On the effective properties of suspensions

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Abstract

Einstein's formula for the viscosity of dilute suspensions describes how rigid particles immersed in a Stokes-fluid increase its macroscopic viscosity in terms of the particle volume density ϕ. However, up to now, a rigorous justification has only been obtained for dissipation functionals of the flow field. In this thesis, a cloud of N spherical rigid particles of radius R suspended in a fluid of viscosity μ is considered. It is rigorously shown that the homogenized fluid in the regime $NR^3 \to 0$ as $N \to \infty$ has, in accordance with Einstein's formula, the viscosity

$$\mu' = \mu \left(1 + \frac{5}{2}\phi\right)$$

to first order in ϕ. This is done by establishing L^∞ and L^p estimates for the difference of the solution to the microscopic problem and the solution to the homogenized equation. Regarding the distribution of the particles, it is assumed that the particles are contained in some bounded region and are well separated in the sense that the minimal distance is comparable to the average one. The main tools for the proof are a dipole approximation of the flow field of the suspension together with the so-called method of reflections and a coarse graining of the volume density.

By a very close mathematical analogy to electrostatics a similar result, regarding Maxwell's formula for the conductivity of suspensions, is proven, namely that the conductivity of the homogenized material is

$$\eta' = \eta\,(1 + 3\phi)$$

to first order in ϕ.

Contents

1. Introduction

Mixtures of different materials occur in all kinds of modern scientific applications. They appear naturally (sand, sea water,...) or as artificially produced materials with predefined properties. In order to understand natural processes on the one hand but more importantly to manufacture special materials on the other hand, it is important to characterize the properties of such mixtures with respect to their individual constituents. In this thesis we consider the special case of the mixture of *two* materials and furthermore assume that one of the materials constitutes the main part of the mixture while the other one is contained in it in many small and dilute inclusions. Examples for such mixtures are bacterial suspensions in which many small bacteria are suspended in a liquid or dilute solutions of large molecules (like sugar) in water.

1.1. Einstein's and Maxwell's formulas

In his annus mirabilis, 1905, Einstein published five seminal works contributing to different areas of physics. One of these works was his dissertation "Eine neue Bestimmung der Moleküldimensionen" [Ein06]. In it he derives a formula for the effective viscosity of a dilute suspension and relates it to the formula for the mass diffusivity in order to obtain a formula for the size of the particles in the suspension. Applying this to a solution of sugar in water he is able to estimate the molecular dimensions of sugar, since both viscosity and diffusivity can be measured experimentally. These findings contributed greatly to the theory of matter as the general idea that materials are constituted of small entities like molecules was still under dispute at that time.

The formula that Einstein derived for the effective viscosity was $\mu_{\text{eff}} = \mu(1 + \phi)$ where ϕ is the concentration (the volume fraction) of the dissolved substance. Later he was made aware that the coefficient of ϕ seemed to be larger in experiments. He asked his assistant to check the calculations and he found an error. Einstein thus revised his formula to the final ([Ein11]):

$$\mu_{\text{eff}} = \mu \left(1 + \frac{5}{2}\phi\right).$$

This formula became known as 'Einstein's formula'. In his thesis, Einstein assumes that the dissolved particles are of spherical shape, rigid and very dilute so that every particle can be considered as being a single particle immersed in the fluid. It is therefore possible to compute the additional energy dissipation caused by one particle explicitly and sum up these contributions. This work has inspired many other works attempting to improve the result considering different shapes, rigidity and density of the particles as well as rigour of the mathematical derivation.

1. Introduction

About a third of a century before Einstein derived his result, Maxwell [Max73, p. 365] gave a formula for the resistance (the inverse of the conductivity) of a suspension of spheres.

For spheres with resistance k_1 and a surrounding material of resistance k_2 Maxwell states that the effective resistance k_{eff} of the mixture must be

$$k_{\text{eff}} = \frac{2k_1 + k_2 + \phi(k_1 - k_2)}{2k_1 + k_2 - 2\phi(k_1 - k_2)} k_2. \tag{1.1}$$

A perfectly conducting sphere has resistance $k_1 = 0$, which reduces the formula to

$$k_{\text{eff}} = k_2 \frac{1 - \phi}{1 + 2\phi}.$$

Taking the inverse gives a formula for the effective conductivity η_{eff} of a suspension of perfectly conducting spheres in a material of conductivity η:

$$\eta_{\text{eff}} = \eta \frac{1 + 2\phi}{1 - \phi}.$$

Expanding the fraction in powers of ϕ gives

$$\frac{1 + 2\phi}{1 - \phi} = 1 + 3\phi + o(\phi),$$

where $o(\phi)$ is a term that satisfies $\frac{o(\phi)}{\phi} \to 0$ as $\phi \to 0$. If we take into account only the terms up to first order of ϕ we obtain

$$\eta_{\text{eff}} = \eta(1 + 3\phi),$$

a formula that looks remarkably similar to Einstein's formula. Of course, this is due to the fact that both are expansions up to first order in ϕ.

Nevertheless, Taylor [Tay32] later found a formula for the effective viscosity when the immersed particles are viscous themselves, that had the same general form as Maxwell's original formula (1.1). This is not by chance. The two situations of a viscous suspension of particles and a conducting suspension of particles are mathematically quite similar.

1.2. The mathematical similarity between Stokes fluids and electrostatics

The state of an incompressible fluid occupying a domain $\Omega \subset \mathbb{R}^3$ can at any given time be described by its velocity $u : \Omega \to \mathbb{R}^3$. The associated local rate of strain is $e = \frac{1}{2}\left(\nabla u + \nabla u^T\right)$. The stress inside the fluid is given by σ satisfying the equilibrium equation $-\operatorname{div}\sigma = f$ where f is some force density. In order to know how the fluid reacts to forces, it is necessary to know what strain is caused by the stress. This is called a constitutive relation. Simple theories often assume that this relation is linear and fluids that behave like this are called Newtonian.

In general, this linear dependence is described by a tensor $\mathbb{R}^{3\times 3} \to \mathbb{R}^{3\times 3}$, however, if the fluid is isotropic and incompressible, the relation is given by a simple scalar:

$$\sigma = 2\mu e - p\,\mathrm{Id},$$

and the factor μ is called the *viscosity* of the fluid. The second term is associated to the pressure p, which is a consequence of the incompressibility.

Using the constitutive equation we obtain a partial differential equation for the velocity, the Stokes equation

$$-\mu\Delta u + \nabla p = f.$$

Taking into account boundary conditions, the solution of this partial differential equation gives the reaction of the fluid to the force.

A very similar theory can be found in electrostatics. There, the potential $u : \Omega \to \mathbb{R}$ in a dielectric material induces an electric field that is given by $E = -\nabla u$. Again, there is a flux, here the electric displacement field D, which in turn satisfies the equilibrium equation $\operatorname{div} D = \rho$ where ρ is the charge density. Looking for a constitutive relation between E and D, the simplest assumption is, that this relation is linear and that for homogeneous media the associated tensor is actually a scalar:

$$D = \epsilon E,$$

where the factor ϵ is called the electric permittivity.

There are theories for other physical quantities like heat, current or mass which are mathematically exactly the same. There, the factor in the linear relationship between the flux and the gradient of the potential is called thermal conductivity, electric conductivity or diffusion coefficient respectively. For simplicity, from now on we will call the factor *conductivity* and denote it by η.

Again, using the constitutive equation we obtain a partial differential equation for the potential. This is the Poisson equation for the electrostatic case:

$$-\eta\Delta u = \rho.$$

The similarity between the fluid and the electrostatic case and their respective partial differential equations extends to situations where particles are present inside the material. Let a rigid, force-free body be present inside Ω. Assume that the body occupies the part $B \subset \Omega$ of the domain and that a fluid occupies $\Omega \setminus B$. Then the correct boundary conditions are,

$$u(x) = V + \omega \wedge x \text{ in } B, \quad \int_{\partial B} \sigma n \, \mathrm{d}S = 0, \quad \int_{\partial B} x \wedge (\sigma n) \, \mathrm{d}S = 0$$

since rigid bodies can undergo only rigid body motions (V is the translational velocity and ω is the angular velocity of the body) and the total force and moment of force on the body is zero. Now let a perfectly conducting, uncharged body be present inside Ω. Again, let the body occupy the part $B \subset \Omega$ of the domain and let a dielectric material occupy $\Omega \setminus B$. The electrostatic boundary conditions are

$$u(x) = c \text{ in } B, \quad \int_{\partial B} D \cdot n \, \mathrm{d}S = 0,$$

since all charges present in the body will have moved to the boundary (constant potential) and the total charge is zero. Here $c \in \mathbb{R}, V, \omega \in \mathbb{R}^3$ are constants which are not a priori known but must be determined as part of solving the problem. Note that in both cases the second boundary condition is the integrated flux through the boundary ∂B. In the first condition a rigid body motion for fluids plays the role of a constant in the electrostatic case.

In the spirit of this similarity, lots of results that were first obtained for the Poisson equation (a partial differential equation that is very well understood), were later transferred to the Stokes equation where one has to deal with the additional constraint of incompressibility.

One example we have discussed already: Maxwell first derived a formula for the effective conductivity and Taylor later found a formula for the effective viscosity that resembled Maxwell's formula. And of course, if the reader was to repeat Einstein's computation for a suspension of ideally conducting spheres she would arrive at

$$\eta_{\text{eff}} = \eta(1 + 3\phi),$$

while the computation would be simpler than the one for Stokes equation. This thesis adopts this parallelism between the two equations. All results will be derived for the Poisson equation and then (with some additional work here and there) be transferred to Stokes equation.

1.3. Heuristic derivation

Let us consider a collection of rigid spherical particles $B_i = B_R(X_i), i = 1, .., N$ where $X_i \in \mathbb{R}^3$ and $|X_i - X_j| > 2R$ for all $i \neq j$. This implies that the particles neither intersect nor touch each other. We set

$$\Omega = \mathbb{R}^3 \setminus \bigcup_{i=1}^{N} \overline{B_i}.$$

Let us start with the electrostatic setting, where the particles are perfectly conducting inclusions in a dielectric material of conductivity η occupying Ω. Let some charge distribution f be given. This leads to the following set of equations for the potential $u : \mathbb{R}^3 \to \mathbb{R}$:

$$-\eta \Delta u = f \quad \text{in } \Omega, \tag{1.2}$$

$$\int_{\partial B_i} \eta \frac{\partial u}{\partial n} \, \mathrm{d}S = 0 \quad \text{for } i = 1, \dots, N, \tag{1.3}$$

$$u = c_i \quad \text{on } \overline{B_i} \text{ for } i = 1, \dots, N, \tag{1.4}$$

$$u(x) \to 0 \quad \text{as } |x| \to \infty, \tag{1.5}$$

where c_i is unknown and must be determined as part of the solution. In problem (1.2)-(1.5) one can replace f by $f' = \chi_\Omega f$ where χ_Ω is the characteristic function of Ω since the equation holds only in Ω. Additionally we can assume that $\eta = 1$ by rescaling f by the factor $\frac{1}{\eta}$. Let us assume that the solution u is already close, in some sense, to the solution of the problem without particles:

$$\begin{aligned} -\Delta v &= f' \quad \text{in } \mathbb{R}^3, \\ v(x) &\to 0 \quad \text{as } |x| \to \infty. \end{aligned}$$

Then, in order to get a better approximation of u, the main point is to satisfy the condition $u =$ const. on the balls. On each ball B_i the function v has to first order the form

$$v(x) = v(X_i) + \nabla v(X_i)(x - X_i) + o(R).$$

So in order to get closer to a constant we subtract the (dipole-)function d_i that is defined by

$$d_i(x) = \begin{cases} \nabla v(X_i) \cdot (x - X_i) & , \text{for } |x - X_i| \le R, \\ R^3 \nabla v(X_i) \cdot \frac{x - X_i}{|x - X_i|^3} & , \text{for } |x - X_i| > R. \end{cases}$$

Then, $v - d_i = \text{const.} + o(R)$ in B_i. Now we want the approximation $\tilde{u}$ to be close to constant on all the balls which means we set

$$\tilde{u} = v - \sum_{i=1}^{N} d_i.$$

Of course for $i \neq j$ the dipole d_i will not vanish on B_j but since the decay of d_i is quadratic we may hope that under some conditions on the particle distribution this effect is comparable to the one coming from higher order terms in the Taylor expansion of v in B_i. Note that the d_i are harmonic outside B_i so that $-\Delta \tilde{u} = f'$ is still valid in Ω. Now let $\phi = RN^3$ (this is a slight abuse of notation since ϕ denoted the physical volume fraction before) and assume that the rescaled volume density

$$\rho^N = \frac{1}{\phi}\frac{4\pi}{3}R^3\sum_{i=1}^N \delta_{X_i},$$

converges in some sense to a function ρ as $N \to \infty$ so that $\phi\rho$ is some kind of virtual limit volume density (assuming that ϕ stays constant while the number of particles tend to infinity,). Then we can write

$$\begin{aligned}\tilde{u}(x) &= v(x) - \sum_{i=1}^N d_i(x) \\ &= v(x) - \int_{\mathbb{R}^3} \frac{3}{4\pi}\phi\rho^N \nabla v(y) \cdot \frac{x-y}{|x-y|^3} \,\mathrm{d}y \\ &\approx v(x) - \int_{\mathbb{R}^3} \frac{3}{4\pi}\phi\rho \nabla v(y) \cdot \frac{x-y}{|x-y|^3} \,\mathrm{d}y \\ &= v(x) - \int_{\mathbb{R}^3} 3\phi\rho\nabla v(y) \cdot \nabla_y \frac{1}{4\pi|x-y|} \,\mathrm{d}y \\ &= v(x) + \int_{\mathbb{R}^3} \mathrm{div}_y\left(3\phi\rho\nabla v(y)\right) \frac{1}{4\pi|x-y|} \,\mathrm{d}y.\end{aligned}$$

Now we use the fact that

$$\Phi^P(x) = \frac{1}{4\pi|x|},$$

is the fundamental solution of the Poisson equation. Taking $-\Delta$ on both sides and using $f' \approx (1-\phi\rho)f$ we arrive at

$$-\Delta\tilde{u} = (1-\phi\rho)f + \mathrm{div}\left(3\phi\rho\nabla v\right).$$

Since $\tilde{u}$ is already close to v, by replacing v by $\tilde{u}$ in the divergence term, we make an error of $\phi o(\phi)$, since ∇v is additionally multiplied by ϕ (which is supposed to be small). Then we obtain the following equation

$$-\mathrm{div}\left((1+3\phi\rho)\nabla\tilde{u}\right) = (1-\phi\rho)f.$$

This suggests that the effective conductivity for a suspension in a material of conductivity η is given by

$$\eta_{\text{eff}} = (1+3\phi\rho)\eta,$$

to first order in ϕ for small volume fractions of the immersed particles, where, we recall, $\phi\rho$ is the physical volume density, that plays the role of ϕ in Einstein's formula. Note that, since ρ is typically non-constant, the effective conductivity is a function of the space variable.

Now assume that the particles are rigid, inertialess and suspended force-free in a surrounding fluid of viscosity μ occupying Ω. Here f is a force density. This entails the following problem for the fluid velocity $u : \mathbb{R}^3 \to \mathbb{R}^3$:

$$-\mu\Delta u + \nabla p = f \quad \text{in } \Omega, \tag{1.6}$$
$$\operatorname{div} u = 0 \quad \text{in } \Omega, \tag{1.7}$$
$$\int_{\partial B_i} \sigma n \, \mathrm{d}S = 0 \quad \text{for } i = 1, \dots, N, \tag{1.8}$$
$$\int_{\partial B_i} (x - X_i) \wedge (\sigma n) \, \mathrm{d}S = 0 \quad \text{for } i = 1, \dots, N, \tag{1.9}$$
$$u(x) = V_i + \omega_i \wedge (x - X_i) \quad \text{on } \overline{B_i} \quad \text{for } i = 1, \dots, N, \tag{1.10}$$
$$u(x) \to 0 \quad \text{as } |x| \to \infty, \tag{1.11}$$

where

$$\sigma = -p \operatorname{Id} + 2\mu e u, \quad eu = \frac{1}{2}(\nabla u + \nabla u^T),$$

and the $V_i, \omega_i \in \mathbb{R}^3$ are a priori unknown. Again we can rescale so that $\mu = 1$. The solution without particles is given by

$$\begin{aligned} -\Delta v + \nabla p &= f' \quad \text{in } \mathbb{R}^3, \\ \operatorname{div} v &= 0 \quad \text{in } \mathbb{R}^3, \\ v(x) &\to 0 \quad \text{as } |x| \to \infty. \end{aligned}$$

In order to approximate u we want $u = V_i + \omega_i \wedge (x - X_i)$ to be satisfied on the balls and pursue the same strategy as for problem (1.2)-(1.5). On the ball B_i the function v, up to first order, has the form

$$v(x) = v(X_i) + \nabla v(X_i)(x - X_i) + o(R).$$

The linear part consists of a skew-symmetric part that induces rotations and that we want to keep, while we need to correct for the symmetric part $ev(X_i) = \epsilon_i$. So this time, in order to get closer to a rigid body motion, we subtract the (dipole-)function d_i that only incorporates the symmetric gradient and is defined by

$$d_i(x) = \begin{cases} \epsilon_i(x - X_i) & \text{, for } x \in \overline{B_i}, \\ \frac{5}{2}R^3 \left(\frac{(x-X_i)((x-X_i)\cdot\epsilon_i(x-X_i))}{|x-X_i|^5} \right) & \\ +R^5 \left(\frac{\epsilon_i(x-X_i)}{|x-X_i|^5} - \frac{5}{2}\frac{(x-X_i)((x-X_i)\cdot\epsilon_i(x-X_i))}{|x-X_i|^7} \right) & \text{, otherwise.} \end{cases}$$

Then $v - d_i = v(X_i) + \omega_i \wedge (x - X_i) + o(R)$ in B_i, where ω_i is determined by the skew-symmetric part of the gradient. Again we set up $\tilde{u} = v - \sum_{i=1}^N d_i$. This time the d_i solve the homogeneous Stokes equation outside $\overline{B_i}$ so that $-\Delta\tilde{u} + \nabla p = f'$ is still valid in Ω. Note that here d_i consists of two parts, one of which decays much more rapidly than the other. Hence we take into account only the first part for the following heuristics. Again assume that the rescaled volume density $\rho^N = \frac{1}{\phi}\frac{4\pi}{3}R^3 \sum_{i=1}^N \delta_{X_i}$ converges in some sense to ρ as $N \to \infty$ so that $\phi\rho$ is the virtual limit volume density. We can write

$$\begin{aligned} \tilde{u}(x) &= v(x) - \sum_{i=1}^N d_i(x) \\ &\approx v(x) - \int_{\mathbb{R}^3} \frac{3}{4\pi}\phi\rho^N(y)\frac{5}{2}\left(\frac{(x-y)\left((x-y)\cdot ev(y)(x-y)\right)}{|x-y|^5} \right) \mathrm{d}y. \end{aligned}$$

Now we introduce the fundamental solution to the Stokes equation

$$\Phi_{ij}^S(x) = \frac{1}{8\pi}\left(\frac{\delta_{ij}}{|x|} + \frac{x_i x_j}{|x|^3} \right).$$

We will see later that the following identity holds for symmetric and traceless matrices ϵ, where here, and in the following we use the Einstein convention to always sum over doubly appearing subscripts:

$$\epsilon_{ki}\partial_k \Phi_{ij}(x) = -\frac{3}{8\pi}\frac{x_j x_k \epsilon_{ki} x_i}{|x|^5}.$$

Using this we arrive at the following approximation:

$$\begin{aligned} \tilde{u}_j(x) &\approx v_j(x) + \int_{\mathbb{R}^3} 5\phi\rho^N(y) ev(y)_{ki} \left(\partial_k \Phi_{ij}^S\right)(x-y)\,\mathrm{d}y \\ &\approx v_j(x) + \int_{\mathbb{R}^3} 5\phi\rho(y) ev(y)_{ki} \left(\partial_k \Phi_{ij}^S\right)(x-y)\,\mathrm{d}y \\ &= v_j(x) + \int_{\mathbb{R}^3} \Phi_{ij}^S(x-y)\,\mathrm{div}_y\left(5\phi\rho ev(y)\right)_i\,\mathrm{d}y. \end{aligned}$$

Here we used the fact that $\left(\partial_k \Phi^S_{ij}\right)(x-y) = -\partial_{y_k}\left(\Phi^S_{ij}(x-y)\right)$. Taking $-\Delta$ on both sides and using $f' \approx (1-\phi\rho)f$ we arrive at

$$-\Delta\tilde{u} + \nabla p = (1-\phi\rho)f + \operatorname{div}(5\phi\rho e v)\,.$$

Again, we replace v by $\tilde{u}$ in the divergence term to obtain the following equation

$$\begin{aligned} -\operatorname{div}(\nabla\tilde{u} + 5\phi\rho e\tilde{u}) + \nabla\tilde{p} &= (1-\phi\rho)f, \\ \operatorname{div}\tilde{u} &= 0. \end{aligned}$$

We can use the fact that $\operatorname{div}\tilde{u} = 0$ (and hence $\operatorname{div}\nabla\tilde{u}^T = 0$) to write

$$\begin{aligned} -\operatorname{div}((2 + 5\phi\rho)\, e\tilde{u}) + \nabla\tilde{p} &= (1-\phi\rho)f, \\ \operatorname{div}\tilde{u} &= 0. \end{aligned}$$

This has the form

$$-\operatorname{div}\sigma = (1-\phi\rho)f,$$

where

$$\sigma = 2\left(1 + \frac{5}{2}\phi\rho\right) e\tilde{u} - p\operatorname{Id}.$$

Comparing to the form of the stress tensor for a homogeneous fluid this suggests that the effective viscosity of a suspension for small volume fractions of the immersed particles in a material of viscosity μ is given by

$$\mu_{\text{eff}} = (1 + \frac{5}{2}\phi\rho)\mu$$

to first order of ϕ. Note that, since ρ is typically non-constant, the effective viscosity is a function of the space variable.

In regions where the density ρ is constant, the divergence acting on the part of the transposed gradient vanishes because $\operatorname{div}\tilde{u} = 0$. In these regions we recover Einstein's formula even for the classical form of the Stokes equation:

$$\begin{aligned} -(1 + \frac{5}{2}\phi\rho)\Delta\tilde{u} + \nabla\tilde{p} &= (1-\phi\rho)f, \\ \operatorname{div}\tilde{u} &= 0. \end{aligned}$$

1.4. Review of literature

In this section we give an overview of what results have been achieved so far regarding the effective viscosity of suspensions and of methods that may be of importance for the derivation of such results.

The first generalization of Einstein's result was undertaken by Jeffery [Jef22] who considers ellipsoidal particles instead of spheres. His approach is quite similar to Einstein's approach in his thesis. The result for spheres is rediscovered and it is shown that, for spheroids (an ellipsoid with two identical semi-diameters), depending on the ellipticity $\frac{a-b}{a}$ of the bodies, the coefficient lies in an interval that is contained in $[2, \infty]$ and contains $\frac{5}{2}$. The fact that the author can only give an interval and not an exact value comes from the problem not being well-posed, since the orientations are not fixed. Hinch and Leal [LH71, HL72] solved this problem by considering the ensemble average and a steady-state distribution of orientations getting explicit numerical values for the coefficient for different ellipticities.

In [Tay32], mentioned in Section 1.1, drops of another fluid (with finite viscosity) suspended in a surrounding fluid are considered for the first time. This is the analogous situation to the one for which Maxwell derived his formula in electrostatics although Taylor must assume that the boundary of the particles stays spherical. And indeed a similar formula is obtained for the effective viscosity.

In [KRM67] the authors establish several extremum principles for the Stokes flow including fairly general boundary conditions and rigid particles. They use those principles to prove, among other results, uniqueness of the solution and to obtain bounds and an asymptotic formula for the effective viscosity in the low concentration regime and for high concentrations when the particles are situated on a lattice. In the same year in [FA67] another result was given for high concentrations. Numerical research can be found in [NK84], in which arbitrary concentrations are considered and also asymptotic formulas for high concentrations are obtained. [BBP05] considers the case of highly concentrated suspensions and uses a so-called network approximation.

A second order correction to the viscosity was first considered by Batchelor and Green. In [BG72] they calculate the second order correction to the viscosity to be $7.6\phi^2$ with an estimated error of the numerical factor of 10% which comes from numerical and asymptotic evaluation of an, in principle, known function.

While the so far mentioned results mostly rely on formal considerations to derive viscosity formulas, [KRM67] rigorously proves

$$\mu_{\text{eff}} = \mu(1 + \frac{5}{2}\phi + o(\phi)).$$

The employed method is the following. On the boundary of the (finite) domain, conditions are imposed that would make a homogeneous fluid undergo a pure shear flow, namely $u(x) = \epsilon x$ on $\partial\Omega$ where ϵ is a symmetric and trace-free matrix. The total (rate of) energy dissipation of the suspension is

$$D[u] = \int_\Omega \mu \, |eu|^2 .$$

This is compared to the energy dissipation of a homogeneous fluid with viscosity μ' for which the solution with pure strain boundary conditions is $u'(x) = \epsilon x$. The dissipation for this homogeneous fluid is thus

$$D[u'] = |\Omega|\, \mu' \, |\epsilon|^2 .$$

Then, the effective viscosity of the suspension is specified to be the viscosity that makes a homogeneous fluid dissipate the same energy as the suspension. This amounts to equating

$$D[u] = |\Omega|\, \mu' \, |\epsilon|^2$$

and gives a formula $\mu' = \mu'(\mu, \phi, \dots)$. Of course this method depends on a good (explicit) computation of $D[u]$. This is done by assuming that the particles are single particles and using explicit solutions to the single particle problem. This method of defining the effective viscosity by equating dissipation functionals is the most prominent one in all articles mentioned here.

The authors of [KRM67] impose the pure strain boundary condition for a domain that becomes infinite in the limit in order to circumvent boundary effects. This disadvantage was overcome only in 2012 by Haines and Mazzucato [HM12] when they proved, simultaneously bounding the power of the next order term:

$$\left| \mu_{\text{eff}} - \mu \left(1 + \frac{5}{2}\phi \right) \right| \le C \mu \phi^{\frac{3}{2}} .$$

They consider a fixed domain with pure strain boundary conditions with particle positions fixed to a lattice. In this sense, their result can also be considered a type of periodic homogenization. Before the proof of this result, with the invention of the so-called two-scale method, a lot of results in periodic homogenization could be obtained. In [LSP85] the periodic homogenization of the Navier-Stokes equation is discussed. For the first time, the effective viscosity is not determined by an asymptotic or a dissipation functional method, but as a prefactor of the strain in the homogenized equation. In their paper the authors derive a homogenized Navier-Stokes equation up to terms of order ϕ that includes the term ([LSP85, p. 13])

$$\operatorname{div} \left(2 \left(1 + \frac{5}{2} \phi \rho \right) e u \right) .$$

Almog and Brenner [AB98] consider non-constant volume fraction and ensemble averages and obtain an effective viscosity field $\mu(x)$ which confirms Einstein's formula. Also here the effective viscosity appears inside the Stokes equation. They also recover the results up to ϕ^2 with a second factor 6.95. Both results are not completely rigorous, though.

The articles [LSP85, AB98] take an approach to the problem of effective viscosity that is different from the dissipation functional approach. Using the comparison of energy dissipation to determine the effective viscosity is physically sound, since the dissipation D is a quantity that can be measured experimentally. Nevertheless one might ask whether the solution to the Stokes equation of the suspension is close in some sense to the solution of the Stokes equation for a homogeneous fluid with the effective viscosity, i.e. the solution of

$$-\operatorname{div}(\mu_{\text{eff}} eu) + \nabla p = f.$$

This looks a bit different from the usual Stokes equation, but recalling that we have

$$\sigma_{\text{eff}} = 2\mu_{\text{eff}} eu - p\,\text{Id}, \quad -\operatorname{div}_{\text{eff}} \sigma = f,$$

this is the natural form of the equation when μ depends on the space variable. The approach just described is also the one that will be taken by this thesis.

Although not directly in the line of research of the effective viscosity we mention [All90a, All90b] where the periodic homogenization of the Stokes equation for suspensions is dealt with rigorously. It is shown that, for Dirichlet boundary conditions, the limit equations are either the Stokes equation if the particles are very small, Darcy's law for large particles or the Brinkman equation for the intermediate regime. The non-periodic homogenization with Dirichlet boundary conditions at the particles is, with increasing levels of generality, discussed in the papers [DGR08, Hil16, HMS17] for bounded domains. They obtain results for the homogenized equation given that the kinetic energy of the empirical measure in phase space is bounded and the first two moments in the momentum-space converge. The results from [DGR08] were reproven by Höfer and Velázquez in [HV18]. They use the so-called method of reflections which will also play an important role in this thesis.

The method of reflections for several particles was first introduced by Smoluchowski in [Smo11] and used extensively in the physical literature to solve all kinds of problems involving several particles. The first mathematically rigorous proof for the convergence of the method with boundary conditions suited for the treatment of sedimenting particles was given in [Luk89]. There the analogy between electrostatics and Stokes fluids was already used to obtain convergence of the method for the electrostatic situation, too. The article also employs extremum principles similar to those used in this thesis. In [HV18] the method is revisited thoroughly and convergence results are proven rigorously, with a version for unbounded domains by means of weighted summation. The method is then used to reprove the result from [DGR08], even extended to unbounded domains. In [JO04] a version of the method of reflections was used to prove bounds for the sedimentation speed of dilute suspensions.

Finally, coming back to vicosity, suspensions of active particles (micro-swimmers) are a very active field of research. In the publications by Haines et al. [HABK08, HABK12] corrections to Einstein's formula are found in a particular case of prolate microswimmers. It turns out that the effective vicosity in their case is lower than the one for non-active particles.

There is a multitude of articles in physics, chemistry and engineering about the effective conductivity of heterogeneous media. Since the result about the conductivity is a byproduct rather than an intended result we do not attempt to give a review of literature in this field.

1.5. Structure of the thesis

In Chapter 2 we first introduce the mathematical framework used in the thesis by defining the appropriate function spaces, the weak formulation of the problem and by stating some basic results. We furthermore state the assumptions on the data and the particle distribution and introduce some specific notations that are necessary to derive the main results which will be presented at the end of the chapter with a short outline of the proof, containing the main ideas.

In Chapter 3 we introduce an abstract and an explicit dipole approximation to the problem and derive closeness to the original problem by proving statements for fixed N. The arguments presented are mostly short calculations, step by step carving out properties of dipoles so that finally the closeness result for the dipoles can be proven.

Chapter 4 is concerned with the closeness of the microscopic to the homogenized equation. The proofs are significantly longer and for the most part concerned with the estimation of convolutions with the fundamental solutions to the Poisson and the Stokes equation.

Finally in Chapter 5 we will discuss in what sense the obtained results are optimal and propose possible further research regarding the effective viscosity of suspensions.

2. Setting of the problem and main result

In this chapter we introduce the setting of the problem regarding function spaces, weak formulation and basic results. We furthermore state all the assumptions that will be used in the derivation of the results. Finally we introduce the notation that is necessary and state the main results of this thesis.

2.1. Setting

2.1.1. Function spaces

In order to obtain meaningful weak formulations of the problems (1.2)-(1.5) and (1.6)-(1.11) it is necessary to overcome the problem that, in $\mathbb{R}^3$, there is no Poincaré inequality, which means it is not possible to control the L^2 norm of a function by the L^2 norm of its gradient. So instead of using the classical Sobolev spaces we use so-called homogeneous Sobolev spaces.

Let $\dot{H}^1$ be the closure of functions in $C_c^\infty(\mathbb{R}^3, \mathbb{R})$ with respect to the L^2 norm of the gradient and let $\dot{H}^{-1}$ be its dual. $\dot{H}^1$ is a Hilbert space and every element of $\dot{H}^1$ is a L^1_{loc} function with a weak gradient bounded in L^2. Elements of the dual space $\dot{H}^{-1}$ are for example expressions of the form $\operatorname{div} g$ (understood in the distributional sense) where $g \in L^2(\mathbb{R}^3)$ and functions in $L^{\frac{6}{5}}(\mathbb{R}^3)$ since all elements of $\dot{H}^1$ are in $L^6(\mathbb{R}^3)$ by the Gagliardo-Nirenberg theorem. We will denote the $\dot{H}^1$ pairing with $\langle \cdot, \cdot \rangle$ while we write $(\cdot, \cdot)$ for the $L^p - L^q$ pairing where $\frac{1}{p} + \frac{1}{q} = 1$. For two functions $u, v \in \dot{H}^1$ this means

$$\langle u, v \rangle = (\nabla u, \nabla v) .$$

In this framework the Laplacian $-\Delta w$ (understood in the distributional sense) of $w \in \dot{H}^1$ is an element in $\dot{H}^{-1}$ since

$$-\Delta w[\varphi] = (\nabla w, \nabla \varphi) = \langle w, \varphi \rangle \text{ for all } \varphi \in \dot{H}^1. \tag{2.1}$$

Take any element $f \in \dot{H}^{-1}$. By the Riesz theorem there is $w \in \dot{H}^1$ such that

$$f[\varphi] = \langle w, \varphi \rangle \text{ for all } \varphi \in \dot{H}^1.$$

By equation (2.1) we then say that $-\Delta w = f$. By this identity the solution operator $(-\Delta)^{-1}$ is an isometric isomorphism from $\dot{H}^{-1}$ to $\dot{H}^1$.

The solution operator $(-\Delta)^{-1}$ is given by $w = \Phi^P * f$ where Φ^P is the fundamental solution of the Poisson equation given by

$$\Phi^P(x) = \frac{1}{4\pi}\frac{1}{|x|}.$$

We will from now on drop the notation $f[\varphi]$ and write (f, φ) instead which coincides with the classical notation if $f \in L^{\frac{6}{5}}(\mathbb{R}^3)$.

Note that any function in $\dot{H}^1$ is in $H^1(U)$ for any open and bounded $U \subset \mathbb{R}^3$ because of the Poincaré inequality.

We now introduce the function spaces connected to the Stokes equation. In order to avoid excessive double notation from now on we will not distinguish spaces that have either $\mathbb{R}$ or $\mathbb{R}^3$ as target space. So instead of $C_c^\infty(\mathbb{R}^3, \mathbb{R})$ and $C_c^\infty(\mathbb{R}^3, \mathbb{R}^3)$ we just write $C_c^\infty(\mathbb{R}^3)$. Which target space is meant should always be clear from the context but is usually $\mathbb{R}$ in the Poisson and $\mathbb{R}^3$ in the Stokes case. In order to incorporate the incompressibility condition we define

$$\dot{H}^1_\sigma = \left\{ w \in \dot{H}^1 : \operatorname{div} w = 0 \right\},$$

the space of all functions in $\dot{H}^1$ whose weak divergence vanishes almost everywhere. Here $\dot{H}^1$ needs to be understood as the closure of functions in $C_c^\infty(\mathbb{R}^3, \mathbb{R}^3)$ with respect to the L^2 norm of the gradient. The dual of $\dot{H}^1_\sigma$ is denoted by $\dot{H}^{-1}_\sigma$.

Note that for functions in $\dot{H}^1_\sigma$ the L^2 pairing of the gradients is the same up to a factor of 2 as the L^2 pairing of the symmetric gradients. We denote $ew = \frac{1}{2}\left(\nabla w + \nabla w^T\right)$. Then for functions $v, w \in \dot{H}^1_\sigma \cap C_c^\infty(\mathbb{R}^3, \mathbb{R}^3)$:

$$\begin{aligned}
\left(\nabla v, \nabla w^T\right) &= \int_{\mathbb{R}^3} \partial_i v_j(x) \partial_j w_i(x) \,\mathrm{d}x \\
&= -\int_{\mathbb{R}^3} \partial_j \partial_i v_j(x) w_i(x) \,\mathrm{d}x \\
&= -\int_{\mathbb{R}^3} \partial_i \partial_j v_j(x) w_i(x) \,\mathrm{d}x \\
&= 0,
\end{aligned}$$

since $\partial_j v_j(x) = \operatorname{div} v(x) = 0$. By density this holds for arbitrary $v, w \in \dot{H}^1_\sigma$ and hence

$$2\,(ev, ew) = \frac{1}{2}\left((\nabla v, \nabla w) + \left(\nabla v, \nabla w^T\right) + \left(\nabla v^T, \nabla w\right) + \left(\nabla v^T, \nabla w^T\right)\right) = (\nabla v, \nabla w).$$

Therefore the two inner products imply exactly the same structure on the space $\dot{H}^1_\sigma$ in terms of orthogonality, norm etc.. Since the decomposition of the gradient into a symmetric and a skew-symmetric part is important for what follows in this thesis we introduce some notation.

Note that there is a bijective linear map T between $\mathbb{R}^3$ and the skew-symmetric matrices $\mathbb{R}^{3\times 3}_{\text{skew}}$ given by

$$T(\omega)_{ij} = -\varepsilon_{ijk}\omega_k,$$

where ε_{ijk} is the Levi-Civita symbol. For any $x \in \mathbb{R}^3$ it holds that $(T\omega)\, x = \omega \wedge x$, where we understand the vector product of $x, y \in \mathbb{R}^3$ to be given by $(x \wedge y)_i = \varepsilon_{ijk}x_j y_k$. We will write $A_\omega = T\omega$ and $\omega_A = T^{-1}A$ for $\omega \in \mathbb{R}^3$ and $A \in \mathbb{R}^{3\times 3}_{\text{skew}}$. We will also deliberately switch between the notation $(\nabla u)^{\text{skew}} = \frac{1}{2}\left(\nabla u - \nabla u^T\right)$ and $\omega u = T^{-1}\,(\nabla u)^{\text{skew}}$ and use that $(\nabla u)^{\text{skew}}\, x = \omega u \wedge x$. In this notation we have $\nabla u\, x = eu\, x + \omega u \wedge x$.

Coming back to function spaces, by the Riesz theorem, for any $f \in \dot{H}^{-1}_\sigma$ there is a $w \in \dot{H}^1_\sigma$ such that

$$(f, \varphi) = \langle w, \varphi\rangle \;\text{ for all } \varphi \in \dot{H}^1_\sigma. \tag{2.2}$$

By [Gal94, Lemma V.1.1] we have that if equation (2.2) holds for all $\varphi \in \dot{H}^1_\sigma$, then, there exists $p \in L^2(\mathbb{R}^3)$ such that

$$(f, \varphi) = \langle w, \varphi\rangle + (\operatorname{div}\varphi, p) \;\text{ for all } \varphi \in \dot{H}^1. \tag{2.3}$$

We then say that

$$-\Delta w + \nabla p = f, \tag{2.4}$$

in the weak sense. The solution operator $S^{-1} : \dot{H}^{-1}_\sigma \to \dot{H}^1_\sigma$ that maps f to w is an isometric isomorphism and its inverse S is the so-called Stokes operator.

The solution operator S^{-1} is given by $S^{-1}f = \Phi^S * f$ where Φ^S is the fundamental solution of the Stokes equation, the so-called Oseen tensor, given by

$$\Phi^S(x) = \frac{1}{8\pi}\left(\frac{\mathrm{Id}}{|x|} + \frac{x\otimes x}{|x|^3}\right).$$

The corresponding pressure such that $-\Delta S^{-1}f + \nabla p = f$ is given by $p = \Pi * f$ where

$$\Pi(x) = \frac{1}{4\pi}\frac{x}{|x|^3}.$$

Since the pressure p is merely a Lagrange multiplier ensuring that the velocity field is solenoidal we will write p for every appearing pressure, so that it may change between different equations but also from line to line in one computation.

At this point we want to state the typical decay properties of the fundamental solutions. There is a constant $C > 0$ such that for all $x \in \mathbb{R}^3 \setminus \{0\}$:

$$\begin{aligned} \left|\Phi^P(x)\right| &\leq C\frac{1}{|x|} &\text{and}\quad \left|\Phi^S(x)\right| &\leq C\frac{1}{|x|}, \\ \left|\nabla\Phi^P(x)\right| &\leq C\frac{1}{|x|^2} &\text{and}\quad \left|\nabla\Phi^S(x)\right| &\leq C\frac{1}{|x|^2}, \\ \left|\nabla^2\Phi^P(x)\right| &\leq C\frac{1}{|x|^3} &\text{and}\quad \left|\nabla^2\Phi^S(x)\right| &\leq C\frac{1}{|x|^3}. \end{aligned}$$

For all spaces we will use a 0 as subscript to indicate that the support of that function lies inside the closure of the given domain. E.g. $w \in \dot{H}^1_0(B_1(0))$ means that $w \in \dot{H}^1$ and that $\operatorname{spt} w \subset \overline{B_1(0)}$. Also, for any classical Sobolev spaces, the subscript σ indicates that the weak divergence vanishes.

2.1.2. Weak formulation of the problem

Let us consider for any $N \in \mathbb{N}$ a collection of rigid spherical particles

$$B_i^N := B_{R^N}(X_i^N), \quad i = 1, \ldots, N$$

where $X_i^N \in \mathbb{R}^3$ are the centres and $R^N > 0$ is the radius of all particles so that they all have the same size. Let

$$d_{ij}^N := \left|X_i^N - X_j^N\right| > 2R^N \quad \text{for all } i \neq j.$$

This implies that the particles do not intersect nor touch each other. For future use we set

$$d^N := \min_{1\leq i,j\leq N} d_{ij}^N.$$

The domain of the suspending material is given by

$$\Omega^N = \mathbb{R}^3 \setminus \bigcup_{i=1}^N \overline{B_i^N}.$$

We will drop the superscript N for $B_i, X_i, R, d_{ij}, d, \Omega$ and all other quantities in the further discussion while it is always implicitly understood that they depend on N, but might still use it where it seems appropriate to highlight this dependence.

Given $f \in L^{\frac{6}{5}}(\mathbb{R}^3) \cap L^2(\mathbb{R}^3)$, we define $f^N = f\chi_\Omega$ where χ is the characteristic function. Here, in accordance with our convention, f might be a scalar or a vector valued function depending on the problem.

We state problem (1.2)-(1.5) after rescaling by $\frac{1}{\eta}$:

$$-\Delta u^N = f \quad \text{in } \Omega, \tag{2.5}$$

$$\int_{\partial B_i} \frac{\partial u^N}{\partial n} \, \mathrm{d}S = 0 \quad \text{for } i = 1, \ldots, N, \tag{2.6}$$

$$u^N = c_i \quad \text{on } \overline{B_i} \text{ for } i = 1, \ldots, N, \tag{2.7}$$

$$u^N(x) \to 0 \quad \text{as } |x| \to \infty. \tag{2.8}$$

When dealing with boundary integrals we will always write $\mathrm{d}S$ for the integration with respect to the two dimensional Hausdorff measure confined to the surface that we integrate over. Even though it is not important in this instance , let us fix that by n we will always mean the outward normal of the ball B_i which is the inward normal to Ω. A function u is a weak solution of problem (2.5)-(2.8) if $u \in \dot{H}^1$ (which implies (2.8)), if u is constant on all $\overline{B_i}$ for $i = 1, .., N$ ((2.7)), if for all $\varphi \in \dot{H}^1_0(\Omega)$

$$\int_\Omega \nabla u \cdot \nabla \varphi \, \mathrm{d}x = \int_\Omega f \varphi \, \mathrm{d}x, \tag{2.9}$$

and if

$$\int_{\partial B_i} \frac{\partial u}{\partial n} \, \mathrm{d}S = 0 \text{ for } i = 1, \ldots, N. \tag{2.10}$$

The last condition might seem a bit ambiguous at first glance because u is only in H^1 locally and the trace of the gradient might not exist. However, by equation (2.9) we have $\operatorname{div} \nabla u \in L^2$ and the following statement holds true ([Gal94, Chap. III, Sec. 2, pp 113ff.])

Lemma 2.1. *Let $U \subset \mathbb{R}^3$ be open with Lipschitz boundary. Define the space*

$$L^2_{\mathrm{div}}(U) = \left\{ g \in L^2(U) : \operatorname{div} g \in L^2(U) \right\}.$$

Then there exists a continuous operator $\gamma_1 : L^2_{\mathrm{div}}(U) \to H^{-\frac{1}{2}}(\partial U)$ such that for all $g \in L^2_{\mathrm{div}}(U), w \in H^1(U)$ we have

$$\int_U (g \cdot \nabla w + \operatorname{div} g \cdot w) \, \mathrm{d}x = \int_{\partial U} \gamma_1 g \cdot w \, \mathrm{d}S,$$

And for $g \in C^1(\overline{U}) \cap L^2_{\mathrm{div}}(U)$ we have $\gamma_1 g = g|_{\partial U} \cdot n$.

Remark 2.2. *In Lemma 2.1, the function g might be $\mathbb{R}^3$ or $\mathbb{R}^{3\times 3}$ valued.*

By Lemma 2.1, the expression (2.10) is well-defined, since $\operatorname{div} \nabla u \in L^2$ and certainly $\chi_{\partial B_i} \in H^{\frac{1}{2}}(\partial B_i)$.

The first question is of course whether problem (2.5)-(2.8) has a unique solution. The following lemma gives the affirmative answer.

Lemma 2.3. *Problem* (2.5)-(2.8) *has a unique weak solution in* $\dot{H}^1$.

Before proving Lemma 2.3 we define the space of functions that are constant inside the particles:

$$W^P := \left\{ w \in \dot{H}^1 : \exists\, c_1, \dots, c_N \in \mathbb{R} \text{ s.t. } w = c_i \text{ on } \overline{B_i},\ i = 1, \dots, N \right\}.$$

Proof of Lemma 2.3. There are several ways to prove existence here. One is, to use standard variational arguments considering the minimization of the energy

$$E(w) = \int_{\mathbb{R}^3} \left(\frac{1}{2} |\nabla w|^2 - f^N w \right) \, \mathrm{d}x,$$

in W^P. Note that the domain of integration is in reality Ω since $f^N = 0$ and $\nabla w = 0$ in $\mathbb{R}^3 \setminus \Omega$. For future use it is nevertheless useful to consider the domain of integration to be the whole $\mathbb{R}^3$. All functions in W^P already satisfy (2.7) and (2.8). Since W^P is a closed subspace of $\dot{H}^1$ the direct method gives a minimizer $u \in W^P$. Then, the Euler-Lagrange equation gives that for all $\varphi \in W^P$ it must hold that

$$0 = \int_{\mathbb{R}^3} \left(\nabla u \cdot \nabla \varphi - f^N \varphi \right) \, \mathrm{d}x.$$

Since this holds in particular for all $\varphi \in \dot{H}^1_0(\Omega)$ we have $-\Delta u = f^N$ in Ω whence (2.9) is satisfied. Finally, for fixed i take $\varphi \in W^P$ such that $c_j = \delta_{ji}$. Such a ϕ exists and can e.g. be obtained by solving

$$-\Delta \varphi = 0 \text{ in } \Omega, \varphi = \delta_{ij} \text{ on } \overline{B_j},$$

which is an outer Laplace problem. Then we obtain

$$0 = \int_{\mathbb{R}^3} \nabla u \cdot \nabla \varphi - f^N \varphi \, \mathrm{d}x = \int_{\Omega} \nabla u \cdot \nabla \varphi - f^N \varphi \, \mathrm{d}x = -\sum_{j=1}^{N} \int_{\partial B_j} \varphi \frac{\partial u}{\partial n} \, \mathrm{d}S = -\int_{\partial B_i} \frac{\partial u}{\partial n} \, \mathrm{d}S.$$

This is (2.10). Uniqueness is also standard but for completeness note that by the Euler-Lagrange equation for all minimizers u it must hold that

$$\|\nabla u\|^2_{L^2(\mathbb{R}^3)} = \int_{\mathbb{R}^3} f^N u \, \mathrm{d}x. \tag{2.11}$$

Suppose there are two minimizers u_1, u_2, then (using the Euler-Lagrange equations and (2.11)):

$$
\begin{aligned}
\|\nabla(u_1-u_2)\|^2_{L^2(\mathbb{R}^3)} &= \|\nabla u_1\|^2_{L^2(\mathbb{R}^3)} - 2\int_{\mathbb{R}^3} \nabla u_1 \cdot \nabla u_2 \, \mathrm{d}x + \|\nabla u_2\|^2_{L^2(\mathbb{R}^3)} \\
&= \int_{\mathbb{R}^3} f^N u_1 \, \mathrm{d}x - \int_{\mathbb{R}^3} f^N u_1 \, \mathrm{d}x - \int_{\mathbb{R}^3} f^N u_2 \, \mathrm{d}x + \int_{\mathbb{R}^3} f^N u_2 \, \mathrm{d}x \\
&= 0.
\end{aligned}
$$

Hence $u_1 = u_2$, that is the minimizer is unique.

□

Now we state problem (1.6)-(1.11) after rescaling by $\frac{1}{\mu}$.

$$
\begin{aligned}
-\Delta u + \nabla p &= f \quad \text{in } \Omega, && (2.12)\\
\operatorname{div} u &= 0 \quad \text{in } \Omega, && (2.13)\\
\int_{\partial B_i} \sigma n \, \mathrm{d}S &= 0 \quad \text{for } i = 1,..,N, && (2.14)\\
\int_{\partial B_i} (x - X_i) \wedge (\sigma n) \, \mathrm{d}S &= 0 \quad \text{for } i = 1,..,N, && (2.15)\\
u &= V_i + \omega_i \wedge (x - X_i) \quad \text{on } \overline{B_i} \text{ for } i = 1,..,N, && (2.16)\\
u(x) &\to 0 \quad \text{as } |x| \to \infty, && (2.17)
\end{aligned}
$$

where

$$
\sigma = -p \operatorname{Id} + 2eu, \quad eu = \frac{1}{2}(\nabla u + \nabla u^T).
$$

A function u is a weak solution of problem (2.12)-(2.17) if $u \in \dot{H}^1_\sigma$ (which implies (2.13),(2.17)), if u is a rigid body motion on all $\overline{B_i}$ for $i = 1,..,N$ (this is (2.16)), if for all $\varphi \in \dot{H}^1_{\sigma,0}(\Omega)$

$$
\int_\Omega \nabla u \cdot \nabla \varphi \, \mathrm{d}x = \int_\Omega f \cdot \varphi \, \mathrm{d}x, \tag{2.18}
$$

and if (2.14), (2.15) are satisfied. Here ∇u and p are a priori only in L^2 and the trace of σ might not exist so that (2.14), (2.15) may not be well-defined. We can use Lemma 2.1 to resolve this problem. In order to see that $\operatorname{div} \sigma \in L^2$ it is is useful to introduce the so-called reciprocal principle (or theorem) (see, e.g. [HB65]). For any $p \in L^2(\mathbb{R}^3)$ and $w \in \dot{H}^1_\sigma$ we write $\sigma = 2ew - p \operatorname{Id}$. Then for $v, w \in \dot{H}^1_\sigma$ we have

$$\begin{aligned}\int_{\mathbb{R}^3} \nabla w \cdot \nabla v \, \mathrm{d}x &= 2 \int_{\mathbb{R}^3} ew \cdot ev \, \mathrm{d}x \\ &= 2 \int_{\mathbb{R}^3} ew \cdot \nabla v \, \mathrm{d}x \\ &= \int_{\mathbb{R}^3} \sigma \cdot \nabla v \, \mathrm{d}x.\end{aligned}$$

In the first step we used that the scalar product of a symmetric and a skew-symmetric matrix is zero ($\nabla v = ev + (\nabla v)^{\text{skew}}$), while in the second, we used, that v is divergence-free whence $\mathrm{Id} \cdot \nabla v = \operatorname{div} v = 0$. The name reciprocal principle comes from the fact that the same equality holds for interchanged w, v. Note that, if w satisfies (2.16), then, because $ew = 0$ in B_i for all $i = 1, \dots, N$ we can write

$$\int_{\mathbb{R}^3} \nabla w \cdot \nabla v \, \mathrm{d}x = \int_{\Omega} \sigma \cdot \nabla v \, \mathrm{d}x.$$

Now take a function $u \in \dot{H}^1_\sigma$ that already satisfies (2.12),(2.16). For all $\varphi \in \dot{H}^1_{\sigma,0}(\Omega)$ we have by (2.18) and by the reciprocal principle

$$\int_{\Omega} \sigma \cdot \nabla \varphi \, \mathrm{d}x = \int_{\Omega} f \cdot \varphi \, \mathrm{d}x,$$

whence $\operatorname{div} \sigma = f$ in Ω and hence $\operatorname{div} \sigma \in L^2(\Omega)$. By Lemma 2.1 $\sigma n \in H^{-\frac{1}{2}}(\partial B_i)$ for all $i = 1, \dots, N$. We certainly have $e_k \in H^{\frac{1}{2}}(\partial B_i)$ for $k = 1, 2, 3$ and therefore (2.14) is well-defined since all three components are well-defined. Now we write

$$e_k \cdot ((x - X_i) \wedge (\sigma n)) = (\sigma n) \cdot (e_k \wedge (x - X_i))$$

where we used the vector rule $A \cdot (B \wedge C) = C \cdot (A \wedge B)$ for $A, B, C \in \mathbb{R}^3$. Since $e_k \wedge (x - X_i) \in H^1(B_i)$ we have $e_k \wedge (x - X_i) \in H^{\frac{1}{2}}(\partial B_i)$ and we obtain that (2.15) is well-defined componentwise.

Before proving existence let us state a consequence of the reciprocal principle:

Lemma 2.4. *Let $u \in \dot{H}^1_\sigma$ satisfy* (2.12),(2.16)*. Then for any $\varphi \in \dot{H}^1_\sigma$:*

$$\int_{\mathbb{R}^3} \nabla u \cdot \nabla \varphi \, \mathrm{d}x = -\sum_{i=1}^N \int_{\partial B_i} \varphi \cdot (\sigma n) \, \mathrm{d}S + \int_{\mathbb{R}^3} f^N \cdot \varphi \, \mathrm{d}x.$$

Proof. We use the reciprocal principle.

$$\begin{aligned}\int_{\mathbb{R}^3} \nabla u \cdot \nabla \varphi \, \mathrm{d}x &= \int_\Omega \sigma \cdot \nabla \varphi \, \mathrm{d}x \\ &= -\sum_{i=1}^{N} \int_{\partial B_i} \varphi \cdot (\sigma n) \, \mathrm{d}S + \int_{\mathbb{R}^3} f^N \cdot \varphi \, \mathrm{d}x,\end{aligned}$$

integrating by parts and using that $\operatorname{div} \sigma = f^N$ weakly in Ω and that n is the outward normal of the ball. □

The proof of existence of a weak solution to (2.12)-(2.17) is analogous to the one for the Poisson equation. We need to consider the subspace of functions that are rigid body motions inside the particles:

$$W^S := \left\{ w \in \dot{H}^1_\sigma : \exists \, V_1, \ldots, V_N, \omega_1, \ldots, \omega_N \in \mathbb{R}^3 \text{ s.t. } w(x) = V_i + \omega_i \wedge (x - X_i) \text{ on } \overline{B_i}, i = 1, \ldots, N \right\}.$$

Lemma 2.5. *Problem* (2.12)-(2.17) *has a unique weak solution in* $\dot{H}^1_\sigma$.

Proof. Consider

$$E(w) = \int_{\mathbb{R}^3} \left(|ew|^2 - f^N \cdot w \right) \, \mathrm{d}x,$$

in W^S. Note that for $w \in \dot{H}^1_\sigma$,

$$E(w) = \int_{\mathbb{R}^3} \left(\frac{1}{2} |\nabla w|^2 - f^N \cdot w \right) \, \mathrm{d}x,$$

which also justifies the use of the same symbol as for the Poisson energy. All functions in W^S already satisfy (2.13),(2.16) and (2.17). Since W^S is a closed subspace of $\dot{H}^1_\sigma$ the direct method gives a minimizer $u \in W^S$. Then, the Euler-Lagrange equation gives that for all $\varphi \in W^S$ it must hold that

$$0 = \int_{\mathbb{R}^3} \nabla u \cdot \nabla \varphi - f^N \cdot \varphi \, \mathrm{d}x.$$

Now take all $\varphi \in \dot{H}^1_{\sigma,0}(\Omega)$ to get that u is a weak solution of $-\Delta u + \nabla p = f^N$ in Ω. Next, fix i and take a function $\varphi \in W^S_0(\mathbb{R}^3 \setminus \cup_{j \neq i} \overline{B_j})$. Now, using Lemma 2.4:

$$
\begin{aligned}
0 &= \int_{\mathbb{R}^3} \nabla u \cdot \nabla \varphi \, \mathrm{d}x - \int_{\mathbb{R}^3} f^N \cdot \varphi \, \mathrm{d}x \\
&= -\sum_{j=1}^{N} \int_{\partial B_j} \varphi \cdot (\sigma n) \, \mathrm{d}S + \int_{\mathbb{R}^3} f^N \cdot \varphi \, \mathrm{d}x - \int_{\mathbb{R}^3} f^N \cdot \varphi \, \mathrm{d}x \\
&= -\int_{\partial B_i} \varphi \cdot (\sigma n) \, \mathrm{d}S.
\end{aligned}
$$

By taking $\varphi = e_1, e_2, e_3$ on $\overline{B_i}$ we obtain

$$
\int_{\partial B_i} \sigma n \, \mathrm{d}S = 0.
$$

Again, the existence follows from solving the homogeneous Stokes equation outside the particles. For $\omega \in \mathbb{R}^3$ choosing $\varphi = \omega \wedge (x - X_i)$ we have

$$
\begin{aligned}
0 &= \int_{\partial B_i} (\omega \wedge (x - X_i)) \cdot (\sigma n) \, \mathrm{d}S \\
&= \int_{\partial B_i} \omega \left((x - X_i) \wedge (\sigma n) \right) \, \mathrm{d}S.
\end{aligned}
$$

By the choices $\omega = e_1, e_2, e_3$ we get

$$
\int_{\partial B_i} \left((x - X_i) \wedge (\sigma n) \right) \, \mathrm{d}S = 0.
$$

The proof of uniqueness works exactly the same way as for problem (2.5)-(2.8).

□

We will from now on ignore the superscripts P and S whenever the argument or statement is the same for both cases in order to minimize unnecessary repetitions.

2.2. Assumptions

We set $\phi = NR^3$. Then, this is, up to the factor $\frac{1}{L^3}$ the volume fraction of the particles in the large ball $B_L(0)$. In this thesis we will assume that the following requirements are met by the sequence of particle configurations:

(1) There is some $L > 0$ such that $|X_i| + R < L$ for all $i = 1, \dots, N$.

(2) There is some constant $C > 0$ such that $N^{-\frac{1}{3}} \leq Cd$.

(3) The particles are well separated in the sense that $d \geq 4R$.

(4) The quantity $\phi \log N \to 0$ as $N \to \infty$.

Condition **(1)** ensures that all particles are contained in some large ball $B_L(0)$. Condition **(2)** implies that the minimal particle distance is comparable to the mean particle distance. This is a very common assumption (see, e.g. [DGR08]). Condition **(3)** ensures that the balls $B_{2R}(X_i)$ are still disjoint and we can modify functions in the vicinity of the particles without those modifications influencing each other. In principle the factor 4 can be replaced by any number > 2 but for ease of computations we use 4. Note that **(3)** is implied by **(2)** if $\phi < \frac{1}{64C}$. In particular this is the case for large N if $\phi \to 0$ which is implied by **(4)**. The last condition is a bit stronger than the minimal assumption $\phi \to 0$ as $N \to \infty$ describing the regime we consider. The stronger version **(4)** is necessary so that certain sums stay negligible.

We will make the following assumptions for f:

(i) $f \in L^{\frac{6}{5}}(\mathbb{R}^3)$;

(ii) $f \in C^{0,\alpha}(\mathbb{R}^3)$ for some $\alpha > 0$.

Here, we see $C^{0,\alpha}(\mathbb{R}^3)$ as a normed space. In consequence $f \in C^{0,\alpha}(\mathbb{R}^3)$ does not only mean, that f is continuous and the corresponding Hölder seminorm is bounded but also that $f \in L^\infty(\mathbb{R}^3)$. Together with **(i)** this implies that $f \in L^p(\mathbb{R}^3)$ for every $p \in [\frac{6}{5}, \infty]$. In particular $f \in L^2(\mathbb{R}^3)$.

In the following universal constants $C > 0$ will often appear in statements. They never depend on N, R, d and $X_1, \dots, X_N$ and other N-dependent quantities but possibly on f unless otherwise stated. When constants appear they might change their value from line to line without indication.

2.3. The main result

In order to state the result that compares the microscopic solutions of problem (2.5)-(2.8) and (2.12)-(2.17) to the solutions of certain homogenized problems, it is necessary to define some kind of limit volume density. It will prove useful to use a coarse grained density like in [NV06] where this was applied in the context of a capacity density.

Definition 2.6. *Let $s^N > 0$ be a sequence such that $s^N \log N \to 0$ as $N \to \infty$. In particular $s^N \to 0$ as $N \to \infty$. Let $\mathbb{R}^3$ be decomposed into half-open disjoint cubes A_j of side length s^N where $j \in \mathbb{Z}^3$. Let $n(A_j)$ be the number of particles (particle centres X_i) in A_j, i.e.*

$$n(A_j) = \int_{A_j} \sum_{i=1}^{N} \delta_{X_i}.$$

Then we define the rescaled averaged particle volume density ρ^N to be constant on each of the cubes A_j and for $x \in A_j$ the value $\rho^N(x)$ is given by

$$\rho^N(x) = \frac{4\pi}{3} \frac{1}{N(s^N)^3} n(A_j).$$

Notice that $\phi\rho^N = \frac{4\pi}{3}\frac{R^3}{s^3}n(A_j)$ is the local volume density of the particles in each cube. This vanishes in the limit $N \to \infty$, since by assumption **(2)** $n(A_j) \le C\frac{s^3}{d^3} \le Cs^3N$ and hence $\phi\rho^N \le CNR^3 = C\phi$. Therefore it is necessary to rescale by the volume fraction, in order to obtain a quantity that does not converge to zero. On the other hand ρ^N is, up to numerical factors, the averaged number density of the particles. Since all particles are contained in a big ball (assumption **(1)**), ρ^N will, for large N be compactly supported in $B_{L+1}(0)$. By assumption **(2)** ρ^N is uniformly bounded in L^∞ and, combining both properties, will therefore for all $p \in [1,\infty]$ have a subsequence with a weak(-$*$) limit in L^p. We need some additional regularity of the limit density ρ. We will assume

(5) $\rho^N \rightharpoonup \rho$ in some $L^p(\mathbb{R}^3)$, $p > 3$;

(6) $\rho \in W^{1,\infty}(\mathbb{R}^3)$.

Assumption **(5)** ensures that the whole sequence converges. Otherwise the results hold for a subsequence. Assumption **(6)** is really only an assumption about $\nabla\rho$ since by the bounds derived on ρ^N, the function ρ must be in L^∞.

As the dipoles used for approximation of the microscopic problem are singular, we must define a domain that leaves a bit more space for the particles. Let $r^N = \max(2R, \delta^N)$ where $\delta^N > 0$ such that $\frac{1}{(\delta^N)^2 N} \to 0$ and $\frac{\delta}{d} \to 0$ as $N \to \infty$. We introduce the following domain: $\Omega_\delta^N = \mathbb{R}^3 \setminus \cup_{i=1}^N B_r(X_i)$.

The goal of the thesis is to prove a (non-periodic) result, that shows that Einstein's formula indeed appears in the Stokes equation for a homogenized fluid. We know that $\rho^N \rightharpoonup \rho$. One might think that leaving the volume density ϕ constant and letting $N \to \infty$ should lead (by Einstein's result) to a homogenized equation of the form

$$-\operatorname{div}\left(\left(1 + \frac{5}{2}\phi\rho\right)e\bar{u}\right) + \nabla p = f, \tag{2.19}$$

or something similar. However, Einstein's result is just a linear approximation and therefore, by

$$\mu_{\text{eff}} = \mu\left(1 + \frac{5}{2}\phi\rho + o(\phi)\right), \tag{2.20}$$

the homogenized equation should have the form

$$-\operatorname{div}\left(\left(1 + \frac{5}{2}\phi\rho + o(\phi)\right)e\bar{u}\right) + \nabla p = f.$$

Proving a result that relates the microscopic solutions to the solution of this equation for fixed ϕ would amount to proving a functional dependence of the form $\mu_{\text{eff}} = \mu_{\text{eff}}(\phi)$ for finite values of ϕ. Even for the second order term of the expansion in $\phi = 0$ different formal results exist and no rigorous result is available. Therefore proving a such a functional dependence for finite ϕ is out of the scope of this thesis.

So instead of keeping ϕ fixed, we let it approach 0 as $N \to \infty$. Then of course the candidate for the homogenized equation (2.19) is not fixed anymore but depends on N (through ϕ). The limit of this equation as $\phi \to 0$ is

$$-\Delta \bar{u} + \nabla p = f,$$

which is again the Stokes equation with unchanged viscosity and is not useful as a limit equation. Therefore we are forced to dismiss the idea of *one* limiting equation and instead compare the solutions $\bar{u}$ of (2.19) to the microscopic solutions u for each N. Since the viscosity in (2.19) is $1 + \frac{5}{2}\phi\rho$ and we know that for the suspension the effective viscosity should be given by (2.20).

$$\mu_{\text{eff}} = \mu \left(1 + \frac{5}{2}\phi\rho + o(\phi) \right),$$

the viscosities differ on scale $o(\phi)$. Therefore, we would expect that not only

$$\left\| u^N - \bar{u}^N \right\| \to 0 \text{ as } N \to \infty,$$

for some appropriate norm $\|\cdot\|$, but that the stronger result

$$\left\| u^N - \bar{u}^N \right\| = o(\phi) \text{ as } N \to \infty,$$

holds, or equivalently

$$\frac{1}{\phi} \left\| u^N - \bar{u}^N \right\| \to 0 \text{ as } N \to \infty.$$

Indeed, this result is true, when one takes the L^p norm. The two main results of this thesis read as follows:

Theorem 2.7. *The solution $\bar{u} \in \dot{H}^1$ to the equation*

$$-\operatorname{div}\left((1 + 3\phi\rho) \nabla \bar{u} \right) = (1 - \phi\rho) f,$$

is close to u, the weak solution of (2.5)-(2.8), in the following sense

$$\frac{1}{\phi} \|u - \bar{u}\|_{L^\infty(\Omega_\delta^N)} \to 0, N \to \infty.$$

Furthermore, if $U \subset \mathbb{R}^3$ is of finite measure and $p \in [1, \frac{3}{2}]$, then

$$\frac{1}{\phi} \|u - \bar{u}\|_{L^p(U)} \to 0, N \to \infty.$$

Theorem 2.8. *The solution $\bar{u} \in \dot{H}^1_\sigma$ to the equation*

$$-\operatorname{div}\left((2 + 5\phi\rho)\, e\bar{u}\right) + \nabla p = (1 - \phi\rho) f, \tag{2.21}$$
$$\operatorname{div} \bar{u} = 0, \tag{2.22}$$

is close to u, the solution of (2.12)-(2.17), in the following sense

$$\frac{1}{\phi} \left\| u - \bar{u} \right\|_{L^\infty(\Omega_\delta^N)} \to 0, N \to \infty.$$

Furthermore, if $U \subset \mathbb{R}^3$ is of finite measure and $p \in [1, \frac{3}{2}]$, then

$$\frac{1}{\phi} \left\| u - \bar{u} \right\|_{L^p(U)} \to 0, N \to \infty.$$

Remark 2.9. *Note that*

$$\left|\mathbb{R}^3 \setminus \Omega_\delta^N\right| = Nr^3 \leq C \max(2R, \delta)^3 \leq CNd^3 \max(\frac{R}{d}, \frac{\delta}{d}) \to 0, N \to \infty.$$

Therefore, even in L^∞, the solution of the homogenized equation is close to u on scale ϕ in a volume that is asymptotically the whole $\mathbb{R}^3$.

2.3.1. Strategy of the proof

The strategy is in principle to make the computations of Section 1.3 rigorous. To that end we use successive approximations $u \to v_1 \to \tilde{u} \to \hat{u} \to \bar{u}$ and split the proof in two parts.

The first part is to prove that the dipole approximation $\tilde{u}$ is actually close to the microscopic solution u which is achieved in Chapter 3. For this we take a small detour and first define a related but abstract dipole approximation v_1 defined via projections to subspaces of $\dot{H}^1$ and $\dot{H}^1_\sigma$ incorporating the constant and rigid body boundary conditions on the particles. This method was first used in [Luk89]. In [Hoe16] it is used in a way that is also employed here. Closeness of v_1 to u can be obtained by first proving closeness inside the particles using variational methods and the method of reflections. Closeness in the region outside the particles can then be established using decay properties of the fundamental solutions of the Poissson and the Stokes equation. The structure of the proof closely follows [Hoe16]. Nevertheless, since we take into account rotations for the particles in the Stokes case, some adjustments have to be made and it is necessary to establish a Korn and a Korn-Poincaré inequality for balls with integrated boundary conditions. Using carefully obtained characterizations of the projections we then show that inside the particles v_1 and $\tilde{u}$ are already close and that, again, using the decay of the dipoles, this can be extended to the domain outside the particles.

The second part consists in proving the closeness of $\tilde{u}$ to the solution $\bar{u}$ of the Stokes equation with Einstein viscosity in Chapter 4. This is done by means of the intermediate approximation $\hat{u}$ which is the solution to the equation

$$- \operatorname{div} (\nabla \hat{u} + 5\phi\rho e v) + \nabla p = (1 - \phi\rho) f. \tag{2.23}$$

To prove that $\tilde{u}$ is close to $\hat{u}$ we use the fact that for every point in space the contributions of the particles in a moderately large region around this point are negligible. But further away the number density ρ^N looks approximately like ρ which allows passage from sum to integral. The proof relies heavily on the representation of solutions as convolutions with the fundamental solutions and involves various estimates regarding these convolution integrals.

In order to replace v by $\bar{u}$ in the homogenized equation we first prove that v is already close to $\bar{u}$ namely that

$$\|v - \bar{u}\| \leq C\phi.$$

This is achieved by standard regularity arguments and estimates of the solutions of the homogenized equation in terms of the right hand side. With the same methods it is then possible to prove that the solution to (2.23), $\hat{u}$, is close to the solution of the final equation (2.21), $\tilde{u}$, since their difference satisfies an equation with a right hand side that is already small.

3. The dipole approximation

It is useful to start not with the explicit dipole approximation, but to consider another approximation, using slightly different dipoles, first. In contrast to the explicit dipoles, these can be characterized using variational formulations thereby simplifying the comparison to the microscopic solution. In order to do so we adapt the theory developed in [Hoe16]. With the exception of the Korn and Korn-Poincaré inequality in Lemma 3.9 and Corollary 3.11, which are not needed in [Hoe16], all statements in Section 3.1, Section 3.2 and Section 3.3 concerning the abstract dipoles are statements/ideas from [Hoe16] adapted to the situation of electrostatics and rigid body motions instead of constants respectively.

3.1. Approximation by abstract dipoles

Let us recall the definitions of the solutions for the particle-free problem. The solution v^N to the particle-free Poisson problem is given by

$$-\Delta v^N = f^N \quad \text{in } \mathbb{R}^3, \tag{3.1}$$
$$v^N(x) \to 0 \quad \text{as } |x| \to \infty. \tag{3.2}$$

By the theory for the Poisson equation $v = \Phi^P * f^N$ with $v^N \in H^2_{\mathrm{loc}}(\mathbb{R}^3)$ and $\nabla v^N \in H^1(\mathbb{R}^3)$.

The solution v^N to the particle-free Stokes problem is given by

$$-\Delta v^N + \nabla p = f^N \quad \text{in } \mathbb{R}^3, \tag{3.3}$$
$$\operatorname{div} v^N = 0 \quad \text{in } \mathbb{R}^3, \tag{3.4}$$
$$v^N(x) \to 0 \quad \text{as } |x| \to \infty. \tag{3.5}$$

By the theory for the Stokes equation $v = \Phi^S * f^N$ with $v^N \in H^2_{\mathrm{loc}}(\mathbb{R}^3)$ and $\nabla v^N \in H^1(\mathbb{R}^3)$.

The solutions to the particle-free problems are minimizers of the energies introduced in Subsection 2.1.2.

Lemma 3.1. *The solution v^N of problem* (3.1),(3.2) *is the minimizer in $\dot{H}^1$ of the energy*

$$E(w) = \int_{\mathbb{R}^3} \left(\frac{1}{2} |\nabla w|^2 - f^N w \right) \, \mathrm{d}x.$$

Proof. This follows from the fact that (3.1) is the Euler-Lagrange equation of E. □

Lemma 3.2. *The solution v^N of problem* (3.3)-(3.5) *is the minimizer in $\dot{H}^1_\sigma$ of the energy*

$$E(w) = \int_{\mathbb{R}^3} \left(|ew|^2 - f^N \cdot w \right) \, \mathrm{d}x.$$

Proof. We have for $w \in \dot{H}^1_\sigma$:

$$E(w) = \frac{1}{2} \int_{\mathbb{R}^3} |\nabla w|^2 - \int_{\mathbb{R}^3} f^N w.$$

This implies that the minimizer w of E in $\dot{H}^1_\sigma$ satisfies the following Euler-Lagrange equation:

$$0 = \int_{\mathbb{R}^3} \nabla w \cdot \nabla \varphi - \int_{\mathbb{R}^3} f^N \varphi \text{ for all } \varphi \in \dot{H}^1_\sigma.$$

But this is exactly the weak formulation of (3.3)-(3.5)) and hence $w = v$. □

On the other hand, the solution u to problem (2.5)-(2.8) and (2.12)-(2.17) is the minimizer of E in the space W^P and W^S respectively as seen in Lemma 2.3 and Lemma 2.5.

The fact that u minimizes E in W means that u is the orthogonal projection of v from $\dot{H}^1$ and $\dot{H}^1_\sigma$ to the subspace W^P and W^S respectively. We call $P^P : \dot{H}^1 \to W^P$ and $P^S : \dot{H}^1_\sigma \to W^S$ the orthogonal projections. Then $u = Pv$. That this is the case can be seen by taking any $w \in W$ and observing that, by the weak formulations that v and u satisfy, we have

$$\langle v - u, w \rangle = \int_{\mathbb{R}^3} \nabla (v - u) \cdot \nabla w \, \mathrm{d}x = \int_{\mathbb{R}^3} (f^N - f^N) w \, \mathrm{d}x = 0.$$

This implies that

$$\|v - u\|_{\dot{H}^1} \leq \|v - w\|_{\dot{H}^1} \text{ for all } w \in W.$$

By choosing a suitable function w one can thus get an estimate for $\|u - v\|_{\dot{H}^1}$. This will be part of the proof of Theorem 3.28.

It will become clear that we need to consider the L^∞ norm to obtain $\|u - v\| \leq C\phi$, though. In order to get L^∞ estimates it is useful to work with the so-called method of reflections. This is due to the fact that the projection onto W is not so easy to characterize. The method of reflections works with solutions of single particle problems. The single particle spaces involved, are much easier to characterize than W. For this we first define the particle wise versions of W^P and W^S:

$$W_i^P = \left\{ w \in \dot{H}^1 : w = c \text{ on } \overline{B_i},\ c \in \mathbb{R} \right\},$$
$$W_i^S = \left\{ w \in \dot{H}^1_\sigma : w = V + \omega \wedge (x - X_i) \text{ on } \overline{B_i},\ V, \omega \in \mathbb{R}^3 \right\}.$$

Since W_i is a closed subspace there is an orthogonal projection $P_i^P : \dot{H}^1 \to W_i^P$ and $P_i^S : \dot{H}^1_\sigma \to W_i^S$ respectively. Notice that $W = \cap_{i=1}^N W_i$. The orthogonal complement of W_i has a useful characterization:

Lemma 3.3.

$$\left(W_i^P\right)^\perp = \left\{ w \in \dot{H}^1 : -\Delta w = 0 \textit{ in } \mathbb{R}^3 \setminus \overline{B_i}, \quad \int_{\partial B_i} \frac{\partial w}{\partial n} \, \mathrm{d}S = 0 \right\},$$
$$\left(W_i^S\right)^\perp = \left\{ w \in \dot{H}^1_\sigma : -\Delta w + \nabla p = 0 \textit{ in } \mathbb{R}^3 \setminus \overline{B_i}, \quad \int_{\partial B_i} \sigma n \, \mathrm{d}S = 0, \quad \int_{\partial B_i} (x - X_i) \wedge (\sigma n) \, \mathrm{d}S = 0 \right\}.$$

Proof. Any element s of the right hand side of the first line is orthogonal to $w \in W_i^P$:

$$\begin{aligned}
\langle w, s \rangle &= \int_{\mathbb{R}^3} \nabla w \cdot \nabla s \, \mathrm{d}x \\
&= \int_{\mathbb{R}^3 \setminus \overline{B_i}} \nabla w \cdot \nabla s \, \mathrm{d}x \\
&= -\int_{\partial B_i} w \frac{\partial s}{\partial n} \, \mathrm{d}S - \int_{\mathbb{R}^3 \setminus \overline{B_i}} w \Delta s \, \mathrm{d}x \\
&= -c \int_{\partial B_i} \frac{\partial s}{\partial n} \, \mathrm{d}S \\
&= 0
\end{aligned}$$

On the other hand, observe that for $w \in W_i^P$ and $s \in \left(W_i^P\right)^\perp$, using the computation above, we have:

$$0 = \langle w, s \rangle = -\int_{\partial B_i} w \frac{\partial s}{\partial n} \, \mathrm{d}S - \int_{\mathbb{R}^3 \setminus \overline{B_i}} w \Delta s \, \mathrm{d}x.$$

By first considering all $w \in \dot{H}^1_0(\mathbb{R}^3 \setminus \overline{B_i})$ we obtain $-\Delta w = 0$ in $\mathbb{R}^3 \setminus \overline{B_i}$. Then taking $w = 1$ on $\overline{B_i}$ we get

$$\int_{\partial B_i} \frac{\partial s}{\partial n} = 0.$$

Such a w exists, take e.g. $w(x) = \frac{R}{|x - X_i|}$ for $x \in \mathbb{R}^3 \setminus \overline{B_i}$. Now, take any element s of the right hand side of the second line. Then for any $w \in W_i^S$, using the reciprocal principle, we obtain (let $w = V + \omega \wedge (x - X_i)$ on $\overline{B_i}$):

$$\begin{aligned}
\langle w, s\rangle &= \int_{\mathbb{R}^3\setminus\overline{B_i}} \nabla w \cdot \sigma[s] \,\mathrm{d}x \\
&= -\int_{\partial B_i} w \left(\sigma[s]n\right)\,\mathrm{d}S - \int_{\mathbb{R}^3\setminus\overline{B_i}} w \operatorname{div}\sigma[s]\,\mathrm{d}x \\
&= -\int_{\partial B_i} \left(V + \omega\wedge(x - X_i)\right)\left(\sigma[s]n\right)\,\mathrm{d}S \\
&= -V\int_{\partial B_i} \left(\sigma[s]n\right)\,\mathrm{d}S - \omega\int_{\partial B_i} (x - X_i)\wedge\left(\sigma[s]n\right)\,\mathrm{d}S \\
&= 0.
\end{aligned}$$

On the other hand, by the same computation, for any $s \in \left(W_i^S\right)^\perp$ we have

$$0 = \langle w, s\rangle = -\int_{\partial B_i} w\left(\sigma[s]n\right)\,\mathrm{d}S - \int_{\mathbb{R}^3\setminus\overline{B_i}} w\operatorname{div}\sigma[s]\,\mathrm{d}x.$$

First consider all $w \in \dot H^1_{\sigma,0}(\mathbb{R}^3\setminus\overline{B_i})$ to obtain $0 = -\operatorname{div}\sigma[s] = -\Delta s + \nabla p$ in $\mathbb{R}^3\setminus\overline{B_i}$. Then use that

$$0 = -V\int_{\partial B_i}\left(\sigma[s]n\right)\,\mathrm{d}S - \omega\int_{\partial B_i}(x - X_i)\wedge\left(\sigma[s]n\right)\,\mathrm{d}S.$$

with $V, \omega = e_1, e_2, e_3$. □

A function with the property that

$$0 = \int_{\partial B_i} w\frac{\partial s}{\partial n}\,\mathrm{d}S = \int_{\overline{B_i}} \Delta w\,\mathrm{d}x$$

or, in the Stokes case,

$$0 = \int_{\partial B_i} \sigma n\,\mathrm{d}S = \int_{\overline{B_i}} \operatorname{div}\sigma\,\mathrm{d}x,$$

is usually called a dipole since the first moment of the charge and the force distribution respectively vanishes inside the ball.

We come back to our goal to approximate u by v. We already know that, following the idea from Section 1.3, it makes sense to subtract from v at every ball the dipole preventing v from beeing constant or a rigid body motion respectively. Let $Q_i = \mathrm{Id} - P_i$ be the orthogonal projection onto $W_i^\perp$. We know that $v - Q_i v = P_i v \in W_i$ is a constant or a rigid body motion respectively on the ball B_i, hence $Q_i v$ is the dipole we are looking for. As explained in Section 1.3, subtracting $Q_i v$ only helps with the boundary condition on B_i, so we have to subtract the dipole for all balls which gives rise to the first approximation

$$v_1 = \left(\mathrm{Id} - \sum_{i=1}^{N} Q_i \right) v.$$

Of course this approximation will not be constant on all balls, since the additional $Q_j v$ for $j \neq i$ will have nonvanishing contributions on B_i. But since the dipoles solve the homogeneous Stokes equation outside B_i, the function v_1 still satisfies the same equation as v and u in Ω. The approximation v_1 is already good enough for our purposes. Let us nevertheless repeat the process of subtracting dipoles in order to make the functions closer to constant or to rigid body motions on the balls. This leads to approximations v_k given by

$$v_k = \left(\mathrm{Id} - \sum_{i=1}^{N} Q_i \right)^k v. \tag{3.6}$$

The idea is that taking $k \to \infty$ one should have

$$P = \lim_{k \to \infty} \left(\mathrm{Id} - \sum_{i=1}^{N} Q_i \right)^k .$$

This would imply that $v_k \to Pv = u$ as $k \to \infty$.

Note that one could also take another approach and project onto W_i successively instead of simultaneously, defining

$$v^k = \left(P_N P_{N-1} \dots P_2 P_1 \right)^k v,$$

Also in this case, one could hope that

$$P = \lim_{k \to \infty} \left(P_N P_{N-1} \dots P_2 P_1 \right)^k .$$

This method was used in [Luk89], and convergence of the method was shown, by proving a lower bound on the angle between the different single particle subspaces relying on the geometry of the particle arrangement. However, definition (3.6) is better suited for our purposes because we have good control of the error term $v_1 - P_i v = -\sum_{j \neq i} Q_i v$ next to the particle B_i.

Before attempting to prove that the v_k converge to u we need a better understanding of the projections P_i and Q_i respectively.

3.2. Chracterization of $W_i^\perp$ and P_i

Lemma 3.4. *For $w \in \left(W_i^P\right)^\perp$ we have*

$$\fint_{\partial B_i} w \, \mathrm{d}S = 0, \tag{3.7}$$

and hence for $w \in \dot{H}^1$ the projection to W_i^P satisfies

$$P_i^P w(x) = \fint_{\partial B_i} w \, \mathrm{d}S \quad \text{for all } x \in \overline{B_i}.$$

Remark 3.5. *Here and in the following, for some μ-measurable set $U \subset \mathbb{R}^3$ with bounded μ-measure and a μ-measurable function g we set*

$$\fint_U g \, \mathrm{d}\mu = \frac{1}{\mu(U)} \int_U g \, \mathrm{d}\mu.$$

In particular

$$\fint_{\partial B_i} w \, \mathrm{d}S = \frac{1}{4\pi R^2} \int_{\partial B_i} w \, \mathrm{d}S.$$

Proof of Lemma 3.4. The idea is to use an explicit test function, for which we know the Neumann boundary data on the sphere. Take φ such that $-\Delta\varphi = 0$ in $\mathbb{R}^3 \setminus \overline{B_i}$, $\varphi = 1$ on $\overline{B_i}$. Then $\varphi \in W_i^P$ and

$$\varphi(x) = \frac{R}{|x - X_i|},$$

for $|x - X_i| > R$. For $w \in \left(W_i^P\right)^\perp$ we have

$$0 = \langle w, \varphi \rangle = \int_{\mathbb{R}^3 \setminus \overline{B_i}} \nabla w \cdot \nabla \varphi = -\int_{\partial B_i} \frac{\partial \varphi}{\partial n} w \, \mathrm{d}S = -\int_{\partial B_i} \frac{1}{R} w \, \mathrm{d}S.$$

This implies (3.7). Now, for $w \in \dot{H}^1$, we use that $P_i^P w = w - Q_i^P w$, with $Q_i^P w \in \left(W_i^P\right)^\perp$ and that $P_i^P w$ is constant on $\overline{B_i}$ and in particular on ∂B_i. Then, for $x \in \overline{B_i}$

$$P_i^P w(x) = \fint_{\partial B_i} P_i^P w \, \mathrm{d}S = \fint_{\partial B_i} w - Q_i^P w \, \mathrm{d}S = \fint_{\partial B_i} w \, \mathrm{d}S.$$

□

Lemma 3.6. *For $w \in \left(W_i^S\right)^{\perp}$ we have*

$$\fint_{\partial B_i} w \, \mathrm{d}S = 0 \quad \textit{and} \quad \fint_{\partial B_i} (x - X_i) \wedge w \, \mathrm{d}S = 0. \tag{3.8}$$

Hence, for $w \in \dot{H}^1_\sigma$ the projection to W_i^S satisfies $P_i^S w(x) = V + \omega \wedge (x - X_i)$ for all $x \in \overline{B_i}$ with

$$V = \fint_{\partial B_i} w \, \mathrm{d}S \quad \textit{and} \quad \omega = \frac{3}{2R^2} \fint_{\partial B_i} (x - X_i) \wedge w \, \mathrm{d}S. \tag{3.9}$$

Proof. Again we use, for linear motion and rotation seperately, explicit testfunctions. This time the drag σn plays the role of the Neumann boundary data. Let $w \in \left(W_i^S\right)^{\perp}$. Take φ such that $-\Delta\varphi + \nabla p = 0$, $\operatorname{div}\varphi = 0$ in $\mathbb{R}^3 \setminus \overline{B_i}$ and $\varphi = V$, $V \in \mathbb{R}^3$ on $\overline{B_i}$. Then $\varphi \in W_i^S$ and

$$\varphi(x) = \frac{3}{4} R \left(\frac{V}{|x - X_i|} + \frac{(V \cdot (x - X_i))\,(x - X_i)}{|x - X_i|^3} \right) + \frac{1}{4} R^3 \left(\frac{V}{|x - X_i|} - 3 \frac{(V \cdot (x - X_i))\,(x - X_i)}{|x - X_i|^3} \right),$$

for $|x - X_i| > R$. The drag on the sphere is then given by $\sigma[\varphi] n = \frac{3}{2R} V$ (Stokes drag). Then by the same computation as in Lemma 3.3 we know

$$\begin{aligned} 0 &= \langle w, \varphi \rangle \\ &= - \int_{\partial B_i} w \cdot (\sigma[\varphi] n) \, \mathrm{d}S - \int_{\mathbb{R}^3 \setminus \overline{B_i}} w \cdot \operatorname{div} \sigma[\varphi] \, \mathrm{d}x \\ &= - \frac{3}{2R} V \cdot \int_{\partial B_i} w \, \mathrm{d}S. \end{aligned}$$

By setting $V = e_1, e_2, e_3$ this proves the first part of (3.8).

Now we take φ such that $-\Delta\varphi + \nabla p = 0$, $\operatorname{div}\varphi = 0$ in $\mathbb{R}^3 \setminus \overline{B_i}$ and $\varphi = \omega \wedge (x - X_i)$, $\omega \in \mathbb{R}^3$ on $\overline{B_i}$. Then $\varphi \in W_i^S$ and

$$\varphi(x) = R^3 \frac{\omega \wedge (x - X_i)}{|x|^3},$$

for $|x - X_i| > R$. The drag on the sphere is then given by $(\sigma[\varphi] n) = -\frac{3}{R} \omega \wedge (x - X_i)$. Then

$$\begin{aligned}
0 &= \langle w, \varphi \rangle \\
&= -\int_{\partial B_i} w \cdot (\sigma[\varphi] n) \; \mathrm{d}S - \int_{\mathbb{R}^3 \setminus \overline{B_i}} w \cdot \operatorname{div} \sigma[\varphi] \; \mathrm{d}x \\
&= \frac{3}{R} \int_{\partial B_i} w \cdot (\omega \wedge (x - X_i)) \; \mathrm{d}S \\
&= \frac{3}{R} \omega \cdot \int_{\partial B_i} (x - X_i) \wedge w \; \mathrm{d}S.
\end{aligned}$$

Setting $\omega = e_1, e_2, e_3$ we arrive at the second part of (3.8).

Now for $w \in \dot{H}^1_\sigma$ we know that $w - P_i w = Q_i w \in \left(W_i^S\right)^\perp$. On the other hand there are $V, \omega \in \mathbb{R}^3$ such that $P_i w(x) = V + \omega \wedge (x - X_i)$ for all $x \in \overline{B_i}$. But then

$$\begin{aligned}
0 &= \fint_{\partial B_i} Q_i w \; \mathrm{d}S \\
&= \fint_{\partial B_i} w - P_i w \; \mathrm{d}S \\
&= \fint_{\partial B_i} w \; \mathrm{d}S - \fint_{\partial B_i} V + \omega \wedge (x - X_i) \; \mathrm{d}S \\
&= \fint_{\partial B_i} w \; \mathrm{d}S - V,
\end{aligned}$$

and hence the first part of (3.9) is true.

Also, using the vector rule $A \wedge (B \wedge C) = (A \cdot C)B - (A \cdot B)C$ for $A, B, C \in \mathbb{R}^3$:

$$\fint_{\partial B_i} (x - X_i) \wedge (\omega \wedge (x - X_i)) \; \mathrm{d}S = \fint_{\partial B_i} \omega \, |x - X_i|^2 - (x - X_i) \, (\omega (x - X_i)) \; \mathrm{d}S = \frac{2R^2}{3} \omega,$$

and therefore

$$\begin{aligned}
0 &= \fint_{\partial B_i} (x - X_i) \wedge Q_i w \; \mathrm{d}S \\
&= \fint_{\partial B_i} (x - X_i) \wedge (w - P_i w) \; \mathrm{d}S \\
&= \fint_{\partial B_i} (x - X_i) \wedge w \; \mathrm{d}S - \fint_{\partial B_i} (x - X_i) \wedge (V + \omega \wedge (x - X_i)) \; \mathrm{d}S \omega \\
&= \fint_{\partial B_i} (x - X_i) \wedge w \; \mathrm{d}S - \frac{2R^2}{3} \omega,
\end{aligned}$$

whence

$$\omega = \frac{3}{2R^2} \fint_{\partial B_i} (x - X_i) \wedge w \,\mathrm{d}S.$$

□

As a consequence of these characterizations we obtain a Poincaré inequality on the space $\left(W_i^P\right)^\perp$, as well as a (first) Korn inequality and consequently a Korn-Poincaré inequality on $\left(W_i^S\right)^\perp$.

Lemma 3.7. *Let $r > 0$ and $X \in \mathbb{R}^3$. Let $p \in (1, \infty]$ and let*

$$H^p_{X,r} := \left\{ w \in W^{1,p}(B_r(X)) : \int_{\partial B_r(X)} w \,\mathrm{d}S = 0 \right\}.$$

There is a constant $C > 0$ that does not depend on X or r such that for all $w \in H^p_{X,r}$:

$$\|w\|_{L^p(B_r(X))} \leq Cr \|\nabla w\|_{L^p(B_r(X))}$$

Remark 3.8. *This lemma holds for the target space $\mathbb{R}$ as well as for $\mathbb{R}^3$. In the proof we will use the notation for $\mathbb{R}$ with obvious adjustments for the vector case.*

Proof of Lemma 3.7. It is well known, that for closed cones in which $\nabla w = 0$ implies that $w = 0$, such a Poincaré inequality holds. For instructiveness we give the proof here nevertheless. Let us first prove the case $p = \infty$. If $w \in H^\infty_{X,r}$, then w is Lipschitz and hence absolutely continuous on each line whence

$$w(x) = w(X) + \int_0^1 \frac{\mathrm{d}}{\mathrm{d}t} w(tx + (1-t)X) \,\mathrm{d}t = w(X) + (x - X) \cdot \int_0^1 \nabla w(tx + (1-t)X) \,\mathrm{d}t. \quad (3.10)$$

Taking $\fint_{\partial B_r(X)}$ of this expression we arrive at

$$0 = w(X) + \fint_{\partial B_r(X)} (x - X) \cdot \int_0^1 \nabla w(tx + (1-t)X) \,\mathrm{d}t \,\mathrm{d}S.$$

This gives us the estimate

$$|w(X)| \leq r \|\nabla w\|_{L^\infty(B_r(X))}.$$

Applying this to (3.10) we obtain

$$\|w\|_{L^\infty(B_r(X))} \leq 2r \|\nabla w\|_{L^\infty(B_r(X))}.$$

The proof for general $p \in (1, \infty)$ will be done indirectly here (although it can, in this case, be done directly without too much effort). We first show that $\nabla w = 0$ implies $w = 0$. Assume $w \in H^p_{X,r}$ with $\nabla w = 0$. Then, since $B_r(X)$ is connected, there is some $c \in \mathbb{R}$ such that $w = c$ on $B_r(X)$. But then

$$c = \fint_{\partial B_r(X)} w \, \mathrm{d}S = 0,$$

and therefore $w = 0$. We first prove the result for $X = 0$, $r = 1$. $H^p_{0,1}$ is a closed cone, i.e. it is convex and invariant under multiplication with arbitrary $\alpha \in \mathbb{R}$ (this is due to the fact that $\int_{\partial B_r(X)} w \, \mathrm{d}S = 0$ is a linear and closed condition). In the following we will omit the domain $B_1(0)$ in the spaces. Now, for the sake of contradiction, assume there is no $C > 0$ such that for all $w \in H^p_{0,1}$:

$$\|w\|_{L^p} \leq C \|\nabla w\|_{L^p}.$$

Then there is a sequence $w_k \in H^p_{0,1}$ such that $\|w_k\|_{L^p} \geq k \|\nabla w_k\|_{L^p}$. By rescaling we can arrange that $\|w_k\|_{W^{1,p}} = 1$ for all $k \in \mathbb{N}$ (since $H^p_{0,1}$ is a cone, the w_k will still be in $H^p_{0,1}$). But then the sequence is bounded and there is a subsequence (again denoted by w_k) and $w_* \in W^{1,p}$ such that $w_k \rightharpoonup w_*$ in $W^{1,p}$. Note that, since $H^p_{0,1}$ is convex, we have that $w_* \in H^p_{0,1}$. On the other hand, we know that

$$\|\nabla w_k\|_{L^p} \leq \frac{1}{k} \|w_k\|_{L^p} \leq \frac{1}{k}.$$

Hence $\nabla w_k \to 0$ in L^p and $\nabla w_* = 0$ since at the same time $\nabla w_k \rightharpoonup \nabla w_*$. By our foregoing considerations $\nabla w_* = 0$ implies that $w_* = 0$.

But this is of course a contradiction, since by the Rellich embedding theorem $w_k \to w_* = 0$ strongly in L^p and hence $w_k \to 0$ in $W^{1,p}$ contradicting $\|w_k\|_{W^{1,p}} = 1$ for all $k \in \mathbb{N}$.

The case of general X, r follows by noticing that for $w \in H^p_{X,r}$ the translated and rescaled version $w'(x) = w(rx + X)$ is in $H^p_{0,1}$, and $\|w'\|_{L^p} = \|w\|_{L^p}$ while $\|\nabla w'\|_{L^p} = r \|\nabla w\|_{L^p}$.

□

Lemma 3.9. *Let $r > 0$ and $X \in \mathbb{R}^3$. Let $p \in (1, \infty)$ and let*

$$H^p_{\sigma,X,r} := \left\{ w \in W^{1,p}(B_r(X)) : \int_{\partial B_r(X)} w \, \mathrm{d}S = 0, \int_{\partial B_r(X)} (x - X) \wedge w \, \mathrm{d}S = 0 \right\}.$$

There is a constant $C > 0$ that does not depend on X or r, such that for all $w \in H^p_{\sigma,X,r}$:

$$\|\nabla w\|_{L^p(B_r(X))} \leq C \|ew\|_{L^p(B_r(X))}$$

In order to prove this, we need the following lemma, known as Korn's second inequality for L^p.

Lemma 3.10 (see [KO88], §2, Theorem 8)**.** *Let $U \subset \mathbb{R}^n$ be bounded and connected with Lipschitz boundary. Let $p \in (1, \infty)$. Then there is a constant $C > 0$ that depends only on p and U, such that for all $u \in W^{1,p}(U)$ there is a skew-symmetric matrix $A \in \mathbb{R}^{3\times 3}_{\text{skew}}$ with:*

$$\|\nabla u - A\|_{L^p(U)} \leq C \|eu\|_{L^p(U)} .$$

Proof of Lemma 3.9. Note that we only need to prove the result for $X = 0, r = 1$, the case of general X and r follows simply from the fact that for $w \in H^p_{\sigma,X,r}$ the translated and rescaled version $w'(x) = w(rx + X)$ is in $H^p_{\sigma,0,1}$, and $\|ew'\|_{L^p} = r\|ew\|_{L^p}, \|\nabla w'\|_{L^p} = r\|\nabla w\|_{L^p}$.

$H^p_{\sigma,0,1}$ is a closed cone since the conditions are linear and closed. The proof of the Korn inequality is indirect and uses the same idea as the proof of the Poincaré inequality in Lemma 3.7. Hence we must first prove that for $w \in H^p_{\sigma,0,1}$ we have that $ew = 0$ implies $w = 0$. This is understood by noting the well known result that $ew = 0$ implies that w must be skew-symmetric affine, i.e. ωw is constant with $A = T\omega w$ and $w(x) = w(0) + Ax = w(0) + \omega w \wedge x$ for all $x \in B_1(0)$ (this is also a direct consequence of Lemma 3.10). But now we can use the two integral boundary conditions in the cone to show that any skew-symmetric affine function in $H^p_{\sigma,0,1}$ already vanishes:

$$\begin{aligned}
0 &= \fint_{\partial B_1(0)} w \,\mathrm{d}S \\
&= w(0) + \fint_{\partial B_1(0)} Ax \,\mathrm{d}S \\
&= w(0), \\
0 &= \fint_{\partial B_1(0)} x \wedge w \,\mathrm{d}S \\
&= \fint_{\partial B_1(0)} x \wedge (\omega w \wedge x) \,\mathrm{d}S \\
&= \frac{2}{3}\omega w.
\end{aligned}$$

This shows that $w = 0$.

In the following we will omit the domain $B_1(0)$ in the spaces. Now, for the sake of contradiction, assume that there is no $C > 0$ such that

$$\|\nabla w\|_{L^p} \leq C \|ew\|_{L^p} ,$$

for all $w \in H^p_{\sigma,0,1}$. Then there is a sequence $w_k \in H^p_{\sigma,0,1}$ such that $\|\nabla w_k\|_{L^p} \geq k\|ew_k\|_{L^p}$. By rescaling we can arrange that $\|w_k\|_{W^{1,p}} = 1$ for all $k \in \mathbb{N}$ (since $H^p_{\sigma,0,1}$ is a cone the w_k

will still be in $H^p_{\sigma,0,1}$). But then the sequence is bounded and there is a subsequence (again denoted by w_k) and $w_* \in W^{1,p}$ such that $w_k \rightharpoonup w_*$ in $W^{1,p}$. Note that since $H^p_{\sigma,0,1}$ is convex we have that $w_* \in H^p_{\sigma,0,1}$. On the other hand we know that

$$\|ew_k\|_{L^p} \leq \frac{1}{k} \|\nabla w_k\|_{L^p} \leq \frac{1}{k}.$$

Hence $ew_k \to 0$ in L^p and $ew_* = 0$ since at the same time $ew_k \rightharpoonup ew_*$. By our foregoing considerations this implies that $w_* = 0$.

By compact embedding we know that $w_k \to 0$ strongly in L^p. But to reach a contradiction we also need that the full gradient $\nabla w_k \to 0$ strongly, not only the symmetrized part. The key idea is, that by Korn's second inequality the gradient is already close to some constant skew-symmetric matrix and for constant matrices we have strong compactness.

Indeed, by Korn's second inequality (Lemma 3.10) there exist matrices $A_k \in \mathbb{R}^{3\times 3}_{\text{skew}}$ such that

$$\|\nabla w_k - A_k\|_{L^p} \leq C \|ew\|_{L^p} \leq C\frac{1}{k}.$$

Since ∇w_k is bounded in L^p this implies that the sequence $(A_k)_k$ must be bounded in $\mathbb{R}^{3\times 3}_{\text{skew}}$. But then there is a subsequence (again denoted A_k) such that $A_k \to A_*$. Furthermore we can pick the subsequence in such a way that

$$|B_1(0)|^{1/p} |A_k - A_*| \leq \frac{1}{k}.$$

Then we have

$$\|\nabla w_k - A_*\|_{L^p} \leq \|\nabla w_k - A_k\|_{L^p} + \|A_k - A_*\|_{L^p} \leq (C+1)\frac{1}{k}.$$

Therefore, ∇w_k converges strongly to A_* in L^p. But at the same time $\nabla w_k \rightharpoonup 0$ weakly in L^p. This yields $A_* = 0$ and $\nabla w_k \to 0$ strongly in L^p and $w_k \to 0$ strongly in $W^{1,p}$. This is a contradiction since $\|w_k\|_{W^{1,p}} = 1$ for all $k \in \mathbb{N}$. □

Corollary 3.11. *Let $p \in (1, \infty]$. There is a constant $C > 0$ that does not depend on X or r such that for all $w \in H^p_{\sigma,X,r}$:*

$$\|w\|_{L^p(B_r(X))} \leq Cr \|ew\|_{L^p(B_r(X))}.$$

Remark 3.12. *Note, that this Korn-Poincaré inequality* **does** *hold for $p = \infty$.*

Proof of Corollary 3.11. If $p < \infty$ we can use Lemma 3.7 (in the vector version) on $H^p_{\sigma,X,r} \subset H^p_{X,r}$ and Lemma 3.9 to obtain

$$\|w\|_{L^p(B_r(X))} \leq Cr \|\nabla w\|_{L^p(B_r(X))} \leq Cr \|ew\|_{L^p(B_r(X))} .$$

If $p = \infty$ we further use the Sobolev-embedding $W^{1,q} \hookrightarrow L^\infty$ for $q > 3$ to get for $X = 0$ and $r = 1$:

$$\begin{aligned}\|w\|_{L^\infty(B_1(0))} &\leq C \|w\|_{W^{1,q}(B_1(0))} \leq C \|\nabla w\|_{L^q(B_1(0))} \\ &\leq C \|ew\|_{L^q(B_1(0))} \leq C |B_1(0)|^{1/q} \|ew\|_{L^\infty(B_1(0))}\end{aligned}$$

The general inequality follows by translation and scaling as seen in Lemma 3.7.

□

We know that elements of $w \in W_i^\perp$ solve the homogeneous Poisson and Stokes equation respectively outside $\overline{B_i}$. In both cases this coincides with minimizing the respective norm.

Lemma 3.13. *Let $s \in \dot{H}^1$ and $-\Delta s = 0$ on $\mathbb{R}^3 \backslash V$ for some closed V with Lipschitz boundary. Then s minimizes $\|\nabla w\|_{L^2(\mathbb{R}^3 \backslash V)^2}$ among all $w \in \dot{H}^1_\sigma$ with $w = s$ on V.*

This is well-known and we will not prove it, but it works the same way as

Lemma 3.14. *Let $s \in \dot{H}^1_\sigma$ and $-\Delta s + \nabla p = 0$ on $\mathbb{R}^3 \setminus V$ for some closed V with Lipschitz boundary. Then s minimizes $\|ew\|_{L^2(\mathbb{R}^3 \backslash V)^2}$ among all $w \in \dot{H}^1_\sigma$ with $w = s$ on V.*

Proof. Let s be the minimizer. Then, by taking the variation for all $w \in \dot{H}^1_\sigma$ with $w \in \dot{H}^1_{\sigma,0}(\mathbb{R}^3 \setminus U)$ we have (n is the outer normal of V)

$$\begin{aligned}0 &= \langle s, w \rangle \\ &= -\int_{\partial U} w \, (\sigma[s] n) \, \mathrm{d}S - \int_{\mathbb{R}^3 \backslash U} w \operatorname{div} \sigma[s] \, \mathrm{d}x \\ &= -\int_{\mathbb{R}^3 \backslash U} w \operatorname{div} \sigma[s] \, \mathrm{d}x.\end{aligned}$$

This gives $-\Delta s + \nabla p = 0$ on $\mathbb{R}^3 \setminus U$. □

By choosing a suitable competitor we can therefore get estimates of the norm of $w \in W_i^\perp$ in terms of its values in $\overline{B_i}$. This competitor can be constructed using the following extension lemmas:

Lemma 3.15. *There is a constant $C > 0$ that does not depend on X or r, and an extension operator $E^P_{X,r} : H_{X,r} \to H^1_0(B_{2r}(X))$ such that*

$$\left\|\nabla E^P_{X,r} w\right\|_{L^2(B_{2r}(X))} \leq C \left\|\nabla w\right\|_{L^2(B_r(X))} \text{ for all } w \in H^2_{X,r}.$$

Proof. Let us first consider $X = 0$, $r = 1$. Let $E_{0,1} : H^1(B_1(0)) \to H^1_0(B_2(0))$ be a continuous extension operator, i.e. $\left\|E_{0,1} w\right\|_{H^1(B_2(0))} \leq C \left\|E_{0,1} w\right\|_{H^1(B_1(0))}$ for all $w \in H^1(B_1(0))$. The existence of $E_{0,1}$ follows from the existence of a bounded extension operator $H^1(B_1(0)) \to H^1(\mathbb{R}^3)$ (see for example [Eva10]) and the use of a cutoff $\varphi \in C^\infty_c(B_2(0))$ with $\varphi = 1$ in $B_1(0)$. Then

$$\left\|\nabla E_{0,1} w\right\|_{L^2(B_2(0))} \leq \left\|E_{0,1} w\right\|_{H^1(B_2(0))} \leq C \left\|w\right\|_{H^1(B_1(0))} \leq C \left\|\nabla w\right\|_{L^2(B_1(0))},$$

where the last inequality came from the Poincaré inequality on $H^2_{0,1}$ (Lemma 3.7). Now define

$$E^P_{X,r} w(x) = (E_{0,1} w_{X,r}) \left(\frac{1}{r}(x - X)\right),$$

where $w_{X,r}(x) = w(r(x + X))$. Then

$$\left\|\nabla E^P_{X,r} w\right\|_{L^2(B_{2r}(X))} = \frac{1}{r} \left\|\nabla E_{0,1} w_{X,r}\right\|_{L^2(B_2(0))} \leq C \frac{1}{r} \left\|\nabla w_{X,r}\right\|_{L^2(B_1(0))} = C \left\|\nabla w\right\|_{L^2(B_r(X))}.$$

□

In order to construct an extension operator for the divergence free spaces we will use the following statement by Bogovskii, that allows the construction of a function with given divergence and Dirichlet boundary data:

Lemma 3.16 ([Bog80])**.** *Let U be a bounded domain with Lipschitz-boundary. Let*

$$L^2_0(U) = \left\{g \in L^2(U) : \int_U g \, \mathrm{d}x = 0\right\}.$$

There is a continuous operator $\mathcal{B} : L^2_0(U) \to H^1_0(U)$ such that

$$-\operatorname{div} \mathcal{B}[g] = g \text{ in } U.$$

Lemma 3.17. *There is a constant $C > 0$ that does not depend on X or r, and an extension operator $E^S_{X,r} : H^2_{\sigma,X,r} \to H^1_{\sigma,0}(B_{2r}(X))$ such that*

$$\left\|\nabla E^S_{X,r} w\right\|_{L^2(B_{2r}(X))} \leq C \left\|ew\right\|_{L^2(B_r(X))} \text{ for all } w \in H^2_{\sigma,X,r}.$$

Proof. As in the proof of Lemma 3.15 we can get $E^S_{X,r}$ from $E^S_{0,1}$ by translation and scaling without changing C. Take $E^P_{0,1} : H_{0,1} \to H^1_0(B_2(0))$ from Lemma 3.15. Let

$$\mathcal{B} : L^2_0(B_2(0) \setminus \overline{B_1(0)}) \to H^1_0(B_2(0) \setminus \overline{B_1(0)}),$$

be the Bogowskii-operator from Lemma 3.16 associated to the annulus. Then set

$$E^S_{0,1} w = E^P_{0,1} w + \mathcal{B}[\operatorname{div} E^P_{0,1} w].$$

To show that this is indeed the right choice let us first verify that $\operatorname{div} E^P_{0,1} w \in L^2_0(B_2(0) \setminus \overline{B_1(0)})$. We have

$$\int_{B_2(0)\setminus\overline{B_1(0)}} \operatorname{div} E^P_{0,1} w \, \mathrm{d}x = \int_{\partial B_2(0)} E^p_{0,1} w \, \mathrm{d}S - \int_{\partial B_1(0)} E^P_{0,1} w \, \mathrm{d}S = 0 - \int_{\partial B_1(0)} w \, \mathrm{d}S = 0.$$

Then $E^S_{0,1} w = w$ on $\overline{B_1(0)}$ and $\operatorname{div} E^S_{0,1} w = \operatorname{div} E^P_{0,1} w - \operatorname{div} E^P_{0,1} w = 0$. Finally we have

$$\begin{aligned}\left\|\nabla E^S_{0,1} w\right\|_{L^2(B_2(0))} &\leq \left\|E^S_{0,1} w\right\|_{H^1(B_2(0))} \leq C \left\|E^P_{0,1} w\right\|_{H^1(B_2(0)} + C \left\|\operatorname{div} E^P_{0,1} w\right\|_{L^2(B_2(0))} \\ &\leq C \left\|E^P_{0,1} w\right\|_{H^1(B_2(0))} \leq C \left\|w\right\|_{H^1(B_1(0))} \leq C \left\|\nabla w\right\|_{L^2(B_1(0))} \leq C \left\|ew\right\|_{L^2(B_1(0))},\end{aligned}$$

where the second to last inequality came from the Poincaré inequality on $H_{\sigma,0,1} \subset H_{0,1}$ (Lemma 3.7) while the last is the Korn inequality from Lemma 3.9. □

Corollary 3.18. *There is a constant $C > 0$ such that for all $w \in \left(W^P_i\right)^\perp$*

$$\left\|w\right\|_{\dot{H}^1} \leq C \left\|\nabla w\right\|_{L^2(B_i)}.$$

Proof. By Lemma 3.13 w minimizes the L^2-norm of the gradient among all functions in $\dot{H}^1$ that are equal to w on $\overline{B_i}$. Take $E^P_{X_i,R} w$ as a competitor, then, due to Lemma 3.15,

$$\left\|w\right\|_{\dot{H}^1} \leq \left\|E^P_{X_i,R} w\right\|_{\dot{H}^1} = \left\|\nabla E^P_{X_i,R} w\right\|_{L^2(B_{2R}(X_i))} \leq C \left\|\nabla w\right\|_{L^2(B_i)}.$$

□

Corollary 3.19. *There is a constant $C > 0$ such that for all $w \in \left(W_i^S\right)^\perp$*

$$\|w\|_{\dot{H}^1} \le C \|ew\|_{L^2(B_i)} .$$

Proof. By Lemma 3.14 w minimizes the L^2-norm of the symmetrized gradient among all functions in $\dot{H}^1_\sigma$ that are equal to w on $\overline{B_i}$. Take $E^S_{X_i,R} w$ as a competitor, then, due to Lemma 3.17,

$$\begin{aligned}\|w\|_{\dot{H}^1} = 2\|ew\|_{L^2(\mathbb{R}^3)} &\le 2\left\|eE^S_{X_i,R}w\right\|_{L^2(\mathbb{R}^3)} = 2\left\|eE^S_{X_i,R}w\right\|_{L^2(B_{2R}(X_i))} \\ &\le 2\left\|\nabla E^S_{X_i,R}w\right\|_{L^2(B_{2R}(X_i))} \le C\|ew\|_{L^2(B_i)} .\end{aligned}$$

□

Using the extension lemmas we can furthermore prove the following decay properties of the dipoles

Lemma 3.20. *There is a constant $C > 0$ such that for all $w \in \left(W_i^P\right)^\perp$ and for all $x \in \mathbb{R}^3 \setminus B_{2R}(X_i)$ we have*

$$|w(x)| \le C\frac{R^{\frac{3}{2}}}{|x - X_i|^2}\|w\|_{\dot{H}^1} , \tag{3.11}$$

$$|\nabla w(x)| \le C\frac{R^{\frac{3}{2}}}{|x - X_i|^3}\|w\|_{\dot{H}^1} . \tag{3.12}$$

Proof. The main idea of the proof is to represent w as the convolution of its Laplacian with the fundamental solution. For any function with compactly supported Laplacian this gives a $\frac{1}{|x|}$-like decay. The fact that the Laplacian of w integrates to zero due to the dipole property gives rise to the additional power in the decay.

Let $f = -\Delta w$. This means $f \in \dot{H}^{-1}$ and

$$\|f\|_{\dot{H}^{-1}} = \|w\|_{\dot{H}^1} , \quad \operatorname{spt} f \subset \overline{B_i}, \quad \int_{\overline{B_i}} f = -\int_{\partial B_i} \frac{\partial w}{\partial n}\,\mathrm{d}s = 0.$$

In particular combining the last two statements, we have $\int_{\mathbb{R}^3} f = 0$.

We have $w = \Phi^P * f$. Let

$$(\Phi^P)_{x-X_i,R} = \fint_{\partial B_R(x-X_i)} \Phi^P.$$

Then we obtain

$$
\begin{aligned}
|w(x)| &= \left|\left(\Phi^P * f\right)(x)\right| \\
&= \left|\left(\left(\Phi^P - (\Phi^P)_{x-X_i,R}\right) * f\right)(x)\right| \\
&= \left|\left(E^P_{x-X_i,R}\left(\Phi^P - (\Phi^P)_{x-X_i,R}\right) * f\right)(x)\right| \\
&\le \|f\|_{\dot H^{-1}} \left\|E^P_{x-X_i,R}\left(\Phi^P - (\Phi^P)_{x-X_i,R}\right)\right\|_{\dot H^1}.
\end{aligned}
$$

Note that it was only possible to use E^P because we first subtracted the mean value, so that the argument of the extension operator is actually in the right space $H^2_{x-X_i,R}$. Now by Lemma 3.15 we have

$$
\left\|E_{x-X_i,R}\left(\Phi^P - (\Phi^P)_{x-X_i,R}\right)\right\|_{\dot H^1} \le C\left\|\nabla\Phi^P\right\|_{L^2(B_R(x-X_i))} \le C\frac{R^{\frac{3}{2}}}{|x-X_i|^2}.
$$

Here, in the last step we just used that $|x-X_i| \ge 2R$ and hence for $y \in B_R(x-X_i)$, $|y| \ge \frac{1}{2}|x-X_i|$.

Since $\nabla w = \nabla\Phi^P * f$ we can do the same computation with $\nabla\Phi$, using the decay of $\nabla^2\Phi^P$ to obtain inequality (3.12). ☐

Lemma 3.21. *There is a constant $C > 0$ such that for all $w \in \left(W_i^S\right)^\perp$ and for all $x \in \mathbb{R}^3 \setminus B_{2R}(X_i)$ we have*

$$
|w(x)| \le C\frac{R^{\frac{3}{2}}}{|x-X_i|^2}\|w\|_{\dot H^1}, \tag{3.13}
$$

$$
|\nabla w(x)| \le C\frac{R^{\frac{3}{2}}}{|x-X_i|^3}\|w\|_{\dot H^1}. \tag{3.14}
$$

Proof. The proof is analogous to the one of Lemma 3.20. Let $f = Sw$ (see section 2.1.1) so that $-\Delta w + \nabla p = f$. This means $f \in \dot H^{-1}_\sigma$ and

$$
\|f\|_{\dot H^{-1}_\sigma} = \|w\|_{\dot H^1_\sigma} = \|w\|_{\dot H^1}, \quad \operatorname{spt} f \subset \overline{B_i}, \quad \int_{\overline{B_i}} f = -\int_{\partial B_i} \sigma n = 0.
$$

In particular $\int_{\mathbb{R}^3} f = 0$. Now we use $w = \Phi^S * f$ and $\nabla w = \nabla\Phi^S * f$ to repeat the same computations as in the proof of Lemma 3.20, using the decay of $\nabla\Phi^S$ and $\nabla^2\Phi^S$. Note, that even though we are in the Stokes case it is enough to use the Poisson extension operator E^P since we have a good estimate of the full gradient of the fundamental solution. Also, we did not actually use all conditions on w, so that this decay is valid for a broader class of functions.

☐

3.3. Convergence of the method of reflections

The ultimate goal of the method of reflections is, to prove that the gradients and the symmetrized gradients respectively of the approximations v_k approach 0 in L^∞ inside the particles. If $w \in \dot{H}^1$ has an uniformly bounded gradient in B_i we can get the following estimates

Corollary 3.22. *There is a constant $C > 0$ such that for all $w \in \dot{H}^1 \cap W^{1,\infty}(B_i)$ and for all $x \in \mathbb{R}^3 \setminus B_{2R}(X_i)$ we have*

$$\left|Q_i^P w(x)\right| \leq C \frac{R^3}{|x - X_i|^2} \|\nabla w\|_{L^\infty(B_i)}, \tag{3.15}$$

$$\left|\nabla Q_i^P w(x)\right| \leq C \frac{R^3}{|x - X_i|^3} \|\nabla w\|_{L^\infty(B_i)}. \tag{3.16}$$

Proof. We have $Q_i^P w = w - P_i^P w$. Since $P_i^P w$ is constant on B_i we have $\nabla Q_i^P w = \nabla w$ on B_i.

Since $Q_i^P w \in W_i^\perp$ we know from Corollary 3.18 that:

$$\left\|Q_i^P w\right\|_{\dot{H}^1} \leq C \left\|\nabla Q_i^P w\right\|_{L^2(B_i)} = \|\nabla w\|_{L^2(B_i)} \leq C R^{\frac{3}{2}} \|\nabla w\|_{L^\infty(B_i)}. \tag{3.17}$$

Since $Q_i^P w \in W_i^\perp$, we can use Lemma 3.20 to obtain (3.15) and (3.15). □

Corollary 3.23. *There is a constant $C > 0$ such that for all $w \in \dot{H}_\sigma^1 \cap W^{1,\infty}(B_i)$ and for all $x \in \mathbb{R}^3 \setminus B_{2R}(X_i)$ we have*

$$\left|Q_i^S w(x)\right| \leq C \frac{R^3}{|x - X_i|^2} \|ew\|_{L^\infty(B_i)}, \tag{3.18}$$

$$\left|\nabla Q_i^S w(x)\right| \leq C \frac{R^3}{|x - X_i|^3} \|ew\|_{L^\infty(B_i)}. \tag{3.19}$$

Proof. We know that $Q_i^S w = w - P_i^S w$ and since $P_i^S w$ is a rigid body motion on B_i we have $eP_i^S w(x) = 0$ for all $x \in B_i$ and hence $eQ_i^S w = ew$ on B_i.

Since $Q_i^S w \in W_i^\perp$ we know from Corollary 3.19 that:

$$\left\|Q_i^S w\right\|_{\dot{H}^1} \leq C \left\|eQ_i^S w\right\|_{L^2(B_i)} = \|ew\|_{L^2(B_i)} \leq C R^{\frac{3}{2}} \|ew\|_{L^\infty(B_i)}. \tag{3.20}$$

Since $Q_i^S w \in W_i^\perp$, we can use Lemma 3.21 to obtain (3.18) and (3.18). □

In order to prove the main approximation statement of this chapter we need some estimates for recurring sums (also see Lemma 2.1 of [JO04] for the first two inequalities):

Lemma 3.24. *There is a constant $C > 0$ such that*

$$\sum_{j\neq i} \frac{1}{d_{ij}} \leq C\frac{N^{\frac{2}{3}}}{d} \leq CN, \tag{3.21}$$

$$\sum_{j\neq i} \frac{1}{d_{ij}^2} \leq C\frac{N^{\frac{1}{3}}}{d^2} \leq CN, \tag{3.22}$$

$$\sum_{j\neq i} \frac{1}{d_{ij}^3} \leq C\frac{\log N}{d^3} \leq CN\log N, \tag{3.23}$$

$$\sum_{j\neq i} \frac{1}{d_{ij}^4} \leq C\frac{1}{d^4} \leq CN^{\frac{4}{3}}. \tag{3.24}$$

The proof can be found in the appendix.

Corollary 3.25. *There is a constant $C > 0$ such that for all $x \in \mathbb{R}^3$ the following holds. Let $1 \leq i \leq N$ such that $|x - X_i| \leq |x - X_j|$ for any $1 \leq j \leq N$, i.e. X_i is the centre of the particle closest to x. Then we have*

$$\sum_{j\neq i} \frac{1}{|x - X_j|} \leq C\frac{N^{\frac{2}{3}}}{d} \leq CN, \tag{3.25}$$

$$\sum_{j\neq i} \frac{1}{|x - X_j|^2} \leq C\frac{N^{\frac{1}{3}}}{d^2} \leq CN, \tag{3.26}$$

$$\sum_{j\neq i} \frac{1}{|x - X_j|^3} \leq C\frac{\log N}{d^3} \leq CN\log N, \tag{3.27}$$

$$\sum_{j\neq i} \frac{1}{|x - X_j|^4} \leq C\frac{1}{d^4} \leq CN^{\frac{4}{3}}. \tag{3.28}$$

$$\tag{3.29}$$

Proof. We have to show that $|x - X_j| \geq \frac{1}{2}|X_i - X_j|$. Then the inequalities (3.25), (3.26), (3.27) and (3.28) follow from Lemma 3.24. Suppose that the inequality is false, i.e. $|x - X_j| < \frac{1}{2}|X_i - X_j|$, then

$$|x - X_i| \geq ||X_i - X_j| - |x - X_j|| = |X_i - X_j| - |x - X_j| \geq \frac{1}{2}|X_i - X_j| > |x - X_j|,$$

which is a contradiction to the fact that X_i is the closest centre point to x. □

We will also use a maximum modulus theorem for the Stokes equation:

Lemma 3.26 ([MRS99]). *Let $\Omega \subset \mathbb{R}^3$ be an exterior domain (a domain with bounded complement) and assume that $g \in C^0(\Omega^C)$ satisfies*

$$\int_{\partial\Omega} g \cdot n \, \mathrm{d}S = 0.$$

Then, there is a constant $C > 0$ that depends only on Ω, such that the unique solution $u \in \dot{H}^1_\sigma(\mathbb{R}^3)$ of the Dirichlet problem

$$\begin{aligned} -\Delta u + \nabla p &= 0 \text{ in } \Omega, \\ \operatorname{div} u &= 0 \text{ in } \Omega, \\ u &= g \text{ on } \Omega^C, \end{aligned}$$

satisfies

$$\|u\|_{L^\infty} \leq C \, \|g\|_{L^\infty} \, .$$

Remark 3.27. *We will use this statement only for Ω being the exterior of a ball. The constant is invariant under translation and scaling of the ball: Let $g \in C(\overline{B_r(X)})$ with $\int_{\partial B_r(X)} g \cdot n \, \mathrm{d}S = 0$ be given. Define $g'(x) = g(r(x+X))$. Then $g' \in C^0(\overline{B_1(0)})$ and $\int_{\partial B_1(0)} g' \cdot n \, \mathrm{d}S = 0$ and the solution u' to the homogeneous Stokes equation outside $\overline{B_1(0)}$ satisfies*

$$\|u'\|_{L^\infty} \leq C \, \|g'\|_{L^\infty} \, .$$

But then we have

$$\|u\|_{L^\infty} = \|u'\|_{L^\infty} \leq C \, \|g'\|_{L^\infty} = C \, \|g\|_{L^\infty} \, ,$$

with the same constant C.

We are now able to prove the convergence of the method of reflections. We recall the definition of the dipole approximations (3.6):

$$v_k = \left(\mathrm{Id} - \sum_{i=1}^N Q_i \right)^k v.$$

Theorem 3.28. *There is $\varepsilon > 0$ such that if $\phi \log N < \varepsilon$, we have $v_k \to u$ in $\dot{H}^1$ and $L^\infty(\mathbb{R}^3)$ for $k \to \infty$ and in particular*

$$\|u - v_1\|_{L^\infty} \leq \phi o(1), \text{ as } N \to \infty.$$

Proof. The strategy of the proof is the following. We first establish that for all v_k we have $Pv_k = u$ where P is the projection onto W. Using this we know that $\|u - v_k\|_{\dot{H}^1} \leq \|w - v_k\|_{\dot{H}^1}$ for all $w \in W$. With the use of the extension operators E^P and E^S we then construct suitable competitors in W to obtain (in the Poisson case)

$$\|u - v_k\|^2_{\dot{H}^1} \leq C \sum_{i=1}^{N} R^3 \|\nabla v_k\|^2_{L^\infty(\cup B_i)}.$$

Using that $\nabla v_k - \nabla Q_i^P v_k = 0$ on B_i and the decay of $\nabla Q_j^P v_k$ for $j \neq i$ (Corollary 3.22) together with Corollary 3.25 gives an estimate of the form $\|\nabla v_{k+1}\|_{L^\infty(\cup B_i)} \leq C\phi \log N \|\nabla v_k\|_{L^\infty(\cup B_i)}$. By the smallness of $\phi \log N$ we obtain $v_k \to u$ in $\dot{H}^1$. For the convergence in L^∞ we can use the same decay techniques together with the maximum principle and the maximum modulus theorem respectively.

We first prove that $Pv_k = u$ for all $k \in \mathbb{N}$. We already know that $v_0 = v$ satisfies this equality. Now assume that is holds for some $k \in \mathbb{N}$. Since $v_{k+1} = v_k - \sum_{i=1}^N Q_i v_k$ we need only show that $PQ_i = 0$ for all $i = 1, \dots, N$. We know that $Q_i w \in W_i^\perp$ for all $w \in \dot{H}^1$ and $\dot{H}^1_\sigma$ respectively. But since $W \subset W_i$ we have that $W_i^\perp \subset W^\perp$. Since P is the orthogonal projection onto W and $Q_i w \in W^\perp$ it follows that $PQ_i = 0$. This implies that

$$\|u - v_k\|_{\dot{H}^1} \leq \|w - v_k\|_{\dot{H}^1} \text{ for all } w \in W. \tag{3.30}$$

By choosing a suitable w we obtain a bound for $\|u - v_k\|_{\dot{H}^1}$.

We first consider the Poisson case. Let

$$w = v_k - \sum_{i=1}^{N} E^P_{X_i,R} Q_i^P v_k.$$

By Assumption **(3)** ($d > 4R$), we know that the supports of the $E^P_{X_i,R} Q_i^P v_k$ are disjoint. Hence we have for $x \in B_i$: $w(x) = v_k(x) - Q_i^P v_k(x) = P_i^P v_k(x)$. Therefore $w \in W^P$. Then we have

$$\begin{aligned}\|w-v_k\|_{\dot H^1}^2 &= \left\|\sum_{i=1}^N E^P_{X_i,R}Q_i^P v_k\right\|_{\dot H^1}^2 = \sum_{i=1}^N \left\|E^P_{X_i,R}Q_i^P v_k\right\|_{\dot H^1}^2 \\ &\le C\sum_{i=1}^N \left\|\nabla Q_i^P v_k\right\|_{L^2(B_i)}^2 = C\sum_{i=1}^N \|\nabla v_k\|_{L^2(B_i)}^2 \\ &\le C\sum_{i=1}^N R^3 \|\nabla v_k\|_{L^\infty(\cup B_j)}^2 = C\phi \|\nabla v_k\|_{L^\infty(\cup B_j)}^2 .\end{aligned}$$

At this point it is not yet clear that $\nabla v_k \in L^\infty(\cup B_i)$. We prove this now. To estimate ∇v_k in $L^\infty(B_i)$ suppose we already know that $\nabla v_{k-1} \in L^\infty(\cup B_j)$ and observe that for $x \in B_i$:

$$\nabla v_k(x) = \nabla\left(v_{k-1} - \sum_{j=1}^N Q_j^P v_{k-1}\right) = \sum_{j\neq i} \nabla Q_j^P v_{k-1},$$

since $v_{k-1} - Q_i^P v_{k-1} = P_i^P v_{k-1}$ is constant in B_i. Using (3.16) in Corollary 3.22 and Corollary 3.25 this leads to the following estimate:

$$|v_k(x)| \le C\sum_{j\neq i} \frac{R^3}{|x-X_j|^3} \|\nabla v_{k-1}\|_{L^\infty(B_j)} \le CN\log N R^3 \|\nabla v_{k-1}\|_{L^\infty(\cup B_j)} .$$

Therefore we have

$$\|\nabla v_k\|_{L^\infty(B_i)} \le C\phi \log N \|\nabla v_{k-1}\|_{L^\infty(\cup B_j)} .$$

In particular $\nabla v_k \in L^\infty(\cup B_j)$. By iterating this argument we see that there is $C > 0$ such that

$$\|\nabla v_k\|_{L^\infty(\cup B_i)} \le (C\phi \log N)^k \|\nabla v\|_{L^\infty(\cup B_i)} .$$

It remains to see that $\nabla v \in L^\infty(\cup B_j)$. For this we estimate

$$
\begin{aligned}
|\nabla v(x)| &= \left| \int_{\mathbb{R}^3} (1 - \sum_{k=1}^{N} \chi_{B_k}(y)) \nabla \Phi^P(x-y) f(y) \, \mathrm{d}y \right| \\
&\leq C \int_{\mathbb{R}^3} |f(y)| \frac{1}{|x-y|^2} \, \mathrm{d}y \\
&\leq C \|f\|_{L^2} \int_{\mathbb{R}^3 \setminus B_1(0)} \frac{1}{|y|^4} \, \mathrm{d}y + C \|f\|_{L^\infty} \int_{B_1(0)} \frac{1}{|y|^2} \, \mathrm{d}y \\
&\leq C \|f\|_{L^2} + C \|f\|_{L^\infty} \\
&\leq C
\end{aligned}
$$

Therefore actually $\nabla v \in L^\infty(\mathbb{R}^3)$. Thus all estimates made before are well-defined. Recalling (3.30) this means that

$$
\|u - v_k\|_{\dot{H}^1}^2 \leq \phi \left(C\phi \log N\right)^k \|\nabla v\|_{L^\infty(\cup B_j)}
$$

and hence that $v_k \to u$ in $\dot{H}^1$ if $\phi \log N < \frac{1}{C} = \varepsilon$.

The convergence also holds in L^∞. To prove this we take some $x \in \mathbb{R}^3$ and take X_i to be the centre of the particle closest to x. Then, employing (3.15) and the maximum principle, we obtain

$$
\begin{aligned}
|v_{k+1}(x) - v_k(x)| &= \left| \sum_{j=1}^{N} Q_j^P v_k(x) \right| \\
&\leq \sum_{j \neq i} \left| Q_j^P v_k(x) \right| + \left| Q_i^P v_k(x) \right| \\
&\leq C \sum_{j \neq i} \frac{R^3}{|x - X_i|^2} \|\nabla v_k\|_{L^\infty(\cup B_j)} + C \left\| v_k - \fint_{\partial B_i} v_k \right\|_{L^\infty(B_i)} \\
&\leq C R^3 N \|\nabla v_k\|_{L^\infty(\cup B_j)} + C R \|\nabla v_k\|_{L^\infty(B_i)} \\
&\leq C(\phi + R) \left(C\phi \log N\right)^k \|\nabla v\|_{L^\infty(\cup B_j)} .
\end{aligned}
$$

We used Corollary 3.25 and the Poincaré inequality on $H^\infty_{X_i,R}$ (Lemma 3.7) in the third line.

Summing up we obtain

$$
\begin{aligned}
\sum_{j=k}^{\infty} (v_{j+1}(x) - v_j(x)) &\leq \sum_{j=k}^{\infty} |v_{j+1}(x) - v_j(x)| \\
&\leq C \sum_{j=k}^{\infty} (\phi + R) \left(C\phi \log N\right)^j \\
&\leq C(\phi + R) \left(C\phi \log N\right)^k .
\end{aligned}
$$

Here we already assumed that $\phi \log N < \frac{1}{C}$. Since the sum is absolutely convergent this means in particular that $\lim_{k\to\infty} v_k(x)$ exists for every x. But since $v_k \to u$ in $\dot{H}^1$ this implies $u(x) = \lim_{k\to\infty} v_k(x)$ whence

$$|u(x) - v_k(x)| = \left| \sum_{j=k}^{\infty} (v_{j+1}(x) - v_j(x)) \right| \leq C(\phi + R) \left(C\phi \log N \right)^k .$$

Since $R \log N \leq CRN^{\frac{1}{3}} = C\phi^{\frac{1}{3}}$ we know that $R \log N \to 0$ as $N \to \infty$ and hence we obtain

$$\|u - v_1\|_{L^\infty(\mathbb{R}^3)} \leq \phi o(1), N \to \infty.$$

Now we come to the Stokes case. Let

$$w = v_k - \sum_{i=1}^{N} E^S_{X_i,R} Q^S_i v_k.$$

By Assumption **(3)** we know that the supports of the $E^S_{X_i,R} Q^S_i v_k$ are disjoint. Hence, for $x \in B_i$ we have: $w(x) = v_k(x) - Q^S_i v_k(x) = P^S_i v_k(x)$. Therefore $w \in W^S$. Then we have

$$\begin{aligned} \|w - v_k\|^2_{\dot{H}^1} &= \left\| \sum_{i=1}^{N} E^S_{X_i,R} Q^S_i v_k \right\|^2_{\dot{H}^1} = \sum_{i=1}^{N} \left\| E^S_{X_i,R} Q^S_i v_k \right\|^2_{\dot{H}^1} \\ &\leq C \sum_{i=1}^{N} \left\| e Q^S_i v_k \right\|^2_{L^2(B_i)} = C \sum_{i=1}^{N} \|e v_k\|^2_{L^2(B_i)} \\ &\leq C \sum_{i=1}^{N} R^3 \|e v_k\|^2_{L^\infty(\cup B_i)} = C\phi \|e v_k\|^2_{L^\infty(\cup B_i)} . \end{aligned}$$

Again, it is not clear a priori whether $e v_k \in L^\infty(\cup B_i)$. But exactly the same computation as for ∇v in the Poisson case with Φ^P replaced by Φ^S shows that $\nabla v \in L^\infty(\mathbb{R}^3)$ and hence $e v \in L^\infty(\mathbb{R}^3)$. We now iteratively estimate $e v_k$ in L^∞ by the L^∞ norm of $e v_{k-1}$ on $\cup B_j$. We observe that for $x \in B_i$:

$$e v_k(x) = e \left(v_{k-1} - \sum_{j=1}^{N} Q^S_j v_{k-1} \right) = \sum_{j \neq i} e Q^S_j v_{k-1},$$

since $v_{k-1} - Q^S_i v_{k-1} = P^S_i v_{k-1}$ has vanishing symmetrized gradient in B_i. Using (3.19) and Corollary 3.25 this leads to:

$$|ev_k(x)| \leq C \sum_{j \neq i} \frac{R^3}{|x - X_j|^3} \|ev_{k-1}\|_{L^\infty(B_j)} \leq CN \log NR^3 \|ev_{k-1}\|_{L^\infty(\cup B_j)} .$$

Therefore

$$\|ev_k\|_{L^\infty(\cup B_j)} \leq C\phi \log N \|ev_{k-1}\|_{L^\infty(\cup B_j)} .$$

By iterating this argument we see that

$$\|ev_k\|_{L^\infty(\cup B_j)} \leq (C\phi \log N)^k \|ev\|_{L^\infty(\cup B_j)} .$$

This means that

$$\|u - v_k\|_{\dot{H}^1}^2 \leq \phi (C\phi \log N)^k \|ev\|_{L^\infty(\cup B_j)}$$

and hence that $v_k \to u$ in $\dot{H}^1$ if $\phi \log N < \frac{1}{C} = \varepsilon$.

Again, the convergence also holds in L^∞. To demonstrate this, we take some $x \in \mathbb{R}^3$ and take X_i to be the centre of the particle closest to x. Note that for $x \in \overline{B_i}$ by the Korn-Poincaré inequality from Corollary 3.11 we have

$$\left|Q_i^S v_k(x)\right| \leq CR \|ev_k\|_{L^\infty(\cup B_j)} .$$

But by the maximum modulus Lemma 3.26 we have the same inequality (with a larger constant) for general x. Then

$$\begin{aligned}
|v_{k+1}(x) - v_k(x)| &= \left| \sum_{j=1}^{N} Q_j v_k(x) \right| \\
&\leq \sum_{j \neq i} |Q_j v_k(x)| + |Q_i v_k(x)| \\
&\leq C \sum_{j \neq i} \frac{R^3}{|x - X_i|^2} \|ev_k\|_{L^\infty(\cup B_j)} + CR \|ev_k\|_{L^\infty(\cup B_j)} \\
&\leq C(\phi + R) (C\phi \log N)^k \|ev\|_{L^\infty(\cup B_j)} .
\end{aligned}$$

The rest of the argument is exactly the same as for the Poisson case so that we obtain

$$\|u(x) - v_k(x)\|_{L^\infty(\mathbb{R}^3)} \leq C(\phi + R) (C\phi \log N)^k .$$

This implies

$$\|u - v_1\|_{L^\infty} \leq \phi o(1), N \to \infty.$$

□

3.4. The explicit dipole approximation

We now have a dipole approximation that is close enough to u. The next step is to show that this approximation is close to another, more explicit, dipole approximation. We need this explicit approximation in order to relate the dipoles to the fundamental solution. To do so we need to handle terms of the form $\frac{(x-X_k)^{\otimes l}}{|x-X_k|^n}$. If one considers the difference of such terms with arguments that are close then one gets an additional power in the denominator.

Lemma 3.29. *There is a constant $C > 0$ such that for $x, z \in \mathbb{R}^3$ with $|x - X_i| \leq \frac{1}{2}|X_i - z|$ and all $n > l \geq 0$ and $k \neq i$ we have the following estimate:*

$$\left| \frac{(x-z)^{\otimes l}}{|x-z|^n} - \frac{(X_i - z)^{\otimes l}}{|X_i - z|^n} \right| \leq C \frac{|x - X_i|}{|X_i - z|^{n-l+1}}.$$

Proof. We have $|x - z| \geq \frac{1}{2}|X_i - z|$. First we set

$$g(y) = \frac{(y-z)^{\otimes l}}{|y-z|^n}.$$

Observe that for $y \in B_{|x-X_i|}(X_i)$:

$$|\nabla g(y)| \leq C \frac{1}{|y-z|^{n-l+1}} \leq C \frac{1}{|X_i - z|^{n-l+1}},$$

since $|y - z| \geq \frac{1}{2}|X_i - z|$. Then

$$\begin{aligned} |g(x) - g(X_i)| &= \int_0^1 \frac{\mathrm{d}}{\mathrm{d}t} g((1-t)X_i + tx)\, \mathrm{d}t \\ &\leq |x - X_i| \int_0^1 |\nabla g((1-t)X_i + tx)|\, \mathrm{d}t \\ &\leq C \frac{|x - X_i|}{|X_i - z|^{n-l+1}}. \end{aligned}$$

□

The gradient of the dipoles d_i and $Q_i^P v$ inside the balls is $\nabla v(X_i)$ and $\nabla v(x)$ respectively. We need the difference of those two terms to be small.

Lemma 3.30. *There is a constant $C > 0$ such that for $x \in B_{2L}(0)$ and $a = |x - X_i|$*

$$|\nabla v(X_i) - \nabla v(x)| \leq C\left(\phi^{\frac{1}{4}} + a + a^{\alpha}\right).$$

Remark 3.31. *In particular, taking $a < R$ we have*

$$\|\nabla v(X_i) - \nabla v\|_{L^\infty(B_i)} \leq C\left(\phi^{\frac{1}{4}} + R + R^{\alpha}\right) \leq o(1), \text{ as } N \to \infty.$$

Proof of Lemma 3.30. The idea of the proof is the following. If v was in C^2 this could be proved easily by Taylor expansion. But we cannot expect v to be two times differentiable since f^N is zero on the particles and hence in general not continuous. Still, the function that is subtracted from f to obtain f^N is supported only on the particles and hence should have a contribution vanishing with ϕ. This can be made clear by writing $\nabla v = \nabla\Phi * f^N$. For the part that remains we expect Hölder continuity, since f is Hölder continuous.

We know:

$$|\nabla v(X_i) - \nabla v(x)| = \left|\int_{\mathbb{R}^3} (1 - \sum_{k=1}^{N} \chi_{B_k}(y)) f(y) \left(\nabla\Phi(X_i - y) - \nabla\Phi(x - y)\right) \, \mathrm{d}y\right|.$$

As proposed above we split the term into a part with the pure f and a part that incorporates the characteristic functions of the particles. Let us first estimate the latter

$$\begin{aligned}
&\left|\int_{\mathbb{R}^3} \sum_{k=1}^{N} \chi_{B_k}(y) f(y) \left(\nabla\Phi(X_i - y) - \nabla\Phi(x - y)\right) \, \mathrm{d}y\right| \\
&= \left|\int_{B_L(0)} \sum_{k=1}^{N} \chi_{B_k}(y) f(y) \left(\nabla\Phi(X_i - y) - \nabla\Phi(x - y)\right) \, \mathrm{d}y\right| \\
&\leq \|f\|_{L^\infty} \left\|\sum_{k=1}^{N} \chi_{B_k}\right\|_{L^4} \|\nabla\Phi\|_{L^{\frac{4}{3}}(B_{3L}(0))} \\
&\leq C\phi^{\frac{1}{4}}.
\end{aligned}$$

Here we used that $|\nabla\Phi(x)| \leq C\frac{1}{|x|^2} \in L^{\frac{4}{3}}(B_{3L}(0))$. For the other terms we compute:

$$
\begin{aligned}
&\left|\int_{\mathbb{R}^3} f(y)\left(\nabla\Phi(X_i-y)-\nabla\Phi(x-y)\right)\,\mathrm{d}y\right| \\
&=\left|\int_{\mathbb{R}^3\setminus B_{3L}(0)} f(y)\left(\nabla\Phi(X_i-y)-\nabla\Phi(x-y)\right)\,\mathrm{d}y\right| \\
&+\left|\int_{B_{3L}(0)} f(y)\left(\nabla\Phi(X_i-y)-\nabla\Phi(x-y)\right)\,\mathrm{d}y\right| \\
&\leq \|f\|_{L^2}\left\|\frac{a}{|X_i-y|^3}\right\|_{L^2(\mathbb{R}^3\setminus B_{3L}(0))}+\left|\int_{B_{3L}(X_i)\cap B_{3L}(x)}(f(X_i-y)-f(x-y))\nabla\Phi(y)\,\mathrm{d}y\right| \\
&+\left|\int_{B_{3L}(X_i)\setminus B_{3L}(x)} f(X_i-y)\nabla\Phi(y)\,\mathrm{d}y\right|+\left|\int_{B_{3L}(x)\setminus B_{3L}(X_i)} f(x-y)\nabla\Phi(y)\,\mathrm{d}y\right| \\
&\leq C\left(a+a^\alpha[f]_\alpha\|\nabla\Phi\|_{L^1(B_{3L}(X_i)\cap B_{3L}(x))}+a(3L)^2\|f\|_{L^\infty}\frac{1}{L^2}\right) \\
&\leq C\left(a+a^\alpha\right).
\end{aligned}
$$

Here we used Lemma 3.29 in the second line and Hölder continuity of f as well as decay properties of $\nabla\Phi$. This gives us

$$|\nabla v(X_i)-\nabla v(x)|\leq C\left(\phi^{\frac{1}{4}}+a+a^\alpha\right).$$

□

We recall the explicit dipoles

$$d_i^P(x)=\begin{cases}\nabla v(X_i)(x-X_i) & \text{, for } |x-X_i|\leq R,\\ \nabla v(X_i)R^3\frac{x-X_i}{|x-X_i|^3} & \text{, otherwise.}\end{cases}$$

and

$$d_i^S(x)=\begin{cases}ev(X_i)(x-X_i) & \text{, for } |x-X_i|\leq R,\\ \frac{5}{2}R^3\left(\frac{(x-X_i)((x-X_i)\cdot ev(X_i)(x-X_i))}{|x-X_i|^5}\right) & \\ +R^5\left(\frac{ev(X_i)(x-X_i)}{|x-X_i|^5}-\frac{5}{2}\frac{(x-X_i)((x-X_i)\cdot ev(X_i)(x-X_i))}{|x-X_i|^7}\right) & \text{, otherwise.}\end{cases}$$

Corollary 3.32. *For $x\in\mathbb{R}^3\setminus B_{2R}(X_i)$ we have*

$$|d_i(x)-Q_iv(x)|\leq\frac{R^3}{|x-X_i|^2}o(1),\ \text{as } N\to\infty.$$

Proof. By computation we see that $d_i \in W_i^\perp$ and hence $d_i(x) - Q_i v \in W_i^\perp$. For the Poisson case, applying Corollary 3.22 we obtain

$$\left|d_i^P(x) - Q_i^P v(x)\right| \leq C \frac{R^3}{|x - X_i|^2} \left\|\nabla d_i^P - \nabla Q_i^P v\right\|_{L^\infty(B_i)}.$$

But for $x \in B_i$, using Lemma 3.30 (and Remark 3.31), we know:

$$\left|\nabla d_i^P(x) - \nabla Q_i^P v(x)\right| = |\nabla v(X_i) - \nabla v(x)| = o(1).$$

For the Stokes case, applying Corollary 3.23 we obtain

$$\left|d_i^S(x) - Q_i^S v(x)\right| \leq C \frac{R^3}{|x - X_i|^2} \left\|e d_i^S - e Q_i^S v\right\|_{L^\infty(B_i)}.$$

And for $x \in B_i$, with Lemma 3.30, we have:

$$\left|e d_i^S(x) - e Q_i^S v(x)\right| = |ev(X_i) - ev(x)| \leq |\nabla v(X_i) - \nabla v(x)| = o(1).$$

□

It remains to use Corollary 3.32 to prove that the difference between the whole dipole approximations is small. The singularities of the dipoles are quite strong, which forces us to keep some distance from the particle centres when attempting to prove closeness. Therefore take a sequence $\delta^N > 0$ such that $\frac{1}{(\delta^N)^2 N} \to 0$ and $\frac{\delta}{d} \to 0$ as $N \to \infty$. In particular $N^{-\frac{1}{2}} \leq C\delta^N \leq CN^{-\frac{1}{3}}$. For example one could take $\delta^N = N^{-\beta} d$ for some small $0 < \beta < \frac{1}{6}$. Then we define $r = \max(2R, \delta)$. Furthermore we define the domain with extended holes

$$\Omega_\delta^N = \mathbb{R}^3 \setminus \cup_{i=1}^N \overline{B_r(X_i)}.$$

Finally we recall the definition

$$\tilde{u} = v - \sum_{i=1}^N d_i. \tag{3.31}$$

Lemma 3.33. *We have*

$$\|v_1 - \tilde{u}\|_{L^\infty(\Omega_\delta^N)} \leq \phi o(1), \text{ as } N \to \infty.$$

Proof. We have

$$v_1 - \tilde{u} = \sum_{i=1}^{N} Q_i v - d_i.$$

Take $x \in \Omega_\delta^N$ and take X_i to be the next centre point. Then by Corollary 3.32

$$|Q_i v(x) - d_i(x)| \leq \frac{R^3}{|x - X_i|^2} o(1) \leq R^3 \delta^{-2} o(1) \leq R^3 N o(1) \leq \phi o(1).$$

For the dipoles, that are further away, we use Corollary 3.32 and Corollary 3.25 to get

$$\left| \sum_{j \neq i} (Q_j v(x) - d_j(x)) \right| \leq o(1) \sum_{j \neq i} \frac{R^3}{|x - X_i|^2} \leq R^3 N o(1) = \phi o(1).$$

□

We now want to get a similar estimate inside the particles. We have just seen, for $x \in B_i$:

$$|d_i(x) - Q_i v(x)| \leq R \left\| \nabla d_i - \nabla Q_i v \right\|_{L^\infty(B_i)} \leq CR(\phi^{\frac{1}{4}} + R + R^\alpha).$$

Even if we use that both the first and the second derivative of v are Hölder continuous and hence bounded if one takes f instead of f^N as a right hand side, this improves the (optimal) estimate only slightly to contain R^2 as one of the smallest terms. This is, in general, not of the type $\phi o(1)$. So instead of an estimate in L^∞ we aim for an estimate in L^p for some $p \geq 1$. In order to estimate the difference $v_1 - \tilde{u}$ in this space we only need to consider the difference of the dipoles originated at the closest particle. So we find approximately:

$$\begin{aligned}
\|v_1 - \tilde{u}\|_{L^p(\cup_{i=1}^N B_i)} &= \left(\sum_{i=1}^{N} \int_{B_i} |v_1(x) - \tilde{u}(x)|^p \; \mathrm{d}x \right)^{\frac{1}{p}} \\
&\sim \left(\sum_{i=1}^{N} \int_{B_i} |Q_i v(x) - d_i(x)|^p \; \mathrm{d}x \right)^{\frac{1}{p}} \\
&\leq C \left(\sum_{i=1}^{N} \int_{B_i} (R o(1))^p \; \mathrm{d}x \right)^{\frac{1}{p}} \\
&\leq C \left(\sum_{i=1}^{N} R^{3+p} \right)^{\frac{1}{p}} o(1) \\
&= \left(N R^{3+p} \right)^{\frac{1}{p}} o(1) \\
&= \phi^{\frac{1}{3} + \frac{1}{p}} N^{-\frac{1}{3}} o(1),
\end{aligned}$$

where we used Lemma 3.7 and Remark 3.31 in the third line. For this to be of type $\phi o(1)$ we need $p \leq \frac{3}{2}$. Notice that on the other hand the dipoles decay like $\frac{1}{|x|^2}$. This is only in $L^p(\mathbb{R}^3 \setminus B_L(0))$ for $p > \frac{3}{2}$. Therefore, the powers just fall short of each other so that, using the explicit form of the $o(1)$ term, it might be possible to find a common $p = \frac{3}{2} + \varepsilon$ for which we have closeness on the whole space. But it is clear what the decay is far away from the particles anyway and going from $p = \frac{3}{2}$ to $p = \frac{3}{2} + \varepsilon$ next to the particles does not seem particularly important. So we refrain from trying to prove closeness on the whole space for L^p. If we ignore the decay at infinity and just look at sets of bounded measure the answer above is correct.

Lemma 3.34. *Let $U \subset \mathbb{R}^3$ be be of finite measure. For $p \in [1, \frac{3}{2}]$ it holds:*

$$\|v_1 - \tilde{u}\|_{L^p(U)} \leq \phi o(1).$$

Proof. First of all note that by Lemma 3.33 we only need to prove the statement for $U = \cup_{i=1}^N B_r(X_i)$. Let $x \in U$, then $x \in B_r(X_i)$ for one and only one i because of Assumption **(2)**. By the proof of Theorem 3.33 we only need to consider the dipole $d_i - Q_i v$ because the other dipoles behave exactly the same as outside $B_r(X_i)$, giving L^∞ estimates. If $x \in B_{2R}(X_i)$, by the maximum principle plus Lemma 3.7 as well as Lemma 3.30, we have:

$$\left|Q_i^P v(x) - d_i^P(x)\right| \leq CR\left|\nabla Q_i^P v(x) - \nabla d_i^P(x)\right| = R\left|\nabla v(x) - \nabla v(X_i)\right| \leq Ro(1).$$

and by Lemma 3.26 plus 3.11 as well as Lemma 3.30:

$$\left|Q_i^S v(x) - d_i^S(x)\right| \leq CR\left|eQ_i^P v(x) - ed_i^P(x)\right| = R\left|ev(x) - ev(X_i)\right| \leq R\left|\nabla v(x) - \nabla v(X_i)\right| \leq Ro(1).$$

By the computation done before the lemma this gives $\|v_1 - \tilde{u}\|_{L^p(\cup_{i=1}^N B_{2R}(X_i))} \leq \phi o(1)$. If $r = 2R$ we are done. Otherwise observe that for $|x - X_i| \in (2R, \delta)$ we can use Corollary 3.32 and get

$$\begin{aligned}
\|v_1 - \tilde{u}\|_{L^p(\cup B_\delta(X_i) \setminus B_{2R}(X_i))} &\leq o(1)\left(N\int_{2R}^{\delta} R^{3p}\frac{1}{|x|^{2p}}\,\mathrm{d}x\right)^{\frac{1}{p}} + o(1)\phi \\
&\leq o(1)\left(N\delta^{3-2p}R^{3p}\right)^{\frac{1}{p}} + o(1)\phi \\
&\leq o(1)N^{\frac{1}{p}}R^3 + o(1)\phi \\
&\leq \phi o(1).
\end{aligned}$$

This was the computation for $p \neq \frac{3}{2}$. If $p = \frac{3}{2}$ we get

$$\begin{aligned}\|v_1 - \tilde{u}\|_{L^p(\cup B_\delta(X_i)\setminus B_{2R}(X_i))} &\leq C\left(\phi^{\frac{1}{4}} + R^\alpha + R\right) N^{\frac{1}{p}} R^3 \left(\log\delta - \log 2R\right) \\ &= \phi N^{-\frac{1}{3}} \left(-\log R\right)\left(\phi^{\frac{1}{4}} + R^\alpha + R\right). \\ &= \phi o(1).\end{aligned}$$

Here we used that $-R^\alpha \log R \to 0$ and

$$-N^{-\frac{1}{3}}\phi^{\frac{1}{4}} \log R \leq C R^{-\frac{1}{2}} N^{-\frac{1}{3}} \phi^{\frac{1}{4}} \leq C\phi^{-\frac{1}{6}} N^{-\frac{1}{6}} \phi^{\frac{1}{4}} = o(1).$$

□

4. Homogenization

In this chapter we relate the approximation $\tilde{u}$ to the solution of a homogenized equation on the whole space.

4.1. From the microscopic approximation to a homogenized equation

Assumption **(5)** tells us that the rescaled volume density $\rho^N \rightharpoonup \rho$ in some $L^p(\mathbb{R}^3), p > 3$. The solution v to the reference problem without particles involves the individual particles on the right hand side and is therefore not in a good form for the treatment of the limit problem. Therefore we first prove that v is close to the solution $\hat{v}$ of the following problems:

$$-\Delta \hat{v} = (1 - \phi\rho) f \text{ in } \mathbb{R}^3, \tag{4.1}$$

$$\hat{v}(x) \to 0 \quad \text{as } |x| \to \infty, \tag{4.2}$$

in the Poisson case and

$$-\Delta \hat{v} + \nabla p = (1 - \phi\rho) f \text{ in } \mathbb{R}^3, \tag{4.3}$$

$$\operatorname{div} \hat{v} = 0 \text{ in } \mathbb{R}^3, \tag{4.4}$$

$$\hat{v}(x) \to 0 \quad \text{as } |x| \to \infty, \tag{4.5}$$

in the Stokes case.

Lemma 4.1. *For the solution v of problem* (3.1)-(3.2) *or problem* (3.3)-(3.5) *and the solution $\hat{v}$ of problem* (4.1)-(4.2) *or problem* (4.3)-(4.5) *respectively it holds that*

$$\|v - \hat{v}\|_{W^{1,\infty}(\Omega_\delta^N)} \leq \phi o(1).$$

Let $U \subset \mathbb{R}^3$ be of finite measure and $p \in [1, 3]$. Then

$$\|v - \hat{v}\|_{L^p(U)} \leq \phi o(1).$$

Proof. The idea of the proof is to represent $v, \hat{v}$ in terms of the fundamental solution. Since there is a ϕ in front of ρ and ρ^N is close to ρ in a weak sense (when convoluted with the fundamental solution), we can replace ρ by ρ^N. We can furthermore ignore terms at regions that are close to the point in question since they are small anyway. For regions further away from the point, the number density ρ^N looks approximately like the rescaled sum of the characteristic functions of the particles.

We write $v, \hat{v}$ by means of the fundamental solution to see that

$$|v(x) - \hat{v}(x)| = \left| \int_{\mathbb{R}^3} \left(\sum_{k=1}^{N} \chi_{B_k}(y) - \phi\rho(y) \right) \Phi(x-y) f(y) \, \mathrm{d}y \right|.$$

The proof comes in four parts. The first part shows that we can replace ρ by ρ^N and that particles close to x can be ignored, the second part establishes the closeness of the functions in L^∞ while the third part is concerned with the closeness of the gradients in L^∞. In the last part the necessary L^p estimates are shown.

Part 1: We can replace ρ by ρ^N since

$$\left| \int_{\mathbb{R}^3} (\phi\rho^N(y) - \phi\rho(y)) \Phi(x-y) f(y) \, \mathrm{d}y \right| = \phi \left| \int_{B_{L+1}(0)} (\rho_N(y) - \rho(y)) \Phi(x-y) f(y) \, \mathrm{d}y \right| = \phi o(1),$$

because $\rho^N \rightharpoonup \rho$ in $L^p(B_{L+1}(0))$ and $f\Phi(x - \cdot) \in L^q(B_{L+1}(0))$ where q is the Hölder dual of p and hence $q < \frac{3}{2}$.

Let X_i be the closest centre point to x. Then we can ignore the ith term in the sum:

$$\left| \int_{\mathbb{R}^3} \chi_{B_i}(y) \Phi(x-y) f(y) \, \mathrm{d}y \right| \leq CR^3 \frac{1}{\delta} \leq CR^3 N^{\frac{1}{2}} = \phi N^{-\frac{1}{2}} = \phi o(1),$$

and we can replace χ_{B_k} by $\frac{4\pi}{3} R^3 \delta_{X_k}$ using Lemma 3.29 and Corollary 3.25:

$$\begin{aligned} \left| \int_{\mathbb{R}^3} \sum_{k \neq i} (\chi_{B_k}(y) - \frac{4\pi}{3} R^3 \delta_{X_k}) \Phi(x-y) f(y) \, \mathrm{d}y \right| &\leq C\phi \frac{1}{N} \sum_{k \neq i} \fint_{B_k} |\Phi(x-y) - \Phi(x - X_k)| \, \mathrm{d}y \\ &\leq C\phi \frac{1}{N} \sum_{k \neq i} \frac{R}{|x - X_k|^2} \, \mathrm{d}y \\ &\leq C\phi R = \phi o(1). \end{aligned}$$

Therefore, what is left to show is

$$\left| \frac{1}{N} \sum_{k \neq i} \frac{4\pi}{3} \Phi(x - X_k) f(X_k) - \int_{\mathbb{R}^3} \rho^N(y) \Phi(x-y) f(y) \, \mathrm{d}y \right| = o(1).$$

We can ignore the contributions by particles in the range s :

$$\begin{aligned} \left| \int_{B_s(x)} \rho^N(y) \Phi(x-y) f(y) \, \mathrm{d}y \right| &\leq C \int_{B_s(x)} \frac{1}{|x-y|} \, \mathrm{d}y \\ &\leq C s^2 = o(1), \\ \left| \frac{1}{N} \sum_{k \neq i, |x - X_k| \leq s} \Phi(x - X_k) f(X_k) \right| &\leq C \frac{1}{N} \sum_{k \neq i, |x - X_k| \leq s} \frac{1}{|x - X_k|} \\ &\leq C \frac{1}{N} \frac{1}{d} \left(\frac{s^3}{d^3} \right)^{\frac{2}{3}} \\ &\leq C s^2 = o(1). \end{aligned}$$

Here we used that ρ^N is uniformly bounded in the range s there can only be a number of particles $\leq C \frac{s^3}{d^3}$ and then applied Corollary 3.25 with $N = \frac{s^3}{d^3}$.

The above reasoning applies to particles in the range of $3s$ in the same way. This means we can ignore all cubes A_j that intersect the boundary $\partial B_s(x)$ since they will be included in $B_{3s}(x)$ anyway.

Part 2: Therefore estimating the difference above reduces to estimating the difference of appropriately grouped terms in the sum to its corresponding parts (the cube A_j) of the integral. This means we want to estimate

$$\sum_{j : \mathrm{dist}(A_j, x) > s} \left| \frac{4\pi}{3} \frac{1}{N s^3} \sum_{X_k \in A_j} \int_{A_j} f(X_k) \Phi(x - X_k) - f(y) \Phi(x - y) \, \mathrm{d}y \right| = o(1).$$

Using Hölder-continuity of f and Lemma 3.29 we have

$$\begin{aligned} |(f(X_k) - f(y)) \Phi(x - X_k)| &\leq C \frac{s^\alpha}{|x - X_k|}, \\ |f(y) \left(\Phi(x - X_k) - \Phi(x - y) \right)| &\leq C \frac{s}{|x - X_k|^2}. \end{aligned}$$

Hence, using Corollary 3.25:

$$\begin{aligned}
&\sum_{j:\mathrm{dist}(A_j,x)>s} \left| \frac{4\pi}{3}\frac{1}{Ns^3} \sum_{X_k\in A_j} \int_{A_j} f(X_k)\Phi(x-X_k) - f(y)\Phi(x-y)\, \mathrm{d}y \right| \\
&\leq C \sum_{j:\mathrm{dist}(A_j,x)>s} \left| \frac{4\pi}{3}\frac{1}{N} \sum_{X_k\in A_j} \frac{s^\alpha}{|x-X_k|} + \frac{4\pi}{3}\frac{1}{N} \sum_{X_k\in A_j} \frac{s}{|x-X_k|^2} \right| \\
&\leq C\frac{s^\alpha}{N}\sum_{k\neq i}\frac{1}{|x-X_k|} + C\frac{s}{N}\sum_{k\neq i}\frac{1}{|x-X_k|^2} \\
&\leq C\left(s^\alpha + s\right) = o(1).
\end{aligned}$$

Part 3: In order to understand that the estimate holds for the gradient note that

$$|\nabla v(x) - \nabla \hat{v}(x)| = \left| \int_{\mathbb{R}^3} (\sum_{k=1}^N \chi_{B_k}(y) - \phi\rho(y))\nabla\Phi(x-y)f(y)\, \mathrm{d}y \right|.$$

We can now reproduce all the steps from the proof above using the fact that

- $\rho_N \rightharpoonup \rho$ in $L^p(B_{L+1}(0))$ and $f\nabla\Phi(x-\cdot) \in L^q(B_{L+1}(0))$, since $\frac{1}{|x|^2}$ is q-integrable for $q < \frac{3}{2}$;
- $\frac{1}{\delta^2 N} \to 0$, so
$$\left| \int_{\mathbb{R}^3} \chi_{B_i}(y)\nabla\Phi(x-y)f(y)\, \mathrm{d}y \right| \leq CR^3\frac{1}{\delta^2} = R^3\frac{1}{\delta^2 N}N = \phi o(1);$$
- $$\phi\frac{1}{N}\sum_{k\neq i}\frac{R}{|x-X_k|^3}\, \mathrm{d}y \leq C\phi\log NR \leq C\phi N^{\frac{1}{3}}R = \phi^{\frac{4}{3}} = \phi o(1);$$
- $$\int_{B_s(x)} \frac{1}{|x-y|^2}\, \mathrm{d}y \leq Cs = o(1),$$
$$\frac{1}{N}\sum_{k\neq i, |x-X_k|\leq s}\frac{1}{|x-X_k|^2} \leq C\frac{1}{N}\frac{1}{d^2}\left(\frac{s_N^3}{d^3}\right)^{\frac{1}{3}} \leq Cs = o(1);$$
- $$\frac{s^\alpha}{N}\sum_{k\neq i}\frac{1}{|x-X_k|^2} \leq Cs^\alpha = o(1);$$

- $$\frac{s}{N}\sum_{k\neq i}\frac{1}{|x-X_k|^3}\leq Cs\log N=o(1).$$

This gives

$$|\nabla v(x)-\nabla\hat{v}(x)|=\phi o(1).$$

Part 4: In order to get the L^p result simply notice that we can use the L^∞ results everywhere even where $x\in B_r(X_i)$ as long as we did not use that $|x-X_i|>r$. In fact this was used only once so that we have to look at the following term again when $x\in B_r(X_i)$:

$$\left|\int_{\mathbb{R}^3}\chi_{B_i}(y)\Phi(x-y)f(y)\,\mathrm{d}y\right|\leq C\int_{B_i}\frac{1}{|x-y|}\,\mathrm{d}y.$$

If $|x-X_i|>2R$ then this is smaller than $CR^3\frac{1}{|x-X_i|}$. If $|x-X_i|\leq 2R$ it scales like R^2. Integrating the pth power of the left hand side over the union of the $B_r(X_i)$ gives

$$\begin{aligned}\left(\int_{\cup_{i=1}^N B_r(X_i)}\left|\int_{\mathbb{R}^3}\chi_{B_i}(y)\Phi(x-y)f(y)\,\mathrm{d}y\right|^p\mathrm{d}x\right)^{\frac{1}{p}}&\leq C\left(N\left(R^{2p}R^3+R^{3p}\int_{2R}^{\delta}t^{-p+2}\,\mathrm{d}t\right)\right)^{\frac{1}{p}}\\&\leq C\left(N\left(R^{2p+3}+R^{3p}\left(\delta\right)^{-p+3}\right)\right)^{\frac{1}{p}}\\&\leq C\left(N^{\frac{1}{p}}R^{2+\frac{3}{p}}+R^3N^{\frac{1}{p}}\delta^{-1+\frac{3}{p}}\right)\\&\leq\phi o(1).\end{aligned}$$

□

Now we can establish the first closeness result for the solution of the homogenized equation.

Lemma 4.2. *Let $\hat{u}\in\dot{H}^1$ be the solution to*

$$-\operatorname{div}\left(\nabla\hat{u}+3\phi\rho\nabla\hat{v}\right)=(1-\phi\rho)f,\tag{4.6}$$

and $\tilde{u}$ be the explicit electrostatic dipole approximation from equation (3.31)*. Then we have*

$$\|\tilde{u}-\hat{u}\|_{L^\infty(\Omega_\delta^N)}\leq\phi o(1).$$

Let $U\subset\mathbb{R}^3$ be of finite measure and $p\in[1,\frac{3}{2}]$. Then

$$\|\tilde{u}-\hat{u}\|_{L^p(U)}\leq\phi o(1).$$

Proof. In priciple we employ the same strategy as in the proof of Lemma 4.1. We represent $\hat{u}$ in terms of the fundamental solution and then we use Lemma 4.1 as well as $\rho^N \rightharpoonup \rho$ to show, that the difference of the sum and the integral is small.

Note, that

$$\hat{u}(x) = \hat{v}(x) + \int_{\mathbb{R}^3} 3\phi\rho(y)\nabla\hat{v}(y)(\nabla\Phi^P)(x-y)\,\mathrm{d}y. \tag{4.7}$$

Integrating by parts and applying $-\Delta$ to both sides gives:

$$-\Delta\hat{u} = (1-\phi\rho)f + \operatorname{div}(3\phi\rho\nabla\hat{v}).$$

This means the function $\hat{u}$ from (4.7) is indeed the solution to (4.6). The main point of the proof is to show that $\tilde{u}$ is close to the representation of $\hat{u}$ that is given in (4.7).

In order to show the closeness let $x \in \Omega_\delta^N$ be given. We have

$$|\tilde{u}(x) - \hat{u}(x)| \leq |v(x) - \hat{v}(x)| + \left| -\sum_{k=1}^{N} d_k(x) - \int_{\mathbb{R}^3} 3\phi\rho(y)\nabla\hat{v}(y)\nabla\Phi^P(x-y)\,\mathrm{d}y \right|.$$

In Lemma 4.1 it was shown that $|v(x) - \hat{v}(x)| = \phi o(1)$. It remains to prove that

$$\left| -\sum_{k=1}^{N} d_k(x) - \int_{\mathbb{R}^3} 3\phi\rho(y)\nabla\hat{v}(y)\nabla\Phi^P(x-y)\,\mathrm{d}y \right| = \phi\cdot o(1). \tag{4.8}$$

Again, the proof is divided in four parts. The first part shows that we can replace $\hat{v}$ by v. In the second part we show that particles close to x can be ignored and that we can replace ρ by ρ^N. The third part establishes the closeness of the functions in L^∞ while the fourth part is concerned with the L^p result.

Part 1: Since the function appearing in the dipoles d_i is v we would like to replace $\hat{v}$ by v in (4.8). But by Lemma 4.1 we have

$$
\begin{aligned}
&\left|\int_{\mathbb{R}^3} 3\phi\rho(y)(\nabla v(y) - \nabla \hat{v}(y))\nabla\Phi^P(x-y)\,\mathrm{d}y\right| \\
&= \left|\int_{B_L(0)\cap\Omega_\delta^N} 3\phi\rho(y)(\nabla v(y) - \nabla \hat{v}(y))\nabla\Phi^P(x-y)\,\mathrm{d}y\right| \\
&+ \left|\int_{B_L(0)} \sum_{k=1}^N \chi_{B_r(X_k)}(y) 3\phi\rho(y)(\nabla v(y) - \nabla \hat{v}(y))\nabla\Phi^P(x-y)\,\mathrm{d}y\right| \\
&\le C\phi^2 o(1) \int_{B_L(0)\cap\Omega_\delta^N} \frac{1}{|x-y|^2}\,\mathrm{d}y + C\phi \left\|\nabla\Phi^P\right\|_{L^{\frac{4}{3}}(B_L(0))} \left\|\sum_{k=1}^N \chi_{B_r(X_k)}\right\|_{L^4} \\
&= \phi^2 o(1) + C\phi\left(N\delta^3\right)^{\frac{1}{4}} \\
&= \phi^2 o(1) + \phi\left(Nd^3\right)^{\frac{1}{4}} o(1) \\
&= \phi\left(\phi o(1) + o(1)\right) \\
&= \phi o(1).
\end{aligned}
$$

Thus we in fact need to prove that

$$
\left| -\sum_{k=1}^N d_k(x) - \int_{\mathbb{R}^3} 3\phi\rho(y)\nabla v(y)\nabla\Phi^P(x-y)\,\mathrm{d}y \right| = \phi\cdot o(1).
$$

Part 2: Let X_i be the closest centre point to x. Then we can ignore the ith term in the sum:

$$
|d_i(x)| \le C\frac{R^3}{|x-X_i|^2} \le CR^3\delta^{-2} = R^3 N\frac{1}{\delta^2 N} = \phi o(1).
$$

Let us replace $R^3 = \phi\frac{1}{N}$. Then we are left to show that

$$
\left|\phi\frac{1}{N}\sum_{k\neq i}\nabla v(X_k)\cdot\frac{x-X_k}{|x-X_k|^3} - \phi\int_{\mathbb{R}^3} 3\rho(y)\nabla v(y)\cdot\frac{x-y}{4\pi|x-y|^3}\,\mathrm{d}y\right| = \phi\cdot o(1),
$$

or equivalently

$$
\left|\frac{1}{N}\sum_{k\neq i}\nabla v(X_k)\cdot\frac{x-X_k}{|x-X_k|^3} - \int_{\mathbb{R}^3} 3\rho(y)\nabla v(y)\frac{x-y}{4\pi|x-y|^3}\,\mathrm{d}y\right| = o(1).
$$

We can replace ρ by ρ^N. Since ρ is supported in $B_{2L}(0)$, this is due to the fact that

$$\left| \int_{B_{2L}(0)} \frac{3}{4\pi} (\rho^N - \rho(y)) \nabla v(y) \cdot \frac{x-y}{4\pi |x-y|^3} \, \mathrm{d}y \right| = o(1).$$

The integral vanishes for $N \to \infty$ since $\rho^N \rightharpoonup \rho$ in L^p for $p > 3$ and the rest is in the dual space since $\nabla v \in L^\infty$.

We can leave out a ball of size s around x in the integral since

$$\left| \int_{B_s(x)} \frac{3}{4\pi} \rho^N(y) \nabla v(y) \cdot \frac{x-y}{4\pi |x-y|^3} \, \mathrm{d}y \right| \leq C \int_{B_s(x)} \frac{1}{|x-y|^2} \, \mathrm{d}y \leq Cs = o(1).$$

Here we used that the ρ^N are uniformly bounded in L^∞ due to assumption **(2)**.

In the same manner we can ignore the parts of the sum where $|X_k - x| \leq s$. For the terms in this range we get:

$$\begin{aligned} \left| \frac{1}{N} \sum_{k \neq i, |x - X_k| \leq s} \nabla v(X_k) \cdot \frac{x - X_k}{|x - X_k|^3} \right| &\leq C \frac{1}{N} \sum_{k \neq i, |x - X_k| \leq s} \frac{1}{|x - X_k|^2} \\ &\leq C \frac{1}{N} \left(\frac{s^3}{d^3} \right)^{\frac{1}{3}} \frac{1}{d^2} \leq Cs = o(1), \end{aligned}$$

using Corollary 3.25 with $N = \frac{s^3}{d^3}$, since the number of particles involved in this sum is $\leq C \frac{s^3}{d^3}$.

It remains to prove the following estimate:

$$\left| \frac{1}{N} \sum_{k : |x - X_k| > s} \nabla v(X_k) \cdot \frac{x - X_k}{|x - X_k|^3} - \int_{\mathbb{R}^3 \setminus B_s(x)} \frac{3}{4\pi} \rho^N(y) \nabla v(y) \cdot \frac{x-y}{|x-y|^3} \, \mathrm{d}y \right| = o(1).$$

We can employ the same reasoning as above to exclude all particles in the range of $3s$. This means we can ignore all cubes A_j that intersect the boundary $\partial B_s(x)$ since they will eventually be included in $B_{3s}(x)$ anyway.

Part 3: Therefore estimating the difference above comes down to estimating the difference of appropriately grouped terms in the sum to its corresponding parts (the cube A_j) of the integral. I.e. estimating

$$\sum_{j : \mathrm{dist}(A_j, x) > s} \left| \frac{1}{N} \sum_{X_k \in A_j} \nabla v(X_k) \cdot \frac{x - X_k}{|x - X_k|^3} - \int_{A_j} \frac{3}{4\pi} \rho_N(y) \nabla v(y) \cdot \frac{x-y}{|x-y|^3} \, \mathrm{d}y \right| = o(1).$$

Notice that the cubes with $A_j \cap B_{2L}(0) = \emptyset$ have no contribution since there $\rho_N = 0$. Looking at one term of the sum we are left to estimate

$$\left| \frac{1}{N} \sum_{X_k \in A_j} \nabla v(X_k) \cdot \frac{x - X_k}{|x - X_k|^3} - \int_{A_j} \frac{3}{4\pi} \rho^N(y) \nabla v(y) \cdot \frac{x-y}{|x-y|^3} \, \mathrm{d}y \right|.$$

We now use the definition of ρ^N to write this as

$$\begin{aligned} &\left| \frac{1}{N} \sum_{X_k \in A_j} \nabla v(X_k) \cdot \frac{x - X_k}{|x - X_k|^3} - \int_{A_j} \frac{3}{4\pi} \frac{1}{Ns^3} \frac{4\pi}{3} n(A_k) \nabla v(y) \cdot \frac{x-y}{|x-y|^3} \, \mathrm{d}y \right| \\ &= \frac{1}{N} \left| \sum_{X_k \in A_j} \left(\nabla v(X_k) \cdot \frac{x - X_k}{|x - X_k|^3} - \frac{1}{s^3} \int_{A_j} \nabla v(y) \cdot \frac{x-y}{|x-y|^3} \, \mathrm{d}y \right) \right| \\ &\leq C \frac{1}{Ns^3} \sum_{X_k \in A_j} \left| \int_{A_j} \nabla v(X_k) \cdot \frac{x - X_k}{|x - X_k|^3} - \nabla v(y) \cdot \frac{x-y}{|x-y|^3} \, \mathrm{d}y \right|. \end{aligned}$$

We can replace $\nabla v(y)$ by $\nabla v(X_k)$ in the integral since for the difference, by Lemma 3.30, we have:

$$|\nabla v(y) - \nabla v(X_k)| \leq C \left(s^\alpha + s + \phi^{\frac{1}{4}} \right) = o(1),$$

and hence

$$\left| \int_{A_j} \nabla v(X_k) \cdot \frac{x-y}{|x-y|^3} - \nabla v(y) \cdot \frac{x-y}{|x-y|^3} \, \mathrm{d}y \right| \leq o(1) \int_{A_j} \frac{1}{|x-y|^2} \, \mathrm{d}y.$$

Since the number of particles in one A_j is bounded by Ns^3, summing up we obtain

$$\begin{aligned} \sum_{j:\mathrm{dist}(A_j,x)>s} \frac{1}{Ns^3} \sum_{X_k \in A_j} & \left| \int_{A_j} \nabla v(X_k) \cdot \frac{x-y}{|x-y|^3} - \nabla v(y) \cdot \frac{x-y}{|x-y|^3} \, \mathrm{d}y \right| \\ &\leq \sum_{j:\mathrm{dist}(A_j,x)>s, n(A_j)\neq 0} o(1) \int_{A_j} \frac{1}{|x-y|^2} \, \mathrm{d}y \\ &\leq o(1) \int_{B_{L+s}(0)} \frac{1}{|x-y|^2} \, \mathrm{d}y \\ &\leq o(1). \end{aligned}$$

Then, by Lemma 3.29, for $y, X_k \in A_j$ we have

$$\left| \frac{(x-X_k)^{\otimes 3}}{|x-X_k|^5} - \frac{(x-y)^{\otimes 3}}{|x-y|^5} \right| \le C \frac{s}{|x-X_k|^3}.$$

Using this, we obtain

$$\begin{aligned}
&\frac{1}{Ns^3} \sum_{X_k \in A_j} \nabla v(X_k) \cdot \frac{x-X_k}{|x-X_k|^3} - \nabla v(X_k) \cdot \frac{x-y}{|x-y|^3} \\
&\le C \frac{1}{Ns^3} \sum_{X_k \in A_j} \left| \int_{A_j} \|\nabla v\|_{L^\infty} \frac{s}{|x-X_k|^3} \right| \\
&\le C \frac{s}{N} \sum_{X_k \in A_j} \frac{1}{|x-X_k|^3}.
\end{aligned}$$

Summing up over j gives

$$\begin{aligned}
\sum_{j:\mathrm{dist}(A_j,x)>s} \frac{1}{Ns^3} \sum_{X_k \in A_j} \left| \int_{A_j} \nabla v(X_k) \left(\frac{x-X_k}{|x-X_k|^3} - \frac{x-y}{|x-y|^3} \right) \right| &\le C \frac{s}{N} \sum_{k \ne i} \frac{1}{|x-X_k|^3} \\
&\le C \frac{s}{N} \frac{\log N}{d^3} \\
&\le C s \log N \\
&= o(1).
\end{aligned}$$

This finishes the L^∞ part of the proof.

Part 4: In order to get the L^p result simply notice that we can use the L^∞ results everywhere even where $x \in B_r(X_i)$ as long as we did not use that $|x - X_i| > r$. In fact this was only used once so that we have to look at $d_i(x)$ again when $x \in B_r(X_i)$: If $|x - X_i| > R$, this is smaller than $CR^3 \frac{1}{|x-X_i|^2}$. If $|x - X_i| \le R$ it scales like R. Integrating the pth power of this over the union of the $B_r(X_i)$ gives

$$\begin{aligned}
\left(\sum_{i=1}^N \int_{B_r(X_i)} |d_i(x)|^p \, \mathrm{d}x \right)^{\frac{1}{p}} &\le C \left(N \left(R^p R^3 + R^{3p} \int_R^\delta t^{-2p+2} \, \mathrm{d}t \right) \right)^{\frac{1}{p}} \\
&\le C \left(N \left(R^{p+3} + R^{3p} \delta^{-2p+3} \right) \right)^{\frac{1}{p}} \\
&\le C N^{\frac{1}{p}} R^{1+\frac{3}{p}} + R^3 \delta^{-2+\frac{3}{p}} \\
&\le C N^{\frac{1}{p}} R^{1+\frac{3}{p}} + R^3 N \frac{1}{N\delta^2} \delta^{\frac{3}{p}} \\
&\le \phi o(1).
\end{aligned}$$

□

Lemma 4.3. *Let $\hat{u}$ be the solution to*

$$- \operatorname{div} (\nabla \hat{u} + 5\phi\rho\, e\hat{v}) + \nabla p = (1 - \phi\rho) f, \tag{4.9}$$
$$\operatorname{div} \hat{u} = 0. \tag{4.10}$$

and let $\tilde{u}$ be the explicit Stokes dipole approximation. Then we have

$$\|\tilde{u} - \hat{u}\|_{L^\infty(\Omega_\delta^N)} \leq \phi o(1).$$

Let $U \subset \mathbb{R}^3$ be of finite measure and $p \in [1, \frac{3}{2}]$. Then

$$\|\tilde{u} - \hat{u}\|_{L^p(U)} \leq \phi o(1).$$

Proof. The proof is completely analogous to the one for the Poisson case, except that the dipoles have a faster decaying part that can be ignored and that the gradient of the fundamental solution is a bit more involved.

Again we write $\hat{u}$ (componentwise) in terms of the fundamental solution:

$$\hat{u}_j(x) = \hat{v}_j(x) + \int_{\mathbb{R}^3} 5\phi\rho(y) e\hat{v}(y)_{ki} \partial_k \Phi^S_{ij}(x - y)\, \mathrm{d}y. \tag{4.11}$$

Integrating by parts with respect to y in the second integral shows that $\hat{u}$ is the convolution of Φ with some function and therefore $\operatorname{div} \hat{u} = 0$. Applying $-\Delta$ to both sides then gives:

$$-\Delta \hat{u} + \nabla p = (1 - \phi\rho) f + \operatorname{div} (5\phi\rho\, e\hat{v}),$$

showing that $\hat{u}$ is indeed the solution to (4.9).

In order to show the closeness of $\tilde{u}$ to the representation from (4.11) let $x \in \Omega_\delta^N$ be given. In Lemma 4.1 it was shown that $|v(x) - \hat{v}(x)| = \phi o(1)$. It remains to prove that

$$\left| -\sum_{k=1}^{N} d_k(x) - \int_{\mathbb{R}^3} 5\phi\rho(y) e\hat{v}(y) \nabla \Phi^S(x - y)\, \mathrm{d}y \right| = \phi o(1).$$

The proof is divided in several parts. The first part shows that we can replace $\hat{v}$ by v. In the second part we show that the closest particle as well as the fast decaying parts of the dipoles can be ignored. The next part determines the explicit form of the gradient of the fundamental solution when applied to a symmetric,trace-free matrix. In the fourth part it is shown that we can replace ρ by ρ^N and ignore close particles. The fifth part establishes the closeness of the functions in L^∞ while the last part is concerned with the L^p result.

Part 1: Using the fact that by Lemma 4.1 we have

$$\begin{aligned}
&\left|\int_{\mathbb{R}^3} 5\phi\rho(y)(ev(y) - e\hat{v}(y))\nabla\Phi^S(x-y)\,\mathrm{d}y\right| \\
&= \left|\int_{B_L(0)\cap\Omega_\delta^N} 5\phi\rho(y)(ev(y) - e\hat{v}(y))\nabla\Phi^S(x-y)\,\mathrm{d}y\right| \\
&+ \left|\int_{B_L(0)} \sum_{k=1}^N \chi_{B_\delta}(y) 5\phi\rho(y)(ev(y) - e\hat{v}(y))\nabla\Phi^S(x-y)\,\mathrm{d}y\right| \\
&\lesssim \phi^2 o(1) \int_{B_L(0)\cap\Omega_\delta^N} \frac{1}{|x-y|^2}\,\mathrm{d}y + C\phi \left\|\nabla\Phi^S\right\|_{L^{\frac{4}{3}}(B_L(0))} \left\|\sum_{k=1}^N \chi_{B_r}\right\|_{L^4} \\
&\le \phi^2 o(1) + C\phi \left(N(\delta)^3\right)^{\frac{1}{4}} \\
&= \phi(\phi o(1) + o(1)) \\
&= \phi o(1),
\end{aligned}$$

we will actually prove that

$$\left| -\sum_{k=1}^N d_k(x) - \int_{\mathbb{R}^3} 5\phi\rho(y) ev(y)\nabla\Phi^S(x-y)\,\mathrm{d}y \right| = \phi o(1).$$

Part 2: Let X_i be the closest centre point to x. Then we can ignore the ith term in the sum:

$$|d_i(x)| \le C\frac{R^3}{|x-X_i|^2} + C\frac{R^5}{|x-X_i|^4} \le CR^3\delta^{-2} = CR^3 N\frac{1}{N\delta^2} = \phi o(1).$$

Next we look at the fast decaying terms of d_k:

$$\begin{aligned}
\sum_{k\neq i} R^5 \left(\frac{ev(X_k)(x-X_k)}{|x-X_k|^5} - \frac{5}{2}\frac{(x-X_k)\left((x-X_k)\cdot ev(X_k)(x-X_k)\right)}{|x-X_k|^7} \right) &\le C\sum_{k\neq i}\frac{R^5}{|x-X_k|^4} \\
&\le CR^5 N^{\frac{4}{3}} \\
&= R\phi^{\frac{4}{3}} \\
&= \phi o(1).
\end{aligned}$$

Thus we can leave out these terms and only consider the terms of the form

$$\frac{5}{2}R^3\left(\frac{(x-X_k)\left((x-X_k)\cdot ev(X_k)(x-X_k)\right)}{|x-X_k|^5}\right).$$

Part 3: Now we want to know what $ev(y)\nabla\Phi^S(x-y)$ actually looks like. We compute the derivative of the Oseen-Tensor to be

$$\partial_k\Phi_{ij}(x)=\frac{1}{8\pi}\left(-\frac{\delta_{ij}x_k}{|x|^3}+\frac{\delta_{ik}x_j+\delta_{jk}x_i}{|x^3|}-3\frac{x_ix_jx_k}{|x|^5}\right).$$

Take any symmetric, trace free matrix ϵ. Then

$$\begin{aligned}(\epsilon\nabla\Phi)_j &:= \epsilon_{ki}\partial_k\Phi_{ij}(x)\\ &=\frac{1}{8\pi}\left(-\frac{\epsilon_{ki}x_k}{|x|^3}+\frac{\epsilon_{kk}x_j+\epsilon_{ij}x_i}{|x^3|}-3\frac{\epsilon kix_ix_jx_k}{|x|^5}\right)\\ &=-\frac{3}{8\pi}\frac{\epsilon kix_ix_jx_k}{|x|^5}\\ &=-\frac{3}{8\pi}\left(\frac{x\,(x\cdot\epsilon x)}{|x|^5}\right)_j.\end{aligned}$$

Let us replace $R^3=\phi\frac{1}{N}$. Then we are left to show that

$$\left|\phi\frac{1}{N}\sum_{k\neq i}\frac{5}{2}\frac{(x-X_k)\left((x-X_k)\cdot ev(X_k)(x-X_k)\right)}{|x-X_k|^5}-\phi\int_{\mathbb{R}^3}\frac{15}{8\pi}\rho(y)\frac{(x-y)\left((x-y)\cdot ev(y)(x-y)\right)}{|x-y|^5}\,\mathrm{d}y\right|$$
$$=\phi\cdot o(1),$$

or equivalently

$$\left|\frac{1}{N}\sum_{k\neq i}\frac{(x-X_k)\left((x-X_k)\cdot ev(X_k)(x-X_k)\right)}{|x-X_k|^5}-\int_{\mathbb{R}^3}\frac{3}{4\pi}\rho(y)\frac{(x-y)\left((x-y)\cdot ev(y)(x-y)\right)}{|x-y|^5}\,\mathrm{d}y\right|$$
$$=o(1).$$

Part 4: We can replace ρ by ρ^N. ρ is supported in $B_{2L}(0)$ and we have

$$\left|\int_{B_{2L}(0)}\frac{3}{4\pi}(\rho^N-\rho(y))\frac{(x-y)\left((x-y)\cdot ev(y)(x-y)\right)}{|x-y|^5}\,\mathrm{d}y\right|=o(1).$$

The integral vanishes for $N \to \infty$ since $\rho^N \rightharpoonup \rho$ in L^p for $p > 3$ and the other terms are in the dual space.

We can leave out a ball of size s around x in the integral since

$$\left| \int_{B_s(x)} \frac{3}{4\pi} \rho^N(y) \frac{(x-y)\left((x-y)\cdot ev(y)(x-y)\right)}{|x-y|^5} \, \mathrm{d}y \right| \leq C \int_{B_s(x)} \frac{1}{|x-y|^2} \, \mathrm{d}y$$
$$\leq Cs = o(1).$$

Here we used again that the ρ^N are uniformly bounded in L^∞.

In the same way we can ignore the parts of the sum where $|X_k - x| \leq s$. For the terms in this range we get:

$$\left| \frac{1}{N} \sum_{k \neq i, |x - X_k| \leq s} \frac{(x - X_k)\left((x - X_k) \cdot ev(X_k)(x - X_k)\right)}{|x - X_k|^5} \right| \leq C \frac{1}{N} \sum_{k \neq i, |x - X_k| \leq s} \frac{1}{|x - X_k|^2}$$
$$\leq C \frac{1}{N} \left(\frac{s_N^3}{d^3} \right)^{\frac{1}{3}} \frac{1}{d^2}$$
$$\leq Cs = o(1).$$

Here we used 3.25. The number of particles involved in the sum is $\leq C \frac{s_N^3}{d^3}$ since every particle occupies a volume that is at least proportional to d^3.

It remains to show:

$$\frac{1}{N} \sum_{k: |x - X_k| > s} \frac{(x - X_k)\left((x - X_k) \cdot ev(X_k)(x - X_k)\right)}{|x - X_k|^5}$$
$$- \int_{\mathbb{R}^3 \setminus B_s(x)} \frac{3}{4\pi} \rho^N(y) \frac{(x-y)\left((x-y)\cdot ev(y)(x-y)\right)}{|x-y|^5} \, \mathrm{d}y = o(1).$$

We can employ the same reasoning as above to exclude all particles in the range of $3s$. This means we can ignore all cubes A_j that intersect the boundary $\partial B_s(x)$ since they will eventually be included in $B_{3s}(x)$ anyway.

Part 5: Therefore estimating the difference above reduces to estimating the difference of appropriately grouped terms in the sum to its corresponding parts (the cube A_j) of the integral. I.e. we need to estimate

$$\sum_{j: \mathrm{dist}(A_j, x) > s} | \frac{1}{N} \sum_{X_k \in A_j} \frac{(x - X_k)\left((x - X_k) \cdot ev(X_k)(x - X_k)\right)}{|x - X_k|^5}$$
$$- \int_{A_j} \frac{3}{4\pi} \rho^N(y) \frac{(x-y)\left((x-y)\cdot ev(y)(x-y)\right)}{|x-y|^5} \, \mathrm{d}y | = o(1).$$

Notice that the cubes with $A_j \cap B_{2L}(0) = \emptyset$ have no contribution since there $\rho^N = 0$.
Looking at one term of the sum we are left to estimate

$$\left| \frac{1}{N} \sum_{X_k \in A_j} \frac{(x - X_k)\left((x - X_k) \cdot ev(X_k)(x - X_k)\right)}{|x - X_k|^5} - \int_{A_j} \frac{3}{4\pi} \rho^N(y) \frac{(x - y)\left((x - y) \cdot ev(y)(x - y)\right)}{|x - y|^5} \, \mathrm{d}y \right|.$$

We now use the definition of ρ^N to write this as

$$\begin{aligned}
&\left| \frac{1}{N} \sum_{X_k \in A_j} \frac{(x - X_k)\left((x - X_k) \cdot ev(X_k)(x - X_k)\right)}{|x - X_k|^5} - \int_{A_j} \frac{3}{4\pi} \frac{1}{Ns^3} \frac{4\pi}{3} n(A_k) \frac{(x - y)\left((x - y) \cdot ev(y)(x - y)\right)}{|x - y|^5} \, \mathrm{d}y \right| \\
&= \frac{1}{N} \left| \sum_{X_k \in A_j} \left(\frac{(x - X_k)\left((x - X_k) \cdot ev(X_k)(x - X_k)\right)}{|x - X_k|^5} - \frac{1}{s^3} \int_{A_j} \frac{(x - y)\left((x - y) \cdot ev(y)(x - y)\right)}{|x - y|^5} \, \mathrm{d}y \right) \right| \\
&\leq C \frac{1}{Ns^3} \sum_{X_k \in A_j} \left| \int_{A_j} \frac{(x - X_k)\left((x - X_k) \cdot ev(X_k)(x - X_k)\right)}{|x - X_k|^5} - \frac{(x - y)\left((x - y) \cdot ev(y)(x - y)\right)}{|x - y|^5} \, \mathrm{d}y \right|.
\end{aligned}$$

We can replace $ev(y)$ by $ev(X_k)$ in the integral since for the difference, by Lemma 3.30, we have:

$$|ev(y) - ev(X_k)| \leq |\nabla v(y) - \nabla v(X_k)| \leq C \left(s^\alpha + s + \phi^{\frac{1}{4}} \right) = o(1),$$

and hence

$$\begin{aligned}
&\left| \int_{A_j} \frac{(x - y)\left((x - y) \cdot ev(X_k)(x - y)\right)}{|x - y|^5} \, \mathrm{d}y - \int_{A_j} \frac{(x - y)\left((x - y) \cdot ev(y)(x - y)\right)}{|x - y|^5} \, \mathrm{d}y \right| \\
&\qquad \leq o(1) \int_{A_j} \frac{1}{|x - y|^2} \, \mathrm{d}y.
\end{aligned}$$

Since the number of particles in one A_j is bounded by Ns^3, adding this up we obtain

$$\begin{aligned}
&\sum_{j:\mathrm{dist}(A_j, x) > s} \frac{1}{Ns^3} \sum_{X_k \in A_j} \left| \int_{A_j} \frac{(x - y)\left((x - y) \cdot ev(X_k)(x - y)\right)}{|x - y|^5} \, \mathrm{d}y - \int_{A_j} \frac{(x - y)\left((x - y) \cdot ev(y)(x - y)\right)}{|x - y|^5} \, \mathrm{d}y \right| \\
&\qquad \leq \sum_{j:\mathrm{dist}(A_j, x) > s, n(A_j) \neq 0} o(1) \int_{A_j} \frac{1}{|x - y|^2} \, \mathrm{d}y \\
&\qquad \leq o(1) \int_{B_{L+s}(0)} \frac{1}{|x - y|^2} \, \mathrm{d}y \\
&\qquad \leq o(1).
\end{aligned}$$

Then, by Lemma 3.29, for $y, X_k \in A_j$ we have

$$\left| \frac{(x-X_k)^{\otimes 3}}{|x-X_k|^5} - \frac{(x-y)^{\otimes 3}}{|x-y|^5} \right| \leq C \frac{s}{|x-X_k|^3}.$$

Using this, we obtain

$$\begin{aligned}
&\frac{1}{Ns^3} \sum_{X_k \in A_j} \left| \int_{A_j} \frac{(x-X_k)\left((x-X_k)\cdot ev(X_k)(x-X_k)\right)}{|x-X_k|^5} - \frac{(x-y)\left((x-y)\cdot ev(X_k)(x-y)\right)}{|x-y|^5} \right| \\
&\leq C \frac{1}{Ns^3} \sum_{X_k \in A_j} \left| \int_{A_j} \|\nabla v\|_{L^\infty} \frac{s}{|x-X_k|^3} \right| \\
&\leq C \frac{s}{N} \sum_{X_k \in A_j} \frac{1}{|x-X_k|^3}.
\end{aligned}$$

Summing up over j gives

$$\begin{aligned}
\sum_{j:\operatorname{dist}(A_j,x)>s} \frac{1}{Ns^3} \sum_{X_k \in A_j} \left| \int_{A_j} \nabla v(X_k) \left(\frac{x-X_k}{|x-X_k|^3} - \frac{x-y}{|x-y|^3} \right) \right| &\leq C \frac{s}{N} \sum_{k \neq i} \frac{1}{|x-X_k|^3} \\
&\leq C \frac{s}{N} \frac{\log N}{d^3} \\
&\leq Cs \log N \\
&= o(1).
\end{aligned}$$

This finishes the L^∞ part.

Part 6: In order to get the L^p result simply notice that we can use the L^∞ results everywhere even where $x \in B_r(X_i)$ as long as we did not use that $|x - X_i| > r$. In fact this was only used once so that we have to look at $d_i(x)$ again when $x \in B_r(X_i)$: If $|x - X_i| > R$, this is smaller than $CR^3 \frac{1}{|x-X_i|^2}$. If $|x - X_i| \leq R$ it scales like R. Integrating the pth power of this over the union of the $B_r(X_i)$ gives

$$\begin{aligned}
\left(\sum_{i=1}^N \int_{B_r(X_i)} |d_i(x)|^p \, \mathrm{d}x \right)^{\frac{1}{p}} &\leq C \left(N \left(R^p R^3 + R^{3p} \int_R^{dN^{-\beta}} t^{-2p+2} \, \mathrm{d}t \right) \right)^{\frac{1}{p}} \\
&\leq C \left(N \left(R^{p+3} + R^{3p} \left(dN^{-\beta} \right)^{-2p+3} \right) \right)^{\frac{1}{p}} \\
&\leq CN^{\frac{1}{p}} R^{1+\frac{3}{p}} + CR^3 N^{\frac{2}{3}+2\beta-\frac{3}{p}\beta} \\
&\leq \phi o(1).
\end{aligned}$$

□

4.2. Passage to the Stokes equation with variable viscosity

In order to obtain the final result we want to replace the $\hat{v}$ in equation (4.6) and equation (4.9), respectively by $\hat{u}$. First we establish a regularity lemma:

Lemma 4.4. *There is a constant $C > 0$ such that the following holds. Let $g \in L^2(\mathbb{R}^3)$ compactly supported in $B_{2L}(0)$. Let $w \in \dot{H}^1$ solve the equation*

$$-\Delta w = g \text{ in } \mathbb{R}^3,$$

or $w \in \dot{H}^1_\sigma$ solve

$$\begin{aligned} -\Delta w + \nabla p &= g \text{ in } \mathbb{R}^3, \\ \operatorname{div} w &= 0 \text{ in } \mathbb{R}^3, \end{aligned}$$

respectively. Then, $w \in L^\infty(\mathbb{R}^3)$ and $\|w\|_{L^\infty(\mathbb{R}^3)} \leq C \|g\|_{L^2(\mathbb{R}^3)}$.

Proof. We apply the fundamental solution to write

$$\begin{aligned} |w(x)| &= \left| \int_{R^3} \Phi(x-y) g(y) \, \mathrm{d}y \right| \\ &= \left| \int_{B_L(0)} \Phi(x-y) g(y) \, \mathrm{d}y \right| \\ &\leq C \int_{B_L(0)} \frac{1}{|x-y|} |g(y)| \, \mathrm{d}y \\ &\leq C \|g\|_{L^2} \left\| \frac{1}{|y|} \right\|_{L^2(B_{2L}(x))} \\ &\leq C \|g\|_{L^2}. \end{aligned}$$

□

Now we establish existence and estimates for the final equation:

Lemma 4.5. *There is a constant $C > 0$ such that the following holds. The equation*

$$-\operatorname{div}\left((1 + 3\phi\rho) \nabla \bar{u}\right) = (1 - \phi\rho) f$$

has a solution in $\dot{H}^1$ and for small ϕ we have $\|\nabla \bar{u}\|_{L^2} \leq \|f\|_{L^{\frac{6}{5}}}$. Moreover, the gradient of the solution satisfies $\nabla \bar{u} \in H^1(\mathbb{R}^3)$. The estimate for $\nabla^2 \bar{u}$ is given by

$$\left\|\nabla^2 \bar{u}\right\|_{L^2} \leq C \left(\phi \|\nabla \rho\|_{L^\infty} \|f\|_{L^{\frac{6}{5}}} + (1 + \phi \|\rho\|_{L^\infty}) \|f\|_{L^2} \right)$$

Proof. We consider the weak formulation

$$\int_{\mathbb{R}^3} (1 + 3\phi\rho)\, \nabla\bar{u}\nabla\varphi = \int_{\mathbb{R}^3} (1 - \phi\rho) f\varphi$$

where $\varphi \in \dot{H}^1$. Since ρ is non-negative as the limit of non-negative functions, the left hand side is a bounded and coercive bilinear form on $\dot{H}^1$ (it is even symmetric). The right hand side is a linear form, so by Lax-Milgram there is a solution $\bar{u}$ to the weak form of the equation. For $\phi \le \|\rho\|_{L^\infty}^{-1}$ we get the estimate for the gradient by setting $\varphi = \bar{u}$ and estimating the right hand side like

$$\int_{\mathbb{R}^3} (1 - \phi\rho) f\varphi \le \|f\|_{L^{\frac{6}{5}}} \|\bar{u}\|_{L^6} \le \|f\|_{L^{\frac{6}{5}}} \|\nabla\bar{u}\|_{L^2}.$$

It is classical regularity theory of elliptic equations with variable coefficients to see that $\nabla^2\bar{u} \in L^2(\mathbb{R}^2)$. We nevertheless give a proof here. We test the equation with $\varphi = -D_k^{-h} D_k^h \bar{u}$ where $D_k^h\varphi(x) = \frac{1}{h}\left(\varphi(x + he_k) - \varphi(x)\right)$ is the usual difference quotient. This test function is in $\dot{H}^1$ since $\bar{u}$ is. This gives the following equation:

$$-\int_{\mathbb{R}^3} (1 + 3\phi\rho)\, \nabla\bar{u} D_k^{-h} D_k^h \nabla\bar{u} = -\int_{\mathbb{R}^3} (1 - \phi\rho) f D_k^{-h} D_k^h \bar{u}.$$

By the partial integration rule for difference quotients this leads to

$$\int_{\mathbb{R}^3} (1 + 3\phi\rho)^h\, D_k^h \nabla\bar{u} D_k^h \nabla\bar{u} + \int_{\mathbb{R}^3} D_k^h\left(1 + 3\phi\rho\right) \nabla\bar{u} D_k^h \nabla\bar{u} = -\int_{\mathbb{R}^3} (1 - \phi\rho) f D_k^{-h} D_k^h \bar{u}.$$

Here we denote $\varphi^h(x) = \varphi(x + he_k)$. The first term on the left hand side is obviously bigger than $\left\|D_k^h \nabla\bar{u}\right\|_{L^2}^2$ while for the second term we have the estimate:

$$\begin{aligned}\left|\int_{\mathbb{R}^3} D_k^h\left(1 + 3\phi\rho\right) \nabla\bar{u} D_k^h \nabla\bar{u}\right| &\le \left\|D_k^h\left(1 + 3\phi\rho\right)\right\|_{L^\infty} \|\nabla\bar{u}\|_{L^2} \left\|D_k^h \nabla\bar{u}\right\|_{L^2} \\ &\le \frac{1}{4}\left\|D_k^h \nabla\bar{u}\right\|_{L^2}^2 + 9\phi^2 \|\nabla\rho\|_{L^\infty}^2 \|\nabla\bar{u}\|_{L^2}^2.\end{aligned}$$

Now it is left to estimate the right hand side. We have

$$\begin{aligned}\left|\int_{\mathbb{R}^3} (1 - \phi\rho) f D_k^{-h} D_k^h \bar{u}\right| &\le \|1 - \phi\rho\|_{L^\infty} \|f\|_{L^2} \left\|D_k^h \nabla u\right\|_{L^2} \\ &\le \frac{1}{4}\left\|D_k^h \nabla\bar{u}\right\|_{L^2}^2 + \left(2 + 2\phi^2 \|\rho\|_{L^\infty}^2\right) \|f\|_{L^2}^2.\end{aligned}$$

Here we used that $\left\|D_k^{-h}\varphi\right\|_{L^2} \leq \|\nabla\varphi\|_{L^2}$ for any $\varphi \in \dot{H}^1$. By absorbing the two $\frac{1}{4}\left\|D_k^h\nabla\bar{u}\right\|_{L^2}^2$ terms we obtain the estimate

$$\frac{1}{2}\left\|D_k^h\nabla\bar{u}\right\|_{L^2}^2 \leq 9\phi^2\|\nabla\rho\|_{L^\infty}^2\|\nabla\bar{u}\|_{L^2}^2 + \left(2+2\phi^2\|\rho\|_{L^\infty}^2\right)\|f\|_{L^2}^2 .$$

Since this estimate is independent of h we obtain $\nabla^2 u \in L^2$ with

$$\left\|\nabla^2\bar{u}\right\|_{L^2} \leq C\left(\phi\|\nabla\rho\|_{L^\infty}\|f\|_{L^{\frac{6}{5}}} + (1+\phi\|\rho\|_{L^\infty})\|f\|_{L^2}\right)$$

□

Lemma 4.6. *There is a constant $C > 0$ such that the following holds. The equation*

$$\begin{aligned} -\operatorname{div}\left((2+5\phi\rho)\,e\bar{u}\right) + \nabla p &= (1-\phi\rho)f \ \textit{in}\ \mathbb{R}^3 \\ \operatorname{div}\bar{u} &= 0 \ \textit{in}\ \mathbb{R}^3, \end{aligned}$$

has a solution in $\dot{H}^1_\sigma$ and for small ϕ we have $\|\nabla\bar{u}\|_{L^2} \leq \|f\|_{L^{\frac{6}{5}}}$. Moreover, the gradient of the solution satisfies $\nabla\bar{u} \in H^1(\mathbb{R}^3)$. The estimate for $\nabla^2\bar{u}$ is given by

$$\left\|\nabla^2\bar{u}\right\|_{L^2} \leq \left(\phi\|\nabla\rho\|_{L^\infty}\|f\|_{L^{\frac{6}{5}}} + (1+\phi\|\rho\|_{L^\infty})\|f\|_{L^2}\right).$$

Proof. The proof is completely analogous to the one of Lemma 4.5. Consider the weak formulation

$$\int_{\mathbb{R}^3}(2+5\phi\rho)\,e\bar{u}e\varphi\,\mathrm{d}x = \int_{\mathbb{R}^3}(1-\phi\rho)f\varphi\,\mathrm{d}x, \tag{4.12}$$

where $\varphi \in \dot{H}^1_\sigma$. The left hand side is a bounded and coercive bilinear form on $\dot{H}^1_\sigma$. Coerciveness follows from

$$\int_{\mathbb{R}^3}(2+5\phi\rho)\,|e\varphi|^2\,\mathrm{d}x \geq 2\int_{\mathbb{R}^3}|e\varphi|^2\,\mathrm{d}x = \int_{\mathbb{R}^3}|\nabla\varphi|^2\,\mathrm{d}x.$$

The right hand side of (4.12) is a linear form (since every $\phi \in \dot{H}^1_\sigma$ is also in $L^6(\mathbb{R}^3)$ and $f \in L^{\frac{6}{5}}(\mathbb{R}^3)$), so by Lax-Milgram there is a solution $\bar{u}$ to the weak form of the equation. For $\phi \leq \|\rho\|_{L^\infty}^{-1}$ we get the estimate for the gradient by setting $\varphi = \bar{u}$ and estimating the right hand side like

$$\int_{\mathbb{R}^3} (1-\phi\rho) f\varphi \le \|f\|_{L^{\frac{6}{5}}} \|\bar{u}\|_{L^6} \le \|f\|_{L^{\frac{6}{5}}} \|\nabla\bar{u}\|_{L^2}.$$

Note that we can rewrite the weak formulation as

$$\int_{\mathbb{R}^3} \nabla\bar{u}\nabla\varphi \,\mathrm{d}x + \int_{\mathbb{R}^3} 5\phi\rho e\bar{u}e\varphi \,\mathrm{d}x = \int_{\mathbb{R}^3} (1-\phi\rho) f\varphi \,\mathrm{d}x.$$

To see that $\nabla^2\bar{u} \in L^2(\mathbb{R}^2)$ we use again the method of difference quotients and test the equation with $\varphi = -D_k^{-h} D_k^h \bar{u}$. This test function is in $\dot{H}^1_\sigma$ since $\bar{u}$ is in $\dot{H}^1_\sigma$. This gives the following equation:

$$-\int_{\mathbb{R}^3} \nabla\bar{u} D_k^{-h} D_k^h \nabla\bar{u} \,\mathrm{d}x - \int_{\mathbb{R}^3} 5\phi\rho e\bar{u} D_k^{-h} D_k^h e\bar{u} \,\mathrm{d}x = -\int_{\mathbb{R}^3} (1-\phi\rho) f D_k^{-h} D_k^h \bar{u}.$$

By partial integration this leads to

$$\int_{\mathbb{R}^3} D_k^h \nabla\bar{u} D_k^h \nabla\bar{u} \,\mathrm{d}x + \int_{\mathbb{R}^3} (5\phi\rho)^h D_k^h e\bar{u} D_k^h e\bar{u} + \int_{\mathbb{R}^3} D_k^h (5\phi\rho)\, e\bar{u} D_k^h e\bar{u} = -\int_{\mathbb{R}^3} (1-\phi\rho) f D_k^{-h} D_k^h \bar{u}.$$

The first term on the left hand side is obviously equal to $\left\|D_k^h \nabla\bar{u}\right\|_{L^2}^2$, the second term is positive while for the third term we have the estimate:

$$\begin{aligned}\left|\int_{\mathbb{R}^3} D_k^h (5\phi\rho)\, e\bar{u} D_k^h e\bar{u}\right| &\le \left\|D_k^h (5\phi\rho)\right\|_{L^\infty} \|\nabla\bar{u}\|_{L^2} \left\|D_k^h \nabla\bar{u}\right\|_{L^2} \\ &\le \frac{1}{4} \left\|D_k^h \nabla\bar{u}\right\|_{L^2}^2 + C\phi^2 \|\nabla\rho\|_{L^\infty}^2 \|\nabla\bar{u}\|_{L^2}^2.\end{aligned}$$

Now we are left to estimate the right hand side. We have

$$\begin{aligned}\left|\int_{\mathbb{R}^3} (1-\phi\rho) f D_k^{-h} D_k^h \bar{u}\right| &\le \|1-\phi\rho\|_{L^\infty} \|f\|_{L^2} \left\|D_k^h \nabla u\right\|_{L^2} \\ &\le \frac{1}{4} \left\|D_k^h \nabla\bar{u}\right\|_{L^2}^2 + C\left(1+\phi^2 \|\rho\|_{L^\infty}^2\right) \|f\|_{L^2}^2.\end{aligned}$$

Here we used that $\left\|D_k^{-h}\varphi\right\|_{L^2} \le \|\nabla\varphi\|_{L^2}$ for any $\varphi \in \dot{H}^1$. By absorbing the two $\frac{1}{4}\left\|D_k^h \nabla\bar{u}\right\|_{L^2}^2$ terms we obtain the estimate

$$\frac{1}{2}\left\|D_k^h \nabla \bar{u}\right\|_{L^2}^2 \leq C\left(\phi^2 \left\|\nabla \rho\right\|_{L^\infty}^2 \left\|\nabla \bar{u}\right\|_{L^2}^2 + \left(1+\phi^2 \left\|\rho\right\|_{L^\infty}^2\right)\left\|f\right\|_{L^2}^2\right).$$

Since this estimate is independent of h we obtain $\nabla^2 u \in L^2$ with

$$\left\|\nabla^2 \bar{u}\right\|_{L^2} \leq C\left(\phi \left\|\nabla \rho\right\|_{L^\infty} \left\|f\right\|_{L^{\frac{6}{5}}} + \left(1+\phi \left\|\rho\right\|_{L^\infty}\right)\left\|f\right\|_{L^2}\right).$$

□

We can now establish the final estimate.

Lemma 4.7. *There is a constant $C > 0$ such that the following holds. The weak solutions $\hat{u}$ and $\bar{u}$ in $\dot{H}^1$ to the equations*

$$-\operatorname{div}\left(\nabla \hat{u} + 3\phi\rho\nabla\hat{v}\right) + \nabla p = (1-\phi\rho) f, \tag{4.13}$$

$$-\operatorname{div}\left(\left(1+3\phi\rho\right)\nabla \bar{u}\right) + \nabla p = (1-\phi\rho) f, \tag{4.14}$$

differ on scale ϕ^2, i.e. $\left\|\hat{u}-\bar{u}\right\|_{L^\infty(\mathbb{R}^3)} \leq C\phi^2$.

Proof. By subtracting equation (4.1) from equation (4.14) we obtain:

$$-\operatorname{div}\left(\nabla \bar{u} - \nabla \hat{v} + 3\phi\rho\nabla\bar{u}\right) = 0.$$

Hence, for the difference $\bar{u}-\hat{v}$, we get:

$$-\Delta\left(\bar{u}-\hat{v}\right) = \phi \operatorname{div}\left(3\rho\nabla\bar{u}\right). \tag{4.15}$$

Testing with $\bar{u}-\hat{v}$ gives

$$\left\|\nabla\bar{u}-\nabla\hat{v}\right\|_{L^2} \leq 3\phi\left\|\rho\right\|_{L^\infty}\left\|\nabla\hat{u}\right\|_{L^2} \leq C\phi\left\|\rho\right\|_{L^\infty}\left\|f\right\|_{L^{\frac{6}{5}}}. \tag{4.16}$$

On the other hand we know that $\nabla\bar{u} \in H^1$ and by the same argument $\nabla\hat{v} \in H^1$ so that we can test equation (4.15) by $-\Delta\left(\bar{u}-\hat{v}\right)$ in order to obtain

$$\left\|\nabla^2\left(\bar{u}-\hat{v}\right)\right\|_{L^2}^2 \leq C\phi\left\|\nabla^2\left(\bar{u}-\hat{v}\right)\right\|_{L^2}\left\|\rho\right\|_{W^{1,\infty}}\left\|\nabla\bar{u}\right\|_{H^1},$$

$$\left\|\nabla^2\left(\bar{u}-\hat{v}\right)\right\|_{L^2} \leq C\phi\left\|\rho\right\|_{W^{1,\infty}}\left(1+\phi\left\|\rho\right\|_{W^{1,\infty}}\right)\left(\left\|f\right\|_{L^2}+\left\|f\right\|_{L^{\frac{6}{5}}}\right).$$

This proves that $\|\nabla(\bar{u}-\hat{v})\|_{H^1} \leq C\phi$.

Now we subtract the equations for $\bar{u}$ and $\hat{u}$ to obtain for the difference $w = \bar{u} - \hat{u}$:

$$-\operatorname{div}\left(\nabla w + 3\phi\rho\left(\nabla\bar{u} - \nabla\hat{v}\right)\right) = 0.$$

This means that

$$-\Delta w = \operatorname{div}\left(3\phi\rho\left(\nabla\bar{u} - \nabla v\right)\right).$$

The right hand side is compactly supported in $B_{2L}(0)$ and in L^2. By Lemma 4.4 this means that

$$\begin{aligned}\|w\|_{L^\infty} &\leq C\phi\,\|\rho\|_{W^{1,\infty}}\,\|\nabla\bar{u} - \nabla v\|_{H^1}\,. \\ &\leq C\phi^2.\end{aligned}$$

□

Lemma 4.8. *There is a constant $C > 0$ such that the following holds. The weak solutions in $\dot{H}^1_\sigma$ to the equations*

$$-\operatorname{div}\left(2e\hat{u} + 5\phi\rho e\hat{v}\right) + \nabla p = (1-\phi\rho)f, \tag{4.17}$$
$$-\operatorname{div}\left(\left(2+5\phi\rho\right)e\bar{u}\right) + \nabla p = (1-\phi\rho)f, \tag{4.18}$$

differ on scale ϕ^2, i.e. $\|\hat{u} - \bar{u}\|_{L^\infty(\mathbb{R}^3)} \leq C\phi^2$.

Proof. By subtracting equation (4.3) from equation (4.18) we obtain:

$$-\operatorname{div}\left(2e\bar{u} - 2e\hat{v} + 5\phi\rho e\bar{u}\right) + \nabla p = 0.$$

Hence, for the difference $\bar{u} - \hat{v}$, we get:

$$-\Delta\left(\bar{u} - \hat{v}\right) + \nabla p = \phi\operatorname{div}\left(5\rho e\bar{u}\right). \tag{4.19}$$

Testing with $\bar{u} - \hat{v}$ gives

$$\|\nabla\bar{u} - \nabla\hat{v}\|_{L^2} \leq 5\phi\,\|\rho\|_{L^\infty}\,\|\nabla\hat{u}\|_{L^2} \leq C\phi\,\|\rho\|_{L^\infty}\,\|f\|_{L^{\frac{6}{5}}}\,. \tag{4.20}$$

On the other hand we know that $\nabla \bar{u} \in H^1$ and by the same argument $\nabla \hat{v} \in H^1$ so that we can test equation (4.19) by $-\Delta(\bar{u}-\hat{v})$ in order to obtain

$$\begin{aligned}
\left\|\nabla^2(\bar{u}-\hat{v})\right\|_{L^2}^2 &\le C\phi \left\|\nabla^2(\bar{u}-\hat{v})\right\|_{L^2} \|\rho\|_{W^{1,\infty}} \|\nabla\bar{u}\|_{H^1}, \\
\left\|\nabla^2(\bar{u}-\hat{v})\right\|_{L^2} &\le C\phi \|\rho\|_{W^{1,\infty}} \left(1+\phi\|\rho\|_{W^{1,\infty}}\right)\left(\|f\|_{L^2}+\|f\|_{L^{\frac{6}{5}}}\right).
\end{aligned}$$

This proves that $\|\nabla(\bar{u}-\hat{v})\|_{H^1} \le C\phi$.

Now we subtract the equations for $\bar{u}$ and $\hat{u}$ to obtain for the difference $w = \bar{u}-\hat{u}$:

$$-\operatorname{div}\left(\nabla w + 5\phi\rho\left(\nabla\bar{u}-\nabla\hat{v}\right)\right) + \nabla p = 0.$$

This means that

$$-\Delta w + \nabla p = \operatorname{div}\left(5\phi\rho\left(\nabla\bar{u}-\nabla v\right)\right).$$

The right hand side is compactly supported in $B_{2L}(0)$ and in L^2. By Lemma 4.4 this means that

$$\begin{aligned}
\|w\|_{L^\infty} &\le C\phi \|\rho\|_{W^{1,\infty}} \|\nabla\bar{u}-\nabla v\|_{H^1}. \\
&\le C\phi^2.
\end{aligned}$$

□

Proof of Theorem 2.7. The statement follows by combining Theorem 3.28, Lemma 3.33, Lemma 3.34, Lemma 4.2 and Lemma 4.7. Note that we do not need a separate L^p statement in Theorem 3.28 and in 4.7 since we have control over the L^∞ norm of the difference on the whole space. □

Proof of Theorem 2.8. The statement follows by combining Theorem 3.28, Lemma 3.33, Lemma 3.34, Lemma 4.3 and Lemma 4.8. □

5. Discussion

From an experimental viewpoint, Einstein's formula seems to hold for concentrations of up to $2 - 3\%$. It might be worthwhile to think about rigorous justifications of alternative formulas for the effective viscosity. The authors of [SK13] discuss the possibility that the linear approximation is far better for the inverse of the viscosity, the so-called fluidity (i.e. the formula holds for much larger concentrations) leading to the formula

$$\mu_{\text{eff}} = \frac{1}{1 - \frac{5}{2}\phi}\mu$$

Also the formula given by Taylor [Tay32] seems to have a validity for suspensions of medium concentration.

One might wonder why a Brinkman type term does not appear in the limiting equations of the problem considered in this thesis since the capacity density, crucial for the onset of the additional term, is of order NR which is larger than the small parameter $\phi = NR^3$ of our problem. The answer to this question is, that the boundary conditions here are not of Dirichlet type and allow the particles to move freely with the fluid rather than restricting the fluid velocity to be zero or more generally a given value at the boundary of the particles.

5.1. Optimality

First of all, since the viscosity is intrinsically related to dissipation one might expect to have closeness results in terms of the dissipation norm $\|eu\|_L^2$. But by using the comparison construction from Chapter 3 we see that we can only expect closeness of order $\phi^{\frac{1}{2}}$ due to the decay properties of the dipoles in the vicinity of the particles. Secondly, there is no hope of getting the L^∞ estimate on the whole domain for the small radius regime since just one dipole is too singular to obtain this. In retrospective it should be possible to obtain some result for the closeness in the L^2 norm of the gradient away from the particles, but the L^∞ norm is the most natural one when using the method of reflections and obtaining closeness in another space away from the particles does not change the result qualitatively.

The assumptions on the particle configurations can probably be weakened. In generalization of Assumption **(1)**, if the particles are distributed in the whole space the result should still be valid if the decay of the particle density is strong enough at infinity. This assumption was only used in Chapter 4. Assumption **(2)** can probably be replaced by an integrated condition similar to the one used in [Hil16]. For even weaker assumptions a probabilistic setting might be of use, where the result would then hold with high probability. In any case it is not realistic that the result holds for arbitrary geometries since for (locally) high concentrations the interaction between the particles is expected to dominate. Assumption **(3)** is already weak and since one can take any factor strictly larger than 2 there is not much room for improvement. Finally Assumption **(4)** is a bit unsatisfying but getting rid of the logarithmic term, if at all possible, is a matter of very carefully analyzing the sums involving third powers of the distance. If it is possible to find cancellation effects at least away from the boundary of the cloud, then one might hope to weaken this assumption.

Regarding the assumptions on f, both were used heavily and since we need both the decay at infinity (Assumption **(i)**) and the local continuity behaviour of f (Assumption **(ii)**) substantial improvements do not seem to be possible.

Finally there were two assumptions on ρ^N. Assumption **(5)** is merely there to avoid passing to a subsequence while Assumption **(6)** can probably be weakened to $\rho \in L^\infty$, the space we expect ρ to be in anyway. This seems to be possible by using L^p regularity theory of the gradient instead of L^2 regularity of the second gradient in order to obtain bounds on the L^∞ norm of the function itself.

5.2. Higher Orders

With respect to [BG72] it would be interesting to try and prove the next order term, along the way removing some ambiguity concerning the exact value of the prefactor. The approach taken in this thesis seems to be well suited for this task. The first idea that comes to mind, is to look at the approximation v_2 discussed in chapter 3. By the method of the proof we basically have that $u - v_2 \sim o(\phi^2)$. The two steps that have to be carried out are to find explicit expressions that are close to the dipoles $P_i P_j v$ for $i \neq j$ in the same sense in which d_i is close to $P_i v$ and then to decide whether the terms lead to a term of the form $\phi^2 eu$ in the limit. One could alternatively consider $P_i v_1$ directly and use that v_1 is close to $\bar{u}$. In this setting $\bar{u}$ would replace v as the 'zero order approximation'. This is work in progress of the author. At this point the author proposes the conjecture that the coefficient of the second order term is given by $6.25 = \frac{25}{4} = \left(\frac{5}{2}\right)^2$ which is well in the range of the coefficients mentioned so far in the physics literature. If the second order approximation works, it is probable that the higher order approximations v_n for $n = 3, 4, \ldots$ will give higher order approximations to the effective viscosity.

5.3. Dynamics

By setting $\dot{X}_i = V_i$ the system becomes dynamic in the sense that the particle centres depend on time. Therefore the particle distribution changes over time and it would be interesting to see whether and in what sense the particle density moves with the flow given by the Stokes equation with effective viscosity. Also the time scale in which no critical particle aggregation takes place would be interesting. Short time existence and closeness can probably be established with methods similar to the ones used in [Hoe16]. A second possibility is to consider the Navier-Stokes equation, thus taking into account the inertia of the fluid while a third option would be to take into account the inertia of the particles leading to an acceleration term in the force balance.

A. Appendix

Lemma A.1 (Lemma 3.24)**.** *There is a constant $C > 0$ such that*

$$\sum_{j\neq i} \frac{1}{d_{ij}} \leq C\frac{N^{\frac{2}{3}}}{d} \leq CN, \tag{A.1}$$

$$\sum_{j\neq i} \frac{1}{d_{ij}^2} \leq C\frac{N^{\frac{1}{3}}}{d^2} \leq CN, \tag{A.2}$$

$$\sum_{j\neq i} \frac{1}{d_{ij}^3} \leq C\frac{\log N}{d^3} \leq CN\log N, \tag{A.3}$$

$$\sum_{j\neq i} \frac{1}{d_{ij}^4} \leq C\frac{1}{d^4} \leq CN^{\frac{4}{3}}. \tag{A.4}$$

Proof. Without loss of generality we assume i=1 and $X_1 = 0$. We order the balls in such a way that $|X_1| \leq \cdots \leq |X_N|$. Since $d_{ij} \geq d$ the balls $B\left(X_i, \frac{d}{2}\right)$ and $B\left(X_j, \frac{d}{2}\right)$ do not intersect. Moreover for any $2 \leq i \leq N$ we have

$$\bigcup_{j=1}^{i} B\left(X_j, \frac{d}{2}\right) \subset B\left(0, \frac{d}{2} + |X_i|\right) \subset B(0, 2\,|X_i|).$$

We compare the left and the right volume to obtain

$$i\left(\frac{d}{2}\right)^3 \leq (2\,|X_i|)^3,$$
$$|X_i| \geq \frac{1}{4}d\, i^{\frac{1}{3}}.$$

Now for $n = 1, 2$ (inequalities (A.1) and (A.2)):

$$\begin{aligned}
\sum_{i=2}^{N} \frac{1}{d_{1i}^{n}} &= \sum_{i=2}^{N} \frac{1}{|X_i|^n} \\
&\leq \frac{16}{d^n} \sum_{i=2}^{N} i^{-\frac{n}{3}} \\
&\leq \frac{16}{d^n} \int_0^N x^{-\frac{n}{3}} \, \mathrm{d}x \\
&\leq 48 \frac{N^{1-\frac{n}{3}}}{d^n} \\
&\leq CN.
\end{aligned}$$

For inequality (A.3) we estimate:

$$\begin{aligned}
\sum_{i=2}^{N} \frac{1}{d_{1i}^{3}} &= \sum_{i=2}^{N} \frac{1}{|X_i|^3} \\
&\leq C \frac{1}{d^3} \sum_{i=2}^{N} i^{-1} \\
&\leq C \frac{1}{d^3} \int_1^N x^{-1} \, \mathrm{d}x \\
&\leq C \frac{\log N}{d^3} \\
&\leq CN \log N.
\end{aligned}$$

For inequality (A.4) we have:

$$\begin{aligned}
\sum_{i=2}^{N} \frac{1}{d_{1i}^{4}} &= \sum_{i=2}^{N} \frac{1}{|X_i|^4} \\
&\leq C \frac{1}{d^4} \sum_{i=2}^{N} i^{-\frac{4}{3}} \\
&\leq C \frac{1}{d^4} \int_1^N x^{-\frac{4}{3}} \, \mathrm{d}x \\
&\leq C \frac{1}{d^4} \left(1 - N^{-\frac{1}{3}}\right) \\
&\leq C \frac{1}{d^4} \\
&\leq CN^{\frac{4}{3}}.
\end{aligned}$$

□

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