

On the geometry of nodal sets and nodal domains

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Abstract

In the present work we study and prove results related to the nodal geometry of Laplacian eigenfunctions on closed Riemannian manifolds, as well as solutions to more general classes of elliptic partial differential equations.

Briefly put, the text aims to present the following results:

- Upper estimates for nodal sets of solutions to more general elliptic PDE, thus including Steklov eigenfunctions as a special subclass (cf. Theorems 3.4.1 and 3.4.2). The presentation is partly based on our work [GRF17]:

G., Roy-Fortin, Polynomial upper bound on interior Steklov nodal sets, 2017, submitted.

- Two-sided volume bounds for tubular neighbourhoods around Laplacian nodal sets in the smooth setting - a result towards a question addressed by M. Sodin, C. Fefferman, Jakobson-Mangoubi (cf. Theorem 4.1.2). The presentation is partly based on our work [GM17b]:

G., Mukherjee, Tubular neighbourhoods, cone conditions and inscribed balls: Some remarks on nodal geometry in the smooth setting, 2017, submitted.

- A localization refinement of a celebrated result of E. Lieb concerning almost inscribed wavelength balls (cf. Theorem 6.3.1). The presentation is partly based on our work [GM18b]:

G., Mukherjee, Nodal Geometry, Heat Diffusion and Brownian Motion, 2018, Analysis and PDE, 11(1):133–148.

- Various bounds for nodal domains - straightness, two-sided inner radius bounds, cone conditions, etc (cf. for instance, Theorem 6.4.2 and the results in Chapter 6). The presentation is partly based on our work [Geo16]:

G., On the lower bound of the inner radius of nodal domains, 2018, to appear in Journal of Geometric Analysis.

- Results along the lines of well-known obstacle placement problems and eigenvalue optimization (cf. Theorem 7.2.1, Corollary 7.2.1 and Chapter 7). The presentation is partly based on our work [GM17a]:

G., Mukherjee, On maximizing the fundamental frequency of the complement of an obstacle, 2017, Comptes Rendus Mathematique 356:4, 406-411.

For a more detailed overview of the structure of the present text we refer to Section 1.5.

Most of the results in the present work have been presented and discussed at various seminars and conferences - e.g. as the author was visiting and giving talks at Yale University; Northwestern University; Indiana University and the corresponding AMS Sectional meeting, Spring, 2017; ICTP, Trieste; Münster and Cologne Universities; the conference MDS, Sofia, 2017; the seminar on Algebra, Geometry and Physics, Max Planck Institute for Mathematics, Bonn; etc.

Moreover, to our knowledge some of our results have already been utilized and extended in various settings - for example, we refer to [RS17], [LS], [Bis17], [Zel17], etc.

Acknowledgements

I heartily wish to thank my teachers and mentors:

Werner Ballmann, Herbert Koch, Steve Zelditch, Stefan Müller, Stefan Ivanov, Nadejda Ribarska, George Dimitrov and Vesselin Zlatilov.

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It has been a great pleasure and an honour to work on the present results and discuss problems with amazing people. The experience will remain with me for life.

To Nora and our Parents

Дълги алеи шепнат спомени безброй,

Светът различен е след всеки нов
порой...

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Chapter 1

Introduction

In this Chapter we present a few standpoints, which provide an initial physical motivation behind nodal set problems and eigenfunction questions that we would like to address later on. We also describe general ideas how to approach these issues. For a very broad and far-reaching introduction to the subject, we refer to the surveys [Zel08], [Zel17]. At the end of this Chapter we discuss the organization of the text with an emphasis on the central results of the present work.

1.1 Visualizing sound

Since antiquity, questions related to vibration phenomena have been addressed in various forms - vibration of strings and membranes; harmonics; oscillation of various shapes and the way they resonate and produce sound.

The German physicist and musician E. Chladni (also known as one of the fathers of modern acoustics) was at the heart of one of the first deep studies of acoustics at the end of 18th century. Chladni analyzed the various ways in which differently-shaped plates vibrate.

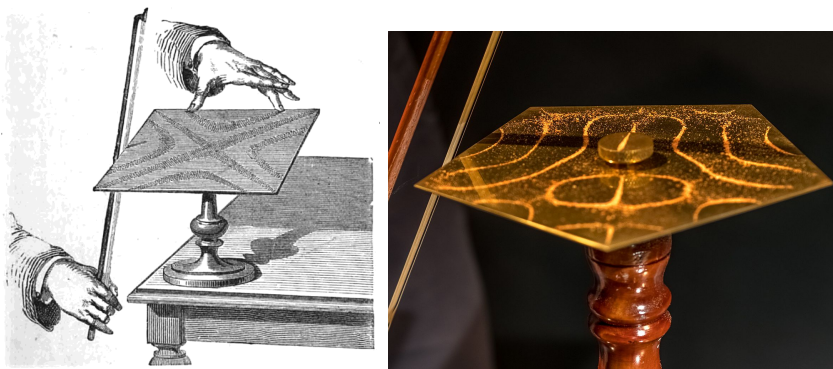


Figure 1.1: Chladni's experiments.

Among other things, he conducted the following type of experiments: a small amount of sand would be uniformly distributed upon a certain thin metal plate; then, one of the edges of the plate

would be scratched by a violin bow; thus, the plate would start to vibrate and the sand upon it would gather in an interesting way, forming curious patterns (cf. Figure 1.1). Moreover, depending on the speed of the bow, the sand patterns would change - high bow speeds would generate more complicated sand patterns, whereas low speeds would produce simpler ones. Chladni recorded the outcome of the experiments by drawing the observed patterns (cf. Figure 1.2).

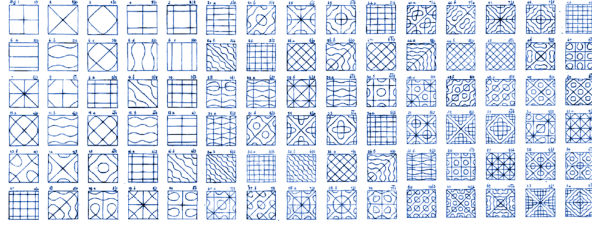


Figure 1.2: Chladni's diagrams: the simpler ones on the left indicate slow bow speed (low frequency), whereas the ones on the right show higher bows speeds.

E. Chladni presented such experiments in Paris and, upon observing these, Napoleon announced a competition and prize for the best rigorous mathematical explanation of how such sand-patterns would form. From a modern point of view, a relevant mathematical model would be the following.

Suppose the metal plate is modeled as a two-dimensional domain Ω . As the bow touches the edge, the plate vibrates - to describe the vibration, we model the profile of the plate by a function

$$u_\lambda : \mathbb{R} \times \Omega \rightarrow \mathbb{R}, \quad (t, x) \mapsto u_\lambda(t, x), \quad (1.1)$$

where the parameter λ accounts for the frequency which depends on the speed of the bow. Now, the main idea is to think of $u_\lambda(t, x)$ as a standing wave/mode. That is, $u_\lambda(t, x)$ solves the wave equation and moreover splits the variables as

$$u_\lambda(t, x) = \sin(t\sqrt{\lambda})\phi_\lambda(x). \quad (1.2)$$

Thus, u_λ indeed appears to be "standing" and only oscillating in vertical direction. Furthermore, the function $\phi_\lambda(x)$ is specific to Ω and solves the elliptic eigenvalue problem

$$\begin{cases} \Delta\phi_\lambda = \lambda\phi_\lambda & x \in \Omega, \\ \phi_\lambda(x) = 0 & x \in \partial\Omega. \end{cases} \quad (1.3)$$

Note that we somewhat simplify and change the problem by assuming that no oscillation occurs at the boundary - this, however, is still a reasonable model. Here, Δ denotes the standard Euclidean Laplace operator on \mathbb{R}^2 (i.e. trace of the Hessian). With this description, it is well-known (by an appropriate spectral theorem) that the set of such numbers λ forms a countable collection $\{\lambda_i\}_{i=0}^\infty$ of non-negative numbers with $\lambda_i \rightarrow \infty$ and there exist corresponding finite dimensional eigenfunction spaces (cf. the Appendix - Chapter A).

With this model in mind, having fixed the frequency λ (i.e. the speed of the bow), the sand will gather at places where $u_\lambda = 0$ for each moment t - that is, the sand gathers at the vanishing set

$$\mathcal{N}_{\phi_\lambda} := \{x \in \Omega : \phi_\lambda(x) = 0\}. \quad (1.4)$$

The set $\mathcal{N}_{\phi_\lambda}$ is also known and referred to as the **nodal set** of ϕ_λ .

Thus, in order to understand the sand patterns for different bow speeds (i.e. different frequencies), one would like to understand the behaviour of nodal sets as λ changes. We will be particularly interested in the geometry of nodal sets for large λ .

From another purely practical perspective, understanding nodal sets and the landscape of ϕ_λ in various such vibration models aids, for instance, the design of musical instruments and the acoustics of halls and buildings (cf. Figure 1.3).

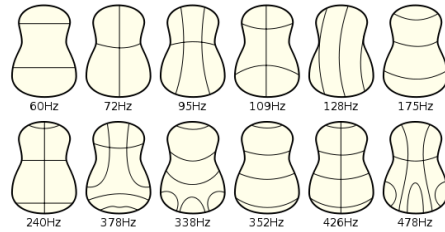


Figure 1.3: Nodal sets on a guitar plate.

1.2 Understanding high energy quantum particles

Another rich source of physical motivation for the study of nodal sets, and eigenfunctions in general, can be found in quantum mechanics (cf. also the motivation in [Zel08]).

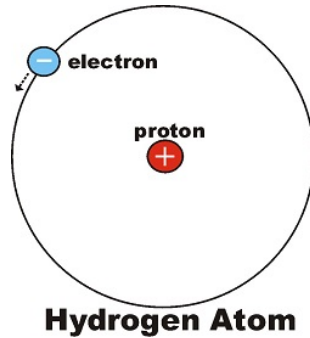


Figure 1.4: Bohr's model.

Prior to quantum mechanics, the hydrogen atom was modeled as a two-body planetary system with Hamiltonian

$$H(x, \xi) = \frac{1}{2}|\xi|^2 + V(x), \quad (1.5)$$

that is, a sum of kinetic and potential energy (which is often taken as $V(x) := \frac{1}{|x|}$). However, due to some electro-dynamical effects, this model exhibits flaws - the electron would have to radiate energy and spiral down, hitting the nucleus almost instantaneously.

A resolution was suggested in the celebrated work of Schrödinger on quantization as an eigenvalue problem. Roughly speaking, from this point of view, the electron is a "fuzzy" object which should appear at a point x with an appropriate probability $|\phi_j|^2(x)$.

Here the probability density ϕ_j is the so called energy state (i.e. an L^2 -normalized eigenfunction of the Schrödinger operator):

$$\hat{H}\phi_j := \left(-\frac{\hbar^2}{2}\Delta + V\right)\phi_j = E_j(\hbar)\phi_j, \quad (1.6)$$

where $\Delta = \sum \frac{\partial^2}{\partial x_j^2}$ is the standard Euclidean Laplace operator; V is the potential, considered as a multiplication operator on $L^2(\mathbb{R}^3)$; \hbar is Planck's constant. The eigenvalue E_j is referred to as the energy and the corresponding eigenfunction ϕ_j is known as the energy state. We note that in the case of a free particle (i.e. $V = 0$) one gets precisely Laplacian eigenfunctions, similarly to the sand-pattern model above.

However, the model of Schrödinger comes at the price of trading the geometric Hamiltonian model with eigenfunctions of \hat{H} . In order to retain the geometric intuition, one would like to have an idea of the eigenfunction's profile.

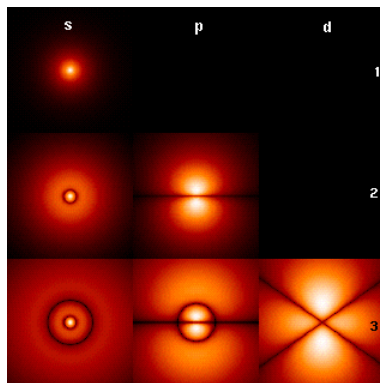


Figure 1.5: Intensity plots of eigenfunctions.

In Figure 1.5 one observes a few initial eigenfunctions - the brighter regions indicate larger values.

Roughly speaking, in order to understand the hydrogen atom of a given energy, one would like to know where the electron is most likely to be, i.e. one poses the following

Question 1.2.1. *Where is the eigenfunction ϕ_j concentrated? How large is it on this set? How large is the set where ϕ_j is "substantially" large (i.e. what is the geometry of super-level sets and domains of positivity/negativity)?*

Such concentration questions can be made precise and discussed in various ways - e.g. in terms of bounds on L^p norms.

On the other hand, one could also "ask" the dual question - that is

Question 1.2.2. *Where is an electron least likely to be? In other words, what is the geometry of the nodal (vanishing) set of the eigenfunction ϕ_j ? Where is the nodal set concentrated? How large is it?*

Questions about the "largeness" of nodal sets can be formulated in terms of surface (i.e. an appropriate Hausdorff) measure. These, in particular, include a conjecture of S.-T. Yau concerning the asymptotics of nodal set volumes (cf. Chapter 3).

Going further in the direction of eigenfunctions' geometry, if one considers the complement of the nodal set, it consists of disjoint connected components known as **nodal domains** - that is, regions of positivity/negativity of ϕ_j whose boundary is the nodal set (cf. Figure 1.6).

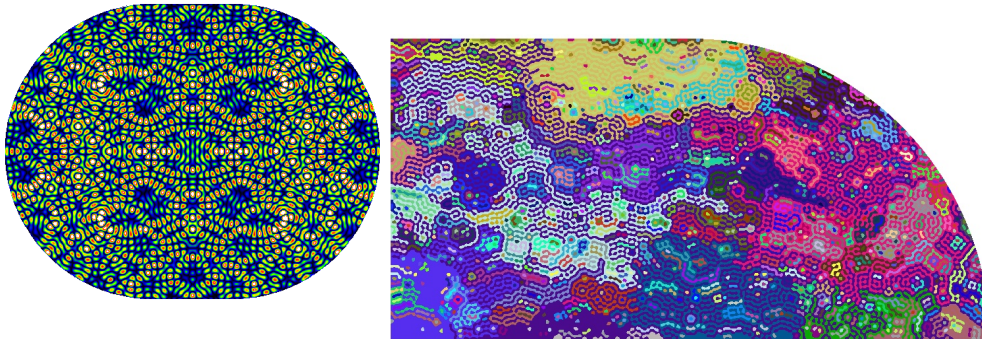


Figure 1.6: Different color coding indicates different nodal domains.

From the perspective of nodal domains, one could, for instance, ask

Question 1.2.3. *How many nodal domains are there? How large/wide can a nodal domain be? What kind of shapes can one expect as nodal domains?*

Such questions can be formalized in terms of local volumes of nodal domains, inner radii, etc.

To conclude this Section we recall an interesting picture, due to the work of Stodolna et al (*Hydrogen Atoms under Magnification: Direct Observation of the Nodal Structure of Stark States* A. S. Stodolna, A. Rouzée, F. Lépine, S. Cohen, F. Robicheaux, A. Gijbetsen, J. H. Jungmann, C. Bordas, and M. J. J. Vrakking *Phys. Rev. Lett.* 110, 2013), which sheds light on the nodal structure of a hydrogen atom (cf. Figure 1.7)

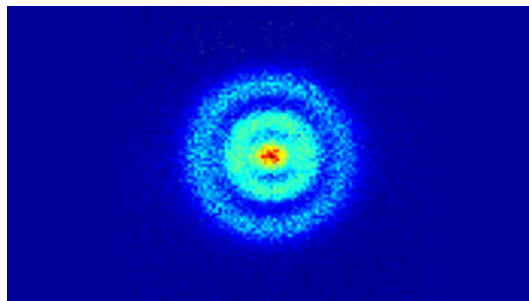


Figure 1.7: The hydrogen atom under the microscope: photoionization microscopy reveals the nodal structure of the electronic orbital of a hydrogen atom placed in a static electric field.

1.3 The framework

We now discuss the formal set-up.

We consider a closed Riemannian manifold (M, g) of dimension n . In a standard fashion, we denote the corresponding volume form by $d \text{Vol}$ and the de Rham exterior derivative by d . Moreover, we will denote the standard interior product (contraction) by i and the Lie derivative by \mathcal{L} .

We define the gradient $\nabla_g f$ of a smooth function $f \in C^\infty(M)$ in the usual way by requiring that

$$g(\nabla_g f, X) = df(X), \quad \forall X \in \Gamma(TM). \quad (1.7)$$

In local coordinates, the gradient $\nabla_g f$ is given by

$$\nabla_g f = \sum_{i,j=1}^n g^{ij} \frac{\partial f}{\partial x_i} \frac{\partial}{\partial x_j}, \quad (1.8)$$

where $\{g^{ij}\}_{i,j=1}^n$ is the inverse of the metric g .

Furthermore, one defines the divergence of a smooth vector field X as the function $\text{div}_g X$ which satisfies

$$d(i_X d \text{Vol}) = \mathcal{L}_X d \text{Vol} =: (\text{div}_g X) d \text{Vol}. \quad (1.9)$$

In local coordinates, the divergence assumes the form

$$\text{div}_g X = \frac{1}{\sqrt{|g|}} \sum_{i=1}^n \frac{\partial}{\partial x_i} \left(\sqrt{|g|} X_i \right), \quad (1.10)$$

where as usual $|g|$ denotes the absolute value of the determinant of the metric g .

The Laplace-Beltrami operator acting on functions is defined as:

$$\Delta = -\text{div}_g \circ \nabla_g. \quad (1.11)$$

In coordinates, one can derive the formula

$$\Delta = -\frac{1}{\sqrt{|g|}} \frac{\partial}{\partial x_j} \left(\sqrt{|g|} g^{jj} \frac{\partial}{\partial x_i} \right). \quad (1.12)$$

By a corresponding spectral theorem (Theorem A.2.1), the Laplacian possesses a discrete spectrum of eigenvalues $0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \dots \rightarrow \infty$ with associated finite dimensional spaces of eigenfunctions. Given an eigenfunction ϕ_λ which solves

$$\Delta \phi_\lambda = \lambda \phi_\lambda, \quad (1.13)$$

we would like to understand the geometry of ϕ_λ (level sets, localization, etc). In particular, we will be interested in the so-called **high-energy limit**, i.e. we will focus on the case where λ is large. A partial motivation for this lies in the correspondence principle, according to which the behaviour of eigenfunctions having large eigenvalues is influenced by the underlying geometry of (M, g) .

The central geometric objects of interest will be the following.

Definition 1.3.1. The *nodal set* of an eigenfunction ϕ_λ as

$$\mathcal{N}_{\phi_\lambda} := \{x \in M : \phi_\lambda(x) = 0\}. \tag{1.14}$$

As before, the *nodal domains*, usually denoted by Ω_λ , are defined to be the connected components of the complement of the nodal set (cf. Figure 1.8).

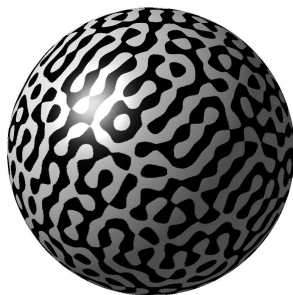


Figure 1.8: Nodal portrait of an eigenfunction on \mathbb{S}^2 : the black and white regions denote positivity and negativity sets of the eigenfunction, whose connected components are nodal domains; the boundary between the black and white regions is the nodal set, i.e. the vanishing set.

In fact, we will also consider solutions to different elliptic PDE problems (such as the Steklov problem, for instance) and also discuss nodal sets in this context.

We make an important remark on notation that will be used throughout the text.

1. With the perspective of quantum mechanics, sometimes we refer to the eigenvalue λ as **energy**.
2. The quantity $\sqrt{\lambda}$ is occasionally referred to as **frequency**.
3. The quantity $\frac{1}{\sqrt{\lambda}}$ is referred to as **wavelength**.

1.4 Ways of approaching the problems

When one studies the literature on high-energy eigenfunction analysis and nodal set/domains results, two major schools of thought can be discerned.

1.4.1 Global methods

The first school deals with global methods. For instance, one sees eigenfunctions from the perspective of being stationary points with respect to the wave group on the manifold (M, g) . In this direction, one investigates delicate properties of the wave group U_t applying the machinery of Fourier Integral Operators (FIOs). In particular, the geodesic flow and the geometry of (M, g) appear naturally in the discussion - e.g. via central statements as Egoroff's theorem, which relates the conjugation of a pseudo-differential operator A by the wave group U_t (propagation in the "quantum world") with

the composition of the corresponding symbol $\sigma(A)$ and the geodesic flow of (M, g) (propagation in the "classical world").

Of course, the literature on global methods is quite vast and we are unable to even scratch on the surface of it. However, for a very extensive and actual treatment, which, more or less, includes details on everything we mention below, we refer to the texts [Zel08], [Zel17].

For completeness, we mention a few celebrated manifestations of such techniques. These include, for example, the famous Quantum Ergodicity Theorem of Schnirelmann, Colin de Verdiere and Zelditch, which, roughly speaking, states, that the Laplacian eigenfunctions equidistribute, provided the geodesic flow is sufficiently chaotic (ergodic). Furthermore, global techniques reveal delicate properties about concentration of eigenfunctions and their growth. For instance, they reveal, that L^2 -normalized eigenfunctions cannot exceed the value $C\lambda^{\frac{n-1}{4}}$ pointwise, where C is a constant depending only on (M, g) . Such bounds are seen to be saturated by spherical harmonics. Furthermore, results of Sogge deliver sharp L^p -bounds as well. In fact, Zelditch-Sogge were able to show that if such bounds are saturated then there must be a large set of geodesic loops starting at a certain point x . On the other hand, Zelditch-Toth were able to prove that if all of the eigenfunctions are pointwise bounded by a constant, then (M, g) must be flat provided some additional integrability conditions hold.

Moreover, there are plenty of intriguing results on eigenvalue concentration in terms of semi-classical measures - e.g. the works of Anantharaman, Dyatlov-Jin, etc; as well as eigenfunction restriction theorems - Bourgain-Rudnick, Burq-Gerard-Tzvetkov, Toth-Zelditch, Zelditch, etc.

1.4.2 Local methods

On the other hand, one can approach the mentioned questions by studying eigenfunctions on sufficiently small balls. In such small regions the eigenfunction is suitably adjusted (e.g. by rescaling) so that it is close to being harmonic - for details on this procedure we refer to Section 2.1.

An upshot of this modification, is that one can rely on the rich theory of harmonic functions and be aided by tools such as mean value properties, maximum principles, doubling properties and unique continuation estimates (cf. Chapter 2), Harnack inequalities, etc.

Again the literature on local methods is large. To mention a few classical examples we refer to the works of [Che76], [DF88], [Bru78], [NPS05], [CM11], etc. Furthermore, local methods have proven a reliable strategy for tackling many interesting nodal problems - bounds on nodal set measure, various estimates on nodal domains, distributions of nodal domains, etc (cf. [Log18a], [Log18b], [Man08b]). In certain cases, local methods have proven to yield sharper estimates. A naive and rough interpretation of this is the fact that working on small scales lends information of the fine structure of eigenfunctions; whereas global methods work mostly with integral quantities and suitable aggregations.

However, unlike the global methods above, one of the difficulties in the local picture is to relate the global dynamics of the geodesic flow of (M, g) to properties of the eigenfunctions (i.e. following the correspondence principle). In some sense, such a relation can be seen as a unification of global and local methods. An illustration of this can be seen, for example, in the recent works of Hezari (cf. [Hez16]), where global and local methods work in some sense together - a global ergodicity result is translated in terms of doubling properties.

In the present text, we present mostly results in the spirit of the local methods and techniques on small scales.

1.5 Overview and organization

We now briefly discuss the organization of the text.

In **Chapter 2** we begin by introducing basic local techniques such as frequency functions and doubling conditions. Most of the ideas here are well-known with the exception of the precise control on constants in several relevant statements (e.g. the frequency function almost monotonicity result in Theorem 2.3.1). Such precise control on the constants is of importance when we discuss Steklov nodal sets. For this reason, we provide detailed computations and proofs. Chapter 2 is somewhat technical and could be, on first read, omitted and referred to whenever needed.

In **Chapter 3** we discuss nodal sets of elliptic problems. We overview some of the central results for Laplacian nodal set bounds. Afterwards, we prove bounds for more general elliptic problems with rougher coefficients (cf. Theorem 3.4.1) and, in particular, Steklov nodal sets (cf. Theorem 3.4.2) - this follows partly our work in [GRF17].

Then, in **Chapter 4** we study tubular neighbourhoods around nodal sets in accordance to a question addressed by Sodin, Fefferman and Jakobson-Mangoubi. Obtaining volume bounds for such tubular neighbourhoods would reveal certain regularity properties of nodal sets. We discuss this issue and prove two-sided bounds for such tubular neighbourhoods in the smooth setting (cf. Theorem 4.1.2). The results were announced in our work [GM17b].

In **Chapter 5** we provide some background in Brownian motion. This will be useful as we discuss certain bounds on nodal domains. Furthermore, in this Chapter we also prove a couple of statements, that seem to be known among experts in Brownian motion. However, to our knowledge, the precise formulation of these statements could not be traced back in the literature - these include a certain hitting probability comparability result (cf. Theorem 5.3.1) and a version of the Feynman-Kac formula on manifolds (cf. Theorem 5.2.1).

In **Chapter 6** we prove several bounds on nodal domains. Our first results (cf. Theorem 6.1.1, Theorem 6.2.1) extend previous work of Steinerberger and show that a nodal domain cannot be contained in a small wavelength-like tubular neighbourhood around sufficiently flat submanifolds.

Afterwards, we prove a refinement of a celebrated theorem of Lieb - our result states that given a domain, there exists a wavelength almost inscribed ball situated at a point of maximum of the first Dirichlet eigenfunction - this statement is also applied to nodal domains (cf. Theorem 6.3.1).

Further, we investigate the width of a nodal domain in terms of its inner radius (i.e. the radius of the largest ball, which is fully inscribed inside). In the case of a real-analytic manifold, we prove two sided bounds on the inner radius, which is an improvement upon previous best known bounds of Mangoubi (cf. Theorem 6.4.2).

At the end of **Chapter 6** we discuss various further estimates on the inner radius in terms of growth and appropriate distribution of L^2 -norm over good/bad cubes (cf. Theorem 6.4.3, Theorem 6.4.4); we also present an observation concerning opening angles and inscribed cone conditions (cf. Theorem 6.4.8).

Most of the results in **Chapter 6** have been announced and presented in our works [Geo16], [GM18b], [GM17b].

In **Chapter 7** we present a few results related to the following well-known problem: what is an optimal placement of an obstacle D along a domain Ω , so as to minimize/maximize the first Dirichlet eigenvalue of the complement $\Omega \setminus D$? Roughly speaking, we show that such a placement should occur near maximum points of the first Dirichlet eigenfunction (cf. Theorem 7.2.1, Theorem 7.4.1, Corollary 7.2.1). The results here are presented in our work [GM17a].

For convenience, at the end of the text we provide a short **Appendix**, which contains some

basic and background material such as Hausdorff measures, basic statements from Sobolev space theory, etc.

Chapter 2

Frequency functions and doubling conditions

As discussed in Chapter 1, when one studies the geometry of solutions to elliptic PDEs, a couple of general strategies could be identified - that is, one could take a local, or a global point of view.

In this section we bring forward the basic ideas behind the local approach of study. We start our discussion by collecting various technical tools that will be useful later on. For instance, these include ideas and objects such as:

- "Harmonization" of Laplacian eigenfunctions.
- The frequency function adapted to various solutions of elliptic PDEs.
- Doubling conditions and growth control.

First, we would like to emphasize that harmonic functions represent a valuable test model. For example, thinking in terms of the local geometry of a Laplacian eigenfunction ϕ_λ , there are a couple of ways in which one could "harmonize" ϕ_λ and concentrate on studying an appropriate associated harmonic function.

Further on, a central idea one would like to keep in mind is that we wish to understand the geometry of solutions to elliptic PDEs in terms of their growth properties. We will develop appropriate tools to keep track and study such relations between growth/doubling and corresponding level sets. A major notion here is the frequency function, which is also understood in terms of doubling.

It turns out that certain solutions of elliptic PDE, such as Laplacian and Steklov eigenfunctions, possess doubling characteristics that distinguishes them from a general harmonic function. More precisely, one can show that the corresponding eigenvalue controls the doubling - a crucial fact for understanding certain features of level sets and nodal domains.

As already mentioned, this Section primarily includes technical material and could be used as a reference whenever needed. However, a brief overview of the material here would improve the intuition behind the results later on.

2.1 Reducing the problem to harmonic functions

We begin by outlining a rather straight-forward scaling procedure.

Let (M, g) be a smooth closed Riemannian manifold of dimension n and let Δ be the corresponding Laplace operator. Suppose ϕ is a Laplacian eigenfunction of eigenvalue λ , i.e.

$$\Delta\phi = \lambda\phi. \quad (2.1)$$

Let us fix a point $p \in M$. Suppose that the eigenvalue λ is sufficiently large, so that a wavelength geodesic ball $B_{\frac{r_0}{\sqrt{\lambda}}}(p) \subseteq M$ is sufficiently small and contained in a normal coordinate neighbourhood around p . Here r_0 is a fixed positive number that we will appropriately determine below.

Using the normal coordinate neighbourhood around p we can identify $B_{\frac{r_0}{\sqrt{\lambda}}}(p)$ with a Euclidean ball $B_{\frac{r_0}{\sqrt{\lambda}}}^e(\bar{0})$ where $\bar{0}$ represents the origin in \mathbb{R}^n . In other words, by a slight abuse of notation, the eigenfunction ϕ can be thought of as a solution to

$$\tilde{\Delta}\phi = \lambda\phi, \quad (2.2)$$

in $B_{\frac{r_0}{\sqrt{\lambda}}}^e(\bar{0})$. Here $\tilde{\Delta}$ is a small perturbation of the Euclidean Laplacian

$$\Delta^e = \sum_{i=1}^n \frac{\partial^2}{\partial x_i^2}. \quad (2.3)$$

Indeed, having in mind the expression (1.12), recall that in normal coordinates the metric g is Euclidean at the origin $\bar{0}$ and is slightly perturbed in a small neighbourhood around $\bar{0}$, more precisely

$$g_{ij}(x) = \delta_i^j - \frac{1}{3} \sum_{k,l=1}^n R_{ijkl} x_k x_l + O(|x|^3), \quad \forall x \in B_{\frac{r_0}{\sqrt{\lambda}}}^e(\bar{0}). \quad (2.4)$$

Now we blow the ball $B_{\frac{r_0}{\sqrt{\lambda}}}^e(\bar{0}) \subset \mathbb{R}^n$ up to the unit ball. To this end, we consider the following scaling map

$$s : B_1(0) \rightarrow B_{\frac{r_0}{\sqrt{\lambda}}}^e(\bar{0}), \quad x \mapsto \frac{r_0}{\sqrt{\lambda}}x. \quad (2.5)$$

We consider the rescaled eigenfunction ϕ_s and rescaled Laplace operator $\tilde{\Delta}_s$ defined as follows:

$$\phi_s : B_1(0) \rightarrow \mathbb{R}, \quad \phi_s(x) = \phi(s(x)), \quad (2.6)$$

$$\tilde{\Delta}_s : C^2(B_1(\bar{0})) \rightarrow C^0(B_1(\bar{0})), \quad \tilde{\Delta}_s|_x = \tilde{\Delta}|_{s(x)}. \quad (2.7)$$

Using (1.12) and applying the chain rule in an elementary way, we observe that

$$\tilde{\Delta}_s(\phi_s(x)) = \frac{r_0^2}{\lambda}(\tilde{\Delta}\phi)(s(x)) = \frac{r_0^2}{\lambda}(\lambda\phi)(s(x)) = r_0^2\phi_s(x). \quad (2.8)$$

We now have the freedom to choose r_0 sufficiently small, so that ϕ_s would solve an equation which is a slight perturbation of the Euclidean harmonic equation.

To summarize: the advantage that one obtains here is the absence of λ - rescaling the eigenfunction in the above manner from a wavelength to the unit ball makes the treatment of high-energy eigenfunctions uniform by forgetting the corresponding eigenvalue and bringing us to work with "almost" harmonic functions. For the latter, one could rely on rich theory to study the function's geometry - this includes mean value estimates; control over higher derivatives in terms of lower ones (i.e. elliptic estimates); Harnack inequalities; etc.

In addition to rescaling, we would also like to point out another useful way of harmonizing a Laplacian eigenfunction. Let (M, g) and ϕ be as above. Consider the product space $N := \mathbb{R} \times M$ equipped with the cylinder metric $\tilde{g} := dt^2 + g$ (one can come up with other suitable choices of \tilde{g} , e.g. a cone metric). Now define the function $\tilde{\phi}(t, x) := e^{-t\sqrt{\lambda}}\phi(x)$. By a direct calculation one sees

$$\tilde{\Delta}\tilde{\phi} = 0, \tag{2.9}$$

where $\tilde{\Delta}$ denotes the Laplacian on N .

One can now focus the discussion, e.g. on a compact subset such as $K := [-1, 1] \times M$. Again we can consider small balls in K and blow them up as described above. We note that here we do not even need wavelength small balls since we have already removed λ - a uniform collection of sufficiently small balls would suffice for all high-energy eigenfunctions. However, we should keep in mind that $\tilde{\phi}$ differs from ϕ by an exponential factor which is bounded over K . In terms of level sets, the analysis of $\tilde{\phi}$ also sheds light on the geometry of ϕ .

2.2 The frequency function and doubling conditions for harmonic functions

We introduce the frequency function - a useful tool that will allow us to measure growth and doubling with respect to a given solution of an elliptic PDE. This will, furthermore, allow one to estimate zero sets and eigenfunction's landscape.

In order to illustrate the main properties of the frequency function and doubling without immediate overwhelming technical details, we will first consider harmonic functions and construct the objects in this situation. Later on, we will provide the necessary details behind frequency functions and doubling for more general forms of PDE - this will be useful, e.g. as we discuss Steklov eigenfunctions.

Important properties of the frequency function include certain monotonicity formulas (cf. [FJA79]); control over the vanishing order, as well as over the doubling indices; etc - we outline most of these results below. These have been studied in a variety of works (e.g. [GL86], [GL87], [Lin91], [HL], [BL15], etc). For some rudiment parts of our discussion, we will follow [HL].

As stated we first discuss the frequency function constructed with respect to a harmonic function. Let $u : B_1(\bar{0}) \rightarrow \mathbb{R}$ be a non-identically vanishing function in the unit ball $B_1(\bar{0}) \subset \mathbb{R}^n$ satisfying

$$\Delta u = 0, \tag{2.10}$$

where (ommiting the superscript) Δ denotes the standard Euclidean Laplacian given by (2.3).

Definition 2.2.1. For any number r in the interval $(0, 1)$ we define the quantities:

$$D(r) := \int_{B_r(\bar{0})} |\nabla u|^2, \quad \text{and} \quad H(r) := \int_{\partial B_r(\bar{0})} u^2. \tag{2.11}$$

The frequency function $N(r)$ is defined as

$$N(r) := \frac{rD(r)}{H(r)}. \quad (2.12)$$

We remark that the so defined frequency function is non-negative.

We now observe that the frequency function actually keeps track of the degree of a homogeneous harmonic polynomial.

Proposition 2.2.1. *If u is a homogeneous harmonic polynomial of degree k , then the frequency function $N(r)$ is identically equal to k for every r in $(0, 1)$.*

Proof. First, integration by parts (Green's formula) yields

$$D(r) = \int_{B_r(\bar{0})} |\nabla u|^2 = \int_{\partial B_r(\bar{0})} u \frac{\partial u}{\partial \nu} - \int_{B_r(\bar{0})} u \Delta u = \int_{\partial B_r(\bar{0})} u \frac{\partial u}{\partial \nu}, \quad (2.13)$$

where ν denotes the outer normal to the boundary.

Due to homogeneity we can set

$$u = r^k f(\theta), \quad \theta \in \mathbb{S}^{n-1}, \quad (2.14)$$

and hence

$$\frac{\partial u}{\partial \nu} = kr^{k-1} f(\theta). \quad (2.15)$$

Plugging in the definition of $N(r)$ we obtain

$$N(r) = \frac{r \int_{\partial B_r(\bar{0})} kr^{2k-1} f^2(\theta)}{\int_{\partial B_r(\bar{0})} r^{2k} f^2(\theta)} = k. \quad (2.16)$$

□

In other words, instead of studying zeros of harmonic polynomials in terms of their degree, one could utilize the frequency function instead, thus making use of several further properties. A central feature is the following

Theorem 2.2.1. *$N(r)$ is a non-decreasing function of r over the interval $(0, 1)$.*

Proof. The proof proceeds by establishing a suitable expression for the derivative $N'(r) = \frac{d}{dr} N(r)$. To this end, one needs to compute the derivatives of $D(r)$ and $H(r)$. Again, using integration by parts, it follows via a direct computation that

$$D'(r) = \frac{n-2}{r} \int_{\partial B_r(\bar{0})} u \frac{\partial u}{\partial \nu} + 2 \int_{\partial B_r(\bar{0})} \left(\frac{\partial u}{\partial \nu} \right)^2, \quad (2.17)$$

$$H'(r) = \frac{n-1}{r} \int_{\partial B_r(\bar{0})} u^2 + 2 \int_{\partial B_r(\bar{0})} u \frac{\partial u}{\partial \nu}. \quad (2.18)$$

Finally, using the definition of $N(r)$ and the obtained derivatives, we have

$$N'(r) = \frac{D(r)}{H(r)} + \frac{rD'(r)}{H(r)} - \frac{rD(r)H'(r)}{H^2(r)} \quad (2.19)$$

$$= N(r) \left(\frac{1}{r} + \frac{D'(r)}{D(r)} - \frac{H'(r)}{H(r)} \right) \quad (2.20)$$

$$= 2N(r) \left(\frac{\int_{\partial B_r(\bar{0})} \left(\frac{\partial u}{\partial \nu} \right)^2}{\int_{\partial B_r(\bar{0})} u \frac{\partial u}{\partial \nu}} - \frac{\int_{\partial B_r(\bar{0})} u \frac{\partial u}{\partial \nu}}{\int_{\partial B_r(\bar{0})} u^2} \right). \quad (2.21)$$

Noting that all the terms in the nominators and denominators are non-negative (see also (2.13)), by the Cauchy-Bunyakovski-Schwartz inequality, it follows that

$$N'(r) \geq 0. \quad (2.22)$$

□

We now put forward a few further useful properties of the frequency function. As already mentioned the frequency function may be used to gain some information concerning the vanishing order of u .

Proposition 2.2.2. *One has*

$$N(r) \rightarrow k \quad \text{as } r \rightarrow 0^+, \quad (2.23)$$

where k is the order of vanishing of u at $\bar{0}$.

Proof. Due to Theorem 2.2.1 we already know that the limit exists as $N(r)$ is monotone. Moreover, since u is an analytic function, the order of vanishing of u at $\bar{0}$ cannot be infinite. Thus in a Taylor expansion one could write

$$u = P_k + R_{k+1}, \quad (2.24)$$

where P_k denotes a non-zero homogeneous polynomial of degree k and R_{k+1} represents the remainder term which decays at least as fast as $O(r^{k+1})$. Using again the analyticity of u and the fact that Δ maps the class of homogeneous polynomials of degree s to the class of homogeneous polynomials of degree $s - 2$, it follows that both P_k and R_{k+1} must be harmonic.

We now have

$$\lim_{r \rightarrow 0^+} N(r) = \lim_{r \rightarrow 0^+} \frac{r \int_{B_r(\bar{0})} |\nabla P_k + \nabla R_{k+1}|^2}{\int_{\partial B_r(\bar{0})} (P_k + R_{k+1})^2} \quad (2.25)$$

$$= \lim_{r \rightarrow 0^+} \frac{r \int_{B_r(\bar{0})} |\nabla P_k|^2 + 2\langle \nabla P_k, \nabla R_{k+1} \rangle + |\nabla R_{k+1}|^2}{\int_{\partial B_r(\bar{0})} P_k^2 + 2P_k R_{k+1} + R_{k+1}^2}. \quad (2.26)$$

We can make the Ansatz $P_k = r^k f(\theta)$, $\theta \in \mathbb{S}^{n-1}$ and $R_{k+1} \in O(r^{k+1})$. Thus by evaluating lowest order terms, one sees that the last expression reduces to

$$\lim_{r \rightarrow 0^+} \frac{r \int_{B_r(\bar{0})} |\nabla P_k|^2}{\int_{\partial B_r(\bar{0})} (P_k)^2} = k, \quad (2.27)$$

where we have also used Proposition 2.2.1. □

The frequency function might be difficult to work with in a direct fashion. That is why one would like to find a comparable ("equivalent") quantity that is more approachable in certain situations. We will see that such a quantity is established when one investigates how the function u grows from a smaller to a larger concentric ball in terms of an L^p -norm. In other words, one studies the so-called doubling conditions.

In the next Proposition we show how the frequency function controls the doubling rate of u .

Proposition 2.2.3. *For any numbers R in $(0, \frac{1}{2})$ and η in $(1, 2]$ one has*

$$\int_{\partial B_{\eta R}(\bar{0})} u^2 \leq \eta^{2N(1)} \int_{\partial B_R(\bar{0})} u^2, \quad (2.28)$$

$$\int_{B_{\eta R}(\bar{0})} u^2 \leq \eta^{2N(1)} \int_{B_R(\bar{0})} u^2. \quad (2.29)$$

Proof. We start by obtaining a formula which will appear and play a role later on.

Lemma 2.2.1. *We have*

$$2 \frac{N(r)}{r} = \frac{d}{dr} \log \left(\frac{H(r)}{r^{n-1}} \right). \quad (2.30)$$

Proof of Lemma. We already know from (2.18) and (2.13) that

$$\frac{H'(r)}{H(r)} = \frac{n-1}{r} + 2 \frac{D(r)}{H(r)}. \quad (2.31)$$

Thus

$$2 \frac{N(r)}{r} = 2 \frac{D(r)}{H(r)} = \frac{H'(r)}{H(r)} - \frac{n-1}{r}. \quad (2.32)$$

Using elementary manipulations one deduces the needed result. \square

Onwards, integrating the formula (2.30) from r_1 to r_2 with $0 < r_1 < r_2 < 1$, we obtain

$$2 \int_{r_1}^{r_2} \frac{N(r)}{r} = \log \left(\frac{H(r_2)}{r_2^{n-1}} \right) - \log \left(\frac{H(r_1)}{r_1^{n-1}} \right). \quad (2.33)$$

After exponentiation we get

$$\frac{H(r_2)}{r_2^{n-1}} = \frac{H(r_1)}{r_1^{n-1}} \exp \left(2 \int_{r_1}^{r_2} \frac{N(r)}{r} \right). \quad (2.34)$$

As $N(r)$ is monotone,

$$\frac{H(r_2)}{r_2^{n-1}} \leq \frac{H(r_1)}{r_1^{n-1}} \exp (2N(1) \log(r)|_{r_1}^{r_2}) = \frac{H(r_1)}{r_1^{n-1}} \left(\frac{r_2}{r_1} \right)^{2N(1)} \quad (2.35)$$

Furthermore, making the Ansatz $r_1 = R$ and $r_2 = \eta R$, we obtain

$$\frac{H(\eta R)}{(\eta R)^{n-1}} \leq \frac{H(R)}{R^{n-1}} \eta^{N(1)}. \quad (2.36)$$

This is precisely (2.28). The second inequality follows by integrating the first one (2.28) with respect to R and using Fubini's theorem. \square

Definition 2.2.2. We will refer to quantities of the type

$$\gamma(\eta, R, \bar{0}) = \log_\eta \left(\frac{\int_{B_{\eta R}(\bar{0})} u^2}{\int_{B_R(\bar{0})} u^2} \right). \quad (2.37)$$

as doubling indices.

A couple of important comments are in place. First, note that γ depends on η , R and $\bar{0}$ (one could consider concentric balls with an offset). In order to simplify notation, whenever the context allows, we may sometimes omit to write all of these dependences explicitly. Second, changing the base of the logarithm \log_η would only introduce another constant in our discussions - thus there is no serious obstruction in working with the natural logarithm (or any other base). Third, as we will observe later, from elliptic theory it follows that we could also define γ in terms of sup-norms (or $L^p, p \geq 2$), instead of the introduced above L^2 -norms.

Remark 2.2.1. Currently, it seems that there is no convention on the precise notation for frequency functions and doubling. For instance, in some sources one finds that the frequency function is denoted by β , whereas other texts prefer to use N . In our discussion, we mostly use N for the frequency function and γ for the doubling (as in Chapter 2). However, in order to be consistent with some pieces in the literature (such as [Log18a]), we will, for instance, use $N(Q)$ to also refer to a certain uniform doubling index over a cube Q . That is the reason why we will restate the definitions of doubling/frequency every time we refer to these, so that no confusion regarding notation might arise in the particular context.

To conclude this Subsection, we briefly address the natural question: how does the frequency function change if one slightly shifts the origin $\bar{0}$? To make the notation clearer we have

Definition 2.2.3. For any point p in the unit ball $B_1(\bar{0})$ and any number r in the interval $(0, 1 - |p|)$ we define the frequency function $N(p, r)$ at p as

$$N(p, r) = \frac{r \int_{B_r(p)} |\nabla u|^2}{\int_{\partial B_r(p)} u^2}. \quad (2.38)$$

We have the following

Proposition 2.2.4. Let R be a number in $(0, 1)$ and let p be an arbitrary point in the ball $B_{\frac{R}{2}}(\bar{0})$. Further, let r be an arbitrary number in the interval $(|p|, \frac{R}{2})$. Then there exist constants C_1, C_2 , which depend on R, r, n , such that

$$N(p, r) \leq C_1 N(\bar{0}, R) + C_2. \quad (2.39)$$

Proof. A common tactic to estimate the frequency function at neighboring points p_1, p_2 , is to carefully select inscribed/circumscribed balls at p_1, p_2 at which one can estimate the integrals appearing the definition of the frequency function. We illustrate this procedure now.

First, the frequency $N(\bar{0}, R)$ is estimated through the doubling condition in Proposition 2.2.3, i.e. we have

$$\int_{B_R(\bar{0})} u^2 \leq \left(\frac{R}{r - |p|} \right)^{2N(\bar{0}, R) + n} \int_{B_{r - |p|}(\bar{0})} u^2. \quad (2.40)$$

Now, by the assumptions, we have the inscribed/circumscribed balls

$$B_{r-|p|}(\bar{0}) \subset B_r(p), \quad \text{and} \quad B_{\frac{R}{2}}(p) \subset B_R(\bar{0}). \quad (2.41)$$

This implies

$$\int_{B_{\frac{R}{2}}(p)} u^2 \leq \left(\frac{R}{r-|p|} \right)^{2N(\bar{0},R)+n} \int_{B_r(p)} u^2. \quad (2.42)$$

We now wish to estimate both integrals in terms of surface integrals, so that one can bring in the formula (2.30) and thus also relate $N(p, r)$.

To this end, we first observe that by the obtained formula (2.30), the function

$$r \mapsto \int_{\partial B_r(p)} u^2 \quad (2.43)$$

is non-decreasing. For convenience, let us set the arithmetic mean of $\frac{R}{2}$ and r as \tilde{r} ,

$$\tilde{r} = \frac{r}{2} + \frac{R}{4}. \quad (2.44)$$

Then, by Fubini's theorem,

$$\int_{B_{\frac{R}{2}}(p)} u^2 \geq \int_{B_{\frac{R}{2}}(p) \setminus B_{\tilde{r}}(p)} u^2 = \int_{\tilde{r}}^{\frac{R}{2}} \rho^{n-1} \int_{\partial B_\rho(p)} u^2 d\sigma d\rho \quad (2.45)$$

$$\geq \left(\frac{\left(\frac{R}{2}\right)^n - \tilde{r}^n}{n} \right) \int_{\partial B_{\tilde{r}}(p)} u^2 d\sigma. \quad (2.46)$$

On the other hand,

$$\int_{B_r(p)} u^2 = \int_0^r \rho^{n-1} \int_{\partial B_\rho(p)} u^2 d\sigma d\rho \leq \frac{r^n}{n} \int_{\partial B_r(p)} u^2 \quad (2.47)$$

Hence, due to (2.42) one has

$$\int_{\partial B_{\tilde{r}}(p)} u^2 d\sigma \leq \left(\frac{\left(\frac{R}{2}\right)^n - \tilde{r}^n}{n} \right)^{-1} \left(\frac{R}{r-|p|} \right)^{2N(\bar{0},R)+n} \left(\frac{r^n}{n} \right) \int_{\partial B_r(p)} u^2 d\sigma \quad (2.48)$$

Finally, via integration of (2.30) we conclude

$$\log \int_{\partial B_{\tilde{r}}(p)} u^2 - \log \int_{\partial B_r(p)} u^2 = \int_r^{\tilde{r}} \frac{2N(p, \rho)}{\rho} d\rho \geq 2N(p, r) \log \left(\frac{\tilde{r}}{r} \right). \quad (2.49)$$

With respect to (2.48) this implies

$$\log \left(\left(\frac{\left(\frac{R}{2}\right)^n - \tilde{r}^n}{n} \right)^{-1} \left(\frac{R}{r-|p|} \right)^{2N(\bar{0},R)+n} \left(\frac{r^n}{n} \right) \right) \geq 2N(p, r) \log \left(\frac{\tilde{r}}{r} \right). \quad (2.50)$$

After elementary algebraic manipulations one gets

$$\log\left(\frac{R}{r-|p|}\right) \log\left(\frac{\tilde{r}}{r}\right)^{-1} N(\bar{0}, R) + \frac{1}{2} \log\left(\left(\frac{\left(\frac{R}{2}\right)^n - \tilde{r}^n}{n}\right)^{-1} \left(\frac{r^n}{n}\right)\right) \log\left(\frac{\tilde{r}}{r}\right)^{-1} + \quad (2.51)$$

$$+ \frac{n}{2} \log\left(\frac{R}{r-|p|}\right) \log\left(\frac{\tilde{r}}{r}\right)^{-1} \geq N(p, r). \quad (2.52)$$

From the above coefficients we deduce the constants C_1, C_2 . □

A couple of immediate corollaries are in place.

Corollary 2.2.1. *The vanishing order of u in $B_{\frac{R}{2}}(\bar{0})$ does not exceed $C_1 N(\bar{0}, R) + C_2$, where R, C_1, C_2 are as in Proposition 2.2.4.*

Proof. From Proposition 2.2.4 one deduces $N(p, r) \leq C_1 N(\bar{0}, R) + C_2$ with p, R as above. Proposition 2.2.2 together with the monotonicity of $N(p, r)$ yields the claim. □

A central theme that we will see later is that frequency and doubling present us with a powerful tool to estimate the function's vanishing set - quite roughly speaking, one should expect that small frequency/doubling would imply a simple/small zero set, whereas a large frequency/doubling would allow the function to have a larger, more complicated zero set. Again, this goes well with the intuition coming from harmonic polynomials. As we have already seen, in a certain sense the frequency/doubling recover the polynomials degree. We give the first simple illustration of these ideas.

Corollary 2.2.2. *There exists a small constant N_0 depending on n, R , such that if $N(\bar{0}, R) \leq N_0$, then u does not vanish in the ball $B_{\frac{R}{2}}(\bar{0})$.*

Proof. As we are interested in the zeros of u , without loss of generality we may normalize u so that

$$\int_{\partial B_R(\bar{0})} u^2 = 1. \quad (2.53)$$

The definition of the frequency function then implies

$$R \int_{B_R(\bar{0})} |\nabla u|^2 = N(\bar{0}, R). \quad (2.54)$$

Basic elliptic estimates for harmonic functions give us

$$\sup_{B_{\frac{R}{2}}(\bar{0})} |\nabla u| \leq C(n, R) \|\nabla u\|_{L^2(B_R(\bar{0}))} = C(n, R) N(\bar{0}, R)^{\frac{1}{2}}. \quad (2.55)$$

By the doubling conditions in Proposition 2.2.3 we have

$$1 = \int_{\partial B_R(\bar{0})} u^2 \leq 2^{2N(\bar{0}, R) + n - 1} \int_{\partial B_{\frac{R}{2}}(\bar{0})} u^2. \quad (2.56)$$

It follows that one can find a point $p_0 \in \partial B_{\frac{R}{2}}(\bar{0})$, such that

$$|u(p_0)| \geq 2^{-N(\bar{0}, R) - \frac{n-1}{2}} \left(\text{Vol}_{n-1}(\partial B_{\frac{R}{2}}(\bar{0})) \right)^{-\frac{1}{2}}. \quad (2.57)$$

From the fundamental theorem of calculus, we conclude that for any $p \in B_{\frac{R}{2}}(\bar{0})$ one has

$$|u(p)| \geq |u(p_0)| - |p - p_0| \sup_{B_{\frac{R}{2}}(\bar{0})} |\nabla u| \quad (2.58)$$

$$\geq 2^{-N(\bar{0}, R) - \frac{n-1}{2}} \left(\text{Vol}_{n-1}(\partial B_{\frac{R}{2}}(\bar{0})) \right)^{-\frac{1}{2}} - RC(n, R)N(\bar{0}, R)^{\frac{1}{2}} \quad (2.59)$$

$$> 0, \quad (2.60)$$

provided $N(\bar{0}, R)$ is sufficiently small, depending on n, R . \square

2.3 A generalized frequency function

In the spirit Subsection 2.2 we are now interested in constructing an appropriate frequency function and doubling conditions for more general type of elliptic PDEs. Central properties of such generalized frequency functions were investigated, e.g. in [GL86], [GL87], [BL15], etc. Such statements will be useful when we address Laplacian and Steklov eigenfunctions later on.

We point out that, although similar statements exist in the literature (for instance, cf. Theorem 2.1, [GL87]), we were unable to locate precise and complete statements in the respective formulations we need. So, we take the time to carry out the needed proofs. In terms of exposition we will also partly follow our work in [GRF17].

We consider the following type of second order elliptic PDE:

$$Lu = \sum_{i,j=1}^n \frac{\partial}{\partial x_i} \left(a^{ij}(x) \frac{\partial u}{\partial x_j} \right) + \sum_{i=1}^n b^i(x) \frac{\partial u}{\partial x_i} + c(x)u = 0, \quad (2.61)$$

for points x in a smooth (i.e. possessing regular boundary) bounded domain $\Omega \subset \mathbb{R}^n$. We denote the leading coefficient matrix $\{a^{ij}(x)\}_{i,j=1}^n$ as $A(x)$ and the drift vector $\{b^i\}_{i=1}^n$ as \mathbf{b} . Moreover, we require the following conditions on the coefficients of L :

1. L is uniformly elliptic, i.e. there exists a positive number η in the interval $(0, 1)$, such that

$$\eta |\xi|^2 \leq \sum_{i,j=1}^n a^{ij}(x) \xi_i \xi_j \leq \eta^{-1} |\xi|^2, \quad \forall \xi \in \mathbb{R}^n, \quad \forall x \in \Omega. \quad (2.62)$$

2. The coefficients of L are bounded, i.e. there exists a positive number Λ with

$$\sum_{i,j=1}^n |a^{ij}(x)| + \sum_{i=1}^n |b^i(x)| + |c(x)| \leq \Lambda. \quad (2.63)$$

3. The leading coefficients are Lipschitz, i.e. there exists a positive number Γ with

$$\sum_{i,j=1}^n |a^{ij}(x) - a^{ij}(y)| \leq \Gamma |x - y|. \quad (2.64)$$

We will in fact without loss of generality assume that the domain of definition Ω contains the ball $B_R(\bar{0})$ and focus our discussion there - such an assumption is easily achieved after a translation. Moreover, we assume the radius R to be 1 as this also does not bring significant effect on the statements to follow.

Now, unless otherwise stated we will assume that $u \in W^{1,2}(\Omega)$ is a non-identically vanishing weak solution of the above PDE, that is

$$\int_{\Omega} \sum_{i,j=1}^n a^{ij}(x) \frac{\partial u}{\partial x_j} \frac{\partial \phi}{\partial x_i} + \sum_{i=1}^n b^i(x) \frac{\partial u}{\partial x_j} \phi + c(x)u\phi \, dx = 0, \quad (2.65)$$

where ϕ is an arbitrary test function from the Sobolev space $W_0^{1,2}(\Omega)$. Via elliptic regularity, it is well-known that such a weak solution u is in the space $W_{\text{loc}}^{2,2}(\Omega)$ (cf. Theorem 8.8, [GT01]).

2.3.1 Finding appropriate coordinates

Before we introduce the generalized frequency function, we first make an appropriate coordinate change, tailored along the matrix $A(x)$. This transformation will actually reduce the operator L to an operator with diagonal leading coefficient matrix. We note that such transformations are a standard tool in unique continuation arguments (cf. [GL86], [GL87], [AKS62], etc).

For n at least 3 (the case $n = 2$ can be handled by an appropriate isothermal coordinate system, but we do not pursue this here), we define the metric $\{\tilde{g}_{ij}\}_{i,j=1}^n$ on the unit ball $B_1(\bar{0})$, whose components are given by:

$$\tilde{g}_{ij}(x) := (\det A(x))^{\frac{1}{n-2}} a^{ij}(x). \quad (2.66)$$

We have

Lemma 2.3.1. *The metric \tilde{g} is Lipschitz whose Lipschitz constant \tilde{l} depends only on n, Γ, Λ . Moreover,*

$$\operatorname{div}_{\tilde{g}}(\nabla_{\tilde{g}} u) = (\det A)^{-\frac{1}{n-2}} \operatorname{div}(A \nabla u), \quad (2.67)$$

where the operators $\operatorname{div}, \nabla$ on the right hand side are taken with respect to the Euclidean metric.

Proof. Concerning the Lipschitz property - the determinant $\det A(x)$ is a sum of products of bounded Lipschitz functions and the inverse is again term-wise given by cofactor matrices of a similar form. It follows that $\tilde{g}_{ij}(x)$ is Lipschitz with Lipschitz constant \tilde{l} , depending only on n, Γ, Λ .

By definition

$$\tilde{g} = (\det A)^{\frac{1}{n-2}} A^{-1}, \quad (2.68)$$

hence

$$\tilde{g}^{-1} = (\det A)^{-\frac{1}{n-2}} A. \quad (2.69)$$

Moreover, the exponent $\frac{1}{n-2}$ is chosen in such a way, that

$$|g| = (\det A)^{\frac{n}{n-2}} \det(A^{-1}) = (\det A)^{\frac{2}{n-2}}. \quad (2.70)$$

Now, recalling the formulae (1.10), (1.8) one concludes the needed claim. \square

The metric \tilde{g} already diagonalizes our operator. However, we would also prefer that the geodesic balls around $\bar{0}$ are not deformed, i.e. they coincide with the geodesic balls induced by the constant metric $\tilde{g}(\bar{0})$. This is achieved through normal coordinates. However, we will take the point of view that \tilde{g} is conformally deformed - to this end, we introduce an appropriate conformal change. First, we define the first order approximation of the distance function:

$$r(x) := (\tilde{g}_{ij}(\bar{0})x_i x_j)^{\frac{1}{2}}, \quad (2.71)$$

where we also apply the Einstein summation convention over repeated indices. The corresponding conformal factor we need is

$$f(x) := \tilde{g}^{ij}(x) \frac{\partial r}{\partial x_i}(x) \frac{\partial r}{\partial x_j}(x) = \frac{1}{r^2(x)} \tilde{g}^{ij}(x) \tilde{g}_{ik}(\bar{0}) \tilde{g}_{jl}(\bar{0}) x_k x_l \quad (2.72)$$

$$= (\nabla r)^T \tilde{g}^{-1} \nabla r. \quad (2.73)$$

Lemma 2.3.2. *The function f is a positive Lipschitz function.*

Proof. Suppose that the positive numbers $\kappa(x), K(x)$ are the smallest, resp. largest, eigenvalues of the matrix $\tilde{g}(x)$. By definition of \tilde{g} and the bounds on $A(x)$, it follows that κ, K are uniformly bounded away from 0 in terms of n, η, Γ, Λ . This implies that

$$f(x) = \frac{x^T (\tilde{g}(\bar{0})^T \tilde{g}(x)^{-1} \tilde{g}(\bar{0})) x}{x^T \tilde{g}(\bar{0}) x} \geq \frac{\frac{1}{K(x)} |\tilde{g}(\bar{0}) x|^2}{K(0) |x|^2} = \frac{\frac{1}{K(x)} \langle \tilde{g}(\bar{0})^T \tilde{g}(\bar{0}) x, x \rangle}{K(0) |x|^2} \quad (2.74)$$

$$\geq \frac{\kappa(0)^2}{K(0)K(x)} > 0. \quad (2.75)$$

In a similar way one can also obtain an upper bound for $f(x)$. Furthermore, we can express the difference $f(x) - f(y)$ and use the well-known Lipschitz property of the ordered eigenvalues with respect to the matrix *sup*-norm (cf. also [HW53], [Wil88]) to derive the Lipschitz continuity of f . \square

We finally define the required conformal metric on $B_1(\bar{0})$ as

$$g(x) := f(x) \tilde{g}(x). \quad (2.76)$$

As usual, we denote the components of $g(x)$ as $g_{ij}(x)$ and $\{g^{ij}(x)\}_{i,j=1}^n$ will represent the inverse matrix. Furthermore, for every vector ξ in \mathbb{R}^n the following bounds hold:

$$\tau_1 |\xi|^2 \leq |\xi|_g^2 := \xi^T g \xi = \xi^T \left(f (\det A)^{\frac{1}{n-2}} A^{-1} \right) \xi \leq \tau_2 |\xi|^2, \quad (2.77)$$

where the positive τ_1, τ_2 depend only on bounds on the matrix A , i.e. on η, Λ, Γ, n (the conformal factor f is also bounded in terms of the A , as we saw in Lemma 2.3.2). Moreover, we remark that τ_1, τ_2 are close to 1 if the matrix A is close to being the identity matrix.

An important statement we will utilize is the following:

Proposition 2.3.1. *For any positive number r in the interval $(0, 1)$, the geodesic balls $B_r^g(\bar{0})$ and $B_r^{\tilde{g}(\bar{0})}(\bar{0})$ induced by g and $\tilde{g}(\bar{0})$ respectively, coincide. In particular, if $A(\bar{0})$ is the identity matrix, then the geodesic balls $B_r^g(\bar{0})$ coincide with the Euclidean balls $B_r(\bar{0})$.*

Proof. We will show that the function $r(x)$ defined above actually measures the Riemannian geodesic distance $d_g(\bar{0}, x)$ with respect to the metric g . A possible way to achieve this is to determine the radial geodesics through the origin via a Christoffel symbols' computation. However, we will determine the radial geodesics ad hoc in the spirit of [AKS62].

Let $B_{r'}^{\tilde{g}(\bar{0})}(\bar{0})$ denote the open ball centered at $\bar{0}$ and of radius r' with respect to the metric $\tilde{g}(\bar{0})$. In other words,

$$B_{r'}^{\tilde{g}(\bar{0})}(\bar{0}) = \{x : r(x) \leq r'\}. \quad (2.78)$$

If r' is sufficiently small (depending only η, Λ), then $B_{r'}^{\tilde{g}(\bar{0})}(\bar{0}) \subseteq B_r(\bar{0})$, where the number r is a fixed number in the interval $(0, 1)$.

Let c_0 be a point in $\partial B_{r'}^{\tilde{g}(\bar{0})}(\bar{0})$ and consider the ODE:

$$\dot{c}(t) = g^{-1}\nabla r|_{c(t)}, \quad c(r') = c_0, \quad (2.79)$$

where ∇ denotes the Euclidean gradient. Using the theory of ODEs with Lipschitz right hand side, it follows that there exists a unique solution $c(t)$ of class C^1 which is defined in an open interval containing r' . By the explicit metric construction and the chain rule, we observe

$$\frac{d}{dt}r(c(t)) = \frac{\partial r}{\partial x_i}(c(t)) \frac{dc_i}{dt}(t) = (\nabla r)^T g^{-1}\nabla r|_{c(t)} \quad (2.80)$$

$$= f^{-1}(\nabla r)^T \tilde{g}^{-1}\nabla r|_{c(t)} = 1. \quad (2.81)$$

This implies that $r(c(t)) = t + C$ for some constant C . However,

$$r' + C = r(c(r')) = r(c_0) = r', \quad (2.82)$$

hence $C = 0$. It follows that one can extend the curve $c(t)$ at least over the interval $(0, r']$ (the right hand side of the ODE is sufficiently regular on this interval and one can apply the standard extension procedure there).

We can apply this construction to every point p in $\partial B_{r'}^{\tilde{g}(\bar{0})}(\bar{0})$ to get a family of disjoint simple arcs which sweep out the entire ball $B_{r'}^{\tilde{g}(\bar{0})}(\bar{0})$. Now, as the $r(x)$ is sufficiently regular (for $x \neq \bar{0}$), the level set $\partial B_{r'}^{\tilde{g}(\bar{0})}(\bar{0})$ forms an embedded smooth $n - 1$ -dimensional manifold, upon which we can select local coordinates $\theta_1, \dots, \theta_{n-1}$.

This allows us to introduce the coordinates $(r, \theta_1, \dots, \theta_n)$ on the product space $(0, r') \times \partial B_{r'}^{\tilde{g}(\bar{0})}(\bar{0})$. Moreover, using the ODE (2.79) and the computation (2.80) we observe

$$g_{ij} \frac{\partial x_i}{\partial r} \frac{\partial x_j}{\partial r} |_x = \left(\frac{\partial x}{\partial r} \right)^T g \frac{\partial x}{\partial r} |_x = (g^{-1}\nabla r)^T g g^{-1}\nabla r|_x \quad (2.83)$$

$$= (\nabla r)^T g^{-1}\nabla r|_x = 1. \quad (2.84)$$

Similarly, for any integer s in $[1, n - 1]$, we get

$$g_{ij} \frac{\partial x_i}{\partial \theta_s} \frac{\partial x_j}{\partial r} |_x = \left(\frac{\partial x}{\partial \theta_s} \right)^T g \frac{\partial x}{\partial r} |_x = \left(\frac{\partial x}{\partial \theta_s} \right)^T g g^{-1}\nabla r|_x \quad (2.85)$$

$$= \left(\frac{\partial x}{\partial \theta_s} \right)^T \nabla r|_x = \frac{\partial r}{\partial \theta_s} = 0. \quad (2.86)$$

This means that the metric g is represented in the new coordinates as

$$g_{ij}dx_i dx_j = g_{ij} \left(\frac{\partial x_i}{\partial r} dr + \frac{\partial x_i}{\partial \theta_p} d\theta_p \right) \left(\frac{\partial x_j}{\partial r} dr + \frac{\partial x_j}{\partial \theta_q} d\theta_q \right) \quad (2.87)$$

$$= dr^2 + r^2 b_{pq} d\theta_p d\theta_q, \quad (2.88)$$

where we have set

$$b_{pq} := \frac{1}{r^2} g_{ij} \frac{\partial x_i}{\partial \theta_p} \frac{\partial x_j}{\partial \theta_q}. \quad (2.89)$$

We conclude that the lines $\theta = \text{const}$ are geodesics and, moreover, $b_{pq}(r, \theta)$ is the restriction of the metric g on the concentric hypersurfaces $\partial B_{r'}^{\tilde{g}(\bar{0})}(\bar{0})$ up to a factor of $\frac{1}{r^2}$. In particular, one can also view the coordinates (r, θ) as geodesic normal coordinates. \square

We also need the following control on the coefficients b_{pq} :

Lemma 2.3.3. *The functions $\{b_{pq}\}_{p,q=1}^{n-1}$ defined in Proposition 2.3.1 satisfy:*

$$\left| \frac{\partial}{\partial r} b_{pq}(r, \theta) \right| \leq B, \quad (2.90)$$

where B is a positive number that depends only on n, η, Λ .

Proof. This follows essentially from the construction of the matrix b . For complete details we refer to Sections V and VI from [AKS62].

The above estimate on $\frac{\partial}{\partial r} b_{pq}(r, \theta)$ was also discussed in [GL86], [GL87], [HL]. \square

To finalize the construction of the appropriate coordinates, we check how the operator L transforms.

Proposition 2.3.2. *The function u is a weak solution to $Lu = 0$ if and only if u is a weak solution of the following operator:*

$$L_g u := -\operatorname{div}_g(\omega(x)\nabla_g u) + \mathbf{b}_g \cdot \nabla_g u + c_g u = 0, \quad (2.91)$$

that is, for every test function ϕ in the Sobolev space $W^{1,2}(B_1(\bar{0}))$ one has

$$\int_{B_1(\bar{0})} \omega(x) \langle \nabla_g u, \nabla_g \phi \rangle_g + \phi \mathbf{b}_g \cdot \nabla_g u + c_g u \phi = 0. \quad (2.92)$$

Here $\omega(x) = f^{-\frac{n-2}{2}}$, the vector \mathbf{b}_g is given by $|g|^{-\frac{1}{2}}(g\mathbf{b})$ and c_g is $|g|^{-\frac{1}{2}}c$. In particular, the function $\omega(x)$ is a bounded Lipschitz function with

$$w_1 \leq \omega(x) \leq w_2, \quad \left| \frac{\partial}{\partial r} \omega(x) \right| \leq W, \quad (2.93)$$

where the constants w_1, w_2 and W depend only on n, Γ, η, Λ .

Proof. One computes

$$\sqrt{|g|} = -f^{\frac{n}{2}} (\det A)^{\frac{1}{n-2}}. \quad (2.94)$$

Hence, plugging into the formulae (1.10) and (1.8) one obtains:

$$-\operatorname{div}_g(\omega(x)\nabla_g u) + \mathbf{b}_g \cdot \nabla_g u + c_g u = (|g|)^{-\frac{1}{2}} Lu, \quad (2.95)$$

which yields the first claim. The properties of $w(x)$ follow from those of $f(x)$. \square

These facts also imply

Proposition 2.3.3. *The operator L_g satisfies the same assumptions as the operator L after an eventual modifications of the corresponding constants $\eta_g, \Lambda_g, \Gamma_g$.*

2.3.2 Defining the generalized frequency function and obtaining basic estimates

In the spirit of Subsection 2.2 we now define the generalized frequency function.

Definition 2.3.1. *For any number r in the interval $(0, R)$ we define the quantities:*

$$D_g(r) := \int_{B_r^g(\bar{0})} \omega(x) |\nabla_g u|_g^2 dx, \quad \text{and} \quad H_g(r) := \int_{\partial B_r^g(\bar{0})} \omega(x) u^2 d\sigma, \quad (2.96)$$

$$I_g(r) := D_g(r) + \int_{B_r^g(\bar{0})} (\mathbf{b}_g \cdot \nabla_g u) u + c_g u^2 dx, \quad (2.97)$$

where integration, unless otherwise stated, is with respect to the volume form of the metric g .

The generalized frequency function $N_g(r)$ with respect to the metric g is defined as

$$N_g(r) := \frac{r I_g(r)}{H_g(r)}, \quad (2.98)$$

whenever $H_g(r)$ is not vanishing.

A few remarks are in order.

First we remind that although the function u is in the space $W_0^{1,2}(\Omega)$ (and $W_{\text{loc}}^{2,2}(\Omega)$) one can still define surface integrals (in particular, $H_g(r)$ is well defined) via the trace operator and, furthermore, integration by parts over appropriate subdomains also holds (divergence theorem). For background, we refer to Section A, Theorems A.3.1, A.3.2. In our case however, the function u is a weak solution to an elliptic PDE and hence, Hölder continuous by the techniques of De Giorgi-Nash-Moser. We refer to Theorem 8.24, [GT01].

Second, in contrast to Subsection 2.2 the sign of $N_g(r)$ is no longer clearly determined. Furthermore, it is not clear whether $H_g(r)$ is not vanishing for a large set of radii r . We address these issues below.

Third, there are other possibilities of defining the frequency function - for example, instead of $I_g(r)$ one may directly consider the energy associated to the operator L , i.e. without undergoing the coordinate transformation above and obtaining the metric g . Furthermore, one could also try

replacing $D_g(r)$ by the Euclidean gradient, as we did in the case of harmonic functions, and use this quantity instead. These options are reasonable. For our purposes, $A(x)$ will be derived from the Laplace operator, and hence, in normal coordinates, $A(\bar{0}) = I$ where I denotes the $n \times n$ -identity matrix. This will imply that all of the mentioned options for a definition of the frequency function will be comparable up to constants close to 1 in a sufficiently small neighbourhood around $\bar{0}$.

In fact, in order to reduce the amount of technicalities, from now on we make the following:

Assumption 2.3.1. *We assume that $A(x)$ is sufficiently close to the identity, i.e.*

$$\|A(x) - I\|_{L^\infty(B_1(\bar{0}))} \leq \delta, \quad (2.99)$$

where δ is a small positive number. This means, in particular, that all bounds on the metrics g, \tilde{g} above are close to the bounds for the identity matrix. The amount of "closeness", i.e. the number δ , will be given in the particular context whenever needed.

We now address the vanishing $H_g(r)$. It will turn out that for a sufficiently small radii (depending only on $n, \Lambda_g, \Gamma_g, \eta_g$) the quantity $H_g(r)$ is positive and $N_g(r)$ is well-defined. We start with the following comparison between the quantities $D_g(r), H_g(r)$ and $I_g(r)$:

Proposition 2.3.4. *Suppose ϵ is an arbitrary number in the interval $(0, 1)$. Then there exists a positive number r_0 , depending on $\epsilon, n, \Lambda_g, \tau_1, w_1$, so that for any number r in the interval $(0, r_0)$ one has*

$$D_g(r) \leq \frac{1}{1-\epsilon} I_g(r) + \frac{\epsilon}{1-\epsilon} H_g(r), \quad (2.100)$$

$$D_g(r) \geq \frac{1}{1+\epsilon} I_g(r) - \frac{\epsilon}{1+\epsilon} H_g(r). \quad (2.101)$$

We begin by first establishing the following form of the Heisenberg uncertainty principle:

Lemma 2.3.4. *For any positive number ρ in $(0, 1)$ we have*

$$\int_{B_\rho^g(\bar{0})} u^2 \leq \frac{2\rho}{n} \int_{\partial B_\rho^g(\bar{0})} u^2 + \frac{4\rho^2}{n^2} \int_{B_\rho^g(\bar{0})} |\nabla_g u|_g^2. \quad (2.102)$$

Proof of Lemma 2.3.4. The idea is to use Fubini's theorem (or co-area formula) and split the integral in radial/tangential parts, followed by integration by parts:

$$\int_{B_\rho^g(\bar{0})} u^2 = \int_0^\rho t^{n-1} \int_{\partial B_1^g(\bar{0})} u^2(t\sigma) d\sigma dt \quad (2.103)$$

$$= \int_0^\rho \int_{\partial B_1^g(\bar{0})} u^2(t\sigma) d\sigma \frac{t^n}{n} \quad (2.104)$$

$$= \frac{\rho^n}{n} \int_{\partial B_1^g(\bar{0})} u^2(\rho\sigma) d\sigma - \frac{2}{n} \int_0^\rho t^n \int_{\partial B_1^g(\bar{0})} u(t\sigma) u_\nu(t\sigma) d\sigma dt \quad (2.105)$$

$$= \frac{\rho}{n} \int_{\partial B_\rho^g(\bar{0})} u^2(\sigma) d\sigma - \frac{2}{n} \int_{B_\rho^g(\bar{0})} |x| u(x) u_\nu(x) dx, \quad (2.106)$$

where as usual the subscript ν denotes differentiation along the normalized radial direction. Finally, Young's inequality with parameter μ gives us

$$\int_{B_g^g(\bar{0})} u^2 \leq \frac{\rho}{n} \int_{\partial B_g^g(\bar{0})} u^2(\sigma) d\sigma + \frac{\mu}{n} \int_{B_g^g(\bar{0})} u^2 + \frac{1}{n\mu} \int_{B_g^g(\bar{0})} |x|^2 |\nabla_g u|_g^2. \quad (2.107)$$

Here we also observe that $|\nabla_g u|_g^2$ dominates the radial part $|u_\nu|^2$ with respect to the metric g . Setting η as $\frac{n}{2}$ finishes the proof of the Lemma. \square

Proof of Proposition 2.3.4. Onwards, using Definition 2.3.1, Young's inequality with parameter μ and the assumptions on the coefficients (2.63) we have

$$D_g(r) = \int_{B_r^g(\bar{0})} \omega(x) |\nabla_g u|_g^2 = I_g(r) - \int_{B_r^g(\bar{0})} (\mathbf{b}_g \cdot \nabla_g u) u + c_g u^2 dx \quad (2.108)$$

$$\leq I_g(r) + \int_{B_r^g(\bar{0})} |\mathbf{b}_g| |\nabla_g u| |u| + |c_g| u^2 dx \quad (2.109)$$

$$\leq I_g(r) + \frac{1}{2\mu} \int_{B_r^g(\bar{0})} |\nabla_g u|^2 + \left(\frac{\mu}{2} \Lambda_g^2 + \Lambda_g\right) \int_{B_r^g(\bar{0})} u^2. \quad (2.110)$$

Now, we bound the last expression employing the Heisenberg uncertainty Lemma 2.3.4 to obtain

$$D_g(r) \leq I_g(r) + \left(\frac{1}{2\mu} + \left(\frac{\mu}{2} \Lambda_g^2 + \Lambda_g\right) \frac{4r^2}{n^2}\right) \int_{B_r^g(\bar{0})} |\nabla_g u|^2 + \left(\frac{\mu}{2} \Lambda_g^2 + \Lambda_g\right) \frac{2r}{n} \int_{\partial B_r^g(\bar{0})} u^2 \quad (2.111)$$

$$\leq I_g(r) + \left(\frac{1}{2\mu\tau_1} + \left(\frac{\mu}{2} \Lambda_g^2 + \Lambda_g\right) \frac{4r^2}{n^2}\right) \frac{1}{w_1} \int_{B_r^g(\bar{0})} \omega(x) |\nabla_g u|_g^2 dx \quad (2.112)$$

$$+ \left(\frac{\mu}{2} \Lambda_g^2 + \Lambda_g\right) \frac{2r}{nw_1} \int_{\partial B_r^g(\bar{0})} \omega(x) u^2, \quad (2.113)$$

where, in the last inequality, we have also used (2.77) and the bounds on $\omega(x)$ from Proposition 2.3.2.

First, we take μ to be sufficiently large, i.e. let

$$\mu = \frac{1}{\epsilon\tau_1 w_1}. \quad (2.114)$$

Now we choose r sufficiently small, so that the coefficients in front of the integrals are small:

$$\left(\frac{\epsilon w_1}{2} + \left(\frac{1}{2\epsilon} \Lambda^2 + \Lambda\right) \frac{4r^2}{n^2}\right) \frac{1}{w_1} \leq \epsilon \quad \text{and} \quad \left(\frac{1}{2\epsilon\tau_1 w_1} \Lambda^2 + \Lambda\right) \frac{2r}{nw_1} \leq \epsilon. \quad (2.115)$$

This holds whenever

$$r \leq r_0, \quad (2.116)$$

where r_0 depends on $\epsilon, \Lambda, n, \tau_1, w_1$.

Hence, we conclude from our latter estimate

$$D_g(r) \leq I_g(r) + \epsilon \int_{B_r^g(\bar{0})} |\nabla_g u|_g^2 + \epsilon \int_{\partial B_r^g(\bar{0})} u^2, \quad (2.117)$$

and after elementary algebraic manipulation

$$D_g(r) \leq \frac{1}{1-\epsilon} I_g(r) + \frac{\epsilon}{1-\epsilon} H_g(r). \quad (2.118)$$

This gives the first bound in the Proposition. To obtain the second bound, instead of adding the absolute value of the integral in the first inequality after (2.108), we subtract it, in order to obtain a reversed inequality, and proceed further in an analogous way. \square

One obtains the following immediate Corollaries:

Corollary 2.3.1. *There exists a positive number (threshold) t_0 in the interval $(0, 1)$ which depends only on $n, \Lambda_g, \Gamma_g, \tau_1$ and has the following property: if the restriction $u|_{B_r^g(\bar{0})}$ is not identically vanishing for any choice of radius r in the interval $(0, t_0)$ (see also Remark 2.3.1 below), then one has*

$$H_g(r) > 0, \quad \forall r \in (0, t_0). \quad (2.119)$$

Proof of Corollary 2.3.1. We fix $\epsilon = \frac{1}{2}$ and plug it in Proposition 2.3.4 to obtain a corresponding radius t_0 that depends only on Λ, n, τ_1 . We will show that t_0 is the required threshold. To this end, let us assume the contrary, i.e.

$$H_g(r) = 0, \quad (2.120)$$

for some radius r in the interval $(0, t_0)$. This implies that u vanishes almost everywhere on $\partial B_r^g(\bar{0})$. Furthermore, since u is in $W_{\text{loc}}^{2,2}(B_1^g(\bar{0}))$, it is in particular, also in the space $W^{2,2}(B_r^g(\bar{0}))$. Furthermore, our assumptions on the metric g dictate that the leading order coefficients, which appear in front of the partial derivatives $\partial_i u$ in the expression $\omega(x)|\nabla_g u|_g^2$, are Lipschitz, hence absolutely continuous, whose derivatives are also square integrable. This allows for integration by parts which yields

$$I_g(r) = \int_{B_r^g(\bar{0})} \omega(x)|\nabla_g u|_g^2 + (\mathbf{b}_g \cdot \nabla_g u)u + c_g u^2 dx \quad (2.121)$$

$$= 0 + \int_{B_r^g(\bar{0})} (-\text{div}_g(\omega(x)|\nabla_g u|_g^2) + (\mathbf{b}_g \cdot \nabla_g u) + c_g u) u dx \quad (2.122)$$

$$= 0. \quad (2.123)$$

The last integral is vanishing, due to the fact that

$$\int_{B_r^g(\bar{0})} (-\text{div}_g(\omega(x)|\nabla_g u|_g^2) + (\mathbf{b}_g \cdot \nabla_g u) + c_g u) \phi dx = 0, \quad (2.124)$$

for an arbitrary test function ϕ in $C_0^\infty(B_r^g(\bar{0}))$ and one can find a sequence

$$\{\phi_n\}_{n=1}^\infty \subset C_0^\infty(B_r(\bar{0})), \quad \phi_n \rightarrow_{L^2(B_r^g(\bar{0}))} u. \quad (2.125)$$

Now the choice of t_0 in combination with the first bound in Proposition 2.3.4 also yields

$$D_g(r) = 0. \quad (2.126)$$

Using the Heisenberg Uncertainty as in Lemma 2.3.4 this yields

$$u \equiv 0, \quad (2.127)$$

in the ball $B_r^g(\bar{0})$ - a contradiction with the non-identically vanishing assumption on u . \square

Remark 2.3.1. We will see that the "clumsily-formulated" non-vanishing assumption on u in Corollary 2.3.1 can be replaced by requiring that u is not identically vanishing on $B_{t_0}^g(\bar{0})$.

Indeed, with this requirement, let us assume the contrary, i.e. $H_g(r)$ vanishes for some radius r in $(0, t_0)$ and suppose that \hat{r} is the supremum of all such r .

First, the proof of Corollary 2.3.1 implies that u identically vanishes on the ball $B_{\hat{r}}^g(\bar{0})$. Thus, $\hat{r} < t_0$ and, in particular, $H_g(r)$ is positive on (\hat{r}, t_0) , so the generalized frequency $N_g(r)$ is well-defined there.

Second, in the statements below, we will establish a doubling condition by means of the frequency function, which tells us that $H_g(r_1)$ controls $H_g(r_2)$ for radii $r_1 < r_2$. This will imply that u vanishes identically on $B_{t_0}^g(\bar{0})$, a contradiction.

Such arguments seem common in the study of unique continuation principles.

From now on, we will assume that $H_g(r)$ is positive on a given interval $(0, t_0)$, where t_0 is the threshold from Corollary 2.3.1. For this to hold, as pointed out in Remark 2.3.1, it is sufficient that u is not identically vanishing on $B_{t_0}^g(\bar{0})$.

We also have a bound on the generalized frequency function indicating how small it can become. It turns out that it is "almost non-negative". Namely,

Corollary 2.3.2. Let ϵ be an arbitrary number in $(0, 1)$. There exists a number ρ_0 in $(0, 1)$ depending only on n, Λ_g, Γ_g , such that for any number r in $(0, \rho_0)$,

$$\frac{N_g(r)}{r} \geq -\epsilon. \quad (2.128)$$

Proof. Let r_0 be the number outputted from Proposition 2.3.4 with respect to ϵ . We set

$$\rho_0 := \min(r_0, t_0), \quad (2.129)$$

where t_0 is the number from Corollary 2.3.1. We claim that ρ_0 satisfies the required property. Indeed, the frequency function is well-defined on $(0, t_0)$ and the bounds from Proposition 2.3.4 imply

$$\frac{N_g(r)}{r} = \frac{I_g(r)}{H_g(r)} \geq \frac{(1 - \epsilon)D_g(r) - \epsilon H_g(r)}{H_g(r)} \geq -\epsilon. \quad (2.130)$$

\square

2.3.3 Almost monotonicity

We now come to a crucial property of the generalized frequency function: it exhibits an almost monotonicity property.

Theorem 2.3.1. *There exist positive numbers R, α_1, α_2 , depending only on n, Λ_g, w_1, W (i.e. on the bounds on the operator L), such that*

$$N(r_1) \leq \alpha_1 N(r_2) + \alpha_2, \quad (2.131)$$

for any choice of positive numbers r_1, r_2 satisfying

$$0 < r_1 < r_2 < R. \quad (2.132)$$

Moreover, if one chooses an arbitrarily small positive number ϵ , then for a sufficiently small r_2 , one may take

$$\alpha_1 = 1 + \epsilon. \quad (2.133)$$

We first require the following

Lemma 2.3.5. *The quantities $H_g(r)$ and $I_g(r)$ are absolutely continuous functions in r . In particular, since $H_g(r)$ is positive for r in $(0, t_0)$, it follows that $N_g(r)$ is absolutely continuous (as a quotient of absolutely continuous functions).*

Proof. With the help of the co-area formula, it is well-known that if f is an integrable function, then $F(r) = \int_{B_r(\bar{0})} f dx$ is absolutely continuous and $F'(r) = \int_{\partial B_r(\bar{0})} f dx$ for almost every r . Then

$$I'_g(r) = \int_{\partial B_r^g(\bar{0})} \omega(x) |\nabla_g u|_g^2 + (\mathbf{b}_g \cdot \nabla_g u) u + cu^2 d\sigma, \quad (2.134)$$

where we also implicitly use the trace operator (cf. Section A, Theorem A.3.1 - one can verify that taking the trace operator yields the result for smooth functions and then use density to deduce the result for Sobolev spaces as well).

Furthermore, one writes

$$H_g(r) = \int_{\partial B_1^g(\bar{0})} r^{n-1} u^2(r\sigma) d\sigma. \quad (2.135)$$

In a standard procedure, one can differentiate under the integral sign (again first verifying the result for smooth functions and using density) to obtain

$$H'_g(r) = \frac{n-1}{r} H_g(r) + 2 \int_{\partial B_r^g(\bar{0})} \omega u u_\nu d\sigma + \int_{\partial B_r^g(\bar{0})} \omega_\nu u^2 d\sigma \quad (2.136)$$

$$= \left(\frac{n-1}{r} + f_0(r) \right) H_g(r) + 2 \int_{\partial B_r^g(\bar{0})} \omega u u_\nu d\sigma \quad (2.137)$$

where, as usual, ν denotes differentiation in the normalized radial direction with respect to the metric g , and where the function $f_0(r)$ might depend on u but is uniformly bounded with respect to r - its bounds depend on the bounds of ω , in particular the number W . We remind that since u is in $W_{\text{loc}}^{2,2}(B_r(\bar{0}))$, the trace Theorem A.3.1 and Cauchy's inequality imply that the last integral is finite. \square

We are now in a position to start the proof of the monotonicity result.

Proof of Theorem 2.3.1. The proof is based on a couple of tactics. As in the case for harmonic functions, the main aim is to estimate the derivative of $N_g(r)$. To do this, we can discern three main steps:

1. Utilize the continuity of $N_g(r)$ to discard the set of radii where $N_g(r)$ is small and on this set the frequency function is already controlled.
2. Obtain new expressions for the derivative $I'_g(r)$. This segment is somewhat technical - we will utilize a radial deformation procedure in the spirit [GL86].
3. Combine the results of the previous steps and obtain the required bounds.

Step 1 - Truncating the frequency function.

We define the set

$$E_{r_0} := \{r \in (0, r_0) : N_g(r) > \max(1, N_g(r_0))\}. \quad (2.138)$$

As above, here the positive number r_0 is the one from Proposition 2.3.4 with a prescribed fixed positive ϵ .

We focus on estimating $N_g(r)$ on E_{r_0} as, by definition, for points in $(0, r_0) \setminus E_{r_0}$ the function $N_g(r)$ is already bounded.

Lemma 2.3.5 implies that E_{r_0} is an open set, hence a disjoint union of open intervals

$$E_{r_0} = \bigcup_{i=1}^{\infty} (a_i, b_i). \quad (2.139)$$

We note that at the endpoints of each interval (a_i, b_i) the frequency function is either 1 or $N_g(r_0)$. Furthermore, on E_{r_0} one has $N_g(r) > 1$ which, according to the definition, means

$$H_g(r) < rI_g(r). \quad (2.140)$$

Moreover, combining this with the first estimate in Proposition 2.3.4 brings us

$$D_g(r) \leq \frac{1}{1-\epsilon} I_g(r) + \frac{\epsilon}{1-\epsilon} H_g(r) < \frac{r\epsilon+1}{1-\epsilon} I_g(r) =: C_1(\epsilon, r) I_g(r). \quad (2.141)$$

Lastly, using Lemma 2.3.4 we also have

$$\int_{B_r^g(\bar{0})} u^2 \leq \frac{2r}{nw_1} H_g(r) + \frac{4r^2}{n^2w_1} D_g(r) < \left(\frac{2}{nw_1} + \frac{4}{n^2w_1} \left(\frac{r\epsilon+1}{1-\epsilon} \right) \right) r^2 I_g(r) \quad (2.142)$$

$$=: C_2(n, \epsilon, r, w_1) r^2 I_g(r). \quad (2.143)$$

For small numbers r, ϵ the constants C_1, C_2 are uniformly bounded.

Step 2 - Computing the derivatives.

Via a direct computation as in the case of harmonic functions, one gets

$$N'_g(r) = N_g(r) \left(\frac{1}{r} + \frac{I'_g(r)}{I_g(r)} - \frac{H'_g(r)}{H_g(r)} \right). \quad (2.144)$$

From Lemma 2.3.5 we already have expressions for the derivatives of H_g and I_g . However, we wish to further analyze the term

$$\int_{\partial B_r^g(\bar{0})} \omega(x) |\nabla_g u|_g^2, \quad (2.145)$$

which appears in the formula for $I'_g(r)$. This term is actually $D'_g(r)$ and appears to be difficult to handle in this form. In order to further estimate this expression we will use a certain variational method which involves the following:

- Construct a variational family

$$u^t : B_{r_0}^g(\bar{0}) \rightarrow \mathbb{R}, \quad u^t|_{t=1} = u. \quad (2.146)$$

via an appropriate rescaling map in radial directions.

- Study a suitable functional $K(u^t)$ along the family u^t . Actually, we set K as the pure kinetic energy, and it turns out that $\frac{d}{dt}|_{t=1} K(u^t)$ encodes the needed quantity (2.145).
- Refine and obtain further estimates for $I'_g(r), D'_g(r)$. This part is crucial when we finally address the bounds on $N'_g(r)$.

Here are the details.

First, we construct the following family of piecewise constant functions: let $r, \Delta r$ be fixed numbers in the interval $(0, r_0)$ with $r + \Delta r < r_0$. Now, for any t in the interval $(0, 1 + \frac{\Delta r}{r + \Delta r})$ we define the function $w^t : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ as being the constant t on $(0, r)$, the constant 1 on $(r + \Delta r, \infty)$ and a linear function on $(r, r + \Delta r)$, so that w^t is continuous. Formally,

$$w^t(\rho) = \begin{cases} t & \text{for } \rho \leq r, \\ 1, & \text{for } \rho \geq r + \Delta r, \\ t \frac{r + \Delta r - \rho}{\Delta r} + \frac{\rho - r}{\Delta r}, & \text{for } r \leq \rho \leq r + \Delta r. \end{cases} \quad (2.147)$$

Using the family of functions w^t one can define a family of bi-Lipschitz scaling maps

$$s^t : B_{r_0}^g(\bar{0}) \rightarrow B_{r_0}^g(\bar{0}), \quad s^t(x) := w^t(|x|)x. \quad (2.148)$$

This finally gives rise to our variation family: we set

$$u^t : B_{r_0}^g(\bar{0}) \rightarrow \mathbb{R}, \quad u^t := u \circ (s^t)^{-1}. \quad (2.149)$$

Second, we define and study a kinetic energy functional along u^t . To this end, one sets

$$K(u^t) := \int_{B_{r_0}^g(\bar{0})} \omega(x) |\nabla_g u^t|_g^2 dx. \quad (2.150)$$

We will compute $\frac{d}{dt}|_{t=1}K(u^t)$ in two ways. At some point this will involve differentiation with respect to t under the integral sign. To justify such an operation, note that $u^t(x)$ does not depend on t for $x \in B_{r_0}^g(\bar{0}) \setminus B_{r+\Delta r}^g(\bar{0})$ and as u is in $W_{\text{loc}}^{2,2}(B_1(\bar{0}))$, it is in particular in $W^{2,2}(B_{r+\Delta r}^g(\bar{0}))$. These facts imply that $\frac{d}{dt}|_{t=1}u^t$ belongs to $W_0^{1,2}(B_{r_0}^g(\bar{0}))$, so differentiation under the integral sign will make sense.

Now, on one hand, we can decompose the total kinetic energy in three pieces

$$K(u^t) = \int_{B_{rt}^g(\bar{0})} + \int_{B_{r+\Delta r}^g(\bar{0}) \setminus B_{rt}^g(\bar{0})} + \int_{B_{r_0}^g(\bar{0}) \setminus B_{r+\Delta r}^g(\bar{0})} =: K_1 + K_2 + K_3. \quad (2.151)$$

By definition of the deformation, it is clear that

$$\frac{d}{dt}|_{t=1}K_3 = 0. \quad (2.152)$$

Concerning K_1 , for convenience let us set

$$y := s^t(x), \quad x \in B_r^g(\bar{0}), \quad y \in B_{rt}^g(\bar{0}), \quad (2.153)$$

and notice that $u^t(y) = u(\frac{y}{t}) = u(x)$. According to the chain rule (cf. Theorem A.3.3) one deduces

$$\frac{\partial u^t}{\partial y_i}(y) = \frac{1}{t} \frac{\partial u}{\partial x_i}(x). \quad (2.154)$$

Hence, changing variables implies

$$K_1 = \int_{B_{rt}^g(\bar{0})} \omega(x) |\nabla_g u^t|_g^2 dx = t^{n-2} \int_{B_r^g(\bar{0})} \omega(tx) |\nabla_g u|_g^2 dx. \quad (2.155)$$

It follows

$$\frac{d}{dt}|_{t=1}K_1 = (n-2) \int_{B_r^g(\bar{0})} \omega(x) |\nabla_g u|_g^2 dx + \int_{B_r^g(\bar{0})} \omega_\nu(x) |\nabla_g u|_g^2 dx \quad (2.156)$$

$$= (n-2)D_g(r) + f_1(r)D_g(r), \quad (2.157)$$

where $f_1(r)$ is a function that may depend on u , but is uniformly bounded in r (see the bounds on ω, ω_ν from Proposition 2.3.2). It remains to compute K_2 . By definition, in this setting we have

$$y = \left(t \frac{r + \Delta r - |x|}{\Delta r} + \frac{|x| - r}{\Delta r} \right) x, \quad tr \leq |y| \leq r + \Delta r, \quad r \leq |x| \leq r + \Delta r. \quad (2.158)$$

We use geodesic polar coordinates (see also Proposition 2.3.1): the variable y is represented as $(|y|, \theta)$, whereas the variable x is given as $(|x|, \theta)$. Switching to geodesic polar coordinates, the gradient is given by

$$|\nabla_g u^t(y)|_g^2 = |\partial_{|y|} u(|y|, \theta)|^2 + \frac{1}{|y|^2} \sum_{i=1}^{n-1} |\partial_{\theta_i} u(|y|, \theta)|^2 \quad (2.159)$$

$$= |\partial_{|x|} u(|x|, \theta)|^2 \left(\frac{\partial |x|}{\partial |y|} \right)^2 + \frac{1}{|y|^2} \sum_{i=1}^{n-1} |\partial_{\theta_i} u(|x|, \theta)|^2. \quad (2.160)$$

Furthermore, the corresponding volume elements satisfy

$$dy = |y|^{n-1} d|y| d\theta = |y|^{n-1} \left(\frac{\partial|y|}{\partial|x|} \right) d|x| d\theta. \quad (2.161)$$

So, we can write

$$K_2 = \int_{B_{r+\Delta r}^g(\bar{0}) \setminus B_r^g(\bar{0})} \omega(x) |\nabla_g u^t(y)|_g^2 dy \quad (2.162)$$

$$= \int_r^{r+\Delta r} \int_{\mathbb{S}^{n-1}} \omega(x) |y|^{n-1} \left(\frac{\partial|x|}{\partial|y|} \right) (\partial_{|x|} u(|x|, \theta))^2 \quad (2.163)$$

$$+ \omega(x) |y|^{n-3} \left(\frac{\partial|y|}{\partial|x|} \right) (\partial_{\theta_i} u(|x|, \theta))^2 d|x| d\theta. \quad (2.164)$$

We now differentiate in t . To do so, we indicate that the objects that depend on t are $|y|$, $\frac{\partial|x|}{\partial|y|}$, $\frac{\partial|y|}{\partial|x|}$ with

$$|y| = t|x| \frac{r + \Delta r - |x|}{\Delta r} + |x| \frac{|x| - r}{\Delta r}, \quad (2.165)$$

$$\frac{\partial|y|}{\partial|x|} = t \frac{r + \Delta r - 2|x|}{\Delta r} + \frac{2|x| - r}{\Delta r}, \quad (2.166)$$

$$\frac{\partial|x|}{\partial|y|} = \frac{\Delta r}{t(r + \Delta r - 2|x|) + 2|x| - r}. \quad (2.167)$$

Plugging-in and differentiating in t , we obtain

$$\frac{d}{dt} \Big|_{t=1} K_2 = \int_r^{r+\Delta r} \int_{\mathbb{S}^{n-1}} \omega(x) |x|^{n-1} \left((n-1) \frac{r + \Delta r - |x|}{\Delta r} - \frac{r + \Delta r - 2|x|}{\Delta r} \right) (\partial_{|x|} u(|x|, \theta))^2 \quad (2.168)$$

$$+ \omega(x) |x|^{n-3} \left((n-3) \frac{r + \Delta r - |x|}{\Delta r} + \frac{r + \Delta r - 2|x|}{\Delta r} \right) (\partial_{\theta_i} u(|x|, \theta))^2 d|x| d\theta \quad (2.169)$$

$$= \frac{1}{\Delta r} \int_{B_{r+\Delta r}^g(\bar{0}) \setminus B_r^g(\bar{0})} \omega(x) ((n-1)(r + \Delta r - |x|) - (r + \Delta r - 2|x|)) (\partial_{|x|} u(x))^2 \quad (2.170)$$

$$+ \omega(x) ((n-3)(r + \Delta r - |x|) + r + \Delta r - 2|x|) (|\nabla u|^2 - (\partial_{|x|} u(x))^2) dx. \quad (2.171)$$

Taking the limit as $\Delta r \rightarrow 0^+$, one concludes

$$\lim_{\Delta r \rightarrow 0^+} \frac{d}{dt} \Big|_{t=1} K_2 = r \int_{\partial B_r^g(\bar{0})} 2\omega(x) u_\nu^2 - \omega(x) |\nabla_g u|_g^2 d\sigma = 2r \int_{\partial B_r^g(\bar{0})} \omega(x) u_\nu^2 d\sigma - r D'_g(r). \quad (2.172)$$

Putting the obtained derivatives for K_1 , K_2 and K_3 together, we obtain

$$\lim_{\Delta r \rightarrow 0^+} \frac{d}{dt} \Big|_{t=1} K = \frac{d}{dt} \Big|_{t=1} K_1 + \frac{d}{dt} \Big|_{t=1} K_2 + \frac{d}{dt} \Big|_{t=1} K_3 \quad (2.173)$$

$$= ((n-2) + f_1(r)) D_g(r) + 2r \int_{\partial B_r^g(\bar{0})} \omega(x) u_\nu^2 d\sigma - r D'_g(r). \quad (2.174)$$

In the computation so far, we have not yet used the fact that u is a weak solution to an elliptic PDE. We now compute the derivative of K making use of this information. We have

$$\lim_{\Delta r \rightarrow 0^+} \frac{d}{dt} \Big|_{t=1} K = \lim_{\Delta r \rightarrow 0^+} 2 \int_{B_{r_0}^g(\bar{0})} \omega(x) \langle \nabla_g u, \nabla_g \frac{d}{dt} \Big|_{t=1} u^t \rangle_g dx. \quad (2.175)$$

Now, using that u is a weak solution of L , $\frac{d}{dt} \Big|_{t=1} u^t$ belongs to $W_0^{1,2}(B_{r+\Delta r}^g(\bar{0}))$ and noting that, as Δr approaches 0, only the integral over $B_r^g(\bar{0})$ remains, we integrate by parts to get

$$\lim_{\Delta r \rightarrow 0^+} \frac{d}{dt} \Big|_{t=1} K = 2 \int_{B_r^g(\bar{0})} \left(\frac{d}{dt} \Big|_{t=1} u^t \right) ((\mathbf{b}_g \cdot \nabla_g u) + c_g u) dx \quad (2.176)$$

$$= 2 \int_{B_r^g(\bar{0})} (|x|u_\nu) ((\mathbf{b}_g \cdot \nabla_g u) + c_g u) dx, \quad (2.177)$$

where we have also used the explicit construction of u^t on $B_r^g(\bar{0})$ and the chain rule (cf. also Theorem A.3.3).

Finally, using our first computation (2.173) we conclude

$$D'_g(r) = \left(\frac{n-2}{r} + f_1(r) \right) D_g(r) + 2 \int_{\partial B_r^g(\bar{0})} \omega u_\nu^2 d\sigma - \frac{2}{r} \int_{B_r^g(\bar{0})} (|x|u_\nu) ((\mathbf{b}_g \cdot \nabla_g u) + c_g u) dx. \quad (2.178)$$

Substituting the last expression for $D'_g(r)$ in our formula for $I'_g(r)$ from Lemma 2.3.5 and completing $D_g(r)$ to $I_g(r)$ by adding/subtracting extra terms, we get

$$I'_g(r) = \left(\frac{n-2}{r} + f_1(r) \right) I_g(r) + 2 \int_{\partial B_r^g(\bar{0})} \omega u_\nu^2 d\sigma - \frac{2}{r} \int_{B_r^g(\bar{0})} (|x|u_\nu) ((\mathbf{b}_g \cdot \nabla_g u) + c_g u) dx \quad (2.179)$$

$$+ \int_{\partial B_r^g(\bar{0})} (\mathbf{b}_g \cdot \nabla_g u) u + c_g u^2 d\sigma - \left(\frac{n-2}{r} + f_1(r) \right) \int_{B_r^g(\bar{0})} (\mathbf{b}_g \cdot \nabla_g u) u + c_g u^2 dx. \quad (2.180)$$

One can further reduce the terms on the right hand side of the last expression. Indeed, suppose that r is a number from the set E_{r_0} . Then

$$\left(\frac{n-2}{r} + f_1(r) \right) \int_{B_r^g(\bar{0})} |\mathbf{b}_g \cdot \nabla_g u| |u| + |c_g| u^2 dx + \frac{2}{r} \int_{B_r^g(\bar{0})} (|x|u_\nu) (|(\mathbf{b}_g \cdot \nabla_g u)| + |c_g| |u|) dx \quad (2.181)$$

$$\leq (n-2 + r f_1(r)) \Lambda \int_{B_r^g(\bar{0})} |\nabla_g u|_g \frac{|u|}{r} + \frac{u^2}{r} dx + \frac{2}{r} \Lambda \int_{B_r^g(\bar{0})} (r|u_\nu|) (|\nabla_g u|_g + |u|) dx \quad (2.182)$$

$$\leq (n-2 + r f_1(r)) \Lambda \int_{B_r^g(\bar{0})} \frac{1}{2} |\nabla_g u|_g^2 + \frac{u^2}{2r^2} + \frac{u^2}{r} dx + 2\Lambda \int_{B_r^g(\bar{0})} \left(|\nabla_g u|_g^2 + \frac{1}{2} u^2 + \frac{1}{2} |\nabla_g u|_g^2 \right) dx \quad (2.183)$$

$$\leq \left(\frac{n-2 + r f_1(r)}{2} + 3 \right) \frac{\Lambda}{w_1} D_g(r) + \left(\frac{(n-2 + r f_1(r))}{r} \left(1 + \frac{1}{2r} \right) + 1 \right) \frac{\Lambda}{w_1} \int_{B_r^g(\bar{0})} u^2 dx, \quad (2.184)$$

where we have used the bounds on the coefficients via Λ and Young's inequality. Now, using the obtained inequalities (2.141), (2.142), we estimate the last expression from above via

$$\left(\frac{n-2+f_1(r)}{2}+3\right)\Lambda C_1(r,\epsilon)I_g(r)+\left((n-2+rf_1(r))\left(r+\frac{1}{2}\right)+r^2\right)\Lambda C_2(n,\epsilon,r,w_1)I_g(r) \quad (2.185)$$

$$=: C_3(n,r,\epsilon,\Lambda,w_1,W)I_g(r), \quad (2.186)$$

where C_3 is uniformly bounded for all numbers r in $(0,r_0)$.

Hence, we can write the formula (2.179) as

$$I'_g(r)=\left(\frac{n-2}{r}+f_2(r)\right)I_g(r)+2\int_{\partial B_r(\bar{0})}\omega u_\nu^2 d\sigma+\int_{\partial B_r^g(\bar{0})}(\mathbf{b}_g\cdot\nabla_g u)+c_g u^2 d\sigma, \quad (2.187)$$

where, similarly to the way we defined $f_1(r)$, $f_2(r)$ denotes a uniformly bounded by $C_3(n,r,\epsilon,\Lambda,w_1,W)$ function.

Further on, by the definition of $I_g(r)$ and integration by parts we obtain

$$I_g(r)=\int_{B_r^g(\bar{0})}\omega(x)|\nabla_g u|_g^2+(\mathbf{b}_g\cdot\nabla_g u)u+c_g u^2 dx \quad (2.188)$$

$$=\int_{\partial B_r^g(\bar{0})}\omega(x)u u_\nu d\sigma. \quad (2.189)$$

Utilizing this observation and the inequality of Cauchy-Bunyakovski-Schwarz, one gets

$$I_g(r)^2\leq\left(\int_{\partial B_r^g(\bar{0})}\omega u u_\nu d\sigma\right)^2\leq\int_{\partial B_r^g(\bar{0})}\omega u^2 d\sigma\int_{\partial B_r^g(\bar{0})}\omega u_\nu^2 d\sigma \quad (2.190)$$

$$=H_g(r)\int_{\partial B_r^g(\bar{0})}\omega u_\nu^2 d\sigma\leq rI_g(r)\int_{\partial B_r^g(\bar{0})}\omega u_\nu^2 d\sigma, \quad (2.191)$$

where we have also used (2.140). This implies

$$I_g(r)\leq r\int_{\partial B_r^g(\bar{0})}\omega u_\nu^2 d\sigma. \quad (2.192)$$

Plugging this into the formula (2.187) we have

$$\int_{\partial B_r^g(\bar{0})}|\nabla_g u|_g^2 d\sigma=\left(\frac{n-2}{r}+f_2(r)\right)I_g(r)+2\int_{\partial B_r^g(\bar{0})}\omega u_\nu^2 d\sigma \quad (2.193)$$

$$\leq((n-2+rf_2(r))+2)\int_{\partial B_r^g(\bar{0})}\omega u_\nu^2 d\sigma \quad (2.194)$$

$$=: C_5(n,\epsilon,r,w_1,W)\int_{\partial B_r(\bar{0})}\omega u_\nu^2 d\sigma. \quad (2.195)$$

Step 3 - Obtaining estimates on the frequency function.

As we have gathered enough material on the participating derivatives, we are now in a position to estimate $N'_g(r)$ on the set E_{r_0} having the expression (2.144) in mind. To this end, one should estimate the term $\frac{I'_g(r)}{I_g(r)}$ (we remind that on the set E_{r_0} the quantity $I_g(r)$ is positive). Glancing over (2.187) and (2.142), it is clear that we can control the integral of u^2 in terms of $I_g(r)$. However, we also we need control over the term

$$\int_{\partial B_r^g(\bar{0})} (\mathbf{b}_g \cdot \nabla u) u d\sigma, \quad (2.196)$$

in terms of $I_g(r)$.

To do this, we wish to keep track how far u is from being a multiple of its radial derivative u_ν . That is, we define a number l which is at least 1 (due to Cauchy-Bunyakovski-Schwarz) and satisfies

$$\int_{\partial B_r^g(\bar{0})} \omega u_\nu^2 d\sigma \int_{\partial B_r^g(\bar{0})} \omega u^2 d\sigma =: l \left(\int_{\partial B_r^g(\bar{0})} \omega u u_\nu \right)^2 = l (I_g(r))^2, \quad (2.197)$$

It turns out that if l is small, then our control on (2.196) in terms of I becomes better, whereas l large means that the term (2.196) could be absorbed easily and the information is sufficient.

More precisely, suppose that l is at most 2.

We have by (2.193),

$$\left| \int_{\partial B_r^g(\bar{0})} (\mathbf{b}_g \cdot \nabla_g u) u d\sigma \right| \leq \frac{\Lambda_g}{w_1} \int_{\partial B_r^g(\bar{0})} \omega |\nabla_g u| |u| d\sigma \quad (2.198)$$

$$\leq \frac{\Lambda_g}{w_1} \left(\int_{\partial B_r^g(\bar{0})} \omega |\nabla_g u|^2 d\sigma \int_{\partial B_r^g(\bar{0})} \omega u^2 d\sigma \right)^{\frac{1}{2}} \quad (2.199)$$

$$\leq \frac{\Lambda_g}{w_1} \left(C_5 \int_{\partial B_r^g(\bar{0})} \omega u_\nu^2 d\sigma \int_{\partial B_r^g(\bar{0})} \omega u^2 d\sigma \right)^{\frac{1}{2}}. \quad (2.200)$$

As l is at most 2, this yields

$$\left| \int_{\partial B_r^g(\bar{0})} (\mathbf{b}_g \cdot \nabla u) u d\sigma \right| \leq \frac{\Lambda_g \sqrt{2C_5}}{w_1} I(r). \quad (2.201)$$

Now, using (2.187) we get

$$\frac{I'_g(r)}{I_g(r)} = \left(\frac{n-2}{r} + f_2(r) \right) + 2 \frac{\int_{\partial B_r^g(\bar{0})} u_\nu^2 d\sigma}{I_g(r)} + \frac{\int_{\partial B_r^g(\bar{0})} (\mathbf{b}_g \cdot \nabla_g u) u}{I_g(r)} + \frac{\int_{\partial B_r^g(\bar{0})} c_g u^2 d\sigma}{I_g(r)} \quad (2.202)$$

$$= \left(\frac{n-2}{r} + f_3(r) \right) + 2 \frac{\int_{\partial B_r^g(\bar{0})} u_\nu^2 d\sigma}{I_g(r)} = \left(\frac{n-2}{r} + f_3(r) \right) + 2 \frac{\int_{\partial B_r^g(\bar{0})} u_\nu^2 d\sigma}{\int_{\partial B_r^g(\bar{0})} u u_\nu d\sigma}, \quad (2.203)$$

where we set similarly $f_3(r)$ to be a uniformly bounded function in terms of r . We also recall our expression for the derivative of $H_g(r)$ from Lemma 2.3.5. We deduce from (2.144) and the derivative of $H_g(r)$ from Lemma 2.3.5 that

$$\frac{N'_g(r)}{N_g(r)} = f_3(r) + 2 \frac{\int_{\partial B_r^g(\bar{0})} u_\nu^2 d\sigma}{\int_{\partial B_r^g(\bar{0})} uu_\nu d\sigma} - 2 \frac{\int_{\partial B_r^g(\bar{0})} uu_\nu d\sigma}{\int_{\partial B_r^g(\bar{0})} u^2 d\sigma} \quad (2.204)$$

$$\geq f_3(r) \geq -C_6(n, \Lambda_g, \epsilon, w_1, W), \quad (2.205)$$

where we have used the inequality of Cauchy-Bunyakovski-Schwarz and where we have introduced the lower bound of f_3 as the constant C_6 .

Now, on the other hand, if l is greater than 2, then the term (2.196) is estimated as follows. As above we have

$$\left| \int_{\partial B_r^g(\bar{0})} (\mathbf{b}_g \cdot \nabla_g u) u d\sigma \right| \leq \frac{\Lambda_g}{w_1} \left(C_5 \int_{\partial B_r^g(\bar{0})} \omega u_\nu^2 d\sigma H_g(r) \right)^{\frac{1}{2}} \quad (2.206)$$

However, now we apply Young's inequality with an appropriate parameter to deduce

$$\left| \int_{\partial B_r^g(\bar{0})} (\mathbf{b}_g \cdot \nabla_g u) u d\sigma \right| \leq \int_{\partial B_r^g(\bar{0})} \omega u_\nu^2 d\sigma + C_7(\Lambda_g, n, r, \epsilon, w_1, W) H(r) \quad (2.207)$$

$$\leq \int_{\partial B_r^g(\bar{0})} \omega u_\nu^2 d\sigma + C_7 I(r), \quad (2.208)$$

where we have also used (2.140). We substitute this estimate as above to get

$$\frac{N'_g(r)}{N_g(r)} = f_3(r) + \frac{\int_{\partial B_r^g(\bar{0})} u_\nu^2 d\sigma}{\int_{\partial B_r^g(\bar{0})} uu_\nu d\sigma} - 2 \frac{\int_{\partial B_r^g(\bar{0})} uu_\nu d\sigma}{\int_{\partial B_r^g(\bar{0})} u^2 d\sigma} \quad (2.209)$$

$$\geq f_3(r) \geq -C_6(n, \Lambda_g, \epsilon, w_1, W), \quad (2.210)$$

noting the advantage of the largeness of l . In conclusion, this implies

$$\frac{d}{dr} \log(N_g(r)) = \frac{N'_g(r)}{N_g(r)} \geq -C_6, \quad (2.211)$$

for every r in the set E_{r_0} . Suppose that r is in an interval (a_i, b_i) (recall the structure of the set E_{r_0} from (2.139)). Integrating over the interval $[r, b_i]$, this shows that

$$\log \left(\frac{N_g(b_i)}{N_g(r)} \right) \geq \int_r^{b_i} (-C_6) dr = -C_6(b_i - r) \geq -C_6(r_0 - r). \quad (2.212)$$

In particular, after taking exponents

$$N_g(r) \leq e^{C_6(r_0 - r)} N_g(b_i). \quad (2.213)$$

We claim that the choice

$$\alpha_1 := e^{C_6(r_0 - r)}, \quad \alpha_2 := (1 + r\epsilon)e^{C_6(r_0 - r)}, \quad (2.214)$$

satisfy the claim of the Theorem.

1. Indeed, if $N_g(r_0) \geq 1$, then for r in E_{x_0} one has

$$N_g(r) \leq e^{C_6(r_0-r)} N_g(b_i) = e^{C_6(r_0-r)} N_g(r_0) \leq \alpha_1 N_g(r_0) + \alpha_2. \quad (2.215)$$

Furthermore, if r is not in E_{x_0} , by definition one has

$$N_g(r) \leq N_g(r_0) \leq \alpha_1 N_g(r_0) + \alpha_2. \quad (2.216)$$

2. On the other hand, if $N_g(r_0) < 1$, then similarly for r in E_{r_0}

$$N_g(r) \leq e^{C_6(r_0-r)} N_g(b_i) = e^{C_6(r_0-r)} \quad (2.217)$$

$$= e^{C_6(r_0-r)} (-r\epsilon + (1+r\epsilon)) \quad (2.218)$$

$$\leq e^{C_6(r_0-r)} (N_g(r_0) + (1+r\epsilon)) \quad (2.219)$$

$$= \alpha_1 N_g(r_0) + \alpha_2, \quad (2.220)$$

where we have also used the bound (2.128). Finally, if r is not in E_{r_0} , then

$$N_g(r) \leq 1 \leq e^{C_6(r_0-r)} \leq \alpha_1 N_g(r_0) + \alpha_2, \quad (2.221)$$

as in the previous estimate.

It follows that

$$N_g(r) \leq \alpha_1 N_g(r_0) + \alpha_2, \quad (2.222)$$

for any r in the interval $(0, r_0)$.

The above arguments will hold if we substitute r_0 by a smaller number r_2 from the interval $(0, r_0)$ (thus replacing E_{r_0} by E_{r_2} , etc). Similarly one will obtain

$$N_g(r) \leq \alpha_1 N_g(r_2) + \alpha_2, \quad (2.223)$$

for any r in $(0, r_2)$.

Finally, observe that α_1 is given by $e^{C_6(r_0-r)}$. The proof of the Theorem is completed. \square

2.3.4 Doubling conditions for the generalized frequency function

Having the almost monotonicity at hand, we are now in a position to derive doubling conditions in view of the generalized frequency function. The strategy is quite similar to the harmonic function case.

First, we have

Proposition 2.3.5. *There exists a constant β_1 , depending only on the bounds of L , such that for all sufficiently small positive numbers r , the function*

$$\frac{e^{\beta_1 r} H_g(r)}{r^{n-1}} \quad (2.224)$$

is non-decreasing.

Proof. First, we recall that due to the formulas (2.136), (2.188) we have

$$H'_g(r) = \left(\frac{n-1}{r} + f_0(r) \right) H_g(r) + 2I_g(r), \quad (2.225)$$

and after algebraic manipulations one writes

$$\frac{d}{dr} \log \left(\frac{H_g(r)}{r^{n-1}} \right) = f_0(r) + 2 \frac{N_g(r)}{r}. \quad (2.226)$$

Now we select β_0 as the lower bound on $f_0(r)$ (we remind that it depends on the bounds on $\omega(x)$, i.e. on the bounds of A). Via a direct check the claim follows. \square

Proposition 2.3.6. *For sufficiently small radii $r_1 < r_2$ (depending only on the bounds on L) one has the following doubling condition*

$$H_g(r_2) \leq \left(\frac{r_2}{r_1} \right)^{2\alpha_1 N_g(r_2) + (2\alpha_2 + n - 1)} e^{\int_{r_1}^{r_2} f_0(r) dr} H_g(r_1), \quad (2.227)$$

where α_1, α_2 are the numbers from Theorem 2.3.1. A similar doubling condition can be obtained for solid balls after one further integration (cf. harmonic case in Section 2.2).

We remark that, since $f_0(r)$ is uniformly bounded in terms of the bounds on L , one can replace the middle term on the right hand side by a constant close to 1.

Proof. After integrating (2.226) from r_1 to r_2 one gets

$$H_g(r_2) = \left(\frac{r_2}{r_1} \right)^{n-1} e^{\int_{r_1}^{r_2} f_0(r) + 2 \frac{N_g(r)}{r} dr} H_g(r_1). \quad (2.228)$$

In combination with the almost monotonicity Theorem 2.3.1 we get

$$H_g(r_2) \leq \left(\frac{r_2}{r_1} \right)^{n-1} e^{\int_{r_1}^{r_2} f_0(r) dr + 2(\alpha_1 N_g(r_2) + \alpha_2) \log \left(\frac{r_2}{r_1} \right)} H_g(r_1) \quad (2.229)$$

$$= \left(\frac{r_2}{r_1} \right)^{2\alpha_1 N_g(r_2) + (2\alpha_2 + n - 1)} e^{\int_{r_1}^{r_2} f_0(r) dr} H_g(r_1). \quad (2.230)$$

\square

Remark 2.3.2. *Proposition 2.3.6 and its proof also give the following two-sided bound:*

$$\left(\frac{r_2}{r_1} \right)^{2\alpha_1^{-1} N_g(r_1) - (2\alpha_2 - n + 1)} e^{\int_{r_1}^{r_2} f_0(r) dr} \leq \frac{H_g(r_2)}{H_g(r_1)} \leq \left(\frac{r_2}{r_1} \right)^{2\alpha_1 N_g(r_2) + (2\alpha_2 + n - 1)} e^{\int_{r_1}^{r_2} f_0(r) dr}. \quad (2.231)$$

We now remind again of Remark 2.3.1, as we have established the required doubling condition.

2.3.5 Doubling indices and scaling properties

As at the beginning of this Subsection let us now come back to the initial formulation of our problem in terms of the operator L acting on functions defined over Ω . Recall also the Assumption 2.3.1.

In the spirit of Definition 2.2.2, we define the following quantity:

Definition 2.3.2. *Given a fixed (Euclidean) ball $B \subseteq \Omega$, such that the concentric ball of twice the radius $2B$ is also contained in Ω we set*

$$\gamma(B) := \log_2 \frac{\sup_{2B} |u|}{\sup_B |u|}. \quad (2.232)$$

We refer to γ as the doubling index of u at the ball B . We also write $\gamma(x, r)$ to refer to the doubling index of the ball $B_r(x)$.

As we saw in the doubling conditions arising from monotonicity formulae, the doubling index is related to the (generalized) frequency function. Furthermore, it turns out that the doubling index actually controls the growth over any choice of a concentric ball - i.e. if one chooses a ball tB instead of $2B$, where t is at least 2. We conclude the discussion on the generalized frequency function by establishing this particular scaling dependency.

Lemma 2.3.6. *Let ϵ be an arbitrary fixed number in the interval $(0, \epsilon_0)$, where ϵ_0 is a sufficiently small number, depending only on n and the bounds on L . Suppose that the Assumption 2.3.1 holds with a given positive number δ . There exist positive constants c, r_1 which depend only on ϵ, δ , such that we have*

$$t^{\gamma(x, \rho)(1-\epsilon)-c} \leq \frac{\sup_{B_{t\rho}(x)} |u|}{\sup_{B_\rho(x)} |u|} \leq t^{\gamma(x, \rho)(1+\epsilon)+c}, \quad (2.233)$$

for any choice of numbers $\rho > 0, t > 2$ and a point x which satisfy

$$B_{t\rho}(x) \subseteq B_{r_1}(\bar{0}). \quad (2.234)$$

Furthermore, there is a threshold $\gamma_0 = \gamma_0(\epsilon)$, such that if $\gamma(x, \rho) > \gamma_0$, then the constant c can be dropped in the above estimate and one has

$$t^{\gamma(x, \rho)(1-\epsilon)} \leq \frac{\sup_{B_{t\rho}(x)} |u|}{\sup_{B_\rho(x)} |u|} \leq t^{\gamma(x, \rho)(1+\epsilon)} \quad (2.235)$$

Proof. We suppose that x, t are fixed and via translation we assume that x coincides with the origin $\bar{0}$. We first bound the quotient

$$\frac{\sup_{B_{t\rho}(x)} |u|}{\sup_{B_\rho(x)} |u|} \quad (2.236)$$

in an appropriate way. To this end, we bound the numerator and denominator.

First, having the Assumption 2.3.1 in mind, we introduce the metric g as we did before in Subsection 2.3.1 and observe that g is close to the identity (depending on δ). In particular, the geodesic balls $B_r^g(\bar{0})$ are "close" to the Euclidean balls $B_r(\bar{0})$.

We now recall the following elliptic estimate:

$$\sup_{B_\rho(\bar{0})} |u|^2 \leq c_1 \int_{B_{\rho(1+\frac{\delta}{2})}(\bar{0})} u^2, \quad (2.237)$$

where the constant c_1 depends on n, ϵ and the bounds on the operator L . We note that the elliptic estimate is a consequence of a De Giorgi-Nash-Moser result (cf. Theorem 8.24, [GT01]).

Now, using the comparability of metrics (i.e. assuming δ is sufficiently small depending on ϵ) and via Proposition 2.3.5 we have

$$\int_{B_{\rho(1+\frac{\epsilon}{2})}(\bar{0})} u^2 \leq \int_{B_{\rho(1+\epsilon)}(\bar{0})} u^2 \leq \frac{c_2}{\rho^{n-1}} H_g(\rho(1+\epsilon)), \quad (2.238)$$

where c_2 depends on ϵ, β, c_1 and on the upper bound for ρ .

On the other hand, using that u is continuous, in combination with metric comparability and Proposition 2.3.5 we also obtain

$$\sup_{B_{2\rho}(\bar{0})} |u|^2 \geq \int_{B_{2\rho}(\bar{0})} u^2 \geq \frac{c_3}{\rho^n} \int_{2\rho(1-\epsilon)}^{2\rho(1-\frac{\epsilon}{2})} H_g(\rho) d\rho \geq c_4 \frac{H_g(2\rho(1-\epsilon))}{\rho^{n-1}}, \quad (2.239)$$

where c_3, c_4 depend on n, ϵ and an upper bound for ρ .

Using the latter, we estimate the doubling indices as follows

$$\gamma(\bar{0}, \rho) := \log_2 \frac{\sup_{B(\bar{0}, 2\rho)} |u|}{\sup_{B(\bar{0}, \rho)} |u|} \geq \frac{1}{2} \log_2 \frac{H_g(2\rho(1-\epsilon))}{H_g(\rho(1+\epsilon))} + c_5, \quad (2.240)$$

where we have set

$$c_5 := \log \frac{c_4}{c_1 c_2}. \quad (2.241)$$

The last quotient is controlled via the generalized frequency as given in Remark 2.3.2. Further, assume that r_1 is sufficiently small, so that the Monotonicity Theorem 2.3.1 gives $\alpha_1 = 1 + \epsilon$. Then, we have

$$\frac{1}{2} \log \frac{H_g(2\rho(1-\epsilon))}{H_g(\rho(1+\epsilon))} \geq \log_2 \left[\left(\frac{2(1-\epsilon)}{1+\epsilon} \right)^{\frac{N_g(\rho(1+\epsilon))}{1+\epsilon} - \frac{2\alpha_2 - n + 1}{2}} \right] \quad (2.242)$$

$$\geq \left(\frac{N_g(\rho(1+\epsilon))}{1+\epsilon} - \frac{2\alpha_2 - n + 1}{2} \right) \log_2 \left(\frac{2(1-\epsilon)}{1+\epsilon} \right) \quad (2.243)$$

$$\geq \frac{N_g(\rho(1+\epsilon))}{1+\epsilon} \log_2 \left(\frac{2(1-\epsilon)}{1+\epsilon} \right) - c_6, \quad (2.244)$$

where we have collected terms as

$$c_6 := \left(\frac{2\alpha_2 - n + 1}{2} \right) \log_2 \left(\frac{2(1-\epsilon)}{1+\epsilon} \right). \quad (2.245)$$

Now, we recall that for sufficiently small ρ , the frequency function is "almost non-negative" in the sense that (cf. Corollary 2.3.2).

$$\frac{N_g(r)}{r} \geq -\epsilon. \quad (2.246)$$

Thus, using the Taylor expansion of the logarithm (provided ϵ is sufficiently small), we can find an appropriate constant c_7 (depending on c_6, ϵ), so that

$$\frac{N_g(\rho(1+\epsilon))}{1+\epsilon} \log_2 \left(\frac{2(1-\epsilon)}{1+\epsilon} \right) - c_6 \geq N_g(\rho(1+\epsilon))(1-3\epsilon) - c_7. \quad (2.247)$$

Putting everything together, we see that

$$\gamma(\bar{0}, \rho) \geq N_g(\rho(1 + \epsilon))(1 - 100\epsilon) - c_7. \quad (2.248)$$

In a similar fashion one also sees

$$\gamma(\bar{0}, \rho) \leq N_g(2\rho(1 + \epsilon))(1 + 100\epsilon) + c_7, \quad (2.249)$$

where one might need to increase the constant c_7 if needed (however, it will still depend on the parameters as before).

We have established the following comparison between the generalized frequency function and the doubling index:

Claim 2.3.1. *Suppose ϵ and ρ are sufficiently small (whose smallness depends only on n and the bounds on L). Then*

$$N_g(\rho(1 + \epsilon))(1 - 100\epsilon) - c_7 \leq \gamma(\bar{0}, \rho) \leq N_g(2\rho(1 + \epsilon))(1 + 100\epsilon) + c_7. \quad (2.250)$$

We now proceed showing the lower bound in Lemma 2.3.6. First, we can assume that t is bounded away from 2. Indeed, if t is close to 2, i.e. $t \leq 2^{1+\epsilon}$, then since $t > 2$ we have,

$$t^{\gamma(\bar{0}, \rho)(1-\epsilon)} \leq 2^{\gamma(\bar{0}, \rho)}, \quad (2.251)$$

and hence,

$$\sup_{B_{t\rho}(\bar{0})} |u| \geq \sup_{B_{2\rho}(\bar{0})} |u| = 2^{\gamma(\bar{0}, \rho)} \sup_{B_\rho(\bar{0})} |u| \geq t^{\gamma(\bar{0}, \rho)(1-\epsilon)} \sup_{B_\rho(\bar{0})} |u|, \quad (2.252)$$

which gives the lower bound and the additional statement as well.

Thus, we assume that $t > 2^{1+\epsilon}$. Let us also set $\tilde{\epsilon} := \epsilon/1000$, so that $(1 - \tilde{\epsilon})t > 2(1 + \tilde{\epsilon})$. Using the estimate (2.239) and definition of the doubling index, we have

$$\frac{\sup_{B_{t\rho}(\bar{0})} |u|^2}{\sup_{B_\rho(\bar{0})} |u|^2} \geq \frac{c_4(t\rho)^{1-n} H_g((1 - \tilde{\epsilon})t\rho)}{2^{-2\gamma(\bar{0}, \rho)} \sup_{B_{2\rho}(\bar{0})} u^2}. \quad (2.253)$$

To bound the numerator further from below we use the estimate from Remark 2.3.2 over balls with radii $(2\rho(1 + \tilde{\epsilon}))$ and $(t\rho(1 - \tilde{\epsilon}))$, followed by an application of Claim 2.3.1. This way we can also absorb the term t^{1-n} in the constants. To bound the denominator we use the elliptic estimate (2.237). Hence, we have

$$\frac{\sup_{B_{t\rho}(\bar{0})} |u|^2}{\sup_{B_\rho(\bar{0})} |u|^2} \geq c_{11} \frac{\left(\frac{(1-\tilde{\epsilon})t}{2(1+\tilde{\epsilon})}\right)^{(2\gamma(\bar{0}, \rho)/(1+100\tilde{\epsilon})(1+\tilde{\epsilon})) - c_9} H_g(2\rho(1 + \tilde{\epsilon}))}{2^{-2\gamma(\bar{0}, \rho)} H_g(2\rho(1 + \tilde{\epsilon}))} \quad (2.254)$$

$$= c_{11} 2^{2\gamma(\bar{0}, \rho)} \left(\frac{(1 - \tilde{\epsilon})t}{2(1 + \tilde{\epsilon})}\right)^{(2\gamma(\bar{0}, \rho)/(1+100\tilde{\epsilon})(1+\tilde{\epsilon})) - c_9}. \quad (2.255)$$

Now, we can absorb the powers of 2 from the numerator and denominator by further adjusting the constants c_{11}, c_9 if needed. The latter is thus bounded from below by

$$c_{12} \left(\frac{(1 - \tilde{\epsilon})t}{(1 + \tilde{\epsilon})}\right)^{(2\gamma(\bar{0}, \rho)/(1+100\tilde{\epsilon})(1+\tilde{\epsilon})) - c_{13}} \geq c_{14} t^{2\gamma(\bar{0}, \rho)(1-\epsilon) - c_6}, \quad (2.256)$$

where we have absorbed the quotient $(1-\tilde{\epsilon})/(1+\tilde{\epsilon})$ by using the smallness of $\tilde{\epsilon}$ and further adjusting the participating constants. In particular, we reduce the power of t by a small multiple of $\gamma(\bar{0}, \rho)$.

This concludes the proof of the lower bound. To show the additional statements in the Lemma, it suffices to take $\epsilon/2$ instead of ϵ and repeat the arguments above: one obtains a corresponding new constant $c = c(\frac{\epsilon}{2})$ and requires that

$$\gamma(\bar{0}, \rho) > \frac{2}{\epsilon}c =: \gamma_0(\epsilon/2). \quad (2.257)$$

□

We will also need the following comparison for doubling numbers at nearby points.

Lemma 2.3.7. *Let ϵ be a small given positive number. There exists a radius $r_0 > 0$ and a threshold γ_0 (depending only on n, ϵ and bounds on L) such that, for any $x_1, x_2 \in B_r(p)$ and a positive number ρ for which*

$$d(x_1, x_2) < \rho < r_0, \quad \gamma(x_1, \rho) > \gamma_0, \quad (2.258)$$

there exists a positive constant C (depending only on n, ϵ and bounds on L) such that

$$\gamma(x_2, C\rho) > (1 - \epsilon)\gamma(x_1, \rho). \quad (2.259)$$

Proof. The proof proceeds by using the definition of $\gamma(p, r)$ and shifting the concentric balls upon which it is defined from x_1 to x_2 (cf. Proposition 2.2.4 and its proof; see also Lemma 7.4, [Log18a] for further details). □

2.4 Bounds on doubling indices for Laplacian and Steklov eigenfunctions

Let (M, g) be a an n -dimensional closed Riemannian manifold.

We point out a special property of doubling indices of Laplacian eigenfunctions ϕ_λ . It turns out that the doubling indices are controlled by the corresponding eigenvalue λ . Of course, when one harmonizes such a Laplacian eigenfunction (in the spirit of Section 2.1, the eigenvalue becomes somewhat hidden, and one works with an almost harmonic function. However, it is this control on the doubling index in terms λ that allows one to distinguish an eigenfunction from a general harmonic function.

Furthermore, the control on the doubling index will play an important role in the other results on nodal sets/nodal domains, which are presented in this text.

Now, the mentioned doubling index estimate is formulated as:

Theorem 2.4.1. *Let ϕ_λ be a Laplacian eigenfunction on (M, g) and let us select an arbitrary point x in M and a positive radius r . Then one has*

$$\frac{\sup_{B_{3r}(x)} |\phi_\lambda|}{\sup_{B_{2r}(x)} |\phi_\lambda|} \leq C_1 e^{C_2 \sqrt{\lambda}}, \quad (2.260)$$

where C_1, C_2 are constants that depend only on (M, g) (the dimension n and certain bounds on the metric g such as injectivity radius/curvature estimates and diameter estimates).

We remark here that the radii $3r, 2r$ of the concentric balls could be taken as $\alpha r, \beta r$ with $\alpha > \beta$. However, the constants C_1, C_2 will be different and will become worse as α approaches β .

Theorem 2.4.1 appeared in the celebrated work of Donnelly-Fefferman where the authors addressed Yau's conjecture and proved sharp bounds on nodal set volumes in the real-analytic case (cf. [DF88]). The initial proof of Theorem 2.4.1 used delicate Carleman type bounds with special (geometric) choices of cut-off and weight functions. Later on, Lin and Mangoubi derived simpler proofs of the doubling bounds (cf. [HL], [Man13] and the references therein). Their techniques relied on the analysis of the function

$$H(r) := \int_{\partial B_r(x)} \phi_\lambda^2 d\sigma_r, \quad (2.261)$$

which was also used in Sections 2.2, 2.3. For further details, we refer to Theorem 4.4, [Man13] and Lemma 6.1.1, [HL].

Bounds in the spirit of Theorem 2.4.1 can also be proved for Steklov eigenfunctions (cf. [BL15], [Zhu]) - we will refer to such bounds later on as well.

Chapter 3

Estimates on nodal sets

In this Chapter we address bounds on nodal sets of elliptic problems. These will be formulated in terms their $(n - 1)$ -dimensional Hausdorff measure - for a short overview of this concept we refer to Section A.1 at the end of the text.

3.1 A brief recollection of results

Let (M, g) be a closed Riemannian manifold of dimension n and let φ_λ be a Laplacian eigenfunction. Considering the works of [HS89] and [DF88] it is known that the nodal set $\mathcal{N}_{\varphi_\lambda}$ has a finite $(n - 1)$ -Hausdorff measure.

We first recall the problem of estimating the size of the $(n - 1)$ -Hausdorff measure of the nodal set - the question was raised by S.-T. Yau in 1982, who conjectured that

$$C_1\sqrt{\lambda} \leq \mathcal{H}^{n-1}(\mathcal{N}_{\varphi_\lambda}) \leq C_2\sqrt{\lambda}, \quad (3.1)$$

where C_1, C_2 are fixed constants that depend only on (M, g) - e.g. the dimension n and certain estimates on the geometry of M .

In a celebrated paper (cf. [DF88]), Donnelly and Fefferman were able to confirm Yau's conjecture whenever (M, g) is a real-analytic manifold. Roughly speaking, their techniques relied on analytic continuation and delicate estimates concerning growth of polynomials.

Later on, in the smooth case, the question of Yau was extensively investigated further: to name a few, we refer to the works by Sogge-Zelditch, Colding-Minicozzi, Lin, Mangoubi, Hezari-Sogge, Hezari-Riviere, etc (for an exhaustive reference list we refer to [Zel08], [Zel17]) - these works were based around the discussion of the lower bound. Further, the works of Hardt-Simon, Dong, etc (cf. [Zel08], [Zel17]) studied the upper bound. As an outcome, non-sharp estimates were obtained. The corresponding methods of study were quite broad in nature, utilizing both local and global methods (cf. Chapter 1).

Recently, in [Log18a], [Log18b], Logunov made a significant breakthrough which delivered the lower bound in Yau's conjecture for closed smooth manifolds (M, g) as well as a polynomial upper bound in terms of λ . In a nutshell, his methods utilized delicate combinatorial estimates based on doubling numbers of harmonic functions - as pointed out in [Log18a] and [Log18b], some of these techniques were also developed in collaboration with Malinnikova (cf. also the references provided in [Log18a]).

3.2 Laplacian nodal sets in the real-analytic case

In this Section we give a quick overview of the Laplacian nodal set bounds presented in [DF88]. Some of the statements in the work of Donnelly-Fefferman will also be of importance when we discuss nodal domain bounds in Chapter 6 - we will give precise references when needed.

3.2.1 Lower bound

The idea of the lower bound could be summarized in the following steps.

Step 1: density of the nodal set and an appropriate covering of M .

It is a well-known fact that the nodal set is **wavelength dense**, i.e. there exists a constant C , depending only on (M, g) (i.e. independent of λ), such that each ball of radius less than $C/\sqrt{\lambda}$ contains a zero of the eigenfunction (cf. [Bru78]). Thus, one can select a covering of suitable wavelength balls, whose centers lie on the nodal set.

Step 2: uniform control on the doubling index on most balls.

Through a technically demanding argument, real analyticity implies that a fixed proportion of the balls in the covering above have a controlled doubling index - that is, bounded by a universal constant, independent of λ . Such balls are called **good**. This statement is the place where one utilizes the intuition that in the real-analytic setting eigenfunctions resemble polynomials of degree $\sqrt{\lambda}$.

Step 3: comparability of positivity/negativity in a good ball.

Provided that one considers a good ball B (i.e. where the doubling index of the eigenfunction is bounded above by a constant), one can use elliptic estimates (cf. Proposition 6.4.1) to show comparability of $\text{Vol}(\{\varphi_\lambda > 0\} \cap B)$ and $\text{Vol}(\{\varphi_\lambda < 0\} \cap B)$ in each such good ball up to a constant, independent of λ .

Step 4: application of isoperimetric inequalities.

One applies the following form of isoperimetric inequality. For open sets $U, V \subset B$ one has

$$\mathcal{H}^{n-1}(\partial U \cap \partial V) \geq C \min \left\{ (\text{Vol}(U))^{\frac{n-1}{n}}, (\text{Vol}(V))^{\frac{n-1}{n}} \right\}, \quad (3.2)$$

where C is a constant depending only on (M, g) . Replacing U, V by $\{\varphi_\lambda > 0\} \cap B$, $\{\varphi_\lambda < 0\} \cap B$ and using the comparability from the previous Step, one obtains

$$\mathcal{H}^{n-1}(\{\varphi_\lambda = 0\} \cap B) \geq \widehat{C} (\text{Vol}(B))^{\frac{n-1}{n}}, \quad (3.3)$$

where \widehat{C} is a constant depending only on (M, g) and independent of λ .

Step 5: conclusion.

Finally, one simply sums the estimate (3.3) over all good balls. This yields the lower bound in Yau's conjecture.

A few remarks are in place. Apart from Step 2, all of the other Steps are robust in the sense that they do not require real-analyticity and go through in the smooth case as well. However, an analogue of Step 2 in the smooth case at the present moment is still unknown.

We also note that Step 2, as presented in [DF88], is rather non-trivial in terms of technical details. One actually proves that, similarly to polynomials, the eigenfunction is close to its average in most small regions. This is achieved through a delicate induction procedure with respect to the dimension n .

3.2.2 Upper bound

The upper bound could be roughly summarized as follows (for a very accessible presentation we also refer to [HL]).

Step 1: The complex one-dimensional case.

If one considers an one-dimensional complex analytic function with a doubling index over a ball B bounded by a number N , one can show that the number of zeros inside B is bounded in terms of N . This is achieved in a direct manner, e.g., via a standard Blaschke factorization.

Step 2: Shooting lines in higher dimensions.

Now, we consider the higher dimensional case. Suppose one works in a given sufficiently small chart. For suitable points x_i one can consider the lines passing through x_i , i.e. of the form $x_i + t\nu$, where $t \in \mathbb{R}, \nu \in \mathbb{S}^{n-1}$. One can complexify the eigenfunction and consider its restriction on the complex plane given by $x_i + z\nu$ where $z \in \mathbb{C}$. By Step 1 and Theorem 2.4.1 the number of zeros over a disk in this complex plane $\{x_i + z\nu\}$ is bounded by $C\sqrt{\lambda}$, for some C depending only on (M, g) . In particular, the number of zeros on the line $x_i + z\nu$ does not exceed $C\sqrt{\lambda}$.

Step 3: Crofton's formula and conclusion.

The formula of Crofton is an integral geometric statement and tells us that the $(n-1)$ -Hausdorff measure of the nodal set $\mathcal{N}_{\varphi_\lambda}$ can be bounded above by integrating the number of intersection points $l \cap \mathcal{N}_{\varphi_\lambda}$ over all lines l passing through the points x_i . However, the number of intersection points $l \cap \mathcal{N}_{\varphi_\lambda}$ is just the number of zeros on l , which we know to be bounded by $C\sqrt{\lambda}$.

Integrating over all lines l and summing over all such charts on M yields the upper bound in Yau's conjecture.

We note that the real-analyticity, although not required for the doubling bound in Theorem 2.4.1, is still used in an essential way in order to take complexifications.

3.3 Laplacian nodal sets in the smooth case

In a recent breakthrough, Logunov was able to prove the lower bound in Yau's conjecture in the smooth case, as well as polynomial upper bounds on Laplacian nodal sets in the smooth case (cf. [Log18a], [Log18b]). His methods built upon a certain combinatorial technique, which as stated in [Log18a] stems also from a joint work with Malinnikova.

A central step in these works is to obtain appropriate sufficient substitutes for good/bad balls (or equivalently, cubes) statements in the spirit of Step 2 from the lower bound of Yau's conjecture in the real-analytic setting above.

This is achieved through a subtle investigation of the frequency function and doubling conditions for harmonic functions (cf. Chapter 2 for background) which roughly speaking focuses on the question whether the doubling/frequency accumulates additively in some way. In other words, if

certain bounds on the doubling index are known on a set, what can be said about the doubling index at suitable nearby points.

The core statements in this direction are referred to as the simplex (or barycenter accumulation) lemma and the hyperplane lemma. The first lemma tells us that if the doubling indices at a vertices of a given simplex are large, then the doubling index at the barycenter of the simplex is also suitably bounded below. The latter hyperplane lemma addresses doubling index bounds in terms of propagation of smallness with Cauchy data in the spirit of Lin (cf. [Log18a] and the references therein).

Below we show analogues of these lemmata in the case of more general elliptic equations with rougher coefficients which allows us to prove nodal set bounds in the spirit of [Log18a] -cf. Propositions 3.4.1 and 3.4.2.

3.3.1 Lower bound

We provide a brief sketch of the lower bound in the smooth setting. Using the perspective from Section 2.1 one restricts the discussion to a harmonic function u .

Step 1: Large doubling index implies the existence of many zero points.

First, we note that the "Good/Bad cube" substitutes are not as strong as the claim in Step 2 from the lower bound in the real-analytic case.

To briefly elaborate on this, let B be a fixed geodesic ball and let us subdivide it into sufficiently smaller cubes. In the real-analytic case, we have the strong statement in Step 2, that if the smaller cubes are sufficiently small, then a fixed proportion of them will be good (i.e. will have universally bounded doubling index).

Now, roughly speaking, in the smooth case one observes that the proportion of good cubes depends on the doubling index over B (cf. Theorem 3.4.3). This is achieved with the aid of the simplex and hyperplane lemmata mentioned above.

Further, utilizing the Harnack inequality and the last relaxed good/bad proportion result, one shows that if the doubling index is sufficiently large, then the considered function u has a lot of well-separated zeros (cf. Theorem 4.3.1).

Step 2: A "compactness" argument.

In the spirit of previous work of Nadirashvili, Nazarov-Polterovich-Sodin (cf. [Log18a]), one considers the quantity

$$f(N) := \inf \frac{\mathcal{H}^{n-1}(\{u = 0\} \cap B_r(x))}{r^{n-1}}, \tag{3.4}$$

where the infimum is taken over all $B_r(x) \subseteq B$ and all harmonic functions u with $u(x) = 0$ whose doubling index over B does not exceed N . One shows that there exists a universal positive constant c , so that

$$f(N) \geq c, \tag{3.5}$$

for every positive N . Roughly speaking, for a small doubling index N the nodal geometry is under control via the methods of Proposition 6.4.1 and for a large doubling index N , one finds many well-separated zeros from Step 1 which should still lead to a large nodal set.

Step 3: conclusion.

One obtains the lower bound in Yau's conjecture by similarly summing the bound on $f(N)$ over the balls where the eigenfunction was "harmonized".

3.3.2 Upper bound

As before, one works mostly with a harmonic function u over a geodesic ball B .

To show the upper bound on the nodal set, the relaxed statement for the proportion of good/bad cubes (cf. Theorem 3.4.3) is sufficient - one does not need to use the existence of many well-separated zeros of u provided the doubling is large.

One similarly defines

$$f(N) := \sup \frac{\mathcal{H}^{n-1}(\{u=0\} \cap Q)}{\text{diam}^{n-1}(Q)}, \quad (3.6)$$

where the supremum is taken over all cubes $Q \subset B$ and harmonic functions u whose doubling index does not exceed N on Q .

Using Theorem 3.4.3 one can show in a direct manner that the positive numbers N for which

$$f(N) \geq 4Af\left(\frac{N}{1+c}\right), \quad (3.7)$$

form a bounded set - here the positive numbers A, c are appropriately chosen and come from Theorem 3.4.3 below. This implies that $f(N) \leq CN^\alpha$, for some constants C, α which depend only on (M, g) .

Summing over initial balls B and using Theorem 2.4.1 yields the polynomial upper bound in Yau's conjecture.

3.4 Upper bounds on nodal sets for more general elliptic PDE

We now turn our attention to non trivial solutions u to the following general second order elliptic equation

$$Lu := \sum_{i,j=1}^N \frac{\partial}{\partial x_i} \left(a^{ij}(x) \frac{\partial u}{\partial x_j} \right) + \sum_{i=1}^N b^i(x) \frac{\partial u}{\partial x_i} + c(x)u = 0. \quad (3.8)$$

in some smooth bounded domain $\Omega \subset \mathbb{R}^n$. We make the following assumptions on the coefficients of L :

1. L is uniformly elliptic, that is for a fixed $\eta > 0$ we have

$$a^{ij}(x)\xi_i\xi_j \geq \eta|\xi|^2, \quad \forall \xi \in \mathbb{R}^n, x \in \Omega. \quad (3.9)$$

2. The coefficients of L are bounded

$$\sum_{i,j} |a^{ij}(x)| + \sum_i |b^i(x)| + |c(x)| \leq \Lambda, \quad x \in \Omega. \quad (3.10)$$

3. The leading coefficients are Lipschitz

$$\sum_{ij} |a^{ij}(x) - a^{ij}(y)| \leq \Gamma|x - y|. \quad (3.11)$$

We focus our interest on the relation between the zero set and the local growth properties of a solution u . In particular, we will address nodal sets of functions whose leading order coefficients $A(x) = \{a^{ij}\}_{i,j=1}^n$ are derived from the Laplace operator. Using normal coordinates we will hence assume that

$$A(0) = I, \tag{3.12}$$

where I denotes the $n \times n$ -identity matrix. This assumption allows us to reduce the amount of technicalities when we utilize the generalized frequency (cf. Section 2.3).

3.4.1 Doubling indices and nodal set

Given a fixed ball B such that $2B \subset \Omega$, the *doubling index* $\gamma(B)$ is a measure of the local growth of u on B defined as before by

$$\gamma(B) = \log_2 \frac{\sup_{2B} |u|}{\sup_B |u|} \tag{3.13}$$

We will often write $\gamma(x, r)$ for the doubling index of u on the ball $B(x, r)$.

As discussed above, the doubling index is a useful tool to study nodal volumes in the case of Laplacian eigenfunctions. Here, we show that the size of the nodal set of solutions to the equation (3.8) with the prescribed control on coefficients is estimated by the doubling index in the following way:

Theorem 3.4.1. *There exist positive numbers $r_0 = r_0(M, g)$, $C = C(M, g)$ and $\alpha = \alpha(n)$ such that for any solution u of equation (3.8) in a domain Ω satisfying the conditions (3.9), (3.10), (3.11), we have*

$$\mathcal{H}^{n-1}(Z_u \cap Q) \leq C \text{diam}^{n-1}(Q) N^\alpha(Q), \tag{3.14}$$

where $Q \subset B(p, r_0)$ is an arbitrary cube in Ω .

Here, $N(Q)$ is the uniform doubling index of u on a cube Q as defined by

$$N(Q) := \sup_{x \in Q, r \in (0, \text{diam}(Q))} \gamma(x, r). \tag{3.15}$$

We remind the reader of Remark 2.2.1 concerning the different conventions on notation for frequency functions and doubling.

The proof of Theorem 3.4.1 adapts the machinery of the generalized frequency functions and doubling conditions developed in Chapter 2. These tools are implemented in the methods of Logunov ([Log18a]), in order to obtain nodal bounds for solutions of more general elliptic equations.

In the next few Subsections we will indicate the appropriate modifications in the work [Log18a] when one deals with more general elliptic PDE. From our standpoint, these modifications include an adaptation of the doubling scaling (cf. Subsection 2.3.4), a propagation of smallness estimate (referred to as Hyperplane lemma in [Log18a]) and an accumulation of growth (referred to as Simplex lemma in [Log18a]).

Having these appropriately modified statements at hand allows one to obtain the relaxed version of the good/bad cube proportions as mentioned above in Sections 3.2 and 3.3. This leads to nodal set upper bounds as mentioned at the end of Section 3.3.

3.4.2 Application: interior nodal sets of Steklov eigenfunctions

Before elaborating more on the doubling index analysis and good/bad-cube-proportion estimates, we take a slight detour to discuss an interesting subcase of the above more general PDE - that is, Steklov nodal sets.

Let M be a smooth, connected and compact manifold of dimension $n \geq 2$ with non-empty smooth boundary ∂M and denote by $\Delta = \Delta_g$ the Laplace-Beltrami operator on M . The Steklov eigenfunctions on M are solutions to

$$\begin{cases} \Delta\phi = 0 & \text{in } M, \\ \partial_\nu\phi = \lambda\phi & \text{on } \partial M. \end{cases} \quad (3.16)$$

In this setting, the spectrum is discrete and is composed of the eigenvalues

$$0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \dots \nearrow \infty.$$

Given a Steklov eigenfunction $u = u_\lambda$, we distinguish the codimension 1 *interior nodal set*

$$Z_\lambda = \{x \in M : \phi(x) = 0\} \quad (3.17)$$

and the codimension 2 *boundary nodal set*

$$N_\lambda = \{p \in \partial M : \phi(p) = 0\}. \quad (3.18)$$

As mentioned earlier, we are interested in measuring the size of these nodal sets. It is expected that their size is controlled by the Steklov eigenvalue. More precisely, it is conjectured that

$$c_1\lambda \leq \mathcal{H}^{n-1}(Z_\lambda) \leq c_2\lambda \quad (3.19)$$

and

$$c_3\lambda \leq \mathcal{H}^{n-2}(N_\lambda) \leq c_4\lambda, \quad (3.20)$$

where \mathcal{H}^n is the n -dimensional Hausdorff measure. In the above, the c_i are positive constants that may only depend on the geometry of the manifold M . These conjectures are similar to Yau's conjecture for nodal sets of eigenfunctions of the Laplace operator. We now briefly present the current best results present in the literature, starting with the interior nodal set:

Table 3.1: Current best bounds for $\mathcal{H}^{n-1}(Z_\lambda)$

Regularity and dimension	Current Best Lower Bound	Current Best Upper Bound
$C^\omega, n = 2$	$c\lambda$ [PST] ✓	$c\lambda$ [PST, Zhu] ✓
$C^\omega, n \geq 3$		$c\lambda$ [Zhu] ✓
$C^\infty, n = 2$	c [SWZ16]	$c\lambda^{\frac{3}{2}}$ [Zhu16]
$C^\infty, n \geq 3$	$c\lambda^{\frac{2-n}{2}}$ [SWZ16]	

Here the symbol ✓ indicates that the bounds agree with the predicted sharp bounds (i.e. of the order of λ).

In the case of the boundary nodal set, we have

Table 3.2: Current best bounds for $\mathcal{H}^{n-2}(N_\lambda)$

Regularity and dimension	Current Best Lower Bound	Current Best Upper Bound
$C^\omega, n \geq 2$		$c\lambda$ [Zel15] ✓
$C^\infty, n = 2$	$c\lambda$ [WZ]	
$C^\infty, n \geq 3$	$c\lambda^{\frac{4-n}{2}}$ [WZ]	

We use Theorem 3.4.1 to provide a polynomial upper bound for interior nodal sets in the smooth case in any dimension $n \geq 2$.

Theorem 3.4.2. *Let M be a smooth, connected and compact manifold of dimension $n \geq 2$ with non-empty smooth boundary ∂M . Let ϕ_λ be a Steklov eigenfunction on M corresponding to the eigenvalue λ . Then*

$$\mathcal{H}^{n-1}(Z_\lambda) \leq c\lambda^\alpha, \quad (3.21)$$

where $c = c(M, g)$ and $\alpha = \alpha(n)$.

The proof of Theorem 3.4.2 is based on a gluing procedure that transforms M into a compact manifold without boundary. Doing so and working locally then allows to transfer the study of the nodal set of ϕ to that of a solution u to the elliptic Equation (3.8). The details are presented in Section 3.4.5.

3.4.3 Additivity of frequency

We now address the simplex and hyperplane lemmata in the case of more general elliptic PDEs.

We have already obtained the essential estimates for the generalized frequency function in Section 2.3. In contrast to the case of Laplacian eigenfunctions (as in [Log18a]), the generalized frequency function needs to be treated with somewhat more care, but it turns out that, similarly to the harmonic case, an analogous scaling of the doubling index holds - that is, Lemma 2.3.6.

Onwards, using the generalized frequency function and Lemma 2.3.6, we verify the simplex and hyperplane lemmata stated below. Here one also needs to introduce appropriate gradient estimates and propagation of smallness for equations with rougher coefficients, whereas [Log18a] exploits bounds pertinent to harmonic functions.

Now, the obtained two lemmata work together to investigate the additivity properties of the frequency. The underlying principal idea can be roughly summarized as follows: if the frequency of u on a big cube Q is high, then it cannot be high in too many disjoint subcubes $q_i \subset Q$. That is, one obtains estimates for the good-bad-cube proportions as mentioned in Sections 3.2, 3.3.

For the rest of the discussion, we essentially refer to [Log18a], as the statements follow directly.

Barycenter accumulation

Roughly speaking, we will assert the following: suppose that the doubling exponents at the vertices $\{x_1, \dots, x_{n+1}\}$ of a simplex are large (i.e. bounded below by a fixed $N_0 > 0$). Then, the doubling exponent at the barycenter of the simplex $x_0 := \frac{1}{n} \sum_{i=1}^{n+1} x_i$ is bounded below by $(1+c)N_0$, where $c > 0$ is a fixed constant. Heuristically, the growth "accumulates" at the barycenter. The proof proceeds via direct use of the frequency/doubling properties discussed in Section 2.3.

Definition 3.4.1. Given a simplex $S := \{x_1, \dots, x_{n+1}\}$, we define the relative width $w(S)$ of S as

$$w(S) := \frac{\text{width}(S)}{\text{diam}(S)}, \quad (3.22)$$

where $\text{diam}(S)$ is the diameter of S and $\text{width}(S)$ is the smallest possible distance between two parallel hyperplanes, containing S in the region between them.

Further on, we will consider simplices S whose relative width is bounded below as $w(S) \geq w_0 := w_0(n) > 0$ - the specific bound w_0 will be specified later.

Now, in order to apply the scaling of frequency we will need the following covering lemma.

Lemma 3.4.1. Let $S := \{x_1, \dots, x_{n+1}\}$ be an arbitrary simplex satisfying $w(S) \geq w_0$. There exist a constant $\alpha := \alpha(n, w_0) > 0$ and a number (ratio) $K := K(n, w_0) \geq \frac{2}{w_0}$, so that if one takes a radius $\rho := K \text{diam}(S)$, then one has

$$B(x_0, (1 + \alpha)\rho) \subset \cup_{i=1}^{n+1} B(x_i, \rho). \quad (3.23)$$

Moreover, for $t > 2$ there exists $\delta(t) \in (0, 1)$ with $\delta(t) \rightarrow 0$ as $t \rightarrow \infty$, so that

$$B(x_i, t\rho) \subset B(x_0, (1 + \delta)t\rho). \quad (3.24)$$

The main result of this subsection is the following proposition.

Proposition 3.4.1 (also known as *Simplex lemma* in [Log18a]). Let $\{B_i\}_{i=1}^{n+1}$ be a collection of balls centered at the vertices $\{x_i\}_{i=1}^{n+1}$ of the simplex S and radii not exceeding $\frac{\rho}{2}$, where $\rho = \rho(n, w_0)$ comes from Lemma 3.4.1. Then, there exist the positive constants $c := c(n, w_0)$, $C := C(n, w_0) \geq K$, $r := r(w_0, L)$ and $N_0 := N_0(w_0, L)$ with the following property:

If $S \subset B(p, r)$ and if $\gamma(B_i) > N > N_0$, $i = 1, \dots, n + 1$, then

$$\gamma(x_0, C \text{diam} S) > (1 + c)N. \quad (3.25)$$

Proof. First, Lemma 2.3.6 shows that by taking larger balls, the doubling exponents essentially increase, so we can assume that all balls B_i have the radius ρ .

Let us set

$$M := \sup_{\cup_{i=1}^{n+1} B(x_i, \rho)} |u|, \quad (3.26)$$

and let us suppose that M is achieved on the ball $B(x_{i_0}, \rho)$ for a fixed index i_0 .

In particular, by Lemma 3.4.1 we have

$$\sup_{B(x_0, (1+\alpha)\rho)} |u| \leq M. \quad (3.27)$$

Further, let us introduce parameters $t > 2, \epsilon > 0$ to be specified below and assume that the second statement in Lemma 2.3.6 holds for the ball $B(x_{i_0}, t\rho)$, by which we see

$$\sup_{B(x_{i_0}, t\rho)} |u| \geq Mt^{N(1-\epsilon)}. \quad (3.28)$$

Moreover, assuming that the scaling in Lemma 2.3.6 is functional at the barycenter x_0 and recalling Lemma 3.4.1, we conclude

$$\left(\frac{t(1+\delta)}{1+\alpha}\right)^{\gamma(x_0, t(1+\delta)\rho)(1+\epsilon)+c_6} \geq \frac{\sup_{B(x_0, t(1+\delta)\rho)} |u|}{\sup_{B(x_0, (1+\alpha)\rho)} |u|} \geq \frac{\sup_{B(x_{i_0}, t\rho)} |u|}{\sup_{B(x_0, (1+\alpha)\rho)} |u|} \quad (3.29)$$

$$\geq \frac{Mt^{N(1-\epsilon)}}{M} = t^{N(1-\epsilon)}. \quad (3.30)$$

Specifying the parameters, we select $t > 2$ large enough to ensure $\delta(t) \leq \frac{\alpha}{2}$, and hence

$$\frac{t(1+\delta)}{1+\alpha} \leq t^{1-\mu}, \quad (3.31)$$

for some $\mu = \mu(t, \alpha) \in (0, 1)$. Thus, putting the last estimates together we see

$$t^{(1-\mu)\gamma(x_0, t(1+\delta)\rho)(1+\epsilon)+c_6} \geq t^{N(1-\epsilon)} \quad (3.32)$$

and therefore

$$\gamma(x_0, t(1+\delta)\rho) \geq \frac{1-\epsilon}{(1+\epsilon)(1-\mu)}N - c_3. \quad (3.33)$$

Selecting an $\epsilon = \epsilon(\mu) > 0$ we can arrange that

$$\frac{1-\epsilon}{(1+\epsilon)(1-\mu)} > 1 + 2c, \quad (3.34)$$

for some $c := c(\mu) > 0$. Hence, we conclude

$$\gamma(x_0, t(1+\delta)\rho) \geq N(1+2c) - c_3 \geq (1+c)N + (cN_0 - c_3) > (1+c)N, \quad (3.35)$$

provided that N_0 is sufficiently big. \square

Propagation of smallness

We use propagation of smallness to derive estimates on the doubling exponents. The main auxiliary result in this discussion is the propagation of smallness for Cauchy data. In contrast to [Log18a], here we essentially need to address the appropriate tools for operators with rough coefficients and lower regularity instead of the standard Laplacian and smooth coefficients.

Lemma 3.4.2 (cf. Lemma 4.3, [Lin91]). *Let u be a solution of (3.8) in the half-ball B_1^+ where the conditions (3.9), (3.10), (3.11) are satisfied. Let us set*

$$F := \{(x', 0) \in \mathbb{R}^n \mid x' \in \mathbb{R}^{n-1}, |x'| < \frac{3}{4}\}. \quad (3.36)$$

If the Cauchy conditions

$$\|u\|_{H^1(F)} + \|\partial_n u\|_{L^2(F)} \leq \epsilon < 1 \quad \text{and} \quad \|u\|_{L^2(B_1^+)} \leq 1. \quad (3.37)$$

are satisfied, then

$$\|u\|_{L^2(\frac{1}{2}B_1^+)} \leq C\epsilon^\beta, \quad (3.38)$$

where the constants C, β depend on n, η, Λ, Γ .

It is convenient to introduce the following doubling index.

Definition 3.4.2. *The doubling index $N(Q)$ of a cube Q is defined as*

$$N(Q) := \sup_{x \in Q, r \in (0, \text{diam}(Q))} \gamma(x, r). \quad (3.39)$$

Again, we remind the reader of Remark 2.2.1 concerning the different conventions on notation for frequency functions and doubling.

An immediate observation is that

$$N(q) \leq N(Q), \quad \text{if } q \subseteq Q, \quad (3.40)$$

and if $Q \subseteq \cup_i Q_i$ with $\text{diam}(Q) \leq \text{diam}(Q_i)$, then there exists an index i_0 such that

$$N(Q) \leq N(Q_{i_0}). \quad (3.41)$$

Proposition 3.4.2 (also known as *Hyperplane lemma* in [Log18a]). *Let Q be a cube $[-R, R]^n$ in \mathbb{R}^n and let us divide Q into $(2A + 1)^n$ equal sub-cubes q_i with side-length $\frac{2R}{2A+1}$. Let $\{q_{i,0}\}$ be the collection of sub-cubes which intersect the hyperplane $\{x_n = 0\}$ and suppose that for each $q_{i,0}$ there exist centers $x_i \in q_{i,0}$ and radii $r_i < 10 \text{diam}(q_{i,0})$ so that the doubling index is bounded below, i.e. $\gamma(x_i, r_i) > N$ where N is fixed. Then there exist constants $A_0 = A_0(n), R_0 = R_0(L), N_0 = N_0(L)$ (here we mean dependence on the bounds of the operator L) with the following property:*

If $A > A_0, N > N_0, R < R_0$, then

$$N(Q) > 2N. \quad (3.42)$$

Proof. We assume that R_0 is small enough, so that Lemma 2.3.6 holds with $\epsilon = \frac{1}{2}$ and the equation (3.8) at this scale is satisfied along with the conditions (3.9), (3.10), (3.11). Moreover, at this scale we can also use Lemma 3.4.2.

To ease notation, without loss of generality by scaling we may assume that $R = \frac{1}{2}, R_0 \geq \frac{1}{2}$. Let B be the unit ball centered at 0. We consider the half ball $\frac{1}{32}B^+ \subset \frac{1}{8}B$ and wish to apply the propagation of smallness for Cauchy data problems. To this end, we need to bound u and ∇u on $F := \frac{1}{32}B^+ \cap \{x_n = 0\}$.

Step 1 - Bound on u .

First, let us set

$$M := \sup_{\frac{1}{8}B} |u|, \quad (3.43)$$

by which we have

$$\sup_{B(x_i, \frac{1}{32})} |u| \leq M, \quad \forall x_i \in \frac{1}{16}B. \quad (3.44)$$

Hence, for $x_i \in \frac{1}{16}B$ Lemma 2.3.6 and the assumption that $\gamma(x_i, r_i) > N$ imply

$$\sup_{8q_{i,0}} |u| \leq \sup_{B(x_i, \frac{16\sqrt{n}}{2A+1})} |u| \leq C \left(\frac{512\sqrt{n}}{2A+1} \right)^{\frac{N}{2}} \sup_{B(x_i, \frac{1}{32})} |u| \leq e^{-c_1 N \log A} M, \quad (3.45)$$

where $c_1 = c_1(n) > 0$ and we have assumed in the last step that N, A are sufficiently large.

Step 2 - Bound on ∇u .

Further, we wish to bound the gradient $|\nabla u|$. We recall the following facts.

Lemma 3.4.3. *Let u be a solution of equation (3.8) in a domain Ω satisfying the conditions (3.9), (3.10), (3.11). Then, if $\Omega' \subset \subset \Omega$, we have*

$$\|u\|_{W^{2,2}(\Omega')} \leq C\|u\|_{L^2(\Omega)}, \quad (3.46)$$

where C depends on the parameters in (3.9), (3.10), (3.11) and $d(\Omega', \partial\Omega)$.

For a proof of Lemma 3.4.3 we refer to Theorem 8.8, the remark thereafter and Problem 8.2, [GT01].

Lemma 3.4.4. *Let $u \in W^{2,2}(\mathbb{R}^n)$ and let us consider the trace of u onto the hyperplane $\{x_n = 0\} \cong \mathbb{R}^{n-1}$ which, abusing of notation, we also denote by u . Then*

$$\|\nabla u\|_{L^2(\mathbb{R}^{n-1})} \leq C(\|u\|_{W^{2,2}(\mathbb{R}^n)} + \|u\|_{L^2(\mathbb{R}^{n-1})}), \quad (3.47)$$

where $C = C(n)$.

For a proof of Lemma 3.4.4 we refer to Lemma 23, [BL15]. Using Lemma 3.4.4 for functions of the form χu , where χ is a standard smooth cut-off function and $u \in W^{2,2}$ we see that

$$\|\nabla u\|_{L^2(\mathbb{R}^{n-1} \cap B_r)} \leq C(\|u\|_{W^{2,2}(B_{2r})} + \|u\|_{L^2(\mathbb{R}^{n-1} \cap B_{2r})}), \quad (3.48)$$

where χ is supported in B_{2r} .

Using these last lemmas along with the standard Sobolev trace estimate, we have

$$\|\nabla u\|_{L^2(F \cap q_{i,0})} \leq C(\|u\|_{W^{2,2}(2q_{i,0})} + \|u\|_{L^2(F \cap 2q_{i,0})}) \quad (3.49)$$

$$\leq C_1 \|u\|_{W^{2,2}(4q_{i,0})} \leq C_2 \|u\|_{L^2(8q_{i,0})}. \quad (3.50)$$

Again using the trace estimate, this shows that

$$\|u\|_{W^{1,2}(F \cap q_{i,0})} + \|\partial_n u\|_{L^2(F \cap q_{i,0})} \leq C_3 \|u\|_{W^{2,2}(4q_{i,0})} + \|\nabla u\|_{L^2(F \cap q_{i,0})} \quad (3.51)$$

$$\leq C_4 \|u\|_{L^2(8q_{i,0})} \leq \frac{C_5}{(2A+1)^n} \sup_{8q_{i,0}} |u|. \quad (3.52)$$

Summing up over the cubes $q_{i,0}$ and using the bound in the first step, we get

$$\|u\|_{W^{1,2}(F)} + \|\partial_n u\|_{L^2(F)} \leq \frac{C_5}{(2A+1)^{n-1}} \sup_{8q_{i,0}} |u| \leq e^{-c_2 N \log A} M. \quad (3.53)$$

Step 3 - Propagation of smallness.

Let us observe that

$$\|u\|_{L^2(\frac{1}{32}B^+)} \leq C_6 M. \quad (3.54)$$

and set

$$v := \frac{u}{C_6 M}, \quad (3.55)$$

by which we have

$$\|v\|_{L^2(\frac{1}{32}B^+)} \leq 1. \quad (3.56)$$

Hence, by the bounds in Steps 1 and 2 and propagation of smallness from Lemma 3.4.2 we get

$$\|v\|_{L^2(\frac{1}{64}B^+)} \leq \epsilon^\beta, \quad (3.57)$$

where $\epsilon = e^{-c_3 N \log A}$. Let us now select a ball $B(p, \frac{1}{256}) \subset \frac{1}{64}B^+$ and observe that by (2.237)

$$\sup_{B(p, \frac{1}{256})} |v| \leq \epsilon^\beta, \quad (3.58)$$

which implies

$$\sup_{B(p, \frac{1}{256})} |u| \leq e^{-c_4 \beta N \log A} M. \quad (3.59)$$

Moreover, as $\frac{1}{8}B \subset B(p, \frac{1}{2})$, we have by definition $\sup_{B(p, \frac{1}{2})} |u| \geq M$. This implies

$$\frac{\sup_{B(p, \frac{1}{2})} |u|}{\sup_{B(p, \frac{1}{256})} |u|} \geq e^{c_4 \beta N \log A}. \quad (3.60)$$

Finally, applying the doubling scaling Lemma 2.3.6 we have

$$\frac{\sup_{B(p, \frac{1}{2})} |u|}{\sup_{B(p, \frac{1}{256})} |u|} \leq (128)^{\tilde{N}/2}, \quad (3.61)$$

where \tilde{N} is the doubling index for $B(p, \frac{1}{2})$. Therefore,

$$\tilde{N} \geq c_5 N \log A \geq 2N, \quad (3.62)$$

where A is assumed to be sufficiently large. \square

3.4.4 Counting Good/Bad cubes and application to nodal geometry

Using the results of Section 3.4.3, one can deduce the following result.

Theorem 3.4.3. *There exist constants $c > 0$, an integer A depending on the dimension d only and positive numbers $N_0 = N_0(M, g)$, $r = r(M, g)$ such that for any cube $Q \in B(p, r)$ the following holds:*

If Q is partitioned into A^n equal sub-cubes q_i , then

$$\# \left\{ q_i \mid N(q_i) \geq \max \left(\frac{N(Q)}{1+c}, N_0 \right) \right\} \leq \frac{A^{n-1}}{2}. \quad (3.63)$$

The proof is combinatorial in nature and we refer to Theorem 5.1, [Log18a] for complete details. As an application we also have

Theorem 3.4.4. *There exist positive numbers $r_0 = r_0(M, g)$, $C = C(M, g)$ and $\alpha = \alpha(n)$ such that for any solution u of equation (3.8) in a domain Ω satisfying the conditions (3.9), (3.10), (3.11), we have*

$$\mathcal{H}^{n-1}(\{u = 0\} \cap Q) \leq C \text{diam}^{n-1}(Q) N^\alpha(Q), \quad (3.64)$$

where $Q \subset B(p, r_0)$ is an arbitrary cube in Ω .

For details, we refer to Theorem 6.1, [Log18a].

This concludes the proof of Theorem 3.4.1.

3.4.5 Application to Steklov eigenfunctions

Our goal is to transform a solution ϕ_λ to the Steklov problem (3.16) on a manifold M into a solution u to Equation (3.8) on some domain $\Omega \subset \mathbb{R}^n$.

Getting rid of the boundary

There exists a procedure (cf. [BL15], [Zhu16, Zhu]) to transform M into a compact manifold without boundary, which we highlight here. We first let $d(x) := \text{dist}(x, \partial M)$ be the distance between a point $x \in M$ and the boundary. We then define

$$\delta(x) = \begin{cases} d(x) & x \in M_\rho, \\ l(x) & x \in M \setminus M_\rho, \end{cases} \quad (3.65)$$

where $\rho = \rho(M) > 0$ is such that $d(x)$ is smooth in a ρ neighborhood M_ρ of ∂M in M . We choose $l \in C^\infty(M \setminus M_\rho)$ in such a way that makes δ smooth on M . It now follows that

$$v(x) := e^{\lambda\delta(x)}\phi_\lambda(x), \quad (3.66)$$

identifies with ϕ_λ on M and satisfies a Neumann boundary condition. More precisely, v solves

$$\begin{cases} \Delta_g v + b(x) \cdot \nabla_g v + q(x)v = 0 & \text{in } M, \\ \partial_\nu v = 0 & \text{on } \partial M, \end{cases} \quad (3.67)$$

where $\nu = -\nabla\delta$ is the unit outward normal and with

$$\begin{cases} b(x) = -2\lambda\nabla_g\delta(x), \\ q(x) = \lambda^2|\nabla\delta(x)|^2 - \lambda\Delta_g\delta(x). \end{cases} \quad (3.68)$$

The fact that v satisfies a Neumann boundary condition now allows us to get rid of the boundary by gluing to copies of M together along the boundary and extend v in the natural way. Denote by $\bar{M} = M \cup M$ the compact boundaryless manifold obtained by doing so. We remark that the induced metric \bar{g}_{ij} on \bar{M} is Lipschitz on ∂M . Using the canonical isometric involution that interchanges the two copies M of \bar{M} , we can then extend v, b and q to \bar{M} . Abusing notation and writing v for the extension, we obtain that v satisfies the elliptic equation

$$\Delta_{\bar{g}}v + \bar{b}(x) \cdot \nabla_{\bar{g}}v + \bar{q}(x)v = 0 \quad (3.69)$$

in \bar{M} and we have the following bounds

$$\begin{cases} \|\bar{b}\|_{W^{1,\infty}(N)} \leq C\lambda, \\ \|\bar{q}\|_{W^{1,\infty}(N)} \leq C\lambda^2. \end{cases} \quad (3.70)$$

Fix a point O in \bar{M} . In local coordinates around O , we have

$$\Delta_{\bar{g}}f = \frac{1}{\sqrt{|\bar{g}|}}\partial_i(\sqrt{|\bar{g}|}\bar{g}^{ij}\partial_jf), \quad (\nabla_{\bar{g}}f)^i = \bar{g}^{ij}\partial_jf. \quad (3.71)$$

where $\sqrt{|\bar{g}|}$ is the determinant of the extended metric tensor \bar{g} . Since the extended metric is Lipschitz and recalling the boundedness of \bar{b} and \bar{q} , it then follows that v is a solution of Equation (3.8) with L satisfying the conditions (3.9, 3.10, 3.11).

In order to get uniform control over the coefficients, we now work at wavelength scale and consider the ball $B(x_0, 1/\lambda) \subset \bar{M}$. We introduce

$$v_{x_0, \lambda}(x) := v\left(x_0 + \frac{x}{\lambda}\right),$$

for $x \in B(0, 1)$. Then, $v_{x_0, \lambda}$ satisfies Equation (3.8) where the coefficients $(a^{ij}), b^i$ and c are uniformly bounded in L^∞ by a constant not depending on λ . Moreover, the ellipticity constant of the (a^{ij}) does not change and the Lipschitz constant Γ can only improve. It is clear that the family of $v_{x_0, \lambda}$ solves Equation 3.8 and satisfies the conditions (3.9), (3.10), (3.11) without any dependence on λ . In what follows, we will thus be able to apply Theorem 3.4.4 uniformly on this family. For more details on the above, we refer the reader to Section 3.2 of [BL15].

3.4.6 Upper bound for the nodal set

Remark 3.4.1. *Many of the results we collect in this subsection work only within a small enough scale $r < r_0$. Since we work locally at wavelength scale $r = \frac{1}{\lambda}$, all those results hold for λ big enough.*

We now fix a point x_0 in \bar{M} , let $r_0 = \lambda^{-1}$ and choose normal coordinates in a geodesic ball $B_{\bar{g}}(x_0, r_0)$. Without loss of generality, we assume r_0 is smaller than the injectivity radius of \bar{M} . For x, y in $B_{\bar{g}}(x_0, r_0)$, we respectively denote the Euclidean and Riemannian distance by $d(x, y)$ and $d_{\bar{g}}(x, y)$. For λ big enough, we have

$$d_{\bar{g}}(x, y) \leq 2d(x, y) \tag{3.72}$$

for any two distinct points $x, y \in B_{\bar{g}}(x_0, r_0)$. By construction, the nodal sets of the eigenfunction ϕ_λ and its extension v coincide in M . Combining this observation with Equation (3.72) allows to compare the size of the corresponding nodal sets on small balls. Indeed, for any $r < r_0/2$, one has

$$\mathcal{H}^{n-1}(Z_{\phi_\lambda} \cap B_{\bar{g}}(O, r)) \leq \mathcal{H}^{n-1}(Z_v \cap B(x, 2r)) \tag{3.73}$$

Denoting by $Z_{v_{x_0, \lambda}}$ the nodal set of $v_{x_0, \lambda}$, we then remark that

$$\mathcal{H}^{n-1}(Z_v \cap B(x, 2r)) \leq \lambda^{1-n} \mathcal{H}^{n-1}(Z_{v_{x_0, \lambda}}) \tag{3.74}$$

Also, by Proposition 1 in [Zhu], there exists $c_1 > 0$ such that the doubling index of $N_{x_0, \lambda}(x, r)$ of $v_{x_0, \lambda}$ on the ball $B(x, r) \subset B(0, 1)$ satisfies

$$N_{x_0, \lambda}(x, r) \leq c_1 \lambda \tag{3.75}$$

for any $r < r_0$. We choose $r < r_0/4$ and let Q be the cube centered at origin and of side length r so that the above now implies

$$N_{x_0, \lambda}(Q) = \sup_{x \in Q, r \in (0, \text{diam}(Q))} N_{x_0, \lambda}((x, r)) \leq c_1 \lambda. \tag{3.76}$$

Collecting all of the above, noticing that $B(0, 2r) \subset Q$ and using Theorem 3.4.4, we finally get that

$$\begin{aligned} \mathcal{H}^{n-1}(Z(\phi_\lambda) \cap B_{\bar{g}}(x_0, r)) &\leq \lambda^{1-n} \mathcal{H}^{n-1}(Z_{v_{x_0, \lambda}} \cap Q) \\ &\leq c_1(n) \lambda^{1-n} N^\alpha(Q) \\ &\leq c_2(n) \lambda^{\alpha-n+1}. \end{aligned}$$

Covering M with $\sim \lambda^n$ balls $B(x_0, r)$ of radius $r = \frac{1}{4\lambda}$ finally yields

$$\mathcal{H}^{n-1}(Z_\lambda) \leq c \lambda^{\alpha+1} \tag{3.77}$$

and thus concludes the proof of Theorem 3.4.2.

Chapter 4

Estimates on tubular neighbourhoods around nodal sets

In this Chapter we study tubular neighbourhoods around nodal sets. We begin by providing a brief motivation.

4.1 Motivation

Let (M, g) be closed n -dimensional Riemannian manifold and let ϕ_λ be Laplacian eigenfunction with nodal (vanishing) set denoted by N_{ϕ_λ} .

Let $T_{\phi_\lambda, \delta} := \{x \in M : \text{dist}(x, N_{\phi_\lambda}) < \delta\}$, which is the δ -tubular neighbourhood of the nodal set N_{ϕ_λ} . We are interested in deriving upper and lower bounds on the volume of $T_{\phi_\lambda, \delta}$ in the setting of a smooth manifold. In terms of exposition, we partly follow our work in [GM17b].

Now, with the perspective of Yau's conjecture on nodal sets, one can ask about further "stability" properties of the nodal set - for example, how is the volume of the tubular neighbourhood $T_{\phi_\lambda, \delta}$ of radius δ around the nodal set behaving in terms of λ and δ ? According to Jakobson-Mangoubi (cf. [JM09]), it such a question was initially addressed by M. Sodin and C. Fefferman. Furthermore, such bounds describe a certain regularity property of the nodal set - the upper suggests that the nodal set does not have "too many needles or very narrow branches"; the lower bound hints that the nodal set "does not curve too much".

In the real analytic setting, the question about the volume of a tubular neighbourhood $T_{\phi_\lambda, \delta}$ was studied by Jakobson and Mangoubi. They were able to obtain the following sharp bounds:

Theorem 4.1.1 ([JM09]). *Let M be a real-analytic closed Riemannian manifold. Then we have*

$$\sqrt{\lambda}\delta \lesssim |T_{\phi_\lambda, \delta}| \lesssim \sqrt{\lambda}\delta, \quad (4.1)$$

where $\delta \lesssim \frac{1}{\sqrt{\lambda}}$. Here the symbol \lesssim denotes an inequality up to a constant depending only on (M, g) (such as the dimension n and estimates on the Riemannian metric g); the notation $|\cdot|$ refers to the Riemannian volume.

Concerning the proof, Theorem 4.1.1 extends the techniques of Donnelly and Fefferman from [DF88] by adding an extra parameter δ to the proofs of [DF88], and verifying that the key arguments still hold.

With that said, it seems natural to ask the question of obtaining similar bounds on the tubular neighbourhood in the smooth case as well - in this setting one can no longer fully exploit the analytic continuation and polynomial approximation techniques in the spirit of Donnelly and Fefferman.

Our result states that in the smooth setting we have the following:

Theorem 4.1.2. *Let (M, g) be a smooth closed Riemannian manifold and let $\epsilon > 0$ be a given sufficiently small number. Then there exist constants $r_0 = r_0(M, g) > 0$ and $C_1 = C_1(\epsilon, M, g) > 0$ such that*

$$|T_{\varphi, \delta}| \geq C_1 \lambda^{1/2 - \epsilon} \delta, \quad (4.2)$$

where $\delta \in (0, \frac{r_0}{\sqrt{\lambda}})$ is arbitrary.

On the other hand, there exist positive real numbers $\kappa = \kappa(M, g)$ and $C_2 = C_2(M, g)$, such that

$$|T_{\varphi, \delta}| \leq C_2 \lambda^\kappa \delta, \quad (4.3)$$

where again δ can be any number in the interval $(0, \frac{r_0}{\sqrt{\lambda}})$.

As one sees in the course of the proof of (4.2), the constant C_1 goes to 0 as ϵ approaches 0.

Our methods for proving Theorem 4.1.2 are inspired by the techniques of [DF88] and [JM09], along with some new insights provided by [Log18a], [Log18b]. Particularly, in view of the lower bound in Yau's conjecture and of the results in [JM09], it seems that the bound (4.2) is still not optimal. Of course, the upper bound in (4.3) is just polynomial, and, as would be clear from the proof, improvement of the upper bound would be affected by the corresponding improvement of the upper bound in Yau's conjecture.

4.2 Doubling indices and frequency functions

For convenience we recall a couple of relevant statements for the doubling index and frequency function. For background, we refer to Chapter 2.

Let B_1 denote the unit ball in \mathbb{R}^n , and let φ satisfy

$$L\varphi = 0 \quad (4.4)$$

on B_1 , where L is a second order elliptic operator with smooth coefficients. Moreover, we assume that L is of the form

$$L\varphi = L_1\varphi - \varepsilon q\varphi, \quad (4.5)$$

where

$$L_1\varphi = -\partial_i(a^{ij}\partial_j\varphi). \quad (4.6)$$

Similarly to Section 2.3 we make the following assumptions:

1. The leading coefficient matrix a^{ij} is symmetric and satisfies the ellipticity bounds:

$$\kappa_1|\xi|^2 \leq a^{ij}\xi_i\xi_j \leq \kappa_2|\xi|^2. \quad (4.7)$$

2. The coefficients a^{ij} , q are bounded by $\|a^{ij}\|_{C^1(\overline{B_1})} \leq K$, $|q| \leq K$, and we assume that $\varepsilon < \varepsilon_0$, with ε_0 being sufficiently small.
3. The leading coefficient matrix a^{ij} is the identity at the origin.

In particular, we observe that the operator L and the corresponding solution φ are of the form one gets after a Laplacian eigenfunction ϕ_λ is rescaled from a wavelength to a unit ball - for details on the rescale procedure, we refer to Section 2.1.

For φ satisfying (4.4) in B_1 , for $r < 1$ we define and use the following r -doubling index (also sometimes referred to as doubling exponent):

$$\gamma_r(\varphi) := \log \frac{\sup_{B_1} |\varphi|}{\sup_{B_r} |\varphi|}, \quad (4.8)$$

The fundamental result of Theorem 2.4.1 can also be stated as follows:

Theorem 4.2.1. *Let ϕ_λ be a Laplacian eigenfunction on a closed Riemannian manifold (M, g) . There exist constants $C = C(M, g) > 0$ such that for every point p in M and every $r > 0$ the following growth exponent holds:*

$$\sup_{B(p, r)} |\phi_\lambda| \leq \left(\frac{r}{r'}\right)^{C\sqrt{\lambda}} \sup_{B(p, r')} |\phi_\lambda|, \quad 0 < r' < r. \quad (4.9)$$

In particular, for φ being a rescaled Laplacian eigenfunction, we have

$$\frac{\gamma_r(\varphi)}{\log(1/r)} \lesssim \sqrt{\lambda}. \quad (4.10)$$

In the spirit of Chapter 2, we will use frequency functions. In our methods, we will encounter slightly different types of frequency functions. Here we briefly discuss the comparability of these functions.

Similarly to Chapter 2, for a solution φ satisfying $L\varphi = 0$ in B_1 , define for $a \in B_1$, $r \in (0, 1]$ and $B(a, r) \subset B_1$,

$$D(a, r) = \int_{B(a, r)} |\nabla \varphi|^2 dV,$$

$$H(a, r) = \int_{\partial B(a, r)} \varphi^2 dS.$$

Then, define the generalized frequency of φ by

$$\tilde{N}(a, r) = \frac{rD(a, r)}{H(a, r)}. \quad (4.11)$$

The methods in [Log18a] and [Log18b] use a variant of $\tilde{N}(a, r)$, defined as follows:

$$N(a, r) = \frac{rH'(a, r)}{2H(a, r)}. \quad (4.12)$$

To compare and pass between $\gamma_r(\varphi)$, $\tilde{N}(a, r)$ and $N(a, r)$, we record the following facts.

First, from Lemma 2.3.5 (cf. also equation (3.1.22) of [HL]), we have that

$$\frac{d}{dr} H(a, r) = \left(\frac{n-1}{r} + O(1)\right) H(a, r) + 2D(a, r), \quad (4.13)$$

where $O(1)$ is a function of geodesic polar coordinates (r, θ) bounded in absolute value by a constant C independent of r . More precisely, in Lemma 2.3.5 a certain normalizing factor ω is introduced in the integrand in the definitions of $H(a, r)$ and $D(a, r)$. As it turns out by the construction, $C_1 \leq \omega \leq C_2$ where C_1, C_2 depend on the ellipticity constants of the PDE, the dimension n and a bound on the coefficients (cf. also 3.1.11, [HL]).

This gives us that when $\tilde{N}(a, r)$ is large, we have,

$$N(a, r) \sim \tilde{N}(a, r), \quad (4.14)$$

where \sim denotes comparability up to constants depending on (M, g) and r .

Also, it is clear from Proposition 2.3.6 (cf. also Remark 3.1.4 of [HL]) that

$$\tilde{N}(a, r) \gtrsim \gamma_r(\varphi). \quad (4.15)$$

In fact, in Chapter 2 we saw a couple of statements which imply that the frequency function controls the doubling exponents.

We also remind that the frequency $N(a, r)$ is almost-monotonic in the following sense: for any $\varepsilon > 0$, there exists $R > 0$ such that if $r_1 < r_2 < R$, then

$$N(a, r_1) \leq N(a, r_2)(1 + \varepsilon). \quad (4.16)$$

For background on this statement we refer again to Chapter 2 and [GL86].

Following Section 2.1 we consider a Laplacian eigenfunction ϕ_λ on M and convert it into a harmonic function. To this end, let us consider the Riemannian product manifold $\bar{M} := M \times \mathbb{R}$ - a cylinder over M , equipped with the standard product metric \bar{g} . By a direct check, the function

$$u(x, t) := e^{\sqrt{\lambda}t} \phi_\lambda(x) \quad (4.17)$$

is harmonic.

Hence, by Theorem 4.2.1, the harmonic function u in (4.17) has a doubling exponent (i.e. $\gamma_{\frac{1}{2}}$) which is also bounded by $C\sqrt{\lambda}$ for a constant C depending on (M, g) .

It is well-known that doubling conditions imply upper bounds on the frequency (cf. Lemma 6, [BL15]):

Lemma 4.2.1. *For each point $p = (x, t) \in \bar{M}$ the harmonic function $u(p)$ satisfies the following frequency bound:*

$$\tilde{N}(p, r) \leq C\sqrt{\lambda}, \quad (4.18)$$

where $C > 0$ is a fixed constant depending only on M, g .

For a proof of Lemma 4.2.1 we refer to Lemma 6, [BL15].

4.3 Proof of Theorem 4.1.2

4.3.1 Idea of Proof

We first focus on the proof of the lower bound. Since the proof is somewhat long and technical, we begin by giving a brief sketch of the overall idea of the proof.

Let φ_λ be a Laplacian eigenfunction on the closed Riemannian manifold (M, g) . It is well-known by a Harnack inequality argument (see [Bru78] for example), that the nodal set of φ_λ is wavelength dense in M , which means that one can find $\sim \lambda^{n/2}$ many disjoint balls $B_{\frac{r}{\sqrt{\lambda}}}^i := B(x_i, \frac{r}{\sqrt{\lambda}}) \subset M$ such that $\varphi_\lambda(x_i) = 0$. Now, to obtain a lower bound on $|T_{\varphi, \delta}|$ we wish to estimate $|T_{\varphi, \delta} \cap B(x_i, \frac{r}{\sqrt{\lambda}})|$ from below. The strategy is to consider separately those balls $B_{\frac{r}{\sqrt{\lambda}}}^i$ on which φ_λ has controlled doubling exponent, which we deal with using the tools of [DF88, JM09], and those on which φ_λ has high doubling exponent, for which we bring in the tools of [Log18a, Log18b]. In other words we distinguish two options:

1. First, for a ball $B(x, \rho)$ of controlled doubling exponent (where $\rho \sim \frac{1}{\sqrt{\lambda}}$, and $\varphi_\lambda(x) = 0$), we show that

$$\frac{|T_{\varphi, \delta} \cap B(x, \rho)|}{\rho^{n-1} \delta} \geq c. \quad (4.19)$$

To verify this, we essentially verify that the argument of Jakobson and Mangoubi, [JM09] from the real-analytic case goes through. The main observation is the fact that the volumes of positivity and negativity of φ_λ inside such $B(x, \rho)$ are comparable. A further application of the Brunn-Minkowski inequality then yields (4.19).

2. Now, to continue the idea of the proof, for a ball $B(x, \rho)$ of high doubling exponent N (where $\rho \sim \frac{1}{\sqrt{\lambda}}$, and $\varphi_\lambda(x) = 0$), we prove that

$$\frac{|T_{\varphi, \delta} \cap B(x, \rho)|}{\rho^{n-1} \delta} \geq \frac{1}{N^\varepsilon}, \text{ where } N \lesssim \sqrt{\lambda}. \quad (4.20)$$

To prove this, we use the following sort of iteration procedure. Using the methods of Logunov, [Log18a], [Log18b], one first sees that in such a ball $B(x, \rho)$ of large doubling exponent one can find a large collection of smaller disjoint balls $\{B^j\}_j$, whose centers are again zeros of φ_λ . We then focus on estimating $|T_{\varphi, \delta} \cap B^j|$ and again distinguish the same two options - either the doubling exponent of B^j is small, which brings us back to the previous case (1) where we have appropriate estimates on the tube, or the doubling exponent of B^j is large. Now, in case the doubling exponent of B^j is large, we similarly discover another large subcollection of even smaller disjoint balls inside B^j , whose centers are zeros of φ_λ and so forth.

We repeat this iteration either until the current small ball has a controlled doubling exponent, or until the current small ball is of radius comparable to the width δ of the tube $T_{\varphi, \delta}$. In both situations we have a lower estimate on the volume of the tube which brings us to (4.20).

Once this is done, (4.2) follows by adding (4.19) and (4.20) over $\sim \lambda^{n/2}$ balls $B_{\frac{r}{\sqrt{\lambda}}}^i$, as mentioned above.

Remark 4.3.1. *We make a quick digression here and recall that in the real analytic setting, it is known that one can find $\sim \lambda^{n/2}$ many balls of wavelength (comparable) radius, as mentioned above, such that all of them have controlled doubling exponent - in other words, the first case above is the only one that needs to be considered. However, in the smooth setting, it is still a matter of investigation how large a proportion of the wavelength balls possesses controlled doubling exponent.*

For example, it is shown in [CM11], that one can arrange that $\lambda^{\frac{n+1}{4}}$ such balls possess controlled growth. More explicitly, the following question seems to be of interest and may also have substantial applications in the study of nodal geometry: given a closed smooth manifold M , how many disjoint balls $B(x_\lambda^i, \frac{r}{\sqrt{\lambda}})$ of controlled doubling exponent can one find inside M such that $\varphi_\lambda(x_\lambda^i) = 0$, where r is a suitably chosen constant depending only on the geometry of (M, g) ?

The idea of proof of the upper bound (4.3) is quite simple. We take a cube Q inside M of side-length 1, say, and we chop it up into subcubes Q_k of side-length δ . Observe that due to Logunov's resolution of the Nadirashvili conjecture (cf. [Log18a], [Log18b]), for each subcube Q_k which intersects the nodal set (which we call nodal subcubes following [JM09]), we have a local lower bound of the kind $\mathcal{H}^{n-1}(N_\varphi \cap Q_k) \gtrsim \delta^{n-1}$. Summing this up, we get an upper bound on the number of nodal subcubes, and in turn, an upper bound on the volume of all nodal subcubes in terms of $\mathcal{H}^{n-1}(N_\varphi)$. Now, since $T_{\varphi, \delta}$ is contained inside the union of all such nodal subcubes, combined with the upper bound on $\mathcal{H}^{n-1}(N_\varphi)$ due to [Log18b], we have (4.3).

4.3.2 The lower bound in Theorem 4.1.2

Proof of (4.2). We use the notation above and work in the product manifold \bar{M} with the harmonic function $u(x, t) = e^{\sqrt{\lambda}t} \varphi_\lambda(x)$. For the purpose of the proof of (4.2), we will assume that M is $n - 1$ dimensional. All this is strictly for notational convenience and ease of presentation, as we will now work with the tubular neighbourhood of u , which then becomes n dimensional. Considering the tubular neighbourhood of u instead of φ_λ does not create any problems because the nodal set of u is a product, i.e.

$$\{u = 0\} = \{\varphi_\lambda = 0\} \times \mathbb{R}. \quad (4.21)$$

As the tubular neighbourhoods we are considering are of at most wavelength radius and at the this scale the Riemannian metric is almost the Euclidean one, we have

$$T_{u, \frac{\delta}{2}} \subseteq T_{\varphi, \delta} \times \mathbb{R}. \quad (4.22)$$

Hence, to obtain a lower bound for $|T_{\varphi, \delta}|$ it suffices to bound $|T_{u, \frac{\delta}{2}}|$ below. To this end, we consider a strip $S := M \times [0, R_0]$ where $R_0 > 0$ is sufficiently large.

We will obtain lower bound on $|B_{\frac{r}{\sqrt{\lambda}}}^i(p_i) \cap T_{u, \frac{\delta}{2}}|$, which will give the analogous statements for (4.19) and (4.20) for the function u . As mentioned before, depending on the doubling exponent of u in the ball $B_{\frac{r}{\sqrt{\lambda}}}^i(p_i)$ we distinguish two cases, and we will prove that $\frac{|T_{u, \frac{\delta}{2}} \cap B(x, \rho)|}{\rho^{n-1} \delta} \geq c$ in the case of controlled doubling exponent, and $\frac{|T_{u, \frac{\delta}{2}} \cap B(x, \rho)|}{\rho^{n-1} \delta} \geq \frac{1}{N^\varepsilon}$, where $N \lesssim \sqrt{\lambda}$ in the case of high doubling exponent.

Case I : Controlled doubling exponent:

In the regime of controlled doubling exponent, in which case it is classically known that the nodal geometry is well-behaved, we essentially follow the proof in [JM09]. Let $B := B(p, \rho)$ be a ball such that $u(p) = 0$ and u has bounded doubling exponent on $B(p, \rho)$, that is, $\frac{\sup_{B(p, 2\rho)} |u|}{\sup_{B(p, \rho)} |u|} \leq C$ (ultimately we will set $\rho \sim \frac{1}{\sqrt{\lambda}}$). Then, by symmetry results (see Proposition 6.4.1 below), we have that $C_1 < \frac{|B^+|}{|B^-|} < C_2$, where $B^+ = \{u > 0\} \cap B$, $B^- = \{u < 0\} \cap B$.

Let $\delta := \tilde{c}\rho$, where \tilde{c} is a small constant to be selected later. Denoting by B_δ^+ the δ -neighbourhood of B^+ , and similarly for B^- , and $2B := B(p, 2\rho)$, we have that since $T_{u,\delta} \supset B_\delta^+ \cap B_\delta^-$,

$$|T_{u,\delta} \cap 2B| \geq |B_\delta^+| + |B_\delta^-| - |B(p, \rho + \delta)|. \quad (4.23)$$

By the Brunn-Minkowski inequality, we see that $|B_\delta^+| \geq |B^+| + n\omega_n^{1/n}\delta|B^+|^{1-1/n}$, where ω_n is the volume of the n -dimensional unit ball. Setting $|B^+| = \alpha|B|$, $|B^-| = (1-\alpha)|B|$, we have

$$|T_{u,\delta} \cap 2B| \geq \omega_n \left(\rho^n - (\rho + \delta)^n + n\rho^{n-1}\delta(\alpha^{1-1/n} + (1-\alpha)^{1-1/n}) \right). \quad (4.24)$$

By asymmetry, α is bounded away from 0 and 1, meaning that $\alpha^{1-1/n} + (1-\alpha)^{1-1/n} > 1 + C$. Now, taking \tilde{c} small enough, the right hand side of (4.24) is actually $\gtrsim \rho^{n-1}\delta$, giving us

Lemma 4.3.1. *Let the tubular distance δ and the radius of the ball ρ be in proportion $\frac{\delta}{\rho} \leq \tilde{c}$ where $\tilde{c} > 0$ is a small fixed number. Assume that the doubling index of u over the ball B_ρ is small. Then*

$$|T_{u,\delta} \cap 2B| \gtrsim \rho^{n-1}\delta. \quad (4.25)$$

Case II: Large doubling exponent:

Now, let us consider a ball $B(p, \rho)$ with radius ρ comparable to the wavelength, and let $B' = B(p, \frac{\rho}{2})$. Let us assume that initially we take ρ such that $\frac{\delta}{\rho} \leq \tilde{c}$.

Suppose $\frac{\sup_{B'} |u|}{\sup_{\frac{1}{2}B'} |u|}$ is large. By (4.14) and (4.15), the frequency function $N(p, \frac{\rho}{2})$ is also large. Recall also the almost monotonicity of the frequency function $N(x, r)$, given by (4.16), which will be implicit in our calculations below.

We will make use of the following fact:

Theorem 4.3.1. *Consider a harmonic function u on $B(p, 2\rho)$. If $N(p, \rho)$ is sufficiently large, then there is a number N with*

$$N(p, \rho)/10 < N < 2N(p, \frac{3}{2}\rho). \quad (4.26)$$

such that the following holds: Suppose that $\epsilon \in (0, 1)$ is fixed. Then there exists a constant $C = C(\epsilon) > 0$ and at least $[N^\epsilon]^{n-1} 2^{C \log N / \log \log N}$ disjoint balls $B(x_i, \frac{\rho}{N^\epsilon \log^6 N}) \subset B(p, 2\rho)$ such that $u(x_i) = 0$. Here $[.]$ denotes the integer part of a given number.

Theorem 4.3.1 is a straight-forward modification of the methods in Section 6 of [Log18b] - for completeness and convenience, we give full details of the proof of Theorem 4.3.1 in Section 4.4.

We will now use Theorem 4.3.1 in an iteration procedure. The first step of the iteration proceeds as follows.

Let us denote by ζ_1 the radius of the small balls prescribed by Theorem 4.3.1, i.e.

$$\zeta_1 := \frac{\rho}{N^\epsilon \log^6 N}. \quad (4.27)$$

Further, let \mathcal{B}_1 denote the collection of these small balls inside $B(p, 2\rho)$. Let $F_1 := \inf_{B \in \mathcal{B}_1} \frac{|T_{u,\delta} \cap B|}{\zeta_1^{n-1}\delta}$ and let us assume that it is attained on the ball $B^1 \in \mathcal{B}_1$.

We then have that

$$\begin{aligned} |T_{u,\delta} \cap B(p, 2\rho)| &\geq \sum_{B_i \in \mathcal{B}_1} |T_{u,\delta} \cap B_i| \geq [N^\epsilon]^{n-1} 2^{C \log N / \log \log N} F_1 \zeta_1^{n-1} \delta \geq \\ &\geq [N^\epsilon]^{n-1} 2^{C \log N / \log \log N} \frac{\rho^{n-1} \delta}{(2N^\epsilon \log^6 N)^{n-1}} F_1, \end{aligned}$$

which implies that

$$\frac{|T_{u,\delta} \cap B(p, 2\rho)|}{\rho^{n-1} \delta} \geq 2^{C \log N / \log \log N} F_1 \geq F_1, \quad (4.28)$$

by reducing the constant C , if necessary, and assuming that N is large enough. Recalling that by assumption $F_1 = \frac{|T_{u,\delta} \cap B^1|}{\zeta_1^{n-1} \delta}$, we obtain

$$\frac{|T_{u,\delta} \cap B(p, 2\rho)|}{\rho^{n-1} \delta} \geq 2^{C \log N / \log \log N} \frac{|T_{u,\delta} \cap B^1|}{\zeta_1^{n-1} \delta}. \quad (4.29)$$

This concludes the first step of the iteration.

Now, the second step of the iteration process proceeds as follows. We inspect three sub-cases.

- First, suppose that δ and ζ_1 are comparable in the sense that

$$\frac{8\delta}{\zeta_1} > \tilde{c}, \quad (4.30)$$

where \tilde{c} is the constant from Lemma 4.3.1. As there is a ball of radius δ centered at x_1 (the center of B^1) that is contained in the tubular neighbourhood, we obtain

$$\frac{|T_{u,\delta} \cap B^1|}{\zeta_1^{n-1} \delta} \geq \frac{C(\tilde{c}\zeta_1)^n}{\zeta_1^{n-1} \delta} \geq C\tilde{c}^n \frac{\zeta_1}{\delta}. \quad (4.31)$$

Furthermore, initially we assumed that $\frac{\delta}{\rho} \leq \tilde{c}$, hence

$$\frac{|T_{u,\delta} \cap B^1|}{\zeta_1^{n-1} \delta} \geq C_1 \tilde{c}^{n-1} \frac{\zeta_1}{\rho} = C_1 \tilde{c}^{n-1} \frac{1}{N^\epsilon \log^6 N} \geq C_2 \frac{1}{N^{\epsilon_1}}, \quad (4.32)$$

where $\epsilon_1 > 0$ is slightly larger than ϵ . In combination with the frequency bound of Lemma 4.2.1 and the fact that N is comparable to the frequency by (4.26) we get

$$\frac{|T_{u,\delta} \cap B(p, 2\rho)|}{\rho^{n-1} \delta} \geq \frac{|T_{u,\delta} \cap B^1|}{\zeta_1^{n-1} \delta} \geq \frac{C_3}{\lambda^{\epsilon_1/2}}. \quad (4.33)$$

The iteration process finishes.

- Now suppose that the tubular radius is quite smaller in comparison to the radius of the ball, i.e.

$$\frac{8\delta}{\zeta_1} \leq \tilde{c}. \quad (4.34)$$

Suppose further that the doubling exponent of u in $\frac{1}{8}B^1$ is small. We can revert back to Case I and Lemma 4.3.1 by which we deduce that

$$\frac{|T_{u,\delta} \cap B(p, 2\rho)|}{\rho^{n-1}\delta} \geq \frac{|T_{u,\delta} \cap B^1|}{\zeta_1^{n-1}\delta} \geq \frac{|T_{u,\delta} \cap \frac{1}{8}B^1|}{\zeta_1^{n-1}\delta} \geq C, \quad (4.35)$$

whence the iteration process stops.

- Finally, let us suppose that

$$\frac{8\delta}{\zeta_1} \leq \tilde{c}, \quad (4.36)$$

and further that the doubling exponent of u in B^1 is sufficiently large. We can now replace the initial starting ball $B(p, 2\rho)$ by B^1 and then repeat the first step of the iteration process for $\frac{1}{8}B^1$. As above, we see that there has to be a ball \tilde{B}^1 of radius $\tilde{\zeta}_1 \in (\frac{1}{4}\zeta_1, \frac{1}{2}\zeta_1)$ upon which the frequency is comparable to a sufficiently large number N_1 . Now, we apply Theorem 4.3.1 and within B^1 discover at least $[N_1^\epsilon]^{n-1} 2^{C \log N_1 / \log \log N_1}$ balls of radius

$$\zeta_2 := \frac{\zeta_1}{N_1^\epsilon \log^6 N_1}, \quad (4.37)$$

such that φ_λ vanishes at the center of these balls.

As before, we denote the collection of these balls by \mathcal{B}_2 and put $F_2 := \inf_{B \in \mathcal{B}_2} \frac{|T_{u,\delta} \cap B|}{\zeta_1^{n-1}\delta}$. Analogously we also obtain

$$F_1 = \frac{|T_{u,\delta} \cap B^1|}{\zeta_1^{n-1}\delta} \geq 2^{C \log N_1 / \log \log N_1} F_2 \geq F_2. \quad (4.38)$$

Again, we reach the three sub-cases. If either of the two first sub-cases holds, then we bound F_2 in the same way as F_1 - this yields a bound on $\frac{|T_{u,\delta} \cap B^1|}{\rho^{n-1}\delta}$. If the third sub-case holds, then we repeat the construction and eventually get F_3, F_4, \dots

Notice that the iteration procedure eventually stops. Indeed, it can only proceed if the third sub-case is constantly iterated. However, at each iteration the radius of the considered balls drops sufficiently fast and this ensures that either of the first two sub-cases is eventually reached.

This finally gives us

$$\frac{|T_{u,\delta} \cap B(p, 2\rho)|}{\rho^{n-1}\delta} \geq F_1 \geq F_2 \geq \dots \geq \frac{C_3}{\lambda^{\epsilon_1/2}}. \quad (4.39)$$

At last, we are done with the iteration, and this also brings us to the end of the discussion about Case I and Case II. To summarize what we have established, the most “unfavourable” situation is that scenario in Case II, where we at every level of the iteration we encounter balls of high doubling exponent, and we have to carry out the iteration all the way till the radius of the smaller balls (whose existence at every stage is guaranteed by Theorem 4.3.1) drops below δ . The lower bound for $\frac{|T_{u,\delta} \cap B(p, 2\rho)|}{\rho^{n-1}\delta}$ in such a “worst” scenario is given by (4.39).

We are now ready to finish the proof. Letting $\rho = \frac{r}{2\sqrt{\lambda}}$ and by summing (4.39) over the $\sim \lambda^{n/2}$ many wavelength balls $B_{\frac{r}{\sqrt{\lambda}}}^i$ (as mentioned at the beginning of this Section), we have that

$$|T_{u,\delta}| \geq \frac{C_3}{\lambda^{\epsilon_1/2}} \rho^{n-1} \delta \lambda^{n/2} \gtrsim \lambda^{1/2 - \epsilon_1/2} \delta. \quad (4.40)$$

Using the relationship between the nodal sets of φ_λ and u , this yields (4.2). □

4.3.3 The upper bound in Theorem 4.1.2

Now we turn to the proof of the upper bound.

Proof of (4.3). We start by giving a formal statement of the main result of [?]:

Theorem 4.3.2. *Let (M, g) be a compact smooth Riemannian manifold without boundary. Then there exists a number κ , depending only on $n = \dim M$ and $C = C(M, g)$ such that*

$$\mathcal{H}^{n-1}(N_\varphi) \leq C\lambda^\kappa. \quad (4.41)$$

As remarked before, we assume that M has sufficiently large injectivity radius. Consider a finite covering Q_k of M by cubes of side length 1, say. Consider a subdivision of each cube Q_k into subcubes $Q_{k,\nu}$ of side length δ , where $\delta \leq \frac{1}{3}$. Call a small subcube $Q_{k,\nu}$ a nodal cube if $N_\varphi \cap Q_{k,\nu} \neq \emptyset$. Also, denote by $Q_{k,\nu}^*$ the union of $Q_{k,\nu}$ with its $3^n - 1$ neighbouring subcubes. Then, it is clear that

$$T_{\varphi,\delta} \subset \bigcup_{\text{Nod}} Q_{k,\nu}^*, \quad (4.42)$$

where Nod denotes the set of all nodal subcubes $Q_{k,\nu}$. By Theorem 1.2 of [?], we have that

$$\mathcal{H}^{n-1}(N_\varphi \cap Q_{k,\nu}^*) \gtrsim \delta^{n-1}. \quad (4.43)$$

Summing up (4.43), we get that

$$\begin{aligned} 3^n \mathcal{H}^{n-1}(N_\varphi) &\geq \sum_{\text{Nod}} \mathcal{H}^{n-1}(N_\varphi \cap Q_{k,\nu}^*) \\ &\gtrsim \#(\text{nodal } Q_{k,\nu}) \delta^{n-1}, \end{aligned}$$

which means that the number of nodal subcubes is $\lesssim \mathcal{H}^{n-1}(N_\varphi)/\delta^{n-1}$. Using (4.42), this means that

$$|T_{\varphi,\delta}| \lesssim \mathcal{H}^{n-1}(N_\varphi) \delta.$$

Finally, we invoke Theorem 4.3.2 to finish our proof. □

4.4 Number of zeros over balls with large doubling exponent

We address the proof of Theorem 4.3.1. We will essentially follow and appropriately adjust Section 6, [Log18b] and, for completeness, we will recall all the relevant statements.

Let us briefly give an overview of how the proof proceeds.

First, we consider a harmonic function in a ball and gather a few estimates on the way u grows near a point of maximum. The discussion here involves classical harmonic function estimates as well as scaling of the frequency function $N(p, r)$ (cf. Subsection 4.2) and the doubling numbers.

Second, let us consider a cube Q and divide it into small equal subcubes. We recall a combinatorial result (Theorem 5.2, [Log18b]) which, roughly speaking, gives quantitative estimate on the number of small bad subcubes (i.e., subcubes with large doubling exponent) of a given cube Q .

Third, we utilize the results in the first two steps to prove Theorem 4.3.1.

A few words regarding notation: given a point $O \in M$, we take a small enough coordinate chart (U, ψ) around O such that the Riemannian metric g on the chart is comparable to the Euclidean metric in the following sense: given $\nu > 0$, there is a sufficiently small $R_0 = R_0(\nu, M, g, O)$ such that $(1 - \nu)d_g(x, y) < d_{\text{Euc}}(\psi(x), \psi(y)) < (1 + \nu)d_g(x, y)$ for any two distinct points $x, y \in B_g(O, R_0)$. Under this metric comparability, we will drop the subscript “ g ” henceforth, and will describe “cubes” and “boxes” and their partitions, and such combinatorial ideas directly on the manifold M .

4.4.1 Growth of harmonic functions near a point of maximum

Let us start by recalling the following observation (Lemma 3.2, [Log18b]). Let $B(p, 2r) \subset B(O, R_0)$ where the frequency function satisfies $N(p, \frac{r}{2}) > 10$. Then there exists numbers $s \in [r, \frac{3}{2}r]$ and $N \geq 5$ so that

$$N \leq N(p, t) \leq 2eN, \quad (4.44)$$

where the parameter t is any number within the interval I given by

$$I := \left(s \left(1 - \frac{1}{1000 \log^2 N} \right), s \left(1 + \frac{1}{1000 \log^2 N} \right) \right). \quad (4.45)$$

In words, we find and work in a small spherical layer where the frequency is comparable to N .

Recalling the function $H(p, t) = \int_{\partial B(p, t)} u^2 dS$, it follows from the definition of the frequency function that

$$\frac{H(x, r_2)}{H(x, r_1)} = \exp \left(2 \int_{r_1}^{r_2} \frac{N(x, r)}{r} dr \right). \quad (4.46)$$

Combining this with the control over N in the interval I , we obtain

$$\left(\frac{t_2}{t_1} \right)^{2N} \leq \frac{H(p, t_2)}{H(p, t_1)} \leq \left(\frac{t_2}{t_1} \right)^{4eN}, \quad (4.47)$$

where $t_1 < t_2$ and $t_1, t_2 \in I$.

Now, let us consider a point of maximum $x \in \partial B(p, s)$, such that

$$\sup_{y \in \bar{B}(p, s)} |u(y)| = |u(x)| =: K. \quad (4.48)$$

We now look at concentric spheres of radii $s^- := s(1 - \delta)$ and $s^+ := s(1 + \delta)$ where δ is a small number in the interval $[\frac{1}{10^6 \log^{100} N}, \frac{1}{10^6 \log^2 N}]$. We can estimate $\sup_{B(p, s^+)} |u|$ and $\sup_{B(p, s^-)} |u|$ in terms of K :

Lemma 4.4.1 (Lemma 4.1, [Log18b]). *There exist $c, C > 0$ depending on M, g, n, O, R_0 , such that*

$$\sup_{B(p, s^-)} |u| \leq CK 2^{-c\delta N}, \quad (4.49)$$

$$\sup_{B(p, s^+)} |u| \leq CK 2^{C\delta N}. \quad (4.50)$$

Sketch of Proof. The proof uses the above scaling for $H(p, t)$ and classical estimates for harmonic functions. For a detailed discussion we refer to [Log18b]. \square

Let us recall the classical doubling number $\mathcal{N}(x, r)$ (cf. Subsection 4.2), which was defined as

$$2^{\mathcal{N}(x, r)} = \frac{\sup_{B(x, 2r)} |u|}{\sup_{B(x, r)} |u|}. \quad (4.51)$$

Let us recall the following result (cf. Appendix, [Log18a]):

Lemma 4.4.2. *Let $\epsilon > 0$ be fixed. There exist numbers $R_0 > 0, C > 0$ such that for r_1, r_2 with $2r_1 \leq r_2$ and $B(x, r_2) \subset B(O, R_0)$, we have the following estimate*

$$\left(\frac{r_2}{r_1}\right)^{\mathcal{N}(x, r_1)(1-\epsilon)-C} \leq \frac{\sup_{B(x, r_2)} |u|}{\sup_{B(x, r_1)} |u|} \leq \left(\frac{r_2}{r_1}\right)^{\mathcal{N}(x, r_2)(1+\epsilon)+C}. \quad (4.52)$$

In particular,

$$\mathcal{N}(x, r_1)(1-\epsilon) - C \leq \mathcal{N}(x, r_2)(1+\epsilon) + C. \quad (4.53)$$

As a straightforward corollary of the above discussion we obtain

Lemma 4.4.3. *There is a constant $C = C(M, g, n) > 0$ such that*

$$\sup_{B(x, \delta s)} |u| \leq K 2^{C\delta N + C}. \quad (4.54)$$

Moreover, for any \tilde{x} with $d(x, \tilde{x}) \leq \frac{\delta s}{4}$, we have

$$\mathcal{N}(\tilde{x}, \frac{\delta s}{4}) \leq C\delta N + C, \quad (4.55)$$

$$\sup_{B(\tilde{x}, \frac{\delta s}{10N})} |u| \geq K 2^{-C\delta N \log N - C}. \quad (4.56)$$

For a proof we refer to Lemma 4.2, [Log18b].

4.4.2 An estimate on the number of bad cubes

Let Q be a given cube. We define the doubling index $N(Q)$ of the cube Q by

$$N(Q) := \sup_{x \in Q, r \leq \text{diam}(Q)} \log \frac{\sup_{B(x, 10nr)} |u|}{\sup_{B(x, r)} |u|}. \quad (4.57)$$

Clearly, $N(Q)$ is monotonic in the sense that if a cube Q_1 is contained in the cube Q_2 , then $N(Q_1) \leq N(Q_2)$. Furthermore, if a cube Q is covered by a collection of cubes $\{Q_i\}$ with $\text{diam}(Q_i) \geq \text{diam}(Q)$, then there exists a cube Q_i from the collection, such that $N(Q_i) \geq N(Q)$.

The main result in this subsection is

Theorem 4.4.1 (Theorem 5.3, [Log18b]). *There exist constants $c_1, c_2, C > 0$ and a positive integer B_0 , depending only on the dimension n , and positive numbers $N_0 = N_0(M, g, n, O), R = R(M, g, n, O)$ such that for any cube $Q \subset B(O, R)$ the following holds:*

If we partition Q into B^n equal subcubes, where $B > B_0$, then the number of subcubes with doubling exponent greater than $\max(N(Q) 2^{-c_1 \log B / \log \log B}, N_0)$ is less than CB^{n-1-c_2} .

The last theorem uses and refines a previous result (Theorem 5.1, [Log18b]) where roughly speaking the dynamic relation between the size of the small cubes and their doubling index is not estimated with that precision. The discussion proceeds through an iteration argument.

4.4.3 Proof of Theorem 4.3.1

Step 1 - the set-up. We consider the same setting as in Subsection 4.4.1: we have a ball $B(p, 2r) \subset B(O, R_0)$, numbers $s \in [r, \frac{3}{2}r]$, $N \geq 5$, such that

$$N \leq N(p, t) \leq 2eN, \quad (4.58)$$

for any $t \in I$ where I is the interval defined above.

We also consider a point of maximum $x \in \partial B(p, s)$, $\sup_{\partial B(p, s)} |u| = |u(x)| =: K$ and a point $\tilde{x} \in \partial B(p, s(1 - \delta))$, such that $d(x, \tilde{x}) = \delta s$. Here we have introduced the small number $\delta := \frac{1}{10^{8n^2} \log^2 N}$ (we follow the notation in [?], but to avoid confusion, we note that the δ chosen here is much smaller compared to the δ used in Subsection 4.4.1). By construction, we have that $d(x, \tilde{x}) \sim \frac{r}{\log^2 N}$ up to constants depending only on dimension.

Let us denote by T a (rectangular) box, such that x and \tilde{x} are centers of the opposite faces of T - one side of T is $d(x, \tilde{x})$ and the other $n - 1$ sides are equal to $\frac{d(x, \tilde{x})}{[\log N]^4}$, where $[\cdot]$ denotes the integer part of a given number.

Now, let $\epsilon \in (0, 1)$ be given. By cutting along the long side of T , we subdivide T into equal subboxes (referred to as ‘‘tunnels’’) $T_i, i = 1, \dots, [N^\epsilon]^{n-1}$, so that each T_i has one side of length $d(x, \tilde{x})$ and the other $n - 1$ sides of length $\frac{d(x, \tilde{x})}{[N^\epsilon][\log N]^4}$.

Further, by cutting perpendicularly to the long side, we divide T_i into equal cubes $q_{i,t}, t = 1, \dots, [N^\epsilon][\log N]^4$ all of which have side-length of $\frac{d(x, \tilde{x})}{[N^\epsilon][\log N]^4}$ and whose centers are denoted by $x_{i,t}$. We also arrange the parameter t so that $d(q_{i,t}, x) \geq d(q_{i,t+1}, x)$.

We will assume that N is sufficiently large, i.e. bounded below by $N_0(n, M, g) > 0$.

Step 2 - growth along a tunnel. We wish to relate how large u is at the first and last cubes - $q_{i,1}$ and $q_{i,[N^\epsilon][\log N]^4}$. To this end we will use the lemmata from Subsection 4.4.1.

First, let us observe that $q_{i,1} \subset B(p, s(1 - \frac{\delta}{4}))$. Indeed, for sufficiently large N we have

$$d(p, q_{i,1}) \leq d(p, \tilde{x}) + d(\tilde{x}, q_{i,1}) \leq s(1 - \delta) + \frac{C\delta s\sqrt{n}}{[\log N]^4} \leq s \left(1 - \frac{\delta}{2}\right). \quad (4.59)$$

The estimate (4.49) yields

$$\sup_{q_{i,1}} |u| \leq \sup_{B(p, s(1 - \frac{\delta}{4}))} |u| \leq K 2^{-c_1 \frac{N}{\log^2 N} + C_1}. \quad (4.60)$$

On the other hand, let us denote the last index along the tunnel by τ , i.e. $\tau := [N^\epsilon][\log N]^4$. As the cube $q_{i,\tau}$ is of size comparable to $\frac{1}{[N^\epsilon][\log^6 N]}$ and N is assumed to be large enough, we can find an inscribed geodesic ball $B_{i,\tau} \subset \frac{1}{2}q_{i,\tau}$, centered at $x_{i,\tau}$ and of radius $\frac{s}{N}$.

Now, by definition $d(x_{i,\tau}, x) \leq \frac{Cs}{[\log N]^6}$. Hence, the inequality (4.56) implies (taking \tilde{x} there to be $x_{i,\tau}$)

$$\sup_{q_{i,\tau}} |u| \geq \sup_{B_{i,\tau}} |u| \geq K 2^{-C_3 \frac{N}{\log^6 N} - C_3}. \quad (4.61)$$

Putting the last two estimates together, we obtain

Lemma 4.4.4. *There exist positive constants c, C such that*

$$\sup_{\frac{1}{2}q_{i,[N^\epsilon][\log N]^4}} |u| \geq \sup_{\frac{1}{2}q_{i,1}} |u| 2^{cN/\log^2 N - C}. \quad (4.62)$$

Step 3 - bound on the number of good tunnels. Next, we show that there are sufficiently many tunnels, such that the doubling exponents of the contained cubes are controlled (cf. Claim 6.2, [Log18b]). More precisely,

Lemma 4.4.5. *There exist constants $c = c(\epsilon) > 0, N_0 > 0$ such that at least half of the tunnels T_i are “good” in the sense that they have the following property:*

For each cube $q_{i,t} \in T_i, t \in 1, \dots, [N^\epsilon][\log N]^4$ we have

$$N(q_{i,t}) \leq \max\left(\frac{N}{2^{c \log N / \log \log N}}, N_0\right). \quad (4.63)$$

Proof. We assume that N is sufficiently big. We focus on the cubes that fail to satisfy this condition, i.e. we consider the “bad” cubes $q_{i,t}$ for which

$$N(q_{i,t}) > N 2^{-c \log N / \log \log N}. \quad (4.64)$$

The constant $c = c(\epsilon)$ stems from Theorem 4.4.1 and is addressed below. As the number of all tunnels is $[N^\epsilon]^{n-1}$, by the pigeonhole principle, the claim of the lemma will follow if one shows that the number of bad cubes does not exceed $\frac{1}{2}[N^\epsilon]^{n-1}$.

To this end, we apply Theorem 4.4.1 in the following way. We divide T into equal Euclidean cubes $Q_t, t = 1, \dots, [\log N]^4$ of side-length $\frac{d(x, \tilde{x})}{[\log N]^4}$. We need to control $N(Q_t)$ via N . To do this, observe that

$$d(x, y) \leq 4d(x, \tilde{x}) \leq \frac{s}{10^7 \log^2 N}, \quad (4.65)$$

that is y is not far from the maximum point. Hence, we can apply (4.55) and obtain

$$\frac{\sup_{B(y, \frac{s}{10^7 \log^2 N})} |u|}{\sup_{B(y, \frac{1}{2} \frac{s}{10^7 \log^2 N})} |u|} \leq 2^{C \frac{N}{\log^2 N} + C}. \quad (4.66)$$

The definition and monotonicity of $N(Q_t)$ as well as the assumption that $N > N_0$ imply that

$$N(Q_t) \leq N, \quad t = 1, \dots, [\log N]^4. \quad (4.67)$$

Now, the application of Theorem 4.4.1 with $B = [N^\epsilon]$ gives that the number of bad cubes contained in Q_t (i.e., cubes whose doubling exponent is greater than $\max(N(Q_t) 2^{-c_1 \log(N^\epsilon) / \log \log(N^\epsilon)}, N_0)$) is less than $C[N^\epsilon]^{n-1-c_2}$. Note that we can absorb the ϵ term in the constant c_1 and deduce that the bad cubes have a doubling exponent greater than $\max(N(Q_t) 2^{-c(\epsilon) \log N / \log \log N}, N_0)$.

Summing over all cubes Q_t we obtain that the number of all bad cubes in T is no more than

$$C[N^\epsilon]^{n-1-c_2} [\log N]^4 \leq \frac{1}{2} [N^\epsilon]^{n-1}. \quad (4.68)$$

□

Step 4 - zeros along a good tunnel. Finally, we will count zeros of u along a good tunnel. Roughly, the harmonic function u has tame growth along a good tunnel. If u does not change sign, one could use the Harnack inequality to bound the growth of u in a suitable way. Summing up the growth over all cubes along a tunnel and using the estimate in Step 2 we obtain (cf. Claim 6.3, [Log18b]):

Lemma 4.4.6. *There exists a constant $c_2 = c_2(\epsilon) > 0$ such that if N is sufficiently large and T_i is a good tunnel, then there are at least $2^{c_2 \log N / \log \log N}$ closed cubes $\bar{q}_{i,t}$ that contain a zero of u .*

Proof. As the tunnel is good, Lemma 4.4.5 gives that for every $t = 1, \dots, [N^\epsilon][\log N]^4 - 1$ we have

$$\log \frac{\sup_{\frac{1}{2}q_{i,t+1}} |u|}{\sup_{\frac{1}{2}q_{i,t}} |u|} \leq \log \frac{\sup_{4q_{i,t}} |u|}{\sup_{\frac{1}{2}q_{i,t}} |u|} \leq \frac{N}{2^{c_1 \log N / \log \log N}}. \quad (4.69)$$

We split the index set $\{1, \dots, [N^\epsilon][\log N]^4 - 1\}$ into two disjoint subsets S_1, S_2 : an index t is in S_1 provided u does not change sign in $\bar{q}_{i,t} \cup \bar{q}_{i,t+1}$. The advantage in S_1 is that one can use the Harnack inequality. For $t \in S_1$ we have

$$\log \frac{\sup_{\frac{1}{2}q_{i,t+1}} |u|}{\sup_{\frac{1}{2}q_{i,t}} |u|} \leq C_1. \quad (4.70)$$

Using Lemma 4.4.4 and summing-up we obtain

$$c \frac{N}{\log^2 N} - C \leq \log \frac{\sup_{\frac{1}{2}q_{i,[N^\epsilon][\log N]^4}} |u|}{\sup_{\frac{1}{2}q_{i,1}} |u|} = \sum_{S_1} \log \frac{\sup_{\frac{1}{2}q_{i,t+1}} |u|}{\sup_{\frac{1}{2}q_{i,t}} |u|} + \sum_{S_2} \log \frac{\sup_{\frac{1}{2}q_{i,t+1}} |u|}{\sup_{\frac{1}{2}q_{i,t}} |u|} \leq \quad (4.71)$$

$$\leq C_1 |S_1| + \frac{N}{2^{c_1 \log N / \log \log N}} |S_2| \leq C_1 [N^\epsilon] \log^4 N + \frac{N}{2^{c_1 \log N / \log \log N}} |S_2| \leq \quad (4.72)$$

$$\leq \frac{c}{2} \frac{N}{\log^2 N} - C + \frac{N}{2^{c_1 \log N / \log \log N}} |S_2|. \quad (4.73)$$

This shows that

$$|S_2| \geq 2^{\frac{c_1}{2} \log N / \log \log N}. \quad (4.74)$$

□

We have already seen that there are at least $\frac{1}{2}[N^\epsilon]^{n-1}$ good tunnels, which, by summing-up, means that the number of small cubes, where u changes sign is at least $\frac{1}{2}[N^\epsilon]^{n-1} 2^{c_2 \log N / \log \log N}$.

Finally, in each cube $\bar{q}_{i,t}$ let us fix a zero $x_{i,t} \in \bar{q}_{i,t}$, $u(x_{i,t}) = 0$ and note that

$$\text{diam}(\bar{q}_{i,t}) \sim \frac{r}{N^\epsilon \log^6 N}. \quad (4.75)$$

Each ball $B(x_{i,t}, \frac{r}{N^\epsilon \log^6 N})$ intersects at most $\kappa = \kappa(n)$ other balls of this kind. By taking a maximal disjoint collection of such balls and reducing the constant c_2 to $c_3 = c_3(\epsilon)$ we conclude the proof of Theorem 4.3.1.

Chapter 5

Some background on Brownian motion and hitting probabilities

In this Chapter we gather the necessary background in order to apply Brownian motion techniques to study eigenfunctions. These include, for instance, the Feynman-Kac formula and estimates on hitting probabilities.

5.1 Brownian motion on manifolds

Let (M, g) be a closed connected Riemannian manifold of dimension n .

We begin by recalling the basic constructions of Brownian motion and the corresponding Wiener measures. A thorough discussion of such material could be found in numerous sources - for example, in [BP11] and Chapter 11, [Tay11], where the presentation hardly assumes any knowledge in stochastics. As pointed out in [BP11], most treatments of Brownian motion (e.g. [Hsu02], [Gri09]) tend to introduce the relevant objects in Euclidean space and then use a specific method (e.g. embeddings or the Eells-Elworthy-Malliavin frame bundle construction) to transfer the stochastic process from \mathbb{R}^n to manifolds. In our brief presentation, we follow the treatment in [BP11] which directly uses the heat kernel as a transition function to construct the Wiener measure on the space $C_{x_0}([0, T], M)$ of continuous paths starting from a fixed point x_0 in M and being parametrized on the interval $[0, T]$.

We describe how a suitable transition function gives rise to stochastic process. To this end, we first collect the necessary definitions.

Definition 5.1.1. *Let (X, \mathcal{B}) be a measurable space (i.e. X is a set and \mathcal{B} is a σ -algebra on X) and let (Ω, \mathcal{E}, P) be a probability space (i.e. (Ω, \mathcal{E}) is a measurable space and P is a probability measure on it). For an arbitrary index set $I \subseteq \mathbb{R}$, a corresponding family $\{S_t\}_{t \in I}$ of measurable maps*

$$S_t : (\Omega, \mathcal{E}) \rightarrow (X, \mathcal{B}), \quad \forall t \in I, \quad (5.1)$$

is called a stochastic process on Ω with values in X .

Definition 5.1.2. *Let (X, \mathcal{B}, μ) be a measure space and let us fix a positive number T . A function*

$$f : (0, T] \times X \times X \rightarrow [0, \infty], \quad (t, x, y) \mapsto f_t(x, y), \quad (5.2)$$

is called a stochastic transition function if the following conditions are satisfied:

1. For every number t in the interval $(0, T]$, the mapping

$$f_t : X \times X \rightarrow [0, \infty], \quad (x, y) \mapsto f_t(x, y), \quad (5.3)$$

is measurable with respect to the product σ -algebra of $X \times X$.

2. The following conservation property holds

$$\int_X f_t(x, y) d\mu(y) = 1, \quad \forall x \in X, \quad \forall t \in (0, T]. \quad (5.4)$$

3. The following transition criterion holds

$$\int_X f_t(x, y) f_s(y, z) d\mu(y) = f_{t+s}(z, x), \quad \forall x, z \in X, \quad \forall t, s, t+s \in (0, T]. \quad (5.5)$$

We now recall that on the Riemannian manifold (M, g) the Laplace operator given by (1.12) induces the heat semi-group $e^{t\Delta}$ (for a positive number t), i.e. the solution operator to the heat-equation. Moreover, $e^{t\Delta}$ is a bounded selfadjoint operator acting on the space $L^2(M)$. The construction of the operator $e^{t\Delta}$ can proceed using functional calculus and the spectral theorem. We also recall that $e^{t\Delta}$ is a smoothing operator and its Schwartz kernel (also known as the *heat kernel*)

$$h : (0, \infty) \times M \times M \rightarrow \mathbb{R}, \quad (t, x, y) \mapsto h_t(x, y), \quad (5.6)$$

depends smoothly on t, x, y . Furthermore, the heat kernel has the following properties

$$h_t(x, y) = h_t(y, x), \quad (5.7)$$

$$h_{t+s}(x, y) = \int_M h_t(x, z) h_s(z, y) dz, \quad (5.8)$$

$$\int_M h_t(x, y) dy = 1. \quad (5.9)$$

The last property is also referred to as stochastic completeness. Observe that these properties imply that $h_t(x, y)$ is a stochastic transition function in the sense of Definition 5.1.2. With this in mind we have the following

Theorem 5.1.1. *For an arbitrary point x_0 in M , the heat kernel induces a stochastic process B_t (also known as Brownian motion) where t ranges over the interval $[0, t]$ and takes values in M . Moreover, a probability measure \mathbb{W}_{x_0} (also known as the Wiener measure) is induced on the space $C_{x_0}([0, T], M)$ of continuous paths starting from x_0 and being parametrized on the interval $[0, T]$. The Wiener measure \mathbb{W}_{x_0} satisfies*

$$\mathbb{W}_{x_0}(\{w \in C_{x_0}([0, T], M) | w(t_1) \in U_1, \dots, w(t_m) \in U_m\}) \quad (5.10)$$

$$= \int_{U_m} \dots \int_{U_1} h_{t_m - t_{m-1}}(x_m, x_{m-1}) \dots h_{t_2 - t_1}(x_2, x_1) h_{t_1}(x_1, x_0) dx_1 \dots dx_m, \quad (5.11)$$

for any natural number m , any choice of $0 < t_1 < \dots < t_m = T$ and any open subsets U_1, \dots, U_m of M .

Moreover, for any positive number α in the interval $(0, \frac{1}{2})$, the subset of Hölder continuous paths of order α is a subset of full measure (w.r.t. \mathbb{W}_{x_0}).

Sketch of Proof. The proof proceeds by first constructing an appropriate family of measures adopted to the spaces of maps whose domain is a finite subset of $[0, T]$, i.e.

$$C^F : F \rightarrow M, \quad F \subset [0, T], \quad \#\{x \in F\} < \infty. \quad (5.12)$$

This is achieved by using (5.10) as a definition. Moreover, one sees that the family of such measures is consistent, i.e. the measures associated to C^F and C^G for some finite subsets F, G of $[0, T]$ respect restrictions whenever $F \subset G$.

Afterwards, a well-known result of Kolmogorov allows one to find a unique measure which extends the above family and is adopted to maps whose domain is $[0, T]$. Thus, the stochastic process Brownian motion B_t can be associated with this probability space and be defined by taking the restriction at time t . Furthermore, this probability space also gives rise to the Wiener measure \mathbb{W}_{x_0} .

The Hölder continuity statement follows using a result due to Kolmogorov and Chentsov.

For complete details we refer to Theorem 2.5 and Corollary 3.5, [BP11].

□

We remark that the construction of the Wiener measure is possible not necessarily with the heat kernel as a transition function. One can show that the arguments above hold for certain abstract metric measure spaces and transition functions satisfying a certain integral bound (Theorem 2.5, [BP11]).

Moreover, we recall that a similar construction is utilized when one constructs the standard Brownian motion in \mathbb{R}^n . The transition function one uses in this situation is the standard heat kernel on \mathbb{R}^n .

5.2 The Feynman-Kac formula

In this Section we discuss the the Feynman-Kac formula for open connected domains in compact manifolds where we consider the heat equation with Dirichlet boundary conditions. In principle, the Feynman-Kac formula allows one to express the solution of a certain diffusion process in terms of a path integral, i.e. an integral over the space of continuous paths equipped with the Wiener measure.

More precisely, we have the following

Theorem 5.2.1. *Let (M, g) be a closed connected Riemannian manifold. For an open connected subset $\Omega \subset M$ and a square-integrable function $f \in L^2(\Omega)$, the heat semigroup $e^{t\Delta}$ with imposed Dirichlet boundary conditions on $\partial\Omega$ we have that*

$$e^{t\Delta}f(x) = \mathbb{E}_x f(\omega(t))\phi_\Omega(\omega, t)d\mathbb{W}_x, \quad \forall t > 0, \quad \forall x \in \Omega, \quad (5.13)$$

where $\omega(t)$ denotes an element of the probability space of Brownian motions starting at x , \mathbb{E}_x is the expectation with regards to the Wiener measure \mathbb{W}_x on that probability space, and

$$\phi_\Omega(\omega, t) = \begin{cases} 1, & \text{if } \omega([0, t]) \subset \Omega \\ 0, & \text{otherwise.} \end{cases} \quad (5.14)$$

Sketch of Proof. A proof can be constructed in three steps.

First, one considers the boundaryless case and proves a corresponding statement when $\Omega = M$. The central strategy is to use the Trotter product formula, in combination with dominated convergence. The form of Feynman-Kac formula one gets this way says that the semigroup generated by a Schrödinger operator $H = \Delta - V$ is given as a path integral:

$$e^{tH} f(x) = \mathbb{E}_x f(\omega(t)) e^{-\int_0^t V(\omega(s)) ds} d\mathbb{W}_x, \quad (5.15)$$

We refer to Theorem 6.2, [BP11], for details.

Second, one can consider a domain Ω with Lipschitz boundary and introduce appropriate barrier potentials V_n (growing rapidly outside of the domain as n becomes large). One can apply the last Feynman-Kac formula to each V_n and pass to the limit. This yields the statement of the Theorem with Ω having a Lipschitz boundary. For further details we refer to Proposition 3.3, Chapter 11, [Tay11].

Third, for general open domains Ω with no assumption on boundary regularity, one approximates Ω by compactly contained regular-boundary domains $\Omega_k \subset \subset \Omega, k \in \mathbb{N}$. In a similar fashion one can pass to the limit and obtain the needed formula. For details we refer to Propositions 3.5 and 3.6, Chapter 11, [Tay11]. \square

5.3 Hitting probabilities and comparability

A central notion we will utilize is the following.

Definition 5.3.1. *Given a compact subset K of M , let $\psi_K(t, x)$ denote the probability that a Brownian motion process starting at the point x will hit K by time t . In other words*

$$\psi_K(t, x) = \mathbb{W}_x(\exists s \in [0, t] : \omega(s) \in K). \quad (5.16)$$

There is an extensive literature on hitting probabilities for Brownian motion and related stochastic processes (we refer, for example, to [GSC02], [BPP95] and the references therein).

Implicit in some of our calculations is the following heuristic: if the metric is perturbed slightly, hitting probabilities of compact sets by Brownian motions are also perturbed slightly, provided one is looking at small distances r and at small time scales $t = O(r^2)$.

To describe the particular statement we need, let us cover M by normal-coordinate charts (U_k, ϕ_k) such that in these charts g is bi-Lipschitz comparable to the Euclidean metric. Consider an open ball $B_r(p) \subset M$, where r is considered sufficiently small, and in particular, smaller than the injectivity radius of M . Let $B_r(p)$ be covered by a normal coordinate chart (U, ϕ) centered at p and let us assume that $\phi(p) = q$ with $\phi(B_r(p)) = B_r^e(p) \subset \mathbb{R}^n$ (as before, here the superscript e indicates that we are considering Euclidean balls). Let K be a compact set inside $B_r(p)$ and let $\tilde{K} := \phi(K) \subset B_r^e(p)$.

Now, let us use a superscript M in the notation $\psi_K^M(T, p)$ to denote the probability that a Brownian motion on (M, g) started at p and killed at a fixed time T hits K within time T . The quantity $\psi_{\tilde{K}}^e(t, q)$ is defined similarly for the standard Brownian motion in \mathbb{R}^n started at q and killed at the same fixed time T . Now, we fix the time $T = cr^2$, where c is a constant. The following is the comparability result:

Theorem 5.3.1. *There exists constants c_1, c_2 , depending only on c , the dimension n and certain bounds on the metric g such that*

$$c_1 \psi_{\tilde{K}}^{\varepsilon}(T, q) \leq \psi_K^M(T, p) \leq c_2 \psi_{\tilde{K}}^{\varepsilon}(T, q). \quad (5.17)$$

Such a statement seems to be well-known in the community and used implicitly in several works (cf. [Ste14] and the references therein). However, for convenience we provide a sketch of proof. We follow our work in [GM18b].

Sketch of proof of Theorem 5.3.1. The following strategy uses the concept of Martin capacity (see Definition 2.1, [BPP95]):

Definition 5.3.2. *Let Λ be a set and \mathcal{B} - a σ -algebra of subsets of Λ . Given a measurable function $F : \Lambda \times \Lambda \rightarrow [0, \infty]$ and a finite measure μ on (Λ, \mathcal{B}) , the F -energy of μ is*

$$I_F(\mu) = \int_{\Lambda} \int_{\Lambda} F(x, y) d\mu(x) d\mu(y).$$

The capacity of Λ in the kernel F is

$$Cap_F(\Lambda) = \left[\inf_{\mu} I_F(\mu) \right]^{-1}, \quad (5.18)$$

where the infimum is over probability measures μ on (Λ, \mathcal{B}) , and by convention, $\infty^{-1} = 0$.

In order to state the next result we briefly and informally recall the following notions. The transiency/recurrency property of stochastic process refers to the likelihood that the process will eventually return to its initial state. Further, the Markovian property of a stochastic process, loosely speaking, asserts that the future states can be predicted only by referring to the present state (and not on past states). With this in mind, we observe that the constructed Brownian motion on (M, g) (and also the standard one in \mathbb{R}^n) is a transient Markov stochastic process. For further details, we refer to a complete treatment of Brownian motion - e.g. in [Hsu02], [MP10], [GSC02].

Now we quote the following general result, which is Theorem 2.2 in [BPP95].

Theorem 5.3.2. *Let $\{X_n\}$ be a transient Markov chain on the countable state space Y with initial state ρ and transition probabilities $p(x, y)$. For any subset Λ of Y , we have*

$$\frac{1}{2} Cap_M(\Lambda) \leq \mathbb{P}_{\rho}[\exists n \geq 0 : X_n \in \Lambda] \leq Cap_M(\Lambda), \quad (5.19)$$

where M is the Martin kernel $M(x, y) = \frac{G(x, y)}{G(\rho, y)}$, and $G(x, y)$ denotes the Green's function.

For the special case of Brownian motions, this reduces to (see Proposition 1.1 of [BPP95] and Theorem 8.24 of [MP10]):

Theorem 5.3.3. *Let $\{B(t) : 0 \leq t \leq T\}$ be a transient Brownian motion in \mathbb{R}^n starting from the point ρ , and $A \subset D$ be closed, where D is a bounded domain. Then,*

$$\frac{1}{2} Cap_M(A) \leq \mathbb{P}_{\rho}\{B(t) \in A \text{ for some } 0 < t \leq T\} \leq Cap_M(A). \quad (5.20)$$

An inspection of the proofs reveals that they go through with basically no changes on a compact Riemannian manifold M , when the Brownian motion is killed at a fixed time $T = cr^2$, and the Martin kernel $M(x, y)$ is defined as $\frac{G(x, y)}{G(\rho, y)}$, with $G(x, y)$ being the “cut-off” Green’s function defined as follows: if $h_M(t, x, y)$ is the heat kernel of M as above,

$$G(x, y) := \int_0^T h_M(t, x, y) dt. \quad (5.21)$$

Now, to state it formally, in our setting, we have

Theorem 5.3.4. *We have the following bounds on the Brownian motion hitting probabilities in terms of the Martin capacity*

$$\frac{1}{2} \text{Cap}_M(K) \leq \psi_K^M(T, p) \leq \text{Cap}_M(K). \quad (5.22)$$

Now, let $h_{\mathbb{R}^n}(t, x, y)$ denote the heat kernel on \mathbb{R}^n . To prove Theorem 5.3.1, it suffices to show that for $y \in K$, and $\tilde{y} = \phi(y) \in \tilde{K}$, we have constants C_1, C_2 (depending on c and M) such that

$$C_1 \int_0^T h_{\mathbb{R}^n}(t, q, \tilde{y}) dt \leq \int_0^T h_M(t, p, y) dt \leq C_2 \int_0^T h_{\mathbb{R}^n}(t, q, \tilde{y}) dt. \quad (5.23)$$

In other words, we need to demonstrate comparability of Green’s functions “cut off” at time $T = cr^2$. Recall that we have the following Gaussian two-sided heat kernel bounds on a compact manifold (see, for example, Theorem 5.3.4 of [Hsu02] for the lower bound and Theorem 4 of [CLY81] for the upper bound, also (4.27) of [GSC02]): for all $(t, p, y) \in (0, 1) \times M \times M$, and positive constants c_1, c_2, c_3, c_4 depending only on the geometry of M ,

$$\frac{c_3}{t^{n/2}} e^{-\frac{c_1 d(p, y)^2}{4t}} \leq h_M(t, p, y) \leq \frac{c_4}{t^{n/2}} e^{-\frac{c_2 d(p, y)^2}{4t}},$$

where d denotes the distance function on M . Then, using the comparability of the distance function on M with the Euclidean distance function (which comes via metric comparability in local charts), for establishing (5.23), it suffices to observe that for any positive constant c_5 , we have that

$$\int_0^{cr^2} t^{-\frac{n}{2}} e^{-\frac{c_5 r^2}{4t}} dt = \frac{2^{n-2}}{c_5^{\frac{n}{2}-1}} \frac{1}{r^{n-2}} \Gamma\left(\frac{n}{2} - 1, \frac{c_5}{4c}\right),$$

where $\Gamma(s, x)$ is the (upper) incomplete Gamma function. Since r is a small constant chosen independently of λ , we observe that C_1, C_2 are constants in (5.23) depending only on $c, c_1, c_2, c_3, c_4, c_5, r$ and M , which finally proves (5.17). □

5.4 Hitting probabilities of spheres

Now, we consider an m -dimensional Brownian motion $B(s)$ of a particle starting at the origin in \mathbb{R}^m , and calculate the probability of the particle hitting a sphere $\{x \in \mathbb{R}^m : \|x\| \leq r\}$ of radius r

within time t . By a well known formula first derived in [Ken80], we see that such a probability is given as follows:

$$\mathbb{P}(\sup_{0 \leq s \leq t} \|B(s)\| \geq r) = 1 - \frac{1}{2^{\nu-1}\Gamma(\nu+1)} \sum_{k=1}^{\infty} \frac{j_{\nu,k}^{\nu-1}}{J_{\nu+1}(j_{\nu,k})} e^{-\frac{j_{\nu,k}^2 t}{2r^2}}, \quad \nu > -1, \quad (5.24)$$

where $\nu = \frac{m-2}{2}$ is the ‘‘order’’ of the Bessel process, J_ν is the Bessel function of the first kind of order ν , and $0 < j_{\nu,1} < j_{\nu,2} < \dots$ is the sequence of positive zeros of J_ν .

We will be interested in the regime when t is of the order of r^2 (i.e. respecting the parabolic scaling). In this direction let us, for a large positive number λ , choose $t = \lambda^{-1}$, and let $r = C\lambda^{-1/2}$, where C is a constant to be chosen later, independently of λ . Plugging this in (5.24) then reads,

$$\mathbb{P}(\sup_{0 \leq s \leq \lambda^{-1}} \|B(s)\| \geq C\lambda^{-1/2}) = 1 - \frac{1}{2^{\nu-1}\Gamma(\nu+1)} \sum_{k=1}^{\infty} \frac{j_{\nu,k}^{\nu-1}}{J_{\nu+1}(j_{\nu,k})} e^{-\frac{j_{\nu,k}^2}{2C^2}}, \quad \nu > -1. \quad (5.25)$$

We need to make a few comments about the asymptotic behaviour of $j_{\nu,k}$ here. For notational convenience, we write $\alpha_k \sim \beta_k$, as $k \rightarrow \infty$ if we have $\alpha_k/\beta_k \rightarrow 1$ as $k \rightarrow \infty$. The result in [Wat44], p. 506, gives the asymptotic expansion

$$j_{\nu,k} = (k + \nu/2 + 1/4)\pi + o(1) \text{ as } k \rightarrow \infty, \quad (5.26)$$

which tells us that $j_{\nu,k} \sim k\pi$. Also, from [Wat44], p. 505, we have that

$$J_{\nu+1}(j_{\nu,k}) \sim (-1)^{k-1} \frac{\sqrt{2}}{\pi} \frac{1}{\sqrt{k}}. \quad (5.27)$$

These asymptotic estimates, in conjunction with (5.25), tell us that keeping ν bounded, and given a small $\eta > 0$, one can choose the constant C small enough (depending on η) such that

$$\mathbb{P}(\sup_{0 \leq s \leq \lambda^{-1}} \|B(s)\| \geq C\lambda^{-1/2}) > 1 - \eta. \quad (5.28)$$

In this context, see also refer to Proposition 5.1.4 of [Hsu02].

Chapter 6

Estimates on nodal domains

In this Chapter we study some aspects of the geometry of nodal domains of Laplacian eigenfunctions.

First, we address the following loosely-formulated question:

Question 6.0.1. *Can a nodal domain be "thin and straight", thus resembling a long thin cylinder?*

The more precise form of the question we will address is the following:

Question 6.0.2. *Suppose that $\Sigma \subset M$ is sufficiently flat (where flatness is defined in an appropriate way) submanifold. Can a nodal domain be contained in sufficiently small wavelength neighbourhood of Σ ?*

We will apply Brownian motion techniques to address these last questions.

We will afterwards address the width of nodal domains. To this end we will study the inner radius of nodal domains (i.e. the radius of the largest inscribed geodesic ball inside of a nodal domain). We will be interested in the following

Question 6.0.3. *Given a nodal domain Ω_λ corresponding to an eigenfunction ϕ_λ , how is $\text{inrad}(\Omega_\lambda)$ compared to wavelength (i.e. $1/\sqrt{\lambda}$)?*

Simple examples (such as tori) suggest that nodal domains possess approximately wavelength inner radius. In order to approach such a question, one could try to simplify the discussion and focus only on a particular nodal domain Ω_λ with the corresponding eigenfunction ϕ_λ being restricted on Ω_λ . In fact, since ϕ_λ does not vanish on Ω_λ , it is the first Dirichlet eigenfunction when restricted to Ω_λ . However, using only this information (i.e. forgetting about the complement of Ω_λ) is not sufficient to obtain inner radius bounds. It turns out that in dimension $n \geq 3$ that one can introduce thin spikes which do not change the first Dirichlet eigenvalue, but have a significant impact on the the domain's inner radius (cf. [Hay78]). From this point of view nodal domains seem to be sensitive objects and sharper inner radius bounds would require some further "global" information (i.e. outside of the particular nodal domain).

6.1 Thin and straight nodal domains

Problems, similar in spirit to Question 6.0.2, have been addressed in several results appearing in the works of Jerison-Grieser, Steinerberger, etc (cf. [GJ98], [Ste14] and the references therein).

By considering Brownian motion and applying the Feynman-Kac formula, followed by suitable hitting probability estimates, we further extend the results in these directions. In terms of presentation, we partly follow our work in [GM18b].

We consider a closed n -dimensional smooth Riemannian manifold (M, g) . For an eigenvalue λ of the Laplacian Δ and a corresponding eigenfunction ϕ_λ , we consider a nodal domain Ω_λ (recall Definition 1.3.1).

We start by discussing the problem of whether a nodal domain can be squeezed in a tubular neighbourhood around a certain subset $\Sigma \subseteq M$. A result of Steinerberger (see Theorem 2 of [Ste14]) states that for some constant $r_0 > 0$ a nodal domain Ω_λ cannot be contained in a $\frac{r_0}{\sqrt{\lambda}}$ -tubular neighbourhood of hypersurface Σ , provided that Σ is sufficiently flat in the following sense: the hypersurface Σ must admit a unique metric projection in a wavelength (i.e. $\sim \frac{1}{\sqrt{\lambda}}$) tubular neighbourhood. The proof involves the study of a heat process associated to the nodal domain, where one also uses estimates for Brownian motion.

We relax the conditions imposed on Σ . Our first result is a direct extension of Theorem 2 of [Ste14]. Before stating the result, we begin with the following definition:

Definition 6.1.1 (Admissible Collections). *For each fixed eigenvalue λ , we consider a natural number $m_\lambda \in \mathbb{N}$ and a collection $\Sigma_\lambda := \cup_{i=1}^{m_\lambda} \Sigma_\lambda^i$, where Σ_λ^i is an embedded smooth submanifold (without boundary) of dimension k , ($1 \leq k \leq n - 1$).*

We call Σ_λ admissible up to a distance r if the following property is satisfied: for any $x \in M$ with $\text{dist}(x, \Sigma_\lambda) \leq r$ there exists a unique index $1 \leq i_x(\lambda) \leq m_\lambda$ and a unique point $y \in \Sigma_\lambda^{i_x(\lambda)}$ realizing $\text{dist}(x, \Sigma_\lambda)$ - that is, $\text{dist}(x, y) = \text{dist}(x, \Sigma_\lambda)$.

We note that if Σ_λ consists of one submanifold which is admissible up to distance r , then Definition 6.1.1 means that r is smaller than the normal injectivity radius of Σ_λ . Moreover, if Σ_λ consists of more submanifolds, then these submanifolds must be disjoint and the distance between every two of them must be greater than r .

Let us also remark that, in contrast to Theorem 2 of [Ste14], we also allow Σ_λ to vary with respect to λ in a controlled way, which is made precise by Definition 6.1.1. With that clarification in place, we have the following result:

Theorem 6.1.1. *There is a constant r_0 depending only on (M, g) such that if a submanifold $\Sigma_\lambda \subset M$ is admissible up to distance $\frac{1}{\sqrt{\lambda}}$, then no nodal domain Ω_λ can be contained in a $\frac{r_0}{\sqrt{\lambda}}$ -tubular neighbourhood of Σ_λ .*

Proof. We begin by outlining the strategy.

First, one considers a point $x_0 \in \Omega_\lambda$ where the eigenfunction achieves a maximum on the nodal domain (w.l.o.g. we assume that the eigenfunction is positive on Ω_λ). One then considers the quantity $p(t, x_0)$ - i.e. the probability that a Brownian motion started at x_0 escapes the nodal domain within time t .

The main strategy is to obtain two-sided bounds for $p(t, x_0)$.

On one hand, we have the Feynman-Kac formula (see Section 5.2) which provides a straightforward upper bound only in terms of t .

On the other hand, we provide a lower bound for $p(t, x_0)$ in terms of some geometric data. To this end, we take advantage of various tools some of which are: formulas for hitting probabilities of spheres and the parabolic scaling between the space and time variables (cf. Section 5.4); comparability of Brownian motions on manifolds with similar geometry (cf. Section 5.3).

Step 1 - An associated diffusion process and the Feynman-Kac formula.

Given an open subset $V \subset M$, we consider the solution $p_t(x)$ to the following diffusion process:

$$(\partial_t - \Delta)p_t(x) = 0, \quad x \in V, \quad (6.1)$$

$$p_t(x) = 1, \quad x \in \partial V, \quad (6.2)$$

$$p_0(x) = 0, \quad x \in V. \quad (6.3)$$

By the Feynman-Kac formula (see Section 5.2), this diffusion process can be understood as the probability that a Brownian motion particle started in x will hit the boundary within time t . Indeed, the Feynman-Kac formula yields

$$p_t(x) = 1 - \mathbb{E}_x(\psi_V(\omega, t)), \quad t > 0, \quad (6.4)$$

where we recall that $\omega(t)$ denotes an element of the probability space of Brownian motions starting at x , \mathbb{E}_x is the expectation with regards to the measure on that probability space, and where ψ_V is the cut-off function

$$\psi_V(\omega, t) = \begin{cases} 1, & \text{if } \omega([0, t]) \subset V \\ 0, & \text{otherwise.} \end{cases} \quad (6.5)$$

Now, we adopt this construction to the eigenfunction ϕ_λ (corresponding to the eigenvalue λ) and the nodal domain Ω_λ (replacing the open set V) upon which, without loss of generality, $\phi_\lambda > 0$. Setting $\Phi(t, x) := e^{t\Delta}\phi_\lambda(x)$, we see that Φ solves

$$(\partial_t - \Delta)\Phi(t, x) = 0, \quad x \in \Omega_\lambda, \quad (6.6)$$

$$\Phi(t, x) = 0, \quad x \in \partial\Omega_\lambda \subseteq \{\phi_\lambda = 0\}, \quad (6.7)$$

$$\Phi(0, x) = \phi_\lambda(x), \quad x \in \Omega_\lambda. \quad (6.8)$$

Using the Feynman-Kac formula given by Theorem 5.2.1, we have,

$$e^{t\Delta}\phi_\lambda(x) = \mathbb{E}_x(\phi_\lambda(\omega(t))\psi_{\Omega_\lambda}(\omega, t)), \quad t > 0, \quad (6.9)$$

where the cut-off function is given by

$$\psi_{\Omega_\lambda}(\omega, t) = \begin{cases} 1, & \text{if } \omega([0, t]) \subset \Omega_\lambda \\ 0, & \text{otherwise.} \end{cases} \quad (6.10)$$

Now, let us specify $x_0 \in \Omega_\lambda$ such that $\phi_\lambda(x_0) = \|\phi_\lambda\|_{L^\infty(\Omega_\lambda)}$. We have the following direct bounds:

$$\begin{aligned} \Phi(t, x) &= e^{-\lambda t}\phi_\lambda(x) = \mathbb{E}_x(\phi_\lambda(\omega(t))\psi_{\Omega_\lambda}(\omega, t)) \\ &\leq \|\phi_\lambda\|_{L^\infty(\Omega_\lambda)}\mathbb{E}_x(\psi_{\Omega_\lambda}(\omega, t)) = \|\phi_\lambda\|_{L^\infty(\Omega_\lambda)}(1 - p_t(x)). \end{aligned} \quad (6.11)$$

Setting $t = t_0\lambda^{-1}$ for a positive number t_0 and $x = x_0$, elementary algebraic manipulations imply that the probability $p_t(x)$ of the Brownian motion starting at an extremal point x_0 and leaving Ω within time λ^{-1} is bounded as:

$$p_t(x) \leq 1 - e^{-t_0}. \quad (6.12)$$

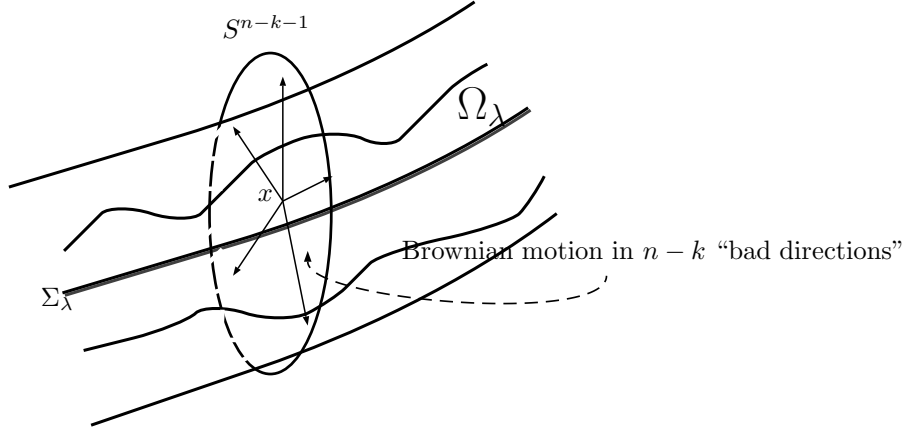


Figure 6.1: The nodal domain "squeezed" in a thin tubular neighbourhood around Σ_λ .

A rough interpretation is that maximal points x are situated deeply into the nodal domain Ω_λ . Using the notation for hitting probabilities introduced in Section 5.3, the last derived upper estimate translates to

$$\psi_{M \setminus \Omega_\lambda}(t_0 \lambda^{-1}, x) \leq 1 - e^{-t_0}. \quad (6.13)$$

Step 2 - A lower bound for the hitting probability.

In order to prove the claimed result in Theorem 6.1.1, we proceed by assuming the contrary. In other words, we assume that Ω_λ is contained in an $(r_0/\sqrt{\lambda})$ -neighbourhood of Σ_λ , where we have the freedom to choose the number r_0 as small as we wish.

First, by the admissibility condition on Σ_λ in Definition 6.1.1 we know that the point of maximum x_0 has a unique metric projection on one and only one $\Sigma_\lambda^{i_{x_0}}$ from the collection Σ_λ .

Further on, let us choose a suitable small radius R and small time parameter t_0 such that the Brownian motion comparability result in Theorem 5.3.1 holds at x_0 . Below we will specify further how small R, t_0 should be taken.

In this direction (as we assumed the contrary) we can choose r_0 to be sufficiently smaller than R (again determined below) and assert that Ω_λ is contained in a $\frac{r_0}{\sqrt{\lambda}}$ -tubular neighbourhood of Σ_λ - for convenience, we denote this tubular neighbourhood by $N_{r_0 \lambda^{-1/2}}(\Sigma_\lambda)$.

Note that from the remarks after Definition 6.1.1, it follows that $\Omega_\lambda \subseteq N_{r_0 \lambda^{-1/2}}(\Sigma_\lambda^{i_{x_0}})$.

Now, we start a Brownian motion at x_0 and, roughly speaking, we see that locally the particle has freedom to wander in $n - k$ "bad directions", namely the directions normal to $\Sigma_\lambda^{i_{x_0}}$, before it hits $\partial\Omega_\lambda$. That means, we may consider an $(n - k)$ -dimensional Brownian motion $B(t)$ starting at x_0 - cf. Figure 6.1.

More formally, we choose a normal coordinate chart (U, ϕ) around $\Sigma_\lambda^{i_{x_0}}$, where the metric is

comparable to the Euclidean metric and where we have that

$$\phi(\Sigma_\lambda^{i_{x_0}}) = \phi(U) \cap \{\mathbb{R}^k \times \{0\}^{n-k}\}, \quad (6.14)$$

$$\phi(N_{2r_0\lambda^{-1/2}}(\Sigma_\lambda^{i_{x_0}})) = \phi(U) \cap \left\{ \mathbb{R}^k \times \left[-\frac{2r_0}{\sqrt{\lambda}}, \frac{2r_0}{\sqrt{\lambda}} \right]^{n-k} \right\}. \quad (6.15)$$

We take a geodesic ball $B \subset U \subset M$ at x_0 of radius $\frac{R}{\sqrt{\lambda}}$. Using the hitting probability notation from Section 5.3 and monotonicity with respect to set inclusion we have

$$\psi_{M \setminus \Omega_\lambda} \left(\frac{t_0}{\lambda}, x_0 \right) \geq \psi_{B \setminus \Omega_\lambda} \left(\frac{t_0}{\lambda}, x_0 \right) \geq \psi_{B \setminus N_{2r_0\lambda^{-1/2}}(\Sigma_\lambda^{i_{x_0}})} \left(\frac{t_0}{\lambda}, x_0 \right), \quad (6.16)$$

and the comparability Theorem 5.3.1 implies that, if $c = \frac{t_0}{R^2}$, then there exists a constant C , depending on c and M , such that

$$\psi_{B \setminus N_{2r_0\lambda^{-1/2}}(\Sigma_\lambda^{i_{x_0}})} \left(\frac{t_0}{\lambda}, x_0 \right) \geq C \psi_{\phi(B \setminus N_{2r_0\lambda^{-1/2}}(\Sigma_\lambda^{i_{x_0}}))} \left(\frac{t_0}{\lambda}, \phi(x_0) \right), \quad (6.17)$$

where, as before, ψ^e denotes the hitting probability in Euclidean space. We denote $N_{r_0\lambda^{-1/2}}^e := \phi(N_{r_0\lambda^{-1/2}}(\Sigma_\lambda^{i_{x_0}}))$.

Let us consider the ‘‘solid cylinder’’ $S = B_{\frac{R_0}{\sqrt{\lambda}}}^{(k)} \times B_{\frac{r_0}{\sqrt{\lambda}}}^{(n-k)} =: B_1 \times B_2$, a product of k dimensional Euclidean ball of radius $R_0/\sqrt{\lambda}$ and $n-k$ dimensional Euclidean ball of radius $r_0/\sqrt{\lambda}$, respectively, both centered at $\phi(x_0)$. By construction, we can appropriately choose R_0 with respect to R , so that S is a cylinder contained in $N_{2r_0\lambda^{-1/2}}^e \cap B$.

If $B(t) = (B_1(t), \dots, B_n(t))$ is an n -dimensional Brownian motion, the components $B_i(t)$'s are independent Brownian motions (see, for example, Chapter 2 of [MP10]). Denoting by $\mathcal{B}_k(t)$ and $\mathcal{B}_{n-k}(t)$ the projections of $B(t)$ onto the first k and last $n-k$ components respectively, it follows that the following lower bound in terms of hitting the B_2 -side of S holds:

$$\begin{aligned} & \psi_{\phi(B \setminus N_{2r_0\lambda^{-1/2}}(\Sigma_\lambda^{i_{x_0}}))}^e \left(\frac{t_0}{\lambda}, \phi(x_0) \right) \\ & \geq \mathbb{P} \left(\sup_{0 \leq s \leq t_0\lambda^{-1}} \|\mathcal{B}_k(t)\| \leq \frac{R_0}{\sqrt{\lambda}} \right) \mathbb{P} \left(\sup_{0 \leq s \leq t_0\lambda^{-1}} \|\mathcal{B}_{n-k}(t)\| \geq \frac{r_0}{\sqrt{\lambda}} \right) \\ & \geq c_k \mathbb{P} \left(\sup_{0 \leq s \leq t_0\lambda^{-1}} \|\mathcal{B}_{n-k}(t)\| \geq \frac{r_0}{\sqrt{\lambda}} \right), \end{aligned}$$

where c_k is a constant depending on k and the ratio t_0/R_0^2 ; moreover, c_k can be calculated explicitly from (5.25).

Using the estimate on hitting probabilities of spheres (5.28), we may take $r_0 \leq R_0$ sufficiently small so that

$$\mathbb{P} \left(\sup_{0 \leq s \leq t_0\lambda^{-1}} \|\mathcal{B}_{n-k}(t)\| \geq \frac{r_0}{\sqrt{\lambda}} \right) > 1 - \varepsilon, \quad (6.18)$$

where ε is a fixed sufficiently small number.

To conclude, let us specify the "smallness" of the parameters as announced at the beginning. Assume for a moment that t_0 is selected. By adjusting R according to t_0 we keep the ratio $c = \frac{t_0}{R^2}$ and, hence, C and $\frac{t_0}{R_0^2}$ fixed. By taking $r_0 \leq R$ appropriately (much smaller than t_0 so that (6.18) holds), the above arguments yield

$$\psi_{M \setminus \Omega_\lambda}(t_0 \lambda^{-1}, x) \geq C c_k (1 - \epsilon), \quad (6.19)$$

where we emphasize that the constants on the right hand side depend only on the ratio $\frac{t_0}{R}$ which is kept fixed.

Combining this with the estimate from the previous Step 1, we obtain

$$1 - e^{-t_0} \geq C c_k (1 - \epsilon). \quad (6.20)$$

Finally, if t_0 is a priori selected small, the left hand side will become less than the expression on the right (which depends on the ratio t_0/R and not on t_0), thus yielding a contradiction. \square

Remark 6.1.1. *Note that the constant r_0 above is independent of Σ_λ (it is selected so that the above Euclidean Brownian motion hitting probabilities hold); in other words, the same constant r_0 will work for Theorem 6.1.1 as long as the surface is admissible up to a wavelength distance. Indeed, this results from the fact that r_0 depends only on the diffusion process associated to the Brownian motion, and is an inherent property of the manifold itself.*

6.2 A further implicit characterization of admissible collections

Now we address the generalizations of Theorem 6.1.1 for collections Σ_λ which are more complicated. It turns out that we can select Σ_λ to be a union of submanifolds of varying dimensions, having relaxed admissibility conditions.

Elaborating on this, we observe that getting entirely rid of the admissibility condition, as in Definition 6.1.1 allows situations where Σ_λ^i is dense in M , for example, $M = \mathbb{T}^2$ and Σ_λ^1 being a generic geodesic. By assuming Σ_λ^i is compact, we avoid such situations. Also, since we are considering unions of (perhaps, intersecting) surfaces, the requirement of "unique projection" of nearby points, as in Definition 6.1.1, makes no sense any more, and one can see that the approach of the proof of Theorem 6.1.1 does not work.

We now introduce the following relaxed notion of admissibility, defined implicitly with the aid of Brownian motion hitting probabilities.

Definition 6.2.1 (α -admissible Collections). *Let $0 < \alpha < 1$ be a constant. For each fixed eigenvalue λ , we consider a natural number $m_\lambda \in \mathbb{N}$ and a collection $\Sigma_\lambda := \cup_{i=1}^{m_\lambda} \Sigma_\lambda^i$, where Σ_λ^i is a compact embedded smooth submanifold (without boundary) of dimension k_i , ($1 \leq k_i \leq n - 1$). Denote the respective tubular neighbourhoods by $N_\epsilon(\Sigma_\lambda^i) := \{x \in M : \text{dist}(x, \Sigma_\lambda^i) < \epsilon\}$, and let $N_\epsilon(\Sigma_\lambda) = \cup_{i=1}^{m_\lambda} N_\epsilon(\Sigma_\lambda^i)$.*

We say that the collection Σ_λ is α -admissible, if for each sufficiently small $\epsilon > 0$ and each $x \in N_\epsilon(\Sigma_\lambda)$ we have

$$\psi_{\partial B(x, 2\epsilon) \setminus N_\epsilon(\Sigma_\lambda)}(4\epsilon^2, x) \geq \alpha \psi_{\partial B(x, 2\epsilon)}(4\epsilon^2, x). \quad (6.21)$$

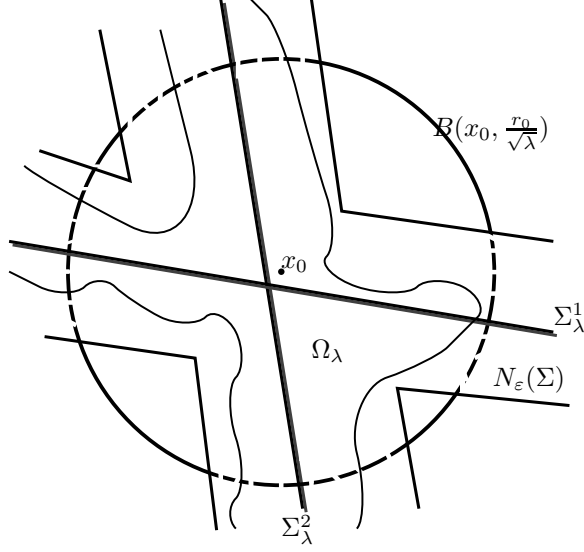


Figure 6.2: An illustration of the α -admissibility assumption: a Brownian particle is supposed to hit the undotted portion of the sphere with a sufficiently large probability.

Intuitively, using the above implicit formulation via Brownian motion hitting probabilities, we wish to ensure that $N_\varepsilon(\Sigma_\lambda)$ does not occupy too large a proportion of each $B(x, 2\varepsilon)$ for $x \in N_\varepsilon(\Sigma_\lambda)$ (cf. Figure 6.2).

In other words, we allow the family Σ_λ to intersect, but the intersections should not be “too dense”. To illustrate the idea, let us for simplicity assume that $M = \mathbb{R}^n$ and let us suppose that each member Σ_λ^i of the collection Σ_λ is a line passing through the origin. If the collection of these lines gets sufficiently close together or in other words “dense”, then no matter how small $\varepsilon > 0$ we take, the tubular neighbourhood $N_\varepsilon(\Sigma_\lambda)$ will contain the ball $B(0, 2\varepsilon)$. In particular, the left hand side of (6.21) is vanishing and so, there is no $\alpha > 0$ for which the collection Σ_λ is α -admissible. Clearly, in the above example, replacing the lines Σ_λ^i by linear subspaces of varying dimensions will deliver a similar example of a collection, which is not α -admissible.

Having this intuition in mind, we have the following result.

Theorem 6.2.1. *Given an α -admissible collection Σ_λ , there exists a constant C , independent of λ , such that $N_{\frac{C}{\sqrt{\lambda}}}(\Sigma_\lambda)$ cannot fully contain a nodal domain Ω_λ .*

Theorem 6.2.1 gives a strong indication as to the “thickness” or general shape of a nodal domain in many situations of practical interest. For example, in dimension 2, numerics show nodal domains to look like a tubular neighbourhood of a tree. We also note that our proof of Theorem 6.2.1 reveals a bit more information, but for aesthetic reasons, we prefer to state the theorem this way. Heuristically, the proof reveals that the nodal domain Ω_λ is thicker at the points where the eigenfunction φ_λ attains its maximum, or at points where $\varphi_\lambda(x) \geq \beta \max_{y \in \Omega_\lambda} |\varphi_\lambda(y)|$, for a fixed constant $\beta > 0$.

Proof of Theorem 6.2.1. The main idea of the proof is similar to the proof of Theorem 6.1.1 - we

repeat Step 1 to obtain an upper bound on the hitting probability $p_t(x_0)$ that a Brownian motion will hit $\partial\Omega_\lambda$ by time t :

$$p_t(x_0) \leq 1 - e^{-t}, \quad (6.22)$$

where x_0 denotes a point where ϕ_λ reaches a maximum on Ω_λ (again without loss of generality ϕ_λ is positive on Ω_λ).

Now, the modification of Step 2 goes as follows. By assumption, we have an α -admissible collection $\Sigma_\lambda := \cup_{i=1}^{m_\lambda} \Sigma_\lambda^i$, and let us again assume the contrary - if the statement is not true, we may select an arbitrarily small $r_0 > 0$ and find a corresponding inscribed nodal domain $\Omega_\lambda \subset N_{r_0\lambda^{-1/2}}(\Sigma_\lambda)$.

Monotonicity of the hitting probability function $\psi_K(.,.)$ with respect to set inclusion in K , as well as the α -admissibility imply that

$$\psi_{M \setminus \Omega_\lambda}(t, x_0) \geq \psi_{B(x_0, 2r_0\lambda^{-1/2}) \setminus \Omega_\lambda}(t, x_0) \quad (6.23)$$

$$\geq \psi_{B(x_0, 2r_0\lambda^{-1/2}) \setminus N_{r_0\lambda^{-1/2}}(\Sigma_\lambda)}(t, x_0) \quad (6.24)$$

$$= \psi_{\partial(B(x_0, 2r_0\lambda^{-1/2}) \setminus N_{r_0\lambda^{-1/2}}(\Sigma_\lambda))}(t, x_0) \quad (6.25)$$

$$\geq \psi_{\partial B(x_0, 2r_0\lambda^{-1/2}) \setminus N_{r_0\lambda^{-1/2}}(\Sigma_\lambda)}(t, x_0) \quad (6.26)$$

$$\geq \alpha \psi_{\partial B(x_0, 2r_0\lambda^{-1/2})}(t, x_0), \quad (6.27)$$

where we introduce the constant $\alpha > 0$ coming from the α -admissibility condition. Moreover, following Definition 6.2.1 of α -admissibility, in (6.23) we also assume that the radius $\frac{r_0}{\sqrt{\lambda}}$ is sufficiently small and that $t := \frac{t_0}{\lambda}$ with $t_0 := 4r_0^2$.

The latter estimate (6.23) implies, in particular, that

$$\frac{\psi_{M \setminus \Omega_\lambda}(t, x_0)}{\psi_{M \setminus B(x_0, 2r_0\lambda^{-1/2})}(t, x_0)} = \frac{\psi_{M \setminus \Omega_\lambda}(t, x_0)}{\psi_{\partial B(x_0, 2r_0\lambda^{-1/2})}(t, x_0)} \geq \alpha. \quad (6.28)$$

We now set $t = \frac{t_0}{\lambda}$, still having the freedom to choose t_0 . We show that we can select t_0 such that (6.28) is violated. To this end we observe that the upper bound on $\psi_{M \setminus \Omega_\lambda}$ from Step 1 along with the hitting probability of spheres from (5.25) and the comparability Theorem 5.3.1 give:

$$\begin{aligned} \frac{\psi_{M \setminus \Omega_\lambda}(\frac{t_0}{\lambda}, x_0)}{\psi_{M \setminus B(x_0, 2r_0\lambda^{-1/2})}(\frac{t_0}{\lambda}, x_0)} &\lesssim \frac{1 - e^{-t_0}}{1 - \frac{1}{2^{\nu-1}\Gamma(\nu+1)} \sum_{k=1}^{\infty} \frac{j_{\nu,k}^{\nu-1}}{J_{\nu+1}(j_{\nu,k})} e^{-\frac{j_{\nu,k}^2 t_0}{2r_0^2}}} \\ &= \frac{1 - e^{-t_0}}{1 - \frac{1}{2^{\nu-1}\Gamma(\nu+1)} \sum_{k=1}^{\infty} \frac{j_{\nu,k}^{\nu-1}}{J_{\nu+1}(j_{\nu,k})} e^{-2j_{\nu,k}^2}} \\ &= \frac{1 - e^{-t_0}}{\tilde{C}}. \end{aligned} \quad (6.29)$$

Now, we choose $t_0 = 4r_0^2$ small enough, so the last estimate yields a contradiction to (6.28). This proves the theorem. \square

Remark 6.2.1. *We wish to comment that in the above proof, it is not essential to look at the nodal domain only around the maximum point x_0 . Given a pre-determined positive constant β , choose a point $y \in \Omega_\lambda$ such that $\varphi_\lambda(y) \geq \beta\varphi_\lambda(x_0)$. Arguing similarly as in Step 1 of Theorems 6.1.1, 6.2.1, we see that $\psi_{M \setminus \Omega_\lambda}(t, y) \leq 1 - \beta e^{-t_0}$. Following the computations in (6.29), we get a constant r_0 (depending on β) such that $\frac{1 - \beta e^{-t_0}}{C} < \alpha$, giving a contradiction. Also, it is clear that in Definitions 6.1.1 and 6.2.1, we do not actually need the submanifolds in the family Σ_λ to be smooth, and the proofs of Theorems 6.1.1 and 6.2.1 work with submanifolds of much lower regularity (for example, C^1 submanifolds).*

A few further comments are in order. An interesting subcase one might also consider is Σ_λ having conical singularities: at its singular points Σ looks locally like $\mathbb{R}^{n-1-k} \times \partial C^k$ for some $k = 1, \dots, n-1$, where ∂C^k denotes the boundary of a generalized cone, i.e. the cone generated by some open set $D \subseteq \mathbb{S}^{n-1}$.

In this situation a useful tool is an explicit heat kernel formula for generalized cones $C \subseteq \mathbb{R}^n$. One denotes the associated Dirichlet eigenfunctions and eigenvalues of the generating set D by m_j, l_j respectively. Using polar coordinates $x = \rho\theta, y = r\eta$, one has that the heat kernel of $P_C(t, x, y)$ of the generalized cone C is given by

$$P_C(t, x, y) = \frac{e^{-\frac{\rho^2 + r^2}{2t}}}{t(\rho r)^{\frac{n}{2}+1}} \sum_{j=1}^{\infty} I_{\sqrt{l_j + (\frac{n}{2}-1)^2}}\left(\frac{\rho r}{t}\right) m_j(\theta) m_j(\eta), \quad (6.30)$$

where $I_\nu(z)$ denotes the modified Bessel function of order ν . For more on the formula (6.30) we refer to [BS97]. An even more general formula can be found in [Che83].

The expression for $P_C(t, x, y)$ provides means for estimating $p_t(x)$ from below as above. However, some features of the conical singularity (i.e. the eigenvalues and eigenfunctions l_j, m_j of the generating set D) enter explicitly in the estimate. Such considerations appear promising in discussing theorems of the following type, for example, and their higher dimensional analogues (see also [Ste14]):

Theorem 6.2.2 (Bers, Cheng). *Let $n = 2$. There exists a constant c such that if $-\Delta u = \lambda u$, then any nodal set satisfies an interior cone condition with opening angle $\alpha \geq c\lambda^{-1/2}$.*

6.3 Almost inscribed balls at max/min points

In this Section we study the problem of how large a ball one may inscribe in a nodal domain Ω_λ at a point where the eigenfunction achieves extremal values on Ω_λ . We show

Theorem 6.3.1. *Let $\dim M \geq 3, \epsilon_0 > 0$ be fixed and $x_0 \in \Omega_\lambda$ be such that $|\phi_\lambda(x_0)| = \max_{\Omega_\lambda} |\phi_\lambda|$. There exists $r_0 = r_0(\epsilon_0)$ and a threshold $\Lambda = \Lambda(M, g)$ such that*

$$\frac{\text{Vol}(B(x_0, r_0\lambda^{-1/2}) \cap \Omega_\lambda)}{\text{Vol}(B(x_0, r_0\lambda^{-1/2}))} \geq 1 - \epsilon_0, \quad (6.31)$$

whenever $\lambda \geq \Lambda$. We refer to such a ball $B(x_0, r_0\lambda^{-1/2})$ as being almost inscribed in the domain Ω_λ .

A celebrated theorem of Lieb (see [Lie83]) considers the case of a domain $\Omega \subset \mathbb{R}^n$ and states that there exists a point $x_0 \in \Omega$, where a ball of radius $\frac{C}{\sqrt{\lambda_1(\Omega)}}$ can almost be inscribed (in the sense of our Theorem 6.3.1). Here $\lambda_1(\Omega)$ denotes the first Dirichlet eigenvalue of the Laplacian.

A further illuminating result was obtained by Maz'ya-Shubin in the paper [MS05] (see, in particular, Theorem 1.1 and Subsection 5.1 of [MS05]). However, the point x_0 was not specified. Physically, one expects that x_0 is close to the point where the first Dirichlet eigenfunction of Ω attains extremal values. This is in fact the essential statement of Theorem 6.3.1 above. Also, in this context, it is illuminating to compare the main Theorem from [CD87].

Now, let us consider a fixed nodal domain Ω_λ corresponding to the eigenfunction ϕ_λ as before. As in the previous Sections, let $x_0 \in \Omega_\lambda$ be such that

$$\phi_\lambda(x_0) = \max_{x \in \Omega_\lambda} |\phi_\lambda|. \quad (6.32)$$

We recall that in the case $\dim M = 2$, it was shown in Section 3 of [Man08b] that at such maximal points x_0 one can fully inscribe a large ball of wavelength radius (i.e $\sim \frac{1}{\sqrt{\lambda}}$) into the nodal domain. In other words for Riemannian surfaces, one has that

$$\frac{C_1}{\sqrt{\lambda}} \leq \text{inrad}(\Omega_\lambda) \leq \frac{C_2}{\sqrt{\lambda}}, \quad (6.33)$$

where C_i are constants depending only on M . Note that the proof for this case, as carried out in [Man08b] by following ideas in [NPS05], makes use of essentially 2-dimensional tools (conformal coordinates and quasi-conformality), which are not available in higher dimensions.

Here we exploit heat equation and Brownian motion techniques to address the question in higher dimensions.

Proof of Theorem 6.3.1. We denote $t' := \frac{t_0}{\lambda}$, and thus

$$\psi_{M \setminus \Omega_\lambda}(t', x) \leq 1 - e^{-t_0}, \quad (6.34)$$

where t_0 is a small constant to be chosen suitably later.

Now, choosing t_0 small enough, and using monotonicity, we have,

$$\psi_{B(x_0, r_0 \lambda^{-1/2}) \setminus \Omega_\lambda}(t, x_0) < \psi_{M \setminus \Omega_\lambda}(t, x_0) < \epsilon. \quad (6.35)$$

For convenience, let us denote $E_{r_0} := B(x_0, r_0 \lambda^{-1/2}) \setminus \Omega_\lambda$ - a relatively compact set. Observe that the comparability Theorem 5.3.1 applies to open balls and compact subsets contained in open balls. To adapt to the setting of Theorem 5.3.1, choose a number $r'_0 < r_0$ such that $B(x_0, r'_0 \lambda^{-1/2})$ satisfies

$$\frac{\text{Vol}(B(x_0, r_0 \lambda^{-1/2}) \setminus B(x_0, r'_0 \lambda^{-1/2}))}{\text{Vol}(B(x_0, r_0 \lambda^{-1/2}))} < \epsilon.$$

Call $E_{r'_0} := \overline{E_{r_0} \cap B(x_0, r'_0 \lambda^{-1/2})}$. Observe that proving that $\frac{\text{Vol}(E_{r'_0})}{\text{Vol}(B(x_0, r_0 \lambda^{-1/2}))} < \epsilon$ will imply that $\frac{\text{Vol}(E_{r_0})}{\text{Vol}(B(x_0, r_0 \lambda^{-1/2}))} < 2\epsilon$, which is what we want.

We are now in a position to apply the comparability Theorem 5.3.1 - we note that if λ is sufficiently large (i.e. $\lambda \geq \Lambda$ where Λ depends only on (M, g)) the ball $B(x_0, r_0 \lambda^{-1/2})$ is sufficiently small and hence, contained in a chart where the metric g is comparable to the Euclidean metric.

Thus, applying Theorem 5.3.1 we can work with sets and Brownian motion in \mathbb{R}^n . This will allow us to apply suitable bounds on hitting probabilities.

Next, we would like to compare the volumes of the two sets $E_{r'_0}$ and $B(x_0, r_0\lambda^{-1/2})$. Let $r = \frac{r_0}{\sqrt{\lambda}}$. We recall from [GSC02], Equation (3.20) and its corollary in Remark 4.1, the following bound on the hitting probability:

$$c \frac{\text{cap}(E_{r'_0})r^2}{\text{Vol}(B(x_0, r_0\lambda^{-1/2}))} e^{-C\frac{r^2}{t'}} \leq \psi_{E_{r_0}}(t', x_0) < \epsilon, \quad (6.36)$$

where $\text{cap}(K)$ denotes the 2-capacity of the set $K \subset M$, and $0 < t' < 2r^2$. Here, the 2-capacity of a set $K \subset M$ is defined as

$$\text{cap}(K) = \inf_{\eta|_K \equiv 1, \eta \in C^\infty(M)} \int_M |\nabla \eta|^2 dM.$$

Formally, (6.36) holds on complete non-compact non-parabolic manifolds, which includes \mathbb{R}^n , $n \geq 3$. For bringing in our comparability result Theorem 5.3.1, we fix the ratio $\frac{t'}{r^2} = \frac{1}{3}$, say, and then choose t_0 small enough that (6.35) still holds. Now (6.36) applies, albeit with a new constant c as determined by the ratio t/r^2 and Theorem 5.3.1.

Now, to rewrite the capacity term in (6.36) in terms of volume, we bring in the following “isocapacity inequality” due to V. Maz’ya (see [Maz11], Section 2.2.3):

$$\text{cap}(E_{r_0}) \geq C' \text{Vol}(E_{r_0})^{\frac{n-2}{n}}, n \geq 3, \quad (6.37)$$

where C' is a constant depending only on the dimension n . We note that the isocapacity inequality (in combination with a suitable Poincare inequality) lies at the heart of the currently optimal inradius estimates, as derived by Mangoubi in [Man08a].

Clearly, (6.36) and (6.37) together give

$$\left(\frac{\text{Vol}(E_{r_0})}{\text{Vol}(B(x_0, r_0\lambda^{-1/2}))} \right)^{\frac{n-2}{n}} \lesssim \frac{\text{cap}(E_{r_0})r^2}{\text{Vol}(B(x_0, r_0\lambda^{-1/2}))} \lesssim \psi_{E_{r_0}}(t, x) < \epsilon. \quad (6.38)$$

The last inequalities contain constants depending only on M , so by taking ϵ even smaller we can arrange $\frac{\text{Vol}(E_{r_0})}{\text{Vol}(B(x_0, r_0\lambda^{-1/2}))} < \epsilon_0$ for any initially given ϵ_0 . \square

Remark 6.3.1. *The condition that λ is bounded below by Λ allows us to work in a small chart and hence, translate the discussion in \mathbb{R}^n via Theorem 5.3.1. In particular, if we work directly in \mathbb{R}^n , where Ω_λ is replaced by an arbitrary domain Ω and λ is replaced by the first Dirichlet eigenvalue of the Laplacian, the above arguments will go through. Thus we recover Lieb’s Theorem, [Lie83], with the additional refinement that an almost inscribed ball is situated at a point of maximum.*

Furthermore, one can also drop the assumption $\lambda \geq \Lambda$ provided some additional information concerning the heat kernel of (M, g) is available.

Remark 6.3.2. *An inspection of the proof of Theorem 6.3.1 reveals that one can take $\epsilon = r_0^{\frac{2n}{n-2}}$. In other words, the relative volume of the error set E_{r_0} decays as $r_0^{\frac{2n}{n-2}}$ as $r_0 \rightarrow 0$. This is slightly better than the scaling prescribed by Corollary 2 of [Lie83].*

Remark 6.3.3. *We note that the heat equation method does not distinguish between a general domain and a nodal domain. This means that we cannot rule out the situation where $B(x_0, \frac{r_0}{\sqrt{\lambda}}) \setminus \Omega_\lambda$ is a collection of “sharp spikes” entering into $B(x_0, \frac{r_0}{\sqrt{\lambda}})$. Indeed the probability of a Brownian particle hitting a spike, no matter how “thin” it is, or how far from x_0 it is, is always non-zero, a fact related to the infinite speed of propagation of heat diffusion. This is consistent with the heuristic discussed in [Hay78] and [Lie83].*

We also note that the proof of Theorem 6.3.1 uses estimates from [GSC02] and a certain isocapacitary estimate (6.37) that works in dimensions $n \geq 3$. As far as dimension $n = 2$ is concerned, it is known (cf. Theorem 1.2 of [Man08b]) that any nodal domain has wavelength inradius.

6.4 The inner radius of nodal domains

Let us again consider a closed smooth Riemannian manifold (M, g) of dimension n . Let ϕ_λ be a Laplacian eigenfunction corresponding to the eigenvalue λ and let Ω_λ be a nodal domain of ϕ_λ .

In this Section we discuss the width of the nodal domain Ω_λ in terms of its inner radius, i.e. the radius of the largest geodesic ball which is entirely contained (inscribed) into the particular domain. Moreover, we are interested in obtaining estimates in the high-energy (or semi-classical) limit, i.e. for large eigenvalues λ .

The problem has been addressed in a variety of works (cf. [Man08b], [Man08a] and the references therein). To our knowledge, the following bounds are known (cf. [Man08a]):

$$\frac{c_1}{\lambda^{\frac{n-1}{4} + \frac{1}{2n}}} \leq \text{inrad}(\Omega_\lambda) \leq \frac{c_2}{\sqrt{\lambda}}, \quad (6.39)$$

where $c_{1,2}$ depend on (M, g) , but not on λ .

We take the time to point out a few remarks.

First, the upper bound follows from a straight-forward argument which uses the monotonicity of the first Dirichlet eigenvalue with respect to inclusion (cf. [Man08b]).

Second, we note that in the case $n = 2$ the bounds are sharp. Moreover, as pointed out in [Man08a] it follows from complex-analysis arguments due to Nazarov-Polterovich-Sodin, that one can find a wavelength inscribed ball at a point where the eigenfunction ϕ_λ reaches a maximum on Ω_λ (again, without loss of generality ϕ_λ is positive on the nodal domain Ω_λ). However, in higher dimensions, no such localization information is available.

Third, an current question of interest is whether the lower bound is optimal. It is speculated that the lower bound should be close to wavelength. We address this problem in an upcoming work (cf. [GM18a]).

In the rest of this Subsection, we address several problems and results in these directions. To a large extent we follow our work in [Geo16].

6.4.1 Localization of an inscribed ball

We now use our Theorem 6.3.1 in order to prescribe the position of a ball whose size is prescribed by the lower bound of Mangoubi above. More precisely, we derive the following:

Corollary 6.4.1. *Let M be a closed manifold of dimension $n \geq 3$, and $\Omega_\lambda \subseteq M$ be a nodal domain upon which the corresponding eigenfunction high-energy eigenfunction ϕ_λ is positive. Let x_0 be a point of maximum of ϕ_λ on Ω_λ . Then there exists a constant $C = C(M, g)$ such that the ball $B(x_0, \frac{C}{\lambda^{\alpha(n)}})$ with $\alpha(n) = \frac{1}{4}(n-1) + \frac{1}{2n}$ is inscribed in Ω_λ .*

In particular, this recovers the above lower bound of D. Mangoubi (Theorem 1.5 of [Man08a]), with the additional information that the ball of radius $\frac{C}{\lambda^{\alpha(n)}}$ is centered around the max point of the eigenfunction ϕ_λ .

Now to establish Corollary 6.4.1, we first recall the following result, which gives a bound on the asymmetry between the volumes of positivity and negativity sets, as developed in [Man08a]:

Theorem 6.4.1. [Man08a] *Let B be a geodesic ball, so that*

$$\left(\frac{1}{2}B \cap \{\phi_\lambda = 0\}\right) \neq \emptyset \quad (6.40)$$

with $\frac{1}{2}B$ denoting the concentric ball of half radius. Then

$$\frac{\text{Vol}(\{\phi_\lambda > 0\} \cap B)}{\text{Vol}(B)} \geq \frac{C}{\lambda^{\frac{n-1}{2}}}. \quad (6.41)$$

Proof of Corollary 6.4.1. It suffices to combine the estimate (6.38) with (6.41). Let $r := \frac{r_0}{\sqrt{\lambda}}$ be the radius of the largest inscribed ball in the nodal domain at x_0 . Noting that $\{\varphi_\lambda < 0\} \subseteq E_{r_0}$ and combining Theorem 6.4.1 for $B_{x_0}(2r)$ with (6.38) and (6.34), we get:

$$\left(\frac{C}{\lambda^{\frac{n-1}{2}}}\right)^{\frac{n-2}{n}} \leq \left(\frac{\text{Vol}(E_{r_0})}{\text{Vol}(B(x_0, r_0\lambda^{-1/2}))}\right)^{\frac{n-2}{n}} \leq 1 - e^{-\sqrt{1/3}r_0^2} \quad (6.42)$$

Expanding the right hand side in Taylor series and rearranging yields the needed lower bound on r_0 and thus, finishes the proof. \square

We remark than an improvement on asymmetry between positivity and negativity sets will lead to a direct improvement of the lower bound on the inner radius. We discuss this question in the upcoming work [GM18a].

6.4.2 The inner radius of nodal domains in the real analytic setting

We now address the lower bound on the inner radius provided that (M, g) is a real-analytic manifold. We are able to prove the following result:

Theorem 6.4.2. *Let (M, g) be a real-analytic closed manifold of dimension at least 3. Let ϕ_λ be an eigenfunction of the Laplace operator Δ and Ω_λ be a nodal domain of ϕ_λ . Then, there exists $r > 0$ and a wavelength ball $B_{\frac{r}{\sqrt{\lambda}}} \subset M$ of radius $\frac{r}{\sqrt{\lambda}}$ with the following property: An initially given proportion (say, 10%) of $\text{Vol}(\Omega_\lambda \cap B_{\frac{r}{\sqrt{\lambda}}})$ is occupied by a collection of inscribed balls $\{B_{c_1\lambda^{-1}}^i\}_{i=0}^{i_0}$, $B_{c_1\lambda^{-1}}^i \subset \Omega_\lambda \cap B_{\frac{r}{\sqrt{\lambda}}}$ of radius $c_1\lambda^{-1}$, where $c_1 = c_1(M, g)$.*

In particular, there exist constants c_1 and c_2 which depend only on (M, g) , such that

$$\frac{c_1}{\lambda} \leq \text{inrad}(\Omega_\lambda) \leq \frac{c_2}{\sqrt{\lambda}}. \quad (6.43)$$

In particular, Theorem 6.4.2 removes the dependence on the dimension n in the lower bound. Moreover, we remark that the initially given proportion of inscribed balls is referred to as 10% only for the ease of presentation. In fact, one has the freedom to select it - however, the constants r, c_1 will be different. As this is not crucial for our present discussion, we do not pursue the investigation of the precise relation between the constants in the present text.

Further, the present lower bound on the inner radius appears to be unoptimal and it seems that a combinatorial argument can lead to a further improvement. This is also reasonable in the smooth setting, having in mind the recent progress on Yau's conjecture (cf. [Log18a]). We address these in a forthcoming note (cf. [GM18a]).

Outline and strategy

Roughly speaking, the argument in the proof of Theorem 6.4.2 consists of two ingredients.

First, we observe that one can almost inscribe a wavelength ball in the nodal domain up to a small in volume error set via our Theorem 6.3.1. One could also utilize the related result of Lieb ([Lie83]) which states that for arbitrary domains Ω in \mathbb{R}^n one can find almost inscribed balls of radius $\frac{1}{\sqrt{\lambda_1(\Omega)}}$.

Second, one would like to somehow rule out the error set that may enter in the almost inscribed ball near a point of maximum $x_0 \in \Omega_\lambda$. One way to argue is as follows. Being in the real-analytic setting, eigenfunctions resemble polynomials of degree $\sqrt{\lambda}$. This observation was utilized in the works of Donnelly-Fefferman (cf. [DF88]) and Jakobson-Mangoubi (cf. [JM09]). What is more, if one takes the unit cube and subdivide it into wavelength-sized small cubes, then these polynomials will be close to their average on most of the small cubes. This implies that the growth of eigenfunctions is controlled on most wavelength-smaller cubes. Now, roughly speaking, we start from a wavelength cube at x_0 and rescale to the unit cube I^n . Further, we subdivide I^n into wavelength cubes Q^ν , most of which will be good (i.e. of controlled growth) by the real-analytic theory. But, if the error set intersects the majority of Q^ν deeply it will gain large volume, as in good cubes the volumes of positivity and negativity are comparable. This will lead to a contradiction with the volume decay of the first step. This means that there is a sufficient proportion of the Q^ν which is not deeply intersected by the error set.

Existence of an almost inscribed ball

Let ϵ be a small fixed positive number. We recall that via Theorem 6.3.1, or via Corollary 2, [Lie83] and a partition of unity argument, one can find a positive number $r = r(\epsilon)$ and an almost inscribed ball $B_{r/\sqrt{\lambda}}(x_0)$ in the sense that

$$\frac{\text{Vol}(B_{\frac{r}{\sqrt{\lambda}}}(x_0) \cap \Omega_\lambda)}{\text{Vol}(B_{\frac{r}{\sqrt{\lambda}}}(x_0))} > 1 - \epsilon. \quad (6.44)$$

We remark that the existence of such an almost inscribed ball does not use the real-analyticity of (M, g) .

A few technical results concerning "Good" cubes of controlled growth

We consider the case of a real analytic manifold (M, g) of dimension at least 3.

As our present discussion is focused on (M, g) being a real-analytic manifold, let us first attempt to briefly motivate the role of real-analyticity towards eigenfunctions and their nodal geometry.

As the eigenequation possesses real-analytic coefficients, a main insight in this situation is that polynomials approximate eigenfunctions sufficiently well, i.e. an eigenfunction ϕ_λ exhibits a behaviour, which is similar to that of a polynomial of degree $\sqrt{\lambda}$. The analogy exhibits itself when it comes to local growth, vanishing orders at the zero set, etc. We remind again of the celebrated work of Donnelly-Fefferman, [DF88], which addresses Yau's conjecture for nodal sets and is a vivid example of these heuristics (cf. also [JM09]).

On the other hand, if (M, g) is assumed to be only smooth, then formal results mimicking certain real-analytic-case facts (Lemmas 6.4.1, 6.4.3 below, for instance) are still not known. Roughly, the difficulty arises from the lack of good polynomial approximation and appropriate holomorphic extensions.

Now let us start describing the real-analytic tools that we will need: we make use of four auxiliary Lemmas (6.4.1, 6.4.2, 6.4.3 and 6.4.4), which are explicitly stated below. The Lemmas originate from the works [DF88] and [JM09].

First, we have the following

Lemma 6.4.1. *Let (M, g) be real-analytic and let us take a sufficiently small number $r > 0$ (to be determined later), and consider an arbitrary ball $B_{\frac{r}{\sqrt{\lambda}}}$ of radius $\frac{r}{\sqrt{\lambda}}$. Furthermore, rescale the ball $B_{\frac{r}{\sqrt{\lambda}}}$ to the unit ball $B_1 \subset \mathbb{R}^n$ and denote the corresponding rescaled eigenfunction on the unit ball by φ_λ^{loc} . There exists a cube $Q \subseteq B_1$, which does not depend on φ_λ^{loc} and λ , and has the following property: suppose $\delta > 0$ is taken, so that $\delta \leq \frac{C_1}{\sqrt{\lambda}}$. We decompose Q into smaller cubes $\{Q_s^\nu\}_\nu$ with sides of size $s \in (\delta, 2\delta)$. Then, for a small number $\epsilon > 0$, there exists a subset $E_\epsilon \subseteq Q$ of measure $|E_\epsilon| \leq C_2\epsilon\sqrt{\lambda}\delta$, so that*

$$\frac{1}{C_3(\epsilon)} \leq \frac{(\varphi_\lambda^{loc}(x))^2}{Av_{(Q_\nu)_x}(\varphi_\lambda^{loc})^2} \leq C_3(\epsilon), \quad \forall x \in Q \setminus E_\epsilon, \quad (6.45)$$

with $C_3(\epsilon) \rightarrow \infty$ as $\epsilon \rightarrow 0$. The constants C_1, C_2, C_3 do not depend on ϕ_λ and λ . The notation $Av_{(Q_\nu)_x} F$ denotes the average of F over a cube Q_s^ν which contains x .

We first remark that Lemma 6.4.1 is a direct adaptation of Proposition 4.1, [JM09], where instead of working in a wavelength ball $B_{\frac{r}{\sqrt{\lambda}}}$ (identified with B_1 as above), Jakobson-Mangoubi are working on an arbitrary small open set V (again identified with a ball) in which the metric can be expanded in power series. A further remark is that rescaling back to the manifold, the cube Q , which is prescribed by the Lemma, is identified with a small wavelength cube inside $B_{\frac{r}{\sqrt{\lambda}}}$, whose side is comparable to $\frac{r}{\sqrt{\lambda}}$ and the cubes $\{Q_s^\nu\}_\nu$ are identified to even smaller subcubes of size comparable to λ^{-1} .

Now, let us briefly sketch the arguments behind Lemma 6.4.1.

Proof of Lemma 6.4.1. As already stated above we essentially follow Proposition 4.1, [JM09].

First, we observe that $\varphi_\lambda^{loc}|_{B_1}$ has an analytic continuation F on a complex ball $B^{\mathbb{C}}(0, \rho_1)$ (complex balls will be denoted by an upper index \mathbb{C}) for some $\rho_1 < 1$, and moreover the function F is bounded as follows:

$$\sup_{B^{\mathbb{C}}(0, \rho_1)} |F| \leq e^{C\sqrt{\lambda}} \sup_{B_1} |\varphi_\lambda^{loc}|. \quad (6.46)$$

We observe that the size ρ_1 does not depend on λ (Lemma 7.1, [DF88], were one uses the fact that on a wavelength scale φ_λ^{loc} is almost harmonic, i.e. it is a solution to slight perturbation of the standard Laplace equation).

Now, we select a fixed $\rho_2 = \rho_2(\rho_1)$ such that the polydisk $B_{2\rho_2}^n := D_{2\rho_2} \times \cdots \times D_{2\rho_2} \subseteq B^{\mathbb{C}}(0, \rho_1) \subset \mathbb{C}^n$. The well-known Donnelly-Fefferman growth bound (cf. [DF88]) gives that

$$\sup_{B_1} |\varphi_\lambda^{loc}| \leq e^{C\sqrt{\lambda}/\rho_2} \sup_{B(0, \rho_2)} |\varphi_\lambda^{loc}|. \quad (6.47)$$

In particular, we obtain

$$\sup_{B_{2\rho_2}^n} |F| \leq e^{C\sqrt{\lambda}} \sup_{B(0, \rho_2)} |\varphi_\lambda^{loc}|. \quad (6.48)$$

By shifting the coordinate system to a point $x \in B(0, \rho_2)$ such that $\varphi_\lambda^{loc}(x) = \sup_{B(0, \rho_2)} |\varphi_\lambda^{loc}|$, we have

$$\sup_{B_{\rho_2}^n} |F| \leq e^{C\sqrt{\lambda}} |F(0)|. \quad (6.49)$$

We now invoke Proposition 3.7, [JM09], applied to the function F^2 , thus inferring Lemma 6.4.1. \square

We now address the notion of "good" cubes.

Let us take the cube Q prescribed by Lemma 6.4.1 and subdivide it into small cubes Q_s^ν for which the statement of the Lemma holds.

Definition 6.4.1. Q_s^ν is called E_ϵ -good, if

$$\frac{|E_\epsilon \cap Q_s^\nu|}{|Q_s^\nu|} < 10^{-2n} \omega_n, \quad (6.50)$$

where ω_n denotes the volume of the unit ball in \mathbb{R}^n . Otherwise, Q_s^ν is E_ϵ -bad.

It turns out that the E_ϵ -good cubes Q_s^ν are characterized also as places where the eigenfunction possesses controlled growth (cf. also Lemma 5.3, [JM09]). We have

Lemma 6.4.2. Let Q_s^ν be an E_ϵ -good cube. Let $B \subseteq 2B \subseteq Q_s^\nu$ be a ball centered somewhere in $\frac{1}{2}Q_s^\nu$, whose size is comparable to the size of Q_s^ν . Then

$$\frac{\int_{2B} (\varphi_\lambda^{loc})^2}{\int_B (\varphi_\lambda^{loc})^2} \leq \tilde{C}_1 C_3(\epsilon), \quad (6.51)$$

where $C_3(\epsilon)$ comes from Lemma 6.4.1 and \tilde{C}_1 depends only on the dimension n .

Lemma 6.4.3. The proportion of bad cubes to all cubes is smaller than $\tilde{C}_2 |E_\epsilon|$, where \tilde{C}_2 depends only on the dimension.

Finally, let us recall a reason why the good cubes of bounded growth are important from the point of view of nodal geometry. We have

Lemma 6.4.4. *Suppose that a cube Q_s^ν from the collection above is good and suppose that ϕ_λ vanishes somewhere in $\frac{1}{2}Q_s^\nu$ (here $\frac{1}{2}Q$ denotes a concentric cube of half-sized side length). Then assuming that λ is sufficiently large, one has*

$$\frac{\text{Vol}(\{\phi_\lambda > 0\} \cap Q_s^\nu)}{\text{Vol}(Q_s^\nu)} \geq C, \quad (6.52)$$

where C depends on $n, \rho, (M, g)$, as well as the control on the doubling number, that is $\tilde{C}_1 C_3(\epsilon)$ from Lemma 6.4.2 above. The same statement holds for the negativity set.

A proof of the last Lemma 6.4.4 for Q_s^ν replaced by a small ball can be found, for example, in Proposition 1, [CM11]. An adaptation for cubes is yielded by essentially following the same argument and using that at small scales

$$B_{\frac{r}{4}}(p) \subseteq Q_r(p) \subseteq B_{\sqrt{nr}}(p), \quad (6.53)$$

where $Q_r(p), B_r(p)$ denote a cube, resp. a ball, of size r and centered at a point p .

Proof of Theorem 6.4.2

We now put all of the tools above together and prove our main result.

Proof of Theorem 6.4.2. Let us assume without loss of generality that ϕ_λ is positive on Ω_λ .

First, let $\epsilon > 0$ be a sufficiently small number to be determined below and let us find a positive number $r = r(\epsilon)$ and an almost inscribed ball $B_{\frac{r}{\sqrt{\lambda}}}(x_0)$ as outlined at the beginning.

Further, we apply the machinery outlined in Subsection 6.4.2 inside the ball $B_{\frac{r}{\sqrt{\lambda}}}(x_0)$. More precisely, by Lemmas 6.4.1, 6.4.2 and 6.4.3 we can find a cube $Q_{\frac{r_1}{\sqrt{\lambda}}} \subset B_{\frac{r}{\sqrt{\lambda}}}(x_0)$ of comparable side length $\frac{r_1}{\sqrt{\lambda}}$ which, using the above notation, is subdivided into a collection $\mathcal{Q} = \{Q_{c\lambda^{-1}}^\nu\}_\nu$ of cubes of side length $c\lambda^{-1}$. For these we know that there is a subset $\mathcal{Q}^g \subseteq \mathcal{Q}$ of E_ϵ -good cubes that consists of a large proportion (say, at least 90%) of all of the small cubes.

Now, let us define the error set (or "spike") $S := B_{\frac{r}{\sqrt{\lambda}}}(x_0) \setminus \Omega_\lambda$, which by our selection of an almost inscribed ball satisfies (6.44)

$$\frac{\text{Vol}(S)}{\text{Vol}(B_{\frac{r}{\sqrt{\lambda}}}(x_0))} \leq \epsilon. \quad (6.54)$$

Let us also define a subcollection of the good cubes whose inner half is intersected by S , i.e.

$$U := \left\{ Q_{c\lambda^{-1}}^\nu \in \mathcal{Q}^g \mid \frac{1}{2}Q_{c\lambda^{-1}}^\nu \cap S \neq \emptyset \right\}. \quad (6.55)$$

In order to get a contradiction, let us suppose that U occupies a very large proportion of \mathcal{Q}^g . Otherwise, there will be a sufficient proportion of cubes \mathcal{Q}^g/U , which all possess inscribed (in the nodal domain Ω_λ) balls of radius $\frac{c}{\lambda}$ - this implies the claim of Theorem 6.4.2.

Now for each cube $Q_{c\lambda^{-1}}^\nu \in U$ we distinguish two cases:

1. Suppose that in a E_ϵ -good cube $Q_{c\lambda^{-1}}^\nu$ the nodal set does not intersect $\frac{1}{2}Q_{c\lambda^{-1}}^\nu$. This means that $\frac{1}{2}Q_{c\lambda^{-1}}^\nu \subseteq S$, hence

$$\frac{\text{Vol}(S \cap Q_{c\lambda^{-1}}^\nu)}{\text{Vol}(Q_{c\lambda^{-1}}^\nu)} \geq \frac{1}{2^n}. \quad (6.56)$$

2. Suppose that the nodal set intersects $\frac{1}{2}Q_{c\lambda^{-1}}^\nu$. Since $Q_{c\lambda^{-1}}^\nu$ is E_ϵ -good, we can then invoke Lemma 6.4.4 which implies that

$$\frac{\text{Vol}(\{\phi_\lambda < 0\} \cap Q_{c\lambda^{-1}}^\nu)}{\text{Vol}(Q_{c\lambda^{-1}}^\nu)} \geq C. \quad (6.57)$$

By definition $\{\phi_\lambda < 0\} \cap Q_{c\lambda^{-1}}^\nu \subseteq S \cap Q_{c\lambda^{-1}}^\nu$, so we get

$$\frac{\text{Vol}(S \cap Q_{c\lambda^{-1}}^\nu)}{\text{Vol}(Q_{c\lambda^{-1}}^\nu)} \geq C. \quad (6.58)$$

Summing up the two cases over all cubes in U , we see that

$$\frac{\text{Vol}(S \cap Q_{\frac{r_+}{\sqrt{\lambda}}})}{\text{Vol}(Q_{\frac{r_+}{\sqrt{\lambda}}})} \geq C. \quad (6.59)$$

By using the estimate (6.54) and selecting ϵ sufficiently small, we arrive at a contradiction to (6.59). This means that U does not occupy a too large proportion of Q^g . The proof is finished. \square

Let us conclude by mentioning a few remarks.

Remark 6.4.1. *Concerning the location of the wavelength ball prescribed in Theorem 6.4.2, Theorem 6.3.1 specifies the location where a ball of wavelength size can almost be inscribed, as well as the way the error set grows in volume nearby. More precisely, wavelength balls can almost be inscribed at points where ϕ_λ achieves $\|\phi_\lambda\|_{L^\infty(\Omega_\lambda)}$.*

We note that the statement extends also to points x_0 at which the eigenfunction almost reaches its maximum on Ω_λ in the sense, that

$$C\phi_\lambda(x_0) \geq \|\phi_\lambda\|_{L^\infty(\Omega_\lambda)}, \quad (6.60)$$

for some fixed constant $C > 0$. In particular, if there are multiple "almost-maximum" points x_0 , there should be an inscribed ball of radius $\frac{1}{\lambda}$ near each of them.

Remark 6.4.2. *Let us observe that the estimates essentially depend on the growth of ϕ_λ at x_0 . We have used the upper bound $C\sqrt{\lambda}$ on the doubling exponent in the worst possible scenario as shown by Donnelly-Fefferman. It is believed that ϕ_λ rarely exhibits such an extremal growth. If the growth is better, this allows to take larger cubes Q_s^ν and the bound on the inner radius improves. In particular, a constant growth implies the existence of a wavelength inscribed ball.*

6.4.3 A further improvement in the ergodic case

Now, utilizing some recent results of Hezari (cf [Hez16]) we get that, if one assumes in addition to real-analyticity that (M, g) is negatively curved, then the inradius improves by a factor of $\log \lambda$. A similar argument works also for (M, g) with ergodic geodesic flow.

We begin by mentioning some of the recent results of H. Hezari (cf. [Hez16]), addressing quantum ergodic sequences of eigenfunctions. Let us assume that (M, g) is a closed Riemannian manifold with negative sectional curvature. Let (ϕ_{λ_i}) be any orthonormal basis of $L^2(M)$, where (ϕ_{λ_i}) are

eigenfunctions with eigenvalues λ_i . Then, for a given $\epsilon > 0$, there exists a density-one subsequence S_ϵ , so that

$$a_1(\log \lambda_j)^{\frac{(n-1)(n-2)}{4n^2} - \epsilon} \lambda_j^{-\frac{1}{2} - \frac{(n-1)(n-2)}{4n}} \leq \text{inrad}(\Omega_\lambda) \quad (6.61)$$

We refer to [Hez16] for further details.

The heart of Hezari's argument lies in observing that growth exponents (i.e. doubling exponents) improve, provided that eigenfunctions equidistribute at small scales. More precisely, if we assume that for some small $r > \frac{1}{\sqrt{\lambda}}$, we have

$$K_1 r^n \leq \int_{B_r(x)} |\phi_\lambda|^2 \leq K_2 r^n, \quad (6.62)$$

for K_1, K_2 fixed constants and all geodesic balls $B_r(x)$, then

$$\log \left(\frac{\sup_{B_{2s}(x)} |\phi_\lambda|^2}{\sup_{B_s(x)} |\phi_\lambda|^2} \right) \leq Cr\sqrt{\lambda}. \quad (6.63)$$

Here the statement holds for all s smaller than $10r$. In particular, in the negatively curved setting, results of Hezari-Riviere give that r above could be taken as $(\log \lambda)^{-k}$ for any $k \in (0, \frac{1}{2n})$.

We have the following observation.

Corollary 6.4.2. *Let (M, g) be a negatively-curved real-analytic closed manifold of dimension at least 3. Then the collection of inscribed balls from Theorem 1.1 can be taken with radius $\frac{C(\log \lambda)^k}{\lambda}$, where k can be selected as any number in $(0, \frac{1}{2n})$. In particular,*

$$\text{inrad}(\Omega_\lambda) \geq \frac{C(\log \lambda)^k}{\lambda}. \quad (6.64)$$

Proof. We note the improvement by a factor of r of Hezari's growth bound (6.63) over the Donnelly-Fefferman growth estimate (6.47), which holds for all wavelength and smaller balls. The discussion after Lemma 6.4.1 indicates that ϕ_λ admits a holomorphic continuation with improved growth control. Hence Lemma 6.4.1 holds with $\delta \leq \frac{C_1}{\sqrt{\lambda}r}$, so while going through the arguments above we can actually take collections $\{Q_s^\nu\}_\nu$ consisting of cubes, whose side length is larger by a factor of $\frac{1}{r}$.

As remarked above r could be taken as $(\log \lambda)^{-k}$ for any $k \in (0, \frac{1}{2n})$. \square

6.4.4 Distribution of good cubes and inner radius bounds

We now investigate the effect of the moderate growth of ϕ_λ on a nodal domain's inner radius. Roughly speaking, we show that if most of the L^2 -norm of ϕ_λ over the nodal domain Ω_λ is contained in good cubes of controlled growth, then this yields a large lower bound on the inner radius of Ω_λ . We do not assume real-analyticity in this discussion.

We exploit a covering by good/bad cubes, inspired by [CM11] and [JM09]. Let us fix a finite atlas (U_i, ϕ_i) of M , such that the transition maps are bounded in C^1 -norm and the metric on each chart domain U_i is comparable to the Euclidean metric in \mathbb{R}^n :

$$\frac{1}{4}e_i \leq g \leq 4e_i, \quad (6.65)$$

where we have denoted $e_i := \phi_i^* e$ with e being the standard Euclidean metric.

We can arrange that M is covered by cubes $K_i \subseteq U_i$, where we decompose K_i into small cubes K_{ij} of size h (to be determined later). Throughout we will denote by δK_{ij} the concentric cube, scaled by some fixed scaling factor $\delta > 1$.

Definition 6.4.2. *Let $\gamma > 0$ be fixed. A cube K_{ij} is called γ -good, if*

$$\int_{\delta K_{ij}} \phi_\lambda^2 \leq \gamma \int_{K_{ij}} \phi_\lambda^2. \quad (6.66)$$

Otherwise, we say that K_{ij} is γ -bad. We also denote by Γ the union of all good cubes (i.e. the good set) and $\Xi := M \setminus \Gamma$.

As we shall see once more below, the motivation behind the "goodness" condition is that the nodal geometry within a good cube is well-behaved.

We have the following

Theorem 6.4.3. *Let (M, g) be a smooth closed Riemannian manifold of dimension at least 3. Let Ω_λ be a fixed nodal domain of the eigenfunction ϕ_λ .*

Then

$$\text{inrad}(\Omega_\lambda) \geq C \gamma^{\frac{2-n}{n}} \tau^{\frac{1}{2}} \lambda^{-\frac{1}{2}}, \quad (6.67)$$

where $\tau := \int_{\Omega_\lambda \cap \Gamma} \phi_\lambda^2 / \int_{\Omega_\lambda} \phi_\lambda^2$ and $C = C(M, g)$.

Remark 6.4.3. *Here*

$$c(\tau) := C \left[\sqrt{\frac{\tau}{\kappa_\delta}} \left(\frac{1}{\gamma} \right)^{\frac{n-2}{n}} \right]. \quad (6.68)$$

where $C = C(M, g)$ and γ, κ_δ are parameters of the good/bad cube covering.

Roughly, Theorem 6.4.3 implies that, if the bulk of the L^2 norm over the nodal domain is contained in good cubes, then the nodal domain possesses large inner radius.

In Section 6.4.4 we deduce the following corollaries:

Corollary 6.4.3. *Let (M, g) be a smooth closed Riemannian manifold of dimension at least 3. For a nodal domain Ω_λ of ϕ_λ , one has*

$$\text{inrad}(\Omega_\lambda) \geq C \frac{\|\phi_\lambda\|_{L^2(\Omega_\lambda)}^{\frac{2(n-2)}{n}}}{\sqrt{\lambda}}, \quad (6.69)$$

with $C = C(M, g)$.

Note that the inequality in Corollary 6.4.3 is useful only in dimensions 3 and 4, as an application of the standard Hölder inequality gives:

$$\text{inrad}(\Omega_\lambda) \geq C \frac{\|\phi_\lambda\|_{L^2(\Omega_\lambda)}}{\sqrt{\lambda}}, \quad (6.70)$$

which is sharper in higher dimensions.

Moreover, we note that as a by-product we obtain

Corollary 6.4.4. *Let (M, g) be a smooth closed Riemannian manifold of dimension at least 3. There exists a nodal domain of ϕ_λ , denoted by $\Omega_{\varphi_\lambda}^*$, such that*

$$\text{inrad}(\Omega_{\varphi_\lambda}^*) \asymp \frac{1}{\sqrt{\lambda}},$$

In other words, there exist constants C_1, C_2 , depending on (M, g) , such that

$$\frac{C_1}{\sqrt{\lambda}} \leq \text{inrad}(\Omega_{\varphi_\lambda}^*) \leq \frac{C_2}{\sqrt{\lambda}}$$

for λ large enough.

As communicated by Dan Mangoubi, Corollary 6.4.4 also follows also by looking at a point, where ϕ_λ achieves its maximum over M , and further using standard elliptic estimates. Indeed, by rescaling we may assume that $\phi_\lambda(x_0) = \|\phi_\lambda\|_{L^\infty(M)} = 1$. Elliptic estimates then imply that $\|\nabla\phi_\lambda\|_{L^\infty(M)} \leq C\sqrt{\lambda}$ which shows that there is a wavelength inscribed ball at x_0 .

We note that the scaling factor $\delta > 0$ also enters in the constants above - however, in our discussion it is fixed and later explicitly set as $\delta := 16\sqrt{n}$ for technical reasons.

The plan for the rest of this note goes as follows.

In Section 6.4.4 we recall the essential steps behind the lower bound in (6.39). Roughly speaking, one cuts a nodal domain Ω_λ into small cubes of size $\text{inrad}(\Omega_\lambda)$. Then, among these, one finds a special cube $K_{i_0j_0}$, over which the Rayleigh quotient is carefully estimated. Here an essential role is played by how one compares the volumes of $\{\phi_\lambda > 0\}$ and $\{\phi_\lambda < 0\}$ in the special cube (also known as asymmetry estimates).

The motivation behind Theorem 6.4.3 comes from the question, whether in the special cube $K_{i_0j_0}$ one has $\text{Vol}(\{\phi_\lambda > 0\}) \sim \text{Vol}(\{\phi_\lambda < 0\})$. Having this would imply that the inner radius is comparable to the wavelength $\frac{1}{\sqrt{\lambda}}$, i.e. the optimal asymptotic bound.

We also introduce a covering by small good and bad cubes, which arises when one investigates the way the local L^2 -norm of an eigenfunction ϕ_λ grows (w.r.t. the domain of integration). Good cubes represent places of controlled L^2 -norm growth. The motivation behind this consideration is the fact that the volumes of the positivity and negativity set of ϕ_λ in a good cube are comparable, i.e. their ratio is bounded by constants.

In Section 6.4.4 we show the statement of Theorem 6.4.3 and its Corollaries 6.4.4 and 6.4.3.

We end the discussion by making some further comments in Section 6.4.4.

Some preliminary notation

We begin by collecting some notions such as local Rayleigh quotients.

Let us consider an eigenfunction ϕ_λ and an associated nodal domain $\Omega_\lambda \subseteq M$.

We denote by ψ the restriction of ϕ_λ to Ω_λ , extended by 0 to M . Then ψ realizes the first Dirichlet eigenvalue of Ω_λ , i.e.

$$\frac{\int_{\Omega_\lambda} |\nabla\psi|^2 d\text{Vol}}{\int_{\Omega_\lambda} |\psi|^2 d\text{Vol}} = \lambda_1(\Omega_\lambda) = \lambda. \quad (6.71)$$

We may assume that $\phi_i(U_i)$ is a cube K_i , whose edges are parallel to the coordinate axes, and we can further cut it into small non-overlapping small cubes $K_{ij} \subseteq K_i$ of appropriately selected side length h , comparable to $\text{inrad}(\Omega_\lambda)$.

Having this construction in mind, we define *local Rayleigh quotients*:

Definition 6.4.3. A local Rayleigh, associated to the eigenfunction ψ and decomposition $\{K_{ij}\}$ as above, is the quantity

$$R_{ij}(\psi) := \frac{\int_{\phi_i^{-1}(K_{ij})} |\nabla\psi|^2 d\text{Vol}}{\int_{\phi_i^{-1}(K_{ij})} |\psi|^2 d\text{Vol}}. \quad (6.72)$$

Good cubes and bad cubes

Let us fix an eigenfunction ϕ_λ of Δ with eigenvalue λ and $\|\phi_\lambda\|_{L^2(M)} = 1$.

As above we consider the finite atlas $\{(U_i, \phi_i)\}$ of M , and arrange that M is covered by cubes $K_i \subseteq U_i$, where K_i is decomposed into small cubes K_{ij} of size h (to be determined later). Again denoting by δK_{ij} the concentric cube, scaled by some fixed scaling factor $\delta > 1$, we may also assume that $\delta K_{ij} \subseteq U_i$.

We note that the metric g is comparable to the Euclidean one on each cube K_i and, moreover, each point $x \in M$ is contained in at most κ_δ of the concentric cubes δK_{ij} , where κ_δ is some constant, not depending on the chosen cube size h .

In the light of Definition 6.4.2, we have that the covering K_{ij} is divided into good and bad cubes.

First, we note that the covering is robust, in the sense that the good cubes can be arranged to consume most of the L^2 norm. Essentially, by using the definition one can show:

Lemma 6.4.5. *We have*

$$\int_{\Gamma} \phi_\lambda^2 \geq 1 - \frac{\kappa_\delta}{\gamma}. \quad (6.73)$$

Proof. Using $\|\phi_\lambda\|_{L^2} = 1$, we have

$$\int_{\Gamma} \phi_\lambda^2 \geq 1 - \int_{\Xi} \phi_\lambda^2 \geq 1 - \sum_{K_{ij}\text{-bad}} \int_{K_{ij}} \phi_\lambda^2 \geq 1 - \sum_{K_{ij}\text{-bad}} \frac{1}{\gamma} \int_{\delta K_{ij}} \phi_\lambda^2 \quad (6.74)$$

$$\geq 1 - \frac{\kappa_\delta}{\gamma} \int_M \phi_\lambda^2 = 1 - \frac{\kappa_\delta}{\gamma}. \quad (6.75)$$

$$(6.76)$$

□

Again, we note that without any dependence on λ or the size of the small cubes h one is able to control how big (in L^2 sense) the good set Γ is.

Proof of Theorem 6.4.3

We now show how the portion of the L^2 norm over a nodal domain, occupied by good cubes, gives a lower bound on the inner radius. Roughly, having a lot of good cubes over a nodal domain increases the chance that $K_{i_0 j_0}$ is a good one. We find a special small cube and estimate the corresponding Rayleigh quotient in an appropriate way. The technique is in the spirit of [Man08a]. However, we will have the advantage that the special cube is also good, which would lead to optimal asymmetry.

Proof of Theorem 6.4.3.

Claim 6.4.1. *There exists a good cube $K_{i_0j_0}$, such that*

$$R_{(\delta K_{i_0j_0})}(\psi) \leq \frac{\kappa_\delta}{\tau} \lambda_1(\Omega_\lambda), \quad (6.77)$$

where ψ is defined as in Section 6.4.4 and $R_{(\delta K_{i_0j_0})}(\psi)$ denotes the local Rayleigh quotient w.r.t. the cube $\delta K_{i_0j_0}$. As above, κ_δ denotes the maximal number of cubes δK_{ij} that can intersect at a given point.

Proof. First, let us denote by $\delta\Gamma$ the union of all good cubes scaled by a factor of $\delta > 1$. Assuming the contrary, we get:

$$\int_{\Omega_\lambda} |\nabla\psi|^2 \geq \int_{\Omega_\lambda \cap \delta\Gamma} |\nabla\psi|^2 \geq \frac{1}{\kappa_\delta} \sum_{K_{ij}\text{-good}} \int_{\Omega_\lambda \cap \delta K_{ij}} |\nabla\psi|^2 \quad (6.78)$$

$$> \frac{1}{\tau} \lambda_1(\Omega_\lambda) \sum_{K_{ij}\text{-good}} \int_{\Omega_\lambda \cap \delta K_{ij}} |\psi|^2 \geq \frac{1}{\tau} \lambda_1(\Omega_\lambda) \int_{\Gamma} |\psi|^2 \quad (6.79)$$

$$\geq \lambda_1(\Omega_\lambda) \int_{\Omega_\lambda} |\psi|^2. \quad (6.80)$$

$$(6.81)$$

Hence, a contradiction with the definition of ψ . \square

Thus we obtain a bound on the local Rayleigh quotient over the cube $\delta K_{i_0j_0}$. However, we have the advantage that $K_{i_0j_0}$ is γ -good - this implies that the asymmetry and the geometry of the nodal set is under control.

From now on let us fix $\delta := 16\sqrt{n}$. The following proposition is similar to Proposition 1 in [CM11] and Proposition 5.4 in [JM09]. We supply the technical details for completeness:

Proposition 6.4.1. *Let $\gamma, \rho > 1$ be given. Then there exists $\Lambda > 0$, such that, if one takes the cube size $r \leq \frac{\rho}{\sqrt{\lambda}}$ for $\lambda \geq \Lambda$ and assumes that ϕ_λ vanishes somewhere in $\frac{1}{2}K_{i_0j_0}$, then*

$$\frac{\text{Vol}(\{\phi_\lambda > 0\} \cap \delta K_{i_0j_0})}{\text{Vol}(\delta K_{i_0j_0})} \geq \frac{C}{\gamma^2}, \quad (6.82)$$

where C depends on $n, \rho, (M, g)$.

The same holds for the negativity set. Hence the asymmetry of Ω_λ in $\delta K_{i_0j_0}$ is bounded below by the constant $C/\gamma^2 > 0$, which essentially depends on the good/bad growth condition and not on λ .

Proof. (of Proposition) We denote by $K_r(p)$ the cube of edge size r centered at p , whose edges are parallel to the coordinate axes. We also denote by $B_r(p)$ a metric ball (w.r.t the metric g) of radius r centered at p . Let us assume that $K_r(p) := K_{i_0j_0}$. By the metric comparability (6.65), we have:

$$B_{\frac{r}{4}} \subseteq K_r(p) \subseteq B_{\sqrt{n}r}(p). \quad (6.83)$$

Recall the following generalization of the mean value principle (Lemma 5, [?]):

Lemma 6.4.6. *There exists $R = R(M, g) > 0$, such that if $r \leq R$ and $\phi_\lambda(p) = 0$, then*

$$\left| \int_{B_r(p)} \phi_\lambda \right| \leq \frac{1}{3} \int_{B_r(p)} |\phi_\lambda|. \quad (6.84)$$

By assumption, there exists a point $q \in \frac{1}{2}K_{i_0j_0}$, $\phi_\lambda(q) = 0$, so the lemma, in combination with metric comparability, implies that

$$\int_{K_r(q)} |\phi_\lambda| \leq \int_{B_{\sqrt{n}r}(q)} |\phi_\lambda| \leq 3 \int_{B_{\sqrt{n}r}(q)} \phi_\lambda^+ \leq 3 \int_{K_{4\sqrt{n}r}(q)} \phi_\lambda^+, \quad (6.85)$$

where $\phi_\lambda^+, \phi_\lambda^-$ respectively denote the positive and negative part of ϕ_λ .

Hence,

$$\frac{1}{9} \left(\int_{K_{2r}(q)} |\phi_\lambda| \right)^2 \leq \left(\int_{K_{8\sqrt{n}r}(q)} \phi_\lambda^+ \right)^2 \quad (6.86)$$

$$\leq \text{Vol}(K_{8\sqrt{n}r}(q) \cap \{\phi_\lambda > 0\}) \int_{K_{16\sqrt{n}r}(q)} \phi_\lambda^2, \quad (6.87)$$

$$(6.88)$$

where we have used the Cauchy-Schwartz inequality.

We estimate further the integral from the last expression:

$$\left(\int_{K_{16\sqrt{n}r}(q)} \phi_\lambda^2 \right)^2 \leq \gamma^2 \left(\int_{K_r(p)} \phi_\lambda^2 \right)^2 \quad (6.89)$$

$$\leq \gamma^2 \left(\int_{K_{2r}(q)} |\phi_\lambda| |\phi_\lambda| \right)^2 \leq \gamma^2 \sup_{K_{2r}(q)} \phi_\lambda^2 \left(\int_{K_{2r}(q)} |\phi_\lambda| \right)^2. \quad (6.90)$$

Note that, since r is comparable to wavelength, elliptic estimates imply:

$$\sup_{K_{2r}(q)} \phi_\lambda^2 \leq \sup_{B_{2\sqrt{n}r}(q)} \phi_\lambda^2 \leq C_0 r^{-n} \int_{B_{4\sqrt{n}r}(q)} \phi_\lambda^2 \leq C_0 r^{-n} \int_{K_{16\sqrt{n}r}(q)} \phi_\lambda^2, \quad (6.91)$$

where $C_0 = C_0(M, g, \rho, n)$.

Plugging (6.91) into (6.89), one gets

$$\int_{K_{16\sqrt{n}r}(q)} \phi_\lambda^2 \leq C_0 \gamma^2 r^{-n} \left(\int_{K_{2r}(q)} |\phi_\lambda| \right)^2 \quad (6.92)$$

and in combination with (6.86) this yields

$$\frac{r^n}{9C_0\gamma^2} \leq \text{Vol}(\{\phi_\lambda > 0\} \cap K_{8\sqrt{n}r}(q)) \leq \text{Vol}(\{\phi_\lambda > 0\} \cap \delta K_{i_0j_0}). \quad (6.93)$$

The statement of the proposition follows after dividing by $\text{Vol}(\delta K_{i_0j_0}) \leq C_1 r^n$. \square

Remark 6.4.4. *One may also exhibit a version of Lemma 6.4.6 for cubes, thus making some of the constants better. However, using balls and comparability as above suffices for our purposes.*

To finish the proof of the main statement, we put together the latter observations.

Again, we consider an atlas and cube decomposition as above. Similarly to [Man08b], we fix the size of the small cube-grid

$$h := 8 \max_i r_i, \quad (6.94)$$

where r_i denotes the inner radius of Ω_λ in the chart (U_i, ϕ_i) with respect to the Euclidean metric.

We consider the cube $K_{i_0 j_0}$, prescribed by Claim 6.4.1. The choice of h ensures that $\phi_\lambda(q) = 0$ for some $q \in \frac{1}{2}K_{i_0 j_0}$. Then the conditions of Proposition 6.4.1 are satisfied. This means that

$$\text{Vol}(\{\psi = 0\} \cap \delta K_{i_0 j_0}) \geq \text{Vol}(\{\phi_\lambda > 0\} \cap \delta K_{i_0 j_0}) \geq \frac{C}{\gamma^2} h^n. \quad (6.95)$$

Now, via a suitable Poincaré and capacity estimate one is able to bound $\text{Vol}(\{\psi = 0\} \cap \delta K_{i_0 j_0})$ from above in terms of the local Rayleigh quotient $R_{i_0 j_0}$ as in [Man08a]. Combining this with (6.77) we get

$$C \frac{1}{h^2} \left(\frac{1}{\gamma^2} \right)^{\frac{n-2}{n}} \leq \frac{\kappa_\delta}{\tau} \lambda_1(\Omega_\lambda). \quad (6.96)$$

A rearrangement gives

$$h \geq C \left[\sqrt{\frac{\tau}{\kappa_\delta}} \left(\frac{1}{\gamma} \right)^{\frac{n-2}{n}} \right] \frac{1}{\sqrt{\lambda}}. \quad (6.97)$$

The proof finishes by recalling that $h \leq 8 \text{inrad}(\Omega_\lambda)$ by assumption. \square

Some further comments and corollaries

Let us briefly explain the Corollaries 6.4.4 and 6.4.3.

Proof of Corollary 6.4.4. Let us fix $\gamma := 4\kappa_\delta$. A simple summation argument, yields

Claim 6.4.2. *There exists a nodal domain $\Omega_{\varphi_\lambda}^*$, such that*

$$\int_{\Gamma \cap \Omega_{\varphi_\lambda}^*} \phi_\lambda^2 \geq 3 \int_{\Xi \cap \Omega_{\varphi_\lambda}^*} \phi_\lambda^2. \quad (6.98)$$

In particular,

$$\int_{\Gamma} (\psi^*)^2 \geq 3 \int_{\Xi} (\psi^*)^2, \quad (6.99)$$

where ψ^* is the function, which realizes $\lambda_1(\Omega_{\varphi_\lambda}^*)$, extended by zero outside $\Omega_{\varphi_\lambda}^*$.

Indeed, assuming the contrary and summing over all nodal domains, one gets a contradiction with Lemma 6.4.5 and the fact that $\|\phi_\lambda\|_{L^2} = 1$.

Now, apply Theorem 6.4.3 with $\Omega_{\varphi_\lambda}^*$. \square

We now prove the energy inequality. The idea is just to tailor γ along Ω_λ .

Proof of Corollary 6.4.3. In the light of Lemma 6.4.5, we just take

$$\gamma := \frac{4\kappa_\delta}{\|\phi_\lambda\|_{L^2(\Omega_\lambda)}^2}, \quad (6.100)$$

thus having

$$\int_\Gamma \phi_\lambda^2 \geq 1 - \frac{\|\phi_\lambda\|_{L^2(\Omega_\lambda)}^2}{4}. \quad (6.101)$$

This ensures that Ω_λ satisfies the condition of Theorem 6.4.3 with $\tau = 1/4$ and the prescribed γ . So, the claim follows from Theorem 6.4.3. \square

In particular,

$$\left(\text{inrad}(\Omega_\lambda)\sqrt{\lambda}\right)^{\frac{2n}{n-2}} \geq C\|\phi_\lambda\|_{L^2(\Omega_\lambda)} \quad (6.102)$$

and summing over all nodal domains yields

$$\sum_{\Omega_\lambda} \text{inrad}(\Omega_\lambda)^{\frac{2n}{n-2}} \geq \frac{C}{\lambda^{\frac{n}{2n-4}}}, \quad (6.103)$$

with the constant C being better than the constant C_1 , appearing in Theorem 6.4.3. This allows one to obtain an estimate on the generalized mean with exponent $\frac{n}{n-2}$ of all the inner radii corresponding to different nodal domains.

Note that the main obstruction against the application of Theorem 6.4.3 is the fact that one needs to know that the L^2 -norm of ϕ_λ over Ω_λ is mainly contained in good cubes and this should be uniform w.r.t. λ (or at least conveniently controlled).

As further questions one might ask whether a dissipation of the bad cubes is to be expected in some special cases (e.g. the case of ergodic geodesic flow) - that is, is it true that a nodal domain should have a well-distributed L^2 norm in the sense of Theorem 6.4.3?

A relaxed version of this question is, of course, a probabilistic statement of the kind - a significant amount of nodal domains should enjoy the property of having well-distributed L^2 norm.

6.4.5 The effect of sub-exponential growth

In this Subsection we discuss the following question: suppose that at a point x_0 a Laplacian eigenfunction φ_λ is positive and sufficiently large. How large a ball can one find at x_0 such that φ_λ is positive on this ball (i.e. the ball is inscribed inside the nodal domain containing x_0)?

Having this discussion in mind, we also recall the following observation:

Claim 6.4.3. *If for a point $x_0 \in M$ we know that $|\varphi_\lambda(x_0)| \geq \beta\|\varphi_\lambda\|_{L^\infty(M)}$, where β is a constant independent of λ , then there exists a ball of radius $\sim 1/\sqrt{\lambda}$ centered at x_0 where φ_λ does not change sign.*

In other words, there exists a fully inscribed ball of wavelength size centered at x_0 . This claim follows directly from elliptic bounds on the gradient $|\nabla\varphi_\lambda|$ (which with appropriate normalization is bounded by $C\sqrt{\lambda}$ where C is a constant depending on (M, g)).

We seek quantitative generalizations of this fact under more relaxed lower bounds on $|\varphi_\lambda(x_0)|$. Theorem 6.4.4 below may be seen as one such quantitative generalization.

First, it is natural to ask how small the local *sup*-norm of an eigenfunction can become. This was addressed by Donnelly-Fefferman ([DF88]), who showed that on any wavelength radius geodesic ball $B(x, \frac{1}{\sqrt{\lambda}})$ in a closed Riemannian manifold M with smooth metric, we have that

$$\sup_{B(x, \frac{1}{\sqrt{\lambda}})} |\varphi_\lambda| \gtrsim e^{-C\sqrt{\lambda}} \|\varphi_\lambda\|_{L^\infty(M)}. \quad (6.104)$$

Such exponential bounds can occur (as certain examples on the sphere suggest - Gaussian beams of highest weight spherical harmonics). However, these exponential bounds are considered to be a rare phenomenon and pertinent to spaces with an abundance of symmetries such as the sphere. In general, the growth of an eigenfunction should average to a constant (cf. [NPS05]), which suggests that much better bounds should hold. Motivated by this, we investigate bounds on the size of inscribed balls which are centered at points x_0 for which $|\varphi_\lambda(x_0)|$ is at most “exponentially” small.

We have the following observation:

Theorem 6.4.4. *Let M be a closed Riemannian manifold of dimension $n \geq 3$ with smooth metric and let φ_λ be a high-energy eigenfunction. Further, let $x_0 \in \Omega_\lambda$ be such that $\varphi_\lambda(x_0) = \|\varphi\|_{L^\infty(\Omega_\lambda)}$. Suppose that*

$$\varphi_\lambda(x_0) \geq 2^{-1/\eta} \|\varphi_\lambda\|_{L^\infty(M)}, \quad (6.105)$$

where $\eta > 0$ is smaller than a fixed constant η_0 . Then there exists an inscribed ball $B(x_0, \rho) \subseteq \Omega_\lambda$ of radius

$$\rho \gtrsim \max\left(\frac{\eta^{\beta(n)}}{\sqrt{\lambda}}, \frac{1}{\lambda^{\alpha(n)}}\right), \quad (6.106)$$

where $\beta(n) = \frac{(n-1)(n-2)}{2n}$, $\alpha(n) = \frac{n-1}{4} + \frac{1}{2n}$. Furthermore, such a ball can be centered around any such *max* point x_0 .

In particular, Theorem 6.4.4 implies the following remark (cf. Claim 6.4.3):

Corollary 6.4.5. *If for x_0 as above, one has that $|\varphi_\lambda(x_0)| \gtrsim e^{-\lambda^\mu} \|\varphi_\lambda\|_{L^\infty(M)}$, where $\mu := 2n\nu/((n-1)(n-2))$, $\nu > 0$, then there exists a ball of radius $\sim \frac{1}{\lambda^{1/2+\nu}}$ centered at x_0 where φ_λ does not change sign.*

The proof of Theorem 6.4.4 is based on a combination of a rapid growth in narrow domains (Theorem 6.4.6), and the existence of an almost inscribed ball (Theorem 6.3.1). We start again by collecting some auxiliary results that we need for the proof of Theorem 6.4.4.

Local elliptic maximum principle

We quote the following local maximum principle, which appears as Theorem 9.20 in [GT01].

Theorem 6.4.5. *Suppose $Lu \leq 0$ on B_1 . Then*

$$\sup_{B(y, r_1)} u \leq C(r_1/r_2, p) \left(\frac{1}{\text{Vol}(B(y, r_2))} \int_{B(y, r_2)} (u^+(x))^p dx \right)^{1/p}, \quad (6.107)$$

for all $p > 0$, whenever $0 < r_1 < r_2$ and $B(y, r_2) \subseteq B_1$.

Local asymmetry of nodal domains

We also once again recall the asymmetry result of Theorem 6.4.1 which yields that

$$\frac{|\{\varphi_\lambda > 0\} \cap B|}{|B|} \gtrsim \frac{1}{\lambda^{\frac{n-1}{2}}}, \quad (6.108)$$

whenever the nodal set intersects inner half of the ball B .

Rapid growth in narrow domains

Heuristically, this means that if φ solves (4.4), and has a deep and narrow positivity component, then φ grows rapidly in the said component. In our discussion, we use an iterated version of this principle, which appears as Theorem 3.2 in [Man08a] (cf. also the references therein). Let φ_λ satisfy (4.4) on the rescaled ball B_1 .

Theorem 6.4.6. *Let $0 < r' < 1/2$. Let Ω be a connected component of $\{\varphi > 0\}$ which intersects $B_{r'}$. Let $\eta > 0$ be small. If $|\Omega \cap B_r|/|B_r| \leq \eta^{n-1}$ for all $r' < r < 1$, then*

$$\frac{\sup_{\Omega} \varphi}{\sup_{\Omega \cap B_{r'}} \varphi} \geq \left(\frac{1}{r'}\right)^{C'/\eta},$$

where C' is a constant depending only on the metric (M, g) .

Idea of proof of Theorem 6.4.4

Before going into the details of the proof, let us first outline the main ideas. Let us define $B := B(x_0, \frac{r_0}{\sqrt{\lambda}})$ where x_0 is a point of maximum as stated in Theorem 6.4.4 and $r_0 > 0$ is a sufficiently small number. Also recall that $\varphi_\lambda(x_0)$ is assumed to be bounded below in terms of η .

Now, roughly speaking, we will see that if r_0 is sufficiently small in terms of η , then φ_λ does not vanish in $\frac{1}{4}B$, a concentric ball of quarter radius. This will imply the claim of the Theorem.

To this end, we argue by contradiction (i.e. we assume that φ_λ vanishes in $\frac{1}{4}B$) and follow the three steps below:

1. First, Theorem 6.3.1 above tells us that we can “almost” inscribe a ball $B = B(x_0, \frac{r_0}{\sqrt{\lambda}})$ inside Ω_λ , up to the error of certain “spikes” of total volume $\epsilon_0|B|$ entering the ball, where according to Remark 6.3.2 ϵ_0 and r_0 are related by

$$r_0 = C\epsilon_0^{\frac{n-2}{2n}}, \quad (6.109)$$

In particular, if we assume w.l.o.g. that φ_λ is positive on Ω_λ , then the volume $|\{\varphi_\lambda < 0\} \cap B|$ is relatively small and does not exceed $\epsilon_0|B|$.

2. The second step consists in showing that the sup norms of φ^- and φ^+ in the spikes are comparable. More formally, observe that on each connected component of $\frac{1}{4}B \setminus \Omega_\lambda$ (i.e., on each spike in $\frac{1}{4}B$), φ_λ can be positive or negative a priori. However, by a straightforward argument involving the mean value property of harmonic functions and standard elliptic maximum principles, we show that on $\frac{1}{4}B \setminus \Omega_\lambda$, $\sup \varphi_\lambda^- \lesssim \sup \varphi_\lambda^+$.

3. For the third step, we begin by noting that if we can show that the doubling exponent of φ_λ in $\frac{1}{8}B$ is bounded above by a constant, then the asymmetry estimate (Proposition 6.4.1) will give us that the set $\{\varphi_\lambda < 0\} \cap \frac{1}{4}B$ has a large volume, which contradicts Step 1 above. This will be a contradiction to our assumption that φ_λ vanishes somewhere in $\frac{1}{4}B$, and thus we finally conclude that $\frac{1}{4}B$ is fully inscribed inside Ω_λ .

Now, the assumed bounded doubling exponent will be ensured, if $\varphi_\lambda(x_0)$ controls (up to a constant) all the values of φ_λ inside $\frac{1}{4}B$. Using the input from Step 2 above as well as the a priori assumption on $\varphi_\lambda(x_0)$, it suffices to ensure that φ^+ is suitably bounded. This is where we bring in the rapid growth in narrow domains result (Theorem 6.4.6).

Proof of Theorem 6.4.4

Proof. Step 1: An almost inscribed ball:

As before, let x_0 denote the max point of φ_λ in the nodal domain Ω_λ . Let us assume the sup-norm bound (6.105) and let us set $B := B(x_0, \frac{r_0}{\sqrt{\lambda}})$ be a ball centered at x_0 and of radius $\frac{r_0}{\sqrt{\lambda}}$, where $r_0 > 0$ is sufficiently small and determined below. Further, let us denote the non-inscribed “error-set” by $X := B \setminus \Omega_\lambda$.

We start by making the following choice of parameters: we select $0 < \epsilon_0 \leq (\eta C')^{n-1}$ with a corresponding $r_0 := C\epsilon_0^{\frac{n-2}{2n}}$ (prescribed by (6.109)), where C' is the constant coming from Theorem 6.4.6; moreover we assume that $0 < \eta \leq \eta_0$ for some fixed small positive number η_0 , so that by Theorem 6.3.1, the relative volume of the “error” set X is sufficiently small, i.e.

$$\frac{|X \cap \frac{1}{4}B|}{|\frac{1}{4}B|} \lesssim \frac{4^n |X \cap B|}{|B|} \leq 4^n \epsilon_0 =: C_2, \quad (6.110)$$

where $C_2 > 0$ is appropriately chosen below. Indeed, the condition (6.105) still insures the application of Theorem 6.3.1 via an inspection of the proof - the only difference is the application of the Feynman-Kac formula, where one introduces the coefficient $2^{-\frac{1}{\eta}}$ on the left hand side.

We now claim that in fact φ_λ does not vanish in $\frac{1}{4}B$, the concentric ball of a quarter radius.

To prove this, we will argue by contradiction - that is, let us suppose that φ_λ vanishes somewhere in $\frac{1}{4}B$.

Step 2: Comparability of φ_λ^+ and φ_λ^- :

By assuming the contrary, let x be a point in $\overline{X} \cap \frac{1}{4}B$ lying on the boundary of a spike, that is, $\varphi_\lambda(x) = 0$. Consider a ball B' around x with radius $\frac{r_0}{2\sqrt{\lambda}}$. Since $\varphi_\lambda(x) = 0$, we have that (up to constants depending on (M, g)),

$$\frac{1}{|B'|} \int_{B'} \varphi_\lambda^- \sim \frac{1}{|B'|} \int_{B'} \varphi_\lambda^+. \quad (6.111)$$

This follows from mean value properties of harmonic functions; for a detailed proof, see Lemma 5 of [?].

Now, let B'' be a ball slightly smaller than and fully contained in B' . Using the local maximum principle (6.107), we have that (up to constants depending on (M, g)),

$$\sup_{B''} \varphi_\lambda^- \lesssim \frac{1}{|B'|} \int_{B'} \varphi_\lambda^- \lesssim \frac{1}{|B'|} \int_{B'} \varphi_\lambda^+ \leq \sup_{B'} \varphi_\lambda^+. \quad (6.112)$$

This shows that in order to bound φ_λ^- , it suffices to bound φ_λ^+ . This finishes Step (2).

Step 3: Controlled doubling exponent and conclusion:

Our aim is to be able to bound $\sup_{\frac{1}{4}B} \varphi_\lambda^+$ in terms of $\varphi_\lambda(x_0)$, as that would give us control of the doubling exponent of φ_λ on $\frac{1}{8}B$. In other words, we wish to establish that

$$\sup_{\frac{1}{4}B} \varphi_\lambda^+ \leq C\varphi_\lambda(x_0), \quad (6.113)$$

where C is a constant independent of λ .

If $X \cap \frac{1}{2}B \cap \{\varphi > 0\} = \emptyset$, then (6.113) follows immediately by definition. Otherwise, calling $X' := X \cap \frac{1}{2}B$, let Ω'_λ represent another nodal domain on which φ_λ is positive and which intersects X' . In other words, $\Omega'_\lambda \cap \frac{1}{2}B$ gives us a spike entering $\frac{1}{2}B$ which φ_λ is positive, and our aim is to obtain bounds on this spike.

Observe that (6.110) implies that the volume of the spike $\Omega'_\lambda \cap \frac{1}{2}B$ is small compared to $\frac{1}{2}B$, and this allows us to invoke Theorem 6.4.6. We see that

$$\begin{aligned} 2^{1/\eta} \varphi_\lambda(x_0) &\gtrsim \|\varphi_\lambda\|_{L^\infty(M)} \text{ (by hypothesis (6.105))} \\ &\geq \sup_{\Omega'_\lambda} \varphi_\lambda \geq 2^{1/\eta} \sup_{\Omega'_\lambda \cap \frac{1}{2}B} \varphi_\lambda \geq 2^{1/\eta} \sup_{\Omega'_\lambda \cap \frac{1}{4}B} \varphi_\lambda \text{ (by applying Theorem 6.4.6)}. \end{aligned}$$

Now (6.113) follows, which implies that the growth is controlled in the ball $\frac{1}{8}B$, that is,

$$\beta_{1/8}(\varphi_\lambda) = \frac{\sup_{\frac{1}{4}B} |\varphi_\lambda|}{\sup_{\frac{1}{8}B} |\varphi_\lambda|} \leq c_1, \quad (6.114)$$

where c_1 depends on (M, g) and not on ϵ_0 or λ (in particular, not on r_0, η).

Now, we bring in the asymmetry estimate (Proposition 6.4.1), which, together with (6.114), tells us that

$$\frac{|\{\varphi_\lambda < 0\} \cap \frac{1}{4}B|}{|\frac{1}{4}B|} \geq c_2, \quad (6.115)$$

where c_2 is a constant depending only on c_1 and (M, g) . But selecting the constant C_2 to be smaller than c_2 we see that (6.115) contradicts (6.110). Hence, we obtain a contradiction with the fact that the function φ_λ vanishes inside $\frac{1}{4}B$.

Finally, this proves that with the initial choice of parameters, there is an inscribed ball of radius $\frac{r_0}{4\sqrt{\lambda}}$ inside Ω_λ . By construction, we had that $r_0 \sim \eta^{\frac{(n-1)(n-2)}{2n}} = \eta^{\beta(n)}$.

Combined with the inner radius estimates in [Man08a], this proves the claim of Theorem 6.4.4. \square

6.4.6 An application: interior cone conditions

In dimension $n = 2$, a famous result of Cheng [Che76] says the following (see also [Ste14] for a proof using Brownian motion):

Theorem 6.4.7. *For a closed Riemannian surface M , the nodal set N_φ satisfies an interior cone condition with opening angle $\alpha \gtrsim \frac{1}{\sqrt{\lambda}}$.*

Furthermore, in dimension 2, the nodal lines form an equiangular system at a singular point of the nodal set.

Setting $\dim M \geq 3$, we discuss the question whether at the singular points of the nodal set N_φ , the nodal set can have arbitrarily small opening angles, or even “cusp”-like situations, or the nodal set has to self-intersect “sufficiently transversally”. We observe that in dimensions $n \geq 3$ the nodal sets satisfies an appropriate “interior cone condition”, and give an estimate on the opening angle of such a cone in terms of the eigenvalue λ .

Now, in order to properly state or interpret such a result, one needs to define the concept of “opening angle” in dimensions $n \geq 3$. We start by defining precisely the notion of tangent directions in our setting.

Definition 6.4.4. *Let Ω_λ be a nodal domain and $x \in \partial\Omega_\lambda$, which means that $\varphi_\lambda(x) = 0$. Consider a sequence $x_n \in N_\varphi$ such that $x_n \rightarrow x$. Let us assume that in normal coordinates around x , $x_n = \exp(r_n v_n)$, where r_n are positive real numbers, and $v_n \in S(T_x M)$, the unit sphere in $T_x M$. Then, we define the space of tangent directions at x , denoted by $\mathcal{S}_x N_\varphi$ as*

$$\mathcal{S}_x N_\varphi = \{v \in S(T_x M) : v = \lim v_n, \text{ where } x_n \in N_\varphi, x_n \rightarrow x\}. \quad (6.116)$$

We note that there are more well-studied variants of the above definition, for example, as due to Clarke or Bouligand (for more details, see [Roc79]). With that in place, we now give the following definition of “opening angle”.

Definition 6.4.5. *We say that the nodal set N_φ satisfies an interior cone condition with opening angle α at $x \in N_\varphi$, if any connected component of $S(T_x M) \setminus \mathcal{S}_x N_\varphi$ has an inscribed ball of radius $\gtrsim \alpha$.*

Now we have the following:

Theorem 6.4.8. *When $\dim M = 3$, the nodal set N_φ satisfies an interior cone condition with angle $\gtrsim \frac{1}{\sqrt{\lambda}}$. When $\dim M = 4$, N_φ satisfies an interior cone condition with angle $\gtrsim \frac{1}{\lambda^{7/8}}$. Lastly, when $\dim M \geq 5$, N_φ satisfies an interior cone condition with angle $\gtrsim \frac{1}{\lambda}$.*

We will use Bers scaling of eigenfunctions near zeros (see [Ber55]). We quote the version as appears in [Zel08], Section 3.11.

Theorem 6.4.9 (Bers). *Assume that φ_λ vanishes to order k at x_0 . Let $\varphi_\lambda(x) = \varphi_k(x) + \varphi_{k+1}(x) + \dots$ denote the Taylor expansion of φ_λ into homogeneous terms in normal coordinates x centered at x_0 . Then $\varphi_k(x)$ is a Euclidean harmonic homogeneous polynomial of degree k .*

We also recall the inradius estimate for real analytic metrics from Theorem 6.4.2.

Since the statement of Theorem 6.4.2 is at first glance asymptotic in nature, we need to note that a nodal domain corresponding to λ will still satisfy $\text{inrad}(\Omega_\lambda) \geq \frac{c_3}{\lambda}$ for some constant c_3 even for small λ . This follows from the inradius estimates of Mangoubi in [Man08a], which hold for all frequencies. Consequently, we can assume that every nodal domain Ω on S^n corresponding to the spherical harmonic $\varphi_k(x)$, as in Theorem 6.4.9, has inradius $\gtrsim \frac{1}{\lambda}$.

Proof of Theorem 6.4.8. We observe that Theorem 6.4.2 applies to spherical harmonics, and in particular the function $\exp^*(\varphi_k)$, restricted to $S(T_{x_0} M)$, where $\varphi_k(x)$ is the homogeneous harmonic polynomial given by Theorem 6.4.9. Also, a nodal domain for any spherical harmonic on S^2 (respectively, S^3) corresponding to eigenvalue λ has inradius $\sim \frac{1}{\sqrt{\lambda}}$ (respectively, $\gtrsim \frac{1}{\lambda^{7/8}}$).

With that in place, it suffices to prove that

$$\mathcal{S}_{x_0}N_\varphi \subseteq \mathcal{S}_{x_0}N_{\varphi_k}. \quad (6.117)$$

By definition, $v \in \mathcal{S}_{x_0}N_\varphi$ if there exists a sequence $x_n \in N_\varphi$ such that $x_n \rightarrow x_0$, $x_n = \exp(r_n v_n)$, where r_n are positive real numbers and $v_n \in S(T_{x_0}M)$, and $v_n \rightarrow v$.

This gives us,

$$\begin{aligned} 0 &= \varphi_\lambda(x_n) = \varphi_\lambda(r_n \exp v_n) \\ &= r_n^k \varphi_k(\exp v_n) + \sum_{m>k} r_n^m \varphi_m(\exp v_n) \\ &= \varphi_k(\exp v_n) + \sum_{m>k} r_n^{m-k} \varphi_m(\exp v_n) \\ &\rightarrow \varphi_k(\exp v), \text{ as } n \rightarrow \infty. \end{aligned}$$

Observing that $\varphi_k(x)$ is homogeneous, this proves (6.117). □

Chapter 7

Obstacles

7.1 Formulation and background

In this Chapter, we consider the problem of placing of an obstacle in a domain so as to maximize the fundamental frequency of the complement of the obstacle. To be more precise, let $\Omega \subset \mathbb{R}^n$ be a bounded domain, and let D be another bounded domain referred to as "obstacle". The problem is to determine the optimal translate $x + D$ so that the fundamental Dirichlet Laplacian eigenvalue $\lambda_1(\Omega \setminus (x + D))$ is maximized/minimized.

In case the obstacle D is a ball, physical intuition suggests that for sufficiently regular domains and sufficiently small balls, Ω , $\lambda_1(\Omega \setminus B_r(x))$ will be maximized when $x = x_0$, a point of maximum of the ground state Dirichlet eigenfunction ϕ_{λ_1} of Ω . Heuristically, such maximum points x_0 seem to be situated deeply in Ω , hence removing a ball around x_0 should be an optimal way of truncating the lowest possible frequency. Our methods give equally good results for Schrödinger operators on a large class of bounded domains sitting inside Riemannian manifolds (see the remarks at the end of Section 7.2). In terms of exposition, we follow our work in [GM17a].

The following well-known result of Harrell-Kröger-Kurata treats the case when Ω satisfies convexity and symmetry conditions:

Theorem 7.1.1 ([HKK06]). *Let Ω be a convex domain in \mathbb{R}^n and B a ball contained in Ω . Assume that Ω is symmetric with respect to some hyperplane H . Then,*

- (a) *at the maximizing position, B is centered on H , and*
- (b) *at the minimizing position, B touches the boundary of Ω .*

The last result of Harrell-Kröger-Kurata seems to work under rather strong symmetry assumption. We also recall that the proof of Harrell-Kröger-Kurata proceeds via a moving planes method which essentially measures the derivative of $\lambda_1(\Omega \setminus B)$ when B is shifted in a normal direction to the hyperplane.

There does not seem to be any result in the literature treating domains without symmetry or convexity properties.

In the following discussion, we consider bounded domains $\Omega \subset \mathbb{R}^n$ which satisfy an asymmetry assumption in the following sense:

Definition 7.1.1. A bounded domain $\Omega \subset \mathbb{R}^n$ is said to satisfy the asymmetry assumption with coefficient α (or Ω is α -asymmetric) if for all $x \in \partial\Omega$, and all $r_0 > 0$,

$$\frac{|B_{r_0}(x) \setminus \Omega|}{|B_{r_0}(x)|} \geq \alpha. \quad (7.1)$$

This condition seems to have been introduced in [Hay78]. We also recall that the α -asymmetry property was utilized by D. Mangoubi in order to obtain inradius bounds for Laplacian nodal domains (cf. Section 6.4 and also [Man08a]) as nodal domains are asymmetric with $\alpha = \frac{C}{\lambda^{(n-1)/2}}$.

From our perspective, the notion of asymmetry is useful as it basically rules out narrow "spikes" (i.e. with relatively small volume) entering deeply into Ω . For example, let us also observe that convex domains trivially satisfy our asymmetry assumption with coefficient $\alpha = \frac{1}{2}$.

7.2 The basic estimate for general obstacles

With the above in mind, we consider any bounded α -asymmetric domain $\Omega \subset \mathbb{R}^n$ and a bounded obstacle domain D . We denote the first positive Dirichlet eigenvalue and eigenfunction of Ω by λ_1 and $\phi_{\lambda_1(\Omega)}$ respectively and let

$$M := \{x \in \Omega \mid \phi_{\lambda_1}(x) = \|\phi_{\lambda_1(\Omega)}\|_{L^\infty(\Omega)}\} \quad (7.2)$$

be the set of maximum points of $\phi_{\lambda_1(\Omega)}$.

Let us also put

$$\mu_\Omega := \max_x \lambda_1(\Omega \setminus (x + D)). \quad (7.3)$$

Finally, for a given translate $x + D$ of the obstacle let us set

$$\rho_x := \max_{y \in M} d(y, x + D), \quad (7.4)$$

measuring the maximum distance from a maximum point of $\phi_{\lambda_1(\Omega)}$ to the translate $x + D$.

We have the following result.

Theorem 7.2.1. *Let us fix a translate $(x + D)$ and assume that $\rho_x > 0$. Then*

$$\lambda_1(\Omega \setminus (x + D)) \leq \beta(\rho_x) \lambda_1(\Omega), \quad (7.5)$$

where β is a continuous decreasing function defined as

$$\beta(\rho) = \begin{cases} \beta_0 = \beta_0(n, \alpha), & \rho \sqrt{\lambda_1(\Omega)} > r_0 := r_0(n, \alpha), \\ \frac{c_0}{\rho^2 \lambda_1(\Omega)}, & \rho \sqrt{\lambda_1(\Omega)} \leq r_0, \quad c_0 = c_0(n), \end{cases} \quad (7.6)$$

where $\beta_0 r_0 = c_0$.

We remark that in particular if ρ_x is of sub-wavelength order (i.e. $\lesssim \frac{1}{\sqrt{\lambda_1(\Omega)}}$), then $\lambda_1(\Omega \setminus (x + D)) \lesssim \frac{1}{\rho_x^2}$. If the obstacle D is convex, we can say more (see Theorem 7.4.1 below).

Proof of Theorem 7.2.1. The proof essentially exploits the fact that there are “almost inscribed” wavelength balls centered at maximum points of $\phi_{\lambda_1(\Omega)}$, utilizing Theorem 6.3.1 for an initially fixed small positive number ϵ_0 . We also recall Remarks 6.3.1 and 6.3.2, noting that the number r_0 can be taken as $r_0 = \epsilon_0^{\frac{n-2}{2n}}$.

Now, with the aid of Theorem 6.3.1 it is clear that under the α -asymmetry assumption, there exists an $r_0 := r_0(\alpha, n)$, such that around each maximum point $x_0 \in \Omega$ of $\phi_{\lambda_1(\Omega)}$ one can find a fully inscribed ball $B_{r_0/\sqrt{\lambda_1(\Omega)}}(x_0) \subseteq \Omega$. By the definition of ρ_x it follows that we can find a maximum point $x_0 \in (\Omega \setminus (x + D))$ and an inscribed ball $B_{\rho_0}(x_0)$ where

$$\rho_0 := \min \left(\frac{r_0}{\sqrt{\lambda_1(\Omega)}}, \rho_x \right). \quad (7.7)$$

As the first eigenvalue is monotonic with respect to inclusion, we see that

$$\lambda_1(\Omega \setminus (x + D)) \leq \lambda_1(B_{\rho_0}(x_0)) = \frac{C}{\rho_0^2}, \quad (7.8)$$

where $C = C(n)$ is a universal constant.

Expressing the right hand side of the last inequality in terms of $\lambda_1(\Omega)$ we define the function $\beta(\rho)$ as above.

This concludes the proof. □

Here, we have considered the obstacle problem in the case of Euclidean spaces, on reasonably well-behaved domains, and for the operator $-\Delta + \lambda_1(\Omega)$, as that seems to be the primary case of interest. However, we also include some remarks outlining some straightforward generalizations.

Remark 7.2.1. *It is clear that removing capacity zero sets from α -asymmetric domains considered in Definition 7.1.1 will lead to the same conclusions. Indeed, in this situation we will not be dealing with fully inscribed balls as above - instead, we will have balls whose first eigenvalue is comparable to the one of an inscribed one.*

Remark 7.2.2. *Lastly, it is clear that the results of [RS17] allow us to extend our discussion here from operators of the form $-\Delta + \lambda_1(\Omega)$ to Schrödinger operators of the form $-\Delta + V$, where V is bounded above. The conclusions are analogous with $\lambda_1(\Omega)$ replaced by $\|V\|_{L^\infty}$ and the proofs are identical.*

Now, as an immediate implication of Theorem 7.2.1 we have the following corollary.

Corollary 7.2.1. *Suppose that $\mu_\Omega = C_0 \lambda_1(\Omega)$, where $C_0 > \frac{c_0}{r_0^2}$ is a given fixed constant and c_0, r_0 are the constants in Theorem 7.2.1. Then, for a maximizer $\bar{x} + D$ of μ_Ω we have*

$$\rho_{\bar{x}} \leq \beta^{-1}(C_0). \quad (7.9)$$

In particular, if C_0 is large,

$$\rho_{\bar{x}} \lesssim \frac{1}{\sqrt{C_0 \lambda_1(\Omega)}}. \quad (7.10)$$

In other words the above corollary can be interpreted as follows: either μ_Ω is comparable to $\lambda_1(\Omega)$, or the maximum points of $\phi_{\lambda_1(\Omega)}$ are near the maximizer sets $\bar{x} + D$ of μ_Ω .

We note that the localization in the Corollary above gets better when C_0 is large. By Faber-Krahn's inequality, straightforward examples with large C_0 are domains Ω for which $|\Omega \setminus (x + D)|$ is sufficiently small for some x .

Particularly, for bounded convex domains in \mathbb{R}^n , by a theorem of Brascamp-Lieb (cf. [BL76]), the level sets of $\phi_{\lambda_1(\Omega)}$ are convex. Since $\phi_{\lambda_1(\Omega)}$ is real analytic and it can be assumed positive on $\Omega \setminus \partial\Omega$ without loss of generality, this means that it has a unique point of maximum. So, in this setting, our result heuristically says that if removal of a ball B_r has a "significant effect" on the vibration of $\Omega \setminus B_r$, then B_r must be centered quite close to the max point of the ground state Dirichlet eigenfunction ϕ_{λ_1} of the domain Ω , where the bound on ρ_x gives the quantitative relation between the "effect" and the order of "closeness". In a sense, this can be seen to be complementary to Corollary II.3 of [HKK06].

7.3 Inscribed balls

Further, we specify our results to the obstacle being a ball D . We point out a few observations related to the possibility to inscribe a large ball at a given point x in Ω .

Proposition 7.3.1. *Let Ω be α -asymmetric and let $\text{inrad}(\Omega)$ denote the inner radius of Ω . If x_0 is a point of maximum of $\phi_{\lambda_1(\Omega)}$, then there exists an inscribed ball $B_{C \text{inrad}(\Omega)}(x_0) \subseteq \Omega$, where $C = C(n, \alpha)$.*

Proof of Proposition 7.3.1. We observe that by Corollary 6.4.1 and its proof, or by the results of [Man08a], α -asymmetric domains Ω satisfy

$$\frac{C_1(\alpha, n)}{\sqrt{\lambda_1(\Omega)}} \leq \text{inrad}(\Omega) \leq \frac{C_2(n)}{\sqrt{\lambda_1(\Omega)}}. \quad (7.11)$$

Moreover, such a ball can be inscribed at a point x_0 where the first eigenfunction ϕ_λ reaches a maximum. \square

In particular, the last proposition applies for convex domains. We mention in this connection that localization results for maximum points of $\phi_{\lambda_1(\Omega)}$ in case Ω is a planar convex domain can be found in the work of Grieser-Jerison (see [GJ98]).

7.4 Relation between maximum points and convex obstacles

Note that Theorem 7.2.1 holds for arbitrary obstacles and gives a bound on the distance ρ_x to maximum points of $\phi_{\lambda_1(\Omega)}$. However, it is desirable to deduce that $\rho_x = 0$, i.e. maximizers actually contain the maximum points of $\phi_{\lambda_1(\Omega)}$.

From Proposition 7.3.1 and Theorem 7.2.1 we deduce the following:

Theorem 7.4.1. *Let D be a convex obstacle, and $\bar{x} + D$ maximize $\lambda_1(\Omega \setminus (x + D))$. Then there exists a constant $C_0 = C_0(\alpha, n)$ such that if $\lambda_1(\Omega \setminus (\bar{x} + D)) \geq C\lambda_1(\Omega)$ for some $C \geq C_0$, then $\rho_{\bar{x}} = 0$.*

In other words, either $\mu_\Omega \sim \lambda_1(\Omega)$ or $\rho_{\bar{x}} = 0$.

Proof. To the contrary let us suppose that $\rho_{\bar{x}} = d(\bar{x} + D, x_0) > 0$ where x_0 is a maximum point of $\phi_{\lambda_1(\Omega)}$ and $\lambda_1(\Omega \setminus (\bar{x} + D)) \geq C\lambda_1(\Omega)$ for an arbitrary large $C > 0$.

We apply the statement of Proposition 7.3.1 and deduce that there is a wavelength inscribed ball B at x_0 . As D is a convex domain, we can find a wavelength half-ball $B^{1/2} \subset \Omega \setminus (\bar{x} + D)$ containing x_0 . By the assumption and eigenvalue monotonicity with respect to inclusion:

$$C\lambda_1(\Omega) \leq \lambda_1(\Omega \setminus (\bar{x} + D)) \leq \lambda_1(B^{1/2}) \leq \frac{C_1}{(\text{inrad}(\Omega))^2} = C_2\lambda_1(\Omega), \quad (7.12)$$

where $C_2 = C_2(n, \alpha)$. Taking C sufficiently large we get a contradiction. □

It is clear that for explicit applications, particularly in the case of convex domains, Theorem 7.4.1 is dependent on a precise knowledge of the location of the maximum point of $\phi_{\lambda_1(\Omega)}$. Localization of the maximum point of $\phi_{\lambda_1(\Omega)}$ (or more generally, the “hot spot”) is a problem which is far from being settled. Here we take the space to augment Theorem 7.4.1 with the recent results of [BMS11].

First we recall the definition of the “heart” of a convex body Ω . The following intuitive definition appears in [SH16], and it is equivalent to the (more technical) definition presented in [BMS11].

Definition 7.4.1. *Let P be a hyperplane in \mathbb{R}^n which intersects Ω so that $\Omega \setminus P$ is the union of two components located on either side of P . The domain Ω is said to have the interior reflection property with respect to P if the reflection through P of one of these subsets, denoted Ω_s , is contained in Ω , and in that case P is called a hyperplane of interior reflection for Ω . When Ω is convex, the heart of Ω , denoted by $\heartsuit(\Omega)$, is defined as the intersection of all such $\Omega \setminus \Omega_s$ with respect to hyperplanes of interior reflection of Ω .*

The following result is contained in Proposition 4.1 of [BMS11].

Theorem 7.4.2 ([BMS11]). *The unique maximum point x_0 of $\phi_{\lambda_1(\Omega)}$ is contained in $\heartsuit(\Omega)$. Furthermore, x_0 is contained in the interior of $\heartsuit(\Omega)$, if the latter is non-empty.*

Appendix A

Auxiliary Material

A.1 Hausdorff measure

A central tool in our discussion is that of a Hausdorff measure. We briefly recall a few definitions. For a full treatment we refer to Chapter 12, [Tay06].

Let A be non-empty subset of a metric space X . For any $\delta \in (0, \infty]$ and $s > 0$ we define

$$\mathcal{H}_\delta^s(A) := \inf \left\{ \sum_{j=1}^{\infty} \omega(s) \left(\frac{\text{diam } C_j}{2} \right)^2 : A \subset \bigcup_{j=1}^{\infty} C_j, \text{diam } C_j \leq \delta \right\}, \quad (\text{A.1})$$

where

$$\omega(s) := \frac{\pi^{\frac{s}{2}}}{\Gamma\left(\frac{n}{2} + 1\right)}, \quad s \in (0, \infty). \quad (\text{A.2})$$

We note that in the case $s \in \mathbb{N}$, the quantity $\omega(s)$ is the volume of the s -dimensional Euclidean unit ball.

The s -dimensional *Hausdorff measure* of A is defined as

$$\mathcal{H}^s(A) = \lim_{\delta \rightarrow 0} \mathcal{H}_\delta^s(A) = \sup_{\delta > 0} \mathcal{H}_\delta^s(A). \quad (\text{A.3})$$

The Hausdorff measure of the empty set is defined to be 0. We note that in the case $X = \mathbb{R}^n$ and $s = n$, the Hausdorff measure coincides with the Lebesgue measure. Furthermore, one can show that, in general, the Hausdorff measure is a Borel regular measure.

We also have the following property. Let us set $0 \leq s < t < \infty$, then:

- if $\mathcal{H}^s(A) < \infty$, it follows $\mathcal{H}^t(A) = 0$;
- if $\mathcal{H}^t(A) > 0$, it follows $\mathcal{H}^s(A) = \infty$.

With this in mind, one defines the *Hausdorff dimension* of A , as

$$\dim_{\mathcal{H}}(A) := \inf \{s \in [0, \infty) : \mathcal{H}^s(A) = 0\}. \quad (\text{A.4})$$

A.2 A spectral theorem for the Laplace operator on closed Riemannian manifolds

Let (M, g) be closed Riemannian manifold of dimension n . We consider the Laplace-Beltrami operator Δ as given in (1.12) from Chapter 1. It is known that Δ is essentially self-adjoint in $L^2(M)$ (i.e. has a self-adjoint closure). In general, when (M, g) is not necessarily closed, one can always consider a self-adjoint extension of Δ known as the Friedrichs extension - for background and a detailed introduction we refer to Chapter 8, [Tay11].

It turns out that similarly to the case of the standard flat Laplace operator on a domain $\Omega \subset \mathbb{R}^n$ with Dirichlet boundary conditions (i.e. working on the functional space $W_0^{1,2}(\Omega)$), one can write a compact self-adjoint resolvent of Δ , and thus obtain the following theorem.

Theorem A.2.1. *The Laplace operator Δ has a discrete spectrum of non-negative eigenvalues $\lambda_i \rightarrow \infty$. Here, each λ_i has a finite multiplicity, giving rise to a corresponding finite dimensional space of smooth eigenfunctions. Moreover, one can extract an orthonormal basis of $L^2(M)$ consisting of eigenfunctions.*

For a detailed discussion and proof we refer to Theorem 1, Section 6.5, [Eva97] and Chapter 8, [Tay11].

A.3 Some facts from the theory of Sobolev spaces

Theorem A.3.1. *Let $1 \leq p < \infty$ and let Ω be a bounded domain in \mathbb{R}^n with C^1 -regular boundary. Then there exists a unique continuous linear mapping*

$$\text{tr} : W^{1,p}(\Omega) \rightarrow L^p(\partial\Omega), \quad (\text{A.5})$$

such that for all smooth functions ϕ in $C_0^\infty(\mathbb{R}^n)$ we have

$$\text{tr}(\phi) = \phi|_{\partial\Omega}. \quad (\text{A.6})$$

Proof. We refer to Theorem 6.3.3, [Wil10]. □

Theorem A.3.2. *Let Ω be a bounded domain in \mathbb{R}^n of with C^1 -regular boundary. Let $V : \Omega \rightarrow \mathbb{R}^n$ be vector field belonging to the space $W^{1,1}(\Omega, \mathbb{R}^n)$. Then*

$$\int_{\Omega} \text{div } V dx = \int_{\partial\Omega} \langle \text{tr}(V), \nu \rangle d\sigma, \quad (\text{A.7})$$

where tr is the operator from Theorem A.3.1 applied component-wise to V and where ν denotes the outer unit normal with respect to Ω .

Proof. We refer to Theorem 6.3.4, [Wil10]. □

Theorem A.3.3. *Let Ω_1, Ω_2 be open subsets of \mathbb{R}^n , $F : \Omega_1 \rightarrow \Omega_2$ be a C^1 -diffeomorphism and $u \in W_{\text{loc}}^{1,1}(\Omega_2)$. Then $u \circ F$ is in $W_{\text{loc}}^{1,1}(\Omega_1)$ and the derivatives are computed via the usual chain rule for smooth functions.*

Furthermore, if f belongs to the Banach space $C^1(\mathbb{R})$, then $f \circ u$ belongs to $W_{\text{loc}}^{1,1}(\Omega_2)$ and the derivatives are again computed via the usual chain rule for smooth functions.

Proof. We refer to Propositions 6.1.11 and 6.1.13, [Wil10]. □

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