Two-row Springer fibers, foams and arc algebras of type D

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Summary

This thesis is concerned with the geometry, topology and combinatorics arising in the study of arc algebras of types B_{m-1} and D_m. These algebras are closely related to infinite-dimensional representation theory of Lie algebras, the geometry of perverse sheaves on isotropic Grassmannians and the representation theory of non-semisimple Brauer algebras. Results of Ehrig and Stroppel show that the center of the arc algebra of type D_m (which is isomorphic to the arc algebra of type B_{m-1}) is isomorphic to the cohomology ring of a certain two-row Springer fiber of type D_m.

In the first part of this thesis we combinatorially construct an explicit topological model for every two-row Springer fiber of type D_m and prove that the respective topological model is homeomorphic to its corresponding Springer fiber. In doing so, we confirm a conjecture by Ehrig and Stroppel concerning the topology of the equal-row Springer fiber for type D_m. Moreover, we show that every two-row Springer fiber of type C_{m-1} is homeomorphic (even isomorphic as an algebraic variety) to a certain two-row Springer fiber of type D_m. This unexpected isomorphism of algebraic varieties can be interpreted as Langlands dual to the known isomorphism between the arc algebras of type B_{m-1} and D_m.

In the second part we explain an elementary, topological construction of the Springer representation on the homology of (topological) Springer fibers of types C_{m-1} and D_m in the two-row case. The Weyl group action and the component group action admit a diagrammatic description in terms of cup diagrams which appear in the definition of the arc algebras of types B_{m-1} and D_m. We determine the decomposition of the representations into irreducibles and relate our construction to classical Springer theory. In addition to that, we give an explicit presentation of the cohomology rings of all two-row Springer fibers in types C_{m-1} and D_m.

In the third part we describe a low-dimensional topology approach to understanding the arc algebras of types B_{m-1} and D_m using two-dimensional surfaces and TQFTs. More precisely, we combinatorially describe the 2-category of singular cobordisms, called (rank one) foams, which governs the functorial version of (type A) Khovanov homology. As an application we use this singular cobordism construction to realize the arc algebras of type B_{m-1} and D_m topologically by establishing an explicit isomorphism to a web algebra arising from foams. This result reduces the proof of the associativity of the arc algebras of type B_{m-1} and D_m (which requires hard combinatorial work and a cumbersome amount of computations) to certain obvious topological equivalences between two-dimensional surfaces. Moreover, it shows how to remove the technical and unnatural condition of having to choose a certain admissible order of surgery moves in order for the multiplication to be well-defined in the original definition of the arc algebra.

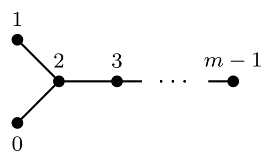
Introduction

Background and Motivation

Let $\mathcal{W}_{D_m}$ be the Weyl group of type D_m, i.e. the Coxeter group generated by the simple reflections $s_0, s_1, \ldots, s_{m-1}$ subject to the relations $(s_i s_j)^{\alpha_{ij}} = e$, where

$$\alpha_{ij} = \begin{cases} 1 & \text{if } i = j, \\ 3 & \text{if } i \text{ — } j \text{ are connected in } \Gamma_{D_m}, \\ 2 & \text{else.} \end{cases}$$

Γ_{D_m} :



Let $l\colon \mathcal{W}_{D_m} \to \mathbb{Z}_{\geq 0}$ be the length function, i.e. $l(w)$ is the number of simple reflections in a reduced expression of $w \in \mathcal{W}_{D_m}$.

Let $\mathcal{H}_{D_m}$ be the generic Hecke algebra for $\mathcal{W}_{D_m}$ over the ring $\mathcal{L} = \mathbb{C}[q, q^{-1}]$ of formal Laurent polynomials. As module over $\mathcal{L}$, $\mathcal{H}_{D_m}$ has a basis H_w indexed by the elements $w \in \mathcal{W}_{D_m}$. These elements are subject to the relations $H_v H_w = H_{vw}$ if $l(vw) = l(v) + l(w)$ and $H_{s_i}^2 = (q^{-1} - q)H_{s_i} + H_e$ for $i \in \{0, \ldots, m-1\}$. As an $\mathcal{L}$-algebra, $\mathcal{H}_{D_m}$ is generated by $C_{s_i} = H_{s_i} - q^{-1}H_e$, where $i \in \{0, \ldots, m-1\}$.

We fix the maximal parabolic subgroup $\mathcal{P}_{D_m} \subseteq \mathcal{W}_{D_m}$ of type A_{m-1} generated by $s_1, \ldots, s_{m-1}$. Associated with $\mathcal{P}_{D_m}$ we have the subalgebra $\mathcal{H}_{\mathcal{P}_{D_m}}$ generated by C_{s_i} for $i \in \{1, \ldots, m-1\}$. Letting C_{s_i}, $i \in \{1, \ldots, m-1\}$, act as 0 turns $\mathcal{L}$ into a right $\mathcal{H}_{\mathcal{P}_{D_m}}$-module. Since $\mathcal{H}_{P_{D_m}}$ acts on $\mathcal{H}_{D_m}$ by restriction, we can define the *parabolic Hecke module* as the tensor product $\mathcal{M}_{D_m} = \mathcal{L} \otimes_{\mathcal{H}_{P_{D_m}}} \mathcal{H}_{D_m}$ which is naturally a right $\mathcal{H}_{D_m}$-module.

Remark 1. The Hecke algebra $\mathcal{H}_{D_m}$ turns into the group algebra $\mathbb{C}[\mathcal{W}_{D_m}]$ by specializing q to 1, i.e. by taking the quotient of $\mathcal{H}_{D_m}$ with respect to the ideal generated by $q-1$. Similarly, the algebra $\mathcal{H}_{P_{D_m}}$ becomes the group algebra $\mathbb{C}[S_m]$ of the symmetric group $\mathcal{P}_{D_m} \cong S_m$. In particular, $\mathcal{M}_{D_m}$ is isomorphic to the induced trivial module $\mathbb{C} \otimes_{\mathbb{C}[S_m]} \mathbb{C}[\mathcal{W}_{D_m}]$ for $\mathbb{C}[\mathcal{W}_{D_m}]$ after specializing q to 1. ▲

The module $\mathcal{M}_{D_m}$ has two interesting $\mathcal{L}$-bases both of which are indexed by the elements of the set $\mathcal{P}_{D_m}^{\min}$ of the representatives of the cosets in $\mathcal{P}_{D_m} \backslash \mathcal{W}_{D_m}$ which have minimal length. Firstly, the *standard basis* is given by the elements $M_w = 1 \otimes H_w$, $w \in \mathcal{P}_{D_m}^{\min}$. Secondly, the *Kazhdan-Lusztig basis* $\underline{M}_w$, $w \in \mathcal{P}_{D_m}^{\min}$, is uniquely determined by the condition that each $\underline{M}_w$ is invariant under a certain involution (cf. Subsection 2.4.1 for details) and the fact that $\underline{M}_w \in M_w + \sum_{v \neq w} q^{-1}\mathbb{Z}[q^{-1}]M_v$. The *parabolic Kazhdan-Lusztig polynomials* $d_{v,w} \in \mathbb{Z}[q^{-1}]$ are defined as the entries of the corresponding base change matrix, i.e. $\underline{M}_w = \sum d_{v,w} M_v$, see [16, 56].

Question 2. Is it possible to explicitly compute the parabolic Kazhdan-Lusztig polynomials using some easy combinatorics? ▲

For general Coxeter groups together with a choice of parabolic subgroup, this seemingly innocent linear algebra problem is incredibly hard to solve and there are no closed formulas for computing these polynomials. However, in our special case, this is possible because the parabolic Kazhdan-Lusztig polynomials turn out to be monomials, see [6] and Proposition 10 below. The idea is to encode the standard basis vectors of $\mathcal{M}_{D_m}$ as combinatorial weights and the Kazhdan-Lusztig basis in terms of cup diagrams. These are the main diagrammatic tools throughout this thesis.

Definition 3. A *combinatorial weight* is a $\{\wedge, \vee\}$-sequence of length m. Let Λ_m^{even} denote the set of combinatorial weights with an even number of $\wedge$'s. ▲

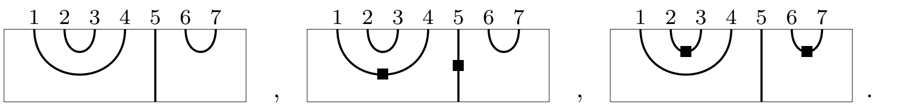
Example 4. The set of Λ_3^{even} consists of the following combinatorial weights: $\vee\vee\vee$, $\vee\wedge\wedge$, $\wedge\vee\wedge$, $\wedge\wedge\vee$. ▲

Definition 5. Consider a rectangle in the plane together with a finite collection of vertices evenly spread along the upper horizontal edge of the rectangle.

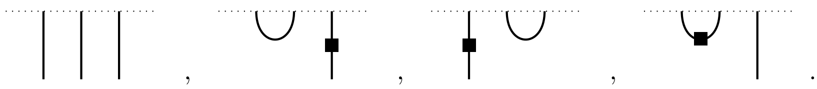
A *cup diagram* is a non-intersecting diagram inside the rectangle obtained by attaching lower semicircles called *cups* and vertical line segments called *rays* to the vertices. We require that every vertex is joined with precisely one endpoint of a cup or ray. Moreover, a ray always connects a vertex with a point on the lower horizontal edge of the rectangle. Additionally, any cup or ray for which there exists a path inside the rectangle connecting this cup or ray to the left vertical edge of the rectangle without intersecting any other part of the diagram may be equipped with one single *marker*, i.e. a small black box. We do not distinguish between diagrams which are related by a planar isotopy fixing the boundary.

Let $C_{\mathrm{KL}}(m)$ denote the set of all cup diagrams on m vertices such that the number of marked rays plus the number of unmarked cups is even. ▲

Example 6. The following first two diagrams are examples of cup diagrams, but the third is not, because the two markers cannot be connected to the right edge of the rectangle with a path not intersecting the diagram:



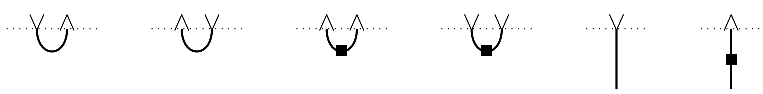
Usually we neither draw the rectangle nor display the numbering of the vertices. Here are all cup diagrams contained in the set $C_{\mathrm{KL}}(3)$:



▲

Since the parabolic Kazhdan-Lusztig polynomials $d_{v,w}$ depend on a choice of a standard basis vector and a Kazhdan-Lusztig basis vector, we have to combine diagrammatic weights and cup diagrams.

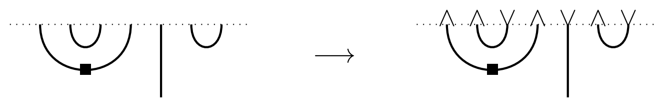
Definition 7. An *oriented circle diagram* is obtained by sticking a combinatorial weight on top of a cup diagram such that the endpoints of the cups and rays align with the symbols of the combinatorial weight and the the resulting oriented cups and rays are of the following form:



A cup of an oriented cup diagram is said to be oriented clockwise (resp. anticlockwise) if the symbol at the right endpoint of the cup is a $\vee$ (resp. a $\wedge$). ▲

Example 8. Here is an example of a weight and a cup diagram put together to form a circle diagram:

$\wedge \wedge \vee \wedge \vee \wedge \vee$,



Note that this diagram is indeed an oriented circle diagram in the sense of Definition 7. The two cups without a marker are oriented clockwise, whereas the marked cup is oriented anticlockwise. ▲

Remark 9. Note that e.g. a marked cup with two different symbols at both endpoints is not an allowed orientation. The marker on a cup can therefore be interpreted as a point where the orientation reverses. We will later discuss a nice topological interpretation of the marker as a singularity of a two-dimensional surface associated with the diagram. ▲

The combinatorics introduced above allow us to give an answer to Question 2 above.

Proposition 10 ([41, Theorem 2.1])**.** There exist explicit bijections (see [41] or Subsection 2.4.1 for their definition)

$$\phi_{\mathrm{wt}} \colon \{M_v \mid v \in \mathcal{P}_{D_m}^{\min}\} \xleftarrow{\cong} \Lambda_m^{\mathrm{even}}$$

$$\phi_{\mathrm{cup}} \colon \{\underline{M}_w \mid w \in \mathcal{P}_{D_m}^{\min}\} \xleftarrow{\cong} C_{\mathrm{KL}}(m)$$

such that for any $v, w \in \mathcal{P}_{D_m}^{\min}$ the Kazhdan-Lusztig polynomial $d_{v,w}$ can be computed diagrammatically as follows:

$$d_{v,w} = \begin{cases} (-q)^{-c(\phi_{\mathrm{wt}}(v),\phi_{\mathrm{KL}}(w))} & \text{if } \phi_{\mathrm{wt}}(v)\phi_{\mathrm{KL}}(w) \text{ is oriented,} \\ 0 & \text{else.} \end{cases}$$

The power $c(v, \phi_{\mathrm{KL}}(w))$ equals the number of clockwise oriented cups in the diagram $\phi_{\mathrm{wt}}(v)\phi_{\mathrm{KL}}(w)$ (the diagram obtained by sticking $\phi_{\mathrm{wt}}(v)$ on top of the cup diagram $\phi_{\mathrm{KL}}(w)$). ▲

Remark 11. We remark that Proposition 10 does not contradict the positivity conjecture for the Kazhdan-Lusztig polynomials. The appearance of the minus signs comes from our preferred choice of conventions. In fact, the setup in which the polynomials are positive can be recovered by replacing q with $-q^{-1}$, see [56, Remark 2.8]. ▲

Question 12. Where else do the parabolic Kazhdan-Lusztig polynomials occur and what do they describe? ▲

▷ **Infinite-dimensional representation theory of Lie algebras.** In the principal block of the Bernstein-Gelfand-Gelfand parabolic category $\mathcal{O}_0^{\mathfrak{p}}(\mathfrak{so}_{2m}(\mathbb{C}))$ of type D_m with parabolic of type A_{m-1}, see [31], the Kazhdan-Lusztig polynomial $d_{v,w}$ (evaluated at -1) counts how often a simple module $L(w)$ appears in a Jordan-Hölder series of the parabolic Verma module $\Delta(v)$. In particular, in view of Proposition 10, all composition factors occur with multiplicity one.

▷ **Geometry of isotropic Grassmannians.** By the Beilinson-Bernstein localization theorem and the Riemann-Hilbert correspondence, see [29], the category $\mathcal{O}_0^{\mathfrak{p}}(\mathfrak{so}_{2m}(\mathbb{C}))$ is equivalent to the category $\mathcal{P}erv\left(\Upsilon_m^D\right)$ of perverse sheaves (constructible with respect to the Schubert stratification) on the isotropic Grassmannian Υ_m^D of m-dimensional isotropic subspaces in $\mathbb{C}^{2m}$ (with a chosen nondegenerate symmetric bilinear form) and we recover the preceding example in a geometric disguise.

▷ **Representation theory of the Brauer algebra.** In the category of finite-dimensional modules over the Brauer algebra $\mathrm{Br}_d(\delta)$ for integral δ, the Kazhdan-Lusztig polynomials count how often a simple module $L(w)$ appears in a Jordan-Hölder series of the standard module $\Delta(v)$, see [14, 19].

▷ **Geometry of Springer fibers.** The intersections of the irreducible components of two-row Springer fibers of type D_m associated with a nilpotent element of Jordan type (m, m) are controlled by Kazhdan-Lusztig polynomials, i.e. the dimension of the (singular) cohomology of an intersection can be expressed as a sum of products of Kazhdan-Lusztig polynomials, see [17].

Question 13. Does there exist a category built out of the Kazhdan-Lusztig combinatorics (the combinatorial weights and cup diagrams) which governs all the examples above? ▲

Indeed, all these categories are governed (in the sense of Remark 14 below) by the category $\mathbb{D}_m$-mod of finite-dimensional modules over the arc algebra $\mathbb{D}_m$ of type D defined in [18]. The algebra $\mathbb{D}_m$ is constructed purely combinatorially in terms of combinatorial weights and cup diagrams. More precisely, the underlying vector space has an explicit basis given by oriented circle diagrams (these are obtained by putting a cup diagram upside down on top of another cup diagram and adding a compatible diagrammatic weight in between) and can be equipped with a diagrammatically defined associative graded algebra structure. Similar algebras in type A were first defined by Khovanov [33],[34] in the context of link and tangle homology and were substantially generalized in subsequent work of Brundan and Stroppel [8],[9] in which case a similar picture as above emerges (involving the walled Brauer algebra instead of the Brauer algebra).

Remark 14. Using Braden's description of the category $\mathcal{P}erv\left(\Upsilon_m^D\right)$, [7], one can establish equivalences of categories

$$\mathbb{D}_m - \mathrm{mod} \cong \mathcal{P}erv\left(\Upsilon_m^D\right) \cong \mathcal{O}_0^{\mathfrak{p}}(\mathfrak{so}_{2m}(\mathbb{C})).$$

Moreover, by [19], the category of finite-dimensional modules over the Brauer algebra is equivalent to a subcategory of $\mathbb{D}_m$-mod. As an application we can study the non-semisimple (and therefore interesting and non-trivial) representation theory of the Brauer algebra $\mathrm{Br}_d(\delta)$, $\delta \in \mathbb{Z}$, from many different perspectives, e.g. from a geometric, combinatorial or Lie theoretic point of view. ▲

The goal of this thesis is to expand the picture explained above. More precisely, we contribute the following:

▷ In [35], Khovanov introduced a combinatorially defined topological model for all Springer fibers of type A corresponding to nilpotent endomorphisms $x \in \mathfrak{gl}_{2m}$ with two equally sized Jordan blocks as a means of showing that the cohomology rings of these Springer fibers are isomorphic to the center of the above-mentioned arc algebras appearing in his categorification of the Jones polynomial [33, 34]. He also conjectured that the topological models are in fact homeomorphic to the corresponding Springer fibers, see [35, Conjecture 1]. This conjecture was proven independently by Wehrli [69] and Russell-Tymoczko [51, Appendix] using results contained in [11]. The constructions and results were generalized to all two-row Springer fibers of type A in [50].

In the first chapter of this thesis we define topological models for all two-row Springer fibers associated with the even orthogonal (type D) and the symplectic group (type C) and prove that they are homeomorphic to their corresponding Springer fiber. The results confirm [17, Conjecture 3].

▷ The second chapter of this thesis connects classical Springer theory [59, 60] with the cup diagram combinatorics of arc algebras. In type A this was achieved by Russell-Tymoczko [51] and Russell [50] in the two-block case. As an application of the results of the first chapter we generalize their construction to types C and D and reconstruct the Springer representation on the homology of the topological Springer fibers with the advantage that the resulting construction admits a combinatorial description accessible for explicit computations (which appears to be difficult using Springer's original theory). As a byproduct, we also compute the cohomology ring of these Springer fibers and provide an explicit description using generators and relations. The surprising fact that two-block Springer theory in types C and D admits a concrete description using the combinatorics of arc algebras of types B and D should be interpreted as some form of Langlands duality, see Remark 20.

▷ In the last chapter we study the web algebra $\mathfrak{W}$ which naturally appears in the context of singular topological quantum field theories (TQFTs) in the sense that its bimodule 2-category $\mathfrak{W}$-**biMod** is equivalent to the 2-category $\mathfrak{F}$ of certain singular surfaces, called *foams*, as originally constructed by Blanchet [4] (see also [21],[20]). Webs are certain planar, trivalent graphs and foams are certain singular surfaces whose boundary are webs. These are pieced together into the web algebra $\mathfrak{W}$. In particular, $\mathfrak{W}$ and $\mathfrak{W}$-**biMod**, are topological by nature with multiplication and action given by gluing of singular surfaces.

The first main result of the third chapter is the construction of a (graded) algebra $c\mathfrak{W}$ which serves as a combinatorial, planar model of $\mathfrak{W}$. The combinatorial model is given by certain decorated webs called *dotted webs* very similar to the circle diagrams appearing in the context of arc algebras. By interpreting the webs as slices through a foam we obtain an isomorphism of graded algebras between $\mathfrak{W}$ and $c\mathfrak{W}$. The second main result shows that the type D arc algebra can be embedded into $\mathfrak{W}$ as a subalgebra. This gives a topological construction of the arc algebra of type D. In particular, the associativity (whose proof requires hard and tedious work in the combinatorial framework) is an immediate consequence of evident topological equivalences.

In the following we discuss the contents of the three chapters of this thesis in more detail and provide a more precise summary of the results.

Part I: Topology of two-row Springer fibers in types C & D

In order to explain the results of the first chapter we begin by fixing an even positive integer $n = 2m$. Let β_D (resp. β_C) be a nondegenerate symmetric (resp. symplectic) bilinear form on $\mathbb{C}^n$ and let $O(\mathbb{C}^n, \beta_D)$ (resp. $Sp(\mathbb{C}^n, \beta_C)$) be the corresponding isometry group with Lie algebra $\mathfrak{so}(\mathbb{C}^n, \beta_D)$ (resp. $\mathfrak{sp}(\mathbb{C}^n, \beta_C)$). The group $O(\mathbb{C}^n, \beta)$ (resp. $Sp(\mathbb{C}^n, \beta_C)$) acts on the affine variety of nilpotent elements $\mathcal{N}_D \subseteq \mathfrak{so}(\mathbb{C}^n, \beta_D)$ (resp. $\mathcal{N}_C \subseteq \mathfrak{sp}(\mathbb{C}^n, \beta_C)$) by conjugation and it is well known that the orbits under this action are in bijective correspondence with partitions of n in which even (resp. odd) parts occur with even multiplicity, see [72],[26]. The parts of the partition associated to the orbit of an endomorphism encode the sizes of the Jordan blocks in Jordan normal form.

Given a nilpotent endomorphism $x \in \mathcal{N}_D$, the associated *(algebraic) Springer fiber $\mathcal{F}l^x_D$ of type D* is defined as the projective variety consisting of all full isotropic (with respect to β_D) flags $\{0\} = F_0 \subsetneq F_1 \subsetneq \ldots \subsetneq F_m$ in $\mathbb{C}^n$ which satisfy the condition $xF_i \subseteq F_{i-1}$ for all $i \in \{1, \ldots, m\}$. Analogously one obtains the *Springer fiber of type C* (simply replace all the D's in the definition by C's). These varieties naturally arise as the fibers of a resolution of singularities of the nilpotent cone, see e.g. [13, Chapter 3]. In general they are not smooth and decompose into many irreducible components.

The goal is to understand the topology of the irreducible components of the Springer fibers and their intersections explicitly and provide a combinatorial description. In general this is a very difficult problem. Even in type A only partial results are known, see e.g. [23–25],[61]. Thus, we restrict ourselves to *two-row Springer fibers*, i.e. we only consider endomorphisms of Jordan type $(n-k,k)$, where $k \in \{1,\dots,m\}$. Note that for type D (resp. type C) we either have $k=m$ or k is odd (resp. even) due to the classification of nilpotent orbits mentioned above. Since the Springer fiber depends (up to isomorphism) only on the conjugacy class of the chosen endomorphism, it makes sense to speak about the $(n-k,k)$ Springer fiber, denoted by $\mathcal{F}l_D^{n-k,k}$ (resp. $\mathcal{F}l_C^{n-k,k}$), without further specifying the nilpotent endomorphism.

Henceforth, we fix a two-row partition $(n-k,k)$ labeling a nilpotent orbit of type D. Note that for every two-row partition labeling a nilpotent orbit of type C, there exists a two-row partition of type D such that subtracting 1 in both parts of the partition yields the given partition of type C.

Topological Springer fibers of type D

Let $\mathbb{B}^{n-k,k}$ denote the set of all cup diagrams on m vertices (cf. Definition 5) with $\lfloor \frac{k}{2} \rfloor$ cups. This set decomposes as a disjoint union $\mathbb{B}^{n-k,k} = \mathbb{B}^{n-k,k}_{\text{even}} \sqcup \mathbb{B}^{n-k,k}_{\text{odd}}$, where $\mathbb{B}^{n-k,k}_{\text{even}}$ (resp. $\mathbb{B}^{n-k,k}_{\text{odd}}$) consists of all cup diagrams with an even (resp. odd) number of markers.

Let $\mathbb{S}^2 \subseteq \mathbb{R}^3$ be the standard unit sphere on which we fix the points $p=(0,0,1)$ and $q=(1,0,0)$. Given a cup diagram $\mathbf{a} \in \mathbb{B}^{n-k,k}$, we define $S_\mathbf{a} \subseteq \left(\mathbb{S}^2\right)^m$ as the submanifold of the m-fold cartesian product of the sphere with itself consisting of all $(x_1,\dots,x_m) \in \left(\mathbb{S}^2\right)^m$ which satisfy the relations $x_i = -x_j$ (resp. $x_i = x_j$) if the vertices i and j are connected by an unmarked cup (resp. marked cup). Moreover, we impose the relations $x_i = p$ if the vertex i is connected to a marked ray, and $x_i = -p$ (resp. $x_i = q$) if i is connected to an unmarked ray which is the leftmost ray in $\mathbf{a}$ (resp. not the leftmost ray). The *topological Springer fiber* $\mathcal{S}_D^{n-k,k}$ *of type* D is defined as the union

$$\mathcal{S}_D^{n-k,k} := \bigcup_{\mathbf{a} \in \mathbb{B}^{n-k,k}} S_\mathbf{a} \subseteq \left(\mathbb{S}^2\right)^m .$$

The above definition generalizes the construction of the topological Springer fiber in [17, §4.1] from the equal-row case to the general two-row case and returns (up to a sign convention) the definition in the equal-row case (see Remark 1.2.6 for the precise relationship).

The first result of this thesis proves a conjecture by Ehrig and Stroppel [17, Conjecture C] on the topology of Springer fibers of type D corresponding to partitions with two equal parts and at the same time extends the result to all two-row Springer fibers of type D.

Theorem A. *There exists a homeomorphism $\mathcal{S}_D^{n-k,k} \cong \mathcal{F}l_D^{n-k,k}$ such that the image of $S_\mathbf{a}$ is an irreducible component of $\mathcal{F}l_D^{n-k,k}$ for all $\mathbf{a} \in \mathbb{B}^{n-k,k}$.* □

The proof is given in Sections 1.3 and 1.4. The main idea is the following: let $N>0$ be a large integer and let $z\colon \mathbb{C}^{2N} \to \mathbb{C}^{2N}$ be a nilpotent linear operator with two equally-sized Jordan blocks. In [11, §2] the authors define a smooth projective variety

$$Y_m = \left\{ (F_1,\dots,F_m) \mid F_i \subseteq \mathbb{C}^{2N} \text{ has dimension } i,\, F_1 \subseteq \ldots \subseteq F_m,\, zF_i \subseteq F_{i-1} \right\}$$

and construct an explicit diffeomorphism $\phi_m \colon Y_m \xrightarrow{\cong} \left(\mathbb{P}^1\right)^m$. The variety Y_m should be seen as a compactification of the preimage of a Slodowy slice of type A under the Springer resolution. The diffeomorphism ϕ_m also plays a crucial role in establishing topological models for the Springer fibers of type A which are naturally embedded in Y_m (cf. [69] and [50]). It turns out that the two-row Springer fibers of type D, resp. of type C, can also be embedded into Y_m, resp. Y_{m-1} (see Section 1.3).

Furthermore, we introduce a diffeomorphism $\gamma_{n-k,k}\colon \left(\mathbb{S}^2\right)^m \to \left(\mathbb{P}^1\right)^m$ (unlike the diffeomorphism ϕ_m this diffeomorphism actually depends on the partition). This diffeomorphism does not play a vital role and it is only introduced for cosmetic reasons.

In order to prove Theorem A one needs to check that the image of $\mathcal{S}_D^{n-k,k} \subseteq \left(\mathbb{S}^2\right)^m$ under the diffeomorphism $\phi_m^{-1} \circ \gamma_{n-k,k}$ is the embedded Springer fiber $\mathcal{F}l_D^{n-k,k} \subseteq Y_m$. In Lemma 1.4.2 we provide the first step by giving an explicit description of the image of $\mathcal{S}_D^{n-k,k}$ under the map $\gamma_{n-k,k}$. The following picture summarizes the results and constructions discussed so far:

$$\begin{tikzcd}
\left(\mathbb{S}^2\right)^m \arrow[r, "\gamma_{n-k,k}", "\cong"'] & \left(\mathbb{P}^1\right)^m \arrow[r, "\phi_m^{-1}", "\cong"'] & Y_m \\
\mathcal{S}_D^{n-k,k} \arrow[u, hook] \arrow[r, "\gamma_{n-k,k}|_{\mathcal{S}_D^{n-k,k}}", "\cong"'] \arrow[d, equal] & \gamma_{n-k,k}\left(\mathcal{S}_D^{n-k,k}\right) \arrow[u, hook] \arrow[r, dashed, "\cong"] \arrow[d, equal] & \mathcal{F}l_D^{n-k,k} \arrow[u, hook] \\
\bigcup\limits_{\mathbf{a}\in\mathbb{B}^{n-k,k}} S_{\mathbf{a}} & \bigcup\limits_{\mathbf{a}\in\mathbb{B}^{n-k,k}} \gamma_{n-k,k}\left(S_{\mathbf{a}}\right) &
\end{tikzcd}$$

described explicitly in Lemma 1.4.2

embedding of Lemma 1.3.4

Does ϕ_m^{-1} restrict to a homeomorphism with image $\mathcal{F}l_D^{n-k,k}$?

In Section 1.4 we answer the remaining question in the picture above by showing that ϕ_m^{-1} does indeed restrict to a homeomorphism $\gamma_{n-k,k}\left(\mathcal{S}_D^{n-k,k}\right) \cong \mathcal{F}l_D^{n-k,k}$. Note that it suffices to prove the following statement (cf. Proposition 1.4.3):

Proposition 15. The preimages of the sets $\gamma_{n-k,k}(S_{\mathbf{a}})$ under ϕ_m are pairwise different irreducible components of $\mathcal{F}l_D^{n-k,k} \subseteq Y_m$ for all $\mathbf{a} \in \mathbb{B}^{n-k,k}$. ▲

Since the irreducible components of $\mathcal{F}l_D^{n-k,k}$ are in bijective correspondence with cup diagrams in $\mathbb{B}^{n-k,k}$ we deduce that the inclusion

$$\phi_m^{-1}\left(\gamma_{n-k,k}\left(\mathcal{S}_D^{n-k,k}\right)\right) = \bigcup_{\mathbf{a}\in\mathbb{B}^{n-k,k}} \phi_m^{-1}\left(\gamma_{n-k,k}\left(S_{\mathbf{a}}\right)\right) \subseteq \mathcal{F}l_D^{n-k,k}$$

is in fact an equality which finishes the proof of Theorem A.

In order to prove Proposition 15 we proceed by induction on the number of unmarked cups in $\mathbf{a} \in \mathbb{B}^{n-k,k}$ which is is more or less the same proof as in type A (cf. [50]). One only needs to be careful about the additional isotropy condition (cf. Lemma 1.3.5). Thus, the main difficulty lies in establishing the induction start, i.e. to prove the claim for cup diagrams without any unmarked cups. This is done in Proposition 1.4.5 (which itself is a proof by induction on the number of marked cups) and is considered the technical heart of the argument because it requires new techniques which are not straightforward generalizations of the type A case. In particular, Wehrli's proof of the analog of Theorem A for the type A Springer fiber in [69] does not generalize to type D because it relies on the explicit description of the components of the two-row Springer fibers of type A given earlier by Fung [25].

On the relation between two-row Springer fibers of types C & D

The Springer fiber $\mathcal{F}l_D^{n-k,k}$ as defined above decomposes into two connected components (cf. Remark 1.1.3). Under the inverse of the homeomorphism in Theorem A the two connected components of $\mathcal{F}l_D^{n-k,k}$ are mapped onto $\mathcal{S}_{D,\mathrm{odd}}^{n-k,k} := \bigcup_{\mathbf{a}\in\mathbb{B}_{\mathrm{odd}}^{n-k,k}} S_{\mathbf{a}}$ and $\mathcal{S}_{D,\mathrm{even}}^{n-k,k} := \bigcup_{\mathbf{a}\in\mathbb{B}_{\mathrm{even}}^{n-k,k}} S_{\mathbf{a}}$, respectively. Let $\mathcal{F}l_{D,\mathrm{odd}}^{n-k,k}$ denote the image of $\mathcal{S}_{D,\mathrm{odd}}^{n-k,k}$ under the homeomorphism $\mathcal{S}_D^{n-k,k} \cong \mathcal{F}l_D^{n-k,k}$, i.e. it is one of the connected components of $\mathcal{F}l_D^{n-k,k}$. The following unexpected result relates the two-row Springer fibers of type C and D:

Theorem B. *There exists an isomorphism of algebraic varieties $\mathcal{F}l_{D,\mathrm{odd}}^{n-k,k} \cong \mathcal{F}l_C^{n-k-1,k-1}$, i.e. the $(n-k-1,k-1)$ Springer fiber of type C is isomorphic (as an algebraic variety) to one of the connected components of the $(n-k,k)$ Springer fiber of type D, which can be written down explicitly. In particular, the topological model of the type D Springer fiber also provides a topological model for the type C Springer fiber. More precisely, we have a homeomorphism $\mathcal{S}_{D,\mathrm{odd}}^{n-k,k} \cong \mathcal{F}l_C^{n-k-1,k-1}$.* □

In [27] two-row Slodowy slices of type C and D were studied via fixed-point subvarieties of certain Nakajima quiver varieties arising from diagram automorphisms. The authors show that the Slodowy slice of type D to the orbit with Jordan type $(n-k,k)$ is isomorphic to the Slodowy slice of type C to the orbit with Jordan type $(n-k-1,k-1)$. They also ask whether there is an isomorphism between the resolutions of these singular affine varieties or an isomorphism between the corresponding Springer fibers (cf. [27, §1.3]). Theorem B provides an affirmative answer to the latter question.

In order to prove Theorem B we consider the surjective morphism of varieties $\pi_m\colon Y_m \twoheadrightarrow Y_{m-1}$ given by $(F_1,\dots,F_m) \mapsto (F_1,\dots,F_{m-1})$ and show (using similar arguments as in the proof of Theorem A) that the restriction of π_m to $\mathcal{F}l_{D,\mathrm{odd}}^{n-k,k} \subseteq Y_m$ yields a homeomorphism (even an isomorphism of varieties) whose image is the embedded Springer fiber $\mathcal{F}l_C^{n-k-1,k-1} \subseteq Y_{m-1}$.

Remark 16. Since nilpotent orbits of the odd orthogonal group (type B) are parameterized by partitions in which even parts occur with even multiplicity, it follows that there cannot exist a two-row partition labeling a nilpotent orbit of type B. Hence, there are no two-row Springer fibers of type B. The only interesting Springer fibers of type B would correspond to nilpotent endomorphisms with at least three Jordan blocks (every one-row Springer fiber consists of a single flag only). Those are much more difficult to treat and so far there has been no significant progress (not even in type A) in determining their topology in an explicit and combinatorially satisfying way (at least the author is not aware of any results comparable to those obtained in e.g. [25] or [50]). ▲

Part II: A diagrammatic approach to Springer theory

Let G be a complex, connected, reductive, algebraic group with Lie algebra $\mathfrak{g}$. Given a nilpotent element $x \in \mathfrak{g}$, the associated *(algebraic) Springer fiber* $\mathcal{B}_G^x$ can be defined as the variety of all Borel subgroups of G whose Lie algebra contains x. This Lie theoretic definition of the Springer fiber coincides with the previous one in terms of flags, i.e. we have isomorphisms of algebraic varieties $\mathcal{B}_{\mathrm{Sp}_n}^x \cong \mathcal{F}l_C^x$ and $\mathcal{B}_{SO_n}^x \cong \mathcal{F}l_{D,\mathrm{even}}^x \cong \mathcal{F}l_{D,\mathrm{odd}}^x$.

Let A_G^x be the component group associated with x, i.e. the quotient of the centralizer of x in G by its identity component. By classical Springer theory the centralizer $Z_G(x)$ acts on $\mathcal{B}_G^x$ from the left and thus on $H^*(\mathcal{B}_G^x,\mathbb{C})$. Since the identity component of $Z_G(x)$ acts trivially on $H^*(\mathcal{B}_G^x,\mathbb{C})$ we obtain an induced left action of the component group A_G^x on $H^*(\mathcal{B}_G^x,\mathbb{C})$.

In [59],[60] Springer additionally constructed a grading-preserving action of the Weyl group $\mathcal{W}_G$ associated with G on the cohomology $H^*(\mathcal{B}_G^x,\mathbb{C})$ of the Springer fiber which commutes with the A_G^x-action. Unlike the component group action, the $\mathcal{W}_G$-action on $H^*(\mathcal{B}_G^x,\mathbb{C})$ is not induced from a geometric action on the Springer fiber. Instead, the construction requires sophisticated machinery from algebraic geometry such as perverse sheaves, Fourier transform, correspondences or monodromy, see e.g. [73, §1.5] for a survey.

According to the Springer correspondence [59, Theorem 6.10] the isotypic subspaces of the A_G^x-module $H^{\mathrm{top}}(\mathcal{B}_G^x, \mathbb{C})$ are irreducible $\mathcal{W}_G$-modules and all irreducible $\mathcal{W}_G$-modules can be constructed in this way. This yields a geometric construction and classification of the irreducible representations of $\mathcal{W}_G$.

Springer representations on the homology of topological Springer fibers

In the second chapter of this thesis we provide a different, elementary approach and reconstruct the Springer representation on the homology of the topological Springer fibers. The following observation is a key tool in the second chapter.

Proposition 17. The map $\gamma_{n-k,k}\colon H_*(\mathcal{S}_{D,\mathrm{even}}^{n-k,k}, \mathbb{C}) \to H_*((\mathbb{S}^2)^m, \mathbb{C})$ induced by the natural inclusion $\mathcal{S}_{D,\mathrm{even}}^{n-k,k} \subseteq (\mathbb{S}^2)^m$ in homology is injective. ▲

Our construction can be summarized as follows: we explicitly define commuting actions of the Weyl group and the component group on $(\mathbb{S}^2)^m$ which induce commuting actions on $H_*((\mathbb{S}^2)^m, \mathbb{C})$. Since the map $\gamma_{n-k,k}$ is injective by Proposition 17 above, we can identify $H_*(\mathcal{S}_{D,\mathrm{even}}^{n-k,k}, \mathbb{C})$ with its image under $\gamma_{n-k,k}$ in $H_*((\mathbb{S}^2)^m, \mathbb{C})$. This turns out to be a stable subspace of $H_*((\mathbb{S}^2)^m, \mathbb{C})$ with respect to both group actions.

This yields a combinatorial model for Springer theory because the vector space $H_*(\mathcal{S}_{D,\mathrm{even}}^{n-k,k}, \mathbb{C})$ can be described as follows:

Proposition 18. The cup diagrams on m vertices with an even number of markers and precisely l cups form a basis of $H_{2l}(\mathcal{S}_{D,\mathrm{even}}^{n-k,k}, \mathbb{C})$. ▲

In fact, we construct the diagrammatic basis of $H_*(\mathcal{S}_{D,\mathrm{even}}^{n-k,k}, \mathbb{C})$ in such a way that the Weyl group action and the component group action can be described combinatorially.

The component group action on homology

The component group A_G^x depends (up to isomorphism) only on the Jordan type λ of x which allows us to use the notation A_G^λ. In the case of classical groups it was proven by Spaltenstein, see [58, I.2.9], that this group is isomorphic to a finite product of copies of $\mathbb{Z}/2\mathbb{Z}$. In the two-block case we have isomorphisms

$$A_{\mathrm{Sp}_{n-2}}^{n-k-1,k-1} \cong \begin{cases} \{e\} & \text{if } m \text{ is even,} \\ \mathbb{Z}/2\mathbb{Z} & \text{if } m = k \text{ is odd,} \\ (\mathbb{Z}/2\mathbb{Z})^2 & \text{if } m \neq k \text{ is odd.} \end{cases} \qquad \text{and} \qquad A_{\mathrm{SO}_n}^{n-k,k} \cong \begin{cases} \{e\} & \text{if } m = k, \\ \mathbb{Z}/2\mathbb{Z} & \text{if } m \neq k, \end{cases}$$

see also Section 2.3 for more details.

In type D the action of the component group on the cohomology of the Springer fiber is trivial (even though the group itself is not trivial if $m \neq k$). Thus, we only consider the type C case and construct an action of $A_{\mathrm{Sp}_{n-2}}^x$ on $(\mathbb{S}^2)^m$ by defining the action of generator $\alpha \in A_{\mathrm{Sp}_{n-2}}^x$ on an element $(x_1, \ldots, x_m) \in (\mathbb{S}^2)^m$ by $\alpha.(x_1, x_2, \ldots, x_m) = (-x_1, x_2, \ldots, x_m)$.

Theorem C. *Setting* $\mathbf{a}.\alpha := \gamma_{n-k,k}^{-1}(\gamma_{n-k,k}(\mathbf{a}).\alpha)$, *where* $\alpha \in A_{\mathrm{Sp}_{n-2}}^{n-k-1,k-1}$ *and* $\mathbf{a}$ *is a cup diagram (viewed as a basis element in homology via Proposition 18), yields a well-defined grading-preserving left action of the component group* $A_{\mathrm{Sp}_{n-2}}^{n-k-1,k-1}$ *on* $H_*(\mathcal{S}_{D,\mathrm{even}}^{n-k,k}, \mathbb{C})$ *which can be described explicitly.*

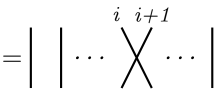
A generator $\alpha \in A^{n-k-1,k-1}_{\mathrm{Sp}_{n-2}}$ acts as the identity on $\mathbf{a}$ *if* $\mathbf{a}$ *has a ray connected to the first vertex. Otherwise it creates (resp. kills) a marker on the cup connected to the first vertex if it is unmarked (resp. marked) and at the same time creates (resp. kills) a marker on the leftmost ray if it is unmarked (resp. marked) (note that if $A^{n-k-1,k-1}_{\mathrm{Sp}_{n-2}}$ is not trivial, i.e. m is odd, there always exists a leftmost ray).* $\square$

The Weyl group action on homology

We proceed as for the component group and first construct a geometric action of the Weyl group on $(\mathbb{S}^2)^m$. There is a right action of the Weyl group $\mathcal{W}_{\mathrm{SO}_n} \cong \mathcal{W}_{D_m}$ on $(\mathbb{S}^2)^m$, where s_i, $i \neq 0$, permutes the coordinates i and $i+1$ and s_0 permutes the first two coordinates and additionally takes their antipodes.

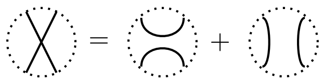
Theorem D. *Setting $\mathbf{a}.s := \gamma^{-1}_{n-k,k}(\gamma_{n-k,k}(\mathbf{a}).s)$, where $s \in \mathcal{W}_{D_m}$ and* $\mathbf{a}$ *is a cup diagram, yields a well-defined, grading-preserving right action of the Weyl group $\mathcal{W}_{D_m}$ on $H_*(\mathcal{S}^{n-k,k}_{D,\mathrm{even}}, \mathbb{C})$ which can be described explicitly using a skein calculus. Given a cup diagram* $\mathbf{a}$ *and an element $s \in \mathcal{W}_{D_m}$ we can write $s = s_{i_1} s_{i_2} \cdots s_{i_l}$ as a product of generators. Diagrammatically, we represent the generators as follows:*

$s_0 :=$ 

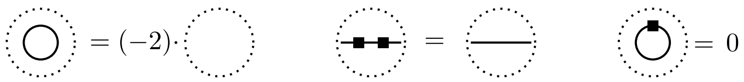
$s_i :=$ 

$i \in \{1, \ldots, m-1\}$.

In order to obtain the linear combination of cup diagrams $\mathbf{a}.s$, we take the cup diagram and stack the pictures corresponding to the generators $s_{i_1}, \ldots, s_{i_l}$ on top of $\mathbf{a}$. *By resolving crossings according to the rule*



and by using the following additional local relations



together with the rule that we kill all diagrams containing a connected component with both endpoints at the bottom of the diagram, we obtain a linear combination of cup diagrams which equals $\mathbf{a}.s$.

The Weyl group $\mathcal{W}_{\mathrm{Sp}_{n-2}} \cong \mathcal{W}_{C_{m-1}}$ can be identified with the subgroup of $\mathcal{W}_{D_m}$ generated by $s_0 s_1$ and s_i, $i \in \{2, \ldots, m-1\}$, see [63]. In particular, the action of the Weyl group of type C obtained by restricting the action of $\mathcal{W}_{D_m}$ (which we identify as the Springer action) admits an explicit description via Theorem D.

Connection to Springer theory

In order to compare our results to classical Springer theory we determine the decomposition of our representations into irreducibles in each homological degree.

Case 1: Two Jordan blocks of equal size:

In [41, Theorem 5.17] the authors identified the parabolic Hecke module $\mathcal{M}_{D_m}$ with the free $\mathcal{L}$-module whose basis is given by all cup diagrams in $C_{\mathrm{KL}}(m)$ (cf. the beginning of the introduction). Via this identification a cup diagram corresponds to a Kazhdan-Lusztig basis vector and the right action of $\mathcal{H}_{D_m}$ on $\mathcal{M}_{D_m}$ admits a diagrammatic description in the spirit of Theorem D. In Subsection 2.4.1 we generalize the combinatorial construction of $\mathcal{M}_{D_m}$ from [41] by providing a similar diagrammatic description of $\mathcal{M}_{C_{m-1}}$. In fact, the Kazhdan-Lusztig basis in type C can be identified with the same cup diagram basis as in type D. The resulting diagrammatic description of the parabolic Hecke modules shows (after specializing q to 1) that

$$\mathrm{Res}^{\mathcal{W}_{D_m}}_{\mathcal{W}_{C_{m-1}}} \mathrm{Ind}^{\mathcal{W}_{D_m}}_{S_m} \mathbb{C} \cong \mathrm{Ind}^{\mathcal{W}_{C_{m-1}}}_{S_{m-1}} \mathbb{C}, \tag{1}$$

which we use to prove the following theorem.

Theorem E. *The right $\mathcal{W}_{D_m}$-module $H^*(\mathcal{S}^{m,m}_{D,\mathrm{even}}, \mathbb{C})$ constructed in Theorem D is isomorphic to the induced trivial module $\mathbb{C} \otimes_{\mathbb{C}[S_m]} \mathbb{C}[\mathcal{W}_{D_m}]$. Similarly, we have an isomorphism $H_*(\mathcal{S}^{m,m}_{D,\mathrm{even}}) \cong \mathbb{C} \otimes_{\mathbb{C}[S_{m-1}]} \mathbb{C}[\mathcal{W}_{C_{m-1}}]$ of $\mathcal{W}_{C_{m-1}}$-modules, where the right $\mathcal{W}_{C_{m-1}}$-action on $H_*(\mathcal{S}^{m,m}_{D,\mathrm{even}})$ is obtained by restricting the $\mathcal{W}_{D_m}$-action.* □

Remark 19. In contrast to the related work of Russell-Tymoczko [51] in type A, we identify the entire homology as an induced representation. This approach has the following advantages:

1. Since the Springer representation $H^*(\mathcal{B}^{m,m}_{D,\mathrm{even}}, \mathbb{C})$ is isomorphic to the induced trivial module, see [43, Theorem 1.3], Theorem E directly implies that we have indeed reconstructed the Springer representation in Theorem D (similarly for type C).

2. Since the decomposition of the induced trivial module into irreducibles is known and multiplicity free, see [17, Lemma 5.19], we can match each of the representations $H_{2i}(\mathcal{S}^{n-k,k}_{D,\mathrm{even}}, \mathbb{C})$, $i \in \{0, \dots, \lfloor \frac{m}{2} \rfloor\}$, with exactly one irreducible of the induced trivial module in type D, see Theorem 2.4.10 for details. However, we are not able to write down an explicit isomorphism and match the cup diagram basis with the tableaux basis of the Specht module. In type A such a result is known, see [48]. In type C the situation is even more complicated (as one might already expect from the nontrivial component group action in case $m = k$ is odd). The representations in each degree, i.e. the Kazhdan-Lusztig cell modules, are not irreducible anymore. But since the $\mathcal{W}_{C_{m-1}}$-action is the restriction of a $\mathcal{W}_{D_m}$-action (for which we have determined the irreducibles) we obtain the desired decomposition by applying results from [63], see Theorem 2.4.11.

3. From the diagrammatic description of $\mathcal{M}_{D_m}$ it follows that acting on a Kazhdan-Lusztig basis element viewed as a cup diagram with a Kazhdan-Lusztig generator may increase the number of cups. In particular, the cup diagram basis of $H_*(\mathcal{S}^{n-k,k}_{D,\mathrm{even}}, \mathbb{C})$ is not the Kazhdan-Lusztig basis (except in top degree) which disproves the type D analog of [51, Conjecture 4.4] (in fact, our methods show that this conjecture does not hold in type A either).

▲

Case 2: Two Jordan blocks of different size:

For the case of two not necessarily equally-sized Jordan blocks, we can identify $H_*(\mathcal{S}^{n-k,k}_{D,\mathrm{even}}, \mathbb{C}) \cong H_{<k}(\mathcal{S}^{m,m}_{D,\mathrm{even}}, \mathbb{C})$ as graded $\mathcal{W}_{D_m}$-module (and thus also as $\mathcal{W}_{C_{m-1}}$-module). In fact, this is

obvious from the diagrammatic description in Theorem D and reduces the general two-block case to *Case 1* above. This phenomenon was also observed in type A, see [50].

The Springer correspondence was explicitly described for the classical groups in work of Shoji, [54],[53]. Given an irreducible $\mathcal{W}_G$-module V, Shoji writes down an algorithm which determines the Jordan type of a nilpotent element $x \in \mathfrak{g}$ as well as the irreducible character ϕ of A_G^x for which $V \cong H_\phi^{\mathrm{top}}(\mathcal{B}_G^x)$ in the Springer correspondence, see [53, Theorem 3.3]. Here, $H_\phi^{\mathrm{top}}(\mathcal{B}_G^x)$ denotes the ϕ-isotypic subspace in $H^{\mathrm{top}}(\mathcal{B}_G^x)$. In Subsection 2.4.3 we recall this algorithm and check that our construction yields the same results for either group actions.

Remark 20. In [18, §9.7] the authors construct an equivalence $\mathcal{P}erv\left(\Upsilon_D^m\right) \simeq \mathcal{P}erv\left(\Upsilon_B^{m-1}\right)$ between the categories of perverse sheaves (both constructible with respect to the Schubert stratification) on an isotropic Grassmannian Υ_m^D (resp. Υ_{m-1}^B) of m-dimensional (resp. $m-1$-dimensional) isotropic subspaces in $\mathbb{C}^{2m}$ (resp. $\mathbb{C}^{2m-1}$). They use this result to deduce that the Kazhdan-Lusztig theory of parabolic type (B_{m-1}, A_{m-2}) is controlled by the Kazhdan-Lusztig theory of parabolic type (D_m, A_{m-1}) which explains the fact that they can be described using the same combinatorics. In particular, if we replace $\mathcal{W}_{C_{m-1}}$ by the isomorphic Weyl group $\mathcal{W}_{B_{m-1}}$ of type B in the formulation of isomorphism (1), then the isomorphism

$$\mathrm{Res}_{\mathcal{W}_{C_{m-1}}}^{\mathcal{W}_{D_m}}\left(H_*(\mathcal{B}_{\mathrm{SO}_n}^{n-k,k},\mathbb{C})\right) \cong H_*(\mathcal{B}_{\mathrm{Sp}_{n-2}}^{n-k-1,k-1},\mathbb{C})$$

can be interpreted as Langlands dual to isomorphism (1). This gives a conceptual perspective on the surprising fact that the cup diagram combinatorics describing Kazhdan-Lusztig theory in types B and D also describe the topology and representation theory associated with the two-block Springer fibers of type C. ▲

Cohomology ring of two-block Springer fibers in types C & D

By [28, Theorem 1.1], the image of the canonical map $H^*(\mathcal{B}_G,\mathbb{C}) \to H^*(\mathcal{B}_G^x,\mathbb{C})$ induced by the inclusion of the Springer fiber into the flag variety $\mathcal{B}_G$ associated with G is given by the A_G^x-invariants in $H_*(\mathcal{B}_G^x,\mathbb{C})$. In particular, the canonical map is surjective if and only if the A_G^x-action is trivial. In type A the component groups are themselves trivial, [13, Lemma 3.6.3], and the canonical map can be used to compute explicit presentations of $H^*(\mathcal{B}_{\mathrm{GL}_n}^x,\mathbb{C})$ as a quotient of $H^*(\mathcal{B}_{\mathrm{GL}_n},\mathbb{C})$, see [15, 62].

Outside of type A the cohomology ring of the Springer fibers have so far been poorly studied. The ring $H^*(\mathcal{B}_{\mathrm{SO}_n}^x,\mathbb{C})$, where $x \in \mathfrak{so}_n(\mathbb{C})$ has Jordan type (m,m), was computed only recently, [17, Theorem B], using methods from equivariant cohomology [40] which in addition to the surjectivity of the canonical map also require the nilpotent operator to be of standard Levi type. Since this assumption does not hold in the general two-block setting we use a different, topological approach to generalize [17, Theorem B] from the case of two Jordan blocks of the same size to the general two-block case and realize the cohomology ring of $\mathcal{S}_{D,\mathrm{even}}^{n-k,k}$ as a quotient of $H^*((\mathbb{S}^2)^m,\mathbb{C})$.

Theorem F. *If $m \neq k$, then we have an isomorphism of graded algebras*

$$H^*\left(\mathcal{S}_{D,\mathrm{even}}^{n-k,k},\mathbb{C}\right) \cong \mathbb{C}[X_1,\ldots,X_m] \Big/ \left\langle X_i^2, X_I \;\middle|\; \begin{array}{c} 1 \leq i \leq m, \\ I \subseteq \{1,\ldots,m\}, |I| = \frac{k-1}{2}+1 \end{array} \right\rangle,$$

where $X_I = \prod_{i\in I} X_i$. In particular, in combination with [17, Theorem B] as well as Theorems A and B this yields an explicit description of the cohomology rings of all two-block Springer fibers of types C and D. □

Part III: Singular TQFTs, foams and type D arc algebras

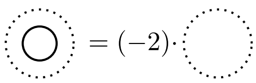
From the Temperley-Lieb category to arc algebras of type A

Fix a rectangle R in the plane with a finite number of vertices evenly spread along the upper and lower horizontal edge of the rectangle. An $\mathfrak{sl}_2$-*web* is an embedded compact 1-manifold inside the rectangle such that its boundary points coincide with the set of vertices in the two horizontal edges. We identify two $\mathfrak{sl}_2$-webs if they differ only by a boundary-preserving isotopy. Note that this definition also allows closed components without boundary (circles). By convention, the empty web $\varnothing$ is also a web.

Definition 21. Let **TL** be the $\mathbb{C}$-linear monoidal category defined as follows:

▷ The set of objects is given by $\mathbb{Z}_{\geq 0}$, the set of non-negative integers.

▷ The morphism space $\mathrm{Hom}_{\mathbf{TL}}(n, m)$ is the $\mathbb{C}$-vector space freely generated by all $\mathfrak{sl}_2$-webs with n bottom boundary and m top boundary points, where we additionally impose the *circle removal relation*:



(2)

▷ The composition $uv = v \circ u$ of webs is the evident gluing of v on top of u and the monoidal product $u \otimes v$ is given by putting u to the left of v.

▲

Remark 22. A classical problem in representation theory is to describe the homomorphism spaces between tensor products of irreducible representations for a given semisimple Lie algebra. Let $\mathrm{Rep}_{\mathfrak{sl}_2}$ be category of finite-dimensional representations of $\mathfrak{sl}_2$ and let $\mathrm{Fund}_{\mathfrak{sl}_2}$ be the full monoidal subcategory whose objects are tensor products of the vector representation $V_{\mathfrak{sl}_2}$ of $\mathfrak{sl}_2$. We have an equivalence of monoidal categories $\mathbf{TL} \xrightarrow{\sim} \mathrm{Fund}_{\mathfrak{sl}_2}$ sending n to the n-fold tensor product of the vector representation with itself. The basic cup and cap morphisms in $\mathrm{Fund}_{\mathfrak{sl}_2}$ are sent to the inclusion $\mathbb{C} \hookrightarrow V_{\mathfrak{sl}_2} \otimes V_{\mathfrak{sl}_2}$ and projection $V_{\mathfrak{sl}_2} \otimes V_{\mathfrak{sl}_2} \twoheadrightarrow \mathbb{C}$, respectively. These maps are unique up to scalar and the scalars can be chosen in such a way that the circle removal relation (2) holds. By taking the Karoubi envelope of $\mathrm{Fund}_{\mathfrak{sl}_2} \simeq \mathbf{TL}$ we recover $\mathrm{Rep}_{\mathfrak{sl}_2}$ (in the $\mathfrak{sl}_2$-case the idempotents that need to be added have an explicit description in terms of the Jones-Wenzl projectors). ▲

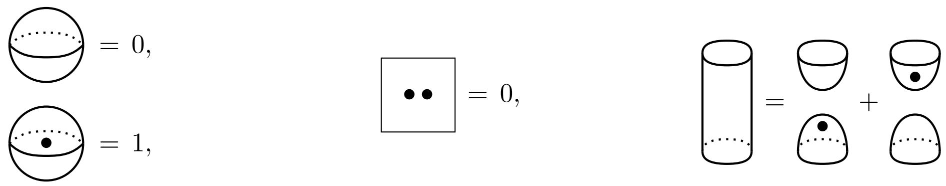
The following 2-category $\mathfrak{F}^{\mathfrak{sl}_2}$ can be seen as a categorification of the Temperley-Lieb category **TL**, [2].

Definition 23. Let $\mathfrak{F}^{\mathfrak{sl}_2}$ denote the additive closure of the $\mathbb{C}$-linear 2-category given by:

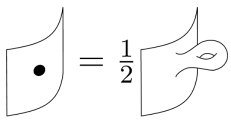
▷ The objects are given by the elements in $\mathbb{Z}_{\geq 0}$.

▷ The 1-morphisms $\mathrm{Hom}_{\mathfrak{F}^{\mathfrak{sl}_2}}(n, m)$ are given by all $\mathfrak{sl}_2$-webs with n bottom and m top boundary points.

▷ Let $u, v \in \mathrm{Hom}_{\mathfrak{F}^{\mathfrak{sl}_2}}(n, m)$ be $\mathfrak{sl}_2$-webs and let P denote the collection of the $n+m$ boundary points in R. The space of 2-morphisms $2\mathrm{Hom}_{\mathfrak{F}^{\mathfrak{sl}_2}}(u, v)$ between u and v is the free $\mathbb{C}$-vector space on basis given by all surfaces embedded in $R \times [0, 1]$ with boundary $u \cup (P \times [0, 1]) \cup v$ viewed up to boundary-preserving isotopy and modulo the local *Bar-Natan relations*



A dot on a surface is a notational abbreviation for half a handle of genus 1, i.e.



In particular, dots can move freely along connected components.

▷ The vertical and the horizontal compositions $\circ$ and $\otimes$ are given by stacking surfaces and placing them side by side, respectively.

▲

Remark 24. The above category can additionally be equipped with a grading coming from the Euler characteristic of the surfaces (we refer the reader to [2, Section 6] for more details). In this case the above 2-category decategorifies (by taking the Grothendieck group of each Hom-category) to the monoidal category $\mathbf{TL}_q$ whose morphism spaces are defined over $\mathbb{C}[q, q^{-1}]$ instead of $\mathbb{C}$ and the factor -2 in relation (2) is replaced by $-(q + q^{-1})$. In the graded setup the 2-category $\mathfrak{F}^{\mathfrak{sl}_2}$ can be used to categorify the Jones polynomial of a link, see [2]. ▲

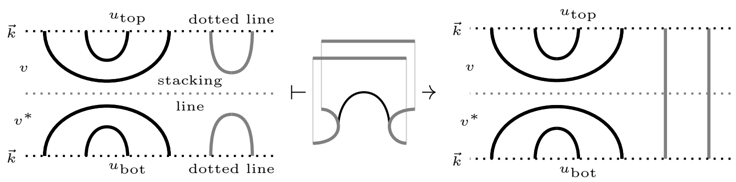
Arc algebras originally appeared in work of Khovanov [34] on the extension of his link homology theory to tangles. In order to recall their definition let m be an even positive integer and let $\mathbb{B}^{m,m}_{\mathrm{undec}}$ be the set of all cup diagrams on m vertices with $m/2$ cups and no markers. If $\mathbf{a}$ is a cup diagram, we write $\mathbf{a}^*$ to denote its corresponding *cap diagram*, i.e. the diagram obtained by reflecting $\mathbf{a}$ in the horizontal line containing the vertices. Given cup webs $\mathbf{a}, \mathbf{b} \in \mathbb{B}^{m,m}_{\mathrm{undec}}$ we can now define the $\mathfrak{sl}_2$-web algebra $\mathfrak{W}^{\mathfrak{sl}_2}_m$ as the algebra whose underlying vector space is given by

$$\mathfrak{W}^{\mathfrak{sl}_2}_m = \bigoplus_{\mathbf{a},\mathbf{b} \in \mathbb{B}^{m,m}_{\mathrm{undec}}} 2\mathrm{Hom}_{\mathfrak{F}^{\mathfrak{sl}_2}}(\emptyset, \mathbf{ab}^*),$$

where we view the collection of circles $\mathbf{ab}^*$ as an object of $\mathrm{End}_{\mathfrak{F}^{\mathfrak{sl}_2}}(\emptyset)$, i.e. an endomorphism of the empty word. The multiplication

$$\mathbf{Mult}^{\mathfrak{W}^{\mathfrak{sl}_2}_m} : \mathfrak{W}^{\mathfrak{sl}_2}_m \otimes \mathfrak{W}^{\mathfrak{sl}_2}_m \to \mathfrak{W}^{\mathfrak{sl}_2}_m, \qquad f \otimes g \mapsto \mathbf{Mult}^{\mathfrak{W}^{\mathfrak{sl}_2}}_m(f, g) \tag{3}$$

is defined using so-called *surgery rules*. That is, the multiplication of two basis vectors $f \in {}_{u_{\mathrm{bot}}}(\mathfrak{W}^{\mathfrak{sl}_2}_m)_v$ and $g \in {}_{v'}(\mathfrak{W}^{\mathfrak{sl}_2}_m)_{u_{\mathrm{top}}}$ is zero if $v \neq v'$. If $v = v'$ we stick f on top of g and inductively apply a sequence of surgery moves to the cup-cap pairs in the middle of the diagram. More precisely, we stick a cobordism on top of the surface glued underneath the stacked circle diagram which is locally a saddle and the identity elsewhere. An example in case $v = v'$ is:



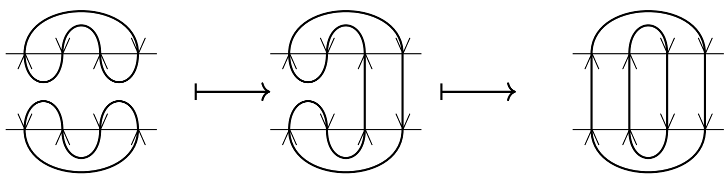
(4)

The reader is also referred to e.g. [21, Definition 2.24] or [45, Definition 3.3] for a detailed account. Note that for topological reasons the multiplication is independent of the order in which the surgeries are performed, which turns $\mathfrak{W}_m^{\mathfrak{sl}_2}$ into an associative algebra (which can additionally be equipped with a grading).

Remark 25. Khovanov's algebra was constructed in a slightly more algebraic setting in [34]. His definition can be recovered by applying the $2d$ TQFT associated with the Frobenius algebra $\mathbb{C}[X]/(X^2)$ which already played a crucial role in the construction of his link homology theory [33]. This $2d$ TQFT is the symmetric monoidal functor $\mathcal{F}$ from the category $\mathrm{End}_{\mathfrak{F}^{\mathfrak{sl}_2}}(\emptyset)$ to the category of finite-dimensional (graded) $\mathbb{C}$-vector spaces (cf. [1] or [39] for details on Frobenius algebras and their relation to TQFTs) which sends a collection of k circles to the k-fold tensor product $\mathbb{C}[X]/(X^2)^{\otimes k}$. The pair of pants cobordism which merges two circles into one circle is mapped to the multiplication m of $\mathbb{C}[X]/(X^2)$ and the reverse pair of pants cobordism is mapped to the comultiplication δ of the Frobenius algebra. This functor is well-defined, i.e. it is well-known that it satisfies the relations between the generating surfaces and it is a straightforward computation to verify that it also respects the Bar-Natan relations, [2, Proposition 7.2]. The first part is true for any given Frobenius algebra. However, the fact that $\mathcal{F}$ also respects the Bar-Natan relations crucially depends on our specific choice of $\mathbb{C}[X]/(X^2)$.

In order to do explicit computations it is useful to fix a basis of each 2Hom-space. In the special case of $2\mathrm{Hom}_{\mathfrak{F}^{\mathfrak{sl}_2}}(\emptyset, \overline{w})$, where $\overline{w} \in \mathrm{End}_{\mathfrak{F}^{\mathfrak{sl}_2}}(\emptyset)$ is a collection of circles, we can choose the basis given by cup foams, i.e. foams obtained by gluing a disk along its boundary to each circle of $\overline{w}$, where we allow each of the disks to be decorated with a single dot. The functor $\mathcal{F}$ identifies $2\mathrm{Hom}(\emptyset, \overline{w})$ with $\mathrm{Hom}(\mathbb{C}, \mathbb{C}[X]/(X^2)^{\otimes k}) \cong \mathbb{C}[x]/(X^2)^{\otimes k}$ by sending such a cup foam to the elementary tensor whose i-th tensor factor is X (resp. 1) if the i-th circle in $\overline{w}$ has a dot (does not have a dot) on the disk glued to it. ▲

In work of Brundan and Stroppel [8], the algebra $\mathfrak{W}_m^{\mathfrak{sl}_2}$ was reconstructed using planar combinatorics only. The underlying vector space of their algebra is the free $\mathbb{C}$-vector space on a basis given by *oriented circle diagrams*, i.e. diagrams obtained by placing a cup diagram upside down on top of another cup diagram and adding a combinatorial weight (a $\{\wedge, \vee\}$-sequence) in between such that each circle is consistently oriented. In order to multiply two such basis elements one proceeds as in the case of the algebra $\mathfrak{W}_m^{\mathfrak{sl}_2}$, i.e. one sticks two circle diagrams on top of each other and applies a sequence of surgery moves to the diagram. Here is an example:



In this case a clockwise (resp. anticlockwise) oriented circle corresponds to the choice of X (resp. 1) in $\mathbb{C}[X]/(X^2)$, e.g. the example above combinatorially encodes the fact that the map

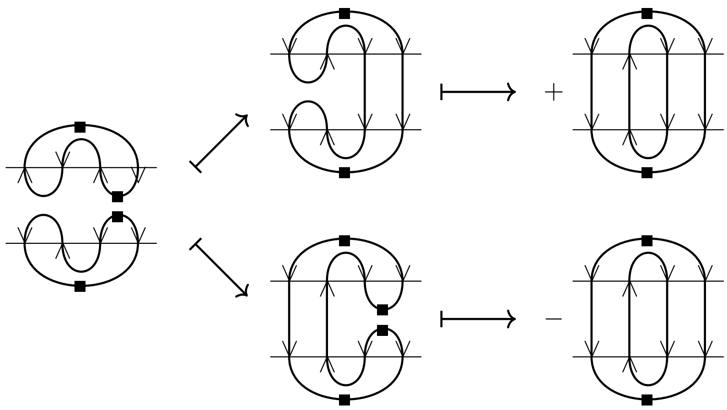
$$\mathbb{C}[X]/(X^2)^{\otimes 2} \xrightarrow{m} \mathbb{C}[X]/(X^2) \xrightarrow{\delta} \mathbb{C}[X]/(X^2)^{\otimes 2}$$

sends $X \otimes 1$ to $X \otimes X$.

Remark 26. The upshot of this combinatorial approach is that it can be generalized substantially leading to the definition of so-called generalized Khovanov algebras (where we also allow rays and not only cups) and the construction of their quasi-hereditary covers, see [8]. In [61] these algebras were defined geometrically as convolution algebras using two-row Springer fibers of type A. ▲

(Sign adjusted) arc algebras of type D

As already mentioned at the beginning of the introduction, in [18] the authors constructed a type D arc algebra $\mathfrak{A}$ and an equivalence of categories between the category $\mathfrak{A}-\mathrm{mod}$ of its finite-dimensional modules and the principal block of the BGG parabolic category $\mathcal{O}^{\mathfrak{p}}(\mathfrak{so}_{2m}(\mathbb{C}))$, where $\mathfrak{p}$ is a parabolic of type A_{m-1}, see [18, Theorem 9.1]. The construction of this algebra is reviewed in detail in Subsection 3.4.1. The main differences to the type A case are that one has to use cup diagrams with markers and there are additional signs appearing in the multiplication which are extremely delicate to determine. Moreover, these signs destroy the locality of the multiplication, see also Remark 3.4.3. Here is an example of the multiplication taken from [18, Example 6.7]:



The example is supposed to show that in order to get a well-defined sign one has to choose an admissible order of the surgery moves. In the case above, the upper sequence of surgeries gives the correct sign. In particular, this multiplication does not seem to be of topological nature at all (exchanging the order of the surgeries should topologically correspond to a height move of the corresponding saddles which is a topologically trivial operation and therefore should not change any signs).

In Subsection 3.4.2 we define an algebra $\overline{\mathfrak{A}}$ which is isomorphic (as a graded algebra) to the arc algebra $\mathfrak{A}$ as originally defined by Ehrig and Stroppel (cf. Proposition 3.4.10). The definitions of the algebras $\mathfrak{A}$ and $\overline{\mathfrak{A}}$ are very similar. In fact, they only differ by the signs appearing in the multiplication rules. The choice of signs for the multiplication in $\overline{\mathfrak{A}}$ surprisingly remedies the above-mentioned problem of having to choose an admissible order of surgery moves when computing the product of two basis elements. We essentially get rid of the unwanted signs by relocating them in the isomorphism

$$\mathtt{sign}\colon \mathfrak{A} \xrightarrow{\cong} \overline{\mathfrak{A}}.$$

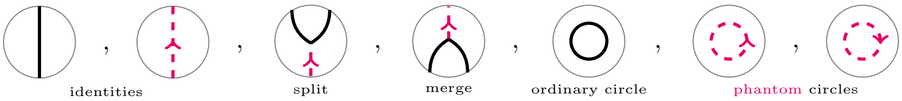
The foam category $\mathfrak{F}$ and its web algebra

We now generalize the notion of an $\mathfrak{sl}_2$-web introduced above. Henceforth, a *web* is a labeled, piecewise linear, one-dimensional CW complex (a graph with vertices and edges) embedded in the plane with boundary supported in two horizontal lines, such that all horizontal slices consists only of a finite number of points. (Hence, it makes sense to talk about the bottom and top boundary of such webs.) Each vertex is either internal and of valency three, or a boundary vertex of valency one.

We assume that each edge carries a label from $\{\mathtt{o}, \mathtt{p}\}$ (we say they are *colored* by $\mathtt{o}$ or $\mathtt{p}$). Moreover, the $\mathtt{p}$-colored edges are assumed to be oriented, and each internal vertex has precisely one attached edge which is $\mathtt{p}$-colored. By convention, the empty web $\varnothing$ is also a web, and we allow circle components which consist of edges only. Webs are considered modulo boundary-preserving planar isotopies.

Throughout we will, not just for webs, consider labelings with $\mathtt{o}$ or $\mathtt{p}$ and always illustrate them directly as colors using the convention that "$\mathtt{p}$=reddish". Moreover, both, webs and (pre)foams as defined below, contain $\mathtt{p}$-colored edges/facets. We call, everything related to these $\mathtt{p}$-colored edges/facets *phantom*, anything else *ordinary*.

Example 27. When we depict webs we omit the edge labels and color the edges instead. Furthermore, for readability, we draw $\mathtt{p}$-colored edges dashed. Using these conventions, such webs are for example locally of the form:

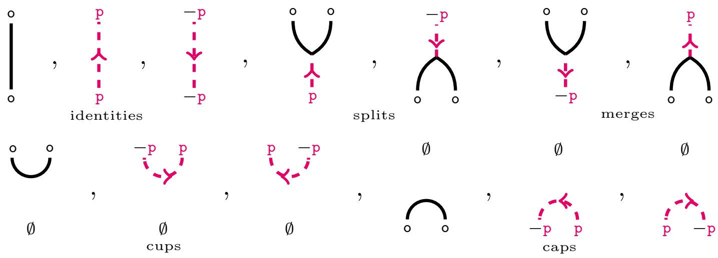


Here the outer circle indicates that these are local pictures. ▲

Definition 28. Let $\mathbf{W}$ be the *monoidal category of webs* given as follows:

▷ Objects are finite words $\vec{k}$ in the symbols $\mathtt{o}$, $\mathtt{p}$ and $-\mathtt{p}$. (The empty word $\emptyset$ is also allowed.)

▷ The morphisms spaces $\mathrm{Hom}_{\mathbf{W}}(\vec{k}, \vec{l})$ are given by all webs with bottom boundary $\vec{k}$ and top boundary $\vec{l}$ using the following local conventions (read from bottom to top) for identities, splits, merges, cups and caps:



(5)

▷ The composition $uv = v \circ u$ is the evident gluing of v on top of u, and monoidal product $\vec{k} \otimes \vec{l}$ or $u \otimes v$ given by putting $\vec{k}$ or u to the left of $\vec{l}$ or v.

▲

Remark 29. We stress that the webs we use in this thesis do not carry orientations on *ordinary* (black) edges. In contrast, *phantom* (reddish, dashed) edges carry orientations, cf. Figure 2. If one interprets the ordinary edges as corresponding to the vector representation $V_{\mathfrak{g}}$ of some associated Lie algebra $\mathfrak{g}$ and phantom edges corresponding to the second exterior power $\bigwedge^2 V_{\mathfrak{g}}$ of it, then this translates to $(V_{\mathfrak{g}})^* \cong V_{\mathfrak{g}}$ as $\mathfrak{g}$-modules, but $(\bigwedge^2 V_{\mathfrak{g}})^* \ncong \bigwedge^2 V_{\mathfrak{g}}$. Thus, if we see webs as $\mathfrak{g}$-intertwiners with "empty" corresponding to the ground field $\mathbb{C}$, then we have the situation as in Figure 1.

For $\mathfrak{sl}_2$-webs (cf. Remark 22) we did not need orientations because

$$(V_{\mathfrak{sl}_2})^* \cong V_{\mathfrak{sl}_2} \quad \text{and} \quad (\textstyle\bigwedge^2 V_{\mathfrak{sl}_2})^* \cong \textstyle\bigwedge^2 V_{\mathfrak{sl}_2}.$$

For $\mathfrak{gl}_2$-webs one would have to orient ordinary edges as well, since

$$(V_{\mathfrak{gl}_2})^* \ncong V_{\mathfrak{gl}_2} \quad \text{and} \quad (\textstyle\bigwedge^2 V_{\mathfrak{gl}_2})^* \ncong \textstyle\bigwedge^2 V_{\mathfrak{gl}_2}.$$

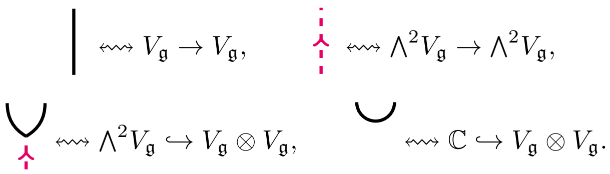
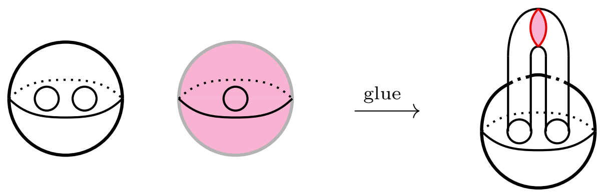


Figure 1: Examples of $\mathfrak{g}$-webs as $\mathfrak{g}$-intertwiners. The two bottom $\mathfrak{g}$-intertwiners are given by the inclusions, the others are identities.

For more on the relation between $\mathfrak{sl}_2$- and $\mathfrak{gl}_2$-web categories see e.g. [66, Remark 1.1]. In fact, we do not have a representation-theoretic interpretation of $\mathbf{W}$ in terms of intertwiners, but $\mathbf{W}$ gives rise to a 2-category (see Definition 30 below) which can be used to construct the type D arc algebras topologically. However, all constructions (which are carried out in detail in Sections 3.1, 3.2 and 3.3) could also be done with orientations on ordinary edges without further complications which would give a $\mathfrak{gl}_2$-foam 2-category $\mathfrak{F}^{\mathfrak{gl}_2}$ as in [21] or [20]. ▲

A *closed prefoam* is a singular surface obtained by gluing the boundary circles of a given set of orientable, compact, two-dimensional real surfaces. Some of these surfaces are called *phantom surfaces* (those are colored red in the following) and we always glue along three circles, where exactly one of the circles comes from a boundary component of a phantom surface, e.g.



Closed prefoams are assumed to be embedded in $\mathbb{R}^3$. As in the $\mathfrak{sl}_2$-case we allow prefoams to be decorated with dots. A (not necessarily closed) *prefoam* is obtained by intersecting a closed prefoam with the subset $\mathbb{R}^2 \times [-1, 1]$ such that the intersection is generic. Here, "generic" means that the intersections of the prefoam with $\mathbb{R}^2 \times \{0\}$ and $\mathbb{R}^2 \times \{1\}$ is a web.

Definition 30. Let $\mathfrak{F}$ denote the additive closure the $\mathbb{C}$-linear 2-category given by:

▷ The underlying structure of objects and morphisms is given by the category $\mathbf{W}$ of webs from Definition 28.

▷ The space of 2-morphisms between two webs u and v is a quotient of the graded, free $\mathbb{C}$-vector space on basis given by all prefoams from u to v.

▷ The quotient is obtained by modding out the relations from Subsection 3.1.2 as well as all relations they induce by closing prefoams.

▷ The vertical and the horizontal compositions are given by stacking and placing surfaces side by side.

▲

The above 2-category $\mathfrak{F}$ is a sign-modified version of Bar-Natan's [2] original $\mathfrak{sl}_2$-foam 2-category $\mathfrak{F}^{\mathfrak{sl}_2}$ from above. The additional signs are crucial for making Khovanov's link homology functorial, see [4],[21] and [20]. Using the category $\mathfrak{F}$ we can define a web algebra $\mathfrak{W}$ by following the construction in the $\mathfrak{sl}_2$-case (cf. Section 3.2). This gives rise to a (structure preserving) equivalence of 2-categories, see Proposition 3.2.11,

$$\mathfrak{F} \xrightarrow{\cong} \mathfrak{W}\textbf{-biMod},$$

where $\mathfrak{W}$**-biMod** is a certain 2-category of $\mathfrak{W}$-bimodules (with details given in Definition 3.2.10).

The first main purpose of the third chapter is to define an algebra $c\mathfrak{W}$ which serves as a combinatorial model of $\mathfrak{W}$. This algebra reduces the complexity of foams in three-dimensional space to planar combinatorics of so-called *dotted webs*. These webs are slightly more complicated than the webs from Definition 28 and can be obtained by "projecting" a foam onto the plane, see Figure 3.1. A rigorous treatment of dotted webs can be found in Subsection 3.3.1.

Theorem G. *There is an isomorphism of graded algebras*

$$\texttt{comb}\colon c\mathfrak{W} \xrightarrow{\cong} \mathfrak{W}.$$

(Consequently, we obtain a combinatorial model of the foam 2*-category* $\mathfrak{F}$*.)* $\square$

The main step to establish this isomorphism is the construction of a cup foam basis (similarly to the cup foam basis in the $\mathfrak{sl}_2$-case mentioned in Remark 25) on which we can calculate the foam multiplication explicitly (see Proposition 3.1.14).

A topological realization of the type D arc algebra

The second main purpose of the third chapter is to realize the original type D arc algebra $\mathfrak{A}$ as defined in [18] as a subalgebra of $\mathfrak{W}$. This yields a natural topological interpretation in terms of foams.

Theorem H. *There is an embedding of (graded) algebras*

$$\texttt{top}\colon \mathfrak{A} \hookrightarrow \mathfrak{W}.$$

(In fact, $\mathfrak{A}$ *is an idempotent truncation of* $\mathfrak{W}$ *giving an embedding between the associated bimodule* 2*-categories.)* $\square$

The map `top` is defined as the map which makes the following diagram commute:

$$\begin{tikzcd} \mathfrak{A} \arrow[r, "\texttt{sign}"] \arrow[rrr, bend right, hook, "\texttt{top}"'] & \overline{\mathfrak{A}} \arrow[r, "\overline{\texttt{top}}"] & c\mathfrak{W} \arrow[r, "\texttt{comb}"] & \mathfrak{W}. \end{tikzcd}$$

The map $\overline{\texttt{top}}\colon \overline{\mathfrak{A}} \hookrightarrow c\mathfrak{W}$ is of combinatorial nature and relates dotted webs with the type D circle diagrams. The picture to keep in mind is given in Figure 2 with a foam on the left, a corresponding (dotted) web in the middle and a type D circle diagram on the right. For a precise treatment we refer the reader to Chapter 3.

We expect that this result might have several implications in future work. Direct consequences which we deduce are that the associativity and the well-definedness of the action on bimodules, which are very involved to prove for the type D (generalized) arc algebras, are immediately clear from the topological realization presented in this thesis, cf. Corollary 3.4.2.

The proofs of Theorems G and H require some involved combinatorial arguments and calculations. For readability we moved all these proofs to the end of the third chapter, see Section 3.5.

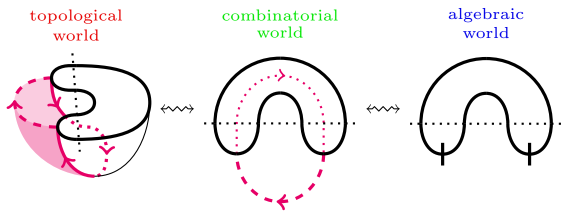


Figure 2: From foams to (dotted) webs to circle diagrams.

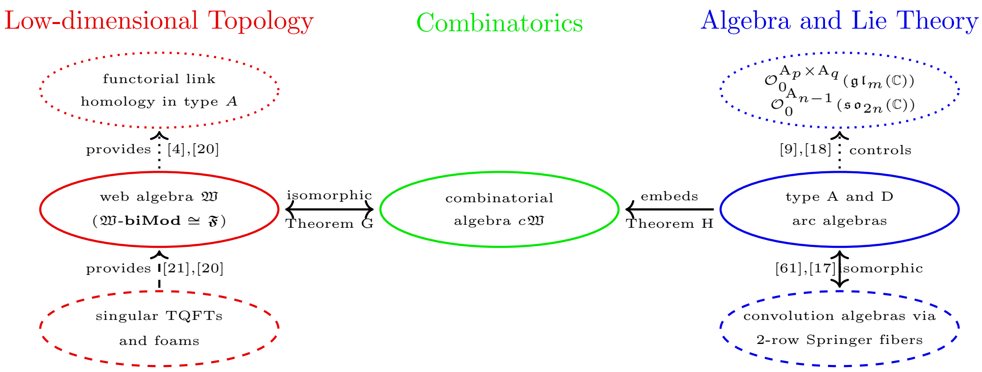


Figure 3: This pictures illustrates the connections between the main objects of study in the third chapter.

Publications and Coauthorships

Over the course of this PhD project all three chapters of this thesis as well as parts of the introduction have appeared in three separate preprints [22, 70, 71] on the ArXiv. The first chapter can be found in the article [70] which was accepted for publication in *Transactions of the AMS* and the second chapter is contained in [71]. The third chapter arose from joint work with Dr. Michael Ehrig and Dr. Daniel Tubbenhauer and appeared in [22].

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First and foremost I would like to thank my advisor Catharina Stroppel for her continuous support and invaluable advice over the years. Her dedication and enthusiasm for mathematics have been a constant source of inspiration. I am grateful for all the freedom she has given me in exploring my mathematical interests and for her willingness to share her profound knowledge about the fascinating interrelations between representation theory, geometry and topology at countless occasions. Her unique perspective on mathematics has had a massive impact on this thesis and has shaped me as a mathematician.

I would like to thank the members of the representation theory group in Bonn for providing such a collegial working atmosphere. I have profited from many helpful and stimulating discussions about mathematics and beyond. In this context I would particularly mention my two coauthors Michael Ehrig and Daniel Tubbenhauer for their interest in my work and for having given me a lot of valuable and detailed feedback throughout the course of my PhD project.

I would like to thank Mikhail Khovanov, Markus Reineke and Ben Webster for inviting me to give a seminar talk at their respective institutions and obtain valuable feedback on this work. In particular, I would like to thank Shrawan Kumar for hosting me during an inspiring two-week research visit in Chapel Hill in April 2017 while this thesis received its finishing touches.

I am grateful to the Bonn International Graduate School in Mathematics and the Max-Planck Institute for Mathematics in Bonn for funding this PhD project by granting me a Hausdorff scholarship and a scholarship of the International Max Planck Research School. I am thankful for some detailed comments on my article [70] by an anonymous referee which indirectly also improved the first chapter of this thesis.

Although mathematics is certainly a passion of mine, I also appreciated all the distractions from my work. I am thankful for my friends who shared my passion for music with me, their private gym or for repeatedly bar-hopping through some major European cities over the last years. In addition to that, I gladly remember some intercontinental trips with my brother which broadened my culinary and cultural horizons. All these things helped me to ground myself in reality and showed me that life is more than just the proof of a theorem.

Last but not least, I would like to thank my parents for all their encouragement and unconditional support over the years. I am particularly grateful for all the back office work and many delicious meals on the house. Even though the actual contents of this dissertation does not lie within their realm of expertise, their indirect contribution to this work cannot be underestimated.

Chapter 1

Topology of two-row Springer fibers in types C and D

1.1 Algebraic Springer fibers

We begin by defining the (algebraic) Springer fibers and provide an overview over some known results concerning the combinatorics of the irreducible components. Unless stated otherwise, $n = 2m$ denotes an even positive integer.

1.1.1 Nilpotent orbits and algebraic Springer fibers

Let $\mathcal{N} \subseteq \mathfrak{sl}(\mathbb{C}^n)$ be the *nilpotent cone* consisting of all nilpotent endomorphisms of $\mathbb{C}^n$ (in the usual sense of linear algebra). The Jordan normal form implies that the orbits under the conjugation-action of the special linear group $\mathrm{SL}(\mathbb{C}^n)$ on $\mathcal{N}$ can be parameterized in terms of *partitions* of n, i.e. r-tuples $\lambda = (\lambda_1, \ldots, \lambda_r) \in \mathbb{Z}^r_{>0}$, $r \in \mathbb{Z}_{>0}$, of positive integers such that $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_r$ and $\lambda_1 + \ldots + \lambda_r = n$, where the *parts* λ_i of λ encode the sizes of the Jordan blocks of the elements contained in an orbit. Let $\mathcal{P}(n)$ denote the set of all partitions of n.

Fix an element $\epsilon \in \{\pm 1\}$ and let β_ϵ be a nondegenerate bilinear form on $\mathbb{C}^n$ which satisfies $\beta_\epsilon(v, w) = \epsilon\beta_\epsilon(w, v)$ for all $v, w \in \mathbb{C}^n$. Let $\mathrm{Aut}(\mathbb{C}^n, \beta_\epsilon)$ denote the isometry group consisting of all linear automorphisms of $\mathbb{C}^n$ preserving β_ϵ. The Lie algebra $\mathfrak{aut}(\mathbb{C}^n, \beta_\epsilon)$ of $\mathrm{Aut}(\mathbb{C}^n, \beta_\epsilon)$ is the subalgebra of $\mathfrak{sl}(\mathbb{C}^n)$ consisting of all endomorphisms x of $\mathbb{C}^n$ which satisfy the equation $\beta_\epsilon(x(v), w) = -\beta_\epsilon(v, x(w))$ for all $v, w \in \mathbb{C}^n$. Note that $\mathrm{Aut}(\mathbb{C}^n, \beta_1) \cong O_n(\mathbb{C})$ and $\mathfrak{aut}(\mathbb{C}^n, \beta_1) \cong \mathfrak{so}_n(\mathbb{C})$ if β_ϵ is nondegenerate and symmetric, whereas $\mathrm{Aut}(\mathbb{C}^n, \beta_{-1}) \cong Sp_n(\mathbb{C})$ and $\mathfrak{aut}(\mathbb{C}^n, \beta_{-1}) \cong \mathfrak{sp}_n(\mathbb{C})$ if β_ϵ is symplectic.

We define $\mathcal{P}_\epsilon(n)$ as the subset of $\mathcal{P}(n)$ consisting of all partitions λ of n for which the cardinality of $\{i \mid \lambda_i = j\}$ is even for all j satisfying $(-1)^j = \epsilon$, i.e. even (resp. odd) parts occur with even multiplicity. We refer to the partitions in $\mathcal{P}_1(n)$ (resp. $\mathcal{P}_{-1}(n)$) as *admissible of type D* (resp. type *C*) since they parameterize nilpotent orbits in the simple Lie algebra of the corresponding type. The following classification of nilpotent orbits is well known [26, 72].

Proposition 1.1.1. The orbits under the conjugation-action of $\mathrm{Aut}(\mathbb{C}^n, \beta_\epsilon)$ on the variety of nilpotent elements $\mathcal{N}_{\mathfrak{aut}(\mathbb{C}^n,\beta_\epsilon)} = \mathcal{N} \cap \mathfrak{aut}(\mathbb{C}^n, \beta_\epsilon)$ are in bijective correspondence with the partitions contained in $\mathcal{P}_\epsilon(n)$. The parts of the partition associated with the orbit of an endomorphism encode the sizes of the Jordan blocks in Jordan normal form. □

Definition 1.1.2. A *full isotropic flag* in $\mathbb{C}^n$ (with respect to β_ϵ) is a sequence $F_\bullet$ of subspaces $\{0\} = F_0 \subsetneq F_1 \subsetneq \ldots \subsetneq F_m$ of $\mathbb{C}^n$ such that F_m is isotropic with respect to β_ϵ, i.e. β_ϵ vanishes on $F_m \times F_m$. The set of all full isotropic flags is denoted by $\mathcal{F}l_{\beta_\epsilon}$. ▲

Since the inclusions of the subspaces of a flag $F_\bullet$ are strict, F_m is maximal isotropic and we have $\dim(F_i) = i$ for all $i \in \{1, \ldots, m\}$. The set $\mathcal{F}l_{\beta_\epsilon}$ can be equipped with the structure of a smooth projective variety, e.g. by identifying it with a homogeneous $\mathrm{Aut}(\mathbb{C}^n, \beta_\epsilon)$-space. Adding the vector spaces $F_{n-i} = F_i^\perp$ to a given full isotropic flag $F_\bullet$ (the orthogonal complement is taken with respect to β_ϵ) defines an embedding of $\mathcal{F}l_{\beta_\epsilon}$ into the *full flag variety $\mathcal{F}l$ of type A*. Given any other nondegenerate symmetric (resp. symplectic) bilinear form β on $\mathbb{C}^n$, the corresponding varieties $\mathcal{F}l_\beta$ and $\mathcal{F}l_{\beta_1}$ (resp. $\mathcal{F}l_{\beta_{-1}}$) are isomorphic which allows us to speak about the *full flag variety of type D* (resp. *type C*), denoted by $\mathcal{F}l_D$ (resp. $\mathcal{F}l_C$), without further specifying a nondegenerate symmetric (resp. symplectic) bilinear form.

Remark 1.1.3. According to our conventions the full flag variety $\mathcal{F}l_D$ of type D is isomorphic to a quotient of $O_n(\mathbb{C})$ (and not $SO_n(\mathbb{C})$). Hence, it consists of two isomorphic connected components. The component containing a given flag $F_\bullet$ is determined by F_m. More precisely, there is a unique flag $F'_\bullet$ such that $F_i = F'_i$ for all $i \in \{1, \ldots, m-1\}$ and $F_m \neq F'_m$ and the two flags lie in different connected components (cf. [67, §1.4] or [17, Remark 2.2]). ▲

Definition 1.1.4. The *(algebraic) Springer fiber* $\mathcal{F}l^x_{\beta_\epsilon}$ associated with β_ϵ and $x \in \mathcal{N}_{\mathfrak{aut}(\mathbb{C}^n, \beta_\epsilon)}$ is the projective subvariety of $\mathcal{F}l_{\beta_\epsilon}$ consisting of all isotropic flags $F_\bullet$ which satisfy the conditions $xF_i \subseteq F_{i-1}$ for all $i \in \{1, \ldots, m\}$. ▲

If β is another nondegenerate symmetric (resp. symplectic) bilinear form on $\mathbb{C}^n$ and y a nilpotent endomorphism of $\mathbb{C}^n$ contained in $\mathfrak{aut}(\mathbb{C}^n, \beta)$, then the Springer fibers $\mathcal{F}l^y_\beta$ and $\mathcal{F}l^x_{\beta_\epsilon}$ are isomorphic if and only if x and y have the same Jordan type. Thus, by Proposition 1.1.1, there is (up to isomorphism) precisely one Springer fiber for every admissible partition λ of type D (resp. type C) which we denote by $\mathcal{F}l^\lambda_D$ (resp. $\mathcal{F}l^\lambda_C$) assuming that some nondegenerate symmetric (resp. symplectic) bilinear form and a compatible nilpotent endomorphism of Jordan type λ have been fixed beforehand.

1.1.2 Irreducible components and combinatorics

Recall that a partition $\lambda = (\lambda_1, \ldots, \lambda_r)$ of n can be depicted as a Young diagram, i.e. a collection of n boxes arranged in r left-aligned rows, where the i-th row consists of λ_i boxes (this is commonly known as "English notation").

Definition 1.1.5. A *standard Young tableau of shape λ* is a filling of the Young diagram of $\lambda \in \mathcal{P}(n)$ with the numbers $1, 2, \ldots, n$ such that each number occurs exactly once and the entries decrease in every row and column. Let $SYT(\lambda)$ be the set of all standard Young tableaux of shape λ. ▲

Example 1.1.6. Here is a complete list of all elements contained in the set $SYT(3, 2)$:

$$\begin{array}{|c|c|c|}\hline 5&4&3\\\hline 2&1\\\cline{1-2}\end{array} \qquad \begin{array}{|c|c|c|}\hline 5&4&2\\\hline 3&1\\\cline{1-2}\end{array} \qquad \begin{array}{|c|c|c|}\hline 5&3&2\\\hline 4&1\\\cline{1-2}\end{array} \qquad \begin{array}{|c|c|c|}\hline 5&3&1\\\hline 4&2\\\cline{1-2}\end{array} \qquad \begin{array}{|c|c|c|}\hline 5&4&1\\\hline 3&2\\\cline{1-2}\end{array}$$

▲

Let $x \in \mathcal{N}$ be a nilpotent endomorphism of $\mathbb{C}^n$ of Jordan type λ and let $\mathcal{F}l^x$ be the associated Springer fiber (of type A) consisting of all sequences $\{0\} = F_0 \subsetneq F_1 \subsetneq \ldots \subsetneq F_n = \mathbb{C}^n$ of subspaces satisfying $xF_i \subseteq F_{i-1}$ for all $i \in \{1, \ldots, n\}$. By work of Spaltenstein [57] and Vargas [68] there exists a surjection $\mathcal{S}^x : \mathcal{F}l^x \twoheadrightarrow SYT(\lambda)$ which can be used to parameterize the irreducible components of the Springer fiber as follows:

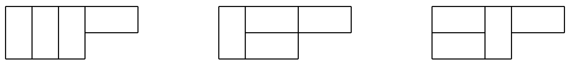
Proposition 1.1.7. The set $SYT(\lambda)$ of standard Young tableaux of shape λ is in bijective correspondence with the irreducible components of the Springer fiber $\mathcal{F}l^x$ via the map which sends a tableau T to the closure of the fiber of the map $\mathcal{S}^x$ over T. □

In order to obtain a parameterization of the irreducible components of the Springer fibers of type C or D one has to replace standard Young tableaux by admissible domino tableaux which we define below.

Definition 1.1.8. A *domino diagram of shape* λ with m dominoes is given by a partitioning of the set of boxes of the Young diagram corresponding to λ into two-element subsets, called *dominoes*, such that any two associated boxes have a common vertical or horizontal edge. ▲

We depict a domino diagram by deleting the common edge of any two boxes forming a domino.

Example 1.1.9. Here is a complete list of all domino diagrams of shape $(5, 3)$:



▲

Definition 1.1.10. An *admissible domino tableau* of shape λ, where $\lambda \in \mathcal{P}_\epsilon(n)$, is obtained from a domino diagram of shape λ by filling the n boxes of its underlying Young diagram with the numbers $1, \ldots, m$ such that:

(ADT1) Each of the numbers $1, \ldots, m$ occurs exactly twice and any two boxes with the same number form a domino.

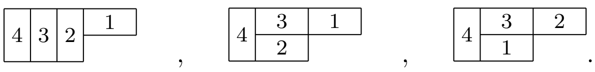
(ADT2) The entries in each row and column are weakly decreasing from left to right and top to bottom.

(ADT3) For every $i \in \{1, \ldots, m\}$, the partition corresponding to the shape of the diagram obtained by deleting all dominoes with label smaller than or equal to i is admissible of type D (resp. type C) if $\epsilon = 1$ (resp. $\epsilon = -1$).

The set of all admissible domino tableaux of shape λ is denoted by $ADT(\lambda)$. ▲

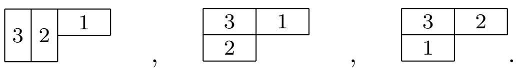
When depicting an admissible domino tableau we draw dominoes instead of paired boxes and only write a single number in every domino.

Example 1.1.11. Here is a complete list of all admissible domino tableaux of shape $(5, 3) \in \mathcal{P}_1(8)$:



Indeed, in case of the rightmost tableau above, removing the domino with label 1 yields a diagram whose shape corresponds to the partition $(5, 1) \in \mathcal{P}_1(6)$, removing the dominoes labeled 1 and 2 yields $(3, 1) \in \mathcal{P}_1(4)$ and removing the dominoes with labels 1, 2 and 3 gives $(1, 1) \in \mathcal{P}_1(2)$. This proves condition (ADT3) for this particular diagram (note that (ADT1) and (ADT2) are evidently satisfied). Similar arguments apply in the other cases.

Furthermore, here are all admissible domino tableaux of shape $(4, 2) \in \mathcal{P}_{-1}(6)$:



▲

Let $x \in \mathcal{N}_{\mathfrak{aut}(\mathbb{C}^n,\beta_\epsilon)}$ be a nilpotent endomorphism of $\mathbb{C}^n$ of Jordan type λ. We define a surjection $\mathcal{S}^x_{\beta_\epsilon} : \mathcal{F}l^x_{\beta_\epsilon} \twoheadrightarrow ADT(\lambda)$ as follows: Given $F_\bullet \in \mathcal{F}l^x_{\beta_\epsilon}$, we obtain a sequence $x^{(m)}, \ldots, x^{(0)}$ of nilpotent endomorphisms, where $x^{(i)} \colon F_i^\perp/F_i \to F_i^\perp/F_i$ denotes the map induced by x (the orthogonal complement is taken with respect to β_ϵ). The Jordan types $J(x^{(m-1)}), \ldots, J(x^{(i)}), \ldots, J(x^{(0)})$ are admissible partitions, where successive Jordan types differ by precisely one domino. We label the new domino by comparing Jordan types of $x^{(i)}$ and $x^{(i+1)}$ with i. More details can be found in [58], [67] or [49].

Example 1.1.12. Let $x \in \mathfrak{sl}(\mathbb{C}^8)$ be a nilpotent endomorphism of Jordan type $(5, 3)$ and let

$$e_1 \leftarrow e_2 \leftarrow e_3 \leftarrow e_4 \leftarrow e_5 \qquad\qquad f_1 \leftarrow f_2 \leftarrow f_3 \tag{1.1}$$

be a Jordan basis (the arrows indicate the action of x, e_1 and f_1 are mapped to zero by x). Let β_1 be the nondegenerate symmetric bilinear form on $\mathbb{C}^8$ given by

$$\beta_1(e_i, f_j) = 0, \qquad \beta_1(e_i, e_{i'}) = (-1)^{i-1}\delta_{i+i',6}, \qquad \beta_1(f_j, f_{j'}) = (-1)^j \delta_{j+j',4},$$

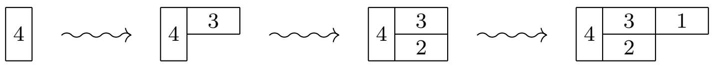
where $i, i' \in \{1, \ldots, 5\}$ and $j, j' \in \{1, 2, 3\}$. Then one easily checks that $x \in \mathcal{N}_{\mathfrak{aut}(\mathbb{C}^8,\beta_1)}$. Consider the flag $F_\bullet \in \mathcal{F}l^x_{\beta_1}$, where

$$F_1 = \operatorname{span}(e_1)\,,\ F_2 = \operatorname{span}(e_1, f_1)\,,\ F_3 = \operatorname{span}(e_1, e_2, f_1)\,,\ F_4 = \operatorname{span}(e_1, e_2, f_1, \mathbf{i}e_3 + f_2)$$

and $\mathbf{i} = \sqrt{-1} \in \mathbb{C}$. Note that $F_1^\perp$ is spanned by all Jordan basis vectors (see (1.1)) except e_5, $F_2^\perp$ is spanned by all vectors except e_5, f_3 and $F_3^\perp = \operatorname{span}(e_1, e_2, e_3, f_1, f_2)$. Hence, one easily computes the Jordan types

$$J(x^{(3)}) = (1, 1)\,,\ J(x^{(2)}) = (3, 1)\,,\ J(x^{(1)}) = (3, 3)\,,\ J(x^{(0)}) = (5, 3)$$

and the tableau associated with $F_\bullet$ via $\mathcal{S}^x_{\beta_1}$ is constructed as follows:



▲

Remark 1.1.13. The closures of the preimages of the elements in $ADT(\lambda)$ under the map $\mathcal{S}^x_{\beta_\epsilon}$ are not necessarily connected and hence cannot be irreducible (as in type A). Nonetheless, the irreducible components of $\mathcal{F}l^x_{\beta_\epsilon}$ contained in the closure of $(\mathcal{S}^x_{\beta_\epsilon})^{-1}(T)$ are precisely its connected components [67, Lemma 3.2.3]. ▲

Example 1.1.14. Consider the Springer fiber $\mathcal{F}l^x_{\beta_1}$ as in Example 1.1.12. Then one can check (e.g. by using the inductive construction in [17, §6.5]) that the fiber of $\mathcal{S}^x_{\beta_1}$ over T, where T denotes the rightmost admissible domino tableau of shape $(5, 3)$ in Example 1.1.11, is the union of the following two disjoint sets of flags

$$\big\{ \operatorname{span}(\mu e_1 + f_1) \subseteq \operatorname{span}(e_1, f_1) \subseteq \operatorname{span}(e_1, e_2, f_1) \subseteq \operatorname{span}(e_1, e_2, f_1, \mathbf{i}e_3 + f_2) \mid \mu \in \mathbb{C} \big\}$$

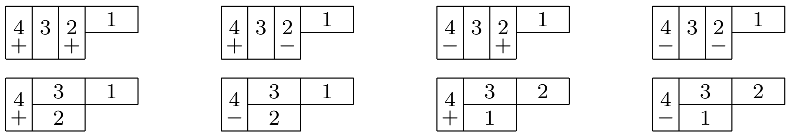
$$\big\{ \operatorname{span}(\mu e_1 + f_1) \subseteq \operatorname{span}(e_1, f_1) \subseteq \operatorname{span}(e_1, e_2, f_1) \subseteq \operatorname{span}(e_1, e_2, f_1, \mathbf{i}e_3 - f_2) \mid \mu \in \mathbb{C} \big\}$$

each of which is isomorphic to an affine space $\mathbb{A}^1$. By taking the closure we add the possibility of choosing $\operatorname{span}(e_1)$ as a one-dimensional subspace of these flags. Thus, the closure of the preimage of T under $\mathcal{S}^x_{\beta_1}$ is isomorphic to a disjoint union of two projective spaces $\mathbb{P}^1$ each of which corresponds to an irreducible component of $\mathcal{F}l^x_{\beta_1}$. ▲

By the above discussion the map $\mathcal{S}^x_{\beta_\epsilon}$ does not provide a parameterization of the irreducible components of $\mathcal{F}l^x_{\beta_\epsilon}$. It is necessary to add more combinatorial data to the tableaux in order to count the connected components.

Definition 1.1.15. A *signed domino tableau* of shape λ is an admissible domino tableau of shape λ together with a choice of sign (an element of the set $\{+,-\}$) for each vertical domino in an odd (resp. even) column if $\lambda \in \mathcal{P}_1(n)$ (resp. $\lambda \in \mathcal{P}_{-1}(n)$). We write $ADT^{\mathrm{sgn}}(\lambda)$ for the set of signed domino tableau of shape λ. We define $ADT^{\mathrm{sgn}}_{\mathrm{odd}}(\lambda)$ (resp. $ADT^{\mathrm{sgn}}_{\mathrm{even}}(\lambda)$) as the set of all signed domino tableaux with an odd (resp. even) number of minus signs. ▲

Example 1.1.16. The set $ADT^{\mathrm{sgn}}((5,3))$ consists of the following eight elements:



The set $ADT^{\mathrm{sgn}}((4,2))$ consists of the following four elements:

▲

The next Proposition summarizes the discussion above and should be seen as the analog of Proposition 1.1.7 for two-row Springer fibers of types C and D:

Proposition 1.1.17 ([67, Lemmas 3.2.3 and 3.3.3])**.** Let $\lambda = (\lambda_1, \lambda_2) \in \mathcal{P}_\epsilon(n)$ be a two-row partition and $x \in \mathcal{N}_{\mathfrak{aut}(\mathbb{C}^n,\beta_\epsilon)}$ a nilpotent endomorphism of Jordan type λ.

For every $T \in ADT(\lambda)$ the closure of the preimage of T under $\mathcal{S}^x_{\beta_\epsilon}$ is a union of disjoint irreducible components of $\mathcal{F}l^x_{\beta_\epsilon}$ indexed by all signed domino tableaux which equal T after forgetting the signs. There exists a bijection between the set of all irreducible components of $\mathcal{F}l^x_{\beta_\epsilon}$ and $ADT^{\mathrm{sgn}}(\lambda)$. □

Remark 1.1.18. For general Jordan types the irreducible components are parameterized by equivalence classes of signed domino tableaux. However, in the case of two-row partitions (which is the case we are primarily interested in) this equivalence relation is trivial. We refer to [67, §3.3] or [49, §3] for details and the more general statement for arbitrary Jordan types. ▲

Example 1.1.19. The two rightmost signed domino tableaux of shape $(5,3)$ in the second row of Example 1.1.16 index the two disjoint irreducible components in Example 1.1.14. The sign on the vertical domino corresponds to the two choices $\mathrm{span}(e_1, e_2, f_1, \mathbf{i}e_3 + f_2)$ respectively $\mathrm{span}(e_1, e_2, f_1, \mathbf{i}e_3 - f_2)$ as the four-dimensional subspace (see also Remark 1.1.3). ▲

1.2 Topological Springer fibers

In this section we fix a partition $(n-k, k)$ of $n = 2m$, $1 \leq k \leq m$, labeling a nilpotent orbit of the orthogonal group, i.e. either $k = m$ or k is odd by Proposition 1.1.1.

1.2.1 Definition of topological Springer fibers

We fix a rectangle in the plane with m vertices evenly spread along the upper horizontal edge of the rectangle. The vertices are labeled by the consecutive integers $1, \ldots, m$ in increasing order from left to right.

Definition 1.2.1. A *cup diagram* is a non-intersecting diagram inside the rectangle obtained by attaching lower semicircles called *cups* and vertical line segments called *rays* to the vertices.

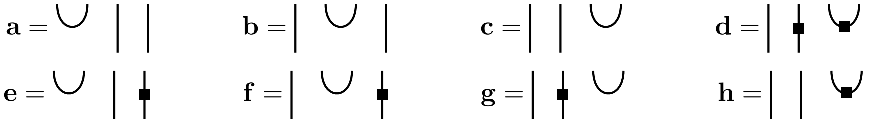
In doing so we require that every vertex is connected with precisely one endpoint of a cup or ray. Moreover, a ray always connects a vertex with a point on the lower horizontal edge of the rectangle. Additionally, any cup or ray for which there exists a path inside the rectangle connecting this cup or ray to the right edge of the rectangle without intersecting any other part of the diagram is allowed to be decorated with a single marker, i.e. a small black box.

If the cups and rays of two given cup diagrams are incident with exactly the same vertices (regardless of the precise shape of the cups) and the distribution of markers on corresponding cups and rays coincides in both diagrams we consider them as equal.

We write $\mathbb{B}^{n-k,k}$ to denote the set of all cup diagrams on m vertices with $\lfloor \frac{k}{2} \rfloor$ cups. This set decomposes as a disjoint union $\mathbb{B}^{n-k,k} = \mathbb{B}^{n-k,k}_{\mathrm{even}} \sqcup \mathbb{B}^{n-k,k}_{\mathrm{odd}}$, where $\mathbb{B}^{n-k,k}_{\mathrm{even}}$ (resp. $\mathbb{B}^{n-k,k}_{\mathrm{odd}}$) consists of all cup diagrams with an even (resp. odd) number of markers. ▲

We usually neither draw the rectangle around the diagrams nor display the vertex labels.

Example 1.2.2. The set $\mathbb{B}^{5,3}$ consists of the cup diagrams



where the diagrams in the first (resp. second) row are the ones in $\mathbb{B}^{n-k,k}_{\mathrm{even}}$ (resp. $\mathbb{B}^{n-k,k}_{\mathrm{odd}}$). ▲

Beware 1.2.3. Note that in this chapter we work with cup diagrams for which the markers are accessible from the right side of the rectangle instead of the left side. Our conventions therefore differ from the ones used in the introduction and in related publications, e.g. [17,18,41]. Moreover, we stress that the decorations of cups and rays are given in terms of black boxes. In related publications the authors usually use a a black dot • instead. Since dots • also turn up in the foam picture in the third chapter of this thesis (as e.g. illustrated in (3.4)), where they have a completely different meaning, this notation had to be altered for our purposes. Also note the difference in terminology, our "marked cups/rays" are called "dotted cups/rays" in the above-mentioned related articles. •

Let $\mathbb{S}^2 \subseteq \mathbb{R}^3$ be the two-dimensional standard unit sphere on which we fix the points $p = (0,0,1)$ and $q = (1,0,0)$.

Given a cup diagram $\mathbf{a} \in \mathbb{B}^{n-k,k}$, define $S_{\mathbf{a}} \subseteq \left(\mathbb{S}^2\right)^m$ as the submanifold consisting of all $(x_1,\ldots,x_m) \in \left(\mathbb{S}^2\right)^m$ which satisfy the relations $x_i = -x_j$ (resp. $x_i = x_j$) if the vertices i and j are connected by an unmarked cup (resp. marked cup). Moreover, we impose the relations $x_i = p$ if the vertex i is connected to a marked ray and $x_i = -p$ (resp. $x_i = q$) if i is connected to an unmarked ray which is the rightmost ray in $\mathbf{a}$ (resp. not the rightmost ray). Note that $S_{\mathbf{a}}$ is homeomorphic to $(\mathbb{S}^2)^{\lfloor \frac{k}{2} \rfloor}$, i.e. each cup of $\mathbf{a}$ contributes a sphere.

Definition 1.2.4. The $(n-k,k)$ *topological Springer fiber* $\mathcal{S}^{n-k,k}_D$ of type D is defined as the union

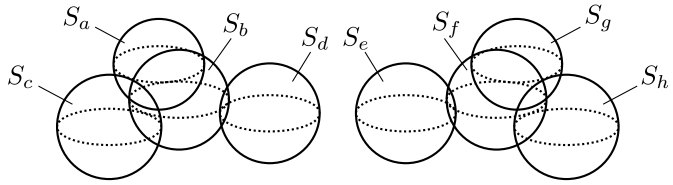
$$\mathcal{S}^{n-k,k}_D := \bigcup_{\mathbf{a}\in\mathbb{B}^{n-k,k}} S_{\mathbf{a}} \subseteq \left(\mathbb{S}^2\right)^m .$$

▲

Example 1.2.5. In the following we discuss in detail the topological Springer fiber $\mathcal{S}^{5,3}_D$. The submanifolds of $\left(\mathbb{S}^2\right)^4$ associated with the cup diagrams in $\mathbb{B}^{5,3}$ (cf. Example 1.2.2) are the following:

- $S_{\mathbf{a}} = \{(x, -x, q, -p) \mid x \in \mathbb{S}^2\}$
- $S_{\mathbf{b}} = \{(q, x, -x, -p) \mid x \in \mathbb{S}^2\}$
- $S_{\mathbf{c}} = \{(q, -p, x, -x) \mid x \in \mathbb{S}^2\}$
- $S_{\mathbf{d}} = \{(q, p, x, x) \mid x \in \mathbb{S}^2\}$
- $S_{\mathbf{e}} = \{(x, -x, q, p) \mid x \in \mathbb{S}^2\}$
- $S_{\mathbf{f}} = \{(q, x, -x, p) \mid x \in \mathbb{S}^2\}$
- $S_{\mathbf{g}} = \{(q, p, x, -x) \mid x \in \mathbb{S}^2\}$
- $S_{\mathbf{h}} = \{(q, -p, x, x) \mid x \in \mathbb{S}^2\}$

Each of these manifolds is homeomorphic to a two-sphere. Their pairwise intersection is either a point or empty, e.g. we have $S_{\mathbf{a}} \cap S_{\mathbf{b}} = \{(q, -q, q, -p)\}$ and $S_{\mathbf{a}} \cap S_{\mathbf{c}} = \emptyset$ (cf. also Proposition 1.2.9). The $(5, 3)$ topological Springer fiber of type D is a disjoint union of two Kleinian singularities of type D_4, [55].

$\mathcal{S}_D^{5,3} \cong$ 

▲

Remark 1.2.6. A topological model for the Springer fibers of type D corresponding to the partition (k, k) using a slightly different sign convention was introduced in [17, §4.1]. In order to see that this model is in fact homeomorphic to our topological Springer fiber one considers the involutory diffeomorphism

$$I_k \colon (\mathbb{S}^2)^k \to (\mathbb{S}^2)^k, \ (x_1, \ldots, x_k) \mapsto (-x_1, x_2, -x_3, \ldots, (-1)^k x_k).$$

Note that if two vertices i and j of a given cup diagram $\mathbf{a} \in \mathbb{B}^{k,k}$ are connected by a cup (marked or unmarked) then either i is odd and j is even, or i is even and j is odd. Moreover, the ray in $\mathbf{a}$ (which exists if and only if k is odd) must be attached to an odd vertex. Thus, the image of the set $S_{\mathbf{a}}$ under I_k, denoted by $S'_{\mathbf{a}}$, consists of all elements $(x_1, \ldots, x_k) \in (\mathbb{S}^2)^k$ which satisfy the relations $x_i = x_j$ (resp. $x_i = -x_j$) if the vertices i and j are connected by an unmarked cup (resp. marked cup), $x_i = -p$ if the vertex i is connected to a marked ray and $x_i = p$ if i is connected to an unmarked ray. In particular, we have

$$I_k \left(\mathcal{S}_D^{k,k} \right) = \bigcup_{\mathbf{a} \in \mathbb{B}^{n-k,k}} I_k \left(S_{\mathbf{a}} \right) = \bigcup_{\mathbf{a} \in \mathbb{B}^{n-k,k}} S'_{\mathbf{a}}$$

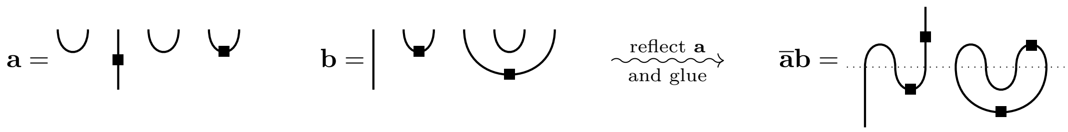
This is precisely the definition of the topological Springer fiber as in [17, §4.1] (after additionally reversing the order of the coordinates, see Beware 1.2.3). ▲

1.2.2 Intersections of components

In the following we provide a combinatorial description of the topology of the pairwise intersections of the submanifolds $S_{\mathbf{a}} \subseteq \left(\mathbb{S}^2\right)^m$, $\mathbf{a} \in \mathbb{B}^{n-k,k}$, using circle diagrams (see Proposition 1.2.9 below). In combination with the homeomorphism $\mathcal{S}_D^{n-k,k} \cong \mathcal{F}l_D^{n-k,k}$ (cf. Theorem 1.4.12) this yields a combinatorial description of the topology of the pairwise intersections of the irreducible components of $\mathcal{F}l_D^{n-k,k}$.

Definition 1.2.7. Let $\mathbf{a}, \mathbf{b} \in \mathbb{B}^{n-k,k}$ be cup diagrams. The *circle diagram* $\overline{\mathbf{a}}\mathbf{b}$ is defined as the diagram obtained by reflecting the diagram $\mathbf{a}$ in the horizontal middle line of the rectangle and then sticking the resulting diagram, denoted by $\overline{\mathbf{a}}$, on top of the cup diagram $\mathbf{b}$, i.e. we glue the two diagrams along the horizontal edges of the rectangles containing the vertices (thereby identifying the vertices of $\overline{\mathbf{a}}$ and $\mathbf{b}$ pairwise from left to right). In general the diagram $\overline{\mathbf{a}}\mathbf{b}$ consists of several connected components each of which is either closed (i.e. it has no endpoints) or a line segment. A line segment which contains a ray of $\mathbf{a}$ and a ray of $\mathbf{b}$ is called a *propagating line*. ▲

Example 1.2.8. Here is an example illustrating the gluing of two cup diagrams in order to obtain a circle diagram:



Hence, $\overline{\mathbf{a}}\mathbf{b}$ consists of one closed connected component and one line segment which is propagating. Note that the circle diagram $\overline{\mathbf{e}}\mathbf{h}$, where $\mathbf{e}$ and $\mathbf{h}$ are the respective cup diagrams from Example 1.2.2, consists of two line segments which are not propagating. ▲

Proposition 1.2.9. Let $\mathbf{a}, \mathbf{b} \in \mathbb{B}^{n-k,k}$ be cup diagrams. We have $S_\mathbf{a} \cap S_\mathbf{b} \neq \emptyset$ if and only if the following conditions hold:

(I1) Every connected component of $\overline{\mathbf{a}}\mathbf{b}$ contains an even number of markers.

(I2) Every line segment in $\overline{\mathbf{a}}\mathbf{b}$ is propagating.

Furthermore, if $S_\mathbf{a} \cap S_\mathbf{b} \neq \emptyset$, then there exists a homeomorphism $S_\mathbf{a} \cap S_\mathbf{b} \cong \left(\mathbb{S}^2\right)^{\mathrm{circ}}$, where circ denotes the number of closed connected components of $\overline{\mathbf{a}}\mathbf{b}$.

Proof. Assume that $S_\mathbf{a} \cap S_\mathbf{b} \neq \emptyset$ and let $(x_1, \ldots, x_m) \in S_\mathbf{a} \cap S_\mathbf{b}$. Consider a connected component of $\overline{\mathbf{a}}\mathbf{b}$ containing the vertices $i_1, \ldots, i_r$ which are ordered such that i_j and i_{j+1} are connected by a cup for all $j \in \{1, \ldots, r-1\}$. If the component is closed (resp. a line segment) the total number of cups on the component equals r (resp. $r-1$). We denote by c (resp. d) the total number of unmarked (resp. marked) cups of this component.

1) If the component is closed we have equalities $x_{i_j} = \pm x_{i_{j+1}}$ for all $j \in \{1, \ldots, r-1\}$ and $x_{i_r} = \pm x_{i_1}$, where the signs depends on whether the cup connecting the respective vertices is marked or unmarked. By successively inserting these equations into each other we obtain $x_{i_1} = (-1)^c x_{i_1} = (-1)^{r-d} x_{i_1}$. In order for this equation to hold, $r-d$ must be even. Since r is even for closed components we deduce that d is even which proves property (I1).

2) If the component is a line segment and i_1 is connected to the rightmost ray of either $\mathbf{a}$ or $\mathbf{b}$ we have $x_{i_1} \in \{\pm p\}$ and thus $x_{i_r} \in \{\pm p\}$ because $x_{i_1} = (-1)^{(r-1)-d} x_{i_r}$. This shows that i_r is also connected to a rightmost ray. In particular, this line segment is propagating which proves (I2).

   Since $r-1$ is even for a propagating line, $x_{i_1} = (-1)^{(r-1)-d} x_{i_r}$ reduces to $x_{i_1} = (-1)^d x_{i_r}$. If d is even, then both rays are either marked or unmarked, and if d is odd, then precisely one of the two rays is marked (otherwise this equation would not be true). In any case, the total number of markers on the line segment is even which shows (I1).

3) If the component is a line segment and i_1 is connected to a ray which is not a rightmost ray in $\mathbf{a}$ nor $\mathbf{b}$ we have $x_{i_1} = q$ and thus also $x_{i_r} = q$ because $x_{i_1} = (-1)^{(r-1)-d} x_{i_r}$ as in the previous case. This implies $(-1)^{(r-1)-d} = 1$. Moreover, $d = 0$ because we already know that the rightmost ray is connected to the rightmost ray. Thus, $r-1$ is even and hence the line is propagating.

In order to prove the implication "$\Leftarrow$" we assume that $\mathbf{a}, \mathbf{b} \in \mathbb{B}^{n-k,k}$ are such that (I1) and (I2) are true. Any element $(x_1, \ldots, x_m) \in S_\mathbf{a} \cap S_\mathbf{b}$ can be constructed according to the following rules:

1) Given a closed connected component of $\overline{\mathbf{a}}\mathbf{b}$ containing the vertices $i_1, \ldots, i_r$ (ordered such that successive vertices are connected by a cup) we fix an element $x_{i_1} \in \mathbb{S}^2$ and then define $x_{i_2}, \ldots, x_{i_r}$ by the relations imposed from $\mathbf{a}$ and $\mathbf{b}$. More precisely, under the assumption that $x_{i_1}, \ldots, x_{i_j}$ are already constructed we define $x_{i_{j+1}} = \pm x_{i_j}$, $j \in \{1, \ldots, r-1\}$, and proceed inductively.

   It remains to check that x_{i_1} and x_{i_r} satisfy the relation imposed by the cup connecting i_1 and i_r (by construction $x_{i_1}, \ldots, x_{i_r}$ automatically satisfy all the other relations imposed by $\overline{\mathbf{a}}\mathbf{b}$). We have $x_{i_1} = (-1)^{(r-1)-d'} x_{i_r} = -(-1)^{d'} x_{i_r}$, where d' denotes the number of marked cups on the component excluding the cup connecting i_1 and i_r. If d' is even (resp. odd) this cup must be unmarked (resp. marked) because of (I1). Hence, the equation gives the correct relation.

2) For every line segment of $\overline{\mathbf{a}}\mathbf{b}$ containing the vertices $i_1, \ldots, i_r$ (again, successive vertices are assumed to be connected by a cup) we define $x_{i_1} = q$ if i_1 is not connected to a rightmost ray and $x_{i_1} = -p$ (resp. $x_{i_1} = p$) if i_1 is connected to an unmarked (resp. marked) rightmost ray. We then define coordinates $x_{i_2}, \ldots, x_{i_r}$ by the relations coming from the cups in $\mathbf{a}$ and $\mathbf{b}$ (as in the case of a closed component).

   We need to check that x_{i_r} satisfies the relation imposed by the ray connected to i_r. We have $x_{i_1} = (-1)^{(r-1)-d} x_{i_r} = (-1)^d x_{i_r}$, where d is the number of marked cups of the line segment ($r-1$ is even because the line segment is propagating by (I2)).

   If $x_{i_1} = -p$ (resp. $x_{i_1} = p$), i.e. i_1 is connected to a rightmost ray which is unmarked (resp. marked), then i_r is connected to a rightmost ray as well (otherwise the diagram would not be crossingless) and we have to show that $x_{i_r} = -p$ (resp. $x_{i_1} = p$) if the ray connected to i_r is unmarked (resp. marked). In case that d is even and i_1 is marked (resp. unmarked), then i_r is also marked (resp. unmarked) and in case that d is even and i_1 is marked (resp. unmarked), then i_r is unmarked (resp. marked). In any case, everything is compatible.

   If $x_{i_1} = q$, i.e. i_1 is connected to a ray which is not a rightmost ray, then i_r is connected to a ray which is not a rightmost ray either and we have to check that $x_{i_r} = (-1)^d x_{i_1}$ equals q. Since there exists a propagating line in $\overline{\mathbf{a}}\mathbf{b}$ containing the two rightmost rays of $\mathbf{a}$ and $\mathbf{b}$ it follows that $d = 0$.

Observe that this construction does not only prove the implication "$\Leftarrow$" but it also shows that $S_{\mathbf{a}} \cap S_{\mathbf{b}}$ is homeomorphic to $\left(\mathbb{S}^2\right)^{\mathrm{circ}}$ if $S_{\mathbf{a}} \cap S_{\mathbf{b}} \neq \emptyset$. ■

Remark 1.2.10. If $\mathbf{a} \in \mathbb{B}^{n-k,k}_{\mathrm{even}}$ and $\mathbf{b} \in \mathbb{B}^{n-k,k}_{\mathrm{odd}}$ there exists a connected component of $\overline{\mathbf{a}}\mathbf{b}$ with an odd number of markers. Thus, it follows directly from Proposition 1.2.9 that $S_{\mathbf{a}} \cap S_{\mathbf{b}} = \emptyset$ because $\overline{\mathbf{a}}\mathbf{b}$ violates (I1). In particular, the topological Springer fiber $\mathcal{S}^{n-k,k}_D$ always decomposes as the disjoint union of the spaces $\mathcal{S}^{n-k,k}_{D,\mathrm{even}} := \bigcup_{\mathbf{a} \in \mathbb{B}^{n-k,k}_{\mathrm{even}}} S_{\mathbf{a}}$ and $\mathcal{S}^{n-k,k}_{D,\mathrm{odd}} := \bigcup_{\mathbf{a} \in \mathbb{B}^{n-k,k}_{\mathrm{odd}}} S_{\mathbf{a}}$. ▲

1.2.3 Some combinatorial bijections

We finish this section by describing some combinatorial bijections which already foreshadow the existence of the homeomorphisms in Theorem 1.4.12 and Theorem 1.4.13 on a purely combinatorial level.

Let $(n-k, k)$ be a two-row partition (not necessarily admissible of type C or D) and let T be a standard Young tableau of shape $(n-k, k)$. Let $\psi(T)$ be the unique undecorated cup diagram on n vertices whose left endpoints of cups are precisely the k entries in the lower row of T.

Lemma 1.2.11 ([61, Proposition 3])**.** The assignment ψ defines a bijection

$$\psi\colon SYT(n-k,k) \overset{1:1}{\longleftrightarrow} \left\{\begin{matrix}\text{undecorated cup diagrams}\\ \text{on } n \text{ vertices with } k \text{ cups}\end{matrix}\right\} \tag{1.2}$$

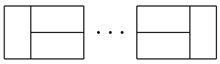
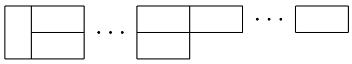
□

Example 1.2.12. Via (1.2) the five standard Young tableaux in Example 1.1.6 correspond (in the same order) to the following undecorated cup diagrams on five vertices with two cups:

▲

In the following we will establish a bijection between signed domino tableaux and cup diagrams similar to the one between standard tableaux and undecorated cup diagrams (cf. [17, Section 5]), thereby providing a precise connection between the combinatorics of tableaux and the combinatorics of cup diagrams involved in the definition of the topological Springer fiber.

Note that condition (ADT3) implies that all horizontal dominoes of an admissible domino tableau of shape $(n-k,k)$ of type D have their left box in an even column. In particular, the domino diagram underlying a given signed domino tableau can be constructed by placing the following "basic building blocks" side by side:

(i)  (ii) 

The left type of building block, called a *closed cluster*, consists of a collection of horizontal dominos enclosed by two vertical dominoes. The vertical domino on the left lies in an odd column, whereas the right one lies in an even column. Note that it might happen that a closed cluster has no horizontal dominoes.

The right building block is called an *open cluster*. It consists of a vertical domino lying in an odd column and a bunch of horizontal dominoes to its right. There is at most one open cluster in a given signed domino tableau and if it exists it has to be the rightmost building block.

When decomposing a signed domino tableau into clusters we number them from right to left.

Example 1.2.13. The signed domino tableau

19	18	17	14	13	12	11	10	8	6	5	4	3	1
−	16	15		−		+	9	7		−	2		

is built out of the following four clusters:

$$\mathcal{C}_1 = \begin{array}{|c|c|c|c|}\hline 5 & 4 & 3 & 1 \\ - & 2 & & \\ \hline\end{array},\ \mathcal{C}_2 = \begin{array}{|c|c|c|c|}\hline 11 & 10 & 8 & 6 \\ + & 9 & 7 & \\ \hline\end{array},\ \mathcal{C}_3 = \begin{array}{|c|c|}\hline 13 & 12 \\ - & \\ \hline\end{array},\ \mathcal{C}_4 = \begin{array}{|c|c|c|c|}\hline 19 & 18 & 17 & 14 \\ - & 16 & 15 & \\ \hline\end{array}.$$

▲

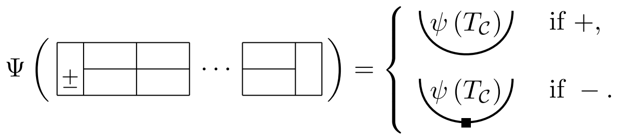
In order to understand the relationship between cup diagrams and signed domino tableaux we first explain how to assign a cup diagram to a cluster of a given tableau T.

Let $\mathcal{C}$ be a cluster (open or closed) of T and consider its *standard tableau part*, i.e. the part of the cluster which remains after removing all vertical dominoes. By viewing the horizontal dominoes as boxes of a Young tableau[1] one can use bijection (1.2) to assign an undecorated cup diagram to the standard tableau part $T_{\mathcal{C}}$ of $\mathcal{C}$.

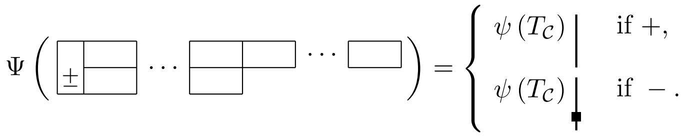
The cup diagram associated with the entire cluster is constructed as follows:

[1]If $\mathcal{C}$ is a closed cluster one has to subtract the number contained in the right vertical domino from the numbers of the horizontal dominoes in order to obtain a correct filling.

- If $\mathcal{C}$ is a closed cluster then we enclose the cup diagram corresponding to the standard tableau part by an additional cup. This cup is marked if and only if the left vertical domino of the cluster has sign $-$:

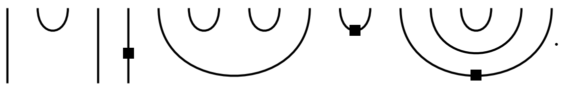


- If $\mathcal{C}$ is an open cluster we add a ray to the right side of the diagram corresponding to the standard tableau part. This ray is marked if and only if the vertical domino of the cluster has sign $-$:



Finally, we define $\Psi(T)$ as the cup diagram obtained by placing the cup diagrams $\Psi(\mathcal{C}_i)$ associated with each cluster side by side, starting with $\Psi(\mathcal{C}_1)$ as the leftmost piece, followed by $\Psi(\mathcal{C}_2)$ to its right, etc. This is clearly a well-defined cup diagram. Note that the numbers contained in the horizontal dominoes in the lower row and the vertical dominoes in an odd column are precisely the left endpoints of the cups in $\Psi(T)$.

Example 1.2.14. To the signed domino tableau from Example 1.2.13, Ψ assigns the cup diagram



▲

Lemma 1.2.15 ([17, Lemma 5.12])**.** Let $(n-k,k)$ be an admissible partition of type D. The assignment Ψ explained above defines bijections

$$ADT_D^{\mathrm{sgn}}(n-k,k) \xleftrightarrow{1:1} \mathbb{B}^{n-k,k}$$

$$ADT_{D,\mathrm{odd}}^{\mathrm{sgn}}(n-k,k) \xleftrightarrow{1:1} \mathbb{B}_{\mathrm{odd}}^{n-k,k}.$$

□

Lemma 1.2.16. Deleting the leftmost vertical domino in a signed domino tableau of shape $(n-k,k)$ gives rise to bijections

$$ADT_{D,\mathrm{odd}}^{\mathrm{sgn}}(n-k,k) \xleftrightarrow{1:1} ADT_C^{\mathrm{sgn}}(n-k-1,k-1)$$

$$ADT_{D,\mathrm{even}}^{\mathrm{sgn}}(n-k,k) \xleftrightarrow{1:1} ADT_C^{\mathrm{sgn}}(n-k-1,k-1).$$

Proof. This follows easily from the above. ■

1.3 A smooth variety containing the algebraic Springer fiber

Let $N > 0$ be a large integer (cf. Remark 1.3.2 for a more accurate description of what is meant by “large”) and let $z\colon \mathbb{C}^{2N} \to \mathbb{C}^{2N}$ be a nilpotent linear endomorphism with two Jordan blocks of equal size, i.e. there exists a Jordan basis

$$e_1 \leftarrow e_2 \leftarrow \dots \leftarrow e_N \qquad\qquad f_1 \leftarrow f_2 \leftarrow \dots \leftarrow f_N. \tag{1.3}$$

on which z acts as indicated (the vectors e_1 and f_1 are sent to zero). We equip $\mathbb{C}^{2N}$ with a hermitian structure by declaring $e_1, \dots, e_N, f_1, \dots, f_N$ to be an orthonormal basis.

Let e, f be the standard basis of $\mathbb{C}^2$ and let $C\colon \mathbb{C}^{2N} \to \mathbb{C}^2$ be the linear map defined by $C(e_i) = e$ and $C(f_i) = f$, $i \in \{1, \dots, N\}$. Note that $\mathbb{C}^2$ has the structure of a unitary vector space coming from the standard Hermitian inner product.

The following lemma is well known [11, Lemma 2.2] (cf. also [69, Lemma 2.1]).

Lemma 1.3.1. Let $U \subseteq \mathbb{C}^{2N}$ be a z-stable subspace, i.e. $zU \subseteq U$, such that $U \subseteq \mathrm{im}(z)$. Then C restricts to a unitary isomorphism $C : z^{-1}U \cap U^{\perp} \xrightarrow{\cong} \mathbb{C}^2$. □

Following [11], we define a smooth projective variety

$$Y_m := \left\{ (F_1, \dots, F_m) \mid F_i \subseteq \mathbb{C}^{2N} \text{has dimension } i,\ F_1 \subseteq \dots \subseteq F_m,\ zF_i \subseteq F_{i-1} \right\}.$$

Remark 1.3.2. Note that the conditions $zF_i \subseteq F_{i-1}$ imply

$$F_m \subseteq z^{-1}F_{m-1} \subseteq \dots \subseteq z^{-m}(0) = \mathrm{span}(e_1, \dots, e_m, f_1, \dots, f_m).$$

In particular, the variety Y_m is independent of the choice of N as long as $N \geq m$ and we can always assume (by increasing N if necessary) that all the subspaces of a flag in Y_m are contained in the image of z. ▲

Proposition 1.3.3 ([11, Theorem 2.1])**.** The map $\phi_m : Y_m \to (\mathbb{P}^1)^m$ defined by

$$(F_1, \dots, F_m) \mapsto \big(C(F_1), C(F_2 \cap F_1^{\perp}), \dots, C(F_m \cap F_{m-1}^{\perp})\big)$$

is a diffeomorphism. □

We fix a partition $(n-k, k)$ of $n = 2m > 0$ labeling a nilpotent orbit of type D, $1 \leq k \leq m$. Let $E_{n-k,k} \subseteq \mathbb{C}^{2N}$ be the subspace spanned by $e_1, \dots, e_{n-k}, f_1, \dots, f_k$. We equip $E_{n-k,k}$ with a bilinear form $\beta_D^{n-k,k}$ defined as follows:

- If $k = m$, we define for all $j, j' \in \{1, \dots, k\}$:
$$\beta_D^{k,k}(e_{j'}, f_j) = \beta_D^{k,k}(f_j, e_{j'}) = (-1)^{j-1}\delta_{j+j',k+1},$$
$$\beta_D^{k,k}(e_j, e_{j'}) = 0 \qquad \text{and} \qquad \beta_D^{k,k}(f_j, f_{j'}) = 0.$$
- If $k < m$, we define
$\beta_D^{n-k,k}(e_i, f_j) = 0\,,\ \beta_D^{n-k,k}(e_i, e_{i'}) = (-1)^{i-1}\delta_{i+i',n-k+1}\,,\ \beta_D^{n-k,k}(f_j, f_{j'}) = (-1)^{j}\delta_{j+j',k+1},$
for all $i, i' \in \{1, \dots, n-k\}$ and $j, j' \in \{1, \dots, k\}$.

Note that the bilinear form $\beta_D^{n-k,k}$ is nondegenerate and symmetric. Moreover, a straightforward computation shows that $\beta_D^{n-k,k}(z(v), w) = -\beta_D^{n-k,k}(v, z(w))$ for all $v, w \in E_{n-k,k}$, i.e. the restriction $z_{n-k,k}$ of z to the subspace $E_{n-k,k}$ is a nilpotent endomorphism in the orthogonal Lie algebra associated with $\beta_D^{n-k,k}$.

Similarly, we equip $E_{n-k-1,k-1} \subseteq \mathbb{C}^{2N}$, which is the span of $e_1, \dots, e_{n-k-1}, f_1, \dots, f_{k-1}$, with a bilinear form $\beta_C^{n-k-1,k-1}$ defined as follows:

- If $k = m$, we define for all $j, j' \in \{1, \dots, k-1\}$:
$$-\beta_C^{k-1,k-1}(e_{j'}, f_j) = \beta_C^{k-1,k-1}(f_j, e_{j'}) = (-1)^{j-1}\delta_{j+j',k},$$
$$\beta_C^{k-1,k-1}(e_j, e_{j'}) = 0 \qquad \text{and} \qquad \beta_C^{k-1,k-1}(f_j, f_{j'}) = 0.$$

- If $k < m$, we define $\beta_C^{n-k-1,k-1}(e_i, f_j) = 0$ and
$$\beta_C^{n-k-1,k-1}(e_i, e_{i'}) = \begin{cases} (-1)^i \delta_{i+i',n-k} & \text{if } i < i', \\ (-1)^{i-1} \delta_{i+i',n-k} & \text{if } i > i', \end{cases}$$
$$\beta_C^{n-k-1,k-1}(f_j, f_{j'}) = \begin{cases} (-1)^{j-1} \delta_{j+j',k} & \text{if } j < j', \\ (-1)^j \delta_{j+j',k} & \text{if } j > j', \end{cases}$$
for all $i, i' \in \{1, \ldots, n-k-1\}$ and $j, j' \in \{1, \ldots, k-1\}$.

Note that the bilinear form $\beta_C^{n-k-1,k-1}$ is nondegenerate and symplectic. We also see that the restriction $z_{n-k-1,k-1}$ of z to $E_{n-k-1,k-1}$ is contained in the symplectic Lie algebra associated with $\beta_C^{n-k-1,k-1}$. The following observation is now evident:

Lemma 1.3.4. We can view the Springer fiber $\mathcal{F}l_D^{n-k,k}$ as a subvariety of Y_m via the following identification
$$\mathcal{F}l_D^{n-k,k} \cong \left\{ (F_1, \ldots, F_m) \in Y_m \;\middle|\; \begin{array}{l} F_m \text{ is contained in } E_{n-k,k} \text{ and} \\ \text{isotropic with respect to } \beta_D^{n-k,k} \end{array} \right\} \tag{1.4}$$
and similarly
$$\mathcal{F}l_C^{n-k-1,k-1} \cong \left\{ (F_1, \ldots, F_{m-1}) \in Y_m \;\middle|\; \begin{array}{l} F_{m-1} \text{ is contained in } E_{n-k-1,k-1} \text{ and} \\ \text{isotropic with respect to } \beta_C^{n-k-1,k-1} \end{array} \right\}. \tag{1.5}$$
□

From now on we will always write $\mathcal{F}l_D^{n-k,k}$ and $\mathcal{F}l_C^{n-k-1,k-1}$ for the embedded Springer varieties via identifications (1.4) and (1.5).

Lemma 1.3.5. Let $U \subseteq \mathbb{C}^{2N}$ be a subspace. If zU is contained in $E_{n-k-2,k-2}$ (resp. $E_{n-k-3,k-3}$) and isotropic with respect to $\beta_D^{n-k-2,k-2}$ (resp. $\beta_C^{n-k-3,k-3}$) then U is contained in $E_{n-k,k}$ (resp. $E_{n-k-1,k-1}$) and isotropic with respect to $\beta_D^{n-k,k}$ (resp. $\beta_C^{n-k-1,k-1}$).

Proof. We only prove the lemma for the type D case since the type C case works completely analogous. By combining the obvious inclusion $E_{n-k-1,k-1} \subseteq E_{n-k,k}$ with the inclusion
$$U \subseteq z^{-1}(zU) \subseteq z^{-1}(E_{n-k-2,k-2}) = E_{n-k-1,k-1} \tag{1.6}$$
we obtain $U \subseteq E_{n-k,k}$. Hence, it suffices to show that U is isotropic with respect to $\beta_D^{n-k,k}$.

Pick two arbitrary elements $v, w \in U \subseteq E_{n-k,k}$ and write
$$v = \sum_{i=1}^{n-k} \lambda_i e_i + \sum_{j=1}^{k} \mu_j f_j \qquad \text{and} \qquad w = \sum_{i=1}^{n-k} \nu_i e_i + \sum_{j=1}^{k} \xi_j f_j.$$
Note that $\lambda_{n-k} = \nu_{n-k} = 0$ and $\mu_k = \xi_k = 0$ because $v, w \in E_{n-k-1,k-1}$ by (1.6). A straightforward calculation (using the definition of the bilinear form) shows that
$$\beta_D^{n-k,k}(v, w) = \begin{cases} \sum_{i=2}^{k-1} (-1)^{i+1} (\xi_i \lambda_{k-i+1} + \mu_i \nu_{k-i+1}) & \text{if } k = m, \\ \sum_{i=2}^{n-k-1} (-1)^{i+1} \lambda_i \nu_{n-k+1-i} + \sum_{i=2}^{k-1} (-1)^i \mu_i \xi_{k+1-i} & \text{if } k < m. \end{cases} \tag{1.7}$$
In order to see that this is zero we apply z to v, w which yields
$$z(v) = \sum_{i=1}^{n-k-2} \lambda_{i+1} e_i + \sum_{j=1}^{k-2} \mu_{j+1} f_j \,, \; z(w) = \sum_{i=1}^{n-k-2} \nu_{i+1} e_i + \sum_{j=1}^{k-2} \xi_{j+1} f_j.$$

Another computation shows that

$$\beta_D^{n-k-2,k-2}(z(v),z(w)) = \begin{cases} \sum_{i=2}^{k-1}(-1)^i(\xi_i\lambda_{k-i+1}+\mu_i\nu_{k-i+1}) & \text{if } k=m, \\ \sum_{i=2}^{n-k-1}(-1)^i\lambda_i\nu_{n-k+1-i}+\sum_{i=2}^{k-1}(-1)^{i-1}\mu_i\xi_{k+1-i} & \text{if } k<m. \end{cases} \tag{1.8}$$

By comparing (1.7) and (1.8) we deduce that $\beta_D^{n-k,k}(v,w) = -\beta_D^{n-k-2,k-2}(z(v),z(w))$. Since, by assumption, zU is isotropic with respect to $\beta_D^{n-k-2,k-2}$, we know that the right hand side of this equality must be zero. This proves the lemma. ∎

1.4 Proof of the isomorphisms

In this section we prove the main results of this chapter, see Theorem 1.4.12 and Theorem 1.4.13. We fix an admissible partition $(n-k,k)$ of $n=2m>0$ of type D, $1 \le k \le m$.

1.4.1 The diffeomorphism $\gamma_{n-k,k}$

In this subsection we define the diffeomorphism $\gamma_{n-k,k}$ and compute the images of the submanifolds $S_{\mathbf{a}} \subseteq (\mathbb{S}^2)^m$ for all $\mathbf{a} \in \mathbb{B}^{n-k,k}$.

Consider the stereographic projection

$$\mathbb{R}^3 \supseteq \mathbb{S}^2 \setminus \{p\} \xrightarrow{\sigma} \mathbb{C}\,,\ (x,y,z) \mapsto \frac{x}{1-z} + \mathbf{i}\frac{y}{1-z}$$

and the map $\theta\colon \mathbb{C} \to \mathbb{P}^1 \setminus \{\mathrm{span}(e)\}$, $\lambda \mapsto \mathrm{span}(\lambda e + f)$, which can be combined to define a diffeomorphism

$$\gamma\colon \mathbb{S}^2 \to \mathbb{P}^1\,,\ (x,y,z) \mapsto \begin{cases} \theta(\sigma(x,y,z)) & \text{if } (x,y,z) \neq p, \\ \mathrm{span}(e) & \text{if } (x,y,z) = p. \end{cases}$$

This induces a diffeomorphism $\gamma_m\colon (\mathbb{S}^2)^m \to (\mathbb{P}^1)^m$ on the m-fold products by setting

$$\gamma_m(x_1,\ldots,x_m) := (\gamma(x_1),\ldots,\gamma(x_m))\,.$$

Moreover, consider the diffeomorphisms $s\colon \mathbb{S}^2 \to \mathbb{S}^2$, $(x,y,z) \mapsto (x,z,y)$, and $t\colon \mathbb{S}^2 \to \mathbb{S}^2$, $(x,y,z) \mapsto (z,y,x)$. These yield diffeomorphisms $s_m, t_m\colon (\mathbb{S}^2)^m \to (\mathbb{S}^2)^m$ by taking m-fold products of the respective maps.

We define the diffeomorphisms $\gamma_{n-k,k}\colon (\mathbb{S}^2)^m \to (\mathbb{P}^1)^m$ as follows:

$$\gamma_{n-k,k} := \begin{cases} \gamma_m & \text{if } m=k, \\ \gamma_m \circ t_m & \text{if } m-k \text{ is odd}, \\ \gamma_m \circ s_m & \text{if } m-k \text{ is even}. \end{cases}$$

Given a cup diagram, we write i—j (resp. i-■-j) if the vertices $i<j$ are connected by a cup (resp. marked cup) and i—| (resp. i-■-|) if there is a ray (resp. marked ray) attached to the vertex i. If $\mathbf{a} \in \mathbb{B}^{n-k,k}$ and $k \neq m$, let $\rho(\mathbf{a}) \in \{1,\ldots,m\}$ denote the vertex connected to the rightmost ray in $\mathbf{a}$.

Definition 1.4.1. Let $\mathbf{a} \in \mathbb{B}^{n-k,k}$ be a cup diagram. We define $T_{\mathbf{a}} \subseteq (\mathbb{P}^1)^m$ as the set consisting of all m-tuples $(l_1,\ldots,l_m) \in (\mathbb{P}^1)^m$ whose entries satisfy the following list of relations:

- If $k=m$, we impose the relations

  (R1) $l_i^\perp = l_j$ if i—j (R3) $x_i = \mathrm{span}(f)$ if i—|

  (R2) $l_i = l_j$ if i-■-j (R4) $x_i = \mathrm{span}(e)$ if i-■-| .

  for all $i,j \in \{1,\ldots,m\}$.

- If $k \neq m$, we impose the relations

  (R1') $l_i^{\perp} = l_j$ if i—j (R2') $l_i = l_j$ if i-■-j (R3') $l_i = \mathrm{span}(e)$ if i—|

  for all $i, j \in \{1, \ldots, m\} \setminus \{\rho(\mathbf{a})\}$ and the additional relation

  $$\text{(R4')}\ l_{\rho(\mathbf{a})} = \begin{cases} \mathrm{span}\left(e + (-1)^{\epsilon} f\right) & \text{if } m-k \text{ is even,} \\ \mathrm{span}\left(\mathbf{i}e + (-1)^{\epsilon} f\right) & \text{if } m-k \text{ is odd,} \end{cases}$$

  where $\epsilon = 0$ if $\rho(\mathbf{a})$-■-| and $\epsilon = 1$ if $\rho(\mathbf{a})$—|.

▲

Lemma 1.4.2. We have an equality of sets $\gamma_{n-k,k}(S_{\mathbf{a}}) = T_{\mathbf{a}}$ for every $\mathbf{a} \in \mathbb{B}^{n-k,k}$.

Proof. A careful comparison of the definitions in question directly proves the lemma. ■

1.4.2 Topology of the irreducible components

The goal of this subsection is to prove that the diffeomorphism

$$\left(\mathbb{S}^2\right)^m \xrightarrow{\gamma_{n-k,k}} \left(\mathbb{P}^1\right)^m \xrightarrow{\phi_m^{-1}} Y_m$$

maps each of the submanifolds $S_{\mathbf{a}} \subseteq \left(\mathbb{S}^2\right)^m$ onto an irreducible component of the Springer fiber $\mathcal{F}l_D^{n-k,k} \subseteq Y_m$. Moreover, if $k > 1$, we check that the composition

$$\left(\mathbb{S}^2\right)^m \xrightarrow{\gamma_{n-k,k}} \left(\mathbb{P}^1\right)^m \xrightarrow{\phi_m^{-1}} Y_m \twoheadrightarrow Y_{m-1},$$

where $\pi_m \colon Y_m \to Y_{m-1}$, $(F_1, \ldots, F_m) \mapsto (F_1, \ldots, F_{m-1})$, is the morphism of algebraic varieties which forgets the last vector space of a flag, maps the submanifolds $S_{\mathbf{a}}$ onto an irreducible component of $\mathcal{F}l_C^{n-k,k} \subseteq Y_{m-1}$. By Lemma 1.4.2 it suffices to prove the following.

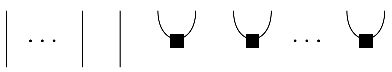
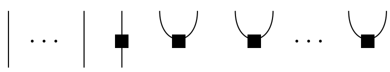
Proposition 1.4.3. The preimage $\phi_m^{-1}(T_{\mathbf{a}}) \subseteq Y_m$ is an irreducible component of the (embedded) Springer fiber $\mathcal{F}l_D^{n-k,k} \subseteq Y_m$ for all cup diagrams $\mathbf{a} \in \mathbb{B}^{n-k,k}$. Moreover, if $k > 1$, $\pi_m\left(\phi_m^{-1}\left(T_{\mathbf{a}}\right)\right) \subseteq Y_{m-1}$ is an irreducible component of the (embedded) Springer fiber $\mathcal{F}l_C^{n-k-1,k-1} \subseteq Y_{m-1}$ for all cup diagrams $\mathbf{a} \in \mathbb{B}^{n-k,k}_{\mathrm{odd}}$. □

In order to prove the above proposition (which will occupy most of the remaining section) we proceed by induction on the number of unmarked cups.

Proof of Proposition 1.4.3: Base case of the induction

In the following we prove Proposition 1.4.3 for all cup diagrams without unmarked cups contained in $\mathbb{B}^{n-k,k}$. It is useful to distinguish two different cases:

1. If k is odd, $\mathbb{B}^{n-k,k}$ contains precisely two such diagrams, namely

    and 

   which consist of $m-k+1$ rays followed by $\frac{k-1}{2}$ marked cups placed side by side.
2. If k is even (which implies $k = m$), there is precisely one such cup diagram in $\mathbb{B}^{k,k}$ (namely the one which consists of $\frac{k}{2}$ successive cups).

The following lemma treats the extremal case in which the cup diagram consists of rays only.

Lemma 1.4.4. The preimage $\phi_m^{-1}(T_{\mathbf{a}}) \subseteq Y_m$ is an irreducible component of the (embedded) Springer fiber $\mathcal{F}l_D^{n-1,1} \subseteq Y_m$ for all cup diagrams $\mathbf{a} \in \mathbb{B}^{n-1,1}$.

Proof. We distinguish between three different cases:

Let $m = 1$ and let $\mathbf{a} \in \mathbb{B}^{1,1}$ be a cup diagram. By Definition 1.4.1 we have $T_{\mathbf{a}} = \{\mathrm{span}(f)\}$ (resp. $T_{\mathbf{a}} = \{\mathrm{span}(e)\}$) if the ray is unmarked (resp. marked). The preimage of $T_{\mathbf{a}}$ under the diffeomorphism

$$\phi_1 \colon \left\{ \begin{matrix} \text{one-dimensional subspaces} \\ F_1 \subset \mathbb{C}^{2N} \text{ contained in } \ker(z) \end{matrix} \right\} \xrightarrow{\cong} \mathbb{P}^1 \,, \; F_1 \mapsto C(F_1)$$

is $\phi_1^{-1}(T_{\mathbf{a}}) = \{\mathrm{span}(f_1)\}$ (resp. $\phi_1^{-1}(T_{\mathbf{b}}) = \{\mathrm{span}(e_1)\}$) because $\phi_1(\mathrm{span}(f_1)) = C(\mathrm{span}(f_1)) = \mathrm{span}(f)$ and $C(\mathrm{span}(e_1)) = \mathrm{span}(e)$. Note that $\mathrm{span}(f_1), \mathrm{span}(e_1) \subseteq E_{1,1}$ are clearly isotropic with respect to $\beta_{1,1}$. Hence, $\phi_1^{-1}(T_{\mathbf{a}})$ is an irreducible component of the (embedded) Springer fiber $\mathcal{F}l_D^{1,1}$.

Assume that $m > 1$ is odd and fix a cup diagram $\mathbf{a} \in \mathbb{B}^{n-1,1}$. Then $T_{\mathbf{a}}$ consists of a single element $(\mathrm{span}(e), \dots, \mathrm{span}(e), \mathrm{span}(e + (-1)^{\epsilon} f)) \in \left(\mathbb{P}^1\right)^m$, where $\epsilon = 0$ if the rightmost ray is marked and $\epsilon = 1$ if it is unmarked. Let $(F_1, \dots, F_m) \in Y_m$ be the preimage of this element under the diffeomorphism ϕ_m. Since $F_1 \subset \ker(z)$ we have $F_1 = \mathrm{span}(\lambda e_1 + \mu f_1)$ for some $\lambda, \mu \in \mathbb{C}$, $\lambda \neq 0$ or $\mu \neq 0$. The condition $C(F_1) = \mathrm{span}(e)$ together with

$$C(F_1) = C\left(\mathrm{span}(\lambda e_1 + \mu f_1)\right) = \mathrm{span}(\lambda e + \mu f)$$

implies $\mu = 0$ and hence $F_1 = \mathrm{span}(e_1)$. By induction we assume that we have already shown

$$F_1 = \mathrm{span}(e_1), \dots, F_j = \mathrm{span}(e_1, \dots, e_j), \dots, F_i = \mathrm{span}(e_1, \dots, e_i),$$

for some $i \in \{1, \dots, m-1\}$. If $i < m-1$ we are looking for F_{i+1} such that $zF_{i+1} \subset F_i$ and

$$\mathrm{span}(e) = C(F_{i+1} \cap F_i^{\perp}) = C\left(F_{i+1} \cap \mathrm{span}(e_{i+1}, \dots, e_{2m-1}, f_1)\right).$$

By Lemma 1.3.1 this subspace is unique. Since $\mathrm{span}(e_1, \dots, e_i, e_{i+1})$ satisfies these properties we deduce $F_{i+1} = \mathrm{span}(e_1, \dots, e_i, e_{i+1})$. If $i = m-1$, we replace the condition $\mathrm{span}(e) = C(F_{i+1} \cap F_i^{\perp})$ by $\mathrm{span}(e + (-1)^{\epsilon} f) = C(F_{i+1} \cap F_i^{\perp})$. Note that $\mathrm{span}(e_1, \dots, e_{m-1}, e_m + (-1)^{\epsilon} f_1)$ is a possible choice. Hence, we deduce that this is F_m.

In order to check that $F_1, \dots, F_m$ are isotropic with respect to $\beta_D^{n-1,1}$ we note that the Gram matrix of $\beta_D^{n-1,1}$ restricted to the span of the vectors $e_1, \dots, e_m, f_1$ is given by

$$\left(\begin{array}{cccc|c} 0 & & & 0 & \\ & & & & \\ & & & 0 & \\ 0 & \dots & 0 & 1 & \\ \hline & & & & -1 \end{array}\right) \tag{1.9}$$

Thus, the vectors $e_1, \dots, e_{m-1}$ are pairwise orthogonal which shows that $F_1, \dots, F_{m-1}$ are isotropic. Since $e_m + (-1)^{\epsilon} f_1$ is clearly orthogonal to $e_1, \dots, e_{m-1}$ it suffices to compute

$$\beta_D^{n-1,1}(e_m + (-1)^{\epsilon} f_1, e_m + (-1)^{\epsilon} f_1) = \beta_D^{n-1,1}(e_m, e_m) + \beta_D^{n-1,1}(f_1, f_1) = 1 + (-1) = 0.$$

This shows that F_m is isotropic, too.

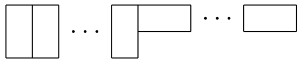
If m is even, we have $T_{\mathbf{a}} = \{(\mathrm{span}(e), \dots, \mathrm{span}(e), \mathrm{span}(\mathbf{i}e + (-1)^{\epsilon} f))\}$ and obtain

$$F_1 = \mathrm{span}(e_1), \dots, F_{m-1} = \mathrm{span}(e_1, \dots, e_{m-1}), F_m = \mathrm{span}(e_1, \dots, e_{m-1}, \mathbf{i}e_m + (-1)^{\epsilon} f_1)$$

by arguing similarly as in the case in which m is odd. Note that the two non-zero entries in the Gram matrix (1.9) are both -1 if m is even. Hence, the additional factor of $\mathbf{i}$ in front of e_m guarantees that F_m is again isotropic. ■

Proposition 1.4.5 (Special Case of Proposition 1.4.3)**.** Let $\mathbf{a} \in \mathbb{B}^{n-k,k}$ be a cup diagram without unmarked cups, $k > 1$.

1. The preimage $\phi_m^{-1}(T_{\mathbf{a}}) \subseteq Y_m$ is an irreducible component of $\mathcal{F}l_D^{n-k,k} \subseteq Y_m$ contained in the closure of $\left(S_D^{z_{n-k,k}}\right)^{-1}(T)$, where $T \in ADT_D(n-k,k)$ is the D-admissible domino tableau obtained by forgetting the signs of $\Psi^{-1}(\mathbf{a})$, i.e. the domino diagram



   consisting of k vertical dominoes followed by $m-k$ successive horizontal dominoes in the first row together with the unique filling such that (ADT1)-(ADT3) are satisfied.

2. Moreover, $\pi_m(\phi_m^{-1}(T_{\mathbf{a}})) \subseteq Y_{m-1}$ is an irreducible component of $\mathcal{F}l_C^{n-k-1,k-1} \subseteq Y_{m-1}$ contained in the closure of $\left(S_C^{z_{n-k-k,k-1}}\right)^{-1}(T')$, where $T' \in ADT_C(n-k-1,k-1)$ is obtained from T by deleting the leftmost vertical domino.

Convention 1.4.6. Let $U \subseteq E_{n-k-1,k-1}$ be a subspace. In the following we write $U^{\perp_D}$ (resp. $U^{\perp_C}$) to denote the orthogonal complement of U with respect to $\beta_D^{n-k,k}$ (resp. $\beta_C^{n-k-1,k-1}$). We write $U^{\perp}$ to denote the orthogonal complement of U in $\mathbb{C}^{2N}$ with respect to the hermitian structure of $\mathbb{C}^{2N}$. ▲

For the proofs of Lemma 1.4.8 and Lemma 1.4.9 below it is useful to introduce a technical definition.

Definition 1.4.7. Let $(F_1, \dots, F_m) \in Y_m$ be a flag such that $F_i \subseteq E_{n-k-1,k-1}$ and F_i is isotropic with respect to both $\beta_D^{n-k,k}$ and $\beta_C^{n-k-1,k-1}$. Moreover, assume that we have Jordan types

$$J(z_{n-k,k}^{(i)}) = (k-i, k-i), \qquad J(z_{n-k-1,k-1}^{(i)}) = (k-i-1, k-i-1),$$

where $z_{n-k,k}^{(i)}$ is the endomorphism of $F_i^{\perp_D}/F_i$ induced by $z_{n-k,k}$ and $z_{n-k-1,k-1}^{(i)}$ is the endomorphism of $F_i^{\perp_C}/F_i$ induced by $z_{n-k-1,k-1}$.

A collection of linearly independent vectors in $\mathbb{C}^{2N}$

$$\begin{array}{ccccccc} e_1^{(i)}, & e_2^{(i)}, & \dots & , & e_{k-i-1}^{(i)}, & e_{k-i}^{(i)}, \\ f_1^{(i)}, & f_2^{(i)}, & \dots & , & f_{k-i-1}^{(i)}, & f_{k-i}^{(i)}, \end{array} \tag{1.10}$$

where z maps each vector to its left neighbor (the leftmost vectors in each row are sent to F_i) is called a *simultaneous Jordan system for* $z_{n-k,k}^{(i)}$ *and* $z_{n-k-1,k-1}^{(i)}$ if their residue classes (modulo F_i) form a Jordan basis of $z_{n-k,k}^{(i)}$ and $z_{n-k-1,k-1}^{(i)}$ (the latter after excluding $e_{k-i}^{(i)}$ and $f_{k-i}^{(i)}$).

A simultaneous Jordan system as above is called *special* if the following additional properties hold:

(SJS1) The restriction of $\beta_D^{n-k,k}$ to the simultaneous Jordan system (1.10) is given by the formulae

$$\beta_D^{n-k,k}\left(f_j^{(i)}, e_{j'}^{(i)}\right) = (-1)^{j-1}\delta_{j+j',k-i+1}$$

for all $j, j' \in \{1, \dots, k-i\}$. Moreover, the restriction of $\beta_C^{n-k-1,k-1}$ is given by

$$-\beta_C^{n-k-1,k-1}\left(e_{j'}^{(i)}, f_j^{(i)}\right) = \beta_C^{n-k-1,k-1}\left(f_j^{(i)}, e_{j'}^{(i)}\right) = (-1)^{j-1}\delta_{j+j',m-i}$$

for all $j, j' \in \{1, \dots, k-i-1\}$.

(SJS2) The vectors in (1.10) form an orthonormal system with respect to the hermitian structure on $\mathbb{C}^{2N}$ and they are all contained in $F_i^{\perp}$.

(SJS3) We have equalities $C(e_j^{(i)}) = C(z(e_j^{(i)}))$ and $C(f_j^{(i)}) = C(z(f_j^{(i)}))$ for all $j \in \{2, \ldots, k-i\}$.

▲

Lemma 1.4.8. Assume that $k > 1$ is odd and let $\mathbf{a} \in \mathbb{B}^{n-k,k}$ be a cup diagram without unmarked cups and $(F_1, \ldots, F_m) \in \phi_m^{-1}(T_\mathbf{a})$.

Then $F_1, \ldots, F_{m-k+1}$ are contained in $E_{n-k-1,k-1}$ and isotropic with respect to $\beta_D^{n-k,k}$ as well as $\beta_C^{n-k-1,k-1}$, and we have

$$J(z_{n-k,k}^{(i)}) = (n-k-2i, k), \qquad J(z_{n-k-1,k-1}^{(i)}) = (n-k-2i-1, k-1),$$

for all $i \in \{1, \ldots, m-k\}$, and

$$J(z_{n-k,k}^{(m-k+1)}) = (k-1, k-1), \qquad J(z_{n-k-1,k-1}^{(m-k+1)}) = (k-2, k-2).$$

Furthermore, there exists a simultaneous Jordan system for $z_{n-k,k}^{(m-k+1)}$ and $z_{n-k-1,k-1}^{(m-k+1)}$ which is special (in the sense of Definition 1.4.7).

Proof. Let $(F_1, \ldots, F_m) \in \phi_m^{-1}(T_\mathbf{a})$, i.e. there exists $(l_1, \ldots, l_m) \in T_\mathbf{a}$ such that

$$C(F_1) = l_1,\ C(F_2 \cap F_1^\perp) = l_2,\ \ldots\ ,\ C(F_m \cap F_{m-1}^\perp) = l_m. \tag{1.11}$$

As in the proof of Lemma 1.4.4 we distinguish between three different cases:

If $k = m$, we have $l_1 = \operatorname{span}(f)$ if $\mathbf{a}$ has an unmarked ray and $l_1 = \operatorname{span}(e)$ if $\mathbf{a}$ has a marked ray. It follows directly from (1.11) that $F_1 = \operatorname{span}(f_1)$ or $F_1 = \operatorname{span}(e_1)$ both of which are obviously isotropic with respect to $\beta_D^{k,k}$ and $\beta_C^{k-1,k-1}$.

If $F_1 = \operatorname{span}(e_1)$, we have

$$F_1^{\perp_D} = \operatorname{span}(e_1, \ldots, e_k, f_1, \ldots, f_{k-1}) \qquad \text{and} \qquad F_1^{\perp_C} = \operatorname{span}(e_1, \ldots, e_{k-1}, f_1, \ldots, f_{k-2})$$

which immediately yields the proposed Jordan types of $z_{k,k}^{(1)}$ and $z_{k-1,k-1}^{(1)}$. Note that the vectors

$$e_j^{(1)} := e_{j+1} \text{ and } f_j^{(1)} := f_j, j \in \{1, \ldots, k-1\}$$

form a special simultaneous Jordan system for $z_{k,k}^{(1)}$ and $z_{k-1,k-1}^{(1)}$. Similarly, if $F_1 = \operatorname{span}(f_1)$, we have $F_1^{\perp_D} = \operatorname{span}(e_1, \ldots, e_{k-1}, f_1, \ldots, f_k)$ as well as $F_1^{\perp_C} = \operatorname{span}(e_1, \ldots, e_{k-2}, f_1, \ldots, f_{k-1})$ and the vectors

$$e_j^{(1)} := f_{j+1} \text{ and } f_j^{(1)} := e_j, j \in \{1, \ldots, k-1\}$$

form a special simultaneous Jordan basis for $z_{k,k}^{(1)}$ and $z_{k-1,k-1}^{(1)}$.

If $k \neq m$ and $m-k$ is even, then we have

$$l_1 = \operatorname{span}(e), \ldots, l_{m-k} = \operatorname{span}(e), l_{m-k+1} = \operatorname{span}(e + (-1)^\epsilon f)$$

where $\epsilon = 0$ if the rightmost ray is marked and $\epsilon = 1$ if it is unmarked. By arguing as in Lemma 1.4.4 we obtain $F_i = \operatorname{span}(e_1, \ldots, e_i)$, $1 \leq i \leq m-k$, and

$$F_{m-k+1} = \operatorname{span}\left(e_1, \ldots, e_{m-k}, e_{m-k+1} + (-1)^\epsilon f_1\right)$$

Note that F_{m-k+1} is indeed isotropic with respect to both $\beta_D^{n-k,k}$ and $\beta_C^{n-k-1,k-1}$ (the vectors $e_1, \ldots, e_{m-k+1}, f_1$ are isotropic and pairwise orthogonal with respect to either of the two forms).

In order to check the Jordan types first note that $F_i^{\perp_D} = \operatorname{span}(e_1, \ldots, e_{n-k-i}, f_1, \ldots, f_k)$ and $F_i^{\perp_C} = \operatorname{span}(e_1, \ldots, e_{n-k-i-1}, f_1, \ldots, f_{k-1})$ for $i \in \{1, \ldots, m-k\}$. The residue classes of the vectors $e_{i+1}, \ldots, e_{n-k-i}, f_1, \ldots, f_k$ clearly form a Jordan basis of $z_{n-k,k}^{(i)}$ of the correct type. Similarly, we obtain a Jordan basis of $z_{n-k-1,k-1}^{(i)}$ after deleting e_{n-k-i} and f_k. Furthermore,

$F_{m-k+1}^{\perp_D}$ is spanned by the linearly independent vectors

$$e_1,\ e_2,\ \ldots,\ e_{m-k},\quad \begin{array}{l} e_{m-k+1}+(-1)^{\epsilon}f_1,\ \ldots\ e_{m-1}+(-1)^{\epsilon}f_{k-1} \\ e_{m-k+1}-(-1)^{\epsilon}f_1,\ \ldots\ e_{m-1}-(-1)^{\epsilon}f_{k-1},\ e_m-(-1)^{\epsilon}f_k. \end{array} \tag{1.12}$$

Note that z sends a vector to its left neighbor (the leftmost vectors in the first and third row are sent to the rightmost vector in the second row). Furthermore, $F_{m-k+1}^{\perp_C}$ is spanned by the same vectors excluding $e_m-(-1)^{\epsilon}f_k$ and $e_{m-1}+(-1)^{\epsilon}f_{k-1}$. It follows directly from (1.12) that $z_{n-k,k}^{(m-k+1)}$ and $z_{n-k-1,k-1}^{(m-k+1)}$ have the correct Jordan type. It is straightforward to check that the vectors

$$e_j^{(m-k+1)} := \frac{1}{\sqrt{2}}\left(e_{m-k+1+j}+(-1)^{\epsilon}f_{j+1}\right) \text{ and } f_j^{(m-k+1)} := \frac{1}{\sqrt{2}}\left(e_{m-k+j}-(-1)^{\epsilon}f_j\right),$$

where $j\in\{1,\ldots,k-1\}$, form a special simultaneous Jordan system.

If $m-k$ is odd, we similarly get $F_i=\operatorname{span}(e_1,\ldots,e_i)$, $i\in\{1,\ldots,m-k\}$, and

$$F_{m-k+1}=\operatorname{span}\left(e_1,\ldots,e_{m-k},\mathbf{i}e_{m-k+1}+(-1)^{\epsilon}f_1\right)$$

which is isotropic with respect to $\beta_D^{n-k,k}$ and $\beta_C^{n-k-1,k-1}$ (again, $e_1,\ldots,e_{m-k+1},f_1$ are isotropic and pairwise orthogonal). As in the case in which $m-k$ is even one can show that the maps $z_{n-k,k}^{(i)}$ have the correct Jordan type for all $i\in\{1,\ldots,m-k+1\}$. We omit the details and claim that

$$e_j^{(m-k+1)} := \frac{1}{\sqrt{2}}\left(\mathbf{i}e_{m-k+1+j}+(-1)^{\epsilon}f_{j+1}\right) \text{ and } f_j^{(m-k+1)} := \frac{1}{\sqrt{2}}\left(\mathbf{i}e_{m-k+j}-(-1)^{\epsilon}f_j\right),$$

where $j\in\{1,\ldots,k-1\}$, yields a special Jordan system. ■

Lemma 1.4.9. Let $\mathbf{a}\in\mathbb{B}^{n-k,k}$ be a cup diagram without unmarked cups and let $(F_1,\ldots,F_m)\in\phi_m^{-1}(T_{\mathbf{a}})$.

If $k>1$ is odd, then the vector spaces $F_{m-k+2},\ldots,F_m$ are isotropic with respect to $\beta_D^{n-k,k}$ and for all $i\in\{m-k+2,\ldots,m\}$ we have

$$J(z_{n-k,k}^{(i)})=(k-i,k-i).$$

Moreover, the vector spaces $F_{m-k+2},\ldots,F_{m-1}$ are isotropic with respect to $\beta_C^{n-k-1,k-1}$ and we have

$$J(z_{n-k-1,k-1}^{(i)})=(k-i-1,k-i-1)$$

for all $i\in\{m-k+2,\ldots,m-1\}$.

Furthermore, there exists a special simultaneous Jordan system for $z_{n-k,k}^{(i)}$ and $z_{n-k-1,k-1}^{(i)}$ for all $i\in\{m-k+3,m-k+5,\ldots,m-2\}$ (that is all right endpoints of cups except the rightmost one).

Proof. Let $(F_1,\ldots,F_m)\in\phi_m^{-1}(T_{\mathbf{a}})$, i.e. we have equalities

$$C(F_1)=l_1,\ C(F_2\cap F_1^{\perp})=l_2,\ \ldots,\ C(F_m\cap F_{m-1}^{\perp})=l_m.$$

for some $(l_1,\ldots,l_m)\in T_{\mathbf{a}}$. Let $i\in\{m-k+2,\ldots,m-1\}$ be a left endpoint of a marked cup in $\mathbf{a}$ and assume by induction that the claims of the lemma are true for $F_{m-k+2},\ldots,F_{i-1}$. The goal is to show the claim for F_i and F_{i+1}.

By induction (or by Lemma 1.4.8 if the cup connecting i and $i+1$ is the leftmost cup) there exists a special simultaneous Jordan system

$$\begin{array}{l} e_1^{(i-1)},\ e_2^{(i-1)},\ \ldots\ ,\ e_{k-i}^{(i-1)},\ e_{k-i+1}^{(i-1)}, \\ f_1^{(i-1)},\ f_2^{(i-1)},\ \ldots\ ,\ f_{k-i}^{(i-1)},\ f_{k-i+1}^{(i-1)}, \end{array} \tag{1.13}$$

for $z^{(i-1)}_{n-k,k}$ and $z^{(i-1)}_{n-k-1,k-1}$. Since $e^{(i-1)}_1$ and $f^{(i-1)}_1$ form an orthonormal basis of $z^{-1}F_{i-1} \cap F^{\perp}_{i-1}$it follows from Lemma 1.3.1 that $C(e^{(i-1)}_1)$ and $C(f^{(i-1)}_1)$ form an orthonormal basis of $\mathbb{C}^2$ and we can write $l_i = \operatorname{span}(C(e^{(i-1)}_1) + \mu^{(i)}C(f^{(i-1)}_1))$, where $\mu^{(i)} \in \mathbb{C}$, or $l_i = \operatorname{span}(C(f^{(i-1)}_1))$.

Check that F_i and F_{i+1} are isotropic: In order to determine the vector spaces F_i and F_{i+1} we first assume that $l_i = \operatorname{span}(C(e^{(i-1)}_1) + \mu^{(i)}C(f^{(i-1)}_1))$. Consider the z-invariant vector space $F_{i-1} \oplus \operatorname{span}(e^{(i-1)}_1 + \mu^{(i)} f^{(i-1)}_1)$. Note that

$$\begin{aligned} C\left(\left(F_{i-1} \oplus \operatorname{span}(e^{(i-1)}_1 + \mu^{(i)} f^{(i-1)}_1)\right) \cap F^{\perp}_{i-1}\right) &= C\left(\operatorname{span}(e^{(i-1)}_1 + \mu^{(i)} f^{(i-1)}_1)\right) \\ &= \operatorname{span}\left(C(e^{(i-1)}_1) + \mu^{(i)} C(f^{(i-1)}_1)\right) \end{aligned}$$

Hence, it follows that $F_i = F_{i-1} \oplus \operatorname{span}(e^{(i-1)}_1 + \mu^{(i)} f^{(i-1)}_1)$ because F_i is the unique z-invariant subspace satisfying $C(F_i \cap F^{\perp}_{i-1}) = l_i$ (cf. Lemma 1.3.1). Next consider the vector space $F_i \oplus \operatorname{span}(e^{(i-1)}_2 + \mu^{(i)} f^{(i-1)}_2)$. Again, this space is z-invariant and we have

$$\begin{aligned} C\left(\left(F_{i-1} \oplus \operatorname{span}(e^{(i-1)}_2 + \mu^{(i)} f^{(i-1)}_2)\right) \cap F^{\perp}_{i-1}\right) &= \operatorname{span}\left(C(e^{(i-1)}_2) + \mu^{(i)} C(f^{(i-1)}_2)\right) \\ &\overset{\text{((SJS3))}}{=} \operatorname{span}\left(C(z(e^{(i-1)}_2)) + \mu^{(i)} C(z(f^{(i-1)}_2))\right) \\ &= \operatorname{span}\left(C(e^{(i-1)}_1) + \mu^{(i)} C(f^{(i-1)}_1)\right). \end{aligned}$$

Hence, $F_{i+1} = F_i \oplus \operatorname{span}(e^{(i-1)}_2 + \mu^{(i)} f^{(i-1)}_2)$ because $C(F_{i+1} \cap F^{\perp}_i) = l_i$.

If $l_i = \operatorname{span}(C(f^{(i-1)}_1))$ we argue as above and obtain $F_i = F_{i-1} \oplus \operatorname{span}(f^{(i-1)}_1)$ as well as $F_{i+1} = F_i \oplus \operatorname{span}(f^{(i-1)}_2)$.

Using formula (SJS1) and the fact that the vectors in (1.13) are perpendicular to F_{i-1} it is now easy to deduce that F_i and F_{i+1} are isotropic with respect to $\beta^{n-k,k}_D$. Moreover, F_i and F_{i+1} are isotropic with respect to $\beta^{n-k-1,k-1}_C$ if $i < m-1$. If $i = m-1$ only F_i is isotropic with respect to $\beta^{n-k-1,k-1}_C$.

Jordan types and special Jordan systems: It remains to compute the Jordan types of the maps $z^{(i)}_{n-k,k}$, $z^{(i+1)}_{n-k,k}$ and construct a special Jordan system.

Assume that $F_i = F_{i-1} \oplus \operatorname{span}(e^{(i-1)}_1 + \mu^{(i)} f^{(i-1)}_1)$ with $\mu^{(i)} \neq 0$ and consider the following linearly independent vectors:

$$\begin{array}{c} e^{(i-1)}_1 - \mu^{(i)} f^{(i-1)}_1, e^{(i-1)}_2 - \mu^{(i)} f^{(i-1)}_2, \ldots, e^{(i-1)}_{m-(i-1)} - \mu^{(i)} f^{(i-1)}_{m-(i-1)} \\ e^{(i-1)}_1 + \mu^{(i)} f^{(i-1)}_1, e^{(i-1)}_2 + \mu^{(i)} f^{(i-1)}_2, \ldots, e^{(i-1)}_{m-(i-1)} + \mu^{(i)} f^{(i-1)}_{m-(i-1)} \end{array} \tag{1.14}$$

Again, note that z maps each vector to its left neighbor (the leftmost vectors are sent to F_{i-1}).

Note that $F^{\perp_D}_i$ is the direct sum of F_{i-1} and the span of all the vectors in (1.14) except the rightmost one in the first row. In particular, $z^{(i)}_{n-k,k}$ has the correct Jordan type. Similarly, we see that $F^{\perp_D}_{i+1}$ is the direct sum of F_{i-1} and the span of all vectors in (1.14) except the two rightmost ones in the first row. Hence, $z^{(i+1)}_{n-k,k}$ also has the correct Jordan type.

Moreover, $F^{\perp_C}_i$ is the direct sum of F_{i-1} and the span of all the vectors in (1.14) except the two rightmost ones in the first row and the last one in the second row. In particular, $z^{(i)}_{n-k-1,k-1}$ has the correct Jordan type. Similarly, if $i < m-1$, we see that $F^{\perp_C}_{i+1}$ is the direct sum of F_{i-1} and the span of all vectors in (1.14) except the three rightmost ones in the first and the two rightmost ones in the second row. Hence, $z^{(i+1)}_{n-k-1,k-1}$ also has the correct Jordan type.

In order to finish the induction step we define linearly independent vectors

$$e_j^{(i+1)} := \frac{1}{\sqrt{2\mu}}\left(e_j^{(i-1)} - \mu f_j^{(i-1)}\right) \quad \text{and} \quad f_j^{(i+1)} := \frac{1}{\sqrt{2\mu}}\left(e_{j+2}^{(i-1)} + \mu f_{j+2}^{(i-1)}\right)$$

for $j \in \{1, \ldots, m-i-1\}$. It is straightforward to check that these vectors form a special simultaneous Jordan system.

If $F_i = F_{i-1} \oplus \operatorname{span}(e_1^{(i-1)})$ or $F_i = F_{i-1} \oplus \operatorname{span}(f_1^{(i-1)})$ we see that $F_i^{\perp_D}$ is the direct sum of F_{i-1} and the span of all vectors in (1.13) except $f_{m-(i-1)}^{(i-1)}$, if $F_i = F_{i-1} \oplus \operatorname{span}(e_1^{(i-1)})$, resp. $e_{m-(i-1)}^{(i-1)}$, if $F_i = F_{i-1} \oplus \operatorname{span}(f_1^{(i-1)})$. Similarly, we have that $F_{i+1}^{\perp_D}$ is the direct sum of F_{i-1} and the span of all vectors in (1.13) except $f_{m-i}^{(i-1)}$ and $f_{m-(i-1)}^{(i-1)}$, if $F_i = F_{i-1} \oplus \operatorname{span}(e_1^{(i-1)})$, resp. $e_{m-i}^{(i-1)}$ and $e_{m-(i-1)}^{(i-1)}$, if $F_i = F_{i-1} \oplus \operatorname{span}(f_1^{(i-1)})$. In both cases we obtain the correct Jordan types for $z_{n-k,k}^{(i)}$ and $z_{n-k,k}^{(i+1)}$.

If $F_i = F_{i-1} \oplus \operatorname{span}(e_1^{(i-1)})$ or $F_i = F_{i-1} \oplus \operatorname{span}(f_1^{(i-1)})$ we see that $F_i^{\perp_C}$ is the direct sum of F_{i-1} and the span of all vectors in (1.13) except the rightmost one in the first (resp. second) row and the two rightmost ones in the second (resp. first) row if $F_i = F_{i-1} \oplus \operatorname{span}(e_1^{(i-1)})$ (resp. $F_i = F_{i-1} \oplus \operatorname{span}(f_1^{(i-1)})$). Similarly, we have that $F_{i+1}^{\perp_C}$ is the direct sum of F_{i-1} and the span of all vectors in (1.13) except the three rightmost vectors in the second (resp. first) row and the rightmost one in the first (resp. second) row if $F_i = F_{i-1} \oplus \operatorname{span}(e_1^{(i-1)})$ (resp. $F_i = F_{i-1} \oplus \operatorname{span}(f_1^{(i-1)})$). In both cases $z_{n-k-1,k-1}^{(i)}$ has Jordan type $(m-i-1, m-i-1)$ and $z_{n-k-1,k-1}^{(i+1)}$ has Jordan type $(m-i-2, m-i-2)$.

In order to finish we set $e_j^{(i+1)} := e_j^{(i-1)}$ and $f_j^{(i+1)} := f_{j+2}^{(i-1)}$ for $j \in \{1, \ldots, m-i-1\}$ in case $F_{i+1} = F_{i-1} \oplus \operatorname{span}(f_1^{(i-1)}, f_2^{(i-1)})$ and we set $e_j^{(i+1)} := e_{j+2}^{(i-1)}$ and $f_j^{(i+1)} := f_j^{(i-1)}$ for $j \in \{1, \ldots, m-i-1\}$ in case $F_{i+1} = F_{i-1} \oplus \operatorname{span}(e_1^{(i-1)}, e_2^{(i-1)})$. ■

Lemma 1.4.10. If k is even, then the vector spaces $F_1, \ldots, F_m$ are isotropic with respect to $\beta_D^{k,k}$ and

$$J(z_{k,k}^{(i)}) = (k-i, k-i)$$

for all $i \in \{1, \ldots, m\}$. Moreover, the vector spaces $F_1, \ldots, F_{m-1}$ are isotropic with respect to $\beta_C^{k-1,k-1}$ and we have

$$J(z_{k-1,k-1}^{(i)}) = (k-i-1, k-i-1)$$

for all $i \in \{1, \ldots, m-1\}$.

Proof. The lemma can be proven by an inductive construction similar to the one in the proof of Lemma 1.4.9. Note that this inductive construction starts with the Jordan basis $e_1, \ldots, e_k, f_1, \ldots, f_k$ of the restriction of z to $E_{k,k}$ as special Jordan system. ■

Proof (Proposition 1.4.5). By Lemma 1.4.8, 1.4.9 and 1.4.10 we see that the vector spaces of an arbitrary flag $(F_1, \ldots, F_m) \in \phi_m^{-1}(T_{\mathbf{a}})$ are isotropic with respect to $\beta_D^{n-k,k}$ and that the Jordan types of the sequence of endomorphisms $z_{n-k,k}^{(m-1)}, z_{n-k,k}^{(m-2)}, \ldots, z_{n-k,k}^{(1)}$ are the following

$$(1,1), (2,2), \ldots, (k-1,k-1), (k,k), (k+2,k), \ldots, (n-k,k),$$

if $k < m$, and

$$(1,1), (2,2), \ldots, (k-1,k-1), (k,k),$$

if $m = k$. Moreover, the vector spaces of the flag $(F_1, \ldots, F_{m-1}) = \pi_m(F_1, \ldots, F_m) \in \pi_m\left(\phi_m^{-1}(T_{\mathbf{a}})\right)$ are isotropic with respect to $\beta_C^{n-k-1,k-1}$ and the maps $z_{n-k-1,k-1}^{(m-2)}, z_{n-k-1,k-1}^{(m-2)}, \ldots, z_{n-k-1,k-1}^{(1)}$ have Jordan types

$$(1,1),(2,2),\ldots,(k-2,k-2),(k-1,k-1),(k+1,k-1),\ldots,(n-k-1,k-1),$$

if $k < m$, and

$$(1,1),(2,2),\ldots,(k-2,k-2),(k-1,k-1),$$

if $m = k$. Thus, it follows that $S_D^{z_{n-k,k}}(F_1, \ldots, F_m) = T$ as well as $S_C^{z_{n-k-1,k-1}}(F_1, \ldots, F_{m-1}) = T'$ and $(F_1, \ldots, F_m)$ is contained in the closure of $\left(S_D^{z_{n-k,k}}\right)^{-1}(T)$ and $(F_1, \ldots, F_{m-1})$ is contained in the closure of $\left(S_C^{z_{n-k-1,k-1}}\right)^{-1}(T')$. In particular, since $(F_1, \ldots, F_m)$ is an arbitrarily chosen flag in $\phi_m^{-1}(T_{\mathbf{a}})$, we have proven the inclusion of $\phi_m^{-1}(T_{\mathbf{a}})$ in the closure of $\left(S_D^{z_{n-k,k}}\right)^{-1}(T)$ and $\pi_m\left(\phi_m^{-1}(T_{\mathbf{a}})\right)$ in the closure of $\left(S_C^{z_{n-k-1,k-1}}\right)^{-1}(T')$.

Finally, note that $\phi_m^{-1}(T_{\mathbf{a}})$ is a closed subvariety of $\mathcal{F}l_D^{n-k,k}$ which is connected because it is the preimage of $T_{\mathbf{a}}$ (which is obviously connected) under a diffeomorphism. Thus, $\phi_m^{-1}(T_{\mathbf{a}})$ must be contained in precisely one of the irreducible components whose disjoint union is the closure of $(S_D^{z_{n-k,k}})^{-1}(T)$. Since the dimension of $\phi_m^{-1}(T_{\mathbf{a}})$ equals the dimension of $\mathcal{F}l^{n-k,k}$, it follows that $\phi_m^{-1}(T_{\mathbf{a}})$ equals the irreducible component of $\mathcal{F}l_D^{n-k,k}$ in which it is contained.

Hence, $\pi_m\left(\phi_m^{-1}(T_{\mathbf{a}})\right)$ is also an irreducible closed subvariety of $\mathcal{F}l_C^{n-k-1,k-1}$ of the correct dimension. ■

Proof of Proposition 1.4.3: Inductive step

In [11, Section 2] the authors introduce a smooth subvariety $X_m^i \subseteq Y_m$, $i \in \{1, \ldots, m-1\}$, defined by

$$X_m^i := \{(F_1, \ldots, F_m) \in Y_m \mid F_{i+1} = z^{-1}F_{i-1}\},$$

and a surjective morphism of varieties $q_m^i \colon X_m^i \twoheadrightarrow Y_{m-2}$ given by

$$(F_1, \ldots, F_m) \mapsto (F_1, \ldots, F_{i-1}, zF_{i+2}, \ldots, zF_m)\,.$$

We want to make use of the following lemma (cf. [11, Theorem 2.1] and [69, Lemma 2.4]):

Lemma 1.4.11. The diffeomorphism ϕ_m maps X_m^i bijectively to the set

$$A_m^i := \{(l_1, \ldots, l_m) \in (\mathbb{P}^1)^m \mid l_{i+1} = l_i^{\perp}\} \tag{1.15}$$

and we have a commutative diagram

$$\begin{tikzcd} X_m^i \arrow[r, two heads, "q_m^i"] \arrow[d, "\phi_m|_{X_m^i}"', "\cong"] & Y_{m-2} \arrow[d, "\phi_{m-2}", "\cong"'] \\ A_m^i \arrow[r, two heads, "f_m^i|_{A_m^i}"] & \left(\mathbb{P}^1\right)^{m-2} \end{tikzcd} \tag{1.16}$$

where $f_m^i \colon \left(\mathbb{P}^1\right)^m \twoheadrightarrow \left(\mathbb{P}^1\right)^{m-2}$ is the map which forgets the coordinates i and $i+1$. The orthogonal complement in (1.15) is taken with respect to the hermitian structure of $\mathbb{C}^2$. □

We prove the proposition by induction on the number of unmarked cups in $\mathbf{a}$. If there is no unmarked cup in $\mathbf{a}$, then the claim follows from Lemma 1.4.4 and Proposition 1.4.5. Hence, we may assume that there exists an unmarked cup in $\mathbf{a}$. Then there exists a cup connecting neighboring vertices i and $i+1$. Let $\mathbf{a}' \in \mathbb{B}^{n-k-2,k-2}$ be the cup diagram obtained by removing this cup.

We have

$$\left(q^i_m\right)^{-1}\left(\phi^{-1}_{m-2}(T_{\mathbf{a}'})\right)=\phi^{-1}_m\left(\left(f^i_m\right)^{-1}(T_{\mathbf{a}'})\right)=\phi^{-1}_m(T_{\mathbf{a}}), \tag{1.17}$$

where the first equality followes directly from the commutativity of the diagram (1.16) and the second one is the obvious fact that $\left(f^i_m\right)^{-1}(T_{\mathbf{a}'})=T_{\mathbf{a}}$. Thus, $\phi^{-1}_m(T_{\mathbf{a}})\subseteq Y_m$ is a closed subvariety because it is the preimage of the closed subvariety $\phi^{-1}_{m-2}(T_{\mathbf{a}'})$ (which is even an irreducible component of $\mathcal{F}l^{n-k-2,k-2}_D$ by induction) under the morphism q^i_m.

Thus, by (1.17), for any given flag $(F_1,\dots,F_m)\in\phi^{-1}_m(T_{\mathbf{a}})$, we have

$$q_{m,i}(F_1,\dots,F_m)=(F_1,\dots,F_{i-1},zF_{i+2},\dots,zF_m)\in\phi^{-1}_{m-2}(T_{\mathbf{a}'})$$

which shows that zF_m is contaid in $E_{n-k-2,k-2}$ and isotropic with respect to $\beta^{n-k-2,k-2}_D$ because $\phi_{m-2}(T_{\mathbf{a}'})\subseteq\mathcal{F}l^{n-k-2,k-2}_D$ by induction. Hence, by Lemma 1.3.5, F_m is contained in $E_{n-k,k}$ and isotropic with respect to $\beta^{n-k,k}_D$. Thus, we have proven that $\phi^{-1}_m(T_{\mathbf{a}})\subseteq Y_m$ is an algebraic subvariety contained in $\mathcal{F}l^{n-k,k}_D$.

Note that $\phi^{-1}_m(T_{\mathbf{a}})$ is smooth and connected because it is the preimage of $T_{\mathbf{a}}$ (which is obviously smooth and connected) under a diffeomorphism. This shows that $\phi^{-1}_m(T_{\mathbf{a}})$ is irreducible. Finally, the dimension of the variety $\phi^{-1}_m(T_{\mathbf{a}})$ obviously equals the dimension of $\mathcal{F}l^{n-k,k}_D$ (because the manifold $T_{\mathbf{a}}$ has the correct dimension). To sum up, $\phi^{-1}_m(T_{\mathbf{a}})$ must therefore be an irreducible component of the (embedded) Springer variety $\mathcal{F}l^{n-k,k}_D$.

Let $\mathbf{a}\in\mathbb{B}^{n-k,k}_{\text{odd}}$. If $\mathbf{a}$ only has one unmarked cup which connects the vertices $m-1$ and m it is easy to see that $\pi_m\left(\phi^{-1}_m(T_{\mathbf{a}})\right)$ equals $\pi_m\left(\phi^{-1}_m(T_{\mathbf{b}})\right)$, where $\mathbf{b}$ denotes the cup diagram obtained by exchanging the unmarked cup connecting $m-1$ and m with a marked cup. Then the claim follows from Proposition 1.4.5. Analogously, if $i\neq m-1$, by applying π_m to (1.17), it is easy to see that we also have an equality of sets

$$\pi_m\left(\phi^{-1}_m(T_{\mathbf{a}})\right)=\pi_m\left((q^i_m)^{-1}\left(\phi^{-1}_{m-2}(T_{\mathbf{a}'})\right)\right)=(q^i_{m-1})^{-1}\left(\pi_{m-2}\left(\phi^{-1}_{m-2}(T_{\mathbf{a}'})\right)\right).$$

Now the claim follows as for the type D Springer fiber. This finishes the proof of Proposition 1.4.3. □

1.4.3 Gluing the irreducible components

Now we state and prove the main results of this chapter.

Theorem 1.4.12. The diffeomorphism $\left(\mathbb{S}^2\right)^m\xrightarrow{\gamma_{n-k,k}}\left(\mathbb{P}^1\right)^m\xrightarrow{\phi^{-1}_m}Y_m$ restricts to a homeomorphism

$$\mathcal{S}^{n-k,k}_D\xrightarrow{\cong}\mathcal{F}l^{n-k,k}_D$$

such that the images of the $S_{\mathbf{a}}$ under this homeomorphism are precisely the irreducible components of $\mathcal{F}l^{n-k,k}_D$ for all $\mathbf{a}\in\mathbb{B}^{n-k,k}$.

Proof. We know that the image of $\mathcal{S}^{n-k,k}_D\subseteq\left(\mathbb{S}^2\right)^m$ under the diffeomorphism $\phi^{-1}_m\circ\gamma_{n-k,k}$ is given by

$$\phi^{-1}_m\left(\gamma_{n-k,k}\left(\mathcal{S}^{n-k,k}_D\right)\right)=\bigcup_{\mathbf{a}\in\mathbb{B}^{n-k,k}}\phi^{-1}_m\left(\gamma_{n-k,k}\left(S_{\mathbf{a}}\right)\right)\overset{\text{Lemma}}{\underset{1.4.2}{=}}\bigcup_{\mathbf{a}\in\mathbb{B}^{n-k,k}}\phi^{-1}_m\left(T_{\mathbf{a}}\right).$$

If $\mathbf{a}\neq\mathbf{b}$, we obviously have $T_{\mathbf{a}}\neq T_{\mathbf{b}}$ and thus also $\phi^{-1}_m(T_{\mathbf{a}})\neq\phi^{-1}_m(T_{\mathbf{b}})$, because ϕ_m is bijective. In combination with Proposition 1.4.3 this yields that $\bigcup_{\mathbf{a}\in\mathbb{B}^{n-k,k}}\phi^{-1}_m(T_{\mathbf{a}})$ is a union of irreducible components of $\mathcal{F}l^{n-k,k}_D$ which are pairwise different. Since the cup diagrams in $\mathbb{B}^{n-k,k}$ are in bijective correspondence with the irreducible components of the Springer fiber $\mathcal{F}l^{n-k,k}_D$ (cf.

Proposition 1.1.17 and Lemma 1.2.15), we deduce that $\bigcup_{\mathbf{a}\in\mathbb{B}^{n-k,k}} \phi_m^{-1}(T_{\mathbf{a}})$ is the entire (embedded) Springer fiber. In particular, the restriction of the diffeomorphism $\phi_m^{-1} \circ \gamma_{n-k,k}$ to $\mathcal{S}_D^{n-k,k}$ yields the desired homeomorphism as claimed in the theorem. ■

We define $\mathcal{F}l_{D,\mathrm{odd}}^{n-k,k}$ as the image of $\mathcal{S}_{D,\mathrm{odd}}^{n-k,k}$ and $\mathcal{F}l_{D,\mathrm{odd}}^{n-k,k}$ as the image of $\mathcal{S}_{D,\mathrm{even}}^{n-k,k}$ under the homeomorphism of Theorem 1.4.12. It follows from Remark 1.2.10 that these are precisely the two connected components of $\mathcal{F}l_D^{n-k,k}$. Since they are isomorphic we restrict ourselves to $\mathcal{F}l_{D,\mathrm{odd}}^{n-k,k}$ (the results are also true for $\mathcal{F}l_{D,\mathrm{even}}^{n-k,k}$).

Theorem 1.4.13. The morphism of algebraic varieties $Y_m \xrightarrow{\pi_m} Y_{m-1}$ restricts to a homeomorphism (even an isomorphism of algebraic varieties)

$$\mathcal{F}l_{D,\mathrm{odd}}^{n-k,k} \xrightarrow{\cong} \mathcal{F}l_C^{n-k-1,k-1},$$

i.e. $\mathcal{F}l_C^{n-k-1,k-1}$ is isomorphic to one of the two (isomorphic) connected components of $\mathcal{F}l_D^{n-k,k}$. In particular, in combination with Theorem 1.4.12, we obtain a homeomorphism

$$\mathcal{S}_{D,\mathrm{odd}}^{n-k,k} \cong \mathcal{F}l_C^{n-k-1,k-1}$$

and thus an explicit topological model for $\mathcal{F}l_C^{n-k-1,k-1}$.

Proof. Since $\mathcal{F}l_{D,\mathrm{odd}}^{n-k,k}$ is a connected components of $\mathcal{F}l_D^{n-k,k}$, it follows directly from Remark 1.1.3 that the restriction of π_m to $\mathcal{F}l_{D,\mathrm{odd}}^{n-k,k} \subseteq Y_m$ defines a continuous injection with image

$$\pi_m\left(\mathcal{F}l_{D,\mathrm{odd}}^{n-k,k}\right) = \bigcup_{\mathbf{a}\in\mathbb{B}_{\mathrm{odd}}^{n-k,k}} \pi_m\left(\phi_m^{-1}\left(T_{\mathbf{a}}\right)\right).$$

By Proposition 1.4.3 this is a union of irreducible components of $\mathcal{F}l_C^{n-k-1,k-1}$ which are pairwise different (because the restriction of π_m to $\mathcal{F}l_{D,\mathrm{odd}}^{n-k,k,}$ is bijective and the $\phi_m^{-1}\left(T_{\mathbf{a}}\right)$ are pairwise different irreducible components of $\mathcal{F}l_D^{n-k,k}$ by Theorem 1.4.12). Recall that the irreducible components of $\mathcal{F}l_C^{n-k-1,k-1}$ are in bijective correspondence with the cup diagrams in $\mathbb{B}_{\mathrm{odd}}^{n-k,k}$ (combine Proposition 1.1.17 with the bijections Lemma 1.2.15 and Lemma 1.2.16). Hence, the image of $\mathcal{F}l_{D,\mathrm{odd}}^{n-k,k}$ under π_m equals $\mathcal{F}l_C^{n-k-1,k-1} \subseteq Y_{m-1}$. In particular, π_m restricts to the desired homeomorphism.

Note that the homeomorphism $\pi_m \colon \mathcal{F}l_{D,\mathrm{odd}}^{n-k,k} \xrightarrow{\cong} \mathcal{F}l_C^{n-k-1,k-1}$ is even an isomorphism of algebraic varieties. ■

Chapter 2

A diagrammatic approach to Springer theory

Convention 2.0.1. In this chapter we return to the conventions used in the introduction and work with cup diagrams for which the markers are accessible from the left instead of the right, i.e. one has to replace the word "right" by "left" in Definition 1.2.1 or work with Definition 5 from the introduction. Accordingly, one has to replace the word "rightmost" by "leftmost" in the definition of the topological Springer fiber in Subsection 1.2.1 or use the definition from the introduction again. Clearly, this does not change any of the results of the previous chapter. In fact, both definitions of the topological Springer fiber yield homeomorphic spaces (the homeomorphism is simply given by reversing the order of the coordinates). However, by changing the conventions, the combinatorics matches the combinatorics used in related publications as e.g. [41],[18]. As in the previous chapter, we fix $n = 2m$, an even positive integer. ▲

Convention 2.0.2. In this chapter the term "vector space" means "complex vector space" and a "graded vector space" is a $\mathbb{Z}$-graded complex vector space. Given a finite set S and a commutative ring R, we write $R[S]$ to denote the free R-module with basis S. If X is a topological space, we write $H_*(X)$ and $H^*(X)$ to denote its singular homology and cohomology with complex coefficients, i.e. $H_*(X) = H_*(X, \mathbb{C})$ and $H^*(X) = H^*(X, \mathbb{C})$. Since we work over the complex numbers, homology and cohomology are dual, i.e. the universal coefficient theorem provides natural isomorphisms $H^*(X) \cong \mathrm{Hom}_{\mathbb{C}}(H_*(X), \mathbb{C})$ of graded vector spaces, which enable us to transfer results obtained for cohomology to homology (as long as they do not involve the ring structure on cohomology) and vice versa. ▲

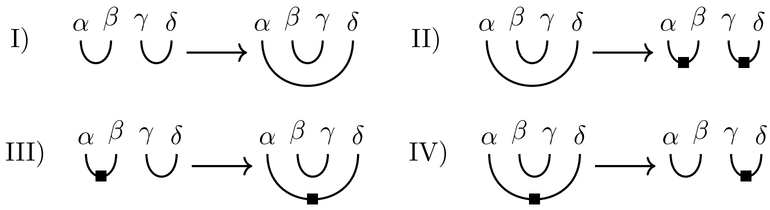
2.1 A cell partition and dimension of cohomology

We begin by constructing an explicit cell partition of the topological Springer fiber $\mathcal{S}^{n-k,k}_{D,\mathrm{even}}$ generalizing a construction in [17]. Similar cell partitions for topological Springer fibers of type A were constructed in [35],[50]. By counting the cells of this partition we obtain an explicit dimension formula for the cohomology (cf. Proposition 2.1.11 below). The Betti numbers of the two-row Springer fibers were independently computed in [38] (using a very different approach).

2.1.1 A partial order on cup diagrams

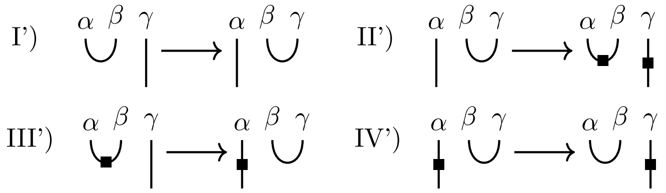
Given cup diagrams $\mathbf{a}, \mathbf{b} \in \mathbb{B}^{n-k,k}_{\mathrm{even}}$, we write $\mathbf{a} \to \mathbf{b}$ if one of the following conditions is satisfied:

- The diagrams $\mathbf{a}$ and $\mathbf{b}$ are identical except at four not necessarily consecutive vertices $\alpha < \beta < \gamma < \delta \in \{1, \ldots, m\}$, where they differ by one of the following local moves:



(2.1)

- The diagrams $\mathbf{a}$ and $\mathbf{b}$ are identical except at three not necessarily consecutive vertices $\alpha < \beta < \gamma \in \{1, \ldots, m\}$, where they differ by one of the following local moves:



(2.2)

We use these loval moves to define a partial order on $\mathbb{B}^{n-k,k}_{\text{even}}$ by setting $\mathbf{a} \prec \mathbf{b}$ if there exists a finite chain of arrows $\mathbf{a} \to \mathbf{c}_1 \to \cdots \to \mathbf{c}_r \to \mathbf{b}$.

Remark 2.1.1. The local moves (2.1) and (2.2) defined in an ad hoc manner above have a natural geometric interpretation in the context of perverse sheaves (constructible with respect to the Schubert stratification) on isotropic Grassmannians, see [7] and [18]. ▲

2.1.2 A cell decomposition compatible with intersections

Let $\mathbf{a} \in \mathbb{B}^{n-k,k}_{\text{even}}$ and let $i_1 < i_2 < \cdots < i_{\lfloor \frac{k}{2} \rfloor}$ be the left endpoints of cups in $\mathbf{a}$.

Definition 2.1.2. Given a vertex $r \in \{1, \ldots, m\}$, we say that a cup connecting vertices $i < j$ or a ray connected to vertex i is to the right of r if $r \leq i$. Let $\sigma(r)$ be the number of marked cups and marked rays to the right of r. ▲

We have the following homeomorphism

$$\begin{array}{cccc} \overline{\xi}_{\mathbf{a}}\colon & S_{\mathbf{a}} & \longrightarrow & (\mathbb{S}^2)^{\lfloor \frac{k}{2} \rfloor} \\ & (x_1, \ldots, x_m) & \longmapsto & (y_1, \ldots, y_{\lfloor \frac{k}{2} \rfloor}) \end{array} \tag{2.3}$$

where $y_r = (-1)^{i_r + \sigma(i_r)} x_{i_r}$ and $1 \leq r \leq \lfloor \frac{k}{2} \rfloor$.

In Subsection 1.4.1 we introduced the homeomorphism $t\colon \mathbb{S}^2 \to \mathbb{S}^2$ induced by restricting the linear transformation $(x, y, z) \mapsto (z, y, x)$ of $\mathbb{R}^3$ to $\mathbb{S}^2 \subseteq \mathbb{R}^3$. Note that $t(p) = q$. Consider the involutive homeomorphism

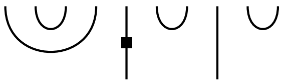
$$\begin{array}{cccc} \Phi_{\mathbf{a}}\colon & (\mathbb{S}^2)^{\lfloor \frac{k}{2} \rfloor} & \longrightarrow & (\mathbb{S}^2)^{\lfloor \frac{k}{2} \rfloor} \\ & (y_1, \ldots, y_{\lfloor \frac{k}{2} \rfloor}) & \longmapsto & \Big(y_1, \ldots, y_s, t(y_{s+1}), \ldots, t(y_{\lfloor \frac{k}{2} \rfloor})\Big) \end{array} \tag{2.4}$$

where s is the number of cups which are not to the right of the second leftmost ray (if $m = k$, in which case there is no second ray, the map (2.4) is the identity).

In the following, the composition of the maps in (2.3) and (2.4) will play a crucial role.

Lemma 2.1.3. The map $\Psi_{\mathbf{a}} \colon S_{\mathbf{a}} \to (\mathbb{S}^2)^{\lfloor \frac{k}{2} \rfloor}$ defined as the composition $\Phi_{\mathbf{a}} \circ \overline{\xi}_{\mathbf{a}}$ is a homeomorphism. $\square$

Example 2.1.4. The preimage of $(p,p,p,p) \in (\mathbb{S}^2)^4$ under $\Psi_{\mathbf{a}}$, where

$\mathbf{a} =$ 

is given by $(p,-p,p,-p,p,p,-p,q,-q,q)$. $\blacktriangle$

In order to define a cell decomposition of $S_{\mathbf{a}}$ for $\mathbf{a} \in \mathbb{B}^{n-k,k}_{\text{even}}$, we proceed as in [17, §4.5] and associate a graph $\Gamma_{\mathbf{a}} = (\mathcal{V}(\Gamma_{\mathbf{a}}), \mathcal{E}(\Gamma_{\mathbf{a}}))$ with each cup diagram $\mathbf{a} \in \mathbb{B}^{n-k,k}_{\text{even}}$ as follows:

- The vertices $\mathcal{V}(\Gamma_{\mathbf{a}})$ of $\Gamma_{\mathbf{a}}$ are given by the cups (i,j) in $\mathbf{a}$.
- Two vertices $(i_1,j_1),(i_2,j_2) \in \mathcal{V}(\Gamma_{\mathbf{a}})$ are connected by an edge $(i_1,j_1)-(i_2,j_2) \in \mathcal{E}(\Gamma_{\mathbf{a}})$ in $\Gamma_{\mathbf{a}}$ if and only if there exists some $\mathbf{b} \in \mathbb{B}^{n-k,k}_{\text{even}}$ with $\mathbf{b} \to \mathbf{a}$ such that $\mathbf{a}$ is obtained from $\mathbf{b}$ by a local move of type **I)-IV)** at the vertices i_1, i_2, j_1, j_2.

As in [17], the graph $\Gamma_{\mathbf{a}}$ is a forest whose roots $\mathcal{R}(\Gamma_{\mathbf{a}})$ are precisely the *outer cups* of $\mathbf{a}$, i.e. those cups which are not nested in any other cup and do not contain any marked cup to their right.

We assign to each subset $J \subseteq \mathcal{R}(\Gamma_{\mathbf{a}}) \cup \mathcal{E}(\Gamma_{\mathbf{a}})$ the subset C'_J of $(\mathbb{S}^2)^{\lfloor \frac{k}{2} \rfloor}$ given by all elements $(y_1, \ldots, y_{\lfloor \frac{k}{2} \rfloor}) \in (\mathbb{S}^2)^{\lfloor \frac{k}{2} \rfloor}$ which satisfy the following relations:

(C1) If $(i_q, j_q) \in \mathcal{R}(\Gamma_{\mathbf{a}}) \cap J$ then $y_q = p$,

(C2) if $(i_q, j_q) \in \mathcal{R}(\Gamma_{\mathbf{a}})$ but $(i_q, j_q) \notin J$ then $y_q \neq p$,

(C3) if $(i_q, j_q) - (i_{q'}, j_{q'}) \in \mathcal{E}(\Gamma_{\mathbf{a}}) \cap J$ then $y_q = y_{q'}$,

(C4) if $(i_q, j_q) - (i_{q'}, j_{q'}) \notin \mathcal{E}(\Gamma_{\mathbf{a}}) \cap J$ then $y_q \neq y_{q'}$.

Lemma 2.1.5. There is a decomposition

$$(\mathbb{S}^2)^{\lfloor \frac{k}{2} \rfloor} = \bigsqcup_{J \subseteq \mathcal{R}(\Gamma_{\mathbf{a}}) \cup \mathcal{E}(\Gamma_{\mathbf{a}})} C'_J \tag{2.5}$$

into disjoint cells C'_J homeomorphic to $\mathbb{R}^{2(\lfloor \frac{k}{2} \rfloor - |J|)}$. Moreover, pushing forward along (2.3) gives a cell decomposition

$$S_{\mathbf{a}} = \bigsqcup_{J \subseteq \mathcal{R}(\Gamma_{\mathbf{a}}) \cup \mathcal{E}(\Gamma_{\mathbf{a}})} C_J, \tag{2.6}$$

where $C_J = (\Psi_{\mathbf{a}})^{-1}(C'_J)$.

Proof. The proof of (2.5) is the same as in the equal-row case [17, Lemma 4.17] because the additional rays do not play any role in the construction of the C'_J. Note that (2.6) is a direct consequence of (2.5). $\blacksquare$

The reason for choosing the homeomorphism $\Psi_{\mathbf{a}}$ in the construction of the cell decomposition (2.6) of $S_{\mathbf{a}}$ is that the resulting cell decompositions are compatible with pairwise intersections in the sense of the next lemma which extends [17, Proposition 4.24] to the general two-row case.

Lemma 2.1.6. Let $\mathbf{a}, \mathbf{b} \in \mathbb{B}^{n-k,k}_{\text{even}}$ such that $\mathbf{b} \to \mathbf{a}$.

(1) If $\mathbf{b} \to \mathbf{a}$ is of type **I)-IV)**, then

$$S_{\mathbf{a}} \cap S_{\mathbf{b}} = \bigcup_{J \subseteq \mathcal{R}(\Gamma_{\mathbf{a}}) \cup \mathcal{E}(\Gamma_{\mathbf{a}}), e \in J} C_J,$$

where $e \in \mathcal{E}(\Gamma_{\mathbf{a}})$ is the edge of $\Gamma_{\mathbf{a}}$ determined by the move $\mathbf{b} \to \mathbf{a}$.

(2) If $\mathbf{b} \to \mathbf{a}$ is of type **I')-IV')**, then there is a unique cup $\alpha \in \operatorname{cups}(\mathbf{a})$ such that $\alpha \notin \operatorname{cups}(\mathbf{b})$. Moreover, $\alpha \in \mathcal{R}(\Gamma_{\mathbf{a}})$ and

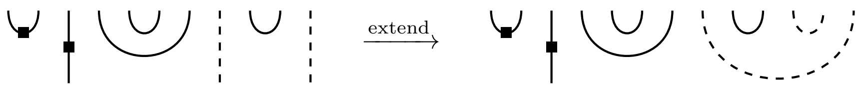
$$S_{\mathbf{a}} \cap S_{\mathbf{b}} = \bigcup_{J \subseteq \mathcal{R}(\Gamma_{\mathbf{a}}) \cup \mathcal{E}(\Gamma_{\mathbf{a}}), \alpha \in J} C_J.$$

Proof. This is a straightforward case-by-case analysis. ■

2.1.3 A cell partition of the topological Springer fiber

Given a cup diagram $\mathbf{a} \in \mathbb{B}^{n-k,k}_{\text{even}}$, define its *extension* $\widetilde{\mathbf{a}} \in \mathbb{B}^{n-k,n-k}_{\text{even}}$ as the unique cup diagram obtained by connecting the loose endpoints of the $m-k$ rightmost rays to $m-k$ newly added vertices to the right of $\mathbf{a}$ via an unmarked cup such that the resulting diagram is crossingless (see [50, Definition 3.11] and [18, §5.3] for a similar but different extension). Let $\mathbb{B}^{n-k,n-k}_{n-k,k}$ denote the cup diagrams in $\mathbb{B}^{n-k,n-k}_{\text{even}}$ which are obtained as an extension of some diagram in $\mathbb{B}^{n-k,k}_{\text{even}}$, i.e. the set of all cup diagrams in $\mathbb{B}^{n-k,n-k}_{\text{even}}$ whose $m-k$ rightmost vertices are right endpoints of unmarked cups. If $m=k$, then the extension procedure returns the same diagram. Note that the leftmost ray is never replaced by a cup.

Example 2.1.7. Here is an example showing the extension of a cup diagram:



For illustrative purposes the components of the diagrams which change during extension are drawn in dashed font. ▲

Inspired by the combinatorial completion procedure we define an embedding of topological Springer fibers

$$\eta_{n-k,k} \colon \mathcal{S}^{n-k,k}_{D,\text{even}} \hookrightarrow \mathcal{S}^{n-k,n-k}_{D,\text{even}}\ ,\ (x_1, \ldots, x_m) \mapsto (x_1, \ldots, x_m, \pm q, \ldots, -q, q),$$

see also [50, Section 5] for a similar map in type A. Note that $\eta_{n-k,k}(S_{\mathbf{a}}) \subseteq S_{\widetilde{\mathbf{a}}}$.

Given $\mathbf{a}, \mathbf{b} \in \mathbb{B}^{n-k,k}_{\text{even}}$, one easily verifies that $\mathbf{b} \to \mathbf{a}$ if and only if $\widetilde{\mathbf{b}} \to \widetilde{\mathbf{a}}$. Thus, we have an isomorphism of posets $\mathbb{B}^{n-k,k}_{\text{even}} \cong \widetilde{\mathbb{B}}^{n-k,n-k}_{\text{even}}$. We equip $\widetilde{\mathbb{B}}^{n-k,n-k}_{\text{even}}$ with the induced total order from some fixed total order (which extends the partial order $\prec$ from Subsection 2.1.1) on $\mathbb{B}^{n-k,n-k}_{\text{even}}$. We pull this total order over to $\mathbb{B}^{n-k,k}_{\text{even}}$ via the isomorphism of posets.

Let $S_{<\mathbf{a}} = \bigcup_{\mathbf{b}<\mathbf{a}} S_{\mathbf{b}}$.

Lemma 2.1.8. For any $\mathbf{a} \in \mathbb{B}^{n-k,k}_{\text{even}}$ we have

$$S_{<\mathbf{a}} \cap S_{\mathbf{a}} \quad = \quad \bigcup_{\mathbf{b} \to \mathbf{a}} (S_{\mathbf{b}} \cap S_{\mathbf{a}}). \tag{2.7}$$

Proof. In case $m=k$ the claimed equality was proven in [17, Lemma 4.23].

Since $\to$ implies $<$, the left side is contained in the right side.

Assume there exists $\mathbf{b} < \mathbf{a} \in \mathbb{B}^{n-k,k}_{\text{even}}$ and $x \in S_{\mathbf{a}} \cap S_{\mathbf{b}}$ such that $x \notin \bigcup_{\mathbf{b} \to \mathbf{a}} S_{\mathbf{b}} \cap S_{\mathbf{a}}$. Then $\eta_{n-k,k}(x) \in S_{\widetilde{\mathbf{b}}} \cap S_{\widetilde{\mathbf{a}}}$ (note that $\widetilde{\mathbf{b}} < \widetilde{\mathbf{a}}$ since $\mathbf{b} < \mathbf{a}$ and by definition of the total order $<$ on

$\mathbb{B}^{n-k,k}_{\text{even}}$). Hence,

$$\bigcup_{\substack{\mathbf{b}<\tilde{\mathbf{a}} \\ \mathbf{b}\in\mathbb{B}^{n-k,n-k}_{\text{even}}}} S_{\mathbf{b}} \cap S_{\tilde{\mathbf{a}}} = \bigcup_{\substack{\mathbf{b}\to\tilde{\mathbf{a}} \\ \mathbf{b}\in\mathbb{B}^{n-k,n-k}_{\text{even}}}} S_{\mathbf{b}} \cap S_{\tilde{\mathbf{a}}} = \bigcup_{\substack{\mathbf{b}\to\tilde{\mathbf{a}} \\ \mathbf{b}\in\widetilde{\mathbb{B}}^{n-k,n-k}_{\text{even}}}} S_{\mathbf{b}} \cap S_{\tilde{\mathbf{a}}}$$

and therefore

$$\eta_{n-k,k}(\mathcal{S}^{n-k,k}_{D,\text{even}}) \cap \bigcup_{\substack{\mathbf{b}<\tilde{\mathbf{a}} \\ \mathbf{b}\in\mathbb{B}^{n-k,n-k}_{\text{even}}}} S_{\mathbf{b}} \cap S_{\tilde{\mathbf{a}}} = \eta_{n-k,k}(\mathcal{S}^{n-k,k}_{D,\text{even}}) \cap \bigcup_{\substack{\mathbf{b}\to\tilde{\mathbf{a}} \\ \mathbf{b}\in\widetilde{\mathbb{B}}^{n-k,n-k}_{\text{even}}}} S_{\mathbf{b}} \cap S_{\tilde{\mathbf{a}}} = \eta_{n-k,k}\left(\bigcup_{\mathbf{b}\to\mathbf{a}} S_{\mathbf{b}} \cap S_{\mathbf{a}}\right).$$

Since $\eta_{n-k,k}(x)$ is contained in the leftmost set above, it is also contained in the rightmost set. By the injectivity of $\eta_{n-k,k}$, we deduce that $x \in \bigcup_{b\to a} S_{\mathbf{b}} \cap S_{\mathbf{a}}$, a contradiction. ■

Fix a cup diagram $\mathbf{a} \in \mathbb{B}^{n-k,k}_{\text{even}}$. A cup in $\mathbf{a}$ is called *special* if there exists some $\mathbf{b} \in \mathbb{B}^{n-k,k}_{\text{even}}$ such that $\mathbf{b} \to \mathbf{a}$, where $\mathbf{a}$ and $\mathbf{b}$ are related by a local move of type **I')-IV')**, and this cup is the unique cup which changes under the move $\mathbf{a} \to \mathbf{b}$. Let $\mathcal{R}(\Gamma_{\mathbf{a}})^{\text{sp}}$ be the set of special cups in $\mathbf{a}$. In particular, we have $\mathcal{R}(\Gamma_{\mathbf{a}})^{\text{sp}} = \emptyset$ if $k = m$ is even, i.e. if there are no rays in $\mathbf{a}$.

Remark 2.1.9. As in [17, Remark 4.25], the moves **I')-IV')** imply that all outer cups (in the sense as defined above) of a given cup diagram $\mathbf{a} \in \mathbb{B}^{n-k,k}_{\text{even}}$ are special if the leftmost ray in $\mathbf{a}$ is marked, whereas only outer cups to the right of the leftmost ray are special if the ray is unmarked. ▲

Corollary 2.1.10. Given $\mathbf{a} \in \mathbb{B}^{n-k,k}_{\text{even}}$, then

$$S_{<\mathbf{a}} \cap S_{\mathbf{a}} \quad = \bigcup_{\substack{J \subset \mathcal{R}(\Gamma_{\mathbf{a}}) \cup \mathcal{E}(\Gamma_{\mathbf{a}}) \\ J \cap (\mathcal{E}(\Gamma_{\mathbf{a}}) \cup \mathcal{R}(\Gamma_{\mathbf{a}})^{\text{sp}}) \neq \emptyset}} C_J.$$

Proof. This follows directly from the previous two lemmas. ■

In order to argue inductively in the proof of the following proposition, we introduce the notion of a *partial/one-step extension.* This is defined as the ordinary extension introduced above, except that we only replace the rightmost ray with a cup.

Proposition 2.1.11. If $k \neq m$, then we have the following dimension formula:

$$\dim H^*(\mathcal{S}^{n-k,k}_{D,\text{even}}) = \sum_{i=0}^{\frac{k-1}{2}} \binom{m}{i}.$$

□

Remark 2.1.12. In the special case $m = k$ we have $\dim H^*(\mathcal{S}^{m,m}_{D,\text{even}}) = 2^{m-1}$ as proven in [17, Prop. 4.28]. ▲

Proof. The dimension of $H^*(\mathcal{S}^{n-k,k}_{D,\text{even}})$ is given by

$$\dim H^*(\mathcal{S}^{n-k,k}_{D,\text{even}}) = \sum_{\mathbf{a}\in\mathbb{B}^{n-k,k}_{\text{even}}} 2^{|\mathcal{R}(\Gamma_{\mathbf{a}})\setminus\mathcal{R}(\Gamma_{\mathbf{a}})^{\text{sp}}|}, \tag{2.8}$$

i.e. we count the cells contained in $S_{\mathbf{a}} \setminus S_{<\mathbf{a}}$ for all $\mathbf{a} \in \mathbb{B}^{n-k,k}_{\text{even}}$ (which by Corollary 2.1.10 correspond bijectively to the outer cups in $\mathbf{a}$ which are not special) and take their sum. This yields a cell partition of $\mathcal{S}^{n-k,k}_{D,\text{even}}$ which can be used to calculate the dimension of cohomology, see [35, Lemma 6].

Since the equal-row case was already proven in [17, Proposition 4.28], we prove that the right hand side of (2.8) equals $\sum_{i=0}^{\frac{k-1}{2}}\binom{m}{i}$ by induction on $m-k$ (the number of rays in the cup diagrams contained in $\mathbb{B}^{n-k,k}_{\mathrm{even}}$ excluding the leftmost ray).

We consider a partition $(n-k,k)$, $m-k>1$, labeling a nilpotent orbit of type D. Then we have

$$2^{|\mathcal{R}(\Gamma_{\mathbf{a}})\setminus\mathcal{R}(\Gamma_{\mathbf{a}})^{\mathrm{sp}}|}=2^{|\mathcal{R}(\Gamma_{\tilde{\mathbf{a}}})\setminus\mathcal{R}(\Gamma_{\tilde{\mathbf{a}}})^{\mathrm{sp}}|} \tag{2.9}$$

for all $\mathbf{a}\in\mathbb{B}^{n-k,k}_{\mathrm{even}}$, where $\tilde{\mathbf{a}}\in\mathbb{B}^{n-k,k+2}_{\mathrm{even}}$ denotes the partial completion of $\mathbf{a}$ (note that the completed cup is always special). We compute

$$\begin{aligned}\sum_{a\in\mathbb{B}^{n-k,k}_{\mathrm{even}}}2^{|\mathcal{R}(\Gamma_{\mathbf{a}})\setminus\mathcal{R}(\Gamma_{\mathbf{a}})^{\mathrm{sp}}|}&=\sum_{\tilde{\mathbf{a}}\in\widetilde{\mathbb{B}}^{n-k,k+2}_{\mathrm{even}}}2^{|\mathcal{R}(\Gamma_{\tilde{\mathbf{a}}})\setminus\mathcal{R}(\Gamma_{\tilde{\mathbf{a}}})^{\mathrm{sp}}|}\\&=\sum_{\mathbf{a}\in\mathbb{B}^{n-k,k+2}_{\mathrm{even}}}2^{|\mathcal{R}(\Gamma_{\mathbf{a}})\setminus\mathcal{R}(\Gamma_{\mathbf{a}})^{\mathrm{sp}}|}-\sum_{\mathbf{a}\in\mathbb{B}^{n-k,k+2}_{\mathrm{even}}\setminus\widetilde{\mathbb{B}}^{n-k,k+2}_{\mathrm{even}}}2^{|\mathcal{R}(\Gamma_{\mathbf{a}})\setminus\mathcal{R}(\Gamma_{\mathbf{a}})^{\mathrm{sp}}|}.\end{aligned}$$

Case $(k+2,k)$, $k>0$ odd (this is $m-k=1$): The diagrams $\mathbf{a}\in\mathbb{B}^{k+2,k+2}_{\mathrm{even}}\setminus\widetilde{\mathbb{B}}^{k+2,k+2}_{\mathrm{even}}$ are precisely those with a ray connected to the rightmost vertex. The summands corresponding to the diagrams $\mathbf{a}$ for which this ray is marked contribute $\frac{1}{2}\binom{k+1}{\frac{k+1}{2}}$, see [17, Remark 3.3], to the sum which is subtracted in the above equation. The summands corresponding to diagrams for which this ray is unmarked contribute $\sum_{i=0}^{\frac{k+1}{2}-1}\binom{k+1}{i}+\frac{1}{2}\binom{k+1}{\frac{k+1}{2}}$.

Case $(n-k,k)$, $k>0$ odd (this is $m-k>1$): The elements $\mathbf{a}\in\mathbb{B}^{n-k,k+2}_{\mathrm{even}}\setminus\widetilde{\mathbb{B}}^{n-k,k+2}_{\mathrm{even}}$ are precisely the cup diagrams with an unmarked ray connected to the rightmost vertex $m+1$.

Thus, using again induction, the above chain of equalities equals

$$\sum_{i=0}^{\frac{k-1}{2}+1}\binom{m+1}{i}-\sum_{i=0}^{\frac{k-1}{2}+1}\binom{m}{i}=\sum_{i=1}^{\frac{k-1}{2}+1}\left(\binom{m+1}{i}-\binom{m}{i}\right)=\sum_{i=1}^{\frac{k-1}{2}+1}\binom{m}{i-1}=\sum_{i=0}^{\frac{k-1}{2}}\binom{m}{i},$$

which proves the claim. ■

Remark 2.1.13. If $k\neq m$, $H^*(\mathcal{S}^{n-k,k}_{D,\mathrm{even}})$ has a basis indexed by all cup diagrams on m vertices with an even number of markers and at most $\frac{k-1}{2}$ cups: in [17, Lemma 5.20], the authors establish for $k\neq m$ a bijection between the set $\mathbb{B}^{n-k,k}_{\mathrm{even}}$ and the set of unordered pairs $\{\lambda,\mu\}$ of one-row Young tableaux, where λ consists of $\frac{k-1}{2}$ and μ of $m-\frac{k-1}{2}$ dots, filled with the numbers $1,\dots,m$ increasing in both rows. Evidently, this set has $\binom{m}{\frac{k-1}{2}}$ elements which is exactly the dimension of the cohomology by Proposition 2.1.11. In the next section we fix an explicit basis element for each cup diagram. ▲

2.2 The cohomology ring and a diagrammatic homology basis of the two-row Springer fibers

Given $\mathbf{a}\in\mathbb{B}^{n-k,k}_{\mathrm{even}}$, let $\phi_{\mathbf{a}}\colon S_{\mathbf{a}}\hookrightarrow\mathcal{S}^{n-k,k}_{D,\mathrm{even}}$, $\psi_{\mathbf{a}}\colon S_{\mathbf{a}}\hookrightarrow(\mathbb{S}^2)^m$ be the inclusions. We obtain a commutative diagram

$$\begin{tikzcd}\bigoplus_{\mathbf{a}\in\mathbb{B}^{n-k,k}_{\mathrm{even}}}H_*(S_{\mathbf{a}})\arrow[r,"\phi_{n-k,k}"]\arrow[rr,bend left,"\psi_{n-k,k}"]&H_*(\mathcal{S}^{n-k,k}_{D,\mathrm{even}})\arrow[r,"\gamma_{n-k,k}"]&H_*((\mathbb{S}^2)^m)\end{tikzcd}, \tag{2.10}$$

where $\phi_{n-k,k}$ (resp. $\psi_{n-k,k}$) is the direct sum of the maps induced by $\phi_{\mathbf{a}}$ (resp. $\psi_{\mathbf{a}}$) and $\gamma_{n-k,k}$ is induced by the inclusion $\mathcal{S}^{n-k,k}_{D,\mathrm{even}} \hookrightarrow \left(\mathbb{S}^2\right)^m$.

2.2.1 Homology bases via cell decompositions

In this subsection we define certain cell decompositions of the manifolds $\left(\mathbb{S}^2\right)^m$ and $S_{\mathbf{a}}$, $\mathbf{a} \in \mathbb{B}^{n-k,k}_{\mathrm{even}}$, which are used to obtain a combinatorial description of the homology of the respective spaces. We begin by introducing the necessary diagrammatic tools.

Line diagrams

We recall the notion of a line diagram [51, Definition 3.1].

Definition 2.2.1. A *line diagram* on m vertices is obtained by attaching vertical rays, each possibly decorated with a single white dot, to m vertices on a horizontal line. Given a subset $U \subseteq \{1, \dots, m\}$, we write l^m_U to denote the corresponding line diagram with white dots precisely on the vertices not contained in U. ▲

Example 2.2.2. Here is a complete list of all line diagrams on two vertices:

▲

The two-sphere has a cell decomposition $\mathbb{S}^2 = \left(\mathbb{S}^2 \setminus \{p\}\right) \cup \{p\}$ consisting of a point and a two-cell. In the following we fix this CW-structure and equip $(\mathbb{S}^2)^m$ with the cartesian product CW-structure. The generators of the homology $H_*((\mathbb{S}^2)^m)$ are in one-to-one correspondence with the cells of the CW-complex $(\mathbb{S}^2)^m$. In particular, the line diagrams on m vertices can be viewed as a basis of $H_*((\mathbb{S}^2)^m)$, where a line with a dot (resp. without a dot) means that we have chosen the 0-cell (resp. 2-cell) for the corresponding sphere. The homological degree of a line diagram (viewed as a homology generator) is given by twice the number of lines without a dot.

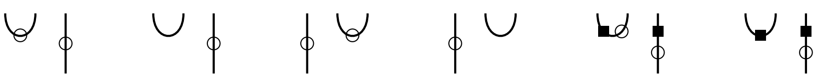
Example 2.2.3. The line diagrams in Example 2.2.2 represent (in the same order) the generators of $H_*(\mathbb{S}^2 \times \mathbb{S}^2)$ corresponding to the cells

$$\{(p,p)\}, \;\; \{(x,p) \mid x \in \mathbb{S}^2 \setminus \{p\}\}, \;\; \{(p,x) \mid x \in \mathbb{S}^2 \setminus \{p\}\}, \;\; \{(x,y) \mid x,y \in \mathbb{S}^2 \setminus \{p\}\}.$$

▲

Definition 2.2.4. An *enriched cup diagram* is a cup diagram in which some of the cups and all of the rays are decorated with a dot. Hereby, we allow at most one dot per component with no accessibility condition like we have for the markers. If a cup (resp. ray) is decorated with both a marker and a dot we place the marker to the left (resp. above) of the dot. Let $\widetilde{\mathbb{B}}^{n-k,k}$ be the set of all enriched cup diagrams on m vertices with $\lfloor \frac{k}{2} \rfloor$ cups. ▲

Example 2.2.5. The enriched cup diagrams corresponding to the cup diagrams in $\mathbb{B}^{3,3}_{\mathrm{even}}$ are given by



▲

In the rest of the chapter we will work with a different (somewhat easier) cell decompositon of $S_{\mathbf{a}}$ than the one introduced in Section 2.1.2.

Given $\mathbf{a} \in \mathbb{B}^{n-k,k}_{\mathrm{even}}$, let $i_1 < i_2 < \cdots < i_{\lfloor \frac{k}{2} \rfloor}$ denote the left endpoints of the cups in $\mathbf{a}$. Then we have a homeomorphism

$$\xi_{\mathbf{a}} \colon S_{\mathbf{a}} \xrightarrow{\cong} (\mathbb{S}^2)^{\lfloor \frac{k}{2} \rfloor},\ (x_1, \ldots, x_m) \mapsto (y_1, \ldots, y_{\lfloor \frac{k}{2} \rfloor}),$$

where $y_j = x_{i_j}$, if i_j is endpoint of an unmarked cup, and $y_j = -x_{i_j}$ if i_j is endpoint of a marked cup, $1 \leq j \leq \lfloor \frac{k}{2} \rfloor$.

We equip $S_{\mathbf{a}}$ with the structure of a CW-complex, where the cells are obtained as the preimages of the cells of $(\mathbb{S}^2)^{\lfloor \frac{k}{2} \rfloor}$ with the cartesian product CW-structure under the homeomorphism. Since the generators of the homology $H_*(S_{\mathbf{a}})$ are in one-to-one correspondence with the cells of $S_{\mathbf{a}}$ we can view the enriched cup diagrams coming from $\mathbf{a}$ as a basis of $H_*(S_{\mathbf{a}})$, where a cup with a dot (resp. a cup without a dot) means that we have chosen the 0-cell (resp. 2-cell) for the corresponding sphere. The homological degree of an enriched cup diagram (viewed as a homology generator) is given as twice the number of cups without a dot.

Beware 2.2.6. We would like to stress that the white dots in this thesis play exactly the same role as the black dots on the cup diagrams in the related work in type A, see [51],[50]. Black dots also appear in Chapter 3. These dots should not be confused with the white dots in the present chapter. •

2.2.2 A combinatorial description of $\psi_{n-k,k}$

In the previous subsection we have chosen bases of the right (resp. left) vector space appearing in the commutative diagram (2.10). Furthermore, we have given a combinatorial description of these bases in terms of line diagrams (resp. enriched cup diagrams). This allows of a combinatorial description of the map $\psi_{n-k,k}$.

Definition 2.2.7. Given an enriched cup diagram M, we define the associated *line diagram sum* L_M as follows

$$L_M = \sum_{U \in \mathcal{U}_M} (-1)^{\Lambda_M(U)} l_U,$$

where $\mathcal{U}_M$ is the set of all subsets $U \subseteq \{1, \cdots, m\}$ containing precisely one endpoint of every cup without a dot and $\Lambda_M(U)$ counts the total number of all endpoints of cups with a marker and no dot, plus the number of right endpoints of cups in U with neither a marker nor a dot. ▲

Similar line diagram sums (for type A) also appear in [51, Definition 3.11]. Their significance lies in the following proposition

Proposition 2.2.8. The map $\psi_{\mathbf{a}} \colon H_*(S_{\mathbf{a}}) \to H_*((\mathbb{S}^2)^m)$ is explicitly given by the assignment $M \mapsto L_M$ (where we view the diagrams as elements of the respective homologies). □

Example 2.2.9. If $m = 1$ note that the map induced by the natural inclusion $S_{\blacklozenge} \hookrightarrow \mathbb{S}^2$ on homology is given by

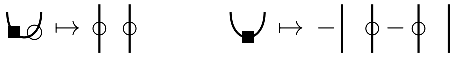
 $\mapsto$ 

since it sends p to the 0-cell of $\mathbb{S}^2$. We also see that the map induced by the inclusion $S_{|} \hookrightarrow \mathbb{S}^2$ is given by

 $\mapsto$ 

because it is clearly homotopic to the map $S_{|} \to \mathbb{S}^2$, $-p \mapsto p$, respecting the 0-skeleta. More generally, the map induced by the inclusion $S_{\mathbf{a}} \hookrightarrow \mathbb{S}^m$, where $\mathbf{a}$ consists of m subsequent rays, sends the generator corresponding to the point $S_{\mathbf{a}}$ to the line diagram consisting of m subsequent rays with a dot.

Furthermore, by arguing similarly as in [51, Lemma 3.2], one can show that the map induced by the inclusion $S_{\cup} \hookrightarrow \mathbb{S}^2 \times \mathbb{S}^2$ is given by



(2.11)

and the map induced by the the inclusion $S_{\cup} \hookrightarrow (\mathbb{S}^2)^2$ is given by

(2.12)

which shows that Proposition 2.2.8 is true for all cup diagrams with one connected component. ▲

Before we prove this proposition in general, we note the following useful combinatorial fact whose (straightforward) proof is omitted (cf. [51, Lemma 3.11] for a similar statement including a proof).

Lemma 2.2.10. Suppose that the enriched cup diagram M' is obtained from M by deleting a cup connecting vertices i and j, $i < j$. If this cup has a dot, then we can write

$$L_M = \sum_{U \in \mathcal{U}_{M'}} (-1)^{\Lambda_{M'}(U)} l_{\widetilde{U_\emptyset}}$$

and if the cup does not have a dot, then we have

$$L_M = (-1)^{\sigma} \sum_{U \in \mathcal{U}_{M'}} (-1)^{\Lambda_{M'}(U)} l_{\widetilde{U_i}} - \sum_{U \in \mathcal{U}_{M'}} (-1)^{\Lambda_{M'}(U)} l_{\widetilde{U_j}},$$

where if $S \subseteq \{i, j\}$, then we have

$$\widetilde{U}_S = S \cup \{x \in U \mid x < i\} \cup \{x + 1 \mid x \in U \text{ and } i \leq x < j\} \cup \{x + 2 \mid x \in U \text{ and } j \leq x\}.$$

and $\sigma = 1$, if the cup has a marker, and $\sigma = 0$, if it has neither a marker nor a dot. □

Proof (Proposition 2.2.8). In the next step we use the small examples above in order to deduce the claim for the cup diagram $\mathbf{a}$ which consists of $\lfloor \frac{k}{2} \rfloor$ subsequent cups connecting neighboring vertices with (possibly) and additional collection of subsequent rays to the right of the cups. Assume that the cup connecting the first two vertices is unmarked and identify $S_{\mathbf{a}}$ with $S_{\cup} \times S_{\mathbf{a}'}$ and the inclusion $S_{\mathbf{a}} \hookrightarrow (\mathbb{S}^2)^m$ with $\psi_{\cup} \times \psi_{\mathbf{a}'}$. Here $\mathbf{a}'$ is the cup diagram obtained by removing the leftmost cup. Via the cross-product isomorphism, the map induced by $\psi_{\cup} \times \psi_{\mathbf{a}'}$ is the tensor product $(\psi_{\cup})_* \otimes (\psi_{\mathbf{a}'})_*$. Hence, if the cup connecting the two leftmost vertices in M has a dot, we compute

$$(\psi_{\cup})_*(\cup) \otimes (\psi_{\mathbf{a}'})_*(M') = l_\emptyset \otimes \left(\sum_{U \in \mathcal{U}_{M'}} (-1)^{\Lambda_{M'}(U)} l_U \right) = \sum_{U \in \mathcal{U}_{M'}} (-1)^{\Lambda_{M'}(U)} l_\emptyset \otimes l_U,$$

where the first equality follows by induction and the small cases above. Here M' denotes the enriched cup diagram obtained by removing the leftmost cup. Applying the Künneth isomorphism to the above equation yields the claim.

On the other hand, if the leftmost cup in M does not have a dot, we compute

$$\begin{aligned}(\psi_{\cup})_*(\cup) \otimes (\psi_{\mathbf{a}'})_*(M') &= \big(l_{\{1\}} - l_{\{2\}}\big) \otimes \left(\sum_{U \in \mathcal{U}_{M'}} (-1)^{\Lambda_{M'}(U)} l_U \right) \\ &= \sum_{U \in \mathcal{U}_{M'}} (-1)^{\Lambda_{M'}(U)} l_{\{1\}} \otimes l_U - \sum_{U \in \mathcal{U}_{M'}} (-1)^{\Lambda_{M'}(U)} l_{\{2\}} \otimes l_U.\end{aligned}$$

If the leftmost cup has a marker we argue similarly.

For the general case let $\mathbf{a}$ be a cup diagram and $\mathbf{a}_0$ the cup diagram obtained by rearranging the cups in such a way that the diagram is completely unnested. Let $\tau_{\mathbf{a}}$ be the permutation of the vertices which realizes the change of passing from $\mathbf{a}$ to $\mathbf{a}'$. This induces a homeomorphism $\tau_{\mathbf{a}} \colon (\mathbb{S}^2)^m \to (\mathbb{S}^2)^m$ which permutes the coordinates accordingly and hence restricts to a homeomorphism $S_{\mathbf{a}} \to S_{\mathbf{a}_0}$. The inclusion $S_{\mathbf{a}} \hookrightarrow (\mathbb{S}^2)^m$ can be written as the composition $\tau_{\mathbf{a}} \circ \psi_{\mathbf{a}_0} \circ \tau_{\mathbf{a}}^{-1}|_{S_{\mathbf{a}}}$. Let M_0 be the cup diagram obtained from $\mathbf{a}_0$ whose i-th component has a dot if and only if the i-th component of M has a dot. Then we have

$$\tau_{\mathbf{a}}(\psi_{\mathbf{a}_0}(\tau_{\mathbf{a}}^{-1}|_{S_{\mathbf{a}}}(M))) = \tau_{\mathbf{a}}(\psi_{\mathbf{a}_0}(M_0)) = \tau_{\mathbf{a}}(\sum_{U \in \mathcal{U}_{M_0}} (-1)^{\Lambda_{M_0}(U)} l_U) = \sum_{U \in \mathcal{U}_M} (-1)^{\Lambda_M(U)} l_U$$

which proves the claim. ■

2.2.3 Standard cup diagrams

In order to obtain a diagrammatic labeling set of a basis of $H_*(\mathcal{S}^{m,m}_{D,\mathrm{even}})$ we introduce the notion of a standard cup diagram.

Definition 2.2.11. A *standard enriched cup diagram*, or shorter *standard cup diagram*, is an enriched cup diagram in which all cups marked with a dot can be connected to the right side of the rectangle by a path which does neither intersect the rest of the diagram nor any given path connecting a marker with the left side. ▲

Definition 2.2.12. The set of all standard cup diagrams on m vertices is denoted by $\widetilde{\mathbb{B}}^{m,\mathrm{std}}$. ▲

Example 2.2.13. Here are the standard cup diagrams contained in $\widetilde{\mathbb{B}}^{3,\mathrm{std}}_{\mathrm{even}}$:

▲

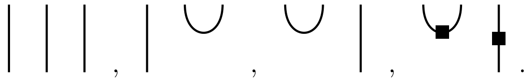
We define a map $\mathsf{C} \colon \widetilde{\mathbb{B}}^{m,\mathrm{std}} \to C(m)$, called "cutting", which takes a standard cup diagram $M \in \widetilde{\mathbb{B}}^{m,\mathrm{std}}$ and replaces each cup with a dot by two unmarked rays (this can be thought of as cutting the cup along the dot). Furthermore, the dots on the rays of M are deleted (again, the reader might think of this as cutting along the dots and throw away the resulting line segment which is not connected to the vertex).

The map $\mathsf{G} \colon C(m) \to \widetilde{\mathbb{B}}^{m,\mathrm{std}}$, called "gluing", is defined as follows: Given a cup diagram $\mathbf{a} \in C(m)$ we replace the two rightmost rays by a cup with a dot (this can be thought of as gluing the two endpoints of the rays resulting in a cup in which the gluing point is marked with a dot). The resulting cup has a marker if and only if one of the rays from which it resulted had a marker. Take the new diagram and continue with this procedure until all rays are glued together and there is at most one ray left.

Lemma 2.2.14. The assignments G and C define mutually inverse bijections which restrict to bijections between $\widetilde{\mathbb{B}}^{m,\mathrm{std}}_{\mathrm{even}}$ and $C_{\mathrm{even}}(m)$.

Proof. This is evident from the definitions. ■

Example 2.2.15. Note that the standard cup diagrams in Example 2.2.13 are obtained by applying G to the following diagrams (in the same order):



▲

2.2.4 Linear independence of line diagram sums

Proposition 2.2.16. The line diagram sums L_M, i.e. the images of M under ψ_m, where M varies over all standard cup diagrams in $\widetilde{\mathbb{B}}^{m,\mathrm{std}}_{\mathrm{even}}$, are linearly independent in $H_*((\mathbb{S}^2)^m)$. $\square$

For the proof of this Proposition we use the following technical lemma.

Lemma 2.2.17. Let $M \in \widetilde{\mathbb{B}}^{m,\mathrm{std}}_{\mathrm{even}}$ be standard cup diagram without markers. Then the line diagram sums L_M, where M varies over all standard cup diagrams in $\widetilde{\mathbb{B}}^{m,\mathrm{std}}_{\mathrm{even}}$ which can be obtained by decorating M with markers, are linearly independent.

Proof. We prove the statement by induction on the total number of components of M which can be decorated with a marker. If there is precisely one decorable component in M, then there exists precisely one cup diagram with the claimed properties and the claim follows. In case that M consists of a single cup without a dot we note that all the possible decorations with markers (without fixing the parity) are linearly independent.

So suppose there is more than one component in M which is allowed to have a marker. In particular, since M is standard, there exists a cup in M without a dot which may be decorated with a marker. In the following we fix such a cup connecting i and j. Let $M = M_1, N_1, M_2, N_2, \cdots, M_r, N_r$ be the standard cup diagrams obtained from all the possible decorations of M with markers and a fixed parity of markers. The diagrams are listed in such a way that M_i and N_i are the same except that M_i does not have a marker on the fixed cup, but a marker on the leftmost component with a dot. On the other hand, N_i has a marker on the fixed cup but no marker on the leftmost component with a dot. In case that M has no dots we simply consider all standard cup diagrams $M_1, N_1, \cdots, M_r, N_r$ (not only the ones with a fixed parity of markers) and drop the condition on the leftmost component with a dot.

Let $M_1', N_1', \cdots, M_r', N_r'$ be the cup diagrams obtained by deleting the cup connecting i and j. We compute

$$\begin{aligned}
0 &= \sum_{l=1}^{r} \lambda_l L_{M_l} + \sum_{l=1}^{r} \mu_l L_{N_l} \\
&= \sum_{l=1}^{r} \lambda_l \left(- \sum_{U \in \mathcal{U}_{M_l'}} (-1)^{\Lambda_{M_l'}(U)} l_{\widetilde{U_j}} + \sum_{U \in \mathcal{U}_{M_l'}} (-1)^{\Lambda_{M_l'}(U)} l_{\widetilde{U_i}} \right) \\
&\quad + \sum_{l=1}^{r} \mu_l \left(- \sum_{U \in \mathcal{U}_{N_l'}} (-1)^{\Lambda_{N_l'}(U)} l_{\widetilde{U_j}} - \sum_{U \in \mathcal{U}_{N_l'}} (-1)^{\Lambda_{N_l'}(U)} l_{\widetilde{U_i}} \right) \\
&= \sum_{U \in \mathcal{U}_{M'}} \left(\sum_{l=1}^{r} (-\mu_l - \lambda_l)(-1)^{\Lambda_{M_l'}(U)} \right) l_{\widetilde{U_j}} + \sum_{U \in \mathcal{U}_{M'}} \left(\sum_{l=1}^{r} (-\mu_l + \lambda_l)(-1)^{\Lambda_{M_l'}(U)} \right) l_{\widetilde{U_i}}.
\end{aligned}$$

Since the family of vectors $l_{\widetilde{U_i}}, l_{\widetilde{U_j}}$ is linearly independent if U varies over all elements in $\mathcal{U}_{M'}$, we deduce the equations

$$\sum_{l=1}^{r} (-\mu_l - \lambda_l)(-1)^{\Lambda_{M_l'}(U)} = 0 \qquad \text{and} \qquad \sum_{l=1}^{r} (-\mu_l + \lambda_l)(-1)^{\Lambda_{M_l'}(U)} = 0 \tag{2.13}$$

for all $U \in \mathcal{U}_{M'}$. Using the left equation in (2.13) we compute

$$0 = \sum_{U \in \mathcal{U}_{M'}} \left(\sum_{l=1}^{r} (-\mu_l - \lambda_l)(-1)^{\Lambda_{M_l'}(U)} \right) l_U = \sum_{l=1}^{r} (-\mu_l - \lambda_l) \left(\sum_{U \in \mathcal{U}_{M'}} (-1)^{\Lambda_{M_l'}(U)} l_U \right) = \sum_{l=1}^{r} (-\mu_l - \lambda_l) L_{M_l'},$$

which implies $\lambda_l = -\mu_l$ for all $l \in \{1, \cdots, r\}$ because the $L_{M'_l}$ are linearly independent by induction. Similarly, we also obtain $\lambda_l = \mu_l$ by repeating the calculation with the right equation in (2.13). Hence, we deduce that $\lambda_l = \mu_l = 0$ for all $l \in \{1, \cdots, r\}$. ■

Proof (Proposition 2.2.16). It suffices to prove that the elements L_M, where M varies over all standard cup diagrams in $\mathbb{B}^{m,l,\text{std}}_{\text{even}}$ with precisely l cups without a dot, are linearly independent in $H_{2l}((\mathbb{S}^2)^m)$, $0 \leq l \leq m$. We define a total order on the subsets of $\{1, \cdots, m\}$ of cardinality l:

$$\{i_1 < \cdots < i_l\} < \{i'_1 < \cdots < i'_l\} :\Leftrightarrow \exists r \colon i_r < i'_r \text{ and } i_{r+1} = i'_{r+1}, \cdots, i_l = i'_l \tag{2.14}$$

which induces a total order on the line diagrams l_U, where $U \subseteq \{1, \cdots, m\}$ consists of l elements, i.e. an order on our basis of $H_{2l}((\mathbb{S}^2)^m)$.

Define $f \colon \mathbb{B}^{m,l,\text{std}}_{\text{even}} \to \mathbb{B}^{m,l,\text{std}}_{\text{undec}}$ as the map which forgets all markers.

Given a standard cup diagram M, we write $U_M \in \mathcal{U}_M$ to denote the set containing all right endpoints of the cups without a dot. Note that U_M is maximal in $\mathcal{U}_M$ with respect to the order (2.14). Assume that

$$0 = \sum_{M \in \widetilde{\mathbb{B}}^{m,\text{std}}_{\text{even}}} \lambda_M L_M = \sum_{N \in \widetilde{\mathbb{B}}^{m,\text{std}}_{\text{undec}}} \sum_{\substack{M \in \widetilde{\mathbb{B}}^{m,\text{std}}_{\text{even}} \\ f(M)=N}} \lambda_M L_M \tag{2.15}$$

for some $\lambda_M \in \mathbb{C}$.

Fix $N_{\max}$ such that $U_{N_{\max}}$ is maximal amongst all $N \in \widetilde{\mathbb{B}}^{m,\text{std}}_{\text{undec}}$. Since the cup diagrams without markers are determined by the right endpoints of cups there is a unique such N.

Note that the basis vector l_{U_M} occurs (with nonzero coefficient) in each line diagram sum $L_{M'}$ with $f(M') = M$ but it does not occur in any $L_{M'}$, where $f(M') \neq M$ because these $L_{M'}$ only contain basis vectors which are strictly smaller with respect to the total order (this follows from our maximality assumption on M and the fact that U_M is maximal in $\mathcal{U}_M$ for every M).

Hence, equation (2.15) decouples into two independent equations

$$\sum_{\substack{M \in \widetilde{\mathbb{B}}^{m,\text{std}}_{\text{even}} \\ f(M)=M'_{\max}}} \lambda_M L_M = 0 \qquad \sum_{\substack{M' \in \widetilde{\mathbb{B}}^{m,\text{std}}_{\text{un}} \\ M' \neq M'_{\max}}} \sum_{\substack{M \in \widetilde{\mathbb{B}}^{m,\text{std}}_{\text{even}} \\ f(M)=M'}} \lambda_M L_M = 0.$$

By iterating the above argument we obtain equations

$$\sum_{\substack{M \in \widetilde{\mathbb{B}}^{m,\text{std}}_{\text{even}} \\ f(M)=N}} \lambda_M L_M = 0,$$

for all $N \in \widetilde{\mathbb{B}}^{m,\text{std}}_{\text{undec}}$. Hence, Proposition 2.2.16 follows from Lemma 2.2.17. ■

2.2.5 An explicit basis in homology

Lemma 2.2.18. The map $\gamma_m \colon H_*(\mathcal{S}^{m,m}_{D,\text{even}}) \to H_*((\mathbb{S}^2)^m)$ induced by the natural inclusion $\mathcal{S}^{m,m}_{D,\text{even}} \hookrightarrow (\mathbb{S}^2)^m$ is injective.

Proof. By considering the commutative diagram (2.10) we deduce that $\operatorname{im}\gamma_m = \operatorname{im}\psi_m$ because ϕ_m is surjective (cf. [17, p. 23]). Since $\dim(\operatorname{im}\psi_m) \geq |\widetilde{\mathbb{B}}^{m,\text{std}}_{\text{even}}| = 2^{m-1}$ (the inequality follows from Proposition 2.2.16 and the equality by combining Lemma 2.2.14 and Remark 2.1.13) we deduce that $\dim(\operatorname{im}\gamma_m) = \dim(\operatorname{im}\psi_m) \geq 2^{m-1}$. Hence, since $\dim H_*(\mathcal{S}^{m,m}_{D,\text{even}}) = 2^{m-1}$ by Remark 2.1.12, the rank nullity theorem implies that $\dim(\operatorname{im}\gamma_m) = 2^{m-1}$ and $\ker(\gamma_m) = \{0\}$ which proves the claim. ■

Proposition 2.2.19. The set of elements $\phi_m(\mathsf{G}(\mathbf{a}))$, where $\mathbf{a}$ varies over all cup diagrams in $C_{\text{even}}(m)$, is a basis of $H_*(\mathcal{S}^{m,m}_{D,\text{even}})$.

Proof. The commutative diagram (2.10) combined with Lemma 2.2.18 gives a commutative diagram

$$\begin{tikzcd} \bigoplus\limits_{\mathbf{a}\in\mathbb{B}^m_{\text{even}}} H_*(S_{\mathbf{a}}) \arrow[r, "\phi_m"] \arrow[rr, bend left, "\psi_m"] & H_*(\mathcal{S}^{m,m}_{D,\text{even}}) \arrow[r, "\gamma_m", "\cong"'] & \operatorname{im}(\psi_m) \subseteq H_*((\mathbb{S}^2)^m), \end{tikzcd}$$

which allows us to write $\phi(\mathsf{G}(\mathbf{a})) = \gamma^{-1}(\psi(\mathsf{G}(\mathbf{a})))$ for all $\mathbf{a}$ cup diagrams on m vertices. Since $\{\mathsf{G}(\mathbf{a}) \mid \mathbf{a} \in C_{\text{even}}(m)\} = \widetilde{\mathbb{B}}^{m,\text{std}}_{\text{even}}$, it follows from Proposition 2.2.16 that the collection of elements $\psi_m(\mathsf{G}(\mathbf{a})) \in H_*((\mathbb{S}^2)^m)$, where $\mathbf{a}$ varies over all cup diagrams in $C_{\text{even}}(m)$, is linearly independent. This remains true after applying the linear isomorphism γ_m^{-1}.

Since

$$|\{\mathsf{G}(\mathbf{a}) \mid \mathbf{a} \in C_{\text{even}}(m)\}| = 2^{m-1} = \dim H_*(\mathcal{S}^{m,m}_{D,\text{even}}),$$

by Remark 2.1.13 and Lemma 2.2.14, we deduce that they form indeed a basis of $H_*(\mathcal{S}^{m,m}_{D,\text{even}})$. ■

From now on we identify $\mathbb{C}[C_{\text{even}}(m)]$ with $H_*(\mathcal{S}^{m,m}_{D,\text{even}})$ via the following isomorphism of vector spaces

$$\mathbb{C}[C_{\text{even}}(m)] \xrightarrow{\cong} H_*(\mathcal{S}^{m,m}_{D,\text{even}}), \; \mathbf{a} \mapsto \phi_m(\mathsf{G}(\mathbf{a})), \tag{2.16}$$

thereby obtaining a diagrammatic basis of the homology. Note that under this identification the cup diagrams with l cups form a basis of $H_{2l}(\mathcal{S}^{m,m}_{D,\text{even}})$, $0 \le l \le \lfloor \frac{m}{2} \rfloor$.

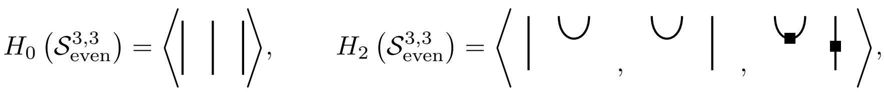
The next proposition (which is an immediate consequence of Proposition 2.2.8 and the commutative diagram (2.10)) provides a combinatorial description of the map γ_m.

Proposition 2.2.20. The isomorphism $\gamma_m \colon H_*(\mathcal{S}^{m,m}_{D,\text{even}}) \xrightarrow{\cong} \operatorname{im}(\gamma_m) \subseteq H_*((\mathbb{S}^2)^m)$ is explicitly given by the assignment

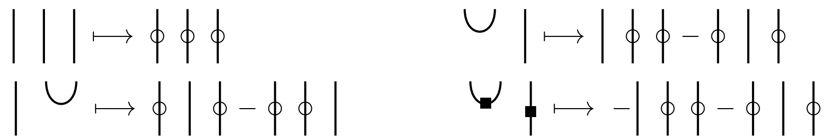
$$\mathbf{a} \mapsto L_{\mathbf{a}} = \sum_{U\in\mathcal{U}_{\mathbf{a}}} (-1)^{\Lambda_{\mathbf{a}}(U)} l_U,$$

where $\mathcal{U}_{\mathbf{a}}$ is the set of all subsets $U \subseteq \{1, \cdots, m\}$ containing precisely one endpoint of every cup of $\mathbf{a}$, and $\Lambda_{\mathbf{a}}(U)$ is the number of right endpoints of unmarked cups plus the number of endpoints of marked cups in the set U. □

Example 2.2.21. In case $m = 3$ we obtain the diagrammatic homology basis



and according to Proposition 2.2.20 the map γ_m is explicitly given by



▲

Now we can use the results of the previous subsection to construct a basis of the homology of $\mathcal{S}^{n-k,k}_{D,\text{even}}$.

Proposition 2.2.22. The set of elements $\phi_{n-k,k}(\mathbf{a})$, where $\mathbf{a}$ varies over all cup diagrams in $\mathbb{B}^{n-k,k}_{\text{even}}$ (viewed as standard cup diagrams), is a basis of the homology $H_*(\mathcal{S}^{n-k,k}_{D,\text{even}})$.

Moreover, we have a commutative diagram

$$\begin{tikzcd} H_{<k}(\mathcal{S}^{m,m}_{D,\text{even}}) \arrow[d, "\Gamma^{n-k,k}"', "\cong"] \arrow[dr, hook, "\gamma_m"] & \\ H_*(\mathcal{S}^{n-k,k}_{D,\text{even}}) \arrow[r, hook, "\gamma_{n-k,k}"] & H_*((\mathbb{S}^2)^m) \end{tikzcd}$$

where the isomorphism of vector spaces $\Gamma^{n-k,k}$ sends a basis element $\phi_m(\mathsf{G}(\mathbf{a}))$ of $H_{<k}(\mathcal{S}^{m,m}_{D,\text{even}})$ to the vector $\phi_{n-k,k}(\mathbf{a}) \in H_*(\mathcal{S}^{n-k,k}_{D,\text{even}})$.

Proof. The dimension of $H^*(\mathcal{S}^{n-k,k}_{D,\text{even}})$ (and hence also the homology because they are dual) is given by the counting the number of all cup diagrams on m vertices with at most $\lfloor \frac{k}{2} \rfloor$ cups. Thus, in order to check that the elements $\phi_{n-k,k}(\mathbf{a})$ form a basis, where $\mathbf{a}$ varies over all cup diagrams in $\mathbb{B}^{n-k,k}_{\text{even}}$, it suffices to see that they are linearly independent. Under the assumption that they are not linearly independent, it follows that the vectors $\gamma_{n-k,k}(\phi_{n-k,k}(\mathbf{a})) = \psi_{n-k,k}(\mathbf{a})$ are not linearly independent either. But Proposition 2.2.8 directly implies that $\psi_{n-k,k}(\mathbf{a}) = \psi_m(\mathsf{G}(\mathbf{a}))$ and the $\psi_m(\mathsf{G}(\mathbf{a}))$ are linearly independent (as a subset of the linearly independent vectors in Proposition 2.2.16), a contradiction. In particular, the map $\Gamma^{n-k,k}$ defines indeed an isomorphism.

Note that the commutativity of the diagram follows easily from $\psi_{n-k,k}(\mathbf{a}) = \psi_m(\mathsf{G}(\mathbf{a}))$. Since γ_m is known to be injective by Lemma 2.2.18, the commutativity of the diagram implies that $\gamma_{n-k,k}$ must be injective, too. ■

Remark 2.2.23. We point out that the isomorphism $H_*(\mathcal{S}^{n-k,k}_{D,\text{even}}) \cong H_{<k}(\mathcal{S}^{m,m}_{D,\text{even}})$ of Proposition 2.2.22 is slightly unnatural in the sense that it is defined ad hoc. It would be preferable to have a description of this map as a map induced by a map of spaces $\mathcal{S}^{n-k,k}_{D,\text{even}} \to \mathcal{S}^{m,m}_{D,\text{even}}$. This could be achieved by closely following [50] but we omit this approach for brevity. ▲

Remark 2.2.24. In [50, 51] the authors had to use a sign-modified inclusion instead of the natural inclusion in order for their construction to work. The reason for this is that in the original definition of topological Springer fibers, [35], the coordinates at endpoints of unmarked cups were identified without twisting with the antipodal map. The sign conventions used in this thesis seem preferable, not only for the purposes of Springer theory, but also from the point of view of functorial Khovanov homology, where passing from one endpoint of an unmarked cup to the other one can be interpreted as passing a singularity (which produces a sign) in the singular cobordism approach developed in [20, 21] based on ideas from [4]. ▲

2.2.6 A presentation of the cohomology rings

Using the results obtained in this section we can provide an explicit presentation of the cohomology ring of $\mathcal{S}^{n-k,k}_{D,\text{even}}$ (or equivalently $\mathcal{S}^{n-k,k}_{D,\text{odd}}$), where $m \neq k$. Using the homeomorphisms $\mathcal{F}l^{n-k,k}_{D,\text{even}} \cong \mathcal{F}l^{n-k-1,k-1}_{C} \cong \mathcal{S}^{n-k,k}_{D,\text{even}}$ from the previous chapter, this yields a description of the cohomology rings of the (algebraic) two-row Springer fibers of types C and D. In case $m = k$ the cohomology rings of the corresponding two-row Springer fibers of type D were computed in [17, Theorem B] (using a very different method).

Firstly, the natural inclusion $\mathcal{S}^{n-k,k}_{D,\text{even}} \subseteq (\mathbb{S}^2)^m$ induces a surjective homomorphism of algebras $H^*((\mathbb{S}^2)^m) \twoheadrightarrow H^*(\mathcal{S}^{n-k,k}_{D,\text{even}})$. The surjectivity follows directly from dualizing Proposition 2.2.22, where we proved that the induced map in homology $H_*(\mathcal{S}^{n-k,k}_{D,\text{even}}) \hookrightarrow H_*((\mathbb{S}^2)^m)$ is injective.

Secondly, the complex dimension of the algebraic variety $\mathcal{F}l^{n-k,k}_{D,\mathrm{even}}$ is known to be $\frac{k-1}{2}$, see e.g. [17, Theorem 6.5], and thus $H^i(\mathcal{S}^{n-k,k}_{D,\mathrm{even}}) \cong H^i(\mathcal{F}l^{n-k,k}_{D,\mathrm{even}}) = 0$ for all $i > 2 \cdot \frac{k-1}{2} = k-1$. In particular, the graded ideal $H^{\geq k}((\mathbb{S}^2)^m) = \oplus_{i \geq k} H^i((\mathbb{S}^2)^m)$ is contained in the kernel of our surjection and we obtain an induced surjective homomorphism of algebras

$$H^*((\mathbb{S}^2)^m) \Big/ H^{\geq k}((\mathbb{S}^2)^m) \twoheadrightarrow H^*(\mathcal{S}^{n-k,k}_{D,\mathrm{even}}), \tag{2.17}$$

which we claim to be an isomorphism. We prove this claim by comparing the dimensions of the two algebras. Recall that

$$H^*((\mathbb{S}^2)^m) \cong \mathbb{C}[X_1, \ldots, X_m] \Big/ \langle X_i^2 \mid 1 \leq i \leq m \rangle \tag{2.18}$$

with $\deg(X_i) = 2$. Thus, the dimension of $H^{2i}((\mathbb{S}^2)^m)$ is given by counting all monomials of degree i with pairwise different factors, i.e. $\dim(H^{2i}(\mathbb{S}^2)^m)) = \binom{m}{i}$. It follows that the dimension of the quotient in (2.17) is $\sum_{i=0}^{\frac{k-1}{2}} \binom{m}{i}$ which equals $\dim H^*(\mathcal{S}^{n-k,k}_{D,\mathrm{even}})$ by Proposition 2.1.11.

Hence, the surjection (2.17) is in fact an isomorphism which we can translate (using isomorphism (2.18) and the fact that the ideal $H^{\geq k}((\mathbb{S}^2)^m)$ is generated by all monomials of degree k) into the following description of $H^*(\mathcal{S}^{n-k,k}_{D,\mathrm{even}})$.

Theorem 2.2.25. If $m \neq k$, then we have an isomorphism of graded algebras

$$H^*\left(\mathcal{S}^{n-k,k}_{D,\mathrm{even}}\right) \cong \mathbb{C}[X_1, \ldots, X_m] \Bigg/ \left\langle X_i^2, X_I \,\middle|\, \begin{array}{c} 1 \leq i \leq m, \\ I \subseteq \{1, \ldots, m\}, |I| = \frac{k-1}{2} + 1 \end{array} \right\rangle,$$

where $X_I = \prod_{i \in I} X_i$. $\square$

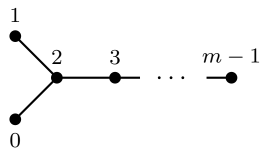
2.3 Diagrammatic description of the Weyl group action and the component group action

2.3.1 The Weyl group action on homology

Assume that $m \geq 4$ and let $\mathcal{W}_{D_m}$ be the Weyl group of type D_m, i.e. the Coxeter group with generators $s_0^D, s_1^D, \ldots, s_{m-1}^D$ and relations $\left(s_i^D s_j^D\right)^{\alpha_{ij}^D} = e$, where

$$\alpha_{ij}^D = \begin{cases} 1 & \text{if } i = j, \\ 3 & \text{if } i \text{ — } j \text{ are connected in } \Gamma_{D_m}, \\ 2 & \text{else.} \end{cases}$$

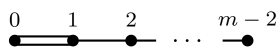
Γ_{D_m} :



The Weyl group $\mathcal{W}_{C_{m-1}}$ of type C_{m-1} (which is isomorphic to the Weyl group of type B_{m-1}) is the Coxeter group with generators $s_0^C, \ldots, s_{m-2}^C$ and relations $\left(s_i^C s_j^C\right)^{\alpha_{ij}^C} = e$, where

$$\alpha_{ij}^C = \begin{cases} 1 & \text{if } i = j, \\ 3 & \text{if } i \text{ — } j \text{ are connected in } \Gamma_{C_{m-1}}, \\ 4 & \text{if } i = j \text{ are connected in } \Gamma_{C_{m-1}}, \\ 2 & \text{else.} \end{cases}$$

$\Gamma_{C_{m-1}}$:



Remark 2.3.1. One easily checks, see also [63, §2.1], that the map

$$\mathcal{W}_{C_{m-1}} \hookrightarrow \mathcal{W}_{D_m}, \; s_i^C \mapsto \begin{cases} s_0^D s_1^D & \text{if } i = 0, \\ s_{i+1}^D & \text{if } i \neq 0, \end{cases}$$

as well as the map

$$\mathcal{W}_{D_m} \hookrightarrow \mathcal{W}_{C_m}\ ,\ s_i^D \mapsto \begin{cases} s_0^C s_1^C s_0^C & \text{if } i = 0, \\ s_i^C & \text{if } i \neq 0, \end{cases}$$

define embeddings of groups. ▲

The group $\mathcal{W}_{D_m}$ (and thus by Remark 2.3.1 also the subgroup $\mathcal{W}_{C_{m-1}}$) acts from the right on $(\mathbb{S}^2)^m$ according to the following rules:

$$(x_1, x_2, x_3, \ldots, x_m).s_0^D = (-x_2, -x_1, x_3, \ldots, x_m),$$

$$(x_1, \ldots, x_i, x_{i+1}, \ldots, x_m).s_i^D = (x_1, \ldots, x_{i+1}, x_i, \ldots, x_m), \qquad i \in \{1, \ldots, m-1\}.$$

In particular, one obtains an induced linear action on $H_*((\mathbb{S}^2)^m)$ which can easily be described explicitly in terms of line diagrams. Given $i \in \{1, \ldots, m-1\}$, let $\tau_i \colon \{1, \ldots, m\} \to \{1, \ldots, m\}$ be the involution of sets which exchanges i and $i+1$ and fixes all other elements. Given a subset $U \subseteq \{1, \ldots, m\}$, we define $\tau_i U$ as the subset of $\{1, \ldots, m\}$ containing the images of the elements of U under the map τ_i. By standard algebraic topology we can then describe the induced action by the formulas

$$l_U.s_0^D = (-1)^{\chi_U} l_{\tau_1 U} \qquad l_U.s_i^D = l_{\tau_i U}\ ,\ i \in \{1, \ldots, m-1\},$$

where χ_U is the cardinality of the set $\{1, 2\} \cap U$.

Proposition 2.3.2. Given a basis element $\mathbf{a} \in H_*(\mathcal{S}^{n-k,k}_{D,\mathrm{even}})$ and a generator $s_i^D \in \mathcal{W}_{D_m}$, $i \in \{0, \ldots, m-1\}$, we claim that

$$\mathbf{a}.s_i^D := \gamma_{n-k,k}^{-1}\left(\gamma_{n-k,k}(\mathbf{a}).s_i^D\right)$$

yields a well-defined right action of $\mathcal{W}_{D_m}$ on $H_*(\mathcal{S}^{n-k,k}_{D,\mathrm{even}})$ which is explicitly given by Table 2.1 and Table 2.2 (the action on $H_*((\mathbb{S}^2)^m)$ is the one defined above).

Proof. We distinguish between four different cases depending on what the cup diagram $\mathbf{a}$ locally looks like at the vertices i and $i+1$.

Case 1: Vertices i and i+1 are both incident with a ray.

If i and $i+1$ are both connected to rays, then we have $\tau_i U = U$ for all $U \in \mathcal{U}_\mathbf{a}$ and compute

$$L_\mathbf{a}.s_i^D = \sum_{U \in \mathcal{U}_\mathbf{a}} (-1)^{\Lambda_\mathbf{a}(U)} l_{\tau_i U} = \sum_{U \in \mathcal{U}_\mathbf{a}} (-1)^{\Lambda_\mathbf{a}(U)} l_U = L_\mathbf{a}, \tag{2.19}$$

$i \in \{1, \ldots, m-1\}$. Now the claim follows from the isomorphism of Proposition 2.2.20. Since $\chi_U = 0$ for all $U \in \mathcal{U}_\mathbf{a}$, calculation (2.19) also proves that s_0^D acts as indicated in the table.

Case 2: Vertices i and i+1 are connected by a cup in a.

We decompose $\mathcal{U}_\mathbf{a} = \mathcal{U}_{\mathbf{a},i} \sqcup \mathcal{U}_{\mathbf{a},i+1}$, where $\mathcal{U}_{\mathbf{a},i}$ (resp. $\mathcal{U}_{\mathbf{a},i+1}$) consists of all sets $U \in \mathcal{U}_\mathbf{a}$ containing i (resp. $i+1$). Then we have

$$L_\mathbf{a} = \sum_{U \in \mathcal{U}_\mathbf{a}} (-1)^{\Lambda_\mathbf{a}(U)} l_U = \sum_{U \in \mathcal{U}_{\mathbf{a},i}} (-1)^{\Lambda_\mathbf{a}(U)} l_U + \sum_{U \in \mathcal{U}_{\mathbf{a},i+1}} (-1)^{\Lambda_\mathbf{a}(U)} l_U.$$

We abbreviate $L_{\mathbf{a},i} = \sum_{U \in \mathcal{U}_{\mathbf{a},i}} (-1)^{\Lambda_\mathbf{a}(U)} l_U$ and $L_{\mathbf{a},i+1} = \sum_{U \in \mathcal{U}_{\mathbf{a},i+1}} (-1)^{\Lambda_\mathbf{a}(U)} l_U$.

Firstly, note that $\tau_i \colon \mathcal{U}_{\mathbf{a},i} \to \mathcal{U}_{\mathbf{a},i+1}$ is a self-inverse bijection. Secondly, for all $U \in \mathcal{U}_\mathbf{a}$ we have equalities $(-1)^{\Lambda_\mathbf{a}(\tau_i U)} = (-1)^{\Lambda_\mathbf{a}(U)-1}$ (resp. $(-1)^{\Lambda_\mathbf{a}(\tau_i U)} = (-1)^{\Lambda_\mathbf{a}(U)}$) if i and $i+1$ are connected

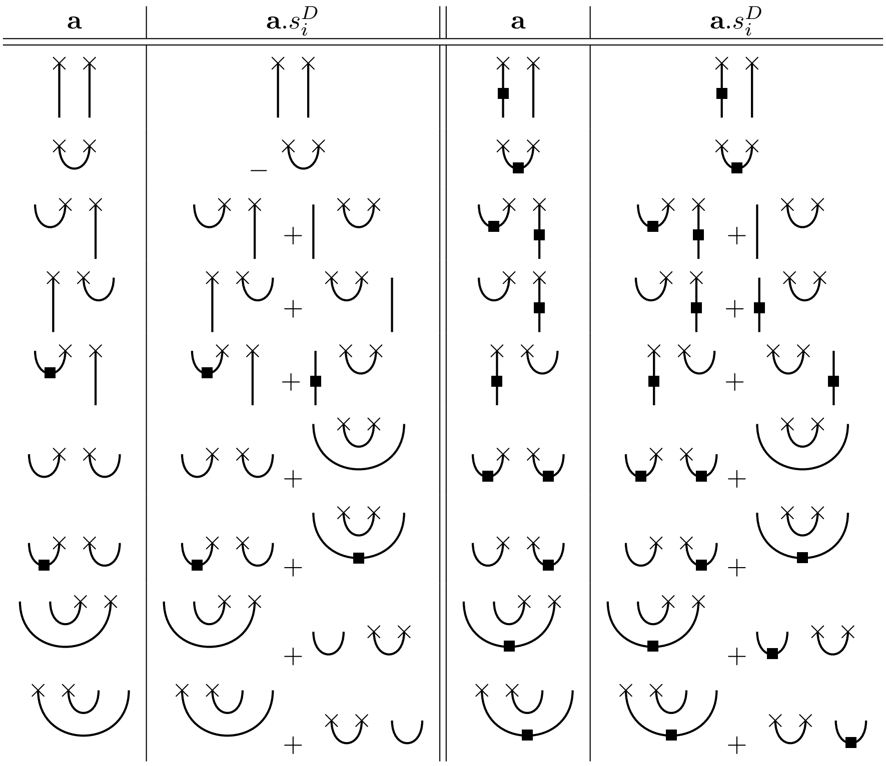


Table 2.1: The action of a generator $s_i^D \in \mathcal{W}_{D_m}$, $i \in \{1, \ldots, m-1\}$, on a cup diagram (crosses indicate vertices i and $i+1$).

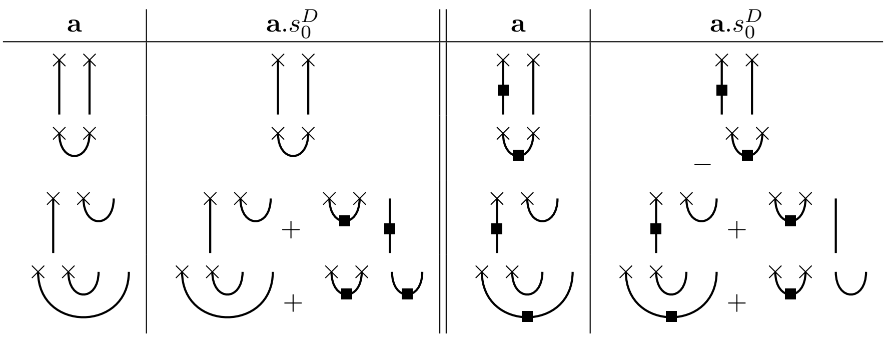


Table 2.2: The action of the generator $s_0^D \in \mathcal{W}_{D_m}$ on a cup diagram (crosses indicate vertices 1 and 2).

by an unmarked cup (resp. marked cup). Now we compute

$$\begin{aligned} L_{\mathbf{a},i}.s_i^D &= \sum_{U\in\mathcal{U}_{\mathbf{a},i}} (-1)^{\Lambda_{\mathbf{a}}(U)} l_{\tau_i U} = \sum_{U\in\mathcal{U}_{\mathbf{a},i+1}} (-1)^{\Lambda_{\mathbf{a}}(\tau_i U)} l_U \\ &= \begin{cases} \sum_{U\in\mathcal{U}_{\mathbf{a},i+1}} (-1)^{\Lambda_{\mathbf{a}}(U)-1} l_U = -L_{\mathbf{a},i+1} & \text{if } i \text{---} i+1, \\ \sum_{U\in\mathcal{U}_{\mathbf{a},i+1}} (-1)^{\Lambda_{\mathbf{a}}(U)} l_U = L_{\mathbf{a},i+1} & \text{if } i \text{-}\blacksquare\text{-} i+1. \end{cases} \end{aligned}$$

Analogously, we obtain $L_{\mathbf{a},i+1}.s_i^D = -L_{\mathbf{a},i}$ (resp. $L_{\mathbf{a},i+1}.s_i^D = L_{\mathbf{a},i}$) if the cup connecting i and $i+1$ is unmarked (resp. marked). Hence, we compute

$$L_{\mathbf{a}}.s_i^D = L_{\mathbf{a},i}.s_i^D + L_{\mathbf{a},i+1}.s_i^D = \begin{cases} -L_{\mathbf{a}} & \text{if } i \text{---} i+1, \\ L_{\mathbf{a}} & \text{if } i \text{-}\blacksquare\text{-} i+1. \end{cases}$$

Note that $l_U.s_0^D = -l_{\tau_1 U}$ for all $U \in \mathcal{U}_{\mathbf{a}}$. Modifying the above calculation with the additional sign yields the claim for the s_0^D-action.

Case 3: Vertices i and i+1 are incident with one cup and one ray.

In the following we assume that i and j, $j < i$, are connected by an unmarked cup (if $i+1$ and j, $i+1 < j$, are connected by a cup we only need to switch i and $i+1$ in the proof below). Decompose $\mathcal{U}_{\mathbf{a}} = \mathcal{U}_{\mathbf{a},i} \sqcup \mathcal{U}_{\mathbf{a},j}$ and $L_{\mathbf{a}} = L_{\mathbf{a},i} + L_{\mathbf{a},j}$ as in the case above.

Let $\mathbf{b}$ be the cup diagram different from $\mathbf{a}$ appearing in the sum $\mathbf{a}.s_i^D$ and decompose $\mathcal{U}_{\mathbf{b}} = \mathcal{U}_{\mathbf{b},i} \sqcup \mathcal{U}_{\mathbf{b},i+1}$ as well as $L_{\mathbf{b}} = L_{\mathbf{b},i} + L_{\mathbf{b},i+1}$.

Since $\tau_i U = U$ for all $U \in \mathcal{U}_{\mathbf{a},j}$, we compute $L_{\mathbf{a},j}.s_i^D = L_{\mathbf{a},j}$ as in (2.19).

Moreover, we see that the assignment $U \mapsto \tau_i U$ defines an involution $\mathcal{U}_{\mathbf{a},i} \to \mathcal{U}_{\mathbf{b},i+1}$ and we have $\Lambda_{\mathbf{a}}(\tau_i U) = \Lambda_{\mathbf{b}}(U)$ for all $U \in \mathcal{U}_{\mathbf{b},i+1}$. Thus, we can compute

$$\begin{aligned} L_{\mathbf{a},i}.s_i^D &= \sum_{U\in\mathcal{U}_{\mathbf{a},i}} (-1)^{\Lambda_{\mathbf{a}}(U)} l_{\tau_i U} = \sum_{U\in\mathcal{U}_{\mathbf{b},i+1}} (-1)^{\Lambda_{\mathbf{a}}(\tau_i U)} l_U \\ &= \sum_{U\in\mathcal{U}_{\mathbf{b},i+1}} (-1)^{\Lambda_{\mathbf{b}}(U)} l_U = L_{\mathbf{b},i+1}, \end{aligned}$$

where the second equation is a simple change of the summation index using the bijection τ_i.

Next, we see that $\mathcal{U}_{\mathbf{a},i} = \mathcal{U}_{\mathbf{b},i}$ and $\Lambda_{\mathbf{a}}(U) = \Lambda_{\mathbf{b}}(U) - 1$ for all $U \in \mathcal{U}_{\mathbf{a},i}$ which implies $L_{\mathbf{a},i} = -L_{\mathbf{b},i}$ and we calculate

$$\begin{aligned} L_{\mathbf{a}}.s_i^D &= L_{\mathbf{a},j}.s_i^D + L_{\mathbf{a},i}.s_i^D = L_{\mathbf{a},j} + L_{\mathbf{b},i+1} = L_{\mathbf{a},j} + L_{\mathbf{b},i+1} + L_{\mathbf{a},i} - L_{\mathbf{a},i} \\ &= L_{\mathbf{a},j} + L_{\mathbf{b},i+1} + L_{\mathbf{a},i} + L_{\mathbf{b},i} = L_{\mathbf{a}} + L_{\mathbf{b}}. \end{aligned}$$

If the cup connecting i and j is marked, we argue similarly. We omit to verify that s_0^D acts as claimed in Table 2.2.

Case 4: Vertices i and i+1 are incident with two different cups.

Let i and j as well as $i+1$ and k be connected by a cup. We decompose $\mathcal{U}_{\mathbf{a}} = \mathcal{U}_{\mathbf{a},j,i+1} \sqcup \mathcal{U}_{\mathbf{a},j,k} \sqcup \mathcal{U}_{\mathbf{a},i,i+1} \sqcup \mathcal{U}_{\mathbf{a},i,k}$ and similarly $\mathcal{U}_{\mathbf{b}} = \mathcal{U}_{\mathbf{b},i,j} \sqcup \mathcal{U}_{\mathbf{b},i,k} \sqcup \mathcal{U}_{\mathbf{b},i+1,j} \sqcup \mathcal{U}_{\mathbf{b},i+1,k}$, where $\mathbf{b}$ is the cup diagram different from $\mathbf{a}$ appearing in the sum $\mathbf{a}.s_i^D$.

Note that $\tau_i\colon \mathcal{U}_{\mathbf{a},j,k} \to \mathcal{U}_{\mathbf{a},j,k}$ and $\tau_i\colon \mathcal{U}_{\mathbf{a},i,i+1} \to \mathcal{U}_{\mathbf{a},i,i+1}$ are identities. Thus, we deduce

$$L_{\mathbf{a},j,k}.s_i^D = \sum_{U\in\mathcal{U}_{\mathbf{a},j,k}} (-1)^{\Lambda_{\mathbf{a}}(U)} l_{\tau_i U} = \sum_{U\in\mathcal{U}_{\mathbf{a},j,k}} (-1)^{\Lambda_{\mathbf{a}}(U)} l_U = L_{\mathbf{a},j,k}$$

and analogously $L_{\mathbf{a},i,i+1}.s_i^D = L_{\mathbf{a},i,i+1}$. Since $(-1)^{\chi(U)} = 1$ in the above cases, these equations are also true if we act with s_0^D.

It is easy to check (by going through the different cases in the table) that $\Lambda_{\mathbf{a}}(\tau_i U) = \Lambda_{\mathbf{b}}(U)$ for all $U \in \mathcal{U}_{\mathbf{b},j,i} \cup U_{\mathbf{b},i+1,k}$ if we act with s_i, $1 \leq i \leq k-1$. If we act with s_0, then we have $\Lambda_{\mathbf{a}}(\tau_1 U) = \Lambda_{\mathbf{b}}(U) \pm 1$. Moreover, $\tau_i \colon \mathcal{U}_{\mathbf{a},j,i+1} \to \mathcal{U}_{\mathbf{b},j,i}$ and $\tau_i \colon \mathcal{U}_{\mathbf{a},i,k} \to \mathcal{U}_{\mathbf{b},i+1,k}$ are self-inverse bijections. Hence, we compute

$$\begin{aligned} L_{\mathbf{a},j,i+1}.s_i^D &= \sum_{U \in \mathcal{U}_{\mathbf{a},j,i+1}} (-1)^{\Lambda_{\mathbf{a}}(U)} l_{\tau_i U} = \sum_{U \in \mathcal{U}_{\mathbf{b},j,i}} (-1)^{\Lambda_{\mathbf{a}}(\tau_i U)} l_U \\ &= \sum_{U \in \mathcal{U}_{\mathbf{b},j,i}} (-1)^{\Lambda_{\mathbf{b}}(U)} l_U = L_{\mathbf{b},j,i}. \end{aligned}$$

If we act with s_0^D, then the additional factor of (-1) is killed by $(-1)^{\chi(U)} = -1$. Similarly, one checks that $L_{\mathbf{a},i,k}.s_i^D = L_{\mathbf{b},i+1,k}$.

Next, we note that there are equalities of sets $\mathcal{U}_{\mathbf{a},i,k} = \mathcal{U}_{\mathbf{b},i,k}$ and $\mathcal{U}_{\mathbf{a},j,i+1} = \mathcal{U}_{\mathbf{b},j,i+1}$. Furthermore, we have $\Lambda_{\mathbf{a}}(U) = \Lambda_{\mathbf{b}}(U) \pm 1$ for all $U \in \mathcal{U}_{\mathbf{a},i,k} \cup \mathcal{U}_{\mathbf{a},j,i+1}$. Again, this follows directly by looking at the table. Putting these two facts together we obtain $L_{\mathbf{b},i,k} = -L_{\mathbf{a},i,k}$ and $L_{\mathbf{b},j,i+1} = -L_{\mathbf{a},j,i+1}$.

We are ready to compute

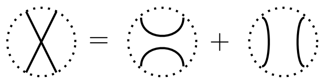
$$\begin{aligned} \mathbf{a}.s_i^D &= L_{\mathbf{a},j,i+1}.s_i^D + L_{\mathbf{a},j,k}.s_i^D + L_{\mathbf{a},i,i+1}.s_i^D + L_{\mathbf{a},i,k}.s_i^D \\ &= L_{\mathbf{b},j,i} + L_{\mathbf{a},j,k} + L_{\mathbf{a},i,i+1} + L_{\mathbf{b},i+1,k} \\ &= L_{\mathbf{b},j,i} + L_{\mathbf{a},j,k} + L_{\mathbf{a},i,i+1} + L_{\mathbf{b},i+1,k} \\ &\quad + L_{\mathbf{a},i,k} - L_{\mathbf{a},i,k} + L_{\mathbf{a},j,i+1} - L_{\mathbf{a},j,i+1} \\ &= L_{\mathbf{b},j,i} + L_{\mathbf{a},j,k} + L_{\mathbf{a},i,i+1} + L_{\mathbf{b},i+1,k} \\ &\quad + L_{\mathbf{a},i,k} + L_{\mathbf{b},i,k} + L_{\mathbf{a},j,i+1} + L_{\mathbf{b},j,i+1} \\ &= L_{\mathbf{a}} + L_{\mathbf{b}}. \end{aligned}$$

■

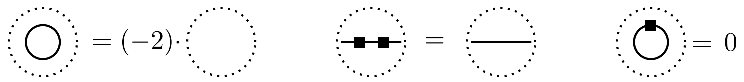
Theorem 2.3.3. Setting $\mathbf{a}.s := \gamma_{n-k,k}^{-1}(\gamma_{n-k,k}(\mathbf{a}).s)$, where $s \in \mathcal{W}_{D_m}$ and $\mathbf{a}$ is a cup diagram, yields a well-defined, grading-preserving right action of the Weyl group $\mathcal{W}_{D_m}$ on $H_*(\mathcal{S}_{D,\mathrm{even}}^{n-k,k}, \mathbb{C})$ which can be described explicitly using a skein calculus. Given a cup diagram $\mathbf{a}$ and an element $s \in \mathcal{W}_{D_m}$ we can write $s = s_{i_1}^D s_{i_2}^D \ldots s_{i_l}^D$ as a product of generators. Diagrammatically, we represent the generators as follows:

$i \in \{1, \ldots, m-1\}$.

In order to obtain the linear combination of cup diagrams $\mathbf{a}.s$, we take the cup diagram and stack the pictures corresponding to the generators $s_{i_1}^D, \ldots, s_{i_l}^D$ on top of $\mathbf{a}$. By resolving crossings according to the rule



and by using the following additional local relations



together with the rule that we kill all diagrams containing a connected component with both endpoints at the bottom of the diagram, we obtain a linear combination of cup diagrams which equals $\mathbf{a}.s$.

Proof. This is a direct consequence of Proposition 2.3.2. In fact, a straightforward case-by-case analysis shows that the $\mathcal{W}_{D_m}$-action given by Table 2.1 and Table 2.2 coincides with the $\mathcal{W}_{D_m}$-action described by the skein calculus. ■

Remark 2.3.4. The isomorphism of vector spaces $\Gamma^{n-k,k}\colon H_*(S^{n-k,k}_{D,\mathrm{even}}) \xrightarrow{\cong} H_{<k}(\mathcal{S}^{m,m}_{D,\mathrm{even}})$ from Proposition 2.2.22 evidently becomes an isomorphism of $\mathcal{W}_{D_m}$-modules. ▲

Remark 2.3.5. By Theorem 1.4.13 $\mathcal{S}^{n-k,k}_{D,\mathrm{even}}$ is homeomorphic to the (algebraic) Springer fiber of type C associated with a nilpotent operator of Jordan type $(n-k-1,k-1)$. Thus, by Springer theory, we expect an interesting $\mathcal{W}_{C_{m-1}}$-action on $H_*(\mathcal{S}^{n-k,k}_{D,\mathrm{even}})$. By restricting the $\mathcal{W}_{D_m}$-action to the subgroup $\mathcal{W}_{C_{m-1}}$ we obtain an $\mathcal{W}_{C_{m-1}}$-action on $H_*(\mathcal{S}^{n-k,k}_{D,\mathrm{even}})$ which admits an explicit skein-theoretic description by Theorem 2.3.3. In the next section we show that this restricted action is indeed the Springer action. ▲

2.3.2 The component group action on homology

Let x be a nilpotent element of Jordan type λ contained in $\mathfrak{sp}_n(\mathbb{C})$ (resp. $\mathfrak{so}_n(\mathbb{C})$) and let r_j denote the number of parts λ_i of λ which equal j. Then the corresponding component group $A^x_{\mathrm{O}_n}$ (resp. $A^x_{\mathrm{Sp}_n}$) is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^t$, where t is the number of even (resp. odd) j such that $r_j > 0$, see [58, I.2.9].

In the two-row case we have

$$A^{n-k,k}_{\mathrm{O}_n} \cong A^{n-k-1,k-1}_{\mathrm{Sp}_n} \cong \begin{cases} \{e\} & \text{if } m \text{ is even,} \\ \mathbb{Z}/2\mathbb{Z} & \text{if } m = k \text{ is odd,} \\ (\mathbb{Z}/2\mathbb{Z})^2 & \text{if } m \neq k \text{ is odd,} \end{cases}$$

and

$$A^{n-k,k}_{\mathrm{SO}_n} \cong \begin{cases} \{e\} & \text{if } m = k, \\ \mathbb{Z}/2\mathbb{Z} & \text{if } m \neq k, \end{cases}$$

where $A^{n-k,k}_{\mathrm{SO}_n} \subseteq A^{n-k,k}_{\mathrm{O}_n}$ is identified with the subgroup generated by products of an even number of generators of $A^{n-k,k}_{\mathrm{O}_n}$, see [54].

Let a_j be the generator of the copy of $\mathbb{Z}/2\mathbb{Z}$ corresponding to an even j with $r_j > 0$. We have a left $A^{n-k-1,k-1}_{\mathrm{Sp}_n}$-action on $(\mathbb{S}^2)^m$, where a generator a_j acts as the antipode on the first component, i.e. $a_j.(x_1, x_2 \ldots, x_m) = (-x_1, x_2 \ldots, x_m)$.

Thus, in homology we have

$$a_j.l_U = \begin{cases} l_U & \text{if } 1 \notin U, \\ -l_U & \text{if } 1 \in U. \end{cases}$$

Note that the special orthogonal component group acts trivially, although it is not always trivial itself.

Lemma 2.3.6. The left $A^{n-k-1,k-1}_{\mathrm{Sp}_n}$-action and the right $\mathcal{W}_{C_{m-1}}$-action on $H_*((\mathbb{S}^2)^m)$ defined above commute. In particular, the induced actions on $H_*(\mathcal{S}^{n-k,k}_{D,\mathrm{even}})$ commute.

Proof. The $A^x_{\mathrm{Sp}_n}$-action and the $\mathcal{W}_{C_{m-1}}$-action on $(\mathbb{S}^2)^m$ defined above obviously commute. ■

Proposition 2.3.7. The $A^x_{\mathrm{Sp}_n}$-action on $H_*((\mathbb{S}^2)^m)$ restricts to an action on $H_*(\mathcal{S}^{n-k,k}_{D,\mathrm{even}})$. This action can be described explicitly as follows: Given a cup diagram $\mathbf{a}$, a generator a_j corresponding to a copy of $\mathbb{Z}/2\mathbb{Z}$ in $A^x_{\mathrm{Sp}_n}$ acts by creating (if the respective component is unmarked) or deleting (if the respective component is already marked) a marker on the cup connected to the first vertex as well as on the leftmost ray. If there is a ray connected to the first vertex, then the action is trivial.

Proof. We denote by $\mathbf{a}^-$ the diagram obtained by acting with a_j on $\mathbf{a}$ according to the combinatorial description of the proposition. Suppose there is a cup connecting the first vertex with some vertex $i > 1$. We decompose $\mathcal{U}_\mathbf{a} = \mathcal{U}_{\mathbf{a},1} \sqcup \mathcal{U}_{\mathbf{a},i}$, where $\mathcal{U}_{\mathbf{a},1}$ (resp. $\mathcal{U}_{\mathbf{a},i}$) consists of all sets $U \in \mathcal{U}_\mathbf{a}$ containing 1 (resp. i). Then we have

$$L_\mathbf{a} = \sum_{U \in \mathcal{U}_\mathbf{a}} (-1)^{\Lambda_\mathbf{a}(U)} l_U = \sum_{U \in \mathcal{U}_{\mathbf{a},1}} (-1)^{\Lambda_\mathbf{a}(U)} l_U + \sum_{U \in \mathcal{U}_{\mathbf{a},i}} (-1)^{\Lambda_\mathbf{a}(U)} l_U$$

and acting by a generator a_j yields

$$a_j.L_\mathbf{a} = \sum_{U \in \mathcal{U}_{\mathbf{a},1}} (-1)^{\Lambda_\mathbf{a}(U)+1} l_U + \sum_{U \in \mathcal{U}_{\mathbf{a},i}} (-1)^{\Lambda_\mathbf{a}(U)} l_U = L_{\mathbf{a}^-}$$

which proves the claim, i.e. the geometric action can be described combinatorially as stated in the proposition. ■

Remark 2.3.8. The action of the component group was described explicitly in [67] (see also [49]) in terms of signed domino tableaux. Our description fits with this if one translates the results into the language of cup diagrams using the bijections from Subsection 1.2.3. ▲

2.4 Relation to Springer theory

2.4.1 A diagrammatic description of the parabolic Hecke module

In this subsection we recall the diagrammatic description of the parabolic Hecke module of type D from [41, Theorem 5.4] and provide a similar result for type C.

Smallest coset representatives

Let Λ^m be the set of all ordered sequences of length m consisting of symbols from the alphabet $\{\wedge, \vee\}$ and let Λ^m_{even} the subset consisting of all sequences in which the symbol "$\wedge$" occurs and even number of times. The Weyl group $\mathcal{W}_{D_m}$ of type D_m acts transitively on Λ^m_{even} from the right, where s_i^D, $i \in \{1, \ldots, m-1\}$, permutes the symbols at positions i and $i+1$ (counted from left to right) and s_0^D exchanges a pair of symbols $\vee\vee$ at positions 1 and 2 for the pair $\wedge\wedge$ (and vice versa) and acts as the identity otherwise. The parabolic subgroup $\mathcal{P}_{D_m}$ of $\mathcal{W}_{D_m}$ generated by $s_1^D, \ldots, s_{m-1}^D$ stabilizes the sequence $\lambda^m_\vee \in \Lambda^m_{\text{even}}$ consisting of m successive $\vee$'s. This yields a bijection

$$\mathcal{P}^{\min}_{D_m} \xrightarrow{\cong} \Lambda^m_{\text{even}}\,, \;\; w \mapsto \lambda^m_\vee.w\,, \tag{2.20}$$

between Λ^m_{even} and the set $\mathcal{P}^{\min}_{D_m}$ of smallest representatives for the right cosets in $\mathcal{P}_{D_m} \setminus \mathcal{W}_{D_m}$ which consists of all $w \in \mathcal{W}_{D_m}$ satisfying $l_{D_m}(s_i^D w) > l_{D_m}(w)$ for all $i \in \{1, \ldots, m-1\}$. Here, l_{D_m} is the length function of the Coxeter group $\mathcal{W}_{D_m}$, see [3],[30] for details.

Remark 2.4.1. Given $w \in \mathcal{P}^{\min}_{D_m}$, we have $ws_i^D \in \mathcal{P}^{\min}_{D_m}$ if and only if the action of s_i^D on the $\wedge\vee$-sequence associated with w changes the sequence. If this is the case, we have $l_{D_m}(ws_i^D) > l_{D_m}(w)$ if and only if the action of s_i^D on the sequence associated with w either changes a pair $\wedge\vee$ to $\vee\wedge$ or $\vee\vee$ changes to $\wedge\wedge$, see [18, Lemma 2.5]. ▲

Similarly, the Weyl group $\mathcal{W}_{C_{m-1}}$ of type C_{m-1} acts transitively on Λ^{m-1} from the right, where s_i^C, $i \in \{1, \ldots, m-2\}$, permutes the symbols at positions i and $i+1$ and s_0^C exchanges a $\wedge$ for a $\vee$ and vice versa at the first position. The parabolic subgroup $\mathcal{P}_{C_{m-1}}$ generated by $s_1^C, \ldots, s_{m-2}^C$ of $\mathcal{W}_{C_{m-1}}$ stabilizes the sequence $\lambda_\vee^{m-1} \in \Lambda^{m-1}$ which yields a bijection

$$\mathcal{P}_{C_{m-1}}^{\min} \xrightarrow{\cong} \Lambda^{m-1}, \ w \mapsto \lambda_\vee^{m-1}.w \,, \tag{2.21}$$

where $\mathcal{P}_{C_{m-1}}^{\min}$ denotes the set of smallest representatives for the right cosets in $\mathcal{P}_{C_{m-1}} \setminus \mathcal{W}_{C_{m-1}}$.

Remark 2.4.2. Given $w \in \mathcal{P}_{C_{m-1}}^{\min}$, we have $ws_i^C \in \mathcal{P}_{C_{m-1}}^{\min}$ if and only if the action of s_i^C changes the sequence associated with w in which case we have $l_{C_{m-1}}(ws_i^C) > l_{C_{m-1}}(w)$ if and only if either $\wedge\vee$ changes to $\vee\wedge$ or $\vee$ changes to $\wedge$. ▲

The map $(\,.\,)^\dagger \colon \Lambda^{m-1} \to \Lambda^m_{\text{even}}$ given by

$$w \mapsto w^\dagger = \begin{cases} \vee w & \text{if } w \text{ contains an even number of } \wedge\text{'s}, \\ \wedge w & \text{if } w \text{ contains an odd number of } \wedge\text{'s}, \end{cases}$$

is clearly a bijection which corresponds to a bijection between the sets of smallest coset representatives $\mathcal{P}_{D_m}^{\min}$ and $\mathcal{P}_{C_{m-1}}^{\min}$ via bijections (2.20) and (2.21) above, see also [18, §9.7].

The parabolic Hecke module

We continue by recalling some facts from [32],[56]. Let $\mathcal{H}_{D_m}$ be the Hecke algebra for $\mathcal{W}_{D_m}$ over the ring of formal Laurent polynomials $\mathcal{L} = \mathbb{C}[q, q^{-1}]$. It has a $\mathcal{L}$-basis H_w indexed by the elements of the Weyl group $w \in \mathcal{W}_{D_m}$ which satisfy the algebra relations $H_v H_w = H_{vw}$ if $l_{D_m}(v) + l_{D_m}(w) = l_{D_m}(vw)$ and the quadratic relations $H_{s_i^D}^2 = H_e + (q^{-1} - q)H_{s_i^D}$, $i \in \{0, \ldots, m-1\}$. As $\mathcal{L}$-algebra, $\mathcal{H}_{D_m}$ is generated by the Kazhdan-Lusztig generators $C_{s_i^D} = H_{s_i^D} - q^{-1}H_e$, $i \in \{0, \ldots, m-1\}$.

Let $\mathcal{H}_{\mathcal{P}_{D_m}} \subseteq \mathcal{H}_{\mathcal{W}_{D_m}}$ be the subalgebra generated by $C_{s_i^D}$, $i \in \{1, \ldots, m-1\}$. The algebra $\mathcal{H}_{\mathcal{P}_{D_m}}$ acts on $\mathcal{L}$ from the right by setting $1_{\mathcal{L}}.H_s = q^{-1}1_{\mathcal{L}}$ which allows us to define the tensor product $\mathcal{M}_{D_m} = \mathcal{L} \otimes_{\mathcal{H}_{\mathcal{P}_{D_m}}} \mathcal{H}_{\mathcal{W}_{D_m}}$ which is naturally a right $\mathcal{H}_{\mathcal{W}_{D_m}}$-module.

The standard basis

This right $\mathcal{H}_{\mathcal{W}_{D_m}}$-module has a *standard basis* consisting of elements $M_w = 1 \otimes H_w$, where $w \in \mathcal{P}_{D_m}^{\min}$ and by [56, §3] the $\mathcal{H}_{\mathcal{W}_{D_m}}$-action on $\mathcal{M}_{D_m}$ is explicitly given by

$$M_w.C_{s_i^D} = \begin{cases} M_{ws_i^D} - q^{-1}M_w & ws \in \mathcal{P}_{D_m}^{\min} \text{ and } l_{D_m}(ws_i^D) > l_{D_m}(w), \\ M_{ws_i^D} - qM_w & ws \in \mathcal{P}_{D_m}^{\min} \text{ and } l_{D_m}(ws_i^D) < l_{D_m}(w), \\ 0 & ws \notin \mathcal{P}_{D_m}^{\min}. \end{cases} \tag{2.22}$$

The Kazhdan-Lusztig basis

The $\mathbb{C}$-linear involution $\overline{(\,.\,)} \colon \mathcal{H}_{\mathcal{W}_{D_m}} \to \mathcal{H}_{\mathcal{W}_{D_m}}$, $H \mapsto \overline{H}$, given by $q \mapsto q^{-1}$ and $H_w \mapsto H_{w^{-1}}^{-1}$ induces an involution on $\mathcal{M}_{D_m}$. For all $w \in \mathcal{P}_{D_m}^{\min}$ there exists precisely one element $\underline{M}_w \in \mathcal{M}_{D_m}$ such that $\overline{\underline{M}_w} = \underline{M}_w$ and $\underline{M}_w = M_w + \sum_{v \neq w} q^{-1}\mathbb{Z}[q^{-1}]M_v$. The set $\{\underline{M}_w \mid w \in \mathcal{P}_{D_m}^{\min}\}$ is the *Kazhdan-Lusztig basis* of $\mathcal{M}_{D_m}$ (see [56, Theorem 3.5]). The *Kazhdan-Lusztig polynomials* $m_{v,w} \in \mathbb{Z}[q, q^{-1}]$ are defined implicitly by $\underline{M}_w = \sum_v m_{v,w}M_v$.

Remark 2.4.3. In [41, Theorem 2.10] the parabolic Kazhdan-Lusztig polynomials were explicitly computed for the parabolic type (D_m, A_{m-1}) (see also the introduction of this thesis). Moreover, by [18, §9.7], for any given $v, w \in \mathcal{P}^{\min}_{C_{m-1}}$ we have $m^C_{v,w} = m^D_{v^\dagger, w^\dagger}$, i.e. the parabolic Kazhdan-Lusztig polynomials of type (C_{m-1}, A_{m-2}) are determined by the Kazhdan-Lusztig polynomials of type (D_{m-1}, A_{m-1}). ▲

Lemma 2.4.4. Let $(\,.\,)^\dagger \colon \mathcal{M}_{C_{m-1}} \to \mathcal{M}_{D_m}$ be the linear isomorphism sending M_w to $M_{w^\dagger}$. Then the linear isomorphism $(\,.\,)^\dagger \colon \mathcal{M}_{C_{n-1}} \to \mathcal{M}_{D_n}$ sends a Kazhdan-Lusztig basis element $\underline{M}_w$ to the Kazhdan-Lusztig element $\underline{M}_{w^\dagger}$ and we have commutative diagrams

$$\begin{tikzcd}
\mathcal{M}_{C_{m-1}} \arrow[r, "(\,.\,)^\dagger", "\cong"'] \arrow[d, "\cdot C_{s_0^C}"'] & \mathcal{M}_{D_m} \arrow[d, "\cdot(C_{s_0^D}+C_{s_1^D})"] \\
\mathcal{M}_{C_{m-1}} \arrow[r, "(\,.\,)^\dagger", "\cong"'] & \mathcal{M}_{D_m}
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{M}_{C_{m-1}} \arrow[r, "(\,.\,)^\dagger", "\cong"'] \arrow[d, "\cdot C_{s_i^C}"'] & \mathcal{M}_{D_m} \arrow[d, "\cdot C_{s_{i+1}^D}"] \\
\mathcal{M}_{C_{m-1}} \arrow[r, "(\,.\,)^\dagger", "\cong"'] & \mathcal{M}_{D_m}
\end{tikzcd}$$

$i \in \{1, \ldots, m-2\}$, where the vertical maps are given by acting from the right with the indicated element of the respective Hecke algebra.

Proof. The first part follows directly from the definition of the Kazhdan-Lusztig polynomials and Remark 2.4.3.

We only prove the commutativity of the right square and omit discussing the left square. We check the commutativity on standard basis vectors. Using (2.22) one easily checks that the composition

$$\mathcal{M}_{C_{m-1}} \xrightarrow{\cdot C_{s_i^C}} \mathcal{M}_{C_{m-1}} \xrightarrow{(\,.\,)^\dagger} \mathcal{M}_{D_m}$$

is given by

$$M_w \longmapsto \begin{cases} M_{(ws_i^C)^\dagger} - q^{-1} M_{w^\dagger} & ws_0^C \in \mathcal{P}^{\min}_{C_{m-1}} \text{ and } l(ws_i^C) > l(w), \\ M_{(ws_i^C)^\dagger} - q M_{w^\dagger} & ws_0^C \in \mathcal{P}^{\min}_{C_{m-1}} \text{ and } l(ws_i^C) < l(w), \\ 0 & ws_i^C \notin \mathcal{P}^{\min}_{C_{m-1}}, \end{cases}$$

for $i \in \{1, \ldots, m-2\}$, and the composition

$$\mathcal{M}_{C_{m-1}} \xrightarrow{(\,.\,)^\dagger} \mathcal{M}_{D_m} \xrightarrow{\cdot C_{s_{i+1}^D}} \mathcal{M}_{D_m}$$

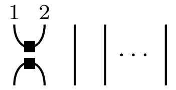
is given by

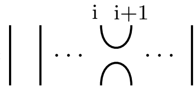
$$M_w \longmapsto \begin{cases} M_{w^\dagger s_{i+1}^D} - q^{-1} M_{w^\dagger} & w^\dagger s_{i+1}^D \in \mathcal{P}^{\min}_{D_m} \text{ and } l(ws_{i+1}^D) > l(w), \\ M_{w^\dagger s_{i+1}^D} - q M_{w^\dagger} & w^\dagger s_{i+1}^D \in \mathcal{P}^{\min}_{D_m} \text{ and } l(ws_{i+1}^D) < l(w), \\ 0 & w^\dagger s_{i+1}^D \notin \mathcal{P}^{\min}_{D_m}. \end{cases}$$

By comparing the two results using the combinatorial description, the claim follows easily. ■

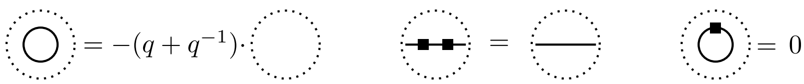
Let $C_{\mathrm{KL}}(m)$ be the set of all cup diagrams on m vertices which have an even number of markers if and only if the total number of cups of the diagram is even, i.e. the parity of markers alternates when passing to diagrams with one more or one less cup. In the following we identify $\mathcal{M}_{D_m}$ with $\mathcal{M}^{\mathrm{comb}}_m = \mathcal{L}[C_{\mathrm{KL}}(m)]$ (as $\mathcal{L}$-module) by sending the Kazhdan-Lusztig basis element $\underline{M}_w$ to the the cup diagram $\mathbf{a}_w$ associated with $w \in \mathcal{P}^{\min}_{D_m}$ which is constructed as follows: Firstly, connect all neighbored $\vee\wedge$ symbols by an unmarked cup successively, thereby ignoring all pairs which are already connected. Secondly, connect the remaining neighbored $\wedge$ pairwise by a marked cup starting from the left. Finally, attach rays to the symbols which are not yet connected to a cup, where a ray is marked if and only if it is connected to a $\wedge$. The assignment $w \mapsto \mathbf{a}_w$ defines a bijection between $\Lambda^m_{\mathrm{even}}$ and $C_{\mathrm{KL}}(m)$, see [41, Section 5.2].

Proposition 2.4.5. 1. The right action of the Hecke algebra $\mathcal{H}_{D_m}$ of type D on $\mathcal{M}_{D_m}$ can be described explicitly via the identification $\mathcal{M}_{D_m} \cong \mathcal{M}_m^{\text{comb}}$ as follows: Given a Kazhdan-Lusztig generator in $\mathcal{H}_{D_m}$ which we represent pictorially as

$C_{s_0^D} :=$ 

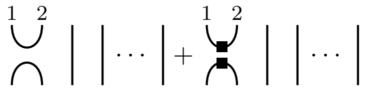
$C_{s_i^D} :=$ 

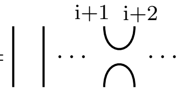
$i \in \{1, \dots, m-1\}$, then we can compute $\mathbf{a}.C_{s_i^D}$, where $\mathbf{a} \in C_{\mathrm{KL}}(m)$ is a cup diagram, by putting the picture corresponding to the generator $C_{s_i^D}$ on top of $\mathbf{a}$ and applying the relations



which yields a new cup diagram in $C_{\mathrm{KL}}(k)$ equal to $\mathbf{a}.C_{s_i^D}$. The relations in the second row only hold for connected components with both endpoints at the bottom of the diagram. The second relation in the second row means that we can simply remove such a component if it is marked, whereas the first relation in the second row always kills the entire diagram.

2. The right action of the Hecke algebra $\mathcal{H}_{C_{m-1}}$ of type C_{m-1} on $\mathcal{M}_{C_{m-1}}$ can be described explicitly via the identification $\mathcal{M}_{C_{m-1}} \cong \mathcal{M}_m^{\text{comb}}$ in a similar way: Given a cup diagram $\mathbf{a} \in C_{\mathrm{KL}}(k)$ and a generator $C_{s_i^C} \in \mathcal{H}_C$ which we represent pictorially as

$C_{s_0^C} :=$ 

$C_{s_i^C} :=$ 

$i \in \{1, \dots, m-2\}$,

then $\mathbf{a}.C_{s_i^C}$ can be computed by putting the picture corresponding to the generator $C_{s_i^C}$ on top of $\mathbf{a}$ and symplifying the resulting pictures using the same relations as above.

Proof. The first part is the analog of [41, Theorem 4.17] for the induced trivial module and can be proven by closely following the argument for the induced sign module. The second part follows from the first one together with Lemma 2.4.4. ∎

Remark 2.4.6. Note that $\mathcal{M}_m^{\text{comb}}$ has a filtration

$$\{0\} \subseteq \mathcal{M}^{\text{comb}}_{m,\lfloor \frac{m}{2} \rfloor} \subseteq \ldots \subset \mathcal{M}^{\text{comb}}_{m,l} \subseteq \ldots \subseteq \mathcal{M}^{\text{comb}}_{m,0} = \mathcal{M}^{\text{comb}}_m \tag{2.23}$$

of $\mathcal{L}$-submodules, where $\mathcal{M}^{\text{comb}}_{m,l}$ is the $\mathcal{L}$-span of all cup diagrams in $C_{\mathrm{KL}}(m)$ with l or more cups, $0 \le l \le \lfloor \frac{m}{2} \rfloor$. It follows from Proposition 2.4.5 that acting with a Kazhdan-Lusztig generator on a cup diagram with l cups always yields a cup diagram with l or more cups (the number of cups changes if and only if we use the right relation in the second row of Proposition 2.4.5). In particular, filtration (2.23) is actually a filtration of both right $\mathcal{H}_{D_m}$- and $\mathcal{H}_{C_{m-1}}$-modules. ▲

2.4.2 Specht modules and the induced trivial representation

In this section we identify our representations in each homology degree for every two-row Springer fiber by providing an explicit decomposition into irreducible Specht modules. In this section we use the results obtained in the previous one but specialized at $q = 1$ in which case the Hecke algebra becomes the group algebra and the parabolic Hecke module becomes the induced trivial representation.

Classification of irreducible representations

The following classification of irreducible $\mathcal{W}_{C_m}$-modules is well known, [44],[52, §8.2].

Proposition 2.4.7. The isomorphism classes of the irreducible representations of the Weyl group $\mathcal{W}_{C_m}$ are in bijective correspondence with pairs (λ, μ) of partitions such that $|\lambda| + |\mu| = m$. $\square$

We fix an irreducible representation $V_{(\lambda,\mu)}$ corresponding to (λ, μ). The reader interested in the construction of these representations is referred to the literature, e.g. [10],[47].

The restriction of $V_{(\lambda,\lambda)}$ to $\mathcal{W}_{D_m}$ decomposes into two non-isomorphic irreducible $\mathcal{W}_{D_m}$-modules of the same dimension which we denote by V_λ^+ and V_λ^-. If $\lambda \neq \mu$ the restriction of $V_{(\lambda,\mu)}$ to $\mathcal{W}_{D_m}$ remains irreducible and the restrictions of $V_{(\lambda,\mu)}$ and $V_{(\mu,\lambda)}$ to $\mathcal{W}_{D_m}$ are isomorphic. We denote this irreducible module by $V_{\{\lambda,\mu\}}$.

Proposition 2.4.8 ([46])**.** The $\mathcal{W}_{D_m}$-representations $V_{\{\lambda,\mu\}}$, where $|\lambda| + |\mu| = m$, together with V_λ^+ and V_λ^-, where $|\lambda| = \frac{m}{2}$, form a complete set of pairwise non-isomorphic irreducible $\mathcal{W}_{D_m}$-representations. $\square$

Irreducibles for two-row characters

In the following we assume that $((m-l),(l))$ is a pair of partitions each of which consists of a single number, $l \in \{0, \dots, m\}$. Then we have $\dim V_{((m-l),(l))} = \binom{m}{l}$, see e.g. [10, Theorem 4.2]. In particular, $\dim V_{\{(m-l),(l)\}} = \binom{m}{l}$ if m is odd and $\dim V^+_{\frac{m}{2},\frac{m}{2}} = \dim V^-_{\frac{m}{2},\frac{m}{2}} = \frac{1}{2}\binom{m}{\frac{m}{2}}$.

Lemma 2.4.9 ([17, Lemma 5.19])**.** The induced trivial representation of type D decomposes as multiplicity free sum of two-row characters. $\square$

If m is odd, it follows from Lemma 2.4.9 and the dimension formulas for the Specht modules, that all simples must occur in the induced trivial representation in order for the dimension to add up to 2^{m-1} (which is the dimension of the induced trivial module because the $\{\wedge\vee\}$-sequences in Λ^m_{even} label a basis).

If m is even, a similar argument shows that all simples must occur exactly once, except one of the modules $V^+_{\frac{m}{2},\frac{m}{2}}$ or $V^-_{\frac{m}{2},\frac{m}{2}}$ is missing (which one depends on the two possible choices of maximal parabolic subgroups of $\mathcal{W}_{D_m}$).

Theorem 2.4.10. The $\mathcal{W}_{D_m}$-module $H_*(\mathcal{S}^{m,m}_{D,\text{even}})$ constructed in the first subsection is isomorphic to the induced trivial module $\mathbb{C} \otimes_{\mathbb{C}[S_m]} \mathbb{C}[\mathcal{W}_{D_m}]$.

If m is odd, then we have isomorphisms

$$H_{2l}(\mathcal{S}^{m,m}_{D,\text{even}}) \cong V_{\{(m-l),(l)\}}$$

for all $l \in \{0, \dots, \lfloor \frac{m}{2} \rfloor\}$.

If m is even, then we have isomorphisms

$$H_m(\mathcal{S}^{m,m}_{D,\text{even}}) \cong V^{+,-}_{\frac{m}{2},\frac{m}{2}} \qquad \text{and} \qquad H_{2l}(\mathcal{S}^{m,m}_{D,\text{even}}) \cong V_{\{(m-l),(l)\}}$$

for all $l \in \{0, \dots, \frac{m}{2} - 1\}$.

If $m \neq k$, then we have isomorphisms

$$H_{2l}(\mathcal{S}^{n-k,k}_{D,\text{even}}) \cong H_{2l}(\mathcal{S}^{m,m}_{D,\text{even}}) \cong V_{\{(m-l),(l)\}}$$

for all $l \in \{0, \dots, \frac{k-1}{2}\}$.

Proof. Recall the filtration from Remark 2.4.6. Note that the subquotients $\mathcal{M}^{\text{comb}}_l / \mathcal{M}^{\text{comb}}_{l-1}$ of this filtration are isomorphic to the representations $H_{2l}(\mathcal{S}^{m,m}_{D,\text{even}})$ constructed in Theorem 2.3.3.

This follows easily by using the relation $H_{s_i^D} = C_{s_i^D} + q^{-1}H_e$ and comparing the local relations for $q = 1$.

The induced trivial representation decomposes into irreducibles corresponding to pairs of one-row partitions

$$\mathbb{C} \otimes_{\mathbb{C}[S_m]} \mathbb{C}[\mathcal{W}_{D_m}] = \bigoplus_{l=0}^{\lfloor \frac{m}{2} \rfloor} V_{\{(m-l),(l)\}}, \tag{2.24}$$

where $V_{\{(m-l),(l)\}}$ is one of the irreducible representations (cf. [17, Lemma 5.19] and [17, Remark 5.21]). In particular, it follows that the filtration (2.23) is already a composition series, i.e. a filtration which cannot be refined. Hence, the subquotients $\mathcal{M}_l^{\mathrm{comb}}/\mathcal{M}_{l-1}^{\mathrm{comb}} \cong H_{2l}(\mathcal{S}_{D,\mathrm{even}}^{m,m})$ are irreducible and isomorphic to the summands in the decomposition (2.24). More precisely, since a basis of $V_{\{(m-l),(l)\}}$ can be labeled by the cup diagrams in $C_{\mathrm{KL}}(m)$ with l cups (cf. [17, Lemma 5.20]) we obtain isomorphisms $V_{\{(m-l),(l)\}} \cong H_{2l}(\mathcal{S}_{D,\mathrm{even}}^{m,m})$ for all $0 \leq l \leq \lfloor \frac{m}{2} \rfloor$ by comparing dimensions.

In particular, $H_*(\mathcal{S}_{D,\mathrm{even}}^{m,m})$ is isomorphic to the induced trivial module because both have the same irreducible constituents.

The second claim follows from the isomorphism in Remark 2.3.4. ■

In the two-row character case the restriction of the simple $\mathcal{W}_{D_m}$-module $V_{\{\lambda,\mu\}}$ to the subgroup $\mathcal{W}_{C_{m-1}}$ decomposes as $V_{(\lambda-1,\mu)} \oplus V_{(\mu-1,\lambda)}$. The simples V_λ^+ and V_λ^- both restrict to $V_{(\lambda-1,\lambda)}$, see [63, §2.1].

Theorem 2.4.11. The $\mathcal{W}_{C_{m-1}}$-module $H_*(\mathcal{S}_{D,\mathrm{even}}^{m,m})$ constructed in the first subsection is isomorphic to the induced trivial module $\mathbb{C} \otimes_{\mathbb{C}[S_{m-1}]} \mathbb{C}[\mathcal{W}_{C_{m-1}}]$.

If m is odd, then we have isomorphisms

$$H_{2l}(\mathcal{S}_{D,\mathrm{even}}^{m,m}) \cong V_{(m-l-1,l)} \oplus V_{(l-1,m-l)}$$

for all $l \in \{0, \dots, \lfloor \frac{m}{2} \rfloor\}$.

If m is even, then we have isomorphisms

$$H_m(\mathcal{S}_{D,\mathrm{even}}^{m,m}) \cong V_{(\frac{m}{2}-1,\frac{m}{2})} \quad \text{and} \quad H_{2l}(\mathcal{S}_{D,\mathrm{even}}^{m,m}) \cong V_{(m-l-1,l)} \oplus V_{(l-1,m-l)}$$

for all $l \in \{0, \dots, \frac{m}{2} - 1\}$.

If $m \neq k$, then we have isomorphisms

$$H_{2l}(\mathcal{S}_{D,\mathrm{even}}^{n-k,k}) \cong H_{2l}(\mathcal{S}_{D,\mathrm{even}}^{m,m}) \cong V_{(m-l-1,l)} \oplus V_{(l-1,m-l)}$$

for all $l \in \{0, \dots, \frac{k-1}{2}\}$.

Proof. By Proposition 2.4.5 (specialized at $q = 1$) the $\mathcal{W}_{C_{m-1}}$-representation $\mathcal{M}_m^{\mathrm{comb}}$ is isomorphic to $\mathbb{C} \otimes_{\mathbb{C}[S_{m-1}]} \mathbb{C}[\mathcal{W}_{C_{m-1}}]$. Thus, it suffices to check that $\mathcal{M}_m^{\mathrm{comb}}$ is isomorphic to $H_*(\mathcal{S}_{D,\mathrm{even}}^{m,m})$ as $\mathcal{W}_{C_{m-1}}$-module. Since $\mathcal{M}_m^{\mathrm{comb}}$ is isomorphic to $H_*(\mathcal{S}_{D,\mathrm{even}}^{m,m})$ as $\mathcal{W}_{D_m}$-module by Theorem 2.4.10, i.e. in type D it is the induced module, it suffices to check that the $\mathcal{W}_{C_{m-1}}$-action on $\mathcal{M}_m^{\mathrm{comb}}$ is obtained by restricting the $\mathcal{W}_{D_m}$-action to the subgroup generated by $s_0^D s_1^D, s_2^D, \dots, s_{m-1}^D$. But this is clear from Proposition 2.4.5. ■

Remark 2.4.12. The identification of our representations with the induced trivial module shows (using [43]) that we have indeed reconstructed the Springer representation (in Lusztig's normalization, see [42], i.e. it is Springer's original representations tensored with the sign representation). ▲

2.4.3 Recovering the Springer correspondence

In this subsection we focus on the type C case. Given a pair of partitions $\tau = (\lambda, \mu)$ such that $|\lambda| + |\tau| = m - 1$, we can use the Shoji algorithm [53, 54] to compute the Jordan type of the nilpotent endomorphism for which the representation $V_{(\lambda,\mu)}$ appears in the top degree of the homology of the corresponding Springer fiber of type C (the type D case is similar and is omitted due to the trivial component group action).

Given a pair of partitions $\tau = (\lambda, \mu)$, define a pair $(d_\tau^{(1)}, \varepsilon_\tau^{(1)})$, where $d_\tau^{(1)} = (d_1^{(1)}, d_2^{(1)}, \dots) \in \mathbb{Z}^{\mathbb{N}}$ and $\varepsilon_\tau^{(1)} = (\varepsilon_1^{(1)}, \varepsilon_2^{(1)}, \dots) \in \{\pm 1\}^{\mathbb{N}}$, as

$$d_{2i-1}^{(1)} = 2\lambda_i, d_{2i}^{(1)} = 2\mu_i, \epsilon_i^{(1)} = 1.$$

Under the assumption that $(d_\tau^{(j)}, \varepsilon_\tau^{(j)})$ is already constructed, we define a new pair $(d_\tau^{(j+1)}, \varepsilon_\tau^{(j+1)})$ by the rules

$$\begin{cases} d_i^{(j+1)} = d_i^{(j)} + 1, d_{i+1}^{(j+1)} = d_i^{(j)} + 1 & \text{if } d_{i+1}^{(j)} = d_i^{(j)} + 2, \\ d_i^{(j+1)} = d_{i+1}^{(j)} - 2, d_{i+1}^{(j+1)} = d_i^{(j)} + 2 & \text{if } d_{i+1}^{(j)} > d_i^{(j)} + 2, \\ d_i^{(j+1)} = d_i^{(j)} & \text{else,} \end{cases}$$

and

$$\begin{cases} \varepsilon_i^{(j+1)} = -\varepsilon_i^{(j)}, \varepsilon_{i+1}^{(j+1)} = -\varepsilon_{i+1}^{(j)} & \text{if } d_{i+1}^{(j)} > d_i^{(j)} + 2, \\ \varepsilon_i^{(j+1)} = \varepsilon_i^{(j)} & \text{else.} \end{cases}$$

This is a well-defined algorithm which stabilizes, i.e. there exists an integer $q > 0$ such that $(d_\tau^{(q)}, \varepsilon_\tau^{(q)}) = (d_\tau^{(q+1)}, \varepsilon_\tau^{(q+1)}) = \cdots$. We denote the stable pair by $(d_\tau, \varepsilon_\tau)$. d_τ is a partition labeling a nilpotent orbit of type C. Moreover, ε_τ can be used to define a character of the associated component group by setting $\phi_\tau(\alpha_{d_i}) = \varepsilon_i$ for all even parts d_i of the partition d. We have $d_i = d_j$ if and only if $\varepsilon_i = \varepsilon_j$ which makes this a well-defined assignment.

Remark 2.4.13. Very similar algorithms exist for the other classical groups, see [53]. ▲

Lemma 2.4.14. Let $\tau = ((\lambda), (\mu))$ be a pair of one-row partitions, i.e. $\lambda, \mu \in \mathbb{Z}_{\geq 0}$. Then we have

$$\begin{cases} d_\tau = (2\lambda + 1, 2\lambda + 1, 0, \dots), \ \epsilon_\tau = (1, \dots) & \text{if } \mu = \lambda + 1, \\ d_\tau = (2\mu - 2, 2\lambda + 2, 0, \dots), \ \epsilon_\tau = (-1, -1, 1, \dots) & \text{if } \mu > \lambda + 1, \\ d_\tau = (2\lambda, 2\mu, 0, \dots), \ \epsilon_\tau = (1, \dots) & \text{if } \mu \leq \lambda, \end{cases}$$

where the non displayed entries of the sequences are all 0 or 1.

Proof. This follows directly from the Shoji algorithm above. ■

Proposition 2.4.15. The Springer representation in top degree cohomology of the $(m-1, m-1)$ algebraic Springer fiber of type C_{m-1} is (in Lusztig's normalization) isomorphic to the irreducible labeled by $(\frac{m}{2} - 1, \frac{m}{2})$ if m is even. If m is odd, then the top degree representation decomposes as the direct sum of $(\frac{m-1}{2} - 1, \frac{m-1}{2} + 1)$ and $(\frac{m-1}{2}, \frac{m-1}{2})$, where the first irreducible is the isotypic component corresponding to the sign representation of $\mathbb{Z}/2\mathbb{Z}$. If $(n-k-1, k-1)$ with $m \neq k$, we have the irreducibles $(\frac{k-3}{2}, \frac{n-k+1}{2})$ and $(\frac{n-k-1}{2}, \frac{k-1}{2})$, where the first simple corresponds to the isotypic component associated with the character $(-1, -1)$ of $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ (and the other one to $(1, 1)$). □

Remark 2.4.16. By comparing the above proposition with Theorem 2.4.11 we see that we have constructed precisely the same representations which also appear in the original Springer correspondence. ▲

Chapter 3

Singular TQFTs, foams and type D arc algebras

Basic conventions

Throughout this chapter we work over a field $\mathbb{K}$ of arbitrary characteristic, and "dimension" is always meant with respect to $\mathbb{K}$. There are two exceptions: our proof of Proposition 3.2.11 requires $\mathbb{K} = \overline{\mathbb{K}}$ (this can be avoided, but extends the proof considerably), while all connections to category $\mathcal{O}$ work over $\mathbb{K} = \mathbb{C}$ only. Apart from these instances, working over $\mathbb{Z}$ instead is entirely possible.

For us, algebras are $\mathbb{K}$-algebras which are not necessarily associative nor finite-dimensional nor unital. (All our algebras are associative, but this is, except for the web algebra, a non-trivial fact.)

By a graded algebra we always mean a $\mathbb{Z}$-graded algebra, where we use the same conventions as in [21, Conventions 1.1 and 1.2] for its graded (finite-dimensional) representation theory. In particular, graded biprojective means graded left and right projective, and $\{\cdot\}$ denotes grading shifts (with conventions as fixed below).

Convention 3.0.1. We briefly discuss our grading conventions for 2-categories: an additive, graded, $\mathbb{K}$-linear 2-category is a category enriched over the category of additive, $\mathbb{Z}$-graded, $\mathbb{K}$-linear categories. Additionally, in our setup, the morphisms of such a 2-category admit grading shifts. That is, given any morphism X and any $s \in \mathbb{Z}$, there is a morphism $X\{s\}$ such that the identity 2-morphism on X gives rise to a degree s homogeneous 2-isomorphism from X to $X\{s\}$. General 2-morphisms in such 2-categories are $\mathbb{K}$-linear combinations of homogeneous ones. Hereby, any 2-morphism of degree d between X and Y becomes a 2-morphism of degree $d + t - s$ when seen between $X\{s\}$ and $Y\{t\}$. ▲

A remark about colors and diagrams 3.0.2. We read all diagrams in this chapter from bottom to top and from left to right, and we often illustrate only local pieces.

Regarding colors we remark that the important colors are the reddish phantom parts of webs and foams. In a black-and-white version of this thesis, these can be distinguished by noting that phantom edges are dashed and phantom facets are shaded. ▲

3.1 Singular TQFTs and foams

In the present section we briefly recall the topological construction of foams via the singular TQFT approach outlined in [21, Section 2] and [20, Section 2]. The reader is referred to these two papers for more details.

3.1.1 Webs and prefoams

First, we recall the construction of prefoams.

The boundary of foams

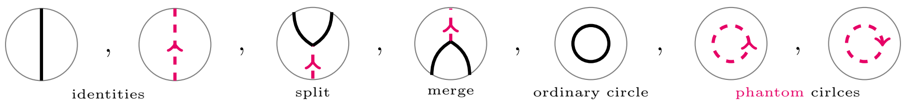
We start by recalling two definitions of webs, which are slightly different than the ones considered in [21] or [20], cf. Remark 29 in the introduction, but the construction of (pre)foams will be, mutatis mutandis, the same as therein.

Definition 3.1.1. A *web* is a labeled, piecewise linear, one-dimensional CW complex (a graph with vertices and edges) embedded in $\mathbb{R}^2 \times \{z\} \subset \mathbb{R}^3$ for some fixed $z \in \mathbb{R}$ with boundary supported in two horizontal lines, such that all horizontal slices consists only of a finite number of points. (Hence, it makes sense to talk about the bottom and top boundary of such webs.) Each vertex is either internal and of valency three, or a boundary vertex of valency one.

We assume that each edge carries a label from $\{\mathtt{o}, \mathtt{p}\}$ (we say they are *colored* by $\mathtt{o}$ or $\mathtt{p}$). Moreover, the $\mathtt{p}$-colored edges are assumed to be oriented, and each internal vertex has precisely one attached edge which is $\mathtt{p}$-colored. By convention, the empty web $\varnothing$ is also a web, and we allow circle components which consist of edges only. Webs are considered modulo boundary preserving isotopies in $\mathbb{R}^2 \times \{z\}$. ▲

Throughout we will, not just for webs, consider labelings with $\mathtt{o}$ or $\mathtt{p}$ and always illustrate them directly as colors using the convention that "$\mathtt{p}$=reddish". Moreover, both, webs and (pre)foams as defined below, contain $\mathtt{p}$-colored edges/facets. We call, everything related to these $\mathtt{p}$-colored edges/facets *phantom*, anything else *ordinary*.

Example 3.1.2. When we depict webs we omit the edge labels and color the edges instead. Furthermore, for readability, we draw $\mathtt{p}$-colored edges dashed. Using these conventions, such webs are for example locally of the form:

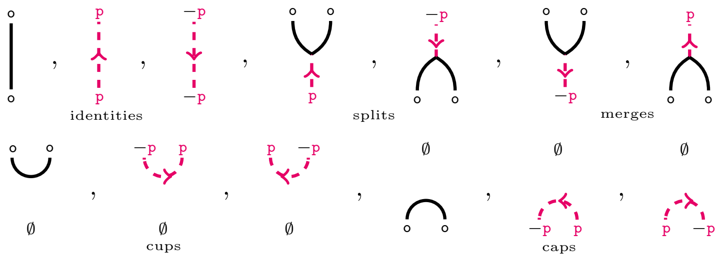


Here the outer circle indicates that these are local pictures. (We omit it in what follows and hope no confusion can arise.) ▲

Definition 3.1.3. Let $\mathbf{W}$ be the *monoidal category of webs* given as follows:

- ▷ Objects are finite words $\vec{k}$ in the symbols $\mathtt{o}$, $\mathtt{p}$ and $-\mathtt{p}$. (The empty word $\emptyset$ is also allowed.)
- ▷ The morphisms spaces $\mathrm{Hom}_{\mathbf{W}}(\vec{k}, \vec{l})$ are given by all webs with bottom boundary $\vec{k}$ and top boundary $\vec{l}$ using the following local conventions (read from bottom to top) for identities,

splits, merges, cups and caps:



(3.1)

▷ The composition $uv = v \circ u$ is the evident gluing of v on top of u, and monoidal product $\vec{k} \otimes \vec{l}$ or $u \otimes v$ given by putting $\vec{k}$ or u to the left of $\vec{l}$ or v.

A *closed* web $\overline{w}$ is an endomorphism of the empty word $\emptyset$, i.e. $\overline{w} \in \mathrm{End}_{\mathbf{W}}(\emptyset)$. ▲

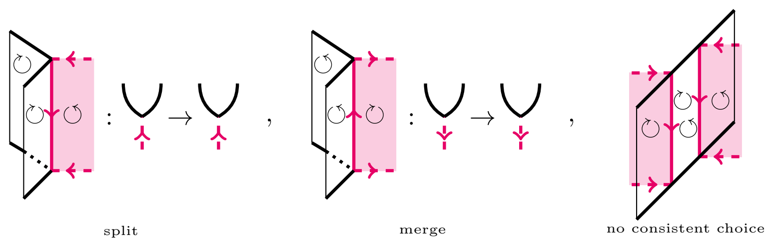
We will use the topological and the algebraic notion of webs interchangeable (e.g. the generators from (3.1) are allowed to have their boundary points far apart).

For later use, we denote by * the involution that mirrors a web along the top horizontal line and reverses orientations. Moreover, since the objects of $\mathbf{W}$ can be read off from the webs, we omit to indicate them.

Prefoams

Instead of giving the details of the construction of prefoams here (the reader is referred to [21, Section 2] and [20, Section 2]), we only stress the important points which we need in this chapter. For our purpose it will be enough to see prefoams as certain singular cobordism whose boundary are webs (including the empty web).

Remark 3.1.4. *Closed prefoams* $\overline{f}$ are certain singular CW complexes embedded in $\mathbb{R}^3$ whose singularities, called *singular seams*, are locally of the form

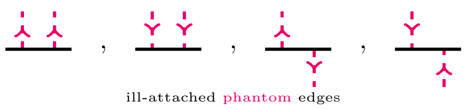


(3.2)

Hereby we stress that we only consider those prefoams which can be embedded into $\mathbb{R}^3$ such that there is a choice of orientation of its facets as illustrated in (3.2) (we fix this orientation); this choice of orientation is consistent in the sense that it induces orientations on the singular seams.

Such prefoams also carry a finite number of markers per connected component which we call *dots* $\bullet$ and which we illustrate as in (3.4). ▲

Remark 3.1.5. Due to these orientation conventions, there are no prefoams bounding closed webs with an odd number of trivalent vertices. There are also no prefoams bounding a local situation which has *ill-attached* phantom edges (cf. (3.2)), i.e.:



All other local situations are said to have *well-attached* phantom edges (see (3.66)). In contrast, their might be closed webs with an odd number of trivalent vertices or ill-attached phantom edges, but these will play no role for us. ▲

Remark 3.1.6. Let $P_{xy}^{\pm 1}$ be the plane spanned by the first two coordinates in $\mathbb{R}^3$, embedded such that the third coordinate is ± 1. A (non-necessarily closed) *prefoam* f is the intersection of $\mathbb{R}^2 \times [-1,+1]$ with some closed prefoam $\overline{f}$ such that $P_{xy}^{\pm 1}$ intersects $\overline{f}$ generically, now with orientation on its boundary induced as in (3.2): the orientation on the phantom facets agrees with the orientation on the phantom edges of the webs which we view as being the target sitting at the top and disagrees at the bottom. Clearly it suffices to indicate the orientations of the singular seams and we do so in the following. In particular, the orientation of the singular seams point into splits and out of merges at the bottom of a prefoam. ▲

3.1.2 Obtaining relations via singular TQFTs

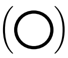
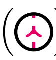
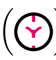
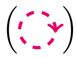
Next, we recall and state the relations which we need, and we sketch where they come from.

Singular TQFTs

The bottom and top of a prefoam f are webs $\overline{w}_b$ and $\overline{w}_t$, and we see f as a cobordism from $\overline{w}_b$ to $\overline{w}_t$, as indicated in (3.2). Using the cobordisms description, the whole data assembles into a symmetric monoidal category which we denote by $\mathbf{pF}$. (Objects are closed webs $\overline{w}_b$ and $\overline{w}_t$, and morphisms are prefoams $f\colon \overline{w}_b \to \overline{w}_t$ between these. Composition is the evident gluing, the monoidal structure is given by juxtaposition.)

Let us denote by $\mathbb{K}$-$\mathbf{VS}$ the symmetric monoidal category of finite-dimensional $\mathbb{K}$-vector spaces. The point now is the existence of a symmetric monoidal functor:

Theorem 3.1.7. (See e.g. [21, Theorem 2.11] or [20, Theorem 2.10].) There exists a symmetric monoidal functor $\mathcal{T}\colon \mathbf{pF} \to \mathbb{K}$-$\mathbf{VS}$ such that

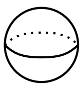
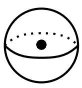
$\mathcal{T}($  $) \cong \mathcal{T}($  $) \cong \mathcal{T}($  $) \cong \mathbb{K}[X]/(X^2)$, $\qquad \mathcal{T}($  $) \cong \mathcal{T}($  $) \cong \mathbb{K}$. ■

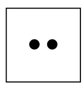
The symmetric monoidal functor $\mathcal{T}$ is called a *singular TQFT*, and its construction is based on ideas from [5] (the universal construction) as well as [4].

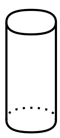
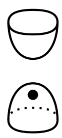
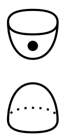
Various foamy relations

A principal objective when dealing with singular TQFTs is to identify sufficiently many relations in the kernel. Hereby we say that a relation $a_1 f_1 + \cdots + a_k f_k = b_1 g_1 + \cdots + b_l g_l$ between formal, finite, $\mathbb{K}$-linear combinations of prefoams lies *in the kernel of* $\mathcal{T}$, if $a_1\mathcal{T}(f_1) + \cdots + a_k\mathcal{T}(f_k) = b_1\mathcal{T}(g_1) + \cdots + b_l\mathcal{T}(g_l)$ holds as $\mathbb{K}$-linear maps. Here are the first examples:

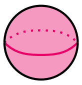
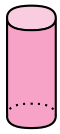
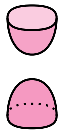
Lemma 3.1.8. (See [21, Lemmas 2.9 and 2.13].) The following relations

(3.3)  $= 0,$  $= 1,$

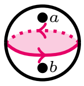
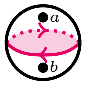
(3.4)  $= 0,$

(3.5)  $=$  $+$ 

called (from left to right) *ordinary sphere, dot and neck cut* relations, as well as the *phantom* *sphere, dot and neck cut* relations

(3.6)  $= -1,$ (3.7) $= -$ (3.8)  $= -$ 

are in the kernel of $\mathcal{T}$. Similarly, if one considers ordinary and phantom parts separately, then all the usual TQFT relations (Frobenius isotopies) as e.g. illustrated in [39, Section 2.3] are in the kernel of $\mathcal{T}$. Furthermore, the *theta foam* relations

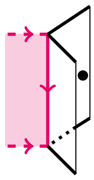
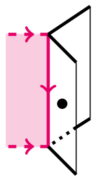
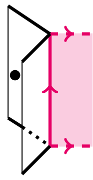
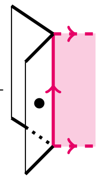
(3.9)  $= \begin{cases} +1, & \text{if } a=1, b=0, \\ -1, & \text{if } a=0, b=1, \\ 0, & \text{otherwise,} \end{cases}$ (3.10)  $= \begin{cases} -1, & \text{if } a=1, b=0, \\ +1, & \text{if } a=0, b=1, \\ 0, & \text{otherwise,} \end{cases}$

are also in the kernel of $\mathcal{T}$. ■

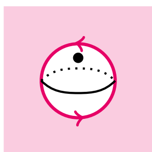
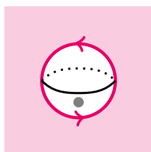
Note that (3.9) and (3.10) are actually the same relation (the reader might want to turn her/his head), but when reading from bottom to top the orientation of the singular seam is reversed when comparing (3.9) to (3.10), which gives an asymmetry.

Moreover, we have the following local relations involving phantom facets.

Lemma 3.1.9. (See [21, Lemma 2.14] and [20, (19)].) The following relations are in the kernel of $\mathcal{T}$. The *dot moving* relations

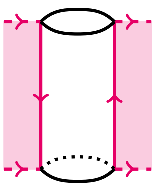
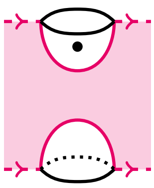
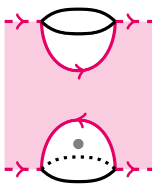
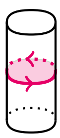
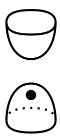
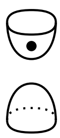
 $= -$  ,  $= -$  (3.11)

the *singular sphere removal* relations

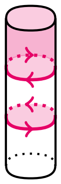
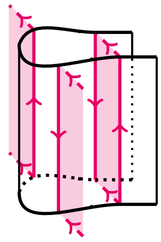
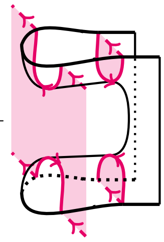
 $=$ $= -$  (3.12)

the *singular neck cutting* and the *closed seam removal* relations (the shaded dots in (3.12) and (3.13) sit on the facets in the back)

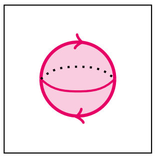
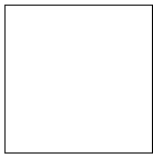
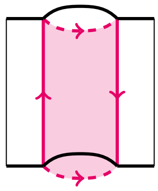
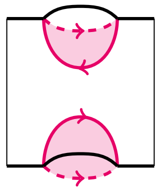
(3.13)

 $=$  $-$  , (3.14)  $=$  $-$ 

the *ordinary-to-phantom neck cutting* and the *ordinary squeeze* relations

(3.15)  $= -$  , (3.16)  $= -$ 

Further, the *phantom cup removal* and *phantom squeeze* relations are also in the kernel of $\mathcal{T}$ (the facets facing towards the reader are phantom facets):

(3.17)  $= -$  , (3.18)  $= -$ 

Relations (3.17) and (3.18) are in fact the only relations which do not directly appear in [21] or [20]. We omit the verification that they are also in the kernel of $\mathcal{T}$ because it can be done, mutatis mutandis, as in the cited lemmas. ■

Remark 3.1.10. The relations from Lemma 3.1.9 also exist in various differently oriented versions, as the reader is encouraged to check (see also [21, Lemma 2.12]). It is crucial that the sign difference in the theta foam relations (3.9) and (3.10) give opposite signs for the reoriented relations (3.12), (3.13), (3.14), (3.17) and (3.18). ▲

Gradings

Note that all finite-dimensional, commutative Frobenius algebras used in the construction of $\mathcal{T}$ carry a grading. In particular, the functor $\mathcal{T}$ from Theorem 3.1.7 can be regarded as a functor to graded, finite-dimensional $\mathbb{K}$-vector spaces. Pulling this degree back to **pF** leads to the following definition.

Definition 3.1.11. Let χ denote the topological Euler characteristic, and let #dots denote the number of dots on some prefoam f. Let $\hat{f}$ be the CW complex obtained from f by removing the phantom edges/facets. Define

$$\deg(f) = -\chi(\hat{f}) + 2 \cdot \#\text{dots}. \tag{3.19}$$

(The empty prefoam $f(\varnothing)$ has, by definition, Euler characteristic zero.) This gives **pF** the structure of a graded category. ▲

By the above, we see that $\mathcal{T}$ is actually a graded, symmetric monoidal functor.

3.1.3 Lineralization of the foam 2-category

Next, we need the notion of an *open prefoam*. These are constructed similarly to closed prefoams, but are embedded in $\mathbb{R} \times [-1, +1]^2 \subset \mathbb{R}^3$, such that its vertical (second coordinate) boundary components are straight lines in $\mathbb{R} \times \{\pm 1\} \times [-1, +1]$, and its horizontal (third coordinate) boundary components are webs embedded in $\mathbb{R} \times [-1, +1] \times \{\pm 1\}$ (using the same conventions for orientation etc. as before, see e.g. (3.2)). Again, we can see these as cobordisms between the (non-necessarily closed) webs u and v. This gives rise to a vertical composition $\circ$ via gluing (and rescaling), as well as a horizontal composition $\otimes$ via juxtaposition (and rescaling). We consider such open prefoams modulo isotopies in $\mathbb{R} \times [-1, +1]^2$ which fix the vertical and horizontal boundary, and the condition that generic slices are webs.

Let #vbound the number of vertical boundary components of some open prefoam f. We extend Definition 3.1.11 to open prefoams f via:

$$\deg(f) = -\chi(\hat{f}) + 2 \cdot \#\mathrm{dots} + \tfrac{1}{2} \cdot \#\mathrm{vbound}. \tag{3.20}$$

(The reader should check that this definition is additive under composition.)

From prefoams to foams

We are now ready to define *foams*, which, in contrast to prefoams, appear in a $\mathbb{K}$-linear setup and all the relations from Subsection 3.1.2 directly make sense.

Definition 3.1.12. Let $\mathfrak{F}$ denote the additive closure of the graded, $\mathbb{K}$-linear 2-category given by:

- ▷ The underlying structure of objects and morphisms is given by the category $\mathbf{W}$ of webs from Definition 3.1.3.
- ▷ The space of 2-morphisms between two webs u and v is a quotient of the graded, free $\mathbb{K}$-vector space on basis given by all open prefoams from u to v. The grading is hereby defined to be induced by (3.20).
- ▷ The quotient is obtained by modding out the relations from Subsection 3.1.2 as well as all relations they induce by closing prefoams.
- ▷ The vertical and the horizontal compositions are $\circ$ and $\otimes$ from above.

Note that these relations are homogeneous which endows $\mathfrak{F}$ with the structure of a graded 2-category (in the sense of Convention 3.0.1). ▲

We call $\mathfrak{F}$ the *(full) foam 2-category.* The 2-morphisms from $\mathfrak{F}$ are called *foams.* (We also use the same notions as we had for prefoams for foams.)

Remark 3.1.13. The 2-category $\mathfrak{F}$ can also be defined as a canopolises in the sense of Bar-Natan [2, Subsection 8.2]. We stay here with the 2-categorical formulation since in this setup we obtain an equivalent, algebraic, description in Proposition 3.2.11. ▲

The cup foam basis

Next, we have a *cup foam basis*:

Proposition 3.1.14. Given $u, v \in \mathrm{Hom}_{\mathfrak{F}}(\vec{k}, \vec{l})$. There is a finite, homogeneous cup foam basis for $2\mathrm{Hom}_{\mathfrak{F}}(u, v)$ in the sense of [21, Definition 4.12]. □

The proof of Proposition 3.1.14 and the construction of the cup foam basis is given in Section 3.5. (The construction is provided by a so-called "cookie-cutter strategy".) As we will see, up to signs, the construction is essentially dictated by our desire to have dotted cups as our basis elements, see e.g. in Example 3.5.5. (For details see Subsection 3.5.1.) We also write ${}_u\mathbb{B}_v$ and $\mathbb{B}(\overline{w}) = {}_{\varnothing}\mathbb{B}_{\overline{w}}$ (for closed webs $\overline{w}$) whenever we mean the fixed cup foam bases given in Definition 3.5.11.

3.2 The (entire) web algebra

In this section we aim at defining an algebra $\mathfrak{W}$ whose 2-category of bimodules **$\mathfrak{W}$-biMod**, see Definition 3.2.10, is equivalent to the foam category $\mathfrak{F}$ from the previous section. This follows a similar strategy as for more general foam 2-categories.

3.2.1 The algebra presenting foams

Recall that $\vec{k},\vec{l}$ etc. denote finite words in the symbols $\mathtt{o}$ and $\mathtt{p},-\mathtt{p}$. We call these *balanced* in case they have an even number of symbols $\mathtt{o}$. The set of such balanced words is denoted by $\mathtt{bl}^{\circ}$. Furthermore, we write $\mathtt{o}_{\vec{k}}$ to denote the total number of $\mathtt{o}$'s in $\vec{k}$. For later use: A *(balanced) block* $\vec{K}$ is a set consisting of all words $\vec{k}$ with $\mathtt{o}_{\vec{k}}=K$, for some fixed, even, non-negative integer K, called the *rank of* $\vec{K}$. (Note that there is only one block of a fixed rank, and we always match this block and its rank notation-wise.) The set of these blocks is denoted by $\mathtt{Bl}^{\circ}$.

Further, denote by $\mathrm{CUP}^{\vec{k}}=\mathrm{Hom}_{\mathbf{W}}(\emptyset,\vec{k})$, whose elements are called *cup webs*. Having two cup webs $u,v\in\mathrm{CUP}^{\vec{k}}$, one obtains a closed web $uv^{*}=v^{*}\circ u$ with composition $\circ$ as in Definition 3.1.3. (In words: we glue v^{*} on top of u.)

Convention 3.2.1. Whenever we work with cup webs $u,v\in\mathrm{CUP}^{\vec{k}}$ or closed webs of the form uv^{*} we fix a line (which we will illustrate as a dotted line, cf. (3.22)) on which $\vec{k}$ lives. This is the monoidal view on webs as in Definition 3.1.3 and breaks some symmetry, which is important to define some notions later. (For example, the notions of a $\mathtt{Ɔ}$ shape and a $\mathtt{C}$ shape make sense.) ▲

Following the terminology of [36, Section 3] (and abusing notation a bit), we define the *web homology* $\mathcal{T}(\overline{w})=2\mathrm{Hom}_{\mathfrak{F}}(\varnothing,\overline{w})$, i.e. the graded, $\mathbb{K}$-linear vector space given by all foams from the empty web $\varnothing$ into the closed web $\overline{w}$.

The web algebra as a $\mathbb{K}$-vector space

Let $d_{\vec{k}}=\frac{1}{2}\mathtt{o}_{\vec{k}}$.

Definition 3.2.2. Given $u,v\in\mathrm{CUP}^{\vec{k}}$ we set

$$_{u}(\mathfrak{W}_{\vec{k}})_{v}=\mathcal{T}(uv^{*})\{d_{\vec{k}}\}.$$

The *web algebra* $\mathfrak{W}_{\vec{k}}$ *for* $\vec{k}\in\mathtt{bl}^{\circ}$ is the graded $\mathbb{K}$-vector space

$$\mathfrak{W}_{\vec{k}}=\bigoplus_{u,v\in\mathrm{CUP}^{\vec{k}}}{}_{u}(\mathfrak{W}_{\vec{k}})_{v}$$

and the *(full) web algebra* $\mathfrak{W}$ is the direct sum of all $\mathfrak{W}_{\vec{k}}$ for $\vec{k}\in\mathtt{bl}^{\circ}$. These $\mathbb{K}$-vector spaces are equipped with the multiplication recalled below. ▲

We also define $\mathfrak{W}_{\vec{K}}=\bigoplus_{\vec{k}\in\vec{K}}\mathfrak{W}_{\vec{k}}$ for all $\vec{K}\in\mathtt{Bl}^{\circ}$ which we will use in Section 3.4.

By Proposition 3.1.14, $_{u}(\mathfrak{W}_{\vec{k}})_{v}$ has a (fixed) cup foam basis which we denote by $_{u}\mathbb{B}(\vec{k})_{v}=\mathbb{B}(uv^{*})$. We also use the evident notation $_{u}\mathbb{B}(\vec{K})_{v}$ later on.

The web algebra as an algebra

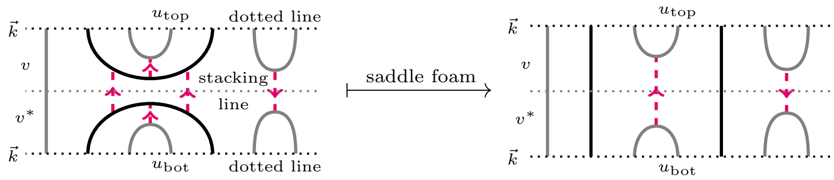
We sketch the multiplication structure. Details (which are easily adapted to our setup) can be found in [45, Section 3].

Convention 3.2.3. We sometimes need more general webs than webs of the form uv^* for $u, v \in \mathrm{CUP}^{\vec{k}}$. Namely, all possible webs which can turn up in multiplication steps which we recall below. We call such webs *stacked webs*, and will use the evident notions of *stacked dotted webs* and *stacked (circle) diagrams* for the two calculi in Sections 3.3 and 3.4 as well. The illustrative example to keep in mind is given in (3.22), where also some terminology (*dotted and stacking line*) is fixed. Note that, as stacked webs, uv^* has a middle part consisting of identity strands only. ▲

The multiplication

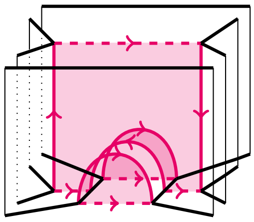
$$\mathbf{Mult}^{\mathfrak{W}}_{\vec{k}} \colon \mathfrak{W}_{\vec{k}} \otimes \mathfrak{W}_{\vec{k}} \to \mathfrak{W}_{\vec{k}}, \qquad f \otimes g \mapsto \mathbf{Mult}^{\mathfrak{W}}_{\vec{k}}(f, g) \tag{3.21}$$

is defined using the *surgery rules*. That is, the multiplication of $f \in {}_{u_{\mathrm{bot}}}(\mathfrak{W}_{\vec{k}})_v$ and $g \in {}_{v'}(\mathfrak{W}_{\vec{k}})_{u_{\mathrm{top}}}$ is zero if $v \neq v'$. An example in case $v = v'$ is:



(3.22)

(the stacking line in (3.22) will be omitted in the following) where the *saddle foam* locally looks as follows (and is the identity elsewhere)

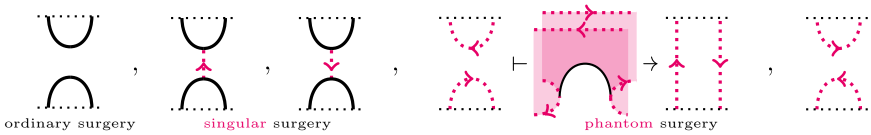


(3.23)

See e.g. [45, Definition 3.3] or [21, Definition 2.24] for a detailed account.

Taking direct sums then defines $\mathbf{Mult}^{\mathfrak{W}}$.

Remark 3.2.4. We stress that the multiplication with a web $u_{\mathrm{bot}}v^*$ at the bottom and a web $v'u^*_{\mathrm{top}}$ at the top is zero in case $v \neq v'$. In particular, one has locally (read as in (3.22)) only the following non-zero surgery configurations:



(3.24)

Hereby the multiplication foams are saddles: either ordinary, in the first case, singular saddles in the second and third cases (as illustrated in (3.23), which is a composition of two saddles as in (3.24)), or phantom in the final two cases (one of which we have illustrated in (3.24)). ▲

By identifying the multiplication in $\mathfrak{W}$ with the composition in $\mathfrak{F}$ (which can be done analogously as in [45, Lemma 3.7] via "clapping") we obtain:

Proposition 3.2.5. The multiplication $\mathbf{Mult}^{\mathfrak{W}}$ is independent of the order in which the surgeries are performed, which turns $\mathfrak{W}$ into an associative, graded algebra. ■

Remark 3.2.6. The web algebras studied in [21] and [20] are subalgebras of the web algebra defined above. This can be seen by closing the diagrams in [21] and [20] in a braid closure fashion.

(Hereby, the reader should keep Remark 29 in mind.) Consequently, the signs that turn up in the combinatorial model presented in Section 3.3 are more sophisticated versions of the ones from e.g. [21, Section 3].

Moreover, Khovanov's original arc algebra from [34] is a subalgebra of $\mathfrak{W}$ in at least two different ways: first by the natural embedding (in which case no phantom facets appear), second by using one of the main results from [20, Theorem 4.18] together with the embedding of the web algebras from [21] and [20] mentioned above. ▲

3.2.2 Its bimodule 2-category

Fix $\vec{k},\vec{l}\in\mathtt{bl}^{\circ}$.

Definition 3.2.7. Given $u\in\mathrm{Hom}_{\mathbf{W}}(\vec{k},\vec{l})$, we consider the graded $\mathbb{K}$-vector space

$$\mathbf{W}(u)=\bigoplus_{v_{\mathrm{bot}},v_{\mathrm{top}}}\mathcal{T}(v_{\mathrm{bot}}uv_{\mathrm{top}}^{*}),$$

with the sum running over all $v_{\mathrm{bot}}\in\mathrm{CUP}^{\vec{k}},v_{\mathrm{top}}\in\mathrm{CUP}^{\vec{l}}$. We endow $\mathbf{W}(u)$ with a left and a right action of $\mathfrak{W}$ as in Definition 3.2.2. ▲

Noting that the left and right actions are far apart, we see that all $\mathbf{W}(u)$'s are graded $\mathfrak{W}$-bimodules referred to as *web bimodules*. In fact:

Proposition 3.2.8. The web bimodules $\mathbf{W}(u)$ are graded biprojective $\mathfrak{W}$-bimodules with finite-dimensional subspaces for all pairs $v_{\mathrm{bot}}\in\mathrm{CUP}^{\vec{k}},v_{\mathrm{top}}\in\mathrm{CUP}^{\vec{l}}$. □

Proof. They are clearly graded. The finite-dimensionality follows from the existence of an explicit cup foam basis, see Proposition 3.1.14. They are biprojective, because they are direct summands of some $\mathfrak{W}_{\vec{k}}$ (of some $\mathfrak{W}_{\vec{l}}$) as left (right) modules for suitable $\vec{k}\in\mathtt{bl}^{\circ}$ (or $\vec{l}\in\mathtt{bl}^{\circ}$), see also [45, Proposition 5.11]. ■

Remark 3.2.9. The web bimodules as well as the web algebras (all of them, i.e. $\mathfrak{W}_{\vec{k}},\mathfrak{W}_{\vec{K}}$ and $\mathfrak{W}$) are infinite-dimensional. However, most of the summands are actually unnecessary since the webs are isomorphic as morphisms of $\mathfrak{F}$, cf. Lemma 3.5.1. If one wants to work with finite-dimensional modules and algebras, then one could choose a basis of the web spaces (as e.g. in [45]). ▲

Taking everything together, we can define:

Definition 3.2.10. Let $\mathfrak{W}$**-biMod** be the following 2-category.

- Objects are the various balanced words $\vec{k}\in\mathtt{bl}^{\circ}$.
- The morphisms are finite sums and tensor products (taken over the algebra $\mathfrak{W}$) of $\mathfrak{W}$-bimodules $\mathbf{W}(u)$, with composition given by tensoring (over $\mathfrak{W}$).
- The 2-morphisms are $\mathfrak{W}$-bimodule homomorphisms, with vertical and horizontal composition given by composition and by tensoring (over $\mathfrak{W}$).

We consider $\mathfrak{W}$**-biMod** as a graded 2-category as in Convention 3.0.1. ▲

For the next proposition we assume $\mathbb{K}=\overline{\mathbb{K}}$. This can be avoided, but additional care needs to be taken in the proof.

Proposition 3.2.11. There is an equivalence of additive, graded, $\mathbb{K}$-linear 2-categories

$$\Upsilon\colon\mathfrak{F}\xrightarrow{\cong}\mathfrak{W}\textbf{-biMod},$$

which is bijective on objects and essential surjective on morphisms. □

The proof is, as usual, given in Subsection 3.5.1.

3.3 The combinatorial model

Foams and the web algebra carry information about two-dimensional topological spaces sitting in three-space. This makes direct (non-local) computations quite involved. The aim of this section is to define a stricter version of the web algebra given by web-like objects sitting in the plane, called the *combinatorial* model. That is, we are going to define an algebra $c\mathfrak{W}$ with multiplication $\mathbf{Mult}^{c\mathfrak{W}}$ and show:

Theorem 3.3.1. There is an isomorphism of graded algebras

$$\mathtt{comb}\colon c\mathfrak{W} \xrightarrow{\cong} \mathfrak{W}.$$

(Similarly, denoted by $\mathtt{comb}_{\vec{k}}$ or $\mathtt{comb}_{\vec{K}}$, on summands.) □

(The proof of Theorem 3.3.1 is given in Subsection 3.5.2.) Note that Theorem 3.3.1 immediately gives the following, which would otherwise be rather involved to prove:

Corollary 3.3.2. The multiplication $\mathbf{Mult}^{c\mathfrak{W}}$ is independent of the order in which the surgeries are performed, turning $c\mathfrak{W}$ into an associative, graded algebra. ■

In order to define $c\mathfrak{W}$ we first need to introduce several combinatorial notions, all of which are dictated by our desire to see $c\mathfrak{W}$ as a "projection" of $\mathfrak{W}$ in the sense of Figure 3.1.

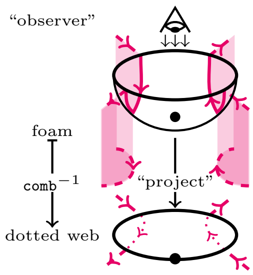


Figure 3.1: From foams to dotted webs: looking from the top to the bottom, a dotted web is obtained from a foam by projection.

These combinatorial notions will assemble into what we call *dotted webs*. The algebra $c\mathfrak{W}$ is then defined very much in the spirit of arc algebras: It has an underlying $\mathbb{K}$-linear structure given by dotted (basis) webs, and its multiplicative structure is inductively defined using a combinatorially defined surgery procedure (in contrast to the topologically defined surgery for web algebras). As usual, the signs turning up are intricate and a major part of this section is just devoted to define combinatorial ways to calculate them. The definition of the mapping $\mathtt{comb}\colon c\mathfrak{W} \to \mathfrak{W}$ is then, up to details which we have migrated to Section 3.5, the inverse of the one from Figure 3.1.

3.3.1 Basic notions

The first step toward the definition of a combinatorial model for foams is to replace foams by a decoration on webs.

Dotted webs

Since it follows from the existence of the cup foam basis, cf. Proposition 3.1.14, that there is a foam basis given by (potentially) dotted cups, such a decoration for us will be a dot • on some component of a web, as well as certain lines keeping track of the singular seams attached to cup foams basis elements.

Remark 3.3.3. For more general web algebras the situation is more delicate, cf. Remark 3.5.7, and combinatorial models are missing for most of them at the moment. We prefer the combinatorially easier model using dots on webs in this chapter, but for e.g. web algebras as in [45] one would need more sophisticated notions as e.g. flows in the sense of [37] as decorations. ▲

Hereby, and throughout, a *component* of a web is meant as a topological space after erasing all phantom edges. Moreover, by our definition of webs, connected components are circles, and we call them *circles* for short. In this spirit (and recalling Convention 3.2.1), we also say *cup* and *cap* in a web meaning the evident notion obtained by erasing phantom edges, while a *phantom cup/cap* are also to be understood in the evident way, cf. (3.31) where several cups and caps appear. Having a circle, we can speak about its *internal/external* by ignoring all other circles.

Convention 3.3.4. Webs can have circles with an odd number of trivalent vertices or ill-attached phantom edges, but their associated foam spaces are zero, cf. Remark 3.1.5. We call such webs *ill-oriented*, all others *well-oriented*. Henceforth, if not stated otherwise, we consider only well-oriented webs (webs for short) with an even number of trivalent vertices and well-attached phantom edges. ▲

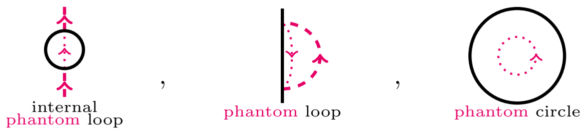
A *path* in a web u is an embedding of $[0,1]$ into the CW complex given by u after erasing all phantom edges.

Given a point i on a web u, then the *segment containing* i is the maximal path containing i which does not cross any phantom edges. Recalling that webs are embedded in $\mathbb{R}\times[-1,+1]$, we make the following definition. Hereby and throughout, *points on* u are always meant to be on ordinary parts of the web u, and are always contained in some segment (which one will be evident).

Definition 3.3.5. Given a web u and a circle C of it. Then the *base point* $\mathtt{B}(C)$ on it is defined to be any point in the bottom right segment of C. ▲

As in [21, Subsection 3.1], $\mathtt{B}(C)$ is a choice of a rightmost point. We also write $\mathtt{B}=\mathtt{B}(C)$ for short if no confusion can arise.

Definition 3.3.6. Given a web u, then a *phantom seam* is a decoration of u with an extra edge starting and ending at some trivalent vertices of u which is oriented in the direction of the adjacent phantom edges of u, e.g. (we illustrate phantom seams dotted and slightly thinner than the other phantom edges):

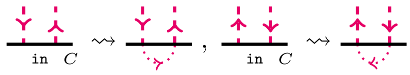


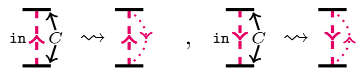
(3.25)

Hereby we also allow phantom circles, which are always assumed to be in some circle of a web, as on the right above in (3.25). Moreover, the phantom seams have to be attached to a web such that the result does not have any intersections, and no trivalent vertex has more than one attached phantom seam. ▲

We are quite free to decorate webs. In order to match decorated webs with cup foam basis elements, we have to chose a decoration. This corresponds to choosing a cup foam basis as we will see in Subsection 3.5.1. In particular, it will depend on a choice of a point for each circle in question. Choosing such a point i, we call a phantom edge *i-closest* if its the first phantom edge one passes when going around anticlockwise, starting at i. (Similarly for other notions.) If these points are the base points, then we call such a decoration $\mathtt{B}$-admissible.

In order to define such decorations, we fix the following choices of how to put phantom seams locally on webs, fixing a circle C of it:

(3.26) 

(3.27) 

where $\mathtt{in}$ denotes the interior of C. (We do not distinguish between putting the phantom seams to the bottom or top in (3.26), or right or left in (3.27), cf. (3.28)).

Now, a $\mathtt{B}$-*admissible* decoration is one obtained by applying Algorithm 1 to a circle C in the web u with chosen base points $i = \mathtt{B}$.

input : a web u, a circle C in it and a chosen point i of it;
output : an i-admissible decoration C_{dec} of C;

initialization, let C_{dec} be the circle without decorations;
while *C has attached phantom edges* **do**
  if *C contains a pair as in* (3.26) **then**
    apply (3.26) to the i-closest such pair;
    add the corresponding phantom seam to C_{dec};
    remove the corresponding pair from C;
  else
    apply (3.27) to any phantom digon not containing i;
    add the corresponding phantom seam to C_{dec};
    remove the corresponding phantom digon from C;
  end
end

Algorithm 1: The i-admissible decoration algorithm.

Then we piece everything together as in Subsection 3.5.1. (Detail are not important for the moment and will follow in Subsection 3.5.1, e.g. phantom digon is defined therein. Furthermore, for the time being, we ignore questions regarding well-definedness etc. The only thing the reader need to know at this point is that the $\mathtt{B}$-admissible decoration are the ones turning up as in Figure 3.1.)

Definition 3.3.7. Given a web u, we allow each circle of it to be decorated by a dot $\bullet$, where we assume that the dot is on a segment of u. Moreover, each trivalent vertex of u is decorated by an attached phantom seam (there can be any finite number of phantom circles), which are not allowed to cross each other. We call such a web with decorations a *dotted web*.

We call a dotted web a *dotted basis web* in case the following are satisfied.

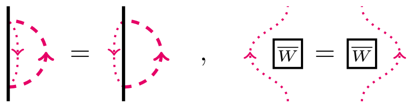
▷ All dots are on the segments of the base points $\mathtt{B}$ for all circles C in u.

▷ All phantom seams decorations are $\mathtt{B}$-admissible.

(If some circle C is not dotted at all, then the first condition is satisfied for it.) ▲

We stress that dotted basis webs will never have phantom circles.

We denote dotted webs using capital letters as e.g. $U, V, \overline{W}$ etc., and we say they are of *shape* $u, v, \overline{w}$ etc. In the following we consider dotted (basis) webs up to isotopies of these seen as decorated (by dots) planar graphs, as well as the relations



(3.28)

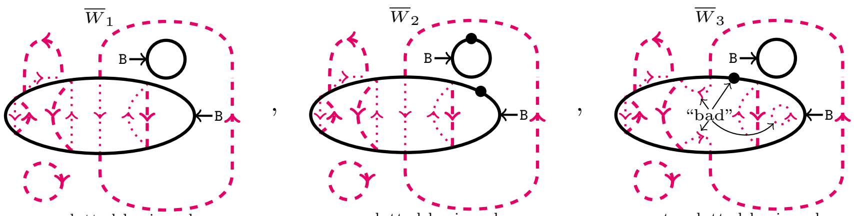
where $\overline{W}$ is some dotted web not connected to the displayed phantom seam. (Similar for all versions of these with different orientations.)

Definition 3.3.8. The *degree* of a dotted web U is defined as

$$\deg(U) = -\#C + 2 \cdot \#\text{dots}, \tag{3.29}$$

where $\#C$ is the number of circles and #dots the number of dots in U. ▲

Example 3.3.9. Below we have illustrated three examples $\overline{W}_1, \overline{W}_2$ and $\overline{W}_3$ of possible decorations of a web $\overline{w}$. The dotted web $\overline{W}_3$ is not a dotted basis web: the dot is not on the B-segment, there is a phantom circle and the two phantom seams are not B-admissible. (We have indicated all of them using the word “bad”.)



a dotted basis web , a dotted basis web , not a dotted basis web

Note hereby that we do not allow more than one dot per circle. From left to right, these are of degree -2, 2 and 0. ▲

Moreover, we define:

Definition 3.3.10. Given a dotted web U we define npcirc(U) to be the total number of anticlockwise (negative) oriented phantom circles. ▲

Example 3.3.11. For the three dotted webs displayed in Example 3.3.9 one has npcirc($\overline{W}_1$) = npcirc($\overline{W}_2$) = 0, but npcirc($\overline{W}_3$) = 1. ▲

Keeping track of the dot moving signs

In the following path from a point i to a point j are denoted by $i \to j$.

Definition 3.3.12. Given web u and two fixed points i, j on u which are connected by a path $i \to j$, we define

$$\text{pedge}(i \to j) = \text{number of phantom edges attached to } i \to j.$$

We extend pedge($\square \to \square$) additively for concatenations of distinct path. (Here and in the following $\square$ plays the role of a place holder.) ▲

Example 3.3.13. A blueprint example is provided by the web from Example 3.3.16, using the same choice of a circle C and points as therein. If we choose the corresponding path going around anticlockwise, then we have

$$\text{pedge}(i \to j) = 3, \qquad \text{pedge}(i \to k) = 5, \qquad \text{pedge}(i \to l) = 8. \qquad ▲$$

In general, pedge($i \to j$) depends on which path connecting i and j is chosen. But we note the following lemma which we need to make the sign assignment below in Subsection 3.3.2 well defined, and whose (very easy) proof is left to the reader (keeping Convention 3.3.4 in mind):

Lemma 3.3.14. The statistics defined in Definition 3.3.12 taken modulo 2 do not depend on the chosen path. ■

Keeping track of the topological signs

We call a situation as in the middle of (3.25) a *phantom loop*, no matter how many other phantom edges are in between the two trivalent vertices in this illustration. In particular, phantom loops in closed webs are always associated to a circle, namely the one they start/end. Thus, we can say whether they are *internal* or *external*, cf. (3.25).

Definition 3.3.15. Given a circle C in some web u and a fixed point i on it. Let L be an internal phantom loop attached to C. Then:

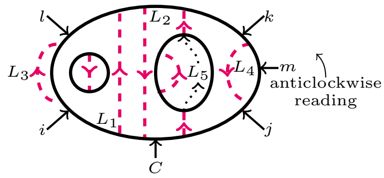
- ▷ The internal phantom loop L is said to be *i-positive*, if it points out of C first when reading from i anticlockwise.
- ▷ The internal phantom loop L is said to be *i-negative*, if it points into C first when reading from i anticlockwise. ▲

(Note that this is an asymmetric property heavily depending on i.)

We sometimes need to consider internal phantom loops attached to some circle C after removing all circles nested in C, cf. Example 3.3.16.

The notion of an *outgoing phantom edge* of some circle C of some web u is, by definition, a phantom edge in the exterior of C, counting the phantom loops in the exterior twice. (For example, the right nested circle in the web from Example 3.3.16 has two outgoing phantom edges and one phantom loop; the circle C in the very same example has two outgoing phantom edges given by the phantom loop L_3.)

Example 3.3.16. Consider the following web with five fixed points on a circle of it.

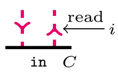


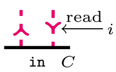
The circle C has four attached loops L_1, L_2, L_3 and L_4 as illustrated. If one removes all circles in C, then there is an extra loop formed by the two outgoing edges of the rightmost nested circle, which is L_5.

With respect to positive or negative phantom loops we have to read anticlockwise starting from the varies points. (Since we always read anticlockwise we just write "read" in the following.) One sees that L_1 is positive with respect to i and l, but negative with respect to j, k and m. For L_2 the situation is vice versa, and L_4 is m-positive, but negative for all other points. The phantom loop L_3 is exterior and we do not need to check whether its □-positive or □-negative.

The reader is encouraged to work out the situation for L_5. ▲

Definition 3.3.17. A local situation (local in the sense that such edges might close to exterior phantom loops, cf. (3.67)) of the following form

i-positive: 

, i-negative: 

(3.30)

is called an *outgoing phantom edge pair of* C. Hereby, $\mathtt{in}$ denotes the interior of C. The notion of these being i-positive respectively i-negative is defined by reading from a point i on C in the anticlockwise fashion, and then seeing if the i-closest of the two outgoing phantom edges points outwards or inwards, see (3.30). ▲

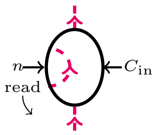
Definition 3.3.18. Given a web u and a circle C in u, and fix a point i on it, and ignore all of its nested circles. With respect to the chosen point i we define:

$$\begin{aligned}\mathtt{np}\mathrm{loop}(C,i) = {} & \text{number of } i\texttt{-negative} \text{ internal phantom loops attached to } C \\ & + \text{ number of } i\texttt{-negative} \text{ outgoing phantom edge pairs of } C.\end{aligned}$$

(Again, this heavily depends on i.) ▲

Example 3.3.19. In Example 3.3.16 one has $\mathtt{np}\mathrm{loop}(C,i) = \mathtt{np}\mathrm{loop}(C,l) = 3$, $\mathtt{np}\mathrm{loop}(C,j) = \mathtt{np}\mathrm{loop}(C,k) = 4$ and $\mathtt{np}\mathrm{loop}(C,m) = 3$. (The external phantom loop L_3 contributes to all of these as a negative outgoing phantom edge pair.)

Next, for the right nested circle C_{in} of the web in the very same example, i.e.

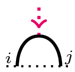
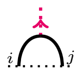


we get $\mathtt{np}\mathrm{loop}(C_{\mathrm{in}},n) = 1$, since the two outgoing phantom edges are n-negative in the sense of (3.30) and the internal phantom loop is n-positive. ▲

Keeping track of the saddle signs

Let u be a stacked web. All three definitions below are with respect to the stacked web u, for which we assume that we are in the situation of a cup-cap pair involved in an ordinary or singular surgery (as e.g. in (3.22)) with fixed points i and j on them.

Definition 3.3.20. A phantom edge attached to the cup of the cup-cap pair is said to be *i-positive* respectively *i-negative* in cases

i-positive:  ,  i-negative:  , 

For j instead of i we swap positive ones with negative ones. ▲

Definition 3.3.21. Then the *saddle type* $\mathtt{s}\mathrm{type}$ is defined to be

$$\mathtt{s}\mathrm{type}(i,j) = \mathtt{s}\mathrm{type}(j,i) = \begin{cases} 0, & \text{if } \mathtt{p}\mathrm{edge}(i \to j) \text{ is even,} \\ 1, & \text{if } \mathtt{p}\mathrm{edge}(i \to j) \text{ is odd.} \end{cases}$$

(We write $\mathtt{s}\mathrm{type} = \mathtt{s}\mathrm{type}(i,j) = \mathtt{s}\mathrm{type}(j,i)$ for short.) ▲

Definition 3.3.22. The *i-saddle width* is defined to be

$$\mathtt{np}\mathrm{sad}(i) = \text{number of } i\texttt{-negative} \text{ phantom edges attached to the cup.}$$

The *j-saddle width* is defined similarly, but using j-negative phantom edges. ▲

Remark 3.3.23. The asymmetries in Definitions 3.3.10, 3.3.18 and 3.3.22 comes from our choice of evaluation in (3.9) and (3.10). We stress this using $\mathtt{np}$ as a prefix. ▲

All of the above can be used for dotted webs as well, and we do so in the following.

Example 3.3.24. Consider the surgery from (3.22). Then one has $\mathtt{np}\mathrm{sad}(i) = 2$ for a point i to the left of it, $\mathtt{np}\mathrm{sad}(j) = 1$ for a point j to the right of it and $\mathtt{s}\mathrm{type} = 1$. More general, the saddle type can be thought of as being 0 (usual) or 1 (singular) with the convention as in (3.31). ▲

3.3.2 Combinatorics of foams

Given $\vec{k}$, and two webs $u, v \in \mathrm{CUP}^{\vec{k}}$, then the set

$$_u\mathbb{B}(\vec{k})_v = \mathbb{B}(uv^*) = \{\text{all dotted basis webs of shape } uv^*\}$$

will play the role of a combinatorial version of the cup foam basis.

The $\mathbb{K}$-linear structure

We start by defining the graded $\mathbb{K}$-vector space structure of the combinatorial model of the web algebra. Recall that $d_{\vec{k}} = \frac{1}{2}\mathtt{o}_{\vec{k}}$.

Definition 3.3.25. Given $u, v \in \mathrm{CUP}^{\vec{k}}$ for $\vec{k} \in \mathtt{bl}^\circ$ we set

$$_u(c\mathfrak{W}_{\vec{k}})_v = \langle {}_u\mathbb{B}(\vec{k})_v \rangle_{\mathbb{K}}\{d_{\vec{k}}\},$$

that is the free $\mathbb{K}$-vector space on basis $_u\mathbb{B}(\vec{k})_v$. The *combinatorial web algebra* $c\mathfrak{W}_{\vec{k}}$ *for* $\vec{k} \in \mathtt{bl}^\circ$ is the graded $\mathbb{K}$-vector space

$$c\mathfrak{W}_{\vec{k}} = \bigoplus_{u,v \in \mathrm{CUP}^{\vec{k}}} {}_u(c\mathfrak{W}_{\vec{k}})_v$$

with grading given on dotted basis webs via (3.29). The *combinatorial (full) web algebra* $c\mathfrak{W}$ is the direct sum of all $c\mathfrak{W}_{\vec{k}}$ for $\vec{k} \in \mathtt{bl}^\circ$. These $\mathbb{K}$-vector spaces are equipped with the multiplication we define below. ▲

For later use in Section 3.4, we also define $c\mathfrak{W}_{\vec{K}} = \bigoplus_{\vec{k} \in \vec{K}} c\mathfrak{W}_{\vec{k}}$ for all $\vec{K} \in \mathtt{Bl}^\circ$. Clearly, a basis of $c\mathfrak{W}_{\vec{K}}$ is given by $_u\mathbb{B}(\vec{K})_v = \coprod_{\vec{k} \in \vec{K}} {}_u\mathbb{B}(\vec{k})_v$.

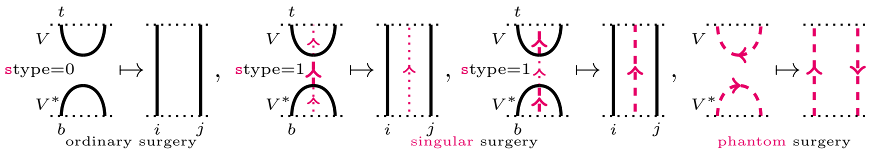
Note the crucial difference to Definition 3.2.2: The multiplicative structure of $\mathfrak{W}$ was naturally given by the foam 2-category $\mathfrak{F}$, but we had (or we will) to construct a basis for the algebra. In contrast, the basis of $c\mathfrak{W}$ is given, but we have to construct the multiplicative structure. That is what we are going to do next.

The multiplication without signs

We define again $\mathbf{Mult}_{\vec{k}}^{c\mathfrak{W}}$ and then take direct sums to obtain $\mathbf{Mult}^{c\mathfrak{W}}$. (We use notation similar to Subsection 3.2.1.)

To define $\mathbf{Mult}_{\vec{k}}^{c\mathfrak{W}}$ as in (3.21) we have to assign to each pair of dotted basis webs $U_{\mathrm{bot}}V^*$ and VU_{top}^* a sum of dotted basis webs of shape $u_{\mathrm{bot}}u_{\mathrm{top}}^*$. We do so by the usual inductive surgery process, where we first only change the shape (very similar to [21, Subsection 3.3]) and reconnect phantom seams.

Now, for any (ordinary, singular or phantom) cup-cap pair in the middle section V^*V we want to model the situation from (3.24) and we do the following local replacements, where we also fix four points on the webs in question, e.g.:



(3.31)

If we are not in a situation as in exemplified in (3.31), then the multiplication is defined to be zero. (See also (3.24).)

Remark 3.3.26. The rules in (3.31) should be read locally in the sense that there might be several unaffected components in between as e.g. in (3.22). These do not matter for what happens to the shape, but the scalars will depend on the precise form as we will see below. ▲

We assume that we perform the local rules from (3.31) for the leftmost available cup-cap pair. Using these conventions, one directly checks that the rules presented below turn $c\mathfrak{W}$ into a graded algebra (not necessarily associative at this point).

Remark 3.3.27. There is a Ↄ shape within the multiplication procedure, cf. Example 3.3.30 (which essentially defines the notion of a Ↄ shape). Its mirror, the C shape, is ruled out by choosing the leftmost available cup-cap pair. Still, below we give the rule for this case as well, since it will follow that one can actually choose any cup-cap pair, see Corollary 3.3.2. This is in contrast to the type D situation as we will see later in Section 3.4. ▲

We are now ready to define the multiplication without signs. Hereby we say for short that we *put a dot on a circle C* and we mean that we put it on the segment of its base point B. Clearly, the procedure from (3.31) in the ordinary or singular case either merges two circles into one, or it splits one into two.

The multiplication without signs is defined as follows, where we always perform the local procedure from (3.31). Hereby, if we write e.g. $C_{\square}$, then the corresponding circle should contain the point $\square$.

Merge. Assume two circles C_b and C_t are merged into a circle C_{af}.

- ▷ If both are undotted, then nothing additionally needs to be done.
- ▷ If one is undotted and the other one dotted, then put a dot on C_{af}.
- ▷ If both circles are dotted, then the result is zero.

Split. Assume a circle C_{be} splits into C_i and C_j.

- ▷ If C_{be} is undotted, then take the sum of two copies of the result, in one put a dot on C_i, in the other on C_j.
- ▷ If C_{be} is dotted, then put a dot on either C_i or C_j such that both are dotted.

In case of a Ↄ or C shape, remove all phantom circles from the result.

Phantom surgery. In this case nothing additionally needs to be done.

"Turning inside out". In the nested case (which we will meet below) the interior of some circle turns into the exterior of another circle after surgery and vice versa. In those cases reconnect the phantom seams until they are B-admissible, cf. Example 3.3.29. (We will show in Subsection 3.5.1 that this can always be done by reconnection locally as illustrated in (3.59).)

The multiplication with signs

We have to define several notions to fix the signs for the multiplication. The signs will crucially depend on the number and the positions of the phantom edges. As in [21, Subsection 3.3] there will be *dot moving signs*, *topological signs*, *saddle signs* and, a new type, *phantom circle signs*. All of the old signs are more general notions than the ones in [21, Subsection 3.3] (due to the fact that we deal here with more flexible notions).

Following [21, Subsection 3.3], there are now several cases for the ordinary and singular surgeries depending on whether a merge or a split involves nested circles or not. In contrast, the phantom surgery only depends on whether the phantom cup-cap pair involved in the surgery form a closed circle.

Then the multiplication result from above is modified as follows. (We use the notation from above. Moreover, the meticulous reader might note that we have to use Lemma 3.3.14 to make sure that the signs are well-defined.) Below all points b, t, i, j are as in (3.31), and we write $\mathtt{B}_\square = \mathtt{B}(C_\square)$ for short.

Non-nested merge. In this case only one modification is made:

▷ If C_b is dotted and C_t undotted, then we multiply by

$$(-1)^{\mathrm{pedge}(\mathtt{B}_b \to \mathtt{B}_{\mathrm{af}})}. \tag{3.32}$$

This sign is called the *(existing) dot moving sign*, and works in the same way if we exchange the roles of b and t.

Nested merge. Denote the inner of the two circles C_b and C_t by C_{in}. Then this case is modified by *(existing) dot moving signs*, *topological signs* and *saddle signs*:

▷ If both circles are undotted, then we multiply the result by

$$(-1)^{\mathtt{np}\mathrm{loop}(C_{\mathrm{in}}, i)}(-1)^{\mathtt{s}\mathrm{type}}(-1)^{\mathtt{np}\mathrm{sad}(i)}. \tag{3.33}$$

▷ If one of them is dotted, say C_b, then we multiply the result by

$$(-1)^{\mathrm{pedge}(\mathtt{B}_b \to \mathtt{B}_{\mathrm{af}})}(-1)^{\mathtt{np}\mathrm{loop}(C_{\mathrm{in}}, i)}(-1)^{\mathtt{s}\mathrm{type}}(-1)^{\mathtt{np}\mathrm{sad}(i)}. \tag{3.34}$$

Similarly for exchanged roles of C_b and C_t.

Non-nested split. Here, both cases are modified by *(new and existing) dot moving signs* and *saddle signs*:

▷ If C_{be} is undotted, then we multiply the summand where C_i is dotted by

$$(-1)^{\mathrm{pedge}(i \to \mathtt{B}_i)}(-1)^{\mathtt{np}\mathrm{sad}(i)}, \tag{3.35}$$

and the one where C_j is dotted by

$$(-1)^{\mathrm{pedge}(j \to \mathtt{B}_j)}(-1)^{\mathtt{s}\mathrm{type}}(-1)^{\mathtt{np}\mathrm{sad}(i)}, \tag{3.36}$$

▷ If C_{be} is dotted, then we multiply the result by

$$(-1)^{\mathrm{pedge}(\square \to \mathtt{B}_\square)}(-1)^{\mathtt{np}\mathrm{sad}(\square)}. \tag{3.37}$$

Here $\square \in \{i, j\}$ is such that $C_\square$ does not contain $\mathtt{B}_{\mathrm{be}}$.

Nested split, Ɔ shape. Let $\overline{W}$ denotes the dotted web after the surgery and before removing the phantom circles. Both cases are modified by *(new and existing) dot moving signs*, *topological signs* and *phantom circle sign*:

▷ If C_{be} is undotted, then we multiply the summand where C_i is dotted by

$$(-1)^{\mathrm{pedge}(i\to\mathtt{B}_i)}(-1)^{\mathtt{n}\mathrm{ploop}(C_j,j)}(-1)^{\mathrm{stype}}(-1)^{\mathtt{n}\mathrm{pcirc}(\overline{W})}, \tag{3.38}$$

and the one where C_j is dotted by

$$(-1)^{\mathrm{pedge}(j\to\mathtt{B}_j)}(-1)^{\mathtt{n}\mathrm{ploop}(C_j,j)}(-1)^{\mathtt{n}\mathrm{pcirc}(\overline{W})}. \tag{3.39}$$

▷ If C_{be} is dotted, then we multiply with

$$(-1)^{\mathrm{pedge}(j\to\mathtt{B}_j)}(-1)^{\mathtt{n}\mathrm{ploop}(C_j,j)}(-1)^{\mathtt{n}\mathrm{pcirc}(\overline{W})}. \tag{3.40}$$

Nested split, $\mathtt{C}$ shape. This is slightly different from the Ↄ shape:

▷ If C_{be} is undotted, then we multiply the summand where C_i is dotted by

$$(-1)^{\mathrm{pedge}(i\to\mathtt{B}_i)}(-1)^{\mathtt{n}\mathrm{ploop}(C_i,i)}(-1)^{\mathtt{n}\mathrm{pcirc}(\overline{W})}, \tag{3.41}$$

and the one where C_j is dotted by

$$(-1)^{\mathrm{pedge}(j\to\mathtt{B}_j)}(-1)^{\mathtt{n}\mathrm{ploop}(C_i,i)}(-1)^{\mathrm{stype}}(-1)^{\mathtt{n}\mathrm{pcirc}(\overline{W})}. \tag{3.42}$$

▷ If C_{be} is dotted, then we multiply with

$$(-1)^{\mathrm{pedge}(i\to\mathtt{B}_i)}(-1)^{\mathtt{n}\mathrm{ploop}(C_i,i)}(-1)^{\mathtt{n}\mathrm{pcirc}(\overline{W})}. \tag{3.43}$$

Phantom surgery. Only one modification has to be made, i.e.:

▷ If the phantom cup-cap pair forms a circle, then we multiply by -1.

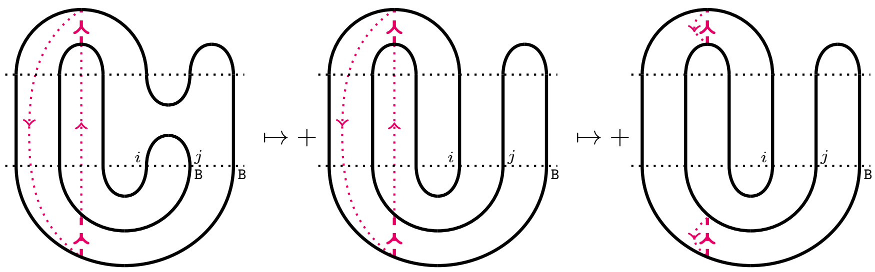
"Turning inside out". No additional changes need to be done.

Remark 3.3.28. The careful reader might wonder why "turning inside out" does not give any signs. It does, but the signs are collected in what we call topological signs. In fact, the phantom seams need to be reconnected precisely because some non-trivial manipulation needs to be done as we will see in Subsection 3.5.2. ▲

Examples of the multiplication

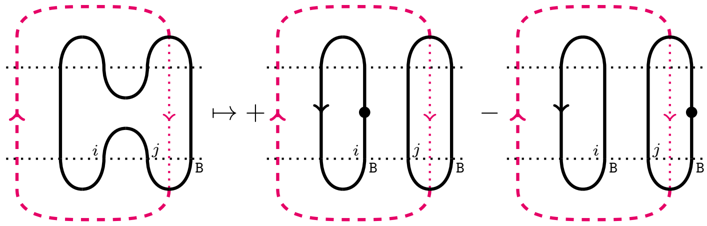
Next, let us give some examples. Note that we always omit the step called *collapsing*, cf. [21, (27)]. Moreover, the reader can find several examples in [21, Examples 3.15 and 3.16] of which we encourage her/him to convert to our situation here, see also Remark 3.2.6.

Example 3.3.29. As already in the setup of [21], the most involved example is a nested merge, which now comes in plenty of varieties. Here is one:

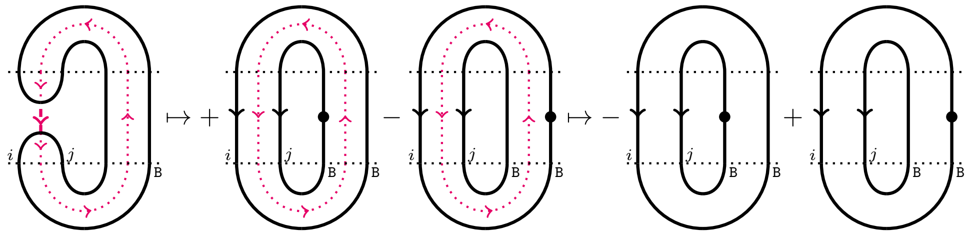


We have $\mathrm{stype} = 0$, $\mathtt{n}\mathrm{ploop}(C_{\mathrm{in}}, i) = 0$ and $\mathtt{n}\mathrm{psad}(i) = 0$, giving a positive sign. The last move, which never gives any signs, cf. Remark 3.3.28, reconnects the phantom seams to fit our choice of basis (which is formally defined in Subsection 3.5.1). ▲

Example 3.3.30. The basic splitting situation are the H and the Ↄ, which come in different flavors (depending on the various attached phantom edges). Here are two small blueprint examples illustrating some new phenomena which do not appear in [21]. First, an H:



Note that stype $= 0$ and the saddle sign npsad(i) is trivial in this case, but the rightmost summand acquires a dot moving sign. Next, a Ↄ:



In the first step the only non-trivial sign comes from stype $= 1$ (which gives the minus sign for the element in the middle). Furthermore, in the last step we have removed the phantom circle at the cost of an overall minus sign. (Again, the phantom seams show us what kind of non-trivial manipulation we need to do to bring the multiplication result in the form of our chosen basis.) ▲

3.3.3 The combinatorial realization

Next, we define the combinatorial isomorphism comb. Morally it is given as in Figure 3.1.

Formally it is given by using our algorithmic construction (where we use subscripts to distinguish between the two cup foam bases):

Definition 3.3.31. Given $\overline{w} = uv^*$ with $u, v \in \mathrm{CUP}^{\vec{k}}$, we define a $\mathbb{K}$-linear map

$$\mathtt{comb}_u^v \colon \langle \mathbb{B}(\overline{w})_{c\mathfrak{W}} \rangle_{\mathbb{K}} \to \langle \mathbb{B}(\overline{w})_{\mathfrak{W}} \rangle_{\mathbb{K}}, \quad \overline{W} \mapsto f(\overline{W}),$$

by sending a dotted basis web $\overline{W}$ with phantom seam structure as obtained from Algorithm 1 (pieced together as in Subsection 3.5.1) to the foam $f(\overline{W})$ of the shape as obtained by using Algorithm 5 with the dot placement matched in the sense that $f(\overline{W})$ has a dot on the facet attached a segment which carries a base point B if and only if $\overline{W}$ has a dot on the very same segment.

Similarly, by taking direct sums, we define $\mathtt{comb}_{\vec{k}}$, $\mathtt{comb}_{\vec{K}}$ and comb. ▲

We will see in Section 3.5 that this $\mathbb{K}$-linear map extends to the isomorphisms of algebras as in Theorem 3.3.1.

Remark 3.3.32. We point out that one could upgrade Theorem 3.3.1 to include combinatorial description for the web bimodules $\mathbf{W}(u)$ as well. In principal, the steps one has to do are the same as for the algebras, but more different local situations than in (3.31) have to be considered, cf. [20, Subsections 4.2, 4.3 and 4.4]. Since we are mainly interested in the algebras and not the bimodules, we omit the rather involved details. ▲

3.4 Foams and type D arc algebra

The purpose of this section is to provide a foam construction of the type D arc algebra. If $\mathfrak{A}$ denotes the type D arc algebra with multiplication $\mathbf{Mult}^{\mathfrak{A}}$, then:

Theorem 3.4.1. There is an embedding of graded algebras

$$\mathtt{top}\colon \mathfrak{A} \longrightarrow \mathfrak{W}.$$

(Similarly, denoted by $\mathtt{top}_{\Lambda}$, on each summand.) $\square$

(Again, the all proofs are given in Subsection 3.5.3.)

In order to define $\mathtt{top}$, we have to sign adjust the multiplication structure of $\mathfrak{A}$, denoted by $\overline{\mathfrak{A}}$, which we do in Subsection 3.4.2, where we also give an isomorphism $\mathtt{sign}\colon \mathfrak{A} \xrightarrow{\cong} \overline{\mathfrak{A}}$. In fact, up to signs, the algebra $\overline{\mathfrak{A}}$ is defined almost verbatim as $\mathfrak{A}$, namely in the usual spirit of arc algebras as a $\mathbb{K}$-linear vector space on certain diagrams called *(marked) arc or circle diagrams*. (With markers displayed as $\mid$.)

Having $\overline{\mathfrak{A}}$ and $\mathtt{comb}\colon c\mathfrak{W} \xrightarrow{\cong} \mathfrak{W}$ from Section 3.3, it is almost a tautology to define an embedding $\overline{\mathtt{top}}\colon \overline{\mathfrak{A}} \hookrightarrow c\mathfrak{W}$. We do the latter in Subsection 3.4.3, but the picture to keep in mind how to go from $\overline{\mathfrak{A}}$ to $c\mathfrak{W}$ is provided by (another) "cookie-cutter strategy" as in Figure 3.2. This gives $\mathtt{top} = \mathtt{comb} \circ \overline{\mathtt{top}} \circ \mathtt{sign}$.

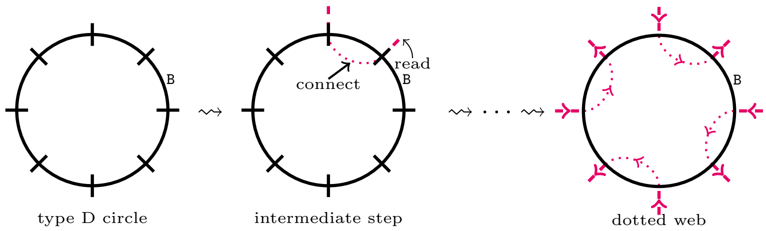


Figure 3.2: From cup diagrams to dotted webs: see a marked circle as a topological space, cut it from the rest, start at a fixed point B and go around anticlockwise, and connect neighboring markers via phantom seams, choosing the first to be oriented outwards.

Note hereby that we will not assume $\mathfrak{A}$ or $\overline{\mathfrak{A}}$ to be associative, and associativity follows immediately from Theorem 3.4.1 and Proposition 3.2.5:

Corollary 3.4.2. The multiplication $\mathbf{Mult}^{\overline{\mathfrak{A}}}$ is independent of the order in which the surgeries are performed, turning $\mathfrak{A}$ and $\overline{\mathfrak{A}}$ into associative, graded algebras. $\blacksquare$

Remark 3.4.3. By [18, Example 6.7], the order of surgeries is important for $\mathfrak{A}$. In contrast, the order is not important for $\overline{\mathfrak{A}}$, cf. Example 3.4.17. The reason is that, by Theorem 3.4.1 and the above, the multiplication rules for $\overline{\mathfrak{A}}$ are obtained by translating the topological multiplication of the web algebra $\mathfrak{W}$ into combinatorics. $\blacktriangle$

We conclude this section by sketching how our results extend to generalized arc and web algebras and the associated bimodule categories (which includes the connection to category $\mathcal{O}$).

3.4.1 Recalling the type D arc algebra

First, we briefly recall the definition of the type D arc algebra $\mathfrak{A}$. All of this closely follows [18], where the reader can find more details (and examples). Note that we simplify the notation here and focus on the type D arc algebra for two types of regular blocks in [18] and only allow cup and cap diagrams without rays. The general case can be recovered from this construction, see Remarks 3.4.18 and 3.4.19.

Weight combinatorics

We call $\mathbb{R} \times \{0\}$ the *dotted line*, cf. (3.44). We identify $\mathbb{Z}$ with the integral points on the dotted line, called *vertices*. (Hence, in contrast to the more flexible setup for the web algebra $\mathfrak{W}$, points i, j etc. are integers.)

A labeling of the vertices $\{1, \ldots, 2K\}$ by the symbols in the set $\{\wedge, \vee\}$ is called a *weight* (of *rank* K). We identify weights of rank K with $2K$-tuples $\lambda = (\lambda_i)_{1 \leq i \leq 2K}$ with entries $\lambda_i \in \{\wedge, \vee\}$. We say that two weights λ, μ are in the same *(balanced) block* Λ if μ is obtained from λ by finitely many combinations of basic linkage moves, i.e. of swapping neighbored labels $\wedge$ and $\vee$, or swapping the pairs $\wedge\wedge$ and $\vee\vee$ at the first two positions. Thus, a block is fixed by the number K and the parity of the occurrence of $\wedge$ in the weight. We denote by $\mathtt{Bl}^{\circ}$ the set of blocks and denote by the rank of a block the rank of the weights in the block.

Diagram combinatorics

Let us recall the notion of cup diagrams. Following [18, Subsection 3.1] we define *cup diagrams* of rank K. This is a collection of crossingless arcs $\{\gamma_1, \ldots, \gamma_K\}$, i.e. embeddings of the interval $[0,1]$ into $\mathbb{R} \times [-1, 0]$, such that the collection of endpoints of all arcs coincides one-to-one with the set $\{1, \ldots, 2K\}$. As in [18, Definition 3.5] we allow arcs whose interior can be connected to $(0,0)$ in $\mathbb{R} \times [-1, 0]$ without crossing any other arcs to carry a decoration (this condition is called *admissibility*), in which case we call the arc *marked* and otherwise *unmarked*.

For short, we simply say *diagram* for any kind of cup, cap or circle diagram (where we also allow stacked circle diagrams, cf. Convention 3.2.3). Moreover, we use the evident notion of a circle C in a diagram D in what follows.

Example 3.4.4. Examples of admissible and non-admissible diagrams can be found in [18, Section 3]. We stress additionally that admissible diagrams will never have circles with markers that are nested in other circles. ▲

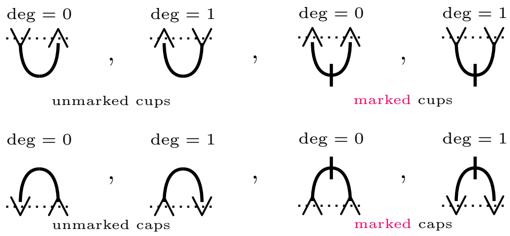
Furthermore, recall that reflecting a cup diagram d along the horizontal axis produces a *cap diagram* d^*, and putting such a cap diagram d^* on top of a cup diagram c of the same rank produces a *circle diagram*, denoted by cd^*, of the corresponding rank. As mentioned above, this is a special case of [18, Definition 3.2]. In all three cases we do not distinguish diagrams whose arcs connect the same points.

For a block Λ of rank K, a triple $c\lambda d^*$ consisting of two cup diagrams c, d of rank K and a weight $\lambda \in \Lambda$ is called an *oriented circle diagram* if all unmarked arcs connect an $\wedge$ and a $\vee$ in λ, while all marked arcs either connect two $\wedge$'s or two $\vee$'s, see e.g. (3.44). In this case we call λ the *orientation* of the diagram cd^*.

By $\mathbb{B}(\Lambda)$ we denote the set of all oriented circle diagrams (with orientations from Λ). Similarly, for cup diagrams c, d of rank K, we denote by ${}_c\mathbb{B}(\Lambda)_d$ the set of all oriented circle diagrams of the form $c\lambda d^*$ with $\lambda \in \Lambda$. In case ${}_c\mathbb{B}(\Lambda)_d = \emptyset$ we say that cd^* is *non-orientable* (by weights in Λ), otherwise it is called *orientable*.

Remark 3.4.5. By direct observation one sees that a circle diagram cd^* is orientable if and only if all of its circles have an even number of decorations on them. ▲

Further, we equip the elements of these sets with a degree by declaring that arcs have the degrees given locally via (which are added globally):



(3.44)

Here, as in the following, the dotted line indicates $\mathbb{R} \times \{0\}$.

The *degree* of an oriented circle diagram is then in turn the sum of the degrees of all arcs contained in it, both in the cup and the cap diagram.

The type D arc algebra as a $\mathbb{K}$-vector space

Very similar as before we define:

Definition 3.4.6. Given a block Λ of rank K and cup diagrams c, d of rank K, we define the graded $\mathbb{K}$-vector space

$$ {}_c(\mathfrak{A}_\Lambda)_d = \langle {}_c\mathbb{B}(\Lambda)_d \rangle_{\mathbb{K}}, $$

that is, the free $\mathbb{K}$-vector space on basis given by all oriented circle diagrams $c\lambda d^*$ with $\lambda \in \Lambda$. (The grading is hereby defined to be the one induced via (3.44).) The *type* D *arc algebra* $\mathfrak{A}_\Lambda$ *for* $\Lambda \in \mathtt{Bl}^\circ$ is the graded $\mathbb{K}$-vector space

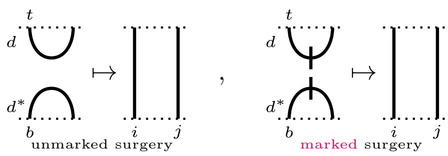
$$ \mathfrak{A}_\Lambda = \bigoplus_{c,d} {}_c(\mathfrak{A}_\Lambda)_d, $$

with the sum running over all pairs of cup diagrams of rank K. Finally the *(full) type* D *arc algebra* $\mathfrak{A}$ is the direct sum of all $\mathfrak{A}_\Lambda$, where Λ varies over all blocks. The multiplication turning $\mathfrak{A}$ into a graded algebra is described in [18, Subsection 4.3] and we summarize it below. ▲

The type D arc algebra as an algebra

As usual, to define $\mathbf{Mult}^{\mathfrak{A}} \colon \mathfrak{A} \otimes \mathfrak{A} \to \mathfrak{A}$, we do it for fixed Λ of rank K. Hereby, the product of two basis elements $c_b\lambda d^*$ and $d'\mu c_t^*$ in $\mathfrak{A}_\Lambda$ is declared to be zero unless $d = d'$. Otherwise, to obtain the product of $c_b\lambda d^*$ and $d\mu c_t^*$, put $d\mu c_t^*$ on top of $c_b\lambda d^*$, producing a diagram that has d^*d as its middle piece. (In this notation, c_b, d, c_t are cup diagrams of rank K.)

For a cup-cap pair (possibly marked) in the middle section d^*d, which can be connected without crossing any arcs and such that to the right of this pair there are no marked arcs, we replace the cup and cap by using the (un)marked surgery:



(3.45)

To avoid questions of well-definedness, we assume that we always pick the leftmost available cup-cap pair as above in what follows. One easily checks that this turns $\mathfrak{A}$ into a graded algebra (not necessarily associative at this point).

The surgery procedure itself is, as usual, performed inductively until there are no cup-cap pairs left in the middle section of the diagram. The final result is a $\mathbb{K}$-linear combination of oriented circle diagrams, all of which have $c_b c_t^*$ as the underlying circle diagram. The result in each step depends on the local situation, i.e. whether two components are merged together, or one is split into two. One then has to add orientations and scalars to the corresponding diagrams. Before we discuss how to obtain these, we need some notions to define the scalars for the multiplication.

For the next few definitions fix a cup or cap $\gamma = i \to j$ in a cup, cap, or circle diagram connecting vertex i and j.

Definition 3.4.7. With the notation as above define

$$\mathrm{utype}(\gamma) = \begin{cases} 0, & \text{if } \gamma \text{ is } \mathtt{marked}, \\ 1, & \text{if } \gamma \text{ is } \mathtt{unmarked}, \end{cases} \qquad \mathtt{m}\mathrm{type}(\gamma) = \begin{cases} 0, & \text{if } \gamma \text{ is unmarked}, \\ 1, & \text{if } \gamma \text{ is } \mathtt{marked}, \end{cases}$$

which we call the *unmarked* respectively *marked saddle type*. ▲

Definition 3.4.8. We define the *unmarked* and *marked distance* of $\gamma = i \to j$ by

$$\mathrm{ulen}_\Lambda(i \to j) = \mathrm{utype}(\gamma) \cdot |i - j|\,, \qquad \mathtt{m}\mathrm{len}_\Lambda(i \to j) = \mathtt{m}\mathrm{type}(\gamma) \cdot |i - j|\,.$$

We extend both additively for sequences of distinct cups and caps. ▲

Similar to Definition 3.3.5 we define:

Definition 3.4.9. The *base point* $\mathtt{B}(C)$ is the rightmost vertex on a circle C inside a circle diagram. ▲

We usually omit the subscript Λ in the following, and we also write $\mathtt{B} = \mathtt{B}(C)$ and $\mathrm{utype} = \mathrm{utype}(\gamma)$ etc. for short. The same works for diagrams D as well.

Let $i < j$ denote the left respectively right vertex for the cup-cap pair where the surgery is performed, see (3.45). Below D_{be} will denote the diagram before the surgery à la (3.45), while D_{af} will denote the diagram after the surgery. Moreover, a circle C is said to be (oriented) clockwise if the rightmost vertex $\mathtt{B}(C)$ it contains is labeled with $\vee$; otherwise it is said to be (oriented) anticlockwise.

Then the multiplication result is defined as follows. (If we write e.g. $C_\square$, then the corresponding circle should contain the vertex $\square$ as in (3.45). Moreover, as in Subsection 3.3.2, we write $\mathtt{B}_\square = \mathtt{B}(C_\square)$ for short.)

Merge. Assume two circles C_b and C_t are merged into a circle C_{af}.

- ▷ If both are anticlockwise, then apply (3.45) and orient the result anticlockwise.
- ▷ If one circle is anticlockwise and one is clockwise, then apply (3.45), orient the result clockwise and also multiply with

$$(-1)^{\mathrm{ulen}(\mathtt{B}_\square \to \mathtt{B}_{\mathrm{af}})}, \tag{3.46}$$

  where $C_\square$ (for $\square \in \{b, t\}$) is the clockwise circle, and $\mathtt{B}_r \to \mathtt{B}_{\mathrm{af}}$ is some concatenation of cups and caps connecting $\mathtt{B}_\square$ and $\mathtt{B}_{\mathrm{af}}$.
- ▷ If both circles are clockwise, then the result is zero.

Split. Assume a circle C_{be} splits into C_i and C_j. If, after applying (3.45), the resulting diagram is non-orientable, the result is zero. Otherwise:

- ▷ If C_{be} is anticlockwise, then apply (3.45) and take two copies of the result. In one copy orient C_i clockwise and C_j anticlockwise, in the other vice versa. Multiply the summand where C_i is oriented clockwise by

$$(-1)^{\mathrm{ulen}(i \to \mathtt{B}_i)}(-1)^{\mathrm{utype}}(-1)^i, \tag{3.47}$$

and the one where C_j is oriented clockwise by

$$(-1)^{\mathrm{ulen}(j\to \mathtt{B}_j)}(-1)^i, \tag{3.48}$$

using a notation similar to (3.46) with $i \to \mathtt{B}_i$ and $j \to \mathtt{B}_j$ appropriately chosen sequences of cups and caps connecting the indicated points.

▷ If C_{be} is clockwise, then apply (3.45) and orient C_i and C_j clockwise. Finally multiply with

$$(-1)^{\mathrm{ulen}(\mathtt{B}_{\mathrm{be}}\to \mathtt{B}_j)}(-1)^{\mathrm{ulen}(i\to \mathtt{B}_i)}(-1)^{\mathrm{utype}}(-1)^i. \tag{3.49}$$

Again, $\mathtt{B}_{\mathrm{be}} \to \mathtt{B}_j$ and $i \to \mathtt{B}_i$ are appropriate concatenations of cups and caps connecting the indicated points.

3.4.2 A sign adjusted version

The construction of the embedding from Theorem 3.4.1 splits into two pieces. First we define a *sign adjusted version* $\overline{\mathfrak{A}}$ of the type D arc algebra $\mathfrak{A}$ and show (the proof is again given in Subsection 3.5.3):

Proposition 3.4.10. There is an isomorphism of graded algebras

$$\mathtt{sign}\colon \mathfrak{A} \xrightarrow{\cong} \overline{\mathfrak{A}}.$$

(Similarly, denoted by $\mathtt{sign}_\Lambda$, on each summand.) □

The sign adjusted type D arc algebra $\overline{\mathfrak{A}}$ is then easy to embed into the web algebra $\mathfrak{W}$, which will be the purpose of the next subsection.

By definition, the algebra $\overline{\mathfrak{A}}$ has the same graded $\mathbb{K}$-vector space structure as given in Definition 3.4.6, but a multiplication modeled on the one from Section 3.3.

The sign adjusted type D arc algebra as an algebra

By definition, up to signs, the surgery procedures for both multiplications $\mathbf{Mult}^{\mathfrak{A}}$ and $\mathbf{Mult}^{\overline{\mathfrak{A}}}$ coincide. The multiplication procedure in contrast follows closely the one from Subsection 3.3.2. (As we will see in Subsection 3.4.3, it is the one from Subsection 3.3.2 specialized to the more rigid setup of the type D arc algebra.) That is, we change the steps in the multiplication as follows. As usual, all vertices b, t, i, j are as in (3.45). (And we also use the same notations and conventions as in 3.3.2 adjusted in the evident way.) Moreover, as in Remark 3.3.27, we give the rules for the Ɔ and the C shapes.

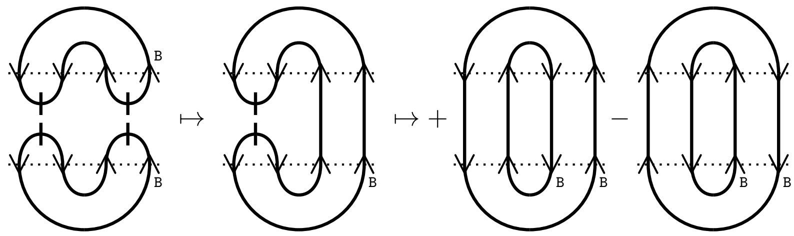
Non-nested merge, nested merge and non-nested split. Take the signs as in (3.32) to (3.37), and change:

$$\begin{aligned} (-1)^{\mathtt{p}\mathrm{edge}(\square\to\square)} &\rightsquigarrow (-1)^{\mathtt{m}\mathrm{len}(\square\to\square)}, \quad \text{(keep dot moving signs)} \\ \text{Rest} &\rightsquigarrow +1, \quad \text{(trivial other signs).} \end{aligned} \tag{3.50}$$

Nested split, Ɔ and C shape. These cases are the most elaborate. That is, take the signs as in (3.38) to (3.43), and change:

$$\begin{aligned} (-1)^{\mathtt{p}\mathrm{edge}(\square\to\square)} &\rightsquigarrow (-1)^{\mathtt{m}\mathrm{len}(\square\to\square)}, \quad \text{(keep dot moving signs)} \\ (-1)^{\mathtt{s}\mathrm{type}} &\rightsquigarrow (-1)^{\mathtt{m}\mathrm{type}}, \quad \text{(keep the saddle type)} \\ \text{Rest} &\rightsquigarrow +1, \quad \text{(trivial other signs).} \end{aligned} \tag{3.51}$$

Example 3.4.11. One of the key examples why one needs to be careful with the multiplication in the (original) type D arc algebra $\mathfrak{A}$ is [18, Example 6.7], which is the case of the following Ↄ shape:



$$(3.52)$$

Here we have used the sign adjusted multiplication. The reader should check that doing the C gives the same result for $\overline{\mathfrak{A}}$, but not for $\mathfrak{A}$. ▲

The isomorphism sign

Next, we define the isomorphism `sign` from Proposition 3.4.10. We stress that `sign` is, surprisingly, quite easy.

To this end, recall that $\mathfrak{A}$ and $\overline{\mathfrak{A}}$ have basis given by orientations on certain diagrams. The isomorphism `sign`, seen as a $\mathbb{K}$-linear map $\mathtt{sign}\colon \mathfrak{A} \to \overline{\mathfrak{A}}$, will be given by rescaling each of these basis elements. In order to give the scalar, we first fix some diagram D and let $\mathbb{B}(D)$ denote the set of all possible orientations of D.

We will write coeff_D^C to indicate the contribution of a circle C inside D to the coefficient, which we define to be

$$\mathrm{coeff}_D^C(D^{\mathrm{or}}) = \begin{cases} 1, & \text{if } C \text{ is anticlockwise in } D^{\mathrm{or}}, \\ -(-1)^{\mathtt{B}(C)}, & \text{if } C \text{ is clockwise in } D^{\mathrm{or}}. \end{cases}$$

Here D^{or} denotes D together with a choice of orientation, which induces a orientation for the circle C.

Definition 3.4.12. We define a $\mathbb{K}$-linear map via:

$$\mathrm{coeff}_D\colon \langle \mathbb{B}(D)\rangle_{\mathbb{K}} \longrightarrow \langle \mathbb{B}(D)\rangle_{\mathbb{K}}, \qquad D^{\mathrm{or}} \longmapsto \Big(\prod_{\text{circles } C \text{ in } D} \mathrm{coeff}_D^C(D^{\mathrm{or}}) \Big) D^{\mathrm{or}}.$$

(With $\mathrm{coeff}_D^C(D^{\mathrm{or}})$ as above.) ▲

Thus, we can use Definition 3.4.12 to define $\mathbb{K}$-linear maps

$$\mathtt{sign}_c^d\colon {}_c(\mathfrak{A}_\Lambda)_d \to {}_c(\overline{\mathfrak{A}}_\Lambda)_d, \qquad \mathtt{sign}_\Lambda\colon \mathfrak{A}_\Lambda \to \overline{\mathfrak{A}}_\Lambda, \qquad \mathtt{sign}\colon \mathfrak{A} \to \overline{\mathfrak{A}}, \tag{3.53}$$

for every blocks Λ of some rank K and all cup and cap diagrams c, d of rank K.

Note that, by construction, the maps in (3.53) are isomorphisms of graded $\mathbb{K}$-vector spaces. We will prove in Section 3.5 that the (two right) isomorphisms of graded $\mathbb{K}$-vector spaces from (3.53) are actually isomorphisms of graded algebras.

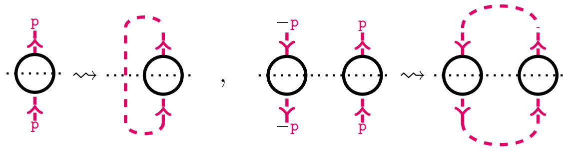
3.4.3 The foamy embedding

To define `top`, we first define $\overline{\mathtt{top}}\colon \overline{\mathfrak{A}} \to c\mathfrak{W}$. Morally, it is defined as in Figure 3.2. The formal definition is not much more elaborate and is obtained from an algorithm, cf. Algorithm 2.

Before we give Algorithm 2, we need to close up the result from Figure 3.2:

Definition 3.4.13. Let $\vec{\mathtt{p}}$ denote a finite word in the symbols $\mathtt{p}$ and $-\mathtt{p}$ only, which alternates in these. Give two webs u, v such that $uv^* \in \mathrm{End}_{\mathbf{W}}(\vec{\mathtt{p}})$. Then we obtain from it a closed web by first connecting neighboring (counting from right to left) outgoing phantom edges of u and v separately, and then finally the remaining outgoing phantom edge of u with the one of v to the very left of uv^*. ▲

Example 3.4.14. Two basic examples of Definition 3.4.13 are:



Observe that one has a phantom edge passing from u to v if and only if $\vec{\mathtt{p}}$ has an odd number of symbols in total. Note also, as in particular the right example illustrates, it is crucial for this to work that $\vec{\mathtt{p}}$ alternates in the symbols $\mathtt{p}$ and $-\mathtt{p}$. ▲

input : a diagram D;
output : a dotted web $\overline{W}(D)$;

initialization, let $\overline{W}(D)$ be the empty web;
for *circles* C *in* D **do**
  if C *is marked* **then**
    run the procedure from Figure 3.2;
    add the corresponding web to $\overline{W}(D)$;
    remove the corresponding circle from D;
  else
    add C as a circle in a web to $\overline{W}(D)$;
    remove the corresponding circle from D;
  end
end
close the phantom edges as in Definition 3.4.13;

Algorithm 2: Turning a marked circle into a web.

For any circle diagram cd^* of rank K we obtain via Algorithm 2 webs

$$\overline{w}(cd^*) \quad \text{and} \quad u(c), u(d) \in \mathrm{CUP}^{\vec{k}}, \quad \vec{k} \in \vec{K} \tag{3.54}$$

by considering the shape of $\overline{W}(cd^*)$. Hence, for an oriented circle diagram $c\lambda d^*$ with $\lambda \in \Lambda$ for a block Λ of rank K, we obtain a dotted basis web

$$\overline{W}(c\lambda d^*) \in {}_{u(c)}\mathbb{B}(\vec{K})_{u(d)} \tag{3.55}$$

by putting a dot on each circle in $\overline{W}(cd^*)$ for which the corresponding circle in $c\lambda d^*$ is oriented clockwise. We call the dotted basis web from (3.55) *the dotted basis web associated to* $c\lambda d^*$. (The careful reader might want to check that this is actually well-defined by observing that Algorithm 2 gives a well-defined result.)

This almost concludes the definition of $\overline{\mathtt{top}}$, but we also need a certain sign which corrects the sign turning up for the nested splits, see Example 3.3.30.

Definition 3.4.15. For a stacked dotted web $\overline{W}$ we define

$$\begin{aligned}\mathtt{np}\mathrm{esci}(\overline{W}) = {} & \text{number of } \mathtt{anticlockwise} \text{ phantom (edge+seam) circles} \\ & \text{touching the top dotted line of } \overline{W},\end{aligned}$$

where phantom (edge+seam) circles are the circles obtained by considering phantom edges and seams. (These have a well-defined notion of being anticlockwise.) ▲

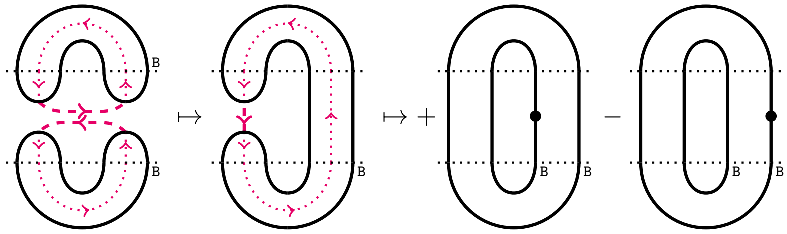
(Note that Definition 3.4.15 is again asymmetric in the sense that we only count anticlockwise phantom (edge+seam) circles.)

Definition 3.4.16. We define a $\mathbb{K}$-linear map via:

$$\overline{\mathtt{top}}_c^d\colon \langle {}_c\mathbb{B}(\Lambda)_d\rangle_{\mathbb{K}} \to \langle {}_{u(c)}\mathbb{B}(\vec{K})_{u(d)}\rangle_{\mathbb{K}}, \quad c\lambda d^* \mapsto (-1)^{\mathrm{npesci}(\overline{W}(c\lambda d^*))}\cdot \overline{W}(c\lambda d^*),$$

by using the notion from (3.55). Taking direct sums then defines $\overline{\mathtt{top}}_\Lambda$ and $\overline{\mathtt{top}}$. ▲

Example 3.4.17. Again, back to [18, Example 6.7]. The diagrams in (3.52) are sent to the following dotted basis webs:



Note the difference to the result calculated in Example 3.3.30, i.e. there is a phantom circle sign turning up, which is corrected by $\overline{\mathtt{top}}$: For the leftmost stacked dotted basis web $\overline{W}_1$ we have $\mathrm{npesci}(\overline{W}_1) = 1$ (since it has an anticlockwise phantom edge-seam circle at the top), the middle stacked dotted basis web $\overline{W}_2$ also has $\mathrm{npesci}(\overline{W}_2) = 1$. In contrast, for the leftmost stacked dotted basis web $\overline{W}_3$ one has $\mathrm{npesci}(\overline{W}_3) = 0$, and the last step is where the sign goes wrong. ▲

The definition of $\mathtt{top}$ is now dictated:

$$\mathtt{top} = \mathtt{comb}\circ\overline{\mathtt{top}}\circ\mathtt{sign}\colon \mathfrak{A} \xrightarrow{\cong} \overline{\mathfrak{A}} \hookrightarrow c\mathfrak{W} \xrightarrow{\cong} \mathfrak{W}, \tag{3.56}$$

and similarly for $\mathtt{top}_c^d$ and $\mathtt{top}_\Lambda$. As usual, we show in Section 3.5 that $\mathtt{top}_\Lambda$ and $\mathtt{top}$ are isomorphisms of graded algebras.

3.4.4 Its bimodules, category $\mathcal{O}$ and foams

Comparing with [18] there are two generalizations that needs to be addressed from the point of view of this section.

Remark 3.4.18. The first generalization is towards more general blocks as they are defined in [18, Section 2.2]. This includes defining weights supported on the positive integers with allowed symbols from the set $\{\circ, \wedge, \vee, \times\}$. As long as one restricts to the situation of balanced blocks, i.e. blocks where the total combined number of $\wedge$ and $\vee$ symbols in a weights is even, the whole construction presented in this section can be used with one key difference: whenever a formula in any of the multiplications (or the isomorphism from Section 3.5.3) includes a power $(-1)^{\square}$ where $\square$ is some index of a vertex this must be interchanged with $(-1)^{\mathrm{p}(\square)}$ (with p as defined in [18, (3.12)]). The rest works verbatim. ▲

Remark 3.4.19. The second generalization is towards the *generalized type* D *arc algebra* $\mathfrak{C}$ (this is the algebra denoted by $\mathbb{D}_m$ in the introduction), and a corresponding *generalized web algebra* $g\mathfrak{W}$. The algebra $\mathfrak{C}$ is the algebra as defined in [18, Section 5] including rays in addition to cups and caps, while, as we explain now, the algebra $g\mathfrak{W}$ can be thought of as a type D foam version of the algebra defined by Chen-Khovanov [12]:

The discussion in [18, Subsection 5.3] is the analog of the (type A) *subquotient construction* from [20, Subsection 5.1], and the analog of [20, Theorem 5.8] holds in the type D setup as well (by using [18, Theorem A.1]). Hereby, the main difference to [20, Subsection 5.1] lies in the fact that for the closure of a weight (as defined in [20, Definition 5.1]) one only uses additional symbols $\wedge$ to the right of the non-trivial vertices of the weight, similar to the type A situation, and one adds a total number of $\wedge$'s equal to the combined number of $\wedge$ and $\vee$ occurring in the weight.

Copying the same subquotient construction on the side of the web algebra (as done in the type A situation in [20, Subsection 5.1]) defines $g\mathfrak{W}$, which then can be seen to be similar to Chen-Khovanov's (type A) construction, cf. [20, Remark 5.7]. The corresponding *generalized foam 2-category* $g\mathfrak{F}$ can then, keeping Proposition 3.2.11 in mind, defined to be $g\mathfrak{W}$-**biMod**. $\blacktriangle$

Using Remarks 3.4.18 and 3.4.19, we see that everything from above generalizes to $\mathfrak{C}$, $g\mathfrak{W}$, $g\mathfrak{F}$ etc. In particular, one gets an embedding

$$g\mathtt{top}\colon \mathfrak{C} \hookrightarrow g\mathfrak{W}.$$

In fact, $g\mathtt{top}(\mathfrak{C})$ is an idempotent truncation of $g\mathfrak{W}$. Hence, we can actually define *type* D *arc bimodules* in the spirit of Definition 3.2.7 for any stacked circle diagram.

Taking everything together and recalling that $\mathfrak{C}$ is the algebra whose category of finite-dimensional modules is equivalent to the category $\mathcal{O}_0^{\mathrm{A}_{n-1}}(\mathfrak{so}_{2n}(\mathbb{C}))$ (see [18, Theorem 9.1]), we obtain a topological interpretation of $\mathcal{O}_0^{\mathrm{A}_{n-1}}(\mathfrak{so}_{2n}(\mathbb{C}))$ in terms of foams.

3.5 Proofs

In this section we give all intricate proofs. There are essentially three things to prove: in the first subsection we construct the cup foam basis, in the second we show that $c\mathfrak{W}$ is a combinatorial model of the web algebra, and in the last we prove that the type D arc algebra embeds into $c\mathfrak{W}$.

Let us stress that we only consider (well-oriented) webs as in Convention 3.3.4, if not stated otherwise. For ill-oriented webs all foam spaces are zero and these also do not show up in the translation from type D to the foam setting. (Hence, there is no harm in ignoring them.)

3.5.1 Proofs of Propositions 3.1.14 and 3.2.11

In this subsection we construct the cup foam basis and prove all the consequences of its existence/construction.

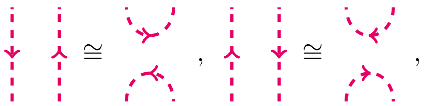
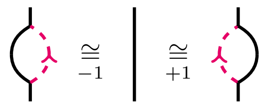
Proof of the existence of the cup foam basis

Our next goal is to describe isomorphisms among the morphisms of $\mathfrak{F}$ which we call *relations among webs.*

Lemma 3.5.1. There exist isomorphisms in $\mathfrak{F}$ realizing the following relations among webs. First, the *ordinary* and *phantom circle removals*:

(3.57) $\bigcirc \cong \varnothing\{-1\} \oplus \varnothing\{+1\}$, (3.58)  $\cong \varnothing \cong$ 

Second, the *phantom saddles* and the *phantom digon removal*:

(3.59)  (3.60) 

(The signs indicated in (3.60) are related to our choice of foams lifting these, see below.) There are isotopy relations of webs as well. □

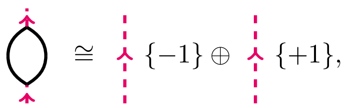
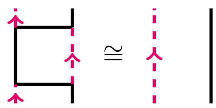
Note that each phantom digon is a phantom loop, but not vice versa since a phantom loop might have additional phantom edges in between its trivalent vertices.

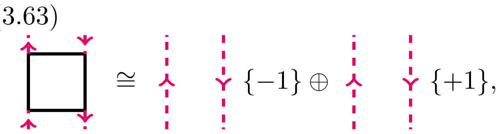
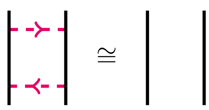
Proof. All of these can be proven in the usual fashion, i.e. by using the corresponding relation of foams and cutting the pictures in half, see e.g. [21, Lemma 4.3].

For the relations among webs the corresponding relations of foams are:

- ▷ The foams corresponding to (3.57) are the ones in (3.3) and (3.5).
- ▷ The foams corresponding to (3.58) are the ones in (3.6) and (3.8).
- ▷ The relations (3.59) among webs are, by (3.8), lifted by phantom saddle foams.
- ▷ The foams corresponding to (3.60) are the ones in (3.17) and (3.18) (as well as their orientation reversed counterparts). ■

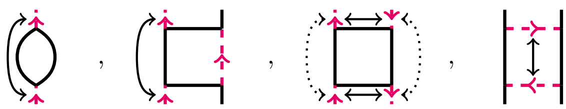
Lemma 3.5.2. The *digon* and *square removals*

(3.61)  (3.62) 

(3.63)  (3.64) 

are consequences of the relations among webs from Lemma 3.5.1. (There are various reoriented versions as well.) □

Proof. We indicate where we can apply phantom saddle relations (3.59):



(For (3.63), there is a choice where to apply the phantom saddles, cf. Example 3.5.6.) One can then continue using the phantom digon (3.60), and removing the circle (3.57) in case of (3.63). The corresponding foams inducing the relations from Lemma 3.5.1 then induce the isomorphisms in $\mathfrak{F}$ realizing the above relations among webs. ■

When referring to these relations among webs we fix the isomorphisms that we have chosen in the proof of Lemma 3.5.1 realizing these relations. (These induce the corresponding isomorphisms lifting the relations from Lemma 3.5.2, except for (3.63) where there is no preferred choice where to apply the phantom saddles.) We call these *evaluation foams*. Note hereby, as indicated in (3.60), the foams realizing the phantom digon removal might come with a plus or a minus sign, cf. Remark 3.1.10.

The point of the relation among webs is that they evaluate closed webs:

Lemma 3.5.3. For closed web $\overline{w}$ there exists a sequence $(\phi_1, \dots, \phi_r)$ of relations among webs and some shifts $s \in \mathbb{Z}$ such that

$$\overline{w} \overset{\phi_1}{\cong} \cdots \overset{\phi_r}{\cong} \bigoplus_s \varnothing\{s\} \qquad (\text{in } \mathfrak{F}).$$

Such a sequence is called *an evaluation of* $\overline{w}$. □

Proof. By induction on the number n of vertices of $\overline{w}$.

If $n \leq 4$, the statement is clear by Lemmas 3.5.1 and 3.5.2. (Recall that we consider well-oriented webs only.) So assume that $n > 4$.

First, we can view a closed (well-oriented) web $\overline{w}$ as a planar, trivalent graph in $\mathbb{R}^2$ with all faces having an even number of adjacent vertices. Thus, by Euler characteristic arguments, $\overline{w}$ must contain at least a circle face (zero adjacent vertices), a digon face (two adjacent vertices) or a square face (four adjacent vertices). By (3.57) and (3.58) we can assume that $\overline{w}$ does not have circle faces. Hence, we are done by induction, since using (3.60), (3.61), (3.62), (3.63) or (3.64) reduces n. (Observe that these are all possibilities of what such digon or square faces could look like.) ■

We are now ready to prove Proposition 3.1.14. The main ingredient is the *cup foam basis algorithm* as provided by Algorithm 3.

input : a closed web $\overline{w}$ and an evaluation $(\phi_1, \dots, \phi_r)$ of it;
output : a sum of evaluation foams in $\mathcal{T}(\overline{w}) = 2\mathrm{Hom}_{\mathfrak{F}}(\varnothing, \overline{w})$;

initialization, let f_0 be the identity foam in $2\mathrm{End}_{\mathfrak{F}}(\overline{w})$;
for $k = 1$ **to** r **do**
  apply the isomorphism lifting ϕ_k to the bottom of f_{k-1} and obtain f_k;
end

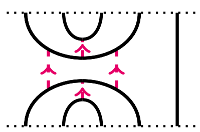
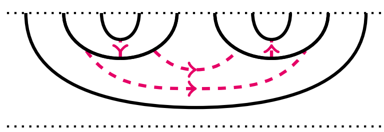
Algorithm 3: The cup foam basis algorithm.

(Hereby, if $\overline{w}$ has more than one connected component, it is important to evaluate nested components first and we do so without saying.)

Proof of Proposition 3.1.14. Given a closed web $\overline{w}$, by Lemma 3.5.3, there exists some evaluation of it which we fix.

Hence, using Algorithm 3, we get a sum of evaluation foams, all of which are $\mathbb{K}$-linear independent by construction. Thus, by taking the set of all summands produced this way, one gets a basis of $2\mathrm{Hom}_{\mathfrak{F}}(\varnothing, \overline{w})$ by Lemmas 3.5.1 and 3.5.2.

For general webs u, v, we use the "bending trick". Define $b(u)$ to be

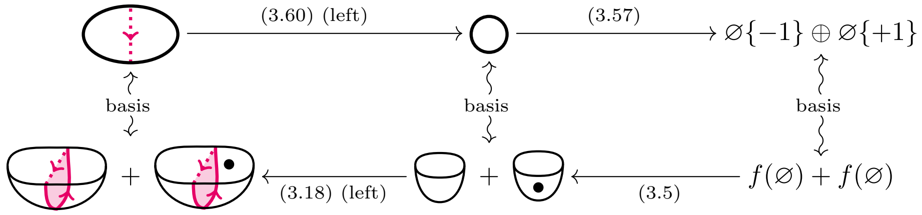
$u =$  , $b(u) =$  (3.65)

Similarly for $b(v)$. Next, using the very same arguments as above, we can write down a basis for $2\mathrm{Hom}_{\mathfrak{F}}(\varnothing, b(u)b(v)^*)$. Bending this basis back proves the statement.

Scrutiny in the above process (keeping track of grading shifts) actually shows that everything works graded as well and the resulting basis is homogeneous. ■

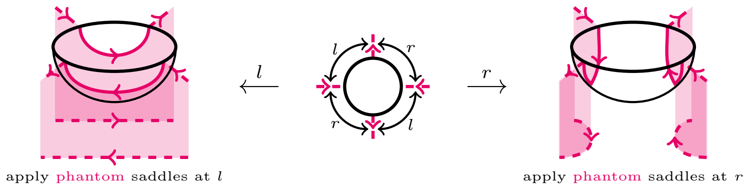
Remark 3.5.4. Indeed, almost verbatim, we also get a dual cup foam basis, called *cap foam basis*, i.e. a basis of $2\mathrm{Hom}(\overline{w}, \varnothing)$, which is dual in the sense that the evident pairing given by stacking a cap foam basis element onto a cup foam basis element gives ± 1 for precisely one pair, and zero else. ▲

Example 3.5.5. Let us consider an easy example, namely:



(Here we apply (3.60) to the left face.) Each summand is a basis element in $2\mathrm{Hom}_{\mathfrak{F}}(\varnothing, \overline{w})$ with the signs depending on whether we apply (3.60) on the left or right face of $\overline{w}$. (Note that the lift of (3.60) gives an overall plus sign in this case.) ▲

Example 3.5.6. The following (local) example illustrates the choice we have to make with respect to the topological shape of our cup foam basis elements:



The two possible cup foam basis elements illustrated above (obtained by applying the phantom saddle relation among webs (3.59) to either the pair indicated by l=left or r=right, and then by applying (3.60) to remove the phantom digons) differ in shape, but after gluing an additional phantom saddle to the bottom of the left (or right) foam, they are the same up to a minus sign. ▲

Remark 3.5.7. Our proof of the existence of a cup foam basis using Algorithm 3 works more general for any kind of web algebra (as e.g. the one studied in [45]).

Note that Algorithm 3 heavily depends on the choice of an evaluation and it is already quite delicate to choose an evaluation such that one can control the structure constants within the multiplication. For general web algebras this is very complicated and basically unknown at the time of writing this thesis, cf. [64] and [65]. This in turn makes e.g. the proof of the analog of Proposition 3.2.11 (as given below) much more elaborate for general web algebras. ▲

Presenting foam 2-categories

Next, we prove Proposition 3.2.11.

Proof of Proposition 3.2.11. Similar to [20, Proof of Proposition 2.43], we can define the 2-functor Υ, which is given by sending a web u to the $\mathfrak{W}$-bimodule $\mathbf{W}(u)$. Moreover, by following [20, Proposition 2.43], one can see that Υ is bijective on objects, essential surjective on morphisms and faithful on 2-morphisms.

In order to see fullness, we fix two webs u, v. We need to compare $\dim(\mathrm{Hom}_{\mathfrak{W}}(\mathbf{W}(u), \mathbf{W}(v)))$ with $\dim(2\mathrm{Hom}_{\mathfrak{F}}(u, v))$. The latter is easy to compute using bending, since we already know that it has a cup foam basis by Proposition 3.1.14. In order to compute $\dim(\mathrm{Hom}_{\mathfrak{W}}(\mathbf{W}(u), \mathbf{W}(v)))$ we need to find the filtrations of the $\mathfrak{W}$-bimodules $\mathbf{W}(u)$ and $\mathbf{W}(v)$ by simple $\mathfrak{W}$-bimodules. (Here we need $\mathbb{K} = \overline{\mathbb{K}}$.)

This is done as follows. By using the cup foam basis for $\mathbf{W}(u)$, we see that $\mathbf{W}(u)$ has one simple $\mathfrak{W}$-sub-bimodule L_1 spanned by the cup foam basis element with a dot on each component corresponding to a circle in $\mathbf{W}(u)$ (called maximally dotted). Then $\mathbf{W}(u)/L_1$ has one $\mathfrak{W}$-sub-bimodule given as the $\mathbb{K}$-linear span of all cup foam basis elements with one dot less than the maximally dotted cup foam basis element. Continue this way computes the filtration of $\mathbf{W}(u)$ by simple $\mathfrak{W}$-bimodules. The same works verbatim for $\mathbf{W}(v)$ which in the end shows that

$$\dim(2\mathrm{Hom}_{\mathfrak{F}}(u,v)) = \dim(\mathrm{Hom}_{\mathfrak{W}}(\mathbf{W}(u),\mathbf{W}(v))).$$

We already know faithfulness and Υ is, by birth, a structure preserving 2-functor, which finishes the proof. ■

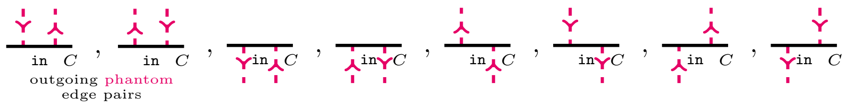
Choosing a cup foam basis

Up to this point, having some basis was enough. For all further applications, e.g. for computing the multiplication explicitly, we have to fix a basis. That is what we are going to do next.

Note hereby, that, as illustrated in Example 3.5.6, the cup foam basis algorithm depends on the choice of an evaluation. Hence, what we have to do is to choose an evaluation for every closed web $\overline{w}$. Then, by choosing to bend to the left as in (3.65), we also get a fixed cup foam basis for $2\mathrm{Hom}_{\mathfrak{F}}(u,v)$ for all webs u,v.

We start by giving an algorithm how to evaluate a fixed circle C in a web u into an empty web, i.e. a web without ordinary edges. This will depend on a choice of a point i on C.

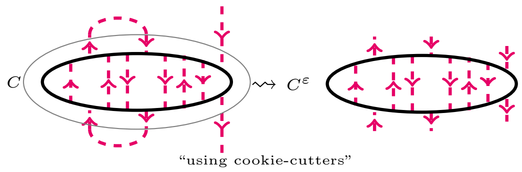
Before giving this algorithm, which we call the *circle evaluation algorithm*, note that one is locally always in one of the following situations (cf. Remark 3.1.5 and (3.30)):



(3.66)

Again, `in` denotes the interior of the circle C. The two leftmost situations are called *outgoing phantom edge pairs*. We say, such a pair is closest to the point i, or *i-closest*, if it is the first such pair reading anticlockwise starting from i.

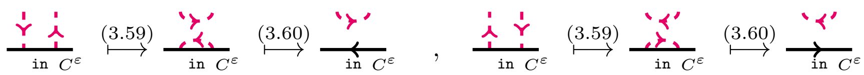
An ε-neighborhood C^{ε} of a circle C is called a *local neighborhood* if C^{ε} contains the whole interior of C and C^{ε} has no phantom loops in the exterior, e.g.



(3.67)

Lemma 3.5.8. Let C be a circle with a point i on it. There is a way to evaluate C^{ε} while keeping i fixed till the end, using only (3.62) followed by (3.60) to detach outgoing phantom edges, and removing all internal phantom edges using (3.60) only. □

Proof. Local situations of the following forms



can always be simplified as indicated above. Thus, we can assume that C^{ε} does not have outgoing phantom edge pairs. But this means that C^{ε} is of the form as in (3.67) (right side), which than can be evaluated recursively using (3.60) only. ■

To summarize, we have two basic situations for C^{ε}'s:

▷ Outgoing phantom edge pairs, cf. (3.66) (left two pictures).

▷ Phantom digons, cf. (3.60).

Now, the *circle evaluation algorithm* is defined in Algorithm 4.

input : a circle C in a web u and a point i on it;
output : an evaluation $\phi = (\phi_1, \dots, \phi_r)$ of the circle C;

initialization; let $\phi = ()$;
while C^{ε} *contains two ordinary edges* **do**
  if C^{ε} *contains an outgoing phantom edge pair* **then**
    apply (3.59) to the i-closest such pair;
    add the corresponding relation among webs to ϕ;
  else
    remove any phantom digon not containing i using (3.60);
    add the corresponding relation among webs to ϕ;
  end
end
remove the circle containing i using (3.57) and all phantom circles using (3.58);
add the corresponding relations among webs to ϕ;

Algorithm 4: The circle evaluation algorithm.

Lemma 3.5.9. Algorithm 4 terminates and is well-defined. □

Proof. That it terminates follows by its very definition via Lemma 3.5.8.

To see well-definedness, observe that the used phantom digon removals (3.60) are far apart and hence, the corresponding foams realizing these commute by height reasons. Similarly, for the relations (3.57) and (3.58). This in total shows that the resulting evaluation foams are the same (as 2-morphisms in $\mathfrak{F}$). ■

Before we can finally define our choice of a cup foam basis, we need to piece Algorithms 3 and 4 together to the *evaluation algorithm*, see Algorithm 5.

Lemma 3.5.10. Algorithm 5 terminates and is well-defined. □

Proof. That the algorithm terminates is clear. That it is well-defined (i.e. that the resulting evaluation foams do not depend on the choice of which circles are taken first to be evaluated) follows because of the "cookie-cutter strategy" (cf. Example 3.5.13) taken within the algorithm which ensures that the resulting foam parts are far apart and thus, height commute. ■

Armed with these notions, we are ready to fix a cup foam basis.

Definition 3.5.11. For any closed web $\overline{w}$ together with a fixed choice of a base point for each of its circles, we define the cup foam basis $\mathbb{B}(\overline{w})$ attached to it to be the evaluation foams turning up by applying Algorithm 3 to the evaluation of $\overline{w}$ obtained by applying Algorithm 5 to $\overline{w}$.

More generally, by choosing to bend to the left as in (3.65), we also fix a cup foam basis ${}_u\mathbb{B}_v$ for any two webs u, v. ▲

input : a closed web $\overline{w}$ and a fixed point on each of its circles;
output : an evaluation $\phi = (\phi_1, \dots, \phi_r)$ of $\overline{w}$;

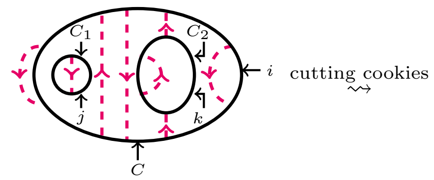
initialization; let $\phi = ()$;
while *$\overline{w}$ contains a circle* **do**
  if *C does not contain a nested circle* **then**
    take the circle C with its fixed point and apply Algorithm 4;
    add the result to ϕ;
    remove C from $\overline{w}$;
  else
    remove all remaining phantom circles using (3.58);
    add the corresponding relation among webs to ϕ;
  end
end

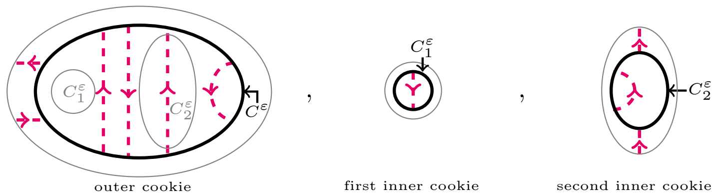
Algorithm 5: The evaluation algorithm.

Note that, by Lemmas 3.5.9 and 3.5.10, the notion of $\mathbb{B}(\overline{w})$ is well-defined, while Proposition 3.1.14 guarantees that $\mathbb{B}(\overline{w})$ is a basis of $\mathcal{T}(\overline{w}) = 2\mathrm{Hom}_{\mathfrak{F}}(\varnothing, \overline{w})$.

Example 3.5.12. Depending on the choice of a base point, the cup foam basis attached to the local situation as in Example 3.5.6 gives either of the two results. ▲

Example 3.5.13. Our construction follows a "cookie-cutter strategy":





To this web the algorithm applies the "cookie-cutter strategy" by first cutting out C_1^ε and C_2^ε and evaluate them using Algorithm 4. (The resulting evaluation foams in the first case are as in Example 3.5.5; the reader is encouraged to work out the resulting evaluation foams in the second case.) Then its cuts out C^ε (with C_1 and C_2 already removed) and applies Algorithm 4 again. The resulting cup foam basis elements are then obtained by piecing everything back together. ▲

3.5.2 Proof of Theorem 3.3.1

The aim is to show that the combinatorial algebra defined in Section 3.3 gives a model for the web algebra. To this end, we follow the ideas from [21, proof of Theorem 4.18], but carefully treat the more flexible situation we are in.

In particular, it is very important to keep in mind that we have fixed a cup foam basis, and we say a foam is (locally) of *cup foam basis shape* if it is topologically as the corresponding foam showing up in our choice of the cup foam basis.

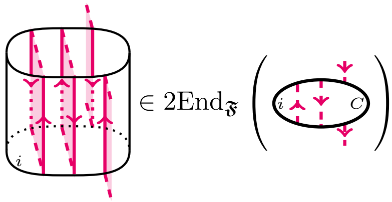
Simplifying foams

First, we give three useful lemmas how to simplify foams. Before we state and prove these lemmas, we need some terminology.

Take a web u, a circle C in it and a local neighborhood C^{ε} of it, and consider the identity foam in $2\mathrm{End}_{\mathfrak{F}}(u)$. Then $C^{\varepsilon} \times [-1,+1]$ is called a *singular cylinder*. Blueprint examples are the foams in (3.5) or (3.13) (seeing bottom/top as webs containing C^{ε}), but also the situation in (3.68).

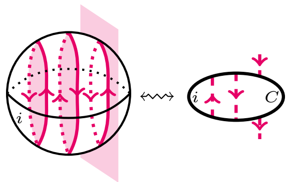
Similarly, a *singular sphere* in a foam is a part of it that is a sphere after removing all phantom edges/facets, cf. (3.3), (3.12) or (3.69).

(3.68)



singular cylinder

(3.69)



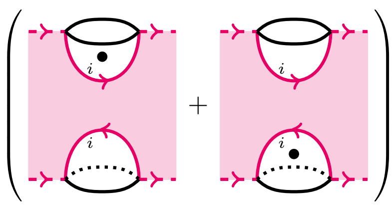
singular sphere

Next, the local situation (3.69) has an associated web with an associated circle given by cutting the pictures in halves. (This is exemplified in (3.69), i.e. cutting the singular sphere around the equator gives the web on the right side.) Hence, from the bottom/top web for singular cylinders, and the webs associated to singular spheres we obtain the numbers as defined in Subsection 3.3.1. Hereby we use the points indicated above, which we also fix for Lemmas 3.5.14 and 3.5.15.

Now we can state the three main lemmas on our way to prove Theorem 3.3.1, namely the signs turning up by simplifying singular cylinders and spheres. For short, we say □-facets for foam facets touching the web segments containing the point □.

Lemma 3.5.14. Given a singular cylinder. Then we can simplify it to

$$(-1)^{\mathtt{np}\mathrm{loop}(C,i)} \cdot$$



(There might be more or fewer attached phantom edges/facets as well - depending on the starting configuration.) Both dots sit on the i-facets and the coefficients are obtained from the associated circle C and base point i on it in the bottom/top web, and the cup and cap are in cup foam basis shape in case $i = \mathtt{B}$. □

Proof. This follows by a recursive squeezing procedure lowering the number of trivalent vertices attached to the circle C in question.

This recursive squeezing procedure should be read as starting from bottom/top of the singular cylinder, applying some foam relations giving a thinner singular cylinder on the next level of the recursion until one ends with a usual cylinder which we can cut using (3.5). (The pictures to keep in mind are (3.16) and (3.18).)

The main technical point is that we want to end with a cup and a cap of cup foam basis shape with respect to the point $\mathtt{B}$. Thus, the squeezing process depends on the particular way how to squeeze the singular cylinder.

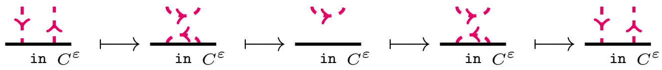
Luckily, an easy trick enables us to always end up with cup foam basis shapes with respect to the point $\mathtt{B}$. Namely, we squeeze the cylinder by first evaluating the bottom circle using Algorithm 4 and then antievaluate the result by reading Algorithm 4 backwards. We obtain from this a sequence of relations among webs and a foam lifting them which correspond to a situation as in the lemma:

$$(\phi_1, \dots, \phi_r, \phi_r^{-1}, \dots, \phi_1^{-1}) \leftrightsquigarrow f \in 2\mathrm{End}_{\mathfrak{F}}(C^{\varepsilon}).$$

(Here ϕ_k^{-1} means the other halves of the foams chosen in Subsection 3.5.1.) By construction, the foam f is the identity and it remains to analyze the signs turning up by the foams lifting the concatenations $\phi_k\phi_k^{-1}$ of the relations among webs.

Now, Algorithm 4 gives us the following:

▷ Outgoing phantom edge pairs are squeezed using

(3.8) and (3.18) $\overset{\text{lift}}{\rightsquigarrow}$  (3.70)

(Or its reoriented version.)

▷ Internal phantom loops are squeezed as in (3.70), but using (3.18) only.

Note that we only use (3.18), which gives a minus or plus sign depending on the local situation (cf. (3.60)), and (3.8), which always gives a minus sign. Carefully keeping track of these signs (e.g. the sign turns around for outgoing edge pairs compared to internal phantom loops since we use both, (3.18) and (3.8)) shows that we get the claimed coefficients.

By construction, the dots sit at the i-facet (since the facet with the point i is the last one to remove in Algorithm 4), and the cup and cap are of cup foam basis shape in case $i = \mathtt{B}$. In total, the lemma follows. ■

Lemma 3.5.15. Given a singular sphere with a dot sitting on some i-facet. Then this singular sphere evaluated to

$$(-1)^{\mathrm{nploop}(C,i)}.$$

The coefficient is obtained from the associated web and its statistics. In case the singular sphere has not precisely one dot, then it evaluates to zero. □

Proof. In fact, the steps for the evaluation of singular spheres are the inverses of the steps for recursively squeezing singular cylinder. Thus, the first statement follows, mutatis mutandis, as in Lemma 3.5.14. The second statement is evident by the described squeezing procedure and (3.3). ■

Lemma 3.5.16. In the setting of Lemma 3.5.14: if $f_{\square}$ denote the foams obtained by cutting the singular cylinder with respect to the points $\square \in \{i, j\}$, then

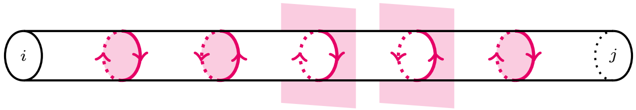
$$f_i = (-1)^{\mathrm{pedge}(i \to j)} \cdot f_j.$$

(Note that f_i and f_j have their dots on different facets and are of different shape.) □

Proof. Take the foam f_i and close its cap/cup at bottom/top such that the result are two singular spheres as in Lemma 3.5.15, that is, with dots on the i-facets. Hence, by Lemmas 3.5.14 and 3.5.15, the result is $+1$ times a foam which consists of parallel phantom facets only. Applying the same to f_j also gives a foam which consists of parallel phantom facets only but with a different sign: The bottom/top singular sphere are topologically equal (not necessarily to the ones for f_i, but equal to each other), but they have a different dot placement. One of them has a dot on the

i-facet, one of them on the j-facet. Thus, after moving the dot from the i-facet to the j-facet (giving the claimed sign), the two created singular spheres can be evaluated in the same way and all other appearing signs cancel. ■

Next, a *singular neck* is a local situation of the form

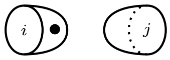
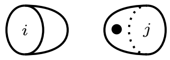


(3.71)

singular neck

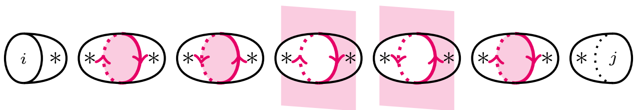
Again, for (3.71) one has an associated web, cup-cap-pair and points, and we get:

Lemma 3.5.17. Given a singular neck. Then we can simplify it to

$(-1)^{\mathtt{np}\mathrm{sad}(i)} \cdot$  $+ (-1)^{\mathtt{s}\mathrm{type}}(-1)^{\mathtt{np}\mathrm{sad}(i)} \cdot$ 

(There might be phantom facets in between as well - depending on the starting configuration.) The coefficients are obtained from the associated web. □

Proof. Assume that the singular neck has n singular seams in total. By using neck cutting (3.5) in between all of these (cutting to the left and the right of the two outermost singular seams as well) we obtain 2^{n+1} summands of the form



with the $*$ indicating that there might be a dot. By (3.9), (3.10), (3.12) as well as the second, reoriented, version of (3.12) we see that all of them but two die. The two remaining summands have a dot on i and j, respectively. The other dots coming from neck cutting (3.5) for these two are always placed on the opposite side of the singular seams in question (looking form i respectively j). So we are left with the foam we want plus a bunch of dotted theta foams and dotted singular spheres.

Next, removing now the theta foams and the singular spheres (again using relations (3.9), (3.10), (3.12) as well as the second, reoriented, version of (3.12)) gives signs depending on the orientations of the singular seams. In total, the sign for the i-dotted respectively j-dotted component is given by $(-1)^{\mathtt{np}\mathrm{sad}(i)}$ respectively by $(-1)^{\mathtt{np}\mathrm{sad}(j)}$. But, clearly, $\mathtt{np}\mathrm{sad}(i) + \mathtt{np}\mathrm{sad}(j) = \mathtt{s}\mathrm{type}$ and we are done. ■

We stress that we abuse language: singular cylinders, spheres and necks might contain no phantom facets at all. The above lemma still work and all appearing coefficients are $+1$.

The combinatorics of the multiplication

First, we complete the definition of dotted basis webs. This is easy: copy almost word-by-word Algorithm 5 and then Definition 3.5.11. The resulting dotted basis webs correspond to our choice of cup foam basis from Definition 3.5.11. Now we prove Theorem 3.3.1:

Proof of Theorem 3.3.1. First note that the $\mathbb{K}$-linear maps $\mathtt{comb}_u^v$ defined in Definition 3.3.31 are isomorphisms of $\mathbb{K}$-vector spaces because dotted basis webs of shape uv^* are clearly in bijection with the cup foam basis elements in $\mathbb{B}(uv^*)$, and the latter is a basis of ${}_u(\mathfrak{W}_{\vec{k}})_v$.

These isomorphisms are homogeneous which basically follows by definition. That is, a cup foam basis element with some dots is, after forgetting phantom edges/facets, topologically just a bunch of dotted cups. Thus, by direct comparison of (3.3.8) and (3.19), we see that all these isomorphisms are homogeneous.

Hence, it remains to show that they intertwine the inductively given multiplication. To this end, similar to [21, Subsection 4.5], we distinguish some cases, with some new cases turning up due to our more flexible setting:

(i) **Non-nested merge**. Two non-nested components are merged.

(ii) **Nested merge**. Two nested components are merged.

(iii) **Non-nested split**. One component splits into two non-nested components.

(iv) **Nested split**. One component splits into two nested components.

(v) **Phantom surgery**. We are in the phantom surgery situation.

(vi) **"Turning inside out"**. Reconnection of phantom seams.

The cases (i) to (iv) are the main cases, and we will start with these. The other cases follow almost directly by construction (as we will see below).

We follow [21, Proof of Theorem 4.18] or [20, Proof of Theorem 4.7]: First, one observes that all components of the webs which are not involved in the multiplication step under consideration can be moved far away (and, consequently, can be ignored). Second, there will be three circles involved in the multiplication. After the multiplication process the resulting foam might not be of the topological form of a basis cup foam and some non-trivial manipulation has to be done:

(I) In all of the main cases, it might be necessary to move existing or newly created dots to the adjacent facets of the chosen base points.

(II) The sign $(-1)^{\mathtt{n}\mathtt{p}\mathrm{circ}(\overline{W})}$ only appears in the nested split case and comes precisely as stated.

(III) In all of the main cases, we cut (one or two) singular cylinders and remove (one or two) singular spheres.

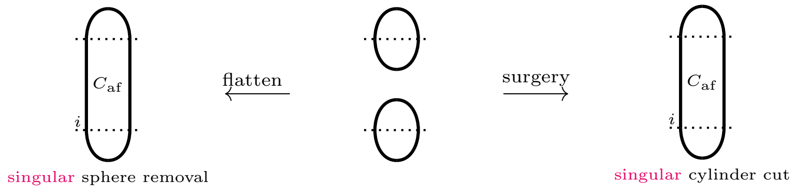
Note that the manipulation that we need to do in (I) is, on the side of foams, given by the dot moving relations (3.11). Clearly, these are combinatorially modeled by the (old and new) dot moving signs, and we ignore these in the following.

Regarding (II): Phantom circles correspond to singular phantom cups which one creates at the bottom of a cup foam basis element and needs to be removed. By (3.17) and its reoriented counterpart, we see that only anticlockwise oriented phantom seams contribute while removing it, giving precisely $(-1)^{\mathtt{n}\mathtt{p}\mathrm{circ}(\overline{W})}$. This sign can only turn up for the nested split, since the corresponding phantom circles have to in the inside of the circle resulting from the surgery (which rules out the non-nested cases as well as the nested merge).

Note also that the operation from (III) is more complicated than the corresponding ones in [21, Proof of Theorem 4.18] or [20, Proof of Theorem 4.7], but ensures that the resulting foam is of cup foam basis shape.

Hence, it remains to analyze what happens case-by-case. The procedure we are going to describe in detail will always basically be the same for all cases. Namely, in order to ensure that the result is of cup foam basis shape, we cut singular cylinders which correspond to circles after the surgery in the way described in Lemma 3.5.14. Since we started already with a foam which is of cup foam basis shape, this creates singular spheres corresponding to circles before the surgery. We will call both of these simplification moves. The total sign will depend on the difference between the signs picked up from the simplification moves.

Non-nested merge. Here the picture (for arbitrary attached phantom edges, topological situations and orientations):

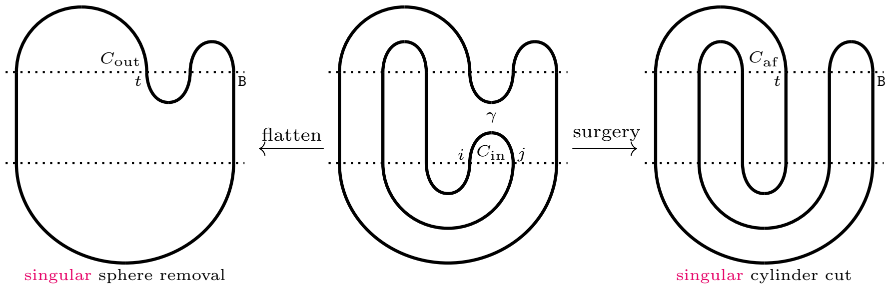


(3.72)

Above in (3.72), we have illustrated the circles (and points) where we perform the simplification moves. Note hereby that we can flatten the singular saddle and the singular sphere removal actually takes place in the left picture in (3.72).

One now directly observes that both simplification moves can be performed with respect to the same circle C_{af} and point i on it. Thus, in this case, by Lemmas 3.5.14 and 3.5.15, all obtained signs cancel and we are left with no signs at all (as claimed). ▶

Nested merge. The picture is as follows (again, for arbitrary attached phantom edges, topological situations and orientations. Note hereby that we can again flatten the situation (also vice versa as in the non-nested merge case) because we can grab the bottom of C_{in} in the created singular sphere and pull it straight to the top, and we get:



(3.73)

(For later use, we have also illustrated the circle C_{in} and point i for which we read off the sign in the combinatorial model, as well as the cup-cap pair γ of the surgery and another point t which will play a role.) Thus, by using Lemmas 3.5.14 and 3.5.15, we end up with the sign

$$(-1)^{\mathtt{np}\mathrm{loop}(C_{\mathrm{af}},\mathtt{B})}(-1)^{\mathtt{np}\mathrm{loop}(C_{\mathrm{out}},\mathtt{B})}. \tag{3.74}$$

Hence, it remains to rewrite the sign from (3.74) in terms of the circle C_{in} and the chosen point i as in (3.73).

Claim: In the setup from (3.73) one has

$$\mathtt{np}\mathrm{loop}(C_{\mathrm{af}},t) + \mathtt{s}\mathrm{type} = \mathtt{np}\mathrm{loop}(C_{\mathrm{out}},t) + \mathtt{np}\mathrm{loop}(C_{\mathrm{in}},i) + \mathtt{np}\mathrm{sad}, \tag{3.75}$$

where $\mathtt{s}$type and $\mathtt{np}$sad are to be calculated with respect to the points i, j. ▷

Proof of the claim: We prove the claim inductively, where the basic case with no phantom edges whatsoever is clear.

If we attach a phantom edge to the situation from (3.73) which does not touch neither C_{in} nor γ, then, clearly, $\mathtt{np}\mathrm{loop}(C_{\mathrm{af}},t)$ changes in the same way as $\mathtt{np}\mathrm{loop}(C_{\mathrm{out}},t)$ does, while everything else stays the same. Hence, the equation (3.75) stays true.

Similarly, if we attach a phantom edge which does not touch neither C_{out} nor γ, then, $\mathtt{np}\mathrm{loop}(C_{\mathrm{af}}, t)$ changes in the same way as $\mathtt{np}\mathrm{loop}(C_{\mathrm{in}}, i)$ does. To see this, note the attached phantom edge is in the internal of C_{af} if and only if it is in the external of C_{in}. If its in the internal of C_{af}, then it is t-positive if and only if its corresponding outgoing phantom edge pair for C_{in} is i-negative. Similarly, when its in the external of C_{af}. Everything else stays the same and thus, (3.75) stays true.

Next, if we attach a phantom edge touching C_{out} and C_{in}, but not γ, then the equation (3.75) still stays true. To see this, note that such a phantom edge will form an outgoing pair for C_{in}, but an internal phantom loop for C_{out} and two internal phantom loops for C_{af}. We observe that precisely one of the new internal phantom loops for C_{af} are counted since they come by splitting the new internal phantom loop of C_{out} into two pieces, one pointing into C_{af}, one out. Hence, $\mathtt{np}\mathrm{loop}(C_{\mathrm{af}}, t)$ always grows by one. Because the new phantom loop for C_{out} is t-positive if and only if its corresponding outgoing phantom edge pair for C_{in} is i-negative, we see that either $\mathtt{np}\mathrm{loop}(C_{\mathrm{out}}, t)$ or $\mathtt{np}\mathrm{loop}(C_{\mathrm{in}}, i)$ grow by one. In total, (3.75) stays true.

Last, the remaining case where γ is affected can be, mutatis mutandis, treated as the preliminary case: Attaching a single phantom edge to C_{out}, we have that either $\mathtt{np}\mathrm{loop}(C_{\mathrm{out}}, t)$ or $\mathtt{np}\mathrm{loop}(C_{\mathrm{in}}, i)$ gets one bigger, while $\mathtt{s}\mathrm{type}$ always gets one bigger. The difference to the preliminary case is that C_{af} now only gets one new internal phantom loop which contributes to $\mathtt{np}\mathrm{loop}(C_{\mathrm{af}}, t)$ if and only if it does not contribute to $\mathtt{np}\mathrm{sad}$. Again, the equation (3.75) stays true.

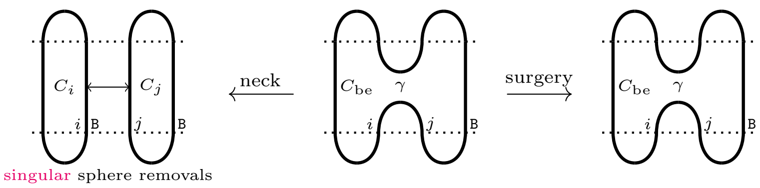
Thus, the claim is proven. ▶

Now, by Lemma 3.5.16, we have

$$
\begin{aligned}
(-1)^{\mathtt{np}\mathrm{loop}(C_{\mathrm{af}}, \mathtt{B})} &= (-1)^{\mathtt{p}\mathrm{edge}(t\to\mathtt{B})}(-1)^{\mathtt{np}\mathrm{loop}(C_{\mathrm{af}}, t)},\\
(-1)^{\mathtt{np}\mathrm{loop}(C_{\mathrm{out}}, \mathtt{B})} &= (-1)^{\mathtt{p}\mathrm{edge}(t\to\mathtt{B})}(-1)^{\mathtt{np}\mathrm{loop}(C_{\mathrm{out}}, t)}.
\end{aligned}
$$

Hence, by the above claim, we get the same signs on both sides. Similar for the horizontal mirror of the situation from (3.73). ▶

Non-nested split. Now the situation looks as follows:



(3.76)

(Again, (3.76) should be seen as a dummy for the general case.) In contrast to the merges, we can not flatten the picture since there is a singular neck appearing around γ, and the singular sphere has to be removed in the leftmost situation in (3.76) by taking the singular neck into account using Lemma 3.5.17.

Moreover, we perform two singular cylinder cuts of which precisely two summands survive. Namely one with the dot on the i-facet, one with the dot on the j-facet. Moreover, the singular sphere in these two cases will have its dot on the j-facet respectively on the i-facet. By Lemmas 3.5.14 and 3.5.15 we obtain the two signs:

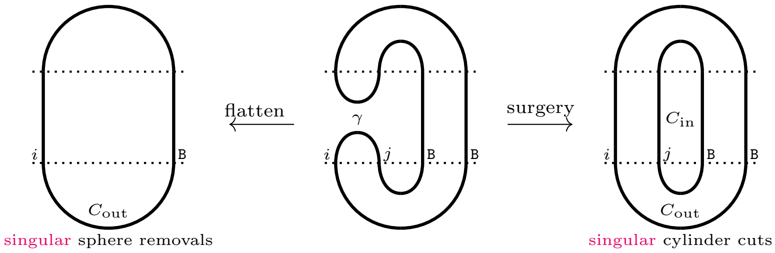
$$
\begin{aligned}
&(-1)^{\mathtt{np}\mathrm{loop}(C_{\mathrm{be}}, j)}(-1)^{\mathtt{np}\mathrm{loop}(C_i, i)}(-1)^{\mathtt{np}\mathrm{loop}(C_j, j)},\\
&(-1)^{\mathtt{np}\mathrm{loop}(C_{\mathrm{be}}, i)}(-1)^{\mathtt{np}\mathrm{loop}(C_i, i)}(-1)^{\mathtt{np}\mathrm{loop}(C_j, j)}.
\end{aligned}
\tag{3.77}
$$

In fact, by cutting the singular neck around γ we can rewrite

$$(-1)^{\mathtt{np}\mathrm{loop}(C_{\mathrm{be}},j)} = (-1)^{\mathtt{np}\mathrm{sad}(i)}(-1)^{\mathtt{np}\mathrm{loop}(C_i,i)}(-1)^{\mathtt{np}\mathrm{loop}(C_j,j)},$$
$$(-1)^{\mathtt{np}\mathrm{loop}(C_{\mathrm{be}},i)} = (-1)^{\mathtt{s}\mathrm{type}}(-1)^{\mathtt{np}\mathrm{sad}(i)}(-1)^{\mathtt{np}\mathrm{loop}(C_i,i)}(-1)^{\mathtt{np}\mathrm{loop}(C_j,j)}.$$

Now, rewriting this in terms of the basis points (using Lemma 3.5.16), and putting it together with (3.77), shows that this case works as claimed. ▶

Nested split. The $\mathtt{Ɔ}$ shape is (with flatten as for the non-nested merge):



(3.78)

Here we use a notation close to the one from the nested merge and the non-nested split. The difference to the nested merge is that our task is way easier now. In fact, by Lemmas 3.5.14 and 3.5.15 we are basically done since the contributions of the C_{out} related simplifications almost cancel. (We also use Lemma 3.5.16 to rewrite everything in terms of the point j. Note also that $(-1)^{\mathtt{p}\mathrm{edge}(i\to j)} = (-1)^{\mathtt{s}\mathrm{type}}$ with $\mathtt{s}$type taken at γ.) The only thing which we can not see in the leftmost picture in (3.78) are phantom circles which contribute the factor $(-1)^{\mathtt{np}\mathrm{circ}(\overline{W})}$. The case of a $\mathtt{C}$ shape works similar and is omitted. ▶

It remains to check cases (v) and (vi). Case (v) is clear (the one special situation comes because we need to apply (3.8)). In the remaining case (vi) one would a priori expect some signs turning up, but these are already build into the four main cases.

The rest then works verbatim as in [21, Proof of Theorem 4.18]. ■

3.5.3 Proofs of Proposition 3.4.10 and Theorem 3.4.1

Last, we prove our foamy realization of the type D arc algebra $\mathfrak{A}$.

Adjusting signs

We need the following simple observations:

Lemma 3.5.18. Let $i \to j$ be a cup or cap connecting i and j, then

$$i \equiv (j+1) \bmod 2. \tag{3.79}$$

Further, we also have

$$\mathtt{u}\mathrm{len}(k\to l) + \mathtt{m}\mathrm{len}(k\to l) \equiv (k+l) \bmod 2, \tag{3.80}$$

where k and l are two points connected by a sequence $k \to l$ of cups and caps. □

Proof. The equation (3.79) is evident, while (3.80) follows by noting that summing up the length of all cups and caps in the sequence $\mathtt{u}\mathrm{len}_\Lambda$ will only contribute to the unmarked ones, while $\mathtt{m}$len only contributes to the marked ones. ■

Proof of Proposition 3.4.10. The maps coeff_D from 3.4.12 are, by birth, homogeneous and $\mathbb{K}$-linear for all diagrams D.

Hence, as in the proof for [20, Proposition 4.15], it remains to show that the maps coeff_D successively intertwine the two multiplication rules for $\mathfrak{A}$ and $\overline{\mathfrak{A}}$. Consequently, we compare two intermediate multiplications steps in the following fashion:

$$\begin{tikzcd} D_m \arrow[r, "\mathbf{Mult}^{\mathfrak{A}}_{D_m,D_{m+1}}"] \arrow[d, "\mathrm{coeff}_{D_m}"'] & D_{m+1} \arrow[d, "\mathrm{coeff}_{D_{m+1}}"] \\ D_m \arrow[r, "\mathbf{Mult}^{\overline{\mathfrak{A}}}_{D_m,D_{m+1}}"'] & D_{m+1}, \end{tikzcd} \tag{3.81}$$

where we denote by $\mathbf{Mult}^{\mathfrak{A}}_{D_m,D_{m+1}}$ and $\mathbf{Mult}^{\overline{\mathfrak{A}}}_{D_m,D_{m+1}}$ the surgery procedure rules as indicated in the two multiplications. Thus, the goal is to show that each such diagrams, i.e. for each appearing D_m and D_{m+1}, commutes.

As usual, this will done by checking the four possible cases that appear in the surgery procedure. But before we start, note that (3.80) immediately implies

$$(-1)^{\mathrm{ulen}(k\to l)} = (-1)^{\mathtt{m}\mathrm{len}(k\to l)} \cdot (-1)^k \cdot (-1)^l \tag{3.82}$$

which we will use throughout below.

Non-nested merge. Assume that circles C_b and C_t are merged into a circle C_{af}. If both circles are oriented anticlockwise, then both multiplication rules yield a factor of $+1$ and $\mathrm{coeff}^{C_b}_{D_m} = \mathrm{coeff}^{C_t}_{D_m} = \mathrm{coeff}^{C_{\mathrm{af}}}_{D_{m+1}} = +1$ as well. The claim follows.

Assume now that the circle $C_\square$ for $\square \in \{b,t\}$ is oriented clockwise and the other circle is oriented anticlockwise. The multiplication in $\mathfrak{A}$ gives the factor $(-1)^{\mathrm{ulen}(\mathtt{B}_\square\to\mathtt{B}_{\mathrm{af}})}$, while the multiplication in $\overline{\mathfrak{A}}$ yields $(-1)^{\mathtt{m}\mathrm{len}(\mathtt{B}_\square\to\mathtt{B}_{\mathrm{af}})}$. We check that

$$\begin{aligned} \mathrm{coeff}^{C_{\mathrm{af}}}_{D_{m+1}} \cdot (-1)^{\mathrm{ulen}(\mathtt{B}_\square\to\mathtt{B}_{\mathrm{af}})} &= -(-1)^{\mathtt{B}_{\mathrm{af}}} \cdot (-1)^{\mathrm{ulen}(\mathtt{B}_\square\to\mathtt{B}_{\mathrm{af}})} \\ &\overset{(3.82)}{=} -(-1)^{\mathtt{B}_\square} \cdot (-1)^{\mathtt{m}\mathrm{len}(\mathtt{B}_\square\to\mathtt{B}_{\mathrm{af}})} = \mathrm{coeff}^{C_b}_{D_m} \cdot \mathrm{coeff}^{C_t}_{D_m} \cdot (-1)^{\mathtt{m}\mathrm{len}(\mathtt{B}_\square\to\mathtt{B}_{\mathrm{af}})}, \end{aligned}$$

which proves the claim in this case.

If both circles are oriented clockwise, the both multiplications are zero. ▶

Nested merge. Due to the definition of $\overline{\mathfrak{A}}$, the signs in the nested merge are exactly as in the non-nested case. Thus, it is verbatim as the non-nested merge. ▶

Non-nested split. Assume that a circle C_{be} is split into circles C_i and C_j at a cup-cap pair connecting i and j.

Assume first that C_{be} is oriented anticlockwise. Note that, by admissibility, it must hold that $\mathrm{utype} = 1$. Hence, the summand where C_i is oriented clockwise and C_j is oriented anticlockwise obtains a factor $(-1)^{\mathrm{ulen}(i\to\mathtt{B}_i)}(-1)(-1)^i$ for $\mathfrak{A}$. In contrast, the summand only gains the factor $(-1)^{\mathtt{m}\mathrm{len}(i\to\mathtt{B}_i)}$ in $\overline{\mathfrak{A}}$. Thus, we check

$$\begin{aligned} &\mathrm{coeff}^{C_i}_{D_{m+1}} \cdot \mathrm{coeff}^{C_j}_{D_{m+1}} \cdot (-1)^{\mathrm{ulen}(i\to\mathtt{B}_i)} \cdot (-1) \cdot (-1)^i \\ &= -(-1)^{\mathtt{B}_i} \cdot (-1)^{\mathrm{ulen}(i\to\mathtt{B}_i)} \cdot (-1) \cdot (-1)^i \\ &\overset{(3.82)}{=} (-1)^{\mathtt{m}\mathrm{len}(i\to\mathtt{B}_i)} = \mathrm{coeff}^{C_{\mathrm{be}}}_{D_m} \cdot (-1)^{\mathtt{m}\mathrm{len}(i\to\mathtt{B}_i)}, \end{aligned}$$

which proves the claim. The second summand is done completely analogous by using (3.79) to see that the factor in $\mathfrak{A}$ is equal to $(-1)(-1)^{\mathtt{ulen}(j \to \mathtt{B}_j)}(-1)^j$.

The clockwise case follows, mutatis mutandis, as the anticlockwise case by incorporating the two additional non-trivial coefficients. ▶

Nested split. Assume the same setup as in the non-nested split case. Note that, due to the definition of the multiplication in $\mathfrak{A}$, we are always looking at the situation of the $\mathtt{Ↄ}$ shape here. Thus, if we assume that the circle C_{be} is oriented anticlockwise, the summand with C_i oriented clockwise and C_j oriented anticlockwise gains the factors $(-1)^{\mathtt{ulen}(i \to \mathtt{B}_i)}(-1)^{\mathtt{utype}}(-1)^i$ in $\mathfrak{A}$ and $(-1)^{\mathtt{mlen}(i \to \mathtt{B}_i)}(-1)^{\mathtt{mtype}}$ in $\overline{\mathfrak{A}}$. Since

$$\mathtt{utype} \equiv (\mathtt{mtype} + 1) \bmod 2$$

the claim follows by the same calculation as in the non-nested case.

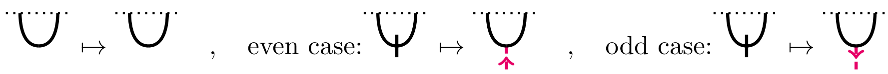
For the other summand there is no difference to the non-nested split, and the case of C_{be} being oriented clockwise is also derived analogously. ▶

This in total proves the proposition. ■

The embedding of the D arc algebras into the web algebras

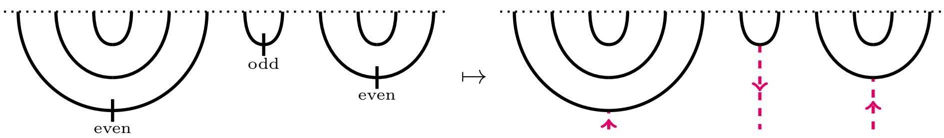
Recall that for the type D arc algebra the multiplication is zero in case the result is non-orientable, i.e. has an odd number of markers on some component, see Remark 3.4.5. Hence, the first thing to make sure is that the isomorphism $\overline{\mathtt{top}}$ preserves this. This is the purpose of the following definition and two lemmas.

Definition 3.5.19. To a cup diagram c we associate a web $\mathtt{u}(c)$ using the rule



where we say a marked cup is even respectively odd if it has an even respectively odd number of marked cups to its right. ▲

Example 3.5.20. Here is one blueprint example:



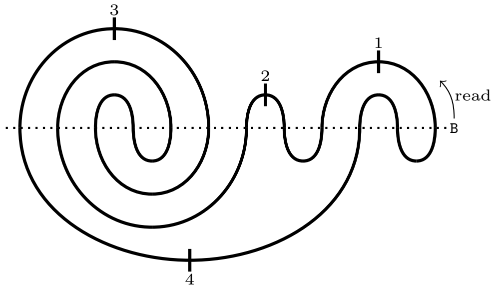
(Recall that we do not orient ordinary web components.) ▲

Lemma 3.5.21. Given two cups diagrams c, d. Then cd^* is orientable if and only if $\mathtt{u}(c)\mathtt{u}(d)^*$ is a (well-oriented) web (cf. Convention 3.3.4). □

Proof. If $\mathtt{u}(c)\mathtt{u}(d)^*$ is a web, then every face has an even number of adjacent trivalent vertices which translate to say that cd^* has an even number of markers per connected component, and we are done.

Conversely, assume (without loss of generality) that cd^* has only one circle C, and that C is orientable. If C is not marked, then we are done since the associated circle in $\mathtt{u}(c)\mathtt{u}(d)^*$ is an ordinary circle. Otherwise, follow C from $\mathtt{B}$ onwards in the anticlockwise fashion. By admissibility, going around C in this way will always pass the marked caps in d^* and then the marked cups in

c. This can be best seen via example (we leave it to the reader to make this rigorous):



(3.83)

Next, the number of markers on C is even since cd^* is orientable. This together with the above observation (and recalling that taking * on webs reverses the orientation of phantom edges) ensures that all neighboring phantom edge pairs of $\mathtt{u}(c)\mathtt{u}(d)^*$ are well-attached, and that $\mathtt{u}(c)\mathtt{u}(d)^*$ has an even number of trivalent vertices, i.e. $\mathtt{u}(c)\mathtt{u}(d)^*$ is well-oriented. ■

Lemma 3.5.22. Given two cups diagrams c, d. Then, up to closing of the phantom edges, we have: $\mathtt{u}(c) = u(c)$ and $\mathtt{u}(d) = u(d)$. (With $u(\square)$ as in (3.54).) □

Proof. By comparing (3.2) and (3.83). ■

Proof of Theorem 3.4.1. By Theorem 3.3.1, Proposition 3.4.10, and summation, it remains to show that $\overline{\mathtt{top}}_\Lambda$ is an embedding of graded algebras.

To this end, fix $\Lambda \in \mathtt{Bl}^\circ$ of rank K. There are four things to be checked, where c_b, d, d', c_t are always cup diagrams of rank K and λ, μ are in Λ:

(1) The $\mathbb{K}$-linear maps $\overline{\mathtt{top}}_{c_b}^{c_t}$ are homogeneous embeddings of $\mathbb{K}$-vector spaces.

(2) We have $\mathbf{Mult}^{\overline{\mathfrak{A}}}(c_b\lambda d^*, d'\mu c_t^*) = 0$ because one has $d \neq d'$ if and only if
$$\mathbf{Mult}^{c\mathfrak{W}}(\overline{\mathtt{top}}_\Lambda(c_b\lambda d^*), \overline{\mathtt{top}}_\Lambda(d\mu c_t^*)) = 0$$
because $u(d) \neq u(d')$.

(3) We have $\mathbf{Mult}^{\overline{\mathfrak{A}}}(c_b\lambda d^*, d\mu c_t^*) = 0$ because $c_b c_t^*$ is not orientable if and only if
$$\mathbf{Mult}^{c\mathfrak{W}}(\overline{\mathtt{top}}_\Lambda(c_b\lambda d^*), \overline{\mathtt{top}}_\Lambda(d\mu c_t^*)) = 0$$
because $u(c_b)u(c_t)^*$ is not a web.

(4) In case $d = d'$ and $c_b c_t^*$ is orientable, the usual diagram (the one very similar to (3.81), but with exchanged notation) commutes.

(1). Note that (3.44) sums up to 0 respectively 2 for anticlockwise respectively clockwise circles. Thus, $\overline{\mathtt{top}}_{c_b}^{c_t}$ is homogeneous by comparing (3.29) and (3.44), while keeping the shift $d(\vec{k})$ in mind. That $\overline{\mathtt{top}}_{c_b}^{c_t}$ is injective follows by definition. ▶

(2)+(3). Directly from Lemmas 3.5.21 and 3.5.22. ▶

(4). The signs for the multiplication process for $\overline{\mathfrak{A}}$ from (3.50) and (3.51) are specializations of the ones for $c\mathfrak{W}$ for dotted basis webs of shape $u(c)u(d)^*$ (with c, d standing for cup diagrams) - up to the phantom circle sign. Thus, it remains to show that the scaling factor $(-1)^{\mathtt{np}\mathrm{esci}}$ accounts for this. To this end, one directly observes that only the phantom circle removal can change the number $\mathtt{np}$esci. Moreover, $\mathtt{np}$esci is defined to count anticlockwise phantom circles, which is what

$\mathtt{np}\mathrm{circ}(\overline{W})$ counts. ▶

Thus, the theorem is proven. ■

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