# Yukawa Couplings from D-branes on non-factorisable Tori

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CHAPTER 1

# Abstract

In this thesis Yukawa couplings from D-branes on non-factorisable tori are computed. In particular intersecting D6-branes on the torus, generated by the $SO(12)$ root lattice, is considered, where the Yukawa couplings arise from worldsheet instantons. Thereby the classical part to the Yukawa couplings are determined and known expressions for Yukawa couplings on factorisable tori are extended.
Further Yukawa couplings for the T-dual setup are computed. Therefor three directions of the $SO(12)$ torus are T-dualized and the boundary conditions of the D6-branes are translated to magnetic fluxes on the torus. Wavefunctions for chiral matter are calculated, where the expressions, known from the factorisable case, get modified in a non-trivial way. Integration of three wavefunctions over the non-factorisable torus yields the Yukawa couplings. The result not only confirms the results from the computations on the $SO(12)$ torus, but also determines the quantum contribution to the couplings.
This thesis also contains a brief review to intersecting D6-branes on $\mathbb{Z}_2\times\mathbb{Z}_2$ orientifolds, with applications to a non-factorisable orientifold, generated by the $SO(12)$ root lattice.

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# CHAPTER 2

# Introduction

### Motivation

Physics at the microscopic level is described with great accuracy by the Standard Model (SM) of particle physics. The embedding space for particles of the SM is given by a four dimensional Minkowski space with an

$$SU(3)_C \times SU(2)_L \times U(1)_Y \tag{2.1}$$

gauge symmetry. Elementary particles are treated as irreducible representations of their little group and gauge groups and are described as fields of a quantum field theory [1–3]. To preserve the gauge symmetry all fields must be apriori massless and the algebra for the little group of massless particles in four dimensional Minkowski space is $\mathfrak{su}(2) \times \mathfrak{su}(2)$. It is distinguished between matter fields and gauge fields, where the first ones have spin 1/2 and the second ones have spin 1. Matter fields occur in three families of the following bifundamental representations of $SU(3) \times SU(2)$

$$(\mathbf{1},\mathbf{2})_{-1/2} \oplus (\mathbf{3},\mathbf{2})_{1/6} \oplus (\mathbf{1},\mathbf{1})_1 \oplus (\mathbf{1},\mathbf{1})_0 \oplus \left(\overline{\mathbf{3}},\mathbf{1}\right)_{-2/3} \oplus \left(\overline{\mathbf{3}},\mathbf{1}\right)_{1/3}\ , \tag{2.2}$$

where the subscript denotes the $U(1)_Y$ charge. The fields belonging to $(\mathbf{1},\mathbf{2})_{-1/2} \oplus (\mathbf{3},\mathbf{2})_{1/6}$ transform in the chiral representation $(\frac{1}{2},0)$ of $\mathfrak{su}(2) \times \mathfrak{su}(2)$ and the other fields in the antichiral representation $(0,\frac{1}{2})$. Further matter fields charged under the $SU(3)$ are identified with quarks, where fields not charged under the $SU(3)$ are called leptons. Gauge fields transform in the vector representation $(\frac{1}{2},\frac{1}{2})$ of their little group and in the adjoint representation of the gauge groups. Hence the SM contains 12 gauge fields belonging to the representation

$$(\mathbf{1},\mathbf{1})_0 \oplus (\mathbf{1},\mathbf{3})_0 \oplus (\mathbf{8},\mathbf{1})_0 \tag{2.3}$$

of $SU(3) \times SU(2) \times U(1)$. In order to allow the particles to gain mass, the $SU(2)_L \times U(1)_Y$ factor hast to be spontaneously broken to a $U(1)_{\text{el}}$ gauge symmetry. The symmetry breaking is triggered by a scalar field, called the Higgs boson and which transforms in the representation

$$(\mathbf{1},\mathbf{2})_{1/2} \tag{2.4}$$

under the gauge symmetry. The Higgs acquiring a non-trivial vacuum expectation value (vev) [4, 5] induces the symmetry breaking. The unbroken $U(1)_{\text{el}}$ is the linear combination of $U(1) \subset SU(2)_L$ and $U(1)_Y$ under which the Higgs boson is uncharged. The gauge boson of the unbroken $U(1)_Y$ is identified with the photon, which is the transmitter of the electromagnetic force. The eight gluons, how the gauge bosons of the $SU(3)$ are called, confine the quarks, thus are responsible for the strong force, which is observed for example as the force binding atomic nuclei. The remaining three gauge bosons of $SU(2)_L \times U(1)_Y/U(1)_{\mathrm{el}}$ mediate the weak force and become massive after symmetry breaking. Therefor the weak force only becomes noticeable in quantum effects, such as $\beta$-decay.

However the force, which is probably the most experienced force in everyday life (except for life in space), is not explained by the SM, namely gravity. Since gravity is to weak to measure its effects at the quantum level, gravity is described in a classical field theory as spacetime curving, due to the back reaction of the coupling to energy densities [6]. At macroscopic levels gravity, as described in the framework of general relativity, has proven itself to describe the universe accurately. But trying to incorporate gravity with quantum field theory fails. The mediator particle of gravity has the properties of a spin 2 symmetric tensor field, called the graviton. Graviton interactions lead to divergences roughly at the Planck scale ($\sim 10^{19}\,GeV$), which can not be handled with the know methods in quantum field theory [7]. But in order to describe for example the early universe or black holes, quantum effects of gravity can no longer be ignored and a quantum theory of gravity is needed. Further the disturbing circumstances as the presence of dark matter and dark energy, which are needed to explain for example the rotation curves of galaxies and the accelerated expansion of the universe, are confirmed by the PLANCK collaboration in 2013 [8]. Their results show that only 4% of the energy in the universe consists of the known particles, where 25% is contained in dark matter and the remaining 70% of the energy has to be dark energy.

A main problem of the SM, besides incorporating gravity, is the inability to explain dark matter and dark energy. Further, since neutrino oscillations have been observed [9], it is clear that the SM particle content needs to be extended in order to explain mass terms for neutrinos. The detection of the Higgs particle at the LHC [10, 11] confronted the SM with another problem: Quantum corrections to the mass of scalar fields push their mass scale towards the Planck scale and the mass of the Higgs field $\sim 125\,GeV$ can be explained in the framework of the SM, only when strong tuning is admitted. These problems lead to the conclusion, that even though the SM is successful to explain many phenomena, it can only be the effective theory of a more fundamental theory.

A new ingredient, which brings promising new features with it, is supersymmetry [12–14]. Supersymmetry is the only possible extension to the Poincare algebra according to the Coleman-Mandula no-go theorem [15]. It is a symmetry relating bosonic and fermionic degrees of freedom and introduces for each SM particle a superpartner, which has the same quantum numbers, except the spin quantum number differing by 1/2. Since no superpartner has been found yet, supersymmetry, if realized in nature, must be broken at an energy scale not yet probed. The superpartners are possible candidates for dark matter and their contribution to the quantum corrections might protect the Higgs mass. Further the gauge coupling constants can get affected in such a way that the couplings unify at a scale $\sim 10^{16}\,GeV$ and the SM gauge symmetry gets enhanced to a bigger gauge group in a Grand Unified Theory (GUT) [16]. Gauged superymmetries, also called supergravity, brings a spin 2 particle with them which has the properties of the graviton. Hence the theory of general relativity is incorporated in supergravity. However the divergences of the quantum contributions from gravitons are not absent in supergravity theories, which is the reason that a fundamental theory even beyond supergravity (if supergravity is realized in nature) needs to exist.

String theory is a candidate for such a fundamental theory. In string theory fundamental particles are considered to be strings, where different quantum numbers of particles are actually just different oscillation modes of the string. Gravity is naturally included into string theory, because closed strings contain states, which behave as expected from the graviton. However consistency requires the string to be embedded into a ten dimensional spacetime, which might seem to be peculiar at first, but compactification opens many possibilities to engineer structures in four dimensions. Further, spacetime supersymmetry is a byproduct of consistency conditions. Actually five consistent descriptions of string theories exist in ten dimensions, which where shown in [17, 18] to be dual to each other and to be ten dimensional limits of an eleven dimensional theory, called M-theory. Effective theories of string theories in the limit of zero string length become ten dimensional supergravity theories. String theories such as $E_8 \times E_8$ and $SO(32)$ heterotic string theory or Type I string theory contain naturally gauge groups, big enough to contain the SM gauge groups, but also Type IIA and IIB string theories are allowed to contain gauge symmetries by including orientifolds.

Four dimensional theories, constructed by compactifying string theory, depend highly on the geometry of the internal space [19–21]. The vast amount of possibilities to compactify string theories, allow them to generate different kinds of four dimensions models, such as (hopefully) the SM among them. That way the SM could by derived by compactifying string theory and the numerical values for parameters of the SM, like masses or coupling strengths, can be derived by the string coupling and geometry of the internal space. Further, desired features such as for example inflation of the early universe, can be manufactured in the framework of string theory [22]. Hence string theory is not only a good candidate for a fundamental theory but also contemplable for a unified theory. As open questions, it still remains to explain how and at which scale supersymmetry is broken and how the particular geometry for string compactification leading to the SM looks like. Further a satisfactory explanation for the right amount of dark energy is still awaited.

### Outline

In this work at first Type IIA string theory compactified on orientifolds with intersecting D6-branes and the resulting four dimensional features are reviewed, where special attention is given to a particular non-factorisable geometry. Later Yukawa couplings on non-factorisable tori are computed.

After briefly introducing the basics to string theory with the main focus on Type II string theories and D-branes in chapter 3, Type IIA compactification on orientifolds are discussed in chapter 4. Chapter 4 is mainly a review of D6-branes on orientifolds, with the application to a specific non-factorisable orientifold: In the first part of chapter 4 the geometry of orientifolds is discussed. They are constructed by projecting out worldsheet parity from orbifolds. The type of orbifolds considered here are given by the quotient space of six dimensional tori divided by discrete subgroups of $SU(3)$. A short insight into the resolution of orbifold fixed points is presented and the geometries of 3-cycles for D6-branes are studied. In the second part of chapter 4, massless states from Type IIA closed strings and open strings on D6-branes in $\mathbb{Z}_2 \times \mathbb{Z}_2$ orientifolds are investigated and the resulting spectra are checked for possible anomalies. In the third part of chapter 4 the introduced concepts are applied to a non-factorisable $\mathbb{Z}_2 \times \mathbb{Z}_2$ orientifold, which is generated by the $SO(12)$ root lattice, and differences to the factorisable case are revealed.

In chapter 5 Yukawa couplings from intersecting branes are considered and extended to the non-factorisable torus, generated by the $SO(12)$ root lattice. Therefor, first Yukawa couplings from intersecting branes on two dimensional tori are reviewed. In the second step the computations are generalized on to the $SO(12)$ torus. It turns out that the couplings involve intersection points labeled by vectors of general three dimensional lattices and the worldsheet instantons, which generate the Yukawa couplings, admit selection rules. A procedure to determine the lattices for labels of intersection points and the selection rules for the instantons is described. Summing over worldsheet instantons yields the classical contribution to the Yukawa couplings on the $SO(12)$ torus.

In chapter 6 the $SO(12)$ torus and the D-brane boundary conditions on it are T-dualized along three directions. In the T-dual picture D9-branes with magnetic fluxes fill out the dual torus. The non-factorisable structure of the torus is mirrored in Wilson lines of the fluxes. The discussion for computing wavefunctions of chiral matter on the factorisable torus is first generalized for the case with magnetic fluxes wrapping non-coprime wrapping numbers. In the next step the discussion is extended to the non-factorisable case by expressing the gauge indices of fields as vectors on general three dimensional lattices. Wavefunctions for massless fields in bifundamental representations are determined by computing zeromodes of the Dirac operator and solving boundary conditions occurring from the Wilson lines. Yukawa couplings follow from calculating the overlap integral of three zeromodes over the non-factorisable torus. The result not only confirms the result from the intersecting D6-brane picture in chapter 5, but also yields the quantum corrections to the Yukawa couplings.

#### List of publications

Parts of this work have been published in

- S. Förste and C. Liyanage,"Yukawa couplings for intersecting D-branes on non-factorisable tori", JHEP **03** (2015) 110, [arXiv:1412.3645 [hep-th]]
- S. Förste and C. Liyanage,"Yukawa couplings from magnetized D-branes on non-factorisable tori", [arXiv:1802.05136 [hep-th]]

CHAPTER 3

# Overview to string theory

In this chapter a short introduction to string theory, especially Type II string theories, is given. For the topics discussed in the present work Type II string theories furnish the necessary framework. In this chapter after introducing the superstring it is explained how Type II string theories are constructed and their relations via T-duality is illustrated by introducing circle compactification. It is also explained how D-branes fit into the context of Type II strings. For that, [19–21, 23–26] are followed and for a more detailed discussion, those references can be consulted.

## 3.1 Superstrings

### 3.1.1 Worldsheet and superstring action

Let $\Sigma$ be a two dimensional Riemannian surface, with one timelike and one spatial direction, parametrized by the coordinates $\sigma^0 \in \mathbb{R}$ and $\sigma^1 \in [0, \ell]$. Let $\Sigma$ be embedded into a $D$ dimensional spacetime $M^D$ via the maps $X : \Sigma \to M^D$ by

$$X : \left(\sigma^0, \sigma^1\right) \mapsto X^\mu(\sigma^0, \sigma^1) \in M^D\,, \tag{3.1}$$

where $X^\mu$ are coordinates in $M^D$, with $\mu \in \{0, 1, ..., D-1\}$ denoting spacetime directions. The Riemannian surface $\Sigma$ is a worldsheet of a superstring, propagating in $M^D$, when the integral

$$\mathcal{S} = -\frac{T}{2}\int_\Sigma \mathrm{d}^2\sigma\, \eta^{\alpha\beta}\partial_\alpha X^\mu \partial_\beta X^\nu \eta_{\mu\nu} - \frac{i}{4\pi}\int_\Sigma \mathrm{d}^2\sigma\, \overline{\psi}^\mu \rho^\alpha \partial_\alpha \psi_\mu\,, \tag{3.2}$$

describes the action for $\Sigma$ in superconformal gauge [27–29]. The indices $\alpha, \beta \in \{0, 1\}$ denote worldsheet coordinates, the parameter $T = (2\pi\alpha')^{-1}$ is the string tension, the operators $\rho^\alpha$ are the two generators of the Clifford algebra with a metric with Lorentzian signature. The functions $\psi^\mu$ are similar to $X^\mu$, maps from the worldsheet to the spacetime, but they transform as Dirac spinors on the worldsheet , with $\overline{\psi}^\mu$ its Dirac conjugated. Hence $\psi^\mu$ has two components $\psi^\mu_\pm$ in the spinor representation of $SO(1, 1)$, where each of them is a map from $\Sigma$ to $M^D$

$$\psi^\mu = \begin{pmatrix} \psi^\mu_+ \\ \psi^\mu_- \end{pmatrix}\,, \quad \text{with} \quad \psi^\mu_\pm : (\sigma^0, \sigma^1) \to M^D\,. \tag{3.3}$$

The action in (3.2) describes the surface of the worldsheet, spread out in $M^D$ while the string is propagating. Furthermore the expression in (3.2) describes the action in superconformal gauge, which means local supersymmetry is imposed on $\Sigma$ and the symmetries on the worldsheet, such as diffeomorphism invariance, Weyl and Super-Weyl invariance, are used to eliminate the degrees of freedom of the worldheet metric and the gravitino. Solutions to the equations of motion for the maps $X^\mu$ and $\psi^\mu$ take the expressions

$$X^\mu(\sigma^0,\sigma^1) = X^\mu_L(\sigma^+) + X^\mu_R(\sigma^-)\,, \quad \psi^\mu_\pm(\sigma^0,\sigma^1) = \psi^\mu_\pm(\sigma^\pm)\,, \tag{3.4}$$

where $\sigma^\pm = \sigma^0 \pm \sigma^1$ are light cone coordinates on the worldsheet. Functions depending solely on $\sigma^\pm$ are named as left- and rightmovers, respectively. However satisfying the equations of motion is not enough to extremize the action in (3.2), but boundary conditions on the strings have to be imposed. The boundary conditions lead to two kinds of strings, given by

- closed strings satisfying

$$X^\mu(\sigma^0,\sigma^1) = X^\mu(\sigma^0,\sigma^1+\ell)\,, \quad \psi^\mu(\sigma^0,\sigma^1) = \eta\psi^\mu(\sigma^0\sigma^1+\ell)\,, \tag{3.5}$$

- open strings, with its end points satisfying Neumann (N) boundary conditions

$$\partial_1 X^\mu\Big|_{\sigma^1\in\{0,\ell\}} = 0\,, \quad \left(\psi^\mu_+ - \eta\psi^\mu_-\right)\Big|_{\sigma^1\in\{0,\ell\}} = 0 \tag{3.6}$$

  or Dirichlet (D) boundary conditions

$$\partial_0 X^\mu\Big|_{\sigma^1\in\{0,\ell\}} = 0\,, \quad \left(\psi^\mu_+ + \eta\psi^\mu_-\right)\Big|_{\sigma^1\in\{0,\ell\}} = 0\,. \tag{3.7}$$

The open string boundary conditions relate the left-and rightmovers at the end points as

$$X^\mu_L\Big|_{\sigma^1\in\{0,\ell\}} = \begin{cases} +\,X^\mu_R\big|_{\sigma^1\in\{0,\ell\}} & \text{for N} \\ -\,X^\mu_R\big|_{\sigma^1\in\{0,\ell\}} & \text{for D}\end{cases}, \quad \psi^\mu_+\Big|_{\sigma^1\in\{0,\ell\}} = \begin{cases} +\eta\,\psi^\mu_-\big|_{\sigma^1\in\{0,\ell\}} & \text{for N} \\ -\eta\,\psi^\mu_-\big|_{\sigma^1\in\{0,\ell\}} & \text{for D}\end{cases}. \tag{3.8}$$

The parameter $\eta$ is allowed to take the values $\eta \in \{\pm 1\}$. The choice $\eta = -1$ leads to the Neveu-Schwarz (NS) string and $\eta = -1$ to the Ramond (R) string [30, 31]. A Fourier expansion of the left-and rightmovers is given by

$$X^\mu_{L/R} = \frac{1}{2}x^\mu + \frac{\pi\alpha'}{\ell}p^\mu\sigma^\pm + i\sqrt{\frac{\alpha'}{2}}\sum_{n\in\mathbb{Z}\setminus\{0\}}\frac{1}{n}\alpha^\mu_n \mathrm{e}^{-2\pi i n\sigma^\pm/\ell}\,, \quad \psi^\mu_\pm = \sqrt{\frac{2\pi}{\ell}}\sum_{n\in\mathbb{Z}} b^\mu_{n+r}\mathrm{e}^{2\pi i\sigma^\pm/\ell} \tag{3.9}$$

for closed strings and

$$X^\mu_{L/R} = \frac{1}{2}x^\mu + \frac{\pi\alpha'}{\ell}p^\mu\sigma^\pm + i\sqrt{\frac{\alpha'}{2}}\sum_{n\in\mathbb{Z}\setminus\{0\}}\frac{1}{n}\alpha^\mu_n \mathrm{e}^{-\pi i n\sigma^\pm/\ell}\,, \quad \psi^\mu_\pm = \sqrt{\frac{2\pi}{\ell}}\sum_{n\in\mathbb{Z}} b^\mu_{n+r}\mathrm{e}^{\pi i(n+r)\sigma^\pm/\ell} \tag{3.10}$$

for open strings, where

$$r = \begin{cases} \frac{1}{2} & \text{for NS sector}\,, \\ 0 & \text{for R sector}\,. \end{cases} \tag{3.11}$$

Usually the set of Fourier modes in the left-and rightmoving sector are distinguished by $\left\{\alpha^\mu_n, b^\mu, b^\mu_{n+1/2}\right\}_{n\in\mathbb{Z}}$ for the rightmovers and $\left\{\tilde{\alpha}^\mu_n, \tilde{b}^\mu_n, \tilde{b}^\mu_{n+1/2}\right\}_{n\in\mathbb{Z}}$ for the leftmovers. However since open string boundary conditions relate the oscillator modes from the leftmoving sector with the ones from the rightmoving sector, s.t. they are not independent and one set of oscillator modes $\{\alpha_n^\mu, b_n^\mu, b_{n+1/2}^\mu\}_{n\in\mathbb{Z}}$ is sufficient to describe oscillations on the open string worldsheet. In the closed string sector on the other hand the two sets of oscillator modes are not independent. The parameters $x^\mu$ and $p^\mu$ denote position and momentum of the center of mass of the string. For the left-and rightmovers to be real fields, the oscillator modes need to behave under complex conjugation as

$$\left(\alpha_n^\mu\right)^* = \alpha_{-n}^\mu\,, \quad \left(\tilde{\alpha}_n^\mu\right)^* = \tilde{\alpha}_{-n}^\mu\,, \quad \left(b_{n+r}^\mu\right)^* = b_{-n-r}^\mu\,, \quad \left(\tilde{b}_{n+r}^\mu\right)^* = \tilde{b}_{-n-r}^\mu\,. \tag{3.12}$$

### 3.1.2 String quantization and D=10 string states

In order to quantize the string, the fields $X_\pm^\mu$ and $\psi_\pm^\mu$ become operators acting on a vacuum state, which describes the ground state of the worldsheet. Therefor the Poisson brackets, satisfied by the fields at the classical level, are replaced by (anti-) commutators. The non vanishing (anti-) commutator relations are given by:

$$[x^\mu, p^\nu] = i\eta^{\mu\nu}\,, \quad \left[\alpha_m^\mu, \alpha_n^\nu\right] = \left[\tilde{\alpha}_m^\mu, \tilde{\alpha}_n^\nu\right] = m\delta_{m+n}\eta^{\mu\nu}\,, \quad \left\{b_{m+r}^\mu, b_{n+r}^\nu\right\} = \left\{\tilde{b}_{m+r}^\mu, \tilde{b}_{n+r}^\nu\right\} = \eta^{\mu\nu}\delta_{m+n+2r}\,, \tag{3.13}$$

and the the algebra for oscillator modes reveal that the oscillators behave as creation and annihilation operators acting on states of a Hilbert space. However the above algebra leads to negative norm states [32–34]. The reason is, that not all oscillators in (3.13) are independent, since the worldsheet contains a remaining superconformal symmetry. Symmetry transformations of the superconformal algebra can be used to gauge away oscillators in two directions, but in order to preserve spacetime Poincare invariance the number of spacetime dimensions has to be fixed to $D = 10$. The procedure is called light cone quantization (see [35] for more details). The algebra for the oscillator modes in light cone gauge is given by

$$[x^\mu, p^\nu] = i\delta^{\mu\nu}\,, \quad \left[\alpha_m^\mu, \alpha_n^\nu\right] = \left[\tilde{\alpha}_m^\mu, \tilde{\alpha}_n^\nu\right] = m\delta_{m+n}\delta^{\mu\nu}\,, \quad \left\{b_{m+r}^\mu, b_{n+r}^\nu\right\} = \left\{\tilde{b}_{m+r}^\mu, \tilde{b}_{n+r}^\nu\right\} \quad = \delta^{\mu\nu}\delta_{m+n+2r}\,, \tag{3.14}$$

with $\mu, \nu \in \{2, ..., 9\}$, where the degrees of freedom in the 0-th and 1st direction for $x^\mu$, $p^\mu$ and oscillators have been gauged away. Now a consistent quantum theory with states, corresponding to oscillations of the string, can be constructed. Let the vacuum state $|0\rangle_{\mathrm{NS}}$ in the NS sector be defined by the state getting annihilated by

$$\alpha_n^\mu|0\rangle_{\mathrm{NS}} = b_{n-1/2}^\mu|0\rangle_{\mathrm{NS}} = 0\,, \quad \forall\, n \geq 1\,, \tag{3.15}$$

where the vacuum state $|0\rangle_{\mathrm{R}}$ in the R sector correspondingly is defined by getting annihilated

$$\alpha_n^\mu|0\rangle_{\mathrm{R}} = b_n^\mu|0\rangle_{\mathrm{R}} = 0\,, \quad \forall\, n \geq 1\,. \tag{3.16}$$

The vacuum state corresponds to the unexcited worldsheet of NS and R strings and are eigenstates of the center of mass momentum $p^\mu$[1]. Acting with creation operators from the rightmoving sector on the

[1] To be more precise, for each value of $p^\mu$ it exists a vacuum state, with $p^\mu$ the eigenvalue to the center of mass momentum operator.

vacuum, the states are constructed, which belong to the Hilbert spaces given by

$$\begin{aligned}\mathcal{H}_{\mathrm{NS}} &= \left\{ \prod_{\mu=2}^{9} \prod_{n=1}^{\infty} \left(\alpha^{\mu}_{-n}\right)^{K^{\mu}_n} \left(b^{\mu}_{-n+1/2}\right)^{S^{\mu}_n} |0\rangle_{\mathrm{NS}} \,\middle|\, K^{\mu}_n \in \mathbb{Z}_{\geq 0}\,, S^{\mu}_n \in \{0,1\} \right\}\,, \\ \mathcal{H}_{\mathrm{R}} &= \left\{ \prod_{\mu=2}^{9} \prod_{n=1}^{\infty} \left(\alpha^{\mu}_{-n}\right)^{K^{\mu}_n} \left(b^{\mu}_{-n}\right)^{S^{\mu}_n} |0\rangle_{\mathrm{R}} \,\middle|\, K^{\mu}_n \in \mathbb{Z}_{\geq 0}\,, S^{\mu}_n \in \{0,1\} \right\}\,,\end{aligned} \tag{3.17}$$

and contain the rightmoving states of the NS and R sector respectively, where by acting with creation operators from the leftmoving sector on the vacuum, the states belonging to the Hilbert spaces

$$\begin{aligned}\tilde{\mathcal{H}}_{\mathrm{NS}} &= \left\{ \prod_{\mu=2}^{9} \prod_{n=1}^{\infty} \left(\tilde{\alpha}^{\mu}_{-n}\right)^{K^{\mu}_n} \left(\tilde{b}^{\mu}_{-n+1/2}\right)^{S^{\mu}_n} |0\rangle_{\mathrm{NS}} \,\middle|\, K^{\mu}_n \in \mathbb{Z}_{\geq 0}\,, S^{\mu}_n \in \{0,1\} \right\}\,, \\ \tilde{\mathcal{H}}_{\mathrm{R}} &= \left\{ \prod_{\mu=2}^{9} \prod_{n=1}^{\infty} \left(\tilde{\alpha}^{\mu}_{-n}\right)^{K^{\mu}_n} \left(\tilde{b}^{\mu}_{-n}\right)^{S^{\mu}_n} |0\rangle_{\mathrm{R}} \,\middle|\, K^{\mu}_n \in \mathbb{Z}_{\geq 0}\,, S^{\mu}_n \in \{0,1\} \right\}\,,\end{aligned} \tag{3.18}$$

which contain the leftmoving NS and R states respectively, are constructed. Those states correspond to excitations of the worldsheet and can be interpreted as oscillations of the string. Open string states are given by elements of $\mathcal{H}_{\mathrm{NS}}$ and $\mathcal{H}_{\mathrm{R}}$, where closed string states $|\mathrm{st}\rangle_{\mathrm{cl.}}$ are constructed by combining a leftmoving state $|\mathrm{st}\rangle_L \in \tilde{\mathcal{H}}_\alpha$ from the $\alpha \in \{\mathrm{NS/R}\}$ sector with a rightmoving state $|\mathrm{st}\rangle_R \in \tilde{\mathcal{H}}_\beta$ from the $\beta \in \{\mathrm{NS/R}\}$ by the tensorproduct

$$|\mathrm{st}\rangle_{\mathrm{cl.}} = |\mathrm{st}\rangle_L \otimes |\mathrm{st}\rangle_R \in \tilde{\mathcal{H}}_\alpha \otimes \mathcal{H}_\beta\,, \tag{3.19}$$

s.t. one can construct four closed string sectors, given by the NS-NS, NS-R, R-NS and R-R sector. Using reparametrization invariance of the worldsheet, the spacetime mass-shell condition can be derived and the mass $M$ of a string state is determined by its eigenvalue to the mass operator given by

$$\alpha' M^2 = \begin{cases} 2\left(N_{\mathrm{B}} + N_{\mathrm{F}} + \tilde{N}_{\mathrm{B}} + \tilde{N}_{\mathrm{F}} + 2a\right) & \text{for closed string}\,, \\ N_{\mathrm{B}} + N_{\mathrm{F}} + a & \text{for open string}\,, \end{cases} \tag{3.20}$$

with the number operators defined by

$$\begin{aligned} N_{\mathrm{B}} &= \sum_{\mu=2}^{9} \sum_{n\in\mathbb{Z}_{>0}} \alpha^{\mu}_{-n}\alpha^{\mu}_{n}\,, \qquad & \tilde{N}_{\mathrm{B}} &= \sum_{\mu=2}^{9} \sum_{n\in\mathbb{Z}_{>0}} \tilde{\alpha}^{\mu}_{-n}\tilde{\alpha}^{\mu}_{n}\,, \\ N_{\mathrm{F}} &= \sum_{\mu=2}^{9} \sum_{n\in\mathbb{Z}_{>0}} b^{\mu}_{-n+r}b^{\mu}_{n-r}\,, & \tilde{N}_{\mathrm{F}} &= \sum_{\mu=2}^{9} \sum_{n\in\mathbb{Z}_{>0}} (n-r)\tilde{b}^{\mu}_{-n+r}\tilde{b}^{\mu}_{n-r}\,, \end{aligned} \tag{3.21}$$

which count the level of excitation, and

$$a = \begin{cases} -\frac{1}{2} & \text{NS sector}\,, \\ 0 & \text{R sector}\,, \end{cases} \tag{3.22}$$

denoting the vacuum energy. Only closed string states satisfying the level matching condition

$$N_{\mathrm{B}} + N_{\mathrm{F}} = \tilde{N}_{\mathrm{B}} + \tilde{N}_{\mathrm{F}}\,, \tag{3.23}$$

are considered to be physical states, which means they satisfy the symmetries of the worldsheet on the quantum level. Applying the mass operator to the NS ground state

$$\alpha' M^2 |0\rangle_{\mathrm{NS}} = -\frac{1}{2}|0\rangle_{\mathrm{NS}}\,, \tag{3.24}$$

reveals that it has negative mass squared and is therefore tachyonic. The tachyonic state indicate an unstable vacuum and needs to be projected out of the spectrum in consistent string theories (as will be discussed in the following section). The algebra for the R zeromodes, given by $\{b_0^\mu, b_0^\nu\} = \delta^{\mu\nu}$, describes the eight dimensional euclidean Clifford algebra up to a factor of 2. Hence the R zeromodes can be related to the eight generators $\Gamma^\mu$ of the Clifford algebra by $b_0^\mu \hat{=} \frac{1}{\sqrt{2}}\Gamma^\mu$ and the operators

$$S_\alpha^\pm = \frac{1}{\sqrt{2}}\left(b_0^{2\alpha} \pm i b_0^{2\alpha+1}\right)\,, \quad \tilde{S}_\alpha^\pm = \frac{1}{\sqrt{2}}\left(\tilde{b}_0^{2\alpha} \pm i\tilde{b}_0^{2\alpha+1}\right)\,, \quad a \in \{1, ..., 4\}\,, \tag{3.25}$$

satisfying the algebra $\left\{S_a^-, S_b^+\right\} = \left\{\tilde{S}_a^-, \tilde{S}_b^+\right\} = \delta_{ab}$, describe lowering and raising operators of weight states in the spinor representation of $SO(8)$. Since acting with the zeromodes onto the groundstate leaves the groundstate energy invariant, the R vacuum has to be degenerate. In particular the groundstate is preserved by the action of $S^\pm$. Hence the R groundstate consists of 16 states, which transform as a Dirac spinor of $SO(8)$. Denoting $|0\rangle_{\mathrm{R}}$ as the lowest weight state of the Dirac spinor, the ground states in the R sector are given by

$$|0\rangle_{\mathrm{R}}\,, \quad S_\alpha^+ S_\beta^+ |0\rangle_{\mathrm{R}}\,, \quad S_1^+ S_2^+ S_3^+ S_4^+ |0\rangle_{\mathrm{R}}\,, \quad S_\alpha^+|0\rangle_{\mathrm{R}}\,, \quad S_\alpha^+ S_\beta^+ S_\gamma^+ |0\rangle_{\mathrm{R}}\,, \quad \alpha \neq \beta \neq \gamma\,. \tag{3.26}$$

The $\gamma^5$ matrix corresponding to the eight dimensional Clifford algebra is given by $\Gamma_{\mathrm{chiral}} = 16\prod_{\mu=2}^{9} b_0^\mu$ and defining the chirality of $|0\rangle_{\mathrm{R}}$ to be given by $\Gamma_{\mathrm{chiral}}|0\rangle_{\mathrm{R}} = +|0\rangle_{\mathrm{R}}$, the eight states $|0\rangle_{\mathrm{R}}$, $S_\alpha^+ S_\beta^+|0\rangle_{\mathrm{R}}$ and $S_1^+ S_2^+ S_3^+ S_4^+|0\rangle_{\mathrm{R}}$ belong to the chiral representation of $SO(8)$, where the other eight states $S_\alpha^+|0\rangle_{\mathrm{R}}$ and $S_\alpha^+ S_\beta^+ S_\gamma^+|0\rangle_{\mathrm{R}}$ belong to the antichiral representation. The group of isometries in ten dimensional Minkowski space contains transformations of $SO(1,9)$. Massless states in ten dimensions transform under the subgroup $SO(8)$, which is the little group of ten dimensional massless fields. Hence the R ground states transform as massless spacetime fermions. The 16 states of the R ground state can be sorted into the chiral and antichiral representation $\mathbf{8}_S$ and $\mathbf{8}_C$ of $SO(8)$ by

$$\prod_{\alpha=1}^{4} (S_\alpha^+)^{K_\alpha}\,|0\rangle_{\mathrm{R}} \in \begin{cases} \mathbf{8}_S & \text{for } \sum_\alpha K_\alpha = \text{even}\,, \\ \mathbf{8}_C & \text{for } \sum_\alpha K_\alpha = \text{odd}\,. \end{cases} \tag{3.27}$$

The massless states in the NS sector are given by the eight states $b_{-1/2}^\mu|0\rangle$, which form the eight states of the vector representation $\mathbf{8}_V$ of $SO(8)$

$$b_{-1/2}^\mu|0\rangle_{\mathrm{NS}} \in \mathbf{8}_V\,. \tag{3.28}$$

Similar massive states in $\mathcal{H}_{\mathrm{NS}}$ transform as bosons in representations of $SO(9)$ and massive states in $\mathcal{H}_{\mathrm{R}}$ transform as fermions of $SO(9)$, where $SO(9)$ is the little group for massive states in ten dimensions.

## 3.2 Type II strings

### 3.2.1 Modular invariance and GSO-projection

Type II string theories are constructed as closed string theories in ten dimensions. The quantum corrections to the spacetime vacuum, coming from one-loop amplitudes of closed string states, lead to a strong condition on the closed string Hilbert spaces called modular invariance [36]: The one-loop vacuum corrections from closed strings arise from worldsheets with the topology of two dimensional tori, where the worldsheet admits the following boundary conditions

$$X^\mu(\sigma^0,\sigma^1) = X^\mu(\sigma^0,\sigma^1+\ell)\,, \quad X^\mu(\sigma^0,\sigma^1) = X^\mu(\sigma^0+\mathrm{Im}\tau,\sigma^1+\mathrm{Re}\tau)\,, \quad \tau\in\mathbb{C}\,, \tag{3.29}$$

s.t. a closed string state $|\mathrm{st}\rangle_{\mathrm{cl.}}$ propagate for a time $\delta\sigma^0 = \mathrm{Im}\tau$ and returns to its initial state $|\mathrm{st}\rangle_{\mathrm{cl.}}$ with a shift $\delta\sigma^1 = \mathrm{Re}\tau$. A two dimensional torus is described by a real Kähler and a complex structure modulus $K\in\mathbb{R}$ and $U\in\mathbb{C}$ (which will be explained in more detail in example (i) of section 4.1.1). The Käher modulus plays no role, since the worldsheet can be rescaled by a Weyl transformation and absorb the Kähler modulus that way. On the other hand the complex structure is given by $U=\tau$. Tori with complex structure moduli differing by $SL(2,\mathbb{Z})$ describe physically equivalent worldsheets. Therefor, transformations of the parameters $\mathrm{Re}\tau$ and $\mathrm{Im}\tau$ in the one-loop amplitude, corresponding to the $SL(2,\mathbb{Z}$ transformations of $U$, must leave the one-loop amplitude invariant. The $SL(2,\mathbb{Z})$ invariance is denoted by modular invariance of the one-loop amplitude. Taking fermionic states on the worldsheet into account, one has to consider that by parallel transporting a fermion on a non-contractible closed loop of a two dimensional surface, the fermion can collect an additional sign unlike to the worldsheet bosons[2]. Hence the boundary conditions for worldsheet fermions on a torus is given by

$$\psi^\mu(\sigma^0,\sigma^1) = \pm\psi^\mu(\sigma^0,\sigma^1+\ell)\,, \quad \psi^\mu(\sigma^0,\sigma^1) = \pm\psi^\mu(\sigma^0+\mathrm{Im}\tau,\sigma^1+\mathrm{Re}\tau)\,. \tag{3.30}$$

The different choices for the sign in the first boundary condition of (3.30) leads to the distinction of the NS and R sector. The choice for the sign in the second boundary condition of (3.30) is implemented by taking only states from (3.17) and (3.18) into account, which survive the GSO-projection, denoted by $|L\rangle_{\mathrm{GSO}}$ and $|R\rangle_{\mathrm{GSO}}$, where [37, 38]

$$\begin{aligned}|R\rangle_{\mathrm{GSO}} &= \begin{cases}\frac{1}{2}\left(1-(-1)^{F_{\mathrm{NS}}}\right)|\mathrm{st}\rangle_R & \text{for } |\mathrm{st}\rangle\in\mathcal{H}_{\mathrm{NS}}\,,\\ \frac{1}{2}\left(1\pm\Gamma_{\mathrm{chiral}}(-1)^{F_{\mathrm{R}}}\right)|\mathrm{st}\rangle_R & \text{for } |\mathrm{st}\rangle\in\mathcal{H}_{\mathrm{R}}\,,\end{cases}\,,\\ |L\rangle_{\mathrm{GSO}} &= \begin{cases}\frac{1}{2}\left(1-(-1)^{\tilde{F}_{\mathrm{NS}}}\right)|\mathrm{st}\rangle_L & \text{for } |\mathrm{st}\rangle\in\tilde{\mathcal{H}}_{\mathrm{NS}}\,,\\ \frac{1}{2}\left(1\pm\Gamma_{\mathrm{chiral}}(-1)^{\tilde{F}_{\mathrm{R}}}\right)|\mathrm{st}\rangle_L & \text{for } |\mathrm{st}\rangle\in\tilde{\mathcal{H}}_{\mathrm{R}}\,,\end{cases}\,,\end{aligned} \tag{3.31}$$

with the fermion number operators defined by

$$\begin{aligned}F_{\mathrm{NS}} &= \sum_{\mu=2}^{9}\sum_{n\in\mathbb{Z}_{\geq0}} b^\mu_{-n-1/2}b^\mu_{n+1/2}\,, \quad F_{\mathrm{R}} = \sum_{\mu=2}^{9}\sum_{n\in\mathbb{Z}_{>0}} b^\mu_{-n}b^\mu_{n}\,,\\ \tilde{F}_{\mathrm{NS}} &= \sum_{\mu=2}^{9}\sum_{n\in\mathbb{Z}_{\geq0}} \tilde{b}^\mu_{-n-1/2}\tilde{b}^\mu_{n+1/2}\,, \quad \tilde{F}_{\mathrm{R}} = \sum_{\mu=2}^{9}\sum_{n\in\mathbb{Z}_{>0}} \tilde{b}^\mu_{-n}\tilde{b}^\mu_{n}\,,\end{aligned} \tag{3.32}$$

[2] See [21] for more details

which count the number of fermionic excitations on the worldsheet. The GSO-projection ensures modular invariance of the one-loop amplitude and projects out the tachyonic state together with half of the other NS states. In the R sector, for each level of excitation only states with the same chirality are preserved by the GSO-projection, where the chirality depends on the sign in front of $\Gamma_{\text{chiral}}$ in the GSO-projection. Closed string theories with states given by $|L\rangle_{\text{GSO}} \otimes |R\rangle_{\text{GSO}}$ are modular invariant and hence consistent at the quantum level. Depending on the choice of the sign for the R sector in the GSO-projection, two inequivalent closed string theories can be constructed that way. String theories, with the same sign for the GSO-projection in the R sector are labeled by Type IIB, where string theories, with opposite sign in the GSO-projection in the R sector, are denoted by Type IIA. It turns out that the GSO-projection preserves the same amount of fermionic and bosonic degrees of freedom in spacetime and the spectra of the two theories exhibit an $\mathcal{N} = 2$ spacetime supersymmetry. The corresponding two supersymmetry generators $Q^L$ and $Q^R$ are 16 dimensional Majorana-Weyl fermions. Their chirality is determined by

$$\Gamma_{\text{chiral}} Q^L = \begin{cases} +\Gamma_{\text{chiral}} Q^R & \text{for Type IIB}\,, \\ -\Gamma_{\text{chiral}} Q^R & \text{for Type IIA}\,, \end{cases} . \tag{3.33}$$

For low energy theories only the massless string states are assumed to play a role. The massless closed string states, constructed from left-and rightmovers, which survive the GSO-projection, are given by

- the $8 \times 8$ NS-NS states
$$\tilde{b}^{\mu}_{-1/2}|0\rangle_{\text{NS}} \otimes b^{\nu}_{-1/2}|0\rangle_{\text{NS}}\,, \tag{3.34}$$
which, decomposed into the trace, antisymmetric and symmetric part, contains a dilaton, antisymmetric background field and a graviton,

- the 64 NS-R states
$$\tilde{b}^{\mu}_{-1/2}|0\rangle_{\text{NS}} \otimes \prod_{\alpha=1}^{4} (S^{+}_{\alpha})^{K_\alpha} |0\rangle_{\text{R}}\,, \tag{3.35}$$
with $K_\alpha \in \{0, 1\}$ and $\sum_\alpha K_\alpha$ = even, containing a dilatino and a gravitino,

- the 64 R-NS states
$$\prod_{\alpha=1}^{4} \left(\tilde{S}^{+}_{\alpha}\right)^{K_\alpha} |0\rangle_{\text{R}} \otimes b^{\mu}_{-1/2}|0\rangle_{\text{NS}}\,, \tag{3.36}$$
with $\sum_\alpha K_\alpha$ = even for Type IIB and $\sum_\alpha K_\alpha$ = odd for Type IIA, containing a dilatino and gravitino, with the same (opposite) chirality as in the NS-R sector for Type IIB (IIA) and

- the 64 R-R states
$$\prod_{\alpha=1}^{4} \left(\tilde{S}^{+}_{\alpha}\right)^{K_\alpha} |0\rangle_{\text{R}} \otimes \prod_{\alpha=1}^{4} (S^{+}_{\alpha})^{L_\alpha} |0\rangle_{\text{R}}\,, \tag{3.37}$$
with $\sum_\alpha K_\alpha$ = even and $\sum_\alpha L_\alpha$ = even for Type IIB or $\sum_\alpha L_\alpha$ = odd for Type IIA, containing a 0-form, 2-form and 4-form gauge potential in Type IIB and a 1-form and 3-form gauge potential in Type IIA.

The massless NS-NS sector describes excitations of the spacetime and hence leads to a theory of gravity. The NS-R and R-NS sector contain the fermionic superpartners in the supergravity multiplet, depending on Type IIB or Type IIA. The R-R sector contains besides the $q$-form gauge potentials $C_q$ also further

$8-p$-form gauge potentials $C_{8-q}$, which are related via the electromagnetic duality

$$\mathrm{d}C_q = *\mathrm{d}C_{8-q}\,. \tag{3.38}$$

The existence of two gravitinos and two dilatinos indicate that the massless spectra for Type IIA and Type IIB fit into $\mathcal{N} = (1,1)$ and $\mathcal{N} = (2,0)$ supergravity multiplets and reflects the relation of the supersymmetry generators in (3.33). The $p$-form gauge potential content from the R-R sector in the two Type II theories are summarized in table 3.1.

| sector | $p$-form | magn. dual |
|---|---|---|
| Type IIA | $C_1$ | $C_7$ |
| | $C_3$ | $C_5$ |
| Type IIB | $C_0$ | $C_8$ |
| | $C_2$ | $C_6$ |
| | $C_4$ | self dual |

Table 3.1: $p$-forms from R-R sector in Type IIA and IIB.

### 3.2.2 Open strings and D-branes

Since the NS-NS states describe deviations of the spacetime from the flat space, the coupling of a worldsheet to spacetime is described by the coupling of a string to the NS-NS fields. Here the coupling of open strings to the spacetime metric and a B-field is considered. Let $\Sigma$ be the worldsheet of an open strings with $\partial\Sigma$ its boundary at the string endpoints. Hence the boundary is tangent to the eigentime of the string (the boundary normal to the worldsheet eigentime is considered to be at $\sigma^0 = -\infty$ and $\sigma^0 = \infty$). The corresponding action is given by [39]

$$\mathcal{S} = -\frac{1}{4\pi\alpha'}\int_\Sigma \mathrm{d}^2\sigma\,\left(g_{\mu\nu}\eta^{\alpha\beta} + B_{\mu\nu}\epsilon^{\alpha\beta}\right)\partial_\alpha X^\mu \partial_\beta X^\nu - \int_{\partial\Sigma}\mathrm{d}\sigma^0\,A_\mu\partial_0 X^\mu\,, \tag{3.39}$$

where $g_{\mu\nu}$ and $B_{\mu\nu}$ denote components of the spacetime metric and B-field. The boundary $\partial\Sigma$ needs to couple to a vector field $A = A_\mu \mathrm{d}x^\mu$, which admits a shift symmetry, in order to preserve the gauge invariance of the B-field[3]. Introducing a gauge invariant field strength $\mathcal{F}$ with the components

$$2\pi\alpha'\mathcal{F}_{\mu\nu} = B_{\mu\nu} + 2\pi\alpha' F_{\mu\nu}, \quad \text{where} \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu\,, \tag{3.40}$$

the action (3.39) can be expressed by

$$S \;=\; -\frac{1}{4\pi\alpha'}\int_\Sigma \mathrm{d}^2\sigma\, g_{\mu\nu}\eta^{\alpha\beta}\partial_\alpha X^\mu\partial_\beta X^\nu - \frac{1}{2}\int_{\partial\Sigma}\mathrm{d}\sigma^0\,\mathcal{F}_{\mu\nu}X^\mu\partial_0 X^\nu\,. \tag{3.41}$$

Extremizing the action (3.41) lead to the boundary conditions at the string end points by

$$\left.\left(g_{\mu\nu}\partial_1 X^\nu + 2\pi\alpha'\mathcal{F}_{\mu\nu}\partial_0 X^\nu\right)\right|_{\partial\Sigma} = 0\,. \tag{3.42}$$

[3] The gauge transformation $B \to B + \mathrm{d}\Lambda$, with $\Lambda$ a 1-form vector field, leads in the first term on the righthand side of (3.39) to a boundary term for open strings. To cancel the boundary term, the gauge field $A$ is introduced, which transforms as $A \to A + \frac{1}{2\pi\alpha'}\Lambda$, s.t. the gauge symmetry for $B$ is preserved for open strings.

The boundary $\partial\Sigma$ is determined by D boundary conditions, thus $\mathcal{F}$, with non vanishing $F$, is restricted on a submanifold $\Sigma_q \subseteq M^{10}$, which purely admits N boundary conditions for the open string. $\Sigma_q$ are worldvolumes spanned by D$p$-branes, with $q = p + 1$ [40]. D$p$-branes are $p$ dimensional dynamical objects, which extend in directions with N boundary conditions and are located at points in the directions with D boundary conditions for open strings. The D boundary conditions for open strings in (3.7) break Poincare invariance. To be more precise, directions in which string end points obey D boundary conditions are no longer invariant under translations and momentum can flow out of string end points. In order to restore momentum conservation, open strings have to be placed on D$p$-branes, which exchange momentum with the string, s.t. total momentum is conserved. D-branes interact with each other via exchanging NS-NS and R-R closed strings [41]. Their coupling to the NS-NS and R-R closed strings can be extracted from the DBI action $\mathcal{S}_{\mathrm{DBI}}$ and Chern-Simons (CS) terms $\mathcal{S}_{\mathrm{CS}}$ in the effective action for D$p$-branes, given by [42, 43]

$$\mathcal{S}_{\mathrm{eff}} = \mathcal{S}_{\mathrm{DBI}} + \mathcal{S}_{\mathrm{CS}} \,. \tag{3.43}$$

The DBI-action is given by

$$\mathcal{S}_{\mathrm{DBI}} = \mu_p \int_{\Sigma_{p+1}} \mathrm{d}^{p+1}\sigma\, \mathrm{e}^{-\phi} \sqrt{\det\left(g_{ij} + B_{ij} + 2\pi\alpha' F_{ij}\right)}\,, \tag{3.44}$$

where $\mu_p$ is the brane tension, $g_{ij}$, $B_{ij}$ and $F_{ij}$ are the components of the induced metric, B-field and fieldstrength $F$ on the branevolume, with $i, j$ denoting directions along $\Sigma_{p+1}$, and $\phi$ the dilaton. The leading order terms with respect to $\alpha'$ in the CS terms, for a vanishing spacetime curvature 2-form, are given by [44], [45]

$$\mathcal{S}_{\mathrm{CS}} = \mu_p \left\{ \int_{\Sigma_{p+1}} C_{p+1} + (2\pi\alpha') \int_{\Sigma_{p+1}} C_{p-1} \wedge \mathrm{Tr}\, F + \frac{1}{2} (2\pi\alpha')^2 \int_{\Sigma_{p+1}} C_{p-3} \wedge \mathrm{Tr}\, F^2 + \ldots \right\} . \tag{3.45}$$

The coupling to the fields $C_{p+1}$ in the CS terms reveal, that D$p$-branes are sources for the R-R $(p + 1)$-form potentials. Comparing the field content in the R-R sector of Type IIA and IIB from table 3.1, it turns out that Type IIA admits D$p$-branes, with $p \in \{0, 2, 4, 6\}$, where Type IIB allows the presence of D$p$-branes with $p \in \{1, 3, 5, 7\}$ and D$(-1)$-branes, which are pointlike instantons. Due to the open string boundary conditions on the branes, the left-and rightmoving sectors get identified according to the N and D conditions. The boundary conditions in (3.6) and (3.7) reveal that each direction $x^a$ admitting D boundary conditions relates the left-and rightmovers with a sign and directions admitting N boundary conditions are related without a sign. Hence the supersymmetry charges from the left-and rightmoving sector $Q_L$ and $Q_R$ are related to each other, s.t. only a linear combination of both is preserved on the branes, s.t. only half of the supersymmetry in the bulk remains on the branevolume [46]. Therefore D$p$-branes are considered as BPS-states of the $\mathcal{N} = 2$ supersymmetry in Type II string theories. Since $F$ is further invariant under $U(1)$ transformations of $A$, each D-brane contains a $U(1)$ gauge symmetry on its worldvolume. String end points attached to the brane are charged under the $U(1)$, since they couple via the boundary term in (3.39) to the gauge field $A$. By stacking D-branes on top of each other, the multiple $U(1)$'s enhance to a non-abelian gauge group such as $U(N)$, $SO(N)$ or $USp(N)$ as it will be encountered for example in section 4.2.2 [47]. Due to the coupling of string end points to the gauge fields on the branes, open string end points can be viewed as states transforming in the fundamental representation $\square_N$ or antifundamental representation $\overline{\square}_N$ of the gauge group on the branevolume, depending on the orientation of the string. The Chan-Paton labels $|ij\rangle$ of an open string, is determined by the states in the gauge representation of its two end points [48]

$$|ij\rangle \in \square_N \otimes \overline{\square}_M\,, \tag{3.46}$$

where the end point corresponding to $i$ transforms in $\square_N$ and the end point corresponding to $j$ transforms in $\overline{\square}_M$. Open string states $|\mathrm{st}\rangle_{\mathrm{op.}}$ from the NS or R sector are then given by

$$|\mathrm{st.}\rangle_{\mathrm{op.}} = |\alpha\rangle \otimes |ij\rangle\,, \quad \text{with} \quad |\alpha\rangle \in \mathcal{H}_\alpha\,, \quad \alpha \in \{NS/R\}\,. \tag{3.47}$$

#### Intersecting D-branes on $\mathbb{C}$

Let the boundary conditions for an open string on a plane be given by by the D and N boundary conditions [49]

$$\begin{aligned} \partial_0\,(\mathrm{ImZ})|_{\sigma^1=0} &= 0\,, \quad \partial_0\left[\mathrm{Im}\left(\mathrm{e}^{i\theta}Z\right)\right]\Big|_{\sigma^1=\ell} = 0\,, \\ \partial_1\,(\mathrm{ReZ})|_{\sigma^1=0} &= 0\,, \quad \partial_1\left[\mathrm{Re}\left(\mathrm{e}^{i\theta}Z\right)\right]\Big|_{\sigma^1=\ell} = 0\,, \end{aligned} \tag{3.48}$$

where $Z$ is the complexified solution to the string equations of motion on the complex plane

$$Z:\left(\sigma^0,\sigma^1\right) \mapsto \mathbb{C}\,, \quad \text{with} \quad Z(\sigma^0,\sigma^1) = X^1(\sigma^0,\sigma^1) + iX^2(\sigma^0,\sigma^1)\,. \tag{3.49}$$

The mode expansion for solutions to the boundary conditions in (3.48) is given by [49]

$$Z(\sigma^+,\sigma^-) = i\sqrt{\frac{\alpha'}{2}}\sum_n\left(\frac{1}{n-\upsilon}\alpha_{n-\upsilon}\mathrm{e}^{\pi i(n-\upsilon)\sigma^+} + \frac{1}{n+\upsilon}\tilde{\alpha}_{n+\upsilon}\mathrm{e}^{\pi i(n+\upsilon)\sigma^-}\right), \quad \text{with} \quad \upsilon = \theta/\pi\,, \tag{3.50}$$

where the moddings for the complex oscillators $\alpha_{n-\upsilon}$ and $\tilde{\alpha}_{n+\upsilon}$ are shifted by a fractional value. Since D boundary conditions require D-branes, two branes $a$ and $b$, with their loci $p_a \in \mathbb{C}$ and $p_b \in \mathbb{C}$ determined by

$$p_a = \{\mathrm{Re}(z)\,|\,z \in \mathbb{C}\}\,, \quad p_b = \left\{z = \mathrm{e}^{i\theta}\mu\,|\,\mu \in \mathbb{R}\right\}\,, \tag{3.51}$$

have to be present in order to preserve momentum conservation. The two branes have the shape of straight lines on the plane and intersect at the origin of $\mathbb{C}$. The open string admitting the boundary conditions (3.48) is attached with its end point $\sigma^1 = 0$ at the brane $a$ and with the end point $\sigma^1 = \ell$ at the brane $b$. Hence the end point at $\sigma^1 = 0$ transforms in the fundamental representation $\square_a$of the gauge group on $a$ and the end point at $\sigma^1 = \ell$ transforms in the antifundamental representation $\overline{\square}_b$of the gauge group on $b$. The Chan-Paton labels then belong to the bifundamental representation $(\square_a, \overline{\square}_b)$. Due to the string tension the open string is localized at the intersection point. Hence chiral matter arises at intersection points of intersecting branes.

### 3.2.3 Circle compactification and T-duality

The simplest approach to compactify a direction $x^\mu$ of the spacetime $M^10$, is to compactify it on a circle with a radius $R$. Formally this means the space for $x^\mu$ is given after circle compactification by the quotient space

$$S^1 = \frac{\mathbb{R}}{R\mathbb{Z}} \tag{3.52}$$

and is to be understood as the identification of all points in the $\mu$-th direction, which differ by the distance of multiples of $2\pi R$

$$x^\mu \simeq x^\mu + 2\pi R\,, \quad x^\mu \in \mathbb{R}\,. \tag{3.53}$$

The boundary conditions in the compactified direction for closed strings have to admit the identification in (3.53) and is given by

$$X^\mu(\sigma^0, \sigma^1) = X^\mu(\sigma^0, \sigma^1 + \ell) + 2\pi R W \tag{3.54}$$

with $W \in \mathbb{Z}$ denoting the number of winding numbers around the $S^1$, and analogous for the worldsheet fermions. The mode expansions solving the boundary conditions in (3.54) is given by [50]

$$X^\mu_{L/R} = \frac{1}{2}x^\mu + \frac{\pi\alpha'}{\ell} p^\mu_{L/R}\sigma^\pm + i\sqrt{\frac{\alpha'}{2}} \sum_{n\in\mathbb{Z}\setminus\{0\}} \frac{1}{n}\alpha'^{i}\mu_n\, \mathrm{e}^{2\pi i n\sigma^\pm/\ell}\,, \tag{3.55}$$

with

$$p^\mu_{L/R} = \frac{K}{R} \pm \frac{WR}{\alpha'}\,, \tag{3.56}$$

where the center of mass momentum $p^\mu = p^\mu_L + p^\mu_R$ is quantized, with $K \in \mathbb{Z}$ denoting Kaluza-Klein excitations, in order to make the quantum mechanical wavefunction single valued. The mass operator for closed strings in (3.20) get an additional contribution of $\frac{K^2}{R^2} + \frac{(WR)^2}{\alpha'^2}$ from the momentum in the $\mu$-th direction and the level matching condition becomes

$$N_\mathrm{B} - \tilde{N}_\mathrm{B} + N_\mathrm{F} - \tilde{N}_\mathrm{F} = WK\,, \tag{3.57}$$

s.t. one gets for each closed string state $|N\rangle_{\mathrm{cl.}} \in \tilde{\mathcal{H}}_{\mathrm{NS/R}} \otimes \mathcal{H}_{\mathrm{NS/R}}$, with a definite excitation level $N$, where $\left(N_\mathrm{B} + \tilde{N}_\mathrm{B} + N_\mathrm{F} + \tilde{N}_\mathrm{F}\right)|N\rangle_{\mathrm{cl.}} = N|N\rangle_{\mathrm{cl.}}$, an infinite tower of new states $|N, W, K\rangle$, with different winding numbers and Kaluza-Klein excitations

$$\alpha' M^2 |N, W, K\rangle = \left(\frac{K^2}{R^2} + \frac{(WR)^2}{\alpha'^2} + 2N + 4a\right)|N, W, K\rangle\,. \tag{3.58}$$

Closed string spectra from circle compactification are invariant under transformations $T$, called T-duality transformation [51]. T-duality transformations exchange Kaluza-Klein excitations with winding numbers by inverting the radius $R$ simultaneously

$$T\,:\, R \to \frac{\alpha'}{R}\,, \quad W \leftrightarrow K\,, \tag{3.59}$$

which leaves (3.58) invariant. At the level of the left- and rightmovers, the transformation in (3.59) translates to giving the rightmovers a sign

$$T\,:\, (X^\mu_L, \psi^\mu_+) \to (X^\mu_L, \psi^\mu_+)\,, \quad (X^\mu_R, \psi^\mu_-) \to -(X^\mu_R, \psi^\mu_-)\,. \tag{3.60}$$

Giving $\psi^\mu_-$ a sign implies that the $\mu$-th zeromode for the rightmovers in the R sector gets a sign and hence flips the GSO-projection in the R sector for the rightmovers. That means performing a T-duality transformation in Type IIA string theory, maps it to Type IIB and vice versa [52]. The theories are said to be T-dual to each other, which means that they actually describe the same physics, but from a different perspective. T-duality transformation for the left-and rightmovers in (3.60) applied to open strings, corresponds to flipping N boundary conditions to D boundary conditions and vice versa. Hence, when the T-dualized direction $x^\mu$ is normal to a D$p$-brane, in the dual picture the brane becomes a D$(p+1)$-brane, extending additionally in the $\mu$-th direction, where when $x^\mu$ is tangent to a D$p$-brane, the brane in the dual theory becomes a D$(p-1)$-brane, localized at a point in the $\mu$-th direction [53]. This is consistent with the brane contents for the Type II string theories, since the D-branes allowed for Type IIB differ by $\pm 1$ dimensions for Type IIA and vice versa.

#### From intersecting to magnetized D-branes on $\mathbb{C}$

Circle compactifying the system of intersecting D-branes in section 3.2.2, for example the direction $x^2$, allows to construct a T-dual theory, by T-dualizing the $x^2$ direction and making the dual radius large. Under the transformation (3.60) the boundaray conditions in (3.48) become for open stirng in the T-dual picture

$$\begin{aligned} \partial_1 X^2\big|_{\sigma^1=0} &= 0\,, \quad \partial_0 X^1 - \tan\theta\,\partial_1 X^2\big|_{\sigma^1=\ell} = 0\,, \\ \partial_0 X^1\big|_{\sigma^1=0} &= 0\,, \quad \partial_1 X^1 + \tan^{-1}\theta\, partial_0 X^2\big|_{\sigma^1=\ell} = 0\,. \end{aligned} \tag{3.61}$$

Comparing with the boundary conditions in (3.42), the term $2\pi\alpha' F_{\mu\nu}$ can be identified with $\tan^{-1}\theta$. Since no B-field is considered the angle $\theta$ between intersecting branes is related in the T-dual picture to a non-vanishing fieldstrength $F$ on the whole $\mathbb{C}$ by

$$F = (2\pi\alpha' \tan\theta)^{-1}\,\mathrm{d}X^1 \wedge \mathrm{d}X^2\,. \tag{3.62}$$

For T-dualizing the direction $x^1$, the flux on the dual theory is given by

$$F = 2\pi\alpha' \tan\theta\,\mathrm{d}X^1 \wedge \mathrm{d}X^2\,. \tag{3.63}$$

The physical interpretation is, that in the T-dual picture the whole $\mathbb{C}$ is filled out with a D-brane, which carries the magnetic flux $F$ on it [49].

CHAPTER 4

# Type IIA compactification on orientifolds

In order for string theory to describe the SM of particle physics in a low energy limit, six of the ten dimensions for the spacetime $M^{10}$ need to be compactified on a six dimensional compact space $X^6$, leaving four dimensional Minkowski space $\mathbb{R}^{1,3}$ uncompact

$$M^{10} \to \mathbb{R}^{1,3} \times X^6 \,. \tag{4.1}$$

In this chapter it is described how to compactify Type IIA string theory on orientifolds with intersecting D-branes. Orientifold compactification already leads in four dimensions to features, which are similar to those described in the SM. Type IIA orientifolds are introduced in [54, 55], where intersecting branes are discussed in [49]. Since then many orientifold models with intersecting D-branes where constructed where [56–58] are just some examples. In section 4.1 geometric properties of orientifolds are discussed and for that [20, 23] is followed.

## 4.1 Geometry of orientifolds

### 4.1.1 Torus

Let $\{\vec{\alpha}_i\}_{i\in\{1,\dots,2n\}}$ be a set of $2n$ linearly independent vectors in $\mathbb{R}^{2n}$. The set $\{\vec{\alpha}_i\}$ generates a $2n$-dimensional lattice $\Lambda^{2n}$, where $\Lambda^{2n}$ is the set of all integer linear combinations of the generators $\vec{\alpha}_i$

$$\Lambda^{2n} = \Big\{ \sum_{i=1}^{2n} n_i \vec{\alpha}_i \Big| n_i \in \mathbb{Z} \Big\} =: \langle \vec{\alpha}_1, ..., \vec{\alpha}_{2n} \rangle \,. \tag{4.2}$$

The quotient space

$$T^{2n} = \frac{\mathbb{R}^{2n}}{\Lambda^{2n}} \,, \tag{4.3}$$

describes a $2n$-dimensional flat torus $T^{2n}$, where the quotient in (4.3) is to be understood as the identification of points, which differ by translation of lattices vectors, as

$$x \sim x + \vec{\lambda}, \quad x \in \mathbb{R}^{2n}, \quad \text{with} \quad \vec{\lambda} \in \Lambda^{2n} \,. \tag{4.4}$$

The neighbourhood in $\mathbb{R}^{2n}$ containing all points, which are inequivalent according to the identification (4.4), is called the fundamental domain of the torus $\mathcal{F}(T^{2n}) \subset \mathbb{R}^{2n}$. A natural basis for a coordinate system on the torus, besides the canonical basis, is given by the set of the generators $\{\vec{\alpha}_i\}$. The basis is named in the following as the lattice basis and coordinates in the lattice basis are denoted by $(y_1, ..., y_{2n}) \in \mathbb{R}^{2n}$. The canonical basis in $\mathbb{R}^{2n}$ is spanned by $2n$ orthonormal basis vectors $\{\vec{e}_i\}_{i\in\{1,...,2n\}}$ and the corresponding coordinates are denoted in the following as $(x_1, ..., x_{2n}) \in \mathbb{R}^{2n}$. The fundamental domain of the torus can then be defined by all points of $\mathbb{R}^{2n}$ within a cell of $\Lambda^{2n}$ and can be expressed by

$$\mathcal{F}(T^{2n}) = \Big\{ \sum_{i=1}^{2n} y_i \vec{\alpha}_i \,\Big|\, \forall\, y^i \in [0,1[ \Big\}, \tag{4.5}$$

s.t. the local geometry of the torus is described by the line element for $\mathbb{R}^{2n}$ given by

$$\mathrm{d}^2 s = \sum_{i=1}^{2n} \mathrm{d}{x_i}^2 . \tag{4.6}$$

The transformation matrix $M \in \left\{ x_i = M_i{}^j y_j \,\middle|\, M \in GL(2n, \mathbb{R}) \right\}$ for a coordinate transformation, contains the vielbein of a metric $g$ on the torus and its components can be deduced by

$$g = M^T \cdot M, \quad \text{s.t.} \quad \delta_{ij}\mathrm{d}x^i \mathrm{d}x^j = g_{ij}\mathrm{d}y^i \mathrm{d}y^j . \tag{4.7}$$

Global properties of the torus, such as lengths and relative angles of the generators $\vec{\alpha}_i$ are captured in $g$. If $g$ has a blockdiagonal form, consisting of $n$ $2 \times 2$ dimensional matrices in the diagonal and the remaining off-diagonal elements vanish, as

$$g = \begin{pmatrix} g^{(1)} & & & \\ & g^{(2)} & & \\ & & \ddots & \\ & & & g^{(n)} \end{pmatrix}, \quad \text{with} \quad g^{(h)} = \begin{pmatrix} g^{(h)}_{11} & g^{(h)}_{12} \\ g^{(h)}_{12} & g^{(h)}_{22} \end{pmatrix}, \tag{4.8}$$

the torus is called factorisable. The generators of the underlying lattice of a factorisable torus can be sorted to $n$ pairs of vectors $\{(\vec{\alpha}_{2h-1}, \vec{\alpha}_{2h})\}_{h\in\{1,...,n\}}$, s.t. each pair is mutually orthogonal to the other pairs, which means the lattice is factorisable into $n$ mutually orthogonal two dimensional lattices $\Lambda^2_{(h)}$, each spanned by a pair $(\vec{\alpha}_{2h-1}, \vec{\alpha}_{2h})$. Thus the quotient space of $\mathbb{R}^{2n}$ by a factorisable lattice, decomposes likewise into a product of $n$ two dimensional tori

$$T^{2n} = \frac{\mathbb{R}^{2n}}{\prod_{h=1}^{n} \Lambda^2_{(h)}} = \prod_{h=1}^{n} T^2_{(h)}, \quad \text{with} \quad T^2_{(h)} = \frac{\mathbb{R}^2}{\Lambda^2_{(h)}} . \tag{4.9}$$

When introducing complex coordinates $(z_h)_{h\in\{1,..,n\}} \in \mathbb{C}^n$, with $z_h = x_{2h-1} + i x_{2h}$ on $\mathbb{R}^{2n}$, each $T^2$ factor can be placed on a complex plane, s.t. a coordinate $z_h$ denotes points on $T^2_{(h)}$. If the generators of the underlying lattice of a torus cannot be sorted to mutually orthogonal pairs, the torus is called non-factorisable. The hermitian metric $h$, Kähler 2-form $\omega_2$ and volume $n$-form $\Omega_n$ for the torus is inherited from its covering space $\mathbb{R}^{2n}$ and given by [59]

$$h = \frac{1}{2}\delta_{i\bar{j}}\mathrm{d}z^i \mathrm{d}\bar{z}^j + \frac{1}{2}\delta_{\bar{i}j}\mathrm{d}\bar{z}^i \mathrm{d}z^j, \quad \omega_2 = \frac{i}{2}\delta_{i\bar{j}}\mathrm{d}z^i \wedge \mathrm{d}\bar{z}^j, \quad \Omega_n = \bigwedge_{i=1}^{n} \mathrm{d}z_i , \tag{4.10}$$

where $z_i = x_{2i-1} + ix_{2i}$ are complex coordinates. The Kähler 2-form and volume $n$-form are calibration forms[1] on the torus. Parameters in the hermitian metric and Kähler form, called moduli, carry global properties of the torus, such as shape and size.

The following two examples shall illustrate and deepen the above discussion:

(i) **Two dimensional Torus $T^2$:**

Let a two dimensional torus $T^2$ be given by

$$T^2 = \frac{\mathbb{R}^2}{\Lambda^2}, \quad \text{with} \quad \Lambda_2 = \langle \vec{\alpha}_1, \vec{\alpha}_2 \rangle . \tag{4.11}$$

By introducing the complex coordinates

$$w = y_1 + U y_2 , \tag{4.12}$$

the torus is fully specified by a Kähler and complex structure modulus $K \in \mathbb{R}$ and $U \in \mathbb{C}$

$$K = \sqrt{\det(g)} , \quad \text{and} \quad U = \frac{g_{12}}{g_{11}} + i\frac{\sqrt{\det(g)}}{g_{11}} , \tag{4.13}$$

where $g$ is the metric on the $T^2$ [61]. Let the two generators of the lattice be given by

$$\vec{\alpha}_1 = R , \quad \vec{\alpha}_2 = R\tau , \quad \text{with} \quad R \in \mathbb{R} , \, \tau \in \mathbb{C} , \tag{4.14}$$

in the complex plane. Then the metric takes the form

$$g = R^2 \begin{pmatrix} 1 & \mathrm{Re}(\tau) \\ \mathrm{Re}(\tau) & |\tau|^2 \end{pmatrix} , \tag{4.15}$$

and the moduli are

$$K = R^2 \mathrm{Im}(\tau) , \quad U = \tau . \tag{4.16}$$

The complex structure modulus contains the information of the angle between $\vec{\alpha}_1$ and $\vec{\alpha}_2$, where the overall size of the torus is encoded in the Kähler structure modulus. Performing a coordinate transformation in (4.10), using the vielbein of $g$, the hermitian metric for the $T^2$ is expressed in the lattice basis by

$$h = R^2 \mathrm{d}w \mathrm{d}\overline{w} , \tag{4.17}$$

where the Kähler 2-form can be given by

$$\omega_2 = R^2 \mathrm{Im}(\tau) \, \mathrm{Re}(\mathrm{d}w) \wedge \mathrm{Im}(\mathrm{d}\overline{w}) , \tag{4.18}$$

s.t. its components is given by the Kähler modulus. The transformations

$$\vec{\alpha}_2 \to \vec{\alpha}_2 + \vec{\alpha}_1 \quad \text{and} \quad \vec{\alpha}_1 \to \vec{\alpha}_2 + \vec{\alpha}_1 \tag{4.19}$$

[1] A calibration $p$-form $\phi$ for an $n$ dimensional complex manifold $M$ is defined by being a closed form $\mathrm{d}\phi = 0$ and minimizing the volume of $p$ dimensional submanifolds $N \subset M$. That means, if the volume of $N$ is given by $\mathrm{Vol}(N) = \int_N \phi$, then $\mathrm{Vol}(N) \leq \mathrm{Vol}(N')$, $\forall N'$ in the same homology class. See [60] for more details.

leave the lattice $\Lambda_2$ invariant. These transformations correspond to the transformations of the complex structure given by

$$\tau \to \tau + 1 \quad \text{and} \quad \tau \to \frac{\tau}{\tau + 1}\,, \tag{4.20}$$

which are the generators of the modular group $SL(2,\mathbb{Z})$. Hence two dimensional tori with complex structure moduli differing by modular transformations are equivalent, since their underlying lattices are equivalent.

(ii) **$SO(12)$-Torus $T^6_{\mathrm{SO}(12)}$:**
The six simple roots of the Lie algebra $SO(12)$

$$\begin{aligned}
&\vec{\alpha}_1 = (1,-1,0,0,0,0)^T, \quad \vec{\alpha}_2 = (0,1,-1,0,0,0)^T, \quad \vec{\alpha}_3 = (0,0,1,-1,0,0)^T,\\
&\vec{\alpha}_4 = (0,0,0,1,-1,0)^T, \quad \vec{\alpha}_5 = (0,0,0,0,1,-1)^T, \quad \vec{\alpha}_6 = (0,0,0,0,1,1)^T\,.
\end{aligned} \tag{4.21}$$

are the generators of the $SO(12)$ root lattice $\Lambda_{\mathrm{SO}(12)}$. The six dimensional torus resulting from the quotient space $\mathbb{R}^6$ divided by the $SO(12)$ root lattice, is denoted in the following as $T^6_{\mathrm{SO}(12)}$:

$$T^6_{\mathrm{SO}(12)} = \frac{\mathbb{R}^6}{\Lambda_{\mathrm{SO}(12)}}\,. \tag{4.22}$$

Since the generators cannot be sorted into three mutually orthogonal pairs the torus is non-factorisable. By turning on moduli, the $T^6_{\mathrm{SO}(12)}$ can be further deformed. In the following the deformation parameters $R_h \in \mathbb{R}$ and $\tau_h \in \mathbb{C}$ are turned on in each complex plane individually, s.t. the hermitian metric and Kähler 2-form is given by

$$h = \sum_{h=1}^{3} R_h{}^2 |1-\tau_h|^2 \mathrm{d}w_h \mathrm{d}\overline{w}_h\,, \quad \omega_2 = \sum_{h=1}^{3} R_h{}^2\, |1-\tau_h|^2\; \mathrm{d}w_h \wedge \mathrm{d}\overline{w}_h\,, \tag{4.23}$$

with the complex coordinates $w_h$ defined by

$$w_1 = y_1 + \frac{\tau_1 y_2}{1-\tau_1},\; w_2 = y_3 - \frac{y_2}{1-\tau_2} + \frac{\tau_2 y_4}{1-\tau_2},\; w_3 = y_5 - \frac{y_4}{1-\tau_3} + \frac{1+\tau_3}{1-\tau_3} y_6\,. \tag{4.24}$$

Due to the diagonal form of the hermitian metric in (4.23), one Kähler structure modulus $K_h$ and one complex structure modulus $U_h$ for each complex plane can be defined similar as for the $T^2$ by[2]

$$w_h = \mathrm{Re}(w_h) + U_h \mathrm{Im}(w_h)\,, \quad \omega_2 = \sum_{h=1}^{3} K_h\, \mathrm{Re}(\mathrm{d}w_h) \wedge \mathrm{Im}(\mathrm{d}w_h)\,. \tag{4.25}$$

The moduli for the deformed $T^6_{\mathrm{SO}(12)}$ depend on the deformation parameters accordingly by

$$U_h = \tau_h\,, \quad K_h = R_h{}^2\, |1-\tau_h|^2\, \mathrm{Im}(\tau_h)\,. \tag{4.26}$$

The deformation parameters capture the lengths and relative angles of the lattice vectors and at the point $(R_h, \tau) = (1, i)$, the undeformed $T^6_{\mathrm{SO}(12)}$ is reconstructed.

[2] Turning on moduli in each plane individually ensures that D6-branes, wrapping the $T^6_{\mathrm{SO}(12)}$ remain supersymmetric. It is further explained in section 4.1.4

### 4.1.2 Orientifolds for type IIA compactification

In this section real six (or complex three) dimensional orientifolds, suitable for Type IIA string compactification are considered. It is required that the orientifolds lead at most to $\mathcal{N} = 1$ supersymmetry in the uncompact space, otherwise a chiral spectrum in four dimensions cannot be achieved. Such orientifold arise, from six dimensional tori quotient out by an orientifold group $G_\Omega$, with $G_\Omega$ containing a discrete subgroup of $SU(3)$ and a worldsheet parity. Let the orientifold group be composed of the discrete group $G$ and the operator $\Omega R$ as

$$G_\Omega = G \cup \Omega R G\,, \tag{4.27}$$

where $G$ is a discrete subgroup of $SU(3)$ and acts on the internal space, where $\Omega R$ acts additionally as an involution on the worldsheet. An orientifold is then defined by the quotient space

$$O^6 = \frac{T^6}{G \cup \Omega R G}\,. \tag{4.28}$$

As an intermediate step before considering the full orientifold projection, orbifolds are discussed in the following. Orbifolds are given by the quotient space $T^6/G$ and are discussed in the context of string theory in [62, 63].

#### Orbifold

The discrete group $G$ is called the orbifolding group and its group elements $g = (\theta, \vec{t}) \in G$ in general consists of a rotation $\theta$ and a translation $\vec{t}$[64]. The action of $g$ on a point $x \in T^6$ is defined by

$$g:\; x \to gx = \theta x + \vec{t}. \tag{4.29}$$

The rotations are called twists and the translations are called roto-translations. The subgroup of $G$ consisting only of the rotations is called the point group $P$ and has to be an automorphism of the underlying lattice of the torus. $G$ is of order $N$, when $g^N$ acts trivially on the torus (but not necessarily on the covering space $\mathbb{R}^6$). That implies, that the action of $g^N$ on a point $x \in T^6$ is trivial up to lattice translations

$$g^N x = x + \vec{\lambda}, \quad \text{with} \quad \vec{\lambda} \in \Lambda^6\,. \tag{4.30}$$

For orbifolding groups with no roto-translations, $g^N$ acts also trivially on the covering space. The group containing the orbifolding group and all lattice translations $\vec{\lambda} \in \Lambda^6$ is called the space group $S = \{\Lambda^6, G\}$ of the orbifold and the quotient space

$$\frac{\mathbb{R}^6}{S} = \frac{T^6}{G} \tag{4.31}$$

is called an orbifold. In the following only orbifolding groups with trivial roto-translations are considered, s.t. the orbifolding group is equivalent to its point group $P$ and the space group is just the semidirect product $S = P \ltimes \Lambda^6$. The rotations of the point group can be expressed in representations of $SO(6)$ transformations. Denoting the three Cartan generators of $SO(6)$ by $H_h$, with $h \in \{1, 2, 3\}$, a rotation $\theta \in P$ is given by

$$\theta = \exp\left(2\pi i \sum_{h=1}^{3} v_h H_v\right)\,. \tag{4.32}$$

The vector $\vec{v} = (v_1, v_2, v_3)^T$ is called the twist vector. Each Cartan generator acts on a complex plane, s.t. the action of $\theta$ on the complex coordinates in the compact space is given by

$$\theta : (z_1, z_2, z_3) \to \left(e^{2\pi i v_1} z_1, e^{2\pi i v_2} z_2, e^{2\pi i v_3} z_3\right) . \tag{4.33}$$

On fermionic fields $\psi$, $\theta$ acts as

$$\theta : |s_1, s_2, s_3\rangle \to e^{2\pi i \sum_h s_h v_h} |s_1, s_2, s_3\rangle, \quad \text{with} \quad s_h = \pm 1/2 , \tag{4.34}$$

where $|s_1, s_2, s_3\rangle$ denotes the state of $\psi$ in the spin representation of $SO(6)$. For $P$ to be of order $N$ the components of the twist vector must satisfy

$$N v_h \in \mathbb{Z}, \quad \forall h \in \{1, 2, 3\} . \tag{4.35}$$

Further, in order to preserve a four dimensional $\mathcal{N} = 1$ supersymmetry generator in the uncompact space from a ten dimensional supersymmetry generator, $P$ must preserve one Killing spinor on the orbifold. For example the state $|\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\rangle$ has to be invariant under the action of $P$. Applying (4.34) on the spinor, it remains invariant only for twist vectors with components satisfying

$$v_1 + v_2 + v_3 = 0 \mod 2 . \tag{4.36}$$

Thus the rotations of $P$ are traceless and hence $P$ is given by a discrete subgroup of $SU(3)$. Some space group elements $(\theta, \vec{\lambda})$ leave some points FIX($g$) on the torus invariant, which are called fixed points. They are determined by the fixed point equation

$$\text{FIX}(g) = \left\{ x = g \cdot x + \vec{\lambda} \,\middle|\, x \in T^6, \, g \in P, \, \vec{\lambda} \in \Lambda^6 \right\} . \tag{4.37}$$

Excluding the fixed points from the orbifold, the local geometry of the remaining space is that of the covering space $\mathbb{R}^6$. That means locally, infinitesimal distances are determined by the euclidean line element

$$d^2 s = \delta_{h\bar{h}} dz^h d\bar{z}^{\bar{h}}, \quad \forall \, z \in \frac{T^6}{P} \backslash \{\text{FIX}(g)\}_{g \in P} . \tag{4.38}$$

However the fixed points contain singular curvature, as can be seen, when tangent vectors are parallel transported around a fixed point: Let $\gamma$ be the path between two points $x_1$ and $x_2$ differing by a spacegroup element

$$\gamma(c)\big|_{c=0} = x_1, \quad \gamma(c)\big|_{c=1} = \theta \cdot x_1 + \vec{\lambda} = x_2, \quad c \in [0, 1] . \tag{4.39}$$

The tangent vector $\frac{d}{dc}\gamma\big|_{c=0} \in T_p M\big|_{x_1}$ parallel transported along $\gamma$ to the point $x_2$ gets rotated, s.t. $\frac{d}{dc}\gamma\big|_{c=0}$ differs by a $\theta$ transformation from the tangent vector $\frac{d}{dc}\gamma\big|_{c=1} \in T_p M\big|_{x_2}$. On the covering space $\mathbb{R}^6$, this would not lead to any consequences, but on the orbifold the tangent spaces $T_p M\big|_{x_1}$ and $T_p M\big|_{x_2}$ are identified and the path $\gamma$ is a closed loop on the orbifold. Hence the holonomy group of the orbifold has to be given by the point group $P \subset SU(3)$, because parallel transporting tangent vectors rotate them by the action of $P$. However the remaining space, without the fixed points, is euclidean, as described in (4.38), so the entire curvature of the orbifold has to be densed at the fixed points. Therefor the curvature becomes singular at the fixed points. Due to the presence of the curvature singularities the orbifold is not a manifold (but an almost manifold so to say).

Next a closer look at supersymmetry from orbifold compactifiaction is taken: Let the supersymmetry charges be denoted by $Q_\alpha$, with $\alpha \in \{1, ..., 16\}$. They are the components of a spinor belonging to the

Majorana-Weyl representation of $SO(1,9)$. The 16 spin states are labeled by

$$Q_\alpha = |s_0, s_1, s_2, s_3\rangle, \quad \text{with} \quad s_i = \pm 1/2, \,. \tag{4.40}$$

After compactification of the ten dimensional spacetime on the orbifold

$$\mathcal{M}^{10} \to \mathbb{R}^{1,3} \times \frac{T^6}{P}, \quad \text{with} \quad P \subset SU(3)\,, \tag{4.41}$$

transformations of $SO(1,9)$ decompose into a $SO(1,3)$ and $SO(6)$ part, where the $SO(1,3)$ is the Lorentz group acting on the uncompact space and the $SO(6)$ part is the rotation group acting on the compact space. The values for $s_1$, $s_2$, $s_3$ correspond to the eigenvalues of the three Cartan generators $H_i$ of the rotation group in the compact space, as described in (4.34), and $s_0$ denotes the spin state in a Weyl representation of the Lorentz group $SO(1,3)$. The holonomy group of the internal space ensures that all states which transform non-trivially under (4.34) gets projected out. The only states of $Q_\alpha$, which are covariantly constant on the orbifold, are

$$Q_\xi = \left|s_0, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}\right\rangle, \quad \text{and} \quad \overline{Q}_{\dot\xi} = \left|s_0, -\tfrac{1}{2}, -\tfrac{1}{2}, -\tfrac{1}{2}\right\rangle, \tag{4.42}$$

with $\xi, \dot\xi \in \{1,2\}$ for $s_0 \in \{\pm\frac{1}{2}\}$. The states $Q_\xi$ serve as the four dimensional $\mathcal{N} = 1$ supersymmetry generator and $\overline{Q}_{\dot\xi}$ its complex conjugated. Hence the orbifold preserves from each ten dimensional supersymmetry generator $Q$, the generators of the four dimensional $\mathcal{N} = 1$ supersymmetry algebra. For string theories with $\mathcal{N} = 1$ supersymmetry in ten dimension, such as heterotic string theories, compactifying on an orbifold with $SU(3)$ holonomy leads to $\mathcal{N} = 1$ supersymmetry in four dimensions. These kinds of models have been intensively studied for example in [65–67]. But for Type II string theories, compactification on orbifolds with $SU(3)$ holonomy leads to $\mathcal{N} = 2$ supersymmetry in four dimensions and brings a non chiral spectrum in four dimensions with it.

#### Strings on orbifolds

Closed string states on the orbifold are inherited by string states on the torus, which survive the point group projection. They are determined by the transformation behaviour of the states under the twist operators. By expressing the twist operators in representations of $SO(6)$ as in (4.32) the transformation on left- and rightmoving string states under the point group can be determined. Closed string states preserved by the point group are those, which transform trivially under all twist elements. Such string states which are preserved from the spectrum on the torus are called untwisted states.

However the non-trivial structure of the orbifold fixed points mirrors in the spectrum from closed strings on the orbifold. Open strings on the torus are closed on the orbifold, when their end points are mapped by a rotation of $g \in P$ to each other. That way new closed string states, called twisted closed strings and which did not exist on the torus, can arise on the orbifold. They wrap around orbifold fixed points and hence are located at the fixed points. Twisted closed strings are determined by twisted boundary conditions (similar to open strings at intersecting D-branes). A twisted closed string, which is closed up to a rotation of $g$ on the orbifold and therefore located at a $g$ fixed point, is determined by the $g$-twisted boundary conditions [63]

$$X^\mu(\sigma^0, \sigma^1 + \ell) = gX^\mu(\sigma^0, \sigma^1) + 2\pi RW\,, \quad \psi^\mu(\sigma^0, \sigma^1 + \ell) = \pm g\psi^\mu(\sigma^0, \sigma^1)\,, \tag{4.43}$$

where $R$ is the compactification radius of the $\mu$-th direction and $W \in \mathbb{R}$ is the winding number of the closed string around the $\mu$-th direction. The necessity of twisted closed strings is realized, when taking a look at the one-loop vacuum corrections from closed strings. The one-loop amplitude is only modular invariant for closed strings in orbifolds, when states from twisted closed strings are included [63]. As an example, in section 4.2.1 and appendix A, the massless Type IIA closed strings states in the untwisted and twisted sector on a $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold are explicitly computed.

### Orientifold

Type II string compactification on orbifolds lead to $\mathcal{N} = 2$ supersymmetry in four dimensions. Hence the supersymmetry charges need to be further halved. The two supersymmetry generators $Q_L$ and $Q_R$ of Type II belong to the left- and rightmoving sector respectively. An operation relating the leftmoving sector with the rightmoving sector and vice versa is given by worldsheet parity. Worldsheet parity is defined by inverting the orientation of the string and by denoting the worldsheet parity operator by $\Omega$, it is acts on the worldsheet coordinates as

$$\Omega : \; (\sigma^0, \sigma^1) \to (\sigma^0, \ell - \sigma^1)\,. \tag{4.44}$$

The symmetry between the left- and rightmoving sector in Type IIB , allows to quotient out the action of $\Omega$ from Type IIB. That means only states invariant under $\Omega$ are preserved, which means for $Q_L$ and $Q_R$, that only a linear combination of those two are preserved, leaving 16 of the 32 supersymmetry charges remaining. Further compactification on an orbifold eliminates 12 degrees of freedom and leads to $\mathcal{N} = 1$ supersymmetry in four dimensions. Hence compactifying Type IIB on an orientifold, with $G_\Omega = G \cup \Omega G$ in (4.27), is a possible choice in order to receive a chiral spectrum in four dimensions. However this is not the only possibility: T-dualiy exposes other possibilities for $\Omega R$. Since $\Omega$ exchanges the left- and rightmoving sector, it acts on the pair of fields $X^\mu = (X^\mu_L, X^\mu_R)$ as

$$\Omega : \; (X^\mu_L, X^\mu_R) \to (X^\mu_R, X^\mu_L)\,. \tag{4.45}$$

The pair $(X^\mu_L, X^\mu_R)$ has to belong to a theory $\mathcal{S}$, which is invariant under $\Omega$, such as a ten dimensional Type IIB. In order to determine the corresponding operator $\tilde{\Omega}$ in the theory $\tilde{\mathcal{S}}$, with a direction $x^a$ T-dualized, one has to transform the pair $\tilde{X}^a = (\tilde{X}^a_L, \tilde{X}^a_R)$ from $\tilde{\mathcal{S}}$ into $\mathcal{S}$ by using T-duality, given in (3.60), apply $\Omega$ and then T-dualize the transformed fields back to $\tilde{\mathcal{S}}$. The computations

$$\tilde{\Omega} : \; (\tilde{X}^a_L, \tilde{X}^a_R) \to T\Omega T\tilde{X}^a = T\Omega(X^a_L, -X^a_R) = T(-X^a_R, X^a_L) = -(\tilde{X}^a_R, \tilde{X}^a_L)\,, \tag{4.46}$$

exposes that worldsheet parity for T-dualized directions, has to be afflicted with a spacetime reflection $R$ along the T-dualized directions

$$\tilde{\Omega} = \Omega R\,, \quad \text{with} \quad R : \; x^a \to -x^a\,. \tag{4.47}$$

By reflecting a spacetime direction the string oscillator modes for that direction also gets a sign. Since the R zeromodes correspond to the eight dimensional Gamma matrices $\Gamma^a$, $\Gamma_{\text{chiral}} = 16 \prod_{\mu=2}^{9} \Gamma^\mu$ obtains for each reflected direction $x^a$ a sign. For $\tilde{\Omega}$ to respect the GSO-projection (3.31), the spacetime involution $R$ has to act on the chirality operator as

$$R : \; \Gamma_{\text{chiral}} \to \begin{cases} \Gamma_{\text{chiral}} & \text{for Type IIB,} \\ -\Gamma_{\text{chiral}} & \text{for Type IIA,} \end{cases} \tag{4.48}$$

and consequently $R$ can be given by

$$R:\ z_i \to \bar{z}_i, \quad \text{for} \quad \begin{cases} \text{even number of } z_i & \text{for Type IIB,} \\ \text{odd number of } z_i & \text{for Type IIA.} \end{cases} \tag{4.49}$$

Further $\Omega R$ has to map spacetime fermions from the leftmoving sector to spacetime fermions in the rightmoving sector and vice versa, s.t. a representation for $\Omega R$ in the spinor space is given by $\prod_a (\Gamma^a \Gamma_{\text{chiral}})$, with $x^a$ the directions on which $R$ acts non-trivially[3]. Hence the 16 supercharges of $Q_L$ from the leftmoving sector get mapped to the 16 supercharges of $Q_R$ of the rightmoving sector and vice versa by $\Omega R$, s.t. quotienting out $\Omega R$ preserves only a linear combination $Q$ of the two supersymmetry generators given by

$$Q = Q_L + \prod_a \Gamma^a \Gamma_{\text{chiral}} Q_R\,, \tag{4.50}$$

with $a$ denoting the T-dualized directions, starting from ten dimensional Type IIB. Compactifying Type II string theory on $T^6/(G \cup \Omega RG)$, with $G \subset SU(3)$, $\Omega R$ cancels 16 of the 32 degrees of freedom of the supersymmetry charges and $G$ eliminates 3/4 of the supercharges, s.t. four supercharges remain and form a Dirac spinor for the $\mathcal{N} = 1$ supersymmetry generator in four dimensions.

Besides fixed points, preserved by the orbifolding group, the orientifold preserves orientifold fixed planes, also named O-planes. For a given group element $\Omega Rg \in G$, where $g \in P$, an O-plane preserved by the group element is defined by the plane consisting of the set of points, which is left invariant under $\Omega Rg$

$$\text{FIX}(\Omega Rg) = \left\{ x = Rgx + \vec{\lambda} \,\middle|\, x \in T^6,\ \vec{\lambda} \in \Lambda^6 \right\}. \tag{4.51}$$

O-planes couple to R-R closed strings and therefore have non-trivial RR charge [41, 45, 69]. The couplings to R-R strings can potentially lead to divergent terms in the vacuum energy called tadpoles. It will be discussed in section 4.2.3 how tadpoles are cancelled by placing D-branes with the opposite R-R charge. In the following of this chapter it will be focused on Type IIA compactified on orientifolds, with $R$ acting non-trivially on three directions $x^5, x^7, x^9$, s.t. O-planes are determined by

$$(z_1, z_2, z_3) = (\mathrm{e}^{-2\pi i v_1}\bar{z}_1, \mathrm{e}^{-2\pi i v_2}\bar{z}_2, \mathrm{e}^{-2\pi i v_3}\bar{z}_3) + \vec{\lambda}\,, \tag{4.52}$$

for $\vec{v} = (v_1, v_2, v_3)^T$ the twist vectors of the point group elements. The O-planes are called O6-planes, because they fill six spatial directions: three in the compact and three in the uncompact space.

### 4.1.3 Fixed point resolution

The orbifold itself is due to the presence of curvature singularities in the fixed points not a manifold. However it exists a definite prescription to resolve the orbifold singularities. In this section it will be briefly sketched how to find the right resolutions to orbifold singularities. For a more detailed decription see for example [70–72]. Orbifolds with the point group given by $\mathbb{Z}_N$ or $\mathbb{Z}_M \times \mathbb{Z}_N$ is considered in the following. The resulting orbifold singularities are locally like $\mathbb{C}^3/\mathbb{Z}_N$ singularities. First the following objects have to be introduced:
Let $N$ be a lattice isomorphic to $\mathbb{Z}^n$. Let $V = \{v_i\}_{i\in\{1,\dots,d\}}$, with $d > n$, be a collection of lattice vectors in

[3] Since $\Gamma_{\text{chiral}}$ anticommutes with $\Gamma^a\Gamma_{\text{chiral}}$, $\prod_a \Gamma^a\Gamma_{\text{chiral}}\psi$ flips its chirality of $\psi$ for each direction $x^a$. For more details see [19, 68].

$N$. A cone $\sigma$ spanned by a subset of lattice vectors $\{v_i\}_{i\subseteq\{1,\dots,d\}}$ is given by

$$\sigma = \langle v_{i_1}, v_{i_2}, \dots\rangle = \Big\{ \sum_i a_i v_i \Big| a_i \in \mathbb{R}_{\geq 0},\ i \subseteq \{1, \dots, d\}\Big\} \tag{4.53}$$

and is called convex if $\omega \in \sigma - \{0\}$, then $-\omega \notin \sigma$. If the cones constructed out of $V$ are all convex and their faces are again lower dimensional cones and furthermore the intersection of two cones are also cones, then the collection of the cones or the set $V$ is called a fan $\Sigma$. To a fan $\Sigma$ a toric variety $X_\Sigma$ can be associated the following way: Let $z_i$ be a homogeneous coordinate in $\mathbb{C}^d$, associated to a vector $v_i$ of the fan. To each vector $v_i$ a codimension one divisor $D_i$ can be assigned by $D_i = \{z_i = 0\}$. The vectors $v_i$ need to satisfy $d - n$ linear equation

$$\sum_{i=1}^{d} l_i^{(a)} v_i = 0, \quad a = 1, \dots, d-n, \quad l_i^{(a)} \in \mathbb{Z}. \tag{4.54}$$

The above linear equation defines a $(\mathbb{C}^*)^{d-n}$ action on the homogeneous coordinates

$$\mathbb{C}^* : (z_1, \dots, z_d) \sim (\lambda_a^{l_1^{(a)}} z_1, \dots, \lambda_a^{l_d^{(a)}} z_d). \tag{4.55}$$

Let the exclusion set $F$ be defined by the set of simultaneous zero loci of the homogeneous coordinates, whose assosiated vectors do not spann a fan

$$F = \bigcup_i \{z_{1_i} = 0, \dots, z_{k_i} = 0\}, \quad \langle v_{1_i}, \dots, v_{1_k}\rangle \notin \Sigma. \tag{4.56}$$

The toric variety is then given by the quotient space

$$X_\Sigma = \frac{\mathbb{C}^d - F}{(\mathbb{C}^*)^{d-n}}. \tag{4.57}$$

The space $X_\Sigma$ is smooth if the set of vectors $V$ cover all points in the lattice $N$. These tools can now be used to resolve orbifold singularities. Let the orbifold action be generated by

$$\theta : (z_1, z_2, z_3) \to (\epsilon^{n_1} z_1, \epsilon^{n_2} z_2, \epsilon^{n_3} z_3), \quad \epsilon = \mathrm{e}^{2\pi i/n}, \quad \sum_{i=1}^{3} n_i = 0 \mod n. \tag{4.58}$$

From the orbifold action it can be read of that three vectors belonging to the fan and which correspond to the coordinates $z_i$ have to satisfy

$$n_1 v_1 + n_2 v_2 + n_3 v_3 = 0 \mod n. \tag{4.59}$$

Due to the condition that the orbifold group is a subgroup of $SU(3)$, the coordinates for the three $v_i$'s can always be set to one in one plane. When the orbifold contains singularities, the three $v_i$'s are not enough to reach all lattice points in $N$, indicating that the orbifold is not smooth. Next one adds the missing vectors $x_i$, s.t. the resulting space gets smooth. Adding new vectors $x_i$ means introducing new coordinates $w_i$ corresponding to exceptional divisors at $D_i = \{w_i = 0\}$. The set of vectors $\{v_i, w_i\}$ and a collection of the cones, which they span, generates the fan of the resolved orbifold $\mathrm{Res}\left(\mathbb{C}^3/\mathbb{Z}_n\right)$. The exclusion set $F$ contains all fixed points and are replaced by the exceptional divisors at $D_i = \{w_i = 0\}$. That way the fixed points are removed and replaced by smooth manifolds, s.t. the resulting space is smooth.

### 4.1.4 3-cycles on tori and orbifolds

#### Homology

In order to introduce cycles on compact manifolds arguments in [59, 73] are followed. Let $M$ be a dimensional compact manifold and the set $\left\{N_p^i\right\}_{i\in\{1,2,\ldots\}}$ contain the $p$ dimensional submanifolds $N_p \subseteq M$, with $k \leq \dim(M)$. They form a vectorspace $C_p$ of $p$-chains, where a $p$-chain is given by linear combinations of the submanifolds

$$C_k(M) = \left\{ a_p = \sum_i x_i Z_p^i \,\middle|\, x_i \in \mathbb{R},\, Z_p^i \subseteq M,\, \dim(N_p^i) = p \leq \dim(M) \right\} . \tag{4.60}$$

Let the interior $\mathrm{Int}(N)$ of a $p$ dimensional manifold $N$ denotes the set of points in $N$, whose neighbourhood is given by an open subset of $\mathbb{R}^p$. The boundary $\partial N$ of $N$ is then given by the complement of the interior of $N$ and maps $p$-chains to $(p-1)$-chains

$$\partial : \, C_p \to C_{p-1} \, , \quad \text{with} \quad \partial N = N \backslash \mathrm{Int}(N) \, . \tag{4.61}$$

A $p$-chain $a_p$ without a boundary is called a $p$-cycle of $M$ and the set of $p$-cycles $Z_p$ are given by

$$Z_p(M) = \left\{ a_p \in C_p \,\middle|\, \partial a_p = 0,\, a_p \in M \right\} . \tag{4.62}$$

Since the boundary of a boundary is always zero a $p$-cycle $a_p$, which is already the boundary of a $(p+1)$-chain is a trivial $p$-cycle. Let $B_p$ denote the set of trivial $p$-cycles

$$B_p(M) = \left\{ \partial a_{p+1} \in C_p \,\middle|\, a_{p+1} \in C_{p+1},\, a_{p+1} \in M \right\} , \tag{4.63}$$

then the space of the $p$-th homology $H_p(M)$ of $M$ is given by

$$H_p(M) = \frac{Z_p(M)}{B_p(M)} , \tag{4.64}$$

and contains the set of non-trivial $p$-cycles on $M$. All $p$-cycles, which are related by trivial $p$-cycles belong to the same homology class $\left[a_p\right]$

$$\left[a_p\right] = \left\{ a_p \simeq a_p + \partial b_{p+1} \,\middle|\, a_p \in H_p,\, \partial b_{p+1} \in B_p \right\} . \tag{4.65}$$

The $p$-th homology has a vectorspace structure with the different homology classes serving as a basis and the dimension of $H_p$ given by the number of homology classes.

For example, the first homology $H_1(T^2)$ of the two dimensional torus $T^2$ is two dimensional. It is spanned by the 1-cycles

$$a = \left\{ \vec{x} = \mu\vec{\alpha}_1 \,\middle|\, \mu \in [0,1) \right\} \quad \text{and} \quad b = \left\{ \vec{x} = \nu\vec{\alpha}_2 \,\middle|\, \nu \in [0,1) \right\} . \tag{4.66}$$

Due to the identification of the points $0 \sim \vec{\alpha}_1$, $a$ has no boundaries and similar due to $0 \sim \vec{\alpha}_2$, $b$ has also no boundaries. A cycle $a'$ parallel to $a$, belongs to the same homology class as $a$, because $a$ and $a'$ form the boundary of a segment of the torus, which is a trivial 2-chain. Since the difference of $a$ and $b$ is not a segment of the $T^2$, they lie in different homology classes.

In the following D6-branes are discussed intensively. They wrap volume minimizing 3-cycles in the compact space which are given by special Lagrangian (sLag) cycles[74]. Hence the third homology of six dimensional tori and orientifolds $H_3(T^6)$ and $H_3(O^6)$ are particularly interesting for this work. First sLag 3-cycles on the torus are discussed. When the orientifold is considered, it is distinguished between two kinds of 3-cycles: Bulk 3-cycles, which are inherited from the underlying torus, and fractional 3-cycles, wrapping the exceptional divisors at orbifold fixed points.

### Special Lagrangian cycles on the $T^6$

A particular class of cycles, which are important in the context of D6-branes, are called special Lagrangian (sLag) cycles. They have the important property of being volume minimizing in their homology class [60]. An $n$ dimensional submanifold $N \subset M$ is volume minimizing, when it is calibrated by a calibration $n$-form [75]. That means, when $J$ is a calibration $n$-form on $M$, then $N$ is volume minimizing, when its volume is given by

$$\mathrm{Vol}(N) = \int_N J\,. \tag{4.67}$$

The sLag conditions for a $p$-cycle $\Pi^p$ in a compact manifold $M$ with Kähler 2-form $\omega_2$ and volume $n$-form $\Omega_n$ are [60, 75, 76]

- $\dim\left(\Pi^k\right) = \frac{1}{2}\dim(M)$,
- $\omega_2\big|_{\Pi^k} = 0$,
- $\mathrm{Im}(\mathrm{e}^{i\vartheta}\Omega_n)\big|_{\Pi^k} = 0$,

with $\vartheta \in [0, 2\pi]$ some phase called the calibration phase. On the $T^6$ the sLag conditions are solved by 3-cycles $\Pi^3$, which factorise into three mutually orthogonal 1-cycles: Let $\Pi^1_{(h)} \in H_1(T^6)$, for $h \in \{1,2,3\}$, with $\Pi^1_{(1)} \perp \Pi^1_{(2)} \perp \Pi^1_{(3)}$. Then $\Pi^3 \in H_3(T^6)$, with

$$\Pi^3 = \prod_{h=1}^{3} \Pi^1_{(h)}\,, \tag{4.68}$$

is a sLag 3-cycle, when $\Pi^1_{(h)}$ has the shape of a straight line in the $h$-th complex plane. The relative angles $\theta_h$ of $\Pi^1_{(h)}$ to the real axis are related to the calibration phase by $\theta_1 + \theta_2 + \theta_3 = \vartheta$, s.t.

$$\sum_{h=1}^{3} \arctan\left(\frac{\left\|\mathrm{Im}\left(\Pi^1_{(h)}\right)\right\|}{\left\|\mathrm{Re}\left(\Pi^1_{(h)}\right)\right\|}\right) = \vartheta\,, \tag{4.69}$$

with $\left\|\mathrm{Re}\left(\Pi^1_{(h)}\right)\right\|$ and $\left\|\mathrm{Im}\left(\Pi^1_{(h)}\right)\right\|$ the length of the real and complex part of $\Pi^1_{(h)}$ in the $h$-th complex plane on the fundamental domain of the $T^6$. On the factorisable torus each 1-cycle $\Pi^1_{(h)}$ can be parametrized by integers $n^h, m^h \in \mathbb{Z}$, which denote the wrapping numbers around the $a$ and $b$ cycle in $H_1(T^2_{(h)})$ on each $T^2$ factor. For the non-factorisable torus the notation can be adopted when it is possible to introducing a factorisable lattice $\Lambda^6_{\text{fact}}$, where the underlying lattice of the $T^6$ is a sublattice $\Lambda^6 \subset \Lambda^6_{\text{fact}}$. Then the integers $n^h, m^h$ parametrize wrappings numbers of $\Pi^1_{(h)}$ along two basis vectors of $\Lambda^6_{\text{fact}}$ in the $h$-plane and the whole 3-cycle is determined by six integers $\left\{n^h, m^h\right\}_{\{} h \in \{1,2,3\}\}$ and its position on the covering space $\mathbb{R}^6$ is determined by

$$\Pi^3 = \prod_{h=1}^{3}(n^h, m^h) =: \left\{ \sum_{h=1}^{3} \left(n^h \vec{\alpha}_{2h-1} + m^h \vec{\alpha}_{2h}\right)\mu_h \,\middle|\, \mu_1,\, \mu_2,\, \mu_3 \in \mathbb{R} \right\}, \tag{4.70}$$

where $\vec{\alpha}_{2h-1}$, $\vec{\alpha}_{2h}$ are the two generators of $\Lambda^6_{\text{fact.}}$, which lie in the $h$-plane. Since the basis vectors of $\Lambda^6_{\text{fact}}$ are in general shorter than the basis vectors of the non factorisable lattice, it can happen that multiple wrappings on the factorisable lattice correspond to one wrapping number around the non factorisable torus.

#### Example: sLag 3-cycles on $T^6_{\mathbf{SO(12)}}$:

Here sLag cycles on the $T^6_{\text{SO}(12)}$ shall be investigated. The Kähler 2-form $\omega_2$ vanishes at a subspace $N$, when each term $\mathrm{d}z_h \wedge \mathrm{d}\bar{z}_h$ vanishes individually as it is the case for example for a straight line in the $h$-th plane: The term $\mathrm{d}z_h \wedge \mathrm{d}\bar{z}_h$ at

$$z_h = \left\{ \left(n^h \vec{e}_{2h-1} + i m^h \vec{e}_{2h}\right)\mu \,\middle|\, \mu \in \mathbb{R} \right\}, \tag{4.71}$$

with $n^h, m^h$ integers, becomes $\left(\left(n^h\right)^2 - \left(m^h\right)^2\right)\mathrm{d}\mu \wedge \mathrm{d}\mu$ and vanishes due to the antisymmetric properties of the wedge product. Consequently the Kähler 2-form vanishes at the subspace given by

$$\vec{z} = \left\{ \sum_{h=1}^{3} \left(n^h \vec{e}_{2h-1} + i m^h \vec{e}_{2h}\right)\mu_h \mid \mu_1, \mu_2, \mu_3 \in \mathbb{R} \right\}. \tag{4.72}$$

The integers $n^h, m^h$ denote wrapping numbers around generators of the lattice $\mathbb{Z}^6 = \langle \vec{e}_1, ..., \vec{e}_6 \rangle$. The $SO(12)$ Lie root lattice is a sublattice of $\mathbb{Z}^6$, and sLag 3-cycles on $T^6_{\text{SO}(12)}$ can therefor be described by (4.72), where (4.71) can be used to describe the 1-cycle factors of the sLag cycles. When $\vec{x}\big|_{\mu=0}$ is identified with $\vec{x}\big|_{\mu=1}$ by some $SO(12)$ roots, the line gets compactified to a 1-cycle on $T^6_{\text{SO}(12)}$. However not all integers integers $n^h, m^h$ describe closed cycles on the torus. Only for $n^h + m^h \in 2\mathbb{Z}$, the points at $\mu = 0$ get identified with points at $\mu = 1$ for (4.71), which means for example a cycle with wrapping numbers $(n^h, m^h) = (2, 0)$ wraps the $T^6_{\text{SO}(12)}$ only once, even though it wraps $\Lambda^6_{\text{fact}}$ twice. For wrapping numbers satisfying

$$n^h + m^h \in 2\mathbb{Z}, \quad \forall h \in \{1, 2, 3\}, \tag{4.73}$$

the corresponding 3-cycle is closed on $T^6_{\text{SO}(12)}$ [77]. If the condition (4.73) is not fulfilled, then $\{\vec{z}\}$ in (4.72) covers for $\forall_{h\in\{1,2,3\}}\ \mu_h \in [0, 1)$ only half of a closed 3-cycle on $T^6_{\text{SO}(12)}$ [77]. By doubling the the wrapping numbers $n^h$, $m^h$ of such a half cycle in a plane $h$, for which (4.73) is violated, one receives a closed cycle on the $T^6_{\text{SO}(12)}$. For the case, where (4.73) is violated in only one plane, doubling the wrapping numbers in that plane, satisfies (4.73) clearly. For the cases where (4.73) is violated for two or three planes, it is more subtle, that only one factor of 2 is enough to close the 3-cycle on the $T^6_{\text{SO}(12)}$. As an example a closer look at

$$\prod_{h=1}^{3}(n^h, m^h) = \left\{ \sum_{h=1}^{3}(n^h \vec{e}_{2h-1} + i m^h \vec{e}_{2h})\mu_h \,\middle|\, \mu_1, \mu_2, \mu_3 \in [0, 1) \right\}, \tag{4.74}$$

with

$$n^1 + n^1 = \text{odd}\,, \quad n^2 + n^2 = \text{odd}\,, \quad n^3 + n^3 = \text{even}\,, \tag{4.75}$$

is taken. The cycle does not satisfy (4.73) and therefor only wraps half of a 3-cycle on $T^6_{\mathrm{SO}(12)}$. However $2\prod_{h=1}^3(n^h, m^h)$ can be expressed by [77]

$$\begin{aligned} &2\prod_{h=1}^{3}(n^h, m^h) = \\ &\left\{(n^1\vec{e}_1 + m^1\vec{e}_2 + n^2\vec{e}_3 + m^2\vec{e}_4)\nu_1 + (n^1\vec{e}_1 + m^1\vec{e}_2 + n^2\vec{e}_3 - m^2\vec{e}_4)\nu_2 + (n^3\vec{e}_5 + m^3\vec{e}_6)\mu_3\right\} , \end{aligned} \tag{4.76}$$

with $\nu_1, \nu_2, \mu_3 \in [0, 1)$, which is indeed closed on the $SO(12)$ lattice, since the vectors $(n^1, m^1, n^2, m^2, 0, 0)$, $(n^1, m^1, n^2, -m^2, 0, 0)$ and $(0, 0, 0, 0, n^3, m^3)$ are $SO(12)$ roots. For wrapping numbers violating (4.73) in all three planes, the doubled 3-cycle can also be expressed by $SO(12)$ lattice vectors in a similar way by

$$\begin{aligned} &2\prod_{h=1}^{3}(n^h, m^h) = \\ &(n^1, m^1, n^2, m^2, 0, 0)^T\mu_1 + (n^1, m^1, n^2, -m^2, 0, 0)^T\mu_2 + (0, 0, n^2, m^2, n^3, m^3)^T\mu_3 , \end{aligned}$$

with $\mu_1, \mu_2, \mu_3 \in [0, 1)$. Turning on deformation parameters $R_h$ and $\tau_h$ as described in section 4.1.1, the position of the 3-cycles is given by

$$\prod_{h=1}^{3}\left(n^h, m^h\right) = \sum_{h=1}^{3} R_h\left(\left(n^h + \mathrm{Re}(\tau_h)m^h\right)\vec{e}_{2h-1} + i\mathrm{Im}(\tau_h)m^h\vec{e}_{2h}\right)\mu_h \tag{4.77}$$

and the calibration phase of the cycle is given by

$$\sum_{h=1}^{3}\arctan\left(\frac{\mathrm{Im}(\tau_h)m^h}{n^h + \mathrm{Re}(\tau_h)m^h}\right) = \vartheta . \tag{4.78}$$

That way, sLag 3-cycles on the $T^6_{\mathrm{SO}(12)}$ remain sLag after turning on deformations. It also means that the factorisable lattice $\Lambda_{\mathrm{fact}}$, containing $\Lambda_{\mathrm{SO}(12)}$, has to be deformed as well and is actually spanned by

$$\begin{aligned} &K_1\,(1, 0, 0, 0, 0, 0)^T , \quad K_1\,(\mathrm{Re}(\tau_1), \mathrm{Im}(\tau_1), 0, 0, 0, 0)^T , \\ &K_2\,(0, 0, 1, 0, 0, 0)^T , \quad K_2\,(0, 0, \mathrm{Re}(\tau_2), \mathrm{Im}(\tau_2), 0, 0)^T , \\ &K_3\,(0, 0, 0, 0, 1, 0)^T , \quad K_3\,(0, 0, 0, 0, \mathrm{Re}(\tau_3), \mathrm{Im}(\tau_3))^T , \end{aligned} \tag{4.79}$$

when deformations are turned on and $\mathbb{Z}^6 = \Lambda_{\mathrm{fact}}$ holds for the point $(R_h, \tau) = (1, i)$.

### Bulk- and fractional cycles on the orbifold

By quotienting the point group out of the torus, only cycles, which are invariant under the point group, remain on the orbifold. A superposition of a torus 3-cycle with all its images under the point group is called a bulk cycle $\Pi^3_{\mathrm{Bulk}}$ [78]. Therefore it is naturally invariant under the point group action and for an orbifold with the point group $P$ given by

$$\Pi^3_{\mathrm{Bulk}} = \sum_{g\in P} g\cdot\Pi^3 \in H_3\left(T^6/P\right) . \tag{4.80}$$

The exceptional divisors may also contain 3-cycles, so called twisted cycles $\Pi^3_{\text{twist}}$, and when they are preserved by the point group they can be used to construct fractional cycles [78, 79]. A fractional cycle consists of a torus cycle $\Pi^3$, whose images under the point group action are collapsed to twisted cycles at the fixed points, which intersect with $\Pi^3$. A fractional cycle $\Pi^3_{\text{frac}}$ is then given by

$$\Pi^3_{\text{frac}} = \Pi^3 + \sum_{g \in P} \epsilon^g_\alpha \Pi^3_{\text{twist},g} , \tag{4.81}$$

where $\alpha$ denote the fixed points, which lie on $\Pi^3$ and $\epsilon^g_\alpha$ encodes the orientation of $\Pi^3_{\text{twist},g}$. Fractional cycles are preserved by the orbifold projection, when discrete torsion is turned on [78, 80]. Discrete torsion denotes the non trivial choice of a phase $\epsilon(g,h)$ in the orbifold partition function [79]

$$Z = \frac{1}{N} \sum_{h \in P} \sum_{g \in P} \epsilon(g,h) Z(h,g). \tag{4.82}$$

The term $Z(g,h)$ denotes the sums over the closed string states from the twisted sector $g$, forming invariant states under the action of $h$ [4]. Let $T$ and $S$ denote the generators of the modular group $SL(2,\mathbb{Z})$ transforming the complex structure modulus $\tau$ of a $T^2$ by

$$T : \tau \to \tau + 1 , \quad S : \tau \to -1/\tau . \tag{4.83}$$

Applying $T$ and $S$ transformations on the closed string partition function, the terms in the partition transform as

$$T : Z(g,h) \to Z(g,hg) , \quad S : Z(g,h) \to Z(h,g^-1) . \tag{4.84}$$

The phase $\epsilon(g,h)$ has to be choosen, s.t. the partition function respects modular invariance. When modular invariance allows $\epsilon(\mathbb{1},\mathbb{1}) \neq \epsilon(g,h)$ for some $g,h \in P$, then the choice for the phase $\epsilon(g,h)$ is called discrete torsion and determines the action of $h$ on the exceptional divisors at the fixed point of $g$.

### 4.1.5 Example: $T^6_{\text{SO(12)}} / (\mathbb{Z}_2 \times \mathbb{Z}_2 \times \Omega R)$

To illustrate the previous discussion, the orientifold, given by

$$O^6 = \frac{T^6_{\text{SO(12)}}}{\mathbb{Z}_2 \times \mathbb{Z}_2 \times \Omega R} . \tag{4.85}$$

shall be investigated in this section. The point group $\mathbb{Z}_2 \times \mathbb{Z}_2$ is generated by two generators, which are denoted by $\theta$ and $\omega$. They act on the complex coordinates of the compact space as

$$\theta : (z_1, z_2, z_3) \to (-z_1, -z_2, z_3), \quad \omega : (z_1, z_2, z_3) \to (z_1, -z_2, -z_3) , \tag{4.86}$$

and the spacetime involution $R$ reflects the directions along all three imaginary axis

$$R : (z_1, z_2, z_3) \to (\bar{z}_1, \bar{z}_2, \bar{z}_3) . \tag{4.87}$$

In order to safely quotient out the discrete group $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \Omega R$, it has to be an automorphism of the underlying lattice. This could possibly constrain the deformation parameters $R_h$ and $\tau_h$ for $T^6_{\text{SO(12)}}$. Invariance of the lattice under the spacetime involution $R$ demands that the real part of the complex

[4] For more details see [23]

strucutre moduli need to be integer $\mathrm{Re}(\tau_h) \in \mathbb{Z}$, since otherwise the lattice vector $1 + \tau_h$ in the $h$-th complex plane gets projected away by the involution[5].

### Fixed points and exceptional divisors

The fixed point equation for fixed points under $\theta$, given by

$$(x_1, x_2, x_3, x_4, x_5, x_6) = (-x_1, -x_2, -x_3, -x_4, x_5, x_6) + \vec{\lambda}, \quad \vec{\lambda} \in \Lambda_{\mathrm{SO}(12)}, \tag{4.90}$$

is solved by

$$\mathrm{FIX}(\theta) = \left\{ \left( \frac{n_1}{2}, \frac{n_2}{2}, \frac{n_3}{2}, \frac{n_4}{2}, x_5, x_6 \right) \middle| n_i \in \mathbb{Z},\ \sum_{i=1}^{4} n_i = 0 \mod 2,\ x_5, x_6 \in \mathbb{R} \right\}. \tag{4.91}$$

The solutions given in (4.91) show that, because $\theta$ acts trivially on the last plane, the coordinates $x_5, x_6$ are unconstrained and taking the action of the $SO(12)$ roots on the last plane into account, the last plane for the fixed loci $\mathrm{FIX}(g)$ are compactified on two dimensional tori, which has the $SO(4)$ Lie lattice underlying

$$T^2_{\mathrm{fix}} = \frac{\mathbb{R}^2}{\Lambda_{\mathrm{SO}(4)}}, \quad \text{with} \quad \Lambda_{\mathrm{SO}(4)} = \langle (1,1)^T, (1,-1)^T \rangle. \tag{4.92}$$

That means the fixed loci are given by fixed tori $T^2_{\mathrm{fix}}$, each located at fixed points in the first two planes. Identifying all fixed tori differing by lattice vectors, one finds that there are eight inequivalent fixed tori on $T^6_{\mathrm{SO}(12)}/(\mathbb{Z}_2 \times \mathbb{Z}_2)$, whose loci are given by

$$\begin{aligned} \mathrm{FIX}(\theta) = &\left(0,0,0,0,T^2_{\mathrm{fix}}\right) \cup \left(\frac{1}{2},\frac{1}{2},0,0,T^2_{\mathrm{fix}}\right) \cup \left(\frac{1}{2},0,\frac{1}{2},0,T^2_{\mathrm{fix}}\right) \cup \left(0,\frac{1}{2},\frac{1}{2},0,T^2_{\mathrm{fix}}\right) \\ &\cup \left(0,0,\frac{1}{2},\frac{1}{2},T^2_{\mathrm{fix}}\right) \cup \left(\frac{1}{2},0,0,\frac{1}{2},T^2_{\mathrm{fix}}\right) \cup \left(0,\frac{1}{2},0,\frac{1}{2},T^2_{\mathrm{fix}}\right) \cup \left(\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},T^2_{\mathrm{fix}}\right). \end{aligned} \tag{4.93}$$

Each of those fixed tori gets twisted by $\omega$

$$\omega:\ T^2_{\mathrm{fix}} \to \frac{T^2_{\mathrm{fix}}}{\mathbb{Z}_2}. \tag{4.94}$$

In a similar fashion the other two twist operators $\omega$ and $\theta\omega$ preserve each eight fixed tori. They are located on fixed points in the second and last plane or in the first and third plane respectively, as listed in table 4.1.

In order to find the exceptional divisors hidden in the orbifold fixed points, one has to resolve the singularities as explained in section 4.1.3. The neighbourhood of a fixed torus $T^2_{\mathrm{fix}}$ in $T^6_{\mathrm{SO}(12)}$ has locally

[5] The action of $R$ on $1 + \tau_h$ is given by

$$R:\ 1 + \tau_h \to 1 + \bar{\tau}_h. \tag{4.88}$$

In order for $R$ to be a lattice automorphism, the lattice has to contain lattice vectors, which compensate the transformation of the involution. Therefore it needs to be found an $SO(12)$ vector, which satisfies

$$1 + \tau_h = 1 + \bar{\tau}_h + u_1 + \tau_h u_2, \tag{4.89}$$

where $u_1, u_2$ are components of $SO(12)$ roots in the $h$-th plane. The solutions are given by $u_2 = 2$ and $u_1 + 2\mathrm{Re}(\tau_h)$. Since $u_1$, $u_2$ have to satisfy $u_1 + u_2 = 0 \mod 2$ (otherwise they do not form an $SO(12)$ root), the real part of $\tau_h$ needs to be integer $\mathrm{Re}(\tau_h) \in \mathbb{Z}$.

| FIX($\theta$) | FIX($\omega$) | FIX($\theta\omega$) |
|---|---|---|
| $(0,0,0,0,T^2_{\text{fix}})$ | $(T^2_{\text{fix}},0,0,0,0)$ | $(0,0,T^2_{\text{fix}},0,0)$ |
| $(\frac{1}{2},0,\frac{1}{2},0,T^2_{\text{fix}})$ | $(T^2_{\text{fix}},\frac{1}{2},0,\frac{1}{2},0)$ | $(\frac{1}{2},0,T^2_{\text{fix}},\frac{1}{2},0)$ |
| $(0,\frac{1}{2},\frac{1}{2},0,T^2_{\text{fix}})$ | $(T^2_{\text{fix}},0,\frac{1}{2},\frac{1}{2},0)$ | $(0,\frac{1}{2},T^2_{\text{fix}},\frac{1}{2},0)$ |
| $(0,\frac{1}{2},0,\frac{1}{2},T^2_{\text{fix}})$ | $(T^2_{\text{fix}},0,\frac{1}{2},0,\frac{1}{2})$ | $(0,\frac{1}{2},T^2_{\text{fix}},0,\frac{1}{2})$ |
| $(\frac{1}{2},0,0,\frac{1}{2},T^2_{\text{fix}})$ | $(T^2_{\text{fix}},\frac{1}{2},0,0,\frac{1}{2})$ | $(\frac{1}{2},0,T^2_{\text{fix}},0,\frac{1}{2})$ |
| $(\frac{1}{2},\frac{1}{2},0,0,T^2_{\text{fix}})$ | $(x_1,x_2,\frac{1}{2},\frac{1}{2},0,0)$ | $(\frac{1}{2},\frac{1}{2},T^2_{\text{fix}},0,0)$ |
| $(0,0,\frac{1}{2},\frac{1}{2},T^2_{\text{fix}})$ | $(T^2_{\text{fix}},0,0,\frac{1}{2},\frac{1}{2})$ | $(0,0,T^2_{\text{fix}},\frac{1}{2},\frac{1}{2})$ |
| $(\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},T^2_{\text{fix}})$ | $(T^2_{\text{fix}},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2})$ | $(\frac{1}{2},\frac{1}{2},T^2_{\text{fix}},\frac{1}{2},\frac{1}{2})$ |

Table 4.1: Fixed tori of $T^6_{\text{SO}(12)}/\mathbb{Z}_2\times\mathbb{Z}_2$..

the geometry of $T^2_{\text{fix}}$ sitting at a $\mathbb{C}^2/\mathbb{Z}_2$ singularity. A $\mathbb{Z}_2$ action on $\mathbb{C}^2$ is given by

$$(z_1,z_2)\to(-z_1,-z_2) \tag{4.95}$$

s.t. the vectors of the fan satisfy

$$v_1+v_2=0 \mod 2\,. \tag{4.96}$$

This is the case for $v_1=(1,1)$ and $v_2=(1,-1)$. Notice that the point $(1,0)$ cannot be reached by $v_1$ and $v_2$ revealing that the space contains singularities. To resolve the singularity, the lattice vector $w=(1,0)$ is included into the fan, which corresponds to a coordinate $x$. The fan consists of the two cones spanned by $\{v_1,w\}$ and $\{v_2,w\}$. The set $\{v_1,v_2\}$ does not spann a cone, in a way that the criteria for a fan are satisfied. Therefore the exclusion set is $F=\{z_1=z_2=0\}$. This is precisely the $\mathbb{C}^2/\mathbb{Z}_2$ singularity, which is cut out. The scaling relation for the toric variety can be extracted by the coefficients of the equation

$$v_1+v_2-2w=0. \tag{4.97}$$

The $(\mathbb{C})^*$ action on the homogeneous coordinates are then given by

$$(\mathbb{C})^*:\ (z_1,z_2,x)\sim(\lambda z_1,\lambda z_2,\lambda^{-2}x)\,. \tag{4.98}$$

At $D_i=\{x\neq0\}$, $x$ can be scaled away by $\lambda=\pm\sqrt{x}$, s.t. at $D_i$ one has $(\pm\sqrt{x}z_1,\pm\sqrt{x}z_2,1)$, which is the $\mathbb{Z}_2$ orbifold. At the point $E=\{x=0\}$, from the scaling relation $(z_1,z_2)\sim(\lambda z_1,\lambda z_2)$ it can be recognized, that a $\mathbb{CP}^1$ is hidden at the fixed point. Each fixed torus in the $T^6_{\text{SO}(12)}$ therefore is actually $\mathbb{CP}^1\times T^2_{\text{fix}}$. Taking the action of the other point group elements, the orbifold group preserves

$$\mathbb{CP}^1\oplus\frac{T^2_{\text{fix}}}{\mathbb{Z}_2} \tag{4.99}$$

at the fixed points.

### O6-planes

The spacetime reflections contained in the group $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \Omega R$ are given by the set $\{R,\ R\theta,\ R\omega,\ R\theta\omega\}$. Their action on the complex coordinates is given by

$$\begin{aligned} R\quad &: (z_1, z_2, z_3) \to (\bar{z}_1, \bar{z}_2, \bar{z}_3), \\ R\theta\quad &: (z_1, z_2, z_3) \to (-\bar{z}_1, -\bar{z}_2, \bar{z}_3), \\ R\omega\quad &: (z_1, z_2, z_3) \to (\bar{z}_1, -\bar{z}_2, -\bar{z}_3), \\ R\theta\omega &: (z_1, z_2, z_3) \to (-\bar{z}_1, \bar{z}_2, -\bar{z}_3). \end{aligned} \tag{4.100}$$

Solving the fixed loci equation for O-planes (4.51), the O6-plane loci on the covering space $\mathbb{R}^6$ are given by

$$\begin{aligned} \mathrm{FIX}(R) &= \left\{ \left( x_4, \frac{n_1}{2}, x_6, \frac{n_2}{2}, x_8, \frac{n_4}{2} \right)^T \middle| n_i \in \mathbb{Z}, \sum_{i=1}^{3} n_i \in 2\mathbb{Z},\ x_4, x_6, x_8 \in \mathbb{R} \right\}, \\ \mathrm{FIX}(R\theta) &= \left\{ \left( \frac{n_1}{2}, x_5, \frac{n_2}{2}, x_7, x_8, \frac{n_4}{2} \right)^T \middle| n_i \in \mathbb{Z}, \sum_{i=1}^{3} n_i \in 2\mathbb{Z},\ x_5, x_7, x_8 \in \mathbb{R} \right\}, \\ \mathrm{FIX}(R\omega) &= \left\{ \left( x_4, \frac{n_1}{2}, \frac{n_2}{2}, x_7, \frac{n_4}{2}, x_9 \right)^T \middle| n_i \in \mathbb{Z}, \sum_{i=1}^{3} n_i \in 2\mathbb{Z},\ x_4, x_7, x_9 \in \mathbb{R} \right\}, \\ \mathrm{FIX}(R\theta\omega) &= \left\{ \left( \frac{n_1}{2}, x_5, x_6, \frac{n_2}{2}, \frac{n_4}{2}, x_9 \right)^T \middle| n_i \in \mathbb{Z}, \sum_{i=1}^{3} n_i \in 2\mathbb{Z},\ x_5, x_6, x_9 \in \mathbb{R} \right\}. \end{aligned} \tag{4.101}$$

Identifying the O-planes, which differ by $SO(12)$ lattice translations, the inequivalent O-planes on the torus are given by

$$\begin{aligned} \mathrm{FIX}(R) = &\ (x_5 = 0, x_7 = 0, x_9 = 0) \cup \left(x_5 = \tfrac{1}{2}, x_7 = \tfrac{1}{2}, x_9 = 0\right) \\ &\cup \left(x_5 = \tfrac{1}{2}, x_7 = 0, x_9 = \tfrac{1}{2}\right) \cup \left(x_5 = 0, x_7 = \tfrac{1}{2}, x_9 = \tfrac{1}{2}\right), \\ \mathrm{FIX}(R\theta) = &\ (x_4 = 0, x_6 = 0, x_9 = 0) \cup \left(x_4 = \tfrac{1}{2}, x_6 = \tfrac{1}{2}, x_9 = 0\right) \\ &\cup \left(x_4 = \tfrac{1}{2}, x_6 = 0, x_9 = \tfrac{1}{2}\right) \cup \left(x_4 = 0, x_6 = \tfrac{1}{2}, x_9 = \tfrac{1}{2}\right), \\ \mathrm{FIX}(R\omega) = &\ (x_5 = 0, x_6 = 0, x_8 = 0) \cup \left(x_5 = \tfrac{1}{2}, x_6 = \tfrac{1}{2}, x_8 = 0\right) \\ &\cup \left(x_5 = \tfrac{1}{2}, x_6 = 0, x_8 = \tfrac{1}{2}\right) \cup \left(x_5 = 0, x_6 = \tfrac{1}{2}, x_8 = \tfrac{1}{2}\right), \\ \mathrm{FIX}(R\theta\omega) = &\ (x_4 = 0, x_7 = 0, x_8 = 0) \cup \left(x_4 = \tfrac{1}{2}, x_7 = \tfrac{1}{2}, x_8 = 0\right) \\ &\cup \left(x_4 = \tfrac{1}{2}, x_7 = 0, x_8 = \tfrac{1}{2}\right) \cup \left(x_4 = 0, x_7 = \tfrac{1}{2}, x_8 = \tfrac{1}{2}\right). \end{aligned} \tag{4.102}$$

The fixed planes in (4.102), preserved by the action of $Rg$, with $g \in \{\theta, \omega, \theta\omega\}$, wrap four sLag 3-cycles of the same homology class on the $T^6_{\mathrm{SO}(12)}$ $\left[\Pi^3\right]_{\mathrm{O6}_{Rg}}$

$$\mathrm{FIX}(Rg) = 4\left[\Pi^3\right]_{\mathrm{O6}_{Rg}}, \tag{4.103}$$

where the cycles in the homology class $\left[\Pi^3\right]_{O6_{Rg}}$ are specified according to (4.70) by the wrapping numbers

$$
\begin{aligned}
[\Pi^3]_{O6_R} &= (2,0)\times(1,0)\times(1,0), \qquad & [\Pi^3]_{O6_{R\theta}} &= (0,1)\times(0,-1)\times(2,0), \\
[\Pi^3]_{O6_{R\omega}} &= (2,0)\times(0,1)\times(0,-1), \qquad & [\Pi^3]_{O6_{R\theta\omega}} &= (0,1)\times(2,0)\times(0,-1)\,.
\end{aligned}
\tag{4.104}
$$

The O6-planes preserved by the whole orientifold group are therefor given by

$$
4\Pi^3_{O6} = 4\left([\Pi^3]_{O6_R} + [\Pi^3]_{O6_{R\theta}} + [\Pi^3]_{O6_{R\omega}} + [\Pi^3]_{O6_{R\theta\omega}}\right). \tag{4.105}
$$

According to (4.69) the calibration phase of $\Pi^3_{O6}$ is given by $\vartheta = 0$.

### sLag bulk 3-cycles

In section 4.1.4 sLag 3-cycles on the $T^6_{SO(12)}$ where introduced. They can be used to construct bulk 3-cycles on the orbifold, which are wrapped by D6-branes. The 1-cycles $\Pi^1_{(h)} = (n^h, m^h)$ of the sLag 3-cycle on the $h$-th plane transforms under the point group elements as

$$
\mathbb{Z}_2 :\ (n^h, m^h) \to -(n^h, m^h)\,, \tag{4.106}
$$

when the group elements have a non-trivial action in that plane. Since each point group element acts non-trivially on two planes, the two signs from the transformations of the two 1-cycles cancel in the product $\prod_{h=1}^3 (n^h, m^h)$ and the corresponding 3-cycle transforms trivially and is mapped to itself. From (4.80), it can be deduced that a bulk 3-cycle on the orbifold $\Pi^3_{\text{Bulk}} \in H_3\left(T^6_{SO(12)}/(\mathbb{Z}_2\times\mathbb{Z}_2)\right)$ is given by

$$
\Pi^3_{\text{Bulk}} = \Pi^3 + \theta\Pi^3 + \omega\Pi^3 + \theta\omega\Pi^3 = 4\Pi^3\,, \tag{4.107}
$$

where $\Pi^3 \in H_3\left(T^6_{SO(12)}\right)$ denotes a sLag 3-cycle on the torus. Let two 3-cycles on the torus be denoted by $\Pi^3_a$ and $\Pi^3_b$ and the corresponding bulk cycles on the orbifold by $\Pi^3_{\text{Bulk},a} = 4\Pi^3_a$ and $\Pi^3_{\text{Bulk},b} = 4\Pi^3_b$. The intersection number $I^{\text{orbi}}_{ab}$ of the two bulk cycles on the orbifold is related to the intersection number $I_{ab}$ of the two cycles $\Pi^3_a$ and $\Pi^3_b$ on the torus by

$$
I^{\text{orbi}}_{ab} = \frac{1}{4}\left[\Pi^3_a\right]_{\text{Bulk}}\cdot\left[\Pi^3_b\right]_{\text{Bulk}} = \frac{1}{4}4\left[\Pi^3_a\right]\cdot 4\left[\Pi^3_b\right] = 4I_{ab}\,. \tag{4.108}
$$

with $I_{ab} = [\Pi^3_a]\cdot[\Pi^3_b]$, where the prefactor of $1/4$ after the first equal sign arises from the identification of intersection points by the point group. According to that the intersection numbers of bulk cycles are always multiples of four and therefore it seems that on $\mathbb{Z}_2\times\mathbb{Z}_2$ orbifolds, there are possibly smaller cycles, given by half of a bulk cycle. But in order not to get projected out by the orbifold projection, they must go through fixed points. The spacetime reflection $R$ maps cycles to their orientfold image, or to be more precisely a cycle with wrapping numbers $\Pi^3 = \prod_{h=1}^3 (n^h, m^h)$ is mapped to $\Pi^{3\prime} = \prod_{h=1}^3 (n^h, -m^h)$. An invariant 3-cycle $\Pi^3_{\text{inv}} \in H_3\left(T^6_{SO(12)}/(\mathbb{Z}_2\times\mathbb{Z}_2\times\Omega R)\right)$ on the orientifold is then given by a superposition of orbifold invariant 3-cycles with their orientifold images

$$
\Pi^3_{\text{inv}} = \frac{1}{2}\left(\Pi^3_{\text{Bulk}} + \Pi^{3\prime}_{\text{Bulk}}\right) = 2\left(\Pi^3 + \Pi^{3\prime}\right)\,, \quad \text{with} \quad R:\ \Pi^3 \to \Pi^{3\prime}\,, \quad \Pi^3 \in H_3\left(T^6_{SO(12)}\right)\,, \tag{4.109}
$$

In section 4.2.3 it is explained that D6-branes share the same supersymmetry, when they are wrapped on cycles with the same calibration phase. The O6-planes fix the calibration phase to $\vartheta = 0$. Therefor, in order to preserve $\mathcal{N} = 1$ supersymmetry, the calibration phase of the D6-branes have to be $\vartheta = 0$ and inserting it into (4.69), one gets the supersymmetry condition

$$\sum_{h=1}^{3} \frac{\mathrm{Im}(\tau_h) m^h}{n^h + \mathrm{Re}(\tau_h) m^h} = 0\,, \quad \mathrm{Re}(\tau_h) \in \mathbb{Z}\,, \mathrm{Im}(\tau_h) \in \mathbb{R}\,, \tag{4.110}$$

for wrapping numbers of D6-brane cycles on the deformed $T^6_{\mathrm{SO}(12)}$.

### Fractional 3-cycles

The fixed point resolution exposes that the fixed loci of the point group are $\mathbb{CP}^1 \oplus \left(T^2_{\mathrm{SO}(4)}/\mathbb{Z}_2\right)$'s. A $\mathbb{CP}^1$ is topologically isomorphic to a 2-sphere $S^2$, which means each $\mathbb{CP}^1$ contains a non-trivial 2-cycle, the $S^2$ itself. The 2-cycles belonging to the exceptional divisors of the orbifold, are denoted by $e^g_{ij}$, where $g \in \{\theta, \omega, \theta\omega\}$ denotes the generator of the twisted sector and $(ij)$, with $i, j \in \{1, 2, 3, 4\}$, denote the position of the fixed point in the two planes, on which $g$ acts non-trivially. The fixed loci are placed on the points $(0, 0)$, $(\frac{1}{2}, 0)$, $(0, \frac{1}{2})$, $(\frac{1}{2}, \frac{1}{2})$ on each plane and when they are labeled by

$$(0,0) \to 1\,, \quad (\tfrac{1}{2}, 0) \to 2\,, \quad (0, \tfrac{1}{2}) \to 3\,, \quad (\tfrac{1}{2}, \tfrac{1}{2}) \to 4\,, \tag{4.111}$$

for example the $S^2$ from the resolution of $\mathrm{FIX}(\theta) = (0, \frac{1}{2}, \frac{1}{2}, 0, T^2_{\mathrm{inv}})$ is wrapped by the 2-cycle, which is denoted by $e^\theta_{21}$. The transformation rules for the terms in the orbifold partition function (4.82) under the modular transformations, allows different phases $\epsilon$ for the two sets

$$\begin{aligned} \left\{Z(\mathbb{1}, \mathbb{1})\right\} \quad \cup \quad & \left\{Z(g_1, g_2), Z(g_1, g_1 \cdot g_2), Z(g_2, g_1), \right. \\ & \left. Z(g_2, g_1 \cdot g_2), Z(g_1 \cdot g_2, g_2), Z(g_1 \cdot g_2, g_1)\right\} \backslash \left\{Z(\mathbb{1}, \mathbb{1})\right\}, \end{aligned}$$

with $\forall\, g_1, g_2 \in \{\mathbb{1}, \theta, \omega, \theta\omega\}$. That means one can choose two different phases $\epsilon$: One for $Z(\mathbb{1}, \mathbb{1})$ and one for all other terms, s.t. the action of the orbifold on the $S^2$'s at the fixed points can be given by [78]

$$S^2 \to \epsilon S^2, \quad \text{with} \quad \epsilon = \pm 1\,. \tag{4.112}$$

The choice $\epsilon = -1$ is called "with discrete torsion" and implies that the orbifold action on the collapsed 2-cycles is given by

$$e^g_{ij} \to -e^g_{ij}\,. \tag{4.113}$$

Together with a 1-cycle wrapping the fixed torus of the corresponding twisted sector, $e^g_{ij}$ forms a 3-cycle which is called a twisted cycle and is invariant under the orbifold action [81]. Therefore the orbifold with discrete torsion contains at each twisted sector, a two dimensional homology of twisted 3-cycles, spanned by the following two homology classes

$$[\alpha^g_{ij,n}] = 2[e^g_{ij}] \otimes [a], \quad [\alpha^g_{ij,m}] = 2[e^g_{ij}] \otimes [b]\,, \tag{4.114}$$

with $a$ and $b$ denoting 1-cycles of the $T^2$ defined as in (4.66). Twisted 3-cycle can then be expressed by

$$\Pi^3_{ij,g} = n[\alpha^g_{ij,n}] + m[\alpha^g_{ij,m}], \quad n, m \in \mathbb{Z} \tag{4.115}$$

with $n, m$ as the wrapping numbers around the $a$ and $b$ cycle of the $T^2_{\text{fix}}$. For the self-intersection number $[e_{ij}] \cdot [e_{kl}] = -2\delta_{ik}\delta_{jl}$ [78], intersection numbers with twisted 3-cycles are given by

$$[\Pi^3_{ij,g}]_a \cdot [\Pi^3_{kl,h}]_h = -4\delta_{ik}\delta_{jl}\delta_{gh}\,(n_a m_b - m_a n_b)\ . \tag{4.116}$$

Then by (4.81) the whole fraction cycle is given by [78]

$$\Pi^3_{\text{frac}} = \frac{1}{4}\Pi^3_{\text{Bulk}} + \frac{1}{4}\sum_{g\in\{\theta,\omega,\theta\omega\}}\ \sum_{(i,j)\in x^g_{\text{fix}}} \epsilon^g_{ij}\Pi^3_{ij,g}\,, \tag{4.117}$$

where $\epsilon^g_{ij} = \pm 1$ defines the charge of the collapsed brane, wrapping $\Pi^3_{ij,g}$, and determines the orientation of the cycle around the $S^2$ at the fixed point labeled by $(ij)$. The charges of the collapsed cycles are not independent from each other but need to satisfy certain relation due to consistency reasons[6]. The fractional cycle in (4.117) describes a cycle on the torus, whose orbifold images are collapsed at fixed loci and wrap twisted cycles.
In the following however orientifolds without discrete torsion are considered and therefor the D6-branes do not wrap fractional cycles. They where introduced for the sake of completeness and to mention the further possibilities there are for string compactification.

### Factorisable vs. non-factorisable

In order to compare orientifolds, their Hodge numbers are a good quantity to look at. The 1-forms $\mathrm{d}z_i$ and $\mathrm{d}\bar{z}_i$ correspond via deRham duality to 1-cycles, wrapping the $T^6_{\text{SO}(12)}$ in the $i$-th plane, hence belonging to the cohomology groups $H^{1,0}\left(T^6_{\text{SO}(12)}\right)$ and $H^{0,1}\left(T^6_{\text{SO}(12)}\right)$. By wedging the 1-forms, the $(p,q)$-forms

$$\mathrm{d}z_{i_1}\wedge ... \wedge \mathrm{d}z_{i_p}\wedge \mathrm{d}\bar{z}_{j_1}\wedge ... \wedge \mathrm{d}\bar{z}_{j_q}\,,\quad \text{with}\quad i_1,...,i_p,j_1,...,j_q \in \left\{1,...,\dim_{\mathbb{C}}\left(T^6_{\text{SO}(12)}\right)\right\}\,, \tag{4.118}$$

span a basis for the $(p,q)$-th cohomology group $H^{p,q}\left(T^6_{\text{SO}(12)}\right)$. Using the antisymmetric behaviour of the wedge product, from straight combinatorics it follows that $\dim(H^{p,q}) = \binom{3}{p}\binom{3}{q}$. The cohomology groups $H^{p,g}$, $H^{q,p}$, $H^{3-p,3-q}$ and $H^{3-q,3-p}$ are dual to each other via the combination of deRham-, Hodge-, and Poincare duality [7]. Hence for a complex three dimensional manifold it is sufficient to look at cohomology groups $H^{1,0}$, $H^{1,1}$, $H^{2,0}$, $H^{2,1}$, $H^{3,0}$ and $H^{3,3}$. On the orbifold only $(p,q)$-forms invariant under the point group are preserved. The action of a twist element $g \in P$ on the cohomology group elements from (4.118) is given by

$$\mathrm{e}^{2\pi i v_{i_1}}\mathrm{d}z_{i_1}\wedge ... \wedge \mathrm{e}^{2\pi i v_{i_p}}\mathrm{d}z_{i_p}\wedge \mathrm{e}^{-2\pi i v_{j_1}}\mathrm{d}\bar{z}_{j_1}\wedge ... \wedge \mathrm{d}\mathrm{e}^{-2\pi i v_{j_q}}\bar{z}_{j_q}\,. \tag{4.119}$$

Due to the condition (4.36) for the twist vectors,the $(2,0)$- and $(1,0)$-forms are always projected away and the $(3,0)$- and $(3,3)$-form are always preserved. Hence $h^{1,0} = h^{2,0} = 0$ and $h^{3,0} = h^{3,3} = 1$ for all orbifolds (with the holonomy given by a discrete subgroup of $SU(3)$), where $h^{p,q} = \dim\left(H^{p,q}\left(\frac{T^6_{\text{SO}(12)}}{\mathbb{Z}_2\times\mathbb{Z}_2}\right)\right)$ are the Hodge numbers on the orbifold. Therefor it is sufficient to investigate the Hodge numbers $h^{1,1}$ and $h^{2,1}$. The $(1,1)$- and $(2,1)$-forms preserved by the $\mathbb{Z}_2\times\mathbb{Z}_2$ point group are

$$\mathrm{d}z_i\wedge \mathrm{d}\bar{z}_i\,,\quad \mathrm{d}z_i\wedge \mathrm{d}x_j\wedge \mathrm{d}\bar{z}_k\,,\quad \text{for}\quad i\neq j\neq k\,,\quad i,j,k\in\{1,2,3\}\,. \tag{4.120}$$

[6] For more details see [78]
[7] For more details, see appendix B of [20].

Therefor the Hodge numbers on the orbifold, which are inherited from the torus, are given by

$$h^{0,0}_{\text{un}} = h^{3,0}_{\text{un}} = 1\,, \quad h^{1,1}_{\text{un}} = h^{2,1}_{\text{un}} = 3\,, \tag{4.121}$$

from the invariant forms of (4.120). But the orbifold further contains exceptional divisors at fixed loci. Each fixed locus is topologically isomorphic to $S^2 \times T^2$, where the sphere contains one $(1,1)$-form. The contribution to the Hodge numbers from the exceptional divisors depend on the choice of discrete torsion $\epsilon = \pm 1$. For the case without discrete torsion $\epsilon = +1$ the point group acts trivially on the sphere and the $(1,1)$-form on the $S^2$ is preserved. For the case with discrete torsion $\epsilon = -1$ the $(1,1)$-form gets a sign, but together with the 1-form on the fixed torus, which also gets a sign, the $(2,1)$-form on the $S^2 \times T^2$ is preserved by the point group. Hence the 24 exceptional divisors on $T^6_{\text{SO(12)}} / (\mathbb{Z}_2 \times \mathbb{Z}_2)$ contribute

$$h^{1,1}_{\text{tw}} = 24 \quad \text{without discrete torsion} \quad h^{2,1}_{\text{tw}} = 24 \quad \text{with discrete torsion}\,, \tag{4.122}$$

s.t. the the Hodge numbers of $T^6_{\text{SO(12)}} / (\mathbb{Z}_2 \times \mathbb{Z}_2)$ are given by

$$\left(h^{1,1}, h^{2,1}\right) = \begin{cases} (27,3)\,, & \epsilon = +1 \\ (3,27)\,, & \epsilon = -1 \end{cases}\,. \tag{4.123}$$

The features resulting from the non-factorisable structure of the orbifold are turned off in the factorisable orbifold $T^6_{\text{fact}} / (\mathbb{Z}_2 \times \mathbb{Z}_2)$, with $T^6_{\text{fact}} = \mathbb{R}^6 / \Lambda^6_{\text{fact}}$, where $\Lambda_{\text{SO(12)}} \subset \Lambda^6_{\text{fact}}$ is defined in section 4.1.4. The Hodge numbers of the factorisabel orbifold are discussed in [78] and given by

$$\left(h^{1,1}, h^{2,1}\right) = \begin{cases} (51,3)\,, & \epsilon = +1 \\ (3,51)\,, & \epsilon = -1 \end{cases}\,. \tag{4.124}$$

The spacetime reflection $R$, acting as complex conjugation, maps $(p,q)$-forms to $(q,p)$-forms, hence the $(p,q)$-th cohomology group gets mapped to the $(q,p)$-th cohomology group

$$R:\ H^{p,q} \rightarrow H^{q,p}\,, \tag{4.125}$$

and forms in $H^{1,1}$ and linear combinations of forms in $H^{2,1} \oplus H^{1,2}$ are preserved on the orientifold. Hence by comparison of Hodge numbers of the factorisable orientifold with the non-factorisable orientifold it seems evident that the non-factorisable orientifold has less structrure than the factorisable orientifold and hence compactification on the non-factorisable orientifold is more restrictive. Further the number of O6-planes on the non-factorisable orientifold is also reduced to half of the number of O6-planes on the factorisable orientifold. For D6-branes on the non-factorisable orientifold that means, their wrapping numbers are more restricted than on the factorisable orientofold, as will be seen in the following of this chapter.

## 4.2 Intersecting D-branes on $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \Omega R$-Orientifolds

In this section Type IIA string theory compactified on $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \Omega R$-orientifolds

$$\text{Type IIA}:\ M^{1,9} \rightarrow \mathbb{R}^{1,3} \times \frac{T^6}{\mathbb{Z}_2 \times \mathbb{Z}_2 \times \Omega R} \tag{4.126}$$

is discussed and the resulting massless spectra in four dimensions is analyzed. The generators of the orientifolding group are given by (4.86) and (4.87). Since the spacetime reflections preserve O6-planes, D6-branes have to be included in order to cancel the total R-R charge [41]. Open string on the D6-branes lead to a chiral spectrum in four dimensions [49]. Models arising from these kind of orientifolds have been discussed mostly for factorisabel tori, for example in [81–83]. But also for non-factorisable orientifolds there has been attempts to construct realistic intersecting D-brane models for example in [77].

### 4.2.1 Massless states from type IIA closed strings

Compactifying the ten dimensional spacetime according to (4.126) on the orientifold, decomposes the group of spacetime isometries $SO(1,9)$ into the Lorentz group $SO(1,3)$ acting on the uncompact space $\mathbb{R}^{1,3}$ and an $SO(6)$ part, acting on the internal space. The point group elements $g \in \{\theta, \omega, \theta\omega\}$ are expressed in an $SO(6)$ representation by

$$g = \exp\left(2\pi i \sum_{h=1}^{3} v_h H_h\right), \quad \text{with} \quad \vec{v} = \begin{cases} \frac{1}{2}(1,-1,0)^T & \text{for } \theta\,, \\ \frac{1}{2}(0,1,-1)^T & \text{for } \omega\,, \end{cases} \tag{4.127}$$

and $H_h$ the three Cartan generators of $SO(6)$. The transformation behaviour of the string states under the $SO(1,3)$ factor of $SO(1,9)$ determines what kind of field the states correspond to in four dimensions. In appendix A the transformation of the massless Type IIA closed string states under the algebra $\mathfrak{su}(2) \otimes \mathfrak{su}(2)$, which is the little group of massless particles in four dimensions, is investigated. Further their transformation under the $\mathbb{Z}_2 \times \mathbb{Z}_2$ point group and the orientifold projection by $\Omega R$ has been coomputed and that way the invariant states on $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \Omega R$ orientifolds are determined.

The dimensional reduction of the eight massless NS states lead to a massless vector field and six scalars. The decomposition of the eight chiral and eight antichiral states in the left- and rightmoving R sector, leads to four chiral and four antichiral spinors in four dimensions. Combining the left- and rightmoving states to closed string states, 64 states in each of the NS-NS, NS-R, R-Ns and R-R sector arise. Their representation under the algebra $\mathfrak{su}(2) \otimes \mathfrak{su}(2)$ for the uncompact space is deduced by decomposing the products of left-and rightmoving states into irreducible representations. The trace, antisymmetric and symmetric part of the product of the four dimensional vectors from the left- and rightmoving NS sector, form a dilaton, B-field and graviton in the uncompact space, where the combinations of the vectors with the six scalars and the scalars with each other in the NS-NS sector provide the four dimensional spectrum with 12 graviphotons and 36 scalars. In the NS-R sector the decomposition of the vector with the four fermions gives four dilatinos and four gravitinos and the combination of the six scalars with the four fermions lead to 24 fermions. The R-NS sector contains also four dilatinos, four gravitinos and 24 fermions, but each has the opposite chirality as in the fields in the NS-R sector. The product of a chiral an antichiral fermion in four dimensions, decompose for the massless case into two scalars and a vector field. Hence the combination of four chiral and four antichiral fermions in the R-R sector lead to 16 vector potentials and 32 scalars. The massless spectrum achieved that way by the dimensional reduction of the massless Type IIA closed string states fit into a $\mathcal{N} = (4,4)$ supergravity multiplet. The trivial holonomy in the compact space, which was assumed by the decompostion of the states, leaves all supercharges unbroken and hence the 32 supercharges of Type IIA form eight supersymmetry generators in four dimensions.

### Untwisted sector

The states from the initial closed string states in ten dimensions, which are invariant under the $\mathbb{Z}_2 \times \mathbb{Z}_2$ projection, form the set of untwisted massless states on the orbifold. The 12 graviphotons and 24 scalars from the NS-NS sector are projected out. Further one gravitino, dilatino and six fermions are preserved in each of the NS-R and R-NS sector. In the R-R sector 1/4 of the states survive the point group projection, s.t. four vector potentials and eight scalars remain. As explained in section 4.1.2 only eight of the 32 supercharges in Type IIA are preserved by the point group. The field content formed by the untwisted states, fit into a griviton multiplet three vector and four hyper multiplets of $\mathcal{N} = (1, 1)$ supersymmetry in four dimensions. By the orientifold projection $\Omega R$ half of those states get further projected out, s.t. in the NS-NS sector ten scalars, including the dilaton, and the graviton are preserved, the NS-R and R-NS sector form invariant linear combinations, s.t. one dilatino, one gravitino and six other fermions remain and in the R-R sector the symmetric part in the tensor product of the antichiral and chiral fermion are projected out, leaving in total three scalars. The orientifold action reduces the supersymmetry further to $\mathcal{N} = 1$ in four dimensions and the states remaining from the massless Type IIA closed string states on the $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \Omega R$ orientifold form a graviton multiplet and seven chiral multiplets. It seems that $\Omega R$ preserves from each of the $\mathcal{N} = (1, 1)$ vector multiplet and hyper multiplet a $\mathcal{N} = 1$ chiral multiplet. Comparing the number of multiplets with the Hodge numbers of a $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold (as in section 4.1.5), the vector multiplets in the untwisted states correspond to the number of non-tivially closed $(1, 1)$-forms in the bulk $h^{1,1}_{\text{un}}$ (as in (4.121)), where the number of hyper multiplets is related to the number of non-trivially closed $(2, 1)$-forms in the bulk by $h^{2,1}_{\text{un}} + 1$ [84, 85]. Since the $(1, 1)$-form get mapped to themselves by the orientifold projection their eigenvalue to $\Omega R$ be $\pm 1$. Since only chiral multiplets remain from each vector multiplet, by the correspondence of the differential forms and the supersymmetry multiplets[8], it seems that the $(1, 1)$-forms from the bulk transform with a sign under $\Omega R$.

### Twisted sector

According to (4.43), twisted states from closed strings, which arise due to the non-trivial structure of the orbifold fixed points can arise. Applying the twisted boundary conditions in (4.43) to the $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold, the moddings of the bosonic and fermionic oscillators from twisted strings at a $g$ fixed point get shifted by $1/2$ in the direction on which the $g$ acts non-trivially. For example $\theta$ from (4.86) acts on the directions $x^i$, with $i \in \{4, ..., 7\}$, non-trivally, thus the bosonic and fermionic oscillators modes in those directions are given by

$$i \in \{4, ..., 7\} \quad \begin{cases} \alpha^i_{n-1/2} & \text{for bos. sector} \\ b^i_n & \text{for NS sector} \\ b^i_{n-1/2} & \text{for R sector} \end{cases} , \quad n \in \mathbb{Z} , \tag{4.128}$$

where for the remaining directions the moddings of the oscillators do not change. The oscillator moddings for the other twisted sectors are determined analogously. The ground state energies $E^{\text{tw}}_0$ for the twisted NS and R sector get modified due to the different moddings of the oscillators. The ground state energies are determined by normal ordering constants arising from commuting oscillator modes in the quantized theory. Therefor the ground state energy gets the contribution $Z_\mu$ from oscillator modes for each directions $\mu$ by

$$Z_\mu = \frac{1}{2} \sum_{w>0} w - \frac{1}{2} \sum_{w+r>0} (w + r) , \tag{4.129}$$

[8] for more details see [86].

with $r \in \{0, 1/2\}$ for the R/NS sector and $w$ the moddings of the oscillators. The first term is the contribution from the bosonic oscillators and the second term the contricbution from the fermionic oscillators. For a $g \in \mathbb{Z}_2$ twisted sectors, $w \in \mathbb{Z}$ for the directions in which $g$ acts trivially and $w \in \mathbb{Z} + \frac{1}{2}$ for directions in which $g$ acts non-trivially and the ground state energies are given by then given by

$$E_0^{\text{tw}} = \sum_{\mu=2}^{9} Z_\mu = 0\,, \tag{4.130}$$

for the NS and R sector. Hence the twisted NS and R ground states are massless and because the both twisted sectors contain each four zeromodes, which satisfy the four dimensional Clifford algebra, the ground states are four fold degenerate. A GSO-projection, which is consistent with the choice for $\Omega R$, preserves in the left-and rightmoving sector the same two massless states in the twisted NS sector, where in the twisted R sector the opposite states are preserved in the left- and rightmoving sector. From the four dimensional point of view, the twisted NS states transform as scalars, where in each of the twisted left- and rightmoving R sector the two states transform as a fermion, but with opposite chiralities. Each $\mathbb{Z}_2$ fixed locus contains four scalars from the NS-NS sector, four fermions from the NS-R and R-NS sector and a vector field and two scalars in the R-R sector. They fit into a $\mathcal{N} = (2, 2)$ vector multiplet. However the second $\mathbb{Z}_2$ of the point group breaks the $\mathcal{N} = (2, 2)$ supersymmetry to $\mathcal{N} = (1, 1)$ by projecting out half of the twisted states. The $\mathcal{N} = (2, 2)$ vector multiplet decomposes into an $\mathcal{N} = (1, 1)$ vector multiplet and a $\mathcal{N} = (1, 1)$ hyper multiplet. Which of those multiplets are preserved depends on the choice of discrete torsion. For the case without discrete torsion the vector multiplet is preserved, where for the case with discrete torsion the hyper multiplet is preserved. Comparing the number of multiplets in the twisted sector with the Hodge numbers from the exceptional divisors of $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifolds, the vector multiplets can be associated to the $(1, 1)$-forms, where the hyper multiplets correspond to $(2, 1)$-forms, which is consistent with the behaviour under discrete torsion.

### 4.2.2 Massless spectrum from intersecting D6-branes

#### Supersymmetric D6-branes

D6-branes are six dimensional dynamical objects spreading out a seven dimensional worldvolume $\Sigma_7$ in spacetime. Their loci are determined by Neumann boundary conditions along the directions tangent to brane volume and Dirichlet boundary conditions along directions normal to the brane volume. D6-branes are spacetimefilling in the uncompact space and wrap 3-cycles in the compact space $X^6$

$$\Sigma_7 = \mathbb{R}^{1,3} \times \Pi^3 \subset \mathbb{R}^{1,3} \times X^6\,, \quad \text{with} \quad \Pi^3 \in H_3(X^6)\,. \tag{4.131}$$

A D6-brane carries a brane tension given by $\mu_6 = \frac{(\alpha')^{-7/2}}{(2\pi)^6}$ and therefor $\Sigma_7$ tends to minimize its volume. Hence the D6-brane has to wrap a volume minimizing 3-cycle, which means $\Pi^3$ has to be a sLag cycle [87–90]. According to the discussion in section 4.1.4, D6-branes on a $T^6$ have in each complex plane the shape of straight lines. Generalizing the case for intersecting branes in section 3.2.2 to three complex dimensions, D6-branes intersect at points on the $T^6$ and unlike to the uncompact example on $\mathbb{C}$, the identification of points differing by lattice vector leads possibly to more than one intersection point. Chan-Paton labels of open string states attached to a single stack of D6-branes form states of the adjoint representation $\mathbf{adj} \subseteq \square \otimes \overline{\square}$ of the gauge group on that stack. String states from open strings attached to two different stacks of D6-branes $a$ and $b$ are located at the intersection points of $a$ and $b$ on the $T^6$ and their Chan-Paton labels transform in the bifundamental representation $(\square_a, \overline{\square}_b)$ of the gauge groups on $a$ and $b$. From the four dimensional point of view the string states on a single stack of D6-branes lead to gauge fields of the corresponding gauge symmetry and strings states on two different stacks of D6-branes behave like chiral matter. Since the D6-branes fill out the whole uncompact space, open strings on D6-branes enjoy N boundary conditions in the uncompact space and can propagate there freely. The mass of an open string state on two D6-banes $a$ and $b$ is given by [49, 91]

$$\alpha' M^2 = \frac{Y^2}{4\pi^2\alpha'} + N_{\mathrm{B}} + N_{\mathrm{F}} + r\sum_{h=1}^{3} v_{ab}^h - r\,, \qquad r = \begin{cases} 0 & \text{R sector} \\ \frac{1}{2} & \text{NS sector} \end{cases}, \tag{4.132}$$

where $v_{ab}^h = \theta_{ab^h/\pi}$, with $\theta_{ab}^h$ the relative angles of the branes $a$ and $b$ on the $h$-th complex plane and $Y$ measuring the distance stretched by the string. The lightest states at intersection points are given by [92]

$$\alpha' M^2 = \frac{(\vec{r} - \vec{v}_{ab})^2}{2} + \sum_{h=1}^{3} \frac{1}{2}|v_{ab}^h|\left(1 - |v_{ab}^h|\right) - \frac{1}{2}\,, \tag{4.133}$$

where $\vec{v}_{ab} = (0, v_{ab}^1, v_{ab}^2, v_{ab}^3)^T$ and $\vec{r} = (r_0, r_1, r_3, r_4)^T$, with $r_i \in \mathbb{Z}$ for NS states and $r_i \in \mathbb{Z} + \frac{1}{2}$ for R states, where $r_0 = 0$ denotes a scalar, $r_0 = \pm\frac{1}{2}$ a chiral, antichiral Weyl spinor and $r_0 = \pm 1$ vector fields in four dimensions. The four dimensional spectrum of light states in four dimensions is listed in table 4.2 for the case where $\theta_{ab}^1, \theta_{ab}^2 > 0$ and $\theta_{ab}^3 < 0$ [92]. At the intersection points there is always a massless fermion in

| sector | $\vec{r} - \vec{v}_{ab}$ | $\alpha' M^2$ | 4d field |
|---|---|---|---|
| NS: | $\left(0, -1 + v_{ab}^1, v_{ab}^2, v_{ab}^3\right)$ | $\frac{1}{2}\left(-v_{ab}^1 + v_{ab}^2 - v_{ab}^3\right)$ | scalar |
| | $\left(0, v_{ab}^1, -1 + v_{ab}^2, v_{ab}^3\right)$ | $\frac{1}{2}\left(v_{ab}^1 - v_{ab}^2 - v_{ab}^3\right)$ | scalar |
| | $\left(0, v_{ab}^1, v_{ab}^2, 1 + v_{ab}^3\right)$ | $\frac{1}{2}\left(v_{ab}^1 + v_{ab}^2 + v_{ab}^3\right)$ | scalar |
| | $\left(0, -1 + v_{ab}^1, -1 + v_{ab}^2, 1 + v_{ab}^3\right)$ | $1 - \frac{1}{2}\left(v_{ab}^1 + v_{ab}^2 - v_{ab}^3\right)$ | scalar |
| R: | $\left(-\frac{1}{2}. -\frac{1}{2} + v_{ab}^1, -\frac{1}{2} + v_{ab}^2, \frac{1}{2} + v_{ab}^3\right)$ | 0 | Weyl spinor |

Table 4.2: Light open string states at intersection points of D6-branes and their representation in the uncompact space.

four dimensions and for

$$\sum_{h=1}^{3} \theta_{ab}^h = 0\,, \tag{4.134}$$

the NS sector provides the four dimensional spectrum with a massless scalar, s.t. an $\mathcal{N} = 1$ chiral multiplet in four dimensions is formed at each intersection point. The condition (4.134) is satisfied, when both D-branes $a$ and $b$ wrap 3-cycles with the same calibration phase. The calibration phase determines which subset of supercharges from the bulk is preserved on the brane [88], since the D6-branes are BPS states in Type IIA [46]. When both branes share the same calibration phase it means they preserve the same supercharges on their volume. Further the condition (4.134) is similar to the condition (4.36) for twist vectors for orbifolds preserving one killing spinor. The killing spinors preserved on the D-branes $a$ and $b$ are related by a $SU(3)$ transformation according to (4.134), which is the supersymmetry condition on orbifolds.

### $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \Omega R$-projection

In section 4.1.5 it was discussed that $\mathbb{Z}_2 \times \mathbb{Z}_2$ invariant cycles wrap twice a torus cycle, which passes through $\mathbb{Z}_2 \times \mathbb{Z}_2$ fixed points. That means a D6-brane $a$ wrapping $N_a$ times a cycle $\Pi_a^3$ on the torus, becomes under the $\mathbb{Z}_2 \times \mathbb{Z}_2$ action a brane wrapping $\frac{N_a}{2}$ times an orbifold cycle. Under the action of the spacetime involution $R$, $a$ gets mapped to an image brane $a'$, which wraps the image cycle $\Pi^3_{a'}$, where $R:\ \Pi_a^3 \to \Pi^3_{a'}$. On the level of the gauge groups this means, that a $U(N_a) \times U(N_a')$ gauge group on the torus, with $U(N_a')$ the gauge symmetry on the image branes to the stack generating $U(N_a)$, gets broken by the point group to a $U(N_a/2) \times U(N_a'/2)$ gauge symmetry and the spacetime refelection further reduces the gauge symmetry to $U(N_a/2)$ [81]:

$$U(N_a) \times U(N_a) \xrightarrow{\mathbb{Z}_2\times\mathbb{Z}_2} U(N_a/2) \times U(N_a/2) \xrightarrow{R} U(N_a/2)\,. \tag{4.135}$$

However if the stacks $a$ and $a'$ lie on top of an O-plane, they can be viewed as a single stack with $2N_a$ branes getting mapped by $R$ to themselves. The gauge symmetry of D-branes on top of O-planes are determined by the effect of worldsheet parity on the Chan-Paton labels: Open string states on a stack on top of an O-plane have to form invariant states under worldsheet parity. Worldsheet parity interchanges the two open string endpoints and therefore $\Omega$ acts on the Chan-Paton labels as transposing them

$$\Omega:\ |ij\rangle \to \gamma|ji\rangle\gamma^{-1}\,, \tag{4.136}$$

with $\gamma$ containing the action of $\Omega$ on the gauge indices. Acting twice with $\Omega$ has to give back initial state

$$\Omega^2:\ |ij\rangle \to \gamma\left(\gamma|ji\rangle\gamma^{-1}\right)^T\gamma^{-1} = |ij\rangle \tag{4.137}$$

leading to $\gamma = \pm\gamma^T$. The condition is solved by $\gamma = \mathbb{1}_N$ for the plus sign and by $\gamma = i\begin{pmatrix} 0 & \mathbb{1}_{N/2} \\ \mathbb{1}_{N/2} & 0 \end{pmatrix}$ for the minus sign. Recalling the open string boundary conditions (3.6), (3.7) and inserting the open string mode expansions into them, one receives for N boundary conditions on both end points

$$X^\mu(\sigma^0,\sigma^1) = x^\mu + \frac{2\pi\alpha'}{\ell}p^\mu\sigma^0 + i\sqrt{2\alpha'}\sum_{n\in\mathbb{Z}\backslash\{0\}}\frac{1}{n}\alpha_n^\mu \mathrm{e}^{-i\pi n\sigma^0/\ell}\cos\left(n\pi\sigma^1/\ell\right)\,, \tag{4.138}$$

and for D boundary conditions on both end points

$$X^\mu(\sigma^0,\sigma^1) = x^\mu + \frac{1}{\ell}c^\mu\sigma^1 + \sqrt{2\alpha'}\sum_{n\in\mathbb{Z}\backslash\{0\}}\frac{1}{n}\alpha_n^\mu \mathrm{e}^{-i\pi n\sigma^0/\ell}\sin\left(n\pi\sigma^1/\ell\right)\,. \tag{4.139}$$

Using the fact that worldsheet partity acts on the strings as

$$\Omega R:\ X^\mu(\sigma^0,\sigma^1) = \begin{cases} X^\mu(\sigma^0,\ell-\sigma^1) & \text{for N} \\ -X^\mu(\sigma^0,\ell-\sigma^1) & \text{for D} \end{cases}, \tag{4.140}$$

it can be deduced that the oscillators transform as $\Omega R:\ \alpha_n^\mu \to \alpha_n^\mu$. Analogous the fermionic oscillators transform as $\Omega R:\ b^\mu_{n+r} \to \mathrm{e}^{i\pi(n+r)}b^\mu_{n+r}$. The massless open string states then transform as

$$\Omega R:\ b^\mu_{-1/2}|0\rangle_{\mathrm{NS}}\otimes|ij\rangle \to i\eta_{\mathrm{NS}}b^\mu_{-1/2}|0\rangle_{\mathrm{NS}}\otimes\gamma|ij\rangle\gamma^{-1}\,,\quad b_0^\mu|0\rangle_{\mathrm{R}}\otimes|ij\rangle \to \eta_{\mathrm{R}}b_0^\mu|0\rangle_{\mathrm{R}}\otimes\gamma|ij\rangle\gamma^{-1}\,, \tag{4.141}$$

with $\eta_{\mathrm{NS/R}}$ capturing the transformation behavior of the ground states $\Omega R:\ |0\rangle_{\mathrm{NS/R}} \to \eta_{\mathrm{NS/R}}|0\rangle_{\mathrm{NS/R}}$. For $\eta_{\mathrm{R}} = -i\eta_{\mathrm{NS}} =: \eta$, the massless open string states stay invariant, when the Chan-Paton factors transform as

$$|ij\rangle = \eta\gamma|ij\rangle\gamma^{-1}\,. \tag{4.142}$$

Since $\Omega$ is an involution on the worldsheet $\eta = \pm 1$. For those two choices, the Chan-Paton factors belonging to fundamental representations of [41]

$$|ij\rangle \in \begin{cases} SO(N) & \text{for } |ij\rangle = \gamma|ij\rangle\gamma^{-1} \\ USp(N) & \text{for } |ij\rangle = -\gamma|ij\rangle\gamma^{-1} \end{cases} \tag{4.143}$$

preserve the massless NS and R open string states. The choice for $\eta$ has to be specified in accord with tadpole cancellation condition. As going to be explained in section 4.2.3, the total R-R charge needs to vanish. That means the amplitudes for interaction of O-planes and D-branes via R-R closed strings among each other needs to interfere destructively [41]. Therefore the amplitude of the interactions between O-planes with D-branes need to have the opposite sign as O-planes with O-planes and D-branes with D-branes. For Type IIA strings on a $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \Omega R$ orientifold this is the case for the Chan-Paton factors belonging to $USp(N)$ [81].

The massless spectrum in four dimensions received from intersection D6-branes $a$ and $b$ on a $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \Omega R$ orientifold is listed in table 4.3 [81]. $I_{ab}$ is the intersection number of the cycles, wrapped by $a$ and $b$, and $G_\alpha \in \{U(N_\alpha), SO(N_\alpha), USp(N_\alpha)\}$, with $\alpha \in \{a, b\}$, the gauge symmetry on the branes. When branes intersect with their orientifold images one gets beside bifundamental matter also chiral matter in the symmetric and antisymmetric irreducible representations $\mathbf{Sym}_\alpha \subset \Box_\alpha \otimes \Box_\alpha$ and $\mathbf{Anti}_\alpha \subset \Box_\alpha \otimes \Box_\alpha$. Their multiplicity depend on the number of O-planes $N_{\mathrm{O6}}$.

| sector | multiplicity | rep. $G_a \times G_b$ |
|---|---|---|
| $a$ | 1 | $(\mathbf{adj}_a, \mathbf{1})$ |
| $b$ | 1 | $(\mathbf{1}, \mathbf{adj}_b)$ |
| $a \cap b + a' \cap b'$ | $I_{ab}$ | $(\Box_a, \overline{\Box}_b)$ |
| $a \cap b' + a' \cap b$ | $I_{ab'}$ | $(\Box_a, \Box_b)$ |
| $a \cap a' + a' \cap a$ | $\frac{1}{2}\left(I_{aa'} - N_{\mathrm{O6}} I_{a,\mathrm{O6}}\right)$<br>$\frac{1}{2}\left(I_{aa'} + N_{\mathrm{O6}} I_{a,\mathrm{O6}}\right)$ | $(\mathbf{Sym}_a, \mathbf{1})$<br>$(\mathbf{Anti}_a, \mathbf{1})$ |

Table 4.3: Gauge representation of massless open strings states on intersecting branes and their number of families in four dimensions.

### 4.2.3 Consistency conditions and anomaly cancellation for D6-branes

O6-planes, as well as D6-branes, are charged under the RR 7-form gauge fields $C_7$ [41]. From Maxwells theory for higher dimensions the equations of motion for RR 7-forms are given by

$$\mathrm{d} * \mathrm{d}C_7 \propto J\,, \tag{4.144}$$

where $J$ is the R-R charge density and $*$ the Hodge star. Integrating (4.144) over a compact space, the left-hand side of (4.144) has to vanish due to Gauss law. Hence the right-hand side must also vanish, s.t. the total R-R charge in the compact space has sum up to zero. Otherwise the interactions of O6-planes and D6-branes via R-R closed strings lead to divergent amplitudes, called tadpoles. The R-R charge of an O6-plane is -4 times the RR charge of a D6-brane and to cancel the charge of an O-plane four D-branes, wrapping cycles from the same homology class as the O-plane cycle, has to be inserted [54, 55, 83]. Hence the tadpole cancelation condition is given by

$$\sum_a N_a \left[\Pi^3_a\right] + \sum_{a'} N_a \left[\Pi^3_{a'}\right] - 4N_{O6}\left[\Pi^3_{O6}\right] = 0\,. \tag{4.145}$$

Due to the presence of chiral matter in the four dimensional spectrum, anomalies in the four dimensional theory are possible. In [45, 93–97] such anomalies are discussed also for orientifold compactifications, where some of the arguments are repeated in the following of this section. The possible four dimensional anomalies are $SU(N)^3$ cubic anomalies, mixed $U(1)-a-SU(N_b)^2$ anomalies and mixed $U(1)$-gravitational anomalies, arising from the $U(1)$ and $SU(N)$ factors in the $U(N)$ gauge groups on the brane volumes[9]. The cubic anomalies $\mathcal{A}^{aaa}$, coming from the current $\overline{\psi}\overline{\sigma}\psi$ with $\psi$ a chiral fermion in the fundamental representation $\Box_a$ of $SU(N_a)$, are proportional to the trace

$$\mathcal{A}^{aaa}_{\Box_a} = \mathrm{Tr}\left(T^a\left\{T^a, T^a\right\}\right)\,, \tag{4.146}$$

for each generators $T^a$ of $SU(N_a)$, where the trace is taken over indices in the gauge representation. Using the relations [10]

$$\mathrm{Tr}_{\mathbf{Sym}_a}F^2 = (N_a+4)\mathrm{Tr}_{\Box_a}\,, \quad \mathrm{Tr}_{\mathbf{Anti}_a}F^2 = (N_a-4)\mathrm{Tr}_{\Box_a}\,, \tag{4.147}$$

with $F$ the fieldstrength of $SU(N_a)$ and $\mathrm{Tr}_\alpha$ the trace taken over the gauge indices in the $\alpha$ representation, the anomalies coming from chiral fermions in the antifundamental, symmetric and antisymmetric representations $\overline{\Box}_a$, $\mathbf{Sym}_a$ and $\mathbf{Anti}_a$ are related to the anomaly from fundamental fields by

$$\mathcal{A}^{aaa}_{\overline{\Box}_a} = -\mathcal{A}^{aaa}_{\Box_a}\,, \quad \mathcal{A}^{aaa}_{\mathbf{Sym}_a} = (N_a+4)\mathcal{A}^{aaa}_{\Box_a}\,, \quad \mathcal{A}^{aaa}_{\mathbf{Anti}_a} = (N_a-4)\mathcal{A}^{aaa}_{\Box_a}\,. \tag{4.148}$$

For an intersecting D6-brane setup with O6-planes, the total cubic $SU(N_a)^3$ anomaly $\mathcal{A}^{aaa}_{\text{total}}$ for coming from all chiral fields with the corresponding multiplicity from table 4.3 is given by

$$\mathcal{A}^{aaa}_{\text{total}} = \left\{\sum_{b\neq a} N_b\left(I_{a'b}-I_{ab}\right) + \frac{N_a-4}{2}\left(I_{aa'}+N_{O6}I_{a,O6}\right) + \frac{N_a+4}{2}\left(I_{aa'}-N_{O6}I_{a,O6}\right)\right\}\mathcal{A}^{aaa}_{\Box_a}\,, \tag{4.149}$$

[9] $U(N) \simeq (SU(N)\times U(1))/\mathbb{Z}_N$ as will be explained in section 6.2.2.

[10] Given in appendix B of [86]

and using $I_{ab} = \left[\Pi_a^3\right] \cdot \left[\Pi_b^3\right]$ one can factor out $\Pi_a^3$, s.t the total cubic anomaly is proportional to

$$\sum_b N_b \left(\left[\Pi_b^3\right] + \left[\Pi_{b'}^3\right]\right) - 4N_{O6}\left[\Pi_{\mathrm{O6}}^3\right] \tag{4.150}$$

and hence vanishes, when tadpole cancelation is satisfied. The mixed $U(1)_a - SU(N_b)^2$ anomalies are given by

$$\mathcal{A}^{abb} = \mathrm{Tr}\left(Q_a T^b T^b\right), \tag{4.151}$$

where $Q_a$ is the $U(1)_a$ generator. Again using the relations (4.147), the mixed $U(1)_a - SU(N_b)^2$ anomalies are proportional to

$$\mathcal{A}^{abb} \propto \sum_b N_b \left(I_{ab} - I_{a'b}\right), \tag{4.152}$$

and hence non zero. However they can get canceled by a four dimensional Green-Schwarz mechanism [92, 98]. In the following the anomaly cancellation is illustrated by using the arguments in [20]. The terms $\int_{\Sigma_7} C_5 \wedge \mathrm{Tr}F$ and $\int_{\Sigma_7} C_3 \wedge \mathrm{Tr}F^2$ in the Chern-Simons action in (3.45) express how the gauge groups on the brane volumes couple to the 5-form and 3-form gauge potentials. First a basis for the $H_3(X^6)$ is introduced by $\left\{[\alpha_k], \left[\beta^k\right]\right\}_{k\in\{1,\ldots,4\}}$, where $[\alpha_k]$ are dual to $\left[\beta^k\right]$ in the sense that $[\alpha_k] \cdot \left[\beta^l\right] = \delta_k^l$. Now let the stack $a$ wrap a cycle in the internal space from the homology class $[\alpha_k]$ and the stack $b$ a cycle $\left[\beta^l\right]$. Then integrating the 5-form $C_5$ on the stack $a$ and the 3-form $C_3$ on the stack $b$ over the internal space leads to a 2-form $B_2$ and scalar $\theta$ in four dimensions

$$B_2^k = \int_{[\alpha_k]} C_5, \quad \theta_l = \int_{[\beta^l]} C_3. \tag{4.153}$$

Remembering that the $C_5$ gauge potential is the magnetic dual of $C_3$ in ten dimensions and they are related via $\mathrm{d}C_5 = *\mathrm{d}C_3$, the four dimensional relation $\theta_l$ and $B_2^k$ follows[11]

$$\partial_\mu B_{\nu\rho}^k = \epsilon_{\mu\nu\rho\sigma}\partial^\sigma \theta_l. \tag{4.155}$$

Let the brane $a$ containing the $U(1)_a$ wrap the cycle $\Pi_a^3$ and the brane $b$, containing the $SU(N_b)$ factor wrap the cycle $\Pi_b^3$. Then integrating out the internal space from the Chern-Simons term $N_a \int_{\mathbb{R}^{1,3}\times\Pi_a^3} C_5 \wedge \mathrm{Tr}F_a$ from the brane $a$ and $\int_{\mathbb{R}^{1,3}\times\Pi_b^3} C_3 \wedge \mathrm{Tr}{F_b}^2$ from the brane $b$ the four dimensional couplings of the $U(1)_a$ to $B_2^k$ and $SU(N_a)$ to $\theta_l$ is given by

$$N_a s_{ak} \int_{\mathbb{R}^{1,3}} B_2^k \wedge \mathrm{Tr}F_a, \quad q_b^l \int_{\mathbb{R}^{1,3}} \theta_l \wedge \mathrm{Tr}{F_b}^2, \tag{4.156}$$

where $s_{ak} = \left[\Pi_a^3\right] \cdot [\alpha_k]$ and $q_b^l = \left[\Pi_b^3\right] \cdot \left[\beta^l\right]$ contain the overlap of the brane cycles with the cycle wrapped by the gauge potentials. The field $B_2^k$ couples to the $U(1)_a$ gauge boson and the field $\theta^l$ to two $SU(N_b)$ gauge bosons and due to the relation (4.154) the $U(1)_a$ can couple to two $SU(N_b)$'s via the exchange

[11] Integrating the equation $\mathrm{d}C_5 = *\mathrm{d}C_3$ over the compact space one receives

$$\partial_\mu \int_{[\alpha_k]} C_5 = \epsilon_{\mu\nu\rho\sigma}\partial^\sigma \int_{[\beta^l]} C_3, \tag{4.154}$$

where $C_5$ fills out the directions $\nu$ and $\rho$ in the uncompact space. Inserting the definitions of $B_2^k$ and $\theta_l$ the relation in four dimensions $\partial_\mu B_2^k = \epsilon_{\mu\nu\rho\sigma}\partial^\sigma\theta_l$ follows.

of the scalar field $\theta^l$, generating four dimensional Green-Schwarz terms. The Green-Schwarz terms interfere destructively with the $U(1)_a - SU(N_b)^2$ anomaly terms and therefor the mixed $U(1)_a - SU(N_b)^2$ anomalies are canceled. Similar the mixed gravitational anomalies also vanish[12]. However the coupling of the $U(1)$ gauge boson to the field $B_2^k$ breaks the $U(1)$ symmetry through a Stückelberg mechanism, where $\theta^l$ behaves as an axion with a shift symmetry[13]. The broken $U(1)$ remains as a discrete symmetry [99]. The non-anomalous $U(1)$, is a linear combination of the other $U(1)$ and its generator $Q_Y$ is given by a linear combination of the other $U(1)$ generators $Q$ by

$$Q_Y = \sum_{a \in \{U(1)\}} c_a Q_a \,, \tag{4.157}$$

where the coefficients $c_a$ are determined by the conditons

$$\sum_a N_a \left(s_{ak} - s_{a'k}\right) c_a = 0 \,, \quad \forall k \in \{1, 2, 3, 4\} \,. \tag{4.158}$$

## 4.3 Model building on $T^6_{\mathrm{SO(12)}}/(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \Omega R)$

### 4.3.1 Towards realistic four dimensional particle physics

| sector | $U(4) \times U(2) \times U(2)$ |
|---|---|
| $ab + ab'$ | $3 \times (\mathbf{4}, \mathbf{2}, \mathbf{1})$ |
| $ac + ac'$ | $3 \times \left(\overline{\mathbf{4}}, \mathbf{1}, \mathbf{2}\right)$ |
| $bc + b'c$ | $(I_{bc} + I_{b'c}) \times (\mathbf{1}, \mathbf{2}, \mathbf{2})$ |

Table 4.4: Pati-Salam from intersecting branes.

The results achieved in [58, 81, 82, 100–102] are just some examples for the progress in constructing realistic four dimensional particle physics models with intersecting D-branes on orientifolds. There has been also some attempts to construct intersecting brane models on non-factorisable orientifolds as in [77, 103–105]. The term "realistic" in the context of string compactifiaction is used for the following six criteria:

(i) The maximal supersymmetry in the massless four dimensional spectrum should be $\mathcal{N} = 1$. Otherwise one receives a non chiral spectrum.

(ii) The gauge group of the theory should be either $SU(3) \times SU(2) \times U(1)$, where the $U(1)$ has the properties of hypercharge, or a GUT group of the SM, s.t. it is possible to break the gauge group to the SM.

(iii) The massless chiral spectrum should contain at least three families of quarks and leptons.

[12] For more details see [86].

[13] The four dimensional coupling of $B_2$ and the $U(1)$ gauge bosonis given by $cB_2 \wedge F$ in the Lagrangian, with $F$ the $U(1)$ fieldstrength. Substituting the field $B_2$ by its magnetic dual scalar $\theta$ by using the relation (4.155), the corresponding term in the Lagrangian becomes $2\,(cA + \mathrm{d}\theta)^2$, which indicate massterms for the gauge bosons. The term is symmetric under $A \to A + \mathrm{d}\Lambda$, when the scalar $\theta$ admits the simultaneous shift symmetry of $\theta \to \theta - c\Lambda$, s.t. only a subset of the $U(1)$ symmetry is preserved.

| sector | $U(3)_{\mathrm{B}} \times U(2)$ | $Q_{\mathrm{B}}$ | $Q_{\mathrm{L}}$ | $Q_b$ | $Q_{\mathrm{I}_1}$ | $Q_{\mathrm{I}_1}$ | Field |
|---|---|---|---|---|---|---|---|
| $a_1 b + a_1 b'$ | $3 \times (\mathbf{3}, \mathbf{2})$ | 1 | 0 | -1 | 0 | 0 | $Q_L$ |
| $a_2 b + a_2 b'$ | $3 \times (\mathbf{1}, \mathbf{2})$ | 0 | 1 | -1 | 0 | 0 | $L$ |
| $a'_1 c_1$ | $3 \times \left(\bar{\mathbf{3}}, \mathbf{1}\right)$ | -1 | 0 | 0 | 1 | 0 | $u$ |
| $a'_1 c_2$ | $3 \times \left(\bar{\mathbf{3}}, \mathbf{1}\right)$ | -1 | 0 | 0 | 0 | 1 | $d$ |
| $a'_2 c_1$ | $3 \times (\mathbf{1}, \mathbf{1})$ | 0 | -1 | 0 | 1 | 0 | $\nu_R$ |
| $a'_2 c_2$ | $3 \times (\mathbf{1}, \mathbf{1})$ | 0 | -1 | 0 | 0 | 1 | $e_R$ |
| $bc_1 + b'c_1$ | $(I_{bc} + I_{b'c}) \times (\mathbf{2}, \mathbf{1})$ | 0 | 0 | 1 | 1 | 0 | $H_u$ |
| $bc_2 + b'c_2$ | $(I_{bc} + I_{b'c}) \times (\mathbf{2}, \mathbf{1})$ | 0 | 0 | -1 | 0 | 1 | $H_d$ |

Table 4.5: Pati-Salam to $SU(3) \times SU(2) \times U(1)$.

(iv) It should be possible to construct Yukawa couplings between chiral fermions and a scalar field, in order for the quarks and leptons to gain masses via a Higgs mechanism. It is further desired that the couplings allow mass hierachies.

(v) Vector-like exotics should be absent in the massless spectrum.

(vi) Absence of anomalies and tadpoles.

The first criterion is satisfied, when the extra dimensions are compactfied on an orientifold described as described in section 4.1.2 and D-branes are placed, which share the same calibration phase. The gauge group and particle spectrum is controlled by the number of branes on a stack and the intersection numbers of the branes. There are two types of models, which can be constructed: In the first type the $SU(2)$ factor comes from a stack of branes with a $U(2)$ symmetry on them and in the second type the $SU(2)$ factor comes from a stack of branes on top of O-planes with a $USp(2)$ symmetry on them. Both choices are possible since $SU(2) \subset U(2)$ and $SU(2) \simeq USp(2)$. The amount and wrapping numbers of D6-branes are restricted by the tadpole cancellation condition. Further, anomaly cancellation leads to the $U(1)$ charges of the fields. In order to allow Yukawa couplings, the D-brane configuration must obey a certain geometry: A Yukawa coupling between three fields accrues from worldsheet instantons connecting the three fields [106]. This is only possible, when the branes on which the three fields sit, form the boundary for worldsheet instantons. When three stacks of branes intersect non trivially and no stack is parallel to the other two stacks, worldsheet instantons can stretch between the D-branes and the fields, sitting at their intersection points, couple via the instanton and allow Yukawa couplings. Since the instanton strength is supressed by its area spread out in the internal space, mass hierachies can naturally arise in intersecting D-brane models and different masses in the low energy mass spectrum can be explained by the global geometry of the internal space. A simple GUT model containing the correct Yukawas is given by $U(4) \times U(2) \times U(2)$ Pati-Salam, for example discussed in [107]. In order to realize Pati-Salam from intersecting branes, one needs at least three stacks of branes. Let the three stacks be denoted by $a$, $b$ and $c$. All three stacks are placed apart from O-planes, s.t. orientifold images $a'$, $b'$, $c'$ need to be included to the setup. The number $N_\alpha$ of D-branes on the stacks $\alpha \in \{a, b, c\}$ are

$$N_a = 4\,, \quad N_b = N_c = 2\,. \tag{4.159}$$

Their intersection numbers have to be

$$I_{ab} + I_{ab'} = 3\,, \quad I_{ac} = -3\,, \quad I_{ac'} = 0\,, \quad I_{bc} + I_{b'c} \neq 0\,. \tag{4.160}$$

Then one gets three generations of bifundamentals in $SU(4) \times SU(2)$ and a non vanishing amount of bifundamentals in $SU(2)^2$:

$$3 \times (\mathbf{4}, \mathbf{2}), \; 3 \times \left(\overline{\mathbf{4}}, \mathbf{2}\right) \in SU(4) \times SU(2), \quad (I_{bc} + I_{b'c}) \times (\mathbf{2}, \mathbf{2}) \in SU(2) \times SU(2). \tag{4.161}$$

By splittint the stack $a$ into two stacks $a = a_1 + a_2$, with $N_{a_1} = 3$ and $N_{a_2} = 1$, the group $U(4)$ breaks to

$$U(4) \xrightarrow{a \to a_1 + a_2} U(3)_{\mathrm{B}} \times U(1)_{\mathrm{L}} \tag{4.162}$$

and the fields $(\mathbf{4}, \mathbf{2})$ and $\left(\overline{\mathbf{4}}, \mathbf{2}\right)$ decompose into

$$(\mathbf{4}, \mathbf{2}) \to (\mathbf{3}, \mathbf{2}) \oplus (\mathbf{1}, \mathbf{2}), \quad \left(\overline{\mathbf{4}}, \mathbf{2}\right) \to (\overline{\mathbf{3}}, \mathbf{2}) \oplus (\mathbf{1}, \mathbf{2}). \tag{4.163}$$

By further splitting the stack $c$ into $c = c_1 + c_2$, with $N_{c_1} = N_{c_2} = 1$, the second $U(2)$ gauge group breaks to

$$U(2) \xrightarrow{c \to c_1 + c_2} U(1)_{\mathrm{I}_1} \times U(1)_{\mathrm{I}_2} \tag{4.164}$$

and the fields $(\mathbf{2}, \mathbf{2})$ and $\left(\overline{\mathbf{3}}, \mathbf{2}\right) \oplus (\mathbf{1}, \mathbf{2})$ decompose into

$$\left(\overline{\mathbf{3}}, \mathbf{2}\right) \oplus (\overline{\mathbf{1}}, \mathbf{2}) \to \left(\overline{\mathbf{3}}, \mathbf{1}\right) \oplus \left(\overline{\mathbf{3}}, \mathbf{1}\right) \oplus (\mathbf{1}, \mathbf{1}) \oplus (\mathbf{1}, \mathbf{1}), \quad (\mathbf{2}, \mathbf{2}) \to (\mathbf{2}, \mathbf{1}) \oplus (\mathbf{2}, \mathbf{1}). \tag{4.165}$$

The total number of chiral fields and its representation under the gauge groups, before and after the Higgsing[14], are listed in table 4.4 and 4.5, where $Q_{\mathrm{B}}$ and $Q_b$ are the generators of the $U(1)$'s in $U(3)_{\mathrm{B}}$ and $U(2)$. When the remaining massless $U(1)_{\mathrm{Y}}$ is generated by the following linear combination

$$Q_{\mathrm{Y}} = \frac{1}{6} Q_{\mathrm{B}} - \frac{Q_{\mathrm{L}} + Q_{\mathrm{I}_1} - Q_{\mathrm{I}_2}}{2}, \tag{4.166}$$

$U(1)_{\mathrm{Y}}$ serves as the SM-like hypercharge and the fields listed in table 4.5 are indeed SM-like chiral matter. In order not to overshoot the tadpole cancellation condition, the cycles $\Pi^3_\alpha$, wrapped by the branes $\alpha \in \{a, b, c\}$, need to satisfy

$$\sum_{\alpha \in \{a,b,c\}} N_\alpha [\Pi^3_\alpha] + \sum_{\alpha \in \{a',b',c'\}} N_\alpha [\Pi^3_\alpha] \;\leq 4 N_{\mathrm{O6}} [\Pi^3_{\mathrm{O6}}]. \tag{4.167}$$

If more branes are needed to saturate the tadpole cancellation, additional fields on the extra branes arise. In order not to spoil the SM-like particle content in low energies, the new fields should not form vector-like exotics and a mechanism, to give them high masses and decouple them from the low energy spectrum, needs to be implemented. In Pati-Salam models arising from intersecting branes, as in (4.5), it is possible to construct the trilinear couplings

$$(Q_L H_u u), \quad (Q_L H_d d), \quad (L H_u \nu_R) \quad \text{and} \quad (Q_L H_d e_R), \tag{4.168}$$

which gives the quarks and leptons masses, proportional to the coupling strength, when vev's for $H_u$ and $H_d$ are turned on.

[14] The seperation of branes from a stack is denoted as Higgsing, since massless states, from strings attached to both stacks, become massive and the process can be viewed as a Higgs mechanism.

### 4.3.2 Supersymmetric toy model on $T^6_{\mathrm{SO}(12)}/(\mathbb{Z}_2\times\mathbb{Z}_2\times\Omega R)$

Let six stacks of D6-branes $\alpha\in\{a,b,c,h_1,h_2,h_3\}$ wrap $N_\alpha$ times the 3-cycles $\Pi^3_\alpha$ on the $T^6_{\mathrm{SO}(12)}$, with wrapping numbers given in table 4.6. The stacks $a$ and $c$ are oblique to the O-planes, where the

| stack $\alpha$ | $\Pi^3_\alpha=\prod_{h=1}^3(n^h_\alpha,m^h_\alpha)$ | $N_\alpha$ |
|---|---|---|
| $a$ | $(2,0)\times(1,1)\times(1,-1)$ | 6 |
| $b$ | $(0,1)\times(2,0)\times(0,-1)$ | 2 |
| $c$ | $(1,1)\times(1,-1)\times(2,0)$ | 2 |
| $h_1$ | $(2,0)\times(0,1)\times(0,-1)$ | 2 |
| $h_2$ | $(0,1)\times(0,-1)\times(2,0)$ | 6 |
| $h_3$ | $(0,1)\times(2,0)\times(0,-1)$ | 6 |

Table 4.6: Intersecting D6-branes on $T^6_{\mathrm{SO}(12)}$

stacks $b$, $h_1$, $h_2$ and $h_3$ are on top of O-planes. The stacks $b$ and $h_3$ are parallel but separated from each other. In order for the D-branes to wrap calibrated cycles with the calibration phase $\vartheta=0$, the deformation parameters of the torus have to be chosen to be $\mathrm{Im}(\tau_1)=\mathrm{Im}(\tau_2)=\mathrm{Im}(\tau_3)=1$ and $\mathrm{Re}(\tau_1)=\mathrm{Re}(\tau_2)=\mathrm{Re}(\tau_3)=0$. The tadpole cancellation condition (4.145) is satisfied and therefore non-abelian and mixed anomalies are absent from the resulting model. The gauge symmetry arising from the D-branes is $U(3)\times SU(2)^2\times USp(6)^2$, where the $SU(2)$ factors are generated by the stacks $b$ and $h_1$. The non-vanishing intersection numbers for the intersecting branes are[15]

$$\begin{aligned}&I_{ab}=2\,,\quad I_{ac}=-4\,,\quad I_{bc}=2\,,\quad I_{ah_2}=-2\,,\\&I_{ah_3}=2\,,\quad I_{ch_1}=2\,,\quad I_{ch_3}=-2\,.\end{aligned}\tag{4.169}$$

The four dimensional spectrum of chiral matter resulting from string states at the intersection points is listed in table 4.7. Two families of lefthanded quark-like fields, four families of righthanded quark-like

| sector | $U(3)_a\times SU(2)_b\times SU(2)_{h_1}\times USp(6)_{h_2}\times USp(6)_{h_3}$ | $U(1)_a$ | $U(1)_c$ | Field |
|---|---|---|---|---|
| $a\cap b$ | $2\times(\mathbf{3},\mathbf{2},\mathbf{1},\mathbf{1},\mathbf{1})$ | +1 | 0 | $Q$ |
| $a\cap c$ | $4\times\left(\overline{\mathbf{3}},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1}\right)$ | -1 | +1 | $q$ |
| $b\cap c$ | $2\times(\mathbf{1},\mathbf{2},\mathbf{1},\mathbf{1},\mathbf{1})$ | 0 | -1 | $H$ |
| $a\cap h_2$ | $2\times\left(\overline{\mathbf{3}},\mathbf{1},\mathbf{1},\mathbf{6},\mathbf{1}\right)$ | -1 | 0 | $\phi_1$ |
| $a\cap h_3$ | $2\times\left(\overline{\mathbf{3}},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{6}\right)$ | +1 | 0 | $\phi_2$ |
| $c\cap h_1$ | $2\times(\mathbf{1},\mathbf{1},\mathbf{2},\mathbf{1},\mathbf{1})$ | 0 | +1 | $h$ |
| $c\cap h_3$ | $2\times(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{6})$ | 0 | -1 | $\varphi$ |

Table 4.7: Four dimensional spectrum of chiral matter.

fields and two Higgs-like fields are contained in the toy model. The remaining fields are exotic chiral matter, because, due to their charge under the $USp(6)$ gage groups, they cannot even remotely be associated to any SM particle. Both $U(1)$ factors are anomaly free as can be seen from the vanishing of the total $U(1)$ charges of the chiral fields. However non of the $U(1)$'s have the properties to be the SM hypercharge $U(1)$. Even though the above model is just a toy model, several aspects of model building on

[15] Remember that the intersection number on the $T^6_{\mathrm{SO}(12)}$ is given by $I_{ab}=\frac{1}{2}\prod_{h=1}^3(n^h_am^h_b-n^h_bm^h_a)$.

the $T^6_{\mathrm{SO}(12)}/\,(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \Omega R)$ orientifold can be elucidated. As already pointed out in [77] it is not possible to have models with an odd number of families, which is why one has to construct models with at least four generations and find a mechanism to make the extra families massive in such a way that only three families remain in the low energy limit. However the tadpole cancellation condition is more restrictive for the non-factorisable orientifold that for the factorisable orientifold, since the number of O-planes is reduced to half of the number as possible on factorisable orientifolds, which consequently reduces the possible number of wrappings for the D-branes. In order not to overshoot the tadpole cancellation condition the wrapping numbers for the stack carrying the $SU(3)$ color symmetry have to be minimized. This was tried for the stack $a$ in table 4.6. The electroweak $SU(2)$ factor can be generated from branes on top of O-planes. That allows to spare half of the amount of branes, which are needed when the $SU(2)$ factor comes from a $U(2)$ gauge group. However at least one further stack is needed, which intersects non-trivially with the stacks containing the $SU(3)$ and $SU(2)$ gauge groups. Otherwise massterms cannot be constructed. If the third stack had a $U(2)$ symmetry on it, by seperating the stack to two stacks and breaking the $U(2)$ to $U(1) \times U(1)$, up- and down-like quarks and two Higgs fields, connecting the left-and righthanded quarks, could be realized. However tadpole cancellation only allowed for the stack $c$ in table 4.6 a $U(1)$ gauge symmetry. The only realistic feature the above model realizes is allowing Yukawa couplings for the fields $Q$, $H$ and $q$. Let the Yukawa coupling of the fields $\{Q_i\}_{i\in\{1,2\}}$, $\{H_j\}_{j\in\{1,2\}}$ and $\{q_k\}_{k\in\{1,\dots,4\}}$ be given by

$$\mathcal{L}_{\mathrm{Yuk}} = Y_{ijk} Q^i H^j q^k \,, \tag{4.170}$$

with $i, j, k$ labeling the families of the three fields. The factor $Y_{ijk}$ gives the coupling strength of the trilinear coupling. The term $\mathcal{L}_{\mathrm{Yuk}}$ can be expressed as the product $\mathbf{3} \otimes \mathbf{1} \otimes \overline{\mathbf{3}}$ in the $SU(3)$ representation and as $\mathbf{2} \otimes \mathbf{2} \otimes \mathbf{1}$ in the $SU(2)$ representation. The antisymmetric part of both tensor products contain a singlet, hence $\mathcal{L}_{\mathrm{Yuk}}$ is invariant under the $SU(3) \times SU(2)$ factor of the model. Since the $U(1)$ charges add up to zero in $\mathcal{L}_{\mathrm{Yuk}}$ and the fields $Q$, $H$ and $q$ are not charged under the remaining gauge groups, $\mathcal{L}_{\mathrm{YUk}}$ is indeed a gauge invariant coupling in the model and the fields $Q$ and $q$ acquire mass by giving the fields $H^j$ vev's through a Higgs mechanism.

It is difficult to get a spectrum with four families and simultaneously gauge groups, which are big enough to reproduce the whole SM gauge group with the SM field content from the particular orientifold. As can already be seen in table 4.7 only the field $q$ has four generations, where for the other fields there are just two families. However turning on discrete torsion might relieve the constraints on the D-brane wrapping numbers. In order to fully turn away from the $T^6_{\mathrm{SO}(12)}$ has to investigate, whether discrete torsion does not bring along more structure for model building [108].

**CHAPTER 5**

# Yukawa couplings from D6-branes on $T^6_{\mathrm{SO}(12)}$

As discussed in section 4.3.1, there is the hope that realistic four dimensional particle physics models can be realised in orientifold models with intersecting branes on them. It is then necessary that one can compute Yukawa couplings in for the models. The couplings in orientifolds are inherited by the couplings in the underlying torus, which is why it is necessary to be able to compute Yukawa couplings on the torus as an intermediat step. In [106] Yukawa couplings from intersecting branes on factorisable tori where already computed. In this chapter the generalization to Yukawa couplings on a non-factorisable is presented. It is based on the discussion in [109].

## 5.1 Yukawa couplings from D6-branes

### 5.1.1 Yukawa couplings from worldsheet instantons

A worldsheet instanton is a worldsheet with the topology of a disc embedded into the inernal space, s.t. it is localized at a point in the uncompact space and can be viewed as an instanton in four dimensions. They are discussed in [52, 76, 110–112]. Since boundaries of worldsheets are attached to D-branes, worldsheet instantons are stretched between D-branes. Let $D$ be a worldsheet with the topology of a disc and $\partial D = d$ its boundary. Let the 3-cycles $\Pi^3_\alpha \in H_3(X^6)$ in the internal space $X^6$, be wrapped by the D6-branes $\alpha$. The embedding $f$ of $D$ into $X^6$ has to be given by the following properties [76]

$$f : d \to f(d) \in \cup_\alpha \Pi^3_\alpha , \quad D \to f(D) \in H_2\left(X^6, \cup_\alpha \Pi^3_\alpha\right) , \tag{5.1}$$

where $H_2\left(X^6, \cup_\alpha \Pi^3_\alpha\right)$ is the space of two dimensional surfaces in $X^6$, with their boundaries on $\cup_\alpha \Pi^3_\alpha$. Further $f(D)$ has to satisfy the classical equations of motion, which for worldsheets translates to the condition that the area, streched out by the worldsheet, has to be minimized. Hence $f(D)$ has to be calibrated by the Kähler 2-form, s.t. its volume is given by

$$\mathrm{Vol}\left(f(D)\right) = \int_{f(D)} \omega_2 , \tag{5.2}$$

which means $f(D)$ wraps holomorphic 2-cycles. Since the worldsheet also couple to the $U(1)$ gauge field on the branes, it can collect phases, due to Wilson lines $\theta_\alpha$ on the branes, which is given by [76]

$$\mathrm{e}^{2\pi i \theta_\alpha} = \mathrm{e}^{2\pi i \int_{f(d)} A_\alpha} , \tag{5.3}$$

where $A_\alpha$ is the $U(1)$ gauge field on the brane $\alpha$. The coupling strength $I$ of the instanton is exponentially supressed by the worldsheet area and proportional to the $U(1)$ phases [113]

$$I \propto \exp\left\{-\frac{1}{2\pi\alpha'}\int_{f(D)}\omega_2 + 2\pi i\sum_\alpha\int_{f(d)}A_\alpha\right\}. \tag{5.4}$$

Since on a compact space cycles can be wrapped multiple times, $f(D)$ is specified by its wrapping number $n$ around the 2-cycle $\Sigma^2 \in H_2\left(X^6, \cup_\alpha \Pi^3_\alpha\right)$. A Yukawa coupling describes the interaction of three chiral fields. As already discussed, in intersecting D-brane setups, chiral fields are located at intersection points of two D6-branes. Let the three chiral fields, denoted by $\phi^i$, $\phi^j$ and $\phi^k$, belong to the following bifundamental representations

$$\phi^i \simeq (\square_a, \overline{\square}_b)\,,\quad \phi^j \simeq (\square_c, \overline{\square}_a)\,,\quad \phi^k \simeq (\square_b, \overline{\square}_c)\,, \tag{5.5}$$

for example in a $U(N_a)\times U(N_b)\times U(N_c)$ gauge theory. Then they sit at the intersection points

$$i \in \Pi^3_a \cap \Pi^3_b\,,\quad j \in \Pi^3_c \cap \Pi^3_a\,,\quad k \in \Pi^3_b \cap \Pi^3_c\,, \tag{5.6}$$

where $\Pi^3_\alpha$, $\alpha \in \{a, b, c\}$, are the 3-cycles wrapped by the branes $\alpha$, which carry the gauge group $U(N_\alpha)$. The Yukawa coupling between the fields $\phi^i$, $\phi^j$ and $\phi^k$ is denoted by $Y_{ijk}$. At the CFT level, tree level contributions to $Y_{ijk}$ are described by correlation functions of three Vertex operators $V^i$, $V^j$ and $V^k$ on a worldsheet with the topology of a disc $D$

$$Y_{ijk} \simeq \langle V^i\, V^j\, V^k\rangle_D\,. \tag{5.7}$$

The vertex operators $V^i, V^j, V^k$ create the CFT states, which correspond to the fields $\phi^i$, $\phi^j$, $\phi^k$ in spacetime. That means, from the spacetime point of view, tree level constributions to the Yukawa coupling $Y_{ijk}$ arise from worldsheet instantons, where the embedding $f$ of $D$ into $X^6$ has to satisfy the conditions:

- $f\,:\, D \to f(D) \in H_2\left(X^6, \cup_{\alpha\in\{a,b,c\}}\Pi^3_\alpha\right)$,
- $f\,:\, d \to f(d) \in \cup_{\alpha\in\{a,b,c\}}\Pi^3_\alpha$,
- $\{i, j, k\} \in f(d)$ and
- $\int_{f(D)}\omega_2 = \mathrm{Vol}(f(D))$.

The sum over all instantons satisfying the above four conditions lead to the tree level contribution of $Y_{ijk}$ [106]

$$Y_{ijk} \propto \sum_{f(D)\in H_2\left(X^6,\cup_{\alpha\in\{a,b,c\}}\Pi^3_\alpha\right)} \exp\Big\{-\frac{1}{2\pi\alpha'}\int_{f(D)}\omega_2 + 2\pi i\sum_{\alpha\in\{a,b,c\}}\int_{f(d)}A_\alpha\Big\}. \tag{5.8}$$

### 5.1.2 Yukawa couplings on the torus

Since the Kähler 2-form on a flat torus is given by the sum $\omega_2 = \frac{i}{2}\sum_h \mathrm{d}z_h \wedge \mathrm{d}\bar{z}_h$, with $z_h = x_{2h-1} + ix_{2h}$, the condition for $f(D)$ to wrap holomorphic 2-cycles in $T^{2n}$ becomes

$$\frac{i}{2}\int_{f(D)}\sum_{h=1}^{n}\mathrm{d}z_h \wedge \mathrm{d}\bar{z}_h = \mathrm{Vol}(f(D))\,, \tag{5.9}$$

which shows that worldsheet instantons on a torus are given by a sum over $n$ two dimensional surfaces, each spreading out in a complex plane. Further the boundary of the instanton has to be placed on D6-branes. As explained in section 4.1.4 D6-branes in tori span in each complex plane a straight line and therefor, to connect three different intersection points, the instantons have the shape of triangles in each complex plane. The edges of the triangle lie on D6-branes and the corners of the triangles are intersection points of the branes. The strategy to compute Yukawa couplings consists of the following steps:

1. Parametrizing the three D6-branes $a, b, c$ on the covering space $\mathbb{R}^6$.
2. Calculating the loci of intersection points of the D-branes and finding a labeling $i, j, k$ for the inequivalent intersection points on the torus.
3. Computing all triangles which connect three intersection points, labeled by the same triplet $(i, j, k)$.
4. Inserting the areas of the triangles into (5.8) and adding Wilson lines if necessary.

## 5.2 Yukawa couplings on $T^2$

### 5.2.1 Computing Yukawa couplings on $T^2$

Yukawa couplings on factorisable $T^6$ have been discussed intensively in [106]. For Yukawa couplings in factorisable tori, one computes Yukawa couplings on each two dimensional $T^2$ factor and considers the product of the two dimensional couplings. In the following it will first be revised how Yukawa couplings on a $T^2$ are computed and some arguments in [106] are repeated, in order to adjust them to the more general case when the non-factorisable torus is discussed. In the following the two dimensional torus $T^2 = \mathbb{R}^2/\Lambda^2$ is considered, whose underlying lattice $\Lambda^2$ is spanned by the basis vectors

$$\vec{\alpha}_1 = R\begin{pmatrix}1\\0\end{pmatrix} \quad \text{and} \quad \vec{\alpha}_2 = R\begin{pmatrix}\mathrm{Re}\tau\\ \mathrm{Im}\tau\end{pmatrix}, \tag{5.10}$$

where $R^2\mathrm{Im}(\tau) \in \mathbb{R}$ and $\tau \in \mathbb{C}$ are the Kähler and complex structure modulus. Let the three branes, which are denoted by $a$, $b$ and $c$, be placed into the torus. Each brane $\alpha \in \{a, b, c\}$ wraps a sLag 1-cycle $\Pi^1_\alpha \in H_1(T^2)$ on the torus. The wrapping numbers of $\alpha$ around the cycle, generated by $\vec{\alpha}_1$, is denoted by $n_\alpha$ and the cycle, generated by $\vec{\alpha}_2$, by $m_\alpha$:

$$\Pi^1_\alpha = (n_\alpha, m_\alpha)\,. \tag{5.11}$$

In order to calculate the Yukawa couplings, the four steps described in section 5.1.2 are followed :

1. **Parametrizing the D-branes in $\mathbb{R}^2$:** The brane $\alpha$ is parametrized in $\mathbb{R}^2$ by the set of points $\{\vec{x}_\alpha\}$, which satisfy the equation

$$\vec{x}_\alpha = \vec{s}_\alpha + R\begin{pmatrix}n_\alpha + m_\alpha\mathrm{Re}\tau\\ m_\alpha\mathrm{Im}\tau\end{pmatrix}\mu_\alpha + \vec{\lambda}_\alpha, \quad \vec{\lambda}_\alpha \in \Lambda^2,\ \mu_\alpha \in \mathbb{R}\,, \tag{5.12}$$

with

$$\vec{s}_\alpha = R\begin{pmatrix}-m_\alpha\mathrm{Im}\tau\\ n_\alpha + m_\alpha\mathrm{Re}\tau\end{pmatrix}\frac{\epsilon_\alpha\mathrm{Vol}(T^2)}{||\Pi^1_\alpha||^2}, \quad \vec{\lambda}_\alpha = R\begin{pmatrix}p_1 + p_2\mathrm{Re}\tau\\ p_2\mathrm{Im}\tau\end{pmatrix}, \quad p_1,\ p_2 \in \mathbb{Z}\,. \tag{5.13}$$

The vector $\vec{s}_\alpha$ is a displacement vector, where $\epsilon_\alpha$ is the shortest distance of the brane to the origin, measured in units of $\mathrm{Vol}(T^2)/||\Pi^1_\alpha||$. $\mathrm{Vol}(T^2)$ is the the volume of the torus and $||\Pi^1_\alpha||$ is the lenght of the cycle $\Pi^1_\alpha$

$$\mathrm{Vol}(T^2) = R^2\mathrm{Im}\tau, \quad ||\Pi^1_\alpha|| = R\sqrt{n_\alpha{}^2 + 2n_\alpha\,, m_\alpha\mathrm{Re}\tau + |\tau|^2 m_\alpha{}^2}\,. \tag{5.14}$$

The displacement vector parametrizes the displacement of the brane to the origin. The lattice vectors $\vec{\lambda}_\alpha$ generate the images of $\alpha$, generated by lattice translations, in $\mathbb{R}^2$.

2. **Intersection points on $\mathbb{R}^2$:** In [106] inequivalent intersection points on a $T^2$ were labeled by an integer $i \in \{0, 1, ..., |I_{ab}| - 1\}$, where $I_{ab} = n_a m_b - n_b m_a$ is the intersection number of the cycles $\Pi^1_a$ and $\Pi^1_b$ on the torus. The notation can be adopted for general tori, but since the aim is to generalize the concepts to non factorisable tori, one needs to keep track of how the lattice vectors $\vec{\lambda}_\alpha$ depend on the intersection points. The loci $(ab)$ of intersection points of the brane $a$ with $b$ is found by solving the equation $\vec{x}_a = \vec{x}_b$. The solutions are given by

$$(ab):\ \mathbb{R}^2\Big|_{a\cap b} = \vec{s}_b + R\begin{pmatrix} n_b + m_b\mathrm{Re}\tau \\ m_b\mathrm{Im}\tau \end{pmatrix}\mu_b + \vec{\lambda}_b\,, \tag{5.15}$$

with

$$\mu_b = (q_1 - p_1)\,\frac{m_a}{I_{ab}} - (q_2 - p_2)\,\frac{n_a}{I_{ab}} + \frac{\epsilon_a}{I_{ab}} - \frac{v_{ab}\epsilon_b}{||\Pi^1_b|| \cdot ||\Pi^1_b||}\,, \tag{5.16}$$

where

$$v_{ab} = \frac{n_a n_b + \mathrm{Re}\tau(n_a m_b + n_b m_a) + |\tau|^2 m_a m_b}{I_{ab}}\,. \tag{5.17}$$

One gets an infinit amount of intersection points in $\mathbb{R}^2$, where each intersection point is generated by the images under lattice translation of $a$ and $b$. However all intersection points, which differ by lattice vectors, are identified on the torus and only those, which cannot be identified, are inequivalent intersection points on the torus. Taking a closer look at (5.16) one finds for $g.c.d.(n_a, m_a) = 1$, that the parameter $\mu_b$ can be expressed by a label $i$ as

$$\mu_b = \frac{i}{I_{ab}} + \frac{\epsilon_a}{I_{ab}} - \frac{v_{ab}\epsilon_b}{||\Pi^1_b|| \cdot ||\Pi^1_b||}, \quad i \in \mathbb{Z}\,, \tag{5.18}$$

where the intersection points with $i \in \{0, 1, ..., |I_{ab}| - 1\}$ form the set of inequivalent intersection points on $T^2$. The index $i$ is used to label the intersection points on the torus and the inequivalent intersection points on the torus are determined by

$$(ab):\ T^2\Big|_{a\cap b} = \vec{s}_b + R\begin{pmatrix} n_b + m_b\mathrm{Re}\tau \\ m_b\mathrm{Im}\tau \end{pmatrix}\left(\frac{i}{I_{ab}} + \frac{\epsilon_a}{I_{ab}} - \frac{v_{ab}\epsilon_b}{||\Pi^1_b|| \cdot ||\Pi^1_b||}\right) \tag{5.19}$$

with

$$i \in \frac{\mathbb{Z}}{I_{ab}\mathbb{Z}}, \quad (q_1 - p_1)m_a - (q_2 - p_2)n_a = i \mod I_{ab}\,. \tag{5.20}$$

It will turn out for six dimensional tori, the intersection points can be denoted by labels, which belong to more general three dimensional lattices. Including the brane $c$ one further gets the intersection points between $a$ and $c$, given by

$$(ac):\ \mathbb{R}^2\Big|_{c\cap a} = \vec{s}_a + R\begin{pmatrix} n_a + m_a\mathrm{Re}\tau \\ m_a\mathrm{Im}\tau \end{pmatrix}\left(\frac{j}{I_{ca}} + \frac{\epsilon_c}{I_{ca}} - \frac{v_{ac}\epsilon_a}{||\Pi^1_a|| \cdot ||\Pi^1_a||}\right) + \vec{\lambda}_a\,, \tag{5.21}$$

and intersection points between $b$ and $c$

$$(bc):\ \mathbb{R}^2\Big|_{b\cap c} = \vec{s}_c + R\begin{pmatrix} n_c + m_c\mathrm{Re}\tau \\ m_c\mathrm{Im}\tau \end{pmatrix}\left(\frac{k}{I_{bc}} + \frac{\epsilon_b}{I_{bc}} - \frac{v_{cb}\epsilon_c}{||\Pi_c^1||\cdot||\Pi_c^1||}\right) + \vec{\lambda}_c\,, \tag{5.22}$$

with

$$j \in \frac{\mathbb{Z}}{I_{ca}\mathbb{Z}},\ k \in \frac{\mathbb{Z}}{I_{bc}\mathbb{Z}} \quad \text{and} \quad \vec{\lambda}_a,\, \vec{\lambda}_c \in \Lambda^2\,. \tag{5.23}$$

The labels $j$ and $k$ depend on lattice vectors analogous to (5.20).

3. **Closed triangles in $\mathbb{R}^2$:** When the branes $a$, $b$ and $c$ form closed triangles, the edges of the triangles lie on the branes and the corners of the triangle are intersection points. The edges $\vec{z}_a$, $\vec{z}_b$ and $\vec{z}_c$ of a triangle $\triangle^{ijk}$, whose corners are labeld by $i$, $j$ and $k$, are computed by subtracting the corresponding intersection points $(ab)$, $(bc)$ and $(ca)$ from each other. That way one gets the expressions

$$\begin{aligned}
\vec{z}_a &= \vec{v}_a + R\begin{pmatrix} n_a + m_a\mathrm{Re}\tau \\ m_a\mathrm{Im}\tau \end{pmatrix}\left(\frac{j}{I_{ca}} + \frac{\epsilon_c}{I_{ca}} - \frac{v_{ac}\epsilon_a}{||\Pi_a^1||\cdot||\Pi_a^1||}\right) \\
&\quad -R\begin{pmatrix} n_b + m_b\mathrm{Re}\tau \\ m_b\mathrm{Im}\tau \end{pmatrix}\frac{i}{I_{ab}} + R\begin{pmatrix} p_1 - q_1 + (p_2 - q_2)\mathrm{Re}\tau \\ (p_2 - q_2)\mathrm{Im}\tau \end{pmatrix}, \\
\vec{z}_b &= \vec{v}_b + R\begin{pmatrix} n_b + m_b\mathrm{Re}\tau \\ m_b\mathrm{Im}\tau \end{pmatrix}\left(\frac{i}{I_{ab}} + \frac{\epsilon_a}{I_{ab}} - \frac{v_{ab}\epsilon_b}{||\Pi_b^1||\cdot||\Pi_b^1||}\right) \\
&\quad -R\begin{pmatrix} n_c + m_c\mathrm{Re}\tau \\ m_c\mathrm{Im}\tau \end{pmatrix}\frac{k}{I_{bc}} + R\begin{pmatrix} q_1 - t_1 + (q_2 - t_2)\mathrm{Re}\tau \\ (q_2 - t_2)\mathrm{Im}\tau \end{pmatrix}, \\
\vec{z}_c &= \vec{v}_c + R\begin{pmatrix} n_c + m_c\mathrm{Re}\tau \\ m_c\mathrm{Im}\tau \end{pmatrix}\left(\frac{k}{I_{bc}} + \frac{\epsilon_b}{I_{bc}} - \frac{v_{cb}\epsilon_c}{||\Pi_c^1||\cdot||\Pi_c^1||}\right) \\
&\quad -R\begin{pmatrix} n_a + m_a\mathrm{Re}\tau \\ m_a\mathrm{Im}\tau \end{pmatrix}\frac{j}{I_{ca}} + R\begin{pmatrix} t_1 - p_1 + (t_2 - p_2)\mathrm{Re}\tau \\ (t_2 - p_2)\mathrm{Im}\tau \end{pmatrix},
\end{aligned} \tag{5.24}$$

with $p_1, q_2, q_1, q_2, t_1, t_2 \in \mathbb{Z}$ and

$$\begin{aligned}
\vec{v}_a &= -R\begin{pmatrix} n_b + m_b\mathrm{Re}\tau \\ m_b\mathrm{Im}\tau \end{pmatrix}\left(\frac{\epsilon_a}{I_{ab}} - \frac{v_{ab}\epsilon_b}{||\Pi_b^1||\cdot||\Pi_b^1||}\right) + \vec{s}_a - \vec{s}_b, \\
\vec{v}_b &= -R\begin{pmatrix} n_c + m_c\mathrm{Re}\tau \\ m_c\mathrm{Im}\tau \end{pmatrix}\left(\frac{\epsilon_c}{I_{bc}} - \frac{v_{bc}\epsilon_c}{||\Pi_c^1||\cdot||\Pi_c^1||}\right) + \vec{s}_b - \vec{s}_c, \\
\vec{v}_c &= -R\begin{pmatrix} n_a + m_a\mathrm{Re}\tau \\ m_a\mathrm{Im}\tau \end{pmatrix}\left(\frac{\epsilon_c}{I_{ca}} - \frac{v_{ac}\epsilon_a}{||\Pi_a^1||\cdot||\Pi_a^1||}\right) + \vec{s}_c - \vec{s}_a\,.
\end{aligned} \tag{5.25}$$

The edges $\vec{z}_\alpha$ have to be parallel to the brane $\alpha$, since the corners connected by $\vec{z}_\alpha$ lie both on that brane. Hence the scalar product of $\vec{z}_\alpha$ with the directional vector orthogonal to $\alpha$ has to vanish

$$\vec{z}_\alpha \cdot \begin{pmatrix} -m_\alpha\mathrm{Im}\tau \\ n_\alpha + m_\alpha\mathrm{Re}\tau \end{pmatrix} = 0\,. \tag{5.26}$$

This is only the case if the parameters $p_1, p_2, ..., t_2$ satisfy the three Diophantine equations

$$\begin{aligned} -i &= (p_1 - q_1)m_a - (p_2 - q_2)n_a, \\ -k &= (q_1 - t_1)m_b - (q_2 - t_2)n_b, \\ -j &= (t_1 - p_1)m_c - (t_2 - p_2)n_c\,. \end{aligned} \tag{5.27}$$

In order to solve (5.27) it is helpful to relabel the intersection points by

$$i \to \frac{iI_{ac}}{d_a}, \quad j \to \frac{jI_{cb}}{d_c}, \quad k \to \frac{kI_{ba}}{d_b}, \tag{5.28}$$

where $d_a = g.c.d.(I_{ab}, I_{ca})$, $d_b = g.c.d.(I_{bc}, I_{ab})$ and $d_c = g.c.d.(I_{ca}, I_{bc})$. After relabeling one can find infinitly many solutions to (5.27), which are given by

$$\begin{aligned} \begin{pmatrix} p_1 - q_1 \\ p_2 - q_2 \end{pmatrix} &= \frac{i}{d_a}\begin{pmatrix} n_c \\ m_c \end{pmatrix} + s_a \begin{pmatrix} n_a \\ m_a \end{pmatrix}, \quad s_a \in \mathbb{Z} \\ \begin{pmatrix} q_1 - t_1 \\ q_2 - t_2 \end{pmatrix} &= \frac{k}{d_b}\begin{pmatrix} n_a \\ m_a \end{pmatrix} + s_b \begin{pmatrix} n_b \\ m_b \end{pmatrix}, \quad s_b \in \mathbb{Z} \\ \begin{pmatrix} t_1 - p_1 \\ t_2 - p_2 \end{pmatrix} &= \frac{j}{d_c}\begin{pmatrix} n_b \\ m_b \end{pmatrix} + s_c \begin{pmatrix} n_c \\ m_c \end{pmatrix}, \quad s_c \in \mathbb{Z}\,. \end{aligned} \tag{5.29}$$

The intersection numbers satisfy the relation

$$\begin{pmatrix} n_c \\ m_c \end{pmatrix} I_{ab} + \begin{pmatrix} n_a \\ m_a \end{pmatrix} I_{bc} + \begin{pmatrix} n_a \\ m_a \end{pmatrix} I_{ca} = 0 \tag{5.30}$$

and for $g.c.d.(n_\alpha, m_\alpha) = 1$ it can be seen, that if for a pair of intersection numbers have $d_\alpha \neq 1$, then the other intersection number contains a factor of $d_\alpha$, s.t. $d = d_a = d_b = d_c = g.c.d.(I_{ab}, I_{bc}, I_{ca})$. For some cases, the relabeling defined in (5.28) labels two inequivalent intersection points by the same label. An example is given by the case with the intersection numbers $I_{ab} = 4$ and $I_{ca} = 6$. The six labels $j \in \{0, 1, ..., 5\}$ are asigned to the new labels $j' = 2j \mod 6$

$$0 \to 0, \quad 1 \to 2, \quad 2 \to 4, \quad 3 \to 0, \quad 4 \to 2, \quad 5 \to 4\,, \tag{5.31}$$

leaving only three ineqivalent labels $j' \in \{0, 2, 4\}$, where the 'old' labels $i \in \{1, 3, 5\}$ are lost. The fact that after relabeling some intersection points have no "new" labels, leads to the problem that triangles connected to those intersection points cannot be computed. However it will turn out that for the case $d \neq 1$, for each triangle there are further $d$ congruent triangels, from couplings fields with different labels $i, j, k$, and each triangle, whose "new" label vanishes, can be calculated by a congruente triangle, of a coupling, whose involved fields have "new" labels. Continuing the computations, one finds the vector $\vec{v}_\alpha$ is parallel to $\alpha$, since the scalar product with $(-m_\alpha \mathrm{Im}\tau,\ n_a + m_a \mathrm{Re}\tau)^T$ vanishes. Therefore $\vec{v}_\alpha$ can also be expressed by the projection onto the directional vector of $\alpha$ by

$$\vec{v}_a = R^2 \begin{pmatrix} n_\alpha + m_\alpha \mathrm{Re}\tau \\ m_\alpha \mathrm{Im}\tau \end{pmatrix} \frac{\vec{v}_\alpha \cdot \begin{pmatrix} n_\alpha + m_\alpha \mathrm{Re}\tau \\ m_\alpha \mathrm{Im}\tau \end{pmatrix}}{\|\Pi^1_\alpha\| \cdot \|\Pi^1_\alpha\|}\,. \tag{5.32}$$

Inserting the results into (5.24) and using the relations (5.30),

$$v_{ab} + v_{ca} = \|\Pi_a^1\| \cdot \|\Pi_a^1\| \frac{I_{cb}}{I_{ab} I_{ca}} \tag{5.33}$$

and

$$\frac{v_{ab}{}^2 I_{ab}}{\|\Pi_a^1\| \cdot \|\Pi_b^1\|} + \frac{I_{ab}{}^2}{\|\Pi_a^1\| \cdot \|\Pi_b^1\| I_{ab}} = \frac{1}{I_{ab}}\,, \tag{5.34}$$

the edges $\vec{z}_\alpha$ can be expressed

$$\begin{aligned}
\vec{z}_a &= R\begin{pmatrix} n_a + m_a \mathrm{Re}\tau \\ m_a \mathrm{Im}\tau \end{pmatrix} I_{cb} \left( \frac{j}{d_c I_{ca}} + \frac{i}{d_a I_{ab}} + \frac{I_{bc}\epsilon_a + I_{ca}\epsilon_b + I_{ab}\epsilon_c}{I_{ca} I_{ab} I_{bc}} + \frac{s_a}{I_{cb}} \right), \\
\vec{z}_b &= R\begin{pmatrix} n_b + m_b \mathrm{Re}\tau \\ m_b \mathrm{Im}\tau \end{pmatrix} I_{ca} \left( \frac{i}{d_a I_{ab}} + \frac{k}{d_b I_{bc}} + \frac{I_{bc}\epsilon_a + I_{ca}\epsilon_b + I_{ab}\epsilon_c}{I_{ca} I_{ab} I_{bc}} + \frac{s_b}{I_{ca}} \right), \\
\vec{z}_c &= R\begin{pmatrix} n_c + m_c \mathrm{Re}\tau \\ m_c \mathrm{Im}\tau \end{pmatrix} I_{ab} \left( \frac{j}{d_c I_{ca}} + \frac{k}{d_b I_{bc}} + \frac{I_{bc}\epsilon_a + I_{ca}\epsilon_b + I_{ab}\epsilon_c}{I_{ca} I_{ab} I_{bc}} + \frac{s_c}{I_{ab}} \right).
\end{aligned} \tag{5.35}$$

For $\triangle^{ijk}$ to be closed, the edges need to satisfy the Diophantine equation

$$\vec{z}_a + \vec{z}_b + \vec{z}_c = 0\,, \tag{5.36}$$

which is solved by

$$s_a = -\frac{k}{d_b} + \frac{I_{bc}}{d}(\ell + l_0), \quad s_b = -\frac{j}{d_c} + \frac{I_{ca}}{d}(\ell + l_0), \quad s_c = -\frac{i}{d_a} + \frac{I_{ab}}{d}(\ell + l_0), \tag{5.37}$$

with $d = d_a = d_b = d_c$ and $\ell \in \mathbb{Z}$. $l_0$ is a function mapping the labels $i, j, k$ to numbers in $\mathbb{Z}_d$

$$l_0 : (i, j, k) \to \left\{ 0, \frac{1}{d}, ..., \frac{d-1}{d} \right\} \tag{5.38}$$

and is needed to ensure that in (5.29) the lefthand side is indeed integer and thus correspond to integer lattice shifts. Further the condition that the lefthand side of (5.29) is integer leads to selection rules for $i$, $j$ and $k$, which were postulated in [106], and unveils that not all combinations of labels belong to closed triangles, when $d \neq 1$.

4. **Computing Yukawa couplings:** After inserting (5.37) into (5.35), the edges $\vec{z}_\alpha$ for designated labels $i, j, k$, depend only on the free parameter $\ell$. $\ell$ corresponds to the winding number of the triangle around the torus and hence denotes the winding number of the instanton. The area $\mathrm{Vol}(\triangle^{ijk})$ of the triangles are calculated via

$$\mathrm{Vol}(\triangle^{ijk}) = \frac{1}{2}\sqrt{|\vec{z}_a|^2 |\vec{z}_b|^2 - (\vec{z}_a \cdot \vec{z}_b)^2} \tag{5.39}$$

and is given by the expression

$$\mathrm{Vol}(\triangle^{ijk}) = \frac{1}{2} A |I_{ab} I_{bc} I_{ca}| \left( \frac{i}{d_a I_{ab}} + \frac{j}{d_c I_{ca}} + \frac{k}{d_b I_{bc}} + \tilde{\epsilon} + \frac{l_0}{d} + \frac{\ell}{d} \right)^2 , \tag{5.40}$$

with $A = \mathrm{Vol}\left(T^2\right)$ and $\tilde{\epsilon} = (I_{bc}\epsilon_a + I_{ca}\epsilon_b + I_{ab}\epsilon_c)/I_{ca}I_{ab}I_{bc}$. Substituting $A$ by $-iK$, where $K = B_2 + iA$ is the complexified Kähler structure modulus, the instanton is allowed to couple to a B-field $B$. Wilson lines on the branes can be further turned on, as in [106], s.t. the instanton collects the $U(1)$ phases $\mathrm{e}^{2\pi I_{ab}\theta_c}$, $\mathrm{e}^{2\pi I_{ca}\theta_b}$ and $\mathrm{e}^{2\pi I_{bc}\theta_a}$, when going around the branes. Inserting the instantons into the sum in (5.8), the Yukawa couplings on the $T^2$ are given by

$$\mathcal{Y}_{ijk} \propto \sum_{\ell\in\mathbb{Z}} \mathrm{e}^{\frac{1}{2}(\delta+\ell/d)^2\kappa} \mathrm{e}^{2\pi i(\delta+\ell/d)\phi} \tag{5.41}$$

with

$$\begin{aligned}
\delta &= \frac{i}{d_a I_{ab}} + \frac{j}{d_c I_{ca}} + \frac{k}{d_b I_{bc}} + \tilde{\epsilon} + \frac{l_0(i,j,k)}{d}, \\
\phi &= I_{ab}\theta_c + I_{ca}\theta_b + I_{bc}\theta_a, \\
\kappa &= \frac{iK}{2\pi\alpha'}|I_{ab}I_{bc}I_{ca}|\,.
\end{aligned} \tag{5.42}$$

### 5.2.2 Example: Branes with non coprime intersection numbers in $T^2$

To illustrate the discussion in section 5.2.1, the following toy model will be investigated: Let the three branes $a, b, c$ wrap cycles, with the following wrapping numbers

$$(n_a, m_a) = (1, 0), \quad (n_a, m_a) = (1, 2), \quad (n_a, m_a) = (1, -4)\,. \tag{5.43}$$

The intersection numbers for the setup are

$$I_{ab} = 2, \quad I_{bc} = -6, \quad I_{ca} = 4\,. \tag{5.44}$$

By relabeling $i, j, k$ under the prescription (5.28) one gets the “new” labels

$$i:\ (0,1) \to (0,-), \quad j:\ (0,1,2,3) \to (0,3,2,1), \quad k:\ (0,1,2,3,4,5) \to (0,5,4,3,2,1)\,, \tag{5.45}$$

and oberves that the “old” label $i = 1$ has no “new” label. But for $d = 2$, couplings with the "old" label $i = 1$ can be expressed by a coupling, which are congruent to couplings with the "old" label $i = 0$. But first one has to determine the selection rules for the labels $i, j, k$. Demanding the lefthand side of (5.29) to be integer for the case at hand, for $i = 0$ the conditions are given by

$$\begin{aligned}
\begin{pmatrix}1\\0\end{pmatrix}\left(\frac{k}{2} - 3l_0\right) &\in \mathbb{Z}^2, \\
\begin{pmatrix}1\\2\end{pmatrix}\left(\frac{j}{2} + 2l_0\right) - \begin{pmatrix}1\\0\end{pmatrix}\frac{k}{2} &\in \mathbb{Z}^2, \\
\begin{pmatrix}1\\-4\end{pmatrix}l_0 - \begin{pmatrix}1\\2\end{pmatrix}\frac{j}{2} &\in \mathbb{Z}^2\,.
\end{aligned} \tag{5.46}$$

The function $l_0(i, j, k)$ can take the values $\{0, \frac{1}{2}\}$. For $l_0 = 0$, $k$ and $j$ have to be even, where for $l_0 = \frac{1}{2}$, $k$ and $j$ have to be odd. The selection rules are hence

$$k + j = 0 \mod 2 \tag{5.47}$$

and an explicit choice for the function $l_0$ is for example $l_0 = \frac{j}{2}$. Then the quantity $x_0 = \frac{i}{I_{ab}} + \frac{k}{I_{bc}} + \frac{j}{I_{ca}} + l_0$ becomes

$$x_0 = \frac{3j}{4} - \frac{k}{6}\,. \tag{5.48}$$

For the triangles with a corner labeled by the "old" label $i = 1$, one needs to find a congruent triangle

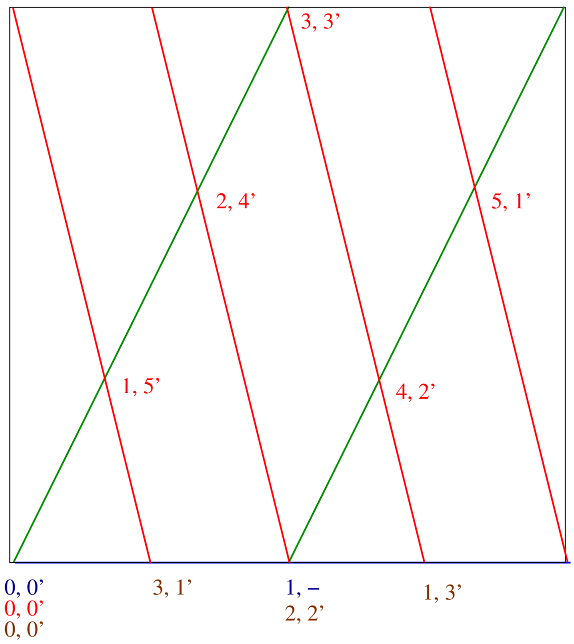


Figure 5.1: Three intersecting branes on a $T^2$: The blue, green and red line indicate the position of the branes $a$, $b$ and $c$ on the fundamental domain of the $T^2$, where the blue, brown and red integers, label the intersection points $i$, $j$ and $k$. The primed labels, are labels after relabeling and the unprimed labels are the "old" labels.

with a corner denoted by the "old" label $i = 0$. This can be achieved by taking the triplet of "old" labels $(i, j, k)$ and shifting them by

$$(i, j, k) \to (i, j, k) + \frac{1}{d}(I_{ab}, I_{ca}, I_{bc}) \tag{5.49}$$

to receive the intersection points of the congruent triangle. Afterwards one needs to relabel the intersection points and inserts them into $x_0$. For example, the intersection points with the old labels $(0, 3, 1)$ and $(1, 1, 4)$ each form three corners of two congruent triangles, as can be seen in figure5.1. Since $(1, 1, 4)$ has no $i$ label after relabeling, one has to shift the triplet by

$$(1, 1, 4) \to (1, 1, 4) + \frac{1}{2}(2, 4, -6) = (0, 3, 1) \tag{5.50}$$

and sees that the shift indeed mapped the triplet of labels to labels, corresponding to the congruent triangle.

## 5.3 Yukawa couplings on $T^6_{\mathbf{SO(12)}}$

### 5.3.1 Labeling inequivalent intersections

In this section the notation used in [106], for labeling intersection points on a $T^6$, is extended. Already in the two dimensional case one could observe the relevance of finding a convenient labeling for the intersection points in order to compute the Yukawa couplings. In [106] the intersection points of 3-cycles on a factorisable $T^6$ are labeled by a triplet of integers $j = (j^{(1)}, j^{(2)}, j^{(3)})$, where each $j^{(h)}$ corresponds to the label of the intersection point on the $T^2$ factor lying in the $h$-th plane. Since the three $T^2$ factors are independent from each other the label $j$ belongs to the factorisable lattice

$$j \in \bigotimes_{h=1}^{3} \frac{\mathbb{Z}}{|I_{ab}^{(h)}|\mathbb{Z}} . \tag{5.51}$$

On the non factorisable torus $T^6_{\mathrm{SO}(12)}$, it is not possible to subdivide the torus into independent $T^2$ factors, but since the brane cycles factorise into mutually orthogonal lines, it is possible to adopt the notation of labeling the intersection points by triplets of integers. However it will be shown that $j$ belongs to a more general three dimensional lattice

$$j \in \Lambda^3_{ab} . \tag{5.52}$$

Let a brane $\alpha$ wrap a 3-cycle $\Pi^3_\alpha = (n^1_\alpha, m^1_\alpha) \times (n^2_\alpha, m^2_\alpha) \times (n^3_\alpha, m^3_\alpha)$. Taking all images under $SO(12)$ lattice translations into account the brane position $\vec{x}_\alpha$ on the covering space $\mathbb{R}^6$ is given by

$$\vec{x}_\alpha = \left(n^1_\alpha \mu^{(1)}_\alpha, m^1_\alpha \mu^{(1)}_\alpha, n^2_\alpha \mu^{(2)}_\alpha, m^2_\alpha \mu^{(2)}_\alpha, n^3_\alpha \mu^{(3)}_\alpha, m^3_\alpha \mu^{(3)}_\alpha\right)^T + \vec{\lambda} , \tag{5.53}$$

with $\vec{\lambda} \in \Lambda_{\mathrm{SO}(12)}$ generating the images under the compactification lattice and $\mu^{(1)}, \mu^{(2)}, \mu^{(3)} \in \mathbb{R}$, denoting the points on each 1-cycles. For now it is implied that $\alpha$ goes trough the origin and a displacement vector is not needed. Since the intersection number is a topological quantity, displacements of the brane do not change the intersection numbers or the labeling of intersection points. The intersection points of two branes $a$ and $b$ are given by the solutions to the equation $\vec{x}_a = \vec{x}_b$, which are

$$\begin{aligned} \mathbb{R}^6\big|_{a\cap b} = \Big\{ \quad & \left(A_{ab} n^1_a, A_{ab} m^1_a, B_{ab} n^2_a, B_{ab} m^2_a, C_{ab} n^3_a, C_{ab} m^3_a,\right)^T , \quad \text{with} \\ & A_{ab} = \tfrac{t_1 m^1_b - t_2 n^1_b}{I^{(1)}_{ab}},\ B_{ab} = \tfrac{t_3 m^2_b - t_4 n^2_b}{I^{(2)}_{ab}},\ C_{ab} = \tfrac{t_5 m^3_b - t_6 n^3_b}{I^{(3)}_{ab}}, \\ & I^{(h)}_{ab} = (n^h_a m^h_b - n^h_b m^h_b),\ \vec{t} = (t_1, t_2, t_3, t_4, t_5, t_6)^T \in \Lambda_{\mathrm{SO}(12)} \Big\} . \end{aligned} \tag{5.54}$$

The parameters $A_{ab}$, $B_{ab}$ and $C_{ab}$ are the same as in (5.16) for the two dimensional case, with the displacements $\epsilon_a$ and $\epsilon_b$ set to zero. That means, by introducing labels $j^{(h)}$ for the intersections in the $h$-th plane, that they can be defined analogous to the factorisable case by (5.20). The crucial difference is however that the components $t_i$ of the lattice vectors in (5.20) are no longer independent of the components in the other planes and therefor the labels $j^{(h)}$ are also not independent of labels from other planes. Thus the triplet $j$ can in general not be factorised. The intersection number on $T^6_{\mathrm{SO}(12)}$ is half of the intersection number on the factorisable torus [77]

$$I_{ab} = \frac{1}{2} \prod_{h=1}^{3} I^{(h)}_{ab} . \tag{5.55}$$

Hence the index of the lattice $\Lambda^3_{ab}$ has to be $\left|\Lambda^3_{ab}\right| = \frac{1}{2}\prod_{h=1}^3 I_{ab}^{(h)}$. The following example illustrates how the lattice for the labels is constructed: Let the wrapping numbers of $a$ and $b$ be given by

$$\forall_{h\in\{1,2,3\}}\,\forall_{\alpha\in\{a,b\}}\,\left(n^h_\alpha + m^i_\alpha = 0 \mod 2, \quad g.c.d.(n^h_\alpha, m^h_\alpha) = 1\right). \tag{5.56}$$

Then the inequivalent labels belong to

$$j \in \left\{\left(t_1 m^1_b - t_2 n^1_b,\, t_3 m^2_b - t_4 n^2_b,\, t_5 m^3_b - t_6 n^3_b\right)\middle|\vec{t} \in \Lambda_{\mathrm{SO}(12)}\right\}. \tag{5.57}$$

The projections of $j$ onto the $h$-th plane are computed via

$$j^{(h)} = t_{2h-1} m^h_b - t_{2h} n^h_b \in \mathbb{Z}, \tag{5.58}$$

with

$$t_{2h-1} m^h_b - t_{2h} n^h_b = \begin{cases} even & \text{for} \quad t_{2h-1} + t_{2h} = even \\ odd & \text{for} \quad t_{2h-1} + t_{2h} = odd \end{cases}. \tag{5.59}$$

When the labels $j^{(h)}$ are composed to the triplet $j$, one has to take into account that $\sum_{h=1}^3 t_{2h-1} + t_{2h}$ has to be even for $\vec{t}$ to be an $SO(12)$ root. Then $j^{(h)}$ can only be odd for an even number of planes, where it can be even for an odd number of planes. The generators of the $SO(6)$ Lie lattice, $\vec{\alpha}_1 = (1,-1,0)^T$, $\vec{\alpha}_2 = (0,1,-1)^T$ and $\vec{\alpha}_3 = (0,1,1)^T$, spann a lattice satisfying the criteria for $j$ and hence $j$ belongs to the $SO(6)$ Lie lattice $\Lambda_{\mathrm{SO}(6)}$. Since $I^{(h)}_{ab}$ are all even the following shifts

$$j^{(h)} \to j^{(h)} + I^{(h)}_{ab} \tag{5.60}$$

in (5.54) can be absorbed by a lattice vector $\vec{\lambda}$, s.t. $j$ and $j + (..., I^{(h)}_{ab}, ...)$ labels equivalent intersection points on the torus. Therefor the labels for inequivalent intersection on the torus belong to the lattice

$$j \in \frac{\Lambda_{\mathrm{SO}(6)}}{\bigotimes_{h=1}^3 I^{(h)}_{ab}\mathbb{Z}}. \tag{5.61}$$

The index of the lattice is computed by the quotient

$$\left|\frac{\Lambda_{\mathrm{SO}(6)}}{\bigotimes_{h=1}^3 I^{(h)}_{ab}\mathbb{Z}}\right| = \left|\frac{E_{\bigotimes_{h=1}^3 I^{(h)}_{ab}\mathbb{Z}}}{E_{\Lambda_{\mathrm{SO}(6)}}}\right|, \tag{5.62}$$

with $E_\Lambda$ the volume of the fundamental cell of $\Lambda$. The volumes for a fundamental cell of a lattice is determined by the determinant of the dreibein of the lattices. Hence

$$\begin{aligned} E_{\Lambda_{\mathrm{SO}(6)}} &= \det\begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 1 & 1 \end{pmatrix} &&= 2, \\ E_{\bigotimes_{h=1}^3 I^{(h)}_{ab}\mathbb{Z}} &= \det\begin{pmatrix} I^{(1)}_{ab} & 0 & 0 \\ 0 & I^{(2)}_{ab} & 0 \\ 0 & 0 & I^{(3)}_{ab} \end{pmatrix} &&= \prod_{h=1}^3 I^{(h)}_{ab}, \end{aligned} \tag{5.63}$$

and the index $\left|\frac{\Lambda_{\mathrm{SO}(6)}}{\bigotimes_{h=1}^{3} I_{ab}^{(h)}\mathbb{Z}}\right| = \frac{1}{2}\prod_{h=1}^{3} I_{ab}^{(h)}$ is indeed the expected intersection number on $T^6_{\mathrm{SO}(12)}$. In appendix B various cases for $\Lambda^3_{ab}$ are discussed and it is concluded that the index of $\Gamma_{ab}$ is always in accordance with (5.55).

### 5.3.2 Computing Yukawa couplings on $T^6_{\mathrm{SO}(12)}$

In the above section the intersection points on $T^6_{\mathrm{SO}(12)}$ where investigated and a labeling for them where introduced. The next step is to compute the triangles formed by three intersecting branes. Let the branes be parametrized as in (4.72), but this time the branes do not need to intersect the origin necessarily. The displacement of the brane $\alpha$ from the origin is captured in the displacement vector $\vec{s}_\alpha$ similar to (5.13) in the the two dimensional case, s.t. the cycle $\Pi^3_\alpha$ is shifted by

$$\Pi^3_\alpha \to \Pi^3_\alpha + \vec{s}_\alpha\,, \tag{5.64}$$

in (4.72), where the displacement vector fot the six dimensional case is given by

$$\vec{s}_\alpha = \left(-m^1_\alpha \frac{\epsilon^{(1)}_\alpha}{\|\Pi^1_\alpha\|^2}, n^1_\alpha \frac{\epsilon^1_\alpha}{\|\Pi^1_\alpha\|^2}, -m^2_\alpha \frac{\epsilon^2_\alpha}{\|\Pi^2_\alpha\|^2}, n^2_\alpha \frac{\epsilon^2_\alpha}{\|\Pi^2_\alpha\|^2}, -m^3_\alpha \frac{\epsilon^3_\alpha}{\|\Pi^3_\alpha\|^2}, n^3_\alpha \frac{\epsilon^3_\alpha}{\|\Pi^3_\alpha\|^2}\right)^T\,, \tag{5.65}$$

with $\|\Pi^h_\alpha\|^2 = (n_\alpha)^2 + (m_\alpha)^2$ the length of the cycle in the $h$-th plane. Since the worldsheet instantons are given by a sum of three triangles, each located on a different plane, one has to investigate the projections of the branes on each plane, because they form the triangles. The projections of the intersection points on the $h$-th plane are corners for the triangles in those planes and are given by

$$\begin{aligned}
\mathbf{p}^{(h)}_{ab} =& \begin{pmatrix} n^h_b \\ m^h_b \end{pmatrix} \left( \frac{i^{(h)}}{I^{(h)}_{ab}} + \frac{\epsilon^h_a}{I^{(h)}_{ab}} - \frac{\left(m^h_b m^h_a + n^h_a n^h_b\right)\epsilon^h_b}{\left((n^h_b)^2 + (m^h_b)^2\right) I^{(h)}_{ab}} \right) + \\
& \begin{pmatrix} -m^h_b \\ n^h_b \end{pmatrix} \frac{\epsilon^h_b}{(n^h_b)^2 + (m^h_b)^2} + \begin{pmatrix} p_{2h-1} \\ p_{2h} \end{pmatrix}, \\
\mathbf{p}^{(h)}_{ac} =& \begin{pmatrix} n^h_a \\ m^h_a \end{pmatrix} \left( \frac{j^{(h)}}{I^{(h)}_{ca}} + \frac{\epsilon^h_c}{I^{(h)}_{ca}} - \frac{\left(m^h_c m^h_a + n^h_c n^h_a\right)\epsilon^h_a}{\left((n^h_a)^2 + (m^h_a)^2\right) I^{(h)}_{ca}} \right) + \\
& \begin{pmatrix} -m^h_a \\ n^h_a \end{pmatrix} \frac{\epsilon^h_a}{(n^h_a)^2 + (m^h_a)^2} + \begin{pmatrix} q_{2h-1} \\ q_{2h} \end{pmatrix}, \\
\mathbf{p}^{(h)}_{bc} =& \begin{pmatrix} n^h_c \\ m^h_c \end{pmatrix} \left( \frac{k^{(h)}}{I^{(h)}_{bc}} + \frac{\epsilon^h_b}{I^{(h)}_{bc}} - \frac{\left(m^h_c m^h_b + n^h_c n^h_b\right)\epsilon^h_c}{\left((n^h_c)^2 + (m^h_c)^2\right) I^{(h)}_{bc}} \right) + \\
& \begin{pmatrix} -m^h_c \\ n^h_c \end{pmatrix} \frac{\epsilon^h_c}{(n^h_c)^2 + (m^h_c)^2} + \begin{pmatrix} t_{2h-1} \\ t_{2h} \end{pmatrix},
\end{aligned} \tag{5.66}$$

with $i, j, k$ labels for the inequivalent intersection points of the branes $a$, $b$, $c$, as explained in section 5.3.1, and $\mathbf{p}^{(h)}_{ab}$ the projection of the intersection point $\mathbf{p}_{ab} =: \mathbb{R}^6\big|_{a\cap b}$ onto the $h$-th plane. In the following triangles, which are located in the $h$-th plane and whose corners are labeld by $i$, $j$, $k$, are denoted by $\triangle^{ijk}_{(h)}$. The edges $z^{(h)}_a$, $z^{(h)}_b$ and $z^{(h)}_c$ of $\triangle^{(ijk}_{(h)}$ are obtained by subtracting the intersection points with the corresponding labels. The edges for the triangles are then expressed by

$$\begin{aligned}
z_a^{(h)} &= \vec{v}_a^h + \begin{pmatrix} n_a^h \\ m_a^h \end{pmatrix} \left( \frac{j^{(h)} I_{cb}^{(h)}}{d_c^{(h)} I_{ac}^{(h)}} + \frac{\epsilon_c^h}{I_{ca}^{(h)}} - \frac{\left(m_a^h m_c^h + n_a^h n_c^h\right) \epsilon_a^h}{\left((n_a^h)^2 + (m_a^h)^2\right) I_{ca}^{(h)}} \right) \\
&\quad - \begin{pmatrix} n_b^h \\ m_b^h \end{pmatrix} \frac{i^{(h)} I_{ac}^{(h)}}{d_a^{(h)} I_{ab}^{(h)}} + \begin{pmatrix} q_{2h-1} - p_{2h-1} \\ q_{2h} - p_{2h} \end{pmatrix}, \\
z_b^{(h)} &= \vec{v}_b^h + \begin{pmatrix} n_b^h \\ m_b^h \end{pmatrix} \left( \frac{i^{(h)} I_{ac}^{(h)}}{d_a^{(h)} I_{ab}^{(h)}} + \frac{\epsilon_a^h}{I_{ab}^{(h)}} - \frac{\left(m_a^h m_b^h + n_a^h n_b^h\right) \epsilon_b^h}{\left((n_b^h)^2 + (m_b^h)^2\right) I_{ab}^{(h)}} \right) \\
&\quad - \begin{pmatrix} n_c^h \\ m_c^h \end{pmatrix} \frac{k^{(h)} I_{ba}^{(h)}}{d_b^{(h)} I_{cb}^{(h)}} + \begin{pmatrix} p_{2h-1} - t_{2h-1} \\ p_{2h} - t_{2h} \end{pmatrix}, \\
z_c^{(h)} &= \vec{v}_c^h + \begin{pmatrix} n_c^h \\ m_c^h \end{pmatrix} \left( \frac{k^{(h)} I_{ba}^{(h)}}{d_b^{(h)} I_{bc}^{(h)}} + \frac{\epsilon_b^h}{I_{bc}^{(h)}} - \frac{\left(m_c^h m_b^h + n_c^h n_b^h\right) \epsilon_c^h}{\left((n_c^h)^2 + (m_c^h)^2\right) I_{bc}^{(h)}} \right) \\
&\quad - \begin{pmatrix} n_a^h \\ m_a^h \end{pmatrix} \frac{j^{(h)} I_{cb}^{(h)}}{d_c^{(h)} I_{ca}^{(h)}} + \begin{pmatrix} t_{2h-1} - q_{2h-1} \\ t_{2h} - q_{2h} \end{pmatrix},
\end{aligned} \tag{5.67}$$

with the two dimensional vectors

$$\begin{aligned}
\vec{v}_a^h &= -\begin{pmatrix} n_b^h \\ m_b^h \end{pmatrix} \left( \frac{\epsilon_a^h}{I_{ab}^{(h)}} - \frac{(m_b^h m_a^h + n_a^h n_b^h) \epsilon_b^h}{\left((n_b^h)^2 + (m_b^h)^2\right) I_{ab}^{(h)}} \right) + \begin{pmatrix} -m_a^h \\ n_a^h \end{pmatrix} \frac{\epsilon_a^h}{(n_a)^2 + (m_a)^2} \\
&\quad - \begin{pmatrix} -m_b^h \\ n_b^h \end{pmatrix} \frac{\epsilon_b^h}{(n_b^h)^2 + (m_b^h)^2}, \\
\vec{v}_b^h &= -\begin{pmatrix} n_c^h \\ m_c^h \end{pmatrix} \left( \frac{\epsilon_b^h}{I_{bc}^{(h)}} - \frac{(m_b^h m_c^h + n_c^h n_b^h) \epsilon_c^h}{\left((n_c^h)^2 + (m_c^h)^2\right) I_{bc}^{(h)}} \right) + \begin{pmatrix} -m_b^h \\ n_b^h \end{pmatrix} \frac{\epsilon_b^h}{(n_b)^2 + (m_b)^2} \\
&\quad - \begin{pmatrix} -m_c^h \\ n_c^h \end{pmatrix} \frac{\epsilon_c^h}{(n_c^h)^2 + (m_c^h)^2}, \\
\vec{v}_c^h &= -\begin{pmatrix} n_a^h \\ m_a^h \end{pmatrix} \left( \frac{\epsilon_c^h}{I_{ca}^{(h)}} - \frac{(m_c^h m_a^h + n_a^h n_c^h) \epsilon_a^h}{\left((n_a^h)^2 + (m_a^h)^2\right) I_{ca}^{(h)}} \right) + \begin{pmatrix} -m_c^h \\ n_c^h \end{pmatrix} \frac{\epsilon_c^h}{(n_c)^2 + (m_c)^2} \\
&\quad - \begin{pmatrix} -m_a^h \\ n_a^h \end{pmatrix} \frac{\epsilon_a^h}{(n_a^h)^2 + (m_a^h)^2},
\end{aligned} \tag{5.68}$$

where the components $i^{(h)}$, $j^{(h)}$ and $k^{(h)}$ where already relabeled by the prescription (5.28). The edges $z_\alpha^{(h)}$ have to be parallel to brane $\alpha$ in the $h$-th plane, yielding the following three Diophantine equations for each plane

$$\begin{aligned} n_a^h(q_{2h}-p_{2h})-m_a^h(q_{2h-1}-p_{2h-1}) &= -\frac{i^{(h)}I_{ac}^{(h)}}{d_a^h}\,, \\ n_b^h(t_{2h}-p_{2h})-m_b^h(t_{2h-1}-p_{2h-1}) &= -\frac{k^{(h)}I_{ba}^{(h)}}{d_b^h}\,, \\ n_c^h(p_{2h}-t_{2h})-m_c^h(p_{2h-1}-t_{2h-1}) &= -\frac{j^{(h)}I_{cb}^{(h)}}{d_c^h}\,, \end{aligned} \tag{5.69}$$

which are equivalent to (5.27) in the two dimensional case in section 5.2.1. The solutions are given by (5.29) for each plane. But in contrast to the factorisable case it is not enough to impose

$$\begin{aligned} -\frac{i^{(h)}}{d_a^{(h)}}\begin{pmatrix} n_c^h \\ m_c^h \end{pmatrix}+s_a^{(h)}\begin{pmatrix} n_a^h \\ m_a^h \end{pmatrix} &= \begin{pmatrix} q_{2h-1}-p_{2h-1} \\ q_{2h}-p_{2h} \end{pmatrix} \in \mathbb{Z}^2, \quad s_a^{(h)}\in\mathbb{Z}\,, \\ -\frac{k^{(h)}}{d_b^{(h)}}\begin{pmatrix} n_a^h \\ m_a^h \end{pmatrix}+s_b^{(h)}\begin{pmatrix} n_b^h \\ m_b^h \end{pmatrix} &= \begin{pmatrix} t_{2h-1}-p_{2h-1} \\ t_{2h}-p_{2h} \end{pmatrix} \in \mathbb{Z}^2, \quad s_b^{(h)}\in\mathbb{Z}\,, \\ -\frac{j^{(h)}}{d_c^{(h)}}\begin{pmatrix} n_b^h \\ m_b^h \end{pmatrix}+s_c^{(h)}\begin{pmatrix} n_c^h \\ m_c^h \end{pmatrix} &= \begin{pmatrix} p_{2h-1}-t_{2h-1} \\ p_{2h}-t_{2h} \end{pmatrix} \in \mathbb{Z}^2, \quad s_c^{(h)}\in\mathbb{Z}\,, \end{aligned} \tag{5.70}$$

but the information, that the pairs $(p_{2h-1},p_{2h})$, $(q_{2h-1},q_{2h})$ and $(t_{2h-1},t_{2h})$ are components of $SO(12)$ lattice vectors, needs to be implemented. That leads to the following additional conditions

$$\begin{aligned} \sum_{h=1}^{3}\left(-s_a^{(h)}(n_a^h+m_a^h)+\frac{i^{(h)}(n_c^h+m_c^h)}{d_a^{(h)}}\right) &= 0 \mod 2\,, \\ \sum_{h=1}^{3}\left(-s_b^{(h)}(n_b^h+m_b^h)+\frac{k^{(h)}(n_a^h+m_a^h)}{d_b^{(h)}}\right) &= 0 \mod 2\,, \\ \sum_{h=1}^{3}\left(-s_c^{(h)}(n_c^h+m_c^h)+\frac{j^{(h)}(n_b^h+m_b^h)}{d_c^{(h)}}\right) &= 0 \mod 2\,. \end{aligned} \tag{5.71}$$

Applying the relations (5.30), (5.33) and (5.34) and inserting the results into $z_a^{(h)}$, $z_b^{(h)}$ and $z_c^{(h)}$ one gets

$$\begin{aligned} z_a^{(h)} &= \begin{pmatrix} n_a^h \\ m_a^h \end{pmatrix} I_{bc}^{(h)}\left(\frac{i^{(h)}}{d_a^{(h)}I_{ab}^{(h)}}+\frac{j^{(h)}}{d_c^{(h)}I_{ca}^{(h)}}+\tilde{\epsilon}^{(h)}+\frac{s_a^{(h)}}{I_{bc}^{(h)}}\right), \\ z_b^{(h)} &= \begin{pmatrix} n_b^h \\ m_b^h \end{pmatrix} I_{ca}^{(h)}\left(\frac{i^{(h)}}{d_a^{(h)}I_{ab}^{(h)}}+\frac{k^{(h)}}{d_b^{(h)}I_{bc}^{(h)}}+\tilde{\epsilon}^{(h)}+\frac{s_b^{(h)}}{I_{ca}^{(h)}}\right), \\ z_c^{(h)} &= \begin{pmatrix} n_c^h \\ m_c^h \end{pmatrix} I_{ab}^{(h)}\left(\frac{j^{(h)}}{d_c^{(h)}I_{ca}^{(h)}}+\frac{j^{(h)}}{d_c^{(h)}I_{ca}^{(h)}}+\tilde{\epsilon}^{(h)}+\frac{s_c^{(h)}}{I_{ab}^{(h)}}\right), \end{aligned} \tag{5.72}$$

with $\tilde{\epsilon}^{(h)}=\frac{I_{bc}^{(h)}\epsilon_a^h+I_{ca}^{(h)}\epsilon_b^h+I_{ab}^{(h)}\epsilon_c^h}{I_{ab}^{(h)}I_{bc}^{(h)}I_{ca}^{(h)}}$. For $\triangle_{(h)}^{ijk}$ to close

$$z_a^{(h)}+z_b^{(h)}+z_c^{(h)}=0\,, \quad \forall h\in\{1,2,3\}\,, \tag{5.73}$$

has to be satisfied. Inserting the expressions from (5.72) into (5.73), the parameters $s_a^{(h)}$, $s_b^{(h)}$ and $s_c^{(h)}$ are allowed to depend on a single independent parameter $\ell^{(h)}$ given by

$$\begin{aligned} s_a^{(h)} &= \frac{k^{(h)}}{d_b^{(h)}} + \frac{I_{bc}^{(h)}\left(\ell^{(h)} + l_0^{(h)}\right)}{d^{(h)}}\,, \\ s_b^{(h)} &= \frac{j^{(h)}}{d_c^{(h)}} + \frac{I_{ca}^{(h)}\left(\ell^{(h)} + l_0^{(h)}\right)}{d^{(h)}}\,, \\ s_c^{(h)} &= \frac{i^{(h)}}{d_a^{(h)}} + \frac{I_{ab}^{(h)}\left(\ell^{(h)} + l_0^{(h)}\right)}{d^{(h)}}\,, \end{aligned} \tag{5.74}$$

where $l_0^{(h)}$ is a function of $i^{(h)}$, $j^{(h)}$ and $k^{(h)}$ as explained in (5.38), and makes sure that $s_a^{(h)}$, $s_b^{(h)}$ and $s_c^{(h)}$ lead to integer lattice shifts in (5.29). Composing the parameters $\ell^{(h)}$ from each plane to a triplet $\ell = (\ell^{(1)}, \ell^{(2)}, \ell^{(3)})$, $\ell$ can be expressed as an element from a three dimensional lattice. For the factorisable torus, $\ell$ is given by [106]

$$\ell \in \mathbb{Z}^3\,. \tag{5.75}$$

For the present case however, inserting (5.74) into (5.70) and (5.71) leads to selection rules for $\ell$, s.t. $\ell$ belongs in general to a generic three dimensional lattice $\Lambda^3$, possibly with a label dependend off-set

$$\ell = \left(\ell^{(1)}, \ell^{(2)}, \ell^{(3)}\right) \in \Lambda_3\,. \tag{5.76}$$

The lattice $\Lambda_3$ is determined by the selection rules

$$\begin{aligned} -\frac{i^{(h)}}{d_a^{(h)}}\begin{pmatrix} n_c^h \\ m_c^h \end{pmatrix} + \frac{k^{(h)}}{d_b^{(h)}}\begin{pmatrix} n_a^h \\ m_a^h \end{pmatrix} + \frac{I_{cb}^{(h)}\left(\ell^{(h)} + l_0^{(h)}\right)}{d^{(h)}}\begin{pmatrix} n_a^h \\ m_a^h \end{pmatrix} &\in \mathbb{Z}^2\,, \\ -\frac{k^{(h)}}{d_b^{(h)}}\begin{pmatrix} n_a^h \\ m_a^h \end{pmatrix} + \frac{j^{(h)}}{d_c^{(h)}}\begin{pmatrix} n_b^h \\ m_b^h \end{pmatrix} + \frac{I_{ca}^{(h)}\left(\ell^{(h)} + l_0^{(h)}\right)}{d^{(h)}}\begin{pmatrix} n_b^h \\ m_b^h \end{pmatrix} &\in \mathbb{Z}^2\,, \\ -\frac{j^{(h)}}{d_c^{(h)}}\begin{pmatrix} n_b^h \\ m_b^h \end{pmatrix} + \frac{j^{(h)}}{d_a^{(h)}}\begin{pmatrix} n_c^h \\ m_c^h \end{pmatrix} + \frac{I_{ab}^{(h)}\left(\ell^{(h)} + l_0^{(h)}\right)}{d^{(h)}}\begin{pmatrix} n_c^h \\ m_c^h \end{pmatrix} &\in \mathbb{Z}^2\,, \end{aligned} \tag{5.77}$$

and

$$\begin{aligned} \sum_{h=1}^{3}\left(-\frac{i^{(h)}}{d_a^{(h)}}N_c^{(h)} + \frac{k^{(h)}}{d_b^{(h)}}N_a^{(h)} + \frac{I_{cb}^{(h)}\left(\ell^{(h)} + l_0^{(h)}\right)}{d^{(h)}}N_a^{(h)}\right) &= 0 \mod 2\,, \\ \sum_{h=1}^{3}\left(-\frac{k^{(h)}}{d_b^{(h)}}N_a^{(h)} + \frac{j^{(h)}}{d_c^{(h)}}N_b^{(h)} + \frac{I_{ca}^{(h)}\left(\ell^{(h)} + l_0^{(h)}\right)}{d^{(h)}}N_b^{(h)}\right) &= 0 \mod 2\,, \\ \sum_{h=1}^{3}\left(-\frac{j^{(h)}}{d_c^{(h)}}N_b^{(h)} + \frac{j^{(h)}}{d_a^{(h)}}N_c^{(h)} + \frac{I_{ab}^{(h)}\left(\ell^{(h)} + l_0^{(h)}\right)}{d^{(h)}}N_c^{(h)}\right) &= 0 \mod 2\,, \end{aligned} \tag{5.78}$$

where $N_\alpha^{(h)}$ is defined as $N_\alpha^{(h)} =: n_\alpha^h + m_\alpha^h$. The surface of the instanton, connecting the intersection points $i, j, k$ and wrapping number $\ell$ around the torus, is denoted by $A_{ijk}(\ell)$. The area of $A_{ijk}(\ell)$ in each plane is then computed by inserting (5.72) into (5.40) and therefore the whole area spread out by the instanton is given by the sum

$$A^{ijk}(\ell^{(h)}) = \frac{1}{2}\sum_{h=1}^{3} |I_{ab}^{(h)} I_{bc}^{(h)} I_{ca}^{(h)}| \left( \frac{i^{(h)}}{d_a^{(h)} I_{ab}^{(h)}} + \frac{j^{(h)}}{d_a^{(h)} I_{ca}^{(h)}} + \frac{k^{(h)}}{d_b^{(h)} I_{bc}^{(h)}} + \tilde{\epsilon}^{(h)} + \frac{\ell^{(h)}}{d^{(h)}} \right)^2 . \tag{5.79}$$

By turning on the deformation parameter $R_h$ and $\tau_h$ in each plane, as explained in example (ii) in section 4.1.1, the $T^6_{\mathrm{SO}(12)}$ gets deformed. By complexifying the Kähler structure modulus as

$$K_h = {R_h}^2 \mathrm{Im}(\tau_h) \to B_h + i{R_h}^2 \mathrm{Im}(\tau_h)\,, \tag{5.80}$$

in the $h$-th plane, the worldsheet instanton is allowed to couple to a B-field, with the components $B_h$ in the $h$-th plane. Using the result for the two dimensional case in section 5.2.1, it can be deduce that the volume of each triangle $\triangle^{ijk}_{(h)}$ is scaled by $-iK_h$. The Yukawa coupling for fields, sitting at the intersection points $i, j, k$, is given, according to (5.8), by the sum of all instantons connecting the corresponding intersection points

$$Y_{ijk} = h_{\mathrm{qu}} \sum_{\ell \in \Lambda^3} \exp\left( -\frac{A_{ijk}(\ell)}{2\pi\alpha'} \right), \tag{5.81}$$

with

$$A_{ijk}(\ell) = -\frac{i}{2}\sum_{h=1}^{3} K^{(h)} |I_{ab}^{(h)} I_{bc}^{(h)} I_{ca}^{(h)}| \left( \frac{i^{(h)}}{d_a^{(h)} I_{ab}^{(h)}} + \frac{j^{(h)}}{d_a^{(h)} I_{ca}^{(h)}} + \frac{k^{(h)}}{d_b^{(h)} I_{bc}^{(h)}} + \tilde{\epsilon}^{(h)} + \frac{\ell^{(h)}}{d^{(h)}} \right)^2 . \tag{5.82}$$

The factor $h_{\mathrm{qu}}$ contains the quantum contribution to the Yukawa couplings.

### 5.3.3 Example

Here an illustrative example on the $T^6_{\mathrm{SO}(12)}$ with three branes $a$, $b$, $c$, whose wrapping numbers are listed in table 5.1, is discussed. Investigating the wrapping numbers, one observes that only the brane $a$ satisfies

| | $h = 1$ | $h = 2$ | $h = 3$ |
|---|---|---|---|
| $(n_a^h, m_a^h)$ | (1,-3) | (1,1) | (1,1) |
| $(n_b^h, m_b^h)$ | (1,0) | (2,0) | (1,-2) |
| $(n_c^h, m_c^h)$ | (2,3) | (4,6) | (1,-1) |
| $I_{ab}^{(h)}$ | 3 | -2 | -3 |
| $I_{ac}^{(h)}$ | 9 | 2 | -2 |
| $I_{bc}^{(h)}$ | 3 | 12 | 1 |
| $d^{(h)}$ | 3 | 2 | 1 |

Table 5.1: Example

the condition (4.73), which is why $g.c.d.(n_b^2, m_b^2) = 2$ and $g.c.d.(n_c^2, m_c^2) = 2$. First one has to compute the labels for the intersection points. Following (5.54), the intersection points $\mathbf{p}_{ab}$ of $a$ and $b$ are

$$\mathbf{p}_{ab} = \left\{ \left( \frac{-3t_1 - t_2}{-3}, 0, 2\frac{t_3 - t_4}{2}, 0, \frac{t_5 - t_6}{3}, -2\frac{t_5 - t_6}{3} \right) \middle| \vec{t} \in \Lambda_{\mathrm{SO}(12)} \right\} \tag{5.83}$$

and their labels are given by

$$i = (-3t_1 - t_2, t_3 - t_4, t_4 - t_6)\ . \tag{5.84}$$

For $t_i$ being components of $SO(12)$ lattice vectors, the labels $i$ belongs to the $SO(6)$ root lattice. Inserting the labels into (5.83) and shifting them by

$$i \to i + (0, 2, 0), \quad i \to i + (3, 1, 0), \quad i \to i + (0, 1, 3) \tag{5.85}$$

lead to equivalent intersection poiints on the torus. Hence the inequivalent labels belong the lattice

$$i \in \frac{\Lambda_{\mathrm{SO}(6)}}{\Gamma_{ab}}\ , \tag{5.86}$$

with

$$\Gamma_{ab} = \left\{ \sum_{i=1}^{3} n_i \vec{e}_i \middle| n_i \in \mathbb{Z},\ \vec{e}_1 = \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix}, \vec{e}_2 = \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix}, \vec{e}_3 = \begin{pmatrix} 0 \\ 1 \\ 3 \end{pmatrix} \right\}. \tag{5.87}$$

The lattice for the labels $i$ correspond to case 7 in appendix B, with the first and second plane permuted. Relabeling the intersection points according to (5.28), the mapping of the “old” labels to the “new” labels is given by

$$(i^{(1)}, i^{(2)}, i^{(3)}) \to (3i^{(1)}, i^{(2)}, -2i^{(3)}) \quad \mathrm{mod}\ \Gamma_{ab}\ . \tag{5.88}$$

In tabel 5.2 the "old" and "new" labels are listed together with the coordinates of the corresponding intersections of $a$ and $b$. When inequivalent labels get mapped to equivalent labels by relabeling, the

| old label $i$ | new label $i'$ | coordinates |
|---|---|---|
| (0,0,0) | (0,0,0) | $(0, 0, 0, 0, 0, 0)$ |
| (1,0,1) | - | $(-\frac{1}{3}, 0, 0, 0, \frac{1}{3}, -\frac{2}{3})$ |
| (2,0,0) | - | $(-\frac{2}{3}, 0, 0, 0, 0, 0)$ |
| (0,0,2) | (0,0,2) | $(0, 0, 0, 0, \frac{2}{3}, -\frac{4}{3})$ |
| (2,0,2) | - | $(-\frac{2}{3}, 0, 0, 0, \frac{2}{3}, -\frac{4}{3})$ |
| (1,1,0) | - | $(-\frac{1}{3}, 0, 1, 0, 0, 0)$ |
| (2,1,1) | (0,0,1) | $(0, 0, 1, 0, \frac{1}{3}, -\frac{2}{3})$ |
| (2,1,1) | - | $(-\frac{2}{3}, 0, 1, 0, \frac{1}{3}, -\frac{2}{3})$ |
| (1,1,2) | - | $(-\frac{1}{3}, 0, 1, 0, \frac{1}{3}, -\frac{4}{3})$ |

Table 5.2: Intersection points $a$ and $b$

corresponding intersection point is not assigned by a "new" label. For example the intersection point with the "old" label $i = (1, 0, 1)$ gets mapped to $i' = (3, 0, -2)$. Shifting the "new" label by the lattice vector $(-3, 0, 3) \in \Gamma_{ab}$,one receives $(0, 0, 1)$. But the intersection point with the "old" label $(2, 1, 1)$ gets also mapped to the "new" label $i' = (0, 0, 1)$. Therefor intersection points with the "old" label $(1, 0, 1)$ lose their label after relabeling. Similar, intersection points initially labeled by $(2, 0, 0)$, $(2, 0, 2)$, $(1, 1, 0)$, $(2, 1, 1)$ and $(1, 1, 2)$ also lose their labels after relabeling and only three new labels remain (see table 5.2). Next the "old" and "new" labels for the intersections between $b$ and $c$ and intersections between $a$ and $c$ are computed. The intersection points $\mathbf{p}_{ac}$ between $a$ and $c$ are given by

$$\mathbf{p}_{ac} = \left\{ \left( \frac{3t_1 - 2t_2}{9}, -3\frac{3t_1 - 2t_2}{9}, \frac{6t_3 - 4t_4}{2}, \frac{6t_3 - 4t_4}{2}, \frac{t_5 + t_6}{2}, \frac{t_5 + t_6}{2} \right) \middle| \vec{t} \in \Lambda_{\mathrm{SO}(12)} \right\}\ , \tag{5.89}$$

and are labeled by

$$j = (3t_1 - 2t_2, 6t_3 - 4t_4, -t_5 - t_6)\,. \tag{5.90}$$

For $\vec{t} \in \Lambda_{\mathrm{SO}(12)}$ the labels $j$ belong to the lattice $\mathbb{Z} \otimes 2\mathbb{Z} \otimes \mathbb{Z}$. Shifts in $j$ which map points in (5.89) to points differing by $SO(12)$ lattice vectors are given by

$$j \to j + (9, 0, 0), \quad j \to j + (0, 2, 0), \quad j \to j + (0, 0, 2)\,, \tag{5.91}$$

s.t. the space of labels for inequivalent intersection points $\mathbf{p}_{ac}$ is

$$j \in \frac{\mathbb{Z} \otimes 2\mathbb{Z} \otimes \mathbb{Z}}{9\mathbb{Z} \otimes 2\mathbb{Z} \otimes 2\mathbb{Z}}\,. \tag{5.92}$$

The lattice for $j$ corresponds to case 6 in appendix B and the prescription for relabeling of $j$ is given by

$$(j^{(1)}, j^{(2)}, j^{(3)}) \to (-j^{(1)}, -6j^{(2)}, -j^{(3)}) \mod 9\mathbb{Z} \otimes 2\mathbb{Z} \otimes 2\mathbb{Z}\,. \tag{5.93}$$

On the torus the intersection points $\mathbf{p}_{bc}$ of $b$ and $c$ are given by the set

$$\mathbf{p}_{bc} = \left\{ \left( 2\frac{t_2}{3}, 3\frac{t_2}{3}, 4\frac{2t_4}{12}, 6\frac{2t_4}{12}, 2t_5 + t_6, -2t_5 - t_6 \right) \middle| \, \vec{t} \in \Lambda_{\mathrm{SO}(12)} \right\}, \tag{5.94}$$

with the corresponding labels

$$k = (t_2, 2t_4, 2t_5 + t_6) \tag{5.95}$$

take values on the lattice $\mathbb{Z} \otimes 2\mathbb{Z} \otimes \mathbb{Z}$. Intersection points $\mathbf{p}_{bc}$ are mapped to equivalent intersection points on the torus by

$$k \to k + (0, 0, 1), \quad k \to k + (0, 12, 0), \quad k \to k + (3, 6, 0)\,. \tag{5.96}$$

Hence the set of inequivalent labels belong to the quotient lattice

$$k \in \frac{\mathbb{Z} \otimes 2\mathbb{Z} \otimes \mathbb{Z}}{\Gamma_{bc}}\,, \tag{5.97}$$

with

$$\Gamma_{bc} = \left\{ \sum_{i=1}^{3} n_i \vec{e}_i \middle| \, n_i \in \mathbb{Z},\ \vec{e}_1 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \vec{e}_2 = \begin{pmatrix} 0 \\ 12 \\ 0 \end{pmatrix}, \vec{e}_3 = \begin{pmatrix} 3 \\ 6 \\ 0 \end{pmatrix} \right\}, \tag{5.98}$$

which corresponds to case 8 in appendix B. The set of inequvialent labels before and after relabeling as well as the coordinates of the intersection points $\mathbf{p}_{ca}$ and $\mathbf{p}_{bc}$ are listed in the tables 5.3 and 5.4. All the information needed to compute Yukawa couplings arising for the setup in table 5.1 is contained in the tables 5.2, 5.3 and 5.4. As an illustrative example, the Yukawa couplings of the field sitting at the intersection point $i = (0, 0, 0)$ is computed. Applying (5.70) and (5.71), the selection rules for the couplingstake the form

$$\begin{pmatrix} s_a^{(1)} \\ -3s_a^{(1)} \\ s_a^{(2)} \\ s_a^{(2)} \\ s_a^{(3)} \\ s_a^{(3)} \end{pmatrix}, \begin{pmatrix} s_b^{(1)} - \frac{k^{(1)}}{3} \\ k^{(1)} \\ 2s_b^{(2)} - \frac{k^{(2)}}{2} \\ -\frac{k^{(2)}}{2} \\ s_b^{(3)} - k^{(3)} \\ -2s_b^{(3)} - k^{(3)} \end{pmatrix}, \begin{pmatrix} 2s_c^{(1)} - \frac{j^{(1)}}{3} \\ 3s_c^{(1)} \\ 4s_c^{(2)} - j^{(2)} \\ 6j^{(2)} \\ s_c^{(3)} - j^{(3)} \\ -s_c^{(3)} + 2j^{(3)} \end{pmatrix} \in \Lambda_{\mathrm{SO}(12)} \tag{5.99}$$

| old label $j$ | new label $j'$ | coordinates |
|---|---|---|
| (0,0,0) | (0,0,0) | $(0,0,0,0,0,0)$ |
| (1,0,0) | (8,0,0) | $(\frac{1}{9},-\frac{1}{3},0,0,0,0)$ |
| (2,0,0) | (7,0,0) | $(\frac{2}{9},-\frac{2}{3},0,0,0,0)$ |
| (3,0,0) | (6,0,0) | $(\frac{1}{3},-1,0,0,0,0)$ |
| (4,0,0) | (5,0,0) | $(\frac{4}{9},-\frac{4}{3},0,0,0,0)$ |
| (5,0,0) | (4,0,0) | $(\frac{5}{9},-\frac{5}{3},0,0,0,0)$ |
| (6,0,0) | (3,0,0) | $(\frac{2}{3},-2,0,0,0,0)$ |
| (7,0,0) | (2,0,0) | $(\frac{7}{9},-\frac{7}{3},0,0,0,0)$ |
| (8,0,0) | (1,0,0) | $(\frac{8}{9},-\frac{8}{3},0,0,0,0)$ |
| (0,0,1) | (0,0,1) | $(0,0,0,0,\frac{1}{2},\frac{1}{2})$ |
| (1,0,1) | (8,0,1) | $(\frac{1}{9},-\frac{1}{3},0,0,\frac{1}{2},\frac{1}{2})$ |
| (2,0,1) | (7,0,1) | $(\frac{2}{9},-\frac{2}{3},0,0,\frac{1}{2},\frac{1}{2})$ |
| (3,0,1) | (6,0,1) | $(\frac{1}{3},-1,0,0,\frac{1}{2},\frac{1}{2})$ |
| (4,0,1) | (5,0,1) | $(\frac{4}{9},-\frac{4}{3},0,0,\frac{1}{2},\frac{1}{2})$ |
| (5,0,1) | (4,0,1) | $(\frac{5}{9},-\frac{5}{3},0,0,\frac{1}{2},\frac{1}{2})$ |
| (6,0,1) | (3,0,1) | $(\frac{2}{3},-2,0,0,\frac{1}{2},\frac{1}{2})$ |
| (7,0,1) | (2,0,1) | $(\frac{7}{9},-\frac{7}{3},0,0,\frac{1}{2},\frac{1}{2})$ |
| (8,0,1) | (1,0,1) | $(\frac{8}{9},-\frac{8}{3},0,0,\frac{1}{2},\frac{1}{2})$ |

Table 5.3: Intersection points $c$ and $a$

and for the triangles to close, the parameters $s_\alpha$ have to be the following functions of instanton winding numbers $\ell$:

$$\begin{aligned} s_a^{(1)} &= \frac{k^{(1)}}{3} + \ell^{(1)}, \quad & s_a^{(2)} &= \tfrac{k^{(2)}}{2} + 6\ell^{(2)}, \quad & s_a^{(3)} &= k^{(3)} + \ell^{(3)}, \\ s_b^{(1)} &= \frac{j^{(1)}}{3} - 3\ell^{(1)}, \quad & s_b^{(2)} &= \tfrac{j^{(2)}}{2} - \ell^{(2)}, \quad & s_b^{(3)} &= j^{(3)} + 2\ell^{(3)}, \\ s_c^{(1)} &= \ell^{(1)}, & s_c^{(2)} &= -\ell^{(2)}, & s_c^{(3)} &= -3\ell^{(3)}\,. \end{aligned} \tag{5.100}$$

Inserting (5.100) into (5.99) yields conditions on $k$ and $j$ as well as on $\ell$. For the vectors in (5.99) to be vectors with integer components, one receives from the selection rules the following conditions

$$\begin{aligned} \ell^{(1)} &= -\frac{k^{(3)}}{3} + l^{(1)}\,, \quad \text{with} \quad l^{(1)} \in \mathbb{Z}\,, \\ \frac{j^{(1)}}{3} &= p + \frac{k^{(1)}}{3}\,, \quad \text{with} \quad p \in \mathbb{Z}\,, \\ \ell^{(2)} &= \frac{l^{(2)}}{2}, \quad \text{with} \quad l^{(2)} \in \mathbb{Z}\,, \\ \ell^{(3)} &= l^{(3)}\,, \quad \text{with} \quad l^{(3)} \in \mathbb{Z}\,. \end{aligned} \tag{5.101}$$

Further imposing (5.99) to be $SO(12)$ lattice vectors yield the condition

$$l^{(1)} + l^{(2)} + p + j^{(3)} = 0 \mod 2\,. \tag{5.102}$$

| old label $k$ | new label $k'$ | coordinates |
|---|---|---|
| (0,0,0) | (0,0,0) | $(0,0,0,0,0,0)$ |
| (1,0,0) | (-1,0,0) | $(-\frac{2}{3},-1,0,0,0,0)$ |
| (2,0,0) | (-2,0,0) | $(-\frac{4}{3},-2,0,0,0,0)$ |
| (0,2,0) | (0,2,0) | $(0,0,\frac{2}{3},-1,0,0)$ |
| (1,2,0) | (-1,2,0) | $(-\frac{2}{3},-1,-\frac{2}{3},-1,0,0)$ |
| (2,2,0) | (-2,2,0) | $(-\frac{4}{3},-2,-\frac{2}{3},-1,0,0)$ |
| (0,4,0) | (0,4,0) | $(0,0,-\frac{4}{3},-2,0,0)$ |
| (1,4,0) | (-1,4,0) | $(-\frac{2}{3},-1,-\frac{4}{3},-2,0,0)$ |
| (2,4,0) | (-2,4,0) | $(-\frac{4}{3},-2,-\frac{4}{3},-2,0,0)$ |
| (0,6,0) | (0,6,0) | $(0,0,-2,-3,0,0)$ |
| (1,6,0) | (-1,6,0) | $(-\frac{2}{3},-1,-2,-3,0,0)$ |
| (2,6,0) | (-2,6,0) | $(-\frac{4}{3},-2,-2,-3,0,0)$ |
| (0,8,0) | (0,8,0) | $(0,0,-\frac{8}{3},-4,0,0)$ |
| (1,8,0) | (-1,8,0) | $(-\frac{2}{3},-1,-\frac{8}{3},-4,0,0)$ |
| (2,6,0) | (-2,6,0) | $(-\frac{4}{3},-2,-\frac{8}{3},-4,0,0)$ |
| (0,10,0) | (0,10,0) | $(0,0,-\frac{10}{3},-5,0,0)$ |
| (1,10,0) | (-1,10,0) | $(-\frac{2}{3},-1,-\frac{10}{3},-5,0,0)$ |
| (2,10,0) | (-2,10,0) | $(-\frac{4}{3},-2,-\frac{10}{3},-5,0,0)$ |

Table 5.4: Intersection points $b$ and $c$

The selection rules in (5.102) obliges the winding numbers of the instanton to belong to the lattice

$$\Lambda_{\mathrm{SO}(4)} \otimes \mathbb{Z} = \operatorname{span}\left((1,1,0)^T,\ (1,-1,0)^T,\ (0,0,1)^T\right), \tag{5.103}$$

with an offset depending on $j^{(3)} + p$. For example, the fields sitting at the intersection points with the "new" labels $i = (0,0,0)$, $j = (8,0,0)$ and $k = (-1,0,0)$ have a non trivial Yukawa coupling, since $j^{(1)}$ and $k^{(1)}$ indeed satisfy the second condition in (5.101) with $p = 3$. The Yukawa coupling for them is given by

$$-h_{\mathrm{qu}} \sum_{l \in (1,0,0)+\Lambda_{\mathrm{SO}(4)}\otimes\mathbb{Z}} \exp\left(-\frac{\pi}{\alpha'}\left[9A^{(1)}\left(8+3l^{(1)}\right)^2 + 12A^{(2)}l^{(2)2} + 6A^{(3)}l^{(3)2}\right]\right) \tag{5.104}$$

The boundaries of the worldsheet instanton, which are attached to the branes, are parametrized by the vectors

$$\begin{pmatrix} z_a^{(1)} \\ z_a^{(2)} \\ z_a^{(3)} \end{pmatrix} = \begin{pmatrix} -\frac{8}{9} + l^{(1)} \\ \frac{8}{3} - 3l^{(1)} \\ 3l^{(2)} \\ 3l^{(2)} \\ l^{(3)} \\ l^{(3)} \end{pmatrix}, \quad \begin{pmatrix} z_b^{(1)} \\ z_b^{(2)} \\ z_b^{(3)} \end{pmatrix} = \begin{pmatrix} 2\frac{2}{3} - 3l^{(1)} \\ 0 \\ -l^{(2)} \\ 0 \\ 2l^{(3)} \\ -4l^{(3)} \end{pmatrix}, \quad \begin{pmatrix} z_c^{(1)} \\ z_c^{(2)} \\ z_c^{(3)} \end{pmatrix} = \begin{pmatrix} -1\frac{7}{9} + 2l^{(1)} \\ -\frac{8}{3} + 3l^{(1)} \\ -2l^{(2)} \\ -3l^{(2)} \\ -3l^{(3)} \\ 3l^{(3)} \end{pmatrix}. \tag{5.105}$$

The leading contribution to the Yukawa coupling has either the winding number $l = (1,0,0)$ or $l = (0,1,0)$ depending on the value for the moduli $A^{(1)}$ and $A^{(2)}$.

Not all intersection points between $a$ and $b$ have "new" labels. To compute Yukawa couplings of fields sitting at those intersection points, one proceeds as in the two dimensional case in section 5.2.2: Let the

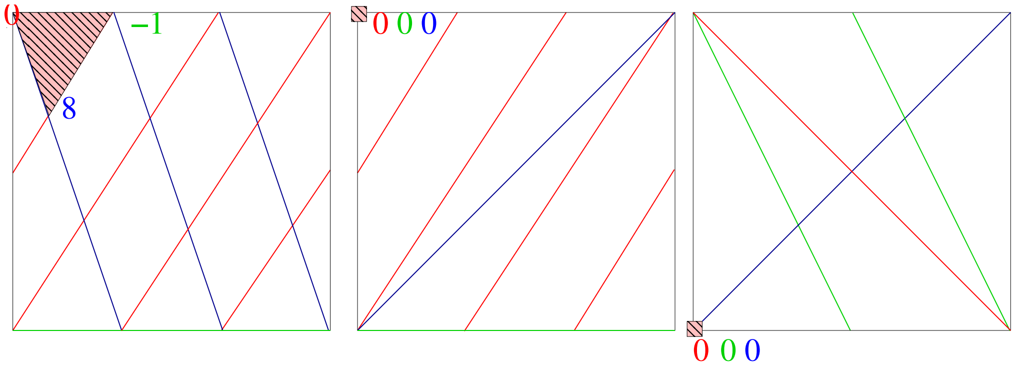


Figure 5.2: Worldsheet Instanton from intersecting branes: The three square bases, depict the three planes and the blue, green and red lines indicate the position of the branes $a$, $b$ and $c$ in the planes. The area spread out by the leading order worldsheet instanton, coupling to the intersection points with the "new" labelsl $i = (0, 0, 0)$, $j = (-1, 0, 0)$ and $k - (8, 0, 0)$ is highlighted by the color pink.

"old" labels, of the coupling considered, be denoted by $i_1$, $j_1$ and $k_1$, where for example $i_1$ has no "new" label. The label $i_1$ is shifted by a fraction of an equivalence shift and mapped to another label $i_2$, which is not removed by relabeling, by

$$i_2 = i_1 - \frac{\delta}{d}\vec{\lambda}_{ab}, \quad \text{with} \quad \delta \in \{1, ..., d-1\}, \quad \vec{\lambda}_{ab} \in \Gamma_{ab}\,. \tag{5.106}$$

The integer $d$ is the greatest common divisor of the three intersection numbers. Next the other two "old" labels $j_1$ and $k_1$, belonging to the coupling, are shifted by the same fraction of the corresponding equivlanece shifts

$$j_1 \to j_2 = j_1 - \frac{\delta}{d}\vec{\lambda}_{ca}, \quad k_1 \to k_2 = k_1 - \frac{\delta}{d}\vec{\lambda}_{bc}, \quad \vec{\lambda}_{ca} \in \Gamma_{ca},\, \vec{\lambda}_{bc} \in \Gamma_{bc}\,. \tag{5.107}$$

The shifts in (5.106) and (5.107) have to be chosen, s.t. the labels $i_2$, $j_2$ and $k_2$ exist and the instanton connecting the corresponding intersection points is congruent to the instanton connecting the points labeled by $i_1$, $j_1$ and $k_1$. On the other hand, if the "new" labels of $i_2$, $j_2$ and $k_2$ do not satisfy the selection rules, the corresponding Yukawa coupling is zero. That way for example the coupling with the "old" labels $i_1 = (2, 0, 0)$, $j_1 = (4, 0, 0)$, $k_1 = (2, 6, 0)$ can be computed, even though relabeling leads to the loss of $i_1$'s label. To compute the coupling, the labels are shifted by

$$i \to i - \frac{1}{3}(6, 0, 0), \quad j \to j - \frac{1}{3}(9, 0, 0), \quad k \to k - \frac{1}{3}(3, 18, 0)\,, \tag{5.108}$$

s.t. one receives the shifted labels $i_2 = (0, 0, 0)$, $j_2 = (1, 0, 0)$ and $k_2 = (1, 0, 0)$. As one can see in figure 5.3, the leading order instanton connecting the labels $i_1$, $j_1$ and $k_1$ is indeed congruent to the instanton connecting $i_2$, $j_2$ and $k_2$. Relabeling $i_2$, $j_2$ and $k_2$ shows that the corresponding Yukawa coupling is given by (5.104).

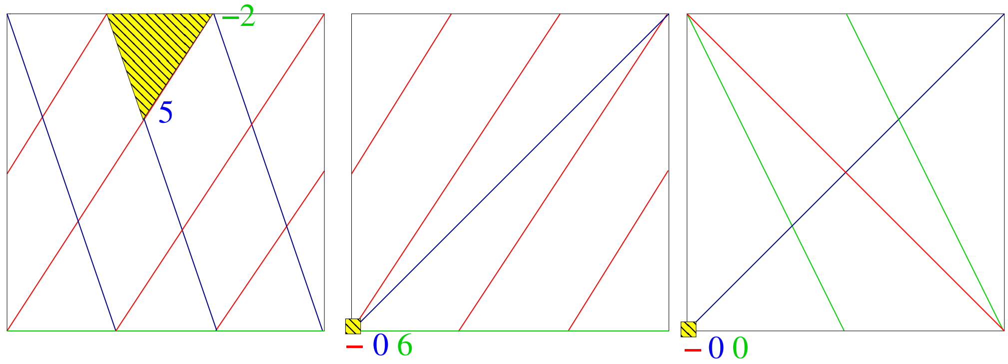


Figure 5.3: Worldsheet instanton coupling to intersection points with no label: The area spread out by worldsheet instanton coupling to the intersection points, labeled by the "old" labels $i = (2, 0, 0)$, $j = (4, 0, 0)$ and $k = (2, 6, 0)$, is highlighted by the color yellow. However after relabeling the intersection points are labeled by $j = (5, 0, 0)$ and $k = (-2, 6, 0)$, but $i$ has no new label. Comparing the area with the worldsheet instanton from figure 5.2, both instantons clearly spread out areas with the same volume.

CHAPTER 6

# Yukawa couplings from D9-branes on the dual $T^6_{\mathrm{SO}(12)}$

In this chapter the T-dual of intersecting D6-branes on $T^6_{\mathrm{SO}(12)}$, with three T-dualized directions, is considered and the Yukawa couplings from a T-dual setup to the one in chapter 5.3 is computed. Besides of serving as a consistency check, the computations lead to the quantum contribution of the Yukawa couplings on the $T^6_{\mathrm{SO}(12)}$. The discussion is based on the work [114] and extends the results of [115].

## 6.1 From $T^6_{\mathrm{SO}(12)}$ to the dual $T^6_{\mathrm{SO}(12)}$

### 6.1.1 Buscher rules

In this section T-duality on more general spaces than discussed in section 3.2.3 is introduced and therefore the results of [53] are revised in order to approach a T-dual torus of $T^6_{\mathrm{SO}(12)}$. Consider a manifold with metric $g_{\mu\nu}$ and a background field $B_{\mu\nu}$.. For open strings on the manifold, the action (3.39) describes the coupling of the open strings to the spacetime. Let the action be invariant under

$$X^\mu \to X^\mu + 2\pi R \tag{6.1}$$

for one spatial direction $\mu = \theta$. Invariance under (6.1) implies circle compactification of the direction $\theta$ on a $S^1$ with radius $R$. The remaining directions are left uncompact. Decomposing (3.39) in components of uncompact directions $i \in \{\mu\}\backslash\{\theta\}$ and circle compactified direction $\theta$, the open string action can be expressed as

$$\begin{aligned} \mathcal{S} \;=\; & -\frac{1}{4\pi\alpha'}\int_\Sigma \mathrm{d}^2\sigma \left[ \left( g_{\theta\theta}\partial_\alpha\theta\partial_\beta\theta + 2g_{\theta i}\partial_\alpha\theta\partial_\beta X^j + g_{ij}\partial_\alpha X^i\partial_\beta X^j \right)\eta^{\alpha\beta} \right. \\ & \left. + \left( 2B_{i\theta}\partial_\alpha X^i\partial_\beta\theta + B_{ij}\partial_\alpha X^i\partial_\beta X^j \right)\epsilon^{\alpha\beta} \right] - \int_{\partial\Sigma}\mathrm{d}\sigma^0 \left( A_i\partial_0 X^i + A_\theta\partial_0\theta \right) . \end{aligned} \tag{6.2}$$

Let the isometry in (6.1) be generated by the infenitesimal translations $\theta \to \theta + \epsilon$. The symmetry can be implemented into the action (6.2) by making it in a covariant. Therefor a covariant derivative $\partial_\alpha\theta \to D_\alpha\theta = \partial_\alpha\theta + V_\alpha$ needs to be introduced, where $V_\alpha$ transforms as $V_\alpha \to V_\alpha - \partial_\alpha\epsilon$. By gauge fixing

$\theta = 0$, one recives

$$\begin{aligned}\mathcal{S}' &= -\frac{1}{4\pi\alpha'}\int_\Sigma \mathrm{d}^2\sigma \Big[ \left(g_{\theta\theta}V_\alpha V_\beta + 2g_{\theta i}V_\alpha\partial_\beta X^j + g_{ij}\partial_\alpha X^i\partial_\beta X^j\right)\eta^{\alpha\beta} \\ &\quad + \left(2B_{i\theta}\partial_\alpha X^i V_\beta + B_{ij}\partial_\alpha X^i\partial_\beta X^j\right)\epsilon^{\alpha\beta} + 2\tilde{\theta}\epsilon^{\alpha\beta}\partial_\alpha V_\beta\Big] \\ &\quad - \int_{\partial\Sigma}\mathrm{d}\sigma^0\left(A_i\partial_{\sigma^0}X^i + A_\theta V_0 + \frac{1}{2\pi\alpha'}\tilde{\theta}V_0\right),\end{aligned} \tag{6.3}$$

where the information $\partial_\alpha\theta = V_\alpha$ is kept in (6.3) by introducing a Lagrange multiplier $\tilde{\theta}$. The equations of motion for $\tilde{\theta}$ lead to the constraint $\partial_\alpha V_\beta\epsilon^{\alpha\beta} = 0$ and inserting the the solutions into (6.3) returns to (6.2). The Lagrange mulitplier $\tilde{\theta}$ corresponds to the T-dual direction of $\theta$. By integrating out the field $V_\alpha$ in (6.3) one receives the T-dual action $\tilde{\mathcal{S}}$to (6.2). The equations of motion for $V_\alpha$ are

$$\left(g_{\theta\theta}V_\beta + g_{\theta i}\partial_\beta X^j\right)\eta^{\alpha\beta} + B_{\theta i}\partial_\beta X^j\epsilon^{\alpha\beta} = -\partial_\beta\tilde{\theta}\epsilon^{\alpha\beta}\,. \tag{6.4}$$

with the additional constraint

$$A_\theta + \frac{1}{2\pi\alpha'}\tilde{\theta} = 0 \tag{6.5}$$

coming from the boundary term. Inserting the equations of motion back into (6.3) leads to

$$\begin{aligned}\tilde{\mathcal{S}} &= -\frac{1}{4\pi\alpha'}\int_\Sigma \mathrm{d}^2\sigma \Big[ \left(\tilde{g}_{\theta\theta}\partial_\alpha\tilde{\theta}\partial_\beta\tilde{\theta} + 2\tilde{g}_{\theta i}\partial_\alpha\tilde{\theta}\partial_\beta X^j + \tilde{g}_{ij}\partial_\alpha X^i\partial_\beta X^j\right)\eta^{\alpha\beta} \\ &\quad + \left(2\tilde{B}_{i\theta}\partial_\alpha X^i\partial_\beta\tilde{\theta} + B_{ij}\partial_\alpha X^i\partial_\beta X^j\right)\epsilon^{\alpha\beta}\Big] - \int_{\partial\Sigma}\mathrm{d}\sigma^0 A_i\partial_{\sigma^0}X^i\,,\end{aligned} \tag{6.6}$$

with

$$\begin{aligned}&\tilde{g}_{\theta\theta} = {g_{\theta\theta}}^{-1}\,, \quad \tilde{g}_{\theta i} = {g_{\theta\theta}}^{-1}B_{\theta i}\,, \quad \tilde{B}_{\theta i} = {g_{\theta\theta}}^{-1}g_{\theta i}\,, \\ &\tilde{g}_{ij} = g_{ij} - {g_{\theta\theta}}^{-1}\left(g_{\theta i}g_{\theta j} - B_{\theta i}B_{\theta j}\right)\,, \quad \tilde{B}_{ij} = B_{ij} - {g_{\theta\theta}}^{-1}\left(g_{\theta i}B_{\theta j} - B_{\theta i}g_{\theta j}\right)\,.\end{aligned} \tag{6.7}$$

The relations (6.7) for the metric and B-field components between the dual theories are called the Buscher rules. Notice that in (6.6) the gauge field $A$ lost its component in the T-dualized direction and in order to perserve the gauge symmetry of the B-field, open string have no longer N boundary conditions in that direction. This consequently means that a D$p$-brane containing the gauge field $A$ on its volume and extending in the direction $\theta$ becomes, by T-dualizing $\theta$, a D$(p-1)$-brane, localised at a point in the direction $\tilde{\theta}$. On the other hand, since T-dualizing twice the same direction is a trivival transformation, T-dualizing a transverse direction of a D$p$-brane, needs to lead in the dual picture to a D$(p+1)$-brane, which fills out the T-dualized direction. In section 3.2.3 T-duality for circle compactification was introduces. T-duality exchanges N with D boundary conditions and vice versa while inverting the radius. In general the flipping of open string boundary holds in T-duality holds due to (6.5). For the case where the direction $\mu = 9$ of the ten dimensional Minkowski space $\mathbb{R}^{1,9}$ is compactified on a $S^1$ with radius $R$, the corresponding metric is given by $g_{\mu\nu} = diag(-1, 1, ..., 2\pi R)$. Applying the Buscher rule $\tilde{g}_{\theta\theta} = g^{-1}_{\theta\theta}$ one receives the metric $\tilde{g}_{\mu\nu} = diag(-1, , 1, ..., 2\pi\frac{\alpha'}{R})$, with $\alpha' = (2\pi)^{-2}$, which leads in the dual theory to an $S^1$ with radius $\frac{\alpha'}{R}$. Hence the Buscher rules are indeed a more general formulation for what was encountered in section 3.2.3.

### 6.1.2 T-dualizing $T^6_{\mathrm{SO(12)}}$ with D6-branes

#### Dual of $T^6_{\mathrm{SO(12)}}$

In section 4.1.1 deformations of $T^6_{\mathrm{SO(12)}}$, where described by turning on the deformation parameters $\tau_h$ and $R_h$. The deformations where chosen, in such a way that the hermitian metric remained diagonal and line element is given by

$$\mathrm{d}^2 s = \sum_{h=1}^{3} {R_h}^2 |1-\tau_h|^2 |\mathrm{d}w_h|^2 \,, \tag{6.8}$$

with the complex coordinates

$$w_1 = y_1 + \frac{\tau_1 y_2}{1-\tau_1}, \quad w_2 = y_3 - \frac{y_2}{1-\tau_2} + \frac{\tau_2 y_4}{1-\tau_2}, \quad w_3 = y_5 - \frac{y_4}{1-\tau_3} + \frac{1+\tau_3}{1-\tau_3} y_6 \,. \tag{6.9}$$

Now a B-field is included, which contains non vanishing components for each complex plane individually, s.t. it is given by [1]

$$B = \frac{i}{2} \sum_{h=1}^{3} {R_h}^2 |1-\tau_h|^2 B_h \mathrm{d}w_h \wedge \mathrm{d}\overline{w}_h \,. \tag{6.10}$$

The Kähler 2-from, deduced from (4.10) and expressed in the lattice basis is given by

$$\omega_2 = \sum_{h=1}^{3} {R_h}^2 |1-\tau_h|^2 \mathrm{d}w_h \wedge \mathrm{d}\overline{w}_h = \sum_{h=1}^{3} {R_h}^2 |1-\tau_h|^2 \mathrm{Im}(\tau_h)\, \mathrm{Re}(\mathrm{d}w_h) \wedge \mathrm{Im}(\mathrm{d}w_h) \,, \tag{6.11}$$

s.t. the complexified Kähler from becomes

$$\omega_2 = \sum_{h=1}^{3} \left\{ {R_h}^2 |1-\tau_h|^2 B_h + i{R_h}^2 |1-\tau_h|^2 \mathrm{Im}(\tau_h) \right\} \mathrm{Re}(\mathrm{d}w_h) \wedge \mathrm{Im}(\mathrm{d}w_h) \,. \tag{6.12}$$

A T-duality transformation in each plane individually is performed, by T-dualizing the 1-cycles, which are generated by

$$y_1 \to y_1 + 1 \,, \quad y_3 \to y_3 + 1 \,, \quad \text{and} \quad y_5 \to y_5 + 1 \,. \tag{6.13}$$

That way each T-duality transformation does not affect the other planes. The line element obtained by applying the Buscher rules is

$$\mathrm{d}^2 s = \sum_{h=1}^{3} \frac{|\mathrm{d}w_h|^2}{{R_h}^2 |1-\tau_h|^2} \,, \tag{6.14}$$

and the B-field becomes

$$\begin{aligned} B \;=\; & \frac{i}{2} \frac{\mathrm{Re}(\tilde\tau_1) - |\tau_1|^2}{|1-\tilde\tau_1|^2} \frac{\mathrm{d}w_1 \wedge \mathrm{d}\overline{w}_1}{\mathrm{Im}(\tilde\tau_1)} + \frac{i}{2} \frac{1-\mathrm{Re}(\tilde\tau_2)}{|1-\tilde\tau_2|^2} \frac{\mathrm{d}w_2 \wedge \mathrm{d}\overline{w}_2}{\mathrm{Im}(\tilde\tau_2)} \\ & + \frac{i}{2} \frac{1-\mathrm{Re}(\tilde\tau_3)}{|1-\tilde\tau_3|^2} \frac{\mathrm{d}w_3 \wedge \mathrm{d}\overline{w}_3}{\mathrm{Im}(\tilde\tau_3)} - \mathrm{d}y_3 \wedge \mathrm{d}y_4 - \mathrm{d}y_5 \wedge \mathrm{d}y_6 \,, \end{aligned} \tag{6.15}$$

[1] Expressing the B-field in orthonormal coordinates $(x_1, ..., x_6)$, it takes the simple form $B = \sum_{h=1}^{3} B_h \mathrm{d}x_{2h-1} \wedge \mathrm{d}x_{2h}$.

for the dual torus $\tilde{T}^6_{\mathrm{SO}(12)}$, where the complex coordinates are defined by

$$w_1 = y_1 + \tilde{\tau}_1 y_2\,, \quad w_2 = y_3 + \tilde{\tau}_2\,(y_4 - y_2)\,, \quad w_3 = y_5 + \tilde{\tau}_3\,(2y_6 - y_4)\,, \tag{6.16}$$

with the complex structure moduli $\tilde{\tau}_h$ defined by

$$\tilde{\tau}_h = B_h + i{R_h}^2 \mathrm{Im}(\tau_h)\,. \tag{6.17}$$

Comparing the moduli of $T^6_{\mathrm{SO}(12)}$ with $\tilde{T}^6_{\mathrm{SO}(12)}$, one observes, that the components of the hermitian metric have been inverted and the complex structure moduli has been exchanged by the complexified Kähler moduli. In the following of this chapter let the six dimensional torus $T^6$ be described by the line element

$$\mathrm{d}^2 s = \sum_{h=1}^{3} {A_h}^2 |\mathrm{d}w_h|^2\,, \tag{6.18}$$

with the complex coordinates defined by

$$w_1 = y_1 + K_1 y_2\,, \quad w_2 = y_3 + K_2(y_4 - y_2)\,, \quad w_3 = y_5 + K_3(2y_6 - y_4)\,. \tag{6.19}$$

Choosing the deformation parameters by $A_h^2 = {R_h}^{-2}|1-\tau_h|^{-2}$ and $K_h = B_h + i{R_h}^2\mathrm{Im}(\tau_h)$ one receives the dual torus of $T^6_{\mathrm{SO}(12)}$ in (6.14).

### Magnetic flux on $\tilde{T}^6_{\mathrm{SO}(12)}$

In this section the T-duality transformations along the directions, given in (6.13), and its effect on the open string sector are investigated. A D6-brane wrapping a sLag cycle $\Pi^3$ on $T^6_{\mathrm{SO}(12)}$, with the wrapping numbers

$$\Pi^3 = (n^1, m^1) \times (n^2, m^2) \times (n^3, m^3)\,, \tag{6.20}$$

is included. The cycle is parametrized in the covering space $\mathbb{R}^6$ by the equations

$$x_{2h} = \frac{m^h}{n^h} x_{2h-1}\,, \quad h \in \{1,2,3\} \tag{6.21}$$

and by performing a change of basis into the lattice basis, one gets

$$y_1 = \frac{n^1}{N^{(1)}} y_2\,, \quad y_3 = \frac{m^2}{N^{(2)}} y_2 + \frac{n^2}{N^{(2)}} y_4\,, \quad y_5 = \frac{m^3}{N^{(3)}} y_4 + \frac{n^3 - m^3}{N^{(3)}} y_6\,, \tag{6.22}$$

where we defined $N^{(h)} = n^h + m^h$. The parametrizations describe N boundary conditions for open strings, because the lines describe the loci in $\mathbb{R}^6$ on which open strirngs can propagat freely. As described in section 6.1.1 N and D boundary conditions are exchanged and the T-dualized directions $y_1$, $y_3$ and $y_5$, correspond in the dual picture to gauge field components. The gauge field corresponding to the boundary conditions in (6.22) is according to (6.5) given by

$$A_1 = -2\pi \frac{n^1}{N^{(1)}} y_2\,, \quad A_3 = -2\pi \frac{m^2 y_2 + n^2 y_4}{N^{(2)}}\,, \quad A_5 = -2\pi \frac{m^3 y_4 + (n^3 - m^3) y_6}{N^{(3)}}\,, \tag{6.23}$$

where $\alpha' = (2\pi)^{-2}$ was set. A 1-form gauge field with the components given in (6.23) can be deduced from a 2-form field strength given by

$$
\begin{aligned}
F &= \frac{i\pi}{\mathrm{Im}(\tilde{\tau}_1)}\frac{n^1}{N^{(1)}}\mathrm{d}w_1 \wedge \mathrm{d}\overline{w}_1 + \frac{i\pi}{\mathrm{Im}(\tilde{\tau}_2)}\frac{m^2}{N^{(2)}}\mathrm{d}w_2 \wedge \mathrm{d}\overline{w}_2 \\
&\quad + \frac{i\pi}{\mathrm{Im}(\tilde{\tau}_3)}\frac{m^3}{N^{(3)}}\mathrm{d}w_3 \wedge \mathrm{d}\overline{w}_3 + 2\pi \mathrm{d}y_3 \wedge \mathrm{d}y_4 + 2\pi \mathrm{d}y_5 \wedge \mathrm{d}y_6 \,.
\end{aligned} \tag{6.24}
$$

Together with the B-field, the gauge invariant fieldstrength defined in (3.40) can be computed:

$$
\begin{aligned}
\mathcal{F} &= \frac{i\pi}{\mathrm{Im}(\tilde{\tau}_1)}\left(\frac{\mathrm{Re}(\tilde{\tau}_1) - |\tau_1|^2}{|1-\tilde{\tau}_1|^2} + \frac{n^1}{N^{(1)}}\right)\mathrm{d}w_1 \wedge \mathrm{d}\overline{w}_1 \\
&\quad + \frac{i\pi}{\mathrm{Im}(\tilde{\tau}_2)}\left(\frac{1-\mathrm{Re}(\tilde{\tau}_2)}{|1-\tilde{\tau}_2|^2} + \frac{m^2}{N^{(2)}}\right)\mathrm{d}w_2 \wedge \mathrm{d}\overline{w}_2 \\
&\quad + \frac{i\pi}{\mathrm{Im}(\tilde{\tau}_3)}\left(\frac{1-\mathrm{Re}(\tilde{\tau}_3)}{|1-\tilde{\tau}_3|^2} + \frac{m^3}{N^{(3)}}\right)\mathrm{d}w_3 \wedge \mathrm{d}\overline{w}_3 \,.
\end{aligned} \tag{6.25}
$$

The terms proportional to $\mathrm{d}y_3 \wedge \mathrm{d}y_4$ and $\mathrm{d}y_5 \wedge \mathrm{d}y_6$ in $F$ and $B$ drop out in $\mathcal{F}$, which means that a simulataneous gauge transformation on $B$ and $F$ can be performed, which brings the field strength tensor into the holomorphic form

$$
\frac{F}{i\pi} = \frac{n^1}{N^{(1)}}\frac{\mathrm{d}w_1 \wedge \mathrm{d}\overline{w}_1}{\mathrm{Im}(\tilde{\tau}_1)} + \frac{m^2}{N^{(2)}}\frac{\mathrm{d}w_2 \wedge \mathrm{d}\overline{w}_2}{\mathrm{Im}(\tilde{\tau}_2)} + \frac{m^3}{N^{(3)}}\frac{\mathrm{d}w_3 \wedge \mathrm{d}\overline{w}_3}{\mathrm{Im}(\tilde{\tau}_3)} \,. \tag{6.26}
$$

The absence of $(2,0)$- and $(0,2)$-forms in $F$ confirms that the D9-brane, dual to the D6-brane, wrapping the cycle (6.20), preserves superymmetry and wraps calibrated cycles on the torus [87, 116]. The sLag condition $\omega_2\big|_{\Pi^3} = 0$ for D6-brane cycles is translated to the condition that $F^{(2,0)} = F^{(0,2)} = 0$ for D9-brane fluxes. The gauge symmetry living on the D9-branes is a $U(N_{\mathrm{D9}})$ symmetry, with $N_{\mathrm{D9}}$ denoting the wrapping number of the D9-brane around the torus. It is consitent to asume that $N_{\mathrm{D9}}$ is equal to the intersections of the corresponding D6-brane in the dual theory with the T-dualized cycle. Let the D6-brane wrap the cycle $\Pi^3$ $N_{\mathrm{D6}}$ times, which means the gauge symmetry on the D6-brane volume is $U(N_{\mathrm{D6}})$. Then, incorporating the multiplicity from $\Pi^3$ wrapping the cycle $(1,-1)\times(1,-1)\times(1,-1)$ [2], the wrapping number $N_{\mathrm{D9}}$ is related to the wrapping number $N_{\mathrm{D6}}$ by

$$
N_{\mathrm{D9}} = \frac{N^{(1)}N^{(2)}N^{(3)}}{2}N_{\mathrm{D6}} =: N N_{\mathrm{D6}} \,, \tag{6.27}
$$

which implies that the gauge symmetry on thr D9-brane is $U(NN_{\mathrm{D6}})$. At first sight this seems like a different gauge symmetry arises on the dual theory, but Wilson lines break the gauge symmetry to

$$
U(N_{\mathrm{D9}}) \to U(N_{\mathrm{D6}}) \,, \tag{6.28}
$$

as will be explained in the following.

[2] Remember that the intersection number $I_{ab}$ of two cycles $\Pi^3_a = \prod_{h=1}^3 (n^h_a, m^h_a)$ and $\Pi^3_b = \ prod_{h=1}^3 (n^h_b, m^h_b)$ on $T^6_{\mathrm{SO(12)}}$ is given by $I_{ab} = \frac{1}{2}\prod_{h=1}^3 (n^h_a m^h_b - n^h_b m^h_a)$.

## 6.2 Magnetic fluxes on the dual $T^6_{\mathbf{SO(12)}}$

### 6.2.1 Symmetry breaking via magnetic fluxes

In this section it is explained how magnetic fluxes break the gauge symmetry on the D9-branes. Here the notation in appendix A of [115] is partly followed. Let a $U(P)$ gauge theory be described by the Lagrangian

$$\mathcal{L} = -\frac{1}{4g^2}\mathrm{Tr}\left\{F^{ij}F_{ij}\right\}, \tag{6.29}$$

where $F_{ij}$ is the corresponding field strength tensor and the trace is taken over the gauge indices. A basis for matrices in the adjoint representation is given by the set of $P$ matrices $U_a = \delta_{ai}\delta_{aj}$ and $e_{ab} = \delta_{ai}\delta_{bj}$, where $U_a$ form a basis for the diagonal elements and $e_{ab} = \delta_{ai}\delta_{bj}$, form a basis for the off diagonal elements. A field $A$ in the adjoint representation of $U(P)$ can then be expanded by

$$A = B^a U_a + W^{ab} e_{ab}\,. \tag{6.30}$$

Magnetic fluxes considered here, arise from giving the $B^a$ components a vev's, s.t. the corresponding fieldstrength is constant. The vev breaks the initial gauge group $U(P)$ to a subgroup of $U(P)$, which is generated by elements, commuting with the flux. Let the wrapping number of the $\prod_\alpha N_\alpha N_{\mathrm{D6}}$ D9-branes on the torus be $P = \sum_{\alpha\in\{a,b,c...\}} N_\alpha N_{\mathrm{D6}_\alpha}$. A vev is turned on, which is given by

$$\langle A_0\rangle = diag(\underbrace{m_a, m_a, ..., m_a}_{N_a N_{\mathrm{D6}_a}\text{ times}}, \underbrace{m_b, ..., m_b}_{N_b N_{\mathrm{D6}_b}\text{ times}}, \underbrace{m_c, ..., m_c}_{Nc N_{\mathrm{D6}_c}\text{ times}}, ...)\,. \tag{6.31}$$

Identifying the generators of $U(P)$, which commute with $\langle A_0\rangle$, one finds that the gauge symmetry is broken to

$$U(P) \xrightarrow{\langle A_0\rangle} \prod_\alpha U(N_\alpha N_{\mathrm{D6}_\alpha})\,. \tag{6.32}$$

In the following discussions, the wrapping number $N_{\mathrm{D6}}$ is set to $N_{\mathrm{D6}} = 1$, that means, only $U(1)$'s symmetries on the T-dual side are considered. As described in section 6.1.2, D6-branes with angles lead to the constant fluxes of the form

$$F = \sum_{h=1}^{3} F_{w_h\bar{w}_h} \mathrm{d}w_h \wedge \mathrm{d}\bar{w}_h\,, \tag{6.33}$$

with

$$F_{w_1\bar{w}_1} = \frac{i\pi}{\mathrm{Im}(\tilde{\tau}_1)}\begin{pmatrix} \frac{n_a^1}{N_a^{(1)}}\mathbb{1}_{N_a} & & \\ & \frac{n_b^1}{N_b^{(1)}}\mathbb{1}_{N_b} & \\ & & \ddots \end{pmatrix}, \qquad \text{and} \tag{6.34}$$

$$F^\alpha_{w_h\bar{w}_h} = \frac{i\pi}{\mathrm{Im}(\tilde{\tau}_h)}\begin{pmatrix} \frac{m_a^h}{N_a^{(h)}}\mathbb{1}_{N_a} & & \\ & \frac{m_b^h}{N_b^{(h)}}\mathbb{1}_{N_b} & \\ & & \ddots \end{pmatrix}, \quad h \in \{2,3\}\,.$$

The corresponding adjoint field is

$$A(\vec{w}) = diag(\underbrace{A_a, ..., A_a}_{N_a \text{ times}}, \underbrace{A_b, ..., A_b}_{N_b \text{ times}}, ...)\,, \tag{6.35}$$

with

$$A_\alpha = \frac{\pi}{\mathrm{Im}(\tilde{\tau}_1)} \frac{n^1_\alpha}{N^{(1)}_\alpha} \mathrm{Im}(\overline{w}_1 \mathrm{d}w_1) + \frac{\pi}{\mathrm{Im}(\tilde{\tau}_2)} \frac{m^2_\alpha}{N^{(2)}_\alpha} \mathrm{Im}(\overline{w}_2 \mathrm{d}w_2) + \frac{\pi}{\mathrm{Im}(\tilde{\tau}_3)} \frac{m^3_\alpha}{N^{(3)}_\alpha} \mathrm{Im}(\overline{w}_3 \mathrm{d}w_3)\,. \tag{6.36}$$

Since $A$ has the same diagonal form as in (6.31), it breaks the $U(P)$ symmetry analogous to (6.32). The unbroken gauge group in the presence of the flux (6.33) is $\prod_\alpha U(N_\alpha) \subset U(P)$.

### 6.2.2 Wilson loops and quantization condition

Next the effect of the compactification lattice on the gauge theory is investigated. D9-branes are placed into $T^6$ wrapping the torus $P$ times. Thus the $T^6$ contains a $U(P)$ gauge symmetry arising from the gauge symmetry on the spacetimefilling branes. By turning on magnetic fluxes, given in (6.33), the gauge symmetry on the branes is broken to $\prod_\alpha U(N_\alpha)$ as explained in the above section. In the following the isomorphism [3]

$$U(n) \simeq \frac{SU(n) \times U(1)}{\mathbb{Z}_n}\,, \tag{6.37}$$

is used to decompose $U(N_\alpha)$ elements into $SU(N_\alpha)$ elements and a $U(1)$ factor. Let $\phi_\alpha$ be a field living on the torus and transforming in the fundamental representation of $U(N_\alpha)$

$$\phi_\alpha :\; T^6 \to \mathbb{C}^{N_\alpha}\,. \tag{6.38}$$

Under translations along the torus, $\phi_\alpha$ transforms in the gauge representation as

$$\phi_\alpha{}^a(x) \to \phi_\alpha{}^a(x') = \exp\left( iq \int_x^{x'} A_\alpha(x) \cdot \mathbb{1}_{N_\alpha} \right) (W_\alpha)_{ab}\, \phi_\alpha{}^b(x)\,, \tag{6.39}$$

where $W_\alpha \in SU(N_\alpha)$ and $a, b \in \{1, ..., N_\alpha\}$ are gauge indices[4]. Since the transformations $w_i \to w_i + 1$ generate isometries on $T^6$, six independent Wilson loops naturally arise on the $T^6$. $\phi$ transforms under the isometries as:

$$\phi_\alpha(..., y_i + 1, ...) = \mathrm{e}^{iq\chi^\alpha_i(...,y_i,...)} \omega^\alpha_i \phi_\alpha(..., y_i, ...)\,, \tag{6.40}$$

[3] Let $M \in SU(n)$ and $\mathrm{e}^{i\varphi} \in U(1)$. Then $K = \mathrm{e}^{i\varphi} M \in U(n)$. But $\tilde{M} = \mathrm{e}^{i\frac{k}{n}} M \in SU(n)$, for $\frac{k}{2\pi} \in \mathbb{Z}$, because $\det\left(\mathrm{e}^{i\frac{k}{n}} M\right) = \mathrm{e}^{ik} \underbrace{\det(M)}_{=1}$. Then also $\mathrm{e}^{i(\varphi - k/n)} \tilde{M} = K$. Since $(\mathrm{e}^{i\frac{k}{n}})^n$, the factors $\mathrm{e}^{i\frac{k}{n}}$ can be seen as $\mathbb{Z}_n$ elements and hence $U(1) \times SU(n)$ contains a $\mathbb{Z}_n$ factor. Quotienting the $\mathbb{Z}_n$ factor out lead to the isomorphism (6.37) .

[4] In the language of gauge bundles, (6.39) describes the transformation in the fibre, when moved along the base space.

with

$$\begin{aligned}
\chi_1^\alpha(\vec{w}) &= \frac{\pi n_\alpha^1}{N_\alpha^{(1)}}\frac{\mathrm{Im}(w_1)}{\mathrm{Im}(K_1)}\,, \quad \chi_2^\alpha(\vec{w}) = \frac{\pi n_\alpha^1}{N_\alpha^{(1)}}\frac{\mathrm{Im}(\overline{K}_1 w_1)}{\mathrm{Im}(K_1)} - \frac{\pi m_\alpha^2}{N_\alpha^{(2)}}\frac{\mathrm{Im}(\overline{K}_2 w_2)}{\mathrm{Im}(K_2)}\,, \\
\chi_3^\alpha(\vec{w}) &= \frac{\pi m_\alpha^2}{N_\alpha^{(2)}}\frac{\mathrm{Im}(w_2)}{\mathrm{Im}(K_2)}\,, \quad \chi_4^\alpha(\vec{w}) = \frac{\pi m_\alpha^2}{N_\alpha^{(2)}}\frac{\mathrm{Im}(\overline{K}_2 w_2)}{\mathrm{Im}(K_1)} - \frac{\pi m_\alpha^3}{N_\alpha^{(3)}}\frac{\mathrm{Im}(\overline{K}_3 w_3)}{\mathrm{Im}(K_3)}\,, \\
\chi_5^\alpha(\vec{w}) &= \frac{\pi m_\alpha^3}{N_\alpha^{(3)}}\frac{\mathrm{Im}(w_3)}{\mathrm{Im}(K_3)}\,, \quad \chi_6^\alpha(\vec{w}) = \frac{2\pi m_\alpha^3}{N_\alpha^{(3)}}\frac{\mathrm{Im}(\overline{K}_3 w_3)}{\mathrm{Im}(K_3)}\,.
\end{aligned} \tag{6.41}$$

The $SU(N_\alpha)$ factors $\omega_i$ are yet undetermined, but the Dirac quantization condition provide constraints on them[115]. Let $\gamma$ be a closed loop wrapping a trivial 1-cycle on a non trivial 2-cycle of the torus, s.t. the 2-cycle is seperated into two surfaces and $\gamma$ is the boundary of both surfaces. The quantization condition demands that the gauge transformations of $\phi$, after its moved along $\gamma$, is trivial [117]. Non trivial 2-cycles on the torus are generated by two lattice vectors: Let $\Pi^2_{ij} \in H_2(T^6)$ be the 2-cycle generated by the two generators of the underlying lattice $\vec{\alpha}_i$ and $\vec{\alpha}_j$. Then a closed contractible loop on $\Pi^2_{ij}$ is generated by the following translation

$$\gamma:\ (y_i, y_j) \xrightarrow{+\vec{\alpha}_i} (y_i, +1 y_j) \xrightarrow{+\vec{\alpha}_j} (y_i+1, y_j+1) \xrightarrow{-\vec{\alpha}_i} (y_i, y_j+1) \xrightarrow{-\vec{\alpha}_j} (y_i, y_j)\,. \tag{6.42}$$

Moving the field $\phi$ along $\gamma$, leads to a gauge transformation, which has to be trivial according to the quantization condition. That means the parameters for the Wilson lines need to satisfy

$$\mathrm{e}^{iq\chi_{ij}^\alpha}\left(\omega_j^\alpha\right)^{-1}\left(\omega_i^\alpha\right)^{-1}\omega_j^\alpha\omega_i^\alpha\phi(\vec{w}) = \phi(\vec{w})\,, \tag{6.43}$$

with

$$\begin{aligned}
\chi_{12}^\alpha &= -2\pi\frac{n_\alpha^1}{N_\alpha^{(1)}}\,, \quad \chi_{32}^\alpha = 2\pi\frac{m_\alpha^2}{N_\alpha^{(2)}}\,, \quad \chi_{34}^\alpha = -2\pi\frac{m_\alpha^2}{N_\alpha^{(2)}}\,, \\
\chi_{54}^\alpha &= 2\pi\frac{m^3}{N^{(3)}}\,, \quad \chi_{56} = -4\pi\frac{m_\alpha^3}{N_\alpha^{(3)}}\,,
\end{aligned} \tag{6.44}$$

and all other $\chi_{ij}^\alpha = 0$.

#### Solving the quantization condition

The solutions to (6.43) lead to expressions for $\omega_i$'s in the Wilson lines. It turns out that the solutions to the following two dimensional problem is helpful: Let $\omega_1$, $\omega_2 \in SU(N)$, with $N \in \mathbb{Z}$ and

$$(\omega_2)^{-1}(\omega_1)^{-1}\omega_2\omega_1 = \mathrm{e}^{2\pi i\frac{k}{N}}\mathbb{1}_N\,, \tag{6.45}$$

with

$$M = k \mod N\,, \qquad M \in \mathbb{Z}\,. \tag{6.46}$$

Solutions to $\omega_1$ and $\omega_2$ are given by [115]

$$\omega_1 = Q^M\,, \quad \omega_2 = P \tag{6.47}$$

with

$$Q = \begin{pmatrix} 1 & & & \\ & \mathrm{e}^{2\pi i/N} & & \\ & & \ddots & \\ & & & \mathrm{e}^{2\pi i(N-1)/N} \end{pmatrix}, \quad P = \begin{pmatrix} & 1 & & \\ & & 1 & \\ & & & \ddots \\ 1 & & & \end{pmatrix}. \tag{6.48}$$

Reminding that $N_\alpha = \frac{1}{2}\prod_{h=1}^{3} N_\alpha^{(h)}$, (6.43) can be rewritten into a similar form as in (6.45)

$$\left(\omega_j^\alpha\right)^{-1}\left(\omega_i^\alpha\right)^{-1}\omega_j^\alpha\omega_i^\alpha = \mathrm{e}^{2\pi i k_{ij}^\alpha/N_\alpha}\cdot \mathbb{1}_{N_\alpha} \tag{6.49}$$

with

$$\begin{aligned} k_{12} &= \frac{n_\alpha^1}{2}N_\alpha^{(2)}N_\alpha^{(3)} \mod N_\alpha\,, \quad k_{32} = -\frac{m_\alpha^2}{2}N_\alpha^{(1)}N_\alpha^{(3)} \mod N_\alpha\,, \\ k_{34} &= \frac{m_\alpha^2}{2}N_\alpha^{(1)}N_\alpha^{(3)} \mod N_\alpha\,, \quad k_{54} = -\frac{m_\alpha^3}{2}N_\alpha^{(1)}N_\alpha^{(2)} \mod N_\alpha\,, \\ k_{56} &= m_\alpha^3 N_\alpha^{(1)}N_\alpha^{(2)} \mod N_\alpha\,. \end{aligned} \tag{6.50}$$

The values for $k_{ij}$ have to be integer. That means if $N_\alpha^{(h)} = 0 \mod 2$ is not satisfied for all planes $h \in \{1,2,3\}$, one needs to double $N_\alpha^{(h)}$ in one plane. This is exactly the condition (4.73) for 3-cycles to be closed on the $T^6_{\mathrm{SO}(12)}$, which should not be a surprise, because the magnetic flux was derived from D-branes wrapping closed 3-cycles on the $T^6_{\mathrm{SO}(12)}$. The two dimensional solutions (6.47) can be used to construct $SU(N_\alpha)$ factors solving (6.43): Let the $\omega$'s for the Wilson loops be given by

$$\begin{aligned} \omega_1 &= Q_{(1)}^{n_\alpha^1}\otimes \mathbb{1}_{N_\alpha^{(2)}}\otimes \mathbb{1}_{N_\alpha^{(3)}}\,, \quad \omega_2 = P_{(1)}\otimes P_{(2)}^{-1}\otimes \mathbb{1}_{N_\alpha^{(3)}}\,, \\ \omega_3 &= \mathbb{1}_{N_\alpha^{(1)}}\otimes Q_{(2)}^{m_\alpha^2}\otimes \mathbb{1}_{N_\alpha^{(3)}}\,, \quad \omega_4 = \mathbb{1}_{N_\alpha^{(1)}}\otimes P_{(2)}\otimes P_{(3)}^{-1}\,, \\ \omega_5 &= \mathbb{1}_{N_\alpha^{(1)}}\otimes \mathbb{1}_{N_\alpha^{(2)}}\otimes Q_{(3)}^{m_\alpha^3}\,, \quad \omega_6 = \mathbb{1}_{N_\alpha^{(1)}}\otimes \mathbb{1}_{N_\alpha^{(2)}}\otimes P_{(3)}^2\,, \end{aligned} \tag{6.51}$$

where the subscript $(h)$ of the matrices $P$ and $Q$ denote that they are $N_\alpha^{(h)}\times N_\alpha^{(h)}$ dimensional. The solutions in (6.51) solve the conditions in (6.49), but the matrices $\omega_i$ are $2N_\alpha\times 2N_\alpha$ dimensional, hence to big. A similar problem was already encountered, when the intersection points of two 3-cycles on the $T^6_{\mathrm{SO}(12)}$ where counted. There the number of projections of the intersection points onto the $h$-th plane is given by $I_{ab}^{(h)}$ and the product $\prod_{h=1}^{3} I_{ab}^{(h)}$ leads to two times the intersection number on $T^6_{\mathrm{SO}(12)}$. It was necessary to place the labels for the intersection points on a general three dimensional lattices. Applying the concept to the solutions in (6.51), the "tensoring" of the matrices has to be defined in a more general way: The components of $\omega_i$'s given by

$$\begin{aligned} (\omega_1)_{ij} &= Q^{n_\alpha^1}_{i^{(1)}j^{(1)}}\delta_{i^{(2)}j^{(2)}}\delta_{i^{(3)}j^{(3)}}\,, \quad (\omega_2)_{ij} = P_{i^{(1)}j^{(1)}}P^{-1}_{i^{(2)}j^{(2)}}\delta_{i^{(3)}j^{(3)}}\,, \\ (\omega_3)_{ij} &= \delta_{i^{(1)}j^{(1)}}Q^{m_\alpha^2}_{i^{(2)}j^{(2)}}\delta_{i^{(3)}j^{(3)}}\,, \quad (\omega_4)_{ij} = \delta_{i^{(1)}j^{(1)}}P_{i^{(2)}j^{(2)}}P^{-1}_{i^{(3)}j^{(3)}}\,, \\ (\omega_3)_{ij} &= \delta_{i^{(1)}j^{(1)}}\delta_{i^{(2)}j^{(2)}}Q^{m_\alpha^3}_{i^{(3)}j^{(3)}}\,, \quad (\omega_6)_{ij} = \delta_{i^{(1)}j^{(1)}}\delta_{i^{(2)}j^{(2)}}P^{2}_{i^{(3)}j^{(3)}}\,, \end{aligned} \tag{6.52}$$

are labeled by two triplets $i = (i^{(1)}, i^{(2)}, i^{(3)})^T$ and $j = (j^{(1)}, j^{(2)}, j^{(3)})^T$, which belong to a general three dimensional lattice $\Lambda_\alpha^3$. As already observed for the 3-cycles on $T^6_{\mathrm{SO}(12)}$, the wrapping numbers $N_\alpha^{(h)}$ can be

(i) even for all planes, with $g.c.d.(n^h_\alpha, m^h_\alpha) = 1$,

(ii) even for all planes, with $g.c.d.(n^h_\alpha, m^h_\alpha) = 2$ for one exactly one $h$ and in the other two planes $g.c.d.(n^h_\alpha, m^h_\alpha) = 1$,

(iii) odd in exactly one plane and $g.c.d.(n^h_\alpha, m^h_\alpha) = 2$ in exactly one plane, in which $N^{(h)}_\alpha$ is even. In the other planes $g.c.d.(n^{h_1}_\alpha, m^h_\alpha) = 1$ and

(iv) even for exactly one plane, where $g.c.d.(n^{h_1}_\alpha, m^h_\alpha) = 2$ in that plane. In the other two planes $N^{(h)}_\alpha$ is odd and $g.c.d.(n^h_\alpha, m^h_\alpha) = 1$.

It will turn out to be consistent to choose $\Lambda^3_\alpha$ as the following: For the $N^{(h)}_\alpha$'s given in (i) and (ii), $\Lambda^3_\alpha$ is given by

$$\Lambda^3_\alpha = \frac{\Lambda_{\mathrm{SO}(6)}}{\bigotimes_{h=1}^3 N^{(h)}_\alpha \mathbb{Z}}\,, \tag{6.53}$$

where for cases (iii) and (iv) it will have the more general form

$$\Lambda^3_\alpha = \frac{\Lambda_{\mathrm{SO}(6)}}{\Gamma_\alpha}\,, \tag{6.54}$$

where the lattice $\Gamma_\alpha$ depends on the wrapping numbers: Without loss of generality $N^{(2)}_\alpha = 1 \mod 2$ and $g.c.d.(n^1_\alpha, m^1_\alpha) = 2$ for case (iii) and $N^{(1)}_\alpha = 0 \mod 2$ and $g.c.d.(n^1_\alpha, m^1_\alpha) = 2$ for case (iv) is set. Then $\Gamma_\alpha$ is spanned by the basis vectors

$$\left(\frac{N^{(1)}_\alpha}{2}, N^{(2)}_\alpha, 0\right)^T\,, \quad \left(\frac{N^{(1)}_\alpha}{2}, -N^{(2)}_\alpha, 0\right)^T\,, \quad \left(0,\, 0,\, N^{(3)}_\alpha\right)^T \tag{6.55}$$

for case (iii) and

$$\left(\frac{N^{(1)}_\alpha}{2}, N^{(2)}_\alpha, 0\right)^T\,, \quad \left(\frac{N^{(1)}_\alpha}{2}, -N^{(2)}_\alpha, 0\right)^T\,, \quad \left(0,\, N^{(2)}_\alpha,\, N^{(3)}_\alpha\right)^T \tag{6.56}$$

for case (iv). For other cases of (iii) and (iv) the basis vectors has to be adjusted accordingly.

The Wilson lines break each $U(N_\alpha)$ gauge group to a $U(1)$ (or $U(N_\alpha N_{\mathrm{D6}})$ to $U(N_{\mathrm{D6}})$ for $N_{\mathrm{D6}} \neq 1$). To see that, one needs to observe how a gauge field transforms under the action of the Wilson lines: Let $A \in \mathbf{adj}(U(N_\alpha))$ be a gauge field of $U(N_\alpha)$. Translations along a lattice vectors $\vec{\alpha}_i$, transforms $A$ in the gauge representation as

$$A \to \omega_i A (\omega_i)^{-1}\,, \tag{6.57}$$

but due to the compactification $A$ has to be identified with $\omega_i A(\omega_i)^{-1}$, which projects $N_\alpha - 1$ degrees of freedom out of $A$, leaving only one degree of freedom left.

### 6.2.3 Boundary conditions for bifundamentals

In this section it is investigated how to deduce the boundary conditions from Wilson lines for bifundamental fields. Let $\phi \in (\square_a, \overline{\square}_b)$ be a field on the torus, transforming in the bifundamental representation of $U(N_a) \times U(N_b)$:

$$\phi:\, T^6 \to \mathbb{C}^{N_a \times N_b}\,. \tag{6.58}$$

In the gauge representation $\phi$ is a $N_a \times N_b$ matrix and its components are denote by $\phi_{k_a,k_b}$, where $k_a$ and $k_b$ are triplets of integers belonging to $\Lambda^3_a$ and $\Lambda^3_b$ as defined in (6.53) and (6.54). Shifting $\phi$ on the torus along the lattice vectors $\vec{\alpha}_i$, the components of $\phi$ transform due to the Wilson lines as the following:

$$\begin{aligned}
\phi_{k_a,k_b}(y_1+1,...) &= e^{i\pi\tilde{I}^{(1)}_{ab}\frac{\mathrm{Im}(w_1)}{\mathrm{Im}(K_1)}} e^{2\pi i\left(k_a^{(1)}\frac{n_a^1}{N_a^{(1)}}-k_b^{(1)}\frac{n_b^1}{N_b^{(1)}}\right)}\phi_{k_a,k_b}(y_1,...)\,, \\
\phi_{k_a,k_b}(...,y_2+1,...) &= e^{i\pi\left(\tilde{I}^{(1)}_{ab}\frac{\mathrm{Im}(\overline{K}_1 w_1)}{\mathrm{Im}(K_1)}-\tilde{I}^{(2)}_{ab}\frac{\mathrm{Im}(\overline{K}_2 w_2)}{\mathrm{Im}(K_2)}\right)}\phi_{k'_a,k'_b}(...,y_2,...)\,, \\
\phi_{k_a,k_b}(...,y_3+1,...) &= e^{i\pi\tilde{I}^{(2)}_{ab}\frac{\mathrm{Im}(w_2)}{\mathrm{Im}(K_2)}} e^{-2\pi i\left(k_a^{(2)}\frac{m_a^2}{N_a^{(2)}}-k_b^{(2)}\frac{m_b^2}{N_b^{(2)}}\right)}\phi_{k_a,k_b}(...,y_3,...)\,, \\
\phi_{k_a,k_b}(...,y_4+1,...) &= e^{i\pi\left(\tilde{I}^{(2)}_{ab}\frac{\mathrm{Im}(\overline{K}_2 w_2)}{\mathrm{Im}(K_2)}-\tilde{I}^{(3)}_{ab}\frac{\mathrm{Im}(\overline{K}_3 w_3)}{\mathrm{Im}(K_3)}\right)}\phi_{k''_a,k''_b}(...,y_4,...)\,, \\
\phi_{k_a,k_b}(...,y_5+1,...) &= e^{i\pi\tilde{I}^{(3)}_{ab}\frac{\mathrm{Im}(w_3)}{\mathrm{Im}(K_3)}} e^{-2\pi i\left(k_a^{(3)}\frac{m_a^3}{N_a^{(3)}}-k_b^{(3)}\frac{m_b^3}{N_b^{(3)}}\right)}\phi_{k_a,k_b}(...,y_5,...)\,, \\
\phi_{k_a,k_b}(...,y_6+1) &= e^{2i\pi\tilde{I}^{(3)}_{ab}\frac{\mathrm{Im}(\overline{K}_3 w_3)}{\mathrm{Im}(K_3)}}\phi_{k'''_a,k'''_b}(...,y_6)\,,
\end{aligned} \tag{6.59}$$

where $\tilde{I}^{(h)}_{ab} = I^{(h)}/N_a^{(h)}N_b^{(h)}$, $k'_\alpha = k_\alpha + (1,-1,0)^T$, $k''_\alpha = k_\alpha + (0,1,-1)^T$ and $k'''_\alpha = k_\alpha + (0,0,2)^T$. One can observe that the Wilson lines shift the indices $k_a$ and $k_b$ by $SO(6)$ root vectors and since there are in total $N_\alpha$ inequivalent indices for $k_a$ and $N_b$ inequivalent indices for $k_b$, the assumption

$$k_a \in \Lambda_a^3\,, \quad k_b \in \Lambda_b^3\,, \tag{6.60}$$

with $\Lambda_a^3$ and $\Lambda_b^3$ defined in (6.53) and (6.54) is consistent.

In [115] bifundamentals of $U(N_a)\times U(N_b)$, with $g.c.d.(N_a,N_b) = 1$ on a factorisbale $T^6$ where discussed and the components of the bifundamentals where labeled by an single index $l$. In the following it is shown that bifundamentals on non factorisable $T^6$ can be labeled in a similar way by two labels, which are denoted by $l$ and $\delta$ and belonging to general three dimensional lattices.

#### Labeling bifundamental states with non coprime $(N_a, N_b)$ on a $T^2$

Before investigating bifundamentals on a $T^6$, they are examined on a $T^2$. A bifundamental field $\phi \in (\Box_a, \overline{\Box}_b)$ on a $T^2$ has in the gauge representation $N_a \times N_b$ components $\phi_{k_a,k_b}$, where the indices $k_a$ and $k_b$ are given by

$$k_a \in \{0,1,...,N_a-1\}\,, \quad \text{and} \quad k_b \in \{0,1,...,N_b-1\}\,. \tag{6.61}$$

In [115], each component was labeled by one index $l$, where

$$l \in \{0,1,...,N_aN_b-1\}\,, \quad \text{with} \quad k_\alpha = l \mod N_\alpha\,, \quad \alpha\in\{a,b\}\,, \tag{6.62}$$

s.t. $\phi_{k_a,k_b} \sim \phi_{l,l}$. This works as long as $g.c.d.(N_a,N_b) = 1$. Let $g.c.d.(N_a,N_b) = d$. Then the equation

$$k_a + sN_a = k_b + tN_b = l\,, \quad \text{with} \quad s,t\in\mathbb{Z}\,, \tag{6.63}$$

can only be solved when $k_a - k_b = 0 \mod d$. For

$$k_a - k_b = \delta \mod d\,, \tag{6.64}$$

with $\delta \in \{1,...,d-1\}$, however (6.63) can be modified by

$$k_a + sN_a = k_b + \delta + tN_b = l\,, \tag{6.65}$$

which is solvable for any given pair of indices $(k_a, k_b)$ and they can be assigned to two indices $(\delta, l)$ by

$$k_a = l \mod N_a\,, \quad k_b + \delta = l \mod N_b\,, \quad \text{for} \quad k_a - k_b = \delta \mod d\,, \tag{6.66}$$

with

$$l \in \left\{0, 1, ..., \frac{N_a N_b}{d}\right\}\,, \quad \delta \in \{0, 1, ..., d-1\}\,. \tag{6.67}$$

Wilson lines on the $T^2$ lead, similar to (6.59), to the boundary conditions for bifundamentals given by

$$\begin{aligned} \phi_{l,l-\delta}(w+1) &= \mathrm{e}^{i\pi \tilde{I}_{ab} \frac{\mathrm{Im}(w)}{\mathrm{Im}(\tau)}} \mathrm{e}^{2\pi i\left(\frac{n_a k_a}{N_a} - \frac{n_b k_b}{N_b}\right)} \phi_{l,l-\delta}(w)\,, \\ \phi_{l,l-\delta}(w+\tau) &= \mathrm{e}^{i\pi \tilde{I}_{ab} \frac{\mathrm{Im}(\bar{\tau} w)}{\mathrm{Im}(\tau)}} \phi_{l+1,l-\delta+1}(w)\,, \end{aligned} \tag{6.68}$$

where $w = y_1 + \tau y_2$, with $\tau$ the complex structure modulus for $T^2$. Since $I_{ab}$ is a multiple of $d$ [5], the following translation leads to a trivial phase for $\phi$:

$$\phi_{l,l-\delta}(y_1, y_2 + N_a N_b/d) = \phi_{l+\frac{N_a N_b}{d}, l-\delta+\frac{N_a N_b}{d}}(y_1, y_2)\,. \tag{6.69}$$

Notice that the boundary conditions in (6.68) relates only components with the same value for $\delta$. That motivates to view $\phi_{k_a,k_b}$'s with different $\delta$'s as independent and decompose $\phi$ into $d$ "irreducible subsets" $\phi^\delta \subset \phi$, where each subset contains the set of states $\phi_{k_a,k_b}$, with the same $\delta$. Each $\phi^\delta$ therefore contains $\frac{N_a N_b}{d}$ components and for $d = 1$, one returns to $\phi^\delta = \phi$. The boundary condition for $y_1 \to y_1 + 1$ in (6.68) can then be expressed in terms of $l$ and $\delta$ by

$$\phi_{l,l-\delta}(w) = \mathrm{e}^{i\pi \tilde{I}_{ab} \frac{\mathrm{Im}(w)}{\mathrm{Im}(\tau)}} \mathrm{e}^{2\pi i\left(\tilde{I}_{ab} l + \frac{n_b \delta}{N_b}\right)} \phi_{l,l-\delta}(w)\,. \tag{6.70}$$

Unlike the case where $N_a, N_b$ are coprime, the boundary conditions involve an additional phase depending on $\delta$.

### Labeling bifundamental states on a the $T^6_{\mathrm{SO(12)}}$

Now the above discussion is generalized in order to label all components of $\phi$ given in (6.58), which are related by the boundary conditions in (6.59). The gauge indices $k_a$ and $k_b$ of $\phi$ belong to the lattices

$$k_a \in \frac{\Lambda_{\mathrm{SO}(6)}}{\Gamma_a}\,, \quad k_a \in \frac{\Lambda_{\mathrm{SO}(6)}}{\Gamma_b}\,, \quad \text{with} \quad \Lambda_a,\, \Lambda_b \in \Lambda_{\mathrm{SO}(6)}\,. \tag{6.71}$$

Let $\Gamma_a$ and $\Gamma_b$ be spanned by three generators $a_i$ and $b_i$ repsectively:

$$A = \mathrm{span}\,(a_1, a_2, a_3)\,, \quad B = \mathrm{span}\,(b_1, b_2, b_3)\,, \tag{6.72}$$

and let $\Gamma_a$ and $\Gamma_b$ be sublattices of $\Gamma_d$, s.t. all other lattices which contain $\Gamma_a$ and $\Gamma_b$ are sublattices of $\Gamma_d$. Then the following equation is only solvable for $k_a - k_b \in \Gamma_d$

$$k_a + As = k_b + Bt = l\,, \quad s, t \in \mathbb{Z}^3\,, \tag{6.73}$$

with

$$A = (a_1,\, a_2,\, a_3)\,, \quad B = (b_1,\, b_2,\, b_3)\,, \tag{6.74}$$

[5] Rewriting $I_{ab} = n_a N_b - n_b N_b$, $I_{ab}$, one can see that $I_a b$ is always a multiple of $g.c.d.(N_a, N_b) = d$.

where $A$ and $B$ are $3 \times 3$ matrices, with the columns given by the components of the corresponding basis vectors. Similar to the non coprime case in two dimensions, by introducing another label

$$\delta \in \frac{\Lambda_{\mathrm{SO}(6)}}{\Gamma_d}\,, \tag{6.75}$$

where $k_a - k_b = \delta \mod \Gamma_d$, the equation (6.73) can be modified by

$$k_a + As = k_b + \delta + Bt = l\,, \quad s, t \in \mathbb{Z}^3\,, \tag{6.76}$$

s.t. it can be solved for any pair $(k_a, k_b)$, similar to (6.65) in the two dimensional case. The label $l$ in (6.76) has to belong to the $SO(6)$ lattice, because $k_a$, $k_b$, $As$, $Bt$ are $SO(6)$ vectors for any $s, t \in \mathbb{Z}^3$. For

$$l \in \frac{\Lambda_{\mathrm{SO}(6)}}{\Gamma_a \cap \Gamma_b}\,, \tag{6.77}$$

any pair $(k_a, k_b)$, with $k_a - k_b = \delta \mod \Gamma_d$, obeys the relation

$$k_a = l \mod \Gamma_a\,, \quad k_b + \delta = l \mod \Gamma_b\,. \tag{6.78}$$

The number of states $\#(k_a, k_b)$ of $\phi$ satisfies the relation[6]

$$\#(k_a, k_b) = \left|\frac{\Lambda_{\mathrm{SO}(6)}}{\Gamma_a}\right| \left|\frac{\Lambda_{\mathrm{SO}(6)}}{\Gamma_b}\right| = \left|\frac{\Lambda_{\mathrm{SO}(6)}}{\Gamma_d}\right| \left|\frac{\Lambda_{\mathrm{SO}(6)}}{\Gamma_a \cap \Gamma_b}\right|\,, \tag{6.79}$$

hence there is indeed a one to one relation between the pairs $(l, \delta)$ and $(k_b, k_b)$. The boundary conditions in (6.59) reveal, that, analogous to the two dimensional case, only states with the same $\delta$ are related by lattice shifts of the $\phi$ and therefor $\phi$ can be decomposed into $\left|\frac{\Lambda_{\mathrm{SO}(6)}}{\Gamma_d}\right|$ independent (or irreducible) subsets $\phi^\delta \subset \phi$, with each $\phi^\delta$ containing $\left|\frac{\Lambda_{\mathrm{SO}(6)}}{\Gamma_a \cap \Gamma_b}\right|$ components. Each component in an irreducible subset $\phi^\delta$ is labeled by a different $l$. Using the labels $\delta$ and $l$, the boundary conditions, generated by $z_1 \to z_1 + 1$, $z_2 \to z_2 + 1$ and $z_3 \to z_3 + 1$, can be expressed as

$$\begin{aligned}
\phi_{l,l-\delta}(y_1 + 1, ...,) &= e^{i\pi \tilde{I}^{(1)}_{ab} \frac{\mathrm{Im}(w_1)}{\mathrm{Im}(K_1)}} e^{2\pi i \left(\tilde{I}^{(1)}_{ab} l^{(1)} + \frac{n^1_b \delta^{(1)}}{N^{(1)}_b}\right)} \phi_{l,l-\delta}(y_1, ...)\,, \\
\phi_{l,l-\delta}(..., y_3 + 1, ...,) &= e^{i\pi \tilde{I}^{(2)}_{ab} \frac{\mathrm{Im}(w_2)}{\mathrm{Im}(K_2)}} e^{2\pi i \left(\tilde{I}^{(2)}_{ab} l^{(2)} + \frac{n^2_b \delta^{(2)}}{N^{(2)}_b}\right)} \phi_{l,l-\delta}(..., y_3, ...)\,, \\
\phi_{l,l-\delta}(..., y_5 + 1, ...,) &= e^{i\pi \tilde{I}^{(3)}_{ab} \frac{\mathrm{Im}(w_3)}{\mathrm{Im}(K_3)}} e^{2\pi i \left(\tilde{I}^{(3)}_{ab} l^{(3)} + \frac{n^3_b \delta^{(3)}}{N^{(3)}_b}\right)} \phi_{l,l-\delta}(..., y_5, ...)
\end{aligned} \tag{6.80}$$

[6] See appendix C for a proof.

## 6.3 Wavefunctions for chiral matter on the $T^6$

### 6.3.1 Massless chiral fermions

Let a $U(N_a+N_b+N_c)$ gauge symmetry on the $T^6$ be broken by the magnetic flux $F = \sum_{h=1}^{3} F_{w_h\overline{w}_h} \mathrm{d}w_h \mathrm{d}\overline{w}_h$ with

$$\begin{aligned} F_{w_1\overline{w}_1} &= \frac{i\pi}{\mathrm{Im}(K_1)} \begin{pmatrix} \frac{n_a^1}{N_a^{(1)}}\mathbb{1}_{N_a} & & \\ & \frac{n_b^1}{N_b^{(1)}}\mathbb{1}_{N_b} & \\ & & \frac{n_c^1}{N_c^{(1)}}\mathbb{1}_{N_c} \end{pmatrix}, \qquad (6.81) \\ F_{w_2\overline{w}_2} &= \frac{i\pi}{\mathrm{Im}(K_2)} \begin{pmatrix} \frac{m_a^2}{N_a^{(2)}}\mathbb{1}_{N_a} & & \\ & \frac{m_b^2}{N_b^{(2)}}\mathbb{1}_{N_b} & \\ & & \frac{m_c^2}{N_c^{(2)}}\mathbb{1}_{N_c} \end{pmatrix}, \\ F_{w_3\overline{w}_3} &= \frac{i\pi}{\mathrm{Im}(K_3)} \begin{pmatrix} \frac{m_a^3}{N_a^{(3)}}\mathbb{1}_{N_a} & & \\ & \frac{m_b^3}{N_b^{(3)}}\mathbb{1}_{N_b} & \\ & & \frac{m_c^3}{N_c^{(3)}}\mathbb{1}_{N_c} \end{pmatrix}, \end{aligned}$$

to a $U(N_a) \times U(N_b) \times U(N_c)$ symmetry. The Lagrangian for the gauge theory is given in (6.29). The corresponding Lagrangian containing the superpartners $\Psi$ to the bosonic fields $A$ is given by [115]

$$\mathcal{L} = \frac{i}{2g^2}\mathrm{Tr}\left\{\overline{\Psi}\Gamma^i D_i \Psi\right\}, \qquad (6.82)$$

with $\Gamma^i$ the six Gamma matrices forming the six dimensional Clifford algebra and $D_i$ the covariant derivative. The fields $\Psi$ are fermions on the torus and since they are the superpartners to $A$, they transforms in the adjoint representation $\mathbf{adj}(U(N_a+N_b+N_c))$. In six dimensions a Dirac spinor contains eight spin states, which are denoted by

$$\psi_{\vec{\epsilon}} : T^6 \to \mathbb{C}^{(N_a+N_b+N_c)^2}, \qquad (6.83)$$

where $\psi_{\vec{\epsilon}}$ is a state of $\Psi$ in the eight dimensional Dirac spinor respresentation of $SO(6)$, and $\vec{\epsilon} = (\epsilon^{(1)}, \epsilon^{(2)}, \epsilon^{(3)})$, with $\epsilon^{(h)} = \pm 1/2$, denote the eigenvalues under the three Cartan generators of $SO(6)$. The equations of motion from (6.82) for massless spinors $\Psi$ leads to the Dirac equation

$$i\sum_{h=1}^{3}\left(\Gamma^{w_h}D_{w_h} - \Gamma^{\overline{w}_h}D_{\overline{w}_h}\right)\Psi(\vec{z}) = 0\,. \qquad (6.84)$$

From the Dirac equation (6.84), three differential equations for each of the eight spin states $\psi_{\vec{\epsilon}}$ can be deduced:

$$\forall\, h \in \{1,2,3\} \quad \begin{cases} D_h\psi_{\vec{\epsilon}}(\vec{w}) = 0, & \text{for} \quad \epsilon^{(h)} = + \\ D_h^{\dagger}\psi_{\vec{\epsilon}}(\vec{w}) = 0, & \text{for} \quad \epsilon^{(h)} = - \end{cases}, \qquad (6.85)$$

where the covariant derivative acts on the fermionic fields as

$$D_h\psi_{\vec{\epsilon}} = \overline{\partial}_h\psi_{\vec{\epsilon}} + \left[A_{\overline{w}_h}, \psi_{\vec{\epsilon}}\right] , \tag{6.86}$$

with the components for $A_{\overline{w}_h}$ deduced from the flux (6.81) and are given by

$$\begin{aligned}
A_{\overline{w}_1} &= -\frac{\pi w_1}{2\mathrm{Im}(K_1)}\begin{pmatrix} \frac{n_a^1}{N_a^{(1)}}\mathbb{1}_{N_a} & & \\ & \frac{n_b^1}{N_b^{(1)}}\mathbb{1}_{N_b} & \\ & & \frac{n_c^1}{N_c^{(1)}}\mathbb{1}_{N_c}\end{pmatrix}, \\
A_{\overline{w}_2} &= -\frac{\pi w_2}{2\mathrm{Im}(K_2)}\begin{pmatrix} \frac{m_a^2}{N_a^{(2)}}\mathbb{1}_{N_a} & & \\ & \frac{m_b^2}{N_b^{(2)}}\mathbb{1}_{N_b} & \\ & & \frac{m_c^2}{N_c^{(2)}}\mathbb{1}_{N_c}\end{pmatrix}, \\
, A_{\overline{w}_3} &= -\frac{\pi w_3}{2\mathrm{Im}(K_3)}\begin{pmatrix} \frac{m_a^3}{N_a^{(3)}}\mathbb{1}_{N_a} & & \\ & \frac{m_b^3}{N_b^{(3)}}\mathbb{1}_{N_b} & \\ & & \frac{m_c^3}{N_c^{(3)}}\mathbb{1}_{N_c}\end{pmatrix}.
\end{aligned}$$

By expressing $\psi_{\vec{\epsilon}}$ in the gauge representation as

$$\psi_{\vec{\epsilon}} = \begin{pmatrix} A & B & C \\ D & E & F \\ G & H & I \end{pmatrix} \tag{6.87}$$

with $A, B, ..., I$ being $N_a \times N_a$- , $N_a \times N_b$- ,..., $N_c \times N_c$-dimensional block matrices. Inserting (6.87) into (6.86), we recive

$$\begin{aligned}
&\overline{\partial}_h A = 0\,, \quad \overline{\partial}_h E = 0\,, \quad \overline{\partial}_h I = 0\,, \quad \text{for} \quad \epsilon^{(h)} = +\,, \\
&\partial_h A = 0\,, \quad \partial_h E = 0\,, \quad \partial_h I = 0\,, \quad \text{for} \quad \epsilon^{(h)} = -\,,
\end{aligned} \tag{6.88}$$

s.t. the components of $A$, $B$ and $C$ are holomorphic or antiholomprphic in the variables $w_h$. They correspond to the gaugeginos of $U(N_a)$, $U(N_b)$ and $U(N_c)$ respectively. The off-diagonal blocks have to satsify

$$\begin{aligned}
&\left(\overline{\partial}_h + \frac{\pi \tilde{I}_{ab}^{(h)}}{2\mathrm{Im}(K_h)}w_h\right)B = 0\,, \qquad \left(\overline{\partial}_h - \frac{\pi \tilde{I}_{ab}^{(h)}}{2\mathrm{Im}(K_h)}w_h\right)D = 0\,, \\
&\left(\overline{\partial}_h + \frac{\pi \tilde{I}_{ca}^{(h)}}{2\mathrm{Im}(K_h)}w_h\right)G = 0\,, \qquad \left(\overline{\partial}_h - \frac{\pi \tilde{I}_{ca}^{(h)}}{2\mathrm{Im}(K_h)}w_h\right)C = 0\,, \\
&\left(\overline{\partial}_h + \frac{\pi \tilde{I}_{bc}^{(h)}}{2\mathrm{Im}(K_h)}w_h\right)F = 0\,, \qquad \left(\overline{\partial}_h - \frac{\pi \tilde{I}_{bc}^{(h)}}{2\mathrm{Im}(K_h)}w_h\right)H = 0\,,
\end{aligned} \tag{6.89}$$

for $\epsilon^{(h)} = +$ and

$$\left(\partial_h - \frac{\pi \tilde{I}_{ab}^{(h)}}{2\mathrm{Im}(K_h)}\overline{w}_h\right) B = 0\,, \qquad \left(\partial_h + \frac{\pi \tilde{I}_{ab}^{(h)}}{2\mathrm{Im}(K_h)}\overline{w}_h\right) D = 0\,, \tag{6.90}$$
$$\left(\partial_h - \frac{\pi \tilde{I}_{ca}^{(h)}}{2\mathrm{Im}(K_h)}\overline{w}_h\right) G = 0\,, \qquad \left(\partial_h + \frac{\pi \tilde{I}_{ca}^{(h)}}{2\mathrm{Im}(K_h)}\overline{w}_h\right) C = 0\,,$$
$$\left(\partial_h - \frac{\pi \tilde{I}_{bc}^{(h)}}{2\mathrm{Im}(K_h)}\overline{w}_h\right) F = 0\,, \qquad \left(\partial_h + \frac{\pi \tilde{I}_{bc}^{(h)}}{2\mathrm{Im}(K_h)}\overline{w}_h\right) H = 0\,,$$

for $\epsilon^{(h)} = -$. The following Ansatz for $B$ can be made:

$$B \propto \begin{cases} \mathrm{e}^{i\frac{\pi \tilde{I}_{ab}^{(h)}}{\mathrm{Im}(K_h)} z_h \mathrm{Im}(w_h)}\,, & \text{for} \quad \epsilon^{(h)} = +\,, \\ \mathrm{e}^{i\frac{\pi \tilde{I}_{ab}^{(h)}}{\mathrm{Im}(K_h)} \overline{w}_h \mathrm{Im}(w_h)}\,, & \text{for} \quad \epsilon^{(h)} = -\,, \end{cases} \tag{6.91}$$

but $B$ diverges for

$$\lim_{\mathrm{Im}(w_h)\to\pm\infty} B = \infty\,, \quad \text{for} \quad \epsilon^{(h)} = -sign\left(I_{ab}^{(h)}\right)\,. \tag{6.92}$$

That means the block $B$ in $\psi_{\vec{\epsilon}}$ can only be non vanishing for

$$\vec{\epsilon} = \left(sign\left(I_{ab}^{(1)}\right), sign\left(I_{ab}^{(2)}\right), sign\left(I_{ab}^{(3)}\right)\right)^T\,. \tag{6.93}$$

Similar the following ansatz for $D$ can be made

$$D \propto \begin{cases} \mathrm{e}^{-i\frac{\pi \tilde{I}_{ab}^{(h)}}{\mathrm{Im}(K_h)} w_h \mathrm{Im}(w_h)}\,, & \text{for} \quad \epsilon^{(h)} = +\,, \\ \mathrm{e}^{-i\frac{\pi \tilde{I}_{ab}^{(h)}}{\mathrm{Im}(K_h)} \overline{z}_h \mathrm{Im}(w_h)}\,, & \text{for} \quad \epsilon^{(h)} = -\,, \end{cases} \tag{6.94}$$

with $D$ only non vanishing for

$$\vec{\epsilon} = \left(-sign\left(I_{ab}^{(1)}\right), -sign\left(I_{ab}^{(2)}\right), -sign\left(I_{ab}^{(3)}\right)\right)^T\,. \tag{6.95}$$

Without loss of generality it will be focused from now on, on the case where

$$I_{ab}^{(h)} > 0\,, \quad \forall\, h \in \{1,2,3\}\,, \tag{6.96}$$

and the non vanishing $B$ and $D$ are contained in $\psi_{+++}$ and $\psi_{---}$ respectively. They are given by

$$B = \exp\left(i\pi \sum_{h=1}^{3} \frac{\tilde{I}_{ab}^{(h)}}{\mathrm{Im}(K_h)} w_h \mathrm{Im}(w_h)\right) \xi^{I_{ab}}(w_1, w_2, w_3)\,, \tag{6.97}$$
$$D = \exp\left(-i\pi \sum_{h=1}^{3} \frac{\tilde{I}_{ab}^{(h)}}{\mathrm{Im}(K_h)} \overline{w}_h \mathrm{Im}(w_h)\right) \xi^{I_{ba}}(\overline{w}_1, \overline{w}_2, \overline{w}_3)\,,$$

where $\xi^{I_{ab}}$ and $\xi^{I_{ba}}$ are $N_a \times N_b$ and $N_b \times N_a$ matrices respectively. Taking a closer look at the fields $B$ and $D$, one can see that they are related by complex conjugation to each other and by defining the chirality of $\psi_{\vec{\epsilon}}$ by the sign of $\epsilon^1\epsilon^2\epsilon^3$, $B$ and $D$ have opposite chiralities. They can be interpret as the field $B$ as a bifundamental field $(\square_a, \overline{\square}_b)$ of $U(N_a) \times U(N_b)$ and $D$ being its conjugate $(\overline{\square}_a, \square_b)$. The intersection numbers satisfy the relation

$$\tilde{I}_{ab}^{(h)} + I_{bc}^{(h)} + I_{ca}^{(h)} = 0\,, \quad \forall\, h \in \{1,2,3\} \tag{6.98}$$

and without loss of generality it will be focused from now on, on the case

$$|I_{ab}^{(h)}| + |I_{ca}^{(h)}| = |I_{bc}^{(h)}|\,, \quad I_{bc}^{(h)} < 0\,, \quad \forall\, h \in \{1,2,3\}\,. \tag{6.99}$$

Having fixed the signs for the intersection numbers the remaining equations in (6.89) and (6.90) can be solved analogous to $B$ and $D$ and one receives expressions for the other blocks in $\psi_{\vec{\epsilon}}$. Defining

$$\psi^{I_{\alpha\beta}}(w_1, w_2, w_3) = \exp\left( i\pi \sum_{h=1}^{3} \frac{\tilde{I}_{\alpha\beta}^{(h)}}{\mathrm{Im}(K_h)} w_h \mathrm{Im}(w_h) \right) \xi^{I_{\alpha\beta}}(w_1, w_2, w_3)\,, \tag{6.100}$$

the fields $C$, $F$, $G$ and $H$ are given by

$$G = C^\dagger = \psi^{I_{ca}}(w_1, w_2, w_3)\,, \quad H = F^\dagger = \psi^{I_{cb}}(w_1, w_2, w_3)\,, \tag{6.101}$$

with $G$ and $H$ ($C$ and $F$) being chiral (antichiral) fermions in the bifundamental representation of $U(N_c) \times U(N_a)$ and $U(N_c) \times U(N_b)$ respectively. For the choice, made in (6.99), only the two spin states

$$\psi_{+++} = \begin{pmatrix} A & \psi^{I_{ab}} & 0 \\ 0 & E & 0 \\ \psi^{I_{ca}} & \psi^{I_{cb}} & I \end{pmatrix}, \quad \text{and} \quad \psi_{---} = \begin{pmatrix} A^\dagger & 0 & \left(\psi^{I_{ca}}\right)^\dagger \\ \left(\psi^{I_{ab}}\right)^\dagger & E^\dagger & \left(\psi^{I_{cb}}\right)^\dagger \\ 0 & 0 & I^\dagger \end{pmatrix}, \tag{6.102}$$

have non vanishing off diagonal blocks. For a different choice of signs in (6.99), the off diagonal blocks of two other spin states $\psi_{\vec{\epsilon}}$, would have non vanishing entries, where $\vec{\epsilon}$ depends on the signs of the intersection numbers similar to (6.93).

### 6.3.2 Light chiral scalars

Here a closer look a the bosonic fields in (6.29) is taken in order to determine the wavefunctions for chiral scalars. As already mentioned, the states commuting with the vev, belong to the adjoint representation of the unbroken gauge group. The remaining degrees of freedom, remain as bifundamental scalars, which correspond to the superpartners of the chiral fermions $\psi^{I_{ab}}$, $\psi^{I_{cb}}$, $\psi^{I_{ca}}$ and their conjugates. To be more precise, when decomposing the $(N_a + N_b + N_c)^2$ states of an adjoint field **adj** $\in U(N_a + N_b + N_c)$ into $N_a \times N_a$, $N_a \times N_b$, etc. blocks similar to (6.87)

$$\Phi_i = \begin{pmatrix} A & B & C \\ D & E & F \\ G & H & I \end{pmatrix}, \tag{6.103}$$

the gauge field $A_i$ of the unbroken gauge group takes in that representation the form

$$A_i = \begin{pmatrix} A_i^a & & \\ & A_i^b & \\ & & A_i^c \end{pmatrix}, \tag{6.104}$$

where $A_i^\alpha$ has $N_\alpha \times N_\alpha$ components for each $\alpha \in \{a, b, c\}$. $A_i$ indeed commutes with the flux given in (6.81). The action of $A_i$ on $\Phi_j$ is given by the commutator

$$\left[A_i, \Phi_j\right] = \begin{pmatrix} \left[A_i^a, A\right] & A_i^a B - B A_i^b & A_i^a C - C A_i^c \\ A_i^b D - D A_i^a & \left[A_i^b, E\right] & A_i^b F - F A_i^c \\ A_i^c G - G A_i^a & A_i^c H - H A_i^b & \left[A_i^c, I\right] \end{pmatrix}. \tag{6.105}$$

From (6.105), one can deduce that $A$, $E$ and $I$ transform in the adjoint of $U(N_a)$, $U(N_b)$ and $U(N_c)$ respectively. But one can further see that the fields $B$, $F$ and $G$ transform in the bifundamental representation of $U(N_a) \times U(N_b)$, $U(N_b) \times U(N_c)$ and $U(N_c) \times U(N_a)$ respectively, where $C$, $D$ and $H$ transform in the conjugate representations. That means in order to find wavefunctions for the bifundamentals in $\Phi$, eigenfunctions of the Laplace operator $D^2 = \sum_{i=1}^6 {D_i}^2$ have to be found, with $D_i$ the covariant derivative. The square of the Dirac operator is related to the Labplace operator as

$$\begin{aligned} (i\not{D})^2 &= \frac{1}{2}\underbrace{\left\{\Gamma^i, \Gamma^j\right\}}_{2g^{ij}} D_i D_j + \frac{1}{2}\Gamma^i\Gamma^j \underbrace{\left[D_i, D_j\right]}_{=F_{ij}} \\ &= D^2 + \frac{1}{2}\Gamma^i\Gamma^j F_{ij}\,. \end{aligned} \tag{6.106}$$

The Clifford algebra, in the basis chosen for (6.81), is given by

$$\left\{\Gamma^h, \Gamma^k\right\} = \frac{4}{{A_h}^2}\delta^{w_h, \overline{w}_k}\,, \tag{6.107}$$

with $A_h$ defined in (6.18). Inserting the flux components from (6.81) into (6.106), one receives

$$(i\not{D})^2 = D^2 + 2\sum_{h=1}^{3} \frac{F_{w_h, \overline{w}_h}}{{A_h}^2}\,. \tag{6.108}$$

Since the eigenfunctions for the Dirac operator where already found by $i\not{D}\psi_{\vec{\epsilon}} = 0$, the eigenfunctions for the Laplace operator are also known, where their eigenvalues are given by

$$D^2 \psi^{I_{\alpha\beta}} = \sum_{h=1}^{3} \frac{2\pi \tilde{I}^{(h)}_{\alpha\beta}}{{A_h}^2 \mathrm{Im}(K_h)} \psi^{I_{\alpha\beta}}\,, \tag{6.109}$$

s.t. $\psi^{I_{\alpha\beta}}$ are not only solutions to bifundamental fermionic fields, but serves as wavefunctions for bifundamental scalar fields with a mass given by

$$m^2 = \sum_{h=1}^{3} \frac{2\pi \tilde{I}^{(h)}_{\alpha\beta}}{{A_h}^2 \mathrm{Im}(K_h)}\,. \tag{6.110}$$

The mass in (6.110) for the scalars can be related to the masses of the light NS open strings at intersection points of D6-branes [61]. Since the components of $F_{w_h\overline{w}_h}$ where derived by the boundary conditions (6.21), the right term in (6.108) is related to the angles in the complex planes of intersecting D6-branes of the T-dual theory. In order for the two branes to share the same supersymmetry charges on their volume they need to be calibrated by the same calibration phase and due to the condition (4.134) one would expect the right term in (6.108) to vanish for supersymmetric theories [118, 119]. Hence the scalar with vanishing mass for (6.110) corresponds in the intersecting branes picture to the massless NS open string state at intersection points. Analogous to the factorisable torus, the two polarization states of the massless field $\Phi_i$ are given by [115]

$$\Phi_+ = \psi_{+++} \qquad \Phi_- = \psi_{---} \tag{6.111}$$

with $\psi_{+++}$ and $\psi_{---}$ defined in (6.102).

### 6.3.3 Massless chiral fields and Wilson lines

Up to now turning on magnetic flux of the form (6.81) in (6.29) and (6.82), led to chiral matter from gauge fields of the unbroken gauge group. The wave functions for the massless (or light) bifundmantals are

$$\psi^{I_{ab}}(w_1, w_2, w_3) = \exp\left( i\pi \sum_{h=1}^{3} \frac{\tilde{I}_{ab}^{(h)}}{\mathrm{Im}(K_h)} z_h \mathrm{Im}(w_h) \right) \xi^{I_{ab}}(w_1, w_2, w_3)\,, \tag{6.112}$$

with $\xi^{I_{ab}} \in \mathbb{C}^{N_a \times N_b}$, and the complex conjugate of $\psi^{I_{ab}}$. In section 6.2.3 the discussed showed how bifundamentals transform under Wilson lines, which occur from the compactification lattice. For the wavefunction in (6.112) to be consistent with the compactification, it has to satisfy the boundary conditions (6.59). Applying (6.59) to $\psi^{I_{ab}}$, boundary conditions for $\xi^{I_{ab}}$ can be deduced and are given by:

$$\begin{aligned}
\xi^{I_{ab}}_{l,l-\delta}(w_1 + 1, w_2, w_3) &= \mathrm{e}^{2\pi i\left(\tilde{I}_{ab}^{(1)} l^{(1)} + \frac{n_b^1 \delta^{(1)}}{N_b^{(1)}}\right)} \xi^{I_{ab}}_{l,l-\delta}(\vec{w})\,, && (6.113)\\
\xi^{I_{ab}}_{l,l-\delta}(w_1 + K_1, w_2 - \tau_2, w_3) &= \mathrm{e}^{-\pi i\left(\tilde{I}_{ab}^{(1)}(2w_1+K_1) - \tilde{I}_{ab}^{(2)}(2w_2 - K_2)\right)} \xi^{I_{ab}}_{l',l'-\delta}(\vec{w})\,, && (6.114)\\
\xi^{I_{ab}}_{l,l-\delta}(w_1, w_2 + 1, w_3) &= \mathrm{e}^{2\pi i\left(\tilde{I}_{ab}^{(2)} l^{(2)} + \frac{n_b^2 \delta^{(2)}}{N_b^{(2)}}\right)} \xi^{I_{ab}}_{l,l-\delta}(\vec{w})\,, && (6.115)\\
\xi^{I_{ab}}_{l,l-\delta}(w_1, w_2 + K_2, w_3 - \tau_3) &= \mathrm{e}^{-\pi i\left(\tilde{I}_{ab}^{(2)}(2w_2+K_2) - \tilde{I}_{ab}^{(3)}(2w_3 - K_3)\right)} \xi^{I_{ab}}_{l'',l''-\delta}(\vec{w})\,, && (6.116)\\
\xi^{I_{ab}}_{l,l-\delta}(w_1, w_2, w_3 + 1) &= \mathrm{e}^{2\pi i\left(\tilde{I}_{ab}^{(3)} l^{(3)} + \frac{n_b^3 \delta^{(3)}}{N_b^{(3)}}\right)} \xi^{I_{ab}}_{l,l-\delta}(\vec{w})\,, && (6.117)\\
\xi^{I_{ab}}_{l,l-\delta}(w_1, w_2, w_3 + 2K_3) &= \mathrm{e}^{-4\pi i \tilde{I}_{ab}^{(3)}(w_3 + K_3)} \xi^{I_{ab}}_{l''',l'''-\delta}(\vec{w})\,, && (6.118)
\end{aligned}$$

with $l' = (1, -1, 0)^T$, $l'' = (0, 1, -1)^T$ and $l''' = (0, 0, 2)^T$. The following ansatz solves (6.113), (6.115) and (6.117)

$$\xi^{I_{ab}}_{l,l-\delta}(\vec{w}) = \sum_{\vec{n} \in \mathbb{Z}^3} \exp\left( 2\pi i \sum_{h=1}^{3} \left( n^{(h)} + \tilde{I}_{ab}^{(h)} l^{(h)} + \varphi^{(h)} \right) w_h \right) \rho_{\vec{n}}(l)\ , \tag{6.119}$$

with

$$\varphi^{(1)} = \begin{cases} \frac{n_b^1 \delta^{(1)}}{N_b^{(1)}}\,, & \text{for } h = 1 \\ -\frac{m_b^h \delta^{(h)}}{N_b^{(h)}}\,, & \text{for } h \in \{2, 3\} \end{cases}\,, \tag{6.120}$$

and $\rho_{\vec{n}}(l)$ a factor independent of the variables $\vec{w}$. To determine $\rho_{\vec{n}}(l)$, the remaining boundary conditions (6.114), (6.116) and (6.118) are imposed on (6.119). After inserting $\xi$ from (6.119) into the boundary conditions, comparing the coefficients with equal $\vec{n}$, one finds the following conditions for $\rho_{\vec{n}}$

$$\begin{aligned}
\frac{\rho_{\vec{n}}(l+(1,-1,0)^T)}{\rho_{\vec{n}}(l)} &= \mathrm{e}^{2\pi i\left\{\left(n^{(1)}+\tilde{I}^{(1)}_{ab}(l^{(1)}+\frac{1}{2})+\frac{n^1_b\delta^{(1)}}{N^{(1)}_b}\right)K_1-\left(n^{(2)}+\tilde{I}^{(2)}_{ab}(l^{(2)}-\frac{1}{2})-\frac{m^2_b\delta^{(2)}}{N^{(2)}_b}\right)K_2\right\}}\,, \\
\frac{\rho_{\vec{n}}(l+(0,1,-1)^T)}{\rho_{\vec{n}}(l)} &= \mathrm{e}^{2\pi i\left\{\left(n^{(2)}+\tilde{I}^{(2)}_{ab}(l^{(2)}+\frac{1}{2})-\frac{m^2_b\delta^{(2)}}{N^{(2)}_b}\right)K_2-\left(n^{(3)}+\tilde{I}^{(3)}_{ab}(l^{(3)}-\frac{1}{2})-\frac{m^3_b\delta^{(3)}}{N^{(3)}_b}\right)K_3\right\}}\,, \\
\frac{\rho_{\vec{n}}(l+(0,0,2)^T)}{\rho_{\vec{n}}(l)} &= \mathrm{e}^{2\pi i\left(n^{(3)}+\tilde{I}^{(3)}_{ab}(l^{(3)}+1)-\frac{m^3_b\delta^{(3)}}{N^{(3)}_b}\right)K_3}\,. 
\end{aligned} \tag{6.121}$$

The above boundary conditions are solved by

$$\rho_{\vec{n}}(l) = \mathcal{N}_{\vec{n}} \exp\left(i\pi\sum_{h=1}^{3}\left(n^{(h)}+\tilde{I}^{(h)}_{ab}l^{(h)}+\varphi^{(h)}\right)^2\frac{K_h}{\tilde{I}^{(h)}_{ab}}\right), \tag{6.122}$$

where $\mathcal{N}_{\vec{n}}$ is a normalization factor.

### 6.3.4 Counting numbers of independent zeromodes

In the previous section the zeromodes for of the Dirac and Laplace operator (when the D9-branes wrap cycles with the same calibration phase) we determined. The zeromodes are given by (6.112)

$$\psi^{I_{ab}}(\vec{w}) = \exp\left(i\pi\sum_{h=1}^{3}\frac{\tilde{I}^{(h)}_{ab}}{\mathrm{Im}(K_h)}w_h\mathrm{Im}(w_h)\right)\xi^{I_{ab}}(\vec{w})\,, \tag{6.123}$$

with the components for $\xi$ given by

$$\xi_{l,l-\delta} = \sum_{\vec{n}\in\mathbb{Z}^3}\mathcal{N}_{\vec{n}}\mathrm{e}^{2\pi i\sum_{h=1}^{3}\left(\left(n^{(h)}+\tilde{I}^{(h)}_{ab}l^{(h)}+\varphi^{(h)}\right)W_h+\frac{1}{2}\left(n^{(h)}+\tilde{I}^{(h)}_{ab}l^{(h)}+\varphi^{(H)}\right)^2\frac{K_h}{\tilde{I}^{(h)}_{ab}}\right)} \tag{6.124}$$

and $\varphi^{(h)}$ defined in (6.120). First notice that, since the label $l$ admits the identification

$$l \sim l+\lambda\,, \quad \text{with} \quad \lambda\in\Gamma_a\cap\Gamma_b\,, \tag{6.125}$$

a transformation of $l$ of the form (6.125) must leave the wavefunction invariant. Comparing the coefficients of $\xi_{l,l-\delta}$ with the coeffients of $\xi_{l+\lambda,l+\lambda_\delta}$, one finds the relation for the normalization factors are not all independent, but obey the relation

$$\mathcal{N}_{\vec{n}} = \mathcal{N}_{\vec{n}+\vec{q}_\lambda}\,, \quad \text{with} \quad \vec{q}_\lambda = \left(\tilde{I}^{(1)}_{ab}\lambda^{(1)}, \tilde{I}^{(2)}_{ab}\lambda^{(2)}, \tilde{I}^{(3)}_{ab}\lambda^{(3)}\right)^T\,. \tag{6.126}$$

Inserting the generators of $\Gamma_a\cap\Gamma_b$, which are given in appendix D, one finds that vectors $\vec{q}_\lambda$ belong to the following lattice:

- $$\Lambda_n = \bigotimes_{h=1}^{3}\frac{I^{(h)}_{ab}}{d^{(h)}_{ab}}\mathbb{Z}\,, \tag{6.127}$$

  for the cases (a)-(c) in appendix D,

- $$\Lambda_n = \text{span}\left(\begin{pmatrix} \frac{I_{ab}^{(1)}}{2d_{ab}^{(1)}} \\ \frac{I_{ab}^{(2)}}{d_{ab}^{(2)}} \\ 0 \end{pmatrix}, \begin{pmatrix} \frac{I_{ab}^{(1)}}{2d_{ab}^{(1)}} \\ -\frac{I_{ab}^{(2)}}{d_{ab}^{(2)}} \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ \frac{I_{ab}^{(3)}}{d_{ab}^{(3)}} \end{pmatrix}\right) \tag{6.128}$$

  for the cases (d) and (e) in appendix D,

- $$\Lambda_n = \text{span}\left(\begin{pmatrix} \frac{I_{ab}^{(1)}}{2d_{ab}^{(1)}} \\ \frac{I_{ab}^{(2)}}{d_{ab}^{(2)}} \\ 0 \end{pmatrix}, \begin{pmatrix} \frac{I_{ab}^{(1)}}{2d_{ab}^{(1)}} \\ -\frac{I_{ab}^{(2)}}{d_{ab}^{(2)}} \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ \frac{I_{ab}^{(2)}}{d_{ab}^{(2)}} \\ \frac{I_{ab}^{(3)}}{d_{ab}^{(3)}} \end{pmatrix}\right) \tag{6.129}$$

  for the cases (e) in appendix D.

The identification (6.126) implies that the summation index $\vec{n}$ of independent normalization factors belong to the lattice

$$\vec{n} \in \frac{\mathbb{Z}^3}{\Lambda_n} \,. \tag{6.130}$$

Hence $\psi^{I_{ab}}_{l,l-\delta}$ in (6.112) can be decomposed into $\left|\frac{\mathbb{Z}^3}{\Lambda_n}\right|$ independend zero modes:

$$\psi^{I_{ab}}_{l-l-\delta}(\vec{w}) = \sum_{\vec{k}\in\frac{\mathbb{Z}^3}{\Lambda_n}} \mathcal{N}_{\vec{k}} \mathrm{e}^{i\pi\sum_{h=1}^{3}\frac{\tilde{I}_{ab}^{(h)}}{\mathrm{Im}(K_h)} w_h \mathrm{Im}(w_h)} \sum_{\lambda_{ab}\in\Gamma_a\cap\Gamma_b} \chi_{n,k,l,\delta}(\vec{w})\,, \tag{6.131}$$

with

$$\begin{aligned}\chi_{n,k,l,\delta}(\vec{w}) =& \exp\left(2\pi i \sum_{h=1}^{3}\left(I_{ab}^{(h)}\lambda_{ab}^{(h)} + \tilde{I}_{ab}^{(h)} l^{(h)} + k^{(h)} + \varphi^{(h)}\right) w_h\right) \\ &\cdot \exp\left(\pi i \left(I_{ab}^{(h)}\lambda_{ab}^{(h)} + \tilde{I}_{ab}^{(h)} l^{(h)} + k^{(h)} + \varphi^{(h)}\right)^2 \frac{K_h}{\tilde{I}_{ab}^{(h)}}\right) .\end{aligned} \tag{6.132}$$

Notice that wavefunctions with different values for $\delta$ are also independent, because they cannot be related by lattice translations. The total number of independent wavefunctions for a bifundamental $(\square_a, \overline{\square}_b) \in U(N_a) \times U(N_b)$ is therefore given by the product of the amount of wavefunctions with independent normalization factors with the number of irreducible subsets

$$\#(\square_a, \overline{\square}_b) = \left|\frac{\Lambda_{\mathrm{SO}(6)}}{\Gamma_d}\right| \left|\frac{\mathbb{Z}^3}{\Lambda_n}\right| = \frac{1}{2}\prod_{h=1}^{3} I_{ab}^{(h)}\,, \tag{6.133}$$

which is exactly the number of families, for chiral matter on the $T^6_{\mathrm{SO}(12)}$. Since $\Gamma_a$ and $\Gamma_b$ are sublattices of $\Gamma_d$, the matrices $A, B$ from (6.74) can be decomposed into

$$A = DM_a\,, \quad B = DM_b\,, \tag{6.134}$$

where $D = (d_1, d_2, d_3)$, with $d_i$ the generators of $\Gamma_d$, and $M_a$ and $M_b$ both being integral $3 \times 3$ matrices. The following Diophantine equation

$$\mathbb{1}_3 = M_a P - M_b Q\,, \tag{6.135}$$

can be solved by integral matrices $P, Q \in \mathbb{Z}^{3\times 3}$ [7]. The phase $\varphi^{(h)}$ can be manipulated by adding a zero, the following way

$$\begin{aligned}\varphi^{(h)} &= \varphi^{(h)} + \tilde{I}^{(h)}_{ab}(M_a P\delta)^{(h)} - \tilde{I}^{(h)}_{ab}(M_a P\delta)^{(h)} \\ &= \begin{cases} -\frac{n^h_b (M_a P\delta)^{(h)} + n^h_b (M_b Q\delta)^{(h)}}{N^{(h)}_b} + (M_a P\delta)^{(h)}\,, & \text{for } h = 1 \\ \frac{m^h_b (M_a P\delta)^{(h)} + m^h_b (M_b Q\delta)^{(h)}}{N^{(h)}_b} + (M_a P\delta)^{(h)}\,, & \text{for } h \in \{2,3\} \end{cases}\,,\end{aligned} \tag{6.136}$$

where the relation $\tilde{I}^{(h)}_{ab} = \frac{n^h_a}{N^{(h)}_a} - \frac{n^h_b}{N^{(h)}_b} = -\frac{m^h_a}{N^{(h)}_a} + \frac{m^h_b}{N^{(h)}_b}$ has been used.The term $(M_a P\delta)^{(h)}$ is an integer and can be absorbed in $k^{(h)}$ in $\chi_{n,k,l,\delta}(\vec{w})$. The vectors $M_a P\delta$ and $M_b Q\delta$ are not necessary $SO(6)$ vectors, but since $\delta = M_a P\delta - M_b Q\delta \in \Lambda_{\mathrm{SO}(6)}$, the components of $M_a P\delta$ and $M_b Q\delta$ satisfy the relation

$$\sum_{h=1}^{3} (M_a P\delta)^{(h)} + (M_b Q\delta)^{(h)} = 0 \mod 2\,. \tag{6.137}$$

Therefor the six dimensional vector given by

$$\left((M_a P\delta)^{(1)}, (M_b Q\delta)^{(1)}, (M_a P\delta)^{(2)}, (M_b Q\delta)^{(2)}, (M_a P\delta)^{(3)}, (M_b Q\delta)^{(3)}\right)^T \tag{6.138}$$

can be identified with $SO(12)$ lattice vectors. Combining the labels $k \in \frac{\mathbb{Z}^3}{\Lambda_n}$ with the $\delta$'s in the exponents of $\chi_{n,k,l,\delta}$ one gets

$$k^{(h)} + \varphi^{(h)} = \begin{cases} \frac{n^h_b (k - M_a P\delta)^{(h)} + n^h_b (k - M_b Q\delta)^{(h)}}{N^{(h)}_b}\,, & \text{for } h = 1 \\ -\frac{m^h_b (k - M_a P\delta)^{(h)} + m^h_b (k - M_b Q\delta)^{(h)}}{N^{(h)}_b}\,, & \text{for } h \in \{2,3\} \end{cases}\,. \tag{6.139}$$

Comparing the expression in (6.139) with (5.57), the numerator on the righthandside of (6.139) can be identified, with components of the label $j$, which where used to label intersection points on $T^6_{\mathrm{SO}(12)}$, where the components of the corresponding lattice shift $\vec{t} \in \Lambda_{\mathrm{SO}(12)}$ in (5.57) can be identified with

$$\begin{aligned} &t_1 = k^{(1)} - (M_a P\delta)^{(1)}\,, \quad t_2 = k^{(1)} - (M_b Q\delta)^{(1)}\,, \quad t_3 = k^{(2)} - (M_a P\delta)^{(2)}\,, \\ &t_4 = k^{(2)} - (M_b Q\delta)^{(2)}\,, \quad t_5 = k^{(1)} - (M_a P\delta)^{(3)}\,, \quad t_6 = k^{(2)} - (M_b Q\delta)^{(3)}\,. \end{aligned} \tag{6.140}$$

An independent zeromode for chiral matter can therefor be expressed by the label $j$, used for intersection points on the dual theory. On the other hand, if $\delta$ and $l$ in (6.78), where defined by

$$k_a - \delta = l \mod \Gamma_a\,, \quad k_b = l \mod \Gamma_b\,, \tag{6.141}$$

[7] A proof is given in appendix C

s.t. $k_a - k_b = \delta = \Gamma_d$ still holds, but the $\delta$ correction in (6.76), was shifted into $k_a$, the phases $\varphi^{(h)}$ would be given by

$$\varphi^{(h)} = \begin{cases} \frac{n_a^h \delta^{(h)}}{N_a^{(h)}}\,, & \text{for } h = 1 \\ -\frac{m_a^h \delta^{(h)}}{N_a^{(h)}}\,, & \text{for } h \in \{2,3\} \end{cases} \tag{6.142}$$

and the components of the labels $j'$, denoting independent zeromodes, are related to the label $j$ in (6.139) by

$$N_a^{(h)} j'^{\,(h)} = N_b^{(h)} j^{(h)} \,. \tag{6.143}$$

The wavefunction for independent zeromodes are given by

$$\psi_l^{j,I_{ab}}(\vec{w}) = \mathcal{N}^{i,I_{ab}} \exp\left\{ i\pi \sum_{h=1}^{3} \frac{\tilde{I}_{ab}^{(h)}}{\mathrm{Im}(K_h)} w_h \mathrm{Im}(w_h) \right\} \cdot \\ \sum_{\lambda_{ab}\in\Gamma_a\cap\Gamma_b} \mathrm{e}^{2\pi i \sum_{h=1}^{3}\left( \left( I_{ab}^{(h)}\lambda_{ab}^{(h)} + \tilde{I}_{ab}^{(h)} l^{(h)} + \frac{j^{(h)}}{N_b^{(h)}} \right) w_h + \frac{1}{2}\left( I_{ab}^{(h)}\lambda_{ab}^{(h)} + \tilde{I}_{ab}^{(h)} l^{(h)} + \frac{j^{(h)}}{N_b^{(h)}} \right)^2 \frac{K_h}{\tilde{I}_{ab}^{(h)}} \right)} \,. \tag{6.144}$$

### 6.3.5 Normalization factor for chiral wavefunctions

In order to get canonical kinetic terms, the zeromodes need to be orthogonal in the way [21, 115]

$$g^{-2} \int_{T^6} \mathrm{d}^6 x \,\mathrm{Tr}\left\{ \psi^{i,I_{ab}} \cdot \left( \psi^{j,I_{ab}} \right)^\dagger \right\} = \delta_{i,j} \,, \tag{6.145}$$

with $g$ a coupling constant. The integral over $T^6$ can be expressed by integrations over the generators of $T^6$ by

$$\int_{T^6} \mathrm{d}^6 x = 2 \prod_{h=1}^{3} {A_h}^2 \mathrm{Im}(K_h) \int_0^1 \mathrm{d}y_1 \int_0^1 \mathrm{d}y_2 \int_0^1 \mathrm{d}y_3 \int_0^1 \mathrm{d}y_4 \int_0^1 \mathrm{d}y_4 \int_0^1 \mathrm{d}y_6 \,. \tag{6.146}$$

The trace over the gauge indices can be expressed by a sum over the index as $l \in \frac{\Lambda_{\mathrm{SO}(6)}}{\Gamma_a \cap \Gamma_b}$

$$\mathrm{Tr}\left\{ \psi^{i,I_{ab}} \cdot \left( \psi^{j,I_{ab}} \right)^\dagger \right\} = \sum_{k_a \in \frac{\Lambda_{SO(6)}}{\Gamma_b}, k_b \in \frac{\Lambda_{SO(6)}}{\Gamma_b}} \psi_{k_a,k_b}^{i,I_{ab}} \cdot \left( \psi_{k_a,k_b}^{j,I_{ab}} \right)^\dagger = \sum_{l \in \frac{\Lambda_{\mathrm{SO}(6)}}{\Gamma_a \cap \Gamma_b}} \delta_{\delta_1,\delta_2}\, \psi_{l,l-\delta_1}^{i,I_{ab}} \cdot \left( \psi_{l,l-\delta_2}^{j,I_{ab}} \right)^\dagger \,. \tag{6.147}$$

In section 6.2.3, boundary conditions relating different gauge indices $l$ via Wilson lines where faced. Applying those boundary conditions to the combined wavefunction $\psi_{l,l-\delta_1}^{i,I_{ab}} \cdot \left( \psi_{l,l-\delta_2}^{j,I_{ab}} \right)^\dagger$, one finds that wavefunctions with different $l$ can be identified through the Wilson lines as:

$$\begin{aligned} \psi_{l,l-\delta_1}^{i,I_{ab}} \cdot \left( \psi_{l,l-\delta_2}^{j,I_{ab}} \right)^\dagger (\ldots y_2 + 1, \ldots) &= \psi_{l',l'-\delta_1}^{i,I_{ab}} \cdot \left( \psi_{l',l'-\delta_2}^{j,I_{ab}} \right)^\dagger (\ldots y_2, \ldots) \,, \\ \psi_{l,l-\delta_1}^{i,I_{ab}} \cdot \left( \psi_{l,l-\delta_2}^{j,I_{ab}} \right)^\dagger (\ldots y_4 + 1, \ldots) &= \psi_{l'',l''-\delta_1}^{i,I_{ab}} \cdot \left( \psi_{l'',l''-\delta_2}^{j,I_{ab}} \right)^\dagger (\ldots y_4, \ldots) \,, \\ \psi_{l,l-\delta_1}^{i,I_{ab}} \cdot \left( \psi_{l,l-\delta_2}^{j,I_{ab}} \right)^\dagger (\ldots y_6 + 2) &= \psi_{l''',l'''-\delta_1}^{i,I_{ab}} \cdot \left( \psi_{l''',l'''-\delta_2}^{j,I_{ab}} \right)^\dagger (\ldots y_6) \,, \end{aligned} \tag{6.148}$$

with $l' = (1,-1,0)^T$, $l'' = (0,1,-1)^T$ and $l''' = (0,0,2)^T$. The boundary conditions in (6.148) can be used to relate terms with different $l$ in the integral of the trace (6.147), by shifting the domain of integration the following way:

$$\int_0^1 \mathrm{d}y_2\, \psi^{i,I_{ab}}_{l',l'-\delta_1} \cdot \left(\psi^{j,I_{ab}}_{l',l'-\delta_2}\right)^\dagger (\vec{y}) = \int_1^2 \mathrm{d}y_2\, \psi^{i,I_{ab}}_{l,l-\delta_1} \cdot \left(\psi^{j,I_{ab}}_{l,l-\delta_2}\right)^\dagger (\vec{y}) \tag{6.149}$$

and similar for the directions $y_4$ and $y_6$. By using the relation (6.149), the sum over $l$ can be completely absorbed into the integration by enlarging the domain of integration. For the three types of lattices of $\Gamma_a \cap \Gamma_b$, considered in appendix D, one finds that the enlarged domain of integration is given by the unit cell $\tilde{C}$ of the lattice spanned by

$$\vec{v}_1 = (1,0,0)^T\,, \quad \vec{v}_3 = (0,1,0)^T\,, \quad \vec{v}_5 = (0,0,1)^T\,, \tag{6.150}$$

and

(a)

$$\vec{v}_2 = \begin{pmatrix} \frac{N_a^{(1)}N_b^{(1)}}{d^{(1)}}K_1 \\ 0 \\ 0 \end{pmatrix}, \quad \vec{v}_4 = \begin{pmatrix} 0 \\ \frac{N_a^{(2)}N_b^{(2)}}{d^{(2)}}K_2 \\ 0 \end{pmatrix}, \quad \vec{v}_6 = \begin{pmatrix} 0 \\ 0 \\ \frac{N_a^{(3)}N_b^{(3)}}{d^{(3)}}K_3 \end{pmatrix}, \tag{6.151}$$

for the cases (a)-(c) for $\Gamma_a \cap \Gamma_b$ in appendix D,

(b)

$$\vec{v}_2 = \begin{pmatrix} \frac{N_a^{(1)}N_b^{(1)}}{2d^{(1)}}K_1 \\ \frac{N_a^{(2)}N_b^{(2)}}{d^{(2)}}K_2 \\ 0 \end{pmatrix}, \quad \vec{v}_4 = \begin{pmatrix} \frac{N_a^{(1)}N_b^{(1)}}{2d^{(1)}}K_1 \\ -\frac{N_a^{(2)}N_b^{(2)}}{d^{(2)}}K_2 \\ 0 \end{pmatrix}, \quad \vec{v}_6 = \begin{pmatrix} 0 \\ 0 \\ \frac{N_a^{(3)}N_b^{(3)}}{d^{(3)}}K_3 \end{pmatrix}, \tag{6.152}$$

for the cases (d)-(e) for $\Gamma_a \cap \Gamma_b$ in appendix D and

(c)

$$\vec{v}_2 = \begin{pmatrix} \frac{N_a^{(1)}N_b^{(1)}}{2d^{(1)}}K_1 \\ \frac{N_a^{(2)}N_b^{(2)}}{d^{(2)}}K_2 \\ 0 \end{pmatrix}, \quad \vec{v}_4 = \begin{pmatrix} \frac{N_a^{(1)}N_b^{(1)}}{2d^{(1)}}K_1 \\ -\frac{N_a^{(2)}N_b^{(2)}}{d^{(2)}}K_2 \\ 0 \end{pmatrix}, \quad \vec{v}_6 = \begin{pmatrix} 0 \\ \frac{N_a^{(2)}N_b^{(2)}}{d^{(2)}}K_2 \\ \frac{N_a^{(3)}N_b^{(3)}}{d^{(3)}}K_3 \end{pmatrix}, \tag{6.153}$$

for the case (f) for $\Gamma_a \cap \Gamma_b$ in appendix D.

Hence the integral in (6.145) can be reduces to an integration of the term with $l = 0$ over $\tilde{C}$

$$\int_{T^6} \mathrm{d}^6 x\, \mathrm{Tr}\left\{\psi^{i,I_{ab}} \cdot \left(\psi^{j,I_{ab}}\right)^\dagger\right\} = 2\prod_{h=1}^{3} {A_h}^2 \mathrm{Im}(K_h) \int_{\tilde{C}} \mathrm{d}^6 y\, F(\vec{y})\,, \tag{6.154}$$

with

$$F(\vec{w}) =: \delta_{\delta_1,\delta_2}\, \psi^{i,I_{ab}}_{0,-\delta_1} \cdot \left(\psi^{j,I_{ab}}_{0,0-\delta_2}\right)^\dagger = \mathcal{N}^{i,I_{ab}} \mathcal{N}^{j,I_{ab}} \mathrm{e}^{-2\pi \sum_{h=1}^3 \frac{\tilde{I}^{(h)}_{ab}}{\mathrm{Im}(K_h)}(\mathrm{Im}(w_h))^2} \sum_{\lambda_{ab}\in\Gamma_a\cap\Gamma_b} \sum_{\rho_{ab}\in\Gamma_a\cap\Gamma_b} \tag{6.155}$$
$$\cdot\, \mathrm{e}^{2\pi i \sum_{h=1}^3 \left(\left(\tilde{I}^{(h)}_{ab}\lambda^{(h)}_{ab} + \frac{i^{(h)}}{N_b^{(h)}}\right) w_h - \left(\tilde{I}^{(h)}_{ab}\rho^{(h)}_{ab} + \frac{j^{(h)}}{N_b^{(h)}}\right)\overline{w}_h\right)} \cdot\, \mathrm{e}^{\pi i \sum_{h=1}^3 \left(\left(\tilde{I}^{(h)}_{ab}\lambda^{(h)}_{ab} + \frac{i^{(h)}}{N_b^{(h)}}\right)^2 \frac{K_h}{\tilde{I}^{(h)}_{ab}} - \left(\tilde{I}^{(h)}_{ab}\rho^{(h)}_{ab} + \frac{j^{(h)}}{N_b^{(h)}}\right)^2 \frac{\overline{K}_h}{\tilde{I}^{(h)}_{ab}}\right)}\,.$$

When integrating over $y_1$, $y_3$ and $y_5$ in (6.145), the integrals given by

$$\int_0^1 \mathrm{d}y_{2h-1} \exp\left\{2\pi i\left(\tilde{I}_{ab}^{(h)}\left(\lambda_{ab}^{(h)}-\rho_{ab}^{(h)}\right)+\frac{i^{(h)}-j^{(h)}}{N_b^{(h)}}\right)y_{2h-1}\right\}\,,\ \forall h\in\{1,2,3\}\,, \tag{6.156}$$

need to be computed. From the definition of the labels $i$, $j$ in (6.139), it can be deduced that the non integer part in the term

$$\tilde{I}_{ab}^{(h)}\left(\lambda_{ab}^{(h)}-\rho_{ab}^{(h)}\right)+\frac{i^{(h)}-j^{(h)}}{N_b^{(h)}} \tag{6.157}$$

is $\frac{\delta_1^{(h)}-\delta_2^{(h)}}{N_b^{(h)}}$ and taking the Kronecker delta $\delta_{\delta_1,\delta_2}$ from the trace (6.147) into account, the non integer part vanishes and the term in (6.157) is integer. Hence the integration over $y_1$, $y_3$ and $y_5$ leads to Kronecker deltas implying the conditions

$$\tilde{I}_{ab}^{(h)}\left(\lambda_{ab}^{(h)}-\rho_{ab}^{(h)}\right)+\frac{i^{(h)}-j^{(h)}}{N_b^{(h)}}=0\,,\quad \forall\, h\in\{1,2,3\}\,. \tag{6.158}$$

The integral in (6.154) simplifies to

$$\begin{aligned}&\left(\mathcal{N}^{i,I_{ab}}\right)^2\int_{\tilde{C}}\mathrm{d}y_2\mathrm{d}y_4\mathrm{d}y_6\,\mathrm{e}^{-2\pi\sum_{h=1}^3\frac{\tilde{I}_{ab}^{(h)}}{\mathrm{Im}(K_h)}(\mathrm{Im}(w_h))^2}\sum_{\lambda_{ab}\in\Gamma_a\cap\Gamma_b}\mathrm{e}^{-4\pi\sum_{h=1}^3\left(\tilde{I}_{ab}^{(h)}\lambda_{ab}^{(h)}+\frac{i^{(h)}}{N_b^{(h)}}\right)\mathrm{Im}(w_h)}\,\mathrm{e}^{-2\pi\sum_{h=1}^3\left(\tilde{I}_{ab}^{(h)}\lambda_{ab}^{(h)}+\frac{i^{(h)}}{N_b^{(h)}}\right)^2\frac{\mathrm{Im}(K_h)}{\tilde{I}_{ab}^{(h)}}}\\&=\left(\mathcal{N}^{i,I_{ab}}\right)^2\int_{\tilde{C}}\mathrm{d}y_2\mathrm{d}y_4\mathrm{d}y_6\sum_{\lambda_{ab}\in\Gamma_a\cap\Gamma_b}\mathrm{e}^{-2\pi\sum_{h=1}^3\frac{\tilde{I}_{ab}^{(h)}}{\mathrm{Im}(K_h)}\left(\mathrm{Im}(w)+\mathrm{Im}(K_h)\lambda_{ab}+\frac{i^{(h)}}{N_b^{(h)}}\frac{\mathrm{Im}(K_h)}{\tilde{I}_{ab}^{(h)}}\right)^2}\,.\end{aligned} \tag{6.159}$$

Notice that the terms $\mathrm{Im}(K_h)\lambda_{ab}^{(h)}$ in the exponent are actually components of lattice vectors of the lattice spanned by $\vec{v}_2$, $\vec{v}_4$ and $\vec{v}_6$ in (6.151), (6.152) and (6.153), as can be seen by inserting the generators of $\Gamma_a\cap\Gamma_b$, which are presented in appendix D. The sum over $\lambda_{ab}$ can be absorbed into the integration by enlarging the domain of integration similar to the method,which was used to get rid of the sum over $l$:

$$\int_{\tilde{C}}\mathrm{d}y_2\mathrm{d}y_4\mathrm{d}y_6\sum_{\lambda_{ab}\in\Gamma_a\cap\Gamma_b}\rightarrow\int_{\mathbb{R}^3}\mathrm{d}y_2\mathrm{d}y_4\mathrm{d}y_6\sum_{\lambda_{ab}\in\Gamma_a\cap\Gamma_b}=\frac{1}{2}\prod_{h=1}^3\int_{\mathbb{R}}\frac{\mathrm{Im}(w_h)}{\mathrm{Im}(K_h)}\,, \tag{6.160}$$

where in the last step a change of basis is performed. The change of basis makes the following computations simpler, since the function in (6.159) depends on $\mathrm{Im}(w_h)$. The remaining integral in (6.159) are just Gaussian integrals and one receives

$$\frac{1}{2}\prod_{h=1}^3\int_{\mathbb{R}}\frac{\mathrm{Im}(w_h)}{\mathrm{Im}(K_h)}\mathrm{e}^{-2\pi\sum_{h=1}^3\frac{\tilde{I}_{ab}^{(h)}}{\mathrm{Im}(K_h)}\left(\mathrm{Im}(w)++\frac{i^{(h)}}{N_b^{(h)}}\frac{\mathrm{Im}(K_h)}{\tilde{I}_{ab}^{(h)}}\right)^2}=\frac{1}{2}\prod_{h=1}^3\left(2\tilde{I}_{ab}^{(h)}\mathrm{Im}(w_h)\right)^{-\frac{1}{2}}\,. \tag{6.161}$$

Inserting the results into (6.145), the normalization factors need to satisfy

$$g^{-2}\left(\mathcal{N}^{i,I_{ab}}\mathcal{N}^{j,I_{ab}}\right)^2\prod_{h=1}^3{A_h}^2\sqrt{\frac{\mathrm{Im}(K_h)}{2\tilde{I}_{ab}^{(h)}}}=\delta_{i,j}\,, \tag{6.162}$$

which leads to the normalization constant

$$\mathcal{N}^{i,I_{ab}} = g \prod_{h=1}^{3} \frac{1}{A_h} \left( \frac{2\tilde{I}_{ab}^{(h)}}{\mathrm{Im}(K_h)} \right)^{\frac{1}{4}} . \tag{6.163}$$

## 6.4 Yukawa couplings from overlapping wavefunctions

### 6.4.1 Yukawa couplings from magnetic fluxes

In the previous sections it was discussed how magnetic fluxes on a torus break the gauge symmetry and lead to chiral matter on the $T^6$. In this section some results of appendix A in [115] are repeated in order to explain how compactifying a ten dimensional spacetime on a $T^6$, with magnetic fluxes on it leads to four dimensional Yukawa couplings. The magnetic fluxes can only be turned on in the internal space in order to preserve Poincare invariance in the uncompact directions. By giving the gauge fields vev's in the internal space, they only break translation invariance in the $T^6$ and correspond to D boundary conditions in the T-dual theory.

Six dimensions of a 10 dimensional spacetime $M^{10}$ with a $U(P)$ gauge symmetry shall be compactified on a torus

$$M^{10} \to M^4 \times T^6 . \tag{6.164}$$

The field content shall be given by gauge vector fields $A$ and gauginos $\Psi$

$$A_I : M^{10} \to \mathbb{C}^{P\times P} , \quad \Psi_{\vec{\epsilon}} \to \mathbb{C}^{P\times P} , \tag{6.165}$$

which are described by the Lagrangians given in (6.29) and (6.82) for 10 dimensions

$$\mathcal{L} = -\frac{1}{4g^2}\mathrm{Tr}\left\{F^{IJ}F_{IJ}\right\} + \frac{i}{2g^2}\mathrm{Tr}\left\{\overline{\Psi}\Gamma^I D_I \Psi\right\} . \tag{6.166}$$

Here $I, J \in \{0, 1, ..., 9\}$ denote the spacetime indices of $M^{10}$ and $F_{IJ}$ are the components of the field strength tensor. The fermionic part in (6.166) contains the term

$$\mathcal{L} \supset \frac{1}{2g^2}\mathrm{Tr}\left\{\overline{\Psi}\Gamma^I [A_I, \Psi]\right\} , \tag{6.167}$$

which leads to Yukawa couplings after turning on vev's in the diagonal elements of $A_I$. As already discussed in section 6.3.1, the gauge fields $A_I$ and $\psi_{\vec{\epsilon}}$ decompose into gauge fields of the unbroken gauge group and scalars and fermions in bifundamental representations. Let $A$ and $\Psi$ be expanded in the basis $U^a$, $e_{ab}$ as in (6.30). The off-diagonal elements of $A$ and $\Psi$ are denoted by $W_{ab}$ and $\chi_{ab}$. Then the term containing the off-diagonal elements in (6.167) is given by

$$\frac{1}{2g^2}\mathrm{Tr}\left\{\overline{\chi}_{cb}\Gamma_I \left[W^I_{ca}, \chi_{ab}\right]\right\} \subset \mathcal{L} . \tag{6.168}$$

and describes triliner couplings of chiral fields. Decomposing $W^I_{ab}$ and $\chi_{ab}$ into eigenmodes of the Dirac and Laplace operator $\not{D}_6 = \Gamma^i D_i$ and $\Delta_6 = D_i D^i$ of the internal space, such as

$$\begin{aligned} \chi^{ab}(x^I) &= \sum_n {\eta_n}^{ab}(x^\mu) \otimes {\psi_n}^{ab}(x^i)\,, \\ W_J^{ab}(x^I) &= \sum_n {\varphi_{n,J}}^{ab}(x^\mu) \otimes {\phi_{n,J}}^{ab}(x^i)\,, \end{aligned} \tag{6.169}$$

with

$$\begin{aligned} i\not{D}_6 {\psi_n}^{ab} &= m_n {\psi_n}^{ab}\,, \\ \Delta_6 {\phi_{n,J}}^{ab} &= M_n^2 {\phi_{n,J}}^{ab}\,, \end{aligned} \tag{6.170}$$

where $m_n$ and $M_n^2$ are the corresponding masses for the Kaluza-Klein modes. Only zeromodes are considered, because the compactification radii of the torus are assumed to be very small and the Kaluza-Klein momenta therefor to be very high. It is further imposed that the magnetic flux is contained on D9-branes, wrapping calibrated cycles on the torus, s.t. a massless scalar is contained in the chiral spectrum. Therefore, for the low energy spectrum in $M^4$, only the terms containing the zeromodes in (6.168) are relevant and by integrating over the internal space, one gets the Yukawa couplings $Y_{ijk}$ of the form [21, 115]

$$Y_{ijk} = \frac{1}{g^2} \int_{T^6} \mathrm{d}^6 x \left(\psi_i^a\right)^\dagger \Gamma^m \phi_{j,m}^b \psi_k^c f_{abc}\,. \tag{6.171}$$

$f_{abc} = \mathrm{Tr}\,\{T_a, [T_b, T_c]\}$ are the structure constants with $\{T_\alpha\}_{\alpha\in\{a,b,c\}}$ the generators of the gauge group and $i, j, k$ denote indepenendent zeromodes. In sections 6.3.1 and 6.3.2, the wavefunctions for the zeromodes $\psi_0$ and $\phi_0$ where computed, and in section 6.3.4, the zeromodes where decomposed into $I_{ab}$ indepenedent zeromodes. Inserting for $\Psi$ and $A$ in (6.167) the results from 6.3.1 and 6.3.2 for the spin and polarization states $\psi_{\vec{\epsilon}}$ and $\Phi_\pm$ and integrating over the internal space, one gets the corresponding terms to (6.168), which are

$$\frac{1}{g^2} \int_{T^6} \mathrm{d}^6 x\, \mathrm{Tr}\,\{\psi_{+++} [\Phi_-, \psi_{+++}]\}\,, \tag{6.172}$$

and its CPT conjugate. The state $\psi_{+++}$ and $\Phi_-$ contain the normalizable wavefunctions for the choice of signs made in (6.99) for $\tilde{I}_{ab}$, $\tilde{I}_{bc}$, $\tilde{I}_{ca}$. By inserting the results for $\psi_{+++}$ and $\Phi_-$ into (6.172), one gets for the Yukawa couplings the following integral over overlapping wavefunctions :

$$Y_{ijk} = \frac{1}{g^2} \int_{T^6} \mathrm{d}^6 x \mathrm{Tr}\left\{\psi^{j,I_{ca}} \cdot \psi^{j,I_{ab}} \cdot \left(\psi^{k,I_{cb}}\right)^\dagger\right\}\,, \tag{6.173}$$

where $i, j, k$ denote independent zeromodes.

### 6.4.2 Computing Yukawa couplings

In this section the expressions from (6.144) are inserted into (6.173) and the integral is computed. The result gives the Yukawa couplings $Y_{ijk}$ for the three chiral fields $\psi^{j,I_{ca}}$ and $\psi^{k,I_{cb}}$.

#### Domain of integration

First the domain of integration in (6.173) has to be specified. The fundamental domain of the $T^6$ can be deduced from the moduli, specified in (6.18) and is given by the fundamental cell $C$ of a lattice spanned by the vectors

$$\vec{v}_1 = (1,0,0)^T\,, \quad \vec{v}_2 = (K_1, -K_2, 0)^T\,, \quad \vec{v}_3 = (0,1,0)^T\,, \qquad (6.174)$$
$$\vec{v}_4 = (0, K_2, -K_3)^T\,, \qquad \vec{v}_5 = (0,0,1)^T\,, \qquad \vec{v}_6 = (0,0,2K_3)^T\,.$$

The integration in (6.173) can be expressed by

$$\int_{T^6} \mathrm{d}^6 x = (A_1 A_2 A_3)^2 \int_C \mathrm{d}^6 x = 2\prod_{h=1}^{3} {A_h}^2 \mathrm{Im}(K_h) \int_0^1 \mathrm{d}y_1 \int_0^1 \mathrm{d}y_2 \ldots \int_0^1 \mathrm{d}y_6\,, \qquad (6.175)$$

where the factor $2\prod_{h=1}^{3} {A_h}^2 \mathrm{Im}(K_h)$ is the determinant of the Jacobi-matrix from the change of basis. The trace in (6.173), taken over the gauge indices, can be expressed by the sum over a single summation index $l$, where $l$ belongs to the lattice $\frac{\Lambda_{SO(6)}}{\Gamma_a \cap \Gamma_b \Gamma_c} <$ as

$$\begin{aligned}\mathrm{Tr}\left\{\psi^{j,I_{ca}} \cdot \psi^{j,I_{ab}} \cdot \left(\psi^{k,I_{cb}}\right)^\dagger\right\} &= \sum_{k_a \in \frac{\Lambda_{SO(6)}}{\Gamma_a}} \sum_{k_b \in \frac{\Lambda_{SO(6)}}{\Gamma_b}} \sum_{k_c \in \frac{\Lambda_{SO(6)}}{\Gamma_c}} \psi^{j,I_{ca}}_{k_c,k_a} \cdot \psi^{j,I_{ab}}_{k_a,k_b} \cdot \left(\psi^{k,I_{cb}}_{k_c,k_b}\right)^\dagger \\ &= \sum_{l \in \frac{\Lambda_{SO(6)}}{\Gamma_a \cap \Gamma_b \cap \Gamma_c}} \delta_{\delta_{ab}+\delta_{ca},\delta_{cb}} \psi^{j,I_{ca}}_{l,l-\delta_{ca}} \cdot \psi^{j,I_{ab}}_{l-\delta_{ca},l-\delta_{ca}-\delta_{ab}} \\ &\quad \cdot \left(\psi^{k,I_{cb}}_{l-\delta_{ca}-\delta_{ab}+\delta_{cb},l-\delta_{ca}-\delta_{ab}}\right)^\dagger \end{aligned} \qquad (6.176)$$

The relations between the labels $i, j, k$ and the parameters for the phases $\delta_{ab}$, $\delta_{ca}$, $\delta_{cb}$ are given by (6.139). Notice that for the wavefunction $\psi^{k,I_{cb}}_{k_c,k_b}$, the parameter $k_c - k_b = \delta_{cb}$ is shifted into the first index of the matrix elements of $\phi^{k,I_{cb}}$ as in (6.141]), s.t. $\phi^{k,I_{cb}}_{k_c,k_b} = \phi^{k,I_{cb}}_{l-\delta_{cb},l}$, with

$$k_c - \delta_{cb} = l \mod \Gamma_c\,, \quad k_b = l \mod \Gamma_b\,, \quad \text{with} \quad k_c - k_b = \delta_{cb} \mod \Gamma_{d_{cb}}\,. \qquad (6.177)$$

This corresponds to relabeling $k$ by the components $\frac{N_b^{(h)}}{N_c^{(h)}} k^{(h)}$ as mentioned in (6.143). The choice of (6.177) will later turn out to be useful. Applying the boundary conditions from (6.59) to the overlapping wavefunction, one finds that $\psi^{j,I_{ca}}_{k_c,k_a} \cdot \psi^{j,I_{ab}}_{k_a,k_b} \cdot \left(\psi^{k,I_{cb}}_{k_b,k_c}\right)^\dagger (\vec{w})$, $\forall\, l$ with definite $i, j, k$, get identified on the torus

$$\begin{aligned}\psi^{j,I_{ca}}_{k_c,k_a} \psi^{j,I_{ab}}_{k_a,k_b} \left(\psi^{k,I_{cb}}_{k_b,k_c}\right)^\dagger (..., y_2 + 1, ...) &= \psi^{j,I_{ca}}_{k_c',k_a'} \psi^{j,I_{ab}}_{k_a',k_b'} \left(\psi^{k,I_{cb}}_{k_b',k_c'}\right)^\dagger (..., y_2, ...)\,, \\ \psi^{j,I_{ca}}_{k_c,k_a} \psi^{j,I_{ab}}_{k_a,k_b} \left(\psi^{k,I_{cb}}_{k_b,k_c}\right)^\dagger (..., y_4 + 1, ...) &= \psi^{j,I_{ca}}_{k_c'',k_a''} \psi^{j,I_{ab}}_{k_a'',k_b''} \left(\psi^{k,I_{cb}}_{k_b'',k_c''}\right)^\dagger (..., y_4, ...)\,, \\ \psi^{j,I_{ca}}_{k_c,k_a} \psi^{j,I_{ab}}_{k_a,k_b} \left(\psi^{k,I_{cb}}_{k_b,k_c}\right)^\dagger (..., y_6 + 1) &= \psi^{j,I_{ca}}_{k_c''',k_a'''} \psi^{j,I_{ab}}_{k_a''',k_b'''} \left(\psi^{k,I_{cb}}_{k_b''',k_c'''}\right)^\dagger (..., y_6)\,,\end{aligned} \qquad (6.178)$$

where $k'_\alpha = k_\alpha + (1,-1,0)^T$, $k''_\alpha = k_\alpha + (0,1,-1)^T$ and $k'''_\alpha = k_\alpha + (0,0,2)^T$ and the relation (6.98) has been used. These boundary conditions can be used to simplify the integral in (6.173), by shifting the domain of integration in (6.173) for each term in the trace, in such a way that each $l$ for the terms in (6.176) is shifted to $l = 0$. That way it only needs to be integrated one term of (6.176) over an enlarged domain of integration

$$\int_C \mathrm{d}^6 y \sum_{l\in\frac{\Lambda_{\mathrm{SO}(6)}}{\Gamma_a\cap\Gamma_b\cap\Gamma_c}} \delta_{\delta_{ab}+\delta_{ca},\delta_{cb}} \psi^{j,I_{ca}}_{l,l-\delta_{ca}} \cdot \psi^{j,I_{ab}}_{l-\delta_{ca},l-\delta_{ca}-\delta_{ab}} \cdot \left(\psi^{k,I_{cb}}_{l-\delta_{ca}-\delta_{ab}+\delta_{cb},l-\delta_{ca}-\delta_{ab}}\right)^\dagger$$

$$= \int_{\tilde{C}} \mathrm{d}^6 y\, \delta_{\delta_{ab}+\delta_{ca},\delta_{cb}} F_{i,j,k}(\vec{y})\,, \tag{6.179}$$

with

$$F_{i,j,k}(\vec{y}) =: \psi^{j,I_{ca}}_{0,-\delta_{ca}} \cdot \psi^{j,I_{ab}}_{-\delta_{ca},-\delta_{ca}-\delta_{ab}} \cdot \left(\psi^{k,I_{cb}}_{-\delta_{ab}-\delta_{ca}+\delta_{cb},-\delta_{ca}-\delta_{ab}}\right)^\dagger \tag{6.180}$$

where $\tilde{C}$ is now a bigger domain of integration and depends on the lattice $\Gamma_a \cap \Gamma_b \cap \Gamma_c$. In appendix D three types of lattices for $\Gamma_a \cap \Gamma_b \cap \Gamma_c$ are discussed. The enlarged domain of integration $\tilde{C}$ is given by the unit cells of the lattice spanned by the vectors $\vec{v}_1$, $v_3$, $v_5$ as in (6.174) and

$$\vec{v}_2 = \begin{pmatrix} N^{(1)}_{abc} K_1 \\ 0 \\ 0 \end{pmatrix}, \quad \vec{v}_4 = \begin{pmatrix} 0 \\ N^{(2)}_{abc} K_2 \\ 0 \end{pmatrix}, \quad \vec{v}_6 = \begin{pmatrix} 0 \\ 0 \\ N^{(3)}_{abc} K_3 \end{pmatrix}, \tag{6.181}$$

for $\Gamma_a \cap \Gamma_b \cap \Gamma_c$ given in (D.12),

$$\vec{v}_2 = \begin{pmatrix} \frac{N^{(1)}_{abc} K_1}{2} \\ N^{(2)}_{abc} K_2 \\ 0 \end{pmatrix}, \quad \vec{v}_4 = \begin{pmatrix} \frac{N^{(1)}_{abc} K_1}{2} \\ N^{(2)}_{abc} K_2 \\ 0 \end{pmatrix}, \quad \vec{v}_6 = \begin{pmatrix} 0 \\ 0 \\ N^{(3)}_{abc} K_3 \end{pmatrix}, \tag{6.182}$$

for $\Gamma_a \cap \Gamma_b \cap \Gamma_c$ given in (D.13) and

$$\vec{v}_2 = \begin{pmatrix} \frac{N^{(1)}_{abc} K_1}{2} \\ N^{(2)}_{abc} K_2 \\ 0 \end{pmatrix}, \quad \vec{v}_4 = \begin{pmatrix} \frac{N^{(1)}_{abc} K_1}{2} \\ N^{(2)}_{abc} K_2 \\ 0 \end{pmatrix}, \quad \vec{v}_6 = \begin{pmatrix} 0 \\ N^{(2)}_{abc} K_2 \\ N^{(3)}_{abc} K_3 \end{pmatrix}, \tag{6.183}$$

for $\Gamma_a \cap \Gamma_b \cap \Gamma_c$ given in (D.14), where $N^{(h)}_{abc} = \frac{N^{(h)}_a N^{(h)}_b N^{(h)}_c}{d^{(h)}_{abc}}$ was defined. The integration over the directions of $\vec{v}_2$, $\vec{v}_4$, $\vec{v}_6$ is then given by

$$l_1 = (N^{(1)}_{abc}, N^{(1)}_{abc}, N^{(1)}_{abc}/2)^T\,, \quad l_2 = (0, N^{(2)}_{abc}, N^{(2)}_{abc}/2)^T\,, \quad l_3 = (0, 0, N^{(3)}_{abc}/2)^T\,, \tag{6.184}$$

for $\tilde{C}$ given by (6.181),

$$l_1 = \begin{pmatrix} N^{(1)}_{abc}/2 \\ N^{(1)}_{abc}/2 + N^{(2)}_{abc} \\ N^{(1)}_{abc}/4 + N^{(2)}_{abc}/2 \end{pmatrix}, \quad l_2 = \begin{pmatrix} N^{(1)}_{abc}/2 \\ N^{(1)}_{abc}/2 - N^{(2)}_{abc} \\ N^{(1)}_{abc}/4 - N^{(2)}_{abc}/2 \end{pmatrix}, \quad l_3 = \begin{pmatrix} 0 \\ 0 \\ N^{(3)}_{abc}/2 \end{pmatrix}, \tag{6.185}$$

for $\tilde{C}$ given by (6.182) and

$$l_1 = \begin{pmatrix} N^{(1)}_{abc}/2 \\ N^{(1)}_{abc}/2 + N^{(2)}_{abc} \\ N^{(1)}_{abc}/4 + N^{(2)}_{abc}/2 \end{pmatrix}, \quad l_2 = \begin{pmatrix} N^{(1)}_{abc}/2 \\ N^{(1)}_{abc}/2 - N^{(2)}_{abc} \\ N^{(1)}_{abc}/4 - N^{(2)}_{abc}/2 \end{pmatrix}, \quad l_3 = \begin{pmatrix} 0 \\ N^{(2)}_{abc} \\ N^{(2)}_{abc} + N^{(3)}_{abc}/2 \end{pmatrix}, \tag{6.186}$$

for $\tilde{C}$ given by (6.183). It can be verified that the components of $l_i$ are indeed integer and the integration of $F_{i,j,k}$ over $\tilde{C}$ indeed covers all terms in (6.176), by determining whether $N_{abc}$ is even or odd for $N_\alpha^{(h)}$ even or odd and $g.c.d.(N_\alpha^h, m_\alpha^h)$ 1 or 2 from the definitions in appendix D.

### Integrating overlapping wavefunctions

In this section the integration given in (6.179) is computed. Inserting the expressions for the wavefunctions and using the relation (6.98), one gets the following expression for $F_{i,j,k}$:

$$\begin{aligned}
&\psi^{j,I_{ca}}_{0,-\delta_{ca}} \cdot \psi^{j,I_{ab}}_{-\delta_{ca},-\delta_{ca}-\delta_{ab}} \cdot \left(\psi^{k,I_{cb}}_{0,-\delta_{cb}}\right)^\dagger (\vec{w}) = \\
&\mathcal{N}^{i,I_{ab}} \mathcal{N}^{j,I_{ca}} \mathcal{N}^{k,I_{cb}} \exp\left\{-2\pi \sum_{h=1}^{3} \frac{\tilde{I}^{(h)}_{cb}}{\mathrm{Im}(K_h)} (\mathrm{Im}(w_h))^2\right\} \sum_{\lambda_{ab}\in\Gamma_a\cap\Gamma_b} \sum_{\lambda_{ca}\in\Gamma_a\cap\Gamma_a} \sum_{\lambda_{cb}\in\Gamma_c\cap\Gamma_b} \\
&\cdot \mathrm{e}^{2\pi i \sum_{h=1}^{3}\left\{\left(\tilde{I}^{(h)}_{ab}\left(\lambda^{(h)}_{ab}-\delta^{(h)}_{ca}\right)+\tilde{I}^{(h)}_{ca}\lambda^{(h)}_{ca}+\frac{i^{(h)}}{N^{(h)}_b}+\frac{j^{(h)}}{N^{(h)}_a}\right)w_h-\left(\tilde{I}^{(h)}_{cb}\left(\lambda^{(h)}_{cb}\delta^{(h)}_{ab}-\delta^{(h)}_{ca}\right)+\frac{k^{(h)}}{N^{(h)}_b}\right)\overline{w}_h\right\}} \\
&\cdot \mathrm{e}^{\pi i \sum_{h=1}^{3}\left\{\left(\tilde{I}^{(h)}_{ab}\left(\lambda^{(h)}_{ab}-\delta^{(h)}_{ca}\right)+\frac{i^{(h)}}{N^{(h)}_b}\right)^2\frac{K_h}{\tilde{I}^{(h)}_{ab}}+\left(\tilde{I}^{(h)}_{ca}\lambda^{(h)}_{ca}+\frac{j^{(h)}}{N^{(h)}_a}\right)^2\frac{K_h}{\tilde{I}^{(h)}_{ca}}-\left(\tilde{I}^{(h)}_{cb}\left(\lambda^{(h)}_{cb}-\delta^{(h)}_{ab}-\delta^{(h)}_{ca}\right)+\frac{k^{(h)}}{N^{(h)}_b}\right)^2\frac{K_h}{\tilde{I}^{(h)}_{cb}}\right\}} .
\end{aligned} \tag{6.187}$$

Taking a closer look at the following term

$$\tilde{I}^{(h)}_{ab}\left(\lambda^{(h)}_{ab}-\delta^{(h)}_{ca}\right)+\tilde{I}^{(h)}_{ca}\lambda^{(h)}_{ca}-\tilde{I}^{(h)}_{cb}\left(\lambda^{(h)}_{cb}-\delta^{(h)}_{ab}-\delta^{(h)}_{ca}\right)+\frac{i^{(h)}}{N^{(h)}_b}+\frac{j^{(h)}}{N^{(h)}_a}-\frac{k^{(h)}}{N^{(h)}_b}\,, \tag{6.188}$$

which is a factor in the exponent of the term depending on $y_1$, $y_3$ and $y_5$, one finds that, when the Kronecker delta $\delta_{\delta_{ca}+\delta_{ab},\delta_{cb}}$ from (6.176) is taken into account, the expression in (6.188) is integer $\forall\, h \in \{1,2,3\}$ [8]. Therefor the integrals over $y_1$, $y_3$ and $y_5$ can be replaced by the following Kronecker deltas

$$\begin{aligned}
&\int_0^1 \mathrm{d}y_{2h-1}\, \mathrm{e}^{2\pi i\left(\tilde{I}^{(h)}_{ab}\left(\lambda^{(h)}_{ab}-\delta^{(h)}_{ca}\right)+\tilde{I}^{(h)}_{ca}\lambda^{(h)}_{ca}-\tilde{I}^{(h)}_{cb}\left(\lambda^{(h)}_{cb}-\delta^{(h)}_{ab}-\delta^{(h)}_{ca}\right)+\frac{i^{(h)}}{N^{(h)}_b}+\frac{j^{(h)}}{N^{(h)}_a}-\frac{k^{(h)}}{N^{(h)}_b}\right)y_{2h-1}} \\
&\qquad = \delta_{\tilde{I}^{(h)}_{ab}\left(\lambda^{(h)}_{ab}-\delta^{(h)}_{ca}\right)+\tilde{I}^{(h)}_{ca}\lambda^{(h)}_{ca}-\tilde{I}^{(h)}_{cb}\left(\lambda^{(h)}_{cb}-\delta^{(h)}_{ca}-\delta^{(h)}_{ab}\right)+\frac{i^{(h)}}{N^{(h)}_b}+\frac{j^{(h)}}{N^{(h)}_a}-\frac{k^{(h)}}{N^{(h)}_b}} \,.
\end{aligned} \tag{6.190}$$

They lead to the conditions

$$\tilde{I}^{(h)}_{ab}\left(\lambda^{(h)}_{ab}-\delta^{(h)}_{ca}\right)+\tilde{I}^{(h)}_{ca}\lambda^{(h)}_{ca}-\tilde{I}^{(h)}_{cb}\lambda^{(h)}_{cb}+\frac{i^{(h)}}{N^{(h)}_b}+\frac{j^{(h)}}{N^{(h)}_a}-\frac{k^{(h)}}{N^{(h)}_b}=0\,, \quad \forall\, h \in \{1,2,3\}\,, \tag{6.191}$$

[8] The terms $\tilde{I}^{(h)}_{\alpha\beta}\lambda^{(h)}_{\alpha\beta}$ are integers $\forall \lambda_{\alpha\beta} \in \Gamma_\alpha \cap \Gamma_\beta$, with $\alpha, \beta \in \{a, b, c\}$ as can be seen by inserting the generators of $\Gamma_\alpha \cap \Gamma_\beta$, which are listed in appendix D, for $\lambda_{\alpha\beta}$. According to (5.57) and (6.177) the non integer part of the term $\frac{i^{(h)}}{N^{(h)}_b}+\frac{j^{(h)}}{N^{(h)}_a}-\frac{k^{(h)}}{N^{(h)}_c}$ is given by $\frac{n^h_b\delta^{(h)}_{ab}}{N^{(h)}_b}+\frac{n^h_a\delta^{(h)}_{ca}}{N^{(h)}_a}-\frac{n^h_c\delta^{(h)}_{cb}}{N^{(h)}_c}$. By applying the condition $\delta_{ca}+\delta_{ab}=\delta_{cb}$ from the Kronecker delta in (6.176), the potentially non integer term in (6.188) vanishes

$$-\tilde{I}^{(h)}_{ab}\delta^{(h)}_{ca}+\frac{n^{(h)}_b}{N^{(h)}_b}\delta^{(h)}_{ab}+\frac{n^{(h)}_a}{N^{(h)}_a}\delta^{(h)}_{ca}-\frac{n^{(h)}_c}{N^{(h)}_c}\delta^{(h)}_{cb}+\tilde{I}^{(h)}_{cb}\delta^{(h)}_{cb}=0\,, \quad \forall\, h \in \{1,2,3\}\,. \tag{6.189}$$

which are three Diophantine equations, similar to those ee already encountered in section 5.3.2. There the Diophantine equations contained the conditions for D6-branes to wrap 3-cycles forming closed triangles. Apparently the condition for open string instantons wrapping triangles on the $T^6_{\mathrm{SO}(12)}$, corresponds to in the dual theory to the trace over the zero modes. In order to solve (6.191), the following relabeling of the indices $i, j, k$ is performed, similar to (5.28),

$$i^{(h)} \to \frac{i^{(h)}}{d_b^{(h)}} I_{cb}^{(h)} , \quad j^{(h)} \to \frac{j^{(h)}}{d_a^{(h)}} I_{ba}^{(h)} , \quad k^{(h)} \to \frac{k^{(h)}}{d_c^{(h)}} I_{ac}^{(h)} . \tag{6.192}$$

Solutions to the Diophantine equations in (6.191) can be given by

$$\begin{aligned}
\lambda_{ab}^{(h)} &= N_{abc}^{(h)} p^{(h)} + N_a^{(h)} N_b^{(h)} M_c^{(h)} q^{(h)} + \frac{j^{(h)}}{d_a^{(h)}} N_b^{(h)} + \delta_{ca} , \\
\lambda_{ca}^{(h)} &= N_{abc}^{(h)} p^{(h)} + N_a^{(h)} M_b^{(h)} N_c^{(h)} q^{(h)} - \frac{k^{(h)}}{d_c^{(h)}} N_a^{(h)} , \\
\lambda_{cb}^{(h)} &= N_{abc}^{(h)} p^{(h)} + M_a^{(h)} N_b^{(h)} N_c^{(h)} q^{(h)} + \frac{i^{(h)}}{d_b^{(h)}} N_c^{(h)} + \delta_{ca} + \delta_{ab} ,
\end{aligned} \tag{6.193}$$

where $p$ and $q$ belong to three dimensional lattices $\Lambda_p^3 \subseteq \mathbb{Z}^3$ and $\Lambda_q^3 \subseteq \mathbb{Z}^3$, which allow the components of $\lambda_{ab}$, $\lambda_{ca}$ and $\lambda_{cb}$ in (6.193) to be in accordance with the lattices $\Gamma_a \cap \Gamma_b$, $\Gamma_c \cap \Gamma_a$ and $\Gamma_c \cap \Gamma_b$. Inserting the solutions into the overlapping wavefunctions in (6.187), one receives

$$\begin{aligned}
&\int_0^1 \int_0^1 \int_0^1 \mathrm{d}y_1 \mathrm{d}y_3 \mathrm{d}y_5 \; \psi^{j,I_{ca}}_{0,-\delta_{ca}} \cdot \psi^{j,I_{ab}}_{-\delta_{ca},-\delta_{ca}-\delta_{ab}} \cdot \left(\psi^{k,I_{cb}}_{0,-\delta_{cb}}\right)^\dagger (\vec{y}) = \mathcal{N}^{i,I_{ab}} \mathcal{N}^{j,I_{ca}} \mathcal{N}^{k,I_{cb}} \\
&\cdot \sum_{p \in \Lambda_p^3} \sum_{q \in \Lambda_q^3} \mathrm{e}^{-2\pi \sum_{h=1}^3 \frac{\tilde{I}_{cb}^{(h)}}{\mathrm{Im}(\tau_h)} \left[ \mathrm{Im}(w_h) + N_{abc}^{(h)} \mathrm{Im}(K_h) p^{(h)} + \left( I_{cb}^{(h)} M_a^{(h)} q^{(h)} + \frac{i^{(h)}}{N_b^{(h)}} \frac{I_{cb}^{(h)}}{d_b^{(h)}} - \frac{k^{(h)}}{N_c^{(h)}} \frac{I_{ca}^{(h)}}{d_c^{(h)}} \right) \frac{\mathrm{Im}(K_h)}{\tilde{I}_{cb}^{(h)}} \right]^2} \\
&\cdot \exp\left\{ i\pi \sum_{h=1}^3 \left( \frac{i^{(h)}}{d_b^{(h)} I_{ab}^{(h)}} + \frac{k^{(h)}}{d_c^{(h)} I_{cb}^{(h)}} + \frac{j^{(h)}}{d_a^{(h)} I_{ca}^{(h)}} + 2q^{(h)} \right)^2 K_h |I_{ab}^{(h)} I_{ca}^{(h)} I_{cb}^{(h)}| \right\} =: H_{ijk}(\mathrm{Im}(\vec{w})) .
\end{aligned} \tag{6.194}$$

As already mentioned the summation indices $p$ and $q$ must be chosen s.t. the components $\lambda_{ab}^{(h)}$, $\lambda_{ca}^{(h)}$ and $\lambda_{cb}^{(h)}$ from (6.193) belong to vectors of the $\Lambda_{\mathrm{S0}(6)}$ lattice. First the lattice $\Lambda_p^3$ shall be determined: When the components of $p$ satisfy

$$\sum_{h=1}^3 N_{abc}^{(h)} p^{(h)} = 0 \mod 2 . \tag{6.195}$$

it is guaranteed that the terms ins (6.193) depending on $p$ do not spoil the $SO(6)$ structure of the $\lambda$'s. From (6.195) it follows that as long as $N_{abc}^{(h)}$ is even, which means at least $N_a^{(h)}$, $N_b^{(h)}$ or $N_c^{(h)}$ is even, the component $p^{(h)}$ is independent from the components of $p$ of the other planes. But when $N_{abc}^{(h)}$ is odd, either $p^{(h)}$ has to be even or $N_{abc}^{(h')} p^{(h')}$ in a different plane $h'$ has to be odd too. Hence the lattice for $\Lambda_p^3$ depends on $\Gamma_a$, $\Gamma_b$ and $\Gamma_c$, and for the cases considered in appendix D, one finds that $\Lambda_p^3$ is spanned by the following generators

(i) $\rho_1 = (1,0,0)^T, \rho_2 = (0,1,0)^T, \rho_3 = (0,0,1)^T$ when $\Gamma_a \cap \Gamma_b \cap \Gamma_c$ is given by (D.12),

(ii) $\rho_1 = (1/2,1,0)^T, \rho_2 = (1/2,-1,0)^T, \rho_3 = (0,0,1)^T$ when $\Gamma_a \cap \Gamma_b \cap \Gamma_c$ is given by (D.13) and

(iii) $\rho_1 = (1/2, 1, 0)^T, \rho_2 = (1/2, -1, 0)^T, \rho_3 = (0, 1, 1)^T$ when $\Gamma_a \cap \Gamma_b \cap \Gamma_c$ is given by (D.14).

Notice that the vectors given by the components $N^{(h)}_{abc}\rho^{(1)}_i$, with $i \in \{1, 2, 3\}$, are exactly the lattice vectors $\vec{v}_2$, $\vec{v}_4$ and $\vec{v}_6$ spanning the lattice of $\tilde{C}$, which are presented in (6.181), (6.182) and (6.183). Therefore, the shift $p \to p + \rho_i$ in the term

$$\int_{\tilde{C}} \mathrm{d}y_2\mathrm{d}y_4\mathrm{d}y_6 \,\exp\left\{-2\pi \sum_{h=1}^{3} \frac{\tilde{I}^{(h)}_{cb}}{\mathrm{Im}(K_h)} \left(\mathrm{Im}(w_h) + N^{(h)}_{abc}p^{(h)} + c^h_{i,k}(q)\right)^2\right\}, \tag{6.196}$$

with

$$c^h_{i,k}(q) =: \left(I^{(h)}_{cb}M^{(h)}_a q^{(h)} + \frac{i^{(h)}}{N^{(h)}_b}\frac{I^{(h)}_{cb}}{d^{(h)}_b} - \frac{k^{(h)}}{N^{(h)}_c}\frac{I^{(h)}_{ca}}{d^{(h)}_c}\right)\frac{\mathrm{Im}(K_h)}{\tilde{I}^{(h)}_{cb}}, \tag{6.197}$$

can be replaced by a shifting the domain of integration by $\vec{v}_{2i}$ to the neighbouring unit cell $\tilde{C} + \vec{v}_{2i}$ just next to $\tilde{C}$. That way the whole sum over $p \in \Lambda^3_p$ can be absrobed into the integration, by enlarging the domain of integration of $y_2$, $y_4$, $y_6$ to $\mathbb{R}^3$

$$\begin{aligned}&\sum_{p\in\Lambda^3_p}\int_{\tilde{C}} \mathrm{d}y_2\mathrm{d}y_4\mathrm{d}y_6 \,\exp\left\{-2\pi \sum_{h=1}^{3} \frac{\tilde{I}^{(h)}_{cb}}{\mathrm{Im}(K_h)} \left(\mathrm{Im}(w_h) + N^{(h)}_{abc}p^{(h)} + c^h_{i,k}(q)\right)^2\right\} \\ &= \int_{\mathbb{R}^3} \mathrm{d}y_2\mathrm{d}y_4\mathrm{d}y_6 \,\exp\left\{-2\pi \sum_{h=1}^{3} \frac{\tilde{I}^{(h)}_{cb}}{\mathrm{Im}(K_h)} \left(\mathrm{Im}(w_h) + N^{(h)}_{abc}p^{(h)} + c^h_{i,k}(q)\right)^2\right\}\Bigg|_{p=\left(\begin{smallmatrix}0\\0\\0\end{smallmatrix}\right)}\end{aligned} \tag{6.198}$$

Since after the integration over $y_1$, $y_3$ and $y_5$, the integrand in (6.194) only depends on $\mathrm{Im}(w_h)$ it will be helpful to rewrite the remaining integration as

$$\int \mathrm{d}y_2\mathrm{d}y_4\mathrm{d}y_6 = \frac{1}{2}\prod_{h=1}^{3}\int \frac{\mathrm{Im}(\mathrm{d}w_h)}{\mathrm{Im}(K_h)}. \tag{6.199}$$

After having absorbed the sum over $p$ into the integration, as in (6.198), the integral in (6.176) becomes

$$\begin{aligned}&\int_0^1\int_0^1\int_0^1 \mathrm{d}y_2\mathrm{d}y_4\mathrm{d}y_6 \; H_{i,j,k}(\mathrm{Im}(\vec{w})) = \mathcal{N}^{i,I_{ab}}\mathcal{N}^{j,I_{ca}}\mathcal{N}^{k,I_{cb}}\sum_{q\in\Lambda^3_q} \\ &\cdot\exp\left\{i\pi\sum_{h=1}^{3}\left(\frac{i^{(h)}}{d^{(h)}_b I^{(h)}_{ab}} + \frac{k^{(h)}}{d^{(h)}_c I^{(h)}_{cb}} + \frac{j^{(h)}}{d^{(h)}_a I^{(h)}_{ca}} + 2q^{(h)}\right)^2 K_h|I^{(h)}_{ab}I^{(h)}_{ca}I^{(h)}_{cb}|\right\} \\ &\cdot\frac{1}{2}\prod_{h=1}^{3}\int_{\mathrm{R}^3}\frac{\mathrm{Im}(\mathrm{d}w_h)}{\mathrm{Im}(K_h)}\exp\left\{-2\pi\sum_{h=1}^{3}\frac{\tilde{I}^{(h)}_{cb}}{\mathrm{Im}(K_h)}\left(\mathrm{Im}(w_h) + c^h_{i,k}(q)\right)^2\right\} \\ &= \frac{1}{2}\mathcal{N}^{i,I_{ab}}\mathcal{N}^{j,I_{ca}}\mathcal{N}^{k,I_{cb}}\prod_{h=1}^{3}\left(2\tilde{I}^{(h)}_{cb}\mathrm{Im}(K_h)\right)^{-\frac{1}{2}}\sum_{q\in\Lambda^3_q}\cdot\mathrm{e}^{i\pi\sum_{h=1}^{3}\left(\frac{i^{(h)}}{d^{(h)}_b I^{(h)}_{ab}} + \frac{k^{(h)}}{d^{(h)}_c I^{(h)}_{cb}} + \frac{j^{(h)}}{d^{(h)}_a I^{(h)}_{ca}} + 2q^{(h)}\right)^2 K_h|I^{(h)}_{ab}I^{(h)}_{ca}I^{(h)}_{cb}|},\end{aligned}$$

where in the last step the Gaussian integrals where computed. After having evaluated the integration in (6.173) and inserting the normalization factors, the Yukawa couplings are given by the expression

$$Y_{ijk} = g \prod_{h=1}^{3} A_h^{-1} \left( \frac{2}{\mathrm{Im}(K_h)} \frac{\tilde{I}_{ab}^{(h)} \tilde{I}_{ca}^{(h)}}{\tilde{I}_{cb}^{(h)}} \right)^{1/4} \sum_{q \in \Lambda_q^3} \mathrm{e}^{i\pi \sum_{h=1}^{3} \left( \frac{i^{(h)}}{d_b^{(h)} I_{ab}^{(h)}} + \frac{k^{(h)}}{d_c^{(h)} I_{cb}^{(h)}} + \frac{j^{(h)}}{d_a^{(h)} I_{ca}^{(h)}} + 2q^{(h)} \right)^2 K_h |I_{ab}^{(h)} I_{ca}^{(h)} I_{cb}^{(h)}|} . \tag{6.200}$$

#### Selection rules for summation index

The Yukawa couplings from (6.200), contain the sum over the summation index $q \in \Lambda_q^3$, where $\Lambda_q^3$ still has not been specified. By taking linear combinations of the equations (6.193), the equations can be made independent of $p^{(h)}$:

$$\begin{aligned}
\lambda_{ab}^{(h)} - \lambda_{ca}^{(h)} - \delta_{ca}^{(h)} &= I_{bc}^{(h)} N_a^{(h)} l^{(h)} + \frac{j^{(h)}}{d_a^{(h)}} N_b^{(h)} + \frac{k^{(h)}}{d_c^{(h)}} N_a^{(h)} , \\
\lambda_{ca}^{(h)} - \lambda_{cb}^{(h)} - \delta_{cb}^{(h)} &= I_{ba}^{(h)} N_c^{(h)} l^{(h)} - \frac{k^{(h)}}{d_c^{(h)}} N_a^{(h)} - \frac{i^{(h)}}{d_b^{(h)}} N_c^{(h)} , \\
\lambda_{cb}^{(h)} - \lambda_{ab}^{(h)} - \delta_{cb}^{(h)} + \delta_{ca}^{(h)} &= I_{ac}^{(h)} N_b^{(h)} l^{(h)} + \frac{i^{(h)}}{d_b^{(h)}} N_c^{(h)} - \frac{j^{(h)}}{d_a^{(h)}} N_b^{(h)} ,
\end{aligned} \tag{6.201}$$

where $2q^{(h)} = "l^{(h)}$ was defined. Since the vectors $\lambda_{ab}$, $\lambda_{ca}$, $\lambda_{cb}$, $\delta_{ca}$ and $\lambda_{cb}$ belong to the $SO(6)$ lattice, the sum of their components must always be even. Therefor $l$ has to satisfy the conditions

$$\begin{aligned}
\sum_{h=1}^{3} I_{bc}^{(h)} N_a^{(h)} l^{(h)} + \frac{j^{(h)}}{d_a^{(h)}} N_b^{(h)} + \frac{k^{(h)}}{d_c^{(h)}} N_a^{(h)} &= 0 \mod 2 , \\
\sum_{h=1}^{3} I_{ba}^{(h)} N_c^{(h)} l^{(h)} - \frac{k^{(h)}}{d_c^{(h)}} N_a^{(h)} - \frac{i^{(h)}}{d_b^{(h)}} N_c^{(h)} &= 0 \mod 2 , \\
\sum_{h=1}^{3} I_{ac}^{(h)} N_b^{(h)} l^{(h)} + \frac{i^{(h)}}{d_b^{(h)}} N_c^{(h)} - \frac{j^{(h)}}{d_a^{(h)}} N_b^{(h)} &= 0 \mod 2 ,
\end{aligned} \tag{6.202}$$

and comparing (6.202) with the selection rules from (5.78) reveals that the the summation index $l$ from (6.202) and the instanton winding number from (5.78) both belong to the same lattice $\Lambda^3$.

### 6.4.3 Quantum contribution to Yukawa couplings

The Yukawa couplings are given by

$$Y_{ijk} = g \prod_{h=1}^{3} A_h^{-1} \left( \frac{2}{\mathrm{Im}(K_h)} \frac{\tilde{I}_{ab}^{(h)} \tilde{I}_{ca}^{(h)}}{\tilde{I}_{cb}^{(h)}} \right)^{1/4} \sum_{l \in \Lambda^3} \exp\left( -\frac{A_{ijk}(l)}{2\pi\alpha'} \right) , \tag{6.203}$$

with

$$A_{ijk}(l) = -\frac{i}{2} \sum_{h=1}^{3} \left( \frac{i^{(h)}}{d_b^{(h)} I_{ab}^{(h)}} + \frac{k^{(h)}}{d_c^{(h)} I_{cb}^{(h)}} + \frac{j^{(h)}}{d_a^{(h)} I_{ca}^{(h)}} + l^{(h)} \right)^2 K_h |I_{ab}^{(h)} I_{ca}^{(h)} I_{cb}^{(h)}| , \tag{6.204}$$

and $\Lambda^3$ satisfying the selection rules from (6.202) with $\alpha'$ set to $\alpha' = (2\pi)^{-2}$. The coupling constant $g$ is given by $g = \mathrm{e}^{\phi_{10}/2}\alpha'^{3/2}$, where $\phi_{10}$ is the ten dimensional dilaton [115]. Decomposing the ten dimensional dilaton into a four dimensional dilaton $\phi_4$ and the part for the internal space as in [115] one gets

$$\mathrm{e}^{\phi_{10}} = \mathrm{e}^{\phi_4} \prod_{h=1}^{3} \mathrm{Im}\,(J_h)^{1/2} \ , \tag{6.205}$$

where $J_h$ is the Kähler modulus in the $h$-plane and the volume for the $T^6$ is given by $\mathrm{Vol}\left(T^6\right) = \prod_{h=1}^{3} \mathrm{Im}\,(J_h)$. For the moduli of $T^6$ defined in (6.18), the geometry part in the Kähler moduli are

$$\mathrm{Im}(J_h) = 4\pi^2\alpha' {A_h}^2 \mathrm{Im}\,(K_h) \ . \tag{6.206}$$

Inserting the definitions into (6.203), the expression for the Yukawa couplings become

$$Y_{ijk} = (2\pi)^{3/4}\alpha'^{3/2}\mathrm{e}^{\phi_4/2} \prod_{h=1}^{3} \left( 4\pi \left(A_h^2/\alpha\right)^{-1} \frac{\tilde{I}^{(h)}_{ab}\tilde{I}^{(h)}_{ca}}{\tilde{I}^{(h)}_{cb}} \right)^{1/4} \sum_{l\in\Lambda^3} \exp\left(-\frac{A_{ijk}(l)}{2\pi\alpha'}\right) . \tag{6.207}$$

The quantities $\tilde{I}_{a\alpha\beta}$ can be related to the slope of the D6-branes in the T-dual picture as mentioned in section 6.3.2 [61, 115] by introducing the angles

$$\theta^{(h)}_{\alpha\beta} = 4\pi \frac{\tilde{I}^{(h)}_{\alpha\beta}}{\left({A_h}^2 \mathrm{Im}\,(K_h)\right)/\alpha'} \ . \tag{6.208}$$

Expressing (6.207) with the angles $\theta^{(h)}_{\alpha\beta}$ and $\alpha' = 1/(4\pi^2)$, the Yukawa couplings take the form

$$Y_{ijk} = \frac{\mathrm{e}^{\phi_4/2}}{(2\pi)^{9/4}} \prod_{h=1}^{3} \mathrm{Im}\,(K_h)^{1/4} \left| \frac{\theta^{(h)}_{ab}\theta^{(h)}_{ca}}{\theta^{(h)}_{cb}} \right|^{1/4} \sum_{l\in\Lambda^3} \exp\left(-\frac{A_{ijk}(l)}{2\pi\alpha'}\right) , \tag{6.209}$$

which allows it to compare with the Yukawa couplings from intersecting D6-branes on the $T^6_{\mathrm{SO}(12)}$. Comparing (6.209) with (5.81), the quantum contribution $h_{\mathrm{qu}}$ for the intersecting branes picture can be extracted from the expression (6.209) to be given by

$$h_{\mathrm{qu}} = \frac{\mathrm{e}^{\phi_4/2}}{(2\pi)^{9/4}} \prod_{h=1}^{3} \mathrm{Im}\,(K_h)^{1/4} \left| \frac{\theta^{(h)}_{ab}\theta^{(h)}_{ca}}{\theta^{(h)}_{cb}} \right| . \tag{6.210}$$

Since according to (6.17) the complex structure moduli $K_h$ become after T-duality transformations the complexified Kähler moduli for the deformed $T^6_{\mathrm{SO}(12)}$, the factor $\prod_{h=1}^{3} \mathrm{Im}(K_h)$ describes the volume of the $T^6_{\mathrm{SO}(12)}$. Hence $\mathrm{e}^{\phi_4/2} \prod_{h=1}^{3} \mathrm{Im}(K_h)^{1/4}$ describes the string coupling constant $\tilde{g} = \mathrm{e}^{\tilde{\phi}_{10}/2}$ on the Type IIA side with $\mathrm{e}^{\tilde{\phi}_{10}} = \mathrm{e}^{\phi_4} \prod_{h=1}^{3} \mathrm{Im}(K_h)^{1/2}$ and the quantum corrections on the Type IIA side are given by

$$h_{\mathrm{qu}} = \frac{\mathrm{e}^{\tilde{\phi}_{10}/2}}{(2\pi)^{9/4}} \prod_{h=1}^{3} \left| \frac{\theta^{(h)}_{ab}\theta^{(h)}_{ca}}{\theta^{(h)}_{cb}} \right| . \tag{6.211}$$

CHAPTER 7

# Conlusion

In the present work Type IIA string compactification to four dimensions with particularly non-factorisable tori has been investigated. The main goal to extend the expressions of Yukawa couplings on non-factorisable tori has been achieved and demonstrated by performing the computations with intersecting D6-branes on the six dimensional torus, which is generated by the $SO(12)$ root lattice. The procedure can be applied straight forward to other non-factorisable tori, whose underlying lattice consist of $SO(2N)$ Lie lattices, but it can also be adopted for other non-factorisable tori, whenever it is possible to find a factorisable lattice, which contains the underlying lattice of the torus as a sublattice. Intersection points of D6-branes on the non-factorisable torus where described by their position on three mutually orthogonal planes, s.t. the notation for the factorisable torus in [106] could be adopted. Due to the non-factorisable structure of the torus, labels for the intersection points belong to more general three dimensional lattice than for the factorisable torus. The worldsheet instantons generating the Yukawa couplings spread out triangles on the planes like on the factorisable torus. However the worldsheet instantons on the non-factorisable torus underly selection rules, s.t. their winding numbers, similar to the intersection points, belong to general three dimensional lattices. A recipe is given to identify the lattices for labels of the intersection points and how to determine the selection rules for the worldsheet instantons. Additionally the computations revealed a straight forward method to determine selection rules for Yukawa couplings when the branes have non-coprime wrapping numbers, which were merely conjectured in [106]. That way the classical part of the Yukawa couplings from intersecting D6-branes on the non-factorisable torus were computed.

The quantum contributions to the Yukawa couplings were determined by studying the T-dual torus with magnetic fluxes and hence extended the known discussions for Yukawa couplings from magnetized branes. Therefor three directions of the $SO(12)$ torus where T-dualized and the the boundary conditions of the D6-branes where translated into magnetic fluxes of D9-branes. The notation of [115] for the factorisable torus was adopted, but the non-factorisable structure of the torus manifested itself in a non-trivial way on the Wilson lines of the fluxes. Consequently gauge indices had to be expressed as lattice vectors of general three dimensional lattices. As an intermediate step wavefunctions of chiral matter on the factorisable torus had to be derived, which solve boundary conditions arising from fluxes with non-coprime wrapping numbers. It was discovered that unlike to the coprime case, not all states of bifundamentals can be related via Wilson lines to each other. Comparing the number of independent zeromodes for the Dirac operator with the number of independent intersection points of D6-branes on the corresponding T-dual torus, it turned out that in general independent zeromodes are not only determined by boundary conditions on their normalization constants but also on additional phases, which depend on the greatest common divisor of the wrapping numbers for the fluxes. In the second step the wavefunctons for zeromodes on the non-factorisable torus were calculated and manipulations of the arguments in the wavefunctions showed that labels for independent zeromodes are indeed related to the labels of intersection points on the T-dual torus. Yukawa couplings were finally computed by integrating three overlapping wavefunctions over the fundamental domain of the non-factorisable torus. The same selection rules as for the worldsheet instantons, generating Yukawa couplings on the corresponding T-dual torus, accrued. The final result showed that the couplings for the three overlapping wavefunctions are proportional to the Yukawa couplings on the intersecting branes picture. The proportionality factor was identified as the quantum contribution. The quantum contribution is the same as for the factorisable torus, which was somehow expected since quantum corrections are considered to be local effects.

The discussion of the $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \Omega R$ orientifold, showed that orientifolds constructed from the torus, which is generated by the $SO(12)$ lattice, put stronger restrictions on the wrapping numbers of intersecting D6-branes as on factorisable orientifolds. More precisely speaking, the amount of O-planes is reduced and therefore the R-R charge, needed to cancel the O-plane charges, is less than on factorisable orientifolds. The same arguments hold for orientifolds constructed from non-factorisable tori, whose underlying lattice contains $SO(2N)$ Lie lattices. As the toy model in chapter 4 showed, the conditions are actually too restrictive to construct realistic models. Therefore from phenomenological point of view, tori, generated by $SO(2N)$ lattices, seem to be uninteresting for model building, at least when discrete torsion is turned off. However this does not mean that non-factorisable tori in general are not good for phenomenological aspects. The computations in [108] show that with discrete torsion, non-factorisable orientifolds of the type discussed in this work might be attractive for model building. Further the work in [103–105] prove that orientifolds with intersecting D-branes on non-factorisable tori are worth to look at and for that reason the computations of Yukawa couplings, presented in this work, might be a necessary step for being able to derive mass hierarchies.

To finalize the discussion of Yukawa couplings from D-branes, it is still left remained to determine the Yukawa couplings, which are inherited on orientifolds, after the point group and orientifold projection is quotiented out of non-factorisable tori. Further the effect of discrete torsion needs to be studied with respect to Yukawa couplings arising from fractional D-branes.

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**APPENDIX A**

# Massless type IIA closed string states on $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \Omega R$-orientifolds

The representation of massless closed string states from Type IIA under the four dimensional Lorentz group and the resulting fields in four dimensions is investigated here. Further it is described, how the $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \Omega R$ projection is taken over the massless states and how the four dimensional spectrum is affected.

## A.1 Four dimensional fields from massless type IIA strings

Massless states in ten dimensions transform in representations of $SO(8)$, which is their corresponding little group. The massless Type IIA states in the rightmoving sector are given by the eight NS states

$$b^M_{-1/2}|0\rangle_{\mathrm{NS}}\,, \quad \text{with} \quad M \in \{2, ..., 9\}\,, \tag{A.1}$$

which transform in the vector representation $\mathbf{8}_V$ of $SO(8)$ and the eight R states

$$\prod_{\alpha=1}^{3} (S^+_\alpha)^{K_\alpha}\,|0\rangle_{\mathrm{R}}\,, \quad \sum_{\alpha=1}^{4} K_\alpha = 0 \mod 2\,, \quad K_\alpha \in \{0,1\}\,, \tag{A.2}$$

which transform in the chiral representation $\mathbf{8}_S$ of $SO(8)$, where $S^\pm_\alpha$ are the ladder operators for the spinorial states of $SO(8)$ as defined in (3.25). The leftmoving massless states are given by the eight NS states

$$\tilde{b}^M_{-1/2}|0\rangle_{\mathrm{NS}}\,, \tag{A.3}$$

which also transform in the vector representation $\mathbf{8}_V$ of $SO(8)$ and the eight R states

$$\prod_{\alpha=1}^{4} \left(\tilde{S}^+_\alpha\right)^{K_\alpha}\,|0\rangle_{\mathrm{R}}\,, \quad \sum_{\alpha=1}^{4} K_\alpha = 1 \mod 2\,, \quad K_\alpha \in \{0,1\}\,, \tag{A.4}$$

which belong to the antichiral representation $\mathbf{8}_C$ of $SO(8)$. Compactifying the six directions $x^4$,$x^5$,...,$x^9$ of the ten dimensional spacetime $M^{10}$, the $SO(1,9)$ acting on $M^{10}$, decomposes into $SO(6)$, which acts on the directions $x^4$,...,$x^9$, and $SO(1,3)$, which acts on the uncompact directions $x^0, x^1, x^2, x^3$. In the following of this chapter the uncompact directions are denoted by $\mu \in \{2,3\}$ and the compact directions by $i \in \{4, ..., 9\}$. The four dimensional spectrum from Type IIA strings is determined by the transformation of the string states under the $SO(1,3)$ factor. Massless states in four dimensions transform in representations of the algebra $\mathfrak{su}(2) \otimes \mathfrak{su}(2)$, which is the algebra of the little group for massless particles in four dimensions. The two NS states

$$b^{\mu}_{-1/2}|0\rangle_{\mathrm{NS}}\,, \quad \mu \in \{2,3\}\,, \tag{A.5}$$

then transform in the representation $(\frac{1}{2},\frac{1}{2})$ of $\mathfrak{su}(2) \otimes \mathfrak{su}(2)$ and hence describe the two polarization states of a massless vector field in four dimensions. The six states

$$b^{i}_{-1/2}|0\rangle_{\mathrm{NS}}\,, \quad i \in \{4,...,9\}\,, \tag{A.6}$$

transforms as a vector field in the compact space, but as six scalars in the uncompact space. The operators $S^{\pm}_{1}$ are raising and lowering operators of $\int\sqcap(2)$. Hence the two states $\prod_{\alpha=1}^{4}\left(\tilde{S}^{+}_{\alpha}\right)^{K_\alpha}|0\rangle_{\mathrm{R}}$ and $\prod_{\alpha=1}^{4}\left(\tilde{S}^{+}_{\alpha}\right)^{1-K_\alpha}|0\rangle_{\mathrm{R}}$, for fixed $K_\alpha$, can be viewed as the basis for the fundamental representation of the algebra $\mathfrak{su}(2)$. Hence the correspond to the two spin states of a Weyl fermion in four dimensions, where $\sum_\alpha K_\alpha$ = even/odd determines, whether they belong to the chiral/antichiral representation of $\mathfrak{su}(2) \otimes \mathfrak{su}(2)$

$$\left(\prod_{\alpha=1}^{3}\left(\tilde{S}^{+}_{\alpha}\right)^{K_\alpha}|0\rangle_{\mathrm{R}}\,,\ \prod_{\alpha=1}^{3}\left(\tilde{S}^{+}_{\alpha}\right)^{1-K_\alpha}|0\rangle_{\mathrm{R}}\right) \in \begin{cases}(\frac{1}{2},0) & \text{for } \sum_\alpha K_\alpha = \text{even}\\ (0,\frac{1}{2}) & \text{for } \sum_\alpha K_\alpha = \text{odd}\end{cases}. \tag{A.7}$$

It is convenient to use the notation

$$|s_1 s_2 s_3 s_4\rangle =: \prod_{\alpha=1}^{3}\left(\tilde{S}^{+}_{\alpha}\right)^{K_\alpha}|0\rangle_{\mathrm{R}}\,, \quad \text{with} \quad s_\alpha = \frac{1}{2}(-1)^{1-K_\alpha}\,, \tag{A.8}$$

to denote the spinor states. Then one can define four Weyl spinors

$$\psi^1_\xi = |\pm\tfrac{1}{2}\pm\tfrac{1}{2}\pm\tfrac{1}{2}\pm\tfrac{1}{2}\rangle\,,\quad \psi^2_\xi = |\pm\tfrac{1}{2}\pm\tfrac{1}{2}\mp\tfrac{1}{2}\mp\tfrac{1}{2}\rangle\,,\quad \psi^3_\xi = |\pm\tfrac{1}{2}\mp\tfrac{1}{2}\mp\tfrac{1}{2}\pm\tfrac{1}{2}\rangle\,,\quad \psi^4_\xi = |\pm\tfrac{1}{2}\mp\tfrac{1}{2}\pm\tfrac{1}{2}\mp\tfrac{1}{2}\rangle \tag{A.9}$$

in the chiral representation $\left(\frac{1}{2},0\right)$ and four Weyl spinors

$$\overline{\psi}^1_\xi = |\pm\tfrac{1}{2}\pm\tfrac{1}{2}\pm\tfrac{1}{2}\mp\tfrac{1}{2}\rangle\,,\quad \overline{\psi}^2_\xi = |\pm\tfrac{1}{2}\pm\tfrac{1}{2}\mp\tfrac{1}{2}\pm\tfrac{1}{2}\rangle\,,\quad \overline{\psi}^3_\xi = |\pm\tfrac{1}{2}\mp\tfrac{1}{2}\pm\tfrac{1}{2}\pm\tfrac{1}{2}\rangle\,,\quad \overline{\psi}^4_\xi = |\pm\tfrac{1}{2}\mp\tfrac{1}{2}\mp\tfrac{1}{2}\mp\tfrac{1}{2}\rangle \tag{A.10}$$

in the antichiral representation $(0,\frac{1}{2})$, where $\xi = \mathrm{sign}(s_1)$ denotes the spin state.

### A.1.1 Type IIA compactified on a $T^6$

The torus compactification leaves all Type IIA string states invariant. The contribution of the massless Type IIA states to the four dimensional spectrum is computed by tensoring the left-and rightmoving states together and decomposing them into irreducible representations of $\mathfrak{su}(2) \otimes \mathfrak{su}(2)$.

#### NS-NS states

Massless NS-NS states are given by the states $\tilde{b}^{M}_{-1/2}b^{N}_{-1/2}|0\rangle_{\mathrm{NS}}$. Decomposing the $\mathbf{8}_V$ into representations of $\mathfrak{su}(2) \otimes \mathfrak{su}(2)$, the NS-NS states

$$\tilde{b}^{\mu}_{-1/2}b^{\nu}_{-1/2}|0\rangle_{\mathrm{NS}} \tag{A.11}$$

belong to the product $(\frac{1}{2},\frac{1}{2})\otimes(\frac{1}{2},\frac{1}{2})$, where the 12 graviphotons

$$\tilde{b}^{\mu}_{-1/2}b^{i}_{-1/2}|0\rangle_{\rm NS}\,,\quad \tilde{b}^{i}_{-1/2}b^{\nu}_{-1/2}|0\rangle_{\rm NS} \tag{A.12}$$

belong to the vector representation $(\frac{1}{2},\frac{1}{2})$ and the 36 states

$$\tilde{b}^{i}_{-1/2}b^{j}_{-1/2}|0\rangle_{\rm NS} \tag{A.13}$$

transform as scalars in four dimensions. The product $(\frac{1}{2},\frac{1}{2})\otimes(\frac{1}{2},\frac{1}{2})$ decomposes into its trace, antisymmetric and symmetric part and is given by the irreducible representations $(0,0)\otimes(1,0)\otimes(0,1)\otimes(1,1)$. Trace $(0,0)$ corresponds to a dilaton in four dimensions, where the antisymmetric part $(1,0)\otimes(0,1)$ is a B-field $b_{\mu\nu}$ and the symmetric part $(1,1)$ is a graviton $g_{\mu\nu}$ in four dimensions. In four dimensions a B-field is magnetically dual to an axion $\theta$ and hence carries only a scalar dof. The dilaton, B-field and gravition therfore have together four on-shell dof, which fits with the four dof of $\tilde{b}^{\mu}_{-1/2}b^{\nu}_{-1/2}|0\rangle_{\rm NS}$. T he 36 scalars correspond to moduli of the compact space.

### NS-R states

Massless NS-R states are given by the product $\tilde{b}^{M}_{-1/2}|0\rangle_{\rm NS}\otimes|s_1s_2s_3s_4\rangle$, with $\sum_\alpha s_\alpha\in\mathbb{Z}$. Decomposing the $\mathbf{8}_V$ and $\mathbf{8}_S$ into representations of $\mathfrak{su}(2)\otimes\mathfrak{su}(2)$ the NS-R states are given by the four states

$$\tilde{b}^{\mu}_{-1/2}|0\rangle\otimes\psi^{a}_{\xi}\,, \tag{A.14}$$

which belong to the representation $(\frac{1}{2},\frac{1}{2})\otimes(\frac{1}{2},0)$, and the 24 states

$$\tilde{b}^{i}_{-1/2}|0\rangle\otimes\psi^{a}_{\xi}\,, \tag{A.15}$$

which transform as chiral fermions in four dimensions. Decomposing the product $(\frac{1}{2},\frac{1}{2})\otimes(\frac{1}{2},0)$ into irreducible representation, the states $\tilde{b}^{\mu}_{-1/2}|0\rangle\otimes\psi^{a}_{\xi}$, for each $a\in\{1,...,4\}$ the product is given by $(0,\frac{1}{2})\otimes(1,\frac{1}{2})$, which corresponds to a dilatino $\overline{\lambda}^{a}_{\xi}$ and gravitino $\overline{\Psi}^{a}_{\mu\xi}$ in the antichiral representation. Hence the NS-R sector provides the massless four dimensional spectrum with four antichiral gravitinos, four antichiral dilatinos and 24 chiral fermions.

### R-NS states

The R-NS sector is similar to the NS-R sector, except, that the chiralities are flipped. Massless NS-R states are given by the product $|s_1s_2s_3s_4\rangle\otimes b^{M}_{-1/2}|0\rangle_{\rm NS}$, with $\sum_\alpha s_\alpha\in\mathbb{Z}+\frac{1}{2}$. Decomposing the $\mathbf{8}_V$ and $\mathbf{8}_C$ into representations of $\mathfrak{su}(2)\otimes\mathfrak{su}(2)$ the R-NS states are given by the four states

$$\overline{\psi}^{a}_{\xi}\otimes b^{\mu}_{-1/2}|0\rangle\,, \tag{A.16}$$

which belong to the representation $(0,\frac{1}{2})\otimes(\frac{1}{2},\frac{1}{2})$, and the 24 states

$$\overline{\psi}^{a}_{\xi}\otimes b^{i}_{-1/2}|0\rangle\,, \tag{A.17}$$

which transform asantichiral fermions in four dimensions. Decomposing the product $(0,\frac{1}{2})\otimes(\frac{1}{2},\frac{1}{2})$ into irreducible representation, the states $\overline{\psi}^{a}_{\xi}\otimes b^{\mu}_{-1/2}|0\rangle$, for each $a\in\{1,...,4\}$ the product is given by $(\frac{1}{2},0)\otimes(\frac{1}{2},1)$, which corresponds to a dilatino $\lambda^{a}_{\xi}$ and gravitino $\Psi^{a}_{\mu\xi}$ with opposite chirality as in the

NS-R sector. Hence the R-NS sector also provides the massless four dimensional spectrum with four gravitinos, four dilatinos and 24 spinors, but each with the opposite chirality as in the NS-R sector.

#### R-R states

Decomposing the R-R states into products of $\mathfrak{su}(2) \otimes \mathfrak{su}(2)$, they are given by 16 times the states

$$\overline{\psi}^a_\xi \otimes \psi^b_\xi , \tag{A.18}$$

which transform in the product representation $(0, \frac{1}{2}) \otimes (\frac{1}{2}, 0)$ and decomposing it into irreducible representation one gets a vector representation $(\frac{1}{2}, \frac{1}{2})$. However massless vector fields in four dimension have only two dof and hence the other two dof in $\overline{\psi}^a_\xi \otimes \psi^b_\xi$ contribute as two scalars, In other words the two states $\frac{1}{\sqrt{2}}\left(\overline{\psi}^a_+ \cdot \psi^b_+\right)$ and $\frac{1}{\sqrt{2}}\left(\overline{\psi}^a_- \cdot \psi^b_-\right)$ are the polarization states of a vector potential, where the symmetric and antisymetric states $\frac{1}{\sqrt{2}}\left(\overline{\psi}^a_+ \cdot \psi^b_- \pm \overline{\psi}^a_- \cdot \psi^b_+\right)$ are the states of two scalar fields. Hence the R-R sector provides the four dimensional spectrum with 16 1-form gauge potentials and 32 scalar fields.

#### SUSY multiplet

In total the massless Type IIA states contain a graviton $g_{\mu\nu}$, four chiral and four antichiral gravitinos $\Psi^a_{\mu\,\xi}$, $\overline{\Psi}^a_{\mu\,\xi}$, 28 vector fields, 28 chiral and 28 antichiral spinors and 70 scalar fields in four dimensions. The field content fills out an $\mathcal{N} = (4, 4)$ supergravity multiplet. Hence the massless spectrum unveils that the ten dimensional $\mathcal{N} = (1, 1)$ susy is decomposed to $\mathcal{N} = (4, 4)$ susy in four dimensions, for compactification on a six dimensional compact space with trivial holonomy.

## A.2 $\mathbb{Z}_2 \times \mathbb{Z}_2$-orbfiold projection

### A.2.1 Untwisted states

The action of the generators $\theta, \omega$ of $\mathbb{Z}_2 \times \mathbb{Z}_2$ on the compact space is given by (4.127)

$$g = \exp\left\{2\pi i \sum_{i=1}^{3} v_i H_i\right\} , \tag{A.19}$$

with $H_i$ the three Cartan generators of $SO(6)$ and $v_i$ the components of the twist vectors of $g$. The eigenvalues of the NS and R states to the operators $\theta$ and $\omega$ are given by

| R states | NS states | $\theta$ | $\omega$ |
|---|---|---|---|
| $\psi^4_\xi, \overline{\psi}^2_\xi$ | $b^\mu_{1/2}\|0\rangle_{\text{NS}}$ | + | + |
| $\psi^3_\xi, \overline{\psi}^1_\xi$ | $b^{4,5}_{1/2}\|0\rangle_{\text{NS}}$ | - | + |
| $\psi^1_\xi, \overline{\psi}^4_\xi$ | $b^{6,7}_{1/2}\|0\rangle_{\text{NS}}$ | - | - |
| $\psi^2_\xi, \overline{\psi}^3_\xi$ | $b^{8,9}_{1/2}\|0\rangle_{\text{NS}}$ | + | - |

Only closed string states are preserved by the $\mathbb{Z}_2 \times \mathbb{Z}_2$ projection, whose left-and rightmovers have the same eigenvalues under $\theta$ and $\omega$. Those states are called untwisted states and are inherited from the initial Type IIA closed string sates (before the point group projection).

### Untwisted NS-NS states

The four dimensional dilaton, B-field and graviton are preserved by the point group, because the twist elements act trivially in the uncompact space. However all 12 graviphotons transform non-trivally and therefor get projected out. Further only half of the scalars, given by the states

$$\tilde{b}^{4,5}_{-1/2} b^{4,5}_{-1/2}|0\rangle\,, \quad \tilde{b}^{6,7}_{-1/2} b^{6,7}_{-1/2}|0\rangle\,, \quad \tilde{b}^{8,9}_{-1/2} b^{8,9}_{-1/2}|0\rangle\,, \tag{A.20}$$

survive the point group projection and therefore only 12 scalars are preserved from the 36 moduli fields.

### Untwisted NS-R states

Combining leftmovinf NS states with rightmoving R states, which have the same eigenvalues under the $\mathbb{Z}_2 \times \mathbb{Z} - 2$ generators, the invariant NS-R states are given by

$$\tilde{b}^{\mu}_{-1/2}|0\rangle_{\text{NS}} \otimes \psi^4_\xi\,, \quad \tilde{b}^{4,5}_{-1/2}|0\rangle_{\text{NS}} \otimes \psi^3_\xi\,, \quad \tilde{b}^{6,7}_{-1/2}|0\rangle_{\text{NS}} \otimes \psi^1_\xi\,, \quad \tilde{b}^{8,9}_{-1/2}|0\rangle_{\text{NS}} \otimes \psi^2_\xi\,, \tag{A.21}$$

so only one gravitino $\overline{\Psi}^1_{\mu\xi}$, one dilatino $\overline{\lambda}^1_\xi$ and six other fermions are preserved.

### Untwisted R-NS states

Similar to the NS-R sector the invariant R-NS states are given by

$$\overline{\psi}^2_\xi \otimes b^{\mu}_{-1/2}|0\rangle_{\text{NS}}\,, \quad \overline{\psi}^1_\xi \otimes b^{4,5}_{-1/2}|0\rangle_{\text{NS}}\,, \quad \overline{\psi}^4_\xi \otimes b^{6,7}_{-1/2}|0\rangle_{\text{NS}}\,, \quad \overline{\psi}^3_\xi \otimes b^{8,9}_{-1/2}|0\rangle_{\text{NS}}\,, \tag{A.22}$$

so again only one gravitino $\Psi^1_{\mu\xi}$, one dilatino $\lambda^1_\xi$ and six other fermions are preserved. However they belong to the opposite chiralities as in the untwisted NS-R sector.

### Untwisted R-R sector

In the R-R sector the combination of the left- and rightmoving R states with the same transformation behaviour under the point group are

$$\overline{\psi}^2_\xi \otimes \psi^4_\xi\,, \quad \overline{\psi}^1_\xi \otimes \psi^3_\xi\,, \quad \overline{\psi}^4_\xi \otimes \psi^1_\xi\,, \quad \overline{\psi}^3_\xi \otimes \psi^2_\xi\,, \tag{A.23}$$

where the other R-R states are projected out. It remains therefor four 1-form gauge potentials and eight scalar from the R-R sector.

### SUSY multiplets

The four dimensional spectrum contains after the $\mathbb{Z}_2 \times \mathbb{Z}_2$ projection one graviton, one chiral and one antichiral gravitino, four vector fields, seven chiral and seven antichiral fermions and 22, which fits into an $\mathcal{N} = (1,1)$ graviton multiplet, three $\mathcal{N} = (1,1)$ vector multiplets and eight $\mathcal{N} = (1,1)$ half-hyper muliplets (or four hyper multiplets). As discussed in section 4.1.2 the point group preserves only one four dimensional susy generator from each ten dimensional susy generator. That can indeed by confirmed at the level of the four dimensional massless spectrum for the discussed $\mathbb{Z}_2 \times \mathbb{Z}_2$ projection.

### A.2.2 Twisted states

The $g$ twisted boundary conditions, with $g \in \{\theta, \omega, \theta\omega\}$, shift the oscillator moddings by $\frac{1}{2}$ in the directions, on which $g$ acts non-trivially. Therefore the oscillators for the $g$ twisted NS sector contains zeromodes $b_0^i$ in the directions $x^l$ on which $g$ acts non-trivially. They are related by $\frac{1}{\sqrt{2}} b_0^l = \Gamma^l$ to the Gamma matrices of the four dimensional Clifford algebra. They act on the spinor representation of $SO(4)$, which is the rotation group in the four directions $x^l$. The twisted NS ground state $|0\rangle_{\text{NS}}^{\text{tw}}$ is massless and therefore also $b_0^i|0\rangle_{\text{NS}}^{\text{tw}}$. Similar to the untwisted R ground state being a spinor of the $SO(8)$, the twisted NS ground state transforms as a spinor of $SO(4)$ and can be given by four states $|s_{l_1} s_{l_2}\rangle_{\text{NS}}^{\text{tw}}$, with $s_l \in \pm\frac{1}{2}$. Since $x^l$ lie entirely in the compact space the $SO(4)$ has no factor in common with the four dimensional Lorentz group and hence the four states $|s_{l_1} s_{l_2}\rangle_{\text{NS}}^{\text{tw}}$ are scalar fields in the uncompact space. On the other hand the twisted R states contains only four zeromodes, two in the uncompact space and two in the directions $x^k$, on which $g$ acts trivially. They also form a four dimensional Clifford algebra. Therefor the twisted R ground state is given by a spinor $|s_1 s_k\rangle_{\text{R}}^{\text{tw}}$ of $SO(4)$, but since $s_1$ is the eigenvalue to the spin in four dimensions, the twisted R ground state transforms as a fermion in the uncompact space. The GSO-projection acting on the left-and right moving sector for twisted NS states have, unlike for the untwisted states, an additional factor of $4\prod_{i=1}^{4} b_0^i$, which is the chirality operators for the spinor representation of $SO(4)$. Under the action of the orientifold projection $\Omega R$, the zeromodes $b_0^l$ in the imaginary directions of the compact space get a sign which cancels in the combination for the chirality operator. Therefore the GSO projection preserves in the twisted NS sector the same two states $|s_{l_1} s_{l_2}\rangle_{\text{NS}}^{\text{R}}$ for the left-and rightmovers, with either $\sum_a s_{i_a} \in \mathbb{Z}$ or $\sum_a s_{i_a} \in \mathbb{Z} + \frac{1}{2}$. The GSO-projection for the twisted R sector has a factor of $4b_0^2 b_0^3 \prod_j b_0^j$ and since $\Omega R$ only flips the sign of $b^j$ in the imaginary direction, the GSO-projection differs by a sign in the left- and rightmoving twisted R sector, thus preserves fermions with different chiralities in the twisted left-and rightmoving R sectors. The massless twisted states preserved by the GSO-projection are chosen in the rightmoving sector to be given by

$$|\pm\tfrac{1}{2} \pm \tfrac{1}{2}\rangle_{\text{NS}}^{\text{tw}}\,, \quad |\pm\tfrac{1}{2} \pm \tfrac{1}{2}\rangle_{\text{R}}^{\text{tw}}\,, \tag{A.24}$$

with $|\pm\frac{1}{2} \pm \frac{1}{2}\rangle_{\text{NS}}^{\text{tw}}$ two scalars and $|\pm\frac{1}{2} \pm \frac{1}{2}\rangle_{\text{R}}^{\text{tw}}$ two spin states of chiral spinor from the four dimensional point of view, and in the leftmoving sector by

$$|\pm\tfrac{1}{2} \pm \tfrac{1}{2}\rangle_{\text{NS}}^{\text{tw}}\,, \quad |\pm\tfrac{1}{2} \mp \tfrac{1}{2}\rangle_{\text{R}}^{\text{tw}}\,, \tag{A.25}$$

with $|\pm\frac{1}{2} \mp \frac{1}{2}\rangle_{\text{R}}^{\text{tw}}$ two spin states of an antichiral spinor. That way each fixed point FIX($g$) contains 16 massless NS and R states arising from twisted closed strings, by combining the four scalars with the two spinors. The action of another twist element $h \neq g$ on FIX($g$), acts on the twisted NS and R states as

$$h: \; |s_{i_1} s_{i_2}\rangle_{\text{NS}}^{\text{tw}} \to e^{\pi i s_{i_a}} |s_{i_1} s_{i_2}\rangle_{\text{NS}}^{\text{tw}}\,, \quad |s_1 s_j\rangle_{\text{R}}^{\text{tw}} \to e^{\pi i s_j} |s_1 s_j\rangle_{\text{R}}^{\text{tw}}\,, \tag{A.26}$$

where $h$ acts on the $i_a$-th and $j$-th plane non-trivially. In order for the twisted closed string states to remain the whole orbifold projection they need to form invariant states also under the other twist elements $h \in P$.

### Twisted NS-NS states

The massless NS-NS states in the twisted sector is given by combining the states $|\pm\frac{1}{2}\pm\frac{1}{2}\rangle^{\rm tw}_{\rm NS}$ from the left-and rightmoving sector. It gives the four states

$$|+\tfrac{1}{2}+\tfrac{1}{2}\rangle^{\rm tw}_{\rm NS}\otimes|+\tfrac{1}{2}+\tfrac{1}{2}\rangle^{\rm tw}_{\rm NS}\,,\quad |+\tfrac{1}{2}+\tfrac{1}{2}\rangle^{\rm tw}_{\rm NS}\otimes|-\tfrac{1}{2}-\tfrac{1}{2}\rangle^{\rm tw}_{\rm NS}\,,\quad |-\tfrac{1}{2}-\tfrac{1}{2}\rangle^{\rm tw}_{\rm NS}\otimes|+\tfrac{1}{2}+\tfrac{1}{2}\rangle^{\rm tw}_{\rm NS}\,,\quad |-\tfrac{1}{2}-\tfrac{1}{2}\rangle^{\rm tw}_{\rm NS}\otimes|-\tfrac{1}{2}-\tfrac{1}{2}\rangle^{\rm tw}_{\rm NS}\,, \tag{A.27}$$

which are four scalars under the four dimensional Lorentz group. The effect of the other $\mathbb{Z}_2$ action on the twisted states preserve only the two states $|\pm\frac{1}{2}\pm\frac{1}{2}\rangle^{\rm tw}_{\rm NS}\otimes|\mp\frac{1}{2}\mp\frac{1}{2}\rangle^{\rm tw}_{\rm NS}$.

### Twisted NS-R states

The massless NS-R states in the twisted sector is given by combining the states $|\pm\frac{1}{2}\pm\frac{1}{2}\rangle^{\rm tw}_{\rm NS}$ from the leftmoving sector with the $|\pm\frac{1}{2}\pm\frac{1}{2}\rangle^{\rm tw}_{\rm R}$ in the rightmoving sector. They form two fermions

$$|+\tfrac{1}{2}+\tfrac{1}{2}\rangle^{\rm tw}_{\rm NS}\otimes|\pm\tfrac{1}{2}\pm\tfrac{1}{2}\rangle^{\rm tw}_{\rm R}\,,\quad |-\tfrac{1}{2}-\tfrac{1}{2}\rangle^{\rm tw}_{\rm NS}\otimes|\pm\tfrac{1}{2}\pm\tfrac{1}{2}\rangle^{\rm tw}_{\rm R}\,, \tag{A.28}$$

which transform in the $(\frac{1}{2},0)$ representation in the uncompact space. The second $\mathbb{Z}_2$ preserves only the two states $|+\frac{1}{2}+\frac{1}{2}\rangle^{\rm tw}_{\rm NS}\otimes|-\frac{1}{2}-\frac{1}{2}\rangle^{\rm tw}_{\rm R}$ and $|-\frac{1}{2}-\frac{1}{2}\rangle^{\rm tw}_{\rm NS}\otimes|+\frac{1}{2}+\frac{1}{2}\rangle^{\rm tw}_{\rm R}$, which can be put to one Weyl spinor in four dimensions together.

### Twisted R-NS states

The massless R-NS states in the twisted sector is given by combining the states $|\pm\frac{1}{2}\mp\frac{1}{2}\rangle^{\rm tw}_{\rm R}$ from the leftmoving sector with the $|\pm\frac{1}{2}\pm\frac{1}{2}\rangle^{\rm tw}_{\rm NS}$ in the rightmoving sector. Analogous to the twisted NS-R sector, they form two fermions

$$|\pm\tfrac{1}{2}\pm\tfrac{1}{2}\rangle^{\rm tw}_{\rm R}\otimes|+\tfrac{1}{2}+\tfrac{1}{2}\rangle^{\rm tw}_{\rm NS}\,,\quad |\pm\tfrac{1}{2}\pm\tfrac{1}{2}\rangle^{\rm tw}_{\rm R}\otimes|-\tfrac{1}{2}-\tfrac{1}{2}\rangle^{\rm tw}_{\rm NS}\,, \tag{A.29}$$

each transforming in the $(0,\frac{1}{2})$ representation. As in the NS-R sector, the other twist elements project out two states and preserve one Weyl spinor.

### Twisted R-R states

The twisted R-R states are given by the product of the chiral and antichiral fermion. Decomposing the product into irreducible representations of $\mathfrak{su}(2)\otimes\mathfrak{su}(2)$, they provide two dof belonging to the $(\frac{1}{2},\frac{1}{2})$ representation and two scalar dof in four dimensions. That means each twisted R-R sector contains a vector field and two scalars. The invariant states under the second $\mathbb{Z}_2$ action are given by $|+\frac{1}{2}+\frac{1}{2}\rangle^{\rm tw}_{\rm R}\otimes|+\frac{1}{2}-\frac{1}{2}\rangle^{\rm tw}_{\rm R}$ and $|-\frac{1}{2}-\frac{1}{2}\rangle^{\rm tw}_{\rm R}\otimes|-\frac{1}{2}+\frac{1}{2}\rangle^{\rm tw}_{\rm R}$, which are the two polarization states of a spin 1 vector field. Hence the orbifold action on the twisted R-R sector projects out the two scalars.

### Discrete torsion

Turning on discrete torsion gives an additional sign to the $g$ twisted states by the action of other twist elements $h\in P$ with $h\neq g$. Hence the action of $h$ on a massless twisted states is given by

$$h:\ |s_a s_b\rangle^{\rm tw}_{\rm NS/R}\otimes|s_c s_d\rangle^{\rm tw}_{\rm NS/R}\to -e^{\pi i(s_b+s_d)}|s_a s_b\rangle^{\rm tw}_{\rm NS/R}\otimes|s_c s_d\rangle^{\rm tw}_{\rm NS/R}\,, \tag{A.30}$$

where $h$ acts non trivially on the $b$-th and $d$-th plane. Therefor the states which are preserved are those which are projected out in the case without discrete torsion. Each fixed point contributes two scalars from the NS-NS sector, two scalars from the R-R sector and two fermions from the NS-R and R-NS sector.

#### SUSY multiplets

Each fixed point contains six scalars, four spinors and one vector field. They fit into an $\mathcal{N} = (1,1)$ half-hyper multiplet and an $\mathcal{N} = (1,1)$ vector multiplet. However the action of the other $\mathbb{Z}_2$ projection breaks the susy to $\mathcal{N} = 1$, projecting out half of the twisted states. For the case without discrete torsion two scalars, two spinors and one vector field is preserved, which fit into an $\mathcal{N} = 1$ vector multiplet. However for the case with discrete torsion four scalars and two fermions are preserved, which form an $\mathcal{N} = 1$ chiral multiplet.

## A.3 Orientifold projection

The orientifold action $\Omega R$ on the string transforms the left- and rightmovers from as

$$\Omega R : X^\mu_{L/R}(\sigma^\pm),\, \psi^\mu_\pm(\sigma^\pm) \to \begin{cases} +X^\mu_{R/L}(\sigma^\mp),\, \psi^\mu_\mp(\sigma^\mp) & \text{for } \mu \in \{2,3,4,6,8\} \\ -X^\mu_{R/L}(\sigma^\mp),\, \psi^\mu_\mp(\sigma^\mp) & \text{for } \mu \in \{5,7,9\} \end{cases} . \tag{A.31}$$

Inserting the mode expansions from (3.9) the oscillator modes transform as

$$\Omega R : \alpha^\mu_n,\, b^\mu_{n+r} \leftrightarrow \begin{cases} \tilde{\alpha}^\mu_n,\, \tilde{b}^\mu_{n+r} \\ -\tilde{\alpha}^\mu_n,\, -\tilde{b}^\mu_{n+r} \end{cases} . \tag{A.32}$$

In order to preserve the graviton in $\tilde{b}^\mu_{-1/2} b^\nu_{-1/2}|0\rangle_{\text{NS}}$, the orientifold action acts on the NS ground state as $\Omega R : |0\rangle_{\text{NS}} \to -|0\rangle_{\text{NS}}$. On the R sector the orientifold projection acts on the operators $S^\pm_\alpha$ as $\Omega R : S^\pm_{2,3,4} \leftrightarrow \tilde{S}^\mp_{2,3,4}$ and $S^\pm_1 \leftrightarrow S^\pm_1$. It is therefore consistent to define the transformation of the lowest weight state of the R ground state by $\Omega R : |-\frac{1}{2} - \frac{1}{2} - \frac{1}{2} - \frac{1}{2}\rangle_{\text{R}} \leftrightarrow |-\frac{1}{2} + \frac{1}{2} + \frac{1}{2} - \frac{1}{2}\rangle_{\text{R}}$. Then the R states transform under $\Omega R$ as

$$\psi^1_\xi \leftrightarrow \overline{\psi}^4_\xi\,, \quad \psi^2_\xi \leftrightarrow \overline{\psi}^3_\xi\,, \quad \psi^3_\xi \leftrightarrow \overline{\psi}^1_\xi\,, \quad \psi^4_\xi \leftrightarrow \overline{\psi}^2_\xi\,. \tag{A.33}$$

### A.3.1 Untwisted states

The orientifold projection preserves the states from the untwisted NS-NS sector which are symmetric under the exchange of the spacetime indices $\mu$ and $i$. That means the states

$$\tilde{b}^2_{-1/2} b^2_{-1/2}|0\rangle_{\text{NS}}\,, \quad \tilde{b}^3_{-1/2} b^3_{-1/2}|0\rangle_{\text{NS}}\,, \quad \frac{1}{2}\left(\tilde{b}^2_{-1/2} b^3_{-1/2}|0\rangle_{\text{NS}} + \tilde{b}^3_{-1/2} b^2_{-1/2}|0\rangle_{\text{NS}}\right), \tag{A.34}$$

correspinding to the dilaton and graviton dof, and the three scalars

$$\tilde{b}^{2i+1}_{-1/2} b^{2i+1}_{-1/2}|0\rangle_{\text{NS}}\,, \quad \tilde{b}^{2i}_{-1/2} b^{2i}_{-1/2}|0\rangle_{\text{NS}}\,, \quad \frac{1}{2}\left(\tilde{b}^{2i+1}_{-1/2} b^{2i}_{-1/2}|0\rangle_{\text{NS}} + \tilde{b}^{2i}_{-1/2} b^{2i+1}_{-1/2}|0\rangle_{\text{NS}}\right), \tag{A.35}$$

for each plane $i \in \{1,2,3\}$ in the compact space, remain after the action of $\Omega R$. Since the NS-R and R-NS sector are not left-right symmetric only linear combinations of both sectors survive the orientifold projection. The states

$$\frac{1}{2}\left(\tilde{b}^{\mu}_{-1/2}|0\rangle_{\mathrm{NS}} \otimes \psi^{4}_{\xi} + \overline{\psi}^{1}_{\xi} \otimes b^{\mu}_{-1/2}|0\rangle_{\mathrm{NS}}\right), \tag{A.36}$$

corresponding to one dilatino and one gravitino, and the two fermions

$$\frac{1}{2}\left(\tilde{b}^{2i,2i+1}_{-1/2}|0\rangle_{\mathrm{NS}} \otimes \psi^{a}_{\xi} + \overline{\psi}^{a}_{\xi} \otimes b^{2i,2i+1}_{-1/2}|0\rangle_{\mathrm{NS}}\right), \tag{A.37}$$

for each plane $i$, are preserved by $\Omega R$. Since spacetime fermions anticommute the action of $\Omega R$ on R-R states is given by

$$\Omega R:\ \overline{\psi}^{a}_{\xi} \otimes \psi^{b}_{\xi} \to \overline{\psi}^{b}_{\xi} \otimes \overline{\psi}^{a}_{\xi} = -\overline{\psi}^{a}_{\xi} \otimes \psi^{b}_{\xi}\,. \tag{A.38}$$

That means only the antisymmetric states in the untwisted R-R sector are left invariant by $\Omega R$ and the four scalar states

$$\frac{1}{2}\left(\overline{\psi}^{2}_{+}\cdot\psi^{4}_{-} - \overline{\psi}^{2}_{-}\cdot\psi^{4}_{+}\right),\quad \frac{1}{2}\left(\overline{\psi}^{1}_{+}\cdot\psi^{3}_{-} - \overline{\psi}^{1}_{-}\cdot\psi^{3}_{+}\right),\quad \frac{1}{2}\left(\overline{\psi}^{4}_{+}\cdot\psi^{1}_{-} - \overline{\psi}^{4}_{-}\cdot\psi^{1}_{+}\right),\quad \frac{1}{2}\left(\overline{\psi}^{3}_{+}\cdot\psi^{2}_{-} - \overline{\psi}^{3}_{-}\cdot\psi^{2}_{+}\right), \tag{A.39}$$

remain on the orientifold. In total the untwisted sector of Type IIA contributes to the four dimensional spectrum, after the whole $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \Omega R$ projection, a graviton, a gravitino, seven fermions and 14 scalar fields, which fit into an $\mathcal{N} = 1$ graviton multiplet and seven $\mathcal{N} = 1$ chiral multiplets.

### A.3.2 Twisted states

The zeromodes in the twisted sector transform under $\Omega R$ as

$$\Omega R:\ b^{M}_{0} \to \begin{cases} \tilde{b}^{2,3,4,6,8} \\ -\tilde{b}^{5,7,9} \end{cases}, \tag{A.40}$$

and vice versa, so that

$$\Omega R:\ S^{\pm}_{1} \leftrightarrow \tilde{S}_{1}\pm\,,\quad S^{\pm}_{j} \leftrightarrow \tilde{S}^{\mp}_{j}\,, \tag{A.41}$$

for the twisted R sector and

$$\Omega R:\ S^{\pm}_{i} \leftrightarrow \tilde{S}_{i}\mp\,, \tag{A.42}$$

for the twisted NS sector. Defining the twisted ground states to transform under $\Omega R$ as

$$\Omega R:\ |-\tfrac{1}{2}-\tfrac{1}{2}\rangle^{\mathrm{tw}}_{\mathrm{R}} \to |-\tfrac{1}{2}+\tfrac{1}{2}\rangle^{\mathrm{tw}}_{\mathrm{R}}\,,\quad |-\tfrac{1}{2}-\tfrac{1}{2}\rangle^{\mathrm{tw}}_{\mathrm{NS}} \to |-\tfrac{1}{2}-\tfrac{1}{2}\rangle^{\mathrm{tw}}_{\mathrm{NS}}\,, \tag{A.43}$$

the transformation of the twisted states under the orientifold can be deduced to be given by

$$\Omega R:\ |s_{i_1}s_{i_2}\rangle^{\mathrm{tw}}_{\mathrm{NS}} \to |s_{i_1}s_{i_2}\rangle^{\mathrm{tw}}_{\mathrm{NS}}\,,\quad |s_1 s_j\rangle^{\mathrm{tw}}_{\mathrm{R}} \to |s_1 - s_j\rangle^{\mathrm{tw}}_{\mathrm{R}}\,. \tag{A.44}$$

Then the states preserved by $\Omega R$ in the twisted NS-NS sector are the two states $|+\frac{1}{2}+\frac{1}{2}\rangle^{\mathrm{tw}}_{\mathrm{NS}} \otimes |+\frac{1}{2}+\frac{1}{2}\rangle^{\mathrm{tw}}_{\mathrm{NS}}$ and $|-\frac{1}{2}-\frac{1}{2}\rangle^{\mathrm{tw}}_{\mathrm{NS}} \otimes |-\frac{1}{2}-\frac{1}{2}\rangle^{\mathrm{tw}}_{\mathrm{NS}}$. Since they are only preserved for the case with discrete torsion, the twisted NS-NS states are only present when discrete torsion is turned on. Similar the twisted R-R states which are invariant under $\Omega R$ are the two polarization states $|+\frac{1}{2}+\frac{1}{2}\rangle^{\mathrm{tw}}_{\mathrm{R}} \otimes |+\frac{1}{2}-\frac{1}{2}\rangle^{\mathrm{tw}}_{\mathrm{R}}$ and $|-\frac{1}{2}-\frac{1}{2}\rangle^{\mathrm{tw}}_{\mathrm{R}} \otimes |-\frac{1}{2}+\frac{1}{2}\rangle^{\mathrm{tw}}_{\mathrm{R}}$ for a vector field. They are preserved on the orbifold withou discrete torsion, hence the twisted R-R sector contributes only for the case when discrete torsion is turned off. The twisted NS-R and twisted R-NS sector form two linear combinations, which are invariant states under $\Omega R$ and provide the four dimensional spectrum with a fermion. That means at each fixed point the states remaining after the $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \Omega R$ projection, are either two scalars and a fermion for the case with discrete torsion or a vector field and a fermion for the case without discrete torsion. The orientifold projection breaks the $\mathcal{N} = (1, 1)$ susy at the twisted sectors to $\mathcal{N} = 1$. The invariant states form a $\mathcal{N} = 1$ chiral multiplet in the presence of discrete torsion or a $\mathcal{N} = 1$ vector multiplet when no discrete torsion is not turned on.

APPENDIX **B**

# Labels for intersection points on $T^6_{\mathrm{SO}(12)}$

Some of the lattices for labels of inequivalent intersection points of D6-branes on $T^6_{\mathrm{SO}(12)}$ are discussed here. The lattice for a label $j$ depends in the following way on the wrapping numbers of the two cycles, which the Branes are wrapping,

$$j \in \left\{ \left(t_1 m_b^1 - t_2 n_b^1,\ t_3 m_b^2 - t_4 n_b^2,\ t_5 m_b^3 - t_6 n_b^3\right)^T \Big| \vec{t} \in \Lambda_{\mathrm{SO}(12)} \right\} , \tag{B.1}$$

with the identification

$$\begin{aligned} &\left( n_a^1 \frac{j^{(1)}}{I_{ab}^{(1)}},\ m_a^1 \frac{j^{(1)}}{I_{ab}^{(1)}},\ n_a^2 \frac{j^{(2)}}{I_{ab}^{(2)}},\ m_a^2 \frac{j^{(2)}}{I_{ab}^{(2)}},\ n_a^3 \frac{j^{(3)}}{I_{ab}^{(3)}},\ m_a^3 \frac{j^{(3)}}{I_{ab}^{(3)}} \right) \sim \\ &\left( n_a^1 \frac{j^{(1)}}{I_{ab}^{(1)}},\ m_a^1 \frac{j^{(1)}}{I_{ab}^{(1)}},\ n_a^2 \frac{j^{(2)}}{I_{ab}^{(2)}},\ m_a^2 \frac{j^{(2)}}{I_{ab}^{(2)}},\ n_a^3 \frac{j^{(3)}}{I_{ab}^{(3)}},\ m_a^3 \frac{j^{(3)}}{I_{ab}^{(3)}} \right) + \vec{\lambda}\,, \quad \vec{\lambda} \in \Lambda_{\mathrm{SO}(12)} \end{aligned} \tag{B.2}$$

#### Case 1

Here the following configuration is considered

$$\forall_{h\in\{1,2,3\}} \forall_{\alpha\in\{a,b\}} \left( n_\alpha^h + m_\alpha^h = 0 \mod 2\,, \quad g.c.d.(n_\alpha^h, m_\alpha^h) = 1 \right) . \tag{B.3}$$

The only way for the term $t_{2h-1} m_\alpha^h - t_{2h} n_\alpha^h$ to be odd is that the lattice vector components satisfy $t_{2h-1} + t_{2h} = 1 \mod 2$. But, because the sum of all six lattice vector components has to be even, the label $j$ can only have either zero or two components, which are odd. On the other hand it is possible to make $t_{2h-1} m_\alpha^h - t_{2h} n_\alpha^h$ even for each plane individually. This is the case for the $SO(6)$ Lie lattice. Hence $j$ belongs to the $SO(6)$ lattice. From the relation

$$I_{ab}^{(h)} = \frac{n_a^h - m_b^h}{2} \left( n_b^h + m_b^h \right) - \frac{n_b^h - m_b^h}{2} (N_a^h + m_a^h)\,, \tag{B.4}$$

one sees that $I_{ab}^{(h)}$ is even and therefor a shift of the components of $j$ by $j^{(h)} \to j^{(h)} + I_{ab}^{(h)}$, can be performed independently from the components of the other planes. Further, since $n_a^h + m_a^h$ is even, the shift of $j^{(h)}$ leads to a $SO(12)$ lattice shift in the term in (B.2), which is can be absorbed by the identification. For the labels $j$ that means that the inequivlanet labels are given by the quotienting out the shifts from the $SO(6)$

lattice

$$j \in \frac{\Lambda_{\mathrm{SO(6)}}}{\bigotimes_{h=1}^{3} I_{ab}^{(h)}\mathbb{Z}}\,. \tag{B.5}$$

**Case 2**

Here the following configuration is considered

$$\begin{aligned} \forall_{h\in\{1,2,3\}}\left(n_a^h + m_a^h = 0 \mod 2\,, \quad g.c.d.(n_a^h, m_a^h) = 1\right)\,, \\ \forall_{h\in\{2,3\}}\left(n_b^h + m_b^h = 0 \mod 2\,, \quad g.c.d.(n_b^h, m_b^h) = 1\right)\,, \\ n_b^1 + m_b^1 = 0 \mod 2\,, \quad g.c.d.(n_b^1, m_b^1) = 2\,. \end{aligned} \tag{B.6}$$

In this case the brane $b$ wraps the first plane twice and since $\frac{m_b^1}{2} + \frac{n_b^1}{2} = 1 \mod 2$, with $\frac{m_b^1}{2}, \frac{n_b^1}{2} \in \mathbb{Z}$, it is possible for $\frac{1}{2}\left(t_1 m_b^1 - t_2 n_b^1\right)$ to take any integer value, independently from the values of the other two components of $j$. Further, it is possible to find lattice vectors $\vec{t} \in \Lambda_{\mathrm{SO(12)}}$, s.t. $\left(\frac{j^{(1)}}{2}, j^{(2)}, j^{(3)}\right)$ takes any value in $\mathbb{Z}^3$. Hence the labels $j$ belong to the lattice $2\mathbb{Z}\otimes\mathbb{Z}\otimes\mathbb{Z}$. Analogous to case 1, (B.4) reveals that $I_{ab}^{(h)}$, $\forall h \in \{1,2,3\}$, is even, and because $n_a^h + m_a^h$ is even, the inequivalent labels $j$ belong to the quotient lattice

$$j \in \frac{2\mathbb{Z}\otimes\mathbb{Z}\otimes\mathbb{Z}}{\bigotimes_{h=1}^{3} I_{ab}^{(h)}\mathbb{Z}}\,. \tag{B.7}$$

**Case 3**

Here the following configuration is considered

$$\begin{aligned} \forall_{h\in\{1,2,3\}}\left(n_b^h + m_b^h = 0 \mod 2\,, \quad g.c.d.(n_b^h, m_b^h) = 1\right)\,, \\ \forall_{h\in\{2,3\}}\left(n_a^h + m_a^h = 0 \mod 2\,, \quad g.c.d.(n_a^h, m_a^h) = 1\right)\,, \\ n_a^1 + m_a^1 = 0 \mod 2\,, \quad g.c.d.(n_a^1, m_a^1) = 2\,. \end{aligned} \tag{B.8}$$

Inserting the wrapping numbers of the brane $b$ into (B.1), one sees analogous to case 1, that $j$ belongs to the $SO(6)$ lattice. Since $I_{ab}^{(h)}$ and $n_a^h + m_a^h$ are even $\forall h \in \{1,2,3\}$, the labels for inequivalent intersections are given by the quotient lattice in (B.5) as in case 1.

**Case 4**

Here the following configuration is considered

$$\begin{aligned} \forall_{h\in\{1,2,3\}}\left(n_a^h + m_a^h = 0 \mod 2\,, \quad g.c.d.(n_a^h, m_a^h) = 1\right)\,, \\ \frac{n_b^1 + m_b^1}{2} = \text{odd}\,, \quad n_b^2 + m_b^2 = \text{odd}\,, \quad n_b^3 + m_b^3 = \text{even}\,, \\ \forall_{h\in\{2,3\}}\left(g.c.d.(n_b^h, m_b^h) = 1\right)\,, \quad g.c.d.(n_b^1, m_b^1) = 2\,. \end{aligned} \tag{B.9}$$

Similar to case 2, it is possible to for $j$ to take values of the lattice $2\mathbb{Z}\otimes\mathbb{Z}\otimes\mathbb{Z}$ and since $n_a^h + m_a^h$ is even, shifting $j$ by $I_{ab}^{(h)}$ in each plane individually, leads to $SO(12)$ lattice shifts in (B.2). Hence inequivalent $j$'s belongs to the quotient lattice given in (B.7).

#### Case 5

Here the following configuration is considered

$$\forall_{h\in\{1,2,3\}}\left(n_b^h+m_b^h=0\mod 2\,,\quad g.c.d.(n_b^h,m_b^h)=1\right)\,, \tag{B.10}$$
$$\frac{n_a^1+m_a^1}{2}=\text{odd}\,,\quad n_a^2+m_a^2=\text{odd}\,,\quad n_a^3+m_a^3=\text{even}\,,$$
$$\forall_{h\in\{2,3\}}\left(g.c.d.(n_a^h,m_a^h)=1\right)\,,\quad g.c.d.(n_a^1,m_a^1)=2\,.$$

Similar to case 1 $j$ belong to the $SO(6)$ lattice. But this time $I_{ab}^{(2)}$ is not necessarily even and since $n_a^2+m_a^2$ is odd, the shift $j\to j+\left(0,I_{ab}^{(2)},0\right)$ does not lead to a lattice shift in (B.2). Instead, shifting $j$ by the following three vectors

$$\left(I_{ab}^{(1)},0,0\right)^T\,,\quad\left(I_{ab}^{(1)}/2,I_{ab}^{(2)},0\right)^T\,,\quad\left(0,0,I_{ab}^{(3)}\right)^T\,, \tag{B.11}$$

leads to equivalent intersection points on the torus, by (B.2). Hence the inequivalent labels belong to the lattice

$$j\in\frac{\Lambda_{\mathrm{SO(6)}}}{\Gamma_{ab}}\,,\quad\Gamma_{ab}=\mathrm{span}\left(\begin{pmatrix}I_{ab}^{(1)}\\0\\0\end{pmatrix},\begin{pmatrix}\frac{I_{ab}^{(1)}}{2}\\I_{ab}^{(2)}\\0\end{pmatrix},\begin{pmatrix}0\\0\\I_{ab}^{(3)}\end{pmatrix}\right)\,. \tag{B.12}$$

#### Case 6

Here the following configuration is considered

$$\forall_{h\in\{1,2,3\}}\left(n_a^h+m_a^h=0\mod 2\,,\quad g.c.d.(n_a^h,m_a^h)=1\right)\,, \tag{B.13}$$
$$\frac{n_b^1+m_b^1}{2}=\text{odd}\,,\quad n_b^2+m_b^2=\text{odd}\,,\quad n_b^3+m_b^3=\text{odd}\,,$$
$$\forall_{h\in\{2,3\}}\left(g.c.d.(n_b^h,m_b^h)=1\right)\,,\quad g.c.d.(n_b^1,m_b^1)=2\,.$$

Similar to case 4, the inequivlanet labels $j$ belong to the lattice $2\mathbb{Z}\otimes\mathbb{Z}\otimes\mathbb{Z}$ and the labels, denoting inequivlanet intersection points on the torus, belong the quotient lattice given in (B.7).

#### Case 7

Here the following configuration is considered

$$\forall_{h\in\{1,2,3\}}\left(n_b^h+m_b^h=0\mod 2\,,\quad g.c.d.(n_b^h,m_b^h)=1\right)\,, \tag{B.14}$$
$$\frac{n_a^1+m_a^1}{2}=\text{odd}\,,\quad n_a^2+m_a^2=\text{odd}\,,\quad n_a^3+m_a^3=\text{odd}\,,$$
$$\forall_{h\in\{2,3\}}\left(g.c.d.(n_a^h,m_a^h)=1\right)\,,\quad g.c.d.(n_a^1,m_a^1)=2\,.$$

Similar to case 5, the labels $j$ belong to the $SO(6)$ lattice. Unlike to case 5, this time $I^{(3)}_{ab}$ and $n^3_a + m^3_a$ are also odd. The lattice for inequivalent labels $j$ for this case is given by

$$j \in \frac{\Lambda_{\mathrm{SO(6)}}}{\Gamma_{ab}}\,, \quad \Gamma_{ab} = \operatorname{span}\left(\begin{pmatrix} I^{(1)}_{ab} \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \frac{I^{(1)}_{ab}}{2} \\ I^{(2)}_{ab} \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ I^{(h)}_{ab} \\ I^{(3)}_{ab} \end{pmatrix}\right). \tag{B.15}$$

### Case 8

Here the following configuration is considered

$$\forall_{\alpha\in\{a,b\}} \begin{cases} \frac{n^1_\alpha + m^1_\alpha}{2} = \text{odd}\,, \quad n^2_\alpha + m^2_\alpha = \text{odd}\,, \quad n^3_\alpha + m^3_\alpha = \text{even}\,, \\ \forall_{h\in\{2,3\}} \left(g.c.d.(n^h_\alpha, m^h_\alpha) = 1\right)\,, \quad g.c.d.(n^1_\alpha, m^1_\alpha) = 2\,. \end{cases} \tag{B.16}$$

Similar to case 2, $j$ belongs to the lattice $2\mathbb{Z}\otimes\mathbb{Z}\otimes\mathbb{Z}$. This time $I^{(1)}_{ab} = 0 \mod 4$ and therefore allows $j$ to be shifted by

$$\left(I^{(1)}_{ab}, 0, 0\right)^T\,, \quad \left(I^{(1)}_{ab}/2, I^{(2)}_{ab}, 0\right)^T\,, \quad \left(0, 0, I^{(3)}_{ab}\right)^T\,, \tag{B.17}$$

which generate the quotientent lattice $\Gamma_{ab}$, for the inequivalent labels

$$j \in \frac{2\mathbb{Z}\otimes\mathbb{Z}\otimes\mathbb{Z}}{\Gamma_{ab}}\,. \tag{B.18}$$

### Case 9

Here the following configuration is considered

$$\begin{aligned} \frac{n^1_a + m^1_a}{2} &= \text{odd}\,, \quad n^2_a + m^2_a = \text{odd}\,, \quad n^3_a + m^3_a = \text{even}\,, \\ &\forall_{h\in\{2,3\}} \left(g.c.d.(n^h_a, m^h_a) = 1\right)\,, \quad g.c.d.(n^1_a, m^1_a) = 2\,, \\ \frac{n^1_b + m^1_b}{2} &= \text{odd}\,, \quad n^2_b + m^2_b = \text{odd}\,, \quad n^3_b + m^3_b = \text{odd}\,, \\ &\forall_{h\in\{2,3\}} \left(g.c.d.(n^h_b, m^h_b) = 1\right)\,, \quad g.c.d.(n^1_b, m^1_b) = 2\,, \end{aligned} \tag{B.19}$$

Again, similar to case 8, the inequivlanet labels $j$ belong to the quotient lattice given in (B.18) with the generators in (B.17) for $\Gamma_{ab}$.

### Case 10

Here the following configuration is considered

$$\begin{aligned} \frac{n^1_b + m^1_b}{2} &= \text{odd}\,, \quad n^2_b + m^2_b = \text{odd}\,, \quad n^3_b + m^3_b = \text{even}\,, \\ &\forall_{h\in\{2,3\}} \left(g.c.d.(n^h_b, m^h_b) = 1\right)\,, \quad g.c.d.(n^1_b, m^1_b) = 2\,, \\ \frac{n^1_a + m^1_a}{2} &= \text{odd}\,, \quad n^2_a + m^2_a = \text{odd}\,, \quad n^3_a + m^3_a = \text{odd}\,, \\ &\forall_{h\in\{2,3\}} \left(g.c.d.(n^h_a, m^h_a) = 1\right)\,, \quad g.c.d.(n^1_a, m^1_a) = 2\,, \end{aligned} \tag{B.20}$$

This time the ineqivalent labels belong to the quotient lattice

$$j \in \frac{2\mathbb{Z} \otimes \mathbb{Z} \otimes \mathbb{Z}}{\Gamma_{ab}}\,, \quad \Gamma_{ab} = \text{span}\left(\begin{pmatrix} I_{ab}^{(1)} \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \frac{I_{ab}^{(1)}}{2} \\ I_{ab}^{(2)} \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ I_{ab}^{(h)} \\ I_{ab}^{(3)} \end{pmatrix}\right). \tag{B.21}$$

Here the following configuration is considered

$$\forall_{\alpha\in\{a,b\}} \begin{cases} \frac{n_\alpha^1 + m_\alpha^1}{2} = \text{odd}\,, \quad n_\alpha^2 + m_\alpha^2 = \text{odd}\,, \quad n_\alpha^3 + m_\alpha^3 = \text{odd}\,, \\ \forall_{h\in\{2,3\}} \left(g.c.d.(n_\alpha^h, m_\alpha^h) = 1\right)\,, \quad g.c.d.(n_\alpha^1, m_\alpha^1) = 2\,. \end{cases} \tag{B.22}$$

The ineqivalent labels for this case is also belongs to the quotient lattice given in (B.21).

# APPENDIX C

# Quotient lattices and integral matrices

On the dual of $T^6_{\mathrm{SO}(12)}$, the gauge indices $k$ for fields in representations of $U(N_a)$ gauge groups, are given by $k \in \frac{\Lambda_{\mathrm{SO}(6)}}{\Gamma_a}$, where $\Gamma_a$ is a sublattice of $\Lambda_{\mathrm{SO}(6)}$ spanned by three generators with integer components. Denoting the three generators by $a_1$, $a_2$ and $a_3$, a $3 \times 3$ integral matrix $A$ can be associated to $\Gamma_a$, where the components for the $i$-th column is given by $a_i$

$$A = (a_1, a_2, a_3)\ . \tag{C.1}$$

Let $A$ and $B$ be two integral $3 \times 3$ matrices, associated to the lattices $\Gamma_a \subset \Lambda_{\mathrm{SO}(6)}$ and $\Gamma_b \subset \Lambda_{\mathrm{SO}(6)}$. Let $\Gamma_d$ be a three dimensional sublattice of $\Lambda_{\mathrm{SO}(6)}$, s.t. $\Gamma_a$ and $\Gamma_b$ are sublattices of $\Gamma_d$ and all other lattices, containing $\Gamma_a$ and $\Gamma_b$ as sublattices, are sublattices of $\Gamma_d$. Let $\Gamma_a \cap \Gamma_b$ be the contained in $\Gamma_a$ and $\Gamma_b$ as a sublattice, s.t. all other common sublattices of $\Gamma_a$ and $\Gamma_b$ are sublattices of $\Gamma_a \cap \Gamma_b$. Let $D$ and $M$ be the associated $3 \times 3$ matrices of $\Gamma_d$ and $\Gamma_a \cap \Gamma_b$ respectively. $D$ is called a left divisor of $A$ and $B$, which means $A$ and $B$ can be expressed by

$$A = DM_a\,, \quad B = DM_b\,, \tag{C.2}$$

where $M_a$ and $M_b$ are integral $3 \times 3$ matrices. Further $D$ is the greatest common left divisor, since from the definition of $\Gamma_d$, it follows, that all other left divisors of $A$ and $B$ are also left divisors of $D$. Since the the representation of the generators of the lattices should not have an effect, the greatest common left divisor of $A$ and $B$ is unique up to unimodular transformations (see e.g.[120]). Then for the following Diophantine equations

$$D = AP - BQ\,, \tag{C.3}$$

solutions exist, where $P$ and $Q$ are integral $3 \times 3$ matrices (see proof of Proposition 3.4 in [121]). Because $M$ can be expressed by

$$M = AN_a \quad \text{and} \quad M = BN_b\,, \tag{C.4}$$

with $N_a$ and $N_b$ being integral $3 \times 3$ matrices, $M$ is called a right multiple of $A$ and $B$. Since from the definition of $\Gamma_a \cap \Gamma_b$, there does not exist any further common right multiple, which is not a right multiple of $M$, $M$ is called the lowest common right multiple of $A$ and $B$. The matrices $D$ and $M$ can be related by (see theorem 5 of [122].)

$$M = AD^{-1}B\,. \tag{C.5}$$

Using (C.5), the volume of the fundamental cell of the quotient lattice $\frac{\Lambda_{SO(6)}}{\Gamma_a \cap \Gamma_b}$ is given by

$$\left|\frac{\Lambda_{SO(6)}}{\Gamma_a \cap \Gamma_b}\right| = \left|\frac{\det(M)}{2}\right| = \frac{1}{2}\left|\frac{\det(A)\det(B)}{\det(D)}\right| , \tag{C.6}$$

from which it follows

$$\begin{aligned} &\frac{1}{2}\left|\det(M)\det(D)\right| = \frac{1}{2}\left|\det(A)\det(B)\right| \\ \Rightarrow &\left|\frac{\Lambda_{SO(6)}}{\Gamma_a \cap \Gamma_b}\right|\left|\frac{\Lambda_{SO(6)}}{\Gamma_d}\right| = \left|\frac{\Lambda_{SO(6)}}{\Gamma_a}\right|\left|\frac{\Lambda_{SO(6)}}{\Gamma_b}\right| . \end{aligned} \tag{C.7}$$

**APPENDIX D**

# Lattices for gauge indices and irreducible subsets

The lattices $\Gamma_\alpha$, with $\alpha \in \{a, b\}$, where already classified in (6.53) and (6.54): There are three different cases, occurring from, whether $N_\alpha^{(h)} = 1 \mod 2$ for one or two planes (remember that at least in one plane $N_\alpha^{(h)} = 0 \mod 2$, with $g.c.d.(n_\alpha^h, m_\alpha^h) = 2$). Here the cases are investigated, in which $\Gamma_a$ and $\Gamma_b$ are given by

(i)

$$\Gamma_\alpha^1 = \mathrm{span}\left(\begin{pmatrix} N_\alpha^{(1)} \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ N_\alpha^{(2)} \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ N_\alpha^{(3)} \end{pmatrix}\right), \tag{D.1}$$

where $N_\alpha^{(h)} = 0 \mod 2$ and $g.c.d.(n_\alpha^h, m_\alpha^h) = 1$, $\forall\, h \in \{1, 2, 3\}$,

(ii)

$$\Gamma_\alpha^2 = \mathrm{span}\left(\begin{pmatrix} \frac{N_\alpha^{(1)}}{2} \\ N_\alpha^{(2)} \\ 0 \end{pmatrix}, \begin{pmatrix} \frac{N_\alpha^{(1)}}{2} \\ -N_\alpha^{(2)} \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ N_\alpha^{(3)} \end{pmatrix}\right), \tag{D.2}$$

where $N_\alpha^{(2)} = 1 \mod 2$, $N_\alpha^{(h)} = 0 \mod 2$ for $h \in \{1, 3\}$, but $g.c.d.(n_\alpha^1, m_\alpha^1) = 2$ and $g.c.d.(n_\alpha^h, m_\alpha^h) = 1$ for $h \in \{2, 3\}$,

(iii)

$$\Gamma_\alpha^3 = \mathrm{span}\left(\begin{pmatrix} \frac{N_\alpha^{(1)}}{2} \\ N_\alpha^{(2)} \\ 0 \end{pmatrix}, \begin{pmatrix} \frac{N_\alpha^{(1)}}{2} \\ -N_\alpha^{(2)} \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ N_\alpha^{(2)} \\ N_\alpha^{(3)} \end{pmatrix}\right), \tag{D.3}$$

where $N_\alpha^{(h)} = 1 \mod 2$ and $g.c.d.(n_\alpha^h, m_\alpha^h) = 1$ for $h \in \{2, 3\}$, but $N_\alpha^{(1)} = 0 \mod 2$ and $g.c.d.(n_\alpha^1, m_\alpha^1) = 2$.

Since the volume of the fundamental cells of $\Gamma_\alpha^i$ is always $\mathrm{Vol}\left(\Gamma_\alpha^i\right) = \prod_{h=1}^3 N_\alpha^{(h)}$, $\forall i \in \{1, 2, 3\}$, the number of gauge indices $k_\alpha \in \frac{\Lambda_{\mathrm{SO}(6)}}{\Gamma_\alpha}$ is

$$\#(k_\alpha) = \left|\frac{\Lambda_{\mathrm{SO}(6)}}{\Gamma_\alpha}\right| = \frac{N_\alpha^{(1)} N_\alpha^{(2)} N_\alpha^{(3)}}{2}\,. \tag{D.4}$$

The smallest common sublattice $\Gamma_a \cap \Gamma_b$ of $\Gamma_a$ and $\Gamma_b$ and the greatest lattice $\Gamma_d$, for which $\Gamma_a$ and $\Gamma_b$ are sublattices are given for the following different cases by

(a)

$$\Gamma_a \cap \Gamma_b = \bigotimes_{h=1}^{3} \frac{N_a^{(h)} N_b^{(h)}}{d^{(h)}} \mathbb{Z}\,, \quad \Gamma_d = \bigotimes_{h=1}^{3} d^{(h)} \mathbb{Z}\,, \tag{D.5}$$

for $\Gamma_a^1$ and $\Gamma_b^1$,

(b)

$$\Gamma_a \cap \Gamma_b = \bigotimes_{h=1}^{3} \frac{N_a^{(h)} N_b^{(h)}}{d^{(h)}} \mathbb{Z}\,, \quad \Gamma_d = \mathrm{span}\left( \begin{pmatrix} \frac{d^{(1)}}{2} \\ d^{(2)} \\ 0 \end{pmatrix}, \begin{pmatrix} \frac{d^{(1)}}{2} \\ -d^{(2)} \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ d^{(3)} \end{pmatrix} \right), \tag{D.6}$$

for $\Gamma_a^1$, $\Gamma_b^2$ and $\Gamma_a^2$, $\Gamma_b^1$,

(c)

$$\Gamma_a \cap \Gamma_b = \bigotimes_{h=1}^{3} \frac{N_a^{(h)} N_b^{(h)}}{d^{(h)}} \mathbb{Z}\,, \quad \Gamma_d = \mathrm{span}\left( \begin{pmatrix} \frac{d^{(1)}}{2} \\ d^{(2)} \\ 0 \end{pmatrix}, \begin{pmatrix} \frac{d^{(1)}}{2} \\ -d^{(2)} \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ d^{(2)} \\ d^{(3)} \end{pmatrix} \right), \tag{D.7}$$

for $\Gamma_a^1$, $\Gamma_b^3$ or $\Gamma_a^3$, $\Gamma_b^1$,

(d)

$$\begin{aligned} \Gamma_a \cap \Gamma_b &= \mathrm{span}\left( \begin{pmatrix} \frac{N_a^{(1)} N_b^{(1)}}{2d^{(1)}} \\ -\frac{N_a^{(2)} N_b^{(2)}}{d^{(2)}} \\ 0 \end{pmatrix}, \begin{pmatrix} \frac{N_a^{(1)} N_b^{(1)}}{2d^{(1)}} \\ \frac{N_a^{(2)} N_b^{(2)}}{d^{(2)}} \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ \frac{N_a^{(3)} N_b^{(3)}}{d^{(3)}} \end{pmatrix} \right), \\ \Gamma_d &= \mathrm{span}\left( \begin{pmatrix} \frac{d^{(1)}}{2} \\ -d^{(2)} \\ 0 \end{pmatrix}, \begin{pmatrix} \frac{d^{(1)}}{2} \\ d^{(2)} \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ d^{(3)} \end{pmatrix} \right), \end{aligned} \tag{D.8}$$

for $\Gamma_a^2$ and $\Gamma_b^2$,

(e)

$$\begin{aligned} \Gamma_a \cap \Gamma_b &= \mathrm{span}\left( \begin{pmatrix} \frac{N_a^{(1)} N_b^{(1)}}{2d^{(1)}} \\ -\frac{N_a^{(2)} N_b^{(2)}}{d^{(2)}} \\ 0 \end{pmatrix}, \begin{pmatrix} \frac{N_a^{(1)} N_b^{(1)}}{2d^{(1)}} \\ \frac{N_a^{(2)} N_b^{(2)}}{d^{(2)}} \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ \frac{N_a^{(3)} N_b^{(3)}}{d^{(3)}} \end{pmatrix} \right), \\ \Gamma_d &= \mathrm{span}\left( \begin{pmatrix} \frac{d^{(1)}}{2} \\ -d^{(2)} \\ 0 \end{pmatrix}, \begin{pmatrix} \frac{d^{(1)}}{2} \\ d^{(2)} \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ d^{(2)} \\ d^{(3)} \end{pmatrix} \right), \end{aligned} \tag{D.9}$$

for $\Gamma_a^2$, $\Gamma_b^3$ or $\Gamma_a^3$, $\Gamma_b^2$,

(f)

$$\Gamma_a \cap \Gamma_b \;=\; \mathrm{span}\left( \begin{pmatrix} \frac{N_a^{(1)}N_b^{(1)}}{2d^{(1)}} \\ -\frac{N_a^{(2)}N_b^{(2)}}{d^{(2)}} \\ 0 \end{pmatrix}, \begin{pmatrix} \frac{N_a^{(1)}N_b^{(1)}}{2d^{(1)}} \\ \frac{N_a^{(2)}N_b^{(2)}}{d^{(2)}} \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ \frac{N_a^{(2)}N_b^{(2)}}{d^{(2)}} \\ \frac{N_a^{(3)}N_b^{(3)}}{d^{(3)}} \end{pmatrix} \right), \tag{D.10}$$

$$\Gamma_d \;=\; \mathrm{span}\left( \begin{pmatrix} \frac{d^{(1)}}{2} \\ -d^{(2)} \\ 0 \end{pmatrix}, \begin{pmatrix} \frac{d^{(1)}}{2} \\ d^{(2)} \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ d^{(2)} \\ d^{(3)} \end{pmatrix} \right),$$

for $\Gamma_a^3$ or $\Gamma_b^3$,

where $d^{(h)} = g.c.d.(N_a^{(h)}, N_b^{(h)})$ was defined. For all cases (a)-(f) the volumes of $\Gamma_d$ and $\Gamma_a \cap \Gamma_b$ are

$$\mathrm{Vol}\,(\Gamma_a \cap \Gamma_b) = \prod_{h=1}^{3} \frac{N_a^{(h)}N_b^{(h)}}{d^{(h)}}\,, \quad \mathrm{Vol}\,(\Gamma_d) = \prod_{h=1}^{3} d^{(h)} \tag{D.11}$$

and the relation (6.79) is indeed satisfied.

The function of three overlapping wavefunctions in section 6.4.2 involves the smallest common sublattice $\Gamma_a \cap \Gamma_b \cap \Gamma_c$. $\Gamma_c$ is considered to be given by $\Gamma_c^1$, $\Gamma_c^2$ or $\Gamma_c^3$. Then $\Gamma_a \cap \Gamma_b \cap \Gamma_c$ is given by

(a)

$$\Gamma_a \cap \Gamma_b \cap \Gamma_c = \bigotimes_{h=1}^{3} \frac{N_a^{(h)}N_b^{(h)}N_c^{(h)}}{d_{abc}^{(h)}}\mathbb{Z}\,, \tag{D.12}$$

for $\Gamma_\alpha = \Gamma_\alpha^1$ for at least one $\alpha \in \{a, b, c\}$,

(b)

$$\Gamma_a \cap \Gamma_b \cap \Gamma_c = \mathrm{span}\left( \begin{pmatrix} \frac{N_a^{(1)}N_b^{(1)}N_c^{(1)}}{2d_{abc}^{(1)}} \\ -\frac{N_a^{(2)}N_b^{(2)}N_c^{(2)}}{d_{abc}^{(2)}} \\ 0 \end{pmatrix}, \begin{pmatrix} \frac{N_a^{(1)}N_b^{(1)}N_c^{(1)}}{2d_{abc}^{(1)}} \\ \frac{N_a^{(2)}N_b^{(2)}N_c^{(2)}}{d_{abc}^{(2)}} \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ \frac{N_a^{(3)}N_b^{(3)}N_c^{(3)}}{d_{abc}^{(3)}} \end{pmatrix} \right), \tag{D.13}$$

for $\Gamma_\alpha \neq \Gamma_\alpha^1$ for all $\alpha \in \{a, b, c\}$ and $\Gamma_\alpha = \Gamma_\alpha^2$ for at least one $\alpha$,

(c)

$$\Gamma_a \cap \Gamma_b \cap \Gamma_c = \mathrm{span}\left( \begin{pmatrix} \frac{N_a^{(1)}N_b^{(1)}N_c^{(1)}}{2d_{abc}^{(1)}} \\ -\frac{N_a^{(2)}N_b^{(2)}N_c^{(2)}}{d_{abc}^{(2)}} \\ 0 \end{pmatrix}, \begin{pmatrix} \frac{N_a^{(1)}N_b^{(1)}N_c^{(1)}}{2d_{abc}^{(1)}} \\ \frac{N_a^{(2)}N_b^{(2)}N_c^{(2)}}{d_{abc}^{(2)}} \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ \frac{N_a^{(2)}N_b^{(2)}N_c^{(2)}}{d_{abc}^{(2)}} \\ \frac{N_a^{(3)}N_b^{(3)}N_c^{(3)}}{d_{abc}^{(3)}} \end{pmatrix} \right), \tag{D.14}$$

for $\Gamma_\alpha = \Gamma_\alpha^3$ for all $\alpha \in \{a, b, c\}$,

where $d_{abc}^{(h)} = g.c.d.(N_a^{(h)}, N_b^{(h)}, N_c^{(h)})$ was defined.

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