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Conceptualising and estimating rationalised agricultural optimisation models

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In Our Darkest Hour
In My Deepest Despair
Will You Still Care?
Will You Be There?
In My Trials
And My Tribulations
Through Our Doubts
And Frustrations
In My Violence
In My Turbulence
Through My Fear
And My Confessions
In My Anguish And My Pain
Through My Joy And My Sorrow
In The Promise Of Another Tomorrow
I'll Never Let You Part
For You're Always In My Heart.

- M.J.

Kurzfassung

Die computerbasierte Modellierung zur quantitativen Analyse der Agrarpolitik in der EU konzentriert sich zunehmend auf die einzelbetriebliche Ebene. Dies folgt der Entwicklung der Politikinstrumente, die direkt auf einzelbetrieblicher Ebene ansetzen und deren Wirkungen von Betriebscharakteristika abhängen. Gleichzeitig unterstützen methodische Entwicklungen wie die Positive Mathematische Programmierung (PMP) die Akzeptanz solcher Modelle in der Politikanalyse. PMP führt nichtlineare Terme in die Zielfunktionen ein und sorgt dadurch für eine elegante Kalibrierung und ein kontinuierliches Simulationsverhalten. Diese Arbeit beschäftigt sich mit der fehlenden ökonomischen Rationalisierung von PMP und der ökonometrischen Schätzung von alternativen Modellformulierungen.

Diese Dissertation analysiert zunächst in wieweit das am häufigsten verwendeten quadratischen PMP Modells aus ökonomischer Sicht rationalisiert werden kann. In der Literatur werden nichtlineare Kapazitätsbeschränkung (KB), die ein Aggregat von Arbeit und Kapital darstellt, als theoretische Motivation nichtlineare Terme in der Zielfunktion genannt. Die Ergebnisse dieser Arbeit zeigen, dass sich die Äquivalenz zwischen einer quadratischen KB und einem quadratischen PMP Modell lediglich auf die Kalibrierung des Programmierungsmodells beschränkt. In Bezug auf das Simulationsverhalten bzw. die Modellschätzung unterscheiden sich die beiden Modelle. Somit kann eine quadratische KB ein quadratisches PMP-Modell nicht vollständig rationalisieren. Nichtsdestotrotz könnte es dazu beitragen, Angebotsmodelle und Marktmodelle in Verbindung zu bringen, um Informationen über den Primärfaktor auszutauschen. Die Arbeit überprüft weiterhin die Konsistenz der Ökonometrischen Mathematischen Programmierungsmodelle (ÖMP). Diese ermöglichen die Parameterschätzung von nichtlinearen Technologien mithilfe mehrfacher Beobachtungen und Optimalitätskriterien erster Ordnung als Schätzungsgleichungen. Das ÖMP für diese Arbeit ist ein einzelbetriebliches Optimierungsmodell mit konstanten Substitutionselastizitäten in den Produktionsfunktionen. Die Konsistenz des Schätzverfahrens wird durch ein Monte Carlo Verfahren mit unterschiedlichen Fehlerstrukturen ausgewertet. Die Ergebnisse zeigen, dass sich die geschätzten Parameter an die wahren Werte mit zunehmendem Stichprobenumfang annähern. Abschließend, wird ein Verfahren zur statistischen Inferenz für ÖMP eingeführt und damit eine Lücke in der Literatur geschlossen. Die Arbeit verwendet Bootstrapping um-Konfidenzintervalle abzuleiten und evaluiert diese, ebenfalls mit Hilfe eines Monte Carlo Verfahrens, hinsichtlich der Genauigkeit der Überdeckungswahrscheinlichkeiten. Im Allgemeinen gelingt es den simulierten Konfidenzintervallen sich mit ausreichender Genauigkeit den korrekten Überdeckungswahrscheinlichkeiten anzunähern. Die Ergebnisse unterscheiden sich jedoch je nach Auswahl des Stichprobenverfahrens und der Berechnungsmethode des Konfidenzintervalls.

***Schlüsselwörter**:* positive mathematische Programmierung, Kapazitätsbeschränkung, ökonometrisches mathematisches Programmierungsmodell, Fehler in der Optimierung, Bootstrap-Konfidenzintervalle

Abstract

Computational modelling for quantitative agricultural policy assessment in the EU employs more farm level oriented approaches in recent years. This follows policy instruments that increasingly target the farm level and have effects varying with farm characteristics. At the same time, methodological advances such as Positive Mathematical Programming (PMP) increased the acceptance of farm level modelling for policy analysis. By introducing non-linear terms into the objective function of programming models, PMP offers an elegant calibration property and smooth simulation response. This thesis addresses the lack of economic rationalisation of PMP and the econometric estimation of alternative model formulation.

First, this dissertation analyses the economic rationality of the most often used quadratic PMP model. One potential rationalisation of non-linear terms in the objective function discussed in the literature is a non-linear capacity constraint (CC) representing some aggregate of labour and capital stock. Results show that the equivalence between a quadratic CC formulation and PMP model is limited to the calibration property of the programming model. In terms of simulation behaviour and estimation, the two models differ. Therefore, a quadratic capacity constraint cannot fully rationalise a quadratic PMP model. Nevertheless, it could effectively connect supply models to market models in order to exchange information on primary factor. Second, the thesis examines the consistency of Econometric Mathematical Programming (EMP) models. They allow estimating parameters of non-linear technologies using multiple observations and first-order conditions as estimating equations. The chosen EMP model is a single farm optimisation model with Constant Elasticity of Substitution production functions. A Monte Carlo setup evaluates the consistency of the estimation procedure under different error structures. Results show that the estimated parameters converge to the true values with increasing sample sizes. Finally, the dissertation addresses the lack of statistical inference procedures for EMP models in the literature. Bootstrapped confidence intervals are suggested here and evaluated with respect to the accuracy of the coverage probabilities, again using a Monte Carlo approach. The simulated confidence intervals generally succeed in approximating correct coverage probabilities with sufficient accuracy but results differ somewhat by sampling approach and choice of confidence interval calculation.

Keywords: positive mathematical programming, capacity constraint, econometric mathematical programming model, errors in optimisation, bootstrapped confidence intervals.

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Abbreviations

ARMSD	Average of Root Mean Squared Deviation
CAP	Common Agricultural Policy
CAPRI	Common Agricultural Policy Regionalised Impact Modelling System
CC	Capacity Constraint
CES	Constant Elasticity of Substitution
CGE	Computable General Equilibrium
CI	Confidence Interval
CR	Case Resampling
DGP	Data-Generating Process
EEM	Econometric Estimation Model
EMP	Econometric Mathematical Programming
EU	European Union
FOC	First-Order Condition
GAMS	General Algebraic Modelling System
GME	Generalised Maximum Entropy
LP	Linear Programming
MATLAB	Matrix Laboratory
ME	Measurement Error
OE	Optimisation Error
OLS	Ordinary Least Square
PMP	Positive Mathematical Programming
RR	Residual Resampling

Chapter 1
Introduction

1.1 Background and motivation

Computational modelling has been employed to assess European agricultural policy over decades. Mathematical programming models are an important and widely used tools applied for economic analysis in agriculture. They can address the multivariate and highly interlinked nature of the agricultural sector while integrating and utilising detailed micro-level data (Hazell and Norton 1986). The theoretical and methodological base of mathematical programming models has advanced greatly in the last decades, such that factors of increasing policy-relevance like individual farm characteristic and interaction between agriculture and the environment may receive an improved treatment in this modelling approach. These developments have been partly triggered by the fact that the focus of agricultural policies has changed considerably. Generally, policy instruments are designed to be more farm level oriented, which is one of the pronounced drivers for significant progress in farm-level modelling. This dissertation is committed to the theoretical and methodological development of farm-level economic modelling. The following section describes the history and development of the European Union (EU) agricultural policies and the modelling in agriculture economics. It highlights the most significant of them and motivates how the research conducted in this dissertation further improves upon this field of quantitative research.

The Common Agricultural Policy (CAP) was introduced in 1962. For the first three decades of the CAP, the priority was to support farm income.

Intervention and border protection measures were applied to raise farm and market prices. Until the late 1980s, model-based quantitative policy analysis did not play a role in the EU (Hendrichsmeyer and Wolf 1992). The fundamental change of agricultural policy in the EU was signified by the MacSharry reform of the CAP in 1992. The focus was shifted from market and trade policies to policy instruments at farm level. This transformation has been pursued in a stepwise approach through the reduction of support prices and the introduction of direct payments[1]. The second pillar of the CAP introduced new measures and policy instruments at farm level. Also, food safety concerns and animal welfare issues surfaced frequently and gained more public attention. These issues have led to the discussion or implementation of regulation and standard (e.g. food safety and animal welfare) in the most recent CAP reform (ENRD, 2015).

The computational modelling continuously evolved to provide better policy analysis, while the process of policy-making became more and more evidence-oriented. Over decades, agricultural policies were designed to be more market oriented and less distorted, even though the support given to the agriculture sector is still large nowadays. Approximately 38% of the EU budget (equivalent to 0.4% of the Union's gross domestic product) has been spent on agriculture and rural development in 2015 (European Commission 2015).

Changes in agricultural policy instruments, changes in relevance and understanding of policy impact indicators and the non-linear nature of key biophysical or economics processes, all these issues were responsible for a boosting demand for results from policy-relevant farm-level modelling. This was accompanied by the improvement from the supply side of

[1] Direct payments were introduced in the 1992 MacSharry CAP reform, which started the shift from product support to producer support. Direct payments are decoupled in the 2003 CAP reform with the introduction of a single payment scheme. (European Commission 2017)

modelling approaches, which was made possible by the progress in data availability and information technology as well as the methodological and theoretical developments in agricultural economics. (Heckelei 2016)

Much political attention has been directed at the market level instead of the farm level despite the objective of the income support policy at the early stage of CAP. Market-level models were the power horse for the policy assessment. Among several reasons for this choice are a) the non-existent data on representative farm households, b) decision makers' disinterest in confirming the inefficiency of the income support policy and c) the logical choice to originally assess the policies solely on market level given the dominant product-based income support (Heckelei 2016). As a result, supply and demand analysis and equilibrium estimation were the primary tools for policy assessment and were widely used to simulate the impact of policies on prices on regional, sector or country scale. Individual reactions at farm level were only modelled implicitly in an aggregated fashion. Linking market- and farm-level models appeared to be too difficult in the infancy stage of quantitative agricultural policy assessment.

The share of policy instruments directly targeted at individual farm management was slowly rising (European Commission 2013). This was a result of increasing environmental problems related to the agriculture sector, which largely depend on farm, local and regional characteristics. Sectoral models are too aggregated to include the details that form the core of the agri-enviromental measures and farm-level models present an alternative (Röhm and Dabert 2003).

The MacSharry reform shifts support in the direction of farm-level policy instruments away from market price support. The agreement on the partially decoupled, compensatory payments created a research need for empirical models to quantify the degree of decoupling and payment impacts. The main classes of mathematical models applied for policy assessment include econometric models, mathematical programming as well as partial and computable general equilibrium models and mainly mathematical programming models and econometric models were used for this purpose by modelling the producers' choice (Salvatici et al. 2000). Econometric models usually focused on the supply side and were largely

used to measure the impact of specific agricultural policy instruments on farmers' production decisions concerning certain commodities. Their size and structure allow the estimation of supply elasticities, which is not the case for many mathematical programming models and equilibrium models (Salvatici et al. 2000). And, these supply elasticity estimates are considered to be their most important outcome for their applications in this context and are often provided as input for other simulation models.

However, econometric models have difficulties in sorting the relationships into sets of constant incentives and behaviour (the constant economic structure necessary for estimation) and changed policy or technology (the impacts of the policy or technology necessary for evaluation of the change). On the other hand, mathematical programming models are capable to incorporate the changing policy or technology to the existing framework. Furthermore, the mathematical programming approach enables much greater regional and commodity disaggregation and provides detailed analysis of the effects of the changes across commodities, regions, types of farms etc. (Preckel et al. 2002)

The development of mathematical programming models during the last decades is strongly connected with the development of agricultural policy and has become more than a pure farm management instrument. They have been extensively used to analyse the impact of agricultural policies on supply and on the socio-economic and environmental systems linked to the farming sector (Salvatici et al. 2000). Calculation of meaningful biophysical or economic indicators requires modelling at disaggregated level, because the linear aggregation of single biophysical or economic processes would lead to incorrect total environmental externalities or require very restrictive and inflexible restrictions. Even at the aggregate level, farm-level modelling is often necessary to reach the desired degree of detail for policy assessment (Heckelei 2016).

Also, the progress in data availability and information technology renders farm-level modelling more inviting. Two categories of mathematical programming models exist: those deriving from the 'classical' mathematical programming and those that have adopted the more recent approach of Positive Mathematical Programming (PMP) (Howitt 1995).

PMP type models thrived in the late 1990s and after 2000, because they allow the combination of a more robust and empirically based behavioural specification with the technology–rich formulation of mathematical programming models. This rendered the technique especially suitable for the assessment of policy measures linked to biophysical indicators at the farm level (Heckelei and Britz 2005; Heckelei et al. 2012). PMP is considered as one of the most important innovations in the field of mathematical programming applied to the agricultural sector.

In mathematical programming models, producers are assumed to behave rationally and optimise the production activities under resource constraints. Mathematical programming models in the early stage, for example linear programming (LP) models, had difficulties to closely reproduce historical results. Howitt's PMP procedure addresses the question of calibrating mathematical programming models. The general idea is to introduce artificial constraints which force the model to reproduce historical observations. The shadow values of the constraints are then used to construct additional non-linear cost terms for the objective function, so that the calibrated model reproduces historical observations without the original artificial constraints. PMP models also produce more realistic, less 'jumpy' simulation responses compared to LP models. While one strand of PMP literatures focuses on calibrating the 'PMP' term with various techniques, the other evolve towards econometric estimation with calibration being used for verification (Heckelei et al. 2012; Mérel and Howitt 2014). Heckelei and Wolff (2003) proposed an alternative to PMP, which this dissertation is centred around. In this approach the optimality conditions of the specific PMP model are directly used as parameterisation constraints for the econometric estimation. The parameters and shadow prices of calibration constraints are estimated simultaneously. Thus, it skips the determination of shadow prices using artificial constraints and avoids the fundamental inconsistencies of the PMP approach. Furthermore, multiple observations could be utilised and the parameter estimates contain more empirical content compared to the traditional PMP approach relying on one single observation. Both strands have contributed to combine econometrics and PMP with their own methodological innovations.

Despite these enormous efforts, the issue concerning the missing economic and technological rationale behind the non-linearity in PMP-type models has not been sufficiently addressed. The lack of rationale causes inconsistency and thus calls the proper use of such models into question. This motivates the research in chapter 2 and raises the research question on how to rationalise the non-linear terms in the objective functions of PMP models. Following the other strand, which is the estimation of PMP models, chapter 3 exploits the general approach of combining econometric models and mathematical programming models. This chapter is devoted to evaluating the consistency of the estimation approach. The computational intensity of such estimation procedure greatly restricts the application of statistical inference. As a result, the empirical reliability of the estimation results cannot be verified. Chapter 4 aims to identify an adequate method which might close this research gap.

The next section is dedicated to a general description of this dissertation. Finally, a concluding section discusses the limitations and the potential for future research.

1.2 Research contribution

This section summarizes the dissertation including the specific literature background and reports research gaps, objective, methodological approach and the main findings for each chapter.

1.2.1 *Rationalising non-linear agricultural programming models with a capacity constraint*

Heckelei et al. (2012) assess the progress with respect to the empirical foundation of PMP approaches and picks up an issue raised by Heckelei (2002) and Heckelei and Wolff (2003): the economic or technological rationale behind non-linearity in typical PMP models is unclear. A (typical quadratic) cost function in the objective function represents the missing explicit formulation of some economic phenomena due to data or analytical insufficiency. This lack of rationale potentially creates an inconsistency between the model structure and the true underlying

technology and behaviour. It also renders the interpretation of model responses caused by the PMP terms difficult to interpret. Risk behaviour, land heterogeneity and unknown resource/technology constraints are often not explicitly modelled and are candidates to rationalise the PMP formulation. If one assumes that non-linearity in PMP models originates from non-linear technologies in a typical profit maximisation behavioural model, it could be captured by an explicit non-linear capacity constraint (Heckelei 2002, p. 30). If the resulting model is equivalent to a PMP model, the non-linearity in PMP models could be rationalised. Alternatively, the mean-variance risk model under gross margin uncertainty offers another possibility for rationalisation (Heckelei 2002; Cortignani and Severini 2009; Severini and Cortignani 2011; Petsakos and Rozakis 2011; Jansson et al. 2014).

Chapter 2 focuses on the potential rationalisation of PMP terms by a non-linear capacity constraint. One could stick to the explicit constraint formulation like in Doole et al. (2011), where non-linear terms in the objective function of a typical PMP are removed and replaced by a quadratic constraint. A non-linear constraint as such could represent a "non-linear level technology" which defines the feasible relationship between activity levels and a fixed, non-allocable operating capacity[2]. In the meantime, it is still possible to preserve the desired technology assumption (such as Leontief technology) for the allocation of variable inputs. The capacity could relate to the primary factors (labour and capital), which are frequently omitted in programming models for agricultural policy analysis due to a lack of data or desire for simplification. Including this capacity constraint not only allows explicit analyses of the impact of changes in labour and capital on production, but also enables the linkage to

[2] For example, data on capital stocks are often not available or need to be derived from investment data through complicated and fallible procedures (Witzke 1996). Also, it is not easy to assume how capital stocks are allocated to different productions.

market models through transmitting information on primary factor market signals. Examining and comparing the first-order conditions derived from the Lagrangian function of Capacity Constraint (CC) and the PMP model show that both models could be calibrated to identical historical observations under the same calibration criterion. However, the different model structures hint at different behaviours for simulation and estimation: the marginal effect of the gross margin on variable input in the CC model is analytically different compared to the one from the PMP model. A fully equivalent form of the CC model with a non-linear objective function instead of the nonlinear CC is presented by analytically solving the Lagrangian multiplier of the CC. The functional form of this model, however, is different from a typical PMP model as the non-linear terms in the objective function are not quadratic. The resulting CC model demonstrates equivalence to the PMP model only in terms of calibration. Once moving away from the calibrated point, these two models will behave differently. Thus, a quadratic CC representing an aggregate and fixed labour and capital stock could not rationalise the use of the quadratic PMP cost function. Nevertheless, a non-linear cost function as part of the objective function can be rationalised by the CC.

Despite not being able to fully rationalise the typical PMP formulation, the CC model is potentially useful to connect agricultural programming models to models that endogenously simulate factor markets, as information on primary factors needs to be passed between supply and market models. The specification of the CC allows to explicitly reflect the changes in the primary factor markets signals. Market models, like Computable General Equilibrium (CGE) models, commonly assume constant returns to scale, while the CC model does not. A general alternative formulation for the CC is introduced, which allows us to explicitly specify returns to scale.

The empirical content offered is still limited, as the calibration approach typically relies only on a single observation. Thus, the future research in this dissertation should consider the estimation of mathematical programming models based on multiple observations on farm-level data.

1.2.2 *Consistency of estimating constrained optimisation models*

Leaving the realm of calibration, chapter 3 focuses on combining econometric estimation and mathematical programming models. Paris and Howitt (1998) make the first attempt to econometrically estimate mathematical programming models, where the Generalised Maximum Entropy (GME) method is used to specify PMP models. Only single observations on two farms are used in their approach and the curvature of the cost function has to be enforced through parameterisation. Apart from these limitations, the lack of rationale discussed in chapter 2 certainly also applies to all estimation practices based on PMP type models[3].

Heckelei and Wolff (2003) introduce an alternative approach for the specification of mathematical programming models and show its theoretical advantages over PMP-based approaches. The optimality conditions of mathematical programming models are directly employed for the estimation. Thus, it bypasses the fundamental inconsistency in typical PMP approaches caused by the first phase of using a linear programming model to identify non-linear parameters of the resulting calibrated/estimated model[4]. Also, it allows for the specification of more complex models and at the same time a more flexible choice of the functional form.

Buysse et al. (2007b) name this type of mathematical programming models Econometric Estimation Programming (EMP) model. Extensive applications of this approach exist in the literature despite its relative early development stage. Buysse et al. (2007a) apply an EMP model to analyse the reform of the common market organisation in the sugar sector of the European Union. A very extensive estimation utilising the same basic approach is demonstrated by Jansson and Heckelei (2011) where they

[3] On estimating PMP type model see, for example, Arfini et al. (2008) and Paris (2010), p. 397-400.

[4] See Heckelei and Wolff (2003) for a detailed illustration on the methodological inconsistency.

estimate the behavioural parameters of regional constrained mathematical programming models in the EU using time-series data. Both studies incorporate non-linear cost function in the objective function of the resulting optimisation models.

The research objective for this study is to evaluate the consistency of EMP models based on the approach advocated by Heckelei and Wolff (2003). Their approach offers flexible choice concerning the functional form and in this exercise we employ the Constant Elasticity of Substitution (CES) production function. The options and motivations for functional forms specifying mathematical programming models are discussed in Mérel and Howitt (2014). According to their review, assuming global concavity in the objective function and global convexity in the constraint set is the most common choice among numerous ways to ensure a unique optimum in mathematical programming models. Global concavity is generally addressed by either a quadratic cost function assuming increasing marginal cost like in Buysse et al. (2007a) and Jansson and Heckelei (2011), or by a CES crop-specific production function assuming decreasing marginal yield. They conclude that both choices are motivated rather by pragmatic consideration from a modelling perspective than strong empirical and theoretical justification. The EMP model used in chapter 3 is based on a single farm optimisation model with CES production functions and variable input allocation. Using CES production functions to specify production technology in the context of mathematical programming models can be traced back to Howitt (1995). The same approach and variations of it are frequently employed in recent and notable studies (Mérel et al. 2011; Frisvold and Konyar 2012; Howitt et al. 2012; Medellín-Azuara et al. 2012; Garnache 2013, pp. 39-76; Graveline and Mérel 2014; Mérel et al. 2014).

Monte Carlo simulation is employed to validate the consistency of the estimation procedure. Statistical errors are introduced to the synthetic data generation process with known model parameters. For each generated data set, the model parameters are estimated by Ordinary Least Squares (OLS) directly using the first-order optimality conditions as data constraints. The whole simulation procedure is carried out repeatedly for increasing sample

sizes and convergence to the true parameter values is checked. One single error structure is used in Heckelei and Wolff (2003) to account for the aggregate effect of multiple factors causing deviations of endogenous model outcomes from observations. Here, two types of errors are explicitly distinguished to allow for a more explicit interpretation in the application. Measurement errors are added to endogenous quantities to represent deviations from true values that occur when observing or recording the variables. In a second set of simulations, optimisation errors are added to the first-order optimality conditions to capture mistakes made in the optimisation process. The results show that parameters under measurement errors can be consistently recovered, while additional information on the land shadow prices is required to render the estimation with optimisation errors consistent. Given the evidence for the consistency of EMP models, the subsequent and final chapter 4 of the thesis focuses on developing and validating a procedure to perform statistical inference when estimating programming models, as such tools are still missing.

1.2.3 *Statistical inference for econometric mathematical programming models*

All applications of estimating mathematical programming models so far have the primary interest to utilise all available information to provide the 'best' estimated parameters with higher empirical content compared to previous calibration approaches (e.g. Buysse et al. 2007a, Jansson and Heckelei 2011). However, a systematic implementation of statistical inference for the estimated parameters is still missing, thus the empirical reliability of the estimation results cannot be evaluated. Chapter 4 aims to provide an approach to fill this research gap. The study explores the possibility of bootstrapping sampling distributions for hypothesis testing and confidence interval estimation of econometrically estimated parameters of mathematical programming models.

The bootstrap method measures the accuracy of parameter estimates by estimating the sampling distribution using a random sampling approach. Many variations of the bootstrap are developed since it was first proposed by Efron (1979). Its basic principle is very straightforward: bootstrap

samples are created by repeatedly resampling the data with replacement, and statistical inference on the bootstrap data employs this representation of the parameter sampling distribution by calculating statistics of interest.

One of the research objectives is thus to construct confidence intervals for the parameter estimates, as a combination of point estimates and interval estimates would give the best guess for the 'true' parameter values. The EMP model from chapter 3 is chosen and it consists of a Data-Generating Process (DGP) with a statistical model and an Econometric Estimation Model (EEM) for the parameter estimation. The same two types of error specifications are considered. The bootstrap algorithm to obtain confidence intervals is described as follows:

1. Generate sample data with DGP and obtain point estimates with EEM using randomly generated synthetic data and 'true' parameter data
2. Create bootstrap sample data by resampling sample data with replacement
3. Obtain bootstrap estimates of parameters with EEM
4. Repeat step 2 and 3 for $b = 1, \ldots, B$ times to obtain the sampling distribution of the bootstrap estimates and calculate the bootstrapped confidence intervals

The quality of the algorithm needs to be examined, before it could be applied for empirical application. Hence, another objective is the evaluation of the bootstrapped confidence intervals. Sufficient replications of this bootstrap algorithm in a Monte Carlo simulation setup allow us to calculate the actual coverage probability. This reflects how often the 'true' parameters are covered by the bootstrapped confidence intervals. According to the "goodness" criterion (Efron and Tibshirani 1994) the coverage probability should be a proper approximation of the chosen confidence level in all situations. The Monte Carlo evaluation procedure could be summarised as follow:

1. Carry out $s = 1, \ldots, S$ Monte Carlo simulations with the EMP model with one set of 'true' parameters, where the error term is

randomly simulated with known distribution. This yields S sets of point estimates.

2. Use the bootstrap algorithm to construct an inner bootstrap procedure for each Monte Carlo point estimate to obtain a bootstrapped confidence interval.
3. Calculate the empirical coverage as a frequency measure (in percentage) for how often the 'true' parameters fall in the confidence interval.

Based on insight from the literature on promising implementations, two resampling approaches and two variations of interval calculations are selected. Residual resampling and case resampling are two common approaches based on different assumptions. The residual resampling approach relies on the functional relationships of the model being correct, while the case resampling approach does not assume a correct model structure. Basic bootstrap confidence intervals and percentile confidence intervals are selected as they represent two standard methods among a range of more advanced bootstrap interval methods[5]. Different resampling approaches and confidence interval methods are included to increase the representativeness of this study. At the same time, the choice is also limited by the computational capacity: the superior methods often require a second layer of bootstrap replications and the Monte Carlo simulations with the chosen methods are already computationally challenging.

The Monte Carlo results show that accuracy of the confidence intervals can be observed in most of the cases. Thus, the bootstrapping procedure is proven to be valid and can be applied to empirical application with EMP models. The confidence intervals obtained by different resampling approaches and confidence interval calculation methods are compared with

[5] See, for example, bootstrap-t (Efron 1981), variance stabilised bootstrap-t (Tibshirani 1988), bias-corrected and accelerated (BCa) method (Efron 1987), approximate bootstrap confidence (ABC) interval (DiCiccio and Efron 1992) and double bootstrap (Beran 1987).

each other in terms of coverage probability. Our findings generally agree with those of other studies, although careful interpretation is necessary, as the comparisons are conducted in different contexts and setups. Some limitations of this study need to be addressed as well. The functional choice of EMP models is flexible. Thus, with more complex model setups the computational challenge might become quite big. Also there is an uncertainty whether the estimation results hold for different model setups. Furthermore, there exists the possibility that other estimation approaches might offer better results.

1.3 Conclusion and outlook

Over the last decades, computational modelling for quantitative agricultural policy assessment has developed towards more farm level oriented approaches. This development is fuelled by multiple factors such as changes in policy instruments, changes in relevance and understanding of policy impact indicators, aggregation issues regarding key biophysical and economic processes and the simultaneous development of databases and information technology. This dissertation is committed to the theoretical and methodological development of farm-level economic modelling. It contributes to the rationalisation of PMP-type models, evaluates estimator consistency for econometric programming models with more complex error specifications and is the first attempt of developing transparent and plausible algorithms for statistical inference procedure in this context. It gives further insights into the application of estimating EMP model and the reliability of the estimated EMP mode parameters and offers a better understanding of applying agricultural optimisation models, especially econometric programming models.

Given the theoretical/experimental nature of the studies, one important missing aspect is the realistic implementation of the approaches advocated in this dissertation. Future research should test the applicability of the theoretical work in real world scenarios based on observed data.

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Chapter 2
Rationalising non-linear agricultural programming models with a capacity constraint[1]

Abstract. Doole et al. (2011) employ a variation of Positive Mathematical Programming (PMP) models using a quadratic constraint. The objective of this chapter is to conceptually analyse their approach. First, the equivalence to PMP models is investigated and the result shows that it indeed holds for calibrating to base year activity levels. However, the equivalence does not extend to simulation, because the dual value of the constraint changes endogenously. Second, this quadratic constraint is interpreted as a capacity constraint (CC) i.e. representing a level technology where activities require resources from an aggregate and fixed labour and capital stock. A more general formulation which allows for an explicit representation of returns to scale is presented. This feature facilitates linking to market models.

Keywords: calibration, farm programming models, capacity constraint

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2.1 Introduction

Positive Mathematical Programming (PMP), which is firstly advocated by Howitt (1995), is a calibration approach introducing non-linear terms in the objective function of (agricultural) programming models. This serves to let optimality conditions being fulfilled at observed levels of activities such that the model solution reproduces those.

Despite its wide use, only limited attention has been paid to the economic or technological rationale behind the non-linear terms in the objective function of the simulation model. Heckelei (2002) raises this question and concludes that under the assumption of a Leontief technology, risk behaviour, land heterogeneity, aggregation errors and other missing unknown resource constraints could be the rationale behind the non-linearity.

Heckelei et al. (2012) review and discuss the more recent literature and application of PMP regarding a) the development of calibration method, b) the estimation of programming models with multiple observations and c) rationalisation of PMP-type models. Regarding the latter, the authors basically find two different possibilities for explicit rationalisation in the literature: first, leaving the profit maximisation behaviour allows to interpret the quadratic objective functions by a mean-variance framework under gross margin uncertainty (Heckelei 2002; Cortignani and Severini 2009; Severini and Cortignani 2011; Petsakos and Rozakis 2011). Second, Doole et al. (2011) apply a programming model with a linear objective function subject to a quadratic constraint calibrating the total milk production on farm as a quadratic function of herd size. Doole et al. were apparently under the impression that this modification was equivalent to the standard PMP approach. If it was, and if one could successfully interpret such a constraint as “capacity constraint (CC)”, then one could fully rationalise the use of quadratic objective functions employed in most PMP type agricultural programming models for policy analysis of recent years.

The objective of this chapter is to conceptually analyse the approach used by Doole et al. (2011). Specifically, this quadratic constraint is interpreted

as a CC, which assumes a “level technology” where production activities use some aggregate of labour and capital stock. We investigate its relationship to standard PMP formulations thereby showing equivalence in calibration and difference in simulation. Finally, we suggest an alternative functional form allowing to explicitly specify returns to scale which might become relevant when agricultural programming models are linked to factor markets. This chapter is organised as follows. In section 2.2 it is briefly explained what the current lack of rationale of a typical PMP model means. Then the approach by Doole et al. (2011) is presented with our interpretation. In section 2.3 the equivalence of the quadratic CC model to a typical PMP model regarding calibration and its deviation in the context of simulation are demonstrated analytically. Subsequently an alternative functional specification is introduced. Finally, section 2.5 concludes and discusses the possibilities for further studies.

2.2 A capacity constrained agricultural programming model

A very important argument for the wide application of PMP type agricultural programming models instead of econometric models is that one can explicitly simulate farm management in detail (use of fertiliser, plan protection, tillage irrigation, etc.) which considerably facilitates the analysis of agri-enviromental interactions. Under the assumption of a Leontief technology, input use increases linearly with increasing activity level. If the non-linearity of the PMP-term relates to non-linearity in the true relationship between output and variable input, then PMP simply corrects for wrong production activity specification not able to reflect differences between average and marginal input application rates (Heckelei et al. 2012, pp. 114). In this case, marginal cost and marginal physical input use as represented in PMP models are inconsistent.

How could one interpret the non-linearity and at the same time preserve the desirable Leontief technology assumption for the definition of the single production activities? Non-linear PMP terms in the objective function may represent economically relevant but empirically missing resource constraints. Moving to a more explicit formulation of such an

interpretation, Doole et al. (2011) extract the non-linear part from the objective function of a typical PMP model and use a quadratic constraint instead. This quadratic constraint is used to calibrate total milk production on farm as a quadratic function of herd size. However, the model is incorrectly interpreted as analogous to a standard PMP model (Doole et al. 2011, pp. 865). Heckelei (2002, pp. 29) already shows that such modification does not render the resulting model equivalent to a typical PMP model. Thus, a quadratic constraint cannot fully rationalise PMP models. Nevertheless, an appropriately chosen quadratic constraint could be a better interpretable alternative to non-linear objective functions while still allowing for the same useful calibration and simulation properties that characterise PMP models.

This study takes the idea from Doole et al. (2011) and consider the non-linear constraint as representing a "level technology" defining the feasible relationship between production activity levels and a (for now) fixed, non-allocable operating capacity while keeping in place the Leontief technology for the allocation of variable inputs. This "level technology" is generally (and likely) non-linear by nature.

Apart from the interpretation of their constraint as a CC, we would like to go one step further beyond Doole et al. (2011) and explicitly link the capacity to the available stocks of labour and capital (or subcategories thereof). In many (aggregate) agricultural programming models used for policy analysis, primary factors are not represented at all. Including it in such a way allows to explicitly reflect and analyse the impact of labour and capital on the production without being forced to represent the heterogeneity of these factors (even at farm level) with a complex set of linear restrictions as is often done in more normative modelling exercises at farm level (Heckelei 2002, pp. 1).

The reflection of labour and capital points to another motivation behind the formulation of a CC model apart from rationalising PMP models: to allow for an explicit and consistent link to factor market models, for example in the form of a Computable General Equilibrium (CGE) model. The linkage requires that information about primary factor use and prices has to pass

between CGE and agricultural supply models (often PMP-type). The explicit inclusion of the labour and capital in the detailed sectoral supply model allows us to directly and appropriately modify the CC to reflect changes in the primary factor market signals from CGE model. In a sequential calibration approach, as demonstrated in Britz (2008), capital and labour quantities may be adjusted and the return to those factors could be appropriately represented by the dual value of the CC.

For the now more formal discussion on the CC model based on the idea from Doole et al. (2011), lower case bold-faced letters are used to represent vectors, upper case bold-faced letters to represent matrices and italic letters to represent scalars. In a typical PMP agricultural programming model, the representative farmer maximises total revenue z by producing with N different production activities subject to M resource constraints.

$$\begin{aligned} \underset{\mathbf{x}}{\text{Max}}\ z &= \mathbf{gm}'\mathbf{x} - \mathbf{d}'_{\text{pmp}}\mathbf{x} - 0.5\mathbf{x}'\mathbf{Q}_{\text{pmp}}\mathbf{x} \\ &\text{subject to} \\ \mathbf{Ax} &\le \mathbf{b} \quad \left[\boldsymbol{\lambda}_{\text{pmp}}\right] \\ \mathbf{x} &\ge 0 \end{aligned} \tag{1}$$

where

z is a scalar representing total profit.

$\mathbf{gm}$ is a $N \times 1$ vector of gross margin

$\mathbf{x}$ is a $N \times 1$ vector of endogenous production activity levels

$\mathbf{d}_{\text{pmp}}$ is a $N \times 1$ vector of parameters associated with the linear PMP term

$\mathbf{Q}_{\text{pmp}}$ is a $N \times N$ positive semi-definite matrix of parameters associated with the quadratic PMP term

$\mathbf{A}$ is a $M \times N$ matrix of input coefficients

$\mathbf{b}$ is a $M \times 1$ vector of resource endowments

$\boldsymbol{\lambda}_{\text{pmp}}$ is a $M \times 1$ vector of dual values associated with the resource constraints.

The gross margin $\mathbf{gm}$ is calculated using a $N \times 1$ vector of output prices $\mathbf{p}$, a $N \times N$ matrix of output coefficients $\mathbf{O}$, the resource use coefficients $\mathbf{A}$ and a $M \times 1$ vector of input costs $\mathbf{c}$ as

$$\mathbf{gm} = \mathbf{Op} - \mathbf{A}'\mathbf{c} \,. \tag{2}$$

In the following, $\mathbf{gm}$ is not expanded for simplicity.

Doole et al. (2011) remove the quadratic term $\mathbf{d}'_{\text{pmp}}\mathbf{x} + 0.5\mathbf{x}'\mathbf{Q}_{\text{pmp}}\mathbf{x}$ in the objective function of (1) and introduce instead a quadratic constraint $g(\mathbf{x})$. The new model can be written as

$$\begin{aligned} &\underset{\mathbf{x}}{\text{Max}}\, z = \mathbf{gm}'\mathbf{x} \\ &\quad \text{subject to} \\ &\mathbf{Ax} \le \mathbf{b} \quad [\boldsymbol{\lambda}] \\ &g(\mathbf{x}) = a - \mathbf{d}'\mathbf{x} - 0.5\mathbf{x}'\mathbf{Qx} = 0 \quad [\gamma] \\ &\mathbf{x} \ge 0 \end{aligned} \tag{3}$$

where

- a is a scalar of parameter associated with the constant term of the CC
- $\mathbf{d}$ is a $N \times 1$ vector of parameters associated with the linear terms of the CC
- $\mathbf{Q}$ is a $N \times N$ positive semi-definite matrix of parameters associated with the quadratic term of the CC
- $\boldsymbol{\lambda}$ is a $M \times 1$ vector of duals associated with the resource constraints
- γ is a scalar of the dual associated with the CC.

The parameter a in $g(\mathbf{x})$ could be seen as the fixed operating capacity implicitly depending on the availability of labour l and capital k. For estimating or simulating changes in capacity caused by adjustments of l and k, this relationship would have to be made explicit as $a(l,k)$. It should be noted here that in order to interpret the non-linear constraint in (3) as a CC where production activities require resources from a non-linear aggregation of fixed labour and capital stocks, two conditions need to be met: 1) the linear objective function covers only the difference between

revenue and variable costs; 2) the linear resource constraints do not include labour and capital.

Apart from a different notation and a specific application context, the model (3) is identical to the model in Doole et al. (2011) and we will refer to it from now on as such. In the next section it is investigated how this model is related to the PMP model (1) in the context of calibration and simulation. It could only be seen as a complete rationalisation of the PMP model if it was fully equivalent in these respects.

Before turning to this, however, it is worth noting that Heckelei et al. (2012) define the condition under which the PMP model (1) may be rationalised by the CC model (3): for this, the PMP related part in the objective function of model (1) must be functionally related to the CC in model (3). To make this explicit, let us define $f(\mathbf{x})$ as an equivalent formulation for the quadratic constraint $g(\mathbf{x})$ in model (3) replacing the PMP terms in model (1). The model can be rewritten as

$$\begin{aligned} \underset{\mathbf{x}}{\text{Max}}\, z &= \mathbf{gm}'\mathbf{x} + f(\mathbf{x}) \\ &\text{subject to} \\ \mathbf{Ax} &\leq \mathbf{b} \quad [\boldsymbol{\lambda}] \\ \mathbf{x} &\geq 0. \end{aligned} \tag{4}$$

For this model to be fully equivalent to (3), first-order conditions of both models need to be the same. This is only the case if

$$\frac{\partial f(\mathbf{x})}{\partial \mathbf{x}_i} = \frac{\partial g(\mathbf{x})}{\partial \mathbf{x}_i}\gamma + g(\mathbf{x})\frac{\partial \gamma}{\partial \mathbf{x}_i} \text{ for all } i. \tag{5}$$

It is already clear now, that merely transforming the non-linear objective function of a PMP model to a non-linear constraint with the same quadratic functional form as done above when moving from (1) to (3) will not satisfy condition (5). Consequently, the model in Doole et al. (2011) with our interpretation of a CC may not rationalise the often applied PMP models with quadratic objective functions.

2.3 Calibration and simulation in comparison with PMP

This section analytically compares the PMP and the CC model regarding calibration and simulation. First the equivalency of the model by Doole et al. to the PMP regarding calibration is shown and then the differences in simulation are illustrated.

Heckelei (2002) presents a programming model with a quadratic constraint to approximate the convex combination constraints advocated by McCarl (1982) and Önal and McCarl (1989 and 1991). Heckelei (2002) compares the first-order conditions of this model with a PMP model and shows that the equivalency to PMP only holds for calibration but not for simulation and estimation. The line of argument is presented here in more detail to compare the CC model by Doole et al. (2011) with the PMP model:

For simplicity of notation, assuming positive optimal quantities for all elements of $\mathbf{x}$, the Lagrangian formulation of the CC model (3) is given by

$$\mathbf{L} = \mathbf{gm}'\mathbf{x} + \gamma\left[a - \mathbf{d}'\mathbf{x} - 0.5\mathbf{x}'\mathbf{Q}\mathbf{x}\right] + \boldsymbol{\lambda}\left[\mathbf{b} - \mathbf{A}\mathbf{x}\right] \tag{6}$$

which implies the first-order necessary conditions as

$$\frac{\partial \mathbf{L}}{\partial \mathbf{x}} = \mathbf{gm} + \gamma\left(-\mathbf{d} - \mathbf{Q}\mathbf{x}\right) - \mathbf{A}'\boldsymbol{\lambda} = 0 \tag{7}$$

$$\frac{\partial \mathbf{L}}{\partial \gamma} = a - \mathbf{d}'\mathbf{x} - 0.5\mathbf{x}'\mathbf{Q}\mathbf{x} = 0 \tag{8}$$

$$\frac{\partial \mathbf{L}}{\partial \boldsymbol{\lambda}} = \mathbf{b} - \mathbf{A}\mathbf{x} = 0\,. \tag{9}$$

The first-order necessary conditions could serve as the calibration conditions. For calibrating to a base year observation, the Lagrange multiplier γ needs to be set to an arbitrary value, because it is not identified and only scales parameters a, $\mathbf{d}$ and $\mathbf{Q}$. Assuming it equals to one, equation (7) can be rewritten as

$$\frac{\partial \mathbf{L}}{\partial \mathbf{x}} = \mathbf{gm} - \mathbf{d} - \mathbf{Q}\mathbf{x} - \mathbf{A}'\boldsymbol{\lambda} = 0\,. \tag{10}$$

The corresponding Lagrangian formulation of the PMP model (1), again assuming positive optimal quantities for all elements of **x,** is given by

$$\mathbf{L} = \mathbf{gm}'\mathbf{x} - \mathbf{d}'_{\text{pmp}}\,\mathbf{x} - \mathbf{x}'\mathbf{Q}_{\text{pmp}}\mathbf{x} + \boldsymbol{\lambda}_{\text{pmp}}\left[\mathbf{b} - \mathbf{A}\mathbf{x}\right] \tag{11}$$

and implies the first-order necessary conditions to obtain

$$\frac{\partial \mathbf{L}}{\partial \mathbf{x}} = \mathbf{gm} - \mathbf{d}_{\text{pmp}} - \mathbf{Q}_{\text{pmp}}\mathbf{x} - \mathbf{A}'\boldsymbol{\lambda}_{\text{pmp}} = 0 \tag{12}$$

$$\frac{\partial \mathbf{L}}{\partial \boldsymbol{\lambda}_{\text{pmp}}} = \mathbf{b} - \mathbf{A}\mathbf{x} = 0\,. \tag{13}$$

Comparing equation (12) with the equation (10) shows the equivalence. Any value, as long as $\mathbf{d}$ equals to $\mathbf{d}_{\text{pmp}}$ and $\mathbf{Q}$ to $\mathbf{Q}_{\text{pmp}}$, will calibrate the two models to the point observation $\mathbf{x}^{\mathbf{o}}$ using the same calibration criterion, the same exogenous $\mathbf{gm}$, $\mathbf{A}$ and prior information on the shadow price of the resource constraint $\boldsymbol{\lambda}$ and $\boldsymbol{\lambda}_{\text{pmp}}$. Note that additionally, equation (8) of the CC model still needs to be fulfilled at $\mathbf{x}^{\mathbf{o}}$ by appropriately choosing the value for parameter a for the given $\mathbf{d}$ and $\mathbf{Q}$.

The equivalence shown is limited, however, as it does not extend to the simulation case. The implied responses of product supply or activity level to changing prices differs between CC and PMP model. Equation (8) of the CC model forces the term $\mathbf{dx} + 0.5\mathbf{x}'\mathbf{QX}$ equal to the parameter a under all economic conditions. This, however, does not apply for the PMP model. The difference in the model structure will result in different simulation behaviours despite having the same values for $\mathbf{d}$ and $\mathbf{Q}$. Expressed differently, when moving away from base year observation in simulations with the specified CC model, the shadow price of the constraint will not stay fixed at calibration value, but change endogenously.

To see this difference, assuming that only the $\mathbf{gm}$ changes for both models and ceteris paribus. Then the simulation behaviour can be described in terms of the marginal effect of the $\mathbf{gm}$ on $\mathbf{x}$ and these can be compared between two models.

The behavioural function for the production activities of the PMP model can be derived by solving (12) for $\mathbf{x}$ as

$$\mathbf{x} = \mathbf{Q}_{\text{pmp}}^{-1}\left(\mathbf{gm} - \mathbf{d}_{\text{pmp}} - \mathbf{A}'\boldsymbol{\lambda}_{\text{pmp}}\right). \tag{14}$$

Substituting (14) into (13), $\boldsymbol{\lambda}_{\text{pmp}}$ can be solved as as

$$\boldsymbol{\lambda}_{\text{pmp}} = \left(\mathbf{A}\mathbf{Q}_{\text{pmp}}^{-1}\mathbf{A}'\right)^{-1}\left(\mathbf{A}\mathbf{Q}_{\text{pmp}}^{-1}\left(\mathbf{gm} - \mathbf{d}_{\text{pmp}}\right) - \mathbf{b}\right). \tag{15}$$

Substituting (15) back into (14) gives the supply function of **x** as a function of exogenous parameters for the PMP model:

$$\mathbf{x} = \mathbf{Q}_{\text{pmp}}^{-1}\left(\mathbf{gm} - \mathbf{d}_{\text{pmp}}\right) - \mathbf{Q}_{\text{pmp}}^{-1}\mathbf{A}'\left(\mathbf{A}\mathbf{Q}_{\text{pmp}}^{-1}\mathbf{A}'\right)^{-1}\left(\mathbf{A}\mathbf{Q}_{\text{pmp}}^{-1}\left(\mathbf{gm} - \mathbf{d}_{\text{pmp}}\right) - \mathbf{b}\right). \tag{16}$$

The marginal effect of **gm** on **x** of the PMP model could thus be expressed as

$$\frac{\partial \mathbf{x}}{\partial \mathbf{gm}} = \mathbf{Q}_{\text{pmp}}^{-1} - \mathbf{Q}_{\text{pmp}}^{-1}\mathbf{A}'\left(\mathbf{A}\mathbf{Q}_{\text{pmp}}^{-1}\mathbf{A}'\right)^{-1}\mathbf{A}\mathbf{Q}_{\text{pmp}}^{-1}. \tag{17}$$

The marginal effect for the CC model is derived in a similar fashion. For the sake of simplicity, only the most important steps are shown here instead of presenting the full derivation. The extended full procedure is provided in the appendix. Rearranging equation (7) gives

$$\mathbf{x} = \gamma^{-1}\mathbf{Q}^{-1}\left(\mathbf{gm} - \gamma\mathbf{d} - \mathbf{A}'\boldsymbol{\lambda}\right). \tag{18}$$

Substituting equation (18) into equation (9), $\boldsymbol{\lambda}$ can be solved as

$$\boldsymbol{\lambda} = \left(\gamma^{-1}\mathbf{A}\mathbf{Q}^{-1}\mathbf{A}'\right)^{-1}\left(\gamma^{-1}\mathbf{A}\mathbf{Q}^{-1}\left(\mathbf{gm} - \gamma\mathbf{d}\right) - \mathbf{b}\right). \tag{19}$$

Substituting equation (19) into equation (18) allows to solve for $\mathbf{x}$ as

$$\begin{aligned} &\mathbf{x} = \gamma^{-1}\mathbf{H}\mathbf{gm} - \mathbf{H}\mathbf{d} + \boldsymbol{\eta} \\ &\quad \text{with} \\ &\mathbf{H} = \left(\mathbf{Q}^{-1} - \mathbf{Q}^{-1}\mathbf{A}'\left(\mathbf{A}\mathbf{Q}^{-1}\mathbf{A}'\right)^{-1}\mathbf{A}\mathbf{Q}^{-1}\right) \\ &\boldsymbol{\eta} = \mathbf{Q}^{-1}\mathbf{A}'\left(\mathbf{A}\mathbf{Q}^{-1}\mathbf{A}'\right)^{-1}\mathbf{b} \end{aligned} \tag{20}$$

Substituting (20) into (8) allows to solve for γ as

$$\gamma = \sqrt{\frac{0.5\mathbf{gm}'\mathbf{Hgm}}{a + 0.5\mathbf{d}'\mathbf{Hd} - 0.5\boldsymbol{\eta}'\mathbf{Q}\boldsymbol{\eta} - \mathbf{d}'\boldsymbol{\eta}}}\,. \tag{21}$$

Finally substituting (21) back into (20), the behavioural function determining production activity levels based only on the exogenous variables and parameters is specified as

$$\mathbf{x} = \sqrt{\frac{a + 0.5\mathbf{d}'\mathbf{Hd} - 0.5\boldsymbol{\eta}'\mathbf{Q}\boldsymbol{\eta} - \mathbf{d}'\boldsymbol{\eta}}{0.5\mathbf{gm}'\mathbf{Hgm}}}\mathbf{Hgm} - \mathbf{Hd} + \boldsymbol{\eta}\,. \tag{22}$$

The marginal effect of $\mathbf{gm}$ on $\mathbf{x}$ is then given by

$$\frac{\partial \mathbf{x}}{\partial \mathbf{gm}} = \gamma^{-1}\frac{\mathbf{H}\left(\mathbf{gm}'\mathbf{Hgm}\right) - \left(\mathbf{Hgm}\right)\left(\mathbf{gm}'\mathbf{H}\right)}{\left(\mathbf{gm}'\mathbf{Hgm}\right)}\,. \tag{23}$$

The marginal change in production activity levels with respect to gross margins for the PMP model (17) and for the CC model (23) are clearly different from each other. Consequently, a different response behaviour is implied for the values of $\mathbf{d}$ and $\mathbf{Q}$ that calibrated both models to the same observation point $\mathbf{x}^{\mathbf{o}}$ as shown above.

Finally, the CC model (3) is presented in a fully equivalent form with a non-linear objective function instead of the non-linear constraint. Based on the derivations above in equation (5), the function $f\left(\mathbf{x}\right)$ has the form

$$f\left(\mathbf{x}\right) = g\left(\mathbf{x}\right)\gamma = \left(a - \mathbf{d}'\mathbf{x} - 0.5\mathbf{x}'\mathbf{Q}\mathbf{x}\right)\gamma\,. \tag{24}$$

Consequently, a programming model with a non-linear objective function fully rationalised by the CC is given as

$$\begin{aligned} &\underset{\mathbf{x}}{\text{Max}}\; z = \mathbf{gm}'\mathbf{x} + \left(a - \mathbf{d}'\mathbf{x} - 0.5\mathbf{x}'\mathbf{Q}\mathbf{x}\right)\gamma \\ &\text{subject to} \\ &\mathbf{Ax} \le \mathbf{b} \qquad [\boldsymbol{\lambda}] \\ &\mathbf{x} \ge 0 \end{aligned} \tag{25}$$

with

$$\gamma = \sqrt{\frac{0.5\mathbf{gm}'\mathbf{Hgm}}{a + 0.5\mathbf{d}'\mathbf{Hd} - 0.5\boldsymbol{\eta}'\mathbf{Q}\boldsymbol{\eta} - \mathbf{d}'\boldsymbol{\eta}}}$$

$$\mathbf{H} = \left(\mathbf{Q}^{-1} - \mathbf{Q}^{-1}\mathbf{A}'\left(\mathbf{A}\mathbf{Q}^{-1}\mathbf{A}'\right)^{-1}\mathbf{A}\mathbf{Q}^{-1}\right)$$

$$\boldsymbol{\eta} = \mathbf{Q}^{-1}\mathbf{A}'\left(\mathbf{A}\mathbf{Q}^{-1}\mathbf{A}'\right)^{-1}\mathbf{b}\,.$$

Obviously, the functional specification of this model is far from a typical PMP model with a quadratic objective function as in (1). We could not come up with any functional form of the capacity constraint replacing the quadratic formulation in (3) such that we would end up with a quadratic objective function in a consistent formulation of the objective function. Even though no proof can be provided at this point, we doubt that the typical PMP with a quadratic objective function can be rationalised with a capacity constraint specification.

2.4 A capacity constrained model with explicit returns to scale

A recent implementation of a quadratic CC in the Common Agricultural Policy Regionalised Impact Modelling System (CAPRI) (Britz 2008) revealed a problem which is ignored so far.

The implementation assumed that the CC represents a capacity defined by an aggregate of the available labour and capital. These primary inputs are not explicitly represented in the CAPRI supply model. For analysing scenarios, where factor market feedbacks were potentially relevant, information on primary factor use and prices was passed between the CGE model and the (partial equilibrium) CAPRI model in an iterative market solution algorithm. Consequently, the dual value of the capacity constraint, γ , was supposed to capture the change in labour and capital from the CGE models. However, passing the simulated changes in labour and capital from the CGE model to the supply model (equivalently shifting the constant term a in our CC model) resulted in non-controllable and non-converging behaviour between the CGE and CAPRI.

This drew our attention to the fact that the CGE model assumes constant returns to scale, while the quadratic CC model does not. Consequently, it is desirable to look for an alternative functional form allowing to define specific returns to scale.

A CC model which allows for the explicit representation of labour and capital defining "capacity" and a measure of returns to scale of the level technology in the programming model may be defined by the following more general form of the CC representing an implicit multi-output-multi-input production function (or transformation function):

$$F\left(l,k,\mathbf{x}\right)=h\left(l,k\right)-v\left(\mathbf{x}\right)=0 \tag{26}$$

The function $h(l,k)$ defines the capacity depending on labour and capital stocks which is "consumed" by the function of activity levels $v(\mathbf{x})$. If both functions are homogeneous of degree 1, then the difference between both functions is homogeneous of degree 1, implying that the level technology exhibits constant returns to scale. A multiplication of labour and capital on the one side and all production activities on the other with the same factor will leave the feasibility of the constraint unchanged.

2.5 Conclusion and outlook

This chapter is embedded in the literature on PMP as a technique to calibrate agricultural optimisation models with non-linear terms in the objective function. The most often applied approach employs a quadratic cost function in activity levels. We addressed the question if a formulation with a non-linear constraint as suggested by Doole et al. (2011) and its interpretation as a constraint on activity levels by operating capacity (level technology) may economically rationalise the use of the non-linear objective function.

It is shown that employing a quadratic constraint instead of a quadratic function to the objective function is fully equivalent in terms of calibrating the programming model to a base year observation on activity levels, because the optimality conditions are the same with respect to the identified parameters. This means that the same parameter values will

calibrate the PMP and the CC model. This equivalence breaks down, however, if one simulates with both models the effect of changing economic conditions. This is due to the fact that the dual value of the CC changes endogenously which is not reflected in the typical PMP formulation. One can derive an optimisation model with just a non-linear objective function and linear constraints that is fully equivalent to the CC model by observing integrability conditions, but it differs from the PMP formulation. The advantage of the latter is clearly its economic interpretability. The question whether it performs better in empirical applications was not within the scope of the analysis.

Another problem arises regarding the usefulness of simple quadratic level technology if one would like to link agricultural programming models to models endogenously simulating factor markets, like CGE models: returns to scale depend on activity levels and are therefore difficult to determine in sequential calibration approaches. A more general formulation of a CC is therefore recommended which allows specifying/estimating explicitly returns to scale in the level technology.

Elaborating on empirical approaches – calibration and estimation – to specify a concrete CC which allows to explicitly represent labour and capital and the returns to scale may improve upon the possibilities to consistently link agricultural sector models with CGEs or other factor market models for policy analysis where such feedbacks are relevant. The subsequent chapters aim at moving into estimation approaches using multiple observations on farm level data.

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2.7 Appendix: complete steps of deriving marginal effect of gm on x

The Lagrange formulation of the CC model is given by

$$\mathbf{L} = \mathbf{gm}'\mathbf{x} + \gamma\left[a - \mathbf{d}'\mathbf{x} - 0.5\mathbf{x}'\mathbf{Q}\mathbf{x}\right] + \boldsymbol{\lambda}'\left[\mathbf{b} - \mathbf{A}\mathbf{x}\right] \tag{1}$$

implying the first-order conditions

$$\frac{\partial \mathbf{L}}{\partial \mathbf{x}} = \mathbf{gm} + \gamma(-\mathbf{d} - \mathbf{Q}\mathbf{x}) - \mathbf{A}'\boldsymbol{\lambda} = 0 \tag{2}$$

$$\frac{\partial \mathbf{L}}{\partial \gamma} = a - \mathbf{d}'\mathbf{x} - 0.5\mathbf{x}'\mathbf{Q}\mathbf{x} = 0\,. \tag{3}$$

Rearranging equation (2) gives

$$\mathbf{x} \;=\; \gamma^{-1}\mathbf{Q}^{-1}\left(\mathbf{gm} - \gamma\mathbf{d} - \mathbf{A}'\boldsymbol{\lambda}\right). \tag{4}$$

The $\boldsymbol{\lambda}$ can be defined as below by substituting equation (4) into equation (3):

$$\boldsymbol{\lambda} = \left(\gamma^{-1}\mathbf{A}\mathbf{Q}^{-1}\mathbf{A}'\right)^{-1}\left(\gamma^{-1}\mathbf{A}\mathbf{Q}^{-1}\left(\mathbf{gm} - \gamma\mathbf{d}\right) - \mathbf{b}\right). \tag{5}$$

Further substituting equation (5) into equation (4) and then rearranging, x can be solved as

$$\begin{aligned}
\mathbf{x} &= \gamma^{-1}\mathbf{Q}^{-1}\left(\mathbf{gm} - \gamma\mathbf{d}\right) - \gamma^{-1}\mathbf{Q}^{-1}\mathbf{A}'\left(\gamma^{-1}\mathbf{A}\mathbf{Q}^{-1}\mathbf{A}'\right)^{-1}\left(\gamma^{-1}\mathbf{A}\mathbf{Q}^{-1}\left(\mathbf{gm} - \gamma\mathbf{d}\right) - \mathbf{b}\right) \\
&= \gamma^{-1}\mathbf{Q}^{-1}\left(\mathbf{gm} - \gamma\mathbf{d}\right) - \mathbf{Q}^{-1}\mathbf{A}'\left(\mathbf{A}\mathbf{Q}^{-1}\mathbf{A}'\right)^{-1}\left(\gamma^{-1}\mathbf{A}\mathbf{Q}^{-1}\left(\mathbf{gm} - \gamma\mathbf{d}\right) - \mathbf{b}\right) \\
&= \mathbf{Q}^{-1}\left(\gamma^{-1}\mathbf{gm} - \mathbf{d}\right) - \mathbf{Q}^{-1}\mathbf{A}'\left(\mathbf{A}\mathbf{Q}^{-1}\mathbf{A}'\right)^{-1}\left(\mathbf{A}\mathbf{Q}^{-1}\left(\gamma^{-1}\mathbf{gm} - \mathbf{d}\right) - \mathbf{b}\right) \\
&= \mathbf{Q}^{-1}\left(\gamma^{-1}\mathbf{gm} - \mathbf{d}\right) - \mathbf{Q}^{-1}\mathbf{A}'\left(\mathbf{A}\mathbf{Q}^{-1}\mathbf{A}'\right)^{-1}\mathbf{A}\mathbf{Q}^{-1}\left(\gamma^{-1}\mathbf{gm} - \mathbf{d}\right) + \mathbf{Q}^{-1}\mathbf{A}'\left(\mathbf{A}\mathbf{Q}^{-1}\mathbf{A}'\right)^{-1}\mathbf{b} \\
&= \left(\mathbf{Q}^{-1} - \mathbf{Q}^{-1}\mathbf{A}'\left(\mathbf{A}\mathbf{Q}^{-1}\mathbf{A}'\right)^{-1}\mathbf{A}\mathbf{Q}^{-1}\right)\left(\gamma^{-1}\mathbf{gm} - \mathbf{d}\right) + \mathbf{Q}^{-1}\mathbf{A}'\left(\mathbf{A}\mathbf{Q}^{-1}\mathbf{A}'\right)^{-1}\mathbf{b}
\end{aligned}\,.$$

$$\begin{aligned} &= \gamma^{-1}\left(\mathbf{Q}^{-1} - \mathbf{Q}^{-1}\mathbf{A}'\left(\mathbf{A}\mathbf{Q}^{-1}\mathbf{A}'\right)^{-1}\mathbf{A}\mathbf{Q}^{-1}\right)\mathbf{gm} \\ &-\left(\mathbf{Q}^{-1} - \mathbf{Q}^{-1}\mathbf{A}'\left(\mathbf{A}\mathbf{Q}^{-1}\mathbf{A}'\right)^{-1}\mathbf{A}\mathbf{Q}^{-1}\right)\mathbf{d} + \mathbf{Q}^{-1}\mathbf{A}'\left(\mathbf{A}\mathbf{Q}^{-1}\mathbf{A}'\right)^{-1}\mathbf{b} \end{aligned} \tag{6}$$

Denoting $\left(\mathbf{Q}^{-1} - \mathbf{Q}^{-1}\mathbf{A}'\left(\mathbf{A}\mathbf{Q}^{-1}\mathbf{A}'\right)^{-1}\mathbf{A}\mathbf{Q}^{-1}\right) = \mathbf{H}$ and $\mathbf{Q}^{-1}\mathbf{A}'(\mathbf{A}\mathbf{Q}^{-1}\mathbf{A}')^{-1}\mathbf{b} = \boldsymbol{\eta}$ due to simplicity, the $\mathbf{x}$ could be rewritten as $\mathbf{x} = \gamma^{-1}\mathbf{H}\mathbf{gm} - \mathbf{H}\mathbf{d} + \boldsymbol{\eta}$.

Directly substituting equation (6) into equation (3) allows to solve for γ as a function of **gm**, a, **d**, **Q**, **A** and **b**. First, rearranging the equation (3) gives

$$0 = a - (\mathbf{d}' + 0.5\mathbf{x}'\mathbf{Q})\mathbf{x}. \tag{7}$$

The transposed **x** could be written as in the equation (8). Note that the transpose of a symmetric matrix is the matrix itself. In the CC model **Q** is symmetric, therefore $\mathbf{Q}^{-1}$, **H** and $(\mathbf{A}\mathbf{Q}^{-1}\mathbf{A})^{-1}$ are symmetric as well.

$$\begin{aligned} \mathbf{x}' &= \left(\gamma^{-1}\mathbf{H}\mathbf{gm}\right)' - \left(\mathbf{H}\mathbf{d}\right)' + \boldsymbol{\eta}' \\ &= \gamma^{-1}\mathbf{g}\mathbf{m}'\mathbf{H} - \mathbf{d}'\mathbf{H} + \boldsymbol{\eta}' \end{aligned} \tag{8}$$

Substituting equations (6) and (8) into equation (7) results in a quadratic function specified as below

$$\begin{aligned} 0 &= a - (\mathbf{d}' + 0.5\mathbf{x}'\mathbf{Q})\mathbf{x} \\ &= a - \left[\mathbf{d}' + 0.5\left(\gamma^{-1}\mathbf{g}\mathbf{m}'\mathbf{H} - \mathbf{d}'\mathbf{H} + \boldsymbol{\eta}'\right)\mathbf{Q}\right]\left(\gamma^{-1}\mathbf{H}\mathbf{gm} - \mathbf{H}\mathbf{d} + \boldsymbol{\eta}\right) \\ &= \left[\mathbf{d}' + 0.5\left(\gamma^{-1}\mathbf{g}\mathbf{m}'\mathbf{H} - \left(\mathbf{d}'\mathbf{H} - \boldsymbol{\eta}'\right)\right)\mathbf{Q}\right]\left(\gamma^{-1}\mathbf{H}\mathbf{gm} - \left(\mathbf{H}\mathbf{d} - \boldsymbol{\eta}\right)\right) - a \\ &= \left[\mathbf{d}' + 0.5\gamma^{-1}\mathbf{g}\mathbf{m}'\mathbf{H}\mathbf{Q} - 0.5\left(\mathbf{d}'\mathbf{H} - \boldsymbol{\eta}'\right)\mathbf{Q}\right]\left(\gamma^{-1}\mathbf{H}\mathbf{gm} - \left(\mathbf{H}\mathbf{d} - \boldsymbol{\eta}\right)\right) - a \\ &= \gamma^{-1}\mathbf{d}'\mathbf{H}\mathbf{gm} - \mathbf{d}'\left(\mathbf{H}\mathbf{d} - \boldsymbol{\eta}\right) + 0.5\mathbf{g}\mathbf{m}'\mathbf{H}\mathbf{Q}\mathbf{H}\mathbf{gm}\left(\gamma^{-1}\right)^2 - \gamma^{-1}0.5\mathbf{g}\mathbf{m}'\mathbf{H}\mathbf{Q}\left(\mathbf{H}\mathbf{d} - \boldsymbol{\eta}\right) - \\ &\quad \gamma^{-1}0.5\left(\mathbf{d}'\mathbf{H} - \boldsymbol{\eta}'\right)\mathbf{Q}\mathbf{H}\mathbf{gm} + 0.5\left(\mathbf{d}'\mathbf{H} - \boldsymbol{\eta}'\right)\mathbf{Q}\left(\mathbf{H}\mathbf{d} - \boldsymbol{\eta}\right) - a \\ &= 0.5\mathbf{g}\mathbf{m}'\mathbf{H}\mathbf{Q}\mathbf{H}\mathbf{gm}\left(\gamma^{-1}\right)^2 + \\ &\quad \left(\mathbf{d}'\mathbf{H}\mathbf{gm} - 0.5\mathbf{g}\mathbf{m}'\mathbf{H}\mathbf{Q}\mathbf{H}\mathbf{d} + 0.5\mathbf{g}\mathbf{m}'\mathbf{H}\mathbf{Q}\boldsymbol{\eta} - 0.5\mathbf{d}'\mathbf{H}\mathbf{Q}\mathbf{H}\mathbf{gm} + 0.5\boldsymbol{\eta}'\mathbf{Q}\mathbf{H}\mathbf{gm}\right)\gamma^{-1} + \\ &\quad 0.5\left(\mathbf{d}'\mathbf{H} - \boldsymbol{\eta}'\right)\mathbf{Q}\left(\mathbf{H}\mathbf{d} - \boldsymbol{\eta}\right) - a - \mathbf{d}'\left(\mathbf{H}\mathbf{d} - \boldsymbol{\eta}\right). \end{aligned} \tag{9}$$

Given a normal quadratic function (10) where ω and θ are coefficients and x is the decision variable:

$$\varpi x^2 + \theta x + \vartheta = 0. \tag{10}$$

And the corresponding expanded parameters from equation (9) can be written as

$$\begin{aligned}\varpi &= 0.5\mathbf{gm}'\mathbf{HQHgm}\\ \theta &= \mathbf{d}'\mathbf{Hgm} - 0.5\mathbf{gm}'\mathbf{HQHd} + 0.5\mathbf{gm}'\mathbf{HQ\eta} - 0.5\mathbf{d}'\mathbf{HQHgm} + 0.5\mathbf{\eta}'\mathbf{QHgm}\\ \vartheta &= 0.5\left(\mathbf{d}'\mathbf{HQHd} - \mathbf{d}'\mathbf{HQ\eta} - \mathbf{\eta}'\mathbf{QHd} + \mathbf{\eta}'\mathbf{Q\eta}\right) - a - \mathbf{d}'\mathbf{Hd} + \mathbf{d}'\mathbf{\eta}.\end{aligned} \tag{11}$$

Substituting $\mathbf{H}$ and $\mathbf{\eta}$ back into equation (11) and by using the associative property yields

$$\begin{aligned}\mathbf{HQH} &= \left(\mathbf{Q}^{-1} - \mathbf{Q}^{-1}\mathbf{A}'\left(\mathbf{AQ}^{-1}\mathbf{A}'\right)^{-1}\mathbf{AQ}^{-1}\right)\mathbf{Q}\left(\mathbf{Q}^{-1} - \mathbf{Q}^{-1}\mathbf{A}'\left(\mathbf{AQ}^{-1}\mathbf{A}'\right)^{-1}\mathbf{AQ}^{-1}\right)\\ &= \left(\mathbf{I} - \mathbf{Q}^{-1}\mathbf{A}'\left(\mathbf{AQ}^{-1}\mathbf{A}'\right)^{-1}\mathbf{A}\right)\left(\mathbf{Q}^{-1} - \mathbf{Q}^{-1}\mathbf{A}'\left(\mathbf{AQ}^{-1}\mathbf{A}'\right)^{-1}\mathbf{AQ}^{-1}\right)\\ &= \mathbf{Q}^{-1} - \mathbf{Q}^{-1}\mathbf{A}'\left(\mathbf{AQ}^{-1}\mathbf{A}'\right)^{-1}\mathbf{AQ}^{-1} - \mathbf{Q}^{-1}\mathbf{A}'\left(\mathbf{AQ}^{-1}\mathbf{A}'\right)^{-1}\mathbf{AQ}^{-1}\\ &\quad + \mathbf{Q}^{-1}\mathbf{A}'\left(\mathbf{AQ}^{-1}\mathbf{A}'\right)^{-1}\mathbf{AQ}^{-1}\mathbf{A}'\left(\mathbf{AQ}^{-1}\mathbf{A}'\right)^{-1}\mathbf{AQ}^{-1}\\ &= \mathbf{Q}^{-1} - 2\mathbf{Q}^{-1}\mathbf{A}'\left(\mathbf{AQ}^{-1}\mathbf{A}'\right)^{-1}\mathbf{AQ}^{-1} + \mathbf{Q}^{-1}\mathbf{A}'\left(\mathbf{AQ}^{-1}\mathbf{A}'\right)^{-1}\mathbf{AQ}^{-1}\\ &= \mathbf{Q}^{-1} - \mathbf{Q}^{-1}\mathbf{A}'\left(\mathbf{AQ}^{-1}\mathbf{A}'\right)^{-1}\mathbf{AQ}^{-1}\\ &= \mathbf{H},\end{aligned} \tag{12}$$

$$\begin{aligned}\mathbf{\eta}'\mathbf{Q\eta} &= \left(\mathbf{b}'\left(\mathbf{AQ}^{-1}\mathbf{A}'\right)^{-1}\mathbf{AQ}^{-1}\right)\mathbf{Q}\left(\mathbf{Q}^{-1}\mathbf{A}'\left(\mathbf{AQ}^{-1}\mathbf{A}'\right)^{-1}\mathbf{b}\right)\\ &= \mathbf{b}'\left(\mathbf{AQ}^{-1}\mathbf{A}'\right)^{-1}\mathbf{A}\left(\mathbf{Q}^{-1}\mathbf{A}'\left(\mathbf{AQ}^{-1}\mathbf{A}'\right)^{-1}\mathbf{b}\right)\\ &= \mathbf{b}'\left(\mathbf{AQ}^{-1}\mathbf{A}'\right)^{-1}\mathbf{b}\end{aligned} \tag{13}$$

and

$$\begin{aligned}\mathbf{HQ\eta} &= \left(\mathbf{Q}^{-1} - \mathbf{Q}^{-1}\mathbf{A}'\left(\mathbf{AQ}^{-1}\mathbf{A}'\right)^{-1}\mathbf{AQ}^{-1}\right)\mathbf{Q}\left(\mathbf{Q}^{-1}\mathbf{A}'\left(\mathbf{AQ}^{-1}\mathbf{A}'\right)^{-1}\mathbf{b}\right)\\ &= \left(\mathbf{I} - \mathbf{Q}^{-1}\mathbf{A}'\left(\mathbf{AQ}^{-1}\mathbf{A}'\right)^{-1}\mathbf{A}\right)\left(\mathbf{Q}^{-1}\mathbf{A}'\left(\mathbf{AQ}^{-1}\mathbf{A}'\right)^{-1}\mathbf{b}\right)\\ &= \mathbf{Q}^{-1}\mathbf{A}'\left(\mathbf{AQ}^{-1}\mathbf{A}'\right)^{-1}\mathbf{b} - \mathbf{Q}^{-1}\mathbf{A}'\left(\mathbf{AQ}^{-1}\mathbf{A}'\right)^{-1}\mathbf{AQ}^{-1}\mathbf{A}'\left(\mathbf{AQ}^{-1}\mathbf{A}'\right)^{-1}\mathbf{b}\\ &= \mathbf{Q}^{-1}\mathbf{A}'\left(\mathbf{AQ}^{-1}\mathbf{A}'\right)^{-1}\mathbf{b} - \mathbf{Q}^{-1}\mathbf{A}'\left(\mathbf{AQ}^{-1}\mathbf{A}'\right)^{-1}\mathbf{b}\\ &= 0.\end{aligned} \tag{14}$$

The transpose of the vector in equation (14) implies

$$\mathbf{\eta}'\mathbf{QH} = 0. \tag{15}$$

Rewriting the equation (11) with the simplified terms gives

$$
\begin{aligned}
\varpi &= 0.5\mathbf{gm}'\mathbf{HQHgm} = 0.5\mathbf{gm}'\mathbf{Hgm} \\
\theta &= \mathbf{d}'\mathbf{Hgm} - 0.5\mathbf{gm}'\mathbf{HQHd} + 0 - 0.5\mathbf{d}'\mathbf{HQHgm} + 0 \\
&= \mathbf{d}'\mathbf{Hgm} - 0.5\mathbf{gm}'\mathbf{Hd} - 0.5\mathbf{d}'\mathbf{Hgm} \\
&= 0 \\
\vartheta &= 0.5\left(\mathbf{d}'\mathbf{HQHd} - \mathbf{d}'\mathbf{HQ}\boldsymbol{\eta} - \boldsymbol{\eta}'\mathbf{QHd} + \boldsymbol{\eta}'\mathbf{Q}\boldsymbol{\eta}\right) - a - \mathbf{d}'\mathbf{Hd} + \mathbf{d}'\boldsymbol{\eta} \\
&= 0.5\left(\mathbf{d}'\mathbf{Hd} - 0 - 0 + \boldsymbol{\eta}'\mathbf{Q}\boldsymbol{\eta}\right) - a - \mathbf{d}'\mathbf{Hd} + \mathbf{d}'\boldsymbol{\eta} \\
&= 0.5\boldsymbol{\eta}'\mathbf{Q}\boldsymbol{\eta} + \mathbf{d}'\boldsymbol{\eta} - a - 0.5\mathbf{d}'\mathbf{Hd}\,.
\end{aligned}
\tag{16}
$$

For the equation of coefficient θ in the equation (16), (**gm′Hd**) is the transpose of (**d′Hgm**) and they are identical scalars.

Solving for γ^{-1} according to equation (10) gives

$$
\begin{aligned}
0 &= \varpi\left(\gamma^{-1}\right)^2 + \vartheta \\
\gamma^{-1} &= \pm\sqrt{\frac{-\vartheta}{\varpi}} = \pm\sqrt{\frac{a + 0.5\mathbf{d}'\mathbf{Hd} - 0.5\boldsymbol{\eta}'\mathbf{Q}\boldsymbol{\eta} - \mathbf{d}'\boldsymbol{\eta}}{0.5\mathbf{gm}'\mathbf{Hgm}}} \\
\gamma &= \pm\sqrt{\frac{0.5\mathbf{gm}'\mathbf{Hgm}}{a + 0.5\mathbf{d}'\mathbf{Hd} - 0.5\boldsymbol{\eta}'\mathbf{Q}\boldsymbol{\eta} - \mathbf{d}'\boldsymbol{\eta}}}\,.
\end{aligned}
\tag{17}
$$

Note that for the above and the following derivation, it is only possible, if ω is great than 0, while ϑ is smaller than 0: ω is greater than 0 due to the positive and definiteness of **Q**, ϑ needs to be smaller than 0 to guarantee the division under the square root is greater than 0; furthermore, later derivation steps require to slip the nominator and the denominator in equation 18. This has been taken into consideration in the programming process.

Now substituting equation (17) into equation (8) **x** can be rewritten as

$$
\mathbf{x} = \pm\sqrt{a + 0.5\mathbf{d}'\mathbf{Hd} - 0.5\boldsymbol{\eta}'\mathbf{Q}\boldsymbol{\eta} - \mathbf{d}'\boldsymbol{\eta}}\,\frac{\mathbf{Hgm}}{\sqrt{\left(0.5\mathbf{gm}'\mathbf{Hgm}\right)}} - \mathbf{Hd} + \boldsymbol{\eta}\,. \tag{18}
$$

Letting $a + 0.5\mathbf{d}'\mathbf{Hd} - 0.5\boldsymbol{\eta}'\mathbf{Q}\boldsymbol{\eta} - \mathbf{d}'\boldsymbol{\eta} = -\vartheta$ again due to simplicity and the marginal effect of **gm** to **x** could be written as

$$
\begin{aligned}
\frac{\partial \mathbf{x}}{\partial \mathbf{gm}} &= \pm\sqrt{-\vartheta}\,\frac{\partial\left(\dfrac{\mathbf{Hgm}}{\sqrt{(0.5\mathbf{gm}'\mathbf{Hgm})}}\right)}{\partial \mathbf{gm}} \\
&= \pm\sqrt{-\vartheta}\,\frac{\mathbf{H}\sqrt{(0.5\mathbf{gm}'\mathbf{Hgm})} - \mathbf{Hgm}0.5\left(\dfrac{1}{\sqrt{(0.5\mathbf{gm}'\mathbf{Hgm})}}\right)(\mathbf{gm}'\mathbf{H})}{(0.5\mathbf{gm}'\mathbf{Hgm})} \\
&= \pm\sqrt{-\vartheta}\,\frac{\mathbf{H}\sqrt{(0.5\mathbf{gm}'\mathbf{Hgm})} - \dfrac{0.5(\mathbf{Hgm})(\mathbf{gm}'\mathbf{H})}{\sqrt{(0.5\mathbf{gm}'\mathbf{Hgm})}}}{(0.5\mathbf{gm}'\mathbf{Hgm})} \\
&= \pm\sqrt{-\vartheta}\,\frac{\dfrac{\mathbf{H}(0.5\mathbf{gm}'\mathbf{Hgm}) - 0.5(\mathbf{Hgm})(\mathbf{gm}'\mathbf{H})}{\sqrt{(0.5\mathbf{gm}'\mathbf{Hgm})}}}{(0.5\mathbf{gm}'\mathbf{Hgm})} \\
&= \left[\pm\frac{\sqrt{-\vartheta}}{\sqrt{(0.5\mathbf{gm}'\mathbf{Hgm})}}\right]\frac{\mathbf{H}(0.5\mathbf{gm}'\mathbf{Hgm}) - 0.5(\mathbf{Hgm})(\mathbf{gm}'\mathbf{H})}{(0.5\mathbf{gm}'\mathbf{Hgm})} \\
&= \gamma^{-1}\frac{\mathbf{H}(\mathbf{gm}'\mathbf{Hgm}) - (\mathbf{Hgm})(\mathbf{gm}'\mathbf{H})}{(\mathbf{gm}'\mathbf{Hgm})}.
\end{aligned}
\tag{19}
$$

By definition, the point elasticity could be written as

$$
\begin{aligned}
\mathbf{E} &= \frac{\partial \mathbf{x}}{\partial \mathbf{gm}} \odot \mathbf{G} \\
&= \left(\gamma^{-1}\frac{\mathbf{H}(\mathbf{gm}'\mathbf{Hgm}) - (\mathbf{Hgm})(\mathbf{gm}'\mathbf{H})}{(\mathbf{gm}'\mathbf{Hgm})}\right) \odot \left(\frac{\mathbf{gm}}{\mathbf{x}}\right).
\end{aligned}
\tag{20}
$$

Chapter 3
Consistency of estimating constrained optimisation models

Abstract. In this chapter we examine the estimation of a mathematical programming model with an explicit production function for its activities. The approach allows a flexible choice in terms of functional form. Monte Carlo simulations with a least-squares estimator are applied to evaluate the consistency of the estimation procedure choosing a CES production function. Two types of error structures are introduced to address different potential data structures.

Keywords: agricultural supply analysis, estimation of mathematical programming model, CES function, Monte Carlo simulation, errors in optimisation.

3.1 Introduction

The typical Positive Mathematical Programming (PMP) approach relies solely on calibration in contrast to the estimation approach of production functions that is based on dual systems of supply and input equations (Heckelei et al. 2012). Over the last two decades these two approaches have converged to each other to a certain degree. For instance, Heckelei and Wolff (2003) argue that the PMP-type model is not suitable for estimation due to its fundamental inconsistency problem. They suggest an alternative approach which allows the direct estimation of programming models with multiple observations.

In a more recent review article, Heckelei et al. (2012) discuss methodological advances of estimating constrained optimisation models during the previous one and a half decades. There are only a few studies applying that method, and they originate from only a few non-associated research groups. Most of the estimation applications adopt the conventional PMP-type quadratic cost function. The lack of rationale behind this formulation (see Heckelei 2002 and Heckelei and Wolff 2003) is still not resolved, as it is not based on an explicit behavioural or technological assumption. The current methods to rationalise PMP models can be broadly divided into two types: (1) models replacing the non-linear PMP term in the objective function with a non-linear capacity constraint (CC) representing an activity level technology like in Doole et al. (2011) and (2) interpreting the quadratic objective function in the context of mean-variance risk analysis (Cortignani and Severini 2009; Severini and Cortignani 2011; Petsakos and Rozakis 2011).

The latest development in estimating programming models not covered in the review by Heckelei et al. (2012) is proposed by Jansson et al. (2014) and several other papers, for example Donati et al. (2013) and Arata et al. (2017), which focus on the estimation and the rationalisation of 'PMP costs'. The model applied by Jansson et al. (2014) is a farm-level agricultural supply model. The typical PMP cost function is applied and partially rationalised using a mean-variance utility type function. A large panel data set from the Farm Accountancy Data Network is used for the estimation. Their transparent Bayesian methodology is proven to be feasible, but a significant amount of effort has been devoted to separate the covariance matrix from the quadratic PMP terms, and to solve the technical and numerical difficulties working with a large unbalanced data set.

Chapter 2 targets the rationalisation of the PMP model by examining the CC model from Doole et al. (2011). And it concludes that 1) the CC model is equivalent to the PMP model only in terms of calibration, while 2) in simulation or estimation the equivalence does not hold anymore, and 3) that a quadratic CC cannot rationalise PMP models. In this study, we focus on the estimation of such mathematical programming models. The general approach advocated by Heckelei and Wolff (2003) allows a more flexible

choice of functional form than the typical PMP parameterisation. A model with crop-specific, constant elasticity of substitution (CES) production functions is applied. The application of CES-type functions in the context of agricultural programming models is not new. The CES-quadratic model, where land input is used as the quadratic term in the nonlinear cost function, is first introduced by Howitt (1995). Mérel and Bucaram (2010) derive the necessary and sufficient conditions for exact calibration based on this precise model specification. Mérel et al. (2011) propose a modified version of the previous model and term it 'generalised[1]' CES model, which demonstrates more flexibility in terms of calibration against exogenous supply elasticities than the quadratic version. The concavity of the objective function is accounted for by a CES production function with decreasing returns to scale. Mérel and Howitt (2014) provide the latest review on theoretical and empirical developments of PMP models. They state that despite the numerous possibilities to define a non-linear model, the most common choice in agricultural programming models is the globally convex model, namely the combination of a concave objective function and a convex set of constraints. Typically, the concavity of the objective function is implemented by decreasing gross margins in variable activity levels. This can be achieved either with a non-linear cost function assuming increasing marginal costs, or with crop-specific production functions assuming decreasing marginal yields. The latter assumption is adopted in many notable studies (Heckelei and Wolff 2003; Mérel et al. 2011; Frisvold and Konyar 2012; Howitt et al. 2012; Medellín-Azuara et al. 2012; Garnache 2013, pp. 39-76; Graveline and Mérel 2014; Mérel et al. 2014). Mérel and Howitt (2014) conclude that neither of the two

[1] "Generalised CES production function" usually refers to a CES production function where the elasticities of substitution among pairs of inputs can vary (Lu and Fletcher 1968). The CES function in the model of Mérel et al. (2011) has in fact a constant elasticity of substitution for all pairs of inputs. They only term it 'generalised' CES model to differentiate it from the quadratic CES model in Howitt (1995).

approaches have solid empirical and theoretical justification, and claim that both are rather motivated by pragmatic considerations.

This chapter aims at examining the statistical consistency of estimating a constrained optimisation model with a CES function. Heckelei and Wolff (2003) conduct a similar study with the same model. The approach presented in this chapter is different from the former in two respects: (1) additional to the measurement error structure, an optimisation error structure is also considered to allow for more explicit interpretation; (2) an Ordinary Least Squares (OLS) estimator is applied instead of Generalised Maximum Entropy (GME). In this 'well-posed' scenario with more observations than parameters to be estimated, the OLS estimator is sufficient. It also avoids the arbitrariness in choosing the number of support points when using GME.

The remainder of the chapter is organised as follows: section 3.2 presents the detailed specification of the economic model. Followed by that, the statistical model and the estimation model which comprise the estimation approach are illustrated in detail in section 3.3. Also, the setup for the evaluation approach with Monte Carlo simulation is presented. The evaluation results are shown and discussed in section 3.4. Finally, section 3.4 concludes by addressing the limitations of the proposed approach and giving a direction for future research.

3.2 **Model description**

The model analysed in this study is a single farm optimisation model with crop-specific, CES production functions:

$$\max_{l_{ij}\geq 0, q_i \geq 0} \pi = \sum_{i=1}^{I} \left(p_i q_i - w_{i2} l_{i2} \right) \tag{1}$$

subject to

$$q_i = \theta_i \left[\sum\nolimits_{j=1}^{J} \beta_{ij} \left(l_{ij} \right)^{-\rho_i} \right]^{-\nu_i/\rho_i} \quad \forall i = 1,\ldots,I \tag{2}$$

$$\sum_{i=1}^{I} l_{i1} = L \quad [\lambda] \tag{3}$$

$$l_{ij} \geq 0 \quad [\gamma_{ij}] \tag{4}$$

where

i is an index for output and $i = 1,\ldots,I$

j is an index for input and $j = 1,\ldots,J$, where $j = 1$ and $j = 2$ stand for the fixed input land and for the variable input fertiliser, respectively.

p_i are the output prices

w_{ij} are the input prices

l_{ij} are the endogenous resource allocations

L is the total land endowment

λ is the shadow price of the land constraint (3)

γ_{ij} are the shadow prices of the non-negativity constraint (4)

π is the profit

q_i are the CES output production functions

θ_i are the efficiency parameters indicating the state of technology and organisational aspects of production and $\theta_i > 0$ for all $i = 1,\ldots I$

β_{ij} are the distribution parameters (or land/fertiliser intensity factor coefficients) expressing relative factor shares in total output and $\beta_{ij} > 0$, $\sum_{j=1}^{J} \beta_{ij} = 1$ for all $i = 1,\ldots I$ and for all $j = 1,\ldots J$

ρ_i are the substitution parameters which determines the elasticity of substitution and $\rho_i > 0$ for all $i = 1,\ldots I$

ν_i are the economies of scale parameters and $0 < \nu_i < 1$ for all $i = 1,\ldots I$

The objective function (1) maximises profit defined as total revenue minus total cost and is subject to the land resource constraint (3). The CES production functions (2) are linearly homogeneous and quasi-concave, which renders the objective function concave. Decreasing returns to scale

is assumed to allow a positive output level for all outputs i. Hence, ν_i should take a value between 0 and 1. The substitution elasticities σ_i should satisfy $0 < \sigma_i < 1$ and are calculated as $\sigma_i = 1/(1+\rho_i)$. Therefore ρ_i must be greater than 0. Together, $0 < \nu_i < 1$ and $\rho_i > 0$ render the CES function strictly concave.

The first-order conditions of the economic model are explicitly formulated below to construct the statistical model for the data-generating process as well as the econometric model for parameter estimation illustrated in the next section. First, one needs to define the Lagrange function of the economic model:

$$\mathcal{L} = \sum_{i=1}^{I}\left(p_i \theta_i \left[\sum\nolimits_{j=1}^{J} \beta_{ij} \left(l_{ij} \right)^{-\rho_i} \right]^{-\nu_i/\rho_i} - w_{i2} l_{i2} \right) + \lambda \left(L - \sum_{i=1}^{I} l_{i1} \right) + \sum_{j=1}^{J} \gamma_{ij} l_{ij} \,. \quad (5)$$

Taking the derivatives of the Lagrange function w.r.t the endogenous variables to obtain the corresponding first-order conditions yields the marginal value product conditions for land (6), the marginal value product conditions for fertiliser (7), the land shadow price equations (8), and the Kuhn-Tucker condition for positive land allocation (9).

$$\frac{\partial \mathcal{L}}{\partial l_{i1}} = p_i \theta_i \nu_i \left[\sum\nolimits_{j=1}^{J} \beta_{ij} \left(l_{ij} \right)^{-\rho_i} \right]^{(-\nu_i/\rho_i)-1} \beta_{i1} \left(l_{i1} \right)^{-\rho_i - 1} - \lambda - \gamma_{i1} = 0 \quad (6)$$

$$\frac{\partial \mathcal{L}}{\partial l_{i2}} = p_i \theta_i \nu_i \left[\sum\nolimits_{j=1}^{J} \beta_{ij} \left(l_{ij} \right)^{-\rho_i} \right]^{(-\nu_i/\rho_i)-1} \beta_{i2} \left(l_{i2} \right)^{-\rho_i - 1} - w_{i2} - \gamma_{i2} = 0 \quad (7)$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} = \sum\nolimits_{i=1}^{I} l_{i1} = L \quad (8)$$

$$\frac{\partial L_i}{\partial \gamma_{ij}} = l_{ij} > 0 \;\perp\; \sum\nolimits_{j=1}^{J} \gamma_{ij} l_{ij} = 0 \quad (9)$$

3.3 **Model estimation and Monte Carlo assessment**

This section describes the estimation approach of the economic model and the Monte Carlo simulations which are employed to evaluate the consistency of the estimation approach. First, the data-generating process employing a statistical model with random errors is introduced. Second, the econometric model and its estimation are explained. Third, the set-up of Monte Carlo simulations is described. These three steps are presented for both two error specifications, namely measurement error and optimisation error.

Heckelei and Wolff (2003) introduce the error terms around the endogenous variable input and output and interpret the errors as "*…a measurement error of the variable or an optimisation error by the farmer, or stem from specific circumstances relevant to the optimal allocation of the respective economic unit unknown to the econometrician, or some combination of these factors…*" Measurement and optimisation errors are considered separately in this study. This differentiation allows a more sophisticated error structure specification and the separation of the errors effects.

3.3.1 *Measurement error*

The measurement error is defined as an error term related to the input of the single farm optimisation model. It is interpreted as counting or observing error made by the farmer *after* the optimisation process. An example would be that after harvest the farmer reported the amount of fertiliser used for agricultural production and he over- or underestimated the amount of fertiliser. Introducing measurement errors to the first-order conditions of the economic model, the statistical model can be formulated as follows:

$$p_{it}^{*}\theta_{i}^{*}\nu_{i}^{*}\left[\sum_{j=1}^{J}\beta_{ij}^{*}\left(l_{ijt}-\varepsilon_{ijt}^{m*}\right)^{-\rho_{i}^{*}}\right]^{\left(-\nu_{i}^{*}/\rho_{i}^{*}\right)-1}\beta_{i1}^{*}\left(l_{i1t}-\varepsilon_{i1t}^{m*}\right)^{-\rho_{i}^{*}-1}-\lambda_{t}-\gamma_{i1t}=0$$

$$p_{it}^{*}\theta_{i}^{*}\nu_{i}^{*}\left[\sum_{j=1}^{J}\beta_{ij}^{*}\left(l_{ijt}-\varepsilon_{ijt}^{m*}\right)^{-\rho_{i}^{*}}\right]^{\left(-\nu_{i}^{*}/\rho_{i}^{*}\right)-1}\beta_{i2}^{*}\left(l_{i2t}-\varepsilon_{i2t}^{m*}\right)^{-\rho_{i}^{*}-1}-w_{i2t}^{*}-\gamma_{i2t}=0$$

$$\sum_{i=1}^{I}\left(l_{i1t}-\varepsilon_{i1t}^{m*}\right)=L^{*}$$

$$\sum_{j=1}^{J}\gamma_{ijt}\left(l_{ijt}-\varepsilon_{ijt}^{m*}\right)=0, \tag{10}$$

where

t is an index for observations and $t=1,...,T$

ε_{ijt}^{m} are the measurement errors

* is a superscript indicating that the current symbol is data.

Given the exogenous 'true' CES parameters $\left(\theta_i^{*},\beta_{ij}^{*},\nu_i^{*},\rho_i^{*}\right)$ and prices $\left(p_{it}^{*},\, w_{ijt}^{*}\right)$, the profit maximisation model reaches its optimum at a certain unobserved resource allocation for each observation. This optimum is represented by the optimal land and fertiliser allocations l_{ijt}^{**} obtained from the data-generating process *without* statistical errors. The difference between the *actual* observed resource allocations and the optimal l_{ijt}^{**} is randomly distributed across all observations. The generated data on the actual observed resource allocations are obtained by subtracting stochastic measurement errors from the optimal resource allocations as $l_{ijt}^{*}=l_{ijt}^{**}-\varepsilon_{ijt}^{m*}$. Note that the shadow prices for land λ_t are implicit functions of l_{ijt} and Lagrange multiplier γ_{ijt}. The latter terms are always equal to zero as long as the optimal solution regarding the input use is found.

Adding an objective function to the statistical model, the econometric estimation model is formulated as

$$\min_{\theta_{it},\beta_{ijt},\nu_{it},\rho_{it}}\sum_{i=1}^{I}\sum_{j=1}^{J}\sum_{t=1}^{T}\left(\varepsilon_{ijt}^{m}\right)^{2}$$

subject to

$$p_{it}^{*}\theta_i\nu_i\left[\sum_{j=1}^{J}\beta_{ij}\left(l_{ijt}^{*}-\varepsilon_{ijt}^{m}\right)^{-\rho_i}\right]^{(-\nu_i/\rho_i)-1}\beta_{i1}\left(l_{i1t}^{*}-\varepsilon_{i1t}^{m}\right)^{-\rho_i-1}-\lambda_t-\gamma_{i1t}=0$$

$$p_{it}^{*}\theta_i \nu_i \left[\sum_{j=1}^{J}\beta_{ij}\left(l_{ijt}^{*}-\varepsilon_{ijt}^{m}\right)^{-\rho_i}\right]^{(-\nu_i/\rho_i)-1}\beta_{i2}\left(l_{i2t}^{*}-\varepsilon_{i2t}^{m}\right)^{-\rho_i-1}-w_{i2t}^{*}-\gamma_{i2t}=0$$

$$\sum_{i=1}^{I}\left(l_{i1t}^{*}-\varepsilon_{i1t}^{m}\right)=L^{*}$$

$$\sum_{j=1}^{J}\gamma_{ijt}\left(l_{ijt}^{*}-\varepsilon_{ijt}^{m}\right)=0\,. \qquad (11)$$

The objective function employs an OLS estimator, which is sufficient for this 'well-posed' estimation problem, and minimises the sum of squared errors. Boundary conditions for the CES parameters, i.e. $\theta_{it}>0$, $0<\beta_{ijt}<1$, $0<\nu_{it}<1$ and $\rho_{it}>0$, need to be satisfied in addition. The exogenous and endogenous variables in the statistical model (10) become parameters and data in the econometric model (11), respectively: the parameters to be estimated are now θ_i, β_{ij}, ν_i, ρ_i, ε_{ijt}^{m}, λ_t and γ_{ijt}, while p_{it}^{*}, w_{ijt}^{*} and l_{ijt}^{*} are data.

The estimation approach of the economic model is completed by combining the statistical model and the econometric estimation model. This estimation approach allows a consistent and simultaneous estimation of CES parameters and shadow prices of land. Due to the distortion created by measurement errors, parameter estimates are bound to deviate from the true values used in the data-generating process. This deviation can be used to measure the consistency of the estimated model.

The data-generating process and the econometric estimation approach are carried out repeatedly in Monte Carlo simulations with random sampling for different sample sizes. The measures Root Mean Squared Deviation (RMSD) between estimates and true values (of both CES parameters and shadow prices of land) are calculated during each iteration. To summarise the results, RMSD are summed across all observations and iterations to obtain the Average Root Mean Squared Deviation (ARMSD). The measures for CES parameters are summed over all CES parameters in addition. The quality of the estimation approach can be evaluated by statistical assessment of the measures ARMSD: a decreasing ARMSD with increasing sample sizes indicates consistency of the estimator.

Indices for output and input are defined as $I=10$ and $J=2$. Six different sample sizes, $T=(10, 20, 30, 50, 100, 200)$, are considered. Two different sizes of standard deviations, $\delta=10\%$ and $\delta=50\%$, are defined for measurement errors to increase the representativeness of Monte Carlo simulations. 10 per cent and 50 per cent are labelled as 'low' and 'high' standard deviations, respectively. Each standard deviation size is mapped with all six sample sizes, and it results in total of twelve categories of Monte Carlo simulations with measurement errors. The number of iterations of the Monte Carlo simulation is chosen to be $K=1000$.

Random measurement errors for the data-generating process are defined as $\varepsilon_{ijt}^{m*} \sim N\left[0,\left(\sigma_{ij}^{m}\right)^2\right]$. The standard deviations σ_{ij}^{m} are made proportional to the standard deviations of optimum resource allocations σ_{ij}^{l} which is defined as

$$\sigma_{ij}^{l}=\sqrt{\left[\sum_{t=1}^{T}\left(l_{ijt}-\bar{l}_{ij}\right)^2\right]/(t-1)}$$

where

$$\bar{l}_{ij}=\sum_{t=1}^{T} l_{ijt}/(t-1). \tag{12}$$

The proportion is defined as a weight $\delta\left(l_{ijt}/\bar{l}_{ij}\right)$, where the term $\left(l_{ijt}/\bar{l}_{ij}\right)$ relativises the value of σ_{ij}^{l} for all observations. Multiplying σ_{ij}^{l} by the weight to obtain σ_{ij}^{m}, the measurement errors for the Monte Carlo simulations are defined as $\varepsilon_{ijt}^{m*} \sim N\left[0,\left(\delta\left(l_{ijt}/\bar{l}_{ij}\right)\cdot\sigma_{ij}^{l}\right)^2\right]$. To guarantee that the land constraint in model (10) and (11) holds, the measurement errors of the last crop is calculated as the residual equal to $0-\sum_{i=1}^{I-1}\varepsilon_{ijt}^{m*}$.

Randomly generated synthetic data are applied for the Monte Carlo simulations. The output prices p_{it}^{*} and input prices w_{ijt}^{*} are normally distributed, whereas the 'true' CES parameters $\left(\theta_i^{*}, \beta_{ij}^{*}, \nu_i^{*}, \rho_i^{*}\right)$ have an uniform distribution. Land endowment *L* is set to 10. Except for ε_{ijt}^{m*},

which are regenerated for each Monte Carlo iteration, all other data are generated only once for all iterations.

3.3.2 *Optimisation error*

Pope and Just (2002) interpret optimisation errors as "*…weather and other vicissitudes of nature that occur after input decisions are made…"* However, it does not necessarily mean 'weather', as assumed behaviour in such models is always reflecting "expected" yields or prices, so that a certain weather or market situation cannot make the optimisation wrong. It could be interpreted, however, as faulty formation of expectations of the first-order conditions. This distortion could then be represented by introducing random statistical errors directly into the equations (6) and (7), i.e. the first-order conditions of the marginal value product conditions for land and fertiliser. The resulting statistical model with the optimisation error structure is formulated as

$$p_{it}^{*}\theta_{i}^{*}\nu_{i}^{*}\left[\sum\nolimits_{j=1}^{J}\beta_{ij}^{*}\left(l_{ijt}\right)^{-\rho_{i}^{*}}\right]^{\left(-\nu_{i}^{*}/\rho_{i}^{*}\right)-1}\beta_{i1}^{*}\left(l_{i1t}\right)^{-\rho_{i}^{*}-1}-\lambda_{t}-\gamma_{i1t}-\varepsilon_{i1t}^{o}=0$$

$$p_{it}^{*}\theta_{i}^{*}\nu_{i}^{*}\left[\sum\nolimits_{j=1}^{J}\beta_{ij}^{*}\left(l_{ijt}\right)^{-\rho_{i}^{*}}\right]^{\left(-\nu_{i}^{*}/\rho_{i}^{*}\right)-1}\beta_{i2}^{*}\left(l_{i2t}\right)^{-\rho_{i}^{*}-1}-w_{i2t}^{*}-\gamma_{i2t}-\varepsilon_{i2t}^{o}=0$$

$$\sum\nolimits_{i=1}^{I}l_{i1t}=L^{*}$$

$$\sum\nolimits_{j=1}^{J}\gamma_{ijt}l_{ijt}=0, \qquad (13)$$

where optimisation errors are denoted by ε_{ijt}^{o}. The data-generating process with the above statistical model generates l_{ijt}^{*}, the *actual* resource allocations of land and fertiliser under the impact of optimisation errors.

The econometric estimation model with optimisation errors is specified as

$$\min_{\theta_{it},\beta_{ijt},\nu_{it},\rho_{it}}\sum_{i=1}^{I}\sum_{j=1}^{J}\sum_{t=1}^{T}\left[\left(\varepsilon_{ijt}^{o}\right)^{2}+\left(\lambda_{t}-\bar{\lambda}_{t}\right)^{2}\right]$$

$$p_{it}^{*}\theta_i\nu_i\left[\sum\nolimits_{j=1}^{J}\beta_{ij}\left(l_{ijt}^{*}\right)^{-\rho_i}\right]^{(-\nu_i/\rho_i)-1}\beta_{i1}\left(l_{i1t}^{*}\right)^{-\rho_i-1}-\lambda_t-\gamma_{i1t}-\varepsilon_{i1t}^{o}=0$$

$$p_{it}^{*}\theta_i\nu_i\left[\sum\nolimits_{j=1}^{J}\beta_{ij}\left(l_{ijt}^{*}\right)^{-\rho_i}\right]^{(-\nu_i/\rho_i)-1}\beta_{i2}\left(l_{i2t}^{*}\right)^{-\rho_i-1}-w_{i2t}^{*}-\gamma_{i2t}-\varepsilon_{i2t}^{o}=0\,. \quad (14)$$

The same boundary conditions as in the case with measurement errors apply for the CES parameters. The land constraint and the positive constraint are omitted from the estimation model, as the optimisation errors are not added to the resource allocation. The OLS estimator is also chosen for the econometric model with optimisation errors. The objective function, however, minimises not only the sum of squared errors, but also the sum of squared deviations between the estimated land shadow prices and the expectation of the true values. The average per-hectare profit, defined as $\lambda_t^{*}=\pi_t^{*}/L^{*}$, is chosen to approximate the expectation of true land shadow prices. The profit π_t^{*} is calculated as in equation (1) where the optimal resource allocations are obtained by solving the statistical model (13) *without* optimisation errors.

The reason for introducing additional information on land shadow prices is as follows: applying the unmodified objective function to the model (14), i.e. only minimising the sum of squared optimisation errors, would result in identification problems for the parameter estimates. This means that the parameters estimates would be over- or underestimated. This inefficiency of estimation might be compensated by choosing a sufficiently large sample size. However, the current setting and the computation capacity do not allow sample sizes beyond T>200. Thus, one cannot clearly observe that the estimates converge to true values with increasing data information. However, since the land shadow prices in the estimation model (14) can also be expressed as implicit functions of the CES parameters, utilising reasonable information on the true land shadow prices could achieve a precise and simultaneous estimation of both land shadow prices and CES parameters. Furthermore, in a real world application it is more likely to find approximations of true value for land shadow prices (e.g. land tenure prices) than for CES parameters.

Apart from the standard deviations for optimisation errors, the same settings for Monte Carlo simulations with measurement errors in terms of output and input dimension, sample size, standard deviation size and number of Monte Carlo iterations are applied here as well. Also, ARMSD is used to evaluate the consistency of the estimation approach.

The normally distributed random optimisation errors are defined as $\varepsilon_{ijt}^{o*} \sim N\left[0, \left(\sigma_{ij}^{o}\right)^2\right]$ with mean 0 and standard deviations σ_{ij}^{o}. The standard deviations are defined proportionally to the standard deviations of the marginal revenue of land and fertiliser σ_{ij}^{g} which is defined as

$$\sigma_{ij}^{g} = \sqrt{\left[\sum_{t=1}^{T}\left(g_{ijt} - \bar{g}_{ij}\right)^2\right] / \left(t-1\right)}$$

with

$$\bar{g}_{ij} = \sum_{t=1}^{T} g_{ijt} / \left(t-1\right)$$

$$g_{ijt} = p_{it}^{*}\theta_{i}^{*}\nu_{i}^{*}\left[\sum_{j=1}^{J}\beta_{ij}^{*}\left(l_{ijt}\right)^{-\rho_i^{*}}\right]^{\left(-\nu_i^{*}/\rho_i^{*}\right)-1}\beta_{ij}^{*}\left(l_{ijt}\right)^{-\rho_i^{*}-1}. \qquad (15)$$

The true optimum resource allocations l_{jit} in equation (15) are obtained by solving the statistical model (13) without optimisation errors. Finally, ε_{ijt}^{o} can be specified as $\varepsilon_{ijt}^{o*} \sim N\left[0, \left(\delta \cdot \sigma_{ij}^{o}\right)^2\right]$ where the multiplier δ proportionates the size of optimisation errors.

3.4 Discussion of results

Monte Carlo simulations are performed for the two estimation approaches with measurement and optimisation errors, respectively. For each error structure, two sizes of standard deviations are considered. The indicator ARMSD is calculated for the estimates of both CES parameters and land shadow prices at each sample size. The values of the indicator are presented and discussed in this section.

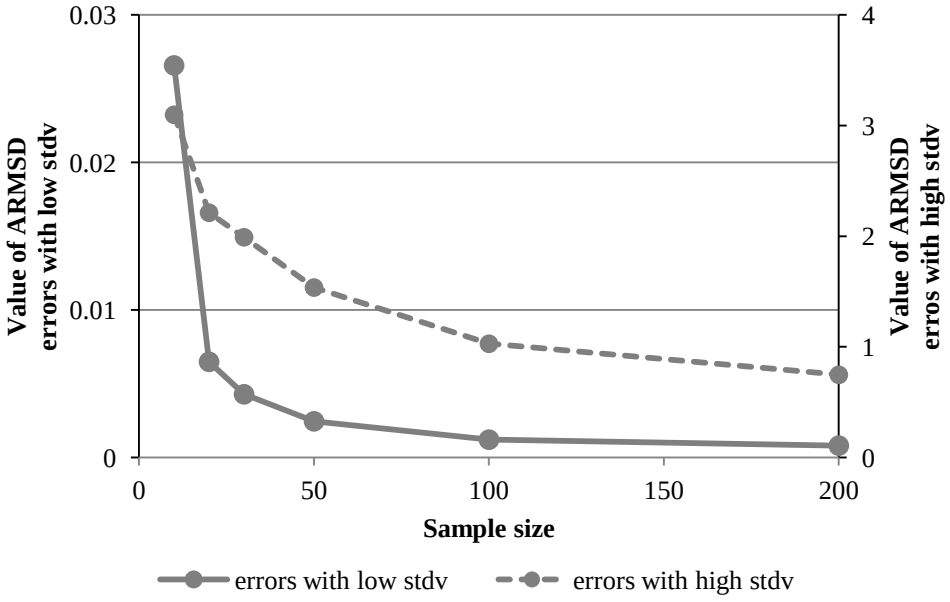

Figure 1. ARMSD of the estimated CES parameters with measurement errors

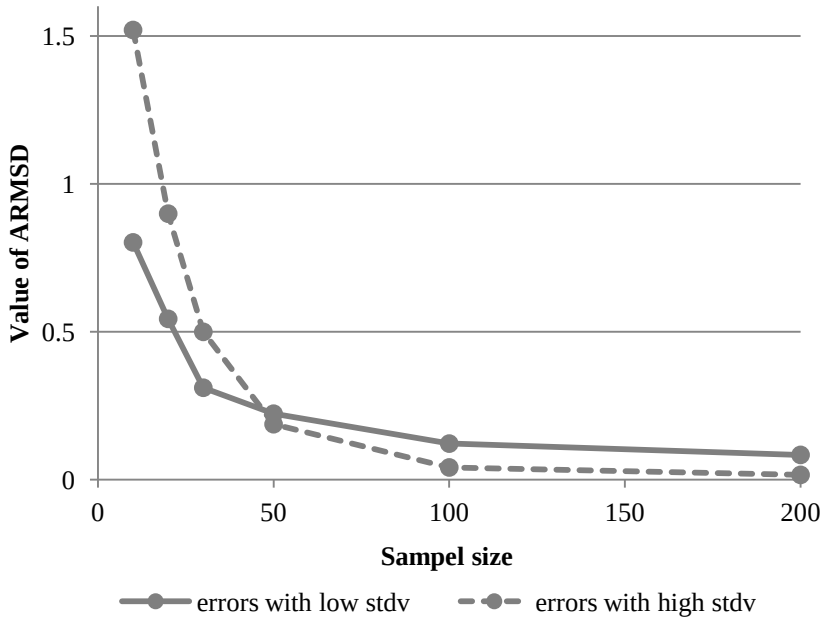

Figure 2. ARMSD of the estimated land shadow prices with measurement errors

Figure 1 and Figure 2 present the evaluation results for the estimation approach with measurement error structure. ARMSD of both CES parameters (Figure 1) and land shadow prices (Figure 2) decrease with increasing sample size, indicating consistency of the estimator. ARMSD in both figures reach negligible values at a sample size of 200 except for the CES parameters with errors with high standard deviation (denoted by the dashed line in Figure 1). It is difficult to judge based on Figure 1 whether it would eventually converge to zero. Evidence from larger sample sizes are missing, as Monte Carlo simulations with $K = 1000$ iterations and $T > 200$ observations requires computing capacity beyond the possibilities of this study. Nevertheless, the results from errors with low standard deviations in both figures indicate the consistency of the estimator. Therefore, it is assumed with confidence that the consistency of the estimator is also ensured for the case with high standard deviations. Errors with higher standard deviations should create larger sampling variance in the estimation. This phenomenon can be well observed in Figure 1 with $T > 10$ but not so clearly in Figure 2 (with $T < 50$).

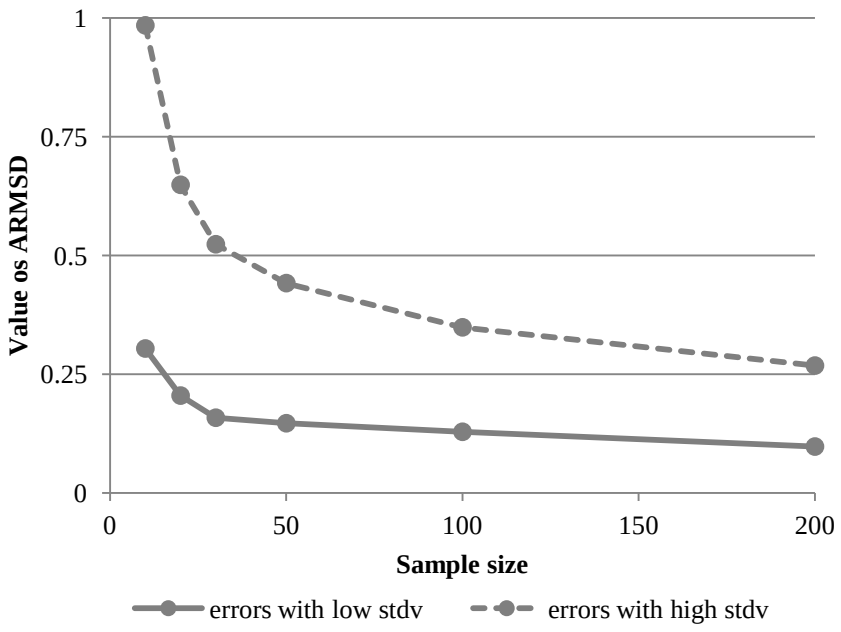

Figure 3. ARMSD of the estimated CES parameters with optimisation errors

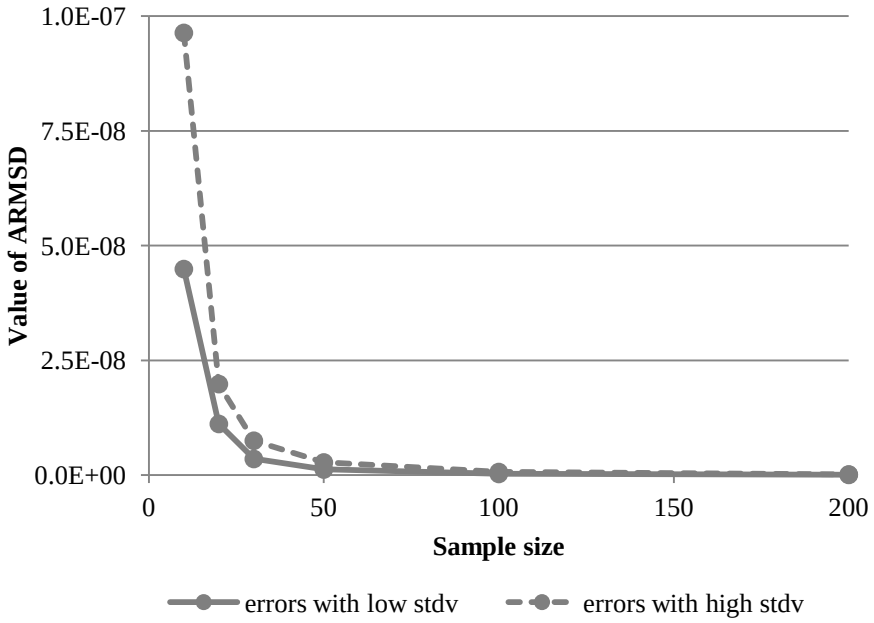

Figure 4. ARMSD of the estimated land shadow prices with optimisation errors

Evaluation results for the estimation approach with optimisation errors are illustrated in Figure 3 and Figure 4. The consistency of the estimator is indicated by the decreasing ARMSD with increasing sample size in both figures. The low magnitude of ARMSD in Figure 4 suggests a very precise estimation of shadow prices of land. This could be explained as the effect of applying additional information on land shadow prices in the estimation approach. Bounded by the limitations in terms of computational capacity as mentioned above, Figure 3 offers no evidence whether ARMSD will eventually converge to zero. However, given the precise estimation of land shadow prices, this is assumed to be the case with sufficiently larger sample size.

Removing the prior information on land shadow prices from the estimation approach will result in identification problems for both CES parameters and land shadow prices. Results with this ‘incorrect’ estimation approach are presented and discussed below.

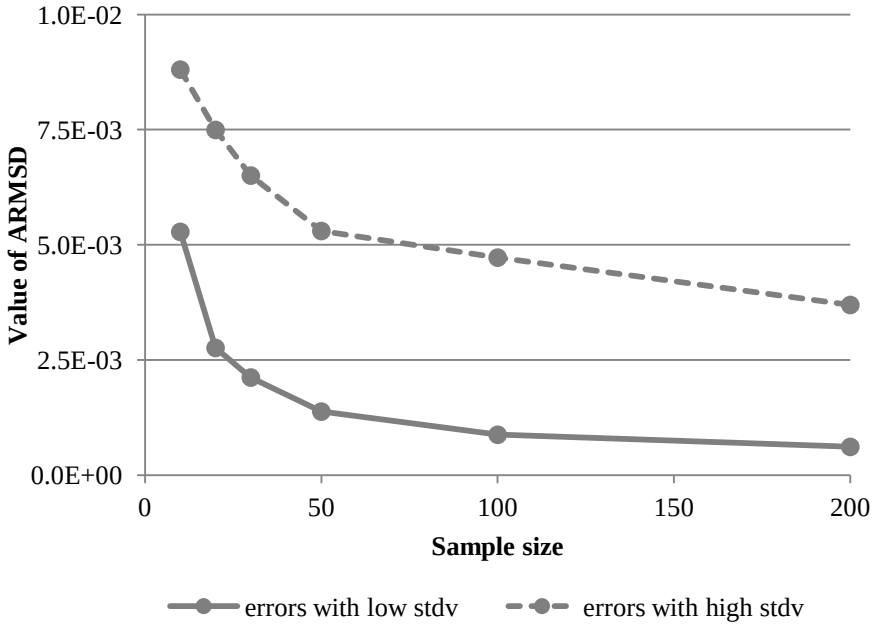

Figure 5. ARMSD of the estimated land shadow prices with identification problem categorised by optimisation errors with high and low standard deviations

The magnitude of ARMSD for the estimates of land shadow prices with identification problem, (0.0E+00, 1.0E-02) as shown in Figure 5, is 10,000 times larger than those obtained from the 'correct' estimates, (0.0E+00, 1.0E-07) as shown in Figure 4. This indicates a much less precise estimation without additional information on land shadow prices, and reflects the mentioned identification problem for the estimated CES parameters.

For the estimated land shadow prices, 410,000 estimates are obtained in total from Monte Carlo simulations with $K = 1000$ iterations for each sample size $T = (10, 20, 30, 50, 100, 200)$. An indicator s_t is calculated for each estimate to measure the accuracy of estimation as $s_t = \left[\left(\lambda_t - \lambda_t^* \right) / \lambda_t^* \right] \cdot 100$. It is defined as the distance between estimates and true values divided by the true values in percentage. A positive s_t suggests overestimation and a negative one means underestimation. The greater its absolute value is, the more severe the identification problem is. We assume [-10%, 10%] to be the tolerance range for a precise or 'correct' estimation.

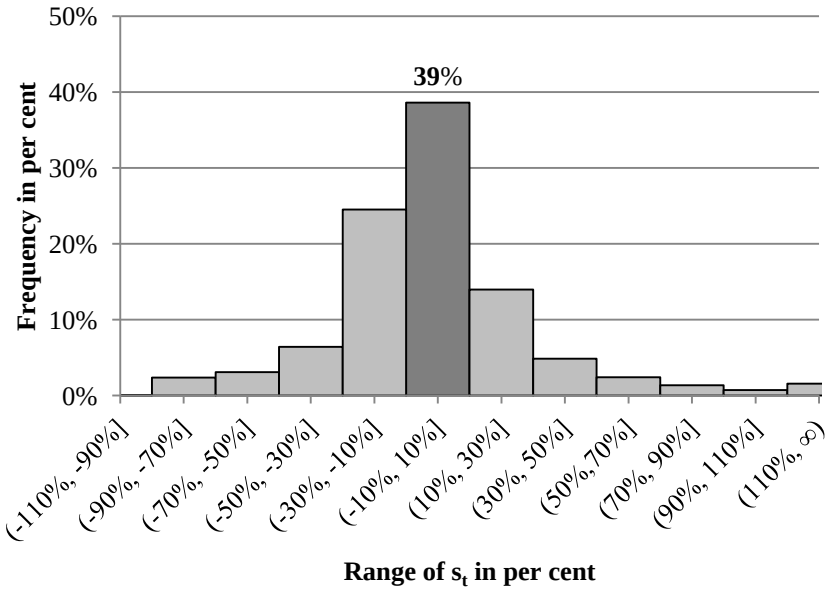

Figure 6. Over- and underestimation of the land shadow prices (optimisation errors with low standard deviations)

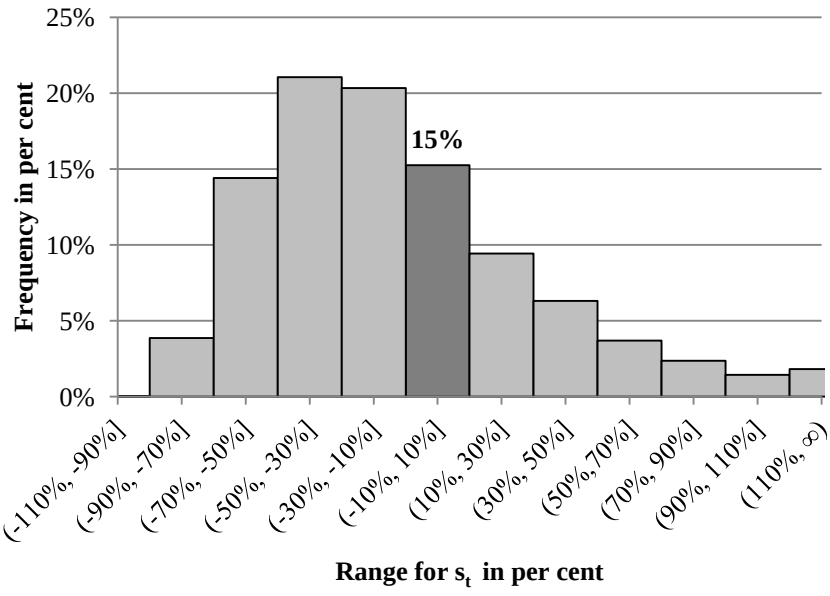

Figure 7. Over- and underestimation of the land shadow prices (optimisation errors with high standard deviations)

The frequency of over- and underestimation are indicated by the histograms in Figure 6 and Figure 7. The horizontal axes specify the range, whereas the vertical axes indicate the percentage[2] of s_t falling into the corresponding range. The highlighted bars in both figures give the percentage of the 'correct' estimation. For errors with low standard deviations it is roughly 39% as shown in Figure 6, which means that about 61% of the land shadow prices are over- or underestimated. Increasing the standard deviation of the errors reduces the portion of 'correct' estimations from 39% to 15% as shown in Figure 7. In other words, 85% of the estimated land shadow prices cannot be 'correctly' identified. The systematic identification problem is thus strongly present among estimates of land shadow prices.

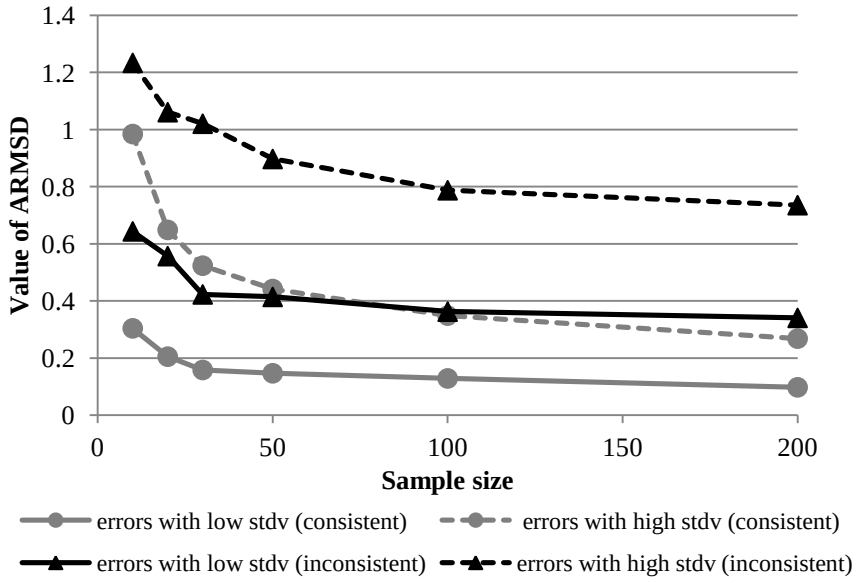

Figure 8. Comparing ARMSD of the estimated CES parameters obtained by estimating with optimisation errors including and excluding additional information on land shadow prices

[2] Normally, frequency is the indicator in a histogram. Here it is converted into a percentage for a simple and clear view. It is calculated as the frequency for the current range divided by the total number of estimates.

The same identification problem can also be observed with the estimated CES parameters. Figure 8 shows that ARMSD are notably higher with the 'incorrect' estimation approach (denoted by black lines with triangle markers) than with the consistent approach (same as in Figure 3 and denoted by grey lines with round markers).

3.5 Conclusion and outlook

In this study, a mathematical programming model with crop-specific CES production functions is econometrically estimated using its first-order conditions directly as estimating equations and applying a least squares procedure. Monte Carlo simulations with measurement and optimisation errors are carried out to evaluate the consistency of the estimation procedure. The two error types are distinguished by interpretation and specification. The Monte Carlo simulation results show the consistency of the estimation procedure with measurement errors. In the case of optimisation errors, modification of the estimator by introducing additional information on land shadow prices is necessary, as it would otherwise result in under- or overestimation problems.

We acknowledge the following limitations of our study. First, we were only concerned with consistency. Yet our estimation approach cannot claim to be 'efficient' in the statistical sense. For this, iterative procedures with inverse covariance weighting are likely to be required. Both theoretical and further stochastic simulation efforts may provide advances in this respect. Second, a statistical inference procedure for the estimated parameters is missing. This is an important issue for empirical application, as no measure for estimator accuracy is offered. Third, this study does not provide a real world application. Chapter 4 further elaborates on the second of the mentioned limitations by developing and evaluating a statistical inference procedure within the estimation context considered here.

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Chapter 4
Statistical inference for econometric mathematical programming models[1]

Abstract: Over the last two decades, the agricultural economics literature introduced and implemented the estimation of constrained optimisation models connecting mathematical programming models and econometrics. Statistical inference on parameter estimates, however, was not considered yet, as the estimation itself was already computationally demanding. Here, we explore this possibility developing and testing bootstrap algorithms for optimisation models with different error structures. This allows to calculate confidence intervals for estimated parameters. Monte Carlo simulation is used to evaluate bootstrap procedure showing promising results regarding the accuracy of the generated confidence intervals.

Keywords: econometric mathematical programming, errors in variables, bootstrap, statistical inference

[1] An early version of this study was presented at the XV EAAE congress as: Zhang, Y. and Heckelei, T. (2017). Statistical inference for Econometric Mathematical Programming Models, *selected paper at XV EAAE congress, August 28 – September 1st, Parma, Italy.*

4.1 Introduction

Heckelei and Wolff (2003) suggest Econometric Mathematical Programming (EMP) as a general alternative to Positive Mathematical Programming (PMP) model. The term EMP originates from Buysse et al. (2007b). The approach econometrically estimates parameters of a mathematical programming model using the model's optimality conditions as estimating equations. Using this approach, Jansson and Heckelei (2011) estimate behavioural parameters of a primal model of regional crop supply in the European Union using the time-series data in the CAPRI (Britz and Witzke 2014) database. Even though their primary goal is to provide an empirical parameterisation rather than to test the functional relationships of the CAPRI model, they point at the desirability of confidence region for the estimated parameters. Buysse et al. (2007a) also argue that EMP mostly focuses on the estimation or calibration of unknown parameters using all available information. However, the empirical reliability of the results is questionable due to the lack of statistical inference. Heckelei et al. (2012) suggest the conceptual possibility of bootstrapping GME models. They also state that the major difficulty of such exercise is the highly demanding nature of computation required.

The bootstrap is originally proposed by Efron (1979). It is a resampling method which assigns measures of accuracy to parameter estimates based on the simulated sampling distribution of the statistic of interest. The objective of this study is to develop a bootstrap procedure for drawing statistical inference from EMP model parameters to assess the performance of the approach. We provide a) the algorithm for bootstrapping confidence intervals and b) the coverage probability of such confidence interval through Monte Carlo simulation. The EMP model introduced in chapter 3 is implemented here. The consistency of the estimation approach with the EMP model has been shown in chapter 3.

Section 4.2 first briefly illustrates the fundamental concept of bootstrap. Then it revisits some basics on different bootstrap sampling procedures and bootstrapping confidence intervals applied in this chapter. Section 4.3 presents the detailed layout of EMP models with two different error

structures. In section 4.4, the bootstrap approaches are applied to the EMP models. The detailed procedure to obtain the bootstrapped confidence intervals for the parameter estimates and its algorithm are presented, followed by the algorithm for evaluating the confidence interval calculation using Monte Carlo simulation. Both algorithms are designed for all four scenarios by combining two different sampling approaches with two error types. Section 4.5 shows the findings on the coverage probability of the bootstrapped confidence interval before concluding.

4.2 **Bootstrap**

4.2.1 *Concept*

The bootstrap is advocated first by Efron (1979) and it is inspired by earlier work on the jackknife by Quenouille (1949, 1956) and Tukey (1958). It creates a new tool for statistical analysis based on simulation. The bootstrap is very intuitive and thus appealed to practitioners. Despite the substantial development since its first appearance, the basic concept of bootstrap remains the same. Sample data is treated as 'population' and one creates a bootstrap sample by resampling the sample data. The inference on the resampled data gives an approximation of the inference on the population. Thus, one could gain some insight into population by utilising only its sample data. Like the name "bootstrapping" adequately suggests: one lifts himself up by pulling his bootstrap. It is a finite sample alternative to calculate the asymptotic distribution of an estimator statistic.

Following notation is applied throughout this chapter: lowercase bold letters are used to denote vectors. Parameters are denoted by Greek letters. A hat on a letter indicates an estimate, while a tilde indicates a bootstrapped sample or bootstrapped estimate. The capital letters F and G stand for populations. Suppose one observes a vector of random sample $\mathbf{x}=(x_1, x_2, \cdots, x_n)$ from an unknown probability distribution F. The goal is to estimate a parameter of interest $\phi = t(F)$ based on the observed data $\mathbf{x}$ which can be calculated as a point estimate $\hat{\phi} = g(\mathbf{x})$. The discrete distribution which assigns probability $1/n$ on each $x_i, i = 1, 2, \ldots, n$ is

defined to be the empirical distribution function $\hat{F}$. A *bootstrap* sample $\tilde{\mathbf{x}} = (\tilde{x}_1, \tilde{x}_2, \ldots, \tilde{x}_n)$ is defined to be a random sample of size n drawn from $\hat{F}$ as

$$\hat{F} \rightarrow (\tilde{x}_1, \tilde{x}_2, \ldots, \tilde{x}_n). \tag{1}$$

The bootstrap data points of $\tilde{\mathbf{x}} = (\tilde{x}_1, \tilde{x}_2, \ldots, \tilde{x}_n)$ are obtained by random sampling *with replacement* from the 'population' $\mathbf{x} = (x_1, x_2, \cdots, x_n)$. By applying the same function $g(\cdot)$ to $\tilde{\mathbf{x}}$, the bootstrap replicate of $\hat{\phi} = s(\mathbf{x})$ is defined as

$$\tilde{\hat{\phi}} = g(\tilde{\mathbf{x}}). \tag{2}$$

By drawing $b = 1, \ldots, B$ independent bootstrap samples $\tilde{\mathbf{x}}_b = (\tilde{\mathbf{x}}_1, \tilde{\mathbf{x}}_2, \ldots, \tilde{\mathbf{x}}_B)$ one can correspondingly obtain B bootstrapped estimates $\tilde{\hat{\phi}}_b = \left(\tilde{\hat{\phi}}_1, \tilde{\hat{\phi}}_2, \ldots, \tilde{\hat{\phi}}_B\right)$.

4.2.2 *Resampling approach*

Bootstrap has many variations regarding the resampling approach, for example parametric versus non-parametric bootstrapping. What differentiates them is where the resampled or 'surrogate' data comes from: parametric bootstrapping relies on the parametric model to generate samples, whereas the non-parametric variation directly resamples the data.

While parametric bootstrapping requires an assumption on the distribution from which the surrogate data is resampled, no assumption is needed for non-parametric bootstrapping. Non-parametric approach treats the sample data as the least prejudiced estimate of the underlying distribution, since anything else might impose biases and thus could be misleading. Thus, surrogate data is generated through random sampling of observations *with replacement*. It is a matter of how much one trusts the parametric model, and whether additional assumptions are desirable. For our study, we assume that there is no knowledge about the distribution and we exclusively elaborate the non-parametric approach. There are also many

variations regarding the non-parametric approach. They yield confidence intervals with different numerical accuracy. Two resampling approaches are considered: 1) residual resampling and 2) case resampling. Suppose a linear regression model is defined as follows,

$$\mathbf{y} = \phi \mathbf{x} + \boldsymbol{\varepsilon} \tag{3}$$

with $\mathbf{y} = (y_1, y_2, \cdots, y_n)$ being the dependent variables, $\mathbf{x} = (x_1, x_2, \cdots, x_n)$ the independent variables, ϕ the parameters and $\boldsymbol{\varepsilon} = (\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_n)$ the unobserved residuals. The detailed steps for these two variations are listed below:

Residual Resampling

1. Estimate model (3) with Ordinary Least Square (OLS) estimator and obtain the residual as $\hat{\boldsymbol{\varepsilon}} = \mathbf{y} - \hat{\phi}\mathbf{x}$.
2. Draw B independent bootstrap samples $\tilde{\boldsymbol{\varepsilon}}_b$ by randomly resampling $\hat{\boldsymbol{\varepsilon}}$ with replacement based on index k [2].
3. Calculate bootstrap samples $\tilde{\mathbf{y}}_b$ as $\tilde{\mathbf{y}}_b = \hat{\phi}\mathbf{x} + \tilde{\boldsymbol{\varepsilon}}_b$.
4. Estimate model (3) with $(\tilde{\mathbf{y}}_b, \mathbf{x}_b)$ to obtain B bootstrapped estimates $\tilde{\hat{\phi}}_b$

Case Resampling

1. Draw B independent bootstrap samples $(\tilde{\mathbf{y}}_b, \tilde{\mathbf{x}}_b)$ by random resampling $(\mathbf{y}, \mathbf{x})$ with replacement (with the same index k from above for both $\mathbf{y}$ and $\mathbf{x}$).

[2] The random index k is a string of N real numbers generated by random sampling with replacement the string of numbers $(1,2,\ldots,N)$. Each number of k stands for the nth element from the original sample. The index k varies for each bootstrap replication.

2. Estimate model (3) with bootstrap samples $\left(\tilde{\mathbf{y}}_b, \tilde{\mathbf{x}}_b\right)$ to obtain B bootstrapped estimates $\tilde{\hat{\phi}}_b$.

Residual resampling trusts that the model has the correct shape of regression function, while case resampling does not. If one compares these two variations based on the same data for the same statistic and assume that the regression model is true, then resampling residuals yields generally better numerical accuracy for the same bootstrap sample size (Horowitz 2001).

4.2.3 *Bootstrapped confidence intervals*

The bootstrap provides the possibility to estimate standard error of the statistic of interest and confidence intervals. The standard error of the bootstrapped estimate $\tilde{\hat{\phi}}_b$ from above can be estimated by the sample standard deviation of the B replications as

$$\text{se} = \sqrt{\sum_{b=1}^{B}\left(\tilde{\hat{\phi}}_b - \bar{\tilde{\hat{\phi}}}\right)^2 / \left(B-1\right)}$$

$$\text{where} \qquad \bar{\tilde{\hat{\phi}}} = \sum\nolimits_{b=1}^{B} \tilde{\hat{\phi}}_b / B\,. \tag{4}$$

The percentile bootstrap confidence interval and the basic bootstrap confidence interval, two textbook standard methods, are chosen for this study. They are referred from now on as *percentile* and *basic* method for the sake of simplicity. Due to already significant computational challenges and little relevance for the evaluation of the general approach, bootstrap confidence interval methods requiring more than one layer of bootstrap iteration are not considered for calculation or evaluation. Alternative methods are discussed in the conclusion.

Consider the following standard normal symmetric confidence interval

$$\left[\hat{\phi} - z^{(1-\alpha)} \cdot \text{se}, \quad \hat{\phi} - z^{(\alpha)} \cdot \text{se}\right] \tag{5}$$

with confidence level being $100\cdot(1-2a)$ in percentage. The term z^{α} denotes the $100\cdot a$ th percentile point of a $N(0,1)$ distribution of a standard normal table. The lower and upper limit of this interval could be interpreted as the $100\cdot a$ th and $100\cdot(1-a)$ th percentile of some random variable $\tilde{\hat{\phi}}$ drawn from the distribution $N\left(\hat{\phi},\text{se}^{2}\right)$, i.e. the lower limit $\hat{\phi}_{lo}$ and upper limit $\hat{\phi}_{up}$ could be defined as

$$\hat{\phi}_{lo}=\tilde{\hat{\phi}}^{*(\alpha)}=100\cdot\alpha^{th}\text{ percentile of }\tilde{\hat{\phi}}^{*}\text{'s distribution}$$

$$\hat{\phi}_{up}=\tilde{\hat{\phi}}^{*(1-\alpha)}=100\cdot(1-\alpha)^{th}\text{ percentile of }\tilde{\hat{\phi}}^{*}\text{'s distribution}. \qquad (6)$$

This approximation of $\hat{\phi}$ can be applied analogously to bootstrapped estimate $\tilde{\hat{\phi}}$. Given the bootstrap data set $\tilde{\mathbf{x}}$, one can obtain the bootstrapped estimate $\tilde{\hat{\phi}}_{b}$. The cumulative distribution function of $\tilde{\hat{\phi}}$ is denoted by $\hat{H}$. The $(1-2\alpha)$ percentile interval could be formulated as

$$\left[\hat{\phi}_{\%,lo},\hat{\phi}_{\%,up}\right]=\left[\hat{H}^{-1}(\alpha),\hat{H}^{-1}(1-\alpha)\right]. \qquad (7)$$

And by definition $\hat{H}^{-1}(\alpha)=\tilde{\hat{\phi}}^{(\alpha)}$ where $\tilde{\hat{\phi}}^{(\alpha)}$ is the $100\cdot a$ th percentile of the distribution of $\tilde{\hat{\phi}}$. Equation (7) could be rewritten as

$$\left[\hat{\phi}_{\%,lo},\hat{\phi}_{\%,up}\right]=\left[\tilde{\hat{\phi}}^{(\alpha)},\tilde{\hat{\phi}}^{(1-\alpha)}\right]. \qquad (8)$$

Note that equations (7) and (8) represent the ideal case with infinite bootstrap replications. When applying finite number of B replications, one obtains B replicates of bootstrapped estimate $\tilde{\hat{\phi}}_{b}$ using B independent bootstrap samples $\tilde{\mathbf{x}}_{b}$. Defining $\tilde{\hat{\phi}}_{B}^{(\alpha)}$ with subscript capital letter B as the $B\cdot a$ th value in the ordered list of the B bootstrapped estimate $\tilde{\hat{\phi}}$. It is also the $100\cdot a$ th empirical percentile of the $\tilde{\hat{\phi}}_{b}$. Similarly, $\tilde{\hat{\phi}}_{B}^{(1-\alpha)}$ is the

$100 \cdot (1-\alpha)$ th empirical percentile. The *approximate* $(1-2\alpha)$ percentile interval is defined as

$$\left[\hat{\phi}_{\%,lo}, \hat{\phi}_{\%,up}\right] \approx \left[\tilde{\hat{\phi}}_B^{(\alpha)}, \tilde{\hat{\phi}}_B^{(1-\alpha)}\right] \tag{9}$$

and it is referred as ***percentile confidence interval***. Even though the percentile interval is only approximation, the central limit theorem implies that the percentile interval would close to the standard normal intervals as $B \to \infty$. Efron and Tibshirani (1994) demonstrate that the percentile interval is generally preferable to the standard normal interval. The percentile method is also range-preserving. It means that it produces lower and upper limits which are inside the theoretical boundary for the parameter.

The ***basic bootstrap confidence interval*** is constructed in the following way. Consider the following equation:

$$1-2\alpha = \Pr\left(\phi^* \in C\right) \tag{10}$$

where C denotes confidence interval. The theoretical probability that C contains the 'true' value ϕ^* is $(1-2\alpha)$. Manipulating and rearranging the following equation for the lower limit $\hat{\phi}_{lo}$:

$$\begin{aligned} \alpha &= \Pr\left(\hat{\phi}_{lo} \geq \phi^*\right) \\ &= \Pr\left(\hat{\phi}_{lo} - \hat{\phi} \geq \phi^* - \hat{\phi}\right) \\ &= \Pr\left(\hat{\phi} - \hat{\phi}_{lo} \leq \hat{\phi} - \phi^*\right). \end{aligned} \tag{11}$$

Similarly for the upper limit:

$$\begin{aligned} \alpha &= \Pr\left(\hat{\phi}_{up} \leq \phi^*\right) \\ &= \Pr\left(\hat{\phi}_{up} - \hat{\phi} \leq \phi^* - \hat{\phi}\right) \\ &= \Pr\left(\hat{\phi} - \hat{\phi}_{up} \geq \hat{\phi} - \phi^*\right). \end{aligned} \tag{12}$$

Adopting the bootstrap principle, the bootstrap estimate $\tilde{\hat{\phi}}$ could approximate the point estimate $\hat{\phi}$, whereas the point estimate $\hat{\phi}$ could represent the 'true' value ϕ^*. Thus, the distribution of $\left(\tilde{\hat{\phi}} - \hat{\phi}\right)$ can then be used to estimate the distribution of $\left(\hat{\phi} - \phi^*\right)$. With knowledge on the distribution of $\left(\tilde{\hat{\phi}} - \hat{\phi}\right)$ and the point estimate $\hat{\phi}$, (11) and (12) can be further rearranged as

$$\begin{aligned} \alpha &= \Pr\left(\hat{\phi} - \hat{\phi}_{lo} \le \hat{\phi} - \phi^*\right) \\ &\approx \Pr\left(\hat{\phi} - \hat{\phi}_{lo} \le \tilde{\hat{\phi}}^{(1-\alpha)} - \hat{\phi}\right) \\ &\approx \Pr\left(2\hat{\phi} - \hat{\phi}_{lo} \le \tilde{\hat{\phi}}^{(1-\alpha)}\right) \\ &\approx \Pr\left(\hat{\phi}_{lo} \ge 2\hat{\phi} - \tilde{\hat{\phi}}^{(1-\alpha)}\right) \end{aligned} \tag{13}$$

and

$$\begin{aligned} \alpha &= \Pr\left(\hat{\phi} - \hat{\phi}_{up} \ge \hat{\phi} - \phi^*\right) \\ &\approx \Pr\left(\hat{\phi} - \hat{\phi}_{up} \ge \tilde{\hat{\phi}}^{(\alpha)} - \hat{\phi}\right) \\ &\approx \Pr\left(2\hat{\phi} - \hat{\phi}_{up} \ge \tilde{\hat{\phi}}^{(\alpha)}\right) \\ &\approx \Pr\left(\hat{\phi}_{up} \le 2\hat{\phi} - \tilde{\hat{\phi}}^{(\alpha)}\right). \end{aligned} \tag{14}$$

And ultimately the basic confidence interval is defined as

$$\left[2\hat{\phi} - \tilde{\hat{\phi}}^{(1-\alpha)}, 2\hat{\phi} - \tilde{\hat{\phi}}^{(\alpha)}\right]. \tag{15}$$

Unlike the percentile method, basic method is not range-preserving. The quantity $\left(\hat{\phi} - \phi^*\right)$ is not pivotal, so the interval (15) is not very accurate (Efron and Tibshirani 1994; Canty et al. 1996). More accurate method, like

the variance stabilised bootstrap-*t* by Tibshirani (1988), replies on $\left(\hat{\phi}-\phi^{*}\right)/\,\mathrm{se}$ instead of $\left(\hat{\phi}-\phi^{*}\right)$.

The confidence interval methods could be evaluated by the so-called "goodness" criteria by Efron and Tibshirani (1994) defined as "…*the bootstrap intervals should closely match exact confidence intervals in those special situations where statistical theory yields an exact answer, and should give dependably accurate coverage probabilities in all situations*..." On the one hand, the basic and percentile methods do not deliver ideal performance in terms of these "goodness" criteria. There exist refinements of these methods providing better confidence intervals. To be more specific, they have the advantage of being second-order accurate[3], while the basic and percentile methods are only first-order accurate (Efron and Tibshirani 1994). Although a second layer of bootstrap replication is often required by these superior methods, whereas one layer is sufficient with the chosen simple methods for this study. Therefore, there is a trade-off between the quality of the interval in terms of the "goodness" criteria and the required computational time.

The seasons for choosing the simpler methods are twofold: 1) for the objectives of this study, the advantage of obtaining better bootstrapped confidence interval does not offer any more general insights; 2) one objective of this study is to evaluate the bootstrapped confidence interval by applying Monte Carlo simulation. Using the superior methods required a second layer of bootstrap would results in computational demand beyond our capacity. This could be shown in the example below:

Taking bootstrap-*t* as an example, it is first introduced by Efron (1981). Singh (1981) applies Edgeworth theory to the bootstrap-*t* interval. This is

[3] Second-order accurate means the errors in matching the true probably coverage decrease to zero at rate $1/n$ with n being the sample size. If the errors in matching is $1/\sqrt{n}$, which is an order of magnitude larger, it is called first-order accurate.

the first bootstrap confidence interval developed with second-order accuracy and it requires calculation of standard error of each bootstrap estimate and of all bootstrap estimates. This implies a second layer of $b_2 = 1, \ldots, B_2$ replications for each of the $b_1 = 1, \ldots, B_1$ replications from the first layer. If $B_2 = 25$ were enough to obtain the standard error and assuming $B_1 = 1000$, the number of total replications would be $B_1 + B_1 \times B_2 = 26,000$, which is feasible for empirical applications. However, applying the bootstrap-*t* method to Monte Carlo simulation would result in totally $S + S \times \left(B_1 + B_1 \times B_2 \right) = 26,001,000$ replications assuming $S = 1000$.

Thus, only after establishing the principle validity of bootstrapping confidence intervals, one can build upon this and apply refined methods in empirical applications where the objective is to obtain better intervals and no Monte Carlo simulation is required. Many efforts have been made to reduce the computational demand. This leads to method like Bias-Corrected and accelerated (BCa) method (Efron 1987). Interestingly, despite having second-order accuracy, methods like bootstrap-*t* and BC_a are not often used. In fact, standard interval is the most preferred choice in practice even by experienced statisticians (Efron 2003).

4.3 **EMP Model**

This section revisits the EMP model from chapter 3 which comprises the statistical model for the Data-Generating Process (DGP) and the Econometric Estimation Model (EEM). First, considering the following single farm economic model:

$$\max_{l_{ij} \geq 0, q_i \geq 0} \pi = \sum_{i=1}^{I} \left(p_i q_i - w_{i2} l_{i2} \right) \tag{16}$$

subject to

$$q_i = \theta_i \left[\sum\nolimits_{j=1}^{J} \beta_{ij} \left(l_{ij} \right)^{-\rho_i} \right]^{-\nu_i / \rho_i} \quad \forall i = 1, \ldots, I \tag{17}$$

$$\sum_{i=1}^{I} l_{i1} = L \quad [\lambda] \tag{18}$$

$$l_{ij} \geq 0 \quad [\gamma_{ij}]. \tag{19}$$

The same notation from chapter 3 is applied here as well: output is denoted by i, input (land and fertiliser) by j, profit by π, output prices by p_i, input fertiliser prices by w_{i2} (land is indicated by $j=1$ and fertiliser by $j=2$), CES production functions by q_i, resource allocations l_{ij}, total resource endowment L, CES parameters $\Phi = (\theta_i, \beta_{ij}, \rho_i, \nu_i)$ and Lagrangian multipliers λ and γ_{ij}. The farmer behaves rationally and maximises his profit by optimising resource allocations under resource constraints. Profit is defined as revenue minus variable fertiliser cost as shown in (16). The production technology is specified by the CES production function (17). Land is the only binding resource constraint, which renders land price endogenous as shown in (18). Resource allocations must be positive as in (19).

The Lagrange function and the First-Order Conditions (FOC) of the economic model function as optimisation conditions or econometric criteria for the statistical model and EEM in the following subsections. They are specified as follow:

$$\mathcal{L} = \sum_{i=1}^{I}\left(p_i \theta_i \left[\sum_{j=1}^{J} \beta_{ij} \left(l_{ij}\right)^{-\rho_i} \right]^{-\nu_i/\rho_i} - w_{i2} l_{i2} \right) + \lambda \left(L - \sum_{i=1}^{I} l_{i1} \right) + \sum_{j=1}^{J} \gamma_{ij} l_{ij} \tag{20}$$

$$\frac{\partial \mathcal{L}}{\partial l_{i1}} = p_i \theta_i \nu_i \left[\sum_{j=1}^{J} \beta_{ij} \left(l_{ij}\right)^{-\rho_i} \right]^{(-\nu_i/\rho_i)-1} \beta_{i1} \left(l_{i1}\right)^{-\rho_i - 1} - \lambda - \gamma_{i1} = 0 \tag{21}$$

$$\frac{\partial \mathcal{L}}{\partial l_{i2}} = p_i \theta_i \nu_i \left[\sum_{j=1}^{J} \beta_{ij} \left(l_{ij}\right)^{-\rho_i} \right]^{(-\nu_i/\rho_i)-1} \beta_{i2} \left(l_{i2}\right)^{-\rho_i - 1} - w_{i2} - \gamma_{i2} = 0 \tag{22}$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} = \sum_{i=1}^{I} l_{i1} = L \tag{23}$$

$$\frac{\partial L_i}{\partial \gamma_{ij}} = l_{ij} > 0 \;\perp\; \sum_{j=1}^{J} \gamma_{ij} l_{ij} = 0 . \tag{24}$$

Measurement and optimisation error structures, as their interpretations and specifications already explained in chapter 3, are considered in the DGP and in the estimation model. Combining with the two resampling approaches introduced above results in totally four scenarios: a) Optimisation Error with Residual Resampling (OE-RR), b) Optimisation Error with Case Resampling (OE-CR), c) Measurement Error with Residual Resampling (ME-RR) and d) Measurement Error with Case Resampling (ME-RR).

4.3.1 *EMP model with optimisation errors*

Optimisation errors occur in the optimisation process and thus are directly inserted into the FOCs of the economic model. Introducing the optimisation errors ε_{ijt}^{o} into the FOCs as optimisation conditions and adding the dimension for observations represented by the index t for sample size to all equations, the statistical model is formulated as follow

$$p_{it}^{*}\theta_i^{*}\nu_i^{*}\left[\sum_{j=1}^{J}\beta_{ij}^{*}\left(l_{ijt}\right)^{-\rho_i^{*}}\right]^{\left(-\nu_i^{*}/\rho_i^{*}\right)-1}\beta_{i1}^{*}\left(l_{i1t}\right)^{-\rho_i^{*}-1}-\lambda_t-\gamma_{i1t}-\varepsilon_{i1t}^{o}=0 \tag{25}$$

$$p_{it}^{*}\theta_i^{*}\nu_i^{*}\left[\sum_{j=1}^{J}\beta_{ij}^{*}\left(l_{ijt}\right)^{-\rho_i^{*}}\right]^{\left(-\nu_i^{*}/\rho_i^{*}\right)-1}\beta_{i2}^{*}\left(l_{i2t}\right)^{-\rho_i^{*}-1}-w_{i2t}^{*}-\gamma_{i2t}-\varepsilon_{i2t}^{o}=0 \tag{26}$$

$$\sum_{i=1}^{I} l_{i1t} = L^{*} \tag{27}$$

$$\sum_{j=1}^{J} \gamma_{ijt} l_{ijt} = 0 . \tag{28}$$

The exogenous variables are the p_{it}^{*}, w_{i2t}^{*}, L^{*} and $\Phi^{*} = \left(\theta_i^{*}, \beta_{ij}^{*}, \rho_i^{*}, \nu_i^{*}\right)$, while the endogenous variables are l_{ijt}, λ_t and γ_{ijt}. The superscript * indicates that the current item is exogenous or data. The DGP is done by drawing random optimisation errors from the statistical model. This step is

referred as ***Data Generation Process with Optimisation Error*** (DGP-OE) and it yields the *actual* resource allocations l_{ijt}^{*}.

An econometric, least squares objective function (29), FOCs (30) and (31) as econometric criteria, the theoretical parameter boundaries (32) together formulate the econometric estimation model:

$$\min_{\theta_{it},\beta_{ijt},\nu_{it},\rho_{it}} \sum_{i=1}^{I}\sum_{j=1}^{J}\sum_{t=1}^{T}\left[\left(\varepsilon_{ijt}^{o}\right)^{2}+\left(\lambda_{t}-\bar{\lambda}_{t}\right)^{2}\right] \tag{29}$$

subject to

$$p_{it}^{*}\theta_{i}\nu_{i}\left[\sum\nolimits_{j=1}^{J}\beta_{ij}\left(l_{ijt}^{*}\right)^{-\rho_{i}}\right]^{(-\nu_{i}/\rho_{i})-1}\beta_{i1}\left(l_{i1t}^{*}\right)^{-\rho_{i}-1}-\lambda_{t}-\gamma_{i1t}-\varepsilon_{i1t}^{o}=0 \tag{30}$$

$$p_{it}^{*}\theta_{i}\nu_{i}\left[\sum\nolimits_{j=1}^{J}\beta_{ij}\left(l_{ijt}^{*}\right)^{-\rho_{i}}\right]^{(-\nu_{i}/\rho_{i})-1}\beta_{i2}\left(l_{i2t}^{*}\right)^{-\rho_{i}-1}-w_{i2t}^{*}-\gamma_{i2t}-\varepsilon_{i2t}^{o}=0 \tag{31}$$

$$\theta_{it}>0;\quad 0<\beta_{ijt}<1;\quad 0<\nu_{it}<1;\quad \rho_{it}>0\,. \tag{32}$$

The land constraint is not necessary and thus excluded. The *actual* land allocations are obtained from the DGP-OE under the land constraint. They are handled as data in the estimation where no error terms are added to them. Thus, the land constraint is redundant.

Given the data p_{it}^{*}, w_{i2t}^{*}, l_{ijt}^{*} L^{*}, and $\bar{\lambda}_{t}$, the parameters to be estimated are $\Phi=\left(\theta_{i},\beta_{ij},\rho_{i},\nu_{i}\right)$, ε_{ijt}^{o} and λ_{t}. This step is referred as ***Econometric Estimation Model with Optimisation Error*** (EEM-OE). Prior information $\bar{\lambda}_{t}$ for the parameter $\bar{\lambda}_{t}$ are introduced in (29) to solve the systematic identification problem elaborated in chapter 3.

4.3.2 *EMP model with measurement errors*

Measurement errors are assumed to be related to the input resource allocations. The statistical model is defined as follow by adding the

dimension for observations and measurement errors ε_{ijt}^{m} to the economic model:

$$p_{it}^{*}\theta_{i}^{*}\nu_{i}^{*}\left[\sum_{j=1}^{J}\beta_{ij}^{*}\left(l_{ijt}-\varepsilon_{ijt}^{m*}\right)^{-\rho_{i}^{*}}\right]^{\left(-\nu_{i}^{*}/\rho_{i}^{*}\right)-1}\beta_{i1}^{*}\left(l_{i1t}-\varepsilon_{i1t}^{m*}\right)^{-\rho_{i}^{*}-1}-\lambda_{t}-\gamma_{i1t}=0 \quad (33)$$

$$p_{it}^{*}\theta_{i}^{*}\nu_{i}^{*}\left[\sum_{j=1}^{J}\beta_{ij}^{*}\left(l_{ijt}-\varepsilon_{ijt}^{m*}\right)^{-\rho_{i}^{*}}\right]^{\left(-\nu_{i}^{*}/\rho_{i}^{*}\right)-1}\beta_{i2}^{*}\left(l_{i2t}-\varepsilon_{i2t}^{m*}\right)^{-\rho_{i}^{*}-1}-w_{i2t}^{*}-\gamma_{i2t}=0 \quad (34)$$

$$\sum_{i=1}^{I}\left(l_{i1t}-\varepsilon_{i1t}^{m*}\right)=L^{*} \quad (35)$$

$$\sum_{j=1}^{J}\gamma_{ijt}\left(l_{ijt}-\varepsilon_{ijt}^{m*}\right)=0. \quad (36)$$

The p_{it}^{*}, w_{i2t}^{*}, L^{*} and $\Phi^{*}=\left(\theta_{i}^{*},\beta_{ij}^{*},\rho_{i}^{*},\nu_{i}^{*}\right)$ are exogenous, while l_{ijt}, λ_{t} and γ_{ijt} are endogenous. The four equations above with random measurement errors serve as optimisation conditions.

The statistical model yields the *optimal* resource allocation l_{ijt}^{**}. By subtracting randomly generated errors ε_{ijt}^{m*} from l_{ijt}^{**}, one can obtain the *actual* resource allocation l_{ijt}^{*} as $l_{ijt}^{*}=l_{ijt}^{**}-\varepsilon_{ijt}^{m*}$.This step is referred as the ***Data Generation Process with Measurement Error*** (DGP-ME).

The EEM with measurement errors is defined as follow

$$\min\sum_{t=1}^{T}\sum_{i=1}^{I}\sum_{j=1}^{J}\left(\varepsilon_{jit}^{m}\right)^{2} \quad (37)$$

subject to

$$p_{jt}^{*}\theta_{jt}\nu_{jt}\sum_{i=1}^{I}\left[\beta_{jit}\left(l_{jit}^{*}-\varepsilon_{jit}^{m}\right)^{-\rho_{jt}}\right]^{\left(-\nu_{jt}/\rho_{jt}\right)-1}\left[\beta_{j1t}\left(l_{j1t}^{*}-\varepsilon_{j1t}^{m}\right)^{-\rho_{jt}-1}\right]-\lambda_{t}-\gamma_{j1t}=0 \quad (38)$$

$$p_{jt}^{*}\theta_{jt}\nu_{jt}\sum_{i=1}^{I}\left[\beta_{jit}\left(l_{jit}^{*}-\varepsilon_{jit}^{m}\right)^{-\rho_{jt}}\right]^{\left(-\nu_{jt}/\rho_{jt}\right)-1}\left[\beta_{j1t}\left(l_{j2t}^{*}-\varepsilon_{j2t}^{m}\right)^{-\rho_{jt}-1}\right]-w_{jt}^{*}-\gamma_{j2t}=0 \quad (39)$$

$$\sum_{j=1}^{J}\left(l_{j1t}^{*}-\varepsilon_{j1t}^{m}\right)=L_{t}^{*} \quad (40)$$

$$\sum_{i=1}^{I} \gamma_{jit} \left(l_{jit}^{*} - \varepsilon_{jii}^{m} \right) = 0 \tag{41}$$

$$\theta_{jt} > 0; \quad 0 < \beta_{jit} < 1; \quad 0 < \nu_{jt} < 1; \quad \rho_{jt} > 0 . \tag{42}$$

The objective function (37) minimises the sum of the squared errors. This is subjected to the FOCs (38) and (39), the resource constraint (40), positive constraint (41) and theoretical boundary on parameters (42). The FOCs function as econometric criteria. Parameters $\Phi = \left(\theta_i, \beta_{ij}, \rho_i, \nu_i \right)$, ε_{ijt}^{m} and λ_t are estimated given the data p_{it}^{*}, w_{i2t}^{*}, l_{ijt}^{*} and L^{*}. This step is termed as ***Econometric Estimation Model with Measurement Error*** (EEM-ME).

4.4 Bootstrapping confidence intervals and evaluation

This section comprehensively illustrates the procedure of boostrapping the EMP models and introduces algorithms for 1) constructing bootstrapped confidence interval with EMP model and for 2) evaluating bootstrapped confidence interval. Both algorithms are listed in a step-by-step fashion for all four scenarios (OE-RR, OE-CR, ME-RR and ME-CR).

4.4.1 *Constructing bootstrapped confidence intervals*

In order to construct the confidence region of the point estimates of the CES parameters $\hat{\Phi} = \left(\hat{\theta}_i, \hat{\beta}_{ij}, \hat{\rho}_i, \hat{\nu}_i \right)$, the sampling distribution of $\hat{\Phi}$ is required. This is done by adopting the bootstrap principle to the EMP models.

For the case resampling approach, l_{ijt}^{*} can be obtained by carrying out DGP-OE and DGP-ME given the data $\left(p_{it}^{*}, w_{i2t}^{*}, L^{*} \right)$ and the 'true' CES parameters Φ^{*}. The observations $\left(p_{it}^{*}, w_{i2t}^{*}, l_{ijt}^{*} \right)$ are now seen as the 'population'. Given that the 'population' is known, the true error in a sample statistic against its population value can be acknowledged and measured. Bootstrapped sample data $\left(\tilde{p}_{it}, \tilde{w}_{i2t}, \tilde{l}_{ijt} \right)$ are created by resampling the observations $\left(p_{it}^{*}, w_{i2t}^{*}, l_{ijt}^{*} \right)$ with replacement. Estimation

with EEM-OE and EEM-ME based on $\left(\tilde{p}_{it}, \tilde{w}_{i2t}, \tilde{l}_{ijt}\right)$ yields the bootstrap estimates of the CES parameters $\tilde{\hat{\Phi}} = \left(\tilde{\hat{\theta}}_i, \tilde{\hat{\beta}}_{ij}, \tilde{\hat{\rho}}_i, \tilde{\hat{\nu}}_i\right)$. With sufficiently large bootstrap sample size, i.e. replicating this whole procedure for sufficient times, one could get a good approximation of the sampling distribution of $\tilde{\hat{\Phi}}_B$. Ultimately, the bootstrapped percentile and basic confidence intervals can be calculated given the sampling distribution.

For the residual resampling approach, "residuals" are the point estimates of measurement and optimisation errors $\left(\hat{\varepsilon}^o_{ijt}, \hat{\varepsilon}^m_{ijt}\right)$ obtained by applying the combination of (DGP-OE, EEM-OE) and (DGP-ME, EEM-ME). These point estimates are treated now as the 'population'. Through sampling with replacement one could get the resampled residuals $\left(\tilde{\hat{\varepsilon}}^o_{ijt}, \tilde{\hat{\varepsilon}}^m_{ijt}\right)$. And these resampled residuals are used in DGP-OE and DGP-ME to produce $\tilde{l}^*_{ijt}$, which are treated as data for the estimation in EEM-OE and EEM-ME. Subsequently, the bootstrap estimates $\tilde{\hat{\Phi}} = \left(\tilde{\hat{\theta}}_i, \tilde{\hat{\beta}}_{ij}, \tilde{\hat{\rho}}_i, \tilde{\hat{\nu}}_i\right)$ are estimated with EEM-OE and EEM-ME using the data on $\left(p^*_{it}, w^*_{i2t}, \tilde{l}_{ijt}\right)$. With sufficient replications of this procedure, one can obtain the sampling distribution of $\tilde{\hat{\Phi}}$ and ultimately the bootstrapped confidence intervals.

The general procedure to construct bootstrapped confidence intervals with EMP model can be summarised as below. The detailed algorithm is presented in Table 1 and Table 2:

1. Generate sample data with DGP and obtain point estimates with EEM using randomly generated synthetic data and 'true' parameters data
2. Create bootstrap sample data by resampling sample data with replacement
3. Obtain bootstrap estimates of parameters with EEM
4. Repeat step 2 and 3 for $b = 1, \ldots, B$ times to obtain the sampling distribution of the bootstrap estimates and calculate the bootstrapped confidence intervals

For the bootstrap procedure the indices are defined as follow: $\alpha = 2.5\%$ for $(1-2\alpha) = 95\%$ confidence level, $I = 10$, $J = 2$, $T = 50$ and $B = 1000$. As *B* approaches ∞ ensures that bootstrap distribution is close to the true distribution. Also, it decreases the error in bootstrap estimates (Efron and Tibshirani 1994) and they recommend that *B* should be $\geq$ 500 or 1000 to make the error in percentile estimation relatively low. The same randomly generated synthetic data $\left(p_{it}^{*}, w_{i2t}^{*}, L^{*}\right)$ and 'true' data for Φ^{*} from chapter 3 are applied for EMP model with both error types. Both optimisation and measurement errors $\left(\varepsilon_{ijt}^{o}, \varepsilon_{ijt}^{m}\right)$ are normally distributed with mean zero, and low error standard deviation defined in chapter 3 are chosen.

OM-RR	**OM-CR**
1. Generate sample data • Obtain endogenous l_{ijt}^{*} with DGP-OE using simulated random errors ε_{ijt}^{o}, exogenous synthetic $\left(p_{it}^{*}, w_{i2t}^{*}, L^{*}\right)$ and exogenous 'true' value Φ^{*} • Obtain point estimates $\hat{\Phi}$ and $\hat{\varepsilon}_{ijt}^{o}$ with EEM-OE using data $\left(p_{it}^{*}, w_{i2t}^{*}, l_{ijt}^{*}, L^{*}\right)$ **2. Generate bootstrap sample** • Obtain bootstrap samples $\tilde{\hat{\varepsilon}}_{ijt}^{o}$ by sampling $\hat{\varepsilon}_{ijt}^{o}$ with replacement • Obtain bootstrap samples $\tilde{l}_{ijt}^{*}$ with DGP-OE using exogenous $\left(p_{it}^{*}, w_{i2t}^{*}, L^{*}\right)$ and bootstrap samples $\tilde{\hat{\varepsilon}}_{ijt}^{o}$ **3. Bootstrap estimation** • Obtain bootstrap estimates $\tilde{\hat{\Phi}}$ with EEM-OE using data $\left(p_{it}^{*}, w_{i2t}^{*}, \tilde{l}_{ijt}^{*}, L^{*}\right)$ **4. Bootstrap replication** • Repeat step 2 and 3 for $b = 1, \ldots, B$ times to obtain the sampling distribution of $\tilde{\hat{\Phi}}_{b}$ and calculate the bootstrapped confidence intervals	**1. Generate sample data** • Obtain endogenous l_{ijt}^{*} with DGP-OE using simulated random error ε_{ijt}^{o}, exogenous synthetic $\left(p_{it}^{*}, w_{i2t}^{*}, L^{*}\right)$ and exogenous 'true' value Φ^{*} • Obtain point estimates $\hat{\Phi}$ and $\hat{\varepsilon}_{ijt}^{o}$ with EEM-OE using data $\left(p_{it}^{*}, w_{i2t}^{*}, l_{ijt}^{*}, L^{*}\right)$ **2. Generate bootstrap sample** • Obtain bootstrap samples $\left(\tilde{p}_{it}, \tilde{w}_{i2t}, \tilde{l}_{ijt}\right)$ by resampling $\left(p_{it}^{*}, w_{i2t}^{*}, l_{ijt}^{*}\right)$ with replacement **3. Bootstrap estimation** • Obtain bootstrap estimates $\tilde{\hat{\Phi}}$ with EEM-OE using data $\left(\tilde{p}_{it}, \tilde{w}_{i2t}, \tilde{l}_{ijt}, L^{*}\right)$ **4. Bootstrap replication** • Repeat step 2 and 3 for $b = 1, \ldots, B$ times to obtain the sampling distribution of $\tilde{\hat{\Phi}}_{b}$ and calculate the bootstrapped confidence intervals

Table 1 Algorithms for constructing bootstrapped confidence intervals with optimisation errors

ME-RR	**ME-CR**
1. Generate sample data • Obtain endogenous l_{ijt}^{**} with DGP-ME using exogenous synthetic $\left(p_{it}^{*}, w_{i2t}^{*}, L^{*}\right)$ and exogenous 'true' value Φ^{*} • Obtain l_{ijt}^{*} by subtracting simulated random error ε_{ijt}^{m} from l_{ijt}^{**} • Obtain point estimates $\hat{\Phi}$ and $\hat{\varepsilon}_{ijt}^{m}$ with EEM-ME using data $\left(p_{it}^{*}, w_{i2t}^{*}, l_{ijt}^{*}, L^{*}\right)$ **2. Generate bootstrap sample** • Obtain bootstrap samples $\tilde{\hat{\varepsilon}}_{ijt}^{m}$ by sampling $\hat{\varepsilon}_{ijt}^{m}$ with replacement • Obtain bootstrap sample $\tilde{l}_{ijt}$ by subtracting $\tilde{\hat{\varepsilon}}_{ijt}^{m}$ from the fitted value $\left(l_{ijt}^{*} - \hat{\varepsilon}_{ijt}^{m}\right)$ **3. Bootstrap estimation** • Obtain bootstrap estimates $\tilde{\hat{\Phi}}$ with EEM-ME using data $\left(p_{it}^{*}, w_{i2t}^{*}, \tilde{l}_{ijt}^{*}, L^{*}\right)$ **4. Bootstrap replication** • Repeat step 2 and 3 for $b = 1, \ldots, B$ times to obtain the sampling distribution of $\tilde{\hat{\Phi}}_{b}$ and calculate the bootstrapped confidence intervals	**1. Generate sample data** • Obtain endogenous l_{ijt}^{**} with DGP-ME using exogenous synthetic $\left(p_{it}^{*}, w_{i2t}^{*}, L^{*}\right)$ and exogenous 'true' value Φ^{*} • Obtain l_{ijt}^{*} by subtracting simulated random error ε_{ijt}^{m} from l_{ijt}^{**} • Obtain point estimates $\hat{\Phi}$ and $\hat{\varepsilon}_{ijt}^{m}$ with EEM-ME using data $\left(p_{it}^{*}, w_{i2t}^{*}, l_{ijt}^{*}, L^{*}\right)$ **2. Generate bootstrap sample** • Obtain $\left(\tilde{p}_{it}, \tilde{w}_{i2t}, \tilde{l}_{ijt}\right)$ by sampling p_{it}^{*}, w_{i2t}^{*} and $\left(l_{ijt}^{*} - \hat{\varepsilon}_{ijt}^{m}\right)$ with replacement **3. Bootstrap estimation** • Obtain bootstrap estimates $\tilde{\hat{\Phi}}$ with EEM-ME using data $\left(\tilde{p}_{it}, \tilde{w}_{i2t}, \tilde{l}_{ijt}, L^{*}\right)$ **4. Bootstrap replication** • Repeat step 2 and 3 for $b = 1, \ldots, B$ times to obtain the sampling distribution of $\tilde{\hat{\Phi}}_{b}$ and calculate the bootstrapped confidence intervals

Table 2 Algorithms for constructing bootstrapped confidence intervals with measurement errors

4.4.2 *Evaluating the bootstrapped confidence intervals*

The confidence interval level is the ideal probability that the true value falls within the interval in infinite repeated sampling. Following this idea, Monte Carlo experiments are conducted with above bootstrap algorithms to test whether the empirical coverage confirms the confidence level. This evaluation is done for all four scenarios from Table 1 and Table 2 and the general steps are listed as follows:

1. Carry out $s=1,\ldots,S$ Monte Carlo simulations with the EMP models with one set of 'true' CES parameters Φ^*, where the error term is randomly simulated with known distribution. This yields S sets of point estimates $\hat{\Phi}_s$.
2. Use the algorithms from Table 1 and Table 2 to construct an inner bootstrap procedure for each Monte Carlo point estimate $\hat{\Phi}_s$ to obtain a bootstrapped confidence interval. Two types of confidence interval are calculated, the basic interval and percentile interval.
3. Obtain the empirical coverage as a frequency measure (in percentage) for how often Φ^* fall in the confidence intervals.

The evaluation procedure is in principal a replication of the algorithms from Table 1 and Table 2 for S number of times. The mechanism for generating the random simulation errors and the bootstrap samples is random and independent for all $s=1,\ldots,S$. The dimension for the indices (I,J,T,B), the 'true' CES parameters, synthetic data for prices and land endowment are the same across all Monte Carlo replications. S is also chosen to be 1000. There are totally four scenarios (OE-RR, OE-CR, ME-RR and ME-CR) and four types of CES parameters $\Phi=\left(\theta_i,\beta_{ij},\rho_i,\nu_i\right)$

with $I=10$ for each type of parameter[4]. For each scenario with each type of parameter, the evaluation procedure generates a large amount of results: $I\times S\times B=10,000,000$ estimates of CES parameters, $I\times S=10,000$ confidence intervals and $I=10$ coverage probabilities obtained by using percentile and basic methods, respectively. The EMP model is the most computational intensive part and it is solved $(S+S\times B)=1,001,000$ times for each scenario. Therefore, the evaluation procedure is very time-consuming and the estimated total computing time for each scenario is 63 hours for OE-RR, 83 hours for ME-RR, 43 hours for OE-CR and 46 hours for ME-CR. The data resampling is exclusively done in MATLAB, while the rest is done in GAMS.

4.5 **Results**

This section presents and discusses the results obtained from the bootstrapped confidence intervals evaluation procedure.

Category	$\beta_{i1}\,(\beta_{i2})$	ρ_i	θ_i	v_i	**Mean**
$C^{o}_{bsc\text{-}RR}$	84%	92%	88%	94%	**89%**
$C^{o}_{prc\text{-}RR}$	92%	92%	92%	93%	**92%**
$C^{o}_{bsc\text{-}CR}$	85%	93%	88%	93%	**90%**
$C^{o}_{prc\text{-}CR}$	91%	92%	91%	93%	**92%**

Table 3 Empirical coverage of basic and percentile intervals based on 95% confidence level (with optimisation errors)

Table 3 presents the coverage probability based on 95% confidence level for EMP model with optimisation errors. The 1st column lists the

[4] Technically there are five types of CES parameter, if the share parameters of land β_{i1} and fertilizer β_{i2} are counted separately. Since they sum up to 1, the parameter value of β_{i1} can be calculated given β_{i2} and vice versa. This renders the coverage probabilities of β_{i1} and β_{i2} identical. Therefore, they count as one type of CES parameter in the results.

categorisation which specifies the coverage probability by the resampling approach and the confidence interval method. The capital letter *C* stands for confidence interval. The superscript indicates the error structure, while the subscript shows the combination of the confidence interval method (*bsc* and *prc* representing basic and percentile methods respectively) and the resampling approach. For each category, 10 coverage probabilities are obtained for each type of the CES parameters. The values (from 2nd to 5th columns) are mean value summed over the index *i* for each type of the CES parameters. The last column sums the mean value again over all four types of parameters for each category. Judging by the mean values, there is a general tendency of undercoverage.

Hypothesis	$\beta_{i1}\,(\beta_{i2})$	ρ_i	θ_i	v_i	**Total**
$C^o_{bsc\text{-}RR} < C^o_{prc\text{-}RR}$	10/10	5/10	10/10	1/10	**26/40**
$C^o_{bsc\text{-}CR} < C^o_{prc\text{-}CR}$	10/10	2/10	8/10	8/10	**28/40**
$C^o_{bsc\text{-}CR} < C^o_{bsc\text{-}RR}$	5/10	4/10	2/10	7/10	**18/40**
$C^o_{prc\text{-}CR} < C^o_{prc\text{-}RR}$	6/10	5/10	6/10	2/10	**19/40**

Table 4 Comparison between coverage probabilities obtained by different resampling approaches and bootstrap confidence interval methods (with optimisation errors)

The smaller the distance between the empirical coverage probabilities and the true confidence level, the more precise the empirical value is. The precision[5], i.e. this distance, is calculated for all 40 coverage probabilities for each category. Table 4 exhibits the comprehensive comparison and the values in Table 4 count how often the hypotheses specified in the 1st column are true. These hypotheses are what generally proved to be true by the empirical comparisons from the literature. They are formulated as

[5] The absolute value is chosen to calculate the distance in order to take both under- and overcoverage into consideration.

follows: 1) the coverage probability obtained with the percentile method is closer to the true confidence level than the one obtained with the basic method (2nd and 3rd rows of 1st column) and 2) the residual resampling approach delivers better results than the case resampling approach (last two rows of 1st column).

Looking at the comparisons between the two different confidence interval methods while applying the same resampling approach, the aggregated results (the last column of 2nd and 3rd rows) suggests that 1) in 26 out of 40 cases the percentile method yields better results than the basic method, if the residual resampling approach is applied and 2) the result is 28 out of 40 with the case resampling approach. Low values are observed for ν_i with the residual resampling approach (1/10) and ρ_i with the case resampling approach (2/10).

Comparing two resampling approaches while considering the same the interval methods, the results (18/40 and 19/40) suggest that the residual resampling approach does not always produce closer coverage probabilities than the case resampling approach. Based on these findings it can be conclude in the context of bootstrapping the EMP model parameters with optimisation errors that applying the percentile methods leads to a slightly better performance than using the basic method, whereas using different resampling approaches does not have a significant influence on the quality of the result.

Category	$\beta_{i1}\,(\beta_{i2})$	ρ_i	θ_i	ν_i	**Mean**
$C^m_{bsc\text{-}RR}$	77%	88%	86%	93%	**86%**
$C^m_{prc\text{-}RR}$	93%	93%	93%	93%	**93%**
$C^m_{bsc\text{-}CR}$	56%	64%	63%	62%	**61%**
$C^m_{prc\text{-}CR}$	96%	95%	96%	94%	**95%**

Table 5 Empirical coverage of basic and percentile intervals based on 95% confidence level (with measurement errors)

Table 5 lists the mean coverage probabilities obtained from the evaluation procedure with EMP model with measurement errors. Same categorisation

from Table 3 is applied. A brief glance at the results indicates again a general undercoverage compared to the desired value 95% except for the results for β_{ij} and θ_i generated by using the percentile method and the case resampling approach (96% in the 2nd and 4th column of the last row).The percentile method provides better or equal coverage probability than the basic method for each type of the CES parameters. Also, the category combining the case resampling approach with the basic method results in notably lower coverage than the other categories.

Hypothesis	$\beta_{i1}\,(\beta_{i2})$	ρ_i	θ_i	v_i	**Total**
$C^m_{bsc\text{-}RR} < C^m_{prc\text{-}RR}$	10/10	10/10	9/10	7/10	**36/40**
$C^m_{bsc\text{-}CR} < C^m_{prc\text{-}CR}$	10/10	10/10	10/10	10/10	**40/40**
$C^m_{bsc\text{-}CR} < C^m_{bsc\text{-}RR}$	10/10	10/10	10/10	10/10	**40/40**
$C^m_{prc\text{-}CR} < C^m_{prc\text{-}RR}$	2/10	2/10	1/10	2/10	**7/40**

Table 6 Comparison between coverage probabilities obtained by different resampling approaches and bootstrap confidence interval methods (with measurement errors)

Table 6 offers a detailed comparison with the same design of Table 4. The results in the 2nd and 3rd rows show the strong advantage of using the percentile method over the basic method, while the results in the last two rows suggest a mixed outcome. The residual resampling approach appears to be superior to the case resampling only in combination with the basic bootstrap method (40/40). In combination with the percentile method, the case resampling approach performs better than the residual resampling approach (7/40). In summary, for bootstrapping the EMP model parameters with measurement errors the percentile method is preferable to the basic method. However, no clear evidence exists to claim a superior resampling approach in this context.

Our results generally agree with those from other literature. Efron and Tibshirani (1994) and Canty et al.(1996) show that the percentile method delivers better coverage probabilities compared to basic method. And the tendency of undercoverage is observed for both methods. Horowitz (2001)

states that the residual sampling has better numerical accuracy than the case resampling. The contexts in which the comparisons are conducted are often different. For example, numerical accuracy is often a test criterion besides empirical coverage, where an *exact* confidence endpoint exists. However, this is not the case for the bootstrapped confidence interval with EMP model. Empirical comparisons in the literature are often made for other bootstrap confidence interval methods, which are not chosen from this study. For example, Davison and Hinkley (1997) suggest that the studentized method yields the best results, if the log scale is used. And only at the larger sample sizes are percentile, BCa and Approximate Bootstrap Confidence (ABC) methods comparable with the studentized method. In their experiment, the lower and upper limits of the above confidence intervals are compared to the exact theoretical lower and upper endpoints, respectively. Canty et al. (1996) conclude based on their empirical comparison that the theoretical analysis of confidence interval methods is not the whole story, as the theory needs to be bolstered by numerical comparisons.

4.6 Conclusion and outlook

In this study, we adopt the bootstrap concept to an EMP model to construct confidence intervals for the estimated EMP model parameters. So far there are no studies offering possibilities to conduct statistical inference in the context of EMP models. This puts the reliability of the empirical results into question, as these estimated parameters are often the major drivers of the model, i.e. they determine how a model behaves in simulation. The simulation results given the uncertainty on the estimated parameter might lead to enormous consequence, if the very model, for example, provides evidence for policy making which would have a substantial socioeconomic and environmental impact on the global or a large scale. Thus, it is crucial to have some degrees of certainty on the value of estimated parameters. The EMP model considered in this chapter consists of a statistical model and an econometric estimation model based on a single farm optimisation economic model with CES crop-specific production functions. The data-generating process with the statistical model provides sample data to the

econometric model to estimate the point estimate of the CES parameters. Stochastic errors are introduced in the data-generating process. Two error structures, measurement and optimisation errors, are considered. The sample data are handled as the 'population'. One bootstrap sample is acquired by randomly resample the sample data with replacement. The residual and case resampling approaches are chosen for this purpose. The inference on the resampled data, i.e. estimating the CES parameters with the EMP model using the resampled data, gives an approximation of the inference on the point estimate. With sufficient bootstrap replications one can obtain the sampling distribution of the point estimates. The percentile and basic bootstrap confidence interval methods are chosen to calculate the bootstrapped confidence intervals. Monte Carlo simulation is implemented to exam the quality of the bootstrapped confidence intervals. The bootstrapping of EMP models is carried out with repeated sampling to determine the empirical coverage probability, i.e. how often the 'true' value is covered by the bootstrapped confidence intervals. Considering that multiple confidence intervals are calculated for multiple parameters, the result suggests that the procedure is in general plausible with exception mentioned above in section 4.5. The contribution of this study is that it is the first application of statistical inference on EMP model and gives some insights into the reliability of the estimated EMP model parameters. Also, it offers two algorithms for bootstrapping the EMP model and the evaluation procedure in a transparent and comprehensive way. And the plausible bootstrap algorithm could be applied for empirical application.

Many other bootstrap confidence interval methods exist. Comparing with the methods chosen in this study, they are superior, at least on a theoretical level, according to the "goodness" criteria, i.e. they produce second-order accurate and correct confidence intervals: variance stabilised bootstrap-*t* (Tibshirani 1988), BCa method (Efron 1987), ABC interval (DiCiccio and Efron 1992) and double bootstrap (Beran 1987), to name a few. The rather simple first-order accurate variations, namely basic and percentile methods, are chosen, as superior methods require enormous computational capacity. Nevertheless, the more important first-order accuracy is covered in this study. These more sophisticated methods are preferable for an

empirical application in the context of bootstrapping EMP model, where Monte Carlo is not required.

An alternative approach to conduct statistical inference on EMP model is outlined in Jansson and Heckelei (2010). They suggest a general Bayesian estimation approach of (inequality) constrained optimisation models with errors in variables. A combination of numerical techniques and out-of-sample information via Bayesian techniques would also ultimately offer statistical inference measures on model parameters.

4.7 References

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