma}$ *is contained in at most two triangulations.*

Proof. See [FST08, Proposition 2.10, Definition 3.1] and [FST12a]. □

Example 5.1.17. The number of arcs in triangulations of the quadrilateral with two orbifold points is $n = -6 + 3 + 4 + 4 = 5$. Two such triangulations are shown below.

Example 5.1.18. Triangulations of the once-punctured triangle have $n = -6 + 3 + 4 + 2 = 3$ arcs. Here are two examples:

The next two propositions reveal the combinatorial nature of triangulations.

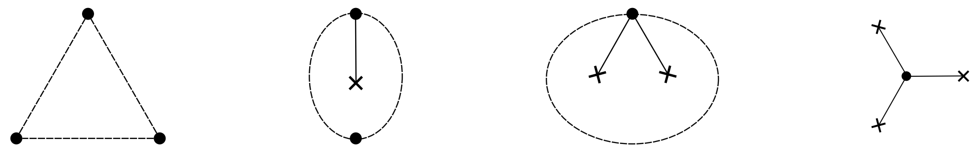


Figure 5.1.1: non-degenerate triangles: ordinary, once-, twice-, and triply-orbifolded.

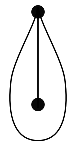


Figure 5.1.2: self-folded triangle.

Proposition 5.1.19. *Let τ be a triangulation of $\mathbf{\Sigma}$ represented by curves $\gamma_1, \ldots, \gamma_n$ that do not intersect in $\Sigma \setminus \mathbb{M}$. Then every closure of a connected component of $\Sigma \setminus (\gamma_1 \cup \cdots \cup \gamma_n)$ (considered up to ambient isotopy) is one of the* triangles *shown in Figures 5.1.1 and 5.1.2.*

Proof. See [FST08, § 2] and [FST12a, § 4] for references. □

Remark 5.1.20. The triply-orbifolded triangle (rightmost in Figure 5.1.1) can only occur in triangulations of the once-punctured sphere with three orbifold points. It is the one and only triangle of each such triangulation.

Definition 5.1.21. The side of a self-folded triangle (Figure 5.1.2) connecting two different marked points is called *folded*; the other side is the *enclosing loop*.

The marked point shared by the folded side and the enclosing loop is referred to as the *basepoint* and the other endpoint of the folded side is called the *enclosed puncture*.

Corollary 5.1.22. *All triangulations of $\mathbf{\Sigma}$ have the same number t of triangles, where*

$$t \;=\; 4(g-1) + 2b + m + p + o\,.$$

Proof. See [FG07, § 2]. □

5.1.3 Puzzle-Piece Decomposition

For a detailed version of the next statement see [FST08, Remark 4.2] and [FST12a, § 4].

Proposition 5.1.23. *Every triangulated orbifold $(\mathbf{\Sigma}, \tau)$ can be obtained as follows:*

(1) Take several copies of the (clockwise oriented) puzzle pieces *shown in Figure 5.A.1.*

(2) Partially pair up the outer sides of the pieces chosen in the first step, but never pair two sides of the same piece.

(3) Glue each paired outer side with its partner, making orientations match. The outer sides without a partner become boundary segments. □

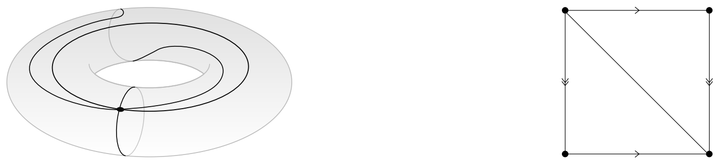


Figure 5.1.3: triangulated once-punctured torus.

Corollary 5.1.24. *Let τ be a triangulation of $\mathbf{\Sigma}$. Every puzzle piece ✤ (see Figure 5.A.1) of τ contains a unique non-degenerate triangle $\Delta($✤$)$ of τ (see Figure 5.1.1). Vice versa, every non-degenerate triangle Δ of τ is contained in a unique puzzle piece ✤$_\tau(\Delta)$ of τ.* $\square$

Remark 5.1.25. By dint of the *puzzle-piece decomposition* described in Proposition 5.1.23 the arcs of any triangulation τ of $\mathbf{\Sigma}$ can be classified as follows: Every arc in τ is either

(a) an inner side of a puzzle piece, called an *unshared arc in τ*; or

(b) an outer side of exactly two puzzle pieces, called a *shared arc in τ*.

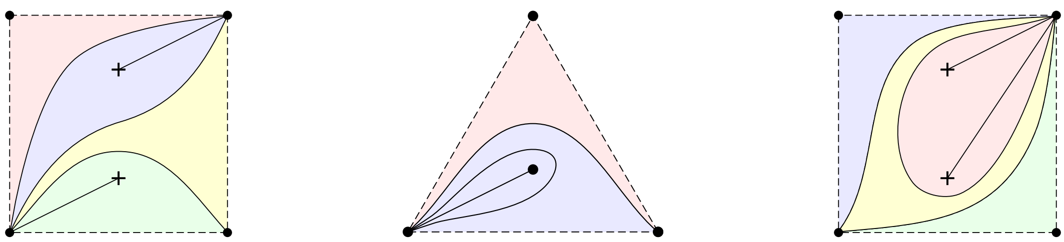
An unshared arc in τ is either pending or belongs to a self-folded triangle of τ. In total there are 18 *arctypes* for non-folded unshared arcs. They are listed in Table 5.A.10.

For a shared arc i in τ let its *kind* be the total number of arcs shared by the two puzzle pieces containing i. The *kind* of an unshared arc i in τ is defined as zero.

The kind of an arc in τ is at most two unless $(\mathbf{\Sigma}, \tau)$ is the triangulated once-punctured torus depicted in Figure 5.1.3 (where the kind of all arcs is three).

All possibilities how two puzzle pieces can share a fixed arc whose kind is one, two, or three, respectively, are listed in Tables 5.A.7 to 5.A.9. Summing up, there are $\binom{9}{2} = 45$ *arctypes* of kind one, $\binom{6}{2} = 21$ *arctypes* of kind two, and just one *arctype* of kind three. All together, we have $85 = 18 + 45 + 21 + 1$ different arctypes for non-folded arcs.

Example 5.1.26. Indicated below are the puzzle-piece decompositions for some of the triangulations from Examples 5.1.17 and 5.1.18:



5.1.4 Flipping Arcs

Removing an arc i from a triangulation τ, one is left with a compatible set of $n-1$ arcs. By Proposition 5.1.16 there can be at least one other triangulation $\mu_i(\tau)$ containing this compatible set. Deciding whether such $\mu_i(\tau)$ exists is simple:

Proposition 5.1.27. *Let τ be a triangulation of $\mathbf{\Sigma}$ and $i \in \tau$. Then $\tau \setminus \{i\}$ is contained in a triangulation $\mu_i(\tau)$ of $\mathbf{\Sigma}$ different from τ if and only if i is a non-folded arc in τ.*

Proof. See [FST08, § 3]. □

Definition 5.1.28. Let i be a non-folded arc in a triangulation τ. The triangulation $\mu_i(\tau)$ is said to be obtained by *flipping* i in τ.

Example 5.1.29. The right triangulation in Example 5.1.18 is obtained from the left one by flipping the enclosing loop. The folded side cannot be flipped.

Remark 5.1.30. Let τ and ς be two triangulations related by flipping an arc, say $\varsigma = \mu_i(\tau)$ and $\tau = \mu_j(\varsigma)$. Then i is pending if and only if j is pending. Denote by $\kappa_\tau(i)$ the kind of i in τ and by $\kappa_\varsigma(j)$ the kind of j in ς. Then $\kappa_\tau(i) = \kappa_\varsigma(j)$ or $\{\kappa_\tau(i), \kappa_\varsigma(j)\} = \{0, 2\}$.

Remark 5.1.31. A set $\{X, Y\}$ of arctypes (see Remark 5.1.25) is called a *flippant pair* if there are triangulations τ and ς of some orbifold $\mathbf{\Sigma}$ that are related by flipping, say $\varsigma = \mu_i(\tau)$ and $\tau = \mu_j(\varsigma)$, such that the arctype of i in τ is X and the arctype of j in ς is Y. Inspection shows that there are 46 flippant pairs. All of them are listed in Table 5.A.11.

Each pair of triangulations of an orbifold can be connected by a finite sequence of flips in the following sense:

Proposition 5.1.32. *For every two triangulations τ and ς of $\mathbf{\Sigma}$ there are arcs $i_1, \ldots, i_\ell$ in $\mathbf{\Sigma}$ such that $\varsigma = \mu_{i_\ell} \cdots \mu_{i_1}(\tau)$ and $\tau \cap \varsigma \subseteq \mu_{i_s} \cdots \mu_{i_1}(\tau)$ for all $0 < s < \ell$.*

Proof. See [FST08, Proposition 3.8] and [FST12a, Theorem 4.2]. □

5.1.5 Tagged Triangulations

The fact that folded arcs cannot be flipped is the reason for introducing a more sophisticated type of triangulation. These *tagged triangulations* arise from triangulations by replacing enclosing loops and by "tagging" arcs at their ends (i.e. by marking some ends with a "notch" $\bowtie$). In a tagged triangulation every arc can be flipped.

Notation 5.1.33. Recall from [FST08, Definition 7.1] that every arc i in $\mathbf{\Sigma}$ has two "ends" and each end e of i contains precisely one point $x(e)$ from $\mathbb{M} \cup \mathbb{O}$.

We denote by $\mathfrak{e}(i)$ the set of ends of i. If i is a *loop enclosing* $x \in \mathbb{P}$, i.e. i cuts out a once-punctured monogon with puncture x, let $i^\sharp$ be the unique arc in $\mathbf{\Sigma}$ such that i and $i^\sharp$ form a self-folded triangle. For arcs i that are not enclosing loops set $i^\sharp := i$.

Definition 5.1.34. A *tagged arc* $\boldsymbol{i}$ in $\mathbf{\Sigma}$ is a pair (i, tag) where i is an arc in $\mathbf{\Sigma}$ that is not an enclosing loop and $\mathfrak{e}(i) \xrightarrow{\mathrm{tag}} \{\circ, \bowtie\}$ is a function with $\mathrm{tag}(e) = \circ$ whenever $x(e) \notin \mathbb{P}$. If $\mathbf{\Sigma}$ is a once-punctured closed orbifold (i.e. $b = 0$, $p = 1$) we require $\mathrm{tag}(e) = \circ$ for all e.

For tagged arcs $\boldsymbol{i} = (i, \mathrm{tag})$ set $\boldsymbol{i}^\flat := i$ and for boundary segments $\boldsymbol{i} \in \mathfrak{s}$ set $\boldsymbol{i}^\flat := \boldsymbol{i}$.

Let $\varepsilon \in \{\pm 1\}^{\mathbb{P}}$ and i an arc in $\boldsymbol{\Sigma}$. For $x \in \mathbb{M} \cup \mathbb{O}$ let

$$\widetilde{\varepsilon}(i, x) \;=\; \begin{cases} +1 & \text{if } x \notin \mathbb{P} \text{ or } b = 0,\ p = 1, \\ -\varepsilon(x) & \text{if } i \text{ is a loop enclosing } x \in \mathbb{P}, \\ \varepsilon(x) & \text{otherwise.} \end{cases}$$

Denote by i^ε the tagged arc $\left(i^\sharp, \mathrm{tag}_\varepsilon\right)$ given by

$$\mathrm{tag}_\varepsilon(e) \;=\; \begin{cases} \circ & \text{if } \widetilde{\varepsilon}(i, x(e)) = +1, \\ \bowtie & \text{if } \widetilde{\varepsilon}(i, x(e)) = -1. \end{cases}$$

A *tagged triangulation* $\boldsymbol{\tau}$ of $\boldsymbol{\Sigma}$ is a set of tagged arcs such that $\boldsymbol{\tau} = \tau^\varepsilon := \left\{ i^\varepsilon \mid i \in \tau \right\}$ for some triangulation τ of $\boldsymbol{\Sigma}$ and some $\varepsilon \in \{\pm 1\}^{\mathbb{P}}$.

For a tagged triangulation $\boldsymbol{\tau}$ the (uniquely determined) triangulation τ with $\boldsymbol{\tau} = \tau^\varepsilon$ for some suitable $\varepsilon \in \{\pm 1\}^{\mathbb{P}}$ is called the *underlying triangulation* of $\boldsymbol{\tau}$. We set $\boldsymbol{\tau}^\flat := \tau$.

A tagged arc $\boldsymbol{i}$ is called *pending* if $\boldsymbol{i}^\flat$ is pending. Define $\boldsymbol{\tau}^\times := \{\boldsymbol{i} \in \boldsymbol{\tau} \mid \boldsymbol{i} \text{ pending}\}$.

Two tagged arcs are *compatible* if there is a tagged triangulation of $\boldsymbol{\Sigma}$ containing both of them. A set of tagged arcs is *compatible* if its elements are pairwise compatible.

The ends e of arcs with $\mathrm{tag}(e) = \bowtie$ are *tagged notched* and will be visualized as $\cdots\!\!\bowtie\!\!-\!\!\bullet$. The ends with $\mathrm{tag}(e) = \circ$ will not be accentuated in any special way.

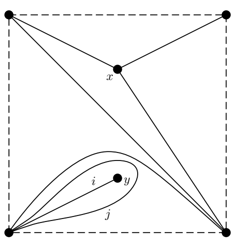
Remark 5.1.35. Let $\varepsilon \in \{\pm 1\}^{\mathbb{P}}$. Then $(i^\varepsilon)^\flat = i^\sharp$ for every arc i in $\boldsymbol{\Sigma}$.

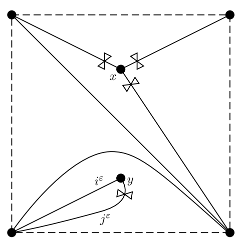
In particular, $\left\{ \boldsymbol{i}^\flat \mid \boldsymbol{i} \in \boldsymbol{\tau} \right\} \subseteq \boldsymbol{\tau}^\flat$ for every tagged triangulation $\boldsymbol{\tau}$. This inclusion is proper if and only if $\boldsymbol{\tau}^\flat$ contains a self-folded triangle.

Remark 5.1.36. To facilitate the formulation of Proposition 5.1.45, Definition 5.1.34 slightly deviates from [FST08; FST12a] for once-punctured closed orbifolds.

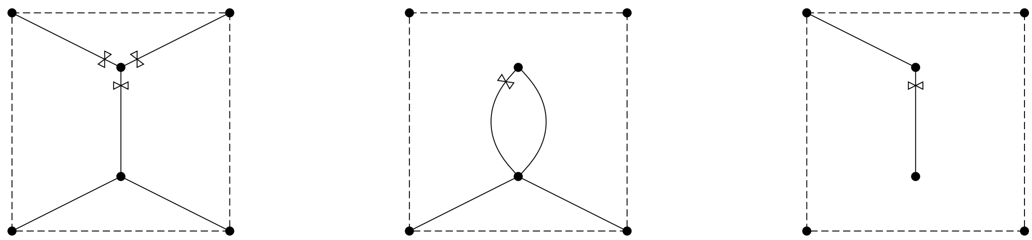
Remark 5.1.37. See the original reference [FST08, § 7] and [FST12a] for various helpful examples and a more illustrative version of Definition 5.1.34.

Example 5.1.38. The triangulation τ of the twice-punctured quadrilateral drawn below on the left gives rise to the tagged triangulation τ^ε on the right where $\varepsilon(x) = -1$ and $\varepsilon(y) = 1$.





Example 5.1.39. The picture on the left and in the middle each shows a set of compatible tagged arcs, whereas on the right picture one sees a pair of incompatible tagged arcs:



Definition 5.1.40. Let $\boldsymbol{\tau}$ be a tagged triangulation and let $\mathbb{P}_{\boldsymbol{\tau}} \subseteq \mathbb{P}$ be the set of punctures that are enclosed by arcs of $\tau = \boldsymbol{\tau}^\flat$. Restriction $\varepsilon \mapsto \varepsilon|_{\mathbb{P}_{\boldsymbol{\tau}}}$ establishes a bijection

$$E_{\boldsymbol{\tau}} = \left\{\varepsilon \in \{\pm 1\}^{\mathbb{P}} \,|\, \tau^\varepsilon = \boldsymbol{\tau}\right\} \longrightarrow \{\pm 1\}^{\mathbb{P}_{\boldsymbol{\tau}}} .$$

The preimage $\varepsilon_{\boldsymbol{\tau}}$ of the function with constant value $+1$ under this bijection is called the *weak signature of* $\boldsymbol{\tau}$ (see [Lab16, Definition 2.10]).

The elements of $E_{\boldsymbol{\tau}}$ are called *sign functions for* $\boldsymbol{\tau}$.

5.1.6 Flipping Tagged Arcs

Here are the analogs of Propositions 5.1.16 and 5.1.27 for tagged triangulations:

Proposition 5.1.41. *Every compatible set of tagged arcs in* $\boldsymbol{\Sigma}$ *is contained in a tagged triangulation of* $\boldsymbol{\Sigma}$*. Every compatible set of* $n-1$ *tagged arcs is contained in exactly two tagged triangulations.*

Proof. See [FST08, Theorem 7.9] and [FST12a]. □

Corollary 5.1.42. *Let* $\boldsymbol{\tau}$ *be a tagged triangulation of* $\boldsymbol{\Sigma}$ *and* $\boldsymbol{i} \in \boldsymbol{\tau}$*. Then* $\boldsymbol{\tau} \setminus \{\boldsymbol{i}\}$ *is contained in exactly one tagged triangulation* $\mu_{\boldsymbol{i}}(\boldsymbol{\tau})$ *of* $\boldsymbol{\Sigma}$ *different from* $\boldsymbol{\tau}$*.* □

Definition 5.1.43. The tagged triangulation $\mu_{\boldsymbol{i}}(\boldsymbol{\tau})$ is said to be obtained by *flipping* $\boldsymbol{i}$.

Corollary 5.1.44. *Let* τ *be a triangulation of* $\boldsymbol{\Sigma}$ *and* $\varepsilon \in \{\pm 1\}^{\mathbb{P}}$*. Then* $(\mu_i(\tau))^\varepsilon = \mu_{i^\varepsilon}(\tau^\varepsilon)$ *for every non-folded arc* i *in* τ*.*

Proof. This follows from Proposition 5.1.27, Corollary 5.1.42, and Definition 5.1.34. □

The following result is often used to see that properties that are invariant under flipping arcs hold for all tagged triangulations of an orbifold if they hold for any.

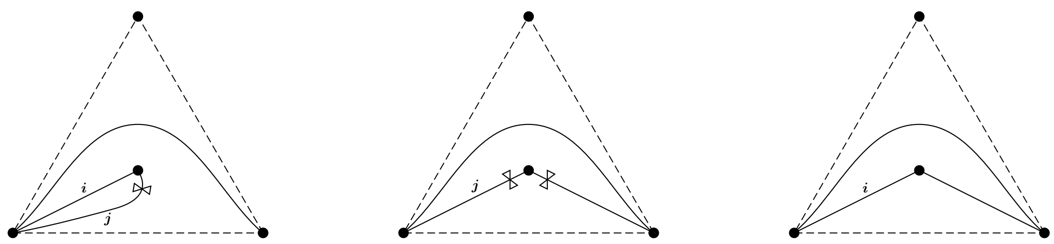
Proposition 5.1.45. *For every two tagged triangulations* $\boldsymbol{\tau}$ *and* $\boldsymbol{\varsigma}$ *of* $\boldsymbol{\Sigma}$ *there are tagged arcs* $\boldsymbol{i}_1, \dots, \boldsymbol{i}_\ell$ *in* $\boldsymbol{\Sigma}$ *with* $\boldsymbol{\varsigma} = \mu_{\boldsymbol{i}_\ell} \cdots \mu_{\boldsymbol{i}_1}(\boldsymbol{\tau})$ *and* $\boldsymbol{\tau} \cap \boldsymbol{\varsigma} \subseteq \mu_{\boldsymbol{i}_s} \cdots \mu_{\boldsymbol{i}_1}(\boldsymbol{\tau})$ *for all* $0 < s < \ell$*.*

Proof. See [FST08, Proposition 7.10] and [FST12a, Theorem 4.2, proof of Lemma 4.3]. □

Remark 5.1.46. The *tagged flip graph* $\mathbf{E}^{\bowtie}(\mathbf{\Sigma})$ of $\mathbf{\Sigma}$ is the simple graph whose vertices are the tagged triangulations of $\mathbf{\Sigma}$ and in which two vertices are joined by an edge if and only if they can be obtained from one another by flipping an arc (see [FST08, § 7]).

By Corollary 5.1.42 and Proposition 5.1.45 the graph $\mathbf{E}^{\bowtie}(\mathbf{\Sigma})$ is n-regular and connected.

Example 5.1.47. Below on the left one sees the tagged triangulation induced by the first triangulation of Example 5.1.18. Flipping the tagged arcs $\boldsymbol{i}$ and $\boldsymbol{j}$, respectively, yields the tagged triangulations shown in the middle and on the right.



5.2 Modulation of a Tagged Triangulation

We will define a modular quiver $Q(\tau)$ for each (tagged) triangulation τ of an orbifold $\mathbf{\Sigma}$.

5.2.1 Adjacency Quiver of a Triangulation

The quiver $Q'(\tau)$ of a triangulation τ defined in [GL16a] carries the same information as the *signed adjacency matrix* used in [FST08; FST12a]. Apart from keeping track of orientation and weights in a slightly different way, they are the *diagrams* of [FZ03].

The adjacency quiver $Q(\tau)$ introduced here is closely related to $Q'(\tau)$ but has additional "frozen" vertices corresponding to the boundary segments.

The purpose of this subsection is to settle notation and to revise how the puzzle-piece decomposition of a triangulation gives rise to a decomposition of the adjacency quiver, the so-called *block decomposition* from [FST08; FST12b].

Notation 5.2.1. We denote by $Q_2(\tau)$ the set of non-degenerate triangles (see Figure 5.1.1) of a triangulation τ of $\mathbf{\Sigma}$. For a triangle $\Delta \in Q_2(\tau)$ and $i, j \in \tau \cup \mathfrak{s}$ we will write

$$(i, j) \;\in\; \Delta$$

if both i and j are sides of Δ and j follows i in Δ with respect to the orientation of $\mathbf{\Sigma}$.

Copying [FST08, Definition 4.1] let $\tau \cup \mathfrak{s} \xrightarrow{\pi_\tau} \tau \cup \mathfrak{s}$ be the idempotent function with

$$\pi_\tau(i) \;=\; \begin{cases} j & \text{if } i \text{ is the folded side of a self-folded triangle in } \tau \text{ with enclosing loop } j, \\ i & \text{otherwise.} \end{cases}$$

Definition 5.2.2. Let τ be a triangulation of $\mathbf{\Sigma}$. The *(adjacency) quiver* $Q(\tau)$ of τ is the modular quiver $(Q(\tau), d, \sigma)$ whose vertices are the arcs in τ and boundary segments of $\mathbf{\Sigma}$.

Formally, $Q_0(\tau) = \tau \cup \mathfrak{s}$. Vertices $i \in Q_0(\tau)$ have weight $d_i = 2$ if i is a pending arc and weight $d_i = 1$ otherwise. The arrow set is

$$Q_1(\tau) \;=\; \left\{ i \xrightarrow{(\Delta,(i,j),r)} j \;\middle|\; \Delta \in Q_2(\tau),\, i,j \in Q_0(\tau) \text{ with } (\pi_\tau(i), \pi_\tau(j)) \in \Delta,\, r \in \mathbb{Z}/d_{ji}\mathbb{Z} \right\}.$$

The tuple $(\sigma_a)_{a \in Q_1(\tau)}$ is defined by $\sigma_a = r$ for each $a = (\Delta, (i,j), r) \in Q_1(\tau)$.

Notation 5.2.3. Set $\Delta(a) := \Delta$ and $\text{🧩}(a) := \text{🧩}_\tau(\Delta)$ for arrows $a = (\Delta, (i,j), r) \in Q_1(\tau)$. We say that a is *induced by* the (non-degenerate) triangle Δ or the puzzle piece $\text{🧩}(a)$.

For each puzzle piece 🧩 of τ we will denote by $Q^{\text{🧩}}$ the subquiver of $Q(\tau)$ spanned by all arrows that are induced by 🧩. More generally, we call every subquiver $Q^{\text{🧩}_1} \oplus \cdots \oplus Q^{\text{🧩}_\ell}$ of $Q(\tau)$ the *subquiver induced by the puzzle pieces* $\text{🧩}_1, \ldots, \text{🧩}_\ell$.

Remark 5.2.4. Obviously, $a = (\Delta(a), (s(a), t(a)), \sigma_a)$ for every $a \in Q_1(\tau)$.

Remark 5.2.5. According to [FST12a, Lemma 4.10] there is a decomposition

$$Q(\tau) \;=\; \bigoplus_{\text{🧩}} Q^{\text{🧩}}$$

where the sum runs over all puzzle pieces 🧩 of τ. The modular quivers $Q^{\text{🧩}}$ for the different types of puzzle pieces are listed in Figures 5.A.2 to 5.A.6. See also Example 5.2.12.

Remark 5.2.6. The quiver $Q(\tau)$ of every triangulation τ is connected, since Σ is connected.

Remark 5.2.7. Proposition 5.1.23 and Remark 5.2.5 lead to the following observations concerning the number q_{ji} of arrows $j \leftarrow i$ in $Q(\tau)$:

- $q_{ji} \leq 2$.
- If $q_{ij} \geq 1$, then $q_{ji} \leq 1$.
- If $d^{ji} = 2$ and $d_{ij} = 1$, then $q_{ji} \leq 1$.
- If $d_{ji} \neq 2$, there is at most one arrow $j \leftarrow i$ induced by the same triangle.
- If $d_{ji} = 2$ and $q_{ji} \geq 1$, then $q_{ij} = 0$ and $q_{ji} = 2$ and $\{\sigma_a \,|\, j \xleftarrow{a} i \in Q_1(\tau)\} = \mathbb{Z}/2\mathbb{Z}$.

The second item shows that $Q(\tau)$ has a unique maximal 2-acyclic subquiver.

Note that $d_{ji} = 2$ means that both i and j are pending, while $d^{ji} = 2$ and $d_{ij} = 1$ means that exactly one of i and j is pending.

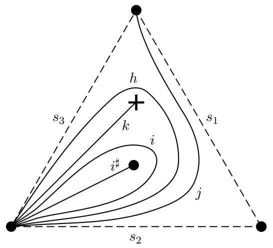
Remark 5.2.8. The full subquiver $Q^\circ(\tau)$ of $Q(\tau)$ spanned by τ (i.e. by all vertices that are not boundary segments) is called the "unreduced weighted quiver of τ" in [GL16a].

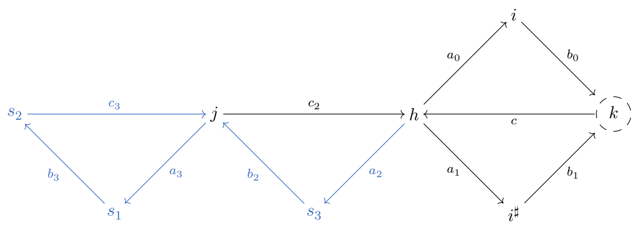
Remark 5.2.9. Let τ be a triangulation and $Q'(\tau)$ the maximal 2-acyclic subquiver of $Q^\circ(\tau)$.

The skew-symmetrizable matrix $B(\tau)$ corresponding to the weighted quiver $Q'(\tau)$ under the bijection of Remark 2.1.6 is one of the matrices described in [FST12a, § 4.3].

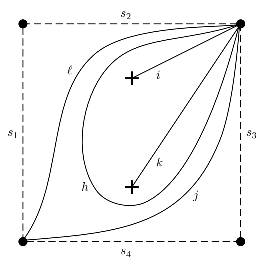
Convention 5.2.10. We illustrate vertices i in $Q(\tau)$ with $d_i = 2$ as (i) and arrows a with $\sigma_a = 2$ as $\overset{a}{\rightsquigarrow}$. The part of $Q(\tau)$ not belonging to $Q^\circ(\tau)$ will be drawn in blue.

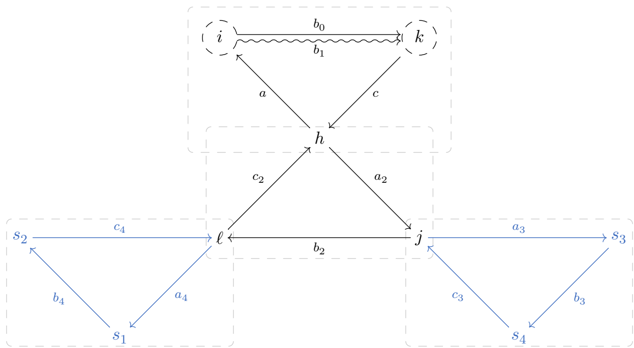
Example 5.2.11. The triangulation τ of the once-punctured triangle with one orbifold point shown below on the left gives rise to the modular quiver $Q(\tau)$ on the right. The arrows a_0, a_1, b_0, b_1, c are all induced by the triangle whose sides are h, i, k.



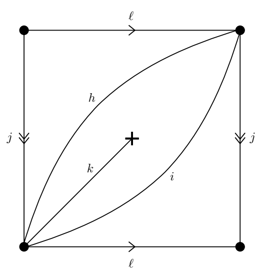


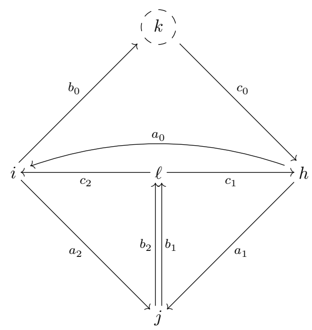
Example 5.2.12. The second triangulation of Example 5.1.17 has the modular quiver $Q(\tau)$ depicted on the right. The dashed gray boxes indicate the decomposition $Q(\tau) = \bigoplus_{✣} Q^{✣}$.





Example 5.2.13. Every triangulation τ of the once-punctured torus with one orbifold point looks like the one below. Its quiver $Q(\tau)$ can be seen on the right.





With respect to the ordering $i < k < h < j < \ell$ on $Q_0(\tau)$ the matrix $B(\tau)$ has the form

$$\begin{pmatrix} 0 & -2 & 1 & -1 & 1 \\ 1 & 0 & -1 & 0 & 0 \\ -1 & 2 & 0 & -1 & 1 \\ 1 & 0 & 1 & 0 & -2 \\ -1 & 0 & -1 & 2 & 0 \end{pmatrix}.$$

5.2.2 Mutating Adjacency Quivers

Given a triangulation $\mu_i(\tau)$ obtained from another triangulation by flipping an arc i, we will identify the modular quiver $Q(\mu_i(\tau))$ with a subquiver of the premutation $\widetilde{\mu}_i(Q(\tau))$.

Convention 5.2.14. The unique maximal 2-acyclic subquiver $Q'(\tau)$ of $Q(\tau)$ is a reduction in the sense of Definition 2.1.14. We define $\widetilde{\mu}_i(Q(\tau)) := \widetilde{\mu}_i(Q'(\tau))$ for all $i \in \tau$.

Notation 5.2.15. For an arc i of a triangulation τ write $Q(\tau, i)$ for the subquiver of $Q(\tau)$ induced by all (at most two) puzzle pieces of τ that contain i.

Let $Q(\tau, \neg i)$ be the subquiver of $Q(\tau)$ induced by all puzzle pieces of τ not containing i.

Remark 5.2.16. It is $Q(\tau) = Q(\tau, \neg i) \oplus Q(\tau, i)$ and $\widetilde{\mu}_i(Q(\tau)) = Q(\tau, \neg i) \oplus \widetilde{\mu}_i(Q(\tau, i))$.

Lemma 5.2.17. *Let τ and ς be two triangulations related by flipping an arc, say $\varsigma = \mu_i(\tau)$ and $\tau = \mu_j(\varsigma)$. Then $Q^\neg := Q(\tau, \neg i) = Q(\varsigma, \neg j)$ and there is a monomorphism*

$$Q(\varsigma) \xhookrightarrow{\Phi} \widetilde{\mu}_i(Q(\tau))$$

of modular quivers with $\Phi|_{Q^\neg} = \mathrm{id}_{Q^\neg}$ and $\Phi(k) = k$ for all $k \in Q_0(\varsigma) \setminus \{j\}$ and $\Phi(j) = i$.

The image of Φ contains a maximal 2-acyclic subquiver of $\widetilde{\mu}_i(Q(\tau))$.

Proof. This follows from [FST12a, Lemma 4.12] and Definitions 2.1.12 and 5.2.2. It can be verified directly by inspecting Tables 5.A.12 to 5.A.57. □

Remark 5.2.18. Additionally, one can demand in Lemma 5.2.17 that the monomorphism Φ has the property that for every path $\xleftarrow{b} i \xleftarrow{a}$ in $Q(\tau)$ ...

(i) ... with $\Delta(b) \neq \Delta(a)$, there is a path $b^\vee c^\vee a^\vee$ in $Q(\varsigma)$ with $\Delta(b^\vee) = \Delta(c^\vee) = \Delta(a^\vee)$ such that $\Phi(b^\vee) = b^*$, $\Phi(c^\vee) = [ba]_0^0$, $\Phi(a^\vee) = a^*$.

(ii) ... with $d_i = 2$, $\sigma_b = \sigma_a = 0$, the arrow $[ba]_0^0$ lies in the image of Φ.

Such Φ exists. It is unique if $\mathbf{\Sigma}$ is neither the once-punctured torus (Figure 5.1.3) nor the once-punctured sphere with three orbifold points (i.e. puzzle piece $\widetilde{D}_3$).

Example 5.2.19. The first weighted quiver appearing in each of the Tables 5.A.12 to 5.A.57 is $Q(\tau, i)$, where i corresponds to the boxed vertex, while the second one is $\Phi(Q(\varsigma, j))$ for a monomorphism Φ satisfying the property in Remark 5.2.18.

5.2.3 Adjacency Quiver of a Tagged Triangulation

The quiver of a tagged triangulation is defined analogously as in the untagged situation.

Notation 5.2.20. For tagged triangulations $\boldsymbol{\tau}$ set $\pi_{\boldsymbol{\tau}}(\boldsymbol{i}) := \pi_{\boldsymbol{\tau}^\flat}(\boldsymbol{i}^\flat)$ for all $\boldsymbol{i} \in \boldsymbol{\tau} \cup \mathfrak{s}$.

Definition 5.2.21. Let $\boldsymbol{\tau}$ be a tagged triangulation of $\mathbf{\Sigma}$. The *(adjacency) quiver* $Q(\boldsymbol{\tau})$ of $\boldsymbol{\tau}$ is the modular quiver with vertex set $Q_0(\boldsymbol{\tau}) = \boldsymbol{\tau} \cup \mathfrak{s}$. Vertices $\boldsymbol{i}$ in this quiver have weight $d_{\boldsymbol{i}} = 2$ if $\boldsymbol{i}^\flat$ is a pending arc and weight $d_{\boldsymbol{i}} = 1$ otherwise. The arrow set is

$$Q_1(\boldsymbol{\tau}) = \left\{ \boldsymbol{i} \xrightarrow{(\Delta,(\boldsymbol{i},\boldsymbol{j}),r)} \boldsymbol{j} \,\middle|\, \Delta \in Q_2(\boldsymbol{\tau}^\flat),\ \boldsymbol{i},\boldsymbol{j} \in Q_0(\boldsymbol{\tau}) \text{ with } (\pi_{\boldsymbol{\tau}}(\boldsymbol{i}), \pi_{\boldsymbol{\tau}}(\boldsymbol{j})) \in \Delta,\ r \in \mathbb{Z}/d_{\boldsymbol{j}\boldsymbol{i}}\mathbb{Z} \right\}.$$

The tuple $(\sigma_a)_{a \in Q_1(\boldsymbol{\tau})}$ is defined by $\sigma_a = r$ for each $a = (\Delta, (\boldsymbol{i},\boldsymbol{j}), r) \in Q_1(\boldsymbol{\tau})$.

The arrow $a = (\Delta, (\boldsymbol{i},\boldsymbol{j}), r)$ is *induced by* $\text{🧩} = \text{🧩}_{\boldsymbol{\tau}^\flat}(\Delta)$. Set $\Delta(a) := \Delta$ and $\text{🧩}(a) := \text{🧩}$.

The *subquiver of* $Q(\boldsymbol{\tau})$ *induced by puzzle pieces* $\text{🧩}_1, \ldots, \text{🧩}_\ell$ is the subquiver of $Q(\boldsymbol{\tau})$ spanned by all arrows that are induced by one of the pieces $\text{🧩}_1, \ldots, \text{🧩}_\ell$.

Remark 5.2.22. Let $\boldsymbol{\tau}$ be a tagged triangulation of $\boldsymbol{\Sigma}$ and ε a sign function for $\boldsymbol{\tau}$. Moreover, let $\tau = \boldsymbol{\tau}^\flat$ be the underlying triangulation of $\boldsymbol{\tau}$.

There is a unique isomorphism of modular quivers

$$Q(\tau) \xrightarrow{\iota_{\boldsymbol{\tau},\varepsilon}} Q(\boldsymbol{\tau})$$

that extends the bijection $Q_0(\tau) \to Q_0(\boldsymbol{\tau})$ given by $i \mapsto i^\varepsilon$ (where $i^\varepsilon := i$ for $i \in \mathfrak{s}$) and satisfies $\Delta(\iota_{\boldsymbol{\tau},\varepsilon}(a)) = \Delta(a)$ for all $a \in Q_1(\tau)$.

5.2.4 Adjacency Modulation

For every (tagged) triangulation τ we have now defined an associated modular quiver $Q(\tau)$. All vertex weights in this quiver are either 1 or 2. To obtain a comfy modulation for $Q(\tau)$, we must therefore choose a degree-2 Galois extension.

Convention 5.2.23. For the rest of the chapter fix a comfy extension $(L/K, -1, v)$

Abbreviate $w := v^2 \in K$.

Write z^* for the *conjugate* of $z \in L$, i.e. $z^* = \rho(z)$ for the non-trivial $\rho \in \mathrm{Gal}(L/K)$.

Remark 5.2.24. Fixing a comfy extension $(L/K, -1, v)$ is the same as choosing a degree-2 field extension L/K with $\mathrm{char}(K) \neq 2$ and picking an element $v \in L$ with $v^2 \in K$.

Remark 5.2.25. Let $z = x + yv \in L$ with $x, y \in K$. From field theory we know $z^* = x - yv$, $\mathrm{N}_{L/K}(z) = zz^* = x^2 - wy^2$, $\mathrm{Tr}_{L/K}(z) = z + z^* = 2x$, and $z - z^* = 2yv$.

Example 5.2.26. For L/K we could take $\mathbb{C}/\mathbb{R}$ or $\mathbb{F}_{q^2}/\mathbb{F}_q$ for any odd prime power q. In the former case, we could also assume that $v^2 = w = -1$.

Definition 5.2.27. The *modulation* $\mathcal{H}(\tau)$ *of a (tagged) triangulation* τ is the modulation of $Q(\tau)$ over $(L/K, -1, v)$.

The ground ring, species, path algebra, and completed path algebra of $\mathcal{H}(\tau)$ will be denoted by $R(\tau)$, $A(\tau)$, $H(\tau)$, $\widehat{H}(\tau)$, respectively.

Remark 5.2.28. Let $\boldsymbol{\tau}$ and ε be as in Remark 5.2.22. Using the isomorphism $\iota_{\boldsymbol{\tau},\varepsilon}$ to identify the modular quivers $Q(\tau)$ and $Q(\boldsymbol{\tau})$, there is an induced isomorphism of K-modulations

$$\mathcal{H}(\tau) \xrightarrow{\iota_{\boldsymbol{\tau},\varepsilon}} \mathcal{H}(\boldsymbol{\tau}).$$

All components $R_i(\tau) \xrightarrow{\iota_{\boldsymbol{\tau},\varepsilon}} R_{\iota_{\boldsymbol{\tau},\varepsilon}(i)}(\boldsymbol{\tau})$ and $A_a(\tau) \xrightarrow{\iota_{\boldsymbol{\tau},\varepsilon}} A_{\iota_{\boldsymbol{\tau},\varepsilon}(a)}(\boldsymbol{\tau})$ are the identity maps.

Hence, $\iota_{\boldsymbol{\tau},\varepsilon}$ induces isomorphisms $H(\tau) \xrightarrow{\iota_{\boldsymbol{\tau},\varepsilon}} H(\boldsymbol{\tau})$ and $\widehat{H}(\tau) \xrightarrow{\iota_{\boldsymbol{\tau},\varepsilon}} \widehat{H}(\boldsymbol{\tau})$ of K-algebras.

5.3 Potential of a Tagged Triangulation

Inspired by the series of articles [Lab09a; Lab09b; CL12; Lab16] we constructed in [GL16a] for each (tagged) triangulation τ a potential for the species $A(\tau)$.

5.3.1 Choosing Coefficients

The potentials we define in this section depend on the following choice of coefficients.

Convention 5.3.1. Fix two functions:

$$\begin{aligned} \mathbb{P} &\xrightarrow{\;u\;} K^\times & \qquad\qquad \mathbb{O} &\xrightarrow{\;z\;} L \setminus K \\ \bullet &\longmapsto u_\bullet & \times &\longmapsto z_\times \end{aligned}$$

Define for arcs i in $\mathbf{\Sigma}$

$$u_i \;:=\; \begin{cases} u_\bullet & \text{if } i \text{ is a loop enclosing } \bullet \in \mathbb{P}, \\ 1 & \text{otherwise,} \end{cases}$$

$$z_i \;:=\; \begin{cases} z_\times & \text{if } i \text{ is pending and contains } \times \in \mathbb{O}, \\ 1 & \text{otherwise.} \end{cases}$$

Furthermore, for triangulations τ of $\mathbf{\Sigma}$ and $i \in \tau$ set $u_i^\tau := \left(-u_{\pi_\tau(i)}\right)^{-\delta_{\pi_\tau(i)\neq i}}$.

Example 5.3.2. One valid choice is $u_\bullet = 1$ for all $\bullet \in \mathbb{P}$ and $z_\times = v$ for all $\times \in \mathbb{O}$.

Remark 5.3.3. In [GL16a] we considered the choice $z_\times = 1 - v$ for all $\times \in \mathbb{O}$.

5.3.2 Potential Components Induced by Puzzle Pieces

Definition 5.3.4. Let τ be a triangulation of $\mathbf{\Sigma}$ and 🧩 a puzzle piece of τ.

A *cyclic path induced by* 🧩 is a cyclic path cba in $Q(\tau)$ with 🧩(a) = 🧩(b) = 🧩(c) = 🧩.

For every cyclic path $i \xleftarrow{c} k \xleftarrow{b} j \xleftarrow{a} i$ in $Q(\tau)$ induced by 🧩 the potential

$$u_k^\tau u_j^\tau u_i^\tau \, c z_k b z_j a z_i$$

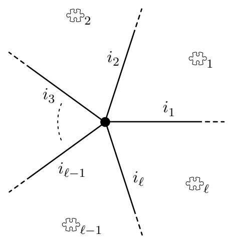
for $A(\tau)$ is called a *potential component induced by* 🧩.

The *potential* $W_{u,z}^{🧩}$ *induced by* 🧩 is the sum of all potential components induced by 🧩.

Example 5.3.5. Figures 5.A.2 to 5.A.6 list $W_{u,z}^{🧩}$ for all types of puzzle pieces 🧩.

5.3.3 Potential Components Induced by Punctures

Definition 5.3.6. Let τ be a triangulation of $\mathbf{\Sigma}$ and $\bullet \in \mathbb{P}$ not enclosed by any arc of τ. Not drawing enclosing loops, an infinitesimal neighborhood of $\bullet$ in $(\mathbf{\Sigma}, \tau)$ looks as follows:



A path $i_1 \xleftarrow{a_\ell} i_\ell \xleftarrow{a_{\ell-1}} \cdots \xleftarrow{a_2} i_2 \xleftarrow{a_1} i_1$ in $Q(\tau)$ is *induced by* $\bullet$ if $\maltese(a_s) = \maltese_s$ for all s.

For every cyclic path $a_\ell \cdots a_1$ in $Q(\tau)$ induced by $\bullet$ the potential

$$u_\bullet\, a_\ell \cdots a_1$$

for $A(\tau)$ is called a *potential component induced by* $\bullet$.

The *potential* $W_u^\bullet(\tau)$ *induced by* $\bullet$ is the sum of all potential components induced by $\bullet$.

5.3.4 Potential of a (Tagged) Triangulation

Definition 5.3.7. The *potential of a triangulation* τ of $\mathbf{\Sigma}$ is defined as

$$W_{u,z}(\tau) \quad = \quad \sum_{\maltese} W_{u,z}^{\maltese} \quad + \quad \sum_{\bullet} W_u^\bullet(\tau)$$

where $\maltese$ runs through all puzzle pieces of τ and $\bullet$ through all punctures in $\mathbf{\Sigma}$ that are not enclosed by any arc of τ.

The *species with potential of* τ is $\mathcal{S}_{u,z}(\tau) = (A(\tau), W_{u,z}(\tau))$.

Lemma 5.3.8. *$\mathcal{S}_{u,z}(\tau)$ is 2-acyclic after reduction for every triangulation τ.*

Proof. If $Q(\tau)$ contains a cyclic path ba whose two vertices are i and j, then i and j are arcs in τ with arctypes $\{X^\uparrow, Y^\downarrow\}$ and $\{X^\downarrow, Y^\uparrow\}$ for some $X, Y \in \{A, B, \widetilde{B}\}$ (see Table 5.A.9). Then $d_i = d_j = 1$, the subquiver Q' of $Q(\tau)$ spanned by a, b is full and $W_{u,z}(\tau)|_{Q'_1} = u_\bullet ba$ for some $\bullet \in \mathbb{P}$. Since $u_\bullet \in K^\times$, we can conclude that $\mathcal{S}_{u,z}(\tau)$ is 2-acyclic after reduction. $\square$

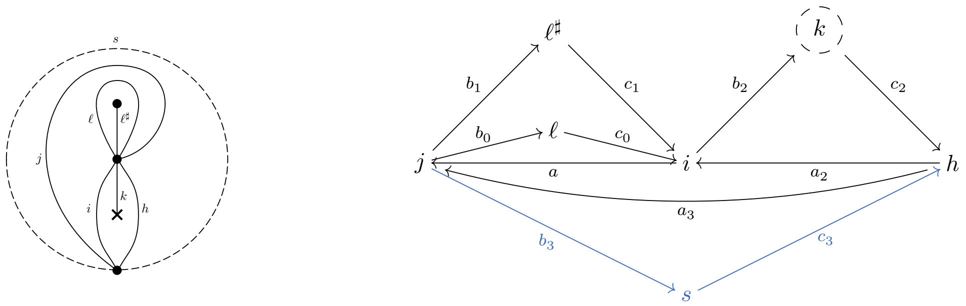
Example 5.3.9. For the triangulations from the examples in the previous subsection it is

$$\begin{aligned}
W_{u,z}(\tau) &= cz_k(b_0a_0 - \tfrac{1}{u_i}b_1a_1) + c_2b_2a_2 + c_3b_3a_3 && \text{(Example 5.2.11)}\,, \\
W_{u,z}(\tau) &= cz_k(b_0 + b_1)z_i a + c_2b_2a_2 + c_3b_3a_3 + c_4b_4a_4 && \text{(Example 5.2.12)}\,, \\
W_{u,z}(\tau) &= c_0z_kb_0a_0 + c_1b_1a_1 + c_2b_2a_2 + u_\bullet\, c_1b_2a_1c_0b_0c_2b_1a_2a_0 && \text{(Example 5.2.13)}\,.
\end{aligned}$$

Example 5.3.10. Every triangulation τ of the sphere with one puncture $\bullet$ and three orbifold points consists of exactly one puzzle piece of type $\widetilde{D}_3$ (see Figure 5.A.1). The modular quiver $Q(\tau) = Q$ and the potential $W_{u,z}$ are shown in Figure 5.A.6. We have

$$W_u^{\bullet}(\tau) \;=\; u_{\bullet}\,(c_0+c_1)(b_0+b_1)(a_0+a_1)\,.$$

Example 5.3.11. The triangulation τ of the twice-punctured monogon with one orbifold point seen below on the left has the quiver $Q(\tau)$ drawn on the right. For the non-enclosed puncture $\bullet$ we have $W_u^{\bullet}(\tau) = u_{\bullet}\, c_2 b_2 c_1 b_1 a_3$.



Definition 5.3.12. Let $\boldsymbol{\tau}$ be a tagged triangulation and ε a sign function for $\boldsymbol{\tau}$. Denote by $\varepsilon \cdot u$ the function $\mathbb{P} \to K^{\times}$ given by pointwise multiplication of ε and u.

The *ε-potential of* $\boldsymbol{\tau}$ is $W^{\varepsilon}_{u,z}(\boldsymbol{\tau}) = \iota_{\boldsymbol{\tau},\varepsilon}\big(W_{\varepsilon\cdot u,z}(\tau)\big)$.

The *potential of* $\boldsymbol{\tau}$ is $W_{u,z}(\boldsymbol{\tau}) = W^{\varepsilon_{\boldsymbol{\tau}}}_{u,z}(\boldsymbol{\tau})$ where $\varepsilon_{\boldsymbol{\tau}}$ is the weak signature of $\boldsymbol{\tau}$.

Define $\mathcal{S}^{\varepsilon}_{u,z}(\boldsymbol{\tau}) = (A(\boldsymbol{\tau}), W^{\varepsilon}_{u,z}(\boldsymbol{\tau}))$ and $\mathcal{S}_{u,z}(\boldsymbol{\tau}) = (A(\boldsymbol{\tau}), W_{u,z}(\boldsymbol{\tau}))$.

Convention 5.3.13. Let τ and ς be (tagged) triangulations with $\mu_i(\tau) = \varsigma$ and $\mu_j(\varsigma) = \tau$.

The bijection $Q_0(\tau) \to Q_0(\varsigma)$ that maps i to j and every $k \in (\tau \cap \varsigma) \cup \mathfrak{s}$ to itself induces an isomorphism $R(\tau) \to R(\varsigma)$ of K-algebras. This isomorphism will be used to consider the path algebra $H(\varsigma)$ and the completed path algebra $\widehat{H}(\varsigma)$ as $R(\tau)$-algebras.

For $X \in \{\tau \cup \mathfrak{s}, R(\tau)\}$ and SPs $\mathcal{S} = (A(\tau), W)$, $\mathcal{S}' = (A(\varsigma), W')$ we write $\mathcal{S}' \approx_X \mu_i(\mathcal{S})$ to indicate that every (equivalently, any) SP in $\mu_i(\mathcal{S})$ is reduced-X-equivalent to $\mathcal{S}'$.

We write $\sim_\tau$ for $\sim_{\tau\cup\mathfrak{s}}$ and $\sim_R$ for $\sim_{R(\tau)}$ and, similarly, $\approx_\tau$ for $\approx_{\tau\cup\mathfrak{s}}$ and $\approx_R$ for $\approx_{R(\tau)}$.

5.3.5 Conjugating the Coefficients

Definition 5.3.14. For (tagged) arcs i in $\boldsymbol{\Sigma}$ let $\mathbb{O} \xrightarrow{z^{*i}} L \setminus K$ be the function $\times \mapsto z^{*i}_{\times}$ given as

$$z^{*i}_{\times} \;=\; \begin{cases} z^{*}_{\times} & \text{if } i \text{ contains } \times, \\ z_{\times} & \text{otherwise.} \end{cases}$$

Remark 5.3.15. Obviously, $(z^{*i})^{*i} = z$ for all i and $z^{*i} = z$ for non-pending i.

Lemma 5.3.16. *For every (tagged) triangulation τ and $i \in \tau$ it is $\mathcal{S}_{u,z^{*i}}(\tau) \sim_\tau \mathcal{S}_{u,z}(\tau)$.*

Proof. Abbreviate $Q = Q(\tau)$, $R = R(\tau)$, $A = A(\tau)$. If $d_i = 1$, there is nothing to show. We will therefore assume $d_i = 2$.

The non-identity element f_i in $\mathrm{Gal}(R_i/K)$ and the identities $f_j = \mathrm{id}_{R_j}$ for $j \neq i$ induce an automorphism $R \xrightarrow{f} R$.

In view of Remark 5.2.7 there is a bijection $Q_1 \xrightarrow{g} Q_1$ such that for $k \xleftarrow{a} j \in Q_1$

$$g(a) = \begin{cases} a' & \text{if } i \in \{j,k\} \text{ and } d_{kj} = 2, \\ & \quad \text{where } a' \text{ is the unique arrow } k \leftarrow j \text{ with } \sigma_{a'} \neq \sigma_a, \\ a & \text{otherwise.} \end{cases}$$

In either case, we have $\rho_{g(a)} f_j = f_k \rho_a$ on $R_k \cap R_j \subseteq L$. Hence, the identities $A_a \to A_{g(a)}$ of K-vector spaces induce an R-bimodule isomorphism $A \xrightarrow{g} f_* A$.

The induced isomorphism $H(\tau) \xrightarrow{g} f_* H(\tau)$ of R-algebras can be regarded as an automorphism of the K^{Q_0}-algebra $H(\tau)$. It is easy to see that $g(W_{u,z}(\tau)) = W_{u,z^{*i}}(\tau)$. □

5.4 Compatibility of Flip and Mutation

We will prove that $\mathcal{S}_{u,z}(\tau)$ and $\mathcal{S}_{u,z}(\varsigma)$ correspond to each other under mutation, if the triangulations τ and ς are related by flipping an arc.

5.4.1 Compatibility for Triangulations

Theorem 5.4.1. *Let τ be a triangulation of $\mathbf{\Sigma}$ and let i be a non-folded arc of τ. Then we have $\mathcal{S}_{u,z^{*i}}(\mu_i(\tau)) \approx_R \mu_i(\mathcal{S}_{u,z}(\tau))$. In particular, $\mathcal{S}_{u,z}(\mu_i(\tau)) \approx_\tau \mu_i(\mathcal{S}_{u,z}(\tau))$.*

Proof. Let $\varsigma = \mu_i(\tau)$, $\tau = \mu_j(\varsigma)$. The last claim follows from the first one by Lemma 5.3.16.

We prove the theorem case by case. In each case, we assume that the arctype of i in τ is a previously fixed one (see Remark 5.1.25). In total, we thus have 85 cases.

Since $\mathcal{S}_{u,z^{*i}}(\varsigma) \approx_R \mu_i(\mathcal{S}_{u,z}(\tau)) \Leftrightarrow \mathcal{S}_{u,z}(\tau) \approx_R \mu_j(\mathcal{S}_{u,z^{*i}}(\varsigma))$ by Theorem 2.6.101, we can swap τ, i, z for ς, j, z^{*i} whenever we like to. This reduces the number of cases to consider to 46 (see Remark 5.1.31). Moreover, we can always assume that i is not an enclosing loop.

Abbreviate $Q = Q(\tau, i)$ and $Q' = Q(\varsigma, j)$ and $\widetilde{Q} = \widetilde{\mu}_i(Q)$. Recall that $Q(\tau) = Q^\neg \oplus Q$ and $Q(\varsigma) = Q^\neg \oplus Q'$ and $\widetilde{\mu}_i(Q(\tau)) = Q^\neg \oplus \widetilde{Q}$ for some $Q^\neg$.

Let $Q(\varsigma) \overset{\Phi}{\hookrightarrow} \widetilde{\mu}_i(Q(\tau))$ be a monomorphism of modular quivers like in Lemma 5.2.17 satisfying the property of Remark 5.2.18. Then Φ restricts to a map $Q' \hookrightarrow \widetilde{Q}$ and induces an injective $R(\tau)$-algebra homomorphism

$$H' = H(\varsigma) \overset{\Phi}{\longhookrightarrow} \widetilde{\mu}_i(H(\tau)) = \widetilde{H}\,.$$

Let $W = (W_{u,z}(\tau))^{Q_1}$ and $W' = (W_{u,z^{*i}}(\varsigma))^{Q_1'}$.

We will proceed as follows:

(1) Construct $\vartheta \in \mathrm{Aut}_{Q(\tau)-Q_1}(H(\tau))$ such that $\vartheta(W)$ is in i-split form.

(2) Compute the premutation $\widetilde{W} = \widetilde{\mu}_i(\vartheta(W))$.

(3) Construct $\widetilde{\vartheta} \in \mathrm{Aut}_{\widetilde{\mu}_i(Q(\tau))-\widetilde{Q}_1}(\widetilde{H})$ such that $\widetilde{\vartheta}(\widetilde{W})$ is in $\widetilde{Q}_1$-split form.

(4) Compute $\widetilde{W}' = \mathrm{red}_{\widetilde{Q}_1}(\widetilde{\vartheta}(\widetilde{W}))$ and $\widetilde{T} = \mathrm{triv}_{\widetilde{Q}_1}(\widetilde{\vartheta}(\widetilde{W})) \subseteq \widetilde{Q}$.

We will choose ϑ and $\widetilde{\vartheta}$ in such a way that $\Phi(W') = \widetilde{W}'$ and $\widetilde{Q} = \widetilde{Q}' \oplus \widetilde{T}$ for $\widetilde{Q}' = \Phi(Q')$. All in all, this will prove the theorem, since then

$$\Phi\big(W_{u,z^{*i}}(\varsigma)\big) \;=\; \mathrm{red}_{\widetilde{Q}_1}\Big(\widetilde{\vartheta}\big(\widetilde{\mu}_i(\vartheta(W_{u,z}(\tau)))\big)\Big)\,.$$

It remains to construct the maps ϑ and $\widetilde{\vartheta}$ such that indeed $\Phi(W') = \widetilde{W}'$ and $\widetilde{Q} = \widetilde{Q}' \oplus \widetilde{T}$. Tables 5.A.12 to 5.A.57 give instructions how to do this. Each table has the following form:

X	W				Q
φ	φ_1	$\rightsquigarrow$	$\cdots$	$\rightsquigarrow$	$\varphi_{\ell(\varphi)}$
ψ	ψ_1	$\rightsquigarrow$	$\cdots$	$\rightsquigarrow$	$\psi_{\ell(\psi)}$
T_1	$\dots$				
$\widetilde{\varphi}$	$\widetilde{\varphi}_1$	$\rightsquigarrow$	$\cdots$	$\rightsquigarrow$	$\widetilde{\varphi}_{\ell(\widetilde{\varphi})}$
$\widetilde{\psi}$	$\widetilde{\psi}_1$	$\rightsquigarrow$	$\cdots$	$\rightsquigarrow$	$\widetilde{\psi}_{\ell(\widetilde{\psi})}$
$\widetilde{T}_1$	$\dots$				
Y	$\widetilde{W}'$				$\widetilde{Q}'$

To begin with, locate the table where X is the arctype of i in τ and Y the arctype of j in ς. If you don't find any such table, or, if $\{X, Y\} = \{\{A, A\}, \{A, A\}\}$ and W is not of the form shown in the first row of Table 5.A.28, swap τ, i, z for ς, j, z^{*i} and try again.

The quivers Q and $\widetilde{Q}'$ as well as the potential W are shown in the table at the indicated places. The boxed vertex in the depiction of Q and $\widetilde{Q}'$ is i.

It is not hard to see that W has the form appearing in the table. The subscripted letters ω represent elements in $H(\tau) \cap \widetilde{H} \cap H'$. They are used as placeholders in summands of W and $\widetilde{W}'$ induced by punctures.

Define $\vartheta = \psi \circ \varphi$ and $\widetilde{\vartheta} = \widetilde{\psi} \circ \widetilde{\varphi}$ where $f = f_{\ell(f)} \circ \cdots \circ f_1$ for $f \in \{\varphi, \psi, \widetilde{\varphi}, \widetilde{\psi}\}$. If the table does not contain a row labeled f, take for f the identity. The maps f_r are represented in the tables by their defining substitution rules $a \mapsto f_r(a)$. The notation $a \rightsquigarrow \nu_a$ is used as an abbreviation for the rule $a \mapsto a + \nu_a$. Sometimes the letters x_h and y_h are used for the scalars in K that appear in the decomposition $z_h = x_h + y_h v$ (see Remark 5.2.25).

To check that $\widetilde{W'}$ has the form written down in the table requires straightforward but in many cases lengthy computation. The identity $\Phi(W') = \widetilde{W'}$ is then obvious in all cases. This concludes the proof. $\square$

Remark 5.4.2. Similar computations as those compiled in Tables 5.A.12 to 5.A.57 were carried out in [Lab09a; Lab09b; CL12; Lab16; GL16a]. Some tables treat triangulations of spheres $\mathbf{\Sigma}$ with $m + o = 4$ not considered in the articles just mentioned.

5.4.2 Compatibility for Tagged Triangulations

Notation 5.4.3. For tagged triangulations $\boldsymbol{\tau}$ and sign functions ε for $\boldsymbol{\tau}$ denote by $\boldsymbol{i}^{\boldsymbol{\tau},\varepsilon}$ the preimage of $\boldsymbol{i} \in \boldsymbol{\tau}$ under the bijection $\iota_{\boldsymbol{\tau},\varepsilon}$ from Remark 5.2.22.

Corollary 5.4.4. *Let $\boldsymbol{\tau}$ be a tagged triangulation of $\mathbf{\Sigma}$ and ε a sign function for $\boldsymbol{\tau}$. Then for all $\boldsymbol{i} \in \boldsymbol{\tau}$ such that $\boldsymbol{i}^{\boldsymbol{\tau},\varepsilon}$ is non-folded in $\boldsymbol{\tau}^\flat$ we have $\mathcal{S}^\varepsilon_{u,z^{*\boldsymbol{i}}}(\mu_{\boldsymbol{i}}(\boldsymbol{\tau})) \approx_R \mu_{\boldsymbol{i}}(\mathcal{S}^\varepsilon_{u,z}(\boldsymbol{\tau}))$.*

Proof. Let $i = \boldsymbol{i}^{\boldsymbol{\tau},\varepsilon}$, $\tau = \boldsymbol{\tau}^\flat$, $\varsigma = \mu_i(\tau)$. Then $\mathcal{S}_{\varepsilon\cdot u,z^{*i}}(\varsigma) \approx_R \mu_i(\mathcal{S}_{\varepsilon\cdot u,z}(\tau))$ by Theorem 5.4.1. Now use $\boldsymbol{\varsigma} := \mu_{\boldsymbol{i}}(\boldsymbol{\tau}) = (\mu_i(\tau))^\varepsilon$, $W^\varepsilon_{u,z}(\boldsymbol{\tau}) = \iota_{\boldsymbol{\tau},\varepsilon}(W_{\varepsilon\cdot u,z}(\tau))$, $W^\varepsilon_{u,z^{*\boldsymbol{i}}}(\boldsymbol{\varsigma}) = \iota_{\boldsymbol{\varsigma},\varepsilon}(W_{\varepsilon\cdot u,z^{*i}}(\varsigma))$. $\square$

The next proposition will be used to derive the compatibility of flip and mutation for tagged triangulations from Corollary 5.4.4. Its proof is taken with suitable adaptations from [Lab16, Proposition 6.4] and [GL16a, Proposition 6.4].

Proposition 5.4.5. *Assume $g > 0$ or $b + m + o \geq 7$. For every tagged triangulation $\boldsymbol{\tau}$ of $\mathbf{\Sigma}$ and all sign functions ε and ε' for $\boldsymbol{\tau}$ we have $\mathcal{S}^\varepsilon_{u,z}(\boldsymbol{\tau}) \sim_R \mathcal{S}^{\varepsilon'}_{u,z}(\boldsymbol{\tau})$.*

Proof. We can reduce to the case in which ε and ε' take the same value at all but one $y \in \mathbb{P}$. Without loss of generality $\varepsilon(y) = +1$ and $\varepsilon'(y) = -1$. The bijection in Definition 5.1.40 shows that there is an arc k in $\tau = \boldsymbol{\tau}^\flat$ enclosing y. In particular, $m \geq 2$.

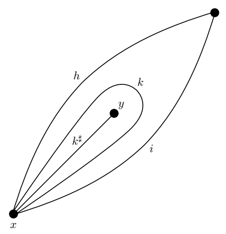
We can replace $\boldsymbol{\tau}$ by any tagged triangulation $\boldsymbol{\varsigma} = \varsigma^\varepsilon = \varsigma^{\varepsilon'}$ where ς is a triangulation of $\mathbf{\Sigma}$ containing k and $k^\sharp$:

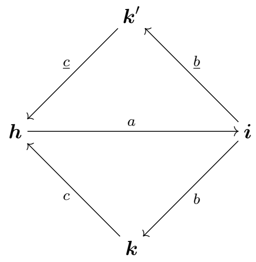
Indeed, $\tau = \mu_{i_\ell} \cdots \mu_{i_1}(\varsigma)$ for arcs $i_1, \ldots, i_\ell$ all different from k and $k^\sharp$ by Proposition 5.1.32. Since $\boldsymbol{i}_s := i^\varepsilon_s = i^{\varepsilon'}_s$ for all s, it is $\boldsymbol{\tau} = \mu_{\boldsymbol{i}_\ell} \cdots \mu_{\boldsymbol{i}_1}(\boldsymbol{\varsigma})$ by Corollary 5.1.44. With Corollary 5.4.4 and Theorem 2.6.101 we see $\big(\forall z : \mathcal{S}^\varepsilon_{u,z}(\boldsymbol{\tau}) \sim_R \mathcal{S}^{\varepsilon'}_{u,z}(\boldsymbol{\tau})\big) \Leftrightarrow \big(\forall z : \mathcal{S}^\varepsilon_{u,z}(\boldsymbol{\varsigma}) \sim_R \mathcal{S}^{\varepsilon'}_{u,z}(\boldsymbol{\varsigma})\big)$.

Abbreviate $Q = Q(\boldsymbol{\varsigma})$ and $H = H(\boldsymbol{\varsigma})$ and $\widehat{H} = \widehat{H}(\boldsymbol{\varsigma})$.

Let x be the basepoint of the self-folded triangle with sides k and $k^\sharp$. Set $\boldsymbol{k} := k^\varepsilon = (k^\sharp)^{\varepsilon'}$ and $\boldsymbol{k}' := k^{\varepsilon'} = (k^\sharp)^\varepsilon$.

If $x \in \partial\Sigma$, take for ς a triangulation with a puzzle piece ✣ of type B containing k and $k^\sharp$. The puzzle piece ✣ and the subquiver $Q^{✣}$ of Q look as follows:





The subpotentials of $W^{\varepsilon}_{u,z}(\varsigma)$ and $W^{\varepsilon'}_{u,z}(\varsigma)$ spanned by $\{c, \underline{c}\}$ are, respectively,

$$\begin{aligned} W &= \big(cb - \tfrac{1}{u_k}\underline{cb}\big)a\,, \\ W' &= \big(\tfrac{1}{u_k}cb + \underline{cb}\big)a\,. \end{aligned}$$

Hence, $W^{\varepsilon'}_{u,z}(\varsigma) = \varphi(W^{\varepsilon}_{u,z}(\varsigma))$ for $\varphi \in \mathrm{Aut}_{Q-\{c,\underline{c}\}}(H)$ given by the rules $c \mapsto \frac{1}{u_k}c$, $\underline{c} \mapsto -u_k\underline{c}$. This concludes the proof for the case $x \in \partial\Sigma$.

We now consider the case $x \in \mathbb{P}$. Then $\boldsymbol{\Sigma}$ admits a triangulation ς containing three puzzle pieces $🧩_1$, $🧩_2$, $🧩_3$ sharing arcs as depicted below:

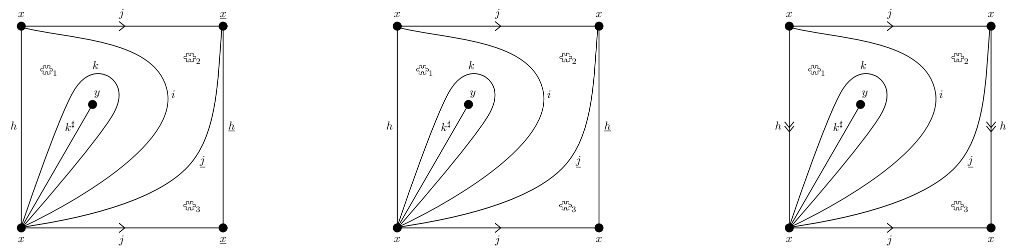
(a) For $g = 1$, $b = 0$, $m = 2$, $o = 0$ (twice-punctured closed torus without orbifold points) such a triangulation ς is shown on the right.

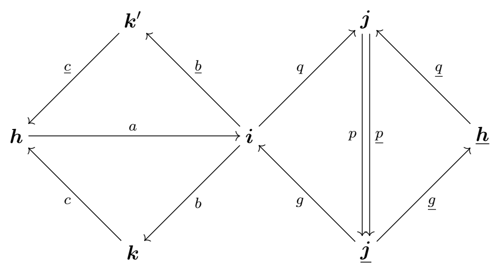
(b) For $g \geq 1$, $g + b + m + o > 3$ (g-fold torus) such a triangulation ς can be easily constructed in the model described in Remark 5.1.3 and Example 5.1.6.

(c) For $g = 0$ (a sphere with boundary) we have by assumption $b + m + o \geq 7$ and the existence of such a triangulation ς is easy to verify.

The subquiver Q' of Q induced by $🧩_1$, $🧩_2$, $🧩_3$ is visualized below (in case (a) with the identification $\underline{h} = h$).

$g = 0$	$g \geq 1$	$g = 1$
$b + m + o \geq 7$	$g + b + m + o > 3$	$b = 0,\ m = 2,\ o = 0$





The subpotentials of $W^{\varepsilon}_{u,z}(\boldsymbol{\varsigma})$ and $W^{\varepsilon'}_{u,z}(\boldsymbol{\varsigma})$ spanned by Q_1' have the respective form

$$\begin{aligned} W &= \big(cb - \tfrac{1}{u_k}\underline{cb}\big)a + gpq + \underline{gpq} + \omega(\underline{g}p\underline{q}) + \underline{cb}(g\underline{p}q)a\alpha\,,\\ W' &= \big(\tfrac{1}{u_k}cb + \underline{cb}\big)a + gpq + \underline{gpq} + \omega(\underline{g}p\underline{q}) + cb(g\underline{p}q)a\alpha\,, \end{aligned}$$

with $\alpha = \underline{\omega} + \underline{\nu}(\underline{g}p\underline{q})\nu$ for some $\omega, \underline{\omega}, \nu, \underline{\nu} \in H|_{Q-Q_1'}$ where $H|_{Q-Q_1'}$ is the path algebra of the submodulation of $\mathcal{H}(\varsigma)$ induced by $Q - Q_1' \subseteq Q$.

The rules $a \mapsto a + u_k(g\underline{p}q)a\alpha$, $c \mapsto \frac{1}{u_k}c$, $\underline{c} \mapsto -u_k\underline{c}$ define an element $\varphi_0 \in \mathrm{Aut}_{Q-Q_1'}(H)$ that maps W to

$$W_0 \;=\; W' - g(\lambda g\underline{p})q$$

for $\lambda = \underline{p}qa(u_k\alpha)^2\underline{cb}$. The unitriangular automorphism $\varphi_1 \in \mathrm{Aut}_{Q-Q_1'}(H)$ defined by the substitution rule $p \mapsto p + \lambda g\underline{p}$ maps W_0 to a potential of the form

$$W_1 \;=\; W' + \gamma_1(g\underline{p}q) + \delta_1(gp\underline{q})$$

for some $\gamma_1, \delta_1 \in H$ with $\mathrm{ord}(\gamma_1) > 0$ (and $\delta_1 = 0$).

The subpotential $(W')^{\{q,\underline{g}\}}$ has the form $gpq + \underline{gpq} + \tilde{\omega}(\underline{g}p\underline{q}) + \tilde{\nu}(g\underline{p}q) + \hat{\omega}(\underline{g}p\underline{q})\hat{\nu}(g\underline{p}q)$ for some $\tilde{\omega}, \hat{\omega}, \tilde{\nu}, \hat{\nu} \in H|_{Q-\{q,\underline{g}\}}$ with $\mathrm{ord}(\tilde{\nu}) > 0$. Given, for some $r \in \mathbb{N}_+$,

$$W_r \;=\; W' + \gamma_r(g\underline{p}q) + \delta_r(gp\underline{q})$$

with $\gamma_r, \delta_r \in H$, it is thus straightforward to verify that the element $\varphi_{r+1} \in \mathrm{Aut}_{Q-Q_1'}(H)$ defined by the substitutions $\underline{g} \mapsto \underline{g} - \gamma_r g$, $q \mapsto q - \underline{q}\delta_r$ maps W_r to a potential of the form

$$W_{r+1} \;=\; W' + \gamma_{r+1}(g\underline{p}q) + \delta_{r+1}(gp\underline{q})$$

where $\gamma_{r+1} = (\varphi_r(\gamma_r) - \gamma_r) + \delta_r\lambda_r$ and $\delta_{r+1} = (\varphi_r(\delta_r) - \delta_r) + \gamma_r\eta_r$ for some $\lambda_r, \eta_r \in H$ with $\mathrm{ord}(\lambda_r) > 0$. In particular, the two inequalities $\mathrm{ord}(\gamma_{r+1}) > \min(\mathrm{ord}(\gamma_r), \mathrm{ord}(\delta_r))$ and $\mathrm{ord}(\delta_{r+1}) \geq \min(\mathrm{ord}(\gamma_r), \mathrm{ord}(\delta_r))$ hold.

By induction we get a sequence $(\varphi_r)_{r\in\mathbb{N}_+}$ of unitriangular elements in $\mathrm{Aut}_{Q-Q_1'}(H)$ such that $\lim_{r\to\infty}\mathrm{depth}(\varphi_r) = \infty$ and the sequence $(W_r)_{r\in\mathbb{N}}$ of potentials $W_r = \widetilde{\varphi}_r(W)$ with $\widetilde{\varphi}_r = \varphi_r \cdots \varphi_1\varphi_0$ converges to W'. Therefore we have $W^{\varepsilon'}_{u,z}(\boldsymbol{\varsigma}) = \widetilde{\varphi}(W^{\varepsilon}_{u,z}(\boldsymbol{\varsigma}))$ for the automorphism $\widetilde{\varphi} = \lim_{r\to\infty}\widetilde{\varphi}_r \in \mathrm{Aut}_{Q-Q_1'}(\widehat{H})$. This proves the proposition. □

Finally, we can state the main result of this chapter:

Theorem 5.4.6. *Assume $g > 0$ or $b + m + o \geq 7$. Let $\boldsymbol{\tau}$ be a tagged triangulation of $\boldsymbol{\Sigma}$. For all $\boldsymbol{i} \in \boldsymbol{\tau}$ it is $\mathcal{S}_{u,z^{*\boldsymbol{i}}}(\mu_{\boldsymbol{i}}(\boldsymbol{\tau})) \approx_R \mu_{\boldsymbol{i}}(\mathcal{S}_{u,z}(\boldsymbol{\tau}))$. In particular, $\mathcal{S}_{u,z}(\mu_{\boldsymbol{i}}(\boldsymbol{\tau})) \approx_{\boldsymbol{\tau}} \mu_{\boldsymbol{i}}(\mathcal{S}_{u,z}(\boldsymbol{\tau}))$.*

Proof. The last claim follows from the first one by Lemma 5.3.16.

Let $\boldsymbol{\varsigma} = \mu_{\boldsymbol{i}}(\boldsymbol{\tau})$, $\boldsymbol{\tau} = \mu_{\boldsymbol{j}}(\boldsymbol{\tau})$ and $\tau = \boldsymbol{\tau}^\flat$, $\varsigma = \boldsymbol{\varsigma}^\flat$ and $i = \boldsymbol{i}^{\boldsymbol{\tau},\varepsilon_{\boldsymbol{\tau}}}$, $j = \boldsymbol{j}^{\boldsymbol{\varsigma},\varepsilon_{\boldsymbol{\varsigma}}}$.

After possibly swapping $\boldsymbol{i}$, $\boldsymbol{\tau}$, z for $\boldsymbol{j}$, $\boldsymbol{\varsigma}$, $z^{*\boldsymbol{i}}$, we can assume that j is non-folded in ς, since $\mathcal{S}_{u,z^{*\boldsymbol{i}}}(\boldsymbol{\varsigma}) \approx_R \mu_{\boldsymbol{i}}(\mathcal{S}_{u,z}(\boldsymbol{\tau})) \Leftrightarrow \mathcal{S}_{u,z}(\boldsymbol{\tau}) \approx_R \mu_{\boldsymbol{j}}(\mathcal{S}_{u,z^{*\boldsymbol{i}}}(\boldsymbol{\varsigma}))$ by Theorem 2.6.101.

If i is a non-folded arc in τ, $\varepsilon_{\boldsymbol{\tau}} = \varepsilon_{\boldsymbol{\varsigma}}$ and Corollary 5.4.4 yields $\mathcal{S}_{u,z^{*\boldsymbol{i}}}(\boldsymbol{\varsigma}) \approx_R \mu_{\boldsymbol{i}}(\mathcal{S}_{u,z}(\boldsymbol{\tau}))$.

If i is folded in τ, $\boldsymbol{i}^{\boldsymbol{\tau},\varepsilon_{\varsigma}}$ is non-folded in τ and $\mathcal{S}_{u,z^{*\boldsymbol{i}}}(\boldsymbol{\varsigma}) \approx_R \mu_{\boldsymbol{i}}(\mathcal{S}^{\varepsilon_{\varsigma}}_{u,z}(\boldsymbol{\tau}))$ by Corollary 5.4.4. With Proposition 5.4.5 and Theorem 2.6.100 we have $\mu_{\boldsymbol{i}}(\mathcal{S}^{\varepsilon_{\varsigma}}_{u,z}(\boldsymbol{\tau})) \approx_R \mu_{\boldsymbol{i}}(\mathcal{S}_{u,z}(\boldsymbol{\tau}))$. □

Corollary 5.4.7. *Assume $g > 0$ or $b + m + o \geq 7$. Then $\mathcal{S}_{u,z}(\boldsymbol{\tau})$ is non-degenerate for every tagged triangulation $\boldsymbol{\tau}$ of $\boldsymbol{\Sigma}$.*

Proof. This is a direct consequence of Theorem 5.4.6 and Lemma 5.3.8. □

5.5 Uniqueness of Potentials

We conclude with the observation that the equivalence class of $\mathcal{S}_{u,z}(\boldsymbol{\tau})$ does not depend on the particular choice of u and z, if $b > 0$ or $p = 0$.

5.5.1 Non-Closed Orbifolds

The next statement is a variant of [Lab16, Proposition 10.2] and [GL16a, Proposition 8.6].

Corollary 5.5.1. *Assume $b > 0$. Then $\mathcal{S}_{u,z}(\tau) \sim_\tau \mathcal{S}_{u',z'}(\tau)$ for all (tagged) triangulations τ of $\boldsymbol{\Sigma}$ and all functions $u, u' : \mathbb{P} \to K^\times$ and $z, z' : \mathbb{O} \to L \setminus K$.*

Proof. We assume that τ is a triangulation. The proof for tagged triangulations is similar.

We clearly can reduce to the case in which $|\{y \in \mathbb{P} \,|\, u_y \neq u'_y\}| + |\{\times \in \mathbb{O} \,|\, z_\times \neq z'_\times\}| = 1$. It suffices to prove $\mathcal{S}_{u,z}(\varsigma) \sim_\varsigma \mathcal{S}_{u',z'}(\varsigma)$ for any triangulation ς of $\boldsymbol{\Sigma}$ by Proposition 5.1.32 and Theorems 2.6.101 and 5.4.1.

Let us first assume that u and u' take the same value at all but one puncture $y \in \mathbb{P}$. Then $m \geq 2$. Let ς be a triangulation containing a puzzle piece ✾ of type B such that the basepoint x of the self-folded triangle in ✾ lies on the boundary. The subquiver $Q^{✾}$ of $Q = Q(\varsigma)$ is the left quiver shown in Figure 5.A.3. By definition

$$(W_{u,z}(\varsigma))^{\{c_1\}} \;=\; -\tfrac{1}{u_y} c_1 b_1 a\,, \qquad\qquad (W_{u',z}(\varsigma))^{\{c_1\}} \;=\; -\tfrac{1}{u'_y} c_1 b_1 a\,.$$

Hence, $W_{u',z}(\varsigma) = \varphi(W_{u,z}(\varsigma))$ for $\varphi \in \mathrm{Aut}_{Q-\{c_1\}}(H(\tau))$ defined by $c_1 \mapsto \frac{u_y}{u'_y} c_1$.

Let us now assume that z and z' take the same value at all but one orbifold point $\times \in \mathbb{O}$. Let ς be a triangulation containing a puzzle piece ✾ of type $\widetilde{B}$ such that the marked point on its pending arc belongs to the boundary. The subquiver $Q^{✾}$ is the right quiver shown in Figure 5.A.3. By definition

$$(W_{u,z}(\varsigma))^{\{c\}} \;=\; c z_\times b a\,, \qquad\qquad (W_{u,z'}(\varsigma))^{\{c\}} \;=\; c z'_\times b a\,.$$

Hence, $W_{u,z'}(\varsigma) = \varphi(W_{u,z}(\varsigma))$ for $\varphi \in \mathrm{Aut}_{Q-\{c\}}(H(\tau))$ defined by $c \mapsto c z'_\times z_\times^{-1}$. □

5.5.2 Unpunctured Orbifolds

Unpunctured orbifolds will be treated in greater detail in the next chapter. We anticipate the next theorem for its relevance in the current context.

Notation 5.5.2. If $p = 0$, the function u is empty and we define $\mathcal{S}_z(\tau) = \mathcal{S}_{u,z}(\tau)$.

Theorem 5.5.3. *Assume $p = 0$ and $\mathbf{\Sigma}$ is not a monogon. Every non-degenerate potential for $A(\tau)$ is $R(\tau)$-equivalent to $W_z(\tau)$. In particular, $\mathcal{S}_z(\tau) \sim_{R(\tau)} \mathcal{S}_{z'}(\tau)$ for all z, z'.*

Proof. This is similar to the proof of Theorem 6.6.8 and uses Convention 5.2.23. □

5.A Appendix

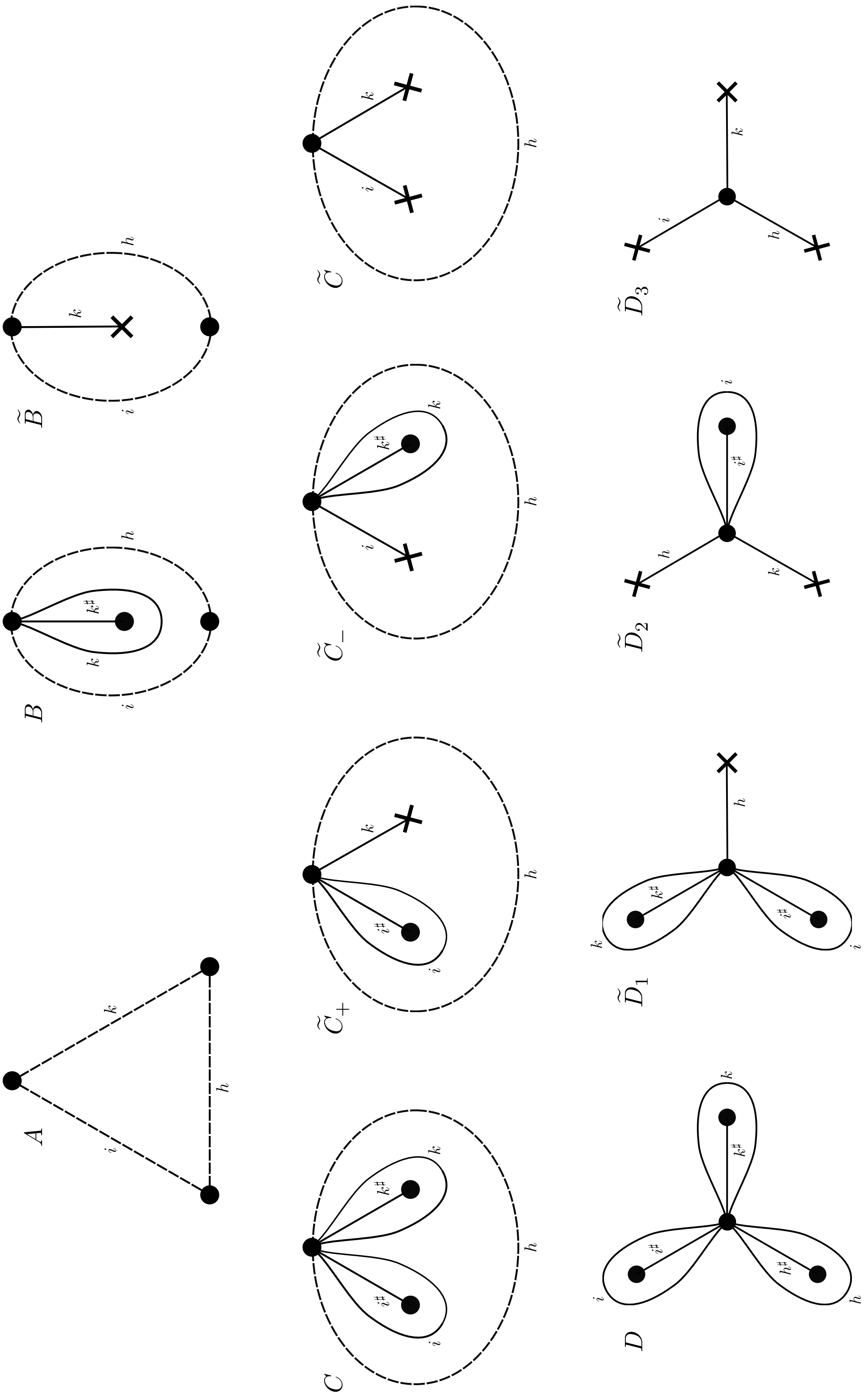


Figure 5.A.1: the eleven puzzle pieces; outer sides dashed.

$$\begin{tikzcd} i \arrow[rr, "b"] & & k \arrow[dl, "c"] \\ & h \arrow[ul, "a"] & \end{tikzcd}$$

$$cba$$

Figure 5.A.2: $Q^{\maltese}$ and $W^{\maltese}_{u,z}$ for pieces of type A.

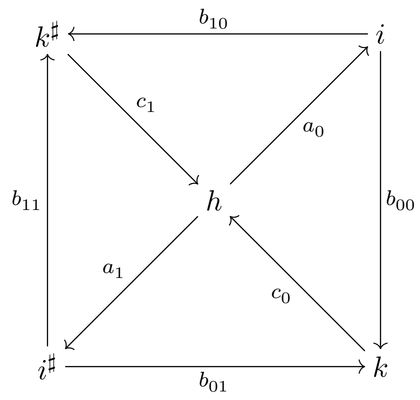
$$\begin{tikzcd} & i \arrow[dl, "b_1"'] \arrow[dr, "b_0"] & \\ k^{\sharp} \arrow[dr, "c_1"'] & & k \arrow[dl, "c_0"] \\ & h \arrow[uu, "a"] & \end{tikzcd}$$

$$c_0 b_0 a - \frac{1}{u_k} c_1 b_1 a$$

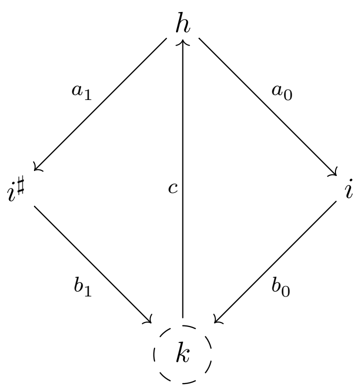
$$\begin{tikzcd} i \arrow[rr, "b"] & & k \arrow[dl, "c"] \\ & h \arrow[ul, "a"] & \end{tikzcd}$$

$$c z_k b a$$

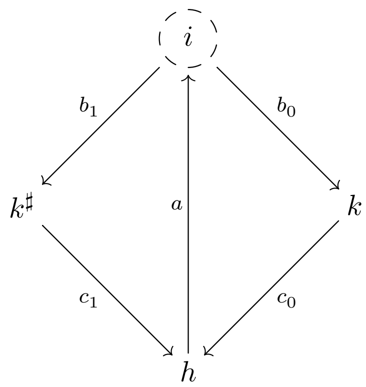
Figure 5.A.3: $Q^{\maltese}$ and $W^{\maltese}_{u,z}$ for pieces of type B (left) and $\widetilde{B}$ (right).



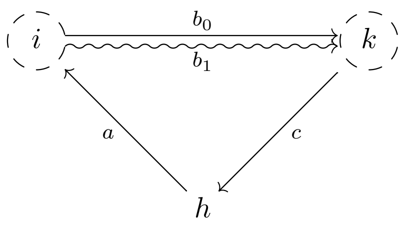
$$c_0b_{00}a_0 - \tfrac{1}{u_i}c_0b_{01}a_1 - \tfrac{1}{u_k}c_1b_{10}a_0 + \tfrac{1}{u_ku_i}c_1b_{11}a_1$$



$$cz_kb_0a_0 - \tfrac{1}{u_i}cz_kb_1a_1$$

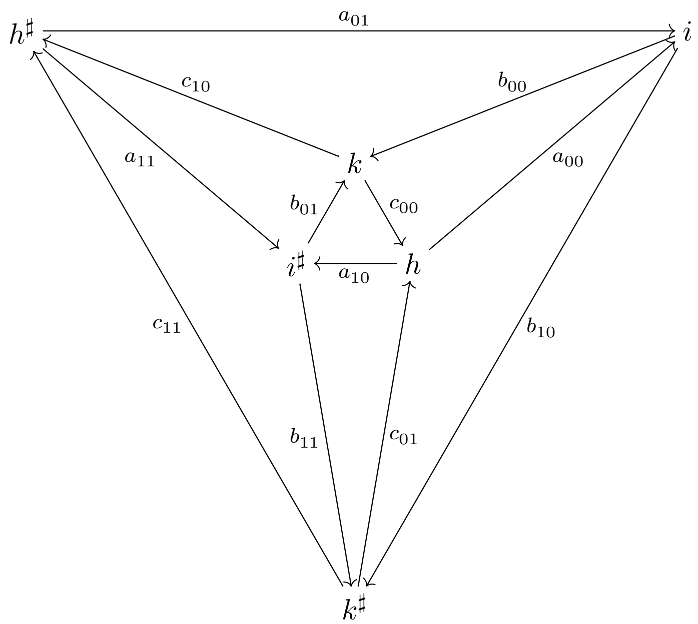


$$c_0b_0z_ia - \tfrac{1}{u_k}c_1b_1z_ia$$



$$cz_k(b_0+b_1)z_ia$$

Figure 5.A.4: $Q^{✿}$ and $W^{✿}_{u,z}$ for pieces of type C (top), $\widetilde{C}_+$ (left), $\widetilde{C}_-$ (right), $\widetilde{C}$ (bottom).



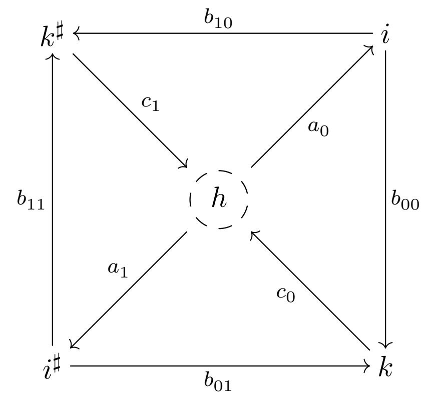
$$c_{00}b_{00}a_{00}$$

$$-\tfrac{1}{u_i}c_{00}b_{01}a_{10} - \tfrac{1}{u_k}c_{01}b_{10}a_{00} - \tfrac{1}{u_h}c_{10}b_{00}a_{01}$$

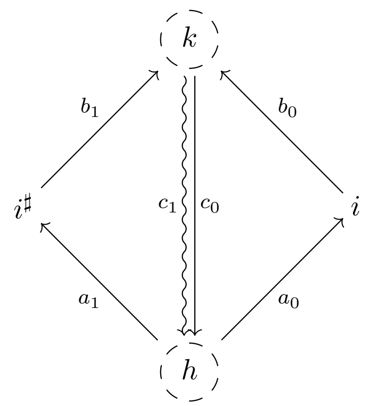
$$+\tfrac{1}{u_k u_h}c_{11}b_{10}a_{01} + \tfrac{1}{u_i u_h}c_{10}b_{01}a_{11} + \tfrac{1}{u_k u_i}c_{01}b_{11}a_{10}$$

$$-\tfrac{1}{u_k u_i u_h}c_{11}b_{11}a_{11}$$

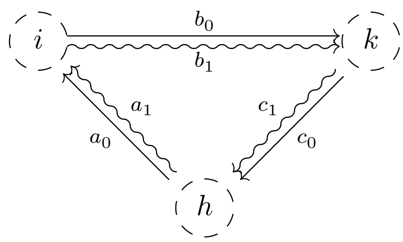
Figure 5.A.5: $Q^{\maltese}$ and $W^{\maltese}_{u,z}$ for pieces of type D.



$$c_0 b_{00} a_0 z_h - \frac{1}{u_i} c_0 b_{01} a_1 z_h - \frac{1}{u_k} c_1 b_{10} a_0 z_h + \frac{1}{u_k u_i} c_1 b_{11} a_1 z_h$$



$$z_h(c_0 + c_1) z_k b_0 a_0 - \frac{1}{u_i} z_h (c_0 + c_1) z_k b_1 a_1$$



$$(c_0 + c_1) z_k (b_0 + b_1) z_i (a_0 + a_1) z_h$$

Figure 5.A.6: $Q^{\maltese}$ and $W^{\maltese}_{u,z}$ for pieces of type $\widetilde{D}_1$ (top), $\widetilde{D}_2$ (middle), $\widetilde{D}_3$ (bottom).

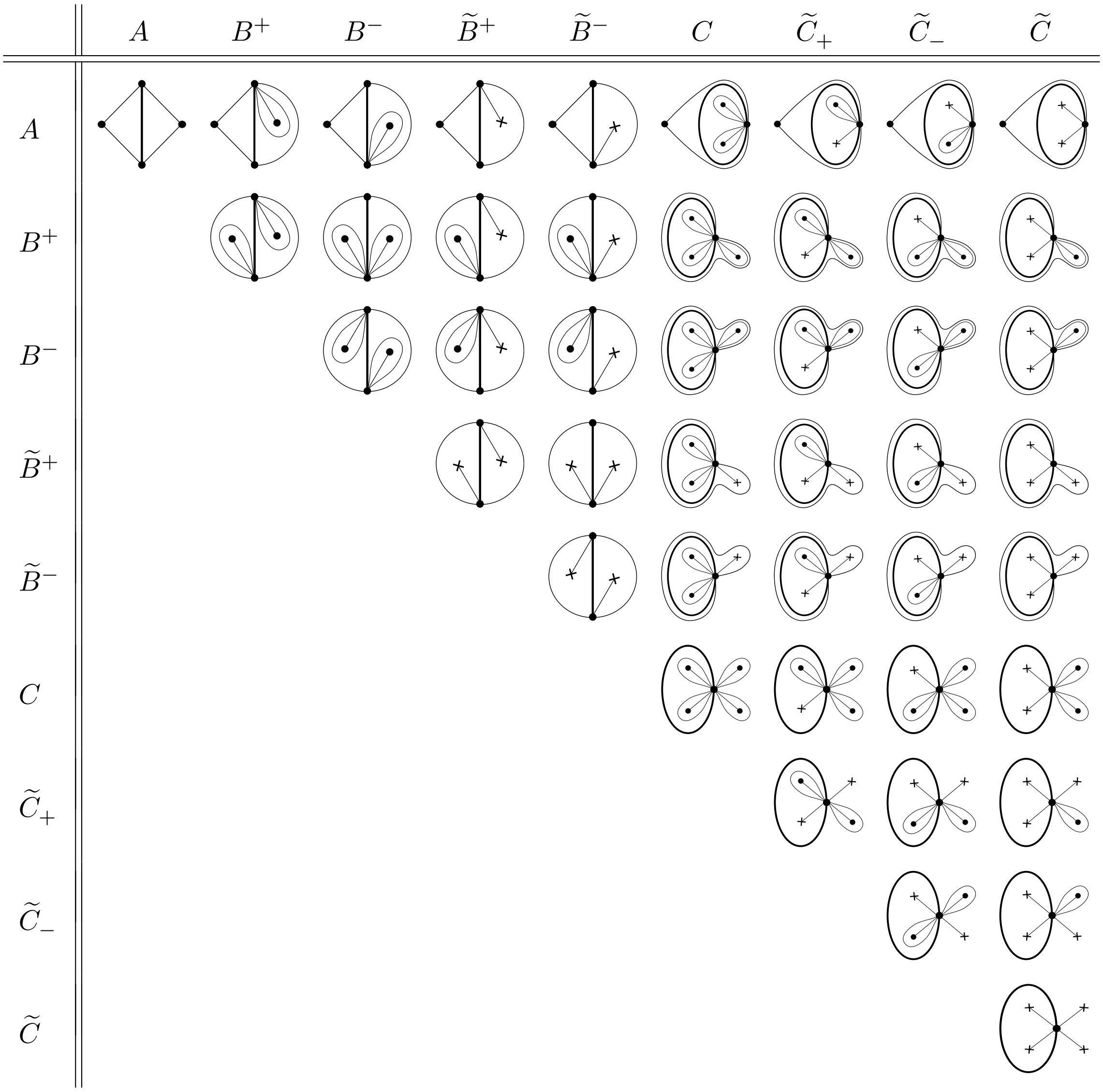


Table 5.A.7: two puzzle pieces sharing exactly one arc (bold).

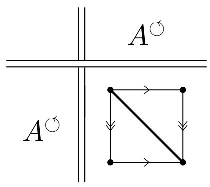


Table 5.A.8: two puzzle pieces sharing exactly three arcs; one shared arc (bold) fixed.

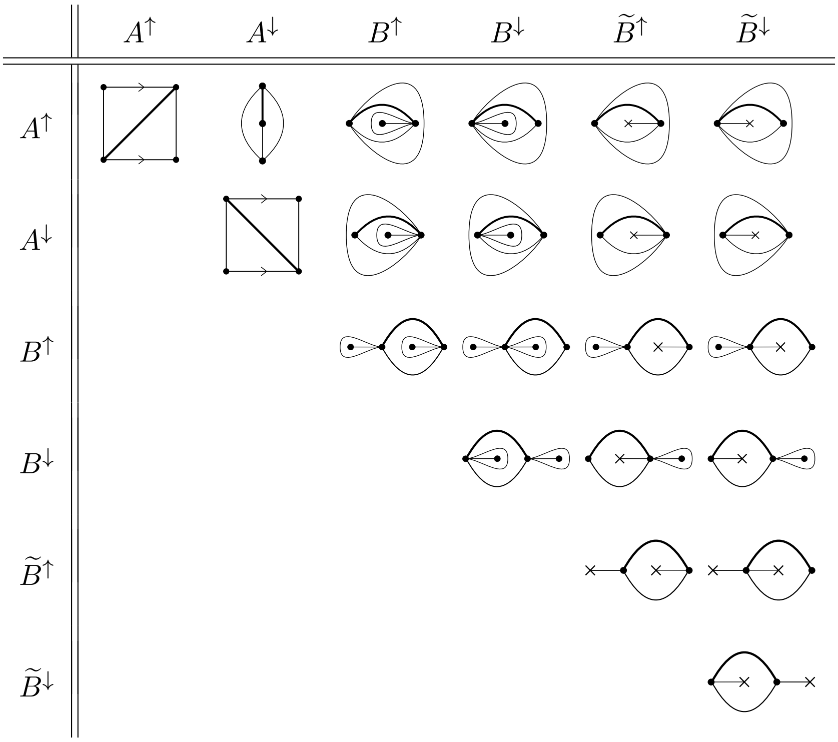


Table 5.A.9: two puzzle pieces sharing exactly two arcs;
one shared arc (bold) fixed.

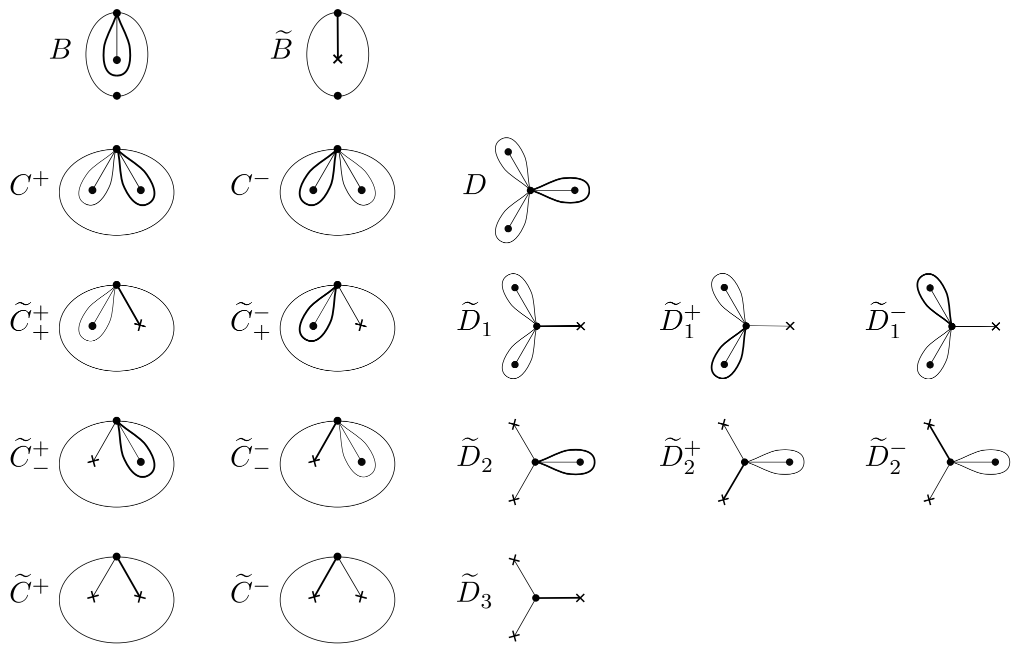


Table 5.A.10: puzzle piece with one non-folded inner side (bold) fixed.

X	Y	Table
$\{A^{\circlearrowleft}, A^{\circlearrowleft}\}$	$\{A^{\circlearrowleft}, A^{\circlearrowleft}\}$	5.A.12

X	Y	Table
$\{A^{\downarrow}, A^{\uparrow}\}$	B	5.A.19
$\{A^{\downarrow}, B^{\uparrow}\}$	C^{+}	5.A.20
$\{A^{\uparrow}, B^{\downarrow}\}$	C^{-}	5.A.21
$\{A^{\downarrow}, \widetilde{B}^{\uparrow}\}$	$\widetilde{C}^{+}_{-}$	5.A.22
$\{A^{\uparrow}, \widetilde{B}^{\downarrow}\}$	$\widetilde{C}^{-}_{+}$	5.A.23
$\{B^{\downarrow}, B^{\uparrow}\}$	D	5.A.24
$\{B^{\downarrow}, \widetilde{B}^{\uparrow}\}$	$\widetilde{D}^{-}_{1}$	5.A.25
$\{B^{\uparrow}, \widetilde{B}^{\downarrow}\}$	$\widetilde{D}^{+}_{1}$	5.A.26
$\{\widetilde{B}^{\downarrow}, \widetilde{B}^{\uparrow}\}$	$\widetilde{D}_{2}$	5.A.27

X	Y	Table
$\{A^{\downarrow}, A^{\downarrow}\}$	$\{A^{\uparrow}, A^{\uparrow}\}$	5.A.13
$\{A^{\downarrow}, B^{\downarrow}\}$	$\{A^{\uparrow}, B^{\uparrow}\}$	5.A.14
$\{A^{\downarrow}, \widetilde{B}^{\downarrow}\}$	$\{A^{\uparrow}, \widetilde{B}^{\uparrow}\}$	5.A.15
$\{B^{\downarrow}, B^{\downarrow}\}$	$\{B^{\uparrow}, B^{\uparrow}\}$	5.A.16
$\{B^{\downarrow}, \widetilde{B}^{\downarrow}\}$	$\{B^{\uparrow}, \widetilde{B}^{\uparrow}\}$	5.A.17
$\{\widetilde{B}^{\downarrow}, \widetilde{B}^{\downarrow}\}$	$\{\widetilde{B}^{\uparrow}, \widetilde{B}^{\uparrow}\}$	5.A.18

X	Y	Table
$\{A, A\}$	$\{A, A\}$	5.A.28
$\{A, B^{+}\}$	$\{A, B^{-}\}$	5.A.29
$\{A, \widetilde{B}^{+}\}$	$\{A, \widetilde{B}^{-}\}$	5.A.30
$\{A, C\}$	$\{B^{+}, B^{-}\}$	5.A.31
$\{A, \widetilde{C}_{+}\}$	$\{B^{-}, \widetilde{B}^{+}\}$	5.A.32
$\{A, \widetilde{C}_{-}\}$	$\{B^{+}, \widetilde{B}^{-}\}$	5.A.33
$\{A, \widetilde{C}\}$	$\{\widetilde{B}^{+}, \widetilde{B}^{-}\}$	5.A.34
$\{B^{+}, B^{+}\}$	$\{B^{-}, B^{-}\}$	5.A.35
$\{B^{+}, \widetilde{B}^{+}\}$	$\{B^{-}, \widetilde{B}^{-}\}$	5.A.36
$\{B^{+}, C\}$	$\{B^{-}, C\}$	5.A.37
$\{B^{+}, \widetilde{C}_{+}\}$	$\{B^{-}, \widetilde{C}_{-}\}$	5.A.38
$\{B^{+}, \widetilde{C}_{-}\}$	$\{\widetilde{B}^{-}, C\}$	5.A.39

X	Y	Table
$\{B^{+}, \widetilde{C}\}$	$\{\widetilde{B}^{-}, \widetilde{C}_{-}\}$	5.A.40
$\{B^{-}, \widetilde{C}_{+}\}$	$\{\widetilde{B}^{+}, C\}$	5.A.41
$\{B^{-}, \widetilde{C}\}$	$\{\widetilde{B}^{+}, \widetilde{C}_{+}\}$	5.A.42
$\{\widetilde{B}^{+}, \widetilde{B}^{+}\}$	$\{\widetilde{B}^{-}, \widetilde{B}^{-}\}$	5.A.43
$\{\widetilde{B}^{+}, \widetilde{C}_{-}\}$	$\{\widetilde{B}^{-}, \widetilde{C}_{+}\}$	5.A.44
$\{\widetilde{B}^{+}, \widetilde{C}\}$	$\{\widetilde{B}^{-}, \widetilde{C}\}$	5.A.45
$\{C, C\}$	$\{C, C\}$	5.A.46
$\{C, \widetilde{C}_{+}\}$	$\{C, \widetilde{C}_{-}\}$	5.A.47
$\{C, \widetilde{C}\}$	$\{\widetilde{C}_{+}, \widetilde{C}_{-}\}$	5.A.48
$\{\widetilde{C}_{+}, \widetilde{C}_{+}\}$	$\{\widetilde{C}_{-}, \widetilde{C}_{-}\}$	5.A.49
$\{\widetilde{C}_{+}, \widetilde{C}\}$	$\{\widetilde{C}_{-}, \widetilde{C}\}$	5.A.50
$\{\widetilde{C}, \widetilde{C}\}$	$\{\widetilde{C}, \widetilde{C}\}$	5.A.51

X	Y	Table
$\widetilde{B}$	$\widetilde{B}$	5.A.52
$\widetilde{C}^{+}_{+}$	$\widetilde{C}^{-}_{-}$	5.A.53
$\widetilde{C}^{-}$	$\widetilde{C}^{+}$	5.A.54
$\widetilde{D}_{1}$	$\widetilde{D}_{1}$	5.A.55
$\widetilde{D}^{-}_{2}$	$\widetilde{D}^{+}_{2}$	5.A.56
$\widetilde{D}_{3}$	$\widetilde{D}_{3}$	5.A.57

Table 5.A.11: all flippant pairs $\{X, Y\}$.

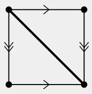
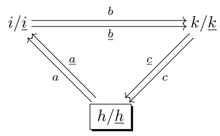
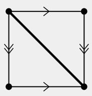
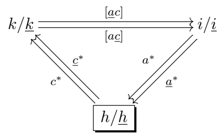
$\{A^{\circlearrowleft}, A^{\circlearrowleft}\}$ 	cba $+\ \underline{cba}$ $+\ u_0 c\underline{b}a\underline{c}b\underline{a}$	
$\widetilde{\psi}$	$b \rightsquigarrow -c^*a^*$ $\underline{b} \rightsquigarrow -\underline{c}^*\underline{a}^*$ $[\underline{ac}] \rightsquigarrow -u_0[a\underline{c}]b[\underline{a}c]$ $\rightsquigarrow$	$[ac] \rightsquigarrow u_0[\underline{a}c]\underline{c}^*\underline{a}^*[a\underline{c}]$ $[\underline{ac}] \rightsquigarrow u_0[a\underline{c}]c^*a^*[\underline{a}c]$
$\widetilde{T}_1$	$b \quad [ac] \quad \underline{b} \quad [\underline{ac}]$	
$\{A^{\circlearrowleft}, A^{\circlearrowleft}\}$ 	$\underline{a}^*[\underline{a}c]c^*$ $+\ a^*[a\underline{c}]\underline{c}^*$ $+\ u_0\underline{a}^*[ac]c^*a^*[\underline{a}c]\underline{c}^*$	

Table 5.A.12: flippant pair $\{\{A^{\circlearrowleft}, A^{\circlearrowleft}\}, \{A^{\circlearrowleft}, A^{\circlearrowleft}\}\}$.

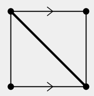
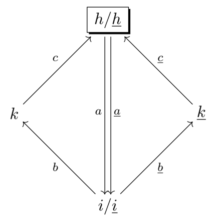
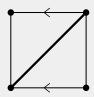
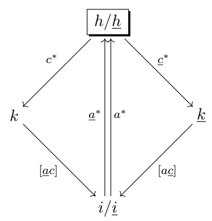
$\{A^{\downarrow}, A^{\downarrow}\}$ 	cba $+\ \underline{cba}$ $+\ \omega_{\underline{kk}}\underline{b}a\underline{c} + \omega_{kk}b\underline{a}c$ $+\ \omega_{\underline{k}k}b\underline{a}c\,\omega_{k\underline{k}}\underline{b}a\underline{c}$ 
$\widetilde{\varphi}$	$c^* \mapsto -c^*$ $\underline{a}^* \mapsto -\underline{a}^*$
$\widetilde{\psi}$	$b \rightsquigarrow c^*a^*$ $[ac] \rightsquigarrow -[\underline{a}c]\,\omega_{kk}$ $\underline{b} \rightsquigarrow \underline{c}^*\underline{a}^*$ $[\underline{ac}] \rightsquigarrow -[a\underline{c}]\,\omega_{\underline{kk}}$ $\quad -[a\underline{c}]\,\omega_{\underline{k}k}b[\underline{a}c]\,\omega_{k\underline{k}}$ $\rightsquigarrow$ $[ac] \rightsquigarrow -[\underline{a}c]\,\omega_{k\underline{k}}\underline{c}^*\underline{a}^*[a\underline{c}]\,\omega_{\underline{k}k}$ $[\underline{ac}] \rightsquigarrow -[a\underline{c}]\,\omega_{\underline{k}k}c^*a^*[\underline{a}c]\,\omega_{k\underline{k}}$
$\widetilde{T}_1$	$b \quad [ac] \quad \underline{b} \quad [\underline{ac}]$
$\{A^{\uparrow}, A^{\uparrow}\}$ 	$[\underline{a}c]c^*\underline{a}^*$ $+\ [a\underline{c}]\underline{c}^*a^*$ $+\ \omega_{\underline{kk}}\underline{c}^*\underline{a}^*[a\underline{c}] + \omega_{kk}c^*a^*[\underline{a}c]$ $+\ \omega_{\underline{k}k}c^*a^*[\underline{a}c]\,\omega_{k\underline{k}}\underline{c}^*\underline{a}^*[a\underline{c}]$ 

Table 5.A.13: flippant pair $\big\{\{A^{\downarrow}, A^{\downarrow}\}, \{A^{\uparrow}, A^{\uparrow}\}\big\}$.

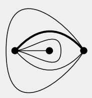
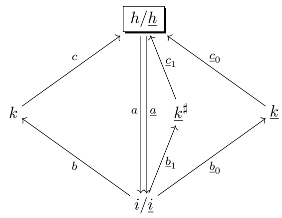
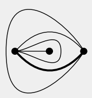
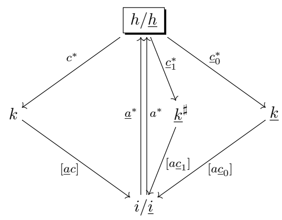
$\{A^{\downarrow}, B^{\downarrow}\}$ 	cba $+\ \underline{c}_0\underline{b}_0\underline{a} - \frac{1}{u_{\underline{k}}}\underline{c}_1\underline{b}_1\underline{a}$ $+\ u_0 a\underline{c}_1\underline{b}_1$ $+\ \omega_{kk} b\underline{a}c$ 
$\widetilde{\varphi}$	$[\underline{ac}_1] \mapsto u_0 u_{\underline{k}}[a\underline{c}_1] - u_{\underline{k}}[\underline{ac}_1] \quad \rightsquigarrow \quad [a\underline{c}_1] \mapsto -\frac{1}{u_{\underline{k}}}[a\underline{c}_1] \quad \rightsquigarrow \quad c^* \mapsto -c^*$, $\underline{a}^* \mapsto -\underline{a}^*$
$\widetilde{\psi}$	$b \rightsquigarrow c^*a^*$ $[ac] \rightsquigarrow -[\underline{a}c]\,\omega_{kk}$ $\underline{b}_0 \rightsquigarrow \underline{c}_0^*\underline{a}^*$ $\underline{b}_1 \rightsquigarrow -u_{\underline{k}}\underline{c}_1^*\underline{a}^*$
$\widetilde{T}_1$	$b \quad [ac] \quad \underline{b}_0 \quad [\underline{ac}_0] \quad \underline{b}_1 \quad [\underline{ac}_1]$
$\{A^{\uparrow}, B^{\uparrow}\}$ 	$[\underline{a}c]c^*\underline{a}^*$ $+\ [a\underline{c}_0]\underline{c}_0^*a^* - \frac{1}{u_{\underline{k}}}[a\underline{c}_1]\underline{c}_1^*a^*$ $+\ u_0\underline{a}^*[a\underline{c}_1]\underline{c}_1^*$ $+\ \omega_{kk}c^*a^*[\underline{a}c]$ 

Table 5.A.14: flippant pair $\{\{A^{\downarrow}, B^{\downarrow}\}, \{A^{\uparrow}, B^{\uparrow}\}\}$.

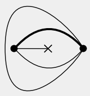
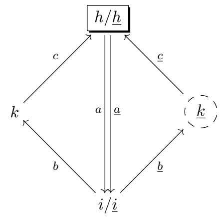
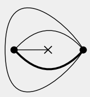
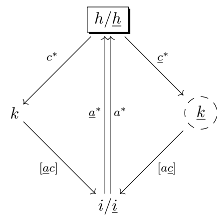
$\{A^{\downarrow}, \widetilde{B}^{\downarrow}\}$ 	cba $+\ \underline{c}z_{\underline{k}}\underline{b}\underline{a}$ $+\ u_0a\underline{c}\underline{b}$ $+\ \omega_{kk}b\underline{a}c$	
$\widetilde{\varphi}$	$[\underline{a}\underline{c}] \mapsto -u_0[a\underline{c}]z_{\underline{k}}^{-1} + [\underline{a}c]z_{\underline{k}}^{-1} \quad \rightsquigarrow \quad [a\underline{c}] \mapsto [a\underline{c}]z_{\underline{k}} \quad \rightsquigarrow$	$c^* \mapsto -c^*$ $\underline{a}^* \mapsto -\underline{a}^*$
$\widetilde{\psi}$	$b \rightsquigarrow c^*a^*$ $[ac] \rightsquigarrow -[\underline{a}c]\,\omega_{kk}$ $\underline{b} \rightsquigarrow z_{\underline{k}}^{-1}\underline{c}^*\underline{a}^*$	
$\widetilde{T}_1$	$b \quad [ac] \quad \underline{b} \quad [\underline{a}\underline{c}]$	
$\{A^{\uparrow}, \widetilde{B}^{\uparrow}\}$ 	$[\underline{a}c]c^*\underline{a}^*$ $+\ [a\underline{c}]z_{\underline{k}}\underline{c}^*a^*$ $+\ u_0\underline{a}^*[a\underline{c}]\underline{c}^*$ $+\ \omega_{kk}c^*a^*[\underline{a}c]$	

Table 5.A.15: flippant pair $\{\{A^{\downarrow}, \widetilde{B}^{\downarrow}\}, \{A^{\uparrow}, \widetilde{B}^{\uparrow}\}\}$.

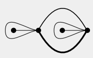
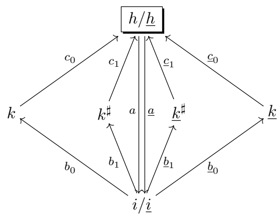
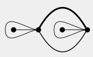
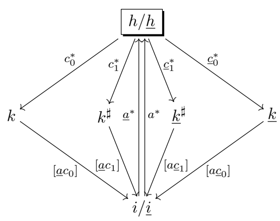
<table>
<tr><td>$\{B^{\downarrow}, B^{\downarrow}\}$
</td><td>$$\begin{aligned} &\quad c_0 b_0 a - \tfrac{1}{u_k} c_1 b_1 a \\ &+ \underline{c}_0 \underline{b}_0 \underline{a} - \tfrac{1}{u_{\underline{k}}} \underline{c}_1 \underline{b}_1 \underline{a} \\ &+ u_1 c_1 b_1 \underline{a} + u_0 a \underline{c}_1 \underline{b}_1 \end{aligned}$$
</td></tr>
<tr><td>$\widetilde{\varphi}$</td><td>$$\begin{aligned} [ac_1] &\mapsto u_1 u_k [\underline{a}c_1] - u_k [ac_1] \\ [\underline{a}\underline{c}_1] &\mapsto u_0 u_{\underline{k}} [a\underline{c}_1] - u_{\underline{k}} [\underline{a}\underline{c}_1] \end{aligned} \quad \rightsquigarrow \quad \begin{aligned} [\underline{a}c_1] &\mapsto -\tfrac{1}{u_k} [\underline{a}c_1] \\ [a\underline{c}_1] &\mapsto -\tfrac{1}{u_{\underline{k}}} [a\underline{c}_1] \end{aligned} \quad \rightsquigarrow \quad \begin{aligned} c_0^* &\mapsto -c_0^* \\ c_1^* &\mapsto -c_1^* \\ \underline{a}^* &\mapsto -\underline{a}^* \end{aligned}$$</td></tr>
<tr><td>$\widetilde{\psi}$</td><td>$$\begin{aligned} b_0 &\rightsquigarrow c_0^* a^* \\ b_1 &\rightsquigarrow -u_k c_1^* a^* \\ \underline{b}_0 &\rightsquigarrow \underline{c}_0^* \underline{a}^* \\ \underline{b}_1 &\rightsquigarrow -u_{\underline{k}} \underline{c}_1^* \underline{a}^* \end{aligned}$$</td></tr>
<tr><td>$\widetilde{T}_1$</td><td>$b_0 \quad [ac_0] \quad b_1 \quad [ac_1] \quad \underline{b}_0 \quad [\underline{a}\underline{c}_0] \quad \underline{b}_1 \quad [\underline{a}\underline{c}_1]$</td></tr>
<tr><td>$\{B^{\uparrow}, B^{\uparrow}\}$
</td><td>$$\begin{aligned} &\quad [\underline{a}c_0] c_0^* \underline{a}^* - \tfrac{1}{u_k} [\underline{a}c_1] c_1^* \underline{a}^* \\ &+ [a\underline{c}_0] \underline{c}_0^* a^* - \tfrac{1}{u_{\underline{k}}} [a\underline{c}_1] \underline{c}_1^* a^* \\ &+ u_1 [\underline{a}c_1] c_1^* a^* + u_0 \underline{a}^* [a\underline{c}_1] \underline{c}_1^* \end{aligned}$$
</td></tr>
</table>

Table 5.A.16: flippant pair $\{\{B^{\downarrow}, B^{\downarrow}\}, \{B^{\uparrow}, B^{\uparrow}\}\}$.

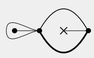
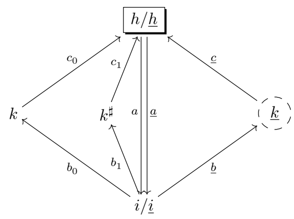
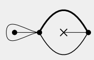
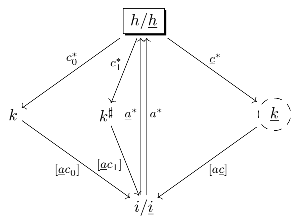
<table>
<tr>
<td>$\{B^{\downarrow}, \widetilde{B}^{\downarrow}\}$
</td>
<td>$c_0b_0a - \frac{1}{u_k}c_1b_1a$
$+\ \underline{c}z_{\underline{k}}\underline{ba}$
$+\ u_1c_1b_1\underline{a} + u_0a\underline{cb}$</td>
<td>
</td>
</tr>
<tr>
<td>$\widetilde{\varphi}$</td>
<td colspan="2">$[ac_1] \mapsto u_1u_k[\underline{a}c_1] - u_k[ac_1]$
$[\underline{ac}] \ \mapsto -u_0[a\underline{c}]z_{\underline{k}}^{-1} + [\underline{ac}]z_{\underline{k}}^{-1}$
$\rightsquigarrow$
$[\underline{a}c_1] \mapsto -\frac{1}{u_k}[\underline{a}c_1]$
$[a\underline{c}] \ \mapsto [a\underline{c}]z_{\underline{k}}$
$\rightsquigarrow$
$c_0^* \mapsto -c_0^*$
$c_1^* \mapsto -c_1^*$
$\underline{a}^* \mapsto -\underline{a}^*$</td>
</tr>
<tr>
<td>$\widetilde{\psi}$</td>
<td colspan="2">$b_0 \rightsquigarrow \ c_0^*a^*$
$b_1 \rightsquigarrow -u_kc_1^*a^*$
$\underline{b} \ \rightsquigarrow \ z_{\underline{k}}^{-1}\underline{c}^*\underline{a}^*$</td>
</tr>
<tr>
<td>$\widetilde{T}_1$</td>
<td colspan="2">$b_0 \quad [ac_0] \quad b_1 \quad [ac_1] \quad \underline{b} \quad [\underline{ac}]$</td>
</tr>
<tr>
<td>$\{B^{\uparrow}, \widetilde{B}^{\uparrow}\}$
</td>
<td>$[\underline{a}c_0]c_0^*\underline{a}^* - \frac{1}{u_k}[\underline{a}c_1]c_1^*\underline{a}^*$
$+\ [a\underline{c}]z_{\underline{k}}\underline{c}^*a^*$
$+\ u_1[\underline{a}c_1]c_1^*a^* + u_0\underline{a}^*[a\underline{c}]\underline{c}^*$</td>
<td>
</td>
</tr>
</table>

Table 5.A.17: flippant pair $\{\{B^{\downarrow}, \widetilde{B}^{\downarrow}\}, \{B^{\uparrow}, \widetilde{B}^{\uparrow}\}\}$.

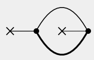
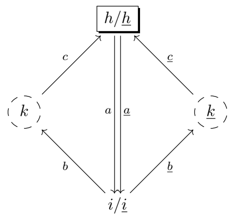
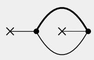
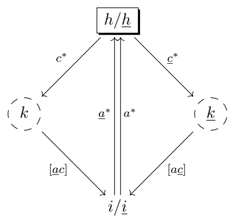
$\{\widetilde{B}^{\downarrow}, \widetilde{B}^{\downarrow}\}$ 	$\ c z_k b a$ $+\ \underline{c} z_{\underline{k}} \underline{b} \underline{a}$ $+\ u_1 c b \underline{a} + u_0 a \underline{c} \underline{b}$ 
$\widetilde{\varphi}$	$[ac] \mapsto -u_1[\underline{a}c]z_k^{-1} + [ac]z_k^{-1}$ $\quad\rightsquigarrow\quad$ $[\underline{a}c] \mapsto [\underline{a}c]z_k$ $\quad\rightsquigarrow\quad$ $c^* \mapsto -c^*$ $[\underline{ac}] \mapsto -u_0[a\underline{c}]z_{\underline{k}}^{-1} + [\underline{ac}]z_{\underline{k}}^{-1}$ $\qquad$ $[a\underline{c}] \mapsto [a\underline{c}]z_{\underline{k}}$ $\qquad$ $\underline{a}^* \mapsto -\underline{a}^*$
$\widetilde{\psi}$	$b \rightsquigarrow \quad z_k^{-1} c^* a^*$ $\underline{b} \rightsquigarrow \quad z_{\underline{k}}^{-1} \underline{c}^* \underline{a}^*$
$\widetilde{T}_1$	$b \quad [ac] \quad \underline{b} \quad [\underline{ac}]$
$\{\widetilde{B}^{\uparrow}, \widetilde{B}^{\uparrow}\}$ 	$\ [\underline{a}c] z_k c^* \underline{a}^*$ $+\ [a\underline{c}] z_{\underline{k}} \underline{c}^* a^*$ $+\ u_1 [\underline{a}c] c^* a^* + u_0 \underline{a}^* [a\underline{c}] \underline{c}^*$ 

Table 5.A.18: flippant pair $\big\{\{\widetilde{B}^{\downarrow}, \widetilde{B}^{\downarrow}\}, \{\widetilde{B}^{\uparrow}, \widetilde{B}^{\uparrow}\}\big\}$.

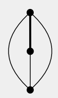
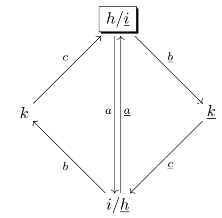
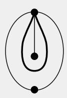
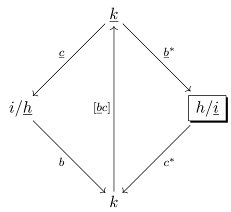
$\{A^{\downarrow}, A^{\uparrow}\}$ 	cba $+\ \underline{cba}$ $+\ u_0 a\underline{a}$ $+\ \omega_{\underline{k}k} b\underline{c} + \omega_{k\underline{k}} \underline{b}c$ $+\ \omega_{\underline{kk}} \underline{b}c\, \omega_{kk} b\underline{c}$	
φ	$\underline{a} \mapsto \frac{1}{u_0}\underline{a}$	
ψ	$a \rightsquigarrow -\frac{1}{u_0}\underline{cb}$ $\underline{a} \rightsquigarrow -cb$	
T_1	$a \quad \underline{a}$	
B 	$c^*\underline{b}^*[\underline{b}c] - \frac{1}{u_0} b\underline{c}[\underline{b}c]$ $+\ \omega_{k\underline{k}}[\underline{b}c]$ $+\ \omega_{\underline{k}k} b\underline{c}$ $+\ \omega_{\underline{kk}}[\underline{b}c]\, \omega_{kk} b\underline{c}$	

Table 5.A.19: flippant pair $\{\{A^{\downarrow}, A^{\uparrow}\}, B\}$.

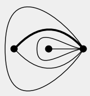
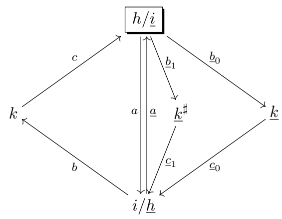
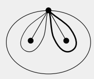
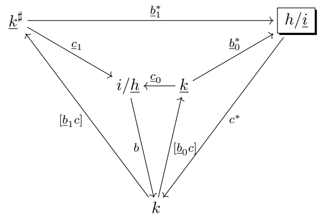
$\{A^{\downarrow}, B^{\uparrow}\}$ 	cba $+\ \underline{c}_0\underline{b}_0\underline{a} - \frac{1}{u_{\underline{k}}}\underline{c}_1\underline{b}_1\underline{a}$ $+\ u_0 a\underline{a}$ $+\ \omega_{kk} b\underline{c}_1\underline{b}_1 c$ 
φ	$\underline{a} \mapsto \frac{1}{u_0}\underline{a}$
ψ	$a \rightsquigarrow -\frac{1}{u_0}\underline{c}_0\underline{b}_0 + \frac{1}{u_0 u_{\underline{k}}}\underline{c}_1\underline{b}_1$ $\underline{a} \rightsquigarrow -cb$
T_1	$a \quad \underline{a}$
$\widetilde{\varphi}$	$\underline{b}_1^* \mapsto -\frac{1}{u_{\underline{k}}}\underline{b}_1^*$
C^+ 	$c^*\underline{b}_0^*[\underline{b}_0 c] - \frac{1}{u_{\underline{k}}}c^*\underline{b}_1^*[\underline{b}_1 c] - \frac{1}{u_0}b\underline{c}_0[\underline{b}_0 c] + \frac{1}{u_0 u_{\underline{k}}}b\underline{c}_1[\underline{b}_1 c]$ $+\ \omega_{kk} b\underline{c}_1[\underline{b}_1 c]$ 

Table 5.A.20: flippant pair $\{\{A^{\downarrow}, B^{\uparrow}\}, C^+\}$.

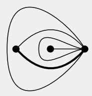
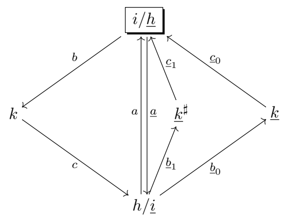
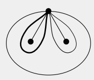
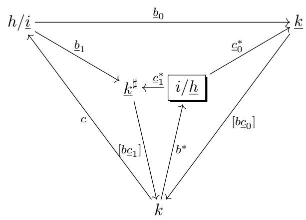
<table>
<tr>
<td>$\{A^{\uparrow}, B^{\downarrow}\}$
</td>
<td>cba
$+\ \underline{c}_0\underline{b}_0\underline{a} - \frac{1}{u_{\underline{k}}}\underline{c}_1\underline{b}_1\underline{a}$
$+\ u_0 a\underline{a}$
$+\ \omega_{kk} b\underline{c}_1\underline{b}_1 c$</td>
<td>
</td>
</tr>
<tr>
<td>φ</td>
<td colspan="2">$\underline{a} \mapsto \frac{1}{u_0}\underline{a}$</td>
</tr>
<tr>
<td>ψ</td>
<td colspan="2">$a \rightsquigarrow -\frac{1}{u_0}\underline{c}_0\underline{b}_0 + \frac{1}{u_{\underline{k}}u_0}\underline{c}_1\underline{b}_1$
$\underline{a} \rightsquigarrow -cb$</td>
</tr>
<tr>
<td>T_1</td>
<td colspan="2">$a \quad \underline{a}$</td>
</tr>
<tr>
<td>$\widetilde{\varphi}$</td>
<td colspan="2">$\underline{c}_1^* \mapsto -\frac{1}{u_{\underline{k}}}\underline{c}_1^*$</td>
</tr>
<tr>
<td>C^-
</td>
<td>$[b\underline{c}_0]\underline{c}_0^* b^* - \frac{1}{u_0}[b\underline{c}_0]\underline{b}_0 c - \frac{1}{u_{\underline{k}}}[b\underline{c}_1]\underline{c}_1^* b^* + \frac{1}{u_{\underline{k}}u_0}[b\underline{c}_1]\underline{b}_1 c$
$+\ \omega_{kk}[b\underline{c}_1]\underline{b}_1 c$</td>
<td>
</td>
</tr>
</table>

Table 5.A.21: flippant pair $\{\{A^{\uparrow}, B^{\downarrow}\}, C^-\}$.

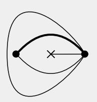
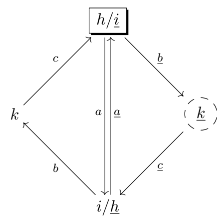
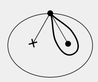
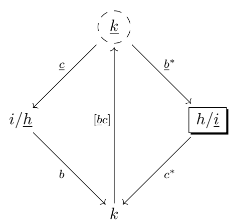
$\{A^{\downarrow}, \widetilde{B}^{\uparrow}\}$ 	cba $+\ \underline{c}z_{\underline{k}}\underline{ba}$ $+\ u_0 a\underline{a}$ $+\ \omega_{kk}b\underline{cb}c$ 
φ	$\underline{a} \mapsto \frac{1}{u_0}\underline{a}$
ψ	$a \rightsquigarrow -\frac{1}{u_0}\underline{c}z_{\underline{k}}\underline{b}$ $\underline{a} \rightsquigarrow -cb$
T_1	$a \quad \underline{a}$
$\widetilde{\varphi}$	$\underline{b}^* \mapsto \underline{b}^* z_{\underline{k}}$
$\widetilde{C}^{+}_{-}$ 	$c^*\underline{b}^* z_{\underline{k}}[\underline{b}c] - \frac{1}{u_0}b\underline{c}z_{\underline{k}}[\underline{b}c]$ $+\ \omega_{kk}b\underline{c}[\underline{b}c]$ 

Table 5.A.22: flippant pair $\{\{A^{\downarrow}, \widetilde{B}^{\uparrow}\}, \widetilde{C}^{+}_{-}\}$.

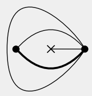
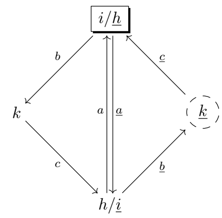
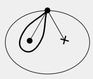
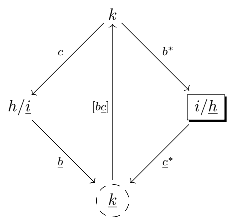
$\{A^{\uparrow}, \widetilde{B}^{\downarrow}\}$ 	cba $+\ \underline{c}z_{\underline{k}}\underline{b}\underline{a}$ $+\ u_0 a\underline{a}$ $+\ \omega_{kk} b\underline{c}\underline{b}c$ 
φ	$\underline{a} \mapsto \frac{1}{u_0}\underline{a}$
ψ	$a \rightsquigarrow -\frac{1}{u_0}\underline{c}z_{\underline{k}}\underline{b}$ $\underline{a} \rightsquigarrow -cb$
T_1	$a \quad \underline{a}$
$\widetilde{\varphi}$	$\underline{c}^* \mapsto z_{\underline{k}}\underline{c}^*$
$\widetilde{C}^{-}_{+}$ 	$[b\underline{c}]z_{\underline{k}}\underline{c}^* b^* - \frac{1}{u_0}[b\underline{c}]z_{\underline{k}}\underline{b}c$ $+\ \omega_{kk}[b\underline{c}]\underline{b}c$ 

Table 5.A.23: flippant pair $\{\{A^{\uparrow}, \widetilde{B}^{\downarrow}\}, \widetilde{C}^{-}_{+}\}$.

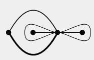
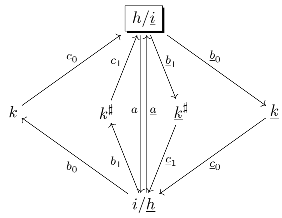
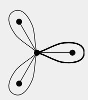
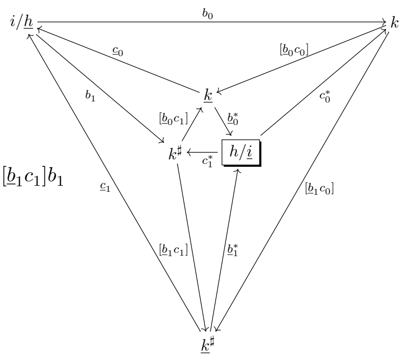
$\{B^{\downarrow}, B^{\uparrow}\}$ 	$c_0b_0a - \frac{1}{u_k}c_1b_1a$ $+ \underline{c}_0\underline{b}_0\underline{a} - \frac{1}{u_{\underline{k}}}\underline{c}_1\underline{b}_1\underline{a}$ $+ u_0a\underline{a} + u_1c_1b_1\underline{c}_1\underline{b}_1$ 
φ	$\underline{a} \mapsto \frac{1}{u_0}\underline{a}$
ψ	$a \rightsquigarrow -\frac{1}{u_0}\underline{c}_0\underline{b}_0 + \frac{1}{u_0u_{\underline{k}}}\underline{c}_1\underline{b}_1$ $\underline{a} \rightsquigarrow -c_0b_0 + \frac{1}{u_k}c_1b_1$
T_1	$a \quad \underline{a}$
$\widetilde{\varphi}$	$c_1^* \mapsto -\frac{1}{u_k}c_1^*$ $\underline{b}_1^* \mapsto -\frac{1}{u_{\underline{k}}}\underline{b}_1^*$
D 	$\underline{b}_0^*[\underline{b}_0c_0]c_0^* - \frac{1}{u_k}\underline{b}_0^*[\underline{b}_0c_1]c_1^* - \frac{1}{u_{\underline{k}}}\underline{b}_1^*[\underline{b}_1c_0]c_0^* + \frac{1}{u_ku_{\underline{k}}}\underline{b}_1^*[\underline{b}_1c_1]c_1^*$ $- \frac{1}{u_0}\underline{c}_0[\underline{b}_0c_0]b_0 + \frac{1}{u_0u_k}\underline{c}_0[\underline{b}_0c_1]b_1 + \frac{1}{u_0u_{\underline{k}}}\underline{c}_1[\underline{b}_1c_0]b_0 - \frac{1}{u_0u_ku_{\underline{k}}}\underline{c}_1[\underline{b}_1c_1]b_1$ $+ u_1\underline{c}_1[\underline{b}_1c_1]b_1$ 

Table 5.A.24: flippant pair $\left\{\left\{B^{\downarrow}, B^{\uparrow}\right\}, D\right\}$.

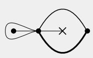
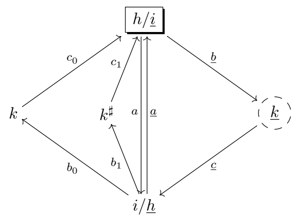
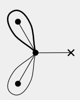
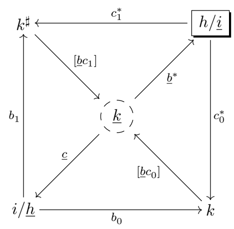
$\{B^{\downarrow}, \widetilde{B}^{\uparrow}\}$ 	c_0b_0a $-\ \frac{1}{u_k}c_1b_1a$ $+\ \underline{c}z_{\underline{k}}\underline{b}\underline{a}$ $+\ u_0a\underline{a}$ $+\ u_1c_1b_1\underline{c}\underline{b}$	
φ	$\underline{a} \mapsto \frac{1}{u_0}\underline{a}$	
ψ	$a \rightsquigarrow -\frac{1}{u_0}\underline{c}z_{\underline{k}}\underline{b}$ $\underline{a} \rightsquigarrow -c_0b_0 + \frac{1}{u_k}c_1b_1$	
T_1	$a \quad \underline{a}$	
$\widetilde{\varphi}$	$c_1^* \mapsto -\frac{1}{u_k}c_1^*$ $\underline{b}^* \mapsto \underline{b}^*z_{\underline{k}}$	
$\widetilde{D}_1^-$ 	$[\underline{b}c_0]c_0^*\underline{b}^*z_{\underline{k}}$ $-\ \frac{1}{u_0}[\underline{b}c_0]b_0\underline{c}z_{\underline{k}}$ $-\ \frac{1}{u_k}[\underline{b}c_1]c_1^*\underline{b}^*z_{\underline{k}}$ $+\ \frac{1}{u_0u_k}[\underline{b}c_1]b_1\underline{c}z_{\underline{k}}$ $+\ u_1[\underline{b}c_1]b_1\underline{c}$	

Table 5.A.25: flippant pair $\left\{\{B^{\downarrow}, \widetilde{B}^{\uparrow}\}, \widetilde{D}_1^-\right\}$.

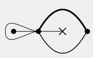
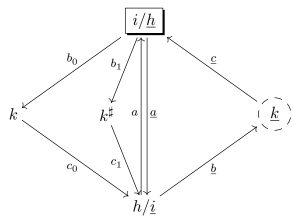
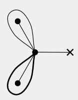
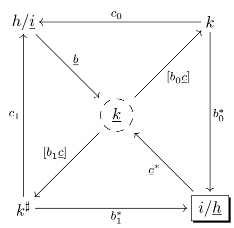
$\{B^{\uparrow}, \widetilde{B}^{\downarrow}\}$ 	c_0b_0a $-\ \frac{1}{u_k}c_1b_1a$ $+\ \underline{c}z_{\underline{k}}\underline{b}a$ $+\ u_0a\underline{a}$ $+\ u_1c_1b_1\underline{c}\underline{b}$ 
φ	$\underline{a} \mapsto \frac{1}{u_0}\underline{a}$
ψ	$a \rightsquigarrow -\frac{1}{u_0}\underline{c}z_{\underline{k}}\underline{b}$ $\underline{a} \rightsquigarrow -c_0b_0 + \frac{1}{u_k}c_1b_1$
T_1	$a \quad \underline{a}$
$\widetilde{\varphi}$	$b_1^* \mapsto -\frac{1}{u_k}b_1^*$ $\underline{c}^* \mapsto z_{\underline{k}}\underline{c}^*$
$\widetilde{D}_1^{+}$ 	$\underline{c}^*b_0^*[b_0\underline{c}]z_{\underline{k}}$ $-\ \frac{1}{u_k}\underline{c}^*b_1^*[b_1\underline{c}]z_{\underline{k}}$ $-\ \frac{1}{u_0}\underline{b}c_0[b_0\underline{c}]z_{\underline{k}}$ $+\ \frac{1}{u_0u_k}\underline{b}c_1[b_1\underline{c}]z_{\underline{k}}$ $+\ u_1\underline{b}c_1[b_1\underline{c}]$ 

Table 5.A.26: flippant pair $\{\{B^{\uparrow}, \widetilde{B}^{\downarrow}\}, \widetilde{D}_1^{+}\}$.

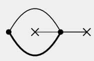
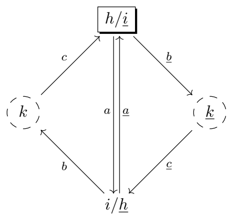
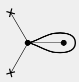
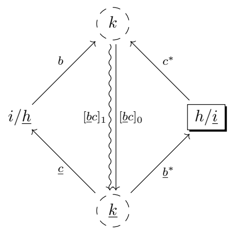
$\{\widetilde{B}^{\downarrow},\widetilde{B}^{\uparrow}\}$ 	cz_kba $+\ \underline{c}z_{\underline{k}}\underline{b}\underline{a}$ $+\ u_0a\underline{a}$ $+\ u_1cb\underline{c}\underline{b}$ 
φ	$\underline{a} \mapsto \frac{1}{u_0}\underline{a}$
ψ	$a \rightsquigarrow -\frac{1}{u_0}\underline{c}z_{\underline{k}}\underline{b}$ $\underline{a} \rightsquigarrow -cz_kb$
T_1	$a \quad \underline{a}$
$\widetilde{\varphi}$	$c^* \mapsto z_kc^*$ $\underline{b}^* \mapsto \underline{b}^*z_{\underline{k}}$
$\widetilde{D}_2$ 	$z_{\underline{k}}\left([\underline{b}c]_0 + [\underline{b}c]_1\right) z_kc^*\underline{b}^*$ $-\ \frac{1}{u_0}z_{\underline{k}}\left([\underline{b}c]_0 + [\underline{b}c]_1\right) z_kb\underline{c}$ $+\ u_1\left([\underline{b}c]_0 + [\underline{b}c]_1\right) b\underline{c}$ 

Table 5.A.27: flippant pair $\left\{\left\{\widetilde{B}^{\downarrow},\widetilde{B}^{\uparrow}\right\},\widetilde{D}_2\right\}$.

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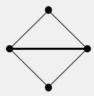
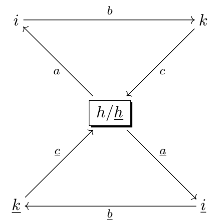
<td>$\{A, A\}$
</td>
<td>

$$\begin{aligned}
& cba \\
+\ & \underline{cba} \\
+\ & \omega_{ik}b + \omega_{\underline{ik}}\underline{b} \\
+\ & \omega_{k\underline{i}}\underline{a}c + \omega_{\underline{k}i}a\underline{c} \\
+\ & \omega_{i\underline{k}}\underline{b}\,\omega_{\underline{i}k}b \\
+\ & \omega_{kk}b\,\omega_{i\underline{i}}\underline{a}c + \omega_{\underline{kk}}\underline{b}\,\omega_{\underline{i}i}a\underline{c} \\
+\ & \omega_{ki}a\underline{c}\,\omega_{\underline{ki}}\underline{a}c \\
+\ & \omega_{kk}b\,\omega_{i\underline{k}}\underline{b}\,\omega_{\underline{ii}}\underline{a}c \\
+\ & \omega_{ki}a\underline{c}\,\omega_{\underline{k}k}b\,\omega_{i\underline{i}}\underline{a}c \\
+\ & \omega_{ki}a\underline{c}\,\omega_{\underline{k}k}b\,\omega_{i\underline{k}}\underline{b}\,\omega_{\underline{ii}}\underline{a}c + \omega_{kk}b\,\omega_{ii}a\underline{c}\,\omega_{\underline{kk}}\underline{b}\,\omega_{\underline{ii}}\underline{a}c \\
+\ & \omega_{\underline{k}k}b\,\omega_{i\underline{i}}\underline{a}c\,\omega_{k\underline{k}}\underline{b}\,\omega_{\underline{i}i}a\underline{c} + \omega_{i\underline{i}}\underline{a}c\,\omega_{ki}a\underline{c}\,\omega_{\underline{kk}}\underline{b}\,\omega_{\underline{i}k}b
\end{aligned}$$



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</tr>
<tr>
<td>$\widetilde{\varphi}$</td>
<td>

$$\begin{aligned}
c^* &\mapsto -c^* \\
\underline{a}^* &\mapsto -\underline{a}^*
\end{aligned}$$

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</tr>
<tr>
<td>$\widetilde{\psi}$</td>
<td>

$$\begin{aligned}
b \quad &\rightsquigarrow \quad c^*a^* \\
[ac] &\rightsquigarrow -\omega_{ik} \\
& \quad -\omega_{i\underline{i}}[\underline{a}c]\,\omega_{kk} \\
& \quad -\omega_{i\underline{i}}[\underline{a}c]\,\omega_{ki}[a\underline{c}]\,\omega_{\underline{k}k} \\
\underline{b} \quad &\rightsquigarrow \quad \underline{c}^*\underline{a}^* \\
[\underline{ac}] &\rightsquigarrow -\omega_{\underline{ik}} \\
& \quad -\omega_{\underline{i}i}[a\underline{c}]\,\omega_{\underline{kk}} \\
& \quad -\omega_{\underline{i}k}b\,\omega_{i\underline{k}} \\
& \quad -\omega_{\underline{i}i}[\underline{a}c]\,\omega_{kk}b\,\omega_{i\underline{k}} \\
& \quad -\omega_{\underline{i}i}[\underline{a}c]\,\omega_{ki}[a\underline{c}]\,\omega_{\underline{k}k}b\,\omega_{i\underline{k}} \\
& \quad -\omega_{\underline{i}i}[\underline{a}c]\,\omega_{kk}b\,\omega_{ii}[a\underline{c}]\,\omega_{\underline{kk}} \\
& \quad -\omega_{\underline{i}i}[a\underline{c}]\,\omega_{\underline{k}k}b\,\omega_{i\underline{i}}[\underline{a}c]\,\omega_{k\underline{k}} \\
& \quad -\omega_{\underline{i}k}b\,\omega_{i\underline{i}}[\underline{a}c]\,\omega_{ki}[a\underline{c}]\,\omega_{\underline{kk}}
\end{aligned}
\quad \rightsquigarrow \quad
\begin{aligned}
[ac] &\rightsquigarrow -\omega_{i\underline{k}}\underline{c}^*\underline{a}^*\,\omega_{\underline{i}k} \\
& \quad -\omega_{i\underline{k}}\underline{c}^*\underline{a}^*\,\omega_{\underline{ii}}[\underline{a}c]\,\omega_{kk} \\
& \quad -\omega_{i\underline{k}}\underline{c}^*\underline{a}^*\,\omega_{\underline{ii}}[\underline{a}c]\,\omega_{ki}[a\underline{c}]\,\omega_{\underline{k}k} \\
& \quad -\omega_{ii}[a\underline{c}]\,\omega_{\underline{kk}}\underline{c}^*\underline{a}^*\,\omega_{\underline{i}i}[\underline{a}c]\,\omega_{kk} \\
& \quad -\omega_{i\underline{i}}[\underline{a}c]\,\omega_{k\underline{k}}\underline{c}^*\underline{a}^*\,\omega_{\underline{i}i}[a\underline{c}]\,\omega_{\underline{k}k} \\
& \quad -\omega_{i\underline{i}}[\underline{a}c]\,\omega_{ki}[a\underline{c}]\,\omega_{\underline{kk}}\underline{c}^*\underline{a}^*\,\omega_{\underline{i}k} \\
[\underline{ac}] &\rightsquigarrow -\omega_{\underline{i}k}c^*a^*\,\omega_{i\underline{k}} \\
& \quad -\omega_{\underline{ii}}[\underline{a}c]\,\omega_{kk}c^*a^*\,\omega_{i\underline{k}} \\
& \quad -\omega_{\underline{ii}}[\underline{a}c]\,\omega_{ki}[a\underline{c}]\,\omega_{\underline{k}k}c^*a^*\,\omega_{i\underline{k}} \\
& \quad -\omega_{\underline{ii}}[\underline{a}c]\,\omega_{kk}c^*a^*\,\omega_{ii}[a\underline{c}]\,\omega_{\underline{kk}} \\
& \quad -\omega_{\underline{i}i}[a\underline{c}]\,\omega_{\underline{k}k}c^*a^*\,\omega_{i\underline{i}}[\underline{a}c]\,\omega_{k\underline{k}} \\
& \quad -\omega_{\underline{i}k}c^*a^*\,\omega_{i\underline{i}}[\underline{a}c]\,\omega_{ki}[a\underline{c}]\,\omega_{\underline{kk}}
\end{aligned}$$

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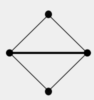
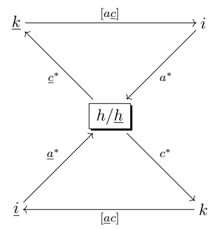
</tr>
<tr>
<td>$\widetilde{T}_1$</td>
<td>$b \quad [ac] \quad \underline{b} \quad [\underline{ac}]$</td>
</tr>
<tr>
<td>$\{A, A\}$
</td>
<td>

$$\begin{aligned}
& a^*[a\underline{c}]\underline{c}^* \\
+\ & \underline{a}^*[\underline{a}c]c^* \\
+\ & \omega_{k\underline{i}}[\underline{a}c] + \omega_{\underline{k}i}[a\underline{c}] \\
+\ & \omega_{ik}c^*a^* + \omega_{\underline{ik}}\underline{c}^*\underline{a}^* \\
+\ & \omega_{ki}[a\underline{c}]\,\omega_{\underline{ki}}[\underline{a}c] \\
+\ & \omega_{kk}c^*a^*\,\omega_{i\underline{i}}[\underline{a}c] + \omega_{\underline{kk}}\underline{c}^*\underline{a}^*\,\omega_{\underline{i}i}[a\underline{c}] \\
+\ & \omega_{i\underline{k}}\underline{c}^*\underline{a}^*\,\omega_{\underline{i}k}c^*a^* \\
+\ & \omega_{ki}[a\underline{c}]\,\omega_{\underline{k}k}c^*a^*\,\omega_{i\underline{i}}[\underline{a}c] \\
+\ & \omega_{kk}c^*a^*\,\omega_{i\underline{k}}\underline{c}^*\underline{a}^*\,\omega_{\underline{ii}}[\underline{a}c] \\
+\ & \omega_{ki}[a\underline{c}]\,\omega_{\underline{k}k}c^*a^*\,\omega_{i\underline{k}}\underline{c}^*\underline{a}^*\,\omega_{\underline{ii}}[\underline{a}c] + \omega_{kk}c^*a^*\,\omega_{ii}[a\underline{c}]\,\omega_{\underline{kk}}\underline{c}^*\underline{a}^*\,\omega_{\underline{ii}}[\underline{a}c] \\
+\ & \omega_{\underline{k}k}c^*a^*\,\omega_{i\underline{i}}[\underline{a}c]\,\omega_{k\underline{k}}\underline{c}^*\underline{a}^*\,\omega_{\underline{i}i}[a\underline{c}] + \omega_{i\underline{i}}[\underline{a}c]\,\omega_{ki}[a\underline{c}]\,\omega_{\underline{kk}}\underline{c}^*\underline{a}^*\,\omega_{\underline{i}k}c^*a^*
\end{aligned}$$



</td>
</tr>
</table>

Table 5.A.28: flippant pair $\{\{A, A\}, \{A, A\}\}$.

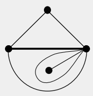
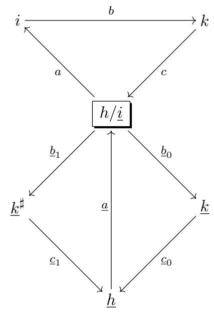
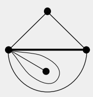
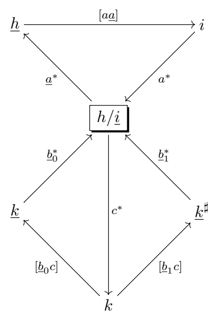
$\{A, B^+\}$ 	cba $+\ \underline{c}_0\underline{b}_0\underline{a} - \frac{1}{u_{\underline{k}}}\underline{c}_1\underline{b}_1\underline{a}$ $+\ \omega_{ik}b$ $+\ \omega_{\underline{h}i}a\underline{a}$ $+\ \omega_{k\underline{h}}\underline{c}_1\underline{b}_1c$ $+\ \omega_{\underline{h}k}b\,\omega_{ii}a\underline{a}$ $+\ \omega_{kk}b\,\omega_{i\underline{h}}\underline{c}_1\underline{b}_1c$ $+\ \omega_{ki}a\underline{a}\,\omega_{\underline{hh}}\underline{c}_1\underline{b}_1c$ $+\ \omega_{ki}a\underline{a}\,\omega_{\underline{h}k}b\,\omega_{i\underline{h}}\underline{c}_1\underline{b}_1c + \omega_{kk}b\,\omega_{ii}a\underline{a}\,\omega_{\underline{hh}}\underline{c}_1\underline{b}_1c$ 
$\widetilde{\varphi}$	$\underline{c}_1 \mapsto -u_{\underline{k}}\underline{c}_1 \quad \rightsquigarrow \quad [\underline{b}_1c] \mapsto -\frac{1}{u_{\underline{k}}}[\underline{b}_1c] \quad \rightsquigarrow \quad \begin{array}{l} c^* \mapsto -c^* \\ \underline{b}_0^* \mapsto -\underline{b}_0^* \\ \underline{b}_1^* \mapsto -\underline{b}_1^* \end{array}$
$\widetilde{\psi}$	$b \rightsquigarrow c^*a^*$ $[ac] \rightsquigarrow -\omega_{ik}$ $\quad -\omega_{ii}[a\underline{a}]\,\omega_{\underline{h}k}$ $\quad -\omega_{i\underline{h}}\underline{c}_1[\underline{b}_1c]\,\omega_{kk}$ $\quad -\omega_{i\underline{h}}\underline{c}_1[\underline{b}_1c]\,\omega_{ki}[a\underline{a}]\,\omega_{\underline{h}k}$ $\quad -\omega_{ii}[a\underline{a}]\,\omega_{\underline{hh}}\underline{c}_1[\underline{b}_1c]\,\omega_{kk}$ $\underline{c}_0 \rightsquigarrow \underline{a}^*\underline{b}_0^*$ $\underline{c}_1 \rightsquigarrow \underline{a}^*\underline{b}_1^*$ $[\underline{b}_1\underline{a}] \rightsquigarrow -[\underline{b}_1c]\,\omega_{k\underline{h}}$ $\quad -[\underline{b}_1c]\,\omega_{ki}[a\underline{a}]\,\omega_{\underline{hh}}$ $\rightsquigarrow$ $[ac] \rightsquigarrow -\omega_{i\underline{h}}\underline{a}^*\underline{b}_1^*[\underline{b}_1c]\,\omega_{kk}$ $\quad -\omega_{i\underline{h}}\underline{a}^*\underline{b}_1^*[\underline{b}_1c]\,\omega_{ki}[a\underline{a}]\,\omega_{\underline{h}k}$ $\quad -\omega_{ii}[a\underline{a}]\,\omega_{\underline{hh}}\underline{a}^*\underline{b}_1^*[\underline{b}_1c]\,\omega_{kk}$ $[\underline{b}_1\underline{a}] \rightsquigarrow -[\underline{b}_1c]\,\omega_{kk}c^*a^*\,\omega_{i\underline{h}}$ $\quad -[\underline{b}_1c]\,\omega_{ki}[a\underline{a}]\,\omega_{\underline{h}k}c^*a^*\,\omega_{i\underline{h}}$ $\quad -[\underline{b}_1c]\,\omega_{kk}c^*a^*\,\omega_{ii}[a\underline{a}]\,\omega_{\underline{hh}}$
$\widetilde{T}_1$	$b \quad [ac] \quad \underline{c}_0 \quad [\underline{b}_0\underline{a}] \quad \underline{c}_1 \quad [\underline{b}_1\underline{a}]$
$\{A, B^-\}$ 	$a^*[a\underline{a}]\underline{a}^*$ $+\ \underline{b}_0^*[\underline{b}_0c]c^* - \frac{1}{u_{\underline{k}}}\underline{b}_1^*[\underline{b}_1c]c^*$ $+\ \omega_{\underline{h}i}[a\underline{a}]$ $+\ \omega_{ik}c^*a^*$ $+\ \omega_{k\underline{h}}\underline{a}^*\underline{b}_1^*[\underline{b}_1c]$ $+\ \omega_{\underline{h}k}c^*a^*\,\omega_{ii}[a\underline{a}]$ $+\ \omega_{ki}[a\underline{a}]\,\omega_{\underline{hh}}\underline{a}^*\underline{b}_1^*[\underline{b}_1c]$ $+\ \omega_{kk}c^*a^*\,\omega_{i\underline{h}}\underline{a}^*\underline{b}_1^*[\underline{b}_1c]$ $+\ \omega_{ki}[a\underline{a}]\,\omega_{\underline{h}k}c^*a^*\,\omega_{i\underline{h}}\underline{a}^*\underline{b}_1^*[\underline{b}_1c] + \omega_{kk}c^*a^*\,\omega_{ii}[a\underline{a}]\,\omega_{\underline{hh}}\underline{a}^*\underline{b}_1^*[\underline{b}_1c]$ 

Table 5.A.29: flippant pair $\{\{A, B^+\}, \{A, B^-\}\}$.

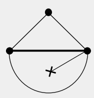
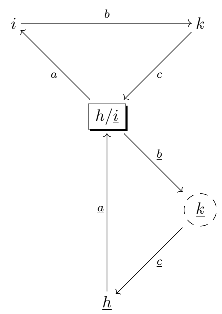
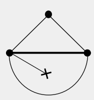
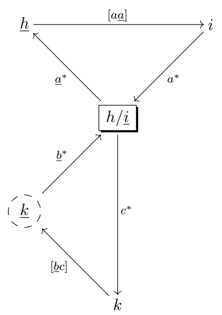
$\{A, \widetilde{B}^{+}\}$ 	cba $+\ \underline{c}z_{\underline{k}}\underline{ba}$ $+\ \omega_{ik}b$ $+\ \omega_{\underline{h}i}a\underline{a}$ $+\ \omega_{k\underline{h}}\underline{cb}c$ $+\ \omega_{\underline{h}k}b\,\omega_{ii}a\underline{a}$ $+\ \omega_{kk}b\,\omega_{i\underline{h}}\underline{cb}c$ $+\ \omega_{ki}a\underline{a}\,\omega_{\underline{hh}}\underline{cb}c$ $+\ \omega_{ki}a\underline{a}\,\omega_{\underline{h}k}b\,\omega_{i\underline{h}}\underline{cb}c + \omega_{kk}b\,\omega_{ii}a\underline{a}\,\omega_{\underline{hh}}\underline{cb}c$ 
$\widetilde{\varphi}$	$\underline{c} \mapsto \underline{c}z_{\underline{k}}^{-1} \quad \rightsquigarrow \quad [\underline{b}c] \mapsto z_{\underline{k}}[\underline{b}c] \quad \rightsquigarrow \quad c^* \mapsto -c^*$, $\underline{b}^* \mapsto -\underline{b}^*$
$\widetilde{\psi}$	$b \rightsquigarrow c^*a^*$ $[ac] \rightsquigarrow -\omega_{ik} - \omega_{ii}[a\underline{a}]\,\omega_{\underline{h}k} - \omega_{i\underline{h}}\underline{c}[\underline{b}c]\,\omega_{kk} - \omega_{i\underline{h}}\underline{c}[\underline{b}c]\,\omega_{ki}[a\underline{a}]\,\omega_{\underline{h}k} - \omega_{ii}[a\underline{a}]\,\omega_{\underline{hh}}\underline{c}[\underline{b}c]\,\omega_{kk}$ $\underline{c} \rightsquigarrow \underline{a}^*\underline{b}^*$ $[\underline{ba}] \rightsquigarrow -[\underline{b}c]\,\omega_{k\underline{h}} - [\underline{b}c]\,\omega_{ki}[a\underline{a}]\,\omega_{\underline{hh}}$ $\rightsquigarrow$ $[ac] \rightsquigarrow -\omega_{i\underline{h}}\underline{a}^*\underline{b}^*[\underline{b}c]\,\omega_{kk} - \omega_{i\underline{h}}\underline{a}^*\underline{b}^*[\underline{b}c]\,\omega_{ki}[a\underline{a}]\,\omega_{\underline{h}k} - \omega_{ii}[a\underline{a}]\,\omega_{\underline{hh}}\underline{a}^*\underline{b}^*[\underline{b}c]\,\omega_{kk}$ $[\underline{ba}] \rightsquigarrow -[\underline{b}c]\,\omega_{kk}c^*a^*\,\omega_{i\underline{h}} - [\underline{b}c]\,\omega_{ki}[a\underline{a}]\,\omega_{\underline{h}k}c^*a^*\,\omega_{i\underline{h}} - [\underline{b}c]\,\omega_{kk}c^*a^*\,\omega_{ii}[a\underline{a}]\,\omega_{\underline{hh}}$
$\widetilde{T}_1$	$b \quad [ac] \quad \underline{c} \quad [\underline{ba}]$
$\{A, \widetilde{B}^{-}\}$ 	$a^*[a\underline{a}]\underline{a}^*$ $+\ \underline{b}^*z_{\underline{k}}[\underline{b}c]c^*$ $+\ \omega_{\underline{h}i}[a\underline{a}]$ $+\ \omega_{ik}c^*a^*$ $+\ \omega_{k\underline{h}}\underline{a}^*\underline{b}^*[\underline{b}c]$ $+\ \omega_{\underline{h}k}c^*a^*\,\omega_{ii}[a\underline{a}]$ $+\ \omega_{ki}[a\underline{a}]\,\omega_{\underline{hh}}\underline{a}^*\underline{b}^*[\underline{b}c]$ $+\ \omega_{kk}c^*a^*\,\omega_{i\underline{h}}\underline{a}^*\underline{b}^*[\underline{b}c]$ $+\ \omega_{ki}[a\underline{a}]\,\omega_{\underline{h}k}c^*a^*\,\omega_{i\underline{h}}\underline{a}^*\underline{b}^*[\underline{b}c] + \omega_{kk}c^*a^*\,\omega_{ii}[a\underline{a}]\,\omega_{\underline{hh}}\underline{a}^*\underline{b}^*[\underline{b}c]$ 

Table 5.A.30: flippant pair $\{\{A, \widetilde{B}^{+}\}, \{A, \widetilde{B}^{-}\}\}$.

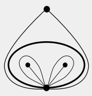
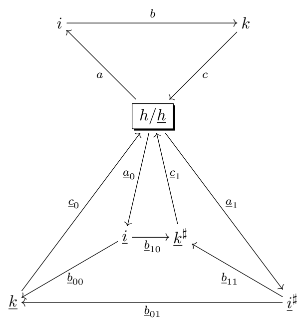
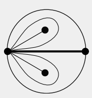
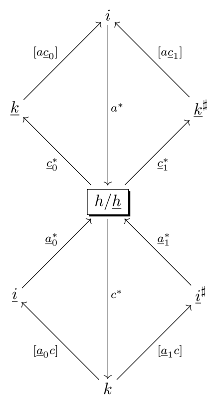
$\{A,C\}$ 	cba $+\ \underline{c}_0\underline{b}_{00}\underline{a}_0 - \frac{1}{u_{\underline{i}}}\underline{c}_0\underline{b}_{01}\underline{a}_1 - \frac{1}{u_{\underline{k}}}\underline{c}_1\underline{b}_{10}\underline{a}_0 + \frac{1}{u_{\underline{i}}u_{\underline{k}}}\underline{c}_1\underline{b}_{11}\underline{a}_1$ $+\ \omega_{ik}b$ $+\ \omega_{ki}a\underline{c}_1\underline{b}_{11}\underline{a}_1c$ $+\ \omega_{kk}b\,\omega_{ii}a\underline{c}_1\underline{b}_{11}\underline{a}_1c$ 
$\widetilde{\varphi}$	$\underline{b}_{01} \mapsto -u_{\underline{i}}\underline{b}_{01}$ $[\underline{a}_0\underline{c}_1] \mapsto -u_{\underline{k}}[\underline{a}_0\underline{c}_1]$ $[\underline{a}_1\underline{c}_1] \mapsto u_{\underline{i}}u_{\underline{k}}[\underline{a}_1\underline{c}_1]$ $\rightsquigarrow$ $[\underline{a}_1c] \mapsto -\frac{1}{u_{\underline{i}}}[\underline{a}_1c]$ $[a\underline{c}_1] \mapsto -\frac{1}{u_{\underline{k}}}[a\underline{c}_1]$ $\rightsquigarrow$ $c^* \mapsto -c^*$ $\underline{a}_0^* \mapsto -\underline{a}_0^*$ $\underline{a}_1^* \mapsto -\underline{a}_1^*$
$\widetilde{\psi}$	$b \rightsquigarrow c^*a^*$ $[ac] \rightsquigarrow -\omega_{ik}$ $\underline{b}_{00} \rightsquigarrow \underline{c}_0^*\underline{a}_0^*$ $\underline{b}_{01} \rightsquigarrow \underline{c}_0^*\underline{a}_1^*$ $\underline{b}_{10} \rightsquigarrow -u_{\underline{k}}\underline{c}_1^*\underline{a}_0^*$ $\underline{b}_{11} \rightsquigarrow u_{\underline{i}}u_{\underline{k}}\underline{c}_1^*\underline{a}_1^*$ $[\underline{a}_1\underline{c}_1] \rightsquigarrow -\frac{1}{u_{\underline{i}}u_{\underline{k}}}[\underline{a}_1c]\,\omega_{ki}[a\underline{c}_1] - \frac{1}{u_{\underline{i}}u_{\underline{k}}}[\underline{a}_1c]\,\omega_{kk}b\,\omega_{ii}[a\underline{c}_1]$ $\rightsquigarrow$ $[ac] \rightsquigarrow -\omega_{ii}[a\underline{c}_1]\underline{c}_1^*\underline{a}_1^*[\underline{a}_1c]\,\omega_{kk}$ $[\underline{a}_1\underline{c}_1] \rightsquigarrow -\frac{1}{u_{\underline{i}}u_{\underline{k}}}[\underline{a}_1c]\,\omega_{kk}c^*a^*\,\omega_{ii}[a\underline{c}_1]$
$\widetilde{T}_1$	b $[ac]$ $\underline{b}_{00}$ $[\underline{a}_0\underline{c}_0]$ $\underline{b}_{01}$ $[\underline{a}_1\underline{c}_0]$ $\underline{b}_{10}$ $[\underline{a}_0\underline{c}_1]$ $\underline{b}_{11}$ $[\underline{a}_1\underline{c}_1]$
$\{B^+,B^-\}$ 	$[a\underline{c}_0]\underline{c}_0^*a^* - \frac{1}{u_{\underline{k}}}[a\underline{c}_1]\underline{c}_1^*a^*$ $+\ \underline{a}_0^*[\underline{a}_0c]c^* - \frac{1}{u_{\underline{i}}}\underline{a}_1^*[\underline{a}_1c]c^*$ $+\ \omega_{ik}c^*a^*$ $+\ \omega_{ki}[a\underline{c}_1]\underline{c}_1^*\underline{a}_1^*[\underline{a}_1c]$ $+\ \omega_{kk}c^*a^*\,\omega_{ii}[a\underline{c}_1]\underline{c}_1^*\underline{a}_1^*[\underline{a}_1c]$ 

Table 5.A.31: flippant pair $\{\{A,C\},\{B^+,B^-\}\}$.

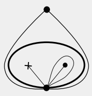
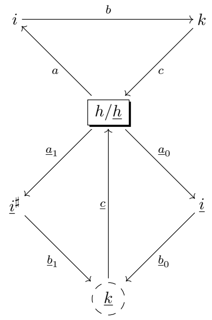
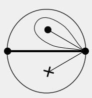
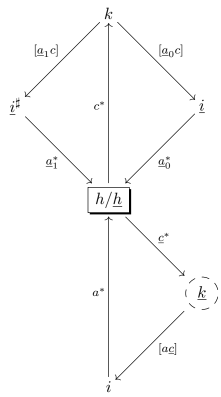
<table>
<tr>
<td>$\{A, \widetilde{C}_+\}$
</td>
<td>$$\begin{aligned} & cba \\ + \; & \underline{c} z_{\underline{k}} \underline{b}_0 \underline{a}_0 - \tfrac{1}{u_{\underline{i}}} \underline{c} z_{\underline{k}} \underline{b}_1 \underline{a}_1 \\ + \; & \omega_{ik} b \\ + \; & \omega_{ki} a \underline{c} \underline{b}_1 \underline{a}_1 c \\ + \; & \omega_{kk} b \, \omega_{ii} a \underline{c} \underline{b}_1 \underline{a}_1 c \end{aligned}$$
</td>
</tr>
<tr>
<td>$\widetilde{\varphi}$</td>
<td>$$\begin{aligned} \underline{b}_0 &\mapsto z_{\underline{k}}^{-1} \underline{b}_0 \\ \underline{b}_1 &\mapsto -u_{\underline{i}} z_{\underline{k}}^{-1} \underline{b}_1 \end{aligned} \quad \rightsquigarrow \quad \begin{aligned} [\underline{a}_1 c] &\mapsto -\tfrac{1}{u_{\underline{i}}} [\underline{a}_1 c] \\ [a\underline{c}] &\mapsto [a\underline{c}] z_{\underline{k}} \end{aligned} \quad \rightsquigarrow \quad \begin{aligned} c^* &\mapsto -c^* \\ \underline{a}_0^* &\mapsto -\underline{a}_0^* \\ \underline{a}_1^* &\mapsto -\underline{a}_1^* \end{aligned}$$</td>
</tr>
<tr>
<td>$\widetilde{\psi}$</td>
<td>$$\begin{aligned} b &\rightsquigarrow c^* a^* \\ [ac] &\rightsquigarrow -\omega_{ik} \\ &\quad\; -\omega_{ii} [a\underline{c}] \underline{b}_1 [\underline{a}_1 c] \, \omega_{kk} \\ \underline{b}_0 &\rightsquigarrow \underline{c}^* \underline{a}_0^* \\ \underline{b}_1 &\rightsquigarrow \underline{c}^* \underline{a}_1^* \\ [\underline{a}_1 \underline{c}] &\rightsquigarrow -[\underline{a}_1 c] \, \omega_{ki} [a\underline{c}] \end{aligned} \quad \rightsquigarrow \quad \begin{aligned} [ac] &\rightsquigarrow -\omega_{ii} [a\underline{c}] \underline{c}^* \underline{a}_1^* [\underline{a}_1 c] \, \omega_{kk} \\ [\underline{a}_1 \underline{c}] &\rightsquigarrow -[\underline{a}_1 c] \, \omega_{kk} c^* a^* \, \omega_{ii} [a\underline{c}] \end{aligned}$$</td>
</tr>
<tr>
<td>$\widetilde{T}_1$</td>
<td>$b \quad [ac] \quad \underline{b}_0 \quad [\underline{a}_0 \underline{c}] \quad \underline{b}_1 \quad [\underline{a}_1 \underline{c}]$</td>
</tr>
<tr>
<td>$\{B^-, \widetilde{B}^+\}$
</td>
<td>$$\begin{aligned} & \underline{a}_0^* [\underline{a}_0 c] c^* - \tfrac{1}{u_{\underline{i}}} \underline{a}_1^* [\underline{a}_1 c] c^* \\ + \; & [a\underline{c}] z_{\underline{k}} \underline{c}^* a^* \\ + \; & \omega_{ik} c^* a^* \\ + \; & \omega_{ki} [a\underline{c}] \underline{c}^* \underline{a}_1^* [\underline{a}_1 c] \\ + \; & \omega_{kk} c^* a^* \, \omega_{ii} [a\underline{c}] \underline{c}^* \underline{a}_1^* [\underline{a}_1 c] \end{aligned}$$
</td>
</tr>
</table>

Table 5.A.32: flippant pair $\big\{\{A, \widetilde{C}_+\}, \{B^-, \widetilde{B}^+\}\big\}$.

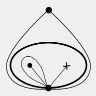
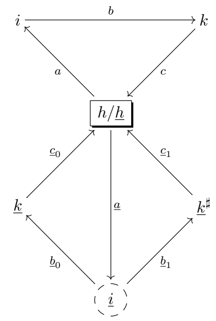
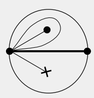
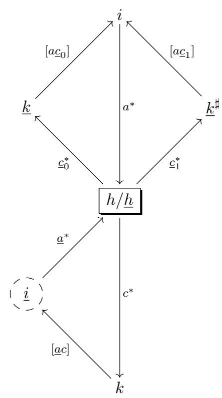
$\{A, \widetilde{C}_-\}$ 	cba $+\ \underline{c}_0\underline{b}_0 z_{\underline{i}}\underline{a} - \frac{1}{u_{\underline{k}}}\underline{c}_1\underline{b}_1 z_{\underline{i}}\underline{a}$ $+\ \omega_{ik}b$ $+\ \omega_{ki}a\underline{c}_1\underline{b}_1\underline{a}c$ $+\ \omega_{kk}b\,\omega_{ii}a\underline{c}_1\underline{b}_1\underline{a}c$ 
$\widetilde{\varphi}$	$[\underline{ac}_0] \mapsto z_{\underline{i}}^{-1}[\underline{ac}_0]$ $[\underline{ac}_1] \mapsto -u_{\underline{k}}z_{\underline{i}}^{-1}[\underline{ac}_1]$ $\rightsquigarrow$ $[\underline{a}c] \mapsto z_{\underline{i}}[\underline{a}c]$ $[a\underline{c}_1] \mapsto -\frac{1}{u_{\underline{k}}}[a\underline{c}_1]$ $\rightsquigarrow$ $c^* \mapsto -c^*$ $\underline{a}^* \mapsto -\underline{a}^*$
$\widetilde{\psi}$	$b \rightsquigarrow c^*a^*$ $[ac] \rightsquigarrow -\omega_{ik}$ $\underline{b}_0 \rightsquigarrow \underline{c}_0^*\underline{a}^*z_{\underline{i}}^{-1}$ $\underline{b}_1 \rightsquigarrow -u_{\underline{k}}\underline{c}_1^*\underline{a}^*z_{\underline{i}}^{-1}$ $[\underline{ac}_1] \rightsquigarrow \frac{1}{u_{\underline{k}}}z_{\underline{i}}[\underline{a}c]\,\omega_{ki}[a\underline{c}_1]$ $\quad + \frac{1}{u_{\underline{k}}}z_{\underline{i}}[\underline{a}c]\,\omega_{kk}b\,\omega_{ii}[a\underline{c}_1]$ $\rightsquigarrow$ $[ac] \rightsquigarrow -\omega_{ii}[a\underline{c}_1]\underline{c}_1^*\underline{a}^*[\underline{a}c]\,\omega_{kk}$ $[\underline{ac}_1] \rightsquigarrow \frac{1}{u_{\underline{k}}}z_{\underline{i}}[\underline{a}c]\,\omega_{kk}c^*a^*\,\omega_{ii}[a\underline{c}_1]$
$\widetilde{T}_1$	$b \quad [ac] \quad \underline{b}_0 \quad [\underline{ac}_0] \quad \underline{b}_1 \quad [\underline{ac}_1]$
$\{B^+, \widetilde{B}^-\}$ 	$[a\underline{c}_0]\underline{c}_0^*a^* - \frac{1}{u_{\underline{k}}}[a\underline{c}_1]\underline{c}_1^*a^*$ $+\ \underline{a}^*z_{\underline{i}}[\underline{a}c]c^*$ $+\ \omega_{ik}c^*a^*$ $+\ \omega_{ki}[a\underline{c}_1]\underline{c}_1^*\underline{a}^*[\underline{a}c]$ $+\ \omega_{kk}c^*a^*\,\omega_{ii}[a\underline{c}_1]\underline{c}_1^*\underline{a}^*[\underline{a}c]$ 

Table 5.A.33: flippant pair $\{\{A, \widetilde{C}_-\}, \{B^+, \widetilde{B}^-\}\}$.

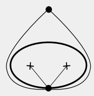
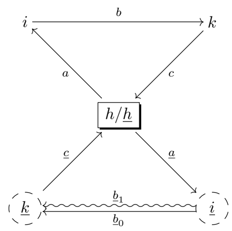
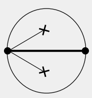
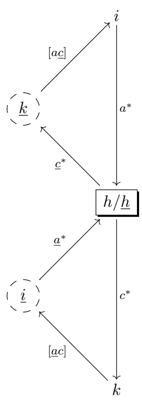
$\{A,\widetilde{C}\}$ 	cba $+\ \underline{c}z_{\underline{k}}(\underline{b}_0+\underline{b}_1)z_{\underline{i}}\underline{a}$ $+\ \omega_{ik}b$ $+\ \omega_{ki}a\underline{c}(\underline{b}_0+\underline{b}_1)\underline{a}c$ $+\ \omega_{kk}b\,\omega_{ii}a\underline{c}(\underline{b}_0+\underline{b}_1)\underline{a}c$ 
$\widetilde{\varphi}$	$[\underline{ac}]_0 \mapsto z_{\underline{i}}^{-1}[\underline{ac}]_0 z_{\underline{k}}^{-1}$ $[\underline{ac}]_1 \mapsto z_{\underline{i}}^{-1}[\underline{ac}]_1 z_{\underline{k}}^{-1}$ $\rightsquigarrow$ $[\underline{a}c] \mapsto z_{\underline{i}}[\underline{a}c]$ $[a\underline{c}] \mapsto [a\underline{c}]z_{\underline{k}}$ $\rightsquigarrow$ $c^* \mapsto -c^*$ $\underline{a}^* \mapsto -\underline{a}^*$
$\widetilde{\psi}$	$b \rightsquigarrow c^*a^*$ $[ac] \rightsquigarrow -\omega_{ik}$ $\underline{b}_0 \rightsquigarrow \pi_0\Big(z_{\underline{i}}^{-1}z_{\underline{k}}^{-1}\underline{c}^*\underline{a}^*\Big)$ $[\underline{ac}]_0 \rightsquigarrow -\pi_0\Big(z_{\underline{i}}z_{\underline{k}}[\underline{a}c]\,\omega_{ki}[a\underline{c}]\Big)$ $-\pi_0\Big(z_{\underline{i}}z_{\underline{k}}[\underline{a}c]\,\omega_{kk}b\,\omega_{ii}[a\underline{c}]\Big)$ $\underline{b}_1 \rightsquigarrow \pi_1\Big(z_{\underline{i}}^{-*}z_{\underline{k}}^{-1}\underline{c}^*\underline{a}^*\Big)$ $[\underline{ac}]_1 \rightsquigarrow -\pi_1\Big(z_{\underline{i}}z_{\underline{k}}^*[\underline{a}c]\,\omega_{ki}[a\underline{c}]\Big)$ $-\pi_1\Big(z_{\underline{i}}z_{\underline{k}}^*[\underline{a}c]\,\omega_{kk}b\,\omega_{ii}[a\underline{c}]\Big)$ $\rightsquigarrow$ $[ac] \rightsquigarrow -\omega_{ii}[a\underline{c}]\underline{c}^*\underline{a}^*[\underline{a}c]\,\omega_{kk}$ $[\underline{ac}]_0 \rightsquigarrow -\pi_0\Big(z_{\underline{i}}z_{\underline{k}}[\underline{a}c]\,\omega_{kk}c^*a^*\,\omega_{ii}[a\underline{c}]\Big)$ $[\underline{ac}]_1 \rightsquigarrow -\pi_1\Big(z_{\underline{i}}z_{\underline{k}}^*[\underline{a}c]\,\omega_{kk}c^*a^*\,\omega_{ii}[a\underline{c}]\Big)$
$\widetilde{T}_1$	$b \quad [ac] \quad \underline{b}_0 \quad [\underline{ac}]_0 \quad \underline{b}_1 \quad [\underline{ac}]_1$
$\{\widetilde{B}^+,\widetilde{B}^-\}$ 	$[a\underline{c}]z_{\underline{k}}\underline{c}^*a^*$ $+\ \underline{a}^*z_{\underline{i}}[\underline{a}c]c^*$ $+\ \omega_{ik}c^*a^*$ $+\ \omega_{ki}[a\underline{c}]\underline{c}^*\underline{a}^*[\underline{a}c]$ $+\ \omega_{kk}c^*a^*\,\omega_{ii}[a\underline{c}]\underline{c}^*\underline{a}^*[\underline{a}c]$ 

Table 5.A.34: flippant pair $\big\{\{A,\widetilde{C}\},\{\widetilde{B}^+,\widetilde{B}^-\}\big\}$.

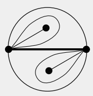
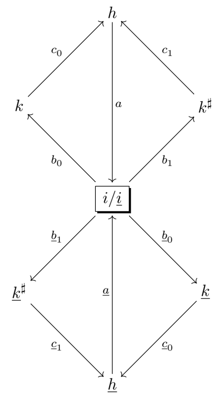
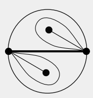
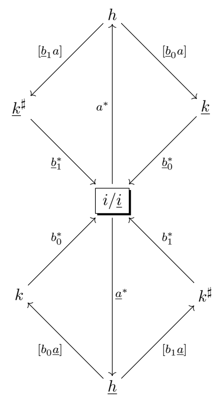
$\{B^+, B^+\}$ 	$c_0 b_0 a - \frac{1}{u_k} c_1 b_1 a$ $+ \underline{c}_0 \underline{b}_0 \underline{a} - \frac{1}{u_{\underline{k}}} \underline{c}_1 \underline{b}_1 \underline{a}$ $+ \ \omega_{h\underline{h}} \underline{c}_1 \underline{b}_1 a + \omega_{\underline{h}h} c_1 b_1 \underline{a}$ $+ \ \omega_{hh} c_1 b_1 \underline{a}\, \omega_{\underline{hh}} \underline{c}_1 \underline{b}_1 a$	
$\widetilde{\varphi}$	$c_1 \mapsto -u_k c_1$ $\underline{c}_1 \mapsto -u_{\underline{k}} \underline{c}_1$ $\rightsquigarrow$ $[\underline{b}_1 a] \mapsto -\frac{1}{u_{\underline{k}}} [\underline{b}_1 a]$ $[b_1 \underline{a}] \mapsto -\frac{1}{u_k} [b_1 \underline{a}]$ $\rightsquigarrow$ $a^* \mapsto -a^*$ $\underline{b}_0^* \mapsto -\underline{b}_0^*$ $\underline{b}_1^* \mapsto -\underline{b}_1^*$	
$\widetilde{\psi}$	$c_0 \rightsquigarrow a^* b_0^*$ $c_1 \rightsquigarrow a^* b_1^*$ $[b_1 a] \rightsquigarrow -[b_1 \underline{a}]\, \omega_{h\underline{h}}$ $\underline{c}_0 \rightsquigarrow \underline{a}^* \underline{b}_0^*$ $\underline{c}_1 \rightsquigarrow \underline{a}^* \underline{b}_1^*$ $[\underline{b}_1 \underline{a}] \rightsquigarrow -[\underline{b}_1 a]\, \omega_{h\underline{h}}$ $\quad -[\underline{b}_1 a]\, \omega_{hh} c_1 [b_1 \underline{a}]\, \omega_{\underline{hh}}$ $\rightsquigarrow$ $[b_1 a] \rightsquigarrow -[b_1 \underline{a}]\, \omega_{\underline{hh}} \underline{a}^* \underline{b}_1^* [\underline{b}_1 a]\, \omega_{hh}$ $[\underline{b}_1 \underline{a}] \rightsquigarrow -[\underline{b}_1 a]\, \omega_{hh} a^* b_1^* [b_1 \underline{a}]\, \omega_{\underline{hh}}$	
$\widetilde{T}_1$	$c_0 \quad [b_0 a] \quad c_1 \quad [b_1 a] \quad \underline{c}_0 \quad [\underline{b}_0 \underline{a}] \quad \underline{c}_1 \quad [\underline{b}_1 \underline{a}]$	
$\{B^-, B^-\}$ 	$\underline{b}_0^* [\underline{b}_0 a] a^* - \frac{1}{u_{\underline{k}}} \underline{b}_1^* [\underline{b}_1 a] a^*$ $+ b_0^* [b_0 \underline{a}] \underline{a}^* - \frac{1}{u_k} b_1^* [b_1 \underline{a}] \underline{a}^*$ $+ \ \omega_{h\underline{h}} \underline{a}^* \underline{b}_1^* [\underline{b}_1 a] + \omega_{\underline{h}h} a^* b_1^* [b_1 \underline{a}]$ $+ \ \omega_{hh} a^* b_1^* [b_1 \underline{a}]\, \omega_{\underline{hh}} \underline{a}^* \underline{b}_1^* [\underline{b}_1 a]$	

Table 5.A.35: flippant pair $\{\{B^+, B^+\}, \{B^-, B^-\}\}$.

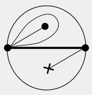
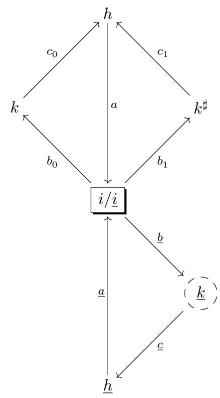
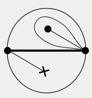
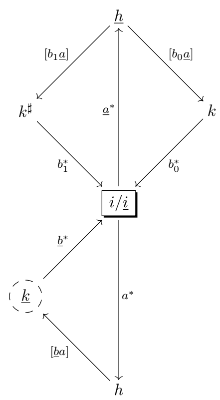
$\{B^+, \widetilde{B}^+\}$ 	$c_0 b_0 a - \frac{1}{u_k} c_1 b_1 a$ $+\ \underline{c} z_{\underline{k}} \underline{b}\underline{a}$ $+\ \omega_{h\underline{h}} \underline{c}\underline{b} a + \omega_{\underline{h}h} c_1 b_1 \underline{a}$ $+\ \omega_{hh} c_1 b_1 \underline{a}\, \omega_{\underline{h}\underline{h}} \underline{c}\underline{b} a$ 
$\widetilde{\varphi}$	$c_1 \mapsto -u_k c_1$ $\underline{c} \mapsto \underline{c} z_{\underline{k}}^{-1}$ $\rightsquigarrow$ $[\underline{b}a] \mapsto z_{\underline{k}} [\underline{b}a]$ $[b_1 \underline{a}] \mapsto -\frac{1}{u_k} [b_1 \underline{a}]$ $\rightsquigarrow$ $a^* \mapsto -a^*$ $\underline{b}^* \mapsto -\underline{b}^*$
$\widetilde{\psi}$	$c_0 \rightsquigarrow a^* b_0^*$ $c_1 \rightsquigarrow a^* b_1^*$ $[b_1 a] \rightsquigarrow -[b_1 \underline{a}]\, \omega_{h\underline{h}}$ $\underline{c} \rightsquigarrow \underline{a}^* \underline{b}^*$ $[\underline{b}a] \rightsquigarrow -[\underline{b}a]\, \omega_{h\underline{h}}$ $\quad -[\underline{b}a]\, \omega_{hh} c_1 [b_1 \underline{a}]\, \omega_{\underline{h}\underline{h}}$ $\rightsquigarrow$ $[b_1 a] \rightsquigarrow -[b_1 \underline{a}]\, \omega_{\underline{h}\underline{h}} \underline{a}^* \underline{b}^* [\underline{b}a]\, \omega_{hh}$ $[\underline{b}a] \rightsquigarrow -[\underline{b}a]\, \omega_{hh} a^* b_1^* [b_1 \underline{a}]\, \omega_{\underline{h}\underline{h}}$
$\widetilde{T}_1$	$c_0 \quad [b_0 a] \quad c_1 \quad [b_1 a] \quad \underline{c} \quad [\underline{b}a]$
$\{B^-, \widetilde{B}^-\}$ 	$b_0^* [b_0 \underline{a}] \underline{a}^* - \frac{1}{u_k} b_1^* [b_1 \underline{a}] \underline{a}^*$ $+\ \underline{b}^* z_{\underline{k}} [\underline{b}a] a^*$ $+\ \omega_{h\underline{h}} \underline{a}^* \underline{b}^* [\underline{b}a] + \omega_{\underline{h}h} a^* b_1^* [b_1 \underline{a}]$ $+\ \omega_{hh} a^* b_1^* [b_1 \underline{a}]\, \omega_{\underline{h}\underline{h}} \underline{a}^* \underline{b}^* [\underline{b}a]$ 

Table 5.A.36: flippant pair $\{\{B^+, \widetilde{B}^+\}, \{B^-, \widetilde{B}^-\}\}$.

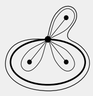
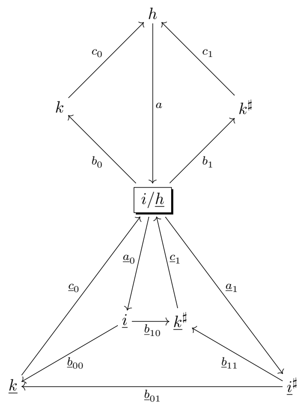
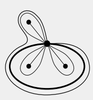
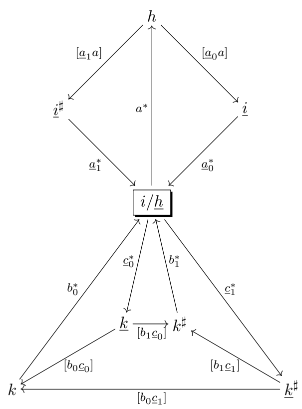
$\{B^+,C\}$ 	$c_0b_0a - \frac{1}{u_k}c_1b_1a$ $+ \underline{c}_0\underline{b}_{00}\underline{a}_0 - \frac{1}{u_{\underline{i}}}\underline{c}_0\underline{b}_{01}\underline{a}_1 - \frac{1}{u_{\underline{k}}}\underline{c}_1\underline{b}_{10}\underline{a}_0 + \frac{1}{u_{\underline{k}}u_{\underline{i}}}\underline{c}_1\underline{b}_{11}\underline{a}_1$ $+\ \omega_{hh}c_1b_1\underline{c}_1\underline{b}_{11}\underline{a}_1a$ 
$\widetilde{\varphi}$	$c_1 \mapsto -u_kc_1$ $\underline{b}_{01} \mapsto -u_{\underline{i}}\underline{b}_{01}$ $[\underline{a}_0\underline{c}_1] \mapsto -u_{\underline{k}}[\underline{a}_0\underline{c}_1]$ $[\underline{a}_1\underline{c}_1] \mapsto u_{\underline{k}}u_{\underline{i}}[\underline{a}_1\underline{c}_1]$ $\rightsquigarrow$ $[\underline{a}_1a] \mapsto -\frac{1}{u_{\underline{i}}}[\underline{a}_1a]$ $[b_1\underline{c}_0] \mapsto -\frac{1}{u_k}[b_1\underline{c}_0]$ $[b_0\underline{c}_1] \mapsto -\frac{1}{u_{\underline{k}}}[b_0\underline{c}_1]$ $[b_1\underline{c}_1] \mapsto \frac{1}{u_{\underline{k}}u_k}[b_1\underline{c}_1]$ $\rightsquigarrow$ $a^* \mapsto -a^*$ $\underline{a}_0^* \mapsto -\underline{a}_0^*$ $\underline{a}_1^* \mapsto -\underline{a}_1^*$
$\widetilde{\psi}$	$c_0 \rightsquigarrow a^*b_0^*$ $c_1 \rightsquigarrow a^*b_1^*$ $\underline{b}_{00} \rightsquigarrow \underline{c}_0^*\underline{a}_0^*$ $\underline{b}_{01} \rightsquigarrow \underline{c}_0^*\underline{a}_1^*$ $\underline{b}_{10} \rightsquigarrow -u_{\underline{k}}\underline{c}_1^*\underline{a}_0^*$ $\underline{b}_{11} \rightsquigarrow u_{\underline{k}}u_{\underline{i}}\underline{c}_1^*\underline{a}_1^*$ $[\underline{a}_1\underline{c}_1] \rightsquigarrow -\frac{1}{u_{\underline{k}}u_{\underline{i}}}[\underline{a}_1a]\,\omega_{hh}c_1[b_1\underline{c}_1]$ $\rightsquigarrow$ $[b_1a] \rightsquigarrow -[b_1\underline{c}_1]\underline{c}_1^*\underline{a}_1^*[\underline{a}_1a]\,\omega_{hh}$ $[\underline{a}_1\underline{c}_1] \rightsquigarrow -\frac{1}{u_{\underline{k}}u_{\underline{i}}}[\underline{a}_1a]\,\omega_{hh}a^*b_1^*[b_1\underline{c}_1]$
$\widetilde{T}_1$	c_0 $[b_0a]$ c_1 $[b_1a]$ $\underline{b}_{00}$ $[\underline{a}_0\underline{c}_0]$ $\underline{b}_{01}$ $[\underline{a}_1\underline{c}_0]$ $\underline{b}_{10}$ $[\underline{a}_0\underline{c}_1]$ $\underline{b}_{11}$ $[\underline{a}_1\underline{c}_1]$
$\{B^-,C\}$ 	$\underline{a}_0^*[\underline{a}_0a]a^* - \frac{1}{u_{\underline{i}}}\underline{a}_1^*[\underline{a}_1a]a^*$ $+ b_0^*[b_0\underline{c}_0]\underline{c}_0^* - \frac{1}{u_{\underline{k}}}b_0^*[b_0\underline{c}_1]\underline{c}_1^* - \frac{1}{u_k}b_1^*[b_1\underline{c}_0]\underline{c}_0^* + \frac{1}{u_{\underline{k}}u_k}b_1^*[b_1\underline{c}_1]\underline{c}_1^*$ $+\ \omega_{hh}a^*b_1^*[b_1\underline{c}_1]\underline{c}_1^*\underline{a}_1^*[\underline{a}_1a]$ 

Table 5.A.37: flippant pair $\{\{B^+,C\},\{B^-,C\}\}$.

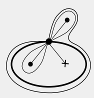
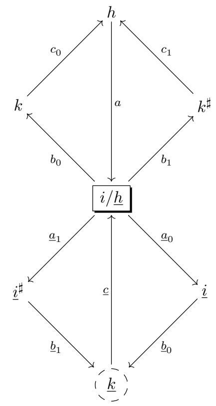
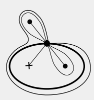
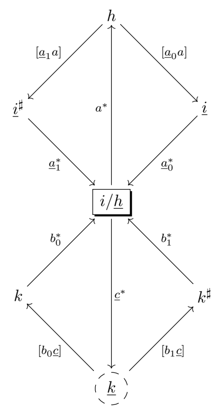
$\{B^+, \widetilde{C}_+\}$ 	$c_0 b_0 a - \frac{1}{u_k} c_1 b_1 a$ $+\ \underline{c} z_{\underline{k}} \underline{b}_0 \underline{a}_0 - \frac{1}{u_{\underline{i}}} \underline{c} z_{\underline{k}} \underline{b}_1 \underline{a}_1$ $+\ \omega_{hh} c_1 b_1 \underline{c}\underline{b}_1 \underline{a}_1 a$ 
$\widetilde{\varphi}$	$c_1 \mapsto -u_k c_1$ $\underline{b}_0 \mapsto z_{\underline{k}}^{-1} \underline{b}_0$ $\underline{b}_1 \mapsto -u_{\underline{i}} z_{\underline{k}}^{-1} \underline{b}_1$ $\rightsquigarrow$ $[\underline{a}_1 a] \mapsto -\frac{1}{u_{\underline{i}}} [\underline{a}_1 a]$ $[b_0 \underline{c}] \mapsto [b_0 \underline{c}] z_{\underline{k}}$ $[b_1 \underline{c}] \mapsto -\frac{1}{u_k} [b_1 \underline{c}] z_{\underline{k}}$ $\rightsquigarrow$ $a^* \mapsto -a^*$ $\underline{a}_0^* \mapsto -\underline{a}_0^*$ $\underline{a}_1^* \mapsto -\underline{a}_1^*$
$\widetilde{\psi}$	$c_0 \rightsquigarrow a^* b_0^*$ $c_1 \rightsquigarrow a^* b_1^*$ $\underline{b}_0 \rightsquigarrow \underline{c}^* \underline{a}_0^*$ $\underline{b}_1 \rightsquigarrow \underline{c}^* \underline{a}_1^*$ $[\underline{a}_1 \underline{c}] \rightsquigarrow -[\underline{a}_1 a]\, \omega_{hh} c_1 [b_1 \underline{c}]$ $\rightsquigarrow$ $[b_1 a] \rightsquigarrow -[b_1 \underline{c}] \underline{c}^* \underline{a}_1^* [\underline{a}_1 a]\, \omega_{hh}$ $[\underline{a}_1 \underline{c}] \rightsquigarrow -[\underline{a}_1 a]\, \omega_{hh} a^* b_1^* [b_1 \underline{c}]$
$\widetilde{T}_1$	c_0 $[b_0 a]$ c_1 $[b_1 a]$ $\underline{b}_0$ $[\underline{a}_0 \underline{c}]$ $\underline{b}_1$ $[\underline{a}_1 \underline{c}]$
$\{B^-, \widetilde{C}_-\}$ 	$\underline{a}_0^* [\underline{a}_0 a] a^* - \frac{1}{u_{\underline{i}}} \underline{a}_1^* [\underline{a}_1 a] a^*$ $+\ b_0^* [b_0 \underline{c}] z_{\underline{k}} \underline{c}^* - \frac{1}{u_k} b_1^* [b_1 \underline{c}] z_{\underline{k}} \underline{c}^*$ $+\ \omega_{hh} a^* b_1^* [b_1 \underline{c}] \underline{c}^* \underline{a}_1^* [\underline{a}_1 a]$ 

Table 5.A.38: flippant pair $\{\{B^+, \widetilde{C}_+\}, \{B^-, \widetilde{C}_-\}\}$.

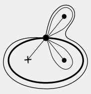
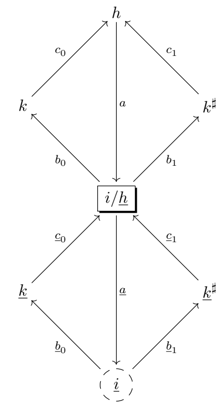
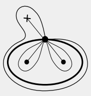
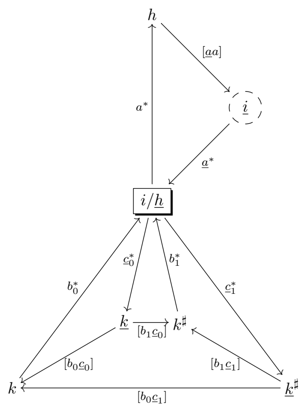
$\{B^+, \widetilde{C}_-\}$ 	$c_0b_0a - \frac{1}{u_k}c_1b_1a$ $+ \underline{c}_0\underline{b}_0z_{\underline{i}}\underline{a} - \frac{1}{u_{\underline{k}}}\underline{c}_1\underline{b}_1z_{\underline{i}}\underline{a}$ $+ \omega_{hh}c_1b_1\underline{c}_1\underline{b}_1\underline{a}a$ 
$\widetilde{\varphi}$	$c_1 \mapsto -u_kc_1$ $[\underline{a}c_0] \mapsto z_{\underline{i}}^{-1}[\underline{a}c_0]$ $[\underline{a}c_1] \mapsto -u_{\underline{k}}z_{\underline{i}}^{-1}[\underline{a}c_1]$ $\rightsquigarrow$ $[\underline{a}a] \mapsto z_{\underline{i}}[\underline{a}a]$ $[b_1\underline{c}_0] \mapsto -\frac{1}{u_k}[b_1\underline{c}_0]$ $[b_0\underline{c}_1] \mapsto -\frac{1}{u_{\underline{k}}}[b_0\underline{c}_1]$ $[b_1\underline{c}_1] \mapsto \frac{1}{u_ku_{\underline{k}}}[b_1\underline{c}_1]$ $\rightsquigarrow$ $a^* \mapsto -a^*$ $\underline{a}^* \mapsto -\underline{a}^*$
$\widetilde{\psi}$	$c_0 \rightsquigarrow a^*b_0^*$ $c_1 \rightsquigarrow a^*b_1^*$ $\underline{b}_0 \rightsquigarrow \underline{c}_0^*\underline{a}^*z_{\underline{i}}^{-1}$ $\underline{b}_1 \rightsquigarrow -u_{\underline{k}}\underline{c}_1^*\underline{a}^*z_{\underline{i}}^{-1}$ $[\underline{a}c_1] \rightsquigarrow \frac{1}{u_{\underline{k}}}z_{\underline{i}}[\underline{a}a]\,\omega_{hh}c_1[b_1\underline{c}_1]$ $\rightsquigarrow$ $[b_1a] \rightsquigarrow -[b_1\underline{c}_1]\underline{c}_1^*\underline{a}^*[\underline{a}a]\,\omega_{hh}$ $[\underline{a}c_1] \rightsquigarrow \frac{1}{u_{\underline{k}}}z_{\underline{i}}[\underline{a}a]\,\omega_{hh}a^*b_1^*[b_1\underline{c}_1]$
$\widetilde{T}_1$	$c_0 \quad [b_0a] \quad c_1 \quad [b_1a] \quad \underline{b}_0 \quad [\underline{a}c_0] \quad \underline{b}_1 \quad [\underline{a}c_1]$
$\{\widetilde{B}^-, C\}$ 	$\underline{a}^*z_{\underline{i}}[\underline{a}a]a^*$ $+ b_0^*[b_0\underline{c}_0]\underline{c}_0^* - \frac{1}{u_{\underline{k}}}b_0^*[b_0\underline{c}_1]\underline{c}_1^* - \frac{1}{u_k}b_1^*[b_1\underline{c}_0]\underline{c}_0^* + \frac{1}{u_ku_{\underline{k}}}b_1^*[b_1\underline{c}_1]\underline{c}_1^*$ $+ \omega_{hh}a^*b_1^*[b_1\underline{c}_1]\underline{c}_1^*\underline{a}^*[\underline{a}a]$ 

Table 5.A.39: flippant pair $\{\{B^+, \widetilde{C}_-\}, \{\widetilde{B}^-, C\}\}$.

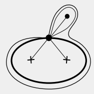
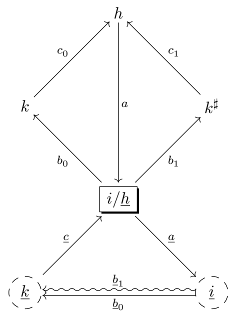
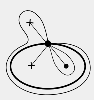
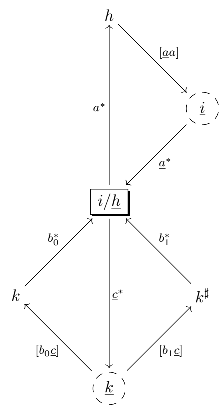
$\{B^+,\widetilde{C}\}$ 	$c_0b_0a - \frac{1}{u_k}c_1b_1a$ $+\ \underline{c}z_{\underline{k}}\,(\underline{b}_0+\underline{b}_1)\,z_{\underline{i}}\underline{a}$ $+\ \omega_{hh}c_1b_1\underline{c}\,(\underline{b}_0+\underline{b}_1)\,\underline{a}a$ 
$\widetilde{\varphi}$	$c_1 \mapsto -u_kc_1$ $[\underline{ac}]_0 \mapsto z_{\underline{i}}^{-1}[\underline{ac}]_0z_{\underline{k}}^{-1}$ $[\underline{ac}]_1 \mapsto z_{\underline{i}}^{-1}[\underline{ac}]_1z_{\underline{k}}^{-1}$ $\rightsquigarrow$ $[\underline{a}a] \mapsto z_{\underline{i}}[\underline{a}a]$ $[b_0\underline{c}] \mapsto [b_0\underline{c}]z_{\underline{k}}$ $[b_1\underline{c}] \mapsto -\frac{1}{u_k}[b_1\underline{c}]z_{\underline{k}}$ $\rightsquigarrow$ $a^* \mapsto -a^*$ $\underline{a}^* \mapsto -\underline{a}^*$
$\widetilde{\psi}$	$c_0 \rightsquigarrow a^*b_0^*$ $c_1 \rightsquigarrow a^*b_1^*$ $\underline{b}_0 \rightsquigarrow \pi_0\Big(z_{\underline{i}}^{-1}z_{\underline{k}}^{-1}\underline{c}^*\underline{a}^*\Big)$ $[\underline{ac}]_0 \rightsquigarrow -\pi_0\Big(z_{\underline{i}}z_{\underline{k}}[\underline{a}a]\,\omega_{hh}c_1[b_1\underline{c}]\Big)$ $\underline{b}_1 \rightsquigarrow \pi_1\Big(z_{\underline{i}}^{-*}z_{\underline{k}}^{-1}\underline{c}^*\underline{a}^*\Big)$ $[\underline{ac}]_1 \rightsquigarrow -\pi_1\Big(z_{\underline{i}}z_{\underline{k}}^*[\underline{a}a]\,\omega_{hh}c_1[b_1\underline{c}]\Big)$ $\rightsquigarrow$ $[b_1a] \rightsquigarrow -[b_1\underline{c}]\underline{c}^*\underline{a}^*[\underline{a}a]\,\omega_{hh}$ $[\underline{ac}]_0 \rightsquigarrow -\pi_0\Big(z_{\underline{i}}z_{\underline{k}}[\underline{a}a]\,\omega_{hh}a^*b_1^*[b_1\underline{c}]\Big)$ $[\underline{ac}]_1 \rightsquigarrow -\pi_1\Big(z_{\underline{i}}z_{\underline{k}}^*[\underline{a}a]\,\omega_{hh}a^*b_1^*[b_1\underline{c}]\Big)$
$\widetilde{T}_1$	$c_0\quad [b_0a]\quad c_1\quad [b_1a]\quad \underline{b}_0\quad [\underline{ac}]_0\quad \underline{b}_1\quad [\underline{ac}]_1$
$\{\widetilde{B}^-,\widetilde{C}_-\}$ 	$\underline{a}^*z_{\underline{i}}[\underline{a}a]a^*$ $+\ b_0^*[b_0\underline{c}]z_{\underline{k}}\underline{c}^* - \frac{1}{u_k}b_1^*[b_1\underline{c}]z_{\underline{k}}\underline{c}^*$ $+\ \omega_{hh}a^*b_1^*[b_1\underline{c}]\underline{c}^*\underline{a}^*[\underline{a}a]$ 

Table 5.A.40: flippant pair $\big\{\{B^+,\widetilde{C}\},\{\widetilde{B}^-,\widetilde{C}_-\}\big\}$.

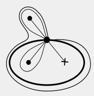
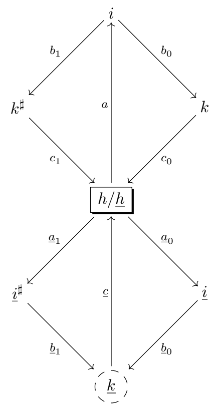
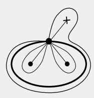
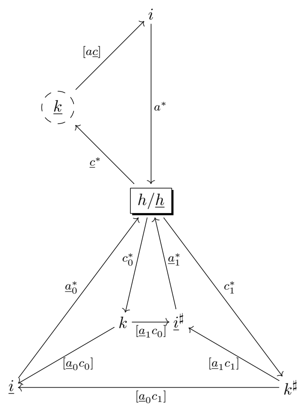
$\{B^-, \widetilde{C}_+\}$ 	$c_0b_0a - \frac{1}{u_k}c_1b_1a$ $+\ \underline{c}z_{\underline{k}}\underline{b}_0\underline{a}_0 - \frac{1}{u_{\underline{i}}}\underline{c}z_{\underline{k}}\underline{b}_1\underline{a}_1$ $+\ \omega_{ii}a\underline{c}\underline{b}_1\underline{a}_1c_1b_1$ 
$\widetilde{\varphi}$	$[ac_1] \mapsto -u_k[ac_1]$ $\underline{b}_0 \mapsto z_{\underline{k}}^{-1}\underline{b}_0$ $\underline{b}_1 \mapsto -u_{\underline{i}}z_{\underline{k}}^{-1}\underline{b}_1$ $\rightsquigarrow$ $[\underline{a}_1c_0] \mapsto -\frac{1}{u_{\underline{i}}}[\underline{a}_1c_0]$ $[\underline{a}_0c_1] \mapsto -\frac{1}{u_k}[\underline{a}_0c_1]$ $[\underline{a}_1c_1] \mapsto \frac{1}{u_{\underline{i}}u_k}[\underline{a}_1c_1]$ $[a\underline{c}] \mapsto [a\underline{c}]z_{\underline{k}}$ $\rightsquigarrow$ $c_0^* \mapsto -c_0^*$ $c_1^* \mapsto -c_1^*$ $\underline{a}_0^* \mapsto -\underline{a}_0^*$ $\underline{a}_1^* \mapsto -\underline{a}_1^*$
$\widetilde{\psi}$	$b_0 \rightsquigarrow c_0^*a^*$ $b_1 \rightsquigarrow -u_kc_1^*a^*$ $[ac_1] \rightsquigarrow \frac{1}{u_k}\omega_{ii}[a\underline{c}]\underline{b}_1[\underline{a}_1c_1]$ $\underline{b}_0 \rightsquigarrow \underline{c}^*\underline{a}_0^*$ $\underline{b}_1 \rightsquigarrow \underline{c}^*\underline{a}_1^*$ $\rightsquigarrow$ $[ac_1] \rightsquigarrow \frac{1}{u_k}\omega_{ii}[a\underline{c}]\underline{c}^*\underline{a}_1^*[\underline{a}_1c_1]$ $[\underline{a}_1\underline{c}] \rightsquigarrow -[\underline{a}_1c_1]c_1^*a^*\omega_{ii}[a\underline{c}]$
$\widetilde{T}_1$	$b_0 \quad [ac_0] \quad b_1 \quad [ac_1] \quad \underline{b}_0 \quad [\underline{a}_0\underline{c}] \quad \underline{b}_1 \quad [\underline{a}_1\underline{c}]$
$\{\widetilde{B}^+, C\}$ 	$[a\underline{c}]z_{\underline{k}}\underline{c}^*a^*$ $+\ \underline{a}_0^*[\underline{a}_0c_0]c_0^* - \frac{1}{u_k}\underline{a}_0^*[\underline{a}_0c_1]c_1^* - \frac{1}{u_{\underline{i}}}\underline{a}_1^*[\underline{a}_1c_0]c_0^* + \frac{1}{u_{\underline{i}}u_k}\underline{a}_1^*[\underline{a}_1c_1]c_1^*$ $+\ \omega_{ii}[a\underline{c}]\underline{c}^*\underline{a}_1^*[\underline{a}_1c_1]c_1^*a^*$ 

Table 5.A.41: flippant pair $\big\{\{B^-, \widetilde{C}_+\}, \{\widetilde{B}^+, C\}\big\}$.

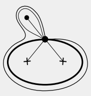
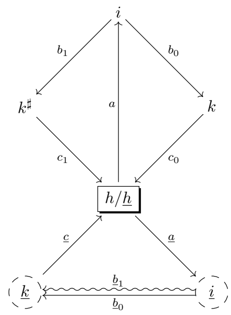
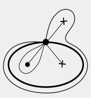
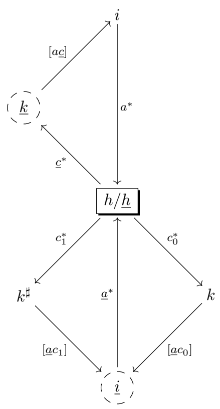
$\{B^-, \widetilde{C}\}$ 	$c_0 b_0 a - \frac{1}{u_k} c_1 b_1 a$ $+\ \underline{c} z_{\underline{k}} \,(\underline{b}_0 + \underline{b}_1)\, z_{\underline{i}} \underline{a}$ $+\ \ \omega_{ii} a\underline{c}\,(\underline{b}_0 + \underline{b}_1)\,\underline{a} c_1 b_1$ 
$\widetilde{\varphi}$	$[ac_1] \mapsto -u_k[ac_1]$ $\quad$ $[\underline{a}c_0] \mapsto z_{\underline{i}}[\underline{a}c_0]$ $\quad$ $c_0^* \mapsto -c_0^*$ $[\underline{ac}]_0 \mapsto z_{\underline{i}}^{-1}[\underline{ac}]_0 z_{\underline{k}}^{-1}$ $\quad\rightsquigarrow\quad$ $[\underline{a}c_1] \mapsto -\frac{1}{u_k} z_{\underline{i}}[\underline{a}c_1]$ $\quad\rightsquigarrow\quad$ $c_1^* \mapsto -c_1^*$ $[\underline{ac}]_1 \mapsto z_{\underline{i}}^{-1}[\underline{ac}]_1 z_{\underline{k}}^{-1}$ $\quad$ $[a\underline{c}] \mapsto [a\underline{c}] z_{\underline{k}}$ $\quad$ $\underline{a}^* \mapsto -\underline{a}^*$
$\widetilde{\psi}$	$b_0 \rightsquigarrow c_0^* a^*$ $b_1 \rightsquigarrow -u_k c_1^* a^*$ $\underline{b}_0 \rightsquigarrow \pi_0\Big(z_{\underline{i}}^{-1} z_{\underline{k}}^{-1} \underline{c}^* \underline{a}^*\Big)$ $[\underline{ac}]_0 \rightsquigarrow \pi_0\Big(\frac{1}{u_k} z_{\underline{i}} z_{\underline{k}} [\underline{a}c_1] b_1\, \omega_{ii} [a\underline{c}]\Big)$ $\underline{b}_1 \rightsquigarrow \pi_1\Big(z_{\underline{i}}^{-*} z_{\underline{k}}^{-1} \underline{c}^* \underline{a}^*\Big)$ $[\underline{ac}]_1 \rightsquigarrow \pi_1\Big(\frac{1}{u_k} z_{\underline{i}} z_{\underline{k}}^* [\underline{a}c_1] b_1\, \omega_{ii} [a\underline{c}]\Big)$ $\rightsquigarrow$ $[ac_1] \rightsquigarrow \frac{1}{u_k}\, \omega_{ii} [a\underline{c}] \underline{c}^* \underline{a}^* [\underline{a}c_1]$ $[\underline{ac}]_0 \rightsquigarrow -\pi_0\Big(z_{\underline{i}} z_{\underline{k}} [\underline{a}c_1] c_1^* a^*\, \omega_{ii} [a\underline{c}]\Big)$ $[\underline{ac}]_1 \rightsquigarrow -\pi_1\Big(z_{\underline{i}} z_{\underline{k}}^* [\underline{a}c_1] c_1^* a^*\, \omega_{ii} [a\underline{c}]\Big)$
$\widetilde{T}_1$	$b_0 \quad [ac_0] \quad b_1 \quad [ac_1] \quad \underline{b}_0 \quad [\underline{ac}]_0 \quad \underline{b}_1 \quad [\underline{ac}]_1$
$\{\widetilde{B}^+, \widetilde{C}_+\}$ 	$[a\underline{c}] z_{\underline{k}} \underline{c}^* a^*$ $+\ \underline{a}^* z_{\underline{i}} [\underline{a}c_0] c_0^* - \frac{1}{u_k} \underline{a}^* z_{\underline{i}} [\underline{a}c_1] c_1^*$ $+\ \ \omega_{ii} [a\underline{c}] \underline{c}^* \underline{a}^* [\underline{a}c_1] c_1^* a^*$ 

Table 5.A.42: flippant pair $\big\{\{B^-, \widetilde{C}\}, \{\widetilde{B}^+, \widetilde{C}_+\}\big\}$.

<table>
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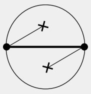
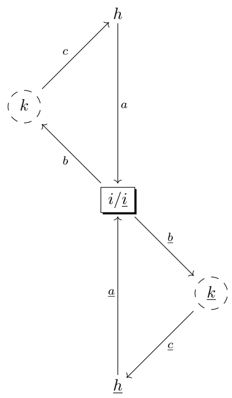
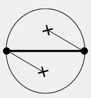
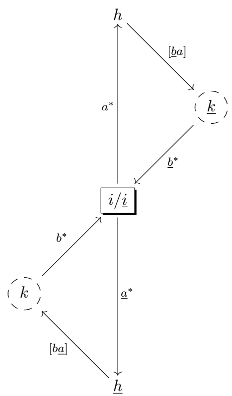
<td>$\{\widetilde{B}^{+}, \widetilde{B}^{+}\}$
</td>
<td>cz_kba
$+\ \underline{c}z_{\underline{k}}\underline{b}\underline{a}$
$+\ \omega_{h\underline{h}}\underline{c}\underline{b}a + \omega_{\underline{h}h}cb\underline{a}$
$+\ \omega_{hh}cb\underline{a}\,\omega_{\underline{h}\underline{h}}\underline{c}\underline{b}a$
</td>
</tr>
<tr>
<td>$\widetilde{\varphi}$</td>
<td>$c \mapsto cz_k^{-1}$
$\underline{c} \mapsto \underline{c}z_{\underline{k}}^{-1}$
$\rightsquigarrow$
$[\underline{b}a] \mapsto z_{\underline{k}}[\underline{b}a]$
$[b\underline{a}] \mapsto z_k[b\underline{a}]$
$\rightsquigarrow$
$a^* \mapsto -a^*$
$\underline{b}^* \mapsto -\underline{b}^*$</td>
</tr>
<tr>
<td>$\widetilde{\psi}$</td>
<td>$c \rightsquigarrow a^*b^*$
$[ba] \rightsquigarrow -[b\underline{a}]\,\omega_{h\underline{h}}$
$\underline{c} \rightsquigarrow \underline{a}^*\underline{b}^*$
$[\underline{b}\underline{a}] \rightsquigarrow -[\underline{b}a]\,\omega_{h\underline{h}}$
$\quad -[\underline{b}a]\,\omega_{hh}c[b\underline{a}]\,\omega_{\underline{h}\underline{h}}$
$\rightsquigarrow$
$[ba] \rightsquigarrow -[b\underline{a}]\,\omega_{\underline{h}\underline{h}}\underline{a}^*\underline{b}^*[\underline{b}a]\,\omega_{hh}$
$[\underline{b}\underline{a}] \rightsquigarrow -[\underline{b}a]\,\omega_{hh}a^*b^*[b\underline{a}]\,\omega_{\underline{h}\underline{h}}$</td>
</tr>
<tr>
<td>$\widetilde{T}_1$</td>
<td>$c \quad [ba] \quad \underline{c} \quad [\underline{b}\underline{a}]$</td>
</tr>
<tr>
<td>$\{\widetilde{B}^{-}, \widetilde{B}^{-}\}$
</td>
<td>$\underline{b}^*z_{\underline{k}}[\underline{b}a]a^*$
$+\ b^*z_k[b\underline{a}]\underline{a}^*$
$+\ \omega_{h\underline{h}}\underline{a}^*\underline{b}^*[ba] + \omega_{\underline{h}h}a^*b^*[\underline{b}\underline{a}]$
$+\ \omega_{hh}a^*b^*[b\underline{a}]\,\omega_{\underline{h}\underline{h}}\underline{a}^*\underline{b}^*[\underline{b}a]$
</td>
</tr>
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Table 5.A.43: flippant pair $\{\{\widetilde{B}^{+}, \widetilde{B}^{+}\}, \{\widetilde{B}^{-}, \widetilde{B}^{-}\}\}$.

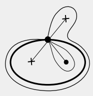
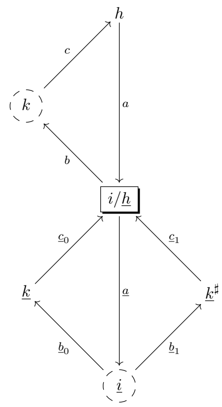
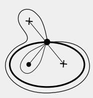
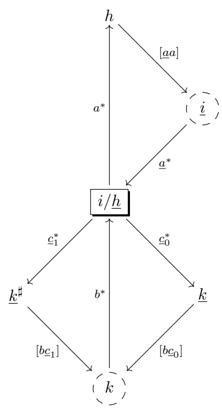
$\{\widetilde{B}^+,\widetilde{C}_-\}$ 	cz_kba $+\ \underline{c}_0\underline{b}_0 z_{\underline{i}}\underline{a} - \frac{1}{u_{\underline{k}}}\underline{c}_1\underline{b}_1 z_{\underline{i}}\underline{a}$ $+\ \omega_{hh} cb\underline{c}_1\underline{b}_1\underline{a}a$ 
$\widetilde{\varphi}$	$c \mapsto cz_k^{-1}$ $[\underline{ac}_0] \mapsto z_{\underline{i}}^{-1}[\underline{ac}_0]$ $[\underline{ac}_1] \mapsto -u_{\underline{k}} z_{\underline{i}}^{-1}[\underline{ac}_1]$ $\rightsquigarrow$ $[\underline{a}a] \mapsto z_{\underline{i}}[\underline{a}a]$ $[b\underline{c}_0] \mapsto z_k[b\underline{c}_0]$ $[b\underline{c}_1] \mapsto -\frac{1}{u_{\underline{k}}} z_k[b\underline{c}_1]$ $\rightsquigarrow$ $a^* \mapsto -a^*$ $\underline{a}^* \mapsto -\underline{a}^*$
$\widetilde{\psi}$	$c \rightsquigarrow a^*b^*$ $\underline{b}_0 \rightsquigarrow \underline{c}_0^*\underline{a}^* z_{\underline{i}}^{-1}$ $\underline{b}_1 \rightsquigarrow -u_{\underline{k}}\underline{c}_1^*\underline{a}^* z_{\underline{i}}^{-1}$ $[\underline{ac}_1] \rightsquigarrow \frac{1}{u_{\underline{k}}} z_{\underline{i}}[\underline{a}a]\,\omega_{hh} c[b\underline{c}_1]$ $\rightsquigarrow$ $[ba] \rightsquigarrow -[b\underline{c}_1]\underline{c}_1^*\underline{a}^*[\underline{a}a]\,\omega_{hh}$ $[\underline{ac}_1] \rightsquigarrow \frac{1}{u_{\underline{k}}} z_{\underline{i}}[\underline{a}a]\,\omega_{hh} a^*b^*[b\underline{c}_1]$
$\widetilde{T}_1$	c $[ba]$ $\underline{b}_0$ $[\underline{ac}_0]$ $\underline{b}_1$ $[\underline{ac}_1]$
$\{\widetilde{B}^-,\widetilde{C}_+\}$ 	$\underline{a}^* z_{\underline{i}}[\underline{a}a]a^*$ $+\ b^* z_k[b\underline{c}_0]\underline{c}_0^* - \frac{1}{u_{\underline{k}}} b^* z_k[b\underline{c}_1]\underline{c}_1^*$ $+\ \omega_{hh} a^*b^*[b\underline{c}_1]\underline{c}_1^*\underline{a}^*[\underline{a}a]$ 

Table 5.A.44: flippant pair $\{\{\widetilde{B}^+,\widetilde{C}_-\},\{\widetilde{B}^-,\widetilde{C}_+\}\}$.

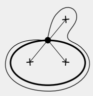
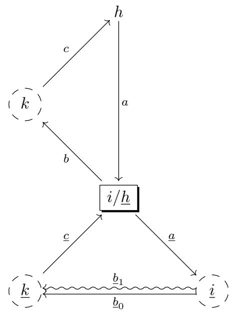
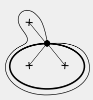
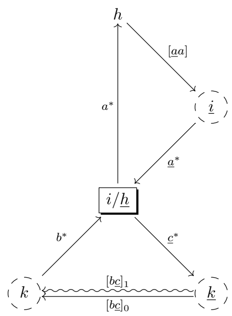
$\{\widetilde{B}^{+}, \widetilde{C}\}$ 	$cz_k ba$ $+\ \underline{c}z_{\underline{k}}\,(\underline{b}_0 + \underline{b}_1)\, z_{\underline{i}}\underline{a}$ $+\ \omega_{hh} cb\underline{c}\,(\underline{b}_0 + \underline{b}_1)\,\underline{a}a$ 
$\widetilde{\varphi}$	$c \mapsto cz_k^{-1}$ $[\underline{a}c]_0 \mapsto z_{\underline{i}}^{-1}[\underline{a}c]_0 z_{\underline{k}}^{-1}$ $[\underline{a}c]_1 \mapsto z_{\underline{i}}^{-1}[\underline{a}c]_1 z_{\underline{k}}^{-1}$ $\rightsquigarrow$ $[\underline{a}a] \mapsto z_{\underline{i}}[\underline{a}a]$ $[b\underline{c}]_0 \mapsto z_k[b\underline{c}]_0 z_{\underline{k}}$ $[b\underline{c}]_1 \mapsto z_k[b\underline{c}]_1 z_{\underline{k}}$ $\rightsquigarrow$ $a^* \mapsto -a^*$ $\underline{a}^* \mapsto -\underline{a}^*$
$\widetilde{\psi}$	$c \rightsquigarrow a^*b^*$ $\underline{b}_0 \rightsquigarrow \pi_0\Big(z_{\underline{i}}^{-1} z_{\underline{k}}^{-1}\underline{c}^*\underline{a}^*\Big)$ $[\underline{a}c]_0 \rightsquigarrow -\pi_0\Big(z_{\underline{i}} z_{\underline{k}}[\underline{a}a]\,\omega_{hh} c\,([b\underline{c}]_0 + [b\underline{c}]_1)\Big)$ $\underline{b}_1 \rightsquigarrow \pi_1\Big(z_{\underline{i}}^{-*} z_{\underline{k}}^{-1}\underline{c}^*\underline{a}^*\Big)$ $[\underline{a}c]_1 \rightsquigarrow -\pi_1\Big(z_{\underline{i}} z_{\underline{k}}^{*}[\underline{a}a]\,\omega_{hh} c\,([b\underline{c}]_0 + [b\underline{c}]_1)\Big)$ $\rightsquigarrow$ $[ba] \rightsquigarrow -([b\underline{c}]_0 + [b\underline{c}]_1)\,\underline{c}^*\underline{a}^*[\underline{a}a]\,\omega_{hh}$ $[\underline{a}c]_0 \rightsquigarrow -\pi_0\Big(z_{\underline{i}} z_{\underline{k}}[\underline{a}a]\,\omega_{hh} a^*b^*\,([b\underline{c}]_0 + [b\underline{c}]_1)\Big)$ $[\underline{a}c]_1 \rightsquigarrow -\pi_1\Big(z_{\underline{i}} z_{\underline{k}}^{*}[\underline{a}a]\,\omega_{hh} a^*b^*\,([b\underline{c}]_0 + [b\underline{c}]_1)\Big)$
$\widetilde{T}_1$	$c \quad [ba] \quad \underline{b}_0 \quad [\underline{a}c]_0 \quad \underline{b}_1 \quad [\underline{a}c]_1$
$\{\widetilde{B}^{-}, \widetilde{C}\}$ 	$\underline{a}^* z_{\underline{i}}[\underline{a}a]a^*$ $+\ b^* z_k\,([b\underline{c}]_0 + [b\underline{c}]_1)\, z_{\underline{k}}\underline{c}^*$ $+\ \omega_{hh} a^*b^*\,([b\underline{c}]_0 + [b\underline{c}]_1)\,\underline{c}^*\underline{a}^*[\underline{a}a]$ 

Table 5.A.45: flippant pair $\big\{\{\widetilde{B}^{+}, \widetilde{C}\}, \{\widetilde{B}^{-}, \widetilde{C}\}\big\}$.

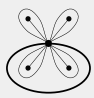
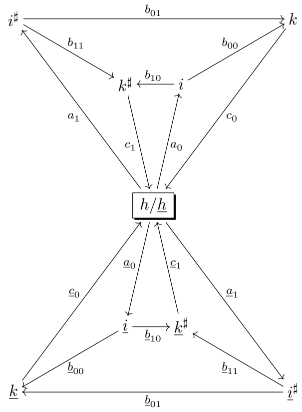
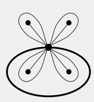
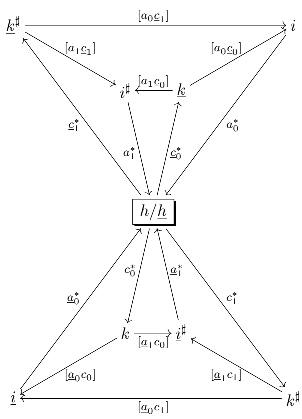
$\{C,C\}$ 	$c_0b_{00}a_0 - \frac{1}{u_i}c_0b_{01}a_1 - \frac{1}{u_k}c_1b_{10}a_0 + \frac{1}{u_iu_k}c_1b_{11}a_1$ $+\ \underline{c}_0\underline{b}_{00}\underline{a}_0 - \frac{1}{u_{\underline{i}}}\underline{c}_0\underline{b}_{01}\underline{a}_1 - \frac{1}{u_{\underline{k}}}\underline{c}_1\underline{b}_{10}\underline{a}_0 + \frac{1}{u_{\underline{k}}u_{\underline{i}}}\underline{c}_1\underline{b}_{11}\underline{a}_1$ $+\ u_0c_1b_{11}a_1\underline{c}_1\underline{b}_{11}\underline{a}_1$ 
$\widetilde{\varphi}$	$b_{01} \mapsto -u_ib_{01}$ $[a_0c_1] \mapsto -u_k[a_0c_1]$ $[a_1c_1] \mapsto u_iu_k[a_1c_1]$ $\underline{b}_{01} \mapsto -u_{\underline{i}}\underline{b}_{01}$ $[\underline{a}_0\underline{c}_1] \mapsto -u_{\underline{k}}[\underline{a}_0\underline{c}_1]$ $[\underline{a}_1\underline{c}_1] \mapsto u_{\underline{k}}u_{\underline{i}}[\underline{a}_1\underline{c}_1]$ $\rightsquigarrow$ $[\underline{a}_1c_0] \mapsto -\frac{1}{u_{\underline{i}}}[\underline{a}_1c_0]$ $[\underline{a}_0c_1] \mapsto -\frac{1}{u_k}[\underline{a}_0c_1]$ $[\underline{a}_1c_1] \mapsto \frac{1}{u_ku_{\underline{i}}}[\underline{a}_1c_1]$ $[a_1\underline{c}_0] \mapsto -\frac{1}{u_i}[a_1\underline{c}_0]$ $[a_0\underline{c}_1] \mapsto -\frac{1}{u_{\underline{k}}}[a_0\underline{c}_1]$ $[a_1\underline{c}_1] \mapsto \frac{1}{u_{\underline{k}}u_i}[a_1\underline{c}_1]$ $\rightsquigarrow$ $c_0^* \mapsto -c_0^*$ $c_1^* \mapsto -c_1^*$ $\underline{a}_0^* \mapsto -\underline{a}_0^*$ $\underline{a}_1^* \mapsto -\underline{a}_1^*$
$\widetilde{\psi}$	$b_{00} \rightsquigarrow c_0^*a_0^*$ $b_{01} \rightsquigarrow c_0^*a_1^*$ $b_{10} \rightsquigarrow -u_kc_1^*a_0^*$ $b_{11} \rightsquigarrow u_iu_kc_1^*a_1^*$ $\underline{b}_{00} \rightsquigarrow \underline{c}_0^*\underline{a}_0^*$ $\underline{b}_{01} \rightsquigarrow \underline{c}_0^*\underline{a}_1^*$ $\underline{b}_{10} \rightsquigarrow -u_{\underline{k}}\underline{c}_1^*\underline{a}_0^*$ $\underline{b}_{11} \rightsquigarrow u_{\underline{k}}u_{\underline{i}}\underline{c}_1^*\underline{a}_1^*$ $[\underline{a}_1\underline{c}_1] \rightsquigarrow -\frac{u_0}{u_{\underline{k}}u_iu_ku_{\underline{i}}}[\underline{a}_1c_1]b_{11}[a_1\underline{c}_1]$ $\rightsquigarrow$ $[a_1c_1] \rightsquigarrow -\frac{u_0}{u_iu_k}[a_1\underline{c}_1]\underline{c}_1^*\underline{a}_1^*[\underline{a}_1c_1]$ $[\underline{a}_1\underline{c}_1] \rightsquigarrow -\frac{u_0}{u_{\underline{k}}u_{\underline{i}}}[\underline{a}_1c_1]c_1^*a_1^*[a_1\underline{c}_1]$
$\widetilde{T}_1$	b_{00} $[a_0c_0]$ b_{01} $[a_1c_0]$ b_{10} $[a_0c_1]$ b_{11} $[a_1c_1]$ $\underline{b}_{00}$ $[\underline{a}_0\underline{c}_0]$ $\underline{b}_{01}$ $[\underline{a}_1\underline{c}_0]$ $\underline{b}_{10}$ $[\underline{a}_0\underline{c}_1]$ $\underline{b}_{11}$ $[\underline{a}_1\underline{c}_1]$
$\{C,C\}$ 	$a_0^*[a_0\underline{c}_0]\underline{c}_0^* - \frac{1}{u_{\underline{k}}}a_0^*[a_0\underline{c}_1]\underline{c}_1^* - \frac{1}{u_i}a_1^*[a_1\underline{c}_0]\underline{c}_0^* + \frac{1}{u_{\underline{k}}u_i}a_1^*[a_1\underline{c}_1]\underline{c}_1^*$ $+\ \underline{a}_0^*[\underline{a}_0c_0]c_0^* - \frac{1}{u_k}\underline{a}_0^*[\underline{a}_0c_1]c_1^* - \frac{1}{u_{\underline{i}}}\underline{a}_1^*[\underline{a}_1c_0]c_0^* + \frac{1}{u_ku_{\underline{i}}}\underline{a}_1^*[\underline{a}_1c_1]c_1^*$ $+\ u_0a_1^*[a_1\underline{c}_1]\underline{c}_1^*\underline{a}_1^*[\underline{a}_1c_1]c_1^*$ 

Table 5.A.46: flippant pair $\{\{C,C\},\{C,C\}\}$.

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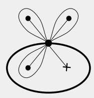
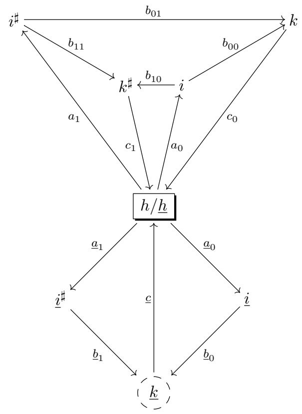
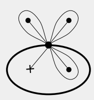
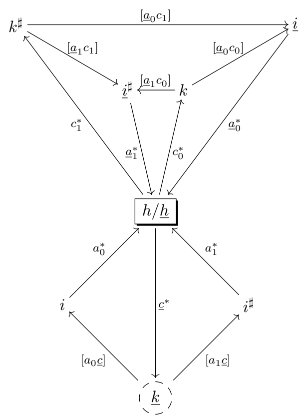
<td>$\{C, \widetilde{C}_+\}$
</td>
<td>$c_0b_{00}a_0 - \frac{1}{u_i}c_0b_{01}a_1 - \frac{1}{u_k}c_1b_{10}a_0 + \frac{1}{u_iu_k}c_1b_{11}a_1$
$+ \underline{c}z_{\underline{k}}\underline{b}_0\underline{a}_0 - \frac{1}{u_{\underline{i}}}\underline{c}z_{\underline{k}}\underline{b}_1\underline{a}_1$
$+ u_0c_1b_{11}a_1\underline{c}\underline{b}_1\underline{a}_1$

</td>
</tr>
<tr>
<td>$\widetilde{\varphi}$</td>
<td>$b_{01} \mapsto -u_ib_{01}$
$[a_0c_1] \mapsto -u_k[a_0c_1]$
$[a_1c_1] \mapsto u_iu_k[a_1c_1]$
$\underline{b}_0 \mapsto z_{\underline{k}}^{-1}\underline{b}_0$
$\underline{b}_1 \mapsto -u_{\underline{i}}z_{\underline{k}}^{-1}\underline{b}_1$
$\rightsquigarrow$
$[\underline{a}_1c_0] \mapsto -\frac{1}{u_{\underline{i}}}[\underline{a}_1c_0]$
$[\underline{a}_0c_1] \mapsto -\frac{1}{u_k}[\underline{a}_0c_1]$
$[\underline{a}_1c_1] \mapsto \frac{1}{u_ku_{\underline{i}}}[\underline{a}_1c_1]$
$[a_0\underline{c}] \mapsto [a_0\underline{c}]z_{\underline{k}}$
$[a_1\underline{c}] \mapsto -\frac{1}{u_i}[a_1\underline{c}]z_{\underline{k}}$
$\rightsquigarrow$
$c_0^* \mapsto -c_0^*$
$c_1^* \mapsto -c_1^*$
$\underline{a}_0^* \mapsto -\underline{a}_0^*$
$\underline{a}_1^* \mapsto -\underline{a}_1^*$</td>
</tr>
<tr>
<td>$\widetilde{\psi}$</td>
<td>$b_{00} \rightsquigarrow c_0^*a_0^*$
$b_{01} \rightsquigarrow c_0^*a_1^*$
$b_{10} \rightsquigarrow -u_kc_1^*a_0^*$
$b_{11} \rightsquigarrow u_iu_kc_1^*a_1^*$
$[a_1c_1] \rightsquigarrow -\frac{u_0}{u_iu_k}[a_1\underline{c}]\underline{b}_1[\underline{a}_1c_1]$
$\underline{b}_0 \rightsquigarrow \underline{c}^*\underline{a}_0^*$
$\underline{b}_1 \rightsquigarrow \underline{c}^*\underline{a}_1^*$
$\rightsquigarrow$
$[a_1c_1] \rightsquigarrow -\frac{u_0}{u_iu_k}[a_1\underline{c}]\underline{c}^*\underline{a}_1^*[\underline{a}_1c_1]$
$[\underline{a}_1\underline{c}] \rightsquigarrow -u_0[\underline{a}_1c_1]c_1^*a_1^*[a_1\underline{c}]$</td>
</tr>
<tr>
<td>$\widetilde{T}_1$</td>
<td>b_{00} $[a_0c_0]$ b_{01} $[a_1c_0]$ b_{10} $[a_0c_1]$ b_{11} $[a_1c_1]$ $\underline{b}_0$ $[\underline{a}_0\underline{c}]$ $\underline{b}_1$ $[\underline{a}_1\underline{c}]$</td>
</tr>
<tr>
<td>$\{C, \widetilde{C}_-\}$
</td>
<td>$\underline{a}_0^*[\underline{a}_0c_0]c_0^* - \frac{1}{u_k}\underline{a}_0^*[\underline{a}_0c_1]c_1^* - \frac{1}{u_{\underline{i}}}\underline{a}_1^*[\underline{a}_1c_0]c_0^* + \frac{1}{u_ku_{\underline{i}}}\underline{a}_1^*[\underline{a}_1c_1]c_1^*$
$+ a_0^*[a_0\underline{c}]z_{\underline{k}}\underline{c}^* - \frac{1}{u_i}a_1^*[a_1\underline{c}]z_{\underline{k}}\underline{c}^*$
$+ u_0\underline{a}_1^*[\underline{a}_1c_1]c_1^*a_1^*[a_1\underline{c}]\underline{c}^*$

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Table 5.A.47: flippant pair $\big\{\{C, \widetilde{C}_+\}, \{C, \widetilde{C}_-\}\big\}$.

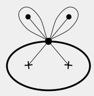
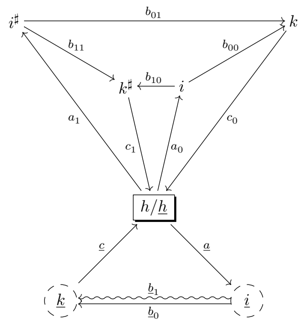
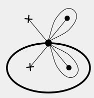
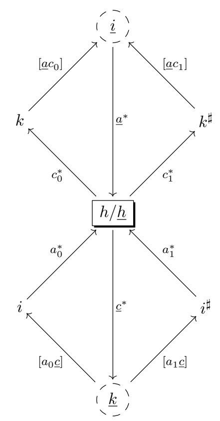
$\{C, \widetilde{C}\}$ 	$c_0 b_{00} a_0 - \frac{1}{u_i} c_0 b_{01} a_1 - \frac{1}{u_k} c_1 b_{10} a_0 + \frac{1}{u_k u_i} c_1 b_{11} a_1$ $+ \underline{c} z_{\underline{k}} (\underline{b}_0 + \underline{b}_1) z_{\underline{i}} \underline{a}$ $+ u_0 c_1 b_{11} a_1 \underline{c} (\underline{b}_0 + \underline{b}_1) \underline{a}$ 
$\widetilde{\varphi}$	$\begin{array}{lcl} b_{01} & \mapsto & -u_i b_{01} \\ [a_0 c_1] & \mapsto & -u_k [a_0 c_1] \\ [a_1 c_1] & \mapsto & u_k u_i [a_1 c_1] \\ [\underline{ac}]_0 & \mapsto & z_{\underline{i}}^{-1} [\underline{ac}]_0 z_{\underline{k}}^{-1} \\ [\underline{ac}]_1 & \mapsto & z_{\underline{i}}^{-1} [\underline{ac}]_1 z_{\underline{k}}^{-1} \end{array} \rightsquigarrow \begin{array}{lcl} [\underline{a} c_0] & \mapsto & z_{\underline{i}} [\underline{a} c_0] \\ [\underline{a} c_1] & \mapsto & -\frac{1}{u_k} z_{\underline{i}} [\underline{a} c_1] \\ [a_0 \underline{c}] & \mapsto & [a_0 \underline{c}] z_{\underline{k}} \\ [a_1 \underline{c}] & \mapsto & -\frac{1}{u_i} [a_1 \underline{c}] z_{\underline{k}} \end{array} \rightsquigarrow \begin{array}{lcl} c_0^* & \mapsto & -c_0^* \\ c_1^* & \mapsto & -c_1^* \\ \underline{a}^* & \mapsto & -\underline{a}^* \end{array}$
$\widetilde{\psi}$	$\begin{array}{lcl} b_{00} & \rightsquigarrow & c_0^* a_0^* \\ b_{01} & \rightsquigarrow & c_0^* a_1^* \\ b_{10} & \rightsquigarrow & -u_k c_1^* a_0^* \\ b_{11} & \rightsquigarrow & u_k u_i c_1^* a_1^* \\ \underline{b}_0 & \rightsquigarrow & \pi_0\Big(z_{\underline{i}}^{-1} z_{\underline{k}}^{-1} \underline{c}^* \underline{a}^*\Big) \\ [\underline{ac}]_0 & \rightsquigarrow & -\pi_0\Big(\frac{u_0}{u_k u_i} z_{\underline{i}} z_{\underline{k}} [\underline{a} c_1] b_{11} [a_1 \underline{c}]\Big) \\ \underline{b}_1 & \rightsquigarrow & \pi_1\Big(z_{\underline{i}}^{-*} z_{\underline{k}}^{-1} \underline{c}^* \underline{a}^*\Big) \\ [\underline{ac}]_1 & \rightsquigarrow & -\pi_1\Big(\frac{u_0}{u_k u_i} z_{\underline{i}} z_{\underline{k}}^* [\underline{a} c_1] b_{11} [a_1 \underline{c}]\Big) \end{array} \rightsquigarrow \begin{array}{lcl} [a_1 c_1] & \rightsquigarrow & -\frac{u_0}{u_k u_i} [a_1 \underline{c}] \underline{c}^* \underline{a}^* [\underline{a} c_1] \\ [\underline{ac}]_0 & \rightsquigarrow & -\pi_0\Big(u_0 z_{\underline{i}} z_{\underline{k}} [\underline{a} c_1] c_1^* a_1^* [a_1 \underline{c}]\Big) \\ [\underline{ac}]_1 & \rightsquigarrow & -\pi_1\Big(u_0 z_{\underline{i}} z_{\underline{k}}^* [\underline{a} c_1] c_1^* a_1^* [a_1 \underline{c}]\Big) \end{array}$
$\widetilde{T}_1$	b_{00} $[a_0 c_0]$ b_{01} $[a_1 c_0]$ b_{10} $[a_0 c_1]$ b_{11} $[a_1 c_1]$ $\underline{b}_0$ $[\underline{ac}]_0$ $\underline{b}_1$ $[\underline{ac}]_1$
$\{\widetilde{C}_+, \widetilde{C}_-\}$ 	$\underline{a}^* z_{\underline{i}} [\underline{a} c_0] c_0^* - \frac{1}{u_k} \underline{a}^* z_{\underline{i}} [\underline{a} c_1] c_1^*$ $+ a_0^* [a_0 \underline{c}] z_{\underline{k}} \underline{c}^* - \frac{1}{u_i} a_1^* [a_1 \underline{c}] z_{\underline{k}} \underline{c}^*$ $+ u_0 \underline{a}^* [\underline{a} c_1] c_1^* a_1^* [a_1 \underline{c}] \underline{c}^*$ 

Table 5.A.48: flippant pair $\big\{\{C, \widetilde{C}\}, \{\widetilde{C}_+, \widetilde{C}_-\}\big\}$.

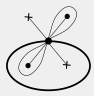
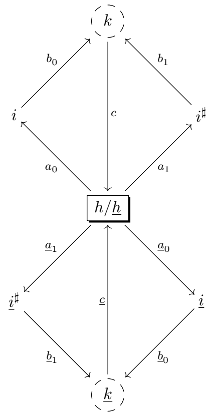
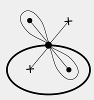
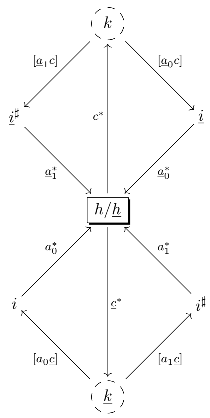
$\{\widetilde{C}_+, \widetilde{C}_+\}$ 	$cz_kb_0a_0 - \frac{1}{u_i}cz_kb_1a_1$ $+\ \underline{c}z_{\underline{k}}\underline{b}_0\underline{a}_0 - \frac{1}{u_{\underline{i}}}\underline{c}z_{\underline{k}}\underline{b}_1\underline{a}_1$ $+\ u_0cb_1a_1\underline{c}\underline{b}_1\underline{a}_1$ 
$\widetilde{\varphi}$	$b_0 \mapsto z_k^{-1}b_0$ $b_1 \mapsto -u_iz_k^{-1}b_1$ $\underline{b}_0 \mapsto z_{\underline{k}}^{-1}\underline{b}_0$ $\underline{b}_1 \mapsto -u_{\underline{i}}z_{\underline{k}}^{-1}\underline{b}_1$ $\rightsquigarrow$ $[\underline{a}_0c] \mapsto [\underline{a}_0c]z_k$ $[\underline{a}_1c] \mapsto -\frac{1}{u_{\underline{i}}}[\underline{a}_1c]z_k$ $[a_0\underline{c}] \mapsto [a_0\underline{c}]z_{\underline{k}}$ $[a_1\underline{c}] \mapsto -\frac{1}{u_i}[a_1\underline{c}]z_{\underline{k}}$ $\rightsquigarrow$ $c^* \mapsto -c^*$ $\underline{a}_0^* \mapsto -\underline{a}_0^*$ $\underline{a}_1^* \mapsto -\underline{a}_1^*$
$\widetilde{\psi}$	$b_0 \rightsquigarrow c^*a_0^*$ $b_1 \rightsquigarrow c^*a_1^*$ $\underline{b}_0 \rightsquigarrow \underline{c}^*\underline{a}_0^*$ $\underline{b}_1 \rightsquigarrow \underline{c}^*\underline{a}_1^*$ $[\underline{a}_1\underline{c}] \rightsquigarrow -u_0[\underline{a}_1c]b_1[a_1\underline{c}]$ $\rightsquigarrow$ $[a_1c] \rightsquigarrow -u_0[a_1\underline{c}]\underline{c}^*\underline{a}_1^*[\underline{a}_1c]$ $[\underline{a}_1\underline{c}] \rightsquigarrow -u_0[\underline{a}_1c]c^*a_1^*[a_1\underline{c}]$
$\widetilde{T}_1$	$b_0 \quad [a_0c] \quad b_1 \quad [a_1c] \quad \underline{b}_0 \quad [\underline{a}_0\underline{c}] \quad \underline{b}_1 \quad [\underline{a}_1\underline{c}]$
$\{\widetilde{C}_-, \widetilde{C}_-\}$ 	$\underline{a}_0^*[\underline{a}_0c]z_kc^* - \frac{1}{u_{\underline{i}}}\underline{a}_1^*[\underline{a}_1c]z_kc^*$ $+\ a_0^*[a_0\underline{c}]z_{\underline{k}}\underline{c}^* - \frac{1}{u_i}a_1^*[a_1\underline{c}]z_{\underline{k}}\underline{c}^*$ $+\ u_0\underline{a}_1^*[\underline{a}_1c]c^*a_1^*[a_1\underline{c}]\underline{c}^*$ 

Table 5.A.49: flippant pair $\{\{\widetilde{C}_+, \widetilde{C}_+\}, \{\widetilde{C}_-, \widetilde{C}_-\}\}$.

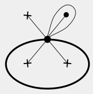
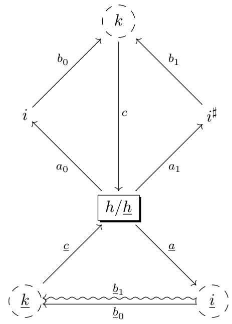
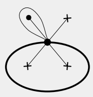
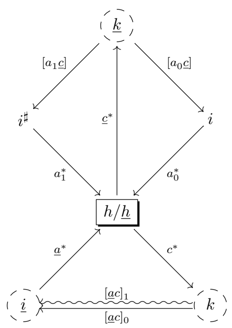
$\{\widetilde{C}_+,\widetilde{C}\}$ 	$cz_kb_0a_0 - \frac{1}{u_i}cz_kb_1a_1$ $+\ \underline{c}z_{\underline{k}}\left(\underline{b}_0+\underline{b}_1\right)z_{\underline{i}}\underline{a}$ $+\ u_0cb_1a_1\underline{c}\left(\underline{b}_0+\underline{b}_1\right)\underline{a}$ 
$\widetilde{\varphi}$	$\begin{array}{ll} b_0 & \mapsto z_k^{-1}b_0 \\ b_1 & \mapsto -u_iz_k^{-1}b_1 \\ [\underline{ac}]_0 & \mapsto z_{\underline{i}}^{-1}[\underline{ac}]_0z_{\underline{k}}^{-1} \\ [\underline{ac}]_1 & \mapsto z_{\underline{i}}^{-1}[\underline{ac}]_1z_{\underline{k}}^{-1} \end{array} \rightsquigarrow \begin{array}{ll} [\underline{ac}]_0 & \mapsto z_{\underline{i}}[\underline{ac}]_0z_k \\ [\underline{ac}]_1 & \mapsto z_{\underline{i}}[\underline{ac}]_1z_k \\ [a_0\underline{c}] & \mapsto [a_0\underline{c}]z_{\underline{k}} \\ [a_1\underline{c}] & \mapsto -\frac{1}{u_i}[a_1\underline{c}]z_{\underline{k}} \end{array} \rightsquigarrow \begin{array}{ll} c^* & \mapsto -c^* \\ \underline{a}^* & \mapsto -\underline{a}^* \end{array}$
$\widetilde{\psi}$	$\begin{array}{ll} b_0 & \rightsquigarrow c^*a_0^* \\ b_1 & \rightsquigarrow c^*a_1^* \\ \underline{b}_0 & \rightsquigarrow \pi_0\Big(z_{\underline{i}}^{-1}z_{\underline{k}}^{-1}\underline{c}^*\underline{a}^*\Big) \\ [\underline{ac}]_0 & \rightsquigarrow -\pi_0\Big(u_0z_{\underline{i}}z_{\underline{k}}\left([\underline{ac}]_0+[\underline{ac}]_1\right)b_1[a_1\underline{c}]\Big) \\ \underline{b}_1 & \rightsquigarrow \pi_1\Big(z_{\underline{i}}^{-*}z_{\underline{k}}^{-1}\underline{c}^*\underline{a}^*\Big) \\ [\underline{ac}]_1 & \rightsquigarrow -\pi_1\Big(u_0z_{\underline{i}}z_{\underline{k}}^*\left([\underline{ac}]_0+[\underline{ac}]_1\right)b_1[a_1\underline{c}]\Big) \end{array} \rightsquigarrow \begin{array}{ll} [a_1c] & \rightsquigarrow -u_0[a_1\underline{c}]\underline{c}^*\underline{a}^*\left([\underline{ac}]_0+[\underline{ac}]_1\right) \\ [\underline{ac}]_0 & \rightsquigarrow -\pi_0\Big(u_0z_{\underline{i}}z_{\underline{k}}\left([\underline{ac}]_0+[\underline{ac}]_1\right)c^*a_1^*[a_1\underline{c}]\Big) \\ [\underline{ac}]_1 & \rightsquigarrow -\pi_1\Big(u_0z_{\underline{i}}z_{\underline{k}}^*\left([\underline{ac}]_0+[\underline{ac}]_1\right)c^*a_1^*[a_1\underline{c}]\Big) \end{array}$
$\widetilde{T}_1$	$b_0 \quad [a_0c] \quad b_1 \quad [a_1c] \quad \underline{b}_0 \quad [\underline{ac}]_0 \quad \underline{b}_1 \quad [\underline{ac}]_1$
$\{\widetilde{C}_-,\widetilde{C}\}$ 	$a_0^*[a_0\underline{c}]z_{\underline{k}}\underline{c}^* - \frac{1}{u_i}a_1^*[a_1\underline{c}]z_{\underline{k}}\underline{c}^*$ $+\ \underline{a}^*z_{\underline{i}}\left([\underline{ac}]_0+[\underline{ac}]_1\right)z_kc^*$ $+\ u_0a_1^*[a_1\underline{c}]\underline{c}^*\underline{a}^*\left([\underline{ac}]_0+[\underline{ac}]_1\right)c^*$ 

Table 5.A.50: flippant pair $\{\{\widetilde{C}_+,\widetilde{C}\},\{\widetilde{C}_-,\widetilde{C}\}\}$.

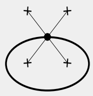
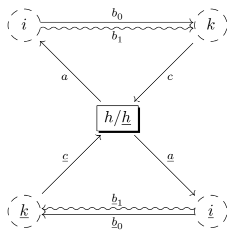
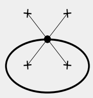
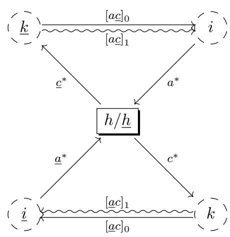
$\{\widetilde{C},\widetilde{C}\}$ 	$cz_k\,(b_0+b_1)\,z_i a$ $+\ \underline{c}z_{\underline{k}}\,(\underline{b}_0+\underline{b}_1)\,z_{\underline{i}}\underline{a}$ $+\ u_0 c\,(b_0+b_1)\,a\underline{c}\,(\underline{b}_0+\underline{b}_1)\,\underline{a}$	
$\widetilde{\varphi}$	$[ac]_0 \mapsto z_i^{-1}[ac]_0 z_k^{-1}$ $[ac]_1 \mapsto z_i^{-1}[ac]_1 z_k^{-1}$ $[\underline{ac}]_0 \mapsto z_{\underline{i}}^{-1}[\underline{ac}]_0 z_{\underline{k}}^{-1}$ $[\underline{ac}]_1 \mapsto z_{\underline{i}}^{-1}[\underline{ac}]_1 z_{\underline{k}}^{-1}$ $\rightsquigarrow$ $[\underline{a}c]_0 \mapsto z_{\underline{i}}[\underline{a}c]_0 z_k$ $[\underline{a}c]_1 \mapsto z_{\underline{i}}[\underline{a}c]_1 z_k$ $[a\underline{c}]_0 \mapsto z_i[a\underline{c}]_0 z_{\underline{k}}$ $[a\underline{c}]_1 \mapsto z_i[a\underline{c}]_1 z_{\underline{k}}$ $\rightsquigarrow$ $c^* \mapsto -c^*$ $\underline{a}^* \mapsto -\underline{a}^*$	
$\widetilde{\psi}$	$b_0 \rightsquigarrow \pi_0\Big(z_i^{-1}z_k^{-1}c^*a^*\Big)$ $b_1 \rightsquigarrow \pi_1\Big(z_i^{-*}z_k^{-1}c^*a^*\Big)$ $\underline{b}_0 \rightsquigarrow \pi_0\Big(z_{\underline{i}}^{-1}z_{\underline{k}}^{-1}\underline{c}^*\underline{a}^*\Big)$ $[\underline{ac}]_0 \rightsquigarrow -\pi_0\Big(u_0[\underline{a}c]z_k b z_i[a\underline{c}]z_{\underline{i}}z_{\underline{k}}\Big)$ $\underline{b}_1 \rightsquigarrow \pi_1\Big(z_{\underline{i}}^{-*}z_{\underline{k}}^{-1}\underline{c}^*\underline{a}^*\Big)$ $[\underline{ac}]_1 \rightsquigarrow -\pi_1\Big(u_0[\underline{a}c]z_k b z_i[a\underline{c}]z_{\underline{i}}^{*}z_{\underline{k}}\Big)$ $\rightsquigarrow$ $[ac]_0 \rightsquigarrow -\pi_0\Big(u_0[a\underline{c}]\underline{c}^*\underline{a}^*[\underline{a}c]z_i z_k\Big)$ $[ac]_1 \rightsquigarrow -\pi_1\Big(u_0[a\underline{c}]\underline{c}^*\underline{a}^*[\underline{a}c]z_i^* z_k\Big)$ $[\underline{a}c]_0 \rightsquigarrow -\pi_0\Big(u_0 z_{\underline{i}}z_{\underline{k}}[\underline{a}c]c^*a^*[a\underline{c}]\Big)$ $[\underline{a}c]_1 \rightsquigarrow -\pi_1\Big(u_0 z_{\underline{i}}z_{\underline{k}}^{*}[\underline{a}c]c^*a^*[a\underline{c}]\Big)$ $\big(b := b_0+b_1\,,\quad [\underline{a}c] := [\underline{a}c]_0+[\underline{a}c]_1\,,\quad [a\underline{c}] := [a\underline{c}]_0+[a\underline{c}]_1\big)$	
$\widetilde{T}_1$	$b_0 \quad [ac]_0 \quad b_1 \quad [ac]_1 \quad \underline{b}_0 \quad [\underline{ac}]_0 \quad \underline{b}_1 \quad [\underline{ac}]_1$	
$\{\widetilde{C},\widetilde{C}\}$ 	$a^*z_i\,([a\underline{c}]_0+[a\underline{c}]_1)\,z_{\underline{k}}\underline{c}^*$ $+\ \underline{a}^*z_{\underline{i}}\,([\underline{a}c]_0+[\underline{a}c]_1)\,z_k c^*$ $+\ u_0 a^*\,([a\underline{c}]_0+[a\underline{c}]_1)\,\underline{c}^*\underline{a}^*\,([\underline{a}c]_0+[\underline{a}c]_1)\,c^*$	

Table 5.A.51: flippant pair $\big\{\{\widetilde{C},\widetilde{C}\},\{\widetilde{C},\widetilde{C}\}\big\}$.

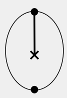
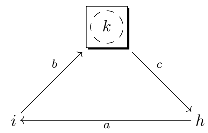
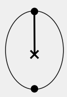
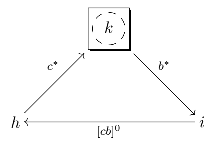
$\widetilde{B}$ 	cz_kba $+\ \omega_{hi}a$ $+\ \omega_{ih}cb$ $+\ \omega_{ii}a\,\omega_{hh}cb$ 
$\widetilde{\varphi}$	$[cb]^1 \mapsto -\frac{x_k}{y_k}[cb]^0 + \frac{1}{y_k}[cb]^1 \quad \rightsquigarrow \quad b^* \mapsto -2y_k b^* v$
$\widetilde{\psi}$	$a \quad \rightsquigarrow \quad b^*c^*$ $[cb]^1 \rightsquigarrow -\omega_{hi}$ $\qquad -\omega_{hh}[cb]^0\,\omega_{ii}$
$\widetilde{T}_1$	$a \quad [cb]^1$
$\widetilde{B}$ 	$b^*z_k^*c^*[cb]^0$ $+\ \omega_{ih}[cb]^0$ $+\ \omega_{hi}b^*c^*$ $+\ \omega_{ii}b^*c^*\,\omega_{hh}[cb]^0$ 

Table 5.A.52: flippant pair $\{\widetilde{B}, \widetilde{B}\}$.

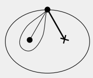
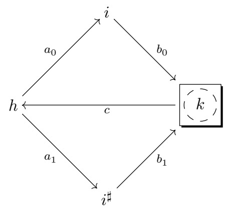
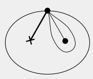
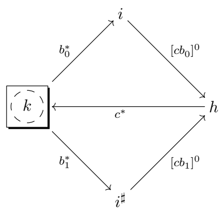
$\widetilde{C}^+_+$ 	$cz_kb_0a_0 - \frac{1}{u_i}cz_kb_1a_1$ $+\ \omega_{hh}cb_1a_1$ 
$\widetilde{\varphi}$	$[cb_0]^1 \mapsto -\frac{x_k}{y_k}[cb_0]^0 + \frac{1}{y_k}[cb_0]^1$ $[cb_1]^1 \mapsto -\frac{x_k}{y_k}[cb_1]^0 - \frac{u_i}{y_k}[cb_1]^1$ $\rightsquigarrow$ $b_0^* \mapsto -2y_kb_0^*v$ $b_1^* \mapsto -2y_kb_1^*v$ $[cb_1]^0 \mapsto -\frac{1}{u_i}[cb_1]^0$
$\widetilde{\psi}$	$a_0 \rightsquigarrow b_0^*c^*$ $a_1 \rightsquigarrow -u_ib_1^*c^*$ $[cb_1]^1 \rightsquigarrow \frac{1}{u_i}\omega_{hh}[cb_1]^0$
$\widetilde{T}_1$	$a_0 \quad [cb_0]^1 \quad a_1 \quad [cb_1]^1$
$\widetilde{C}^-_-$ 	$[cb_0]^0b_0^*z_k^*c^* - \frac{1}{u_i}[cb_1]^0b_1^*z_k^*c^*$ $+\ \omega_{hh}[cb_1]^0b_1^*c^*$ 

Table 5.A.53: flippant pair $\{\widetilde{C}^+_+, \widetilde{C}^-_-\}$.

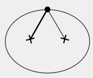
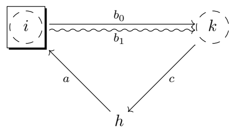
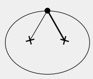
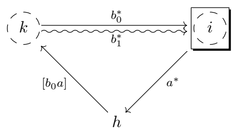
$\widetilde{C}^{-}$ 	$cz_k\,(b_0 + b_1)\,z_i a$ $+\ \omega_{hh} c\,(b_0 + b_1)\,a$	
$\widetilde{\varphi}$	$[b_1 a] \mapsto -z_i z_k [b_0 a] - z_k^{-1}[b_1 a]$ $[b_0 a] \mapsto z_i^* z_k [b_0 a] + z_k^{-1}[b_1 a]$ $\rightsquigarrow$	$c \mapsto \frac{1}{2}\frac{1}{w y_i} cv$ $b_1^* \mapsto -b_1^*$
$\widetilde{\psi}$	$c \rightsquigarrow -a^*\,(b_0^* + b_1^*)\,z_k^{-1}$ $[b_1 a] \rightsquigarrow z_k [b_0 a]\,\omega_{hh}$	
$\widetilde{T}_1$	$c \quad [b_1 a]$	
$\widetilde{C}^{+}$ 	$a^* z_i^*\,(b_0^* + b_1^*)\,z_k [b_0 a]$ $+\ \omega_{hh} a^*\,(b_0^* + b_1^*)\,[b_0 a]$	

Table 5.A.54: flippant pair $\left\{\widetilde{C}^{-}, \widetilde{C}^{+}\right\}$.

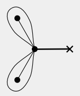
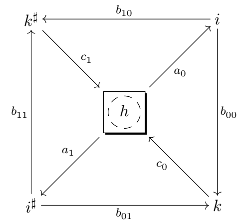
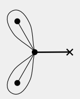
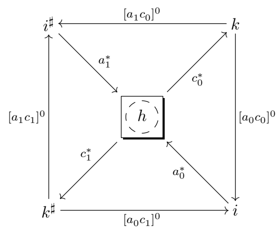
$\widetilde{D}_1$ 	$c_0 b_{00} a_0 z_h$ $- \frac{1}{u_i} c_0 b_{01} a_1 z_h$ $- \frac{1}{u_k} c_1 b_{10} a_0 z_h$ $+ \frac{1}{u_i u_k} c_1 b_{11} a_1 z_h$ $+ u_0 c_1 b_{11} a_1$ 
$\widetilde{\varphi}$	$[a_0c_0]^1 \mapsto -\frac{x_h}{y_h}[a_0c_0]^0 + \frac{1}{y_h}[a_0c_0]^1$ $b_{01} \mapsto -\frac{u_i}{y_h} b_{01}$ $[a_1c_0]^1 \mapsto -\frac{x_h}{y_h}[a_1c_0]^0 + [a_1c_0]^1$ $[a_0c_1]^1 \mapsto -\frac{x_h}{y_h}[a_0c_1]^0 - \frac{u_k}{y_h}[a_0c_1]^1$ $[a_1c_1]^1 \mapsto -\frac{u_i u_k u_0 + x_h}{y_h}[a_1c_1]^0 + \frac{u_i u_k}{y_h}[a_1c_1]^1$ $\rightsquigarrow$ $c_0^* \mapsto -2y_h c_0^* v$ $c_1^* \mapsto -2y_h c_1^* v$ $[a_1c_0]^0 \mapsto -\frac{1}{u_i}[a_1c_0]^0$ $[a_0c_1]^0 \mapsto -\frac{1}{u_k}[a_0c_1]^0$ $[a_1c_1]^0 \mapsto \frac{1}{u_i u_k}[a_1c_1]^0$
$\widetilde{\psi}$	$b_{00} \rightsquigarrow c_0^* a_0^*$ $b_{01} \rightsquigarrow y_h c_0^* a_1^*$ $b_{10} \rightsquigarrow -u_k c_1^* a_0^*$ $b_{11} \rightsquigarrow u_i u_k c_1^* a_1^*$
$\widetilde{T}_1$	b_{00} $[a_0c_0]^1$ b_{01} $[a_1c_0]^1$ b_{10} $[a_0c_1]^1$ b_{11} $[a_1c_1]^1$
$\widetilde{D}_1$ 	$a_0^*[a_0c_0]^0 c_0^* z_h^*$ $- \frac{1}{u_k} a_0^*[a_0c_1]^0 c_1^* z_h^*$ $- \frac{1}{u_i} a_1^*[a_1c_0]^0 c_0^* z_h^*$ $+ \frac{1}{u_i u_k} a_1^*[a_1c_1]^0 c_1^* z_h^*$ $+ u_0 a_1^*[a_1c_1]^0 c_1^*$ 

Table 5.A.55: flippant pair $\{\widetilde{D}_1, \widetilde{D}_1\}$.

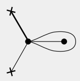
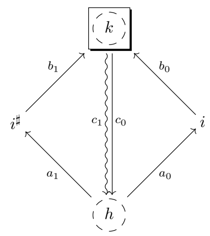
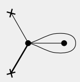
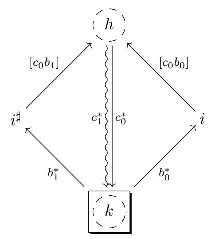
$\widetilde{D}_2^-$ 	$\ z_h\,(c_0+c_1)\,z_k b_0 a_0$ $-\ \frac{1}{u_i} z_h\,(c_0+c_1)\,z_k b_1 a_1$ $+\ u_0\,(c_0+c_1)\,b_1 a_1$ 
$\widetilde{\varphi}$	$[c_1b_0] \mapsto -z_k z_h [c_0b_0] + z_k^{-*} z_h^{-1} [c_1b_0]$ $[c_1b_1] \mapsto \left(\frac{1}{u_i} z_k z_h - u_0\right)[c_0b_1] + \left(-\frac{1}{u_i} z_k^* z_h + u_0\right)^{-1}[c_1b_1]$ $[c_0b_0] \mapsto z_k^* z_h [c_0b_0]$ $[c_0b_1] \mapsto \left(-\frac{1}{u_i} z_k^* z_h + u_0\right)[c_0b_1]$ $\rightsquigarrow \quad c_1^* \mapsto -c_1^*$
$\widetilde{\psi}$	$a_0 \rightsquigarrow \quad b_0^* z_k^{-1} c_1^* z_h^{-1}$ $a_1 \rightsquigarrow \quad b_1^* c_1^* \left(-\frac{1}{u_i} z_k^* z_h + u_0\right)^{-1}$
$\widetilde{T}_1$	$a_0 \quad [c_1b_0] \quad a_1 \quad [c_1b_1]$
$\widetilde{D}_2^+$ 	$\ z_k^*\,(c_0^*+c_1^*)\,z_h [c_0b_0] b_0^*$ $-\ \frac{1}{u_i} z_k^*\,(c_0^*+c_1^*)\,z_h [c_0b_1] b_1^*$ $+\ u_0\,(c_0^*+c_1^*)\,[c_0b_1] b_1^*$ 

Table 5.A.56: flippant pair $\left\{\widetilde{D}_2^-, \widetilde{D}_2^+\right\}$.

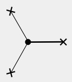
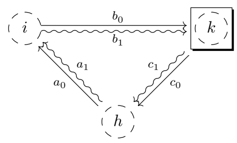
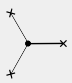
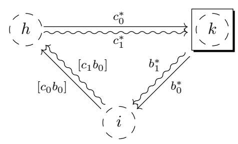
$\widetilde{D}_3$ 	$(c_0+c_1)z_k(b_0+b_1)z_i(a_0+a_1)z_h$ 
$\widetilde{\varphi}$	$[c_1b_1] \mapsto -[c_0b_0]\,(z_hz_iz_k+u_0)+[c_1b_1](z_hz_iz_k^*+u_0)^{-1}$ $[c_0b_1] \mapsto -[c_1b_0]\,(z_h^*z_iz_k+u_0)+[c_0b_1](z_h^*z_iz_k^*+u_0)^{-1}$ $[c_0b_0] \mapsto z_h[c_0b_0]z_iz_k^*+u_0[c_0b_0]$ $[c_1b_0] \mapsto z_h[c_1b_0]z_k^*z_i+u_0[c_1b_0]$ $\quad\rightsquigarrow\quad b_1^* \mapsto -b_1^*$
$\widetilde{\psi}$	$a_0 \rightsquigarrow \quad \pi_0\Big(b_1^*c_1^*(z_hz_iz_k^*+u_0)^{-1}\Big)$ $a_1 \rightsquigarrow \quad \pi_1\Big(b_1^*c_0^*(z_hz_i^*z_k+u_0)^{-1}\Big)$
$\widetilde{T}_1$	$a_0 \quad [c_1b_1] \quad a_1 \quad [c_0b_1]$
$\widetilde{D}_3$ 	$(b_0^*+b_1^*)z_k^*(c_0^*+c_1^*)z_h([c_0b_0]+[c_1b_0])z_i$ 

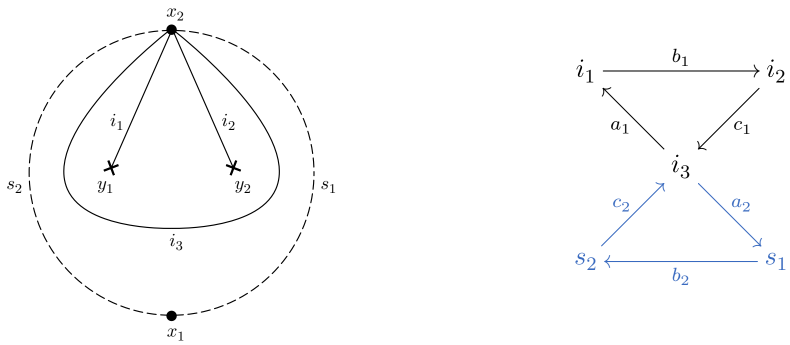
Table 5.A.57: flippant pair $\{\widetilde{D}_3, \widetilde{D}_3\}$.

6 Potentials for Colored Triangulations

Motivation

This chapter continues the investigation of orbifolds. In contrast to the preceding chapter, punctures are not allowed, but orbifold points now carry a *weight*. That is to say, we are dealing with *weighted unpunctured orbifolds*: compact oriented surfaces Σ with *marked points* $\mathbb{M} \subseteq \partial\Sigma$ and *orbifold points* $\mathbb{O} \subseteq \Sigma \setminus \partial\Sigma$ equipped with *weights* $\mathbb{O} \to \{1,4\}$, $y \mapsto d_y$.

Again, we will illustrate the introduction with an example. This time, we consider the digon with $\mathbb{M} = \{x_1, x_2\}$ and $\mathbb{O} = \{y_1, y_2\}$. For the weights we take $d_{y_1} = 1$ and $d_{y_2} = 4$. The three arcs i_1, i_2, i_3 shown below form a triangulation τ:



As before, there is a quiver $X(\tau)$ whose vertices are the sides of the triangles in τ with arrows keeping track of adjacencies. It is a weighted quiver where sides i containing an orbifold point y have weight $d_i = d_y$ and all other sides have weight 2.

In the example, we therefore have $d_{i_1} = 1$, $d_{i_2} = 4$ and $d_{s_1} = d_{s_2} = d_{i_3} = 2$.

The path algebras $R\langle A\rangle$ for $X(\tau)$ are defined over a ground ring $R = R(\tau) = \prod_{i \in X_0(\tau)} R_i$ where all R_i are intermediate fields of a cyclic Galois extension L/K of degree 4.

More precisely, we will assume $L = K(v)$ with $v^4 \in K$. Letting $E = K(v^2)$ this means

$$R_i = \begin{cases} K & \text{if } d_i = 1, \\ E & \text{if } d_i = 2, \\ L & \text{if } d_i = 4. \end{cases}$$

For the bimodule $A = \bigoplus_{a \in X_1(\tau)} A_a$ there are several choices. For each $j \xleftarrow{a} i$ with both weights d_i and d_j divisible by 2 one can take $A_a = R_j \otimes_{\rho_a} R_i$ with $\rho_a|_E \in \mathrm{Gal}(E/K) \cong \mathbb{Z}/2\mathbb{Z}$. To be well-behaved with respect to mutation, it is necessary to impose a "compatibility condition." In the example: $\rho_{c_2}\rho_{b_2}\rho_{a_2} = \mathrm{id}_E$. More generally, the "compatible" choices for the elements ρ_a are parametrized by the set of 1-cocycles $Z^1(\tau)$ of the cocomplex dual to

$$C_\bullet(\tau) \quad : \quad 0 \longrightarrow \mathbb{F}_2\bar{X}_2(\tau) \xrightarrow{\partial_2} \mathbb{F}_2\bar{X}_1(\tau) \xrightarrow{\partial_1} \mathbb{F}_2\bar{X}_0(\tau) \longrightarrow 0$$

where $\big(\bar{X}_0(\tau), \bar{X}_1(\tau)\big)$ is the underlying simple graph of the full subquiver of $X(\tau)$ on the vertices with weight divisible by 2. The "faces" $\bar{X}_2(\tau)$ correspond to triangles of τ.

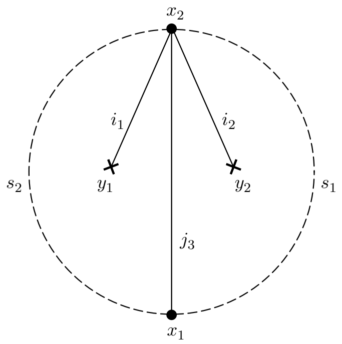
In the example, $\bar{X}_0(\tau) = \{i_2, i_3, s_1, s_2\}$ and $\bar{X}_1(\tau) = \{c_1, c_2, b_2, a_2\}$ and $\bar{X}_2(\tau) = \{\Delta\}$ where Δ is the triangle with sides s_1, s_2, i_3. The differential ∂_2 sends Δ to $c_2 + b_2 + a_2$ such that $\xi(c_2) + \xi(b_2) + \xi(a_2) = 0$ for all $\xi \in Z^1(\tau) \subseteq \mathrm{Hom}_{\mathbb{F}_2}(C_1(\tau), \mathbb{F}_2)$.

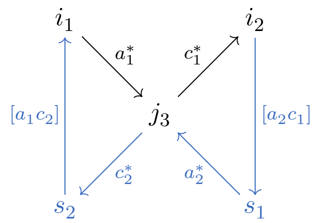
Colored Triangulations

A *colored triangulation* (τ, ξ) is a triangulation τ together with a cocycle $\xi \in Z^1(\tau)$. For all arcs $i \in \tau$ we define another colored triangulation: the *flip* $\mu_i(\tau, \xi) = (\mu_i(\tau), \xi')$.

Every colored triangulation determines a path algebra $R\langle A(\tau, \xi)\rangle$ for $X(\tau)$ over L/K. We will construct an SP $\mathcal{S}(\tau, \xi) = (A(\tau, \xi), W(\tau, \xi))$.

In the running example the potential $W(\tau, \xi)$ is $c_1b_1a_1 + c_2b_2a_2$. Flipping the arc i_3 yields the following triangulation $\mu_{i_3}(\tau)$:





The potential $W(\mu_{i_3}(\tau, \xi))$ has the form $a_1^*[a_1c_2]c_2^* + a_2^*[a_2c_1]c_1^*$.

Results

We will show that the SP $\mathcal{S}(\mu_i(\tau, \xi))$ corresponds to the SP mutation $\mu_i(\mathcal{S}(\tau, \xi))$ for all i. In particular, this will imply the non-degeneracy of $\mathcal{S}(\tau, \xi)$.

Furthermore, we will see that $W(\tau, \xi)$ is up to $R(\tau)$-equivalence the unique non-degenerate potential for $A(\tau, \xi)$ if we assume that the weighted orbifold under consideration is not a monogon with all orbifold points of the same weight.

We prove that the Jacobian algebra $\mathcal{J}(\tau, \xi)$ is finite-dimensional and $\mathcal{J}(\tau, \xi) \cong \mathcal{J}(\tau, \xi')$ as $K^{X_0(\tau)}$-algebras if and only if ξ and ξ' are cohomologous cocycles.

Finally, the set of colored triangulations of a weighted unpunctured orbifold forms the set of vertices of a simple graph in which two colored triangulations are joined with an edge if and only if they are related by flipping an arc. The *flip graph* is obtained from this graph by identifying colored triangulations (τ, ξ) and (τ, ξ') where ξ and ξ' are cohomologous. The flip graph will be shown to be disconnected unless the surface Σ is a disk.

6.1 Triangulated Weighted Orbifolds

The notion of triangulated weighted orbifolds is due to [FST12a]. In the present chapter we only consider weighted orbifolds without punctures. This section discusses consequences of the non-existence of punctures. In addition, it introduces relevant notation.

Definition 6.1.1. A *weighted orbifold* $\mathbf{\Sigma}_d$ is a pair $(\mathbf{\Sigma}, d)$ consisting of an orbifold $\mathbf{\Sigma}$ in the sense of Definition 5.1.1 and a function $\mathbb{O} \xrightarrow{d} \{1,4\}$, $\times \mapsto d_\times$.

A *triangulation* of $\mathbf{\Sigma}_d$ is a triangulation of $\mathbf{\Sigma}$ (see Definition 5.1.14).

Convention 6.1.2. For the rest of the chapter fix a weighted orbifold $\mathbf{\Sigma}_d = (\mathbf{\Sigma}, d)$ such that $\mathbf{\Sigma} = (\Sigma, \mathbb{M}, \mathbb{O})$ has no punctures. This means $\mathbb{M} \subseteq \partial\Sigma \neq \varnothing$. For technical reasons we assume that $\mathbf{\Sigma}$ is not a torus with exactly one boundary marked point.

We use Convention 5.1.2. In particular, we write $\mathfrak{s}$ for the set of *boundary segments*, g for the genus, b for the number of boundary components, m for the number of marked points, and o for the number of orbifold points of $\mathbf{\Sigma}$. Moreover, set $o_1 := |\{\times \in \mathbb{O} \mid d_\times = 1\}|$.

In the drawings we put a circle $d_\times$ close to every orbifold point $\times$ to indicate its weight.

For triangulations τ of $\mathbf{\Sigma}_d$ and $k \in \{1,4\}$ we denote by $\tau^{d=k}$ the set of all pending arcs i in τ such that the orbifold point $\times$ that is an endpoint of i has weight $d_\times = k$.

The set of non-pending arcs in τ will sometimes be written as $\tau^{d=2}$.

Remark 6.1.3. With λ defined as in Convention 5.1.2 the invariant (g, λ, o, o_1) determines the weighted orbifold $\mathbf{\Sigma}_d$ up to diffeomorphism (see Remark 5.1.3).

Remark 6.1.4. Observe that by definition $m \geq b > 0$ and $o \geq o_1 \geq 0$.

Since $\mathbf{\Sigma}$ has no punctures, the ends of all tagged arcs in $\mathbf{\Sigma}$ are tagged in the same way. Therefore the notions of triangulation and tagged triangulation coincide in the setting of this chapter. The puzzle-piece decompositions of triangulations of $\mathbf{\Sigma}_d$ (see Proposition 5.1.23) contain only pieces of those types shown in Figure 6.A.1. In particular, the puzzle pieces of a triangulation of $\mathbf{\Sigma}_d$ are simply the triangles of the triangulation.

Finally, the non-existence of punctures also rules out many of the arctypes listed in Tables 5.A.7 to 5.A.10 an arc in a triangulation of $\mathbf{\Sigma}$ can have. Not taking into account the weights, we are left with only 14 arctypes. They are shown in Tables 6.A.2 to 6.A.4.

6.2 Colored Triangulations

6.2.1 Adjacency Quiver of a Triangulation

The adjacency quivers for triangulations of the weighted orbifold $\mathbf{\Sigma}_d$ are defined in complete analogy to § 5.2.1. However, they are not modular but merely weighted quivers.

Notation 6.2.1. Similar to Notation 5.2.1 we denote by $X_2(\tau)$ the set of triangles of a triangulation τ of $\mathbf{\Sigma}_d$. For $\Delta \in X_2(\tau)$ we write $(i, j) \in \Delta$ if j follows i in Δ.

Definition 6.2.2. The *(adjacency) quiver* $X(\tau) = X(\tau, d)$ of a triangulation τ of $\mathbf{\Sigma}_d$ is a weighted quiver $(X(\tau), d)$ whose vertices are the arcs in τ and boundary segments of $\mathbf{\Sigma}_d$.

Formally, $X_0(\tau) = \tau \cup \mathfrak{s}$. The weight of pending $i \in X_0(\tau)$ is $d_i = d_{\times}$ where $\times$ is the endpoint of i belonging to $\mathbb{O}$. For every other $i \in X_0(\tau)$ define $d_i = 2$. The arrow set is

$$X_1(\tau) = \left\{ i \xrightarrow{(\Delta,(i,j),r)} j \;\middle|\; \begin{array}{l} \Delta \in X_2(\tau),\, i, j \in X_0(\tau) \text{ with } (i, j) \in \Delta, \\ r \in \{0, 1\} \text{ with } r = 1 \text{ only if } d_i = d_j \in \{1, 4\} \end{array} \right\}.$$

The *heavy part* of $X(\tau)$ is the full subquiver $X^{d\neq 1}(\tau)$ of $X(\tau)$ consisting of all vertices i with weight $d_i \neq 1$.

Notation 6.2.3. Set $\Delta(a) := \Delta$ and $r(a) := r$ for every arrow $a = (\Delta, (i, j), r) \in X_1(\tau)$. We say that a is *induced by the triangle* Δ *of* τ.

A cyclic path cba in $X(\tau)$ is *induced by a triangle* Δ *of* τ if $\Delta(a) = \Delta(b) = \Delta(c) = \Delta$.

A path cba in $X(\tau)$ is *triangle-induced* if it is a cyclic path induced by some $\Delta \in X_2(\tau)$.

For a triangle Δ of τ we denote by X^{Δ} the subquiver of $X(\tau)$ spanned by all arrows induced by Δ. More generally, the *subquiver of* $X(\tau)$ *induced by triangles* $\Delta_1, \dots, \Delta_\ell$ is defined as the quiver $X^{\Delta_1} \oplus \cdots \oplus X^{\Delta_\ell}$.

Remark 6.2.4. Assume $\mathbb{O} \xrightarrow{d} \{1, 4\}$ is constant with value 4. Then all vertices of $X(\tau)$ have weight 2 or 4. As quiver $X(\tau)$ coincides with $Q(\tau)$ from Definition 5.2.2. The weight of a vertex in $X(\tau)$ is the double of the weight of the corresponding vertex in $Q(\tau)$.

Remark 6.2.5. The full subquiver $X^{\circ}(\tau)$ of $X(\tau)$ spanned by τ (i.e. by all arcs that are not boundary segments) is the "weighted quiver $Q(\tau, d)$ of τ with respect to d" in [GL16a].

Convention 6.2.6. In the illustrations we indicate the weight of a vertex i in $X(\tau)$ by putting a circle d_i next to it whenever $d_i \in \{1, 4\}$. Vertices of weight 2 are not highlighted in any special way. The part of $X(\tau)$ not belonging to $X^{\circ}(\tau)$ will be drawn in blue.

Remark 6.2.7. Analogously as in Remark 5.2.5 we can rephrase [FST12a, Lemma 4.10] in our context as follows: There is a decomposition of weighted quivers

$$X(\tau) = \bigoplus_{\Delta} X^{\Delta}$$

where the sum runs over all triangles Δ of τ. The weighted quivers X^{Δ} for the different types of triangles (= puzzle pieces) are shown in Figures 6.A.5 to 6.A.7.

Remark 6.2.8. The quiver $X(\tau)$ of every triangulation τ is connected, since Σ is connected.

Remark 6.2.9. We have the following analog of Remark 5.2.7 concerning the number q_{ji} of arrows $j \leftarrow i$ in $X(\tau)$:

- $q_{ji} \leq 2$. More precisely, the outdegree of i and the indegree of j are at most 2.
- If $q_{ij} \geq 1$, then $q_{ji} = 0$.
- If $d_i \neq d_j$, then $q_{ji} \leq 1$.
- There is at most one arrow $j \leftarrow i$ induced by the same triangle unless $d_i = d_j \neq 2$.
- If $d_i = d_j \neq 2$ and $q_{ji} \geq 1$, then $q_{ji} = 2$.

The second item shows that $X(\tau)$ is 2-acyclic.

Note that $d_i = d_j \neq 2$ means that both i and j are pending arcs with the same weight, while $d_i \neq d_j$ implies that at least one of i and j is pending.

Remark 6.2.10. Assume $\mathbb{O} \xrightarrow{d} \{1, 4\}$ is not constant with value 4. Let τ be a triangulation.

The skew-symmetrizable matrix $B = B(\tau, d)$ associated with the weighted quiver $X^{\circ}(\tau)$ via the bijection of Remark 2.1.6 is one of the matrices described in [FST12a, § 4.3].

More precisely, the function $\mathbb{O} \xrightarrow{d} \{1, 4\}$ corresponds to the function $\mathbb{O} \xrightarrow{w} \left\{\frac{1}{2}, 2\right\}$ given as $w(\mathsf{x}) = \frac{2}{d_{\mathsf{x}}}$ in [FST12a, Definition 4.15].[1]

6.2.2 Mutating Adjacency Quivers

The considerations in this subsection are similar to those in § 5.2.2. We discuss how the weighted quiver $X(\mu_i(\tau))$ can be regarded as a subquiver of the premutation $\widetilde{\mu}_i(X(\tau))$.

Notation 6.2.11. Copying Notation 5.2.15 write $X(\tau, i)$ (resp. $X(\tau, \neg i)$) for the subquiver of $X(\tau)$ induced by all triangles of τ containing (resp. not containing) the arc i in τ.

Remark 6.2.12. $X(\tau) = X(\tau, \neg i) \oplus X(\tau, i)$ and $\widetilde{\mu}_i(X(\tau)) = X(\tau, \neg i) \oplus \widetilde{\mu}_i(X(\tau, i))$.

We have the following analogs of Lemma 5.2.17 and Remark 5.2.18:

Lemma 6.2.13. *Let τ and ς be triangulations of $\mathbf{\Sigma}_d$ that are related by a flip, say $\varsigma = \mu_i(\tau)$ and $\tau = \mu_j(\varsigma)$. Then $X^{\neg} := X(\tau, \neg i) = X(\varsigma, \neg j)$ and there is a monomorphism*

$$X(\varsigma) \xhookrightarrow{\Phi} \widetilde{\mu}_i(X(\tau))$$

of weighted quivers with $\Phi|_{X^{\neg}} = \mathrm{id}_{X^{\neg}}$ and $\Phi(k) = k$ for all $k \in X_0(\varsigma) \setminus \{j\}$ and $\Phi(j) = i$.

The image of Φ is a maximal 2-acyclic subquiver of $\widetilde{\mu}_i(X(\tau))$.

Proof. This is similar to [FST12a, Lemma 4.12] and relies on Definitions 2.1.8 and 6.2.2. $\square$

Remark 6.2.14. One can demand in Lemma 6.2.13 that for every $\xleftarrow{b} i \xleftarrow{a}$ in $X(\tau)$ ...

(i) ... with $\Delta(b) \neq \Delta(a)$, there is a triangle-induced cyclic path $b^{\vee}c^{\vee}a^{\vee}$ in $X(\varsigma)$ such that $\Phi(b^{\vee}) = b^*$, $\Phi(c^{\vee}) = [ba]^0_{r(c^{\vee})}$, $\Phi(a^{\vee}) = a^*$.

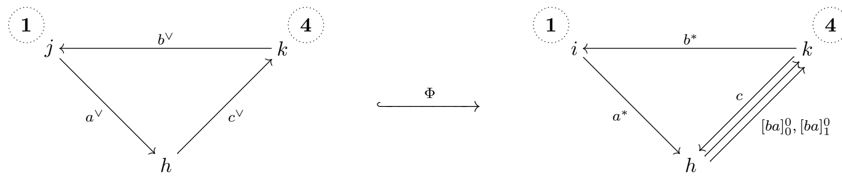
[1]Not to the function $w(\mathsf{x}) = \frac{d_{\mathsf{x}}}{2}$. The reason for this is that we call a diagonal integer matrix D a skew-symmetrizer of $B = B(\tau)$ if DB is skew-symmetric, whilst [FST12a] requires BD to be skew-symmetric.

(ii) ... with $d_i \neq 2$, $r(b) = r(a) = 0$, there is a triangle-induced path $b^\vee c^\vee a^\vee$ in $X(\varsigma)$ with $r(b^\vee) = r(a^\vee) = 0$ such that $\Phi(b^\vee) = b^*$, $\Phi(c^\vee) = [ba]_r^0$, $\Phi(a^\vee) = a^*$ for some r.

This property determines Φ uniquely, if i is not the weight-1 pending arc in a triangle of type $\widetilde{B}_1$, $\widetilde{C}_{14}$, or $\widetilde{C}_{41}$. In the case that i is such an arc, there are two choices for Φ.

These claims can easily be verified case by case. All possibilities for the image of $X(\varsigma, j)$ under Φ are listed in Table 6.A.8 (where i corresponds to the boxed vertex).

Example 6.2.15. Let $X(\tau) = X^\Delta$ be the weighted quiver of type $\widetilde{C}_{14}$ shown in Figure 6.A.7. The premutation $\widetilde{\mu}_i(X(\tau))$ is drawn below on the right. Flipping the arc i in τ yields a triangulation ς consisting of a single triangle of type $\widetilde{C}_{41}$. The quiver $X(\varsigma)$ can be seen on the left. One of the monomorphisms Φ satisfying the property in Remark 6.2.14 sends the arrow $c^\vee$ to $[ba]_0^0$, the other one sends $c^\vee$ to $[ba]_1^0$.



A similar consideration works for the weight-1 arc in triangles Δ of type $\widetilde{B}_1$ and $\widetilde{C}_{41}$.

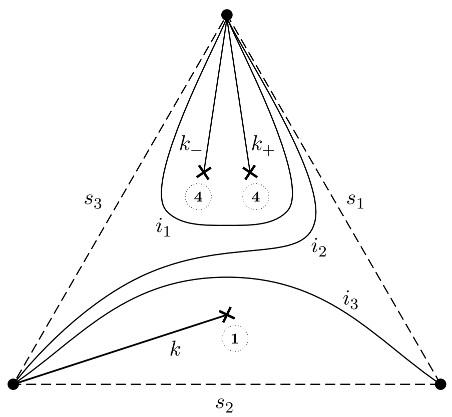
6.2.3 Modular Structures

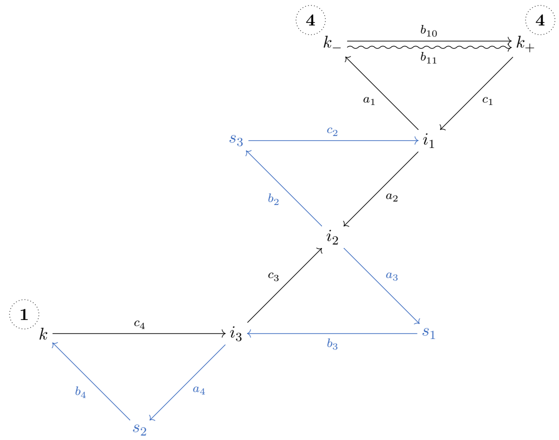
We have defined $X(\tau)$ not as a modular but only as a weighted quiver. Usually, $X(\tau)$ can be turned into a modular quiver in several equally valid ways.

Definition 6.2.16. We call σ a *modular structure* for $X(\tau)$ if $(X(\tau), d, \sigma)$ is a modular quiver. A modular structure σ for $X(\tau)$ is *admissible* if $(X(\tau), d, \sigma)$ is τ-admissible.

Remark 6.2.17. Recall that the admissibility of a modular quiver is a prerequisite for admitting non-degenerate SPs (see Remark 2.6.103).

Example 6.2.18. Depicted below is a triangulation τ of the triangle with one weight-1 and two weight-4 orbifold points. The weighted quiver $X = X(\tau)$ is drawn on the right.





The subquivers X^{Δ} in the decomposition from Remark 6.2.7 are the full subquivers of X spanned by the subsets $\{i_1, k_-, k_+\}$, $\{i_1, i_2, s_3\}$, $\{i_2, s_1, i_3\}$, $\{i_3, s_2, k\} \subseteq X_0$.

We have $\tau^{d=1} = \{k\}$ and $\tau^{d=4} = \{k_-, k_+\}$.

To equip X with the structure of a modular quiver, we have to pick for each $j \xleftarrow{a} i$ in X an element $\sigma_a \in \mathbb{Z}/d_{ji}\mathbb{Z}$. Observe that

$$d_{ji} = \begin{cases} 1 & \text{if } i \in \tau^{d=1} \vee j \in \tau^{d=1}, \\ 4 & \text{if } i \in \tau^{d=4} \wedge j \in \tau^{d=4}, \\ 2 & \text{otherwise.} \end{cases}$$

Therefore we have no choice for $\sigma_{b_4}, \sigma_{c_4}$, four possibilities for each of $\sigma_{b_{10}}, \sigma_{b_{11}}$, and two possibilities for each of $\sigma_{a_1}, \sigma_{c_1}, \sigma_{a_2}, \sigma_{b_2}, \sigma_{c_2}, \sigma_{a_3}, \sigma_{b_3}, \sigma_{c_3}, \sigma_{a_4}$.

In other words, we have to make a choice for all arrows that lie in the heavy part $X^{d\neq 1}(\tau)$. However, not all of these choices define modular structures that are admissible.

Lemma 6.2.19. *A modular structure σ for $X(\tau)$ is admissible if and only if there is an automorphism $X(\tau) \xrightarrow{\pi} X(\tau)$ of Q_0-quivers such that*

(a) $\sigma_{\pi(c)} + \sigma_{\pi(b)} + \sigma_{\pi(a)} = 0$ in $\mathbb{Z}/2\mathbb{Z}$ for all triangle-induced paths cba in $X^{d\neq 1}(\tau)$; and

(b) $\sigma_{c_0} \neq \sigma_{c_1}$ for each pair of parallel arrows $c_0 \neq c_1$ in $X(\tau)$ connecting arcs in $\tau^{d=4}$.

Furthermore, $\sigma_a = 0$ for all arrows a in $X(\tau) - X_1^{d\neq 1}(\tau)$.

Definition 6.2.20. Let τ and ς be triangulations of $\mathbf{\Sigma}_d$ such that $\varsigma = \mu_i(\tau)$. For every modular structure σ for $X(\tau)$ denote by

$$X(\varsigma) \xhookrightarrow{\Phi^{\sigma}} \widetilde{\mu}_i(X(\tau))$$

the monomorphism Φ satisfying the property of Remark 6.2.14 such that, if i is the pending arc of a triangle Δ of type $\widetilde{B}_1$, $\widetilde{C}_{14}$, or $\widetilde{C}_{41}$, it is

$$[ba]^0_{\sigma_c+1} \in \operatorname{im}(\Phi)$$

where $i \xleftarrow{a} \xleftarrow{c} \xleftarrow{b} i$ is the cyclic path in $X(\tau)$ induced by Δ (compare Example 6.2.15).

Proof of Lemma 6.2.19. The last claim is obvious

To prove the "only if" part of the first claim, let us assume that (a) or (b) is violated. Due to Remark 6.2.9 there are at most two arrows between each pair of vertices in $X(\tau)$. Moreover, for every cyclic path cba in $X^{d\neq 1}(\tau)$ at most one of the arrows a, b, c can have a parallel arrow in $X(\tau)$ different from itself (compare Tables 6.A.2 to 6.A.4). Hence, the subquiver X^{cba} of $X(\tau)$ induced by all $\Delta(f)$ with f parallel to a, b, c is full.

If (a) is violated for all π, the preceding discussion shows that there are cyclic paths $c_0b_0a_0$ and $c_1b_1a_1$ in $X^{d\neq 1}(\tau)$ such that c_0 and c_1 are parallel arrows in $X(\tau)$ and $\sigma_{c_p} + \sigma_{b_0} + \sigma_{a_0} \neq 0$ or $\sigma_{c_{1-p}} + \sigma_{b_1} + \sigma_{a_1} \neq 0$ for all $p \in \{0, 1\}$. In this case, let $X' = X^{c_0b_0a_0} = X^{c_1b_1a_1}$.

If (b) is violated, let Δ be the triangle of type $\widetilde{C}_{44}$ inducing the parallel arrows c_0 and c_1 and let X' be the full subquiver X^{Δ} of $X(\tau)$.

In both cases, we consider X' as a (full) modular subquiver of $(X(\tau), d, \sigma)$. It is easy to check that $\widetilde{\mu}_j\widetilde{\mu}_i(X')$ is not 2-acyclic after reduction for some i, j. Thus σ is not admissible.

For the "if" part of the first claim, it is sufficient to verify that for all arcs $i \in \tau$ the premutation $\widetilde{X} = \widetilde{\mu}_i(X(\tau), d, \sigma)$ has the form $\widetilde{X} = \Phi^{\sigma}((X(\varsigma), d, \sigma')) \oplus \widetilde{T}$ with $\widetilde{T}$ trivial and conditions (a) and (b) hold after replacing τ, σ with $\varsigma = \mu_i(\tau)$, $\sigma' = \sigma \circ \Phi^{\sigma}$, where the modular structure for $\widetilde{X}$ induced by σ is still denoted by σ.

Without loss of generality, we will assume that (a) and (b) hold for $\pi = \mathrm{id}$.

Fix $i \in \tau$. First, we consider the case in which i is an arc shared by two triangles Δ_0 and Δ_1 of τ (see Tables 6.A.2 and 6.A.3). In this case, $\Phi = \Phi^{\sigma}$ does not depend on σ.

For $q \in \{0, 1\}$ let $k_q \xleftarrow{b_q} i \xleftarrow{a_q} h_q$ be the paths in $X(\tau)$ with $\Delta(a_q) = \Delta(b_q) = \Delta_q$.

By the properties of Φ there are paths $j \xleftarrow{b_q^{\vee}} k_{1-q} \xleftarrow{c_q^{\vee}} h_q \xleftarrow{a_q^{\vee}} j$ in $X(\varsigma)$ induced each by some $\Delta'_q \in X_2(\varsigma)$ such that $\Phi\big(b_q^{\vee}\big) = b_{1-q}^{*}$, $\Phi\big(c_q^{\vee}\big) = \big[b_{1-q}a_q\big]_0^0$, $\Phi\big(a_q^{\vee}\big) = a_q^{*}$.

If $b_q^{\vee}c_q^{\vee}a_q^{\vee}$ is contained in $X^{d\neq 1}(\varsigma)$, then

$$\sigma'_{b_q^{\vee}} + \sigma'_{c_q^{\vee}} + \sigma'_{a_q^{\vee}} \;=\; \sigma_{b_{1-q}^{*}} + \sigma_{[b_{1-q}a_q]_0^0} + \sigma_{a_q^{*}} \;=\; 0 \;\in\; \mathbb{Z}/2\mathbb{Z}\,.$$

If there is $c'_q \neq c_q^{\vee}$ with $\Delta(c'_q) = \Delta'_q$ parallel to $c_q^{\vee}$ in $X(\varsigma)$, then $h_q, k_{1-q} \in \varsigma^{d=4}$ and the arrows $k_{1-q} \leftarrow h_q$ in $\widetilde{X}$ are $[b_{1-q}a_q]_0^0$ and $[b_{1-q}a_q]_1^0$. Thus $\Phi(c'_q) = [b_{1-q}a_q]_1^0$ and

$$\sigma'_{b_q^{\vee}} + \sigma'_{c'_q} + \sigma'_{a_q^{\vee}} \;=\; \sigma_{b_{1-q}^{*}} + \sigma_{[b_{1-q}a_q]_1^0} + \sigma_{a_q^{*}} \;=\; 0 \;\in\; \mathbb{Z}/2\mathbb{Z}\,.$$

Moreover, $\sigma'_{c_q^{\vee}} = \sigma_{[b_{1-q}a_q]_0^0} \neq \sigma_{[b_{1-q}a_q]_1^0} = \sigma'_{c'_q}$.

Since $X_2(\varsigma) = (X_2(\varsigma) \cap X_2(\tau)) \cup \{\Delta'_0, \Delta'_1\}$ and each arrow in $X_1(\varsigma) \setminus X_1(\tau)$ is induced by either Δ'_0 or Δ'_1, we can conclude that (a) and (b) hold with τ, σ replaced by ς, σ'.

Let $\widetilde{T}$ be the modular subquiver of $\widetilde{X}$ spanned by all arrows not in the image of Φ. This means $\widetilde{T}_1$ consists of the arrows $k_q \to h_q$ induced by Δ_q (for $q \in \{0, 1\}$) and those in

$$\left\{ k_q \xleftarrow{[b_q a_q]_r^s} h_q \;\middle|\; \begin{array}{l} s \in \{0,1\} \text{ with } s = 1 \text{ only if } d_{h_q} = d_{k_q} = 1, \\ r \in \{0,1\} \text{ with } r = 1 \text{ only if } d_{h_q} = d_{k_q} = 4 \end{array} \right\}.$$

If $d_{k_q h_q} = 2$, then $\sigma_{c_q} + \sigma_{[b_q a_q]_0^0} = \sigma_{c_q} + \sigma_{b_q} + \sigma_{a_q} = 0$ in $\mathbb{Z}/2\mathbb{Z}$ by condition (a) for the unique arrow $k_q \xrightarrow{c_q} h_q$ in $\widetilde{X}$ induced by Δ_q.

If $d_{k_q h_q} = 4$, there are precisely two arrows $k_q \xrightarrow{c_{qr}} h_q$ ($r \in \{0, 1\}$) in $\widetilde{X}$ induced by Δ_q. Conditions (a) and (b) imply $\sigma_{c_{qr}} + \sigma_{[b_q a_q]_{f(r)}^0} = 0$ in $\mathbb{Z}/4\mathbb{Z}$ for a permutation f of $\{0, 1\}$.

This shows that $\widetilde{T}$ is a trivial modular quiver. Hence, the image of Φ is a reduction of the modular quiver $\widetilde{X}$ by Lemma 6.2.13.

It remains to consider the case in which i is an inner side of some triangle Δ of τ (see Table 6.A.4). Using Remark 6.2.12 we can assume that Δ is the only triangle of τ. Then ς

consists as well of only one triangle Δ'. The weighted quivers $X(\tau) = X^{\Delta}$ and $X(\varsigma) = X^{\Delta'}$ are among those in Figures 6.A.6 and 6.A.7 and i is one of their vertices of weight 1 or 4.

A straightforward case-by-case inspection shows that $\widetilde{X} = \Phi^{\sigma}((X(\varsigma), d, \sigma')) \oplus \widetilde{T}$, where $\widetilde{T}$ is trivial, and conditions (a) and (b) still hold when τ, σ is replaced by ς, σ'. $\square$

Example 6.2.21. The weighted quivers of type $\widetilde{B}_1$ and $\widetilde{C}_{41}$ in Figures 6.A.6 and 6.A.7 each admit two modular structures $(\sigma_a, \sigma_b, \sigma_c) \in \{(0,0,0), (1,0,0)\}$. Both are admissible.

The only modular structure for the weighted quiver of type $\widetilde{C}_{11}$ in Figure 6.A.7 takes the value zero everywhere. It is admissible.

Example 6.2.22. The modular structure $(\sigma_a, \sigma_b, \sigma_c) = (0,1,1)$ for the weighted quiver of type A in Figure 6.A.5 is admissible, whereas $(\sigma_a, \sigma_b, \sigma_c) = (1,1,1)$ is not.

For the weighted quiver of type $\widetilde{C}_{44}$ in Figure 6.A.7 $(\sigma_a, \sigma_{b_0}, \sigma_{b_1}, \sigma_c) = (0,0,2,0)$ is admissible, while $(\sigma_a, \sigma_{b_0}, \sigma_{b_1}, \sigma_c) \in \{(0,0,0,0), (0,0,0,1), (0,0,1,0)\}$ are not.

6.2.4 Cocycles and Colored Triangulations

We introduce *colored triangulations* (τ, ξ). These are triangulations τ enriched with an additional datum ξ encoding an admissible modular structure σ^{ξ} for $X(\tau)$.

Notation 6.2.23. Denote by $\bar{X}_2(\tau)$ the subset of $X_2(\tau)$ consisting of all triangles Δ such that all sides of Δ belong to the heavy part $X^{d\neq 1}(\tau)$.

Let $\bar{X}(\tau)$ be the quiver obtained from the heavy part $X^{d\neq 1}(\tau)$ by identifying parallel arrows that connect arcs in $\tau^{d=4}$.

Remark 6.2.24. By definition $c_0 = c_1$ in $\bar{X}(\tau)$ whenever $c_0 \neq c_1$ is a pair of parallel arrows in $X^{d\neq 1}(\tau)$ between weight-4 pending arcs.

Example 6.2.25. For $X(\tau) = X^{\Delta}$ with Δ of type A, $\widetilde{B}_4$ or $\widetilde{C}_{44}$ like in Figures 6.A.5 to 6.A.7 the quiver $\bar{X}(\tau)$ is the simple triangle quiver (where $b := b_0 = b_1$ in the $\widetilde{C}_{44}$ case):

$$\begin{tikzcd} \cdot \arrow[rr, "b"] & & \cdot \arrow[dl, "c"] \\ & \cdot \arrow[ul, "a"] & \end{tikzcd}$$

For triangles Δ of type $\widetilde{B}_1$, $\widetilde{C}_{14}$, $\widetilde{C}_{41}$ the quiver $\bar{X}(\tau)$ is the subquiver of X^{Δ} spanned by the single arrow that connects the two vertices of weight unequal one.

For triangles Δ of type $\widetilde{C}_{11}$ the quiver $\bar{X}(\tau)$ consists of just one vertex.

Lemma 6.2.19 suggests to view admissible modular structures for $X(\tau)$ as cocycles of a cochain complex with coefficients in $\mathbb{F}_2 = \mathbb{Z}/2\mathbb{Z}$.

Definition 6.2.26. A *colored triangulation* (τ, ξ) *of* $\mathbf{\Sigma}_d$ consists of a triangulation τ of $\mathbf{\Sigma}_d$ and $\xi \in Z^1(\tau)$ where $Z^1(\tau)$ is the set of 1-cocycles of $C^\bullet(\tau) = \mathrm{Hom}_{\mathbb{F}_2}(C_\bullet(\tau), \mathbb{F}_2)$ and

$$C_\bullet(\tau) \quad : \quad 0 \longrightarrow \mathbb{F}_2\bar{X}_2(\tau) \xrightarrow{\partial_2} \mathbb{F}_2\bar{X}_1(\tau) \xrightarrow{\partial_1} \mathbb{F}_2\bar{X}_0(\tau) \longrightarrow 0\,.$$

The non-zero differentials of $C_\bullet(\tau)$ are defined on basis elements as

$$\begin{aligned} \partial_2(\Delta) &= c + b + a \quad \text{for } \Delta \in \bar{X}_2(\tau), \\ &\qquad\qquad \text{where } cba \text{ is a cyclic path in } X(\tau) \text{ induced by } \Delta, \\ \partial_1(a) &= j + i \qquad\quad \text{for } j \xleftarrow{a} i \in \bar{X}_1(\tau). \end{aligned}$$

Remark 6.2.27. A chain complex similar to $C_\bullet(\tau)$ was considered in [AG16, § 2.2].

Remark 6.2.28. Clearly, $H_0(C_\bullet(\tau)) \cong \mathbb{F}_2$ by Remark 6.2.8 and $H_n(C_\bullet(\tau)) = 0$ for all $n > 1$.

6.2.5 Adjacency Quiver of a Colored Triangulation

Definition 6.2.29. The *quiver* $X(\tau, \xi)$ of a colored triangulation (τ, ξ) of $\mathbf{\Sigma}_d$ is the modular quiver $(X(\tau), d, \sigma^\xi)$ with $\sigma^\xi_a = \xi(a) + 2r(a)$ for all $a \in X_1^{d\neq 1}(\tau)$.

Here, $\mathbb{F}_2$ is regarded as a subset of $\mathbb{Z}/4\mathbb{Z}$ via the inclusion $0 \mapsto 0$, $1 \mapsto 1$.

Lemma 6.2.30. *The modular quiver $X(\tau, \xi)$ is admissible.*

Proof. This is a reformulation of the "if" part of Lemma 6.2.19 for $\pi = \mathrm{id}$. □

Example 6.2.31. For X^Δ of type A like in Figure 6.A.5 the cocycle $\xi = a^* + b^*$ determines the modular structure $\left(\sigma^\xi_a, \sigma^\xi_b, \sigma^\xi_c\right) = (1, 1, 0)$.

For X^Δ of type $\widetilde{C}_{44}$ like in Figure 6.A.7 (where $r(b_s) = s$) the cocycle $\xi = a^* + b_0^* = a^* + b_1^*$ yields the modular structure $\left(\sigma^\xi_a, \sigma^\xi_{b_0}, \sigma^\xi_{b_1}, \sigma^\xi_c\right) = (1, 1, 3, 0)$.

6.2.6 Flipping Colored Arcs

The next lemma formalizes the fact that the admissible modular structures for $X(\tau)$ and those for $X(\varsigma)$ are in canonical bijection whenever τ and ς are related by a flip.

Lemma 6.2.32. *Let τ and ς be triangulations of $\mathbf{\Sigma}_d$ related by flipping an arc, say $\varsigma = \mu_i(\tau)$ and $\tau = \mu_j(\varsigma)$. Then we have a pair of mutually inverse bijections*

$$Z^1(\tau) \underset{\varphi^{\tau,\varsigma}}{\overset{\varphi^{\varsigma,\tau}}{\rightleftarrows}} Z^1(\varsigma)$$

such that for all $\xi \in Z^1(\tau)$ the map $\Phi^\xi = \Phi^{\sigma^\xi}$ from Definition 6.2.20 is a monomorphism of modular quivers $X(\varsigma, \varphi^{\varsigma,\tau}(\xi)) \hookrightarrow \widetilde{\mu}_i(X(\tau, \xi))$.

In particular, $X(\varsigma, \varphi^{\varsigma,\tau}(\xi))$ and $\widetilde{\mu}_i(X(\tau, \xi))$ are reduced-equivalent for all $\xi \in Z^1(\tau)$.

Proof. Let $\xi \in Z^1(\tau)$ and denote by $\widetilde{\mu}_i \sigma^\xi$ the modular structure of $\widetilde{\mu}_i(X(\tau,\xi))$.

According to Lemma 6.2.30 σ^ξ is admissible. The proof of Lemma 6.2.19 shows that there is a unique cocycle $\varphi^{\varsigma,\tau}(\xi) = \xi'$ in $Z^1(\varsigma)$ such that $\widetilde{\mu}_i \sigma^\xi \circ \Phi^\xi = \sigma^{\xi'}$.

Tracing Definitions 2.1.12 and 6.2.20 we have explicitly for $e' \in \bar{X}_1(\varsigma)$

$$\varphi^{\varsigma,\tau}(\xi)(e') \;=\; \begin{cases} \xi(e') & \text{if } e' \in \bar{X}_1(\tau), \\ \xi(a) & \text{if } \Phi^\xi(e') = a^* \text{ for some } a \in \bar{X}_1(\tau), \\ \xi(a)+\xi(b) & \text{if } d_i \neq 1 \text{ and } \Phi^\xi(e') = [ba]^0_r \\ & \quad \text{for some path } ba \text{ in } \bar{X}(\tau) \text{ and some } r, \\ \xi(c)+1 & \text{if } d_i = 1 \text{ and } \Phi^\xi(e') = [ba]^0_r \\ & \quad \text{for some triangle-induced path } cba \text{ in } X(\tau) \text{ and some } r. \end{cases}$$

A similar formula holds for $\varphi^{\tau,\varsigma}(\xi')(e)$ with $e \in \bar{X}_1(\tau)$.

To show that $\varphi^{\tau,\varsigma} \circ \varphi^{\varsigma,\tau}$ is the identity, we must verify $\varphi^{\tau,\varsigma}(\xi')(e) = \xi(e)$ for all $e \in \bar{X}_1(\tau)$.

Clearly, $\varphi^{\tau,\varsigma}(\xi')(e) = \xi(e)$ if $e \in \bar{X}_1(\varsigma)$ or if $\Phi^{\xi'}(e) = a'^*$ (since then $\Phi^\xi(a') = e^*$).

Let us now assume that $\Phi^{\xi'}(e) = [b'a']^0_{r'}$. Then there exist $a, b \in X_1(\tau)$ induced by $\Delta(e)$ with $\Phi^{\xi'}(a) = b'^*$, $\Phi^{\xi'}(b) = a'^*$ and $\Phi^\xi(a') = b^*$, $\Phi^\xi(b') = a^*$.

In case $d_i \neq 1$, we get $\varphi^{\tau,\varsigma}(\xi')(e) = \xi'(a') + \xi'(b') = \xi(b) + \xi(a) = \xi(e)$ where the last equality uses that ξ is a cocycle.

In case $d_i = 1$, let $c'b'a'$ be a triangle-induced cyclic path in $X(\varsigma)$. Then it is $\Phi^\xi(c') = [ba]^0_r$ for some r, so $\varphi^{\tau,\varsigma}(\xi')(e) = \xi'(c') + 1 = (\xi(e)+1)+1 = \xi(e)$.

This proves that $\varphi^{\tau,\varsigma} \circ \varphi^{\varsigma,\tau}$ is the identity. Analogously, $\varphi^{\varsigma,\tau} \circ \varphi^{\tau,\varsigma}$ is the identity. □

Remark 6.2.33. For $\varsigma = \mu_i(\tau)$ the condition

$$\begin{gathered} d_i = 1 \;\wedge \\ \exists\, \xi \in Z^1(\tau),\, e' \in \bar{X}_1(\varsigma),\, cba \text{ triangle-induced in } X(\tau), r \in \{0,1\} : \Phi^\xi(e') = [ba]^0_r \end{gathered}$$

is non-empty if and only if i is a weight-1 arc in a triangle of τ with type $\widetilde{B}_1$, $\widetilde{C}_{14}$, or $\widetilde{C}_{41}$. Moreover, the arrow e' is uniquely determined and independent of ξ.

Corollary 6.2.34. *Let $\varsigma = \mu_i(\tau)$. If i is not the weight-1 pending arc in a triangle of τ with type $\widetilde{B}_1$, $\widetilde{C}_{14}$, or $\widetilde{C}_{41}$, then $\varphi^{\varsigma,\tau}$ and $\varphi^{\tau,\varsigma}$ are mutually inverse vector-space isomorphisms.*

Proof. Use Lemma 6.2.32 and Remark 6.2.33. □

Corollary 6.2.35. *Let $\varsigma = \mu_i(\tau)$ and $\tau = \mu_j(\varsigma)$. Two cocycles ξ_1 and ξ_2 in $Z^1(\tau)$ are cohomologous if and only if the cocycles $\varphi^{\varsigma,\tau}(\xi_1)$ and $\varphi^{\varsigma,\tau}(\xi_2)$ in $Z^1(\varsigma)$ are cohomologous.*

Proof. If j is the weight-1 arc in a triangle Δ of ς with type $\widetilde{B}_1$, $\widetilde{C}_{14}$, or $\widetilde{C}_{41}$, then i is the weight-1 arc in a triangle of τ with type $\widetilde{B}_1$, $\widetilde{C}_{41}$, or $\widetilde{C}_{14}$.

If this is the case, let e' be the unique arrow in $X^{d\neq 1}(\varsigma)$ that is induced by Δ and let $e'^\vee$ be its dual in $C^1(\varsigma)$ (i.e. $e'^\vee(a) = \delta_{a=e'}$ for all $a \in \bar{X}_1(\varsigma)$). Otherwise, let $e'^\vee = 0$.

In either case, $e'^\vee \in Z^1(\varsigma)$ and the formula in the proof of Lemma 6.2.32 together with Remark 6.2.33 show that $f_1 : \xi \mapsto \varphi^{\varsigma,\tau}(\xi) + e'^\vee$ defines an isomorphism $Z^1(\tau) \to Z^1(\varsigma)$ that makes the following diagram commute:

$$\begin{tikzcd}
C^0(\tau) \arrow[r, "\partial^0"] \arrow[d, "f_0"] & Z^1(\tau) \arrow[d, "f_1"] \\
C^0(\varsigma) \arrow[r, "\partial^0"] & Z^1(\varsigma)
\end{tikzcd}$$

Here, ∂^0 are the differentials $\mathrm{Hom}_{\mathbb{F}_2}(\partial_1, \mathbb{F}_2)$ and f_0 is the isomorphism given by $k^\vee \mapsto k^\vee$ for vertices $k \neq i$ in $\bar{X}(\tau)$ and $i^\vee \mapsto j^\vee$. We get an induced isomorphism in cohomology

$$H^1(C^\bullet(\tau)) \xrightarrow{f_*} H^1(C^\bullet(\varsigma)) \,.$$

In particular, $\xi_1 - \xi_2 = 0$ in $H^1(C^\bullet(\tau))$ if and only if $\varphi^{\varsigma,\tau}(\xi_1) - \varphi^{\varsigma,\tau}(\xi_2) = f_*(\xi_1 - \xi_2) = 0$ in $H^1(C^\bullet(\varsigma))$, which proves the corollary. $\square$

Definition 6.2.36. Let (τ, ξ) be a colored triangulation of $\mathbf{\Sigma}_d$ and $\varsigma = \mu_i(\tau)$. The colored triangulation $\mu_i(\tau, \xi) := (\varsigma, \varphi^{\varsigma,\tau}(\xi))$ is obtained by *flipping* the arc i in (τ, ξ).

6.3 Modulation of a Colored Triangulation

The possible weights for vertices in $X(\tau, \xi)$ are 1, 2, 4. Fixing a degree-4 comfy extension determines therefore a modulation of the modular quiver $X(\tau, \xi)$ over this extension.

Convention 6.3.1. For the rest of the chapter fix a degree-4 comfy extension $(L/K, \zeta, v)$.

Abbreviate $w := v^4 \in K$ and $u := v^2 \in K(u) =: E$.

Definition 6.3.2. The *modulation* $\mathcal{H}(\tau, \xi)$ *of a colored triangulation* (τ, ξ) is the modulation of $X(\tau, \xi)$ over $(L/K, \zeta, v)$.

The ground ring, species, path algebra, and completed path algebra of $\mathcal{H}(\tau, \xi)$ will be denoted by $R(\tau, \xi)$, $A(\tau, \xi)$, $H(\tau, \xi)$, $\widehat{H}(\tau, \xi)$, respectively.

6.4 Potential of a Colored Triangulation

6.4.1 Potential Components Induced by Triangles

Definition 6.4.1. Let (τ, ξ) be a colored triangulation of $\mathbf{\Sigma}_d$ and let Δ be a triangle of τ.

The *potential* $W^\Delta(\xi)$ *induced by* Δ is the following potential for $A(\tau, \xi)$:

$$W^\Delta(\xi) \;=\; \begin{cases} cba & \text{if } \Delta \text{ has neither type } \widetilde{C}_{11} \text{ nor type } \widetilde{C}_{44}, \\ & \quad \text{where } cba \text{ is a path in } X(\tau) \text{ induced by } \Delta. \\ cb_0a + cb_1a & \text{if } \Delta \text{ has type } \widetilde{C}_{44}, \\ & \quad \text{where } cb_0a \neq cb_1a \text{ are paths in } X(\tau) \text{ induced by } \Delta, \\ cb_0a + cb_1au & \text{if } \Delta \text{ has type } \widetilde{C}_{11}, \\ & \quad \text{where } cb_0a \neq cb_1a \text{ are paths in } X(\tau) \text{ induced by } \Delta \\ & \quad \text{such that } r(b_0) = 0 \text{ and } r(b_1) = 1. \end{cases}$$

Remark 6.4.2. Apparently, the potential $W^\Delta(\xi)$ looks the same for all ξ.

Example 6.4.3. Figures 6.A.5 to 6.A.7 show $W^\Delta = W^\Delta(\xi)$ for all types of triangles Δ.

6.4.2 Potential of a Colored Triangulation

Definition 6.4.4. The *potential of a colored triangulation* (τ, ξ) of $\mathbf{\Sigma}_d$ is defined as

$$W(\tau, \xi) \;=\; \sum_{\Delta} W^\Delta(\xi)$$

where Δ runs through all triangles of τ.

The *species with potential of* (τ, ξ) is $\mathcal{S}(\tau, \xi) = (A(\tau, \xi), W(\tau, \xi))$.

The *Jacobian algebra of* (τ, ξ) is $\mathcal{J}(\tau, \xi) = \mathcal{J}(W(\tau, \xi))$.

Remark 6.4.5. It is $(W(\tau, \xi))^{X_1^\Delta} = W^\Delta(\xi)$ for all triangles Δ of τ.

Example 6.4.6. For a colored triangulation (τ, ξ) with τ as in Example 6.2.18 it is

$$W(\tau, \xi) \;=\; c_1(b_{10} + b_{11})a_1 + c_2b_2a_2 + c_3b_3a_3 + c_4b_4a_4 \,.$$

Example 6.4.7. We compute the cyclic derivatives of the potential $W = W(\tau, \xi)$ for an arbitrary colored triangulation (τ, ξ). To do this, fix a triangle Δ of τ and let $X = X(\tau)$.

In view of Remark 6.4.5 we have $\partial_{a^\dagger}(W) = \partial_{a^\dagger}(W^\Delta(\xi))$ for all arrows $j \xleftarrow{a} i$ in X^Δ. Note also that $z \cdot a^\dagger(a) = \pi_{\rho_a^{-1}}(z)$ for $z \in {}_i\widehat{H}_j$.

- If Δ has type A or $\widetilde{B}_4$, then X^Δ has the form $i \xleftarrow{c} k \xleftarrow{b} j \xleftarrow{a} i$ and $W^\Delta(\xi) = cba$. We have $d_{ik} = d_{kj} = d_{ji} = 2$ and $\rho_{c^{-1}} = \rho_b\rho_a$, $\rho_{b^{-1}} = \rho_a\rho_c$, $\rho_{a^{-1}} = \rho_c\rho_b$. One computes

$$\partial_{c^\dagger}(W) = ba\,, \qquad \partial_{b^\dagger}(W) = ac\,, \qquad \partial_{a^\dagger}(W) = cb\,.$$

- If Δ has type $\widetilde{C}_{44}$, then X^Δ has the form $i \xleftarrow{c} k \overset{b_0}{\underset{b_1}{\leftleftarrows}} j \xleftarrow{a} i$. Set $b := b_0 + b_1 \in A(\tau, \xi)$ and $b^\dagger := b_0^\dagger + b_1^\dagger \in A^\dagger(\tau, \xi)$ and $\rho_b := \rho_{b_0}|_E = \rho_{b_1}|_E$. Then $W^\Delta(\xi) = cb_0a + cb_1a = cba$. It is $\rho_{c^{-1}} = \rho_b\rho_a$, $\rho_{b^{-1}} = \rho_a\rho_c$, $\rho_{a^{-1}} = \rho_c\rho_b$ and $G_{ji}^{\rho_b} = \{\rho_{b_0}, \rho_{b_1}\}$ such that again

$$\partial_{c^\dagger}(W) = ba\,, \qquad \partial_{b^\dagger}(W) = ac\,, \qquad \partial_{a^\dagger}(W) = cb\,.$$

- If Δ has type $\widetilde{B}_1$, $\widetilde{C}_{14}$, or $\widetilde{C}_{41}$, then X^Δ has the form $i \xleftarrow{c} k \xleftarrow{b} j \xleftarrow{a} i$ with $d_{kj} = 2$ and $d_{ik} = d_{ji} = 1$. It is $W^\Delta(\xi) = cba$ and

$$\partial_{c^\dagger}(W) = ba\,, \qquad \partial_{b^\dagger}(W) = \pi_{\rho_b^{-1}}(ac)\,, \qquad \partial_{a^\dagger}(W) = cb\,.$$

- If Δ has type $\widetilde{C}_{11}$, then X^Δ has the form $i \xleftarrow{c} k \overset{b_0}{\underset{b_1}{\leftleftarrows}} j \xleftarrow{a} i$ with $r(b_0) = 0$ and $r(b_1) = 1$. Then $W^\Delta(\xi) = cb_0a + cb_1au$ and

$$\partial_{c^\dagger}(W) = b_0a + b_1au\,, \quad \partial_{b_0^\dagger}(W) = ac\,, \quad \partial_{b_1^\dagger}(W) = auc\,, \quad \partial_{a^\dagger}(W) = cb_0 + ucb_1\,.$$

6.5 Compatibility of Flip and Mutation

All is said and done to prove the first main result of this chapter: the compatibility of flip and mutation. It is the variant of Theorems 5.4.1 and 5.4.6 for colored triangulations.

Convention 6.5.1. We make use of the obvious generalization of Convention 5.3.13 for colored triangulations (replace Q, τ, $\varsigma = \widetilde{\mu}_i(\tau)$ by X, (τ,ξ), $(\varsigma,\xi') = \widetilde{\mu}_i(\tau,\xi)$, respectively).

Theorem 6.5.2. *$\mathcal{S}(\mu_i(\tau,\xi)) \approx_R \mu_i(\mathcal{S}(\tau,\xi))$ for all colored triangulations (τ,ξ) and $i \in \tau$.*

Proof. The proof is similar to the proof of Theorem 5.4.1.

Let $(\varsigma,\xi') = \mu_i(\tau,\xi)$ and $\tau = \mu_j(\varsigma)$. Abbreviate $X = X(\tau,i)$, $X' = X(\varsigma,j)$, $\widetilde{X} = \widetilde{\mu}_i(X)$. Consider X, X', $\widetilde{X}$ as modular subquivers of $X(\tau,\xi)$, $X(\varsigma,\xi')$, $\widetilde{\mu}_i(X(\tau,\xi))$, respectively.

Recall that $X(\tau) = X^\neg \oplus X$ and $X(\varsigma) = X^\neg \oplus X'$ and $\widetilde{\mu}_i(X(\tau)) = X^\neg \oplus \widetilde{X}$ for some $X^\neg$.

Let $\sigma = \sigma^\xi$ and $\Phi = \Phi^\sigma$ be the map $X(\varsigma,\xi') \hookrightarrow \widetilde{\mu}_i(X(\tau,\xi))$ described in Definition 6.2.20, which is a morphism of modular quivers by Lemma 6.2.32. It restricts to a map $X' \hookrightarrow \widetilde{X}$ and induces an injective $R(\tau)$-algebra homomorphism

$$H' := H(\varsigma,\xi') \xhookrightarrow{\;\Phi\;} \widetilde{\mu}_i(H(\tau,\xi)) =: \widetilde{H}\,.$$

Let $W = (W(\tau,\xi))^X$ and $W' = (W(\varsigma,\xi'))^{X'}$ and $\widetilde{\mathrm{Aut}} = \mathrm{Aut}_{\widetilde{\mu}_i(X(\tau,\xi)) - \widetilde{X}_1}(\widetilde{H})$.

We will proceed as follows:

(1) Compute the premutation $\widetilde{W} = \widetilde{\mu}_i(W)$.

(2) Construct $\widetilde{\vartheta} \in \widetilde{\mathrm{Aut}}$ such that $\widetilde{\vartheta}(\widetilde{W})$ is in $\widetilde{X}_1$-split form.

(3) Compute $\widetilde{W}' = \mathrm{red}_{\widetilde{X}_1}\big(\widetilde{\vartheta}(\widetilde{W})\big)$ and $\widetilde{T} = \mathrm{triv}_{\widetilde{X}_1}\big(\widetilde{\vartheta}(\widetilde{W})\big) \subseteq \widetilde{X}$.

We will choose $\widetilde{\vartheta}$ in such a way that $\Phi(W') = \widetilde{W}'$ and $\widetilde{X} = \Phi(X') \oplus \widetilde{T}$. This will prove the theorem, since then

$$\Phi(W(\varsigma,\xi')) \;=\; \mathrm{red}_{\widetilde{X}_1}\big(\widetilde{\vartheta}\big(\widetilde{\mu}_i(W(\tau,\xi))\big)\big)\,.$$

For the construction of $\widetilde{\vartheta}$ we distinguish four cases: (a) i is a non-pending arc; (b) i is a pending arc in a triangle of type $\widetilde{C}_{11}$ or $\widetilde{C}_{44}$; (c) i is the weight-4 arc in a triangle of type $\widetilde{B}_4$, $\widetilde{C}_{14}$, or $\widetilde{C}_{41}$; (d) i is the weight-1 arc in a triangle of type $\widetilde{B}_1$, $\widetilde{C}_{14}$, or $\widetilde{C}_{41}$.

(a) Let us assume that i is non-pending. Then $d_i = 2$ and i is an arc shared by two triangles Δ_0 and Δ_1 of τ (see Tables 6.A.2 and 6.A.3).

Let $i \xleftarrow{a_q} h_q \xleftarrow{c_{qr}} k_q \xleftarrow{b_q} i$ be the cyclic paths in $X(\tau)$ induced by Δ_q such that $r(c_{qr}) = r$, where $q, r \in \{0,1\}$ with $r = 1$ only if $h_q, k_q \in \tau^{d=4}$ or $h_q, k_q \in \tau^{d=1}$. We set $\bar{q} := 1 - q$ and

$$\begin{aligned} x_q &:= \delta_{h_q,k_q \in \tau^{d=4}}\,, & x'_q &:= \delta_{h_q,k_{\bar{q}} \in \tau^{d=4}}\,, \\ y_q &:= \delta_{h_q,k_q \in \tau^{d=1}}\,, & y'_q &:= \delta_{h_q,k_{\bar{q}} \in \tau^{d=1}}\,. \end{aligned}$$

Then $x_q y_q = x'_q y'_q = x'_q y_q = x_q y'_q = 0$.

Now $X = X^{\Delta_0} \oplus X^{\Delta_1}$ and $W = \sum_{q\in\{0,1\}} W_q$ with

$$W_q \;=\; W^{\Delta_q}(\xi) \;=\; c_{q0} b_q a_q + x_q c_{q1} b_q a_q + y_q c_{q1} b_q u a_q\,.$$

We compute the premutation as $\widetilde{W} = \sum_{q\in\{0,1\}} \widetilde{W}_q$ where

$$\begin{aligned} \widetilde{W}_q \;=\;& c_{q0}[b_q a_q]^0_{!0} \;+\; x_q\; c_{q1}[b_q a_q]^0_{!1} \;+\; y_q \qquad c_{q1}[b_q a_q]^1_0 \\ &+\; a_q^* b_q^* [b_q a_q]^0_0 \;+\; x_q\, a_q^* b_q^* [b_q a_q]^0_1 \;+\; y_q\, a_q^* u^{-1} b_q^* [b_q a_q]^1_0 \\ &+\; a_q^* b_{\bar{q}}^* [b_{\bar{q}} a_q]^0_0 \;+\; x'_q\, a_q^* b_{\bar{q}}^* [b_{\bar{q}} a_q]^0_1 \;+\; y'_q\, a_q^* u^{-1} b_{\bar{q}}^* [b_{\bar{q}} a_q]^1_0\,. \end{aligned}$$

Let $\rho_{qr} := \rho_{c_{qr}} = \big(\rho_{[b_q a_q]^0_{!r}}\big)^{-1}$ and $\widetilde{\psi}_{1q} \in \widetilde{\mathrm{Aut}}$ the element determined by the rules:

$$\begin{aligned} c_{q0} &\mapsto c_{q0} - \quad \pi_{\rho_{q0}}(a_q^* b_q^*) \\ c_{q1} &\mapsto c_{q1} - x_q \pi_{\rho_{q1}}(a_q^* b_q^*) - y_q a_q^* u^{-1} b_q^* \quad (\text{if } x_q \neq 0 \text{ or } y_q \neq 0) \end{aligned}$$

As potentials $\nu[b_q a_q]^0_{!r} = \pi_{\rho_{qr}}(\nu)[b_q a_q]^0_{!r}$ for $\nu \in \widetilde{H}$ (see Corollary 2.6.56). Now we have $\widetilde{\psi}_{1q}(\widetilde{W}_{\bar{q}}) = \widetilde{W}_{\bar{q}}$ and a straightforward calculation shows

$$\begin{aligned} \widetilde{\psi}_{1q}(\widetilde{W}_q) \;=\;& c_{q0}[b_q a_q]^0_{!0} \;+\; x_q\; c_{q1}[b_q a_q]^0_{!1} \;+\; y_q \qquad c_{q1}[b_q a_q]^1_0 \\ &+\; a_q^* b_{\bar{q}}^* [b_{\bar{q}} a_q]^0_0 \;+\; x'_q\, a_q^* b_{\bar{q}}^* [b_{\bar{q}} a_q]^0_1 \;+\; y'_q\, a_q^* u^{-1} b_{\bar{q}}^* [b_{\bar{q}} a_q]^1_0\,. \end{aligned}$$

If $y'_q \neq 0$, let $\widetilde{\psi}_{2q} \in \widetilde{\mathrm{Aut}}$ be the element defined by the substitutions

$$a_q^* \mapsto a_q^* u\,, \qquad [b_{\bar{q}} a_q]^0_0 \mapsto [b_{\bar{q}} a_q]^1_0\,, \qquad [b_{\bar{q}} a_q]^1_0 \mapsto [b_{\bar{q}} a_q]^0_0\,.$$

Otherwise (in particular, if $x'_q \neq 0$), let $\widetilde{\psi}_{2q}$ be the identity.

With $\widetilde{\vartheta} = \widetilde{\psi}_{22} \circ \widetilde{\psi}_{21} \circ \widetilde{\psi}_{12} \circ \widetilde{\psi}_{11}$ it is $\widetilde{W}' = \mathrm{red}_{\widetilde{X}_1}\big(\widetilde{\vartheta}(\widetilde{W})\big) = \sum_{q\in\{0,1\}} \widetilde{W}'_q$ where

$$\widetilde{W}'_q \;=\; a_q^* b_{\bar{q}}^* [b_{\bar{q}} a_q]^0_0 \;+\; x'_q\, a_q^* b_{\bar{q}}^* [b_{\bar{q}} a_q]^0_1 \;+\; y'_q\, a_q^* u b_{\bar{q}}^* [b_{\bar{q}} a_q]^1_0\,.$$

By the properties of Φ (see Remark 6.2.14) there is for each $q \in \{0,1\}$ a triangle Δ'_q of ς inducing a path $b_q^\vee c_q^\vee a_q^\vee$ such that $\Phi(b_q^\vee) = b_{\bar{q}}^*$, $\Phi(c_q^\vee) = [b_{\bar{q}} a_q]^0_{r(c_q^\vee)}$, $\Phi(a_q^\vee) = a_q^*$

This readily implies $\Phi(W') = \sum_{q\in\{0,1\}} \Phi(W^{\Delta'_q}(\xi')) = \widetilde{W}'$ as desired.

Finally, observe that $\widetilde{T} = \mathrm{triv}_{\widetilde{X}_1}\big(\widetilde{\vartheta}(\widetilde{W})\big)$ coincides with the modular quiver $\widetilde{T}$ described in the proof of Lemma 6.2.19. Hence, $\widetilde{X} = \Phi(X') \oplus \widetilde{T}$.

(b) Let i be a pending arc in a triangle Δ of type $\widetilde{C}_{11}$ or $\widetilde{C}_{44}$. Since the other case is symmetric, we will assume that the arctype for i is $\widetilde{C}^-$ (see Table 6.A.4).

There are two paths $i \xleftarrow{a} \xleftarrow{c} \xleftarrow{b_r} i$ with $r(b_r) = r \in \{0,1\}$ in $X(\tau)$ induced by Δ.

We have $X = X^\Delta$ and $W = W^\Delta(\xi) = cb_0a + cb_1au^x$ for $x := \delta_{i\in\tau^{d=1}}$. Thus

$$\begin{aligned}\widetilde{W} \;=\;& \quad c[b_0a]^0_0 \;+\; \quad c[b_1a]^0_0\, u^x \\ &+\; a^*b_0^*[b_0a]^0_0 \;+\; a^*b_1^*[b_1a]^0_0 \quad .\end{aligned}$$

Let $\widetilde{\varphi} \in \widetilde{\mathrm{Aut}}$ be the element defined by $b_1^* \mapsto -b_1^*$, $[b_1a]^0_0 \mapsto ([b_1a]^0_0 - [b_0a]^0_0)u^{-x}$. Then

$$\widetilde{\varphi}\big(\widetilde{W}\big) \;=\; c[b_1a]^0_0 \;+\; a^*b_0^*[b_0a]^0_0 \;-\; a^*b_1^*[b_1a]^0_0\, u^{-x} \;+\; a^*b_1^*[b_0a]^0_0\, u^{-x}\;.$$

Let $\widetilde{\psi}_1 \in \widetilde{\mathrm{Aut}}$ be given by $c \mapsto c + \pi_{\rho_c}(u^{-x}a^*b_1^*)$.

If $x \neq 0$, let $\widetilde{\psi}_2 \in \widetilde{\mathrm{Aut}}$ be given by $a_q^* \mapsto ua_q^*$, $b_0^* \mapsto b_1^*$, $b_1^* \mapsto b_0^*$. Otherwise, let $\widetilde{\psi}_2 = \mathrm{id}$.

Let $\widetilde{\vartheta} = \widetilde{\psi}_2 \circ \widetilde{\psi}_1 \circ \widetilde{\varphi}$. Then $\widetilde{W}' = a^*b_0^*[b_0a]^0_0 + a^*b_1^*[b_0a]^0_0\, u^x = \Phi(W')$ and $\widetilde{X} = \Phi(X') \oplus \widetilde{T}$.

(c) Let i be the weight-4 arc in a triangle Δ of type $\widetilde{B}_4$, $\widetilde{C}_{14}$, or $\widetilde{C}_{41}$.

There is a unique cyclic path $i \xleftarrow{a} \xleftarrow{c} \xleftarrow{b} i$ in $X(\tau)$ induced by Δ.

It is $X = X^\Delta$, $W = W^\Delta(\xi) = cba$, and

$$\begin{aligned}\widetilde{W} \;=\;& \quad c[ba]^0_0 \;+\; \quad\quad c[ba]^1_0 \\ &+\; a^*b^*[ba]^0_0 \;+\; a^*u^{-1}b^*[ba]^1_0\;.\end{aligned}$$

The substitution $[ba]^1_0 \mapsto [ba]^1_0 - [ba]^0_0$ defines $\widetilde{\varphi} \in \widetilde{\mathrm{Aut}}$ such that

$$\widetilde{\varphi}\big(\widetilde{W}\big) \;=\; c[ba]^1_0 \;+\; a^*(1-u^{-1})b^*[ba]^0_0 \;+\; a^*u^{-1}b^*[ba]^1_0\;.$$

Let $\widetilde{\psi} \in \widetilde{\mathrm{Aut}}$ be given by $a^* \mapsto a^*z$, $c \mapsto c - \pi_{\rho_c}(a^*zu^{-1}b^*)$ for $z = (1-u^{-1})^{-1}$.

For $\widetilde{\vartheta} = \widetilde{\psi} \circ \widetilde{\varphi}$ we have $\widetilde{W}' = a^*b^*[ba]^0_0 = \Phi(W')$ and $\widetilde{X} = \Phi(X') \oplus \widetilde{T}$.

(d) Let i be the weight-1 arc in a triangle of type $\widetilde{B}_1$, $\widetilde{C}_{14}$, or $\widetilde{C}_{41}$.

There is a unique cyclic path $i \xleftarrow{a} \xleftarrow{c} \xleftarrow{b} i$ in $X(\tau)$ induced by Δ.

It is $X = X^\Delta$, $W = W^\Delta(\xi) = cba$, and

$$\widetilde{W} \;=\; c[ba]^0_{\sigma_c} \;+\; a^*b^*[ba]^0_0 \;+\; a^*b^*[ba]^0_1\;.$$

Let $\widetilde{\vartheta}$ be given by the substitution $c \mapsto c - \pi_{\rho_c}(a^* b^*)$.

Then $\widetilde{W}' = a^* b^* [ba]^0_{\sigma_c + 1} = \Phi(W')$ and $\widetilde{X} = \Phi(X') \oplus \widetilde{T}$. □

Corollary 6.5.3. *$\mathcal{S}(\tau, \xi)$ is non-degenerate for every colored triangulation (τ, ξ) of $\mathbf{\Sigma}_d$.*

Proof. This is a direct consequence of Theorem 6.5.2 and Remark 6.2.9. □

6.6 Uniqueness of Potentials

In this section we prove that $A(\tau, \xi)$ admits up to R-equivalence exactly one non-degenerate potential, namely $W(\tau, \xi)$, if $\mathbf{\Sigma}_d$ is not a monogon with constant $\mathbb{O} \xrightarrow{d} \{1, 4\}$.

As a preparation for the proof we collect a few combinatorial facts in the lemmas below.

Convention 6.6.1. Fix a colored triangulation (τ, ξ) of $\mathbf{\Sigma}_d$.

Abbreviate $X = X(\tau)$, $H = H(\tau, \xi)$, $\widehat{H} = \widehat{H}(\tau, \xi)$.

Denote by q_{ji} the number of arrows $j \leftarrow i$ in X.

Every cyclic path of length three in X contains at most one arrow $j \leftarrow i$ with $q_{ji} \neq 1$. In other words, the paths $i \xleftarrow{c} k \xleftarrow{b} j \xleftarrow{a} i$ with $q_{kj} = q_{ji} = 1$ are up to rotation all cyclic paths of length three in X. We record this fact in the next lemma.

Lemma 6.6.2. *For all $i, j, k \in X_0$ with $q_{kj}, q_{ji} > 0$ either $q_{kj} = 1$ or $q_{ji} = 1$.*

Proof. This follows easily by inspecting the puzzle-piece decomposition of τ. □

Remark 6.2.9 and Lemma 6.6.2 allow us to label the arrows of each $\Delta \in X_2$ as follows:

(a) Pick a path $i \xleftarrow{c_\Delta} k \xleftarrow{b_\Delta} j \xleftarrow{a_\Delta} i$ in X induced by Δ with $q_{kj} = q_{ji} = 1$ and $r(c_\Delta) = 0$.

(b) Whenever $q_{ik} > 1$ let $i \xleftarrow{\underline{c}_\Delta} k$ be the unique arrow in X unequal c_Δ.

(c) Whenever $q_{ik} = 1$ set $\underline{c}_\Delta := 0$ regarded as an element of H.

Convention 6.6.3. Fix such a labeling for the rest of the section.

Lemma 6.6.4. *If cba is a cyclic path in X with $\Delta(a) \neq \Delta(c) \neq \Delta(b)$, it is $\Delta(b) = \Delta(a)$. More generally, X contains no cyclic path $a_\ell \cdots a_1$ with $\Delta(a_q) \neq \Delta(a_{q-1})$ for all $q \in \mathbb{Z}/\ell\mathbb{Z}$.*

Proof. Assume to the contrary $p = a_\ell \cdots a_1$ is a cyclic path in X with $\Delta(a_q) \neq \Delta(a_{q-1})$ for all $q \in \mathbb{Z}/\ell\mathbb{Z}$. Then all vertices of p are shared arcs in the triangulation τ and by definition of the arrows in X the triangles $\Delta(a_1), \ldots, \Delta(a_\ell)$ must form a configuration as depicted in Definition 5.3.6, a contradiction to the fact that $\mathbf{\Sigma}$ is unpunctured. □

We will need the following more general version of the previous lemma:

Lemma 6.6.5. *Let $p = a_\ell \cdots a_1$ be a cyclic path in X. Then there exists $q \in \mathbb{Z}/\ell\mathbb{Z}$ such that either $\Delta(a_q) = \Delta(a_{q-1})$ is a triangle of type A or $\Delta(a_{q+1}) = \Delta(a_q) = \Delta(a_{q-1})$.*

Proof. Assume p does not verify the property in the lemma. Then, after possibly replacing p by $a_{\ell-1} \cdots a_1 a_\ell$, we can write $p = p_s \cdots p_1$ where each p_q is either an arrow or a path $\xleftarrow{b_q} \xleftarrow{a_q}$ in X that connects two shared arcs in τ and, in the latter case, $\Delta(p_q) := \Delta(b_q) = \Delta(a_q)$ is a triangle of type $\widetilde{B}$. Moreover, we must have $\Delta(p_q) \neq \Delta(p_{q-1})$ for all $q \in \mathbb{Z}/s\mathbb{Z}$. Analogously as in the proof of Lemma 6.6.4 the triangles $\Delta(p_1), \ldots, \Delta(p_s)$ would form a configuration as depicted in Definition 5.3.6 in contradiction to the fact that $\mathbf{\Sigma}$ is unpunctured. $\square$

Lemma 6.6.6. *If $i \xleftarrow{c} k \xleftarrow{b} j \xleftarrow{a} i$ is a path in X with $q_{kj} = q_{ji} = 1$, then either cba is triangle-induced or there is a unique cyclic path $c\underline{ba}$ in X induced by some triangle $\underline{\Delta}$ such that $\underline{c}ba$ is induced by some triangle Δ and both Δ and $\underline{\Delta}$ are triangles of type A.*

Proof. If $\Delta(c) = \Delta(b)$, then we already have $\Delta(c) = \Delta(b) = \Delta(a)$ because of $q_{ji} = 1$. Similarly, $\Delta(a) = \Delta(c)$ implies that cba is triangle-induced. If $\Delta(a) \neq \Delta(c) \neq \Delta(b)$, we must have $\Delta := \Delta(b) = \Delta(a)$ according to Lemma 6.6.4. Since cba is not triangle-induced, necessarily $q_{ik} > 1$ or, equivalently, $q_{ik} = 2$. Because of $\Delta(c) \neq \Delta(b)$ the arcs i and k must be shared by two triangles Δ and $\underline{\Delta}$ of type A (see Table 6.A.3). The claim follows. $\square$

Proposition 6.6.7. *Let W be a non-degenerate potential for $A(\tau,\xi)$. Then there exists a potential $W_{>3}$ with* $\operatorname{ord}(W_{>3}) > 3$ *such that $W \sim_R W(\tau,\xi) + W_{>3}$.*

Proof. For $x \in \{4,1\}$ denote by X_2^x the subset of X_2 consisting of triangles of type $\widetilde{C}_{xx}$. Let $X_2^0 = X_2 \setminus (X_2^4 \cup X_2^1)$.

With Lemmas 6.6.2 and 6.6.6 it is not hard to see that $W = \sum_{q\in\{0,4,1\}} \sum_{\Delta\in X_2^q} W_\Delta + W_{>3}$ where $\operatorname{ord}(W_{>3}) > 3$ and

$$W_\Delta = \begin{cases} x_\Delta c_\Delta b_\Delta a_\Delta + \underline{x}_\Delta \underline{c}_\Delta b_\Delta a_\Delta & \text{for } \Delta \in X_2^0 \text{ and some } x_\Delta, \underline{x}_\Delta \in E, \\ c_\Delta \alpha_\Delta b_\Delta a_\Delta + \underline{c}_\Delta \underline{\alpha}_\Delta b_\Delta a_\Delta & \text{for } \Delta \in X_2^4 \text{ and some } \alpha_\Delta, \underline{\alpha}_\Delta \in L, \\ c_\Delta b_\Delta z_\Delta a_\Delta + \underline{c}_\Delta b_\Delta \underline{z}_\Delta a_\Delta & \text{for } \Delta \in X_2^1 \text{ and some } z_\Delta, \underline{z}_\Delta \in E. \end{cases}$$

Step by step, we will replace W with $R(\tau)$-equivalent potentials, thereby achieving in a first step (a) $x_\Delta = 1$ and $\underline{x}_\Delta = 0$ for $\Delta \in X_2^0$, in a second step (b) $\alpha_\Delta = \underline{\alpha}_\Delta = 1$ for $\Delta \in X_2^4$ and in the final step (c) $z_\Delta = 1$ and $\underline{z}_\Delta = u$ for $\Delta \in X_2^1$. This will prove the proposition.

(a) Let $\Delta \in X_2^0$.

If $\underline{c}_\Delta = 0$, then $(W_\Delta)^{X_1'} = W_\Delta$ for $X' = X^\Delta$. Moreover, $x_\Delta \in E^\times$ because of the non-degeneracy. Replacing W with $\varphi(W)$, where $\varphi \in \operatorname{Aut}_{X-X_1'}(H)$ is given by $c_\Delta \mapsto x_\Delta^{-1} c_\Delta$, we get $x_\Delta = 1$ and can take $\underline{x}_\Delta = 0$.

If $\underline{c}_\Delta \neq 0$, there exists $\underline{\Delta} \in X_2^0$ inducing $\underline{c}_\Delta = c_{\underline{\Delta}}$. We have $(W_\Delta)^{X_1'} = W_\Delta + W_{\underline{\Delta}}$ for $X' := X^\Delta \oplus X^{\underline{\Delta}}$. It is not hard to check that the non-degeneracy of W implies that

the inverse matrix
$$\begin{pmatrix} \nu_\Delta & \underline{\nu}_\Delta \\ \nu_{\underline{\Delta}} & \nu_{\underline{\Delta}} \end{pmatrix} = \begin{pmatrix} x_\Delta & \underline{x}_\Delta \\ \underline{x}_{\underline{\Delta}} & x_{\underline{\Delta}} \end{pmatrix}^{-1}$$
exists. Therefore the substitutions $c_\Delta \mapsto \nu_\Delta c_\Delta + \underline{\nu}_\Delta \underline{c}_\Delta$ and $\underline{c}_\Delta \mapsto \underline{\nu}_{\underline{\Delta}} c_\Delta + \nu_{\underline{\Delta}} \underline{c}_\Delta$ define an element $\varphi \in \mathrm{Aut}_{X-X_1'}(H)$. Replacing W by $\varphi(W)$ we attain $x_\Delta = 1$ and $\underline{x}_\Delta = 0$.

(b) For all $\Delta \in X_2^4$ and $X' := X^\Delta$ we have $(W_\Delta)^{X_1'} = W_\Delta$. The non-degeneracy of W implies $\alpha_\Delta, \underline{\alpha}_\Delta \in L^\times$. Clearly, $c_\Delta \mapsto c_\Delta \alpha_\Delta^{-1}$, $\underline{c}_\Delta \mapsto \underline{c}_\Delta \underline{\alpha}_\Delta^{-1}$ defines $\varphi \in \mathrm{Aut}_{X-X_1'}(H)$ such that, after replacing W with $\varphi(W)$, one has $\alpha_\Delta = \underline{\alpha}_\Delta = 1$.

(c) For all $\Delta \in X_2^1$ and $X' := X^\Delta$ we have $(W_\Delta)^{X_1'} = W_\Delta$. The non-degeneracy implies that $z_\Delta, \underline{z}_\Delta$ are linearly independent over K. Write $z_\Delta^{-1}\underline{z}_\Delta = x + yu$ with $x \in K$, $y \in K^\times$. Define an element $\varphi \in \mathrm{Aut}_{X-X_1'}(H)$ by $b_\Delta \mapsto b_\Delta z_\Delta^{-1}$, $c_\Delta \mapsto c_\Delta - xy^{-1}\underline{c}_\Delta$, $\underline{c}_\Delta \mapsto y^{-1}\underline{c}_\Delta$. Replacing W with $\varphi(W)$ we achieve $z_\Delta = 1$ and $\underline{z}_\Delta = u$. ☐

Theorem 6.6.8. *Assume $\mathbf{\Sigma}_d$ is not a monogon where all orbifold points have the same weight. Then every non-degenerate potential W for $A(\tau,\xi)$ is $R(\tau)$-equivalent to $W(\tau,\xi)$.*

Proof. The assumption guarantees the existence of a triangulation ς of $\mathbf{\Sigma}_d$ without triangles of type $\widetilde{C}_{11}$ and $\widetilde{C}_{44}$.

It is $(\varsigma,\xi') = \mu_{i_\ell}\cdots\mu_{i_1}(\tau,\xi)$ for some ξ' and some arcs $i_1,\dots,i_\ell$ by Proposition 5.1.32. Now $W' := \mu_{i_\ell}\cdots\mu_{i_1}(W) \sim_R W(\varsigma,\xi') \Leftrightarrow W \sim_R W(\tau,\xi)$ by Theorems 2.6.101 and 6.5.2.

Replacing (τ,ξ) and W by (ς,ξ') and W', we can assume that τ does neither contain any triangle of type $\widetilde{C}_{11}$ nor any of type $\widetilde{C}_{44}$. Then
$$W(\tau,\xi) \;=\; \sum_{\Delta\in X_2} c_\Delta b_\Delta a_\Delta \,.$$

Now assume that for some $k \in \mathbb{N}$ we have a potential
$$W_k \;=\; W(\tau,\xi) + W_{>k+3}$$
with $\mathrm{ord}(W_{>k+3}) > k+3$. In view of Lemma 6.6.5 we can write
$$W_{>k+3} \;=\; \sum_{\Delta\in X_2} \left(\nu_{c_\Delta} b_\Delta a_\Delta + c_\Delta \nu_{b_\Delta} a_\Delta + c_\Delta b_\Delta \nu_{a_\Delta}\right)$$
for some elements $\nu_a \in \widehat{H}$ with $\mathrm{ord}(\nu_a) > k+1$ such that $\nu_a = 0$ if $j \xleftarrow{a} i$ is induced by a triangle of type $\widetilde{B}_1$, $\widetilde{C}_{14}$, or $\widetilde{C}_{41}$ and $d_{ji} \neq 1$.

Let $\sigma = \sigma^\xi$. By the choice of ξ (see Lemma 6.2.19) we have $\sigma_a = \sigma_b + \sigma_c$ in $\mathbb{Z}/d_{ji}\mathbb{Z}$ for all triangle-induced cyclic paths $i \xleftarrow{c} k \xleftarrow{b} j \xleftarrow{a} i$ in X such that we can assume $\nu_a = \pi_{\rho_a}(\nu_a)$ (note that, if $\nu_a \neq 0$, either $d_{ji} = 1$ or $d_{ki} = d_{jk} = d_{ji}$; compare Corollary 2.6.56).

The depth of the unitriangular automorphism φ_{k+1} of $\widehat{H}$ given by the rules $a \mapsto a - \nu_a$ for $a \in X_1$ is at least $k+1$. A straightforward computation shows
$$W_{k+1} \;:=\; \varphi_{k+1}(W_k) \;=\; W(\tau,\xi) + W_{>(k+1)+3}$$

with $\operatorname{ord}(W_{>(k+1)+3}) > (k+1)+3$.

Replacing W with an equivalent potential like in Proposition 6.6.7, we can take $W_0 = W$. The construction just described yields a sequence $(\varphi_k)_{k\in\mathbb{N}_+}$ of unitriangular automorphisms with $\lim_k \operatorname{depth}(\varphi_k) = \infty$ and $\lim_k W_k = W(\tau,\xi)$ for $W_k = \widetilde{\varphi}_k(W)$ and $\widetilde{\varphi}_k = \varphi_k \cdots \varphi_1$. Thus $W(\tau,\xi) = \widetilde{\varphi}(W)$ for $\widetilde{\varphi} = \lim_r \widetilde{\varphi}_r$, which proves the theorem. $\square$

6.7 Jacobian Algebras

The Jacobian algebras $\mathcal{J}(\tau,\xi)$ will be shown to be finite-dimensional. Moreover, we prove that for each fixed triangulation τ the isomorphism classes of Jacobian algebras $\mathcal{J}(\tau,\xi)$ are parametrized by a cohomology group. More precisely, there is a bijection

$$\begin{array}{ccc} H^1(C^\bullet(\tau)) & \xrightarrow{\cong} & \{\mathcal{J}(\tau,\xi)\,|\,\xi\in Z^1(\tau)\}/_{\cong_\tau} \\ [\xi] & \longmapsto & \mathcal{J}(\tau,\xi) \end{array}$$

where the set on the right-hand side consists of isomorphism classes of $K^{X_0(\tau)}$-algebras.

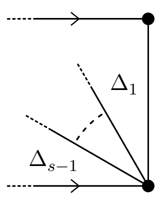
As a preparation, we need the following lemma.

Lemma 6.7.1. *For every path* $i_1 \xleftarrow{a_\ell} i_\ell \xleftarrow{a_{\ell-1}} \cdots \xleftarrow{a_1} i_1$ *in* $X(\tau)$ *of length* $\ell > 3|X_0(\tau)|$ *there is* $1 < q \leq \ell$ *such that* $\Delta(a_q) = \Delta(a_{q-1})$ *and* i_q *is a shared arc in* τ.

Proof. Let $p = a_\ell \cdots a_1$ if $a_\ell \cdots a_1$ starts at a shared arc. Otherwise, let p be one of the cyclic paths $a_1 a_\ell \cdots a_2$ and $a_{\ell-1} \cdots a_1 a_\ell$ such that p starts at a shared arc.

Write $p = p_s \cdots p_1$ such that each path p_r connects two shared arcs and has unshared arcs as its inner vertices. Then all arrows of p_r are induced by the same triangle Δ_r of τ.

Arguing as in the proof of Lemma 6.6.5 we must have either $\Delta_r = \Delta_{r-1}$ for some $1 < r \leq s$ or $\Delta_1 = \Delta_s$. If $\Delta_r \neq \Delta_{r-1}$ for all $1 < r \leq s$, then $\Delta_1, \ldots, \Delta_{s-1}$ would be pairwise different and form a configuration looking as follows:



In particular, we would have $s \leq |X_0(\tau)|$. Since the length of each path p_r is at most 3, the length of p would be at most $3|X_0(\tau)|$ in contradiction to $\ell > 3|X_0(\tau)|$. Therefore we must have $\Delta_r = \Delta_{r-1}$ for some $1 < r \leq s$. This readily implies the lemma. $\square$

Let $\mathfrak{m} = \mathfrak{m}_{\widehat{H}(\tau,\xi)}$. An ideal J of $\widehat{H}(\tau,\xi)$ is *admissible* if there is $\ell \in \mathbb{N}$ with $\mathfrak{m}^\ell \subseteq J \subseteq \mathfrak{m}^2$.

Proposition 6.7.2. *The Jacobian ideal $\partial(W(\tau,\xi))$ is an admissible ideal. In particular, the Jacobian algebra $\mathcal{J}(\tau,\xi)$ is finite-dimensional with* $\operatorname{rad}(\mathcal{J}(\tau,\xi)) = \mathfrak{m}/\partial(W(\tau,\xi))$.

Proof. Let $A = A(\tau,\xi)$ and $W = W(\tau,\xi)$.

Obviously, ∂W is generated by elements in $\mathfrak{m}^2$ (see Example 6.4.7). To show that ∂W is admissible, it is thus sufficient to verify the inclusion $A_{a_\ell} \cdots A_{a_1} \subseteq \partial W$ for every cyclic path $i_1 \xleftarrow{a_\ell} i_\ell \xleftarrow{a_{\ell-1}} \cdots \xleftarrow{a_1} i_1$ in $X(\tau)$ of length $\ell > 3|X_0(\tau)|$.

According to Lemma 6.7.1 for all such paths there is $1 < q \leq \ell$ such that $\Delta(a_q) = \Delta(a_{q-1})$ and i_q is a shared arc in τ. Hence, $d_{i_q} = 2$.

If $d_{i_{q+1}i_q} = 2$ or $d_{i_q i_{q-1}} = 2$, then $a_q a_{q-1}$ generates the R-bimodule $A_{a_q} A_{a_{q-1}}$. Moreover, it is $a_q a_{q-1} \in \partial W$ by Example 6.4.7.

If $d_{i_{q+1}i_q} \neq 2$ and $d_{i_q i_{q-1}} \neq 2$, then $d_{i_{q+1}} = d_{i_{q-1}} = 1$ and $\Delta(a_q)$ is a triangle of type $\widetilde{C}_{11}$. In this case, the generators $a_q a_{q-1}$ and $a_q u a_{q-1}$ of $A_{a_q} A_{a_{q-1}}$ also lie in ∂W by Example 6.4.7.

We can conclude in either case $A_{a_\ell} \cdots A_{a_1} \subseteq \partial W$ as desired. $\square$

Theorem 6.7.3. *For colored triangulations (τ,ξ), (τ,ξ') of $\mathbf{\Sigma}_d$ the following are equivalent:*

(a) $\xi = \xi'$ *in* $H^1(C^\bullet(\tau))$.

(b) $\mathcal{J}(\tau,\xi) \cong \mathcal{J}(\tau,\xi')$ *as* $K^{X_0(\tau)}$*-algebras.*

(c) $A(\tau,\xi) \cong f_* A(\tau,\xi')$ *as R-bimodules over K for some* $f \in \operatorname{Aut}_{K^{X_0(\tau)}}(R(\tau))$.

Proof. Let $R = R(\tau) = \prod_{i\in\tau} L_i$. We write f_i for the i-th component of $f \in \operatorname{End}_{K^{X_0(\tau)}}(R)$.

For $Y \in \{A, W, \widehat{H}, \mathcal{J}\}$ abbreviate $Y := Y(\tau,\xi)$ and $Y' := Y(\tau,\xi')$. Moreover, denote by ρ_a and ρ'_a the unique elements in $\operatorname{Gal}(L_{ji}/K)$ such that $A_a = {}_jL_i^{\rho_a}$ and $A'_a = {}_jL_i^{\rho'_a}$.

We add two more statements:

(a') There is $\bar{X}_0(\tau) \xrightarrow{g} \mathbb{F}_2$ with $\xi(a) + \xi'(a) = g(j) + g(i)$ for all $j \xleftarrow{a} i \in \bar{X}_1(\tau)$.

(c') There is an automorphism f of the $K^{X_0(\tau)}$-algebra R and an automorphism π of the $X_0(\tau)$-quiver $X(\tau)$ such that $\rho'_{\pi(a)} f_i = f_j \rho_a$ on L_{ji} for all $j \xleftarrow{a} i \in X_1(\tau)$.

(a) $\Leftrightarrow$ (a'): True by definition.

(b) $\Rightarrow$ (c): Thanks to Proposition 6.7.2 an $K^{X_0(\tau)}$-algebra isomorphism $\mathcal{J} \xrightarrow{f} \mathcal{J}'$ induces an R-bimodule isomorphism $A \cong \operatorname{rad}(\mathcal{J})/\operatorname{rad}^2(\mathcal{J}) \to f'_* \operatorname{rad}(\mathcal{J}')/\operatorname{rad}^2(\mathcal{J}') \cong f'_* A'$ where f' is the automorphism $R \cong \mathcal{J}/\operatorname{rad}(\mathcal{J}) \to \mathcal{J}'/\operatorname{rad}(\mathcal{J}') \cong R$ induced by f.

(c) $\Rightarrow$ (c'): The isomorphism $A \to f_* A'$ induces for all $i, j \in X_0(\tau)$ an isomorphism

$$\bigoplus_{j \xleftarrow{a} i} {}_jL_i^{\rho_a} = \bigoplus_{j \xleftarrow{a} i} A_a = {}_jA_i \xrightarrow{\cong} {}_j(f_*A')_i = \bigoplus_{j \xleftarrow{a} i} f_* A'_a = \bigoplus_{j \xleftarrow{a} i} {}_jL_i^{f_j^{-1}\rho'_a f_i}.$$

According to Lemma 2.5.14 there must exist π like in (c').

(c') $\Rightarrow$ (b): The identity $\rho_a = f_j^{-1}\rho'_{\pi(a)} f_i$ implies $f_* A'_{\pi(a)} \cong {}_j L_i^{\rho_a}$ as simple R-bimodules over K in view of ($\star$) in § 2.5.3. Let g_a be the isomorphism $A_a = {}_j L_i^{\rho_a} \to f_* A'_{\pi(a)}$ given by $a \mapsto \pi(a)$. We obtain an induced R-bimodule isomorphism

$$A \;=\; \bigoplus_a A_a \xrightarrow{\;\cong\;} \bigoplus_a f_* A'_{\pi(a)} \;=\; f_* A'\,.$$

and thereby an induced isomorphism $\widehat{H} = R\langle\langle A\rangle\rangle \xrightarrow{g} R\langle\langle f_* A'\rangle\rangle = \widehat{H}'$ of $K^{X_0(\tau)}$-algebras.

It is $g(W) \sim_R W'$ (via $b_1 \mapsto -b_1$ for all $i \leftarrow\overset{b_1}{\leftarrow}\leftarrow i$ in $X(\tau)$ with $r(b_1) = 1$ and $f_i \neq \mathrm{id}_{R_i}$ induced by triangles of type $\widetilde{C}_{11}$). Hence, $\mathcal{J} \cong \mathcal{J}'$ as $K^{X_0(\tau)}$-algebras by Proposition 2.6.44.

(a') $\Rightarrow$ (c'): As before, we regard $\mathbb{F}_2$ as a subset of $\mathbb{Z}/4\mathbb{Z}$ via $0 \mapsto 0$, $1 \mapsto 1$. Let $\alpha = \alpha_{\zeta,v}$ be the fixed isomorphism $\mathbb{Z}/4\mathbb{Z} \to \mathrm{Gal}(L/K)$ We use Convention 2.5.11.

Let $f \in \mathrm{Aut}_{K^{X_0(\tau)}}(R)$ given by $f_i = \alpha_i(g(i))$ for all $i \in \bar{X}_0(\tau)$.

If $j \xleftarrow{b_r} i$ with $r(b_r) = r \in \{0,1\}$ are parallel arrows in $X(\tau)$ connecting weight-4 arcs, then $b := b_0 = b_1$ in $\bar{X}(\tau)$ and $\rho_{b_r} = \alpha_{ji}(\xi(b) + 2r)$ and $\rho'_{b_r} = \alpha_{ji}(\xi'(b) + 2r)$.

Since $\xi(b) + \xi'(b) = g(j) + g(i)$ holds in $\mathbb{F}_2$, there is $p \in \{0,1\}$ such that for all $r \in \{0,1\}$ one has the identity $(\xi'(b) + 2(r+p)) + g(i) = g(j) + (\xi(b) + 2r)$ in $\mathbb{Z}/4\mathbb{Z}$.

Then we have $\rho'_{\pi(b_r)} f_i = f_j \rho_{b_r}$ for $\pi(b_r) = b_{|r-p|}$.

Extending π to an automorphism of the quiver $X(\tau)$ with $\pi(a) = a$ for all arrows $j \xleftarrow{a} i$ in $X(\tau)$ not connecting weight-4 arcs, we get f and π as in (c').

(c') $\Rightarrow$ (a'): Let $\bar{X}_0(\tau) \xrightarrow{g} \mathbb{F}_2$ with $g(i) = \alpha_i^{-1}(f_i)$ regarded as element in $\mathbb{F}_2 = \mathbb{Z}/2\mathbb{Z}$.

For $j \xleftarrow{a} i \in \bar{X}_1(\tau)$ the condition $\rho'_{\pi(a)} f_i = f_j \rho_a$ translates to $\xi(a) + \xi'(\pi(a)) = g(j) + g(i)$ in $\mathbb{F}_2$, since $\rho'_{\pi(a)} = \alpha_{ji}(\xi'(\pi(a)) + 2r(a))$ and $\rho_a = \alpha_{ji}(\xi(a) + 2r(a))$.

For single arrows $j \xleftarrow{a} i$ in $\bar{X}(\tau)$ it is $\pi(a) = a$ in $\bar{X}(\tau)$, so $\xi(a) + \xi'(a) = g(j) + g(i)$.

For $\pi(a) \neq a$ in $\bar{X}(\tau)$ let $i \xleftarrow{c} k \xleftarrow{b} j \xleftarrow{a} i$ be the path in $X(\tau)$ induced by $\Delta(a)$. Then it is $\Delta(a) \in \bar{X}_2(\tau)$ and one computes

$$\begin{aligned} \xi(a) + \xi'(a) &= (\xi(c) + \xi(b)) + (\xi'(c) + \xi'(b)) \\ &= (\xi(c) + \xi'(c)) + (\xi(b) + \xi'(b)) \\ &= (g(i) + g(k)) + (g(k) + g(j)) \;=\; g(j) + g(i)\,. \end{aligned}$$

The first equality holds because ξ and ξ' are cocycles and the last equality because c and b are single arrows in $\bar{X}(\tau)$ (see Lemma 6.6.2). □

6.8 Geometric Realization of $\bar{X}_\bullet(\tau)$

In the spirit of [AG16, Lemma 2.3] the cohomology of $C^\bullet(\tau)$ can be identified with the singular cohomology of Σ with $\mathbb{F}_2$-coefficients. This observation is particularly interesting in combination with the parametrization of Jacobian algebras by the first cohomology group $H^1(C^\bullet(\tau)) \cong \mathbb{F}_2^{2g+b-1}$ described in § 6.7. For instance, one can deduce immediately that the isomorphism class of $\mathcal{J}(\tau,\xi)$ does not depend on ξ if Σ is a disk.

Construction of $|\bar{X}(\tau)|$

Fix a triangulation τ of $\mathbf{\Sigma}_d$ and let $(\gamma_i)_{i\in\tau}$ be a family of curves in $\mathbf{\Sigma}$ whose elements do not intersect each other in $\Sigma \setminus \mathbb{M}$ such that γ_i represents i (see Remark 5.1.10).

For boundary segments $i \in \mathfrak{s}$ define $\gamma_i = i$. Abbreviate $\bar{X}_k = \bar{X}_k(\tau)$ for $k \in \{0,1,2\}$.

Construct a subspace $|\bar{X}| = \bigcup_{k\in\{0,1,2\}} \bigcup_{i\in\bar{X}_k} x_i$ of Σ as follows:

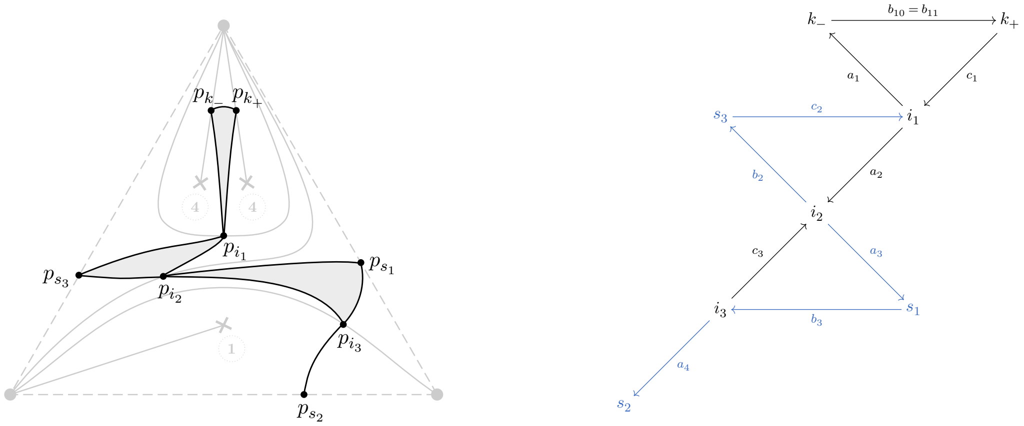
(a) For every $i \in \bar{X}_0$ let $x_i = \{p_i\}$ for some inner point p_i of γ_i.

(b) For every $j \xleftarrow{a} i \in \bar{X}_1$ pick a curve x_a in $(\Sigma, \mathbb{M}\cup\{p_i,p_j\}, \mathbb{O})$ with endpoints $\{p_i,p_j\}$ and inner points in $\Delta(a) \setminus \bigcup_\ell \gamma_\ell$.

Make these choices in such a way that for all $a \neq b$ in $\bar{X}_1$ the curves x_a and x_b do not intersect each other in their inner points.

(c) For every $\Delta \in \bar{X}_2$ the set $c_\Delta = \bigcup_{a\in\bar{X}_1:\Delta(a)=\Delta} x_a$ is the image of a closed simple curve. Let x_Δ be the closure of the component of $\Sigma \setminus c_\Delta$ not intersecting any of the γ_i.

Example 6.8.1. For the triangulation τ of the triangle with one weight-1 and two weight-4 orbifold points from Example 6.2.18 we have visualized below on the left the “geometric realization” $|\bar{X}| \subseteq \Sigma$ of $\bar{X}_\bullet$. On the right one can see the quiver $\bar{X}(\tau)$.



Notation 6.8.2. We denote for topological spaces X by $C_\bullet(X;R)$ the singular complex of X with coefficients in R and by $H_\bullet(X;R)$ its homology. By definition $C_k(X;R)$ is the free R-module generated by all continuous maps $\Delta^k \to X$ where Δ^k is the standard k-simplex.

To relate the homology of $C_\bullet(\tau)$ to the singular homology of $|\bar{X}|$ fix for all $k \in \{0,1,2\}$ and $i \in \bar{X}_k$ a continuous map $\Delta^k \xrightarrow{\theta(i)} |\bar{X}|$ with image x_i that is injective on the interior of Δ^k and has the properties that …

(a) …for each $\Delta \in \bar{X}_2$ and each face F of Δ^2 there is $a \in \bar{X}_1$ with $\theta(\Delta)|_F = \theta(a)$,

(b) ...for each $j \xleftarrow{a} i \in \bar{X}_1$ the image of $\partial\Delta^1$ under $\theta(a)$ is $\{p_i, p_j\}$.

In $C_\bullet(|\bar{X}|;\mathbb{F}_2)$ one has with these choices $\partial_2(\theta(\Delta)) = \theta(c) + \theta(b) + \theta(a)$ for all $\Delta \in \bar{X}_2$, where *cba* is a cyclic path induced by Δ, and $\partial_1(\theta(a)) = \theta(j) + \theta(i)$ for all $j \xleftarrow{a} i \in \bar{X}_1$.

To cut this long story short, the rule $i \mapsto \theta(i)$ with $i \in \bar{X}_k$ induces a chain map

$$C_\bullet(\tau) \xrightarrow{\quad\theta\quad} C_\bullet(|\bar{X}|;\mathbb{F}_2)\,.$$

Proposition 6.8.3. *The map θ induces an isomorphism $H_\bullet(C_\bullet(\tau)) \xrightarrow{\cong} H_\bullet(|\bar{X}|;\mathbb{F}_2)$.*

Proof. This is a standard result in algebraic topology. Compare e.g. [Hat02, Chapter 2.1] where a similar construction is discussed for integral coefficients. □

Postcomposing the map θ with the map $C_\bullet(|\bar{X}|;\mathbb{F}_2) \to C_\bullet(\Sigma;\mathbb{F}_2)$ induced by the canonical inclusion $|\bar{X}| \overset{\iota}{\hookrightarrow} \Sigma$ we get a chain map $C_\bullet(\tau) \to C_\bullet(\Sigma;\mathbb{F}_2)$, which we denote again by θ.

Proposition 6.8.4. *The map θ induces an isomorphism $H_\bullet(C_\bullet(\tau)) \xrightarrow{\cong} H_\bullet(\Sigma;\mathbb{F}_2)$.*

Proof. Having in mind Remark 5.1.3, Example 5.1.6, and Proposition 5.1.23, it is not hard to see that $|\bar{X}|$ is a strong deformation retract of Σ. Consequently, ι induces an isomorphism $H_\bullet(|\bar{X}|;\mathbb{F}_2) \xrightarrow{\cong} H_\bullet(\Sigma;\mathbb{F}_2)$. Now Proposition 6.8.3 implies the claim. □

Corollary 6.8.5. $H_1(C_\bullet(\tau)) \cong \mathbb{F}_2^{2g+b-1}$.

Proof. Use Proposition 6.8.4 and the well-known fact $H_1(\Sigma;\mathbb{F}_2) \cong \mathbb{F}_2^{2g+b-1}$. □

Remark 6.8.6. It is also possible to verify $H_1(C_\bullet(\tau)) \cong \mathbb{F}_2^{2g+b-1}$ directly:

By Remark 6.2.28 the dimension of $H_1(C_\bullet(\tau))$ is

$$r \;:=\; 1 - \chi(C_\bullet(\tau)) \;=\; 1 - |\bar{X}_0(\tau)| + |\bar{X}_1(\tau)| - |\bar{X}_2(\tau)|\,,$$

where $\chi(C_\bullet(\tau)) = \sum_{k\in\mathbb{N}} \dim_{\mathbb{F}_2} H_k(C_\bullet(\tau))$ is the Euler characteristic.

The number n of arcs and t of triangles in τ are by Proposition 5.1.16 and Corollary 5.1.22

$$\begin{aligned} n &= 6(g-1) + 3b + m + 2o\,,\\ t &= 4(g-1) + 2b + m + o\,. \end{aligned}$$

Denote by t_q the number of triangles of τ with exactly q weight-1 orbifold points. Then we can express $t = t_0 + t_1 + t_2$ and $o_1 = t_1 + 2t_2$.

Note that $|\bar{X}_0(\tau)| = n + m - o_1$ and $|\bar{X}_1(\tau)| = 3t_0 + t_1$ and $|\bar{X}_2(\tau)| = t_0$.

All in all, $r = 2t - n - m + 1 = 2g + b - 1$.

6.9 Counting Components of the Flip Graph

Motivated by [FST08, §§ 3 and 7], Proposition 5.1.45, and Remark 5.1.46 one can ask the question whether "the flip graph of colored triangulations is connected." Since colored triangulations with the same underlying triangulation and cohomologous cocycles define isomorphic Jacobian algebras, we take for the vertices of the flip graph "cohomology classes" of colored triangulations instead of colored triangulations themselves.

Definition 6.9.1. The *flip graph* $\mathbf{E}^{\mathrm{H}}(\boldsymbol{\Sigma}_d)$ is the simple graph whose vertices are pairs (τ, x) consisting of a triangulation τ of $\boldsymbol{\Sigma}_d$ and an element $x \in H^1(C^\bullet(\tau))$.

Vertices (τ, x) and (ς, x') are joined by an edge if and only if there are $\xi \in x$ and $\xi' \in x'$ such that the colored triangulations (τ, ξ) and (ς, ξ') are related by flipping an arc.

Theorem 6.9.2. *The flip graph $\mathbf{E}^{\mathrm{H}}(\boldsymbol{\Sigma}_d)$ is disconnected if Σ is not a disk. More precisely, it has at least 2^{2g+b-1} connected components.*

Proof. Let $\mathbb{O}_1 = \{\times \in \mathbb{O} \,|\, d_\times = 1\}$ and let $\widehat{\Sigma}$ be the surface $\Sigma \setminus \mathbb{O}_1$.

Let V be the set of vertices of $\mathbf{E}^{\mathrm{H}}(\boldsymbol{\Sigma}_d)$ and $V_\tau = \{(\tau, x) \,|\, x \in H^1(C^\bullet(\tau))\} \subseteq V$.

We will define $V \xrightarrow{\mathrm{inv}} H^1(\widehat{\Sigma}; \mathbb{F}_2)$ such that $\mathrm{inv}|_{V_\tau}$ is injective for every τ and inv is constant when restricted to the vertex set of any connected component. This will imply the theorem, since $|V_\tau| = |H^1(C^\bullet(\tau))| = |H_1(C_\bullet(\tau))| = 2^{2g+b-1}$ by Corollary 6.8.5.

We begin with the construction of inv. To do this, fix a triangulation τ of $\boldsymbol{\Sigma}_d$.

Call a triangle *exceptional* if its type is $\widetilde{B}_1$, $\widetilde{C}_{14}$, or $\widetilde{C}_{41}$.

For pending arcs i in τ let Δ_i^τ be the triangle of τ containing i. If $i \in \tau^{d=1}$ and Δ_i^τ is exceptional, denote by δ_i^τ the unique arrow in $\bar{X}(\tau)$ induced by Δ_i^τ.

Construction of the Chain Complex $\widehat{C}_\bullet(\tau)$

Let $\widehat{X}_k(\tau) = \bar{X}_k(\tau)$ for $k \neq 1$ and $\widehat{X}_1(\tau) = \bar{X}_1(\tau) \,\dot\cup\, \{\varepsilon_i^\tau \,|\, i \in \tau^{d=1}\}$. Define

$$\partial_1(\varepsilon_i^\tau) \;=\; \begin{cases} \partial_1(\delta_i^\tau) & \text{if } \Delta_i^\tau \text{ is exceptional,} \\ 0 & \text{otherwise.} \end{cases}$$

Let $\widehat{C}_\bullet(\tau) = \big(\mathbb{F}_2 \widehat{X}_k(\tau)\big)_{k \in \mathbb{N}}$ be the chain complex whose differentials for elements in $\bar{X}_k(\tau)$ are given by the same rules as the differentials of the complex $C_\bullet(\tau)$ in Definition 6.2.26. Let $\widehat{C}^\bullet(\tau) = \mathrm{Hom}_{\mathbb{F}_2}(\widehat{C}_\bullet(\tau), \mathbb{F}_2)$ be the dual cochain complex. We have a chain map

$$\begin{tikzcd}
\widehat{C}_\bullet(\tau) \arrow[d, two heads, "\rho_\tau"'] & : & 0 \arrow[r] & \mathbb{F}_2\widehat{X}_2(\tau) \arrow[r, "\partial_2"] \arrow[d, equal] & \mathbb{F}_2\widehat{X}_1(\tau) \arrow[r, "\partial_1"] \arrow[d, two heads, "\rho_\tau"'] & \mathbb{F}_2\widehat{X}_0(\tau) \arrow[r] \arrow[d, equal] & 0 \\
C_\bullet(\tau) & : & 0 \arrow[r] & \mathbb{F}_2\bar{X}_2(\tau) \arrow[r, "\partial_2"] & \mathbb{F}_2\bar{X}_1(\tau) \arrow[r, "\partial_1"] & \mathbb{F}_2\bar{X}_0(\tau) \arrow[r] & 0
\end{tikzcd}$$

where ρ_τ is defined on basis elements in degree one as

$$\rho_\tau(a) \;=\; \begin{cases} a & \text{for } a \in \bar{X}_1(\tau), \\ \delta_i^\tau & \text{for } a = \varepsilon_i^\tau \text{ with } \Delta_i^\tau \text{ exceptional,} \\ 0 & \text{for } a = \varepsilon_i^\tau \text{ with } \Delta_i^\tau \text{ non-exceptional.} \end{cases}$$

The map ρ_τ induces maps $H_1(\widehat{C}_\bullet(\tau)) \xrightarrow{\rho_\tau} H_1(C_\bullet(\tau))$ and $H^1(C^\bullet(\tau)) \xhookrightarrow{\rho_\tau^*} H^1(\widehat{C}^\bullet(\tau))$.

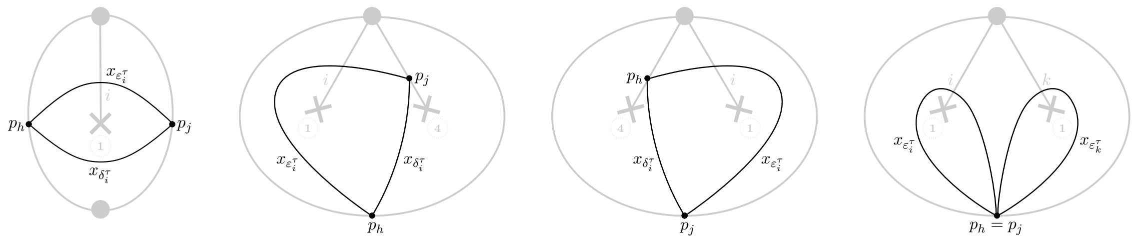
Construction of the Geometric Realization $|\widehat{X}(\tau)|$

Fix a geometric realization $|\bar{X}(\tau)|$ of $\bar{X}(\tau)$ with data $(\gamma_i, x_i, \theta(i))_i$ as described in § 6.8. Construct a subspace $|\widehat{X}(\tau)| = |\bar{X}(\tau)| \cup \bigcup_{i\in\tau^{d=1}} x_{\varepsilon_i^\tau}$ of $\widehat{\Sigma}$ as follows:

(b') For every $i \in \tau^{d=1}$ let $\{h, j\} = \{\ell \in X_1^{d\neq 1}(\tau) \mid \ell \text{ belongs to } \Delta_i^\tau\}$ and pick a curve $x_{\varepsilon_i^\tau}$ in $\big(\Sigma, \mathbb{M} \cup \{p_h, p_j\}, \mathbb{O}\big)$ with endpoints $\{p_h, p_j\}$ and inner points in $\Delta \setminus \bigcup_{\ell\neq i} \gamma_\ell$.

Make these choices in such a way that for all $a \neq b$ in $\widehat{X}_1(\tau)$ the curves x_a and x_b do not intersect each other in their inner points.

Depending on the type of Δ_i^τ the curve $x_{\varepsilon_i^\tau}$ looks as follows in $|\widehat{X}(\tau)| \cap \Delta \subseteq \widehat{\Sigma}$:



Fix for $a = \varepsilon_i^\tau$ a continuous map $\Delta^1 \xrightarrow{\theta(a)} |\widehat{X}(\tau)|$ with image x_a such that it is injective on the interior of Δ^1 and the image of $\partial\Delta^1$ under $\theta(a)$ is $\{p_h, p_j\}$.

Analogously as in § 6.8, we obtain a chain map $\widehat{C}_\bullet(\tau) \xrightarrow{\theta_\tau} C_\bullet(\widehat{\Sigma}; \mathbb{F}_2)$ acting as $i \mapsto \theta(i)$ on $i \in \widehat{X}_k(\tau)$, which induces an isomorphism $H_\bullet(\widehat{C}_\bullet(\tau)) \to H_\bullet(\widehat{\Sigma}; \mathbb{F}_2)$.

Let $H^\bullet(\widehat{\Sigma}; \mathbb{F}_2) \xrightarrow{\theta_\tau^*} H^\bullet(\widehat{C}^\bullet(\tau))$ be the isomorphism induced by θ_τ.

Definition of the Map inv

Let $\varepsilon^\tau = \sum_{i\in\tau^{d=1}} (\varepsilon_i^\tau)^\vee$ where $\{a^\vee \mid a \in \widehat{X}_1(\tau)\}$ is the basis of $\widehat{C}^1(\tau)$ dual to $\widehat{X}_1(\tau)$.

Note that ε^τ is a cocycle and define for $x \in H^1(C^\bullet(\tau))$

$$\operatorname{inv}(\tau, x) \;:=\; (\theta_\tau^*)^{-1}(\rho_\tau^*(x) + \varepsilon^\tau)\,.$$

The restriction $\operatorname{inv}|_{V_\tau}$ is injective because of the injectivity of ρ_τ^*.

The Map inv Is Constant on Flip-Graph Components

To show that inv is constant on the vertex set of every connected component of the flip graph, it is enough to check that $\operatorname{inv}(\tau, \xi) = \operatorname{inv}(\varsigma, \varphi^{\varsigma,\tau}(\xi))$ whenever (τ, ξ) and $(\varsigma, \varphi^{\varsigma,\tau}(\xi))$ are two colored triangulations related by flipping an arc (compare Definition 6.2.36).

Let τ and ς be triangulations of $\mathbf{\Sigma}$ related by a flip, say $\varsigma = \mu_i(\tau)$ and $\tau = \mu_j(\varsigma)$.

By Corollary 6.2.35 the map $\varphi^{\varsigma,\tau}$ induces a map $H^1(C^\bullet(\tau)) \xrightarrow{\varphi^{\varsigma,\tau}} H^1(C^\bullet(\varsigma))$ of sets.

To conclude the proof of the theorem, we have to verify the commutativity of

$$\begin{tikzcd}
& H^1(\widehat{\Sigma};\mathbb{F}_2) & \\
H^1(C^\bullet(\tau)) \arrow[ur, "\mathrm{inv}_\tau"] \arrow[rr, "\varphi^{\varsigma,\tau}"'] & & H^1(C^\bullet(\varsigma)) \arrow[ul, "\mathrm{inv}_\varsigma"']
\end{tikzcd}$$

where $\mathrm{inv}_\tau = \mathrm{inv}(\tau,-)$ and $\mathrm{inv}_\varsigma = \mathrm{inv}(\varsigma,-)$.

This will be achieved by the construction of a chain map $\widehat{C}_\bullet(\varsigma) \xrightarrow{\vartheta_{\tau,\varsigma}} \widehat{C}_\bullet(\tau)$ such that the induced map $\vartheta^*_{\tau,\varsigma}$ in cohomology makes the following diagram commute:

$$\begin{tikzcd}
& H^1(\widehat{\Sigma};\mathbb{F}_2) \arrow[dl, "\theta^*_\tau"'] \arrow[dr, "\theta^*_\varsigma"] & \\
H^1(\widehat{C}^\bullet(\tau)) \arrow[rr, "\vartheta^*_{\tau,\varsigma}"'] & & H^1(\widehat{C}^\bullet(\varsigma)) \\
H^1(C^\bullet(\tau)) \arrow[u, "\rho^*_\tau+\varepsilon^\tau"] \arrow[rr, "\varphi^{\varsigma,\tau}"'] & & H^1(C^\bullet(\varsigma)) \arrow[u, "\rho^*_\varsigma+\varepsilon^\varsigma"']
\end{tikzcd}$$

Using the duality $\mathrm{Hom}_{\mathbb{F}_2}(-,\mathbb{F}_2)$, the commutativity $\vartheta^*_{\tau,\varsigma} \circ \theta^*_\tau = \theta^*_\varsigma$ of the "roof" may be checked in homology instead.

In summary, it suffices to construct a chain map $\widehat{C}_\bullet(\varsigma) \xrightarrow{\vartheta_{\tau,\varsigma}} \widehat{C}_\bullet(\tau)$ making

$$\begin{tikzcd}
& H_1(\widehat{\Sigma};\mathbb{F}_2) & \\
H_1(\widehat{C}_\bullet(\tau)) \arrow[ur, "\theta_\tau"] & & H_1(\widehat{C}_\bullet(\varsigma)) \arrow[ul, "\theta_\varsigma"'] \arrow[ll, "\vartheta_{\tau,\varsigma}"]
\end{tikzcd}$$

commute and the identity $\rho^*_\varsigma(\varphi^{\varsigma,\tau}(x)) + \varepsilon^\varsigma = \vartheta^*_{\tau,\varsigma}(\rho^*_\tau(x) + \varepsilon^\tau)$ hold for all $x \in H^1(C^\bullet(\tau))$.

Construction of the Chain Map $\vartheta_{\tau,\varsigma}$

For the construction of $\vartheta_{\tau,\varsigma}$ we distinguish the following cases:

(1) $i \in \tau^{d=1}$ belongs to an exceptional triangle of τ.

(2) $i \in \tau^{d=1}$ belongs to a non-exceptional triangle of τ.

(3) $i \in \tau^{d=4}$ belongs to an exceptional triangle of τ.

(4) $i \in \tau^{d=4}$ belongs to a non-exceptional triangle of τ.

(5) $i \in \tau^{d=2}$ belongs to two triangles in $\bar{X}_2(\tau)$.

(6) $i \in \tau^{d=2}$ belongs to a triangle in $\bar{X}_2(\tau)$ and to an exceptional triangle of τ.

(7) $i \in \tau^{d=2}$ belongs to two exceptional triangles of τ and is a source or sink in $X(\tau)$.

(8) $i \in \tau^{d=2}$ belongs to two exceptional triangles of τ and is not a source or sink in $X(\tau)$.

(9) $i \in \tau^{d=2}$ belongs to a triangle in $\bar{X}_2(\tau)$ and to a triangle of type $\widetilde{C}_{11}$ in τ

(10) $i \in \tau^{d=2}$ belongs to an exceptional triangle and to a triangle of type $\widetilde{C}_{11}$ of τ

The list is non-redundant and exhaustive, i.e. for fixed τ, i exactly one of the items is true.

For $(\pi, k) \in \{(\tau, i), (\varsigma, j)\}$ let $\bar{X}^k(\pi)$ be the subquiver of $\bar{X}(\pi)$ spanned by all arrows induced by triangles of π with side k.

Then $\bar{X}(\tau) = \bar{X}^i(\tau) \oplus Q$ and $\bar{X}(\varsigma) = \bar{X}^j(\varsigma) \oplus Q$ for some Q (see Lemma 6.2.13).

Tables 6.9.1 and 6.9.2 depict the possible pairs $\big(\bar{X}^i(\tau), \bar{X}^j(\varsigma)\big)$ in all ten cases. The dotted arrows are not part of the quiver $\bar{X}^k(\pi)$ and illustrate the elements of

$$E^k(\pi) \;=\; \left\{\varepsilon_h^\pi \,\middle|\, k \text{ is a side of } \Delta_h^\pi\right\}.$$

Define the map $\widehat{C}_\bullet(\varsigma) \xrightarrow{\vartheta_{\tau,\varsigma}} \widehat{C}_\bullet(\tau)$ on basis elements as

$$\vartheta_{\tau,\varsigma}(x) \;=\; \begin{cases} x & \text{for } x \in (\bar{X}_0(\varsigma) \setminus \{j\}) \cup (\bar{X}_1(\varsigma) \setminus \bar{X}_1^j(\varsigma)) \cup (\bar{X}_2(\varsigma) \cap \bar{X}_2(\tau)), \\ i & \text{for } x = j, \\ \varepsilon_h^\tau & \text{for } x = \varepsilon_h^\varsigma \notin E^j(\varsigma), \\ 0 & \text{for the triangles } x \in \bar{X}_2(\varsigma) \setminus \bar{X}_2(\tau). \end{cases}$$

It remains to specify $\vartheta_{\tau,\varsigma}(x)$ for the elements $x \in \bar{X}_1^j(\varsigma) \cup E^j(\varsigma)$. This is done case by case in Tables 6.9.1 and 6.9.2.

It is not hard to see that $\vartheta_{\tau,\varsigma}$ is a morphism of chain complexes.

The identities $\rho_\varsigma^*(\varphi^{\varsigma,\tau}(x)) + \varepsilon^\varsigma = \vartheta_{\tau,\varsigma}^*(\rho_\tau^*(x) + \varepsilon^\tau)$ for $x \in H^1(C^\bullet(\tau))$ are easily checked via the explicit formula for $\varphi^{\varsigma,\tau}$ given in the proof of Lemma 6.2.32.

Direct inspection shows that in each case $\theta_\tau \circ \vartheta_{\tau,\varsigma} = \theta_\varsigma$ in homology.

This finishes the proof. ☐

Table 6.9.1: $\vartheta_{\tau,\varsigma}$ for pending arcs i in exceptional triangles of τ.

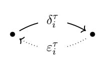
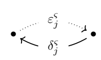
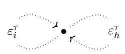
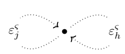
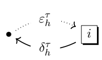
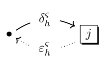
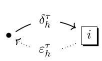
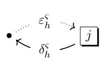
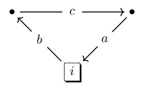
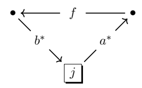
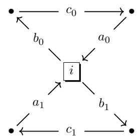
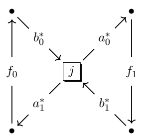
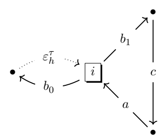
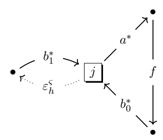
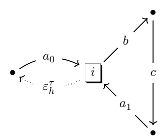
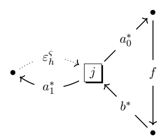
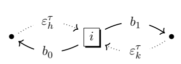
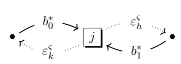
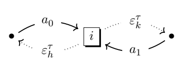
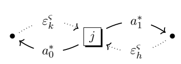
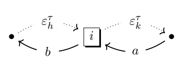
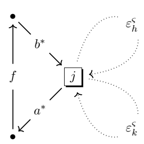
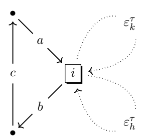
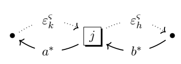
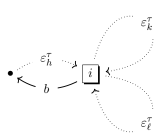
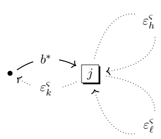
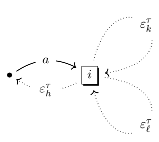
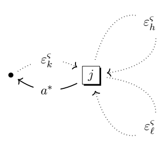
#	$\bar{X}^i(\tau)$	$\bar{X}^j(\varsigma)$	$\vartheta_{\tau,\varsigma}$
1			$\delta_i^\tau \leftarrow\!\shortmid\; \varepsilon_j^\varsigma$ $\varepsilon_i^\tau \leftarrow\!\shortmid\; \delta_j^\varsigma$
2			$\varepsilon_i^\tau \leftarrow\!\shortmid\; \varepsilon_j^\varsigma$ $\varepsilon_h^\tau \leftarrow\!\shortmid\; \varepsilon_h^\varsigma$
3^-			$\delta_h^\tau \leftarrow\!\shortmid\; \delta_h^\varsigma$ $\varepsilon_h^\tau \leftarrow\!\shortmid\; \varepsilon_h^\varsigma$
3^+			

Table 6.9.2: $\vartheta_{\tau,\varsigma}$ for arcs i not pending in an exceptional triangle of τ.

#	$\bar{X}^i(\tau)$	$\bar{X}^j(\varsigma)$	$\vartheta_{\tau,\varsigma}$
4			$a \mapsfrom a^*$ $b \mapsfrom b^*$ $b+a \mapsfrom f$
5			$a_p \mapsfrom a_p^*$ $b_p \mapsfrom b_p^*$ $b_p + a_{1-p} \mapsfrom f_p$
6^-			$a \mapsfrom a^*$ $b_p \mapsfrom b_p^*$ $b_0 + a \mapsfrom f$ $b_1 + \varepsilon_h^\tau + b_0 \mapsfrom \varepsilon_h^\varsigma$
6^+			$a_p \mapsfrom a_p^*$ $b \mapsfrom b^*$ $b + a_0 \mapsfrom f$ $a_0 + \varepsilon_h^\tau + a_1 \mapsfrom \varepsilon_h^\varsigma$
7^-			$b_p \mapsfrom b_p^*$ $b_1 + \varepsilon_h^\tau + b_0 \mapsfrom \varepsilon_h^\varsigma$ $b_0 + \varepsilon_k^\tau + b_1 \mapsfrom \varepsilon_k^\varsigma$
7^+			$a_p \mapsfrom a_p^*$ $a_0 + \varepsilon_h^\tau + a_1 \mapsfrom \varepsilon_h^\varsigma$ $a_1 + \varepsilon_k^\tau + a_0 \mapsfrom \varepsilon_k^\varsigma$
8			$a \mapsfrom a^*$ $b \mapsfrom b^*$ $b + a \mapsfrom f$ $\varepsilon_h^\tau + b \mapsfrom \varepsilon_h^\varsigma$ $a + \varepsilon_k^\tau \mapsfrom \varepsilon_k^\varsigma$
9			$a \mapsfrom a^*$ $b \mapsfrom b^*$ $b + \varepsilon_h^\tau \mapsfrom \varepsilon_h^\varsigma$ $\varepsilon_k^\tau + a \mapsfrom \varepsilon_k^\varsigma$
10^-			$b \mapsfrom b^*$ $\varepsilon_h^\tau + b \mapsfrom \varepsilon_h^\varsigma$ $b + \varepsilon_k^\tau \mapsfrom \varepsilon_k^\varsigma$ $\varepsilon_\ell^\tau \mapsfrom \varepsilon_\ell^\varsigma$
10^+			$a \mapsfrom a^*$ $a + \varepsilon_h^\tau \mapsfrom \varepsilon_h^\varsigma$ $\varepsilon_k^\tau + a \mapsfrom \varepsilon_k^\varsigma$ $\varepsilon_\ell^\tau \mapsfrom \varepsilon_\ell^\varsigma$

Remark 6.9.3. The rules in Table 6.9.2 can be expressed in a more compact and uniform way. Namely, if i is not a pending arc in an exceptional triangle of τ, it is

$$\vartheta_{\tau,\varsigma}(x) = \begin{cases} x^* & \text{for } k \xleftarrow{x} h \text{ in } \bar{X}^j(\varsigma) \text{ with } j \in \{h,k\}, \\ & \quad \text{where } x^* \text{ is the unique arrow } \vartheta_{\tau,\varsigma}(k) \to \vartheta_{\tau,\varsigma}(h) \text{ in } \bar{X}^i(\tau), \\ a^* + b^* & \text{for } k \xleftarrow{x} h \text{ in } \bar{X}^j(\varsigma) \text{ with } j \notin \{h,k\}, \\ & \quad \text{where } \xleftarrow{a} \xleftarrow{x} \xleftarrow{b} \text{ is the path in } \bar{X}^j(\varsigma) \text{ induced by } \Delta(x). \\ \mu_h^\tau + \varepsilon_h^\tau + \nu_h^\tau & \text{for } x = \varepsilon_h^\varsigma \in E^j(\varsigma), \\ & \quad \text{where } \mu_h^\tau \text{ and } \nu_h^\tau \text{ are the elements defined below.} \end{cases}$$

For $\ell^\varsigma \xleftarrow{\varepsilon_h^\varsigma} k^\varsigma \in E^j(\varsigma)$ and $\ell^\tau \xleftarrow{\varepsilon_h^\tau} k^\tau \in E^i(\tau)$ the elements ν_h^τ and μ_h^τ are characterized as:

- ν_h^τ is the unique $k^\tau \longleftarrow \vartheta_{\tau,\varsigma}(k^\varsigma)$ in $\bar{X}_1^i(\tau)$ in case $\vartheta_{\tau,\varsigma}(k^\varsigma) \neq k^\tau$; otherwise, $\nu_h^\tau = 0$.
- μ_h^τ is the unique $\vartheta_{\tau,\varsigma}(\ell^\varsigma) \longleftarrow \ell^\tau$ in $\bar{X}_1^i(\tau)$ in case $\vartheta_{\tau,\varsigma}(\ell^\varsigma) \neq \ell^\tau$; otherwise, $\mu_h^\tau = 0$.

Definition 6.9.4. Let τ be a triangulation of $\mathbf{\Sigma}_d$. Denote by

$$H^1(C^\bullet(\tau)) \xhookrightarrow{\mathrm{inv}_\tau} H^1(\Sigma \setminus \mathbb{O}_1; \mathbb{F}_2)$$

the function constructed in the proof of Theorem 6.9.2.

We call a sequence $(i_1, \dots, i_\ell)$ *τ-admissible* if $i_1 \in \tau$, $i_2 \in \mu_{i_1}(\tau)$, $\dots$, $i_\ell \in \mu_{i_{\ell-1}} \cdots \mu_{i_1}(\tau)$.

The proof of the last theorem has the following interesting consequences:

Corollary 6.9.5. *Let (τ,ξ), (ς,ξ') be colored triangulations of $\mathbf{\Sigma}_d$ with $\mathrm{inv}_\tau(\xi) \neq \mathrm{inv}_\varsigma(\xi')$. Then $\mu_{i_\ell} \cdots \mu_{i_1}(\tau,\xi) \neq (\varsigma,\xi')$ for every τ-admissible sequence $(i_1,\dots,i_\ell)$.* □

Proof. With the notation and arguments used in the proof of Theorem 6.9.2, we have

$$\mathrm{inv}(\mu_{i_\ell} \cdots \mu_{i_1}(\tau,\xi)) = \cdots = \mathrm{inv}(\mu_{i_1}(\tau,\xi)) = \mathrm{inv}(\tau,\xi) = \mathrm{inv}_\tau(\xi)$$

and $\mathrm{inv}(\varsigma,\xi') = \mathrm{inv}_\varsigma(\xi')$. Thus $\mu_{i_\ell} \cdots \mu_{i_1}(\tau,\xi) \neq (\varsigma,\xi')$ because $\mathrm{inv}_\tau(\xi) \neq \mathrm{inv}_\varsigma(\xi')$. □

Corollary 6.9.6. *Let (τ,ξ) and $(\tau,\xi') = \mu_{i_\ell} \cdots \mu_{i_1}(\tau,\xi)$ be colored triangulations of $\mathbf{\Sigma}_d$. Then the Jacobian algebras $\mathcal{J}(\tau,\xi)$ and $\mathcal{J}(\tau,\xi')$ are isomorphic.*

Proof. The proof of Corollary 6.9.5 shows $\mathrm{inv}_\tau(\xi) = \mathrm{inv}_\tau(\xi')$. So $\xi = \xi'$ in $H^1(C^\bullet(\tau))$ because of the injectivity of inv_τ. Now use Theorem 6.7.3. □

6.A Appendix

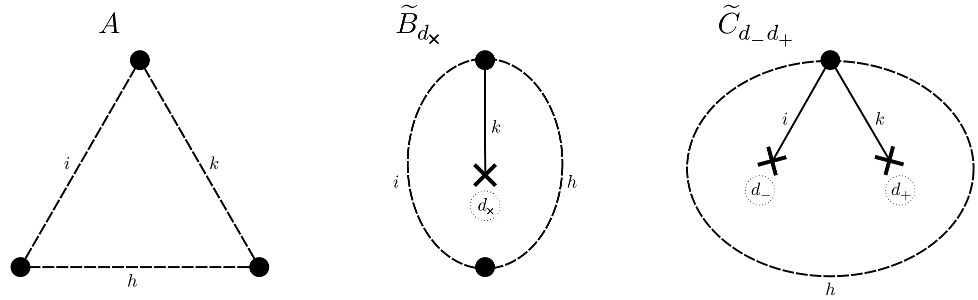


Figure 6.A.1: the seven types of weighted triangles: A, $\widetilde{B}_1$, $\widetilde{B}_4$, $\widetilde{C}_{11}$, $\widetilde{C}_{14}$, $\widetilde{C}_{41}$, $\widetilde{C}_{44}$.

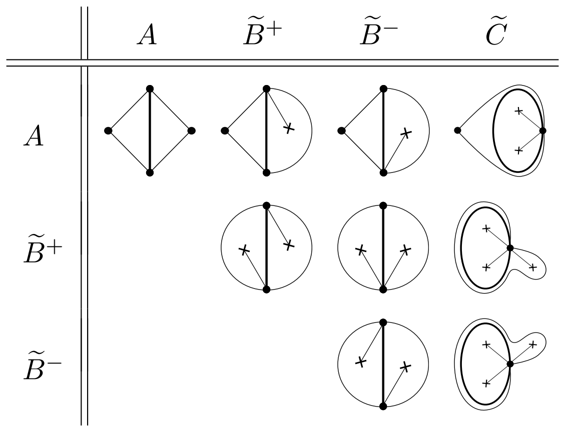


Table 6.A.2: two triangles without punctures sharing exactly one arc (bold).

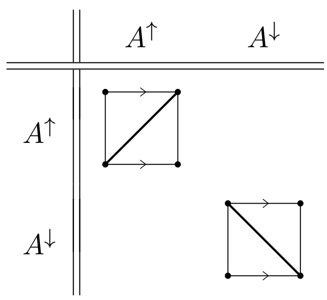


Table 6.A.3: two triangles without punctures sharing exactly two arcs; one shared arc (bold) fixed.

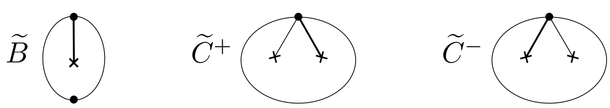
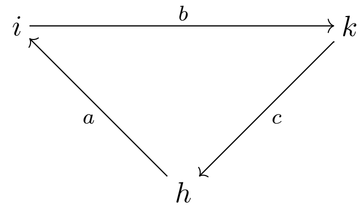
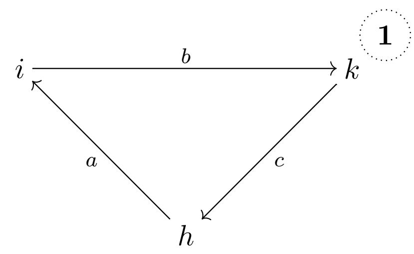


Table 6.A.4: triangles with one inner side (bold) fixed.



cba

Figure 6.A.5: X^Δ and W^Δ for weighted triangles of type A.



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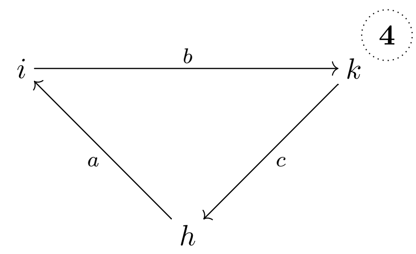
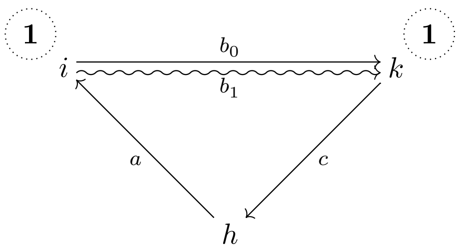
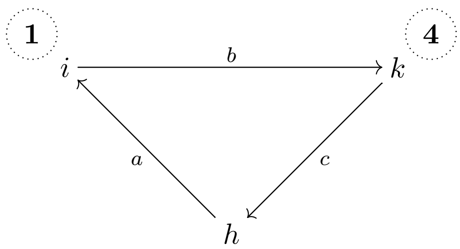


cba

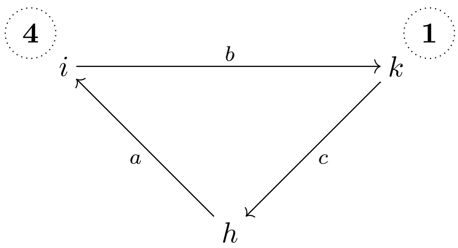
Figure 6.A.6: X^Δ and W^Δ for weighted triangles of type $\widetilde{B}_1$ (left) and $\widetilde{B}_4$ (right).



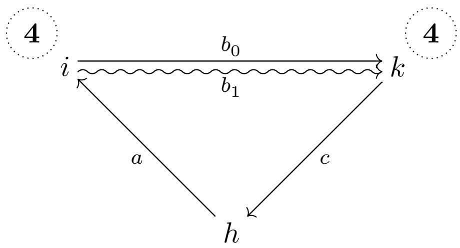
$cb_0a + cb_1au$



cba

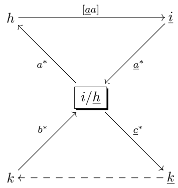


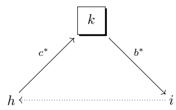
cba

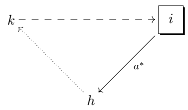


$c(b_0 + b_1)a$

Figure 6.A.7: X^Δ and W^Δ for weighted triangles of type $\widetilde{C}_{11}$, $\widetilde{C}_{14}$, $\widetilde{C}_{41}$, $\widetilde{C}_{44}$ (from top left to bottom right; the wiggling of b_1 indicates $r(b_1) = 1$).

<table>
<tr><th>#</th><th>type of $\square$</th><th>type of j</th><th>$\Phi(X(\varsigma, j))$</th></tr>
<tr><td>1</td><td>$\{A, A\}$</td><td>$\{A, A\}$</td><td>cf. 2nd quiver in Table 5.A.28</td></tr>
<tr><td>2</td><td>$\{A, \widetilde{B}^+\}$</td><td>$\{A, \widetilde{B}^-\}$</td><td>cf. 2nd quiver in Table 5.A.30</td></tr>
<tr><td>3</td><td>$\{A, \widetilde{B}^-\}$</td><td>$\{A, \widetilde{B}^+\}$</td><td>symmetric to #2</td></tr>
<tr><td>4</td><td>$\{A, \widetilde{C}\}$</td><td>$\{\widetilde{B}^+, \widetilde{B}^-\}$</td><td>cf. 2nd quiver in Table 5.A.34</td></tr>
<tr><td>5</td><td>$\{\widetilde{B}^+, \widetilde{B}^+\}$</td><td>$\{\widetilde{B}^-, \widetilde{B}^-\}$</td><td>cf. 2nd quiver in Table 5.A.43</td></tr>
<tr><td>6</td><td>$\{\widetilde{B}^+, \widetilde{B}^-\}$</td><td>$\{A, \widetilde{C}\}$</td><td>

<table>
<tr><th>d_k</th><th>$d_{\underline{k}}$</th><th>$\dashrightarrow$</th></tr>
<tr><td>4</td><td>1</td><td>$[b\underline{c}]$</td></tr>
<tr><td>1</td><td>4</td><td>$[b\underline{c}]$</td></tr>
<tr><td>1</td><td>1</td><td>$[b\underline{c}]^0$ and $[b\underline{c}]^1$</td></tr>
<tr><td>4</td><td>4</td><td>$[b\underline{c}]_0$ and $[b\underline{c}]_1$</td></tr>
</table></td></tr>
<tr><td>7</td><td>$\{\widetilde{B}^+, \widetilde{C}\}$</td><td>$\{\widetilde{B}^-, \widetilde{C}\}$</td><td>looks like #6</td></tr>
<tr><td>8</td><td>$\{\widetilde{B}^-, \widetilde{B}^-\}$</td><td>$\{\widetilde{B}^+, \widetilde{B}^+\}$</td><td>symmetric to #5</td></tr>
<tr><td>9</td><td>$\{\widetilde{B}^-, \widetilde{C}\}$</td><td>$\{\widetilde{B}^+, \widetilde{C}\}$</td><td>symmetric to #7</td></tr>
<tr><td>10</td><td>$\{A^\downarrow, A^\downarrow\}$</td><td>$\{A^\uparrow, A^\uparrow\}$</td><td>cf. 2nd quiver in Table 5.A.13</td></tr>
<tr><td>11</td><td>$\{A^\uparrow, A^\uparrow\}$</td><td>$\{A^\downarrow, A^\downarrow\}$</td><td>symmetric to #10</td></tr>
<tr><td>12</td><td>$\widetilde{B}$</td><td>$\widetilde{B}$</td><td>

<table>
<tr><th>d_k</th><th>$\dashrightarrow$ (dotted)</th></tr>
<tr><td>1</td><td>$[cb]_0$ or $[cb]_1$</td></tr>
<tr><td>4</td><td>$[cb]$</td></tr>
</table></td></tr>
<tr><td>13</td><td>$\widetilde{C}^-$</td><td>$\widetilde{C}^+$</td><td>

<table>
<tr><th>d_k</th><th>d_i</th><th>$\dashrightarrow$ (dotted)</th><th>$\dashrightarrow$</th></tr>
<tr><td>4</td><td>1</td><td>$[ba]_0$ or $[ba]_1$</td><td>b^*</td></tr>
<tr><td>1</td><td>4</td><td>$[ba]$</td><td>b^*</td></tr>
<tr><td>1</td><td>1</td><td>$[b_0a]$</td><td>b_0^* and b_1^*</td></tr>
<tr><td>4</td><td>4</td><td>$[b_0a]$</td><td>b_0^* and b_1^*</td></tr>
</table></td></tr>
<tr><td>14</td><td>$\widetilde{C}^+$</td><td>$\widetilde{C}^-$</td><td>symmetric to #13</td></tr>
</table>

Table 6.A.8: the (at most two) choices for the embedding Φ in Remark 6.2.14; in some cases, there are two possibilities for the dotted arrows.

Bibliography

[AG16] C. Amiot and Y. Grimeland, "Derived invariants for surface algebras," *J. Pure Appl. Algebra*, vol. 220, no. 9, pp. 3133–3155, 2016.
DOI: 10.1016/j.jpaa.2016.02.008. arXiv: 1411.0383 [math.RT].

[AF06] P. S. Aspinwall and L. M. Fidkowski, "Superpotentials for quiver gauge theories," *J. High Energy Phys.*, no. 10, 047, 25 pp. (electronic), 2006.
DOI: 10.1088/1126-6708/2006/10/047. arXiv: hep-th/0506041.

[ABCP10] I. Assem, T. Brüstle, G. Charbonneau-Jodoin, and P.-G. Plamondon, "Gentle algebras arising from surface triangulations," *Algebra Number Theory*, vol. 4, no. 2, pp. 201–229, 2010.
DOI: 10.2140/ant.2010.4.201. arXiv: 0903.3347 [math.RT].

[AS81] M. Auslander and S. O. Smalø, "Almost split sequences in subcategories," *J. Algebra*, vol. 69, no. 2, pp. 426–454, 1981.
DOI: 10.1016/0021-8693(81)90214-3.

[AR91] M. Auslander and I. Reiten, "Applications of contravariantly finite subcategories," *Adv. Math.*, vol. 86, no. 1, pp. 111–152, 1991.
DOI: 10.1016/0001-8708(91)90037-8.

[BGL87] D. Baer, W. Geigle, and H. Lenzing, "The preprojective algebra of a tame hereditary Artin algebra," *Comm. Algebra*, vol. 15, no. 1-2, pp. 425–457, 1987.
DOI: 10.1080/00927878708823425.

[BK12] P. Baumann and J. Kamnitzer, "Preprojective algebras and MV polytopes," *Represent. Theory*, vol. 16, pp. 152–188, 2012.
DOI: 10.1090/S1088-4165-2012-00413-7. arXiv: 1009.2469 [math.RT].

[BL16] R. Bautista and D. López-Aguayo, "Potentials for some tensor algebras," version 6, 2016. arXiv: 1506.05880 [math.RT].

[BP01] C. E. Beasley and M. R. Plesser, "Toric duality is Seiberg duality," *J. High Energy Phys.*, no. 12, Paper 1, 38, 2001.
DOI: 10.1088/1126-6708/2001/12/001. arXiv: hep-th/0109053.

[Ben98] D. J. Benson, *Representations and cohomology. I.* Cambridge University Press, Cambridge, 1998, vol. 30, pp. xii+246.

[BFZ05] A. Berenstein, S. Fomin, and A. Zelevinsky, "Cluster algebras. III. Upper bounds and double Bruhat cells," *Duke Math. J.*, vol. 126, no. 1, pp. 1–52, 2005.
DOI: 10.1215/S0012-7094-04-12611-9. arXiv: math/0305434 [math.RT].

[BP06] A. Bergman and N. Proudfoot, "Moduli spaces for D-branes at the tip of a cone," *J. High Energy Phys.*, no. 3, 073, 9 pp. (electronic), 2006.
DOI: 10.1088/1126-6708/2006/03/073. arXiv: hep-th/0510158.

[BGP73] I. N. Bernšteĭn, I. M. Gel'fand, and V. A. Ponomarev, "Coxeter functors, and Gabriel's theorem," *Uspehi Mat. Nauk*, vol. 28, no. 2(170), pp. 19–33, 1973.

[Bou70] N. Bourbaki, *Eléments de mathématique: Algèbre, chapitres 1 à 3.* Masson, 1970.

[Bou81] N. Bourbaki, *Eléments de mathématique: Algèbre, chapitres 4 à 7.* Masson, 1981.

[Bri05] T. Bridgeland, “t-structures on some local Calabi-Yau varieties,” *J. Algebra*, vol. 289, no. 2, pp. 453–483, 2005.
DOI: 10.1016/j.jalgebra.2005.03.016. arXiv: math/0502050 [math.AG].

[BIRS09] A. B. Buan, O. Iyama, I. Reiten, and J. Scott, “Cluster structures for 2-Calabi-Yau categories and unipotent groups,” *Compos. Math.*, vol. 145, no. 4, pp. 1035–1079, 2009.
DOI: 10.1112/S0010437X09003960. arXiv: math/0701557 [math.RT].

[BIRS11] A. B. Buan, O. Iyama, I. Reiten, and D. Smith, “Mutation of cluster-tilting objects and potentials,” *Amer. J. Math.*, vol. 133, no. 4, pp. 835–887, 2011.
DOI: 10.1353/ajm.2011.0031. arXiv: 0804.3813 [math.RT].

[BMR+06] A. B. Buan, R. Marsh, M. Reineke, I. Reiten, and G. Todorov, “Tilting theory and cluster combinatorics,” *Adv. Math.*, vol. 204, no. 2, pp. 572–618, 2006.
DOI: 10.1016/j.aim.2005.06.003. arXiv: math/0402054 [math.RT].

[CC06] P. Caldero and F. Chapoton, “Cluster algebras as Hall algebras of quiver representations,” *Comment. Math. Helv.*, vol. 81, no. 3, pp. 595–616, 2006.
DOI: 10.4171/CMH/65. arXiv: math/0410187 [math.RT].

[CE56] H. Cartan and S. Eilenberg, *Homological algebra.* Princeton University Press, Princeton, N. J., 1956, pp. xv+390.

[Cec13] S. Cecotti, “Categorical tinkertoys for N=2 gauge theories,” *Internat. J. Modern Phys. A*, vol. 28, no. 5-6, pp. 1330006, 124, 2013.
DOI: 10.1142/S0217751X13300068. arXiv: 1203.6734 [hep-th].

[CV13] S. Cecotti and C. Vafa, “Classification of complete N=2 supersymmetric theories in 4 dimensions,” in *Surveys in differential geometry. Geometry and topology*, vol. 18, Int. Press, Somerville, MA, 2013, pp. 19–101.
DOI: 10.4310/SDG.2013.v18.n1.a2. arXiv: 1103.5832 [hep-th].

[CL12] G. Cerulli Irelli and D. Labardini-Fragoso, “Quivers with potentials associated to triangulated surfaces, part III: Tagged triangulations and cluster monomials,” *Compos. Math.*, vol. 148, no. 6, pp. 1833–1866, 2012.
DOI: 10.1112/S0010437X12000528. arXiv: 1108.1774 [math.RT].

[Cra01] W. Crawley-Boevey, “Geometry of the moment map for representations of quivers,” *Compositio Math.*, vol. 126, no. 3, pp. 257–293, 2001.
DOI: 10.1023/A:1017558904030.

[CH98] W. Crawley-Boevey and M. P. Holland, “Noncommutative deformations of Kleinian singularities,” *Duke Math. J.*, vol. 92, no. 3, pp. 605–635, 1998.
DOI: 10.1215/S0012-7094-98-09218-3.

[Dem10] L. Demonet, “Mutations of group species with potentials and their representations. Applications to cluster algebras,” version 2, 2010.
arXiv: 1003.5078 [math.RT].

[DWZ08] H. Derksen, J. Weyman, and A. Zelevinsky, “Quivers with potentials and their representations. I. Mutations,” *Selecta Math. (N.S.)*, vol. 14, no. 1, pp. 59–119, 2008.
DOI: 10.1007/s00029-008-0057-9. arXiv: 0704.0649 [math.RA].

[DWZ10] H. Derksen, J. Weyman, and A. Zelevinsky, “Quivers with potentials and their representations II: Applications to cluster algebras,” *J. Amer. Math. Soc.*, vol. 23, no. 3, pp. 749–790, 2010.
DOI: 10.1090/S0894-0347-10-00662-4. arXiv: 0904.0676 [math.RA].

[DR76] V. Dlab and C. M. Ringel, “Indecomposable representations of graphs and algebras,” *Mem. Amer. Math. Soc.*, vol. 6, no. 173, pp. v+57, 1976.

[DR75] V. Dlab and C. M. Ringel, “On algebras of finite representation type,” *J. Algebra*, vol. 33, pp. 306–394, 1975.

[DR74] V. Dlab and C. M. Ringel, *Representations of graphs and algebras.* Department of Mathematics, Carleton University, Ottawa, Ont., 1974, pp. iii+86.

[DR80] V. Dlab and C. M. Ringel, "The preprojective algebra of a modulated graph," in *Representation theory, II (Proc. Second Internat. Conf., Carleton Univ., Ottawa, Ont., 1979)*, vol. 832, Springer, Berlin-New York, 1980, pp. 216–231.

[DM96] M. R. Douglas and G. Moore, "D-branes, quivers, and ALE instantons," version 1, 1996. arXiv: `hep-th/9603167`.

[FST12a] A. Felikson, M. Shapiro, and P. Tumarkin, "Cluster algebras and triangulated orbifolds," *Adv. Math.*, vol. 231, no. 5, pp. 2953–3002, 2012. DOI: `10.1016/j.aim.2012.07.032`. arXiv: `1111.3449 [math.CO]`.

[FST12b] A. Felikson, M. Shapiro, and P. Tumarkin, "Cluster algebras of finite mutation type via unfoldings," *Int. Math. Res. Not. IMRN*, no. 8, pp. 1768–1804, 2012. DOI: `10.1093/imrn/rnr072`. arXiv: `1006.4276 [math.CO]`.

[FG07] V. V. Fock and A. B. Goncharov, "Dual Teichmüller and lamination spaces," in *Handbook of Teichmüller theory. Vol. I*, vol. 11, Eur. Math. Soc., Zürich, 2007, pp. 647–684. DOI: `10.4171/029-1/16`. arXiv: `math/0510312 [math.DG]`.

[FST08] S. Fomin, M. Shapiro, and D. Thurston, "Cluster algebras and triangulated surfaces. I. Cluster complexes," *Acta Math.*, vol. 201, no. 1, pp. 83–146, 2008. DOI: `10.1007/s11511-008-0030-7`. arXiv: `math/0608367 [math.RA]`.

[FZ02] S. Fomin and A. Zelevinsky, "Cluster algebras. I. Foundations," *J. Amer. Math. Soc.*, vol. 15, no. 2, 497–529 (electronic), 2002. DOI: `10.1090/S0894-0347-01-00385-X`. arXiv: `math/0104151 [math.RT]`.

[FZ03] S. Fomin and A. Zelevinsky, "Cluster algebras. II. Finite type classification," *Invent. Math.*, vol. 154, no. 1, pp. 63–121, 2003. DOI: `10.1007/s00222-003-0302-y`. arXiv: `math/0208229 [math.RA]`.

[Gab80] P. Gabriel, "Auslander-Reiten sequences and representation-finite algebras," in *Representation theory, I (Proc. Workshop, Carleton Univ., Ottawa, Ont., 1979)*, vol. 831, Springer, Berlin, 1980, pp. 1–71.

[Gab73] P. Gabriel, "Indecomposable representations. II," in *Symposia Mathematica, Vol. XI (Convegno di Algebra Commutativa, INDAM, Rome, 1971)*, Academic Press, London, 1973, pp. 81–104.

[Gab72] P. Gabriel, "Unzerlegbare Darstellungen. I," *Manuscripta Math.*, vol. 6, 71–103, correction, ibid. 6 (1972), 309, 1972.

[GLS11] C. Geiß, B. Leclerc, and J. Schröer, "Kac-Moody groups and cluster algebras," *Adv. Math.*, vol. 228, no. 1, pp. 329–433, 2011. DOI: `10.1016/j.aim.2011.05.011`. arXiv: `1001.3545 [math.RT]`.

[GLS16a] C. Geiß, B. Leclerc, and J. Schröer, "Quivers with relations for symmetrizable Cartan matrices I: Foundations," version 5, 2016, to appear in Invent. Math. arXiv: `1410.1403 [math.RT]`.

[GLS16b] C. Geiß, B. Leclerc, and J. Schröer, "Quivers with relations for symmetrizable Cartan matrices II: Change of symmetrizers," version 3, 2016, to appear in Int. Math. Res. Not. arXiv: `1511.05898 [math.RT]`.

[GLS16c] C. Geiß, B. Leclerc, and J. Schröer, "Quivers with relations for symmetrizable Cartan matrices III: Convolution algebras," *Represent. Theory*, vol. 20, pp. 375–413, 2016. DOI: `10.1090/ert/487`. arXiv: `1511.06216 [math.RT]`.

[GP79] I. M. Gel'fand and V. A. Ponomarev, "Model algebras and representations of graphs," *Funktsional. Anal. i Prilozhen.*, vol. 13, no. 3, pp. 1–12, 1979.

[Geu13] J. Geuenich, “Quivers with potentials on three vertices,” Master’s thesis, Universität Bonn, 2013.

[GL16a] J. Geuenich and D. Labardini-Fragoso, “Species with potential arising from surfaces with orbifold points of order 2, part I: One choice of weights,” version 3, 2016, to appear in Math. Z. DOI: `10.1007/s00209-016-1795-6`. arXiv: `1507.04304 [math.CO]`.

[GL16b] J. Geuenich and D. Labardini-Fragoso, “Species with potential arising from surfaces with orbifold points of order 2, part II: Arbitrary weights,” version 1, 2016. arXiv: `1611.08301 [math.RA]`.

[Hat02] A. Hatcher, *Algebraic topology.* Cambridge University Press, Cambridge, 2002, pp. xii+544.

[Iwa80] Y. Iwanaga, “On rings with finite self-injective dimension. II,” *Tsukuba J. Math.*, vol. 4, no. 1, pp. 107–113, 1980.

[Kna07] A. W. Knapp, *Advanced algebra.* Birkhäuser Boston, Inc., Boston, MA, 2007, pp. xxiv+730.

[Kül16] J. Külshammer, “Prospecies of algebras I: Basic properties,” version 1, 2016. arXiv: `1608.01934 [math.RT]`.

[Lab09a] D. Labardini-Fragoso, “Quivers with potentials associated to triangulated surfaces,” *Proc. Lond. Math. Soc. (3)*, vol. 98, no. 3, pp. 797–839, 2009. DOI: `10.1112/plms/pdn051`. arXiv: `0803.1328 [math.RT]`.

[Lab09b] D. Labardini-Fragoso, “Quivers with potentials associated to triangulated surfaces, part II: Arc representations,” version 2, 2009. arXiv: `0909.4100 [math.RT]`.

[Lab16] D. Labardini-Fragoso, “Quivers with potentials associated to triangulated surfaces, part IV: Removing boundary assumptions,” *Selecta Math. (N.S.)*, vol. 22, no. 1, pp. 145–189, 2016. DOI: `10.1007/s00029-015-0188-8`. arXiv: `1206.1798 [math.CO]`.

[LZ16] D. Labardini-Fragoso and A. Zelevinsky, “Strongly primitive species with potentials I: Mutations,” *Bol. Soc. Mat. Mex. (3)*, vol. 22, no. 1, pp. 47–115, 2016. DOI: `10.1007/s40590-015-0063-9`. arXiv: `1306.3495 [math.RA]`.

[Lam91] T. Y. Lam, *A first course in noncommutative rings.* Springer-Verlag, New York, 1991, vol. 131, pp. xvi+397. DOI: `10.1007/978-1-4684-0406-7`.

[Lam99] T. Y. Lam, *Lectures on modules and rings.* Springer-Verlag, New York, 1999, vol. 189, pp. xxiv+557. DOI: `10.1007/978-1-4612-0525-8`.

[LY15] F. Li and C. Ye, “Representations of Frobenius-type triangular matrix algebras,” version 3, 2015. arXiv: `1511.08040 [math.RA]`.

[Nak39] T. Nakayama, “On Frobeniusean algebras. I,” *Ann. of Math. (2)*, vol. 40, pp. 611–633, 1939.

[Ngu12] B. Nguefack, “Potentials and Jacobian algebras for tensor algebras of bimodules,” version 3, 2012. arXiv: `1004.2213 [math.RT]`.

[Rie80] C. Riedtmann, “Algebren, Darstellungsköcher, Überlagerungen und zurück,” *Comment. Math. Helv.*, vol. 55, no. 2, pp. 199–224, 1980. DOI: `10.1007/BF02566682`.

[RZ13] C. M. Ringel and P. Zhang, “Representations of quivers over the algebra of dual numbers,” version 2, 2013. arXiv: `1112.1924 [math.RT]`.

[Rog75] J. V. Roganov, "The dimension of the tensor algebra of a projective bimodule," *Mat. Zametki*, vol. 18, no. 6, pp. 895–902, 1975.

[RSS80] G.-C. Rota, B. Sagan, and P. R. Stein, "A cyclic derivative in noncommutative algebra," *J. Algebra*, vol. 64, no. 1, pp. 54–75, 1980.
DOI: `10.1016/0021-8693(80)90133-7`.

[Sch85] A. H. Schofield, *Representation of rings over skew fields.* Cambridge University Press, Cambridge, 1985, vol. 92, pp. xii+223.
DOI: `10.1017/CBO9780511661914`.

[Sei95] N. Seiberg, "Electric-magnetic duality in supersymmetric non-abelian gauge theories," *Nuclear Phys. B*, vol. 435, no. 1-2, pp. 129–146, 1995.
DOI: `10.1016/0550-3213(94)00023-8`. arXiv: `hep-th/9411149`.

[War93] S. Warner, *Topological rings.* North-Holland Publishing Co., Amsterdam, 1993, vol. 178, pp. x+498.