Diffusions on Wasserstein Spaces

Dissertation

zur

Erlangung des Doktorgrades (Dr. rer. nat.)

der

Mathematisch-Naturwissenschaftlichen Fakultät

der

Rheinischen Friedrich-Wilhelms-Universität Bonn

vorgelegt von

Lorenzo Dello Schiavo

aus

Rom (Italien)

Bonn, März 2019

Angefertigt mit der Genehmigung der Mathematisch-Naturwissenschaftlichen
Fakultät der Rheinischen Friedrich-Wilhelms-Universität Bonn

1. Gutachter: Prof. Dr. Karl-Theodor Sturm
2. Gutachter: Prof. Maria Gordina, Ph.D.

Tag der Promotion: 26.11.2019
Erscheinungsjahr: 2020

Ai miei insegnanti.

To my teachers.

Typeset by the author using $\mathcal{A}_{\mathcal{M}}\mathcal{S}$-LaTeX, Bib-LaTeX.

Figures by the author using Tikz, Wolfram Mathematica 11.

Acknowledgements

In fond remembrance of Prof. Dr. Kuwada Kazumasa (桒田 和正)

My deepest gratitude goes to everybody, who shared with me their thoughts — mathematical insights, as well as life experiences — and for all the cheerful support and practical help that came, at times in the most unexpected ways.

Foremost, I am grateful to Prof. Dr. Karl-Theodor Sturm for his quiet and observant furtherance of my research, so often prompted, inspired and directed by his questions and remarks.

To Dr.'s Marco Bonacini, Matthias Erbar, Han Bang-Xian (韩 邦先), Martin Huesmann, Flaviana Iurlano, Alessia Nota, Luca Tamanini and Raphael Winter, for their constant mentoring, and for many comments and suggestions about the contents and exposition of the mathematics in this thesis.

To Dr.'s Sebastian Andres, Adolfo Arroyo-Rabasa, Peter Gladbach, Karen Habermann, Eva Kopfer, Chiara Rigoni, Sakurai Yohei (櫻井 陽平), Gerardo Sosa, Suzuki Kohei (鈴木 康平), Prof. Dr. Wu Bo (吴 波), Luigi Borasi, Mathias Braun, Martin Brückerhoff-Plückelmann, Simon Buchholz, Richard Höfer, David Hornshaw, Alessandra Occelli, Angelo Profeta, Walker Stern, Mattia Turra, Immanuel Zachhuber and to all members of the Abteilung für Stochastische Analysis, fellow Ph.D. students, and members of the vast mathematical community of Bonn, with whom I shared every-day life experiences, and very many happy and a few sad memories.

To Prof.'s Sergio Conti, for having been my mentor in the BIGS graduate program; Patrik Ferrari, for his contagious cheerfulness and his several comments about the combinatorics results in Chapter 2; Catharina Stroppel, for her patience with my lingering in the diagonal fallacy and her many insightful suggestions concerning representation theory results in Chapter 2; Massimiliano Gubinelli, for the many carb-free meals we shared, and for his paracontrolled-distributions-thinking which proved a valuable point of view on the approximation techniques used in Chapter 5; Eugene W. Lytvynov (Swansea U.), for his very kind hospitality, and for sharing his understanding of Mecke-type identities which proved to be an invaluable tool in the study of random measures; Sergio Albeverio, for his gentle explanations about direct integrals, Dirichlet forms and particle systems; Maria M. Gordina (U. of Connecticut), for our shared teas, her ль-lectures along the Rhine, and her many mathematical comments and helpful suggestions to Chapters 2, 4 and 5; Federico Bassetti (Politecnico di Milano) for sharing with me the understanding of J. Sethuraman's characterization of the Dirichlet–Ferguson measure as a fixed point problem; Karl-Theodor Sturm, Maria M. Gordina, Christoph Thiele and Markus Cristinziani for being part of my Ph.D. defense Committee.

Since this work would not have been possible without the vaste knowledge and the many references and sources shared with me over the years, my warmest thanks go to Prof.s Sergio Albeverio, Federico Bassetti (Politecnico di Milano), Nicola Gigli (SISSA), Günter Last (KIT) for [101], Willard Miller Jr. (U. of Minnesota) for [124], Mauro Piccioni (Sapienza – U. di Roma) for [107], Anatoly M. Vershik (St. Petersburg State U.) for [161], and to Dr.s Marco Bonacini for [9], Matthias Erbar for [51, 65], Martin Huesmann for [91, 104], Vitalii V. Konarovskyi (U. of Leipzig) for [92], Alessia Nota, Carlo Orrieri (U. degli Studi di Pavia).

Finally, I wish to express my gratitude to:

- the Collaborative Research Center (Sonderforschungsbereich) 1060, the Hausdorff Center for Mathematics, the Bonn International Graduate School in Mathematics and the Bonn Graduate Center, for their financial and non-financial support.
- the Hausdorff Institute for Mathematics in Bonn, for their kind hospitality during the *Follow-up Workshop to Junior Trimester Program "Optimal Transportation"* (August 29$^{\text{th}}$ – September 2$^{\text{nd}}$, 2016);
- the Departments of Mathematics of *Swansea University*, of *Sapienza – Università di Roma*, and of the *Università degli Studi di Pavia* for their kind hospitality during my short research stays;
- the Max Planck Institute for Mathematics in Bonn, for their kind hospitality during the *Intense Activity Period on Metric Measure Spaces and Ricci Curvature* (September 4$^{\text{th}}$ – 29$^{\text{th}}$, 2017)

Summary

The Introduction presents the main subject of the thesis, the construction of a diffusion process on spaces of probability measures. Together with a brief survey of the relevant literature, it collects several tools from the theory of point processes and of optimal transportation.

Chapter 2 appeared as the preprint [39]. It contains a study of the characteristic functionals of Dirichlet–Ferguson measures $\mathcal{D}_\sigma$ with non-negative finite intensity measure σ over locally compact Polish spaces. Firstly, we compute such characteristic functional as the $\mathcal{D}_\sigma$-martingale limit of confluent Lauricella hypergeometric functions ${}_k\Phi_2$ with diverging arity k. Secondly we study the interplay between the self-conjugate prior property of Dirichlet distributions in Bayesian non-parametrics, the dynamical symmetry algebra of ${}_k\Phi_2$ and Pólya Enumeration Theory.

Chapter 3 appeared as the preprint [41], joint work with Eugene W. Lytvynov (Swansea U., Wales, UK). It contains a new proof of J. Sethuraman's fixed point characterization of $\mathcal{D}_\sigma$ [145], providing an understanding of the latter as an integral identity of Mecke- or Georgii–Nguyen–Zessin-type formula.

Chapter 4 appeared as the preprint [38]. It contains the proof of a Rademacher-type result on the L^2-Wasserstein space $\mathscr{P}_2$ over a closed Riemannian manifold M. Namely, sufficient conditions are given for a probability measure $\mathbb{P}$ on $(\mathscr{P}_2, W_2)$, so that real-valued W_2-Lipschitz functions be $\mathbb{P}$-a.e. differentiable in a suitable sense. Some examples of measures satisfying such conditions are provided, mostly in the case when $M = \mathbb{S}^1$.

Chapter 5 appeared as the preprint [40]. It contains two constructions of a Markov diffusion process $\eta_\bullet$ with values in $\mathscr{P}_2$. The process is associated with the Dirichlet integral induced by the L^2-Wasserstein gradient and by $\mathcal{D}_\mathsf{m}$ with intensity the Riemannian volume measure m of M. When $\dim M \geq 2$ we study the properties of $\eta_\bullet$, including its invariant sets, short time asymptotics for the heat kernel and a description by means of a stochastic partial differential equation.

Contents

List of Symbols

Chapter 1
Introduction

As non-trivial translation-invariant measures on any separable infinite-dimensional Banach space do not exist, the quest for well-behaved measures on infinite-dimensional spaces has represented a stimulating problem for decades. Known examples of such measures naturally arise from probability theory: for instance, Wiener measures, Poisson random measures, and more generally the laws of stochastic processes with infinite-dimensional path or state spaces.

The interplay between these laws and the topology and geometry of the underlying spaces is often much subtler than in finite dimensions, even for measures on linear spaces. Again by way of example, this is attested in the case of the Wiener measure by the Cameron–Martin Theorem. Whether finite-dimensional intuitions about basic concepts — like (*Riemannian*) *metric*, *volume measure* or *Brownian motion* — may be applied to infinite-dimensional settings, is usually settled only on a case-by-case basis.

During the last two decades, the space of probability measures over a Riemannian manifold was put forward as a geometrically amenable infinite-dimensional object: Endowed with the L^2-Kantorovich–Rubinshtein[1] distance arising from optimal transportation theory, it inherits metric and geometric curvature properties from the underlying manifold, including Fréchet-type differentiation, Riemannian calculus, a Levi-Civita connection and parallel transport.

This thesis is aimed at the study of a candidate "volume" measure on the space of probability measures over a *closed* Riemannian manifold, and at the construction of a "geometric" diffusion process on the same space. Our interest in these objects stems from three main aspects, roughly corresponding to Chapters 2, 4 and 5 respectively, namely:

- *algebraic* aspects, related to representations of certain large Lie groups naturally acting on spaces of measures, and to the invariance properties, with respect to such actions, of certain random measures on said spaces of measures;
- *geometric* aspects, related to the "differential" and "Riemannian" structures induced on spaces of measures by the above group actions, and to their interplay with the (extended) L^2-Wasserstein distance;
- *stochastic* aspects, related to the existence and properties of Markov diffusion processes generated by the "Laplace–Beltrami" operators associated to the above random measures and "Riemannian" structures.

These correspond to just as many long-term goals, only tangentially addressed in the thesis, and for which the results presented here are but a starting point. Namely,

[1] Whereas we find the term 'Kantorovich–Rubinshtein distance' historically more appropriate (Cf., e.g., [21, p. 796] or [165, p. 119].), we shall in the following adhere to the far more common terminology of 'Wasserstein distance'.

- to provide a Lie-algebraic proof of, and a probabilistic interpretation to, Thoma's classification of the characters of the infinite symmetric group $\mathfrak{S}_\infty$, incarnated as the Weyl group of the dynamical symmetry algebra of the Dirichlet–Ferguson measure;
- to find a natural "Riemannian volume" measure $\mathbf{P}$ on the L^2-Wasserstein space $\mathscr{P}_2(M)$ over any closed Riemannian manifold (M,g) and such that $(\mathscr{P}_2(M),\mathbf{P})$ inherit the lower Ricci curvature bound of M, in the sense of either Bakry–Émery or Lott–Sturm–Villani curvature(-dimension) conditions;
- to prove the existence of the *Wasserstein diffusion* on the space of probability measures over any closed Riemannian manifold of dimension $d \geq 2$, and to understand the relationship between SDEs on $\mathscr{P}_2(M)$ and SPDEs on M in roughly the same spirit as in Otto calculus for PDEs.

In this first Chapter, we shall recall some preliminary results, briefly survey the relevant literature about stochastic processes on spaces of measures, and familiarize the reader with the notation. We alternate the mathematical description of the objects of interest with some heuristics drawn from particle systems.

> For ease of exposition, all such insights are presented in a sans-serif font and enclosed by a side-rule, like the present paragraph.

Throughout the Chapter, we reference text by a combination of section number, e.g., §2.3, page number, and paragraph number, e.g., ¶3.

1.1 Some key ideas

We summarize — informally — the main guidelines to follow.

Let Y be a Polish space, G be a connected Lie group acting freely on Y and $\mathfrak{g}$ be the corresponding Lie algebra. We write $g.y \in Y$ for the action of $g \in G$ on $y \in Y$. An *"increment"* or *"direction"* is any element $a \in \mathfrak{g}$, and the corresponding *"shift"* is its image e^a via the Lie exponential mapping $e\colon \mathfrak{g} \to G$. For fixed $u\colon Y \to \mathbb{R}$, we define the directional derivative of u in direction a by setting

$$\nabla_a u(y) \coloneqq \mathrm{d}_t\big|_{t=0} u\big(e^{t\,a}.y\big) \ ,$$

whenever this exists. For every $y \in Y$ we may define — at this point: arbitrarily — a linear "tangent" space T_yY of admissible directions a. Provided that we endow T_yY with a Hilbert scalar product $\langle \cdot \mid \cdot \rangle_y$ such that $a \mapsto \nabla_a u$ is a continuous linear functional for all $a \in T_yY$, then a gradient $\nabla u(y) \in T_yY$ is induced by the Riesz Representation Theorem and satisfies

$$\nabla_a u(y) = \langle \nabla u(y) \,|\, a \rangle_y \ , \qquad a \in T_yY \ . \tag{1.1.1}$$

Some representation theory. Suppose further that Y be endowed with a Borel measure n.

Definition (*Quasi-invariance*)**.** We term n *quasi-invariant* with respect to the action $G \circlearrowright Y$ if

$$\forall g \in G \qquad (g.)_\sharp \mathsf{n} \sim \mathsf{n} \ ,$$

where by $\mathsf{n}_1 \sim \mathsf{n}_2$ we mean that the measures n_1 and n_2 are mutually absolutely continuous. Further, for any measurable map f we indicate the push-forward measure by

$$f_\sharp \mathsf{n} \coloneqq \mathsf{n} \circ f^{-1} \ .$$

When Y is a G-homogeneous space (i.e., $G \circlearrowleft Y$ is transitive) and n is a $G \circlearrowleft Y$-quasi-invariant (probability) measure, a so called quasi-regular representation of G is induced on $L^2_{\mathsf{n}}(Y)$. This representation is unitary, in the sense that its image is a subgroup of the group of unitary automorphisms of $L^2_{\mathsf{n}}(Y)$. (For a further account of quasi-invariance and group representations, see §2.1.2.)

In the case when G is a Lie group acting as above, we denote by ∇^* the adjoint of ∇ with respect to the natural (pre-)Hilbert scalar product on the space $\Gamma(TY)$ of (continuous) sections to the tangent bundle TY, viz.

$$\langle w^1 \,|\, w^2 \rangle \coloneqq \int_Y \langle w^1_y \,|\, w^2_y \rangle_y \, \mathrm{d}\mathsf{n}(y) \; , \qquad w^i \in \Gamma(TY) \; .$$

Thus, a quasi-regular unitary representation is associated with a quadratic energy functional

$$\mathsf{E}(u) \coloneqq \langle \nabla u \,|\, \nabla u \rangle \; . \tag{1.1.2}$$

As it turns out, the $G \circlearrowleft Y$-quasi-invariance of n is usually sufficient to establish that the quadratic form associated to the functional E in (1.1.2) is closable, and generated by the negative "Laplacian" $\mathsf{L} \coloneqq -\nabla^* \circ \nabla$. (For the details in the case relevant to us, see §4.4.4 ¶1, and Proposition 4.5.6.)

Some metric analysis. When $Y = (Y, \mathsf{r}, \mathsf{n})$ is a metric measure space and ∇ is compatible with the distance r on Y, (in a sense to be made precise) then the $G \circlearrowleft Y$-quasi-invariance of n is a key tool in establishing results about the n-a.e. differentiability of r-Lipschitz functions.

Indeed, provided E be a densely defined and closable pre-Dirichlet form with carré du champ operator $\Gamma(u)(y) \coloneqq |\nabla u(y)|^2_y$, its domain $\mathscr{D}(\mathsf{E})$ may be regarded as a Sobolev space of type H^1. Analogously, the set

$$\{u \in \mathscr{D}_{\mathrm{loc}}(\mathsf{E}) \cap L^\infty_{\mathsf{n}}(Y) \mid \Gamma(u) \in L^\infty_{\mathsf{n}}(Y)\}$$

(properly normed) may be regarded as a Sobolev space of type $W^{1,\infty}$. In this setting, the classical Rademacher Theorem and its converse, the so-called Sobolev-to-Lipschitz property, may be formulated as follows.

Property (Rademacher)**.** *If $u \in \mathrm{Lip}(Y, \mathsf{r}) \cap L^2_{\mathsf{n}}(Y)$, then $u \in \mathscr{D}(\mathsf{E})$ and $\Gamma(u) \leq \mathrm{Lip}[u]^2$.*

Property (Sobolev-to-Lipschitz)**.** *If $u \in \mathscr{D}(\mathsf{E})$ and $\Gamma(u) \in L^\infty_{\mathsf{n}}(Y)$, then there exists an n-version $\tilde{u}$ of u such that $\tilde{u} \in \mathrm{Lip}(Y, \mathsf{r})$ and $\mathrm{Lip}[\tilde{u}]^2 \leq \|\Gamma(u)\|_{L^\infty_{\mathsf{n}}}$.*

Some stochastic analysis. Provided that $(\mathsf{E}, \mathscr{D}(\mathsf{E}))$ be a regular, strongly local Dirichlet form on $L^2_{\mathsf{n}}(Y)$, the standard theory of Dirichlet forms grants the existence of a Markov process $y_\bullet$ with state space Y, uniquely associated to $(\mathsf{E}, \mathscr{D}(\mathsf{E}))$. In general, no pathwise construction of $y_\bullet$ is provided by the theory, and thus it must be achieved by other means. However, several properties of $y_\bullet$, including e.g., conservativeness or ergodicity, may be restated in the language of Dirichlet forms. In particular, the strong locality of $(\mathsf{E}, \mathscr{D}(\mathsf{E}))$ implies the a.s. continuity of the sample paths $t \mapsto y_t(\omega)$, that is, the process $y_\bullet$ is a *diffusion*.

The simplest example of this construction to bear in mind is as follows: Choose $(Y, \mathsf{r}, \mathsf{n}) = \mathbb{R}^d$ the standard Euclidean space, $\mathfrak{g} \cong G \cong \mathbb{R}^d$ the (additive, Abelian) group of translations of Y, and $T_y Y \coloneqq \mathbb{R}^d$ for every $y \in Y$. Then, ∇ is the usual gradient, L the usual Laplacian, and the process $y_\bullet$ is but a d-dimensional Brownian motion.

Different choices for Y and G yield constructions in a variety of contexts. In the following, we shall focus on the case when Y is some space of measures on a Riemannian manifold M and G is the group of diffeomorphisms of M. We start with a complete example.

1.2 A leading example: configuration spaces

We start with a brief survey about the free geometric dynamics on configuration spaces, a (by now) classical example of dynamics on spaces of measures which will be of guidance throughout our study. The related theory, now fully developed, allows us to focus on the anticipated interplay of algebra, geometry and stochastic analysis, while momentarily neglecting the more technical aspects that arise in dealing with richer spaces of measures.

Throughout this chapter, we will adhere to the following notation.

Spaces. We introduce the following topological and measure-theoretical objects:

- X is a second countable locally compact Hausdorff topological space[2];
- M is a (second countable) connected smooth manifold without boundary, possibly endowed with a smooth Riemannian metric g such that (M, g) is additionally complete and stochastically complete;
- $\mathcal{B}$ is the Borel σ-algebra of either X or M, depending on context; $\mathscr{M}^+$ is the space of σ-finite Radon measures on either X or M, depending on context, endowed with the vague topology if not stated otherwise.

Distributions. We shall make use of the following standard probability distributions:

- the *Poisson distribution* with parameter $\gamma \in [0, \infty]$,

$$\mathsf{P}_\gamma(k) \coloneqq \frac{\gamma^k}{k!} e^{-\gamma} \ , \qquad k \in \mathbb{N}_0 \ ;$$

  (If $\gamma \in \{0, \infty\}$, then, conventionally, $\mathsf{P}_\gamma = \delta_\gamma$.)

- the *Gamma distribution* with *shape* parameter $k > 0$ and *scale* parameter $\theta > 0$,

$$\mathsf{G}_{k,\theta}(r) \coloneqq \frac{\theta^{-k}}{\Gamma(k)} r^{k-1} e^{-\frac{r}{\theta}} \ , \qquad r \in \mathbb{R}_+ \ ;$$

- the *Beta distribution* with *shape* parameters $\alpha, \beta > 0$,

$$\mathsf{B}_{\alpha,\beta}(r) \coloneqq \frac{r^{\alpha-1}(1-r)^{\beta-1}}{\mathrm{B}(\alpha, \beta)} \ , \qquad r \in I \ .$$

1.2.1 Analysis on configuration spaces. In the study of random dynamics on spaces of measures several examples naturally arise from physics. In this section, we formalize a simple leading example: Bose gases.

> By a *Bose gas* we shall mean a many-body system of indistinguishable mass-, volume- and charge-less random particles in the Euclidean space. We assume further that
>
> (B_1) each particle is solely identified by its position and any two particles do not occupy the same position with probability 1;
>
> (B_2) the system obeys the *Bose–Einstein statistics*, that is, the particles are non-interacting and the system is invariant under particles' permutation;
>
> (B_3) the system is a *rarefied gas*, that is, each finite volume in $\mathbb{R}^3$ contains only a finite number of particles;

[2] Equivalently, X is a locally compact Polish space.

(B₄) for a volume A of size $|A|$ let γ_A be the number of particles in A. Then,

$$\mathbf{P}(\gamma_A = 0) = e^{-|A|} \ .$$

Mathematically, we shall formalize a Bose gas as a Poisson point process with intensity measure the Lebesgue measure $|\cdot|$ in $\mathbb{R}^3$.

Point processes. (Cf., e.g., [102, Chap. 2].) A measure[3] γ on X is *integral* if $\gamma B \in \overline{\mathbb{N}}_0$ for all $B \in \mathcal{B}$. A *point process* on X is any random element in the space of integral measures on X. For each such γ, the functional $\mathbf{E}(\gamma\,\cdot\,)$ is a measure on X, termed the *intensity measure* of γ.

We say that a point process γ is

- *proper*, if there exists random variables $N \in \overline{\mathbb{N}}_0$ and $(x_i)_{i\leq N} \in X^N$ such that

$$\gamma = \sum_{i=1}^{N} \delta_{x_i} \quad \mathbf{P}\text{-a.e.} \ ; \tag{1.2.1}$$

  (Let us stress that we allow for $x_i = x_j$ when $i \neq j$)
- *simple*, if it is proper and $\mathbf{P}(\gamma\{x\} \leq 1) = 1$ for every $x \in X$;
- *locally finite*, if $\mathbf{P}(\gamma K < \infty) = 1$ for every compact $K \subset X$;
- *completely independent*, if the random variables $\gamma B_1, \ldots, \gamma B_k$ are independent for every pairwise disjoint $B_1, \ldots, B_k \in \mathcal{B}$.

Since particles in a Bose gas are indistinguishable, and exchangeable by property (B_2), they are described by the set (as opposed to: ordered tuple) of their characteristic quantities. By (B_1), the only relevant quantity associated to a particle is its position in space, thus a Bose gas is described by a proper point process γ. Again by (B_1), γ is simple, with *diffuse* (i.e. atomless) intensity. By (B_3), the process γ is additionally locally finite. Finally, property (B_4), together with all of the previous conclusions, implies that γ satisfies to the following definition. (This is by Rényi's Theorem [102, Thm. 6.10].)

Definition (*Poisson measure*, e.g., [102, Dfn. 3.1])**.** A *Poisson point process* with *intensity* (*measure*) $\sigma \in \mathscr{M}^+$ is a completely independent point process on X such that

$$\gamma B \sim \mathsf{P}_{\sigma B} \ , \qquad B \in \mathcal{B} \ .$$

As it turns out, every such process is a proper point process (up to equality in law, [102, Cor. 3.7]). Its law (uniquely determined by σ, [102, Prop. 2.10(ii)]) is termed the *Poisson* (*random*) *measure* $\mathcal{P}_\sigma$. (For further characterizations of $\mathcal{P}_\sigma$, see Eq. (3.1.1) and Thm. 3.1.1 below.)

Configuration spaces. A set $A \subset X$ is termed *locally finite* if $A \cap K$ is finite for any compact $K \subset X$. Any such A has the form $A = \{x_i\}_{i\leq N}$ for some $N \in \overline{\mathbb{N}}_0$, and is uniquely associated to a Radon measure γ of the form (1.2.1), henceforth a *configuration* in X. With customary abuse of notation, we write $x \in \gamma$ whenever $\gamma\{x\} > 0$.

The *configuration space* Υ ([4]) over X is the set of all configurations in X, regarded as a topological subspace of $\mathscr{M}^+$. Provided $\sigma \in \mathscr{M}^+$ be *diffuse*, the Poisson point process γ with intensity σ is additionally simple (See [102, Prop. 6.9].) and locally finite (since σ is Radon). Thus, its law $\mathcal{P}_\sigma$ is concentrated on the configuration space Υ.

[3] Here: not necessarily σ-finite, nor Radon.

[4] This notation, taken from [51], is not the standard one for configuration spaces, usually denoted by Γ. However, it is here very convenient to reserve the symbol Γ for carré du champ operators, as in [51].

Fock spaces. For $\sigma \in \mathscr{M}^+$, we denote by $L^2_\sigma(X;\mathbb{C})^{\odot n}$ the n^{th} symmetric Hilbertian tensor power of $L^2_\sigma(X;\mathbb{C})$, and define the *Bosonic* (or *symmetric*) *Fock space* of $L^2_\sigma(X;\mathbb{C})$

$$\operatorname{Exp}\big(L^2_\sigma(X;\mathbb{C})\big) \coloneqq \widehat{\bigoplus_{n\geq 0}} L^2_\sigma(X;\mathbb{C})^{\odot n} \ . \tag{1.2.2}$$

From now onwards, we shall assume γ to be a Poisson point process with diffuse intensity σ, defined on its sample space Υ. For σ and γ as above set $q \coloneqq \gamma - \sigma$ and define the *compensated Poisson multiple stochastic integral*

$$q^{(n)} f \coloneqq \int_{X^{\times n}} f(x_1,\dots,x_n)\, \mathrm{d}q^{\otimes n}(x_1,\dots,x_n) \ , \qquad n\in\mathbb{N}_0 \ , \ f\in L^2_\sigma(X;\mathbb{C})^{\odot n} \ ,$$

as the square-mean limit of any sequence of bounded simple functions in $L^2_\sigma(X;\mathbb{C})^{\odot n}$. The following is a standard result of the theory. (See e.g., [152, p. 220].)

Theorem (Chaos Expansion)**.** *Let σ, γ and q be as above and let $\mathbf{E}=\mathbf{E}_{\mathcal{P}_\sigma}$. Then,*

(q_1) $\big(\gamma \longmapsto q^{(n)} f\big) \in L^2_{\mathcal{P}_\sigma}(\Upsilon;\mathbb{C})$;

(q_2) $\mathbf{E}\big[q^{(n)} f\big] = 0$;

(q_3) $\mathbf{E}\big[(q^{(n)} f)^2\big] = n!\, \|f\|^2_{\sigma,n}$, *where* $\|\cdot\|_{\sigma,n}$ *is the Hilbert norm of* $L^2_\sigma(X;\mathbb{C})^{\odot n}$;

(q_4) $\mathbf{E}\big[q^{(n)} f\ \overline{q^{(m)} g}\big] = 0$ *for* $n\neq m$, *where the overline indicates the complex conjugate.*

Furthermore, there is an induced unitary isomorphism of Hilbert spaces

$$\begin{aligned} \iota_\sigma \colon \operatorname{Exp}\big(L^2_\sigma(X;\mathbb{C})\big) &\longrightarrow L^2_{\mathcal{P}_\sigma}(\Upsilon;\mathbb{C}) \\ f \coloneqq (f_0, f_1, f_2, \dots) &\longmapsto \left(\gamma \mapsto \sum_{n\geq 0} \frac{q^{(n)} f_n}{n!}\right) \end{aligned} \ . \tag{1.2.3}$$

> An element in $L^2_{\mathcal{P}_\sigma}(\Upsilon;\mathbb{C})$ is but the wave-function of a quantum state of a Bose gas. By the Chaos Expansion Theorem, each such quantum state is uniquely determined by the quantum states of single particles in the gas (i.e., an element of the *first chaos* $L^2(\mathbb{R}^3;\mathbb{C})$) via the isomorphism (1.2.3).

Random dynamics. For a self-adjoint operator $(A,\mathscr{D}(A))$ on $L^2_\sigma(X;\mathbb{C})$ we term *second quantization* of A the operator on the Fock space $\operatorname{Exp}\big(L^2_\sigma(X;\mathbb{C})\big)$ defined by

$$\operatorname{dExp}(A) \coloneqq \bigoplus_{n\geq 0} \underbrace{A\otimes \mathbf{1}\otimes\cdots\otimes\mathbf{1}}_{n} + \underbrace{\mathbf{1}\otimes A\otimes\mathbf{1}\otimes\cdots\otimes\mathbf{1}}_{n} + \cdots + \underbrace{\mathbf{1}\otimes\cdots\otimes\mathbf{1}\otimes A}_{n} \ , \qquad A^{\otimes 0} \coloneqq 0 \ ,$$

that is, the formal differential of $\operatorname{Exp}(A) \coloneqq \oplus_{n\geq 0} A^{\otimes n}$ (the normalization by $n!$ being implicit in the norms of the spaces). The operator $\operatorname{dExp}(A)$ is densely defined on a natural finitary predomain in $\operatorname{Exp}\big(L^2_\sigma(X;\mathbb{C})\big)$. (See e.g., [17, vol. II.6 §1.1].) Furthermore, dExp preserves positivity and (essential) self-adjointness [17, vol. II.6 §1.1], and sub-Markovianity (for self-adjoint operators, see [152, Thm. 5.1]).

Suppose $P_\bullet \coloneqq (P_t)_{t\geq 0}$ is a Markov semigroup on $L^2_\sigma(X)$ associated to a reversible Markov process $x_\bullet \coloneqq (x_t)_{t\geq 0}$ with state space X. As suggested by the Chaos Expansion Theorem, we may lift the stochastic dynamics of $x_\bullet$ to a unique stochastic dynamics

$$\gamma_\bullet \coloneqq \sum_{i=1}^N \delta_{x^i_\bullet} \ , \tag{1.2.4}$$

on the configuration space, generated by i.i.d. copies $x^i_\bullet$ of $x_\bullet$. Namely, $\gamma_\bullet$ is the Markov process with state space Υ associated to the semigroup

$$\mathbf{P}_\bullet \coloneqq (\iota_\sigma)_*\, \operatorname{dExp}(P_\bullet) \ .$$

The dynamics of a Bose gas is given by the second quantization of the dynamics of a single *free* particle. Indeed, let us assume further that particles in a Bose gas are free: By this we shall mean that each particle moves by Brownian motion. Together with assumption (B_2), this implies that the (random) dynamics of a Bose gas is given by (1.2.4), where $x^i_\bullet := \left(x^i_t\right)_{t\geq 0}$ are i.i.d. Wiener processes, each with generator the negative standard Laplacian $-\Delta$ on $\mathbb{R}^3$. Thus, the random dynamics of a Bose gas in $\mathbb{R}^3$ is the reversible Markov dynamics with semigroup

$$\mathbf{P}_t := \iota_* \,\mathrm{dExp}\big(e^{-t\Delta}\big) \;, \tag{1.2.5}$$

where ι is the isomorphism (1.2.3) induced by the Lebesgue measure on $\mathbb{R}^3$.

As a generalization of Bose gases in the Euclidean space, let us now consider a Riemannian manifold (M, g) with volume measure m and Laplace–Beltrami operator Δ^{g}. Equation (1.2.5) hints to the following question:

Question. Are there a "differential structure" and a "Riemannian metric" on Υ such that $\mathcal{P}_{\mathsf{m}}$ may be regarded as the associated "Riemannian volume measure" and

$$\mathbf{\Delta}^{\Upsilon} := -(\iota_{\mathsf{m}})_* \,\mathrm{dExp}(-\Delta^{\mathsf{g}})$$

as the corresponding "Laplace–Beltrami" operator?

1.2.2 Geometry on configuration spaces. A positive answer to the above question was provided in the seminal work [7] by S. Albeverio, Yu. G. Kondrat'ev and M. Röckner. Indeed, it was the far-reaching intuition of [7] that the configuration space Υ over M inherits geometrical properties of M.

We start by showing how self-transformations of M are lifted to self-transformations of Υ.

Representations of large Lie groups. Let $\mathrm{Diff}^\infty_c(M)$ be the group of smooth diffeomorphisms of M equal to the identity outside of a compact set. The natural action of $\mathrm{Diff}^\infty_c(M)$ on M lifts to an action of $\mathrm{Diff}^\infty_c(M)$ on Υ, given by

$$\begin{aligned} \psi_\cdot \colon \Upsilon &\longrightarrow \Upsilon \\ \gamma &\longmapsto \psi_\sharp \gamma \end{aligned} \;, \tag{1.2.6}$$

where $\psi \in \mathrm{Diff}^\infty_c(M)$, a configuration $\gamma \in \Upsilon$ is understood as in (1.2.1), and

$$\psi_\sharp \gamma := \gamma \circ \psi^{-1} \;.$$

As firstly observed by A. M. Vershik, I. M. Gel'fand and M. I. Graev in [162], the Poisson measure is *quasi-invariant* with respect to the action (1.2.6), viz.

$$\frac{\mathrm{d}(\psi_\cdot)_\sharp \mathcal{P}_{\mathsf{m}}}{\mathrm{d}\mathcal{P}_{\mathsf{m}}}(\gamma) = \prod_{x\in\gamma} \frac{\mathrm{d}\psi_\sharp \mathsf{m}}{\mathrm{d}\mathsf{m}}(x) \;,$$

and a unitary representation $U = U_{\mathcal{P}_{\mathsf{m}}}$ of $\mathrm{Diff}^\infty_c(M)$ is induced on $L^2_{\mathcal{P}_{\mathsf{m}}}(\Upsilon)$ by letting

$$\big(U(\psi) f\big)(\gamma) := \left(\frac{\mathrm{d}(\psi_\cdot)_\sharp \mathcal{P}_{\mathsf{m}}}{\mathrm{d}\mathcal{P}_{\mathsf{m}}}(\gamma)\right)^{1/2} f(\psi_\sharp^{-1}\gamma) \;,\; \psi \in \mathrm{Diff}^\infty_c(M) \;.$$

Differentiation on Υ. Let $\mathfrak{X}_c^\infty$ be the algebra of smooth compactly supported vector fields on M and, for $w \in \mathfrak{X}_c^\infty$, let $\left(\psi^{w,t}\right)_{t\in\mathbb{R}} \subset \mathrm{Diff}_c^\infty(M)$ be its flow. In the following, we shall regard $\mathrm{Diff}_c^\infty(M)$ as the Lie group associated to the Lie algebra $\mathfrak{X}_c^\infty$. (This identification has some caveats, see e.g., [125, §§6-7].) For $f \in L^2_{\mathcal{P}_\mathsf{m}}(\Upsilon)$, the action (1.2.6) gives rise to a notion of *directional derivative* w.r.t. $w \in \mathfrak{X}_c^\infty$, viz.

$$(\boldsymbol{\nabla}_w f)(\gamma) \coloneqq \mathrm{d}_t\big|_{t=0} f\left(\psi^{w,t}.\gamma\right) \tag{1.2.7}$$

whenever this exists. In order to show that (1.2.7) is non-void, let $\mathcal{FC}^\infty \subset L^2_{\mathcal{P}_\mathsf{m}}(\Upsilon)$ be the algebra of functions of the form

$$u(\gamma) = F\left(\int_M f_1 \,\mathrm{d}\gamma, \dots, \int_M f_k \,\mathrm{d}\gamma\right) ,$$

for some $k \in \mathbb{N}_0$, $F \in \mathcal{C}_c^\infty(\mathbb{R}^k)$ and $f_1, \dots, f_k \in \mathcal{C}_c^\infty(M)$. A straightforward computation (See e.g., [7, Eqn. (3.7)]) shows that for every $u \in \mathcal{FC}^\infty$ and every $w \in \mathfrak{X}_c^\infty$ there exists

$$(\boldsymbol{\nabla}_w u)(\gamma) = \sum_{i=1}^k (\partial_i F)\left(\int_M f_1 \,\mathrm{d}\gamma, \dots, \int_M f_k \,\mathrm{d}\gamma\right) \cdot \int_M (\mathrm{d}f_i) w \,\mathrm{d}\gamma \ . \tag{1.2.8}$$

Tangent spaces to Υ. Compatibly with the form of the directional derivative (1.2.8), set

$$(\boldsymbol{\nabla} u)(\gamma)(x) \coloneqq \sum_{i=1}^k (\partial_i F)\left(\int_M f_1 \,\mathrm{d}\gamma, \dots, \int_M f_k \,\mathrm{d}\gamma\right) \cdot (\mathrm{d}f_i)_x \ , \qquad u \in \mathcal{FC}^\infty \ , \tag{1.2.9}$$

and notice that, for fixed $\gamma \in \Upsilon$, the function $(\boldsymbol{\nabla} u)(\gamma)(\,\cdot\,)$ is an element of $\mathfrak{X}_c^\infty$, by compactness of $\mathrm{supp} f_i$, $i \leq k$. Let now $\mathfrak{X}_\gamma$ denote the completion of $\mathfrak{X}_c^\infty$ w.r.t. the pre-Hilbert norm

$$\|w\|_{\mathfrak{X}_\gamma} \coloneqq \left(\int_M |w_x|_\mathsf{g}^2 \,\mathrm{d}\gamma(x)\right)^{1/2} .$$

As in [7], we define a *tangent space* $T_\gamma\Upsilon$ to Υ at γ by setting

$$T_\gamma\Upsilon \coloneqq \mathfrak{X}_\gamma \ .$$

As in the abstract case of §1.1, this definition is motivated by the classical Riesz Representation Theorem, in that for every $\gamma \in \Upsilon$ and for any w in the dense subspace $\mathfrak{X}_c^\infty$ of $\mathfrak{X}_\gamma$ one has

$$(\boldsymbol{\nabla}_w u)(\gamma) = \left\langle (\boldsymbol{\nabla} u)(\gamma) \,\middle|\, w \right\rangle_{\mathfrak{X}_\gamma} .$$

> At least for short time, particles in a Bose gas γ behave independently of one another. Heuristically, this suggests that the "space of directions" $T_\gamma\Upsilon$ along which a configuration γ may move is "additive" and "total" w.r.t. the space of velocities of each particle. In mathematical terms
>
> $$T_\gamma\Upsilon = \widehat{\bigoplus_{x\in\gamma}} T_x\mathbb{R}^3 \cong \mathfrak{X}_\gamma(\mathbb{R}^3) \ .$$

The canonical Dirichlet form on Υ. The Poisson measure $\mathcal{P}_\mathsf{m}$, the gradient $\boldsymbol{\nabla}$ defined on the algebra of cylinder functions $\mathcal{FC}^\infty$, and the tangent spaces $\mathfrak{X}_\gamma$ all concur to the definition of a canonical (pre-)Dirichlet form on $L^2_{\mathcal{P}_\mathsf{m}}(\Upsilon)$, viz.

$$\mathcal{E}^\Upsilon(u,v) \coloneqq \int_\Upsilon \left\langle (\boldsymbol{\nabla} u)(\gamma) \,\middle|\, (\boldsymbol{\nabla} v)(\gamma) \right\rangle_{\mathfrak{X}_\gamma} \mathrm{d}\mathcal{P}_\mathsf{m}(\gamma) \ , \ u, v \in \mathcal{FC}^\infty \ . \tag{1.2.10}$$

Theorem (Albeverio–Kondrat'ev–Röckner [7, Thm. 4.1], also cf. [96, Thm. 19], [142, Cor. 3.4])**.** *The pre-Dirichlet form* $(\mathcal{E}^\Upsilon, \mathcal{FC}^\infty)$ *is closable. Its closure* $\left(\mathcal{E}^\Upsilon, \mathscr{D}(\mathcal{E}^\Upsilon)\right)$ *is a quasi-regular strongly local Dirichlet form on* $L^2_{\mathcal{P}_\mathsf{m}}(\Upsilon)$ *with essentially self-adjoint generator* $(\boldsymbol{\Delta}^\Upsilon, \mathcal{FC}^\infty)$.

By the standard theory of Dirichlet forms (e.g., [59, 112]), $(\mathcal{E}, \mathscr{D}(\mathcal{E}))$ is properly associated with a diffusion process with state space Υ, namely the process $\gamma_\bullet$ defined in (1.2.4).

Further developments. The detailed study of the Riemannian geometry of $(\Upsilon, \mathcal{P}_{\mathsf{m}})$ carried out in [7] constitutes the starting point of a prolific theory. Subsequent results include M. Röckner and A. Schied's Rademacher Theorem [142] for general measures on Υ endowed the extended L^2-Wasserstein distance W_2 (described below); S. Albeverio, A. Daletskii and E. W. Lytvynov's work [4], about the de Rham cohomology of $(\Upsilon, \mathcal{P}_{\mathsf{m}})$; L. Decreusfond's work [36], concerned with optimal transport on Υ; M. Erbar and M. Huesmann's work [51], about synthetic Ricci curvature lower bounds for the extended metric measure space $(\Upsilon, W_2, \mathcal{P}_{\mathsf{m}})$.

Onwards. Starting from the next section, we shall adopt the line of reasoning sketched in §1.1 and just detailed for configuration spaces, to lay the foundations for the definition of a diffusion process on the space of probability measures over a closed Riemannian manifold. The general strategy involves three main aspects: differentiation, randomness, and evolution.

1.3 Differentiation

In this section we discuss some notions of differentiability on spaces of measures in the spirit of §1.1. Namely, we choose Y to be a space of (probability) measures and detail natural actions on Y of some (infinite-dimensional) Lie groups G related to either the topology or the geometry of the underlying space.

Let $\mathscr{P}$ be the space of probability measures over a closed Riemannian manifold M. It is a leading intuition in the theory (e.g., [81, 131, 140, 165]), that we may treat $\mathscr{P}$ as a kind of infinite-dimensional Riemannian manifold. However, care should be taken that this intuition has its own limitations. Partly for this reason, and before addressing $\mathscr{P}$, let us briefly recall two basic results about Riemannian manifolds which will be of use in the following.

Manifolds. We start with a (smooth) Riemannian manifold (M, g) with intrinsic distance d_{g}. By $(M, \mathsf{d}_{\mathsf{g}})$ we mean the metric space underlying to (M, g). For the sake of further comparison, we recall a classical result by S. B. Myers and N. E. Steenrod,

Theorem (Myers–Steenrod [126, Thm.s 1 and 8])**.** *Let* $\psi\colon M \to M$. *The following are equivalent:*

- (a) ψ *is a (smooth, bijective) Riemannian isometry of* (M, g)*;*
- (b) ψ *is a (bijective) metric isometry of* $(M, \mathsf{d}_{\mathsf{g}})$*;*
- (c) ψ *is a homeomorphism preserving integral arc-length.*

and its strengthening, due to R. S. Palais,

Theorem (Palais [133])**.** *The differential structure of* (M, g) *may be uniquely reconstructed from* $(M, \mathsf{d}_{\mathsf{g}})$.

1.3.1 Wasserstein spaces and Otto calculus. We say that a metric space (X, d) is *geodesic* if for every choice of $x_i \in X$, $i = 0, 1$, there exists a curve $\gamma\colon I \to X$, henceforth a *geodesic* (*curve*), such that $\gamma_i = x_i$ and

$$\mathsf{d}(\gamma_t, \gamma_s) = |t - s|\, \mathsf{d}(x_0, x_1) \ .$$

Definition (*Wasserstein spaces*)**.** For $p \in [1, \infty)$ we write $\mathscr{P}_p$ for the space of Borel probability measures μ on X with finite p^{th}-moment, viz.

$$\int_X \mathsf{d}(x, x_0)^p \,\mathrm{d}\mu(x) < \infty$$

for some (hence any) $x_0 \in X$. We endow $\mathscr{P}_p$ with the L^p-*Wasserstein* metric W_p defined by

$$W_p(\mu,\nu)^p \coloneqq \inf_{\pi \in \mathrm{Cpl}(\mu,\nu)} \int_{X^{\times 2}} \mathsf{d}(x,y)^p \, \mathrm{d}\pi(x,y) \ , \tag{1.3.1}$$

where $\mathrm{Cpl}(\mu,\nu)$ denotes the set of couplings of the pair (μ,ν). (See Eq. (5.3.13).) The pair $(\mathscr{P}_p, W_p)$ is termed the L^p-*Wasserstein space* (over X).

If $\mathrm{diam}_{\mathsf{d}} X < \infty$, then $\mathscr{P}_p$ coincides, as a set, with $\mathscr{P}$. It is a standard result in the theory (e.g. [10] or [165, Ch. 6]) that (X,d) is separable, resp. complete, compact, geodesic, if and only if so is $(\mathscr{P}_p, W_p)$. The reverse implications are a consequence of the fact that the *Dirac embedding*

$$\delta \colon x \longmapsto \delta_x \tag{1.3.2}$$

is an isometry for every $p \in [1,\infty)$. In the case when (X,d) is a (separable) geodesic space, Lisini's *superposition principle* [110] yields a variational characterization of W_p by the so-called *metric Benamou–Brenier formula* [110, Cor. 4.3]

$$W_p(\mu,\nu)^p = \min_{\boldsymbol{\pi} \in \mathrm{GeoAdm}(\mu,\nu)} \int_{\mathrm{AC}(I;X)} \mathrm{d}\boldsymbol{\pi}(\gamma) \int_I \mathrm{d}t \, |\dot\gamma_t|^p \ . \tag{1.3.3}$$

Here $|\dot\gamma_t|$ denotes the metric speed of a continuous curve $(\gamma_t)_{t\in I}$ and $\mathrm{GeoAdm}(\mu,\nu)$ is the family of Borel probability measures $\boldsymbol{\pi}$ on the space $\mathrm{AC}(I;X)$ of X-valued absolutely continuous curves and such that $(\mathrm{ev}_0)_\sharp \boldsymbol{\pi} = \mu$ and $(\mathrm{ev}_1)_\sharp \boldsymbol{\pi} = \nu$, where $\mathrm{ev}_t \colon \gamma \mapsto \gamma_t$ is the evaluation map. Since the Dirac embedding is an isometry, (1.3.3) yields a version of Myers–Steenrod Theorem, namely:

Corollary. *A complete geodesic metric* d *compatible with a separable topological space* X *is uniquely determined by the metric speed of all* X*-valued absolutely continuous curves.*

Optimal maps and Wasserstein geodesics. In the case when $(X,\mathsf{d}) = (M,\mathsf{d}_\mathsf{g})$ is a closed smooth Riemannian manifold with metric g and volume measure $\mathsf{m} = \mathsf{m}_\mathsf{g}$, the case $p = 2$ acquires particular relevance.

Theorem (Brenier–McCann, see Thm. 4.3.8)**.** *Let* $\mu \ll \mathsf{m}$. *Then for every* $\nu \in \mathscr{P}_2$ *there exists* $\varphi = \varphi_{\mu\to\nu} \colon M \to \mathbb{R}$ *such that*

$$\exp_\cdot(\nabla_\cdot \varphi)_\sharp \mu = \nu$$

and

$$W_2(\mu,\nu)^2 = \int_M \mathsf{d}\big(x, \exp_x(\nabla_x \varphi)\big)^2 \, \mathrm{d}\mu(x) \ .$$

The function φ*, termed a* Kantorovich potential*, is unique up to additive constant among* $\mathsf{d}^2/2$*-convex functions on* M*. (See Dfn. 4.3.5.) Furthermore, the curve of measures*

$$\big(\exp_\cdot(t\,\nabla_\cdot \varphi)_\sharp \mu\big)_{t\in[0,1]}$$

is a W_2*-geodesic connecting* μ *to* ν*.*

The case when $\mu \not\ll \mathsf{m}$ is essentially more difficult and was completely understood in [63, 64] by N. Gigli. (See §§4.3.4-4.3.5.)

Otto calculus. For M as above, a heuristic understanding of $\mathscr{P}_2$ as an infinite-dimensional Riemannian manifold (e.g., F. Otto [131], J. Lott [111], W. Gangbo–H. K. Kim–T. Pacini [60]) is provided by the celebrated Otto calculus (e.g., [131], R. Jordan–D. Kinderlehrer–F. Otto [81].)

Denote by $\mathscr{P}_2^{\mathsf{m}}$ the space of measures in $\mathscr{P}_2$ absolutely continuous w.r.t. m. In a nutshell, solutions $u = u(x,t)$ to some parabolic PDEs (e.g., the heat equation, resp. the porous medium equation) may be regarded as curves of measures in $\mathscr{P}_2^{\mathsf{m}}$ by letting $\mu_t := u(\,\cdot\,,t)\mathsf{m}$. As it turns out, such curves are gradient flows, i.e. solutions to

$$\dot\mu_t = -\operatorname{grad}_{\mu_t} F \ ,$$

where F is a "free energy" functional on $\mathscr{P}_2^{\mathsf{m}}$ (e.g., the Boltzmann entropy, resp. the Rényi entropies). Here, if $\mu_t = \rho_t\mathsf{m}$, then we put $\dot\mu_t := \dot\rho_t$, the total derivative of ρ_t. The gradient $\operatorname{grad}_\mu F$ is defined in duality with the derivatives of F along geodesic curves, as follows. By Brenier–McCann's Theorem, for every $\mu \in \mathscr{P}_2^{\mathsf{m}}$ and every $\nu \in \mathscr{P}_2$ we may uniquely choose a W_2-geodesic $\mu_s := \exp_{\cdot}(s\nabla_{\cdot}\varphi)_\sharp\mu$ for some ${}^{\mathsf{d}^2}\!/_2$-convex φ. Note that, for every $\varphi \in \mathcal{C}^\infty(M)$, the function $s\cdot\varphi$ is ${}^{\mathsf{d}^2}\!/_2$-convex for sufficiently small $s>0$ (See [10, Lem. 1.34]). Thus, we are led to define the *'tangent space'*

$$T_\mu\mathscr{P}_2 := \operatorname{cl}_\mu \{\nabla\varphi \mid \varphi \in \mathcal{C}^\infty(M)\} \ ,$$

where cl_μ denotes the closure w.r.t. F. Otto's "Riemannian metric" on $\mathscr{P}_2$

$$\mathsf{G}_\mu(\nabla\varphi^1, \nabla\varphi^2) := \int_M \left\langle \nabla_x\varphi^1 \,\middle|\, \nabla_x\varphi^2\right\rangle_{\mathsf{g}} \mathrm{d}\mu(x) \ , \qquad \varphi^i \in \mathcal{C}^\infty(M) \ . \tag{1.3.4}$$

Interpreting $\nabla\varphi$ as a "direction" at μ, we define the *'directional derivative'*

$$(\partial_\varphi F)_\mu := \mathrm{d}_s\big|_{s=0} F(\mu_s)$$

with $(\mu_s)_{s\in[0,1]}$ a geodesic as above. Provided that F be sufficiently nice to grant existence for the above derivative, the G_μ-continuity of the linear mapping $\nabla\varphi \mapsto (\partial_\varphi F)_\mu$ yields then the existence of a unique element $\operatorname{grad}_\mu F \in T_\mu\mathscr{P}$ such that

$$(\partial_\varphi F)_\mu = \mathsf{G}_\mu(\operatorname{grad}_\mu F, \nabla\varphi) \ , \qquad \varphi \in \mathcal{C}^\infty(M) \ . \tag{1.3.5}$$

It turns out that the metric G_μ induces the distance W_2 (See [131, §4.3].) and, again if F is nice, then the W_2-metric slope of F coincides with the G_μ-module of its gradient, (Cf. [11, Lem. 10.1.5]) viz.

$$|DF|(\mu) := \limsup_{\nu\to\mu} \frac{|F(\nu) - F(\mu)|}{W_2(\mu,\nu)} = \left|\operatorname{grad}_\mu F\right|_{\mathsf{G}_\mu} \ .$$

1.3.2 Differentiation on spaces of measures. Analogously to the case of configuration spaces, let us firstly introduce some group actions, as follows.

An action in topology. The topology of X is captured by the Abelian Lie algebra $\mathcal{C}_c$. We write $\exp\mathcal{C}_c \circlearrowleft \mathscr{M}^+$ for the action

$$e^f.\colon \mu \longmapsto e^f\cdot\mu \ , \qquad f \in \mathcal{C}_c \ , \tag{1.3.6}$$

where e^f is but the exponential of real-valued functions. On the other hand, $\mathscr{P}$ is a homogeneous space for the action $\exp\mathcal{C}_c \circlearrowleft \mathscr{P}$ (Cf. [73, 143, 147].)

$$e^f.\colon \mu \longmapsto \frac{e^f\cdot\mu}{\displaystyle\int_X e^f\,\mathrm{d}\mu} \ , \qquad f \in \mathcal{C}_c \ . \tag{1.3.7}$$

An action in geometry. When $X = (M, \mathsf{g})$ is a closed smooth Riemannian manifold, its differential structure is captured by the non-Abelian Lie algebra $\mathfrak{X}^\infty$ of smooth vector fields and we write $\exp \mathfrak{X}^\infty \circlearrowleft \mathscr{M}_b^+$ for the action

$$e^w . \colon \mu \longmapsto e^w{}_\sharp \mu \ , \qquad w \in \mathfrak{X}^\infty \ , \tag{1.3.8}$$

where $e^{t\,w} \coloneqq \psi^{w,t}$ is the flow of w at time $t \in \mathbb{R}$. We notice that the latter action restricts naturally to $\mathscr{P}$ and that it is the lift of the natural action $\mathrm{Diff}^\infty_+(M) \circlearrowleft M$.

Linear geometries on $\mathscr{P}$. As detailed in §5.2 below, the action (1.3.6), resp. (1.3.7), naturally relates to those geometries on $\mathscr{M}_b^+$, resp. $\mathscr{P}$, for which the linear combination

$$t \mapsto \mu + t\delta_x \ ,$$

resp. the convex combination

$$t \mapsto (1-t)\mu + t\delta_x \ ,$$

is a geodesic curve for every choice of μ and $x \in X$. Among these geometries are the *linear* geometries inherited by subspaces of the vector space $\mathscr{M}_b$ of finite signed measures, endowed with the total variation norm or the L^1-Kantorovich–Rubinshtein norm $\|\cdot\|_1$. The latter satisfies $\|\mu - \nu\|_1 = W_1(\mu, \nu)$ on $\mathscr{P}_1$ and it is in fact maximal among all (semi-)norms on the space of molecules of X that make the Dirac embedding into an isometry. (See e.g., [120, Thm. 1].)

L^2-Wasserstein geometry on $\mathscr{P}_2$. Similarly to the case of configurations, we define $\mathfrak{X}_\mu$ as the completion of $\mathfrak{X}^\infty$ w.r.t. the pre-Hilbert scalar product

$$\left\langle w^1 \,\middle|\, w^2 \right\rangle_\mu \coloneqq \int_M \left\langle w^1_x \,\middle|\, w^2_x \right\rangle_{\mathsf{g}} \mathrm{d}\mu(x) \ , \qquad w^i \in \mathfrak{X}^\infty \ ,$$

extending G_μ on $T_\mu \mathscr{P}_2$ (1.3.4) to $\mathfrak{X}_\mu$, occasionally (e.g., [60]) termed 'pseudo-tangent space'. A gradient $\boldsymbol{\nabla}$ is induced for functions $u \colon \mathscr{P}_2 \to \mathbb{R}$ by the action (1.3.8), satisfying (1.1.1) for the choice $\mathfrak{g} = \mathfrak{X}^\infty$. This gradient extends grad in (1.3.5), in the sense that

$$\left\langle (\boldsymbol{\nabla}\, u)(\mu) \,\middle|\, \cdot \right\rangle_\mu = \mathsf{G}_\mu(\mathrm{grad}_\mu\, u, \cdot) \quad \text{on} \quad T_\mu \mathscr{P}_2 \ , \tag{1.3.9}$$

for all $u \colon \mathscr{P}_2 \to \mathbb{R}$ such that either exists. For instance, this is the case when $u \in \mathcal{FC}^\infty$, regarded as the algebra of cylinder functions induced by smooth potential energies. (Cf. Dfn. 4.2.1.) In fact, if $u \in \mathcal{FC}^\infty$, then $(\boldsymbol{\nabla}\, u)(\mu) \in T_\mu \mathscr{P}_2 \subsetneq \mathfrak{X}_\mu$.

A heuristic explanation of (1.3.9) is as follows. If $w \in \mathfrak{X}^\infty$, then the geodesic flow $\exp_x(t\,w_x)$ and the flow $\psi^{w,t}(x)$ are tangent to each other at $t = 0$ for all $x \in M$. (See Lem. 4.4.1.) As a consequence, the corresponding lifted flows on measures $\exp_{\cdot}(t\,w_{\cdot})_\sharp$ and $\psi^{w,t}_\sharp$ are tangent to each other at $t = 0$ for every $\mu \in \mathscr{P}_2$. (Cf. the proof of Lem. 4.4.3.) Thus, the induced directional derivatives and gradients coincide. Additionally, $(\mathfrak{X}_\mu, |\cdot|_\mu) \cong (T_\mu \mathscr{P}_2, |\cdot|_{\mathsf{G}_\mu})$ as Hilbert spaces for all μ in the dense set of purely atomic measures with finite support.

In the following we shall however always distinguish between these two gradients and the relative tangent spaces $T_\mu \mathscr{P}_2$ and $\mathfrak{X}_\mu$. Indeed, the "differential structures" induced by $\boldsymbol{\nabla}$, resp. grad, do differ *globally*. The difference between grad on $T_\mu \mathscr{P}_2$ and $\boldsymbol{\nabla}$ on $\mathfrak{X}_\mu$ is accounted for by the corresponding tangent bundles in terms of the global derivations of $\mathfrak{Z}^\infty$ on $\mathscr{P}_2$. (See Prop. 4.6.3).

Postponing rigorous statements to §4.6.1, let us argue here heuristically. Firstly, in the previous section we strove for an understanding of $\mathscr{P}_2^{\mathsf{m}}$ as a Riemannian manifold. Any such understanding for the whole of $\mathscr{P}_2$ is however vitiated, in that $\mathscr{P}_2$ has no local structure. By this we mean that for different points $\mu, \nu \in \mathscr{P}_2$ the spaces $T_\mu \mathscr{P}_2$ and $T_\nu \mathscr{P}_2$ need *not* be isomorphic Hilbert spaces, nor even of the same dimension as real vector spaces. The same holds for $\mathfrak{X}_\mu$ and $\mathfrak{X}_\nu$. Therefore, in spite of the geodesic Myers–Steenrod Theorem, it is a consequence of this "lack of locality" that (any reasonable analogue of) Palais' Theorem need not hold for $\mathscr{P}_2$.

1.4 Randomness

In this section we discuss different choices for reference measures on spaces of measures. Again, we do so in the spirit of §1.1, proposing candidates of measures n on Y satisfying some quasi-invariance properties.

We say that a *random measure* is the law of any $\mathscr{M}^+$-valued random field. Starting from the Poisson measure $\mathcal{P}_\sigma$, we introduce several random measures over X. As in the case of configurations, our intuition is partly taken from the physical description of particle systems.

> By a *non-interacting massive particle system* we shall mean a many-body system of massive, volume- and charge-less random particles in the Euclidean space. We assume further that
>
> (M_1) each particle is uniquely identified by its mass;
>
> (M_2) the system does not undergo condensation shocks, that is, each finite volume in $\mathbb{R}^3$ contains only a finite amount of mass;
>
> (M_3) for a volume $A \subset \mathbb{R}^3$ of size $|A|$ and an interval $[s, t] \subset \mathbb{R}^+$ let $\nu_A^{s,t}$ be the total mass of particles in A of mass comprised between s and t. Then,
>
> $$\ln \mathbf{P}(\nu_A^{s,t} = 0) = -|A| \cdot \lambda[s, t]$$
>
> for some measure λ on $\mathbb{R}_+$.

Analogously to the case of Bose gases, we shall formalize a massive non-interacting particle system as a *marked* point process, a pure-jump measure-valued Lévy process with no drift.

1.4.1 The gamma measure. Let $\hat{X} := X \times \mathbb{R}_+$ and $\hat{\Upsilon}$ be the associated configuration space. A *Lévy measure* λ is any measure on $\mathbb{R}_+$ such that

$$\int_{\mathbb{R}_+} (1 \wedge s)\, \mathrm{d}\lambda(s) < \infty \ .$$

For a Lévy measure λ, and for σ as in the previous sections, we regard $\hat{\sigma} := \sigma \otimes \lambda$ as the intensity measure of the Poisson random measure $\mathcal{P}_{\hat{\sigma}}$ on $\hat{\Upsilon}$. Further let $\mathbf{H} : \mathscr{M}^+_{\mathrm{pa}} \to \hat{\Upsilon}$ be defined as

$$\mathbf{H}\colon \sum_{i=1}^N s_i \delta_{x_i} \longmapsto \sum_{i=1}^N \delta_{(s_i, x_i)} \ .$$

As it turns out, $\mathbf{H}$ is a bi-measurable bijection onto its image, the space of *marked configurations* $\hat{\mathscr{M}}^+_{\mathrm{pa}}$. (See §3.2.) We define the *compound-Poisson random measure* $\mathcal{R}_{\sigma,\lambda}$ on $\mathscr{M}^+_{\mathrm{pa}}$ as

$$\mathcal{R}_{\sigma,\lambda} := (\mathbf{H}^{-1})_\sharp \mathcal{P}_{\hat{\sigma}} \ .$$

A distinguished example among all compound-Poisson random measures is the *gamma measure* $\mathcal{G}_\sigma$. Our interest in $\mathcal{G}_\sigma$ is mainly motivated by four properties, which we comment about in the next paragraphs.

Definition (*Gamma measure*, e.g., [94, 95, 157] or [102, Ex. 15.6])**.** We define the *gamma measure* $\mathcal{G}_\sigma$ as

$$\mathcal{G}_\sigma := \mathcal{R}_{\sigma,\lambda} \ , \qquad \mathrm{d}\lambda(s) := s^{-1} e^{-s} \,\mathrm{d}s \ .$$

(For further characterizations of $\mathcal{G}_\sigma$, see Eq.s (1.4.1), (3.2.3) and Lem. 3.2.1 below.)

Representations of large groups. As shown by N. V. Tsilevich, A. M. Vershik and M. Yor in [157], the gamma measure is quasi-invariant w.r.t. the action (1.3.6) of $\exp \mathcal{C}_c$ on $\mathscr{M}^+$, viz. (Cf. [157, Thm. 3.1].)

$$\frac{\mathrm{d}(e^f.)_\sharp \mathcal{G}_\sigma}{\mathrm{d}\mathcal{G}_\sigma}(\nu) = \exp\left[-\int_X f \,\mathrm{d}\sigma\right] \cdot \exp\left[\int_X \left(e^{-f}-1\right) \mathrm{d}\nu\right] \ , \qquad f \in \mathcal{C}_c \ .$$

It was subsequently shown by Vershik in [161] that there exists a measure $\mathcal{L}^+_\sigma$, unique up to multiplicative constant, mutually absolutely continuous w.r.t. $\mathcal{G}_\sigma$ and invariant w.r.t. the same action. Partly because of this property, $\mathcal{G}_\sigma$ and the so-called *multiplicative infinite-dimensional Lebesgue measure* $\mathcal{L}^+_\sigma$ play a rôle in a long-standing program for the study of representations of measurable SL_2-current groups, i.e., groups of measurable bounded functions on a manifold, and with values in the special linear group SL_2. (We postpone a detailed account about the relevant literature to §2.1.2.)

As already noticed in [162] for Poisson random measures, $\mathcal{G}_\sigma$ too is related to the representation theory of the *infinite symmetric group* $\mathfrak{S}_\infty$, consisting of all permutations of $\mathbb{N}$ with cofinitely many fixed points. The main fact behind this connection is that the Laplace transform $\mathcal{L}[\mathcal{G}_\sigma]$ of the gamma measure satisfies, for all $t \in \mathbb{R}$,

$$\mathcal{L}[\mathcal{G}_\sigma](tf) := \int_{\mathscr{M}^+} \exp\left[t \int_X f \,\mathrm{d}\nu\right] \mathrm{d}\mathcal{G}_\sigma(\nu) = Z_\infty\left(t\int_X f\,\mathrm{d}\sigma, t^2 \int_X f^2 \,\mathrm{d}\sigma, \dots\right) \ , \qquad f \in \mathcal{C}_c \ .$$

Here Z_∞ is the *cycle index polynomial* of $\mathfrak{S}_\infty$, defined by

$$Z_\infty(a_1, a_2, \dots) := \sum_{\mathbf{m}} \prod_{i=1}^\infty \frac{a_i^{m_i}}{i^{m_i} m_i!} \ ,$$

where the sum runs over all non-negative multi-indices $\mathbf{m}$ in an arbitrary number of coördinates and with finite length. For instance, Z_∞ plays an important rôle in A. Yu. Okounkov's proof [130, pp. 28-29] of Thoma's classification of the characters of $\mathfrak{S}_\infty$. (See [154, Satz 2, p. 55].) The finite counterpart of Z_∞, namely the cycle index polynomial Z_n of the symmetric group $\mathfrak{S}_n$, will be of importance throughout Chapter 2.

The extended Fock space. For general $\mathcal{R}_{\sigma,\lambda}$ (the analog of) the multiple stochastic integration ι_σ in (1.2.3), although isometric, is generally not surjective onto $L^2_{\mathcal{R}_{\sigma,\lambda}}(\mathscr{M}^+_{\mathrm{pa}};\mathbb{C})$, and the question arises whether it is possible to give a (Fock) factorization of the latter space. For general Lévy processes, it is known that the answer is affirmative, and in fact ≪*the study of the factorizations generated by an arbitrary process with independent values reduces to the case of the Gaussian and Poisson processes.*≫ [163, p. 442, ¶4]. Nonetheless, since the first chaos space need not coincide with $L^2_\sigma(X;\mathbb{C})$, explicit constructions of such factorizations are generally out of reach. (See [163, §2.6, ¶2].)

In the case of $\mathcal{G}_\sigma$ however, Yu. G. Kondrat'ev, J. L. da Silva, L. Streit and G. F. Us showed in [94] that the standard Fock space (1.2.2) may be enlarged to an explicit *extended Fock space* (e.g., [95, Eq. (8)]) unitarily equivalent to $L^2_{\mathcal{G}_\sigma}(\mathscr{M}^+_{\mathrm{pa}};\mathbb{C})$ (See [95, §2, Thm. 1]). The corresponding creation and annihilation operators in the extended Fock space of $\mathcal{G}_\sigma$ were later fully understood by Kondrat'ev and E. W. Lytvynov in [95].

The factorization property. Let us assume further that $0 < \sigma X < \infty$. Then, it is not difficult to show that $\mathcal{G}_\sigma$ has intensity σ, and thus it is concentrated on the cone of strictly positive finite measures $\mathscr{M}_b^+ \setminus \{0\} \cong \mathscr{P} \times \mathbb{R}_+$. On $\mathscr{M}_b^+ \setminus \{0\}$ we define the *normalization map*

$$\mathbf{N} \colon \nu \longmapsto \overline{\nu} \coloneqq (\nu X)^{-1} \nu \in \mathscr{P} \ .$$

Let now $\beta \coloneqq \sigma X > 0$, so that $\sigma = \beta \overline{\sigma}$. As shown by T. S. Ferguson in [55], the Gamma measure $\mathcal{G}_\sigma$ is (the unique compound-Poisson random measure) factoring via $\mathbf{N}$, i.e. such that

$$\mathcal{G}_\sigma = \mathbf{N}_\sharp \mathcal{G}_\sigma \otimes \lambda' \tag{1.4.1}$$

for some probability distribution λ' on $\mathbb{R}_+$. In fact, it is not difficult to show that $\lambda' = \mathsf{G}_{\beta,1}$. (See e.g., [55, 157] or [158, Thm. 3].)

Partial quasi-invariance. It was shown by Yu. G. Kondrat'ev, E. W. Lytvynov and A. M. Vershik in [96] that $\mathcal{G}_\mathsf{m}$ is *not* $\mathrm{Diff}^\infty(M) \circlearrowleft \mathscr{M}^+$-quasi-invariant. Motivated by the very same guidelines for the construction of diffusion processes as in §1.1, the authors overcame this issue by introducing a notion of 'partial quasi-invariance' (Dfn. 2.1.2). In essence, in the notation of §1.1, the partial $G \circlearrowleft Y$-quasi-invariance of n coincides with the $G \circlearrowleft Y$-quasi-invariance of the restriction of n to each G-invariant σ-algebra $\mathcal{B}_n$ in a filtration $\mathcal{B}_\bullet \coloneqq (\mathcal{B}_n)_{n \geq 0}$ on Y, with $\mathcal{B}$ as terminal σ-algebra. Under suitable measurability conditions, the Radon–Nikodým derivatives $R_\bullet$ form a $(\mathcal{B}_\bullet, \mathsf{n})$-martingale, convergent if n is $G \circlearrowleft Y$-quasi-invariant in the classical sense.

1.4.2 The Dirichlet–Ferguson measure. The factorization property of the gamma measure calls for the following definition of the Dirichlet–Ferguson measure, our candidate for a "volume measure" on $\mathscr{P} = \mathscr{P}(M)$.

Definition. The *Dirichlet–Ferguson measure* with *intensity* σ (e.g., [55]) is the random measure

$$\mathcal{D}_\sigma \coloneqq \mathbf{N}_\sharp \mathcal{G}_\sigma \ .$$

Ever since T. S. Ferguson's seminal paper [55], the Dirichlet–Ferguson measure $\mathcal{D}_\sigma$ has found numerous applications throughout mathematics and beyond, as it is attested by its several names. (See §2.1, ¶2 for a brief account about terminology and applications.) For example, in light of the identification $\mathscr{M}_b^+ \setminus \{0\} \cong \mathscr{P} \times \mathbb{R}_+$, the measure $\mathcal{D}_\sigma$ is occasionally referred to as the *simplicial* part of $\mathcal{G}_\sigma$. (Cf. [158].) Indeed, assume that $X = \{1, \dots, k\}$ consist only of a finite number of points, so that $\sigma = \sum_i^k \alpha_i \delta_i$ for some $\boldsymbol{\alpha} \coloneqq (\alpha_1, \dots, a_k) \in \mathbb{R}_+^k$. Then, $\mathscr{P}$ coincides with the standard $(k-1)$-dimensional simplex Δ^{k-1}, and $\mathcal{D}_\sigma$ is but the *Dirichlet distribution* $\mathsf{D}_{\boldsymbol{\alpha}}$ of parameter $\boldsymbol{\alpha}$, (Dfn. 2.2.1 below.) a multivariate generalization of the Beta distribution $\mathsf{B}_{\alpha_1,\alpha_2}$.

A fixed point characterization. It was firstly shown by J. Sethuraman in [145, §3] that the Dirichlet–Ferguson measure satisfies the following characterization.

Theorem. *Let x be an X-valued $\overline{\sigma}$-distributed random field, r be a $\mathsf{B}_{1,\beta}$-distributed random variable with values in $[0,1]$, and η be a $\mathbf{P}$-distributed $\mathscr{P}$-valued random field. Then, $\mathbf{P}$ is the Dirichlet–Ferguson measure $\mathcal{D}_\sigma$ if and only if*

$$\eta \overset{\mathrm{d}}{=} (1-r)\eta + r\delta_x \ , \tag{1.4.2}$$

where $\overset{\mathrm{d}}{=}$ denotes equality in distribution.

Originally presented as a fixed-point characterization, (Cf. [76, p. 28-279].) the distributional Equation (1.4.2) may be understood as a Mecke- or Georgii–Nguyen–Zessin-type formula, as argued in Chapter 3. Indeed, analogous characterizations by means of Palm distributions were long proven in integral form: for $\mathcal{P}_\sigma$ by J. Mecke in [119], (for $\mathcal{R}_{\sigma,\lambda}$ in e.g., [96]) and for general Gibbs measures by H. O. Georgii and, independently, by X. X. Nguyen and H. Zessin (e.g., [129]).

Among the many consequences of (1.4.2) is the fact that, whenever σ is diffuse, the random measure $\mathcal{D}_\sigma$ is concentrated on the set of purely atomic probability measures with full topological support.

Self-conjugate priors. The characterization (1.4.2) is of use to establish the following result, originally shown in [55] by other means. The precise statement is as follows.

Theorem (Dirichlet-categorical posteriors, e.g., [55, §3, Thm. 1])**.** *Let η be a $\mathcal{D}_\sigma$-distributed random field and let x be a sample from η. Then, the conditional distribution $\mathcal{D}_\sigma^x$ of η given x is $\mathcal{D}_{\sigma+\delta_x}$.*

In the jargon of Bayesian non-parametrics: $\mathcal{D}_\sigma$ is a *self-conjugate prior*, that is, the (*X-categorical*) *posterior* $\mathcal{D}_\sigma^x$ of $\mathcal{D}_\sigma$ is itself a Dirichlet–Ferguson measure. This property is of great importance in Bayesian non-parametrics, since ≪*posterior distributions given a sample of observations from the true probability distribution should be manageable analytically*≫ [55, §1]. Besides, it will also be of importance in relation with SL_2-currents, as detailed in Chapter 2.

1.4.3 The entropic measure. As informally argued in Tsilevich–Vershik–Yor [157, p. 276, ¶2], the multiplicative measures $\mathcal{L}_\sigma^+$ are — among α-stable laws — at the opposite of the spectrum from Wiener processes. Partially pivoting on this antithesis, von Renesse–Sturm [140] constructed the *entropic measure* $\mathbb{P}^\beta$, a Gibbs-like measure on $\mathscr{P}(\mathbb{S}^1)$. Mimicking Feynman's celebrated heuristics of the Wiener measure, the authors replaced the energy functional on curves with the (relative) Boltzmann entropy on measures, formally letting

$$\mathrm{d}\mathbb{P}^\beta(\mu) = \frac{1}{Z_\beta} e^{-\beta\,\mathsf{Ent}(\mu)}\,\mathrm{d}\mathbb{P}(\mu)\ , \qquad \mu\in\mathscr{P}(\mathbb{S}^1)\ , \tag{1.4.3}$$

for some (non-existing!) uniform measure $\mathbb{P}$ and some normalization constant Z_β. (For a rigorous construction on $\mathscr{P}_2(\mathbb{S}^1)$, see [140, §3.2] or §4.5.5 below.)

The conjugation map. Let now M be a closed Riemannian manifold with volume measure m and set $\beta := \mathsf{m}M$. In the general case when $\mathbb{S}^1$ is replaced by M, the construction of the entropic measure $\mathbb{P}_\mathsf{m} = \mathbb{P}^\beta_{\overline{\mathsf{m}}}$ was achieved by K.-T. Sturm in [151, §3], as we detail now.

For $\mu\in\mathscr{P}_2(M)$ let $\varphi_\mu := \varphi_{\overline{\mathsf{m}}\to\mu}$ be the map given by Brenier–McCann's Theorem. Sturm defined the *conjugation map*, a self-homeomorphism of $\mathscr{P}_2(M)$ given by

$$\mathfrak{C}^\mathsf{m}\colon \mu\longmapsto \left(\exp_{.}(\nabla_{.}\,\varphi_\mu^\mathsf{c})\right)_\sharp \overline{\mathsf{m}}\ .$$

Here, φ^c is the *conjugate* Kantorovich potential (Cf. §4.3.4.)

$$\varphi^\mathsf{c}\colon x\longmapsto -\inf_{y\in M}\left(\tfrac{1}{2}\mathsf{d}(x,y)+\varphi(y)\right)\ .$$

Definition. The *entropic measure* with *intensity* β is the random measure

$$\mathbb{P}_\mathsf{m} := \mathfrak{C}^\mathsf{m}_\sharp\,\mathcal{D}_\mathsf{m}\ , \qquad \mathsf{m}=\beta\overline{\mathsf{m}}\ .$$

Consistently with (1.4.3), the parameter β, implicit in the definition of $\mathsf{m} = \beta\overline{\mathsf{m}}$, ought to be understood as an *inverse temperature* in the sense of statistical mechanics. Indeed, it can be shown (See Cor. 2.3.14) that

$$\lim_{\beta\to\infty} \mathcal{D}_{\beta\overline{\mathsf{m}}} = \delta_{\overline{\mathsf{m}}} \qquad \text{and} \qquad \lim_{\beta\to 0} \mathcal{D}_{\beta\overline{\mathsf{m}}} = \delta_\sharp\overline{\mathsf{m}} \ ,$$

where, in the second expression, $\delta_\sharp\overline{\mathsf{m}} = \overline{\mathsf{m}} \circ \delta^{-1}$ and δ is the Dirac embedding (1.3.2).

In statistical mechanical terms, as $\beta \to \infty$ (that is, for low temperatures), the system *crystallizes* to a fixed probability $\overline{\mathsf{m}}$, whereas, as $\beta \to 0$, the system *thermalizes* to a random probability δ_x, where x in an X-valued random variable with law $\overline{\mathsf{m}}$. (Cf. Rmk. 2.3.16 below.)

Support properties. The measure $\mathbb{P}^\beta$ is *not* the law of a point process. Indeed, as noticed in [140], it is possible to show that $\mathbb{P}^\beta$-a.e. μ is supported on a Cantor space[5]. In the general case when M has dimension $d \geq 2$, the study of $\mathbb{P}_{\mathsf{m}}$ turns out to be particularly difficult. This is mainly due to the fact that, for general μ, the Kantorovich potential φ_μ is not more than Lipschitz regular. (Cf. Fig. 1.1.)

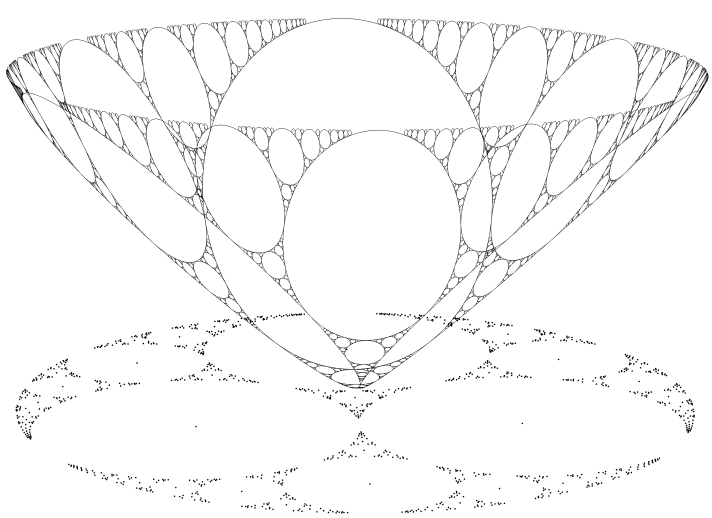

Figure 1.1: The graph of φ_μ on the unit-area disk for μ the purely atomic measure with atoms as displayed, each with mass equal to the area of the corresponding circle.

Quasi-invariance of $\mathbb{P}^\beta$ on $\mathscr{P}_2(\mathbb{S}^1)$. By parallelizability and dimension, the unit circle witnesses the coincidence of the Abelian Lie algebra $\mathcal{C}_c(\mathbb{S}^1)$ of smooth real-valued functions with the Lie algebra $\mathfrak{X}^\infty(\mathbb{S}^1)$ of smooth vector fields. In the form given in §4.5.5, the conjugation map on $\mathscr{P}_2(\mathbb{S}^1)$ interchanges the actions of the two associated groups. As a consequence of this fact, von Renesse–Sturm were able to prove that $\mathbb{P}^\beta$ is quasi-invariant w.r.t. the action (1.3.8). A by now complete understanding of this property, subsequently provided by M.-K. von Renesse,

[5] A topological space homeomorphic to the 1/3-Cantor set.

M. Yor and L. Zambotti's [141] and J. Shao's [147], suggests however that this quasi-invariance property should not be expected in the general case of $\mathbb{P}_{\mathsf{m}}$.

1.5 Evolution

Except for the case of $(\Upsilon, \mathcal{P}_{\mathsf{m}})$, few other cases appear to have been studied in the literature with Dirichlet forms' methods, as we detail below.

1.5.1 Dynamics on spaces of probability measures. We present three main examples of stochastic processes on spaces of probability measures constructed via Dirichlet forms methods.

The Fleming–Viot process on $(\mathscr{P}(X), \mathcal{D}_\sigma)$. When X is a (locally compact) Polish space, L. Overbeck, M. Röckner and B. Schmuland showed in [132] how the gradient induced by the action $\exp \mathcal{C}_c \circlearrowleft \mathscr{P}$ relates to the carré du champ operator of the Dirichlet form associated to the Fleming–Viot process with parent-independent mutation introduced by W. H. Fleming and M. Viot in [57]. Quasi-invariance properties where subsequently shown by K. Handa [73], and Schied [144] proved the Rademacher Theorem.

The Wasserstein diffusion on $(\mathscr{P}_2(\mathbb{S}^1), \mathbb{P}^\beta)$. In the aforementioned study [140], K.-T. Sturm and M.-K. von Renesse proved the closability of the Dirichlet form

$$\mathcal{E}(u,v) := \int_{\mathscr{P}_2(\mathbb{S}^1)} \mathsf{G}_\mu\big((\nabla u)(\mu), (\nabla v)(\mu)\big) \, \mathrm{d}\mathbb{P}^\beta(\mu) \ , \qquad u, v \in \mathcal{FC}^\infty \ ,$$

give a characterization of the generator and a proof of the Rademacher and Sobolev-to-Lipschitz properties. Moreover, they provide some understanding of the associated Markov diffusion process, termed the *Wasserstein diffusion*, by showing that it is a solution to an SPDE of the form

$$\mathrm{d}\mu_t = \mathrm{div}\big(\sqrt{\mu_t}\, \mathrm{d}W_t\big) + \mathbf{L}_2^{\mathrm{WD}}(\mu_t)\, \mathrm{d}t \ , \tag{1.5.1}$$

where $W_\bullet$ is a standard Brownian motion. (See (5.2.23 WD_0) for details, including the definitions of $\mathbf{L}_2^{\mathrm{WD}}$ and of the space of test functions.)

The Modified Massive Arratia Flow on $\mathscr{P}_2([0,1])$. The rather non-amenable form of the "drift" term $\mathbf{L}_2^{\mathrm{WD}}$ in (1.5.1) prompted M. K. von Renesse and V. V. Konarovskyi to investigate, again via Dirichlet form methods, solutions to analogous SPDEs on spaces of probability measures. Their starting point is the Modified Massive Arratia Flow constructed by Konarovskyi in [91].

> The Modified Massive Arratia Flow describes the motion of (a density of) random massive particles additionally
>
> (A_1) *locally-in-time non-interacting* and *free*, that is, each particle moves by Brownian motion until the first meeting time of two or more particles; (Notice that Brownian particles in the unit interval will necessarily meet in finite time.)
>
> (A_2) *turbulent*, in the sense that, when moving by Brownian motion, particles rearrange themselves at a speed inversely proportional to their mass;
>
> (A_3) *sticky*, in the sense that, whenever some massive particles meet, they coalesce to form a single particle with resulting mass the total mass of the meeting particles.

Konarovskyi was able to construct the Modified Massive Arratia Flow as a stochastic process with values in the Skorokhod space $D\big([0,1]; \mathcal{C}([0,T])\big)$, by modifying the quadratic variation functional of the standard Arratia Flow, for which all particles have constant mass 1 and coalescence does not alter the mass of particles. Subsequently, von Renesse–Konarovskyi [93] constructed the process via Dirichlet form and identified as a solution to the SPDE

$$\mathrm{d}\mu_t = \mathrm{div}(\sqrt{\mu_t}\,\mathrm{d}W_t) + \sum_{x\in\mu_t} \delta_x''\,\mathrm{d}t \ ,$$

where $W_\bullet$ is a standard Brownian motion and δ_x'' is the second distributional derivative of δ_x. (See (5.2.23 AF) for details.)

1.5.2 Dynamics on spaces of measures. The same line of reasoning as in [7] was adopted in [96] for the case of Gibbs measures on the cone $\mathscr{M}^+_{\mathrm{pa}}$ of purely atomic Radon measures in $\mathscr{M}^+$.

Stochastic dynamics on $\mathscr{M}^+_{\mathrm{pa}}$. Let (M, g) be a smooth Riemannian manifold with bounded topology and geometry, and m be its volume measure. The actions (1.3.6) and (1.3.8) leave $\mathscr{M}^+_{\mathrm{pa}}$ invariant, thus, the latter is homogeneous for the action of the (suitably defined) semi-direct product $\exp \mathcal{C}_c(M) \rtimes \mathrm{Diff}^\infty_c(M)$. Although the Gamma measure $\mathcal{G}_\mathsf{m}$ is *not* quasi-invariant under this action, Yu. G. Kondrat'ev, E. W. Lytvynov and A. M. Vershik were able to show in [96] the closability of the pre-Dirichlet form

$$\tilde{\mathcal{E}}(u,v) := \int_{\mathscr{M}^+_{\mathrm{pa}}} \big\langle (\tilde{\boldsymbol\nabla} u)(\mu) \,\big|\, (\tilde{\boldsymbol\nabla} v)(\mu) \big\rangle_{\tilde{T}_\mu \mathscr{M}^+_{\mathrm{pa}}} \,\mathrm{d}\mathcal{G}_\mathsf{m}(\mu) \ .$$

Here, u, v belong to a suitable family of cylinder functions (See [96, Eq. (26)]) and the operator $\tilde{\boldsymbol\nabla}$ is defined by Riesz Representation in duality with the directional derivatives induced by shifts in $\exp \mathcal{C}_c(M) \rtimes \mathrm{Diff}^\infty_c(M)$. The tangent space $\tilde{T}_\mu \mathscr{M}^+_{\mathrm{pa}}$ is defined as the closure of $\mathcal{C}^\infty_c(\mathbb{R}_+;\mathbb{R}) \otimes \mathfrak{X}^\infty_c$ w.r.t. to the norm

$$\|f\otimes w\|_{\tilde{T}_\mu \mathscr{M}^+_{\mathrm{pa}}} := \left(\int_M |f(\mu_x)|^2 \, |w_x|^2_\mathsf{g} \,\mathrm{d}\tilde\mu(x) \right)^{1/2} \tag{1.5.2}$$

where, if $\mu = \sum_{i\le N} s_i \delta_{x_i}$ for some $N \in \overline{\mathbb{N}}_0$, we let $\mu_x := \mu\{x\}$ and $\tilde\mu := \sum_{i\le N} \delta_{x_i}$.

The choice of $\tilde{T}_\mu \mathscr{M}^+_{\mathrm{pa}}$ in [96] mimics the analogous choice of tangent spaces to the configuration space Υ. Indeed, the Markov diffusion process $\mu_\bullet$ associated to $\big(\tilde{\mathcal{E}}, \mathscr{D}(\tilde{\mathcal{E}})\big)$ is precisely the free motion of a massive non-interacting particle system as in §1.4, ¶3, satisfying

$$\mu_\bullet = \sum_{i\le N} s^i_t \delta_{x^i_\bullet} \ ,$$

where the $x^i_\bullet$'s are i.i.d. Brownian motions on, say, $\mathbb{R}^3$, and $s^i_\bullet$ are i.i.d. of the form $s^i_\bullet = \exp(y^i_\bullet)$ where $y^i_\bullet$ is a solution to the SDE

$$\mathrm{d}y_t = \mathrm{d}W_t - \tfrac{1}{2}\exp(y_t)\,\mathrm{d}t \ ,$$

driven by a standard Brownian motion $W_\bullet$. (Cf. [96, Eqn. (58)].)

Again in analogy with the case of Υ, the generator of the process $\mu_\bullet$ is constructed as a (kind of) second quantization of the standard Laplacian on $\mathbb{R}^3$, profiting the unitary isomorphism between the Fock space of $\mathcal{P}_{\hat\sigma}$ on $\hat\Upsilon$ and the extended Fock space of $\mathcal{G}_\sigma$ constructed in [94, 95].

1.6 Plan of the work

Let us now summarize some of the results in this thesis in light of the established lexicon.

1.6.1 The results in Chapter 2. The first two chapters are devoted to the study of $\mathcal{D}_\sigma$ measures, mainly over Polish spaces. In light of the program sketched in §1.1, we aim to understand the quasi-invariance properties of $\mathcal{D}_\sigma$ w.r.t. the action (1.3.6) on a general Polish space X, and of $\mathcal{D}_\mathsf{m}$ w.r.t. the action (1.3.8) on a closed manifold M with volume m. This latter goal will however only be achieved in Chapter 5.

One main tool in establishing the (quasi-)invariance of measures on linear spaces is provided by their Fourier transform, or *characteristic functional.* (Cf. e.g. [7, 96, 157, 159].) Whereas $\mathcal{P}_\sigma$ and $\mathcal{G}_\sigma$ are (the laws of) measure-valued Lévy processes, the random measure $\mathcal{D}_\sigma$ does not fall into this class, since the normalization via $\mathbf{N}$ destroys the independent increments property. In particular, it is not possible to compute the characteristic functional $\widehat{\mathcal{D}_\sigma}$ by means of the Lévy–Khintchine formula.

In fact, it is commonly understood that Dirichlet–Ferguson measures belong to a class of ≪*distributions that are difficult to deal with by Fourier transformation, such as relatives of the* [finite-dimensional] *Dirichlet distributions*≫ [79, abstract]. In the specific case of $\mathcal{D}_\sigma$, the main difficulty consists in that the Fourier transform $\widehat{\mathsf{D}_{\boldsymbol{\alpha}_k}}$ of the Dirichlet distribution with parameter $\boldsymbol{\alpha}_k$ — the k-dimensional marginalization of $\mathcal{D}_\sigma$ — coincides with the *confluent* form ${}_k\Phi_2$ of the *Lauricella hypergeometric function of type D*, a kind of k-variate generalization of Gauß' hypergeometric function ${}_2F_1$. (See Dfn. 2.2.3.) In §2.3, we show that

Theorem. *The functional $\widehat{\mathcal{D}_\sigma}$ is the limit for $k \to \infty$ of the discrete $\mathcal{D}_\sigma$-martingale $\big(\widehat{\mathsf{D}_{\boldsymbol{\alpha}_k}}\big)_k$.*

The key observation in order to pass to the limit $k \to \infty$ is that we can rearrange the terms in the infinite-series definition of ${}_k\Phi_2[\boldsymbol{\alpha}_k]$ to express the latter as the exponential generating function of a sequence of *cycle index polynomials Z_n of the symmetric groups* $\mathfrak{S}_n$ (Eq. 2.2.1), computed at suitable expressions in $\boldsymbol{\alpha}_k$. (See Prop. 2.3.5.)

In §2.4, we study the *dynamical symmetry algebra* of the function ${}_k\Phi_2$. The terminology originates in a series of works by M. Ciftan and W. Miller, Jr. (e.g., [30, 121, 122, 123]) concerned with the '*dynamical symmetric group*' of boson-operator realizations of $U(n)$ state vectors. For our purposes however, the dynamical symmetry algebra of ${}_k\Phi_2$ will be the minimal semi-simple Lie algebra $\mathfrak{g}_k$ generated by some integro-differential operators, (See (2.4.10).) acting on ${}_k\Phi_2$ by

$$\begin{aligned} E_{\alpha_i} &\colon {}_k\Phi_2[\boldsymbol{\alpha}] \longmapsto \alpha_i \; {}_k\Phi_2[\boldsymbol{\alpha}+\mathbf{e}_i] \ , \\ E_{-\alpha_i} &\colon {}_k\Phi_2[\boldsymbol{\alpha}] \longmapsto (1-\alpha_1-\cdots-\alpha_k) \; {}_k\Phi_2[\boldsymbol{\alpha}-\mathbf{e}_i] \end{aligned} \ , \qquad i=1,\dots,k \ ,$$

where $\boldsymbol{\alpha} \in \mathbb{R}^k_+$ and $\mathbf{e}_i$ is the i^{th} vector in the canonical basis of $\mathbb{R}^k$.

We show that — on the side of Fourier transforms — the discretization to Δ^{k-1} of the commutative action (1.3.6) extends to a non-commutative action of the semisimple Lie algebra $\mathfrak{g}_k \cong \mathfrak{sl}_{k+1}$. A probabilistic interpretation of this fact is given — on the side of distributions — in terms of the the Bayesian property of $\mathsf{D}_{\boldsymbol{\alpha}}$, corresponding to the computation (Prop. 2.1.5.) of Dirichlet-categorical posteriors. As detailed in §2.1.2, this property of the Dirichlet distributions turns out to be intimately connected with representations of measurable SL_2-currents briefly mentioned above.

1.6.2 The results in Chapter 3. Chapter 3 is a joint work with Prof. E. W. Lytvynov (Swansea University, Swansea, UK). Here, we give an independent proof of Sethuraman's fixed point identity (1.4.2), which we rather understand in integral form as the Mecke-type identity

$$\int_{\mathscr{P}} \mathrm{d}\mathcal{D}_\sigma(\eta) \int_X F(\eta, x, \eta_x)\, \mathrm{d}\eta(x) = \int_{\mathscr{P}} \mathrm{d}\mathcal{D}_\sigma(\eta) \int_X \mathrm{d}\overline{\sigma}(x) \int_I F(\eta^x_r, x, r)\, \mathrm{dB}_{1,\beta}(r) \ , \tag{1.6.1}$$

where $\sigma = \beta\overline{\sigma}$, $F\colon \mathscr{P}\times X\times I \to \mathbb{R}$ is any measurable semi-bounded function, and we set $\eta_x := \eta\{x\}$ and $\eta^x_r := (1-r)\eta + r\delta_x$. Our understanding is derived from the analogous identity (3.1.2) shown by J. Mecke for the Poisson measure $\mathcal{P}_\sigma$. (See [119, Satz 3.1].) Formula (1.6.1) will be an important tool in establishing the closability of the Dirichlet form (1.6.4) in Chapter 5.

1.6.3 The results in Chapter 4. As shown by M. Röckner and A. Schied in [142], the intrinsic distance associated to Albeverio–Kondrat'ev–Röckner's geometry [7] on $\Upsilon = \Upsilon(M)$ is the (extended) L^2-Wasserstein distance. The latter is defined by allowing for μ and ν in (1.3.1) to be arbitrary non-negative (as opposed to: probability) measures[6]. The works by L. Decreusfond [36] and M. Erbar–M. Huesmann [51] confirmed that this choice is also natural with respect to the (curvature) properties of M.

More precisely, Röckner–Schied [142] showed that the Dirichlet form $\big(\mathcal{E}^\Upsilon, \mathscr{D}(\mathcal{E}^\Upsilon)\big)$ defined in (1.2.10) has both the Rademacher and the Sobolev-to-Lipschitz properties defined in §1.1. In Chapter 4 below we advocate that these properties are natural requirements for a to-be "volume measure" on the space $\mathscr{P}_2 = \mathscr{P}_2(M)$. Combining the proof-strategy of [142] with N. Gigli's fine analysis of the metric structure of $\mathscr{P}_2$, we give sufficient assumptions for a probability measure $\mathbb{P}$ on $\mathscr{P}_2$ to satisfy the Rademacher property.

Theorem. *Let $\mathbb{P}$ be a Borel probability measure on $\mathscr{P}_2$, quasi-invariant w.r.t. the action* (1.3.8) *for every $w \in \mathfrak{X}^\infty$, and additionally such that, for all finite $s \leq t$,*

$$\text{for } \mathbb{P}\text{-a.e. } \mu \in \mathscr{P} \qquad \mathrm{Leb}^1\text{-}\operatorname*{essinf}_{r\in[s,t]} R^w_r(\mu) > 0 \ , \qquad \text{where} \quad R^w_r := \frac{\mathrm{d}(\psi^{w,r}.)_\sharp \mathbb{P} \otimes \mathrm{d}r}{\mathrm{d}\mathbb{P}\otimes \mathrm{d}r} \ .$$

Then, the quadratic form

$$\mathcal{E}(u,v) := \int_{\mathscr{P}_2} \mathsf{G}_\mu\big((\boldsymbol{\nabla} u)(\mu), (\boldsymbol{\nabla} v)(\mu)\big)\, \mathrm{d}\mathbb{P}(\mu) \ , \qquad u, v \in \mathcal{FC}^\infty$$

is closable. Its closure $(\mathcal{E}, \mathscr{D}(\mathcal{E}))$ is a regular, strongly local Dirichlet form on $L^2_{\mathbb{P}}(\mathscr{P}_2)$ satisfying the Rademacher property. That is, $u \in \mathscr{D}(\mathcal{E})$ and

$$\mathsf{G}(\boldsymbol{\nabla} u, \boldsymbol{\nabla} u) \leq \mathrm{Lip}[u]^2 \tag{1.6.2}$$

for every $u \in \mathrm{Lip}(\mathscr{P}_2)$.

The rest of Chapter 4 is devoted to the construction of measures satisfying the quasi-invariance assumption in the Theorem, namely: normalized mixed Poisson measures, and the entropic measure and the *Malliavin–Shavgulidze image measure* on $\mathscr{P}_2(\mathbb{S}^1)$. (See Ex. 4.5.18.)

These examples serve two main purposes. Firstly, to show that the Theorem is non-void. Secondly, to comment on the following fact, recently shown by G. De Philippis and F. Rindler in [35]. Namely that, *if μ is a (σ-finite non-negative) measure on $\mathbb{R}^d$ such that every Lipschitz function is differentiable μ-a.e., then $\mu \ll \mathrm{Leb}^d$*. Since the aforementioned example measures are mutually singular, it follows that no analogue of De Philippis–Rindler's result may hold on $\mathscr{P}_2$. (Cf. Rmk. 4.2.9.)

[6] If $\mu X \neq \nu X$, then the set of coupling $\mathrm{Cpl}(\mu,\nu)$ is empty and we let $\inf \varnothing := +\infty$, hence the name 'extended'.

1.6.4 The results in Chapter 5. We show here how the general strategy outlined in §1.1 also applies to the Dirichlet–Ferguson measure $\mathcal{D}_{\mathsf{m}}$ over a closed Riemannian manifold M with volume m and *of dimension* $d \geq 2$.

As it is the case for $(\mathscr{M}^+_{\mathrm{pa}}, \mathcal{G}_{\mathsf{m}})$, the space of purely atomic probability measure $(\mathscr{P}^{\mathrm{pa}}, \mathcal{D}_{\mathsf{m}})$ is hardly approximated by finite-dimensional objects, due to the support properties of Dirichlet–Ferguson measures. In the same spirit as in the treatment of the characteristic functional $\widehat{\mathcal{D}_\sigma}$, we replace standard approximation techniques by martingale approximation techniques. A first step in this direction requires to define a filtration of σ-algebras on $\mathscr{P}_2(M)$ compatible with the geometry of M.

Cylinder functions. In light of [96], we define, for $\hat{f} \in \mathcal{C}^\infty_c(M \times (0,1])$, a (*non-linear*, *non-continuous*) functional on $\mathscr{P}_2$ by setting

$$\hat{f}^\star(\mu) := \int_M \hat{f}(x, \mu_x) \, \mathrm{d}\mu(x) \ ,$$

where $\mu_x := \mu\{x\} > 0$ as in (1.5.2). For each such $\hat{f}$, we define further a *threshold* parameter

$$\varepsilon_{\hat{f}} := \inf_{x \in M} \min \operatorname{supp} \hat{f}(x, \,\cdot\,) > 0 \ .$$

The induced cylinder functions have the form

$$u(\mu) = F\left(\int_M \hat{f}_1(x, \mu_x) \, \mathrm{d}\mu(x), \dots, \int_M \hat{f}_k(x, \mu_x) \, \mathrm{d}\mu(x) \right) \ , \tag{1.6.3}$$

for some $k \in \mathbb{N}_0$, $F \in \mathcal{C}^\infty_c(\mathbb{R}^k)$ and $\hat{f}_1, \dots, \hat{f}_k$ all as above and with threshold $\varepsilon > 0$. It is not difficult to see that every such cylinder function vanishes identically at all measures μ such that $\mu_x \leq \varepsilon$ for every $x \in M$. Informally: *cylinder functions only detect atoms with mass larger than their threshold.*

The canonical Dirichlet form. Further, we let $\widehat{\mathfrak{Z}}^\infty_\varepsilon$ be the algebra of all functions of the form (1.6.3), and $\mathcal{B}_\varepsilon$ be the σ-algebra on $\mathscr{P}_2(M)$ generated by $\widehat{\mathfrak{Z}}^\infty_\varepsilon$. Following [96], it is possible to show that $\mathcal{D}_{\mathsf{m}}$ is partially quasi-invariant (Dfn. 2.1.2) on the filtration $\left(\mathcal{B}_{1/n}\right)_{n \geq 0}$ w.r.t. the action (1.3.8) of $\mathrm{Diff}^\infty(M)$ defining $\boldsymbol{\nabla}$. Combining this result with the Mecke-type identity for $\mathcal{D}_{\mathsf{m}}$ (i.e., the integral version of Sethuraman's characterization (1.4.2) obtained in Chap. 3) yields the first main result of Chapter 5.

Theorem. *The form*

$$\mathcal{E}(u,v) := \int_{\mathscr{P}_2} \mathsf{G}_\mu\big((\boldsymbol{\nabla} u)(\mu), (\boldsymbol{\nabla} v)(\mu) \big) \, \mathrm{d}\mathcal{D}_{\mathsf{m}}(\mu) \ , \qquad u, v \in \bigcup_{\varepsilon > 0} \widehat{\mathfrak{Z}}^\infty_\varepsilon \tag{1.6.4}$$

is closable. Its closure $(\mathcal{E}, \mathscr{D}(\mathcal{E}))$ *is a regular strongly local Dirichlet form on* $\mathscr{P}_2$.

The Theorem is but the starting point for a thorough study of the form (1.6.4), including a description of its carré du champ and generator (Thm. 5.4.11, Rmk. 5.4.12, Prop. 5.5.7), iterated carré du champ (Lem. 5.6.27), and a proof of the Rademacher property in the form (1.6.2) (Prop. 5.5.8).

The associated diffusion process. Since no Fock space representation of $L^2_{\mathcal{D}_{\mathsf{m}}}(\mathscr{P}_2(M))$ is available, it is seemingly not possible to construct the diffusion process associated to $(\mathcal{E}, \mathscr{D}(\mathcal{E}))$ along the same lines as in [96] for the gamma measure $\mathcal{G}_{\mathsf{m}}$. Thus, we develop *ad hoc* methods, relying on three main ingredients.

- The *two-topologies paradigm* of infinite-dimensional analysis. By this we mean that the study of measures on infinite-dimensional spaces often requires dealing with multiple topologies (possibly considered on subspaces). This is foremostly known for the Wiener space and the Cameron–Martin subspace. Other examples include B. K. Driver and M. M. Gordina's *infinite-dimensional Heisenberg* groups (e.g., [47]) and L. Ambrosio, M. Erbar and G. Savaré's study [8] of Cheeger energies on *extended metric-topological spaces*.
- *Quasi-homeomorphisms* of Dirichlet forms: the correct notion of isomorphism for quasi-regular local Dirichlet forms to preserve the associated Markov processes. (See [27].)
- *Kuwae–Shioya's Mosco convergence*: a notion of convergence for Dirichlet (and, more generally, quadratic) forms on *varying* Hilbert spaces, also preserving convergence of the stochastic counterparts (in a suitable sense). (See [90, 99].)

In the present case of $\mathcal{D}_{\mathsf{m}}$, we identify S. N. Ethier and T. G. Kurtz' *weak atomic topology* [52, §2] as the coarsest topology granting the continuity of cylinder functions in $\widehat{\mathfrak{Z}}^\infty_\varepsilon$. This allows us to prove that $(\mathcal{E}, \mathscr{D}(\mathcal{E}))$ is quasi-homeomorphic to a Dirichlet form on an infinite-product space. The latter may be filtered — roughly speaking: by considering truncations of probability measures $\mu = \sum_i^\infty s_i \delta_{x_i}$ to subprobability measures $\mu_n = \sum_i^{N_n} s_i \delta_{x_i}$ such that $s_i \geq 1/n$ for all $i \leq N_n$. Finally, the filtration induces a sequence of Dirichlet forms over finite-dimensional spaces, converging to $(\mathcal{E}, \mathscr{D}(\mathcal{E}))$ in Kuwae–Shioya's generalized Mosco sense.

The final goal to identify the Markov process associated with $(\mathcal{E}, \mathscr{D}(\mathcal{E}))$ is reached as follows.

Theorem. *The Dirichlet form $(\mathcal{E}, \mathscr{D}(\mathcal{E}))$ is associated to a Markov diffusion*

$$\eta_\bullet = \sum_{i \leq N} s_i \delta_{y^i_\bullet} \;, \qquad y^i_t := x^i_{t/s_i} \;,$$

where $x^i_\bullet$ are i.i.d. Brownian motions on $(M, \overline{\mathsf{m}})$. Furthermore, the measure $\mathcal{D}_{\mathsf{m}}$ is a (distinguished) invariant measure for $\eta_\bullet$.

> Since $\mathcal{D}_{\mathsf{m}} = \mathbf{N}_\sharp \mathcal{G}_{\mathsf{m}}$, rather than as a massive particle system we shall interpret a purely atomic probability measure $\mu = \sum_{i \leq N} s_i \delta_{x_i}$ as the *density* of particles in a fluid. The process $\eta_\bullet$ represents then the motion of the particles in the fluid. This motion is
>
> (F_1) *non-interacting* and *free*, that is, each particle moves by Brownian motion, and different particles do not meet;
>
> (F_2) *in dynamical equilibrium*, in the sense that, for every time $t > 0$ and for every region $A \subset M$, the total density $\eta_t A$ of the particles in A coincides in average with the normalized volume $\overline{\mathsf{m}} A$;
>
> (F_3) *turbulent*, in the sense that particles in the fluid rearrange themselves at a speed inversely proportional to their mass.

In mathematical terms: (F_1) is a consequence of the dimension of M: for $d \geq 2$, independent (possibly rescaled) Brownian motions meet with probability 0. (Cf. the capacity estimate Prop. 5.6.24). Property (F_2) holds because of $\mathcal{D}_{\mathsf{m}}$ being invariant for $\eta_\bullet$ and having intensity measure $\overline{\mathsf{m}}$. The 'speed' of particle in (F_3) is the inverse volatility of the corresponding Brownian motion. Because of this property, we do not expect $\eta_\bullet$ to be a *stochastic flow* (in the sense of Y. Le Jan and O. Raimond [105]).

These properties together suggest that the process $\eta_\bullet$ is a multi-dimensional analogue of the Konarovskyi's Modified Massive Arratia Flow: Indeed, as a final result, it is possible to show that $\eta_\bullet$ is a solution to the SPDE

$$\mathrm{d}\eta_t = \mathrm{div}(\sqrt{\eta_t}\, \mathrm{d}W_t) + \tfrac{1}{2} \sum_{x \in \eta_t} \Delta\delta_x \, \mathrm{d}t \ .$$

(Here, $\Delta\delta_x$ is the distributional Laplacian of the measure δ_x. See §5.1.)

Chapter 2
Dirichlet characteristic functionals

In this Chapter, we compute the characteristic functional of the Dirichlet–Ferguson measure over a locally compact Polish space and prove continuous dependence of the random measure on the parameter measure. In finite dimension, we identify the dynamical symmetry algebra of the characteristic functional of the Dirichlet distribution with a simple Lie algebra of type A. We study the lattice determined by characteristic functionals of categorical Dirichlet posteriors, showing that it has a natural structure of weight Lie algebra module and providing a probabilistic interpretation. A partial generalization to the case of the Dirichlet–Ferguson measure is also obtained.

2.1 Introduction and main results

Let X be a locally compact Polish space with Borel σ-algebra $\mathcal{B}$ and let $\mathscr{P}(X)$ be the space of probability measures on $(X, \mathcal{B})$. A $\mathscr{P}(X)$-valued random field P is termed a *Dirichlet–Ferguson process* [55] with *intensity* (*measure*) $\sigma \in \mathscr{P}(X)$ if, for any measurable partition $\mathbf{X} := (X_1, \ldots, X_k)$ of X, the random vector $(PX_1, \ldots, PX_k)$ is distributed according to the Dirichlet distribution with parameter $(\sigma X_1, \ldots, \sigma X_k)$. (See Dfn. 2.2.1.)

For P as above, we term $\mathcal{D}_\sigma := \mathrm{law} P$ the *Dirichlet–Ferguson measure* with intensity σ. We regard $\mathcal{D}_\sigma$ as a probability measure on the linear space $\mathscr{M}_b(X)$ of finite signed measures over $(X, \mathcal{B})$, supported on $\mathscr{P}(X)$. The Dirichlet distribution and the Dirichlet–Ferguson measure have found a wide range of application, including Bayesian non-parametrics [55, 108, 109], genetics [57, 132], representation theory [157, 161], number theory [44, 45].

The characteristic functional of $\mathcal{D}_\sigma$ is commonly recognized as hardly tractable [79] and any approach to $\mathcal{D}_\sigma$ based on characteristic functional methods appears de facto ruled out in the literature. Notably, this led to the introduction of different characterizing transforms (e.g. the *Markov–Krein transform* [86, 159] or the *c-transform* [79, 80]), inversion formulas based on characteristic functionals of other random measures, (in particular, the Gamma measure, as in [138]) and, at least in the case $X = \mathbb{R}$, to the celebrated *Markov–Krein identity.* (See, e.g., [109].)

These investigations are based on complex analysis techniques and integral representations of special functions, in particular the Lauricella hypergeometric function ${}_kF_D$ [103] and Carlson's R function [26]. The novelty in the results presented in this chapter consists in the combinatorial/algebraic approach adopted, allowing for broader generality and far reaching connections, especially with Lie algebra theory.

2.1.1 Fourier analysis. Let $\mathsf{D}_{\boldsymbol{\alpha}_k}$ be the *Dirichlet distribution* on the standard simplex Δ^{k-1} with parameter $\boldsymbol{\alpha}_k \in \mathbb{R}^k_+$. (See Dfn. 2.2.1.) We regard $\mathsf{D}_{\boldsymbol{\alpha}_k}$ as the discretization of $\mathcal{D}_\sigma$ induced by a measurable k-partition $\mathbf{X}_k$ of X. Our first result is the following.

Theorem 2.1.1 (See Thm. 2.3.10). *The characteristic functional $\widehat{\mathcal{D}_\sigma}$ of $\mathcal{D}_\sigma$ is — for suitable sequences of partitions $\mathbf{X}_k$ — the limit of the discrete $\mathcal{D}_\sigma$-martingale $\big(\widehat{\mathsf{D}_{\boldsymbol{\alpha}_k}}\big)_k$. For every continuous compactly supported real-valued f, it satisfies*

$$\widehat{\mathcal{D}_\sigma}(f) := \int_{\mathscr{M}_b(X)} \mathrm{d}\mathcal{D}_\sigma(\eta)\, e^{\mathrm{i}\,\eta f} = \sum_{n=0}^\infty \frac{\mathrm{i}^n}{n!}\, Z_n(\sigma f^1, \dots, \sigma f^n) ,$$

where $\mathrm{i} = \sqrt{-1}$ is the imaginary unit, Z_n is the cycle index polynomial (2.2.1) *of the symmetric group $\mathfrak{S}_n$ and f^j denotes the j^{th} power of f.*

Furthermore, the map $\sigma \mapsto \mathcal{D}_\sigma$ is continuous with respect to the narrow topologies.

The characteristic functional representation is new. It provides — in the unified framework of Fourier analysis:

- a new (although non-explicit) construction of $\mathcal{D}_\sigma$ as the unique probability measure on $\mathscr{P}(X)$ satisfying $\widehat{\mathcal{D}_\sigma} = \lim_k \widehat{\mathsf{D}_{\boldsymbol{\alpha}_k}}$; (See Cor. 2.3.20. Following [161], we call this construction a *weak Fourier limit.*)
- new proofs of known results on the tightness and asymptotics of families of Dirichlet–Ferguson measures, proved, elsewhere in the literature, with ad hoc techniques; (See Cor.s 2.3.13 and 2.3.14. Cf. Rmk. 2.3.12.)
- the continuity statement in the Theorem, which strengthens [146, Thm. 3.2] concerned with norm-to-narrow continuity. This last result is sharp, in the sense that the domain topology cannot be relaxed to the vague topology.

2.1.2 Representations of SL_2-currents. The Dirichlet–Ferguson measure $\mathcal{D}$, the *gamma measure* $\mathcal{G}$ [94, 157] and the *'multiplicative infinite-dimensional Lebesgue measure'* $\mathcal{L}^+$ [157, 161] (both defined below) play an important rôle in a longstanding program [96, 157, 163] for the study of representations of (measurable) $SL_2(\mathbb{R})$-current groups, i.e., spaces of $SL_2(\mathbb{R})$-valued (bounded measurable) functions on a smooth manifold X. In the following, we shall identify some *special linear* objects naturally acting on Dirichlet measures, and translate probabilistic properties of these measures into the language of Lie theory.

We start by briefly recalling the setting and introducing our motivations.

Quasi-invariance and representation theory. Write $h_\sharp\nu := \nu \circ h^{-1}$ for the push-forward of a measure ν via a measurable function h. Consider now a group G acting measurably, freely and transitively on a measurable space $(\Omega, \mathcal{F})$ and write $g.\omega \in \Omega$ for the action of $g \in G$ on $\omega \in \Omega$.

Definition 2.1.2 (Invariance properties). We say that a finite measure ν on $(\Omega, \mathcal{F})$ is

(*i*) *G-quasi-invariant* if for every $g \in G$ there exists a ($\mathcal{F}$-measurable, ν-a.e. finite) Radon–Nikodým derivative $R_g\colon \Omega \to [0, \infty]$ such that $\mathrm{d}(g.)_\sharp\nu(\omega) = R_g(\omega) \cdot \mathrm{d}\nu(\omega)$;

(*ii*) *projectively G-invariant* if, additionally, R_g is constant on Ω for every $g \in G$;

(*iii*) *G-invariant* if, additionally, $R_g \equiv \mathbb{1}$ for every $g \in G$;

(*iv*) *partially G-quasi-invariant* [96, Dfn. 9] if there exists a filtration $(\mathcal{F}_k)_k$ of $\mathcal{F}$ such that

  - $\mathcal{F}$ is the minimal σ-algebra generated by $(\mathcal{F}_k)_k$;
  - for each $g \in G$ and $k \in \mathbb{N}_0$ there exists $n \in \mathbb{N}_0$ such that $g.\mathcal{F}_k \subset \mathcal{F}_n$;

- for each $g \in G$ and $k \in \mathbb{N}_0$ there exists an $\mathcal{F}_k$-measurable function $R_g^{(k)} \colon \Omega \to [0,\infty]$ such that $\mathrm{d}(g.)_\sharp \nu_k(\omega) = R_g^{(k)}(\omega) \cdot \mathrm{d}\nu_k(\omega)$, where ν_k denotes the restriction of ν to $\mathcal{F}_k$.

Notice that *(iii)* $\Longrightarrow$ *(ii)* $\Longrightarrow$ *(i)* $\Longrightarrow$ *(iv)*.

These properties are related to the theory of representations of G. Indeed, each G-quasi-invariant measure ν on Ω induces a so-called 'quasi-regular' unitary representation of G on the Hilbert space $L^2_\nu(\Omega)$ by setting

$$U_\nu(g) \colon f \mapsto (R_g)^{1/2} \cdot f \circ (g^{-1}.) \ .$$

(For the heuristics about partial quasi-invariance, see the Introduction to [96].)

We are interested in the invariance properties of $\mathcal{D}$, $\mathcal{G}$ and $\mathcal{L}^+$ under the following actions.

Two groups of transformations. For $\sigma \in \mathscr{P}(X)$, we define the Abelian Lie algebra $\mathfrak{m} := \mathcal{C}_c(X)$ and its 'σ-traceless' subalgebra $\mathfrak{m}_\sigma := \{f \in \mathfrak{m} \mid \sigma f = 0\}$. The corresponding Abelian Lie groups are the *group of multipliers* $\mathfrak{M} := \{e^f \mid f \in \mathfrak{m}\}$, endowed with the pointwise product, and the subgroup of *σ-traceless multipliers* $\mathfrak{M}_\sigma := \{e^f \mid f \in \mathfrak{m}_\sigma\}$. Both $\mathfrak{M}$ and $\mathfrak{M}_\sigma$ act on $\mathscr{M}_b^+(X)$ by

$$g. \colon \mu \mapsto g \cdot \mu \ , \qquad g \in \mathfrak{M} \ , \ \mu \in \mathscr{M}_b^+(X) \ , \tag{2.1.1}$$

hence the name 'multipliers'. Additionally, we denote by $\mathfrak{S}$ the *group of shifts*, i.e., of bi-measurable transformations of X, and by $\mathfrak{S}_\sigma$ the subgroup of $\mathfrak{S}$ that leaves σ invariant, i.e., $\psi \in \mathfrak{S}_\sigma$ is such that $\psi_\sharp \sigma = \sigma$. Both $\mathfrak{S}$ and $\mathfrak{S}_\sigma$ act naturally on $\mathscr{M}_b^+(X)$, or on $\mathscr{P}(X)$, by

$$\psi. \colon \mu \mapsto \psi_\sharp \mu \ , \qquad \psi \in \mathfrak{S} \ , \ \mu \in \mathscr{M}_b^+(X) \ . \tag{2.1.2}$$

Definition 2.1.3 (Factorizations)**.** Let $\mathbf{N} \colon \mathscr{M}_b^+(X) \setminus \{0\} \to \mathscr{P}(X)$ be the *normalization map* $\mu \mapsto \overline{\mu} := \mu/\mu X$, and let G be a group acting on $\mathscr{M}_b^+(X)$. We say that the action *factors* over the decomposition

$$\mathscr{M}_b^+(X) \setminus \{0\} \cong \mathscr{P}(X) \times \mathbb{R}_+ \tag{2.1.3}$$

if and only if $\mathbf{N} \circ g. = g. \circ \mathbf{N}$ for every $g \in G$. We say that a measure $\mathcal{Q}$ on $\mathscr{M}_b^+(X)$ *factors* (over (2.1.3)) if and only if there exists a Borel measure λ on $\mathbb{R}_+$ such that $\mathcal{Q} = \mathbf{N}_\sharp \mathcal{Q} \otimes \lambda$.

It was shown in [157, Thm. 3.1] that $\mathcal{G}_\sigma := \mathcal{D}_\sigma \otimes e^{-s}\, \mathrm{d}s$ is an $\mathfrak{M}$-quasi-invariant measure on $\mathscr{M}_b^+(X)$, and that $\mathcal{L}^+_\sigma := \mathcal{D}_\sigma \otimes \mathrm{d}s$ is a projectively $\mathfrak{M}$-invariant measure on $\mathscr{M}_b^+(X)$ with Radon–Nikodým derivative $R_g = \exp(-\sigma \ln g)$, and thus it is $\mathfrak{M}_\sigma$-invariant. The importance of $\mathcal{G}$ and $\mathcal{L}^+$ arises from their uniqueness properties w.r.t. these group actions. Indeed, $\mathcal{G}_\sigma$ is the unique measure among the laws of compound Poisson point processes factoring over (2.1.3) (Cf. [157, Cor. 4.2] and [55, §4, Thm. 2].), while $\mathcal{L}^+_\sigma$ is the unique ergodic $(\mathfrak{S}_\sigma \rightthreetimes \mathfrak{M}_\sigma)$-invariant measure equivalent to $\mathcal{G}_\sigma$. (Here, $\mathfrak{S}_\sigma \rightthreetimes \mathfrak{M}_\sigma$ is an appropriate semidirect product of $\mathfrak{S}_\sigma$ and $\mathfrak{M}_\sigma$. See [161, Prop. 4].)

Whereas $\mathcal{D}_\sigma$, $\mathcal{G}_\sigma$ and $\mathcal{L}^+_\sigma$ are trivially $\mathfrak{S}_\sigma$-invariant (by Thm. 2.3.9 below), their $\mathfrak{S}$-(quasi-)invariance does not hold. Indeed, let (X,σ) be a smooth Riemannian manifold with normalized volume σ and $\mathfrak{G} < \mathfrak{S}$ be the group of smooth diffeomorphisms of X. It was shown in [96, §2.4] that $\mathcal{G}_\sigma$ is partially $\mathfrak{G}$-quasi-invariant *not* $\mathfrak{G}$-quasi-invariant. Since the action of (every subgroup of) $\mathfrak{S}$ factors over (2.1.3), the measures $\mathcal{D}_\sigma$ and $\mathcal{L}^+_\sigma$ are *not* $\mathfrak{G}$- (hence not $\mathfrak{S}$-) quasi-invariant as well. However, this property does not transfer immediately to $\mathcal{D}_\sigma$ or $\mathcal{L}^+_\sigma$, since the normalization map is not necessarily measurable on the filtration $(\mathcal{F}_k)_k$ in the definition of partial quasi-invariance. In fact, $\mathcal{D}_\sigma$ too is partially $\mathfrak{G}$-quasi-invariant, (See Chapter 5.) hence so is $\mathcal{L}_\sigma^+$.

When X is a smooth manifold, one main application of the (partial) $\mathfrak{G}$- (or $\mathfrak{G} \leftthreetimes \mathfrak{M}$-) quasi-invariance of these and other random measures is the construction of stochastic dynamics on spaces of measures, for which these random measures are invariant or even ergodic. See e.g., [132, 147] and Chapter 5 for $\mathcal{D}$, [140] for the related *entropic measure* $\mathbb{P}$, [7] for Poisson measures, and [96] for $\mathcal{G}$.

Although inspired by the invariance properties of $\mathcal{L}^+$, in the following we will mostly concentrate on $\mathcal{D}_\sigma$. This is in fact not restrictive, since the discretization of the spaces and measures we are interested in factors over (2.1.3). Whereas Theorem 2.1.1 allows for Bochner–Minlos and Lévy Continuity related results to come into play, the non-multiplicativity of $\widehat{\mathcal{D}_\sigma}$ (corresponding to the non-infinite-divisibility of the measure) immediately rules out the usual approach to quasi-invariance via Fourier transforms [7, 96, 157, 159]. Other approaches to this problem rely on finite-dimensional approximation techniques, variously concerned with approximating the space [140, 141], the σ-algebra [96] or the acting group [65, 161]. The common denominator here is for the approximation to be a filtration (as, e.g., for partial quasi-invariance) — in order to allow for some kind of martingale convergence — and, possibly, for the approximating groups and/or spaces to be (embedded into) linear structures. (Cf., e.g., [140, 161].) In the present case, detailing this approach requires however some preparation.

We shall see in §2.2.3 below how a measurable partition $\mathbf{X}_k := (X_{k,1}, \dots, X_{k,k})$ of X induces a discretization $\mathsf{D}_{\boldsymbol{\alpha}_k}$ of $\mathcal{D}_\sigma$, corresponding to the discretization of X to the space $[k] := \{1, \dots, k\}$. Here, $\boldsymbol{\alpha}_k := (\alpha_{k,1}, \dots, \alpha_{k,k})$ and $\alpha_{k,i} := \sigma X_{k,i}$. Again, since this discretization factors over (2.1.3), the same holds for (the discretizations of) $\mathcal{G}_\sigma$ and $\mathcal{L}_\sigma^+$. Varying $k \in \mathbb{N}$, the family of such discretizations yields the filtration of the $\mathcal{D}_\sigma$-martingale $\big(\widehat{\mathsf{D}_{\boldsymbol{\alpha}_k}}\big)_k$ in Theorem 2.1.1. This is a natural candidate for a filtration $(\mathcal{F}_k)_k$ along which to study the partial $\mathfrak{G}$- or partial $\mathfrak{G} \leftthreetimes \mathfrak{M}_\sigma$-quasi-invariance of $\mathcal{D}_\sigma$. (Notice however that $\mathscr{P}(X)$ is *not* homogeneous for the action of $\mathfrak{M}_\sigma$, thus the $\mathfrak{M}_\sigma$-quasi-invariance of $\mathcal{D}_\sigma$ should be given a precise meaning.)

In the following, we aim to show how the actions of $\mathfrak{G}$ and $\mathfrak{M}$ may be discretized according to the choice of $\mathbf{X}_k$, and to study the quasi-invariance of $\mathsf{D}_{\boldsymbol{\alpha}_k}$ under a general action subsuming the two. We start by recalling the analogous framework for the discretizations of $\mathcal{L}^+$.

Discretizations: The case of $\mathcal{L}^+$. The discretization of the action of $\mathfrak{M}_\sigma$ was given in [161], as we briefly recall now. For $r > 0$, define the $(k-1)$-dimensional *affine sphere of radius r* as

$$M_r^{k-1} := \left\{ \mathbf{s} \in \mathbb{R}_+^k \mid s_1 \cdots s_k = r \right\} .$$

Define the Hadamard product $\mathbf{s} \diamond \mathbf{t} := (s_1 t_1, \dots, s_k t_k)$ and observe that $(M_1^{k-1}, \diamond)$ is a group. Since $\diamond \colon M_1^{k-1} \times M_r^{k-1} \to M_r^{k-1}$, the group M_1^{k-1} acts naturally on M_r^{k-1} for every $r > 0$. It is readily checked that the measure $L_{\boldsymbol{\alpha}_k}$ on M_r^{k-1} with density

$$\mathrm{d}L_{\boldsymbol{\alpha}_k}(\mathbf{y}) = \mathbb{1}_{M_r^{k-1}}(\mathbf{y}) \prod_{i=1}^k \frac{y_i^{\alpha_{k,i}-1}}{\Gamma(\alpha_i)} \, \mathrm{d}y_i \ , \qquad \boldsymbol{\alpha}_k := (\alpha_{k,1}, \dots, \alpha_{k,k}) \ ,$$

is M_1^{k-1}-projectively invariant with Radon–Nikodým derivative

$$R_{\mathbf{s}} := \frac{\mathrm{d}(\mathbf{s}.)_\sharp L_{\boldsymbol{\alpha}_k}}{\mathrm{d}L_{\boldsymbol{\alpha}_k}} = \mathbf{s}^{-\boldsymbol{\alpha}_k} := \prod_{i=1}^k s_i^{-\alpha_{k,i}} \ . \tag{2.1.4}$$

Indeed, $L_{\boldsymbol{\alpha}_k}$ is the discretization of $\mathcal{L}_\sigma^+$ with $\boldsymbol{\alpha}_k$ as above (See [161, Prop. 2].) and, for a suitable sequence of radii $(r_k)_k$, the measure spaces $(M_{r_k}^{k-1}, L_{\boldsymbol{\alpha}_k})$ converge to $(\mathscr{M}_b^+(X), \mathcal{L}_\sigma^+)$ in the weak Fourier sense. (See [161, Thm. 2] for a precise statement.)

The construction of $\mathcal{L}_\sigma^+$ by the aforementioned limiting procedure draws intuition from a parallel with the Maxwell–Poincaré construction of Gaussian measures on $\mathbb{R}^\infty$. (See [161, §2].) In that case, the acting group is the — *non-commutative* — special orthogonal group $SO_k(\mathbb{R})$ and the homogeneous space is the standard sphere $\mathbb{S}_{r_k}^{k-1}$ for some suitable sequence of radii $r_k > 0$.

For $\mathbf{a} \in \mathbb{R}^k$ let now $\operatorname{diag}\mathbf{a}$ be the corresponding diagonal matrix. Let $\mathfrak{h}_{k-1}$ be the diagonal Cartan subalgebra of the real *special linear Lie algebra* $\mathfrak{sl}_k(\mathbb{R})$ of traceless k^2-matrices, and $dSL_k(\mathbb{R})$ be the Abelian Lie group of diagonal matrices with determinant 1. As already noticed in [161], the image of $(M_1^{k-1}, \diamond)$ under diag coincides with the group $dSL_k^+(\mathbb{R})$ of positive definite diagonal matrices with determinant 1, i.e. the connected component of the identity in "the" maximal Abelian subgroup $dSL_k(\mathbb{R})$ of the special linear group $SL_k(\mathbb{R})$. In this language, the Abelian Lie algebra $\mathfrak{m}_\sigma$ of σ-traceless functions is discretized to the Abelian Lie algebra $\mathfrak{h}_{k-1}$ of traceless diagonal k^2-matrices. The resulting acting group is the image $dSL_k^+(\mathbb{R})$ of $\mathfrak{h}_{k-1}$ under the Lie exponential of $\mathfrak{sl}_k(\mathbb{R})$. We summarize the actions and discretizations above in Table 2.1.

In comparison with the Maxwell–Poincaré construction, the following question arises.

Question: Does the action of $M_1^{k-1} \cong dSL_k^+(\mathbb{R})$ on M_r^{k-1} extend to an action of the whole (non-commutative) group $SL_k(\mathbb{R})$? If so, how does the measure $L_{\boldsymbol{\alpha}_k}$ vary under this action?

Table 2.1: Discretizations of multipliers and currents

		∞-dimensional objects			k-discretizations		
		structured	setwise	hom. sp.	structured	setwise	hom. sp.
maximal toral	alg.s	$\mathfrak{m}_\sigma$			$\mathfrak{h}_{k-1}$		
	alg.s	$\mathfrak{m}$	$\mathcal{C}_c(X;\mathbb{R})$	$\mathscr{M}_b^+(X)$	$\operatorname{diag}\mathbb{R}^k$	$\mathbb{R}^{[k]}$	$M_{r_k}^{k-1}$
	grp.s	$\mathfrak{M}_\sigma$			$dSL_k^+(\mathbb{R})$	M_1^{k-1}	
	grp.s	$\mathfrak{M}$	$\mathcal{C}_c(X;\mathbb{R}_+)$		$dGL_k^+(\mathbb{R})$	$\mathbb{R}_+^{[k]}$	
semi-simple	alg.	$\mathcal{C}_c(X, \mathfrak{sl}_2(\mathbb{R}))$		(*)	$\mathfrak{sl}_k(\mathbb{R})$		(*)
	grp.	$\mathcal{C}_c(X, SL_2(\mathbb{R}))$			$SL_k^+(\mathbb{R})$		

* *It seems that it is not possible to consistently identify a homogeneous space for this algebra/group.*

In the following, we will answer in the affirmative — in the conjugate Fourier picture — an analogous question for the simplicial part $\mathsf{D}_{\boldsymbol{\alpha}_k}$ of $L_{\boldsymbol{\alpha}_k}$.

Discretizations: The case of $\mathcal{D}$. For $\boldsymbol{\alpha} \in \mathbb{R}^k_+$, the Fourier transform of $\mathsf{D}_{\boldsymbol{\alpha}}$ satisfies $\widehat{\mathsf{D}_{\boldsymbol{\alpha}}} = {}_k\Phi_2[\boldsymbol{\alpha}]$, the *second Humbert function* or *confluent hypergeometric Lauricella function of type D*. (See Dfn. 2.2.3.) For the purpose of stating the following theorem, let us notice that ${}_k\Phi_2[\boldsymbol{\alpha}]$ is well-defined for every $\boldsymbol{\alpha} \in \mathbb{R}^k$ such that

$$\boldsymbol{\alpha}_\bullet := \alpha_1 + \cdots + \alpha_k \notin \mathbb{Z}_0^- \quad .$$

Let $\boldsymbol{\alpha} \in \operatorname{int} \Delta^{k-1}$ be an interior point of the standard simplex. Set $\Lambda_{\boldsymbol{\alpha}} := \boldsymbol{\alpha} + \mathbb{Z}^k$ and define $\mathcal{O}_{\Lambda_{\boldsymbol{\alpha}}}$ as the real vector space spanned by ${}_k\Phi_2[\boldsymbol{\varepsilon}]$ varying $\boldsymbol{\varepsilon} \in \Lambda_{\boldsymbol{\alpha}}$. Finally, let $\mathfrak{l}_k := \mathfrak{sl}_{k+1}(\mathbb{R})$ with diagonal Cartan subalgebra $\mathfrak{h}_k$. Our second main result is the following.

Theorem 2.1.4 (See Thm. 2.4.14)**.** *Let $\boldsymbol{\alpha} \in \mathbb{R}^k$ be such that $\boldsymbol{\alpha}_\bullet \notin \mathbb{Z}_0^-$. Then, there exists a faithful representation $\rho_{\boldsymbol{\alpha}}$ of $\mathfrak{l}_k$ on $\mathcal{O}_{\Lambda_{\boldsymbol{\alpha}}}$ such that*

(*i*) *$\mathcal{O}_{\boldsymbol{\varepsilon}} := \mathbb{R}_k\Phi_2[\boldsymbol{\varepsilon}]$ is invariant under the action of $\mathfrak{h}_k$ for every $\boldsymbol{\varepsilon} \in \Lambda_{\boldsymbol{\alpha}}$;*

(*ii*) *if additionally $\boldsymbol{\alpha} \in \operatorname{int} \Delta^{k-1}$, then $\mathfrak{h}_k$ acts on $\mathcal{O}_{\boldsymbol{\alpha}}$ by weight $\boldsymbol{\alpha}$;*

(*iii*) *if additionally $\boldsymbol{\alpha}_\bullet \notin \mathbb{Z}$, the representation $\rho_{\boldsymbol{\alpha}}$ extends to a faithful representation on $\mathcal{O}_{\Lambda_{\boldsymbol{\alpha}}}$ of the universal enveloping algebra $\mathfrak{U}(\mathfrak{l}_k)$;*

(*iv*) *the discretization of the action* (2.1.2) *of $\mathfrak{S}$ on $\mathscr{M}_b^+(X)$ is — up to a canonical isomorphism independent of $\boldsymbol{\alpha}$ — the action of a subgroup of the Weyl group W_k of $\mathfrak{l}_k$. (See the proof for details.)*

The theorem provides a rigorous framework for the following *informal* statements. *Up to Fourier transform*:

- assertion (*i*) is the quasi-invariance of $\mathsf{D}_{\boldsymbol{\alpha}}$ under a suitable action of $dSL^+_{k+1}(\mathbb{R})$;
- assertion (*ii*) specifies how the Radon–Nikodým derivative $R_\cdot$ — in algebraic terms, the *weight* of the representation — depends on $\boldsymbol{\alpha}$; (For the dependence on the acting element, see (2.4.12) below.)
- since the action of (non-diagonal elements in) $SL_{k+1}(\mathbb{R})$ leaves $\mathbb{R}\{\mathsf{D}_{\boldsymbol{\varepsilon}}\}_{\boldsymbol{\varepsilon}\in\Lambda_{\boldsymbol{\alpha}}}$ invariant but does not fix $\mathbb{R}\{\mathsf{D}_{\boldsymbol{\alpha}}\}$, assertion (*iii*) specifies iterative applications of the said action;
- assertion (*iv*) describes the discretization of the action of $\mathfrak{S}$ in terms of the Weyl group of $SL_{k+1}(\mathbb{R})$. We summarize this action in Table 2.2;
- together with Theorem 2.1.1, assertion (*i*) yields the partial quasi-invariance of $\mathcal{D}_\sigma$ under the action of traceless multipliers. The filtration in the definition of partial quasi-invariance is exactly the one generated by the martingale $\left(\mathsf{D}_{\boldsymbol{\alpha}_k}\right)_k$, i.e. it is given by the σ-algebras generated by measurable partitions in a monotone null-array of partitions. (See §2.2.3.)

Insights about Theorem 2.1.4 are provided by basic properties of the Fourier transform. Indeed, any discretization of the action of $\mathfrak{M}$ is naturally a *multiplication*. Again informally, the Fourier transform — as opposed to, e.g., the Markov–Krein or the c-transform — maps multiplication by a Lie group element into *differentiation* by the corresponding infinitesimal increment in the Lie algebra of the group. In the case of ${}_k\Phi_2[\boldsymbol{\alpha}] = \widehat{\mathsf{D}_{\boldsymbol{\alpha}}}$, we call the minimal semi-simple Lie algebra generated by these increments the *dynamical symmetry algebra* $\mathfrak{g}_k$ of ${}_k\Phi_2[\boldsymbol{\alpha}]$. The terminology originates in the works [121, 122, 123], concerned with the dynamical symmetry algebras of different Lauricella hypergeometric functions.

Finally, let us notice here that Theorem 2.1.1 allows for a partial generalization of Theorem 2.1.4 to infinite dimensions, the ultimate goal thereof is essentially that to "fill the empty block" in Table 2.2. We shall extensively comment on this point in Remark 2.4.17 below.

Table 2.2: Discretizations of shifts

	Objects			k-discretizations			
	acting obj.	hom. sp.	meas.	acting obj.	homogeneous space		meas.(s)
					structured	setwise	
	$\mathfrak{S}$	X	σ	$\mathfrak{S}_k$	$[k]$		$\boldsymbol{\alpha}$ (*)
		$\mathscr{P}(X)$	$\mathcal{D}_\sigma$		$\mathscr{P}([k])$	Δ^{k-1} (*)	$\mathsf{D}_{\boldsymbol{\alpha}}$
in Fourier				$\mathfrak{S}_k < W_k$		$[k]$ (***)	
				$\mathfrak{h}_k$	$\mathcal{O}_{\boldsymbol{\alpha}}$	$\mathbb{R}_k\Phi_2[\boldsymbol{\alpha}]$	$\boldsymbol{\alpha}$ (**)
				$\mathfrak{U}(\mathfrak{l}_k)$	$\mathcal{O}_{\Lambda_{\boldsymbol{\alpha}}}$	$\mathbb{R}^{\mathbb{Z}^k}$	$\Lambda_{\boldsymbol{\alpha}}$ (**)

* *Under the identification of a measure* $\alpha := \sum_{i=1}^k \alpha_i\delta_i$ *on* $[k]$ *with the vector* $\boldsymbol{\alpha} := (\alpha_1, \ldots, \alpha_k)$*.*

** *Under the correspondence between* $\boldsymbol{\varepsilon}$ *and the weight* $w_{\boldsymbol{\varepsilon}} \in \mathbb{R}^k$ *by which* $\mathfrak{h}_k$ *acts on* $\mathcal{O}_{\boldsymbol{\varepsilon}}$*.*

*** *Understood as the indexing of a basis for the root system of* $\mathfrak{f}_k < \mathfrak{l}_k$*. (See Lem 2.4.7.)*

2.1.3 Bayesian non-parametrics. A *statistical model* on a *sample space* X is any subset $M = \{P_\theta\}_{\theta\in T} \subset \mathscr{P}(X)$. A model M is *parametric* if the *parameter space* T is finite-dimensional, *non-parametric* otherwise. In Bayesian statistics, the parameter θ is modeled as a T-valued random variable Θ. The probability measure $Q := \mathrm{law}\Theta$ is termed a *prior* (*distribution*). Under a Bayesian model, any data W is sampled in two stages, as

$$\begin{aligned} \Theta &\sim Q\ , \\ W_1, W_2, \ldots \mid \Theta &\overset{\text{iid}}{\sim} P_\Theta\ , \end{aligned}$$

and we aim to determine the conditional distribution of Θ given the data, or *posterior* (*distribution*),

$$Q^{\mathbf{w}} := Q[\Theta \in \cdot \mid W_1 = w_1, \ldots, W_n = w_n]\ .$$

In this framework, one remarkable property of Dirichlet measures is the following. (See, e.g., [55, p. 212, property iii°].)

Proposition 2.1.5 (Bayesian property for $\mathsf{D}_{\boldsymbol{\alpha}}$)**.** *Let* $\boldsymbol{\Theta}$ *be a* Δ^{k-1}*-valued random vector,* W *be a* $[k]$*-valued (categorical) random variable, and let* $i \in [k]$*. If the (prior) distribution of* $\boldsymbol{\Theta}$ *is* $\mathsf{D}_{\boldsymbol{\alpha}}$ *and if*

$$\mathbb{P}\{W = i \mid \boldsymbol{\Theta}\} = \Theta_i \quad a.s.\ ,$$

then the posterior distribution of $\boldsymbol{\Theta}$ *given* $W = i$ *is* $\mathsf{D}_{\boldsymbol{\alpha}+\mathbf{e}_i}$*, where* $\mathbf{e}_i$ *is the* i^{th} *vector of the canonical base of* $\mathbb{R}^k$*.*

We term any posterior distribution as in the above proposition a *Dirichlet-categorical posterior*. It is then the content of the proposition that Dirichlet-categorical posteriors are themselves Dirichlet measures with different parameter; that is, Dirichlet measures are *self-conjugate priors*.

This property is implicit in the action of the dynamical symmetry algebra $\mathfrak{g}_k \cong \mathfrak{l}_k$. Indeed, the latter is the minimal semi-simple Lie algebra containing the (nilpotent) *raising* differential operators (See (2.4.10) and Lem. 2.4.8.)

$$E_{\alpha_i} \colon {}_k\Phi_2[\boldsymbol{\alpha}] \longmapsto \alpha_i \ {}_k\Phi_2[\boldsymbol{\alpha} + \mathbf{e}_i] \ , \qquad i \in [k] \ .$$

These correspond, in the conjugate Fourier picture, to take posteriors of $\mathsf{D}_{\boldsymbol{\alpha}}$ given knowledge on the occurrence of categorical random variables in $[k]$ in the sense of Proposition 2.1.5.

Improper priors. Let M be a Bayesian model with parameter Θ and W be some observation sampled from M. It is of high practical interest in statistics to find priors corresponding to known posteriors of Θ given W. By Bayes' formula, any such prior is determined up to a multiplicative constant. If the prior distribution is integrable, then the constant is fixed in such a way that the prior be a probability distribution. If otherwise, the constant is (usually) immaterial, and the prior is termed *improper*.

As a consequence of Theorem 2.1.4, we are able to identify a family of distinguished (possibly improper, hyper-)priors of Dirichlet measures. Indeed, each element E_{α_i} in the Lie algebra $\mathfrak{g}_k$ is paired with a (nilpotent) *lowering* operator $E_{-\alpha_i}$ in the same $\mathfrak{sl}_2$-triple (See Lem. 2.4.12.) and such that (See Lem. 2.4.8.)

$$E_{-\alpha_i} \colon {}_k\Phi_2[\boldsymbol{\alpha}] \longmapsto (1 - \boldsymbol{\alpha}_\bullet) \ {}_k\Phi_2[\boldsymbol{\alpha} - \mathbf{e}_i] \ , \qquad i \in [k] \ .$$

Let $\boldsymbol{\alpha} \in \Delta^{k-1}$ be an interior point of the standard simplex. Set $\Lambda^+_{\boldsymbol{\alpha}} \coloneqq \{\boldsymbol{\varepsilon} \in \Lambda_{\boldsymbol{\alpha}} \mid \boldsymbol{\varepsilon} > 0_\bullet\}$ and define $\mathcal{O}_{\Lambda^+_{\boldsymbol{\alpha}}}$ analogously to $\mathcal{O}_{\Lambda_{\boldsymbol{\alpha}}}$. It is shown in Theorem 2.4.14 that the action of $\mathfrak{g}_k$ on $\mathcal{O}_{\Lambda_{\boldsymbol{\alpha}}}$ fixes $\mathcal{O}_{\Lambda^+_{\boldsymbol{\alpha}}}$. For every $\boldsymbol{\varepsilon} \in \Lambda^+_{\boldsymbol{\alpha}} \setminus \mathbb{R}^k_+$, the function ${}_k\Phi_2[\boldsymbol{\varepsilon}]$ is the Fourier transform of a σ-finite (possibly: finite) measure which we identify as a (non-normalized, possibly: improper) hyper-prior of $\mathsf{D}_{\boldsymbol{\alpha}}$.

Chapter summary. Preliminary results are collected in §2.2, together with the definition and properties of Dirichlet measures and an account of the discretization procedure that we dwell upon in the following. In §2.3 we prove Theorem 2.1.1. As a consequence, by the classical theory of characteristic functionals on linear topological spaces (Cf. e.g., [61, §IV.4] or [160, §IV].) we recover known asymptotic expressions for $\mathcal{D}_{\beta\sigma}$ when $\beta \to 0$ or ∞ is a real parameter (Cor. 2.3.14, cf. [146, p. 311].), propose a *Gibbsean* interpretation thereof (Rem. 2.3.16), and prove analogous expressions for the *entropic measure* $\mathbb{P}^\beta_\sigma$ on compact Riemannian manifolds [151], generalizing the case $X = \mathbb{S}^1$ [140, Prop. 3.14]. In the process of deriving Theorem 2.1.1 we obtain a moment formula for the Dirichlet distribution in terms of the cycle index polynomials Z_n. (Thm. 2.3.3.) In light of *Pólya Enumeration Theory* we interpret this result by means of a coloring problem, §2.4.1. This motivates the study of the dynamical symmetry algebra $\mathfrak{l}_k$ of the Humbert function ${}_k\Phi_2$ resulting in the proof of Theorem 2.1.4. Finally, in §2.4.2 we study the limiting action of the dynamical symmetry algebra $\mathfrak{l}_k$ when k tends to infinity.

Some preliminary results in topology and measure theory are collected in the Appendix.

2.2 Definitions and preliminaries

Notation. Denote by i the imaginary unit, by $\mathbf{G}[a_n](t)$ (resp. by $\mathbf{G}_{\exp}[a_n](t)$) the (exponential) generating function of the sequence $(a_n)_n \subset \mathbb{C}$, computed in the variable t, viz.

$$\mathbf{G}[a_n](t) := \sum_{n\in\mathbb{N}_0} a_n t^n \ , \qquad \text{resp.} \quad \mathbf{G}_{\exp}[a_n](t) := \sum_{n\in\mathbb{N}_0} \frac{a_n}{n!} t^n \ .$$

Whenever not otherwise specified, for $a \in \mathbb{N}_0$ set $a' := a+1$. Let i, k, n be positive integers and set for $1 \le i \le k$ (the position of an element in a vector is stressed by a *left* subscript)

$$\begin{aligned} \mathbf{y} &:= (y_1, \dots, y_k) & \mathbf{e}_i &:= ({}_1 0, \dots, 0, {}_i 1, 0, \dots, {}_k 0) \\ \mathbf{1} &:= ({}_1 1, \dots, {}_k 1) & \mathbf{y}_{\hat{\imath}} &:= (y_1, \dots, y_{i-1}, y_{i+1}, \dots, y_k) \\ \vec{\mathbf{k}} &:= (1, 2, \dots, k) & \mathbf{y}_\bullet &:= y_1 + \dots + y_k \ . \end{aligned}$$

Write $\mathbf{y} > \mathbf{0}$ for $y_1, \dots, y_k > 0$ and analogously for $\mathbf{y} \ge \mathbf{0}$. Further set

$$\begin{aligned} [k] &:= \{1, \dots, k\} & \pi \in \mathfrak{S}_k &:= \{\text{bijections of } [k]\} \\ \mathbf{y}_\pi &:= \big(y_{\pi(1)}, \dots, y_{\pi(k)}\big) & \mathbf{y} \diamond \mathbf{z} &:= (y_1 z_1, \dots, y_k z_k) \\ \mathbf{y}^{\diamond n} &:= \underbrace{\mathbf{y} \diamond \dots \diamond \mathbf{y}}_{n \text{ times}} & \mathbf{y} \cdot \mathbf{z} &:= y_1 z_1 + \dots + y_k z_k \ , \end{aligned}$$

where $\diamond$ denotes the *Hadamard product* and we write $\mathbf{y}^{\diamond \mathbf{z}} = \big(y_1^{z_1}, \dots, y_k^{z_k}\big)$ vs. $\mathbf{y}^{\mathbf{z}} = y_1^{z_1} \cdots y_k^{z_k}$. For $f : \mathbb{C} \to \mathbb{C}$, write

$$f(\mathbf{y}) := f(y_1) \cdots f(y_k) \qquad\qquad f^\diamond(\mathbf{y}) := (f(y_1), \dots, f(y_k)) \ .$$

Denote by Γ the *Euler Gamma function*, by $\langle\alpha\rangle_k := \Gamma(\alpha + k)/\Gamma(\alpha)$ the *Pochhammer symbol* of $\alpha \notin \mathbb{Z}_0^-$, by $\mathrm{B}(y, z) := \Gamma(y)\Gamma(z)/\Gamma(y+z)$, resp. $\mathrm{B}(\mathbf{y}) := \Gamma(\mathbf{y})/\Gamma(\mathbf{y}_\bullet)$, the *Euler Beta function*, resp. its multivariate analogue.

2.2.1 Combinatorial preliminaries.

Set and integer partitions. For a subset $L \subset [n]$ denote by $\tilde{L}$ the ordered tuple of elements in L in the usual order of $[n]$. An *ordered set partition* of $[n]$ is an ordered tuple $\tilde{\mathbf{L}} := (\tilde{L}_1, \tilde{L}_2 \dots)$ of tuples $\tilde{L}_i$ such that the corresponding sets L_i, termed *clusters* or blocks, satisfy $\varnothing \subsetneq L_i \subset [n]$ and $\sqcup_i L_i = [n]$. (By $\sqcup$ we denote the disjoint union.) The order of the tuples in $\tilde{\mathbf{L}}$ is assumed ascending with respect to the cardinalities of the corresponding subsets and, subordinately, ascending with respect to the first element in each tuple. A *set partition* $\mathbf{L}$ of $[n]$ is the family of subsets corresponding to an ordered set partition. This correspondence is bijective. For any set partition write $\mathbf{L} \vdash [n]$ and $\mathbf{L} \vdash_r [n]$ if $\#\mathbf{L} = r$, i.e. if $\mathbf{L}$ has r clusters. A (*integer*) *partition* $\boldsymbol{\lambda}$ *of* n *into* r *parts* (write: $\boldsymbol{\lambda} \vdash_r n$) is an integer solution $\boldsymbol{\lambda} \ge \mathbf{0}$ of the system, $\vec{\mathbf{n}} \cdot \boldsymbol{\lambda} = n$, $\boldsymbol{\lambda}_\bullet = r$; if the second equality is dropped we term $\boldsymbol{\lambda}$ a (*integer*) *partition of* n. (Write: $\boldsymbol{\lambda} \vdash n$.) We always regard a partition in its *frequency representation*, i.e. as the tuple of its ordered frequencies. (Cf. e.g., [12, §1.1].) To a set partition $\mathbf{L} \vdash_r [n]$ one can associate in a unique way a partition $\boldsymbol{\lambda}(\mathbf{L}) \vdash_r n$ by setting $\lambda_i(\mathbf{L}) := \#\{h \mid \#L_h = i\}$.

Permutations and cycle index. A permutation π in $\mathfrak{S}_n$ is said to have *cycle structure* $\boldsymbol{\lambda}$, write $\boldsymbol{\lambda} = \boldsymbol{\lambda}(\pi)$, if λ_i equals the number of cycles in π of length i for each i. Let $\mathfrak{S}_n(\boldsymbol{\lambda}) \subset \mathfrak{S}_n$ be the set of permutations with cycle structure $\boldsymbol{\lambda}$, so that $\mathfrak{S}_n(\boldsymbol{\lambda}(\pi)) = K_\pi$ the conjugacy class of π and $\#\mathfrak{S}_n(\boldsymbol{\lambda}) = M_2(\boldsymbol{\lambda}) \coloneqq n!/(\boldsymbol{\lambda}!\,\vec{\mathbf{n}}^{\boldsymbol{\lambda}})$ [149, Prop. I.1.3.2].

Let now $G < \mathfrak{S}_n$ be any permutation group. The *cycle index polynomial* of G is defined by

$$Z^G(\mathbf{t}) \coloneqq \frac{1}{\#G} \sum_{\pi \in G} \mathbf{t}^{\boldsymbol{\lambda}(\pi)} \ , \ \mathbf{t} \coloneqq (t_1, \dots, t_n) \ .$$

We write $Z_n \coloneqq Z^{\mathfrak{S}_n}$ for the cycle index polynomial of the group $\mathfrak{S}_n$. For $\mathbf{t} \coloneqq (t_1, \dots, t_n)$, and $\mathbf{t}_k \coloneqq (t_1, \dots, t_k)$ with $k \leq n$, it satisfies the identities

$$Z_n(\mathbf{t}) = \frac{1}{n!} \sum_{\boldsymbol{\lambda} \vdash n} M_2(\boldsymbol{\lambda})\, \mathbf{t}^{\boldsymbol{\lambda}} \ , \qquad Z_n\big((a\,\mathbf{1})^{\diamond \vec{\mathbf{n}}} \diamond \mathbf{t}\big) = a^n Z_n(\mathbf{t}) \quad a \in \mathbb{R} \ . \tag{2.2.1}$$

and the recurrence relation

$$Z_n(\mathbf{t}) = \frac{1}{n} \sum_{k=0}^{n-1} Z_k(\mathbf{t}_k)\, t_{n-k} \ , \ Z_0(\varnothing) \coloneqq 1 \ . \tag{2.2.2}$$

2.2.2 The Dirichlet distribution. Denote the *standard*, resp. *corner*, $(k-1)$-dimensional simplex by

$$\Delta^{k-1} \coloneqq \big\{ \mathbf{y} \in \mathbb{R}^k \mid \mathbf{y} \geq \mathbf{0},\, \mathbf{y}_\bullet = 1 \big\} \ , \qquad \Delta_*^{k-1} \coloneqq \big\{ \mathbf{z} \in \mathbb{R}^{k-1} \mid \mathbf{z} \geq \mathbf{0},\, \mathbf{z}_\bullet \leq 1 \big\} \ .$$

Definition 2.2.1 (Dirichlet distribution)**.** We denote by $\mathsf{D}_{\boldsymbol{\alpha}}(\mathbf{y})$ the *Dirichlet distribution with parameter* $\boldsymbol{\alpha} \in \mathbb{R}_+^k$ (e.g., [128]), i.e. the probability measure with density

$$\mathbb{1}_{\Delta^{k-1}}(\mathbf{y})\, \frac{\mathbf{y}^{\boldsymbol{\alpha}-\mathbf{1}}}{\mathrm{B}(\boldsymbol{\alpha})} \tag{2.2.3}$$

with respect to the k-dimensional Lebesgue measure on the hyperplane of equation $\mathbf{y}_\bullet = 1$ in $\mathbb{R}^k$, concentrated on (the interior of) Δ^{k-1}.

Remark 2.2.2. Alternatively, for fixed $\boldsymbol{\alpha} \in \mathbb{R}_+^k$ and any measurable $A \subset \mathbb{R}^{k-1}$,

$$\mathsf{D}_{\boldsymbol{\alpha}}(A) = \frac{1}{\mathrm{B}(\boldsymbol{\alpha})} \int_{\Delta_*^{k-1}} \mathbb{1}_A(\mathbf{z}) \prod_{i=1}^k z_i^{\alpha_i - 1}\, \mathrm{d}\mathbf{z} \quad \text{where} \quad \mathbf{z} \coloneqq (z_1, \dots, z_{k-1}) \ , \ z_k \coloneqq 1 - \mathbf{z}_\bullet \ .$$

Whereas this second description is also common in the literature, the first one makes more apparent property *(ii)* below.

Write '$\sim$' for '*distributed as*' and let $\mathbf{Y}$ be any Δ^{k-1}-valued random vector. The following properties of the Dirichlet distribution are well-known:

(i) *aggregation* (See e.g., [55, p. 211, property i°].) For $i \in [k-1]$ set

$$\mathbf{y}_{+i} \coloneqq (y_1, \dots, y_{i-1}, y_i + y_{i+1}, y_{i+2}, \dots, y_k) \ .$$

Then,

$$\mathbf{Y} \sim \mathsf{D}_{\boldsymbol{\alpha}} \implies \mathbf{Y}_{+i} \sim \mathsf{D}_{\boldsymbol{\alpha}_{+i}} \ . \tag{2.2.4}$$

(ii) *quasi-exchangeability*, or *symmetry.* For all $\pi \in \mathfrak{S}_k$

$$\mathbf{Y} \sim \mathsf{D}_{\boldsymbol{\alpha}} \implies \mathbf{Y}_\pi \sim \mathsf{D}_{\boldsymbol{\alpha}_\pi} \ . \tag{2.2.5}$$

(*iii*) *Bayesian property.* (Iterative generalization of Prop. 2.1.5.) Let $\mathbf{W} \in [k]^r$ be a vector of $[k]$-valued (categorical) random variables and $\mathbf{P} \in \mathbb{N}_0^k$ be the vector of occurrences $P_i := \#\{j \in [r] \mid W_j = i\}$. For $\mathbf{p} \in \mathbb{N}_0^k$ let $\mathbf{Y}$ be such that $\mathbb{P}\{P_i = p_i \mid \mathbf{Y}\} = Y_i$ for all $i \in [k]$ and denote by $\mathsf{D}_{\boldsymbol{\alpha}}^{\mathbf{p}}$ the distribution of $\mathbf{Y}$ given $\mathbf{P} = \mathbf{p}$, termed here the *posterior distribution of* $\mathsf{D}_{\boldsymbol{\alpha}}$ *given atoms with masses* p_i *at points* $i \in [k]$. Then,

$$\mathbf{Y} \sim \mathsf{D}_{\boldsymbol{\alpha}} \implies \mathsf{D}_{\boldsymbol{\alpha}}^{\mathbf{p}} = \mathsf{D}_{\boldsymbol{\alpha}+\mathbf{p}} \ . \tag{2.2.6}$$

Most properties of the Dirichlet distribution may be inferred from its characteristic functional. We recall its definition below.

Definition 2.2.3 (*Confluent* ${}_kF_D$ or (*second*) *Humbert function* ${}_k\Phi_2$ [53, §2.1]). For $\mathbf{b}, \mathbf{s} \in \mathbb{C}^k$, $a \in \mathbb{C}$ and $c \in \mathbb{C} \setminus \mathbb{Z}_0^-$ the *k-variate Lauricella hypergeometric function of type D*, write ${}_kF_D$, is

$$\begin{aligned} {}_kF_D[a, \mathbf{b}; c; \mathbf{s}] :=& \sum_{\mathbf{m} \in \mathbb{N}_0^k} \frac{\langle a \rangle_{\mathbf{m}_\bullet} \langle \mathbf{b} \rangle_{\mathbf{m}} \mathbf{s}^{\mathbf{m}}}{\langle c \rangle_{\mathbf{m}_\bullet} \mathbf{m}!} && \|\mathbf{s}\|_\infty < 1 \\ =& \frac{1}{\mathrm{B}(a, c-a)} \int_0^1 t^{a-1} (1-t)^{c-a-1} (\mathbf{1} - t\mathbf{s})^{-\mathbf{b}} \, \mathrm{d}t && \Re c > \Re a > 0 \ . \end{aligned}$$

For $\mathbf{b}, \mathbf{s} \in \mathbb{C}^k$, its *confluent* form, or *second k-variate Humbert function*, write ${}_k\Phi_2$, is

$${}_k\Phi_2[\mathbf{b}; c; \mathbf{s}] := \lim_{\varepsilon \to 0^+} {}_kF_D[1/\varepsilon; \mathbf{b}; c; \varepsilon\mathbf{s}] = \sum_{\mathbf{m} \in \mathbb{N}_0^k} \frac{\langle \mathbf{b} \rangle_{\mathbf{m}} \mathbf{s}^{\mathbf{m}}}{\langle c \rangle_{\mathbf{m}_\bullet} \mathbf{m}!} \qquad c \in \mathbb{C} \setminus \mathbb{Z}_0^- \ . \tag{2.2.7}$$

Notice that the distribution $\mathsf{D}_{\boldsymbol{\alpha}}$ is moment determinate for any $\boldsymbol{\alpha} > \mathbf{0}$ by compactness of Δ^{k-1}. Its moments are straightforwardly computed via the multinomial theorem as

$$\mu_n'[\mathbf{s}, \boldsymbol{\alpha}] := \int_{\Delta^{k-1}} (\mathbf{s} \cdot \mathbf{y})^n \, \mathrm{d}\mathsf{D}_{\boldsymbol{\alpha}}(\mathbf{y}) = \sum_{\substack{\mathbf{m} \in \mathbb{N}_0^k \\ \mathbf{m}_\bullet = n}} \binom{n}{\mathbf{m}} \mathbf{s}^{\mathbf{m}} \tfrac{\mathrm{B}(\boldsymbol{\alpha}+\mathbf{m})}{\mathrm{B}(\boldsymbol{\alpha})} = \frac{n!}{\langle \boldsymbol{\alpha}_\bullet \rangle_n} \sum_{\substack{\mathbf{m} \in \mathbb{N}_0^k \\ \mathbf{m}_\bullet = n}} \frac{\mathbf{s}^{\mathbf{m}}}{\mathbf{m}!} \langle \boldsymbol{\alpha} \rangle_{\mathbf{m}} \ , \tag{2.2.8}$$

so that the characteristic functional of the distribution indeed satisfies (Cf. [53, §7.4.3])

$$\widehat{\mathsf{D}_{\boldsymbol{\alpha}}}(\mathbf{s}) := \int_{\Delta^{k-1}} \exp(\mathrm{i}\mathbf{s} \cdot \mathbf{y}) \, \mathrm{d}\mathsf{D}_{\boldsymbol{\alpha}}(\mathbf{y}) = \sum_{\mathbf{m} \in \mathbb{N}_0^k} \frac{\langle \boldsymbol{\alpha} \rangle_{\mathbf{m}}}{\mathbf{m}!} \frac{\mathrm{i}^{\mathbf{m}_\bullet} \mathbf{s}^{\mathbf{m}}}{\langle \boldsymbol{\alpha}_\bullet \rangle_{\mathbf{m}_\bullet}} =: {}_k\Phi_2[\boldsymbol{\alpha}; \boldsymbol{\alpha}_\bullet; \mathrm{i}\mathbf{s}] \ . \tag{2.2.9}$$

2.2.3 The Dirichlet–Ferguson measure.

Notation. Everywhere in the following let $(X, \tau(X))$ be a *second countable locally compact Hausdorff* topological space with Borel σ-algebra $\mathcal{B}$. We denote respectively by $\mathrm{cl}A$, $\mathrm{int}\,A$, $\mathrm{bd}A$ the *closure*, *interior* and *boundary* of a set $A \subset X$ with respect to τ. Recall (Prop. 2.2.4) that any space $(X, \tau(X))$ as above is Polish, i.e. there exists a metric d, metrising τ, such that (X, d) is separable and complete; we denote by $\mathrm{diam}A$ the *diameter* of $A \subset X$ with respect to any such metric d (apparent from context and thus omitted in the notation).

Denote by $\mathcal{C}_c(X)$ (resp. $\mathcal{C}_b(X)$) the space of continuous compactly supported (resp. continuous bounded) functions on $(X, \tau(X))$, (both) endowed with the topology of uniform convergence; by $\mathcal{C}_0(X)$ the completion of $\mathcal{C}_c(X)$, i.e. the space of continuous functions on X vanishing at infinity; by $\mathscr{M}_b(X)$ (resp. $\mathscr{M}_b^+(X)$) the space of *finite*, signed (resp. non-negative) *Radon* measures on $(X, \mathcal{B})$ — the topological dual of $\mathcal{C}_c(X)$ and $\mathcal{C}_0(X)$ — endowed with the the vague topology $\tau_v(\mathscr{M}_b(X))$, i.e. the weak* topology, and the induced Borel σ-algebra. Denote further by $\mathscr{P}(X) \subset \mathscr{M}_b^+(X)$ (Cf. Cor. 2.5.3) the space of probability measures on $(X, \mathcal{B})$.

If not otherwise stated, we assume $\mathscr{P}(X)$ to be endowed with the vague topology $\tau_v(\mathscr{P}(X))$ and σ-algebra $\mathcal{B}_v(\mathscr{P}(X))$. On $\mathscr{M}_b^+(X)$ (resp. on $\mathscr{P}(X)$) we additionally consider the narrow topology $\tau_n(\mathscr{M}_b^+(X))$ (resp. $\tau_n(\mathscr{P}(X))$), i.e. the topology induced by duality with $\mathcal{C}_b(X)$. Finally, given any measure $\nu \in \mathscr{M}_b(X)$ and any bounded measurable function g on $(X, \mathcal{B})$, denote by νg the expectation of g with respect to ν and by $g^* \colon \nu \mapsto \nu g$ the linear functional induced by g on $\mathscr{M}_b(X)$ via integration.

The following statement is well-known. (See e.g., [85, Thm. 5.3].) A proof is sketched to establish further notation.

Proposition 2.2.4. *A topological space $(X, \tau(X))$ is second countable locally compact Hausdorff if and only if it is locally compact Polish, i.e. such that $\tau(X)$ is a locally compact separable completely metrizable topology on X. Moreover, if $(X, \mathcal{B})$ additionally admits a fully supported diffuse measure ν, then $(X, \tau(X))$ is perfect, i.e. it has no isolated points.*

Sketch of proof. Let $(\alpha X, \tau(\alpha X))$ denote the Alexandrov compactification of $(X, \tau(X))$ and $\alpha\colon X \to \alpha X$ denote the associated embedding. Notice that αX is Hausdorff, for X is locally compact Hausdorff; hence αX is metrizable, for it is second countable compact Hausdorff, and separable, for it is second countable metrizable, thus Polish by compactness. Finally, recall that X is (homeomorphic via α to) a G_δ-set in αX and every G_δ-set in a Polish space is itself Polish. The converse and the statement on perfectness are trivial. ■

Partitions. Fix $\sigma \in \mathscr{P}(X)$. We denote by $\mathfrak{P}_k(X)$ the family of measurable non-trivial k-partitions of $(X, \mathcal{B}, \sigma)$, i.e. the set of tuples $\mathbf{X} \coloneqq (X_1, \dots, X_k)$ such that

$$X_i \in \mathcal{B}\ ,\ \sigma X_i > 0\ ,\ X_i \cap X_j = \varnothing \quad i,j \in [k], i \neq j\ ,\ \cup_{i\in[k]} X_i = X\ .$$

Given $\mathbf{X} \in \mathfrak{P}_k(X)$ we say that it *refines* A in $\mathcal{B}$ if $X_i \subset A$ whenever $X_i \cap A \neq \varnothing$, respectively that it is a *continuity partition for* σ if $\sigma(\mathrm{bd}X_i) = 0$ for all $i \in [k]$. We denote by $\mathfrak{P}_k(A \subset X)$, resp. $\mathfrak{P}_k(X, \tau(X), \sigma)$ the family of all such partitions. Given $\mathbf{X}_1 \in \mathfrak{P}_{k_1}(X)$ and $\mathbf{X}_2 \in \mathfrak{P}_{k_2}(X)$ with $k_1 < k_2$ we say that $\mathbf{X}_2$ *refines* $\mathbf{X}_1$, write $\mathbf{X}_1 \preceq \mathbf{X}_2$, if for every $i \in [k_2]$ there exists $j_i \in [k_1]$ such that $X_{2,i} \subset X_{1,j_i}$. A sequence $(\mathbf{X}_h)_h$ of partitions $\mathbf{X}_h \in \mathfrak{P}_{k_h}(X)$ is termed a *monotone null-array* if $\mathbf{X}_{h+1} \preceq \mathbf{X}_h$ and $\lim_h \max_{i\in[k_h]} \mathrm{diam} X_{h,i} = 0$. (Recall that $\mathrm{diam} X_{h,i}$ vanishes independently of the chosen metric on $(X, \tau(X))$, cf. [83, §2.1].) We denote the family of all such null-arrays by $\mathfrak{Na}(X)$. Analogously to partitions, we write with obvious meaning of the notation $\mathfrak{Na}(A \subset X)$ and $\mathfrak{Na}(X, \tau(X), \sigma)$. If σ is *diffuse* (i.e. atomless), then $\lim_h \sigma X_{h,i_h} = 0$ for every choice of $X_{h,i_h} \in \mathbf{X}_h$ with $(\mathbf{X}_h)_h \in \mathfrak{Na}(X)$.

Given a (real-valued) simple function f and a partition $\mathbf{X} \in \mathfrak{P}_k(X)$, we say that f is *locally constant on* $\mathbf{X}$ *with values* $\mathbf{s}$ if $f\big|_{X_i} \equiv s_i$ constantly for every $X_i \in \mathbf{X}$. Given a function f in $\mathcal{C}_c(X)$ we say that a sequence of (measurable) simple functions $(f_h)_h$ is a *good approximation* of f if $|f_h| \uparrow_h |f|$ and $\lim_h f_h = f$ pointwise. The existence of good approximations is standard. (See e.g., [42, Prop. III.3.1].)

The Dirichlet–Ferguson measure. By a *random probability* over $(X, \mathcal{B})$ we mean any probability measure on $\mathscr{P}(X)$. For $\mathbf{X} \in \mathfrak{P}_k(X)$ and η in $\mathscr{P}(X)$ set $\eta^\diamond \mathbf{X} \coloneqq (\eta X_1, \dots, \eta X_k)$ and

$$\begin{aligned} \mathrm{ev}^{\mathbf{X}} \colon \mathscr{P}(X) &\longrightarrow \Delta^{k-1} \subset \mathbb{R}^k \\ \eta &\longmapsto \eta^\diamond \mathbf{X}\ . \end{aligned}$$

Recall (Cf. [148]) that, if $\sigma \in \mathscr{P}(X)$ is diffuse, then for every $k \in \mathbb{N}_1$ and $\mathbf{y} \in \mathrm{int}\, \Delta^{k-1}$ there exists $\mathbf{X} \in \mathfrak{P}_k(X)$ such that $\sigma^\diamond \mathbf{X} = \mathbf{y}$.

Definition 2.2.5 (Dirichlet–Ferguson measure)**.** Fix $\beta > 0$ and $\sigma \in \mathscr{P}(X)$. The *Dirichlet–Ferguson measure $\mathcal{D}_{\beta\sigma}$ with intensity $\beta\sigma$* [55, §1, Def. 1] (also: *Dirichlet* [108], *Poisson–Dirichlet* [161], *Fleming–Viot with parent-independent mutation* [54]; see e.g., [145, §2] for an explicit construction) is the unique random probability over $(X, \mathcal{B})$ such that

$$\mathrm{ev}^{\mathbf{X}}_{\sharp}\mathcal{D}_{\beta\sigma} = \mathsf{D}_{\beta\mathrm{ev}^{\mathbf{X}}\sigma} \ , \ \mathbf{X} \in \mathfrak{P}_k(X) \ , \ k \in \mathbb{N}_1 \ . \tag{2.2.10}$$

(Recall that $\sigma^\diamond\mathbf{X} > \mathbf{0}$.) More explicitly, for every bounded measurable function $u\colon \Delta^{k-1} \to \mathbb{R}$

$$\int_{\mathscr{P}(X)} u(\eta^\diamond\mathbf{X}) \,\mathrm{d}\mathcal{D}_{\beta\sigma}(\eta) = \int_{\Delta^{k-1}} u(\mathbf{y}) \,\mathrm{d}\mathsf{D}_{\beta\sigma^\diamond\mathbf{X}}(\mathbf{y}) \ . \tag{2.2.11}$$

Existence was originally proved in [55] by means of Kolmogorov's Extension Theorem, using the aggregation property of Dirichlet distributions to establish the consistency condition. (Cf. Fig. 2.1 below.) A construction on spaces more general than in our assumptions is given in [89]. Other characterizations are available. (See e.g., [145].) Since X is Polish (Prop. 2.2.4), in (2.2.11) it is in fact sufficient to consider u continuous with $|u| < 1$ and, by the Portmanteau Theorem, $\mathbf{X} \in \mathfrak{P}_k(X, \tau(X), \sigma)$. (Cf. e.g., [151, p. 15].)

Let P be a $\mathscr{P}(X)$-valued random field on a probability space $(\Omega, \mathscr{F}, \mathbb{P})$ and recall the following properties of $\mathcal{D}_\sigma$, to be compared with those of $\mathsf{D}_{\boldsymbol{\alpha}}$,

i. *realization properties*: If $P \sim \mathcal{D}_{\beta\sigma}$, then $P(\omega) = \sum_{i\in J} \eta_i(\omega)\delta_{x_i(\omega)}$ is $\mathbb{P}$-a.s. purely atomic (Here: $J \subset \mathbb{N}_0$. See [55, §4, Thm. 2].) with $\mathrm{supp}P(\omega) = \mathrm{supp}\sigma$. (See [55, §3, Prop. 1] or [115].) In particular, if σ is diffuse and fully supported, then J is countable and $\{x_i\}_i$ is $\mathbb{P}$-a.e. dense in X. The sequence $(\eta_i)_i$ is distributed according to the *stick-breaking process*. In particular, $\mathbb{E}\eta_i = \beta^{i-1}/(1+\beta)^i$. (See [67].) The r.v.'s x_i's are i.i.d. (independent also of the η_i's [52]) and σ-distributed.

ii. *σ-symmetry*: for every measurable σ-preserving map $\psi\colon X \to X$, i.e. such that $\psi_\sharp\sigma = \sigma$,

$$P \sim \mathcal{D}_\sigma \implies \psi_\sharp P \sim \mathcal{D}_\sigma \ . \tag{2.2.12}$$

(Consequence of [83, Lem. 9.0] together with (2.2.10) and the quasi-exchangeability of $\mathsf{D}_{\boldsymbol{\alpha}}$) In particular, $P^\diamond\mathbf{X}$ is distributed as a function of $\sigma^\diamond\mathbf{X}$ for every $\mathbf{X} \in \mathfrak{P}_k(X)$ for every k.

iii. *Bayesian property* [55, §3, Thm. 1]: Let $\mathbf{W} := (W_1, \dots, W_r)$ be a sample of size r from P, conditionally i.i.d., and denote by $\mathcal{D}^{\mathbf{W}}_\sigma$ the distribution of P given $\mathbf{W}$, termed the *posterior distribution of $\mathcal{D}_\sigma$ given atoms* $\mathbf{W}$. Then,

$$P \sim \mathcal{D}_\sigma \implies (P \mid \mathbf{W}) \sim \mathcal{D}_{\sigma+\sum_j^r \delta_{W_j}} \ .$$

Discretizations. In order to consider finite-dimensional marginalizations of $\mathcal{D}_{\beta\sigma}$, we introduce the following discretization procedure. (Cf. [139] for a similar construction.) Any partition $\mathbf{X} \in \mathfrak{P}_k(X)$ induces a discretization of X to $[k]$ by collapsing $X_i \in \mathbf{X}$ to an arbitrary point in X_i, uniquely identified by its index $i \in [k]$, i.e. via the map $\mathrm{pr}^{\mathbf{X}}\colon X \supset X_i \ni x \mapsto i \in [k]$. The finite σ-algebra $\sigma_0(\mathbf{X})$ generated by $\mathbf{X}$ induces then a discretization of $\mathscr{P}(X)$ to the space $\mathscr{P}([k])$ via the mapping $\mu \mapsto \sum_i \mu X_i \,\delta_i$. Since the latter space is in turn homeomorphic to the standard simplex Δ^{k-1} via the mapping $\sum_i y_i\delta_i \mapsto \mathbf{y}$, every choice of $\mathbf{X} \in \mathfrak{P}_k(X)$ induces a discretization of $\mathscr{P}(X)$ to Δ^{k-1} via the resulting composition $\mathrm{ev}^{\mathbf{X}} = \mathrm{pr}^{\mathbf{X}}_\sharp$. It is then precisely the content of (2.2.10) that any partition $\mathbf{X}$ as above induces a discretization of the tuple $\big((X,\sigma), (\mathscr{P}(X), \mathcal{D}_{\beta\sigma})\big)$ to the tuple $\big(([k], \boldsymbol{\alpha}), (\Delta^{k-1}, \mathsf{D}_{\boldsymbol{\alpha}})\big)$, where $\boldsymbol{\alpha} := \beta\mathrm{ev}^{\mathbf{X}}\sigma$ is identified with the measure $\sum_i \alpha_i\delta_i$ on $[k]$. (Cf. Fig. 2.1 below.)

Moving further in this fashion, the subgroup $\mathfrak{S}_{\mathbf{X}}$ of bi-measurable isomorphisms ψ of $(X, \mathcal{B})$ respecting $\mathbf{X}$, i.e. such that $\psi^\diamond(\mathbf{X}) := (\psi(X_1), \dots, \psi(X_k))$ coincides with $\mathbf{X}$ up to reordering, is naturally isomorphic to the symmetric group $\mathfrak{S}_k$, the bi-measurable isomorphism group $\mathfrak{S}([k])$ of $[k]$. The canonical action of $\mathfrak{S}_{\mathbf{X}}$ on X, corresponding to the canonical action of $\mathfrak{S}_k$ on $[k]$, lifts to the action of $\mathfrak{S}_k$ on Δ^{k-1} by permutation of its vertices, that is, to the action on $\mathscr{P}([k])$ defined by $\pi.\mathbf{y} := \pi_\sharp \mathbf{y}$ under the identification of $\mathbf{y}$ with the measure $\sum_i y_i \delta_i$.

2.3 Proof of Theorem 2.1.1 and accessory results

2.3.1 Finite-dimensional statements. Thinking of $\boldsymbol{\alpha}$ as a measure on $[k]$ as in §2.2, the aggregation property (2.2.4) may be given a measure-theoretical interpretation too. Indeed with the same notation of §2.2.2, for $i \in [k-1]$ let additionally $\mathfrak{s}^i \colon [k] \to [k-1]$ denote the i^{th} degeneracy map of $[k]$, i.e. the unique weakly order preserving surjection such that $\#(\mathfrak{s}^i)^{-1}(i) = 2$. Then, up to the usual identification of Δ^{k-1} with $\mathscr{P}([k])$, it holds that $\mathfrak{s}^i_\sharp \mathbf{y} = \mathbf{y}_{+i}$ and one has $\mathfrak{s}^i_\sharp \mathbf{Y} \sim \mathbf{Y}_{+i}$. Thus, choosing $\mathbf{Y} \sim \mathsf{D}_{\boldsymbol{\alpha}}$, the aggregation property reads $(\mathfrak{s}^i_\sharp)_\sharp \mathsf{D}_{\boldsymbol{\alpha}} = \mathsf{D}_{\mathfrak{s}^i_\sharp \boldsymbol{\alpha}}$.

The following result is a rather obvious generalization of the latter fact, obtained by substituting degeneracy maps with arbitrary maps. We provide a proof for completeness.

Proposition 2.3.1 (Mapping Theorem for $\mathsf{D}_{\boldsymbol{\alpha}}$)**.** *Fix $\boldsymbol{\alpha} \in \mathbb{R}^k_+$. Then, for every $g \colon [k] \to [k]$*

$$(g_\sharp)_\sharp \mathsf{D}_{\boldsymbol{\alpha}} = \mathsf{D}_{g_\sharp \boldsymbol{\alpha}} \ .$$

Proof. Define the *additive contraction* $\mathbf{y}_{+\boldsymbol{\lambda}}$ of a vector $\mathbf{y}$ with respect to $\boldsymbol{\lambda} \vdash k$ as

$$\begin{aligned} \mathbf{y}_{+\boldsymbol{\lambda}} :=& (\underbrace{y_1, \dots, y_{\lambda_1}}_{\lambda_1}, \underbrace{y_{\lambda_1+1} + y_{\lambda_1+2}, \dots, y_{\lambda_1+2\lambda_2-1} + y_{\lambda_1+2\lambda_2}}_{2\lambda_2}, \dots, \\ & \underbrace{y_{\vec{\mathbf{k}}\cdot\boldsymbol{\lambda}-k\lambda_k+1} + \dots + y_{\vec{\mathbf{k}}\cdot\boldsymbol{\lambda}-(k-1)\lambda_k}, \dots, y_{\vec{\mathbf{k}}\cdot\boldsymbol{\lambda}-\lambda_k+1} + \dots + y_{\vec{\mathbf{k}}\cdot\boldsymbol{\lambda}}}_{k\lambda_k}) \ , \end{aligned} \tag{2.3.1}$$

whence inductively applying (2.2.4) to any Δ^{k-1}-valued random variable $\mathbf{Y}$ yields $\mathbf{Y} \sim \mathsf{D}_{\boldsymbol{\alpha}} \implies \mathbf{Y}_{+\boldsymbol{\lambda}} \sim \mathsf{D}_{\boldsymbol{\alpha}_{+\boldsymbol{\lambda}}}$ for $\boldsymbol{\lambda} \vdash k$. Combining the latter with the quasi-exchangeability (2.2.5), $\mathsf{D}_{\boldsymbol{\alpha}}$ satisfies

$$\mathbf{Y} \sim \mathsf{D}_{\boldsymbol{\alpha}} \implies (\mathbf{Y}_\pi)_{+\boldsymbol{\lambda}} \sim \mathsf{D}_{(\boldsymbol{\alpha}_\pi)_{+\boldsymbol{\lambda}}} \qquad \pi \in \mathfrak{S}_k \ , \ \boldsymbol{\lambda} \vdash k \ . \tag{2.3.2}$$

For $\boldsymbol{\lambda} \vdash k$ set $\lambda_0 := 0$ and define the map $\star\boldsymbol{\lambda} \colon [k] \to [|\boldsymbol{\lambda}|]$ by

$$\star\boldsymbol{\lambda} \colon i \mapsto \lambda_{j-1} + \lceil i/j \rceil \quad \text{if} \quad i \in \{(j-1)\lambda_{j-1}+1, \dots, j\lambda_j\}$$

varying j in $[k]$, where $\lceil \alpha \rceil$ denotes the ceiling of α. It is readily checked that $(\star\boldsymbol{\lambda} \circ \pi)_\sharp \boldsymbol{\alpha} = (\boldsymbol{\alpha}_\pi)_{+\boldsymbol{\lambda}}$ for any π in $\mathfrak{S}_k$. The proof is completed by exhibiting, for fixed $g \colon [k] \to [k]$, the unique partition $\boldsymbol{\lambda}_g \vdash k$ and some permutation $\pi_g \in \mathfrak{S}_k$ such that $g = \star\boldsymbol{\lambda}_g \circ \pi_g$. To this end set $L_{g,(i)} := g^{-1}(i)$ and

$\mathbf{L}_g := \big(L_{g,(1)}, \dots, L_{g,(k)}\big)$, where it is understood that $L_{g,(i)}$ is omitted if empty;

$\tilde{\mathbf{L}}_g := (\tilde{L}_{1,1}, \tilde{L}_{1,2}, \dots, \tilde{L}_{2,1}, \dots)$ the ordered set partition associated to $\mathbf{L}_g$, where

$\tilde{L}_{j,r} := (\ell_{j,r,1}, \dots, \ell_{j,r,j})$ denotes the r^{th} tuple of cardinality j in $\tilde{\mathbf{L}}_g$;

moreover, varying j in $[k]$ and r in $\lfloor k/\lambda_j \rfloor$, where $\lfloor \alpha \rfloor$ denotes the floor of α, define π in $\mathfrak{S}_k$ by

$$\pi\colon i \mapsto \ell_{j,\,r,\,(i-\lambda_{j-1}-1 \mod j)+1} \quad \text{if} \quad \begin{cases} i \in \{(j-1)\lambda_{j-1}+1, \dots, j\lambda_j\} \\ \lceil (i-\lambda_{j-1}-1)/\lambda_j \rceil = r \end{cases} .$$

Finally set $\pi_g := \pi^{-1}$ and $\boldsymbol{\lambda}_g := \boldsymbol{\lambda}(\mathbf{L}_g)$. ■

Remark 2.3.2. Assuming the point of view of conditional expectations rather than that of marginalizations, (2.2.10) may be restated as

$$\mathbb{E}_{\mathcal{D}_{\beta\sigma}}[\,\cdot\,|\sigma_0(\mathbf{X})] = \mathbb{E}_{\mathsf{D}_{\beta\sigma^{\diamond}\mathbf{X}}}[\,\cdot\,] ,$$

where $\sigma_0(\mathbf{X})$ denotes as before the σ-algebra generated by some partition $\mathbf{X} \in \mathfrak{P}_k(X)$. The aggregation property (2.2.4) is but an instance of the tower property of conditional expectations, whereas its generalization (2.3.2) is a consequence of the σ-symmetry of $\mathcal{D}_\sigma$.

Theorem 2.3.3 (Moments of $\mathsf{D}_{\boldsymbol{\alpha}}$)**.** *Fix $\boldsymbol{\alpha} > \mathbf{0}$ and $\mathbf{s} \in \mathbb{R}^k$. Then, the following identity holds*

$$\mu'_n[\mathbf{s}, \boldsymbol{\alpha}] = \frac{n!}{\langle \boldsymbol{\alpha}_\bullet \rangle_n} \sum_{\substack{\mathbf{m} \in \mathbb{N}_0^k \\ \mathbf{m}_\bullet = n}} \frac{\mathbf{s}^{\mathbf{m}}}{\mathbf{m}!} \langle \boldsymbol{\alpha} \rangle_{\mathbf{m}} = \frac{n!}{\langle \boldsymbol{\alpha}_\bullet \rangle_n} Z_n(\mathbf{s}^{\diamond 1} \cdot \boldsymbol{\alpha}, \dots, \mathbf{s}^{\diamond n} \cdot \boldsymbol{\alpha}) =: \zeta_n[\mathbf{s}, \boldsymbol{\alpha}] . \tag{2.3.3}$$

Proof. Let

$$\tilde{\mu}_n[\mathbf{s}, \boldsymbol{\alpha}] := \langle \boldsymbol{\alpha}_\bullet \rangle_n (n!)^{-1} \mu'_n[\mathbf{s}, \boldsymbol{\alpha}] , \qquad \tilde{\zeta}_n[\mathbf{s}, \boldsymbol{\alpha}] := \langle \boldsymbol{\alpha}_\bullet \rangle_n (n!)^{-1} \zeta_n[\mathbf{s}, \boldsymbol{\alpha}] .$$

The statement is equivalent to $\tilde{\mu}_n = \tilde{\zeta}_n$, which we prove in two steps.

Step 1. The following identity holds

$$\tilde{\mu}_{n-1}[\mathbf{s}, \boldsymbol{\alpha} + \mathbf{e}_\ell] = \sum_{h=1}^{n} s_\ell^{h-1} \tilde{\mu}_{n-h}[\mathbf{s}, \boldsymbol{\alpha}] . \tag{2.3.4}$$

By induction on n with trivial (i.e. $1 = 1$) base step $n = 1$. *Inductive step.* Assume for every $\boldsymbol{\alpha} > \mathbf{0}$ and $\mathbf{s}$ in $\mathbb{R}^k$

$$\tilde{\mu}_{n-2}[\mathbf{s}, \boldsymbol{\alpha} + \mathbf{e}_\ell] = \sum_{h=1}^{n-1} s_\ell^{h-1} \tilde{\mu}_{n-1-h}[\mathbf{s}, \boldsymbol{\alpha}] . \tag{2.3.5}$$

Let $\partial_j := \partial_{s_j}$ and notice that

$$\begin{aligned} \partial_j \tilde{\mu}_n[\mathbf{s}, \boldsymbol{\alpha}] &= \sum_{\substack{\mathbf{m} \in \mathbb{N}_0^k \\ \mathbf{m}_\bullet = n}} \frac{m_j\, \mathbf{s}^{\mathbf{m}-\mathbf{e}_j}}{\mathbf{m}!} \langle \boldsymbol{\alpha} \rangle_{\mathbf{m}} = \sum_{\substack{\mathbf{m} \in \mathbb{N}_0^k \\ \mathbf{m}_\bullet = n}} \frac{\mathbf{s}^{\mathbf{m}-\mathbf{e}_j}}{(\mathbf{m}-\mathbf{e}_j)!} \alpha_j \langle \boldsymbol{\alpha} + \mathbf{e}_j \rangle_{\mathbf{m}-\mathbf{e}_j} \\ &= \alpha_j \sum_{\substack{\mathbf{m} \in \mathbb{N}_0^k \\ \mathbf{m}_\bullet = n-1}} \frac{\mathbf{s}^{\mathbf{m}}}{\mathbf{m}!} \langle \boldsymbol{\alpha} + \mathbf{e}_j \rangle_{\mathbf{m}} = \alpha_j \tilde{\mu}_{n-1}[\mathbf{s}, \boldsymbol{\alpha} + \mathbf{e}_j] . \end{aligned} \tag{2.3.6}$$

If $k \geq 2$, we can choose $j \neq \ell$. Applying (2.3.6) to both sides of (2.3.4) yields

$$\begin{aligned} \partial_j \tilde{\mu}_{n-1}[\mathbf{s}, \boldsymbol{\alpha} + \mathbf{e}_\ell] &= \alpha_j \tilde{\mu}_{n-2}[\mathbf{s}, \boldsymbol{\alpha} + \mathbf{e}_j + \mathbf{e}_\ell] \\ \partial_j \sum_{h=1}^{n} s_\ell^{h-1} \tilde{\mu}_{n-h}[\mathbf{s}, \boldsymbol{\alpha}] &= \sum_{h=1}^{n} s_\ell^{h-1} \alpha_j \tilde{\mu}_{n-h-1}[\mathbf{s}, \boldsymbol{\alpha} + \mathbf{e}_j] \\ &= \alpha_j \sum_{h=1}^{n-1} s_\ell^{h-1} \tilde{\mu}_{n-h-1}[\mathbf{s}, \boldsymbol{\alpha} + \mathbf{e}_j] , \end{aligned}$$

where the latter equality holds by letting $\tilde{\mu}_{-1} \coloneqq 0$. Letting now $\boldsymbol{\alpha}' \coloneqq \boldsymbol{\alpha} + \mathbf{e}_j$ and applying the inductive hypothesis (2.3.5) with $\boldsymbol{\alpha}'$ in place of $\boldsymbol{\alpha}$ yields

$$\partial_j \left(\tilde{\mu}_{n-1}[\mathbf{s}, \boldsymbol{\alpha} + \mathbf{e}_\ell] - \sum_{h=1}^{n} s_\ell^{h-1} \tilde{\mu}_{n-h}[\mathbf{s}, \boldsymbol{\alpha}] \right) = 0$$

for every $j \neq \ell$. By arbitrariness of $j \neq \ell$, the bracketed quantity is a polynomial in the sole variables s_ℓ and $\boldsymbol{\alpha}$ of degree at most $n-1$. (Obviously, the same holds also in the case $k=1$.) As a consequence (or trivially if $k=1$), every monomial not in the sole variable s_ℓ cancels out by arbitrariness of $\mathbf{s}$, yielding

$$\tilde{\mu}_{n-1}[\mathbf{s}, \boldsymbol{\alpha} + \mathbf{e}_\ell] - \sum_{h=1}^{n} s_\ell^{h-1} \tilde{\mu}_{n-h}[\mathbf{s}, \boldsymbol{\alpha}] = \frac{s_\ell^{n-1} \langle \alpha_\ell + 1 \rangle_{n-1}}{(n-1)!} - \sum_{h=1}^{n} s_\ell^{h-1} \frac{s_\ell^{n-h}}{(n-h)!} \langle \alpha_\ell \rangle_{n-h} \ .$$

The latter quantity is proved to vanish as soon as

$$\frac{\langle \alpha + 1 \rangle_{n-1}}{(n-1)!} = \sum_{h=1}^{n} \frac{\langle \alpha \rangle_{n-h}}{(n-h)!} \ , \text{ or equivalently} \qquad \langle \alpha + 1 \rangle_{n-1} = \sum_{h=0}^{n-1} \frac{\langle \alpha \rangle_h \, (n-1)!}{h!} \ ,$$

in fact a particular case of the well-known *Chu–Vandermonde identity*

$$\langle \alpha + \beta \rangle_n = \sum_{k=0}^{n} \binom{n}{k} \langle \alpha \rangle_k \langle \beta \rangle_{n-k} \ . \tag{2.3.7}$$

Step 2. It holds that $\tilde{\mu}_n = \tilde{\zeta}_n$. By strong induction on n with trivial (i.e. $1=1$) base step $n=0$. *Inductive step.* Assume for every $\boldsymbol{\alpha} > \mathbf{0}$ and $\mathbf{s}$ in $\mathbb{R}^k$ that $\tilde{\mu}_{n-1}[\mathbf{s}, \boldsymbol{\alpha}] = \tilde{\zeta}_{n-1}[\mathbf{s}, \boldsymbol{\alpha}]$. Then

$$\begin{aligned}
\partial_j \tilde{\zeta}_n[\mathbf{s}, \boldsymbol{\alpha}] =& \sum_{\boldsymbol{\lambda} \vdash n} \frac{M_2(\boldsymbol{\lambda})}{n!} \sum_{h=1}^{n} \frac{\partial_j (\mathbf{s}^{\diamond h} \cdot \boldsymbol{\alpha})^{\lambda_h}}{(\mathbf{s}^{\diamond h} \cdot \boldsymbol{\alpha})^{\lambda_h}} \prod_{i=1}^{n} (\mathbf{s}^{\diamond i} \cdot \boldsymbol{\alpha})^{\lambda_i} \\
=& \sum_{\boldsymbol{\lambda} \vdash n} \frac{M_2(\boldsymbol{\lambda})}{n!} \sum_{h=1}^{n} \frac{h \lambda_h s_j^{h-1} \alpha_j}{\mathbf{s}^{\diamond h} \cdot \boldsymbol{\alpha}} \prod_{i=1}^{n} (\mathbf{s}^{\diamond i} \cdot \boldsymbol{\alpha})^{\lambda_i} \\
=& \alpha_j \sum_{h=1}^{n} s_j^{h-1} \sum_{\boldsymbol{\lambda} \vdash n} \frac{h \lambda_h}{1^{\lambda_1} \lambda_1! \dots h^{\lambda_h} \lambda_h! \dots n^{\lambda_n} \lambda_n!} \frac{1}{\mathbf{s}^{\diamond h} \cdot \boldsymbol{\alpha}} \prod_{i=1}^{n} (\mathbf{s}^{\diamond i} \cdot \boldsymbol{\alpha})^{\lambda_i} \\
=& \alpha_j \sum_{h=1}^{n} s_j^{h-1} \sum_{\boldsymbol{\lambda} \vdash n-h} \frac{M_2(\boldsymbol{\lambda})}{(n-h)!} \prod_{i=1}^{n-h} (\mathbf{s}^{\diamond i} \cdot \boldsymbol{\alpha})^{\lambda_i} \\
=& \alpha_j \sum_{h=1}^{n} s_j^{h-1} \tilde{\zeta}_{n-h}[\mathbf{s}, \boldsymbol{\alpha}] \ .
\end{aligned}$$

The inductive hypothesis, (2.3.4) and (2.3.6) yield

$$\partial_j \tilde{\zeta}_n[\mathbf{s}, \boldsymbol{\alpha}] = \alpha_j \sum_{h=1}^{n} s_j^{h-1} \tilde{\mu}_{n-h}[\mathbf{s}, \boldsymbol{\alpha}] = \partial_j \tilde{\mu}_n[\mathbf{s}, \boldsymbol{\alpha}] \ .$$

By arbitrariness of j this implies that $\tilde{\zeta}_n[\mathbf{s}, \boldsymbol{\alpha}] - \tilde{\mu}_n[\mathbf{s}, \boldsymbol{\alpha}]$ is constant as a function of $\mathbf{s}$ (for fixed $\boldsymbol{\alpha}$), hence vanishing by choosing $\mathbf{s} = \mathbf{0}$. ■

Remark 2.3.4. Here, we gave an elementary combinatorial proof of the moment formula for $\mathsf{D}_{\boldsymbol{\alpha}}$, independently of any property of the distribution. Notice for further purposes that, *defining* $\mu'_n[\mathbf{s}, \boldsymbol{\alpha}]$ as in (2.3.3), the statement holds with identical proof for all $\boldsymbol{\alpha}$ in $\mathbb{C}^k$ such that $\boldsymbol{\alpha}_\bullet \notin \mathbb{Z}_0^-$. For further representations of the moments see Remark 2.3.11 below. Also, notice that a simpler proof of (2.3.4) may be made to follow by expanding $\langle \boldsymbol{\alpha} + \mathbf{e}_\ell \rangle_{\mathbf{m}}$ via the Chu–Vandermonde identity. We opted for the given proof, since we shall need (2.3.6) for future comparison.

Proposition 2.3.5. *The function ${}_k\Phi_2[\boldsymbol\alpha;1;t\mathbf{s}]$ is the exponential generating function of the polynomials Z_n, in the sense that, for all $\boldsymbol\alpha\in\Delta^{k-1}$,*

$$ {}_k\Phi_2[\boldsymbol\alpha;1;t\mathbf{s}] = \mathbf{G}_{\exp}\left[Z_n\left(\mathbf{s}^{\diamond 1}\cdot\boldsymbol\alpha,\dots,\mathbf{s}^{\diamond n}\cdot\boldsymbol\alpha\right)\right](t) \qquad \mathbf{s}\in\mathbb{R}^k\ ,\ t\in\mathbb{R}\ . $$

More generally,

$$ {}_k\Phi_2[\boldsymbol\alpha;\boldsymbol\alpha_\bullet;t\mathbf{s}] = \mathbf{G}_{\exp}\left[\frac{n!}{\langle\boldsymbol\alpha_\bullet\rangle_n} Z_n\left(\mathbf{s}^{\diamond 1}\cdot\boldsymbol\alpha,\dots,\mathbf{s}^{\diamond n}\cdot\boldsymbol\alpha\right)\right](t) \qquad \mathbf{s}\in\mathbb{R}^k\ ,\ t\in\mathbb{R}\ . $$

Proof. Recalling that ${}_k\Phi_2[\boldsymbol\alpha;\boldsymbol\alpha_\bullet;\mathrm{i}\mathbf{s}]=\widehat{\mathsf{D}_{\boldsymbol\alpha}}(\mathbf{s})$ by (2.2.9) and noticing that $\boldsymbol\alpha_\bullet=1$, Theorem 2.3.3 provides an exponential series representation for the characteristic functional of the Dirichlet distribution in terms of the cycle index polynomials of symmetric groups, viz.

$$ \widehat{\mathsf{D}_{\boldsymbol\alpha}}(\mathbf{s}) = \sum_{n=0}^{\infty}\frac{1}{n!}Z_n\left((\mathrm{i}\mathbf{s})^{\diamond 1}\cdot\boldsymbol\alpha,\dots,(\mathrm{i}\mathbf{s})^{\diamond n}\cdot\boldsymbol\alpha\right)\ . $$

Replacing $\mathbf{s}$ with $-\mathrm{i}t\mathbf{s}$ above and using (2.2.1) to extract the term t^n from each summand, the conclusion follows. The second statement has a similar proof. ■

Remark 2.3.6. It is well-known that the characteristic functional of a measure μ on $\mathbb{R}^d$ (or, more generally, on a nuclear space) is always *positive definite*, i.e. it holds that

$$ \forall n\in\mathbb{N}_0\quad \forall\mathbf{s}_1,\dots,\mathbf{s}_n\in\mathbb{R}^d\quad \forall\xi_1,\dots,\xi_n\in\mathbb{C}\qquad \sum_{h,k=1}^{n}\widehat{\mu}(\mathbf{s}_h-\mathbf{s}_k)\,\xi_h\bar\xi_k\geq 0\ , \tag{2.3.8} $$

where $\bar\xi$ denotes the complex conjugate of $\xi\in\mathbb{C}$. Thus, the functional $\mathbf{s}\mapsto{}_k\Phi_2[\boldsymbol\alpha;\boldsymbol\alpha_\bullet;\mathrm{i}\mathbf{s}]$ is positive definite by (2.2.9) for all $\boldsymbol\alpha\in\mathbb{R}^k_+$.

The following Lemma also appeared in [107, Eqn.'s (2), (3)].

Lemma 2.3.7. *There exist the narrow limits*

$$ \lim_{\beta\to 0^+}\mathsf{D}_{\beta\boldsymbol\alpha} = \boldsymbol\alpha_\bullet^{-1}\sum_{i=1}^{k}\alpha_i\delta_{\mathbf{e}_i} \qquad\text{and}\qquad \lim_{\beta\to+\infty}\mathsf{D}_{\beta\boldsymbol\alpha} = \delta_{\boldsymbol\alpha_\bullet^{-1}\boldsymbol\alpha}\ . $$

Proof. Since $\mathsf{D}_{\boldsymbol\alpha}$ is moment determinate, it suffices — by compactness of Δ^{k-1} and Stone–Weierstraß Theorem — to show the convergence of its moments. By Theorem 2.3.3 (cf. also (2.2.1)),

$$ \begin{aligned} \mu'_n[\mathbf{s},\beta\boldsymbol\alpha] :=& \frac{n!}{\langle\beta\boldsymbol\alpha_\bullet\rangle_n}Z_n\left(\beta\mathbf{s}^{\diamond 1}\cdot\boldsymbol\alpha,\dots,\beta\mathbf{s}^{\diamond n}\cdot\boldsymbol\alpha\right)\\ =&\frac{1}{\langle\beta\boldsymbol\alpha_\bullet\rangle_n}\sum_{r=1}^{n}\sum_{\boldsymbol\lambda\vdash_r n}M_2(\boldsymbol\lambda)\prod_i^n(\beta\,\mathbf{s}^{\diamond i}\cdot\boldsymbol\alpha)^{\lambda_i}\\ =&\frac{1}{\langle\beta\boldsymbol\alpha_\bullet\rangle_n}\sum_{r=1}^{n}\sum_{\boldsymbol\lambda\vdash_r n}M_2(\boldsymbol\lambda)\,\beta^{\boldsymbol\lambda_\bullet}\prod_i^n(\mathbf{s}^{\diamond i}\cdot\boldsymbol\alpha)^{\lambda_i}\\ =&\frac{1}{\langle\beta\boldsymbol\alpha_\bullet\rangle_n}\sum_{r=1}^{n}\beta^r\sum_{\boldsymbol\lambda\vdash_r n}M_2(\boldsymbol\lambda)\prod_i^n(\mathbf{s}^{\diamond i}\cdot\boldsymbol\alpha)^{\lambda_i}\\ \underset{\beta\ll 1}{\approx}&\ \frac{1}{\beta\boldsymbol\alpha_\bullet\,\Gamma(n)}\beta M_2(\mathbf{e}_n)\left(\mathbf{s}^{\diamond n}\cdot\boldsymbol\alpha\right)^1 = \boldsymbol\alpha_\bullet^{-1}\boldsymbol\alpha\cdot\mathbf{s}^{\diamond n}\ ,\\ \underset{\beta\gg 1}{\approx}&\ \frac{1}{\beta^n\boldsymbol\alpha_\bullet^n}\beta^n M_2(n\mathbf{e}_1)\left(\mathbf{s}^{\diamond 1}\cdot\boldsymbol\alpha\right)^n = \boldsymbol\alpha_\bullet^{-n}(\mathbf{s}\cdot\boldsymbol\alpha)^n\ . \end{aligned} $$ ■

As a consequence of the Lemma further confluent forms of ${}_k\Phi_2$ may be computed:

Corollary 2.3.8 (Confluent forms of ${}_k\Phi_2$)**.** *There exist the limits*

$$ \lim_{\beta\to 0^+}{}_k\Phi_2[\beta\boldsymbol\alpha;\beta\boldsymbol\alpha_\bullet;\mathbf{s}] = \boldsymbol\alpha_\bullet^{-1}\boldsymbol\alpha\cdot\exp^\diamond(\mathbf{s})\ ,\qquad \lim_{\beta\to+\infty}{}_k\Phi_2[\beta\boldsymbol\alpha;\beta\boldsymbol\alpha_\bullet;\mathbf{s}] = \exp(\boldsymbol\alpha_\bullet^{-1}\boldsymbol\alpha\cdot\mathbf{s})\ . $$

2.3.2 Infinite-dimensional statements. Together with the introductory discussion, Proposition 2.3.1 suggests the following Mapping Theorem for $\mathcal{D}_\sigma$, to be compared with the analogous result for the Poisson random measure $\mathcal{P}_\sigma$ over $(X, \mathcal{B})$. (See e.g., [88, §2.3 and *passim*].) The σ-symmetry of $\mathcal{D}_{\beta\sigma}$ and the quasi-exchangeability and aggregation property of $\mathsf{D}_{\boldsymbol{\alpha}}$ are trivially recovered from the Theorem by (2.2.10).

Theorem 2.3.9 (Mapping theorem for $\mathcal{D}_\sigma$)**.** *Let $(X, \tau(X), \mathcal{B})$ and $(X', \tau(X'), \mathcal{B}')$ be second countable locally compact Hausdorff spaces, ν a non-negative finite measure on $(X, \mathcal{B})$ and $f\colon (X, \mathcal{B}) \to (X', \mathcal{B}')$ be any measurable map. Then,*

$$(f_\sharp)_\sharp \mathcal{D}_\nu = \mathcal{D}_{f_\sharp \nu} \ .$$

Proof. Choosing $\mathbf{X} \coloneqq (g^{-1}(1), \dots, g^{-1}(k))$, the characterization (2.2.11) is equivalent to the requirement that $(g_\sharp)_\sharp \mathcal{D}_\nu = \mathsf{D}_{g_\sharp \nu}$ for any $g\colon X \to [k]$ such that every ν-representative of g is surjective, which makes $\mathbf{X}$ non-trivial for ν. Denote by $\mathcal{S}(X, \nu, k)$ the family of such functions and notice that if $h \in \mathcal{S}(X', f_\sharp\nu, k)$, then $g \coloneqq h \circ f \in \mathcal{S}(X, \nu, k)$. The proof is now merely typographical:

$$(h_\sharp)_\sharp (f_\sharp)_\sharp \mathcal{D}_\nu = (g_\sharp)_\sharp \mathcal{D}_\nu = \mathsf{D}_{g_\sharp \nu} = \mathsf{D}_{h_\sharp(f_\sharp \nu)} \ ,$$

where the second equality suffices to establish that $(f_\sharp)_\sharp \mathcal{D}_\nu$ is a Dirichlet–Ferguson measure by arbitrariness of h, while the third one characterizes its intensity as $f_\sharp \nu$. ∎

We denote by $\mathscr{P}(\mathscr{P}(X))$ the space of probability measures on $(\mathscr{P}(X), \mathcal{B}_n(\mathscr{P}(X)))$, endowed with the narrow topology $\tau_n(\mathscr{P}(\mathscr{P}(X)))$ induced by duality with $\mathcal{C}_b(\mathscr{P}(X))$. We are now able to prove the following more general version of Theorem 2.1.1.

Theorem 2.3.10 (Characteristic functional of $\mathcal{D}_{\beta\sigma}$)**.** *Let $(X, \tau(X), \mathcal{B})$ be a second countable locally compact Hausdorff Borel measurable space, σ a probability measure on $(X, \mathcal{B})$ and fix $\beta > 0$. Then,*

$$\forall f \in \mathcal{C}_c \qquad \widehat{\mathcal{D}_{\beta\sigma}}(tf^*) = \mathbf{G}_{\exp}\left[n! \left\langle\beta\right\rangle_n^{-1} Z_n\left(\beta\sigma f^1, \dots, \beta\sigma f^n\right)\right](\mathrm{i}\, t) \ , \ t \in \mathbb{R} \ . \tag{2.3.9}$$

Moreover, the map $\nu \mapsto \mathcal{D}_\nu$ is narrowly continuous on $\mathscr{M}_b^+(X)$.

Proof. Characteristic functional. Fix f in $\mathcal{C}_c$ and let $(f_h)_h$ be a good approximation of f, locally constant on $\mathbf{X}_h \coloneqq (X_{h,1}, \dots, X_{h,k_h})$ with values $\mathbf{s}_h$ for some $(\mathbf{X}_h)_h \in \mathfrak{Na}(X)$. Fix $n > 0$ and set $\boldsymbol{\alpha}_h \coloneqq \beta\sigma^\diamond \mathbf{X}_h$. Choosing $u\colon \Delta^{k_h - 1} \to \mathbb{R}$, $u\colon \mathbf{y} \mapsto (\mathbf{s}_h \cdot \mathbf{y})^n$ in (2.2.11) yields

$$\mu_n'^{\mathcal{D}_{\beta\sigma}}[f_h^*] \coloneqq \int_{\mathscr{P}(X)} (f_h^* \eta)^n \, \mathrm{d}\mathcal{D}_{\beta\sigma}(\eta) = \int_{\Delta^{k_h - 1}} (\mathbf{s}_h \cdot \mathbf{y})^n \, \mathrm{d}\mathsf{D}_{\beta \mathrm{ev}^{\mathbf{X}_h} \sigma}(\mathbf{y}) = \mu_n'[\mathbf{s}_h, \boldsymbol{\alpha}_h] \ ,$$

hence, by Theorem 2.3.3,

$$\mu_n'^{\mathcal{D}_{\beta\sigma}}[f_h^*] = n! \left\langle\beta\right\rangle_n^{-1} Z_n\left(\mathbf{s}_h^{\diamond 1} \cdot \boldsymbol{\alpha}_h, \dots, \mathbf{s}_h^{\diamond n} \cdot \boldsymbol{\alpha}_h\right) = n! \left\langle\beta\right\rangle_n^{-1} Z_n\left(\beta\sigma f_h^1, \dots, \beta\sigma f_h^n\right) \ ,$$

thus, by Dominated Convergence Theorem, continuity of Z_n and arbitrariness of f,

$$\forall f \in \mathcal{C}_c \qquad \mu_n'^{\mathcal{D}_{\beta\sigma}}[tf^*] = n! \left\langle\beta\right\rangle_n^{-1} Z_n\left(t^1 \beta\sigma f^1, \dots, t^n \beta\sigma f^n\right) \ , \ t \in \mathbb{R} \ .$$

Using (2.2.1) to extract the term t^n from Z_n and substituting t with $\mathrm{i}\, t$ on the right-hand side, the conclusion follows by definition of exponential generating function.

Continuity. Assume first that $(X, \tau(X))$ is compact. By compactness of $(X, \tau(X))$, the narrow and vague topology on $\mathscr{P}(X)$ coincide and $\mathscr{P}(X)$ is compact as well by Prokhorov

Theorem. Let $(\nu_h)_{h\in\mathbb{N}}$ be a sequence of finite non-negative measures narrowly convergent to ν_∞. Again by Prokhorov Theorem and by compactness of $\mathscr{P}(X)$ there exists some $\tau_n(\mathscr{P}(\mathscr{P}(X)))$-cluster point $\mathcal{D}_\infty$ for the family $\{\mathcal{D}_{\nu_h}\}_h$. By narrow convergence of ν_h to ν_∞, continuity of Z_n and absolute convergence of $\widehat{\mathcal{D}_\cdot}(f)$, it follows that $\lim_h \widehat{\mathcal{D}_{\nu_h}} = \widehat{\mathcal{D}_{\nu_\infty}}$ pointwise on $\mathcal{C}_c(X)$, hence, by Corollary 2.5.3, it must be $\mathcal{D}_\infty = \mathcal{D}_{\nu_\infty}$.

In the case when X is not compact, recall the notation established in Proposition 2.2.4, denote by $\mathcal{B}(\alpha X)$ the Borel σ-algebra of $(\alpha X, \tau(\alpha X))$ and by $\mathscr{P}(\alpha X)$ the space of probability measures on $(\alpha X, \mathcal{B}(\alpha X))$. By the Continuous Mapping Theorem there exists the narrow limit $\tau_n(\mathscr{P}(X))$-$\lim_h \alpha_\sharp \nu_h = \alpha_\sharp \nu_\infty$, thus, by the result in the compact case applied to the space $(\alpha X, \mathcal{B}_\alpha)$ together with the sequence $\alpha_\sharp \nu_h$,

$$\tau_n\big(\mathscr{P}(\mathscr{P}(X))\big)\text{-}\lim_h \mathcal{D}_{\alpha_\sharp \nu_h} = \mathcal{D}_{\alpha_\sharp \nu_\infty} \ . \tag{2.3.10}$$

The narrow convergence of ν_h to ν_∞ implies that $\alpha_\sharp \nu_\infty$ does not charge the point at infinity in αX, hence the measure spaces $(X, \mathcal{B}, \nu_*)$ and $(\alpha X, \mathcal{B}(\alpha X), \alpha_\sharp \nu_*)$ are isomorphic for $* = h, \infty$ via the map α, with inverse α^{-1} defined on $\mathrm{im}\alpha \subsetneq \alpha X$. The continuity of α^{-1} and the Continuous Mapping Theorem together yield the narrow continuity of the map $({\alpha^{-1}}_\sharp)_\sharp$. The conclusion follows by applying $({\alpha^{-1}}_\sharp)_\sharp$ to (2.3.10) and using the Mapping Theorem 2.3.9. ■

Remark 2.3.11. Different representations of the univariate moments of the Dirichlet–Ferguson measure have also appeared, without mention to Z_n, in [137, Eq. (17)] (in terms of incomplete Bell polynomials, solely in the case when $X \Subset \mathbb{R}_+$ and $f = \mathrm{id}_\mathbb{R}$) and in [107, proof of Prop. 3.3] (in implicit recursive form). Representations of the multi-variate moments have also appeared in [86, Prop. 7.4] (in terms of summations over 'color-respecting' permutations, in the case $\beta = 1$), in [52, (4.20)] and [54, Lem. 5.2] (in terms of summations over constrained set partitions).

Remark 2.3.12. In the case when ν_h converges to ν_∞ in total variation, the continuity statement in the Theorem and the asymptotics for $\beta \to 0$ in Corollary 2.3.14 below were first shown in [146, Thm. 3.2], relying on Sethuraman's stick-breaking representation. The asymptotic expressions in Corollary 2.3.14 have been subsequently rediscovered many times in different simplified settings: Lastly, in the case $X = \mathbb{R}^d$, in [107, Prop. 3.4 and Thm. 3.5]. The following result was also obtained, again with different methods, in [146].

Corollary 2.3.13 (Tightness of Dirichlet–Ferguson measures [146, Thm. 3.1])**.** *Under the same assumptions of Theorem 2.3.10, let $M \subset \mathscr{M}_b^+(X) \setminus \{0\}$ be such that $\overline{M} := \{\overline{\nu} \mid \nu \in M\}$ is a tight, resp. narrowly compact, family of finite non-negative measures. Then, the family $\{\mathcal{D}_\nu\}_{\nu\in M}$ is itself tight, resp. narrowly compact.*

Corollary 2.3.14 (Asymptotic expressions)**.** *Under the same assumptions of Theorem 2.3.10, for all f in $\mathcal{C}_c$ and complex t there exist the limits*

$$\lim_{\beta\downarrow 0} \widehat{\mathcal{D}_{\beta\sigma}}(tf^*) = \sigma \exp(\mathrm{i}\, tf) \qquad \text{and} \qquad \lim_{\beta\to\infty} \widehat{\mathcal{D}_{\beta\sigma}}(tf^*) = \exp(\mathrm{i}\, t\, \sigma f) \tag{2.3.11}$$

corresponding to the narrow limits

$$\mathcal{D}_\sigma^0 := \lim_{\beta\downarrow 0} \mathcal{D}_{\beta\sigma} = \delta_\sharp \sigma \qquad \text{and} \qquad \mathcal{D}_\sigma^\infty := \lim_{\beta\to\infty} \mathcal{D}_{\beta\sigma} = \delta_\sigma \ , \tag{2.3.12}$$

where, in the first case, $\delta\colon X \to \mathscr{P}(X)$ denotes the Dirac embedding $x \mapsto \delta_x$.

Proof. The existence of $\mathcal{D}^0_\sigma$ and $\mathcal{D}^\infty_\sigma$ as narrow cluster points for $\{\mathcal{D}_{\beta\sigma}\}_{\beta>0}$ follows by Corollary 2.3.13. Retaining the notation established in Theorem 2.3.10, Corollary 2.3.8 yields for all k

$$\lim_{\beta\downarrow 0} \widehat{\mathcal{D}_{\beta\sigma}}(f_k^*) = \sigma \exp(\mathrm{i} f_k) \qquad \text{and} \qquad \lim_{\beta\to\infty} \widehat{\mathcal{D}_{\beta\sigma}}(f_k^*) = \exp(\mathrm{i}\,\sigma f_k) \ ,$$

hence, by Dominated Converge,

$$\lim_k \lim_{\beta\downarrow 0} \widehat{\mathcal{D}_{\beta\sigma}}(f_k^*) = \sigma \exp(\mathrm{i} f) \qquad \text{and} \qquad \lim_k \lim_{\beta\to\infty} \widehat{\mathcal{D}_{\beta\sigma}}(f_k^*) = \exp(\mathrm{i}\,\sigma f) \ . \tag{2.3.13}$$

Furthermore, recalling that $|f_k| \le |f|$ one has

$$\begin{aligned} \left|\widehat{\mathcal{D}_{\beta\sigma}}(f^*) - \widehat{\mathcal{D}_{\beta\sigma}}(f_k^*)\right| \le& e^{\|f\|} \int_{\mathscr{P}(X)} \mathrm{d}\mathcal{D}_{\beta\sigma}(\eta)\, |f - f_k|^* \, \eta \\ =& e^{\|f\|} \left\|f - f_k\right\|_{L^1_\sigma} \le e^{\|f\|} \left\|f_k - f\right\| \ , \end{aligned} \tag{2.3.14}$$

where the equality follows by [55, §3 Prop. 1]. As a consequence, the order of the limits in each left-hand side of (2.3.13) may be exchanged, for the convergence in k is uniform with respect to β. This shows (2.3.11). ■

Remark 2.3.15. By Theorem 2.3.10, $\beta\sigma$ may be substituted with any sequence $(\beta_h\sigma_h)_h$ with $\lim_h \beta_h = 0, \infty$ and $\{\sigma_h\}_h$ a tight family. Observe that, despite the similarity with Lemma 2.3.7, Corollary 2.3.14 is not a direct consequence of the former, since the evaluation map $\mathrm{ev}^{\mathbf{X}}$ is never continuous.

Remark 2.3.16 (A Gibbsean interpretation). Corollary 2.3.14 states that, varying $\beta \in [0,\infty]$, the map $\mathcal{D}_{\beta\,\cdot} : \mathscr{P}(X) \to \mathscr{P}(\mathscr{P}(X))$ is a (continuous) interpolation between the two extremal maps $\mathcal{D}^0_\cdot = \delta^{(0)}_\sharp$ and $\mathcal{D}^\infty_\cdot = \delta^{(1)}$, where $\delta^{(0)} := \delta \colon X \to \mathscr{P}(X)$ and $\delta^{(1)} := \delta \colon \mathscr{P}(X) \to \mathscr{P}(\mathscr{P}(X))$. These asymptotic distributions may be interpreted — at least formally — in the framework of statistical mechanics. In order to establish some lexicon, consider a physical system at *inverse temperature* β driven by a *Hamiltonian* H.

Let $Z^H_\beta := \langle \exp(-\beta H)\rangle$, $F_\beta := -\beta^{-1} \ln Z^H_\beta$ and $G_\beta := (Z^H_\beta)^{-1} \exp(-\beta H)$ respectively denote the *partition function*, the *Helmholtz free energy* and (the distribution of) the *Gibbs measure* of the system. It was heuristically argued in [140, §3.1] that — at least in the case when $(X, \mathcal{B}, \sigma)$ is the unit interval —

$$\mathrm{d}\mathcal{D}_{\beta\sigma}(\eta) = \frac{e^{-\beta\, S(\eta)}}{Z_\beta}\, \mathrm{d}\mathcal{D}^*_\sigma(\eta) \ ,$$

where: S is now an *entropy functional* (rather than an energy functional), Z_β is a normalization constant and β plays the rôle of the inverse temperature. Here, $\mathcal{D}^*_\sigma$ denotes a *non-existing* (!) uniform distribution on $\mathscr{P}(X)$. Borrowing again the terminology, this time in full generality, one can say that for small β (i.e. large *temperature*), the system *thermalizes* towards the “uniform” distribution $\delta_\sharp\sigma$ induced by the reference measure σ on the base space, while for large β it *crystallizes* to δ_σ, so that all randomness is lost. Consistently with property i. of $\mathcal{D}_\sigma$, we see that $\mathbb{E}_{\mathcal{D}^\infty_\sigma} \eta_i = 0$ and $\mathbb{E}_{\mathcal{D}^0_\sigma} \eta_i = \delta_{i1}$ for all i, where δ_{ab} denotes the Kronecker symbol; both statements hold in fact with probability 1.

It is worth noticing that a different interpretation for the parameter β has been given in [107], where the latter is regarded as a ‘time’ parameter in the definition of a *Processus Croissant pour l’Ordre Convexe* (PCOC).

Remark 2.3.17. By the Continuous Mapping Theorem, both the continuity statement in Theorem 2.3.10 and the asymptotic expressions in Corollary 2.3.14 hold, mutatis mutandis, for every narrowly continuous image of $\mathcal{D}_{\beta\sigma}$, hence, for instance, for the *entropic measure* $\mathbb{P}^{\beta}_{\sigma}$ [140, 151]. This generalizes [140, Prop. 3.14] and the discussion for the entropic measure thereafter.

Corollary 2.3.18. *Let $(X, \mathcal{B}, \sigma)$ and β be as in Theorem 2.3.10 and let $h \in \mathcal{B}_b(X; \mathbb{R})$. Then, Equation* (2.3.9) *holds with h in place of f. In particular, $\widehat{\mathcal{D}_{\beta\sigma}}(h)$ does not depend on the choice of the representative of $h \in L^\infty_\sigma(X, \mathcal{B})$.*

Proof. Since h is bounded, $\widehat{\mathcal{D}_{\beta\sigma}}(h^*)$ is well-defined. Let $(f_k)_k \subset \mathcal{C}_c(X)$ be such that $h = L^1_\sigma\text{-}\lim_k f_k$. Observe that we can choose $(f_k)_k$ so that $\sup_k |f_k| \vee \|h\| \leq M$ for some finite $M > 0$. Analogously to (2.3.14) (with e^M in lieu of $e^{\|f\|}$), we have $\lim_k \widehat{\mathcal{D}_{\beta\sigma}}(f^*_k) = \widehat{\mathcal{D}_{\beta\sigma}}(h^*)$. By the same reasoning, $\widehat{\mathcal{D}_{\beta\sigma}}(h^*)$ does not depend on the L^1_σ-representative of h. The L^1_σ-continuity of the right-hand side of (2.3.9) in f is straightforward. Thus, the assertion follows by replacing f with f_k in (2.3.9) and letting $k \to \infty$. ∎

Remark 2.3.19 (Some alternative proofs). Applying [160, §IV.2.2, Prop. 2.4, p. 204] and Corollary 2.3.18 together yields a different proof of the Mapping Theorem 2.3.9, not relying on the marginal distributions of $\mathcal{D}_{\beta\sigma}$. As an immediate consequence, the aggregation property is also recovered by choosing a purely atomic intensity measure.

Corollary 2.3.20 (Alternative construction of $\mathcal{D}_{\beta\sigma}$)**.** *Assume there exists a nuclear function space $\mathcal{S} \subset \mathcal{C}_0(X)$, continuously embedded into $\mathcal{C}_0(X)$ and such that $\mathcal{S} \cap \mathcal{C}_c(X)$ is norm-dense in $\mathcal{C}_0(X)$ and dense in $\mathcal{S}$. Then, there exists a unique Borel probability measure on the dual space $\mathcal{S}'$, namely $\mathcal{D}_{\beta\sigma}$, whose characteristic functional is given by the extension of* (2.3.9) *to $\mathcal{S}$.*

Proof. By the classical Bochner–Minlos Theorem, (See e.g., [61, §4.2, Thm. 2]) it suffices to show that the extension to $\mathcal{S}$, say χ, of the functional (2.3.9) is a characteristic functional. By the convention in (2.2.2), $\chi(\mathbf{0}_{\mathcal{S}}) = \chi(\mathbf{0}_{\mathcal{C}_c(X)}) = 1$. The (sequential) continuity of χ on $\mathcal{S}$ follows by that on $\mathcal{C}_0(X)$ and the continuity of the embedding $\mathcal{S} \subset \mathcal{C}_0(X)$. It remains to show the positivity (2.3.8) of χ, which can be checked only on $\mathcal{S} \cap \mathcal{C}_c(X)$ by $\|\cdot\|$-density of the inclusions $\mathcal{S} \cap \mathcal{C}_c(X) \subset \mathcal{C}_0(X)$. The positivity of χ restricted to $\mathcal{C}_c(X)$ follows from the positivity of ${}_k\Phi_2$ in Remark 2.3.6 by approximation of f with simple functions as in the proof of Theorem 2.3.10. ∎

Remark 2.3.21. Let us notice that the assumption of Corollary 2.3.20 is satisfied, whenever X is (additionally) either finite (trivially), or a differentiable manifold, or a topological group (by the main result in [1]). In particular, when $X = \mathbb{R}^d$, we can choose $\mathcal{S} = \mathcal{S}(\mathbb{R}^d)$, the space of Schwartz functions on $\mathbb{R}^d$.

Consider the map $\mathscr{G} \colon \mathscr{P}(\mathscr{P}(X)) \to \mathscr{P}(X)$ defined by

$$(\mathscr{G}(\mu))A = \int_{\mathscr{P}(X)} \mathrm{d}\mu(\eta)\, \eta A \qquad A \in \mathcal{B}\ ,\ \mu \in \mathscr{P}(\mathscr{P}(X))\ .$$

Since f^* is $\tau_n(\mathscr{P}(X))$-continuous for every $f \in \mathcal{C}_b(X)$ and bounded by $\|f\|$, the map $\mathscr{G}$ is continuous.

Corollary 2.3.22. *Let $(X, \tau(X), \mathcal{B})$ be a locally compact second countable Hausdorff Borel measurable space. For fixed $\beta \in (0, \infty)$, the map $\mathcal{D}_{\beta\,\cdot} \colon \mathscr{P}(X) \to \mathscr{P}(\mathscr{P}(X))$ is a homeomorphism onto its image, with inverse $\mathscr{G}$.*

Proof. The continuity of $\mathcal{D}_{\beta\,\cdot}$ is proven in Theorem 2.3.10. By e.g., [55, Thm. 3] for all $f \in \mathcal{C}_c(X)$ one has $\mathcal{D}_{\beta\sigma} f^* = \beta\sigma f$, hence $\mathscr{G}$ inverts $\mathcal{D}_{\beta\,\cdot}$ on its image. ∎

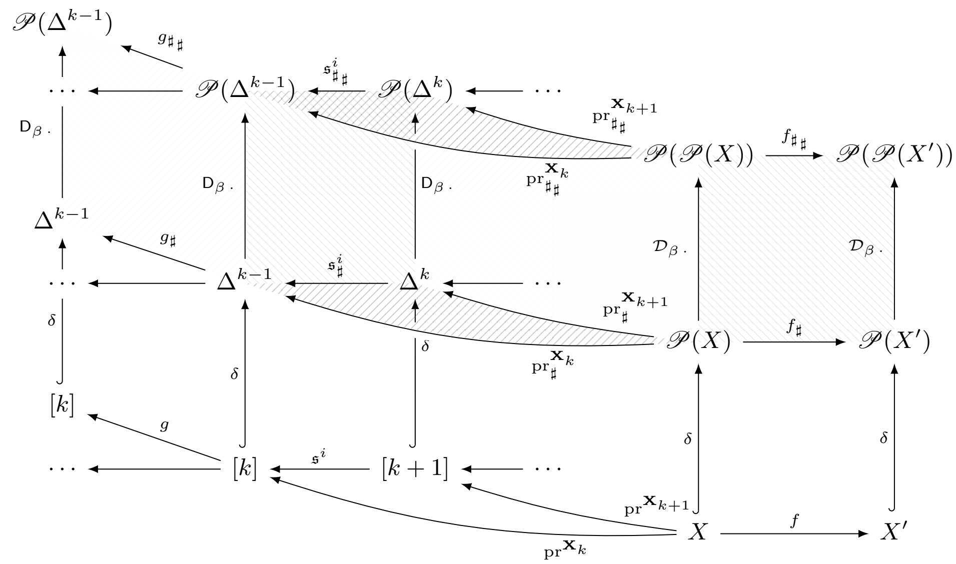

Figure 2.1: Many properties of Dirichlet(–Ferguson) measures can be phrased in terms of the commutation of some diagrams. The commutation of dashed squares of the diagram above, from left to right, respectively corresponds to

- *the symmetry property* (2.2.5) *when* $g = \pi \in \mathfrak{S}_k$ *and, more generally, Proposition 2.3.1;*
- *the aggregation property* (2.2.4)*;*
- *the marginalization* (2.2.10) *(recall that* $\mathrm{pr}^{\mathbf{X}}_{\sharp} = \mathrm{ev}^{\mathbf{X}}$*);*
- *the symmetry property* (2.2.12) *when* $f = \psi$ *is measure preserving and, more generally, Theorem 2.3.9;*

the commutation of the solid sub-diagram delimited by the two dashed triangles corresponds to the requirement of Kolmogorov consistency. (Cf. [55, p. 214].)

As a further application of Theorem 2.3.10, we show how the continuity Theorem extends to *hierarchical Dirichlet processes* [153]. Such processes arise as a natural counterpart to Dirichlet–Ferguson processes in a non-parametric Bayesian approach to the modeling of *grouped* data.

Definition 2.3.23 (Hierarchical Dirichlet Process [153, Eqn.s (13)–(14)])**.** Under the same assumptions as in Theorem 2.3.10, given $\beta > 0$ and η a $\mathscr{P}(X)$-valued $\mathcal{D}_\nu$-distributed random field, a *hierarchical Dirichlet process* is any $\mathscr{P}(X)$-valued random field distributed as $\mathcal{D}_{\beta\eta}$.

As usual, we consider rather the laws of hierarchical Dirichlet processes, namely the measures $\mathcal{D}_{\beta\mathcal{D}_\nu}$ on $\mathscr{P}(\mathscr{P}(X))$. The construction is easily iterated, as in the following definition.

Definition 2.3.24 (Dirichlet–Ferguson-measured Vershik tower)**.** Let $(X, \tau(X), \mathcal{B})$ be a locally compact second countable Hausdorff Borel measurable space. Define inductively for $n \in \mathbb{N}_0$

$$\mathscr{P}^{(0)}(X) \coloneqq X\ , \qquad \mathscr{P}^{(n)}(X) \coloneqq \mathscr{P}\big(\mathscr{P}^{(n-1)}(X)\big)\ ,$$

always endowed with the corresponding narrow topologies, and let $\delta^{(n)}\colon \mathscr{P}^{(n)}(X)\to\mathscr{P}^{(n+1)}(X)$ be the corresponding Dirac embeddings for $n\in\mathbb{N}_0$. Following [21, p. 796] we term the family $\big\{\big(\mathscr{P}^{(n)}(X),\delta^{(n)}\big)\big\}_{n\in\mathbb{N}_0}$ the *Vershik tower* over X. Notice that, since X is Polish, so is $\mathscr{P}^{(n)}(X)$ for every $n\in\mathbb{N}_0$. Analogously, if X is additionally compact, then $\mathscr{P}^{(n)}(X)$ is compact as well, for every $n\in\mathbb{N}_0$.

Let now $\boldsymbol\beta\in\mathbb{R}_+^{\mathbb{N}_0}$ and σ be a probability measure on X. Define inductively for $n\in\mathbb{N}_0$

$$\mathcal{D}^{(0)}_{\boldsymbol\beta,\sigma}:=\sigma\ ,\qquad \mathcal{D}^{(n)}_{\boldsymbol\beta,\sigma}:=\mathcal{D}_{\beta_{n-1}\mathcal{D}^{(n-1)}_{\boldsymbol\beta,\sigma}}\ ,$$

and observe that $\mathcal{D}^{(n)}_{\boldsymbol\beta,\sigma}$ is a Borel probability measure over $\mathscr{P}^{(n)}(X)$ for every $n\in\mathbb{N}_1$. We term the family $\big\{\big(\mathscr{P}^{(n)}(X),\mathcal{D}^{(n)}_{\boldsymbol\beta,\sigma},\delta^{(n)}\big)\big\}_{n\in\mathbb{N}_0}$ the $(\boldsymbol\beta,\sigma)$-*Dirichlet–Ferguson-measured Vershik tower* over X.

Corollary 2.3.25. *Let $(X,\tau(X),\mathcal{B})$ be a compact Hausdorff Borel measurable space and let $\mathbb{R}_+^{\mathbb{N}_0}$ be endowed with the uniform topology. Then, the map $\mathcal{D}^{(n)}_{\cdot}\colon\mathbb{R}_+^{\mathbb{N}_0}\times\mathscr{P}(X)\to\mathscr{P}^{(n)}(X)$ given by $(\boldsymbol\beta,\sigma)\mapsto\mathcal{D}^{(n)}_{\boldsymbol\beta,\sigma}$ is continuous for every $n\in\mathbb{N}_1$.*

Proof. Since $\mathscr{P}^{(n)}$ is a (locally) compact second countable Hausdorff for every $n\in\mathbb{N}_1$, it is sufficient to iteratively apply the continuity statement in Theorem 2.3.10. ∎

Remark 2.3.26. By resorting to the Alexandrov compactification of X, analogously to the proof of Theorem 2.3.10, Corollary 2.3.25 holds even if X is merely *locally* compact. Rigorously, this is however beyond our framework, since $\mathscr{P}(X)$ (hence $\mathscr{P}^{(n)}(X)$, $n\in\mathbb{N}_1$) is locally compact if and only if X is compact. Thus, if X is not compact, $\mathcal{D}_{\beta_0\sigma}$ does not satisfy our definition of intensity measure as a finite measure on a locally compact second countable Hausdorff Borel measurable space.

2.4 Proof of Theorem 2.1.4 and accessory results

2.4.1 Finite-dimensional statements.

Multisets. Given a set S, a (*finite integer-valued*) S-*multi-set* is any function $f\colon S\to\mathbb{N}_0$ such that its *cardinality* $\#f:=\sum_{s\in S}f(s)$ is finite. We denote any such multiset by $[\![\mathbf{s}_{\boldsymbol\alpha}]\!]$, where $\mathbf{s}\in S^{\times k}$ has mutually different entries and $\boldsymbol\alpha:=f^\diamond(\mathbf{s})\in\mathbb{N}_1^k$. We term the set $[\mathbf{s}]:=\{s_1,\dots,s_k\}$ the *underlying set* to $[\![\mathbf{s}_{\boldsymbol\alpha}]\!]$ and put

$$[\mathbf{s}_{\boldsymbol\alpha}]:=\{(s_1,1),\dots,(s_1,\alpha_1),\dots,(s_k,1),\dots,(s_k,\alpha_k)\}\ .$$

A coloring problem. An interpretation of the moments formula (2.3.3) may be given in enumerative combinatorics, by means of *Pólya Enumeration Theory.* (PET, see e.g., [134].) A minimal background is as follows. Let $G<\mathfrak{S}_n$ be a permutation group acting on $[n]$ and $[\mathbf{s}]:=\{s_1,\dots,s_k\}$ denote a set of (distinct) colors.

Definition 2.4.1 (Colorings)**.** A *k-coloring* of $[n]$ is any function f in $[\mathbf{s}]^{[n]}$, where we understand the elements $s_1,\dots,s_k$ of $[\mathbf{s}]$ as placeholders for different colors. Whenever these are irrelevant, given a k-coloring f of $[n]$ we denote by $\tilde f$ the unique function in $[k]^{[n]}$ such that $s_{\tilde f(\,\cdot\,)}=f(\,\cdot\,)$. We say that two k-colorings f_1,f_2 of $[n]$ are *G-equivalent* if $f_1\circ\pi=f_2$ for all π in G. We denote the family of $[k]$-colorings of $[n]$ by $\mathcal{C}^k_n(\mathbf{s})$.

Theorem 2.4.2 (Pólya [134, §4]). *Let $G < \mathfrak{S}_n$ be a permutation group acting on $[n]$ and $a_{h_1,\dots,h_k}$ be the number of G-inequivalent k-colorings of $[n]$ into k colors with exactly h_i occurrences of the i^{th} color. Then, the (multivariate) generating function $\mathbf{G}[a_{h_1,\dots,h_k}](\mathbf{t})$ satisfies*

$$\mathbf{G}[a_{h_1,\dots,h_k}](\mathbf{t}) = Z^G\left(p_{k,1}[\mathbf{t}],\dots,p_{k,n}[\mathbf{t}]\right) \ , \tag{2.4.1}$$

where $p_{k,i}[\mathbf{t}] \coloneqq \mathbf{1}\cdot\mathbf{t}^{\diamond i}$ with $\mathbf{1}\in\mathbb{R}^k$ denotes the i^{th} k-variate power-sum symmetric polynomial.

In the following we consider an extension of PET to multisets of colors, and explore its connections — arising in the case $G = \mathfrak{S}_n$ — with the Dirichlet distribution $\mathsf{D}_{\boldsymbol{\alpha}}$. A different approach in terms of colorings, limited to the case $\boldsymbol{\alpha}_\bullet = 1$, was briefly sketched in [86, §7]. The purpose of this section is that to revisit the key idea of *'color-respecting' permutations* in [86, p. 112] in the well-established framework of PET.

Let now $[\![\mathbf{s}_{\boldsymbol{\alpha}}]\!]$ be an integer-valued multiset with $\boldsymbol{\alpha}\in\mathbb{N}_1^k$, henceforth a *palette*. As before, we understand the elements $s_1,\dots,s_k$ of its underlying set $[\mathbf{s}]$ as placeholders for different colors, and the elements $(s_i,1),\dots,(s_i,\alpha_i)$ of $[\mathbf{s}_{\boldsymbol{\alpha}}]$ as placeholders for different *shades* of the same color s_i.

Definition 2.4.3 (Shadings). An $\boldsymbol{\alpha}$-*shading* of $[n]$ is any function φ in $[\mathbf{s}_{\boldsymbol{\alpha}}]^{[n]}$, where $\boldsymbol{\alpha}\in\mathbb{N}_1^k$. To each $\boldsymbol{\alpha}$-shading of $[n]$ we associate uniquely a $[k]$-coloring of $[n]$ by letting $f(\,\cdot\,)\coloneqq\varphi(\,\cdot\,)_1$. This association (trivially surjective) just amounts to forget information about the shade and only retain information about the color. We say that two $\boldsymbol{\alpha}$-shadings φ_1,φ_2 of $[n]$ are *G-equivalent* if so are the corresponding $[k]$-colorings f_1,f_2 of $[n]$. We denote the family of $\boldsymbol{\alpha}$-shadings of $[n]$ by $\mathcal{S}_n^k(\mathbf{s}_{\boldsymbol{\alpha}})$.

Corollary 2.4.4 (Counting shadings). *Let $G < \mathfrak{S}_n$ be a permutation group acting on $[n]$ and $b^{\boldsymbol{\alpha}}_{h_1,\dots,h_k}$ be the number of G-inequivalent $\boldsymbol{\alpha}$-shadings of $[n]$ with exactly h_i occurrences of the i^{th} color. Then,*

$$\mathbf{G}[b^{\boldsymbol{\alpha}}_{h_1,\dots,h_k}](\mathbf{s}) = Z^G\left(\boldsymbol{\alpha}\cdot\mathbf{s}^{\diamond 1},\dots,\boldsymbol{\alpha}\cdot\mathbf{s}^{\diamond n}\right) \ , \ \mathbf{s}\in\mathbb{R}^k \ .$$

Proof. For each $i\le k$ set $r_i\coloneqq(\alpha_1,\dots,\alpha_i)_\bullet$ and $r_0\coloneqq 0$. Notice that $r_k=\boldsymbol{\alpha}_\bullet$. For every r_k-coloring g of $[n]$ let

$$\varphi_{\boldsymbol{\alpha}}[g](x)\coloneqq(s_i,\tilde g(x)-r_{i-1}) \quad \text{if} \quad \tilde g(x)\in\{r_{i-1}+1,\dots,r_i\}$$

varying $i\in[k]$ and $x\in[n]$. It is readily seen that, for every fixed $\boldsymbol{\alpha}\in\mathbb{N}_1^k$, the map

$$\begin{aligned} Q_{\boldsymbol{\alpha}}\colon \mathcal{C}_n^{\boldsymbol{\alpha}_\bullet}(\mathbf{s}) &\longrightarrow \mathcal{S}_n^k(\mathbf{s}_{\boldsymbol{\alpha}}) \\ g &\longmapsto \varphi_{\boldsymbol{\alpha}}[g] \end{aligned}$$

is bijective and preserves G-equivalence. Thus, the number $a_{h_{1,1},\dots,h_{1,\alpha_1},\dots,h_{k,1},\dots,h_{k,\alpha_k}}$ of G-inequivalent r_k-colorings of $[n]$ with exactly $h_{i,j}$ occurrences of the $(r_{i-1}+j)^{\text{th}}$ color is also the number of G-inequivalent $\boldsymbol{\alpha}$-shadings of $[n]$ with exactly $h_{i,j}$ occurrences of the j^{th} shade of the i^{th} color. By Theorem 2.4.2 this is the coefficient of the monomial

$$t_1^{h_{1,1}}\cdots t_{r_1}^{h_{1,\alpha_1}}\cdots t_{r_{k-1}+1}^{h_{k,1}}\cdots t_{r_k}^{h_{k,\alpha_k}}$$

in $Z^G\left(\mathbf{1}\cdot\mathbf{t},\dots,\mathbf{1}\cdot\mathbf{t}^{\diamond n}\right)$ with $\mathbf{1}\in\mathbb{R}^{r_k}$. By definition,

$$b^{\boldsymbol{\alpha}}_{h_1,\dots,h_k} = \sum_{\substack{h_{1,1},\dots,h_{1,\alpha_1},\dots,h_{k,1},\dots,h_{k,\alpha_k} \\ \sum_j^{\alpha_i} h_{i,j}=h_i}} a_{h_{1,1},\dots,h_{1,\alpha_1},\dots,h_{k,1},\dots,h_{k,\alpha_k}} \ ,$$

which equals the coefficient of the monomial $s_1^{h_1}\dots s_k^{h_k}$ in

$$Z^G\left(\mathbf{1}\cdot\mathbf{t}^{\diamond 1},\dots,\mathbf{1}\cdot\mathbf{t}^{\diamond n}\right) = Z^G\left(\boldsymbol{\alpha}\cdot\mathbf{s}^{\diamond 1},\dots,\boldsymbol{\alpha}\cdot\mathbf{s}^{\diamond n}\right) \ , \ \mathbf{t}\coloneqq(\underbrace{s_1,\dots,s_1}_{\alpha_1},\dots,\underbrace{s_k,\dots,s_k}_{\alpha_k}) \ . \qquad \blacksquare$$

Corollary 2.4.5. *For fixed $\boldsymbol{\alpha} \in \mathbb{N}_1^k$, denote by $\mathcal{S}_n^{\boldsymbol{\alpha}}$ the set of $\mathfrak{S}_n$-equivalence classes $\varphi^\bullet$ of $\boldsymbol{\alpha}$-shadings of $[n]$. Then, the probability $p^{\boldsymbol{\alpha}}_{h_1,\dots,h_k}$ of some $\varphi^\bullet$ uniformly drawn from $\mathcal{S}_n^{\boldsymbol{\alpha}}$ having exactly h_i occurrences of the i^{th} color satisfies*

$$\mathbf{G}[p^{\boldsymbol{\alpha}}_{h_1,\dots,h_k}](\mathbf{s}) = \mu_n'[\mathbf{s}, \boldsymbol{\alpha}] \ .$$

Proof. By Corollary 2.4.4,

$$\#\mathcal{S}_n^{\boldsymbol{\alpha}} = \sum_{\substack{\mathbf{h}\in\mathbb{N}_0^k \\ \mathbf{h}_\bullet = n}} b^{\boldsymbol{\alpha}}_{\mathbf{h}} = \mathbf{G}[b^{\boldsymbol{\alpha}}_{h_1,\dots,h_k}](\mathbf{1}) = \langle \boldsymbol{\alpha}_\bullet \rangle_n / n! \tag{2.4.2}$$

By definition, $p^{\boldsymbol{\alpha}}_{h_1,\dots,h_k} = (\#\mathcal{S}_n^{\boldsymbol{\alpha}})^{-1} b^{\boldsymbol{\alpha}}_{h_1,\dots,h_k}$, hence, by homogeneity,

$$\mathbf{G}[p^{\boldsymbol{\alpha}}_{h_1,\dots,h_k}](\mathbf{s}) = (\#\mathcal{S}_n^{\boldsymbol{\alpha}})^{-1}\, \mathbf{G}[b^{\boldsymbol{\alpha}}_{h_1,\dots,h_k}](\mathbf{s}) \ .$$

The conclusion follows by (2.4.2), Corollary 2.4.4 and Theorem 2.3.3. ■

The study of $\mathsf{D}_{\boldsymbol{\alpha}}$ in the case when $\boldsymbol{\alpha}_\bullet = 1$ is singled out as computationally easiest (as suggested by Theorem 2.3.3, noticing that $\langle 1 \rangle_n = n!$), $\boldsymbol{\alpha}$ representing in that case a *probability* on $[k]$, as detailed in §2.2. For these reasons, this is often the only case considered. (Cf. e.g., [86].) On the other hand though, the general case when $\boldsymbol{\alpha} > \mathbf{0}$ is the one relevant in Bayesian non-parametrics, since posterior distributions of Dirichlet-categorical and Dirichlet-multinomial priors do not have probability intensity. The above coloring problem suggests that the case when $\boldsymbol{\alpha} \in \mathbb{N}_1^k$ is interesting from the point of view of PET, since it allows for some natural operations on palettes, corresponding to functionals of the distribution.

Indeed, we can change the number of colors and shades in a palette $[\![\mathbf{s}_{\boldsymbol{\alpha}}]\!]$ by composing any permutation of the indices $[k]$ with the following elementary operations:

- (i) 'widen', respectively (ii) 'narrow the color spectrum', by adding a color, say s_{k+1}, respectively removing a color, say s_k. That is, we consider new palettes $[\![(\mathbf{s} \oplus s_{k+1})_{\boldsymbol{\alpha}\oplus\alpha_{k+1}}]\!]$, respectively $[\![(s_1, \dots, s_{k-1})_{(\alpha_1,\dots,\alpha_{k-1})}]\!]$;
- (iii) 'reduce color resolution' by regarding two different colors, say s_i and s_{i+1}, as the same, relabeled s_i. In so doing we regard the shades of the former colors as distinct shades of the new one, so that it has $\alpha_i + \alpha_{i+1}$ shades. That is, we consider the new palette $[\![(\mathbf{s}_{\hat{\imath}})_{\boldsymbol{\alpha}_{+i}}]\!]$;
- (iv) 'enlarge', respectively (v) 'reduce the color depth', by adding a shade, say the α_{i+1}^{th}, to the color s_i, respectively removing a shade, say the α_i^{th}, to the color s_i. This latter operation we allow only if $\alpha_i > 1$, so to make it distinct from removing the color s_i from the palette. That is, we consider the new palettes $[\![\mathbf{s}_{\boldsymbol{\alpha}+\mathbf{e}_i}]\!]$, resp. $[\![\mathbf{s}_{\boldsymbol{\alpha}-\mathbf{e}_i}]\!]$ when $\alpha_i > 1$.

Increasing the color resolution of a multi-shaded color, say s_k with $\alpha_k > 1$ shades, by splitting it into two colors, say s_k^* and s_{k+1} with $\alpha_k^* > 0$ and $\alpha_{k+1} > 0$ shades respectively and such that $\alpha_k^* + \alpha_{k+1} = \alpha_k$, is not an elementary operation. It can be obtained by widening the spectrum of the palette by adding a color s_{k+1} with α_{k+1} shades and reducing the color depth of the color s_k to α_k^*. Thus, this operation is not listed above. We do not allow for the number of shades of a color to be reduced to zero: although this is morally equivalent to removing that color, the latter operation amounts more rigorously to remove the color placeholder from the palette.

The said elementary operations are of two distinct kinds: (i)–(iii) alter the number of colors in a palette, while (iv)–(v) fix it. We restrict our attention to the latter ones and ask how the probability $p^{\boldsymbol{\alpha}}_{h_1,\dots,h_k}$ changes under them. By Corollary 2.4.5 this is equivalent to study the corresponding functionals of the n^{th} moment of the Dirichlet distribution. For fixed k, we address all the moments at once, by studying the moment generating function

$$ {}_k\Phi_2[\boldsymbol{\alpha};\boldsymbol{\alpha}_\bullet;t\mathbf{s}] = \mathbf{G}_{\exp}[\mathbf{G}[p^{\boldsymbol{\alpha}}_{h_1,\dots,h_k}](\mathbf{s})](t) \ . $$

Namely, we look for natural transformations yielding the mappings

$$ E_{\pm i}\ {}_k\Phi_2[\boldsymbol{\alpha};\boldsymbol{\alpha}_\bullet;\mathbf{s}] = C_{\boldsymbol{\alpha}}\ {}_k\Phi_2[\boldsymbol{\alpha}\pm\mathbf{e}_i;\boldsymbol{\alpha}_\bullet\pm 1;\mathbf{s}] \ , \tag{2.4.3} $$

where $C_{\boldsymbol{\alpha}}$ is some constant, possibly dependent on $\boldsymbol{\alpha}$. Here 'natural' means that we only allow for meaningful *linear* operations on generating functions: addition, scalar multiplication by variables or constants, differentiation and integration. For practical reasons, it is convenient to consider the following construction.

Definition 2.4.6 (Dynamical symmetry algebra of ${}_k\Phi_2$)**.** Denote by $\mathfrak{g}_k$ the minimal semi-simple Lie algebra containing the linear span of the operators $E_{\pm 1},\dots,E_{\pm k}$ in (2.4.3) and endowed with the bracket induced by their composition. Following [121], we term the Lie algebra $\mathfrak{g}_k$ the *dynamical symmetry algebra* of the function ${}_k\Phi[\boldsymbol{\alpha};\mathbf{s}] \coloneqq {}_k\Phi_2[\boldsymbol{\alpha};\boldsymbol{\alpha}_\bullet;\mathbf{s}]$, characterized below.

Dynamical symmetry algebras. We compute now the dynamical symmetry algebra of the function ${}_k\Phi[\boldsymbol{\alpha};\mathbf{s}] \coloneqq {}_k\Phi_2[\boldsymbol{\alpha};\boldsymbol{\alpha}_\bullet;\mathbf{s}]$, in this section always regarded as the meromorphic extension (2.2.9) of the Fourier transform of $\widehat{\mathsf{D}_{\boldsymbol{\alpha}}}(\mathbf{s})$ in the *complex* variables $\boldsymbol{\alpha},\mathbf{s}\in\mathbb{C}^k$. The choice of complex variables is merely motivated by this identification and every result in the following concerned with complex Lie algebras holds verbatim for their split real form. For dynamical symmetry algebras of Lauricella hypergeometric functions see [121, 122] and references therein; we refer to [77] for the general theory of Lie algebra (representations) and for Weyl groups' theory.

Notation and definitions. Denote by $\mathbf{E}_{i,j}$ varying $i,j\in[k+1]$ the canonical basis of $\mathrm{Mat}_{k+1}(\mathbb{C})$, with $[\mathbf{E}_{i,j}]_{m,n} = \delta_{mi}\,\delta_{nj}$, where δ_{ab} is the Kronecker delta, and by A^* the conjugate transpose of a matrix A.

Lemma 2.4.7 (A presentation of $\mathfrak{sl}_{k+1}(\mathbb{C})$)**.** *For $i,j=0,\dots,k$ with $j>i$ let*

$$ e_{i,j} \coloneqq \mathbf{E}_{i',j'} \ , \ h_{i,j} \coloneqq \mathbf{E}_{i',i'} - \mathbf{E}_{j',j'} \ , \ f_{j,i} \coloneqq e^*_{i,j} \ . $$

Then, the complex Lie sub-algebra $\mathfrak{l}_k$ of $\mathfrak{gl}_{k+1}(\mathbb{C})$ generated by these vectors is $\mathfrak{l}_k = \mathfrak{sl}_{k+1}(\mathbb{C})$, with generators the $\mathfrak{sl}_2$-triples $\{e_{i,i'},h_{i,i'},f_{i',i}\}_{i=0,\dots,k-1}$. Denote further by $\mathfrak{f}_k < \mathfrak{l}_k$ the sub-algebra spanned by $\{e_{i,j},f_{j,i},h_{i,j}\}_{i,j\in[k]}$. Then, $\mathfrak{f}_k \cong \mathfrak{sl}_k(\mathbb{C})$.

Proof. It suffices to verify Serre's relations of type A (See e.g., [77, §18.1].) Let $i,p=0,\dots,k$, $j>i$, $q>p$ and notice that

$$ [h_{i,j}]_{a,b} = (\delta_{i'a}-\delta_{j'a})\delta_{ab} \ , \qquad [e_{i,j}]_{a,b} = \delta_{i'a}\delta_{j'b} \ , \qquad [f_{j,i}]_{a,b} = \delta_{j'a}\delta_{i'b} \ . $$

Then, concerning [77, p. 96, (S_1)],

$$ \begin{aligned} [h_{i,j},h_{p,q}]_{a,b} =& \sum_{c=0}^k [h_{i,j}]_{a,c}[h_{p,q}]_{c,b} - \sum_{c=0}^k [h_{p,q}]_{a,c}[h_{i,j}]_{c,b} \\ =& \sum_{c=0}^k \big((\delta_{i'a}-\delta_{j'a})\delta_{ac}(\delta_{p'c}-\delta_{q'c})\delta_{cb} - (\delta_{p'a}-\delta_{q'a})\delta_{ac}(\delta_{i'c}-\delta_{j'c})\delta_{cb}\big) \\ =& \delta_{ab}(\delta_{i'a}-\delta_{j'a})(\delta_{p'a}-\delta_{q'a}) - \delta_{ab}(\delta_{p'a}-\delta_{q'a})(\delta_{i'a}-\delta_{j'a}) \\ =& 0 \ , \end{aligned} \tag{2.4.4} $$

concerning [77, p. 96, (S_2)],

$$
\begin{aligned}
[e_{i,j}, f_{q,p}]_{a,b} &= \sum_{c=0}^{k} [e_{i,j}]_{a,c}[f_{q,p}]_{c,b} - \sum_{c=0}^{k} [f_{q,p}]_{a,c}[e_{i,j}]_{c,b} \\
&= \sum_{c=0}^{k} \left(\delta_{i'a}\delta_{j'c}\delta_{q'c}\delta_{p'b} - \delta_{q'a}\delta_{p'c}\delta_{i'c}\delta_{j'b} \right) \\
&= \delta_{i'a}\delta_{jq}\delta_{p'b} - \delta_{q'a}\delta_{ip}\delta_{j'b} \\
\overset{(j=i',\ q=p')}{} &= \delta_{ip}[h_{i,j}]_{a,b} \ ,
\end{aligned}
\tag{2.4.5}
$$

concerning [77, p. 96, (S_3)],

$$
\begin{aligned}
[h_{i,j}, e_{p,q}]_{a,b} &= \sum_{c=0}^{k} [h_{i,j}]_{a,c}[e_{p,q}]_{c,b} - \sum_{c=0}^{k} [e_{p,q}]_{a,c}[h_{i,j}]_{c,b} \\
&= \sum_{c=0}^{k} \left((\delta_{i'a} - \delta_{j'a})\delta_{ac}\delta_{p'c}\delta_{q'b} - \delta_{p'a}\delta_{q'c}(\delta_{i'c} - \delta_{j'c})\delta_{cb} \right) \\
&= (\delta_{i'a} - \delta_{j'a})\delta_{p'a}\delta_{q'b} - \delta_{p'a}\delta_{q'b}(\delta_{i'b} - \delta_{j'b}) \\
&= (\delta_{i'p'} + \delta_{j'q'} - \delta_{i'q'} - \delta_{j'p'})[e_{p,q}]_{a,b} \\
\overset{(j=i',\ q=p')}{} &= (2\delta_{ip} - \delta_{ip'} - \delta_{i'p})[e_{p,q}]_{a,b}
\end{aligned}
\tag{2.4.6}
$$

and

$$
\begin{aligned}
[h_{i,j}, f_{q,p}]_{a,b} &= \sum_{c=0}^{k} [h_{i,j}]_{a,c}[f_{q,p}]_{c,b} - \sum_{c=0}^{k} [f_{q,p}]_{a,c}[h_{i,j}]_{c,b} \\
&= \sum_{c=0}^{k} \left((\delta_{i'a} - \delta_{j'a})\delta_{ac}\delta_{p'c}\delta_{q'b} - \delta_{p'a}\delta_{q'c}(\delta_{i'c} - \delta_{j'c})\delta_{cb} \right) \\
&= (\delta_{i'a} - \delta_{j'a})\delta_{p'a}\delta_{q'b} - \delta_{p'a}\delta_{q'b}(\delta_{i'b} - \delta_{j'b}) \\
&= (\delta_{i'p'} + \delta_{j'q'} - \delta_{j'p'} - \delta_{i'q'})[f_{q,p}]_{a,b} \\
\overset{(j=i',\ q=p')}{} &= -(2\delta_{ip} - \delta_{ip'} - \delta_{i'p})[f_{q,p}]_{a,b}
\end{aligned}
\tag{2.4.7}
$$

Finally, concerning [77, p. 96, (S^+_{ij})],

$$
\begin{aligned}
[e_{i,j}, e_{q,p}]_{a,b} &= \sum_{c=0}^{k} [e_{i,j}]_{a,c}[e_{q,p}]_{c,b} - \sum_{c=0}^{k} [e_{q,p}]_{a,c}[e_{i,j}]_{c,b} \qquad (i,j) \neq (p,q) \\
&= \sum_{c=0}^{k} \left(\delta_{i'a}\delta_{j'c}\delta_{p'c}\delta_{q'b} - \delta_{p'a}\delta_{q'c}\delta_{i'c}\delta_{j'b} \right) \\
&= \delta_{i'a}\delta_{j'p'}\delta_{q'b} - \delta_{p'a}\delta_{q'i'}\delta_{j'b} \\
\overset{(j=i',\ q=p')}{} &= \delta_{pa}\delta_{p''b} - \delta_{ia}\delta_{i''b} \qquad (i \neq p) \\
&= \delta_{p''a''}\delta_{p''b} - \delta_{i''a''}\delta_{i''b} \\
&= \delta_{a''b} - \delta_{a''b} \\
&= 0 \ .
\end{aligned}
\tag{2.4.8}
$$

As usual, [77, p. 96, (S^-_{ij})] follows from (S^+_{ij}) by symmetry. ■

Everywhere in the following we regard $\mathfrak{l}_k$ together with the distinguished Cartan sub-algebra $\mathfrak{h}_k < \mathfrak{l}_k$ of diagonal traceless matrices spanned by the basis $\{h_{0,j}\}_{j\in[k]}$; the root system Ψ_k

induced by $\mathfrak{h}_k$, with simple roots γ_j corresponding to the $\mathfrak{sl}_2$-triples of the vectors $e_{i,i'}$ for $i \in [k]$; positive, resp. negative, roots $\Psi_k^\pm$ corresponding to the spaces of strictly upper, resp. strictly lower, triangular matrices $\mathfrak{n}_k^\pm$. The inclusion $\mathfrak{f}_k < \mathfrak{l}_k$ induces the decomposition of vector spaces (*not* of algebras)

$$\mathfrak{l}_k = \mathfrak{r}_k^- \oplus \mathfrak{h}_1 \oplus \mathfrak{f}_k \oplus \mathfrak{r}_k^+ \ , \text{ where } \qquad \mathfrak{r}_k^+ := \mathbb{C}\left\{e_{0,j}\right\}_{j\in[k]} \ , \ \mathfrak{r}_k^- := \mathbb{C}\left\{f_{j,0}\right\}_{j\in[k]} \ , \ \mathfrak{h}_1 = \mathbb{C}\left\{h_{0,1}\right\} \ .$$

The subscript k is omitted whenever apparent from context.

For fixed $\boldsymbol{\alpha} \in \mathbb{C}^k$ regard ${}_k\Phi[\boldsymbol{\alpha}; \cdot\,]$ as a formal power series and let $f_{\boldsymbol{\alpha}} \colon \mathbb{C}^{2k+1}_{\mathbf{s},\mathbf{u},t} \longrightarrow \mathbb{C}$ be

$$f_{\boldsymbol{\alpha}} = f_{\boldsymbol{\alpha}}(\mathbf{s},\mathbf{u},t) := {}_k\Phi[\boldsymbol{\alpha};\mathbf{s}]\mathbf{u}^{\boldsymbol{\alpha}} t^{\boldsymbol{\alpha}_\bullet} \ . \tag{2.4.9}$$

Let $\mathrm{A} \subset \mathbb{C}^k$. It is readily seen that the functions $\{f_{\boldsymbol{\alpha}}\}_{\boldsymbol{\alpha}\in\mathrm{A}}$ are (finitely) linearly independent, since so are the functions $\{f_{\boldsymbol{\alpha}}(\mathbf{1},\mathbf{u},1) \propto \mathbf{u}^{\boldsymbol{\alpha}}\}_{\boldsymbol{\alpha}\in\mathrm{A}}$. Set

$$\mathcal{O}_{\mathrm{A}} := \bigoplus_{\boldsymbol{\alpha}\in\mathrm{A}} \mathbb{C}\{f_{\boldsymbol{\alpha}}\} \ , \ \mathcal{O}_{\boldsymbol{\alpha}} := \mathcal{O}_{\{\boldsymbol{\alpha}\}} \ , \ \mathcal{O} := \mathcal{O}_{\mathbb{C}^k}$$

and define the following differential operators, acting formally on $\mathcal{O}$,

$$\begin{aligned} E_{\alpha_i} &:= u_i t(s_i\partial_{s_i} + u_i\partial_{u_i} - (\mathbf{s}\cdot\nabla^{\mathbf{s}})\partial_{s_i}) \ , \qquad & E_{\alpha_i,-\alpha_j} &:= u_i u_j^{-1}\left((u_i-u_j)\partial_{s_i} + u_i\partial_{u_i}\right) \ , \\ E_{-\alpha_i} &:= (u_i t)^{-1}(s_i - \mathbf{s}\cdot\nabla^{\mathbf{s}} - t\partial_t + 1) \ , & J_{\alpha_i} &:= t\partial_t + u_i\partial_{u_i} - 1 \ , \end{aligned} \tag{2.4.10}$$

where $i,j \in [k]$, $i \neq j$ and $\nabla^{\mathbf{y}} := (\partial_{y_1},\dots,\partial_{y_k})$ for $\mathbf{y} = \mathbf{u},\mathbf{s}$. Term the operators E_{α_i}, resp. $E_{-\alpha_i}$, *raising*, resp. *lowering*, *operators*. Finally, let $\mathfrak{g}_k$ be the complex linear span of the operators (2.4.10) endowed with the bracket induced by their composition.

Actions on spaces of holomorphic functions. For $\boldsymbol{\alpha} \in \mathbb{C}^k$ set $\Lambda_{\boldsymbol{\alpha}} := \boldsymbol{\alpha} + \mathbb{Z}^k$ and, for every $\ell \in \mathbb{R}_+$,

$$\Lambda^+_{\boldsymbol{\alpha}} := \{\boldsymbol{\varepsilon} \in \Lambda_{\boldsymbol{\alpha}} \mid \boldsymbol{\varepsilon}_\bullet > 0\} \ , \ H^\pm_{\boldsymbol{\alpha}} := \boldsymbol{\alpha} \pm \mathbb{N}_0^k \ , \ M_{\boldsymbol{\alpha},\ell} := \left\{\boldsymbol{\varepsilon} \in \Lambda^+_{\boldsymbol{\alpha}} \mid \boldsymbol{\varepsilon}_\bullet = \ell\right\} \ .$$

Notice that if $\Re^\diamond\boldsymbol{\alpha} > \mathbf{0}$, the space $\mathcal{O}_{\Lambda^+_{\boldsymbol{\alpha}}}$ is a space of holomorphic functions $\mathcal{O}(\mathbb{C}^k_{\mathbf{s}} \times (\mathbb{C}\setminus\mathbb{R}_0^-)^{k+1}_{\mathbf{u},t})$, where we choose $\mathbb{R}_0^-$ as branch cut for the complex logarithm in the variables $\mathbf{u}$ and t. The same holds for $\mathcal{O}_{\Lambda_{\boldsymbol{\alpha}}}$ if $\boldsymbol{\alpha}_\bullet \notin \mathbb{Z}$.

Lemma 2.4.8 (Raising/lowering actions)**.** *The operators* (2.4.10) *satisfy, for $i,j \in [k]$, $j \neq i$,*

$$\begin{aligned} E_{\alpha_i} f_{\boldsymbol{\alpha}} &= \alpha_i f_{\boldsymbol{\alpha}+\mathbf{e}_i} \ , \qquad & E_{-\alpha_i} f_{\boldsymbol{\alpha}} &= (1-\boldsymbol{\alpha}_\bullet) f_{\boldsymbol{\alpha}-\mathbf{e}_i} \ , \\ E_{\alpha_i,-\alpha_j} f_{\boldsymbol{\alpha}} &= \alpha_i f_{\boldsymbol{\alpha}+\mathbf{e}_i-\mathbf{e}_j} \ , & J_{\alpha_i} f_{\boldsymbol{\alpha}} &= (\boldsymbol{\alpha}_\bullet + \alpha_i - 1) f_{\boldsymbol{\alpha}} \ . \end{aligned} \tag{2.4.11}$$

Proof. The statement on J_{α_i} is straightforward. Moreover,

$$\begin{aligned} E_{\alpha_i,-\alpha_j} f_{\boldsymbol{\alpha}} =& \mathbf{u}^{\boldsymbol{\alpha}+\mathbf{e}_i-\mathbf{e}_j} t^{\boldsymbol{\alpha}_\bullet} \left(\sum_{\mathbf{m}\geq\mathbf{0}} \frac{\langle\boldsymbol{\alpha}\rangle_{\mathbf{m}} (m_i+\alpha_i)\mathbf{s}^{\mathbf{m}}}{\langle\boldsymbol{\alpha}_\bullet\rangle_{\mathbf{m}_\bullet} \mathbf{m}!} - \frac{\langle\boldsymbol{\alpha}\rangle_{\mathbf{m}} m_i \mathbf{s}^{\mathbf{m}-\mathbf{e}_i+\mathbf{e}_j}}{\langle\boldsymbol{\alpha}_\bullet\rangle_{\mathbf{m}_\bullet} \mathbf{m}!} \right) \\ =& \mathbf{u}^{\boldsymbol{\alpha}+\mathbf{e}_i-\mathbf{e}_j} t^{\boldsymbol{\alpha}_\bullet} \left(\sum_{\mathbf{m}\geq\mathbf{0}} \frac{\langle\boldsymbol{\alpha}\rangle_{\mathbf{m}} (m_i+\alpha_i)\mathbf{s}^{\mathbf{m}}}{\langle\boldsymbol{\alpha}_\bullet\rangle_{\mathbf{m}_\bullet} \mathbf{m}!} - \frac{\langle\boldsymbol{\alpha}\rangle_{\mathbf{m}+\mathbf{e}_i-\mathbf{e}_j} (m_i+1)\mathbf{s}^{\mathbf{m}}}{\langle\boldsymbol{\alpha}_\bullet\rangle_{\mathbf{m}_\bullet} (\mathbf{m}+\mathbf{e}_i-\mathbf{e}_j)!} \right) \\ =& \mathbf{u}^{\boldsymbol{\alpha}+\mathbf{e}_i-\mathbf{e}_j} t^{\boldsymbol{\alpha}_\bullet} \frac{\alpha_i}{\alpha_j - 1} \times \\ & \times \left(\sum_{\mathbf{m}\geq\mathbf{0}} \frac{\langle\boldsymbol{\alpha}+\mathbf{e}_i-\mathbf{e}_j\rangle_{\mathbf{m}-\mathbf{e}_i+\mathbf{e}_j} (m_i+\alpha_i)\mathbf{s}^{\mathbf{m}}}{\langle\boldsymbol{\alpha}_\bullet\rangle_{\mathbf{m}_\bullet} \mathbf{m}!} - \frac{\langle\boldsymbol{\alpha}+\mathbf{e}_i-\mathbf{e}_j\rangle_{\mathbf{m}} \mathbf{s}^{\mathbf{m}}}{\langle\boldsymbol{\alpha}_\bullet\rangle_{\mathbf{m}_\bullet} (\mathbf{m}-\mathbf{e}_j)!} \right) \\ =& \mathbf{u}^{\boldsymbol{\alpha}+\mathbf{e}_i-\mathbf{e}_j} t^{\boldsymbol{\alpha}_\bullet} \frac{\alpha_i}{\alpha_j - 1} \times \end{aligned}$$

$$
\begin{aligned}
&\times\left(\sum_{\mathbf{m}\geq\mathbf{0}} \frac{\langle\boldsymbol{\alpha}+\mathbf{e}_i-\mathbf{e}_j\rangle_{\mathbf{m}}(m_j+\alpha_j-1)\mathbf{s}^{\mathbf{m}}}{\langle\boldsymbol{\alpha}_\bullet\rangle_{\mathbf{m}_\bullet}\,\mathbf{m}!}-\frac{\langle\boldsymbol{\alpha}+\mathbf{e}_i-\mathbf{e}_j\rangle_{\mathbf{m}}\,m_j\mathbf{s}^{\mathbf{m}}}{\langle\boldsymbol{\alpha}_\bullet\rangle_{\mathbf{m}_\bullet}\,\mathbf{m}!}\right)\\
&=\alpha_i f_{\boldsymbol{\alpha}+\mathbf{e}_i-\mathbf{e}_j}\ ,\\
E_{\alpha_i} f_{\boldsymbol{\alpha}} &=\mathbf{u}^{\boldsymbol{\alpha}+\mathbf{e}_i} t^{\boldsymbol{\alpha}_\bullet+1}\left(\sum_{\mathbf{m}\geq\mathbf{0}} \frac{\langle\boldsymbol{\alpha}\rangle_{\mathbf{m}}(m_i+\alpha_i)\mathbf{s}^{\mathbf{m}}}{\langle\boldsymbol{\alpha}_\bullet\rangle_{\mathbf{m}_\bullet}\,\mathbf{m}!}-\frac{\langle\boldsymbol{\alpha}\rangle_{\mathbf{m}}\,m_i(\mathbf{m}_\bullet-1)\mathbf{s}^{\mathbf{m}-\mathbf{e}_i}}{\langle\boldsymbol{\alpha}_\bullet\rangle_{\mathbf{m}_\bullet}\,\mathbf{m}!}\right)\\
&=\mathbf{u}^{\boldsymbol{\alpha}+\mathbf{e}_i} t^{\boldsymbol{\alpha}_\bullet+1}\left(\sum_{\mathbf{m}\geq\mathbf{0}} \frac{\langle\boldsymbol{\alpha}\rangle_{\mathbf{m}+\mathbf{e}_i}\,\mathbf{s}^{\mathbf{m}}}{\langle\boldsymbol{\alpha}_\bullet\rangle_{\mathbf{m}_\bullet}\,\mathbf{m}!}-\frac{\langle\boldsymbol{\alpha}\rangle_{\mathbf{m}}(\mathbf{m}_\bullet-1)\mathbf{s}^{\mathbf{m}-\mathbf{e}_i}}{\langle\boldsymbol{\alpha}_\bullet\rangle_{\mathbf{m}_\bullet}\,(\mathbf{m}-\mathbf{e}_i)!}\right)\\
&=\mathbf{u}^{\boldsymbol{\alpha}+\mathbf{e}_i} t^{\boldsymbol{\alpha}_\bullet+1}\left(\sum_{\mathbf{m}\geq\mathbf{0}} \frac{\langle\boldsymbol{\alpha}\rangle_{\mathbf{m}+\mathbf{e}_i}\,\mathbf{s}^{\mathbf{m}}}{\langle\boldsymbol{\alpha}_\bullet\rangle_{\mathbf{m}_\bullet}\,\mathbf{m}!}-\frac{\langle\boldsymbol{\alpha}\rangle_{\mathbf{m}+\mathbf{e}_i}\,\mathbf{m}_\bullet\mathbf{s}^{\mathbf{m}}}{\langle\boldsymbol{\alpha}_\bullet\rangle_{\mathbf{m}_\bullet+1}\,\mathbf{m}!}\right)\\
&=\mathbf{u}^{\boldsymbol{\alpha}+\mathbf{e}_i} t^{\boldsymbol{\alpha}_\bullet+1}\,\frac{\alpha_i}{\boldsymbol{\alpha}_\bullet}\left(\sum_{\mathbf{m}\geq\mathbf{0}} \frac{\langle\boldsymbol{\alpha}+\mathbf{e}_i\rangle_{\mathbf{m}}\,\mathbf{s}^{\mathbf{m}}}{\langle\boldsymbol{\alpha}_\bullet+1\rangle_{\mathbf{m}_\bullet-1}\,\mathbf{m}!}-\frac{\langle\boldsymbol{\alpha}+\mathbf{e}_i\rangle_{\mathbf{m}}\,\mathbf{m}_\bullet\mathbf{s}^{\mathbf{m}}}{\langle\boldsymbol{\alpha}_\bullet+1\rangle_{\mathbf{m}_\bullet}\,\mathbf{m}!}\right)\\
&=\mathbf{u}^{\boldsymbol{\alpha}+\mathbf{e}_i} t^{\boldsymbol{\alpha}_\bullet+1}\,\frac{\alpha_i}{\boldsymbol{\alpha}_\bullet}\left(\sum_{\mathbf{m}\geq\mathbf{0}} \frac{\langle\boldsymbol{\alpha}+\mathbf{e}_i\rangle_{\mathbf{m}}\,\mathbf{s}^{\mathbf{m}}(\boldsymbol{\alpha}_\bullet+\mathbf{m}_\bullet)}{\langle\boldsymbol{\alpha}_\bullet+1\rangle_{\mathbf{m}_\bullet}\,\mathbf{m}!}-\frac{\langle\boldsymbol{\alpha}+\mathbf{e}_i\rangle_{\mathbf{m}}\,\mathbf{m}_\bullet\mathbf{s}^{\mathbf{m}}}{\langle\boldsymbol{\alpha}_\bullet+1\rangle_{\mathbf{m}_\bullet}\,\mathbf{m}!}\right)\\
&=\alpha_i f_{\boldsymbol{\alpha}+\mathbf{e}_i}\ ,\\
E_{-\alpha_i} f_{\boldsymbol{\alpha}} &=\mathbf{u}^{\boldsymbol{\alpha}-\mathbf{e}_i} t^{\boldsymbol{\alpha}_\bullet-1}\left(\sum_{\mathbf{m}\geq\mathbf{0}} \frac{\langle\boldsymbol{\alpha}\rangle_{\mathbf{m}}\,\mathbf{s}^{\mathbf{m}+\mathbf{e}_i}}{\langle\boldsymbol{\alpha}_\bullet\rangle_{\mathbf{m}_\bullet}\,\mathbf{m}!}-\frac{\langle\boldsymbol{\alpha}\rangle_{\mathbf{m}}\,\mathbf{s}^{\mathbf{m}}}{\langle\boldsymbol{\alpha}_\bullet\rangle_{\mathbf{m}_\bullet}\,\mathbf{m}!}(\mathbf{m}_\bullet+\boldsymbol{\alpha}_\bullet-1)\right)\\
&=\mathbf{u}^{\boldsymbol{\alpha}-\mathbf{e}_i} t^{\boldsymbol{\alpha}_\bullet-1}\left(\sum_{\mathbf{m}\geq\mathbf{0}} \frac{\langle\boldsymbol{\alpha}\rangle_{\mathbf{m}-\mathbf{e}_i}\,m_i\mathbf{s}^{\mathbf{m}}}{\langle\boldsymbol{\alpha}_\bullet\rangle_{\mathbf{m}_\bullet-1}\,\mathbf{m}!}-\frac{\langle\boldsymbol{\alpha}\rangle_{\mathbf{m}}\,\mathbf{s}^{\mathbf{m}}}{\langle\boldsymbol{\alpha}_\bullet\rangle_{\mathbf{m}_\bullet}\,\mathbf{m}!}(\mathbf{m}_\bullet+\boldsymbol{\alpha}_\bullet-1)\right)\\
&=\mathbf{u}^{\boldsymbol{\alpha}-\mathbf{e}_i} t^{\boldsymbol{\alpha}_\bullet-1}\,\frac{\boldsymbol{\alpha}_\bullet-1}{\alpha_i-1}\times\\
&\times\left(\sum_{\mathbf{m}\geq\mathbf{0}} \frac{\langle\boldsymbol{\alpha}-\mathbf{e}_i\rangle_{\mathbf{m}}\,m_i\mathbf{s}^{\mathbf{m}}}{\langle\boldsymbol{\alpha}_\bullet-1\rangle_{\mathbf{m}_\bullet}\,\mathbf{m}!}-\frac{\langle\boldsymbol{\alpha}-\mathbf{e}_i\rangle_{\mathbf{m}}(\alpha_i+m_i-1)\mathbf{s}^{\mathbf{m}}}{\langle\boldsymbol{\alpha}_\bullet-1\rangle_{\mathbf{m}_\bullet}(\boldsymbol{\alpha}_\bullet+\mathbf{m}_\bullet-1)\mathbf{m}!}(\mathbf{m}_\bullet+\boldsymbol{\alpha}_\bullet-1)\right)\\
&=(1-\boldsymbol{\alpha}_\bullet) f_{\boldsymbol{\alpha}-\mathbf{e}_i}\ . \qquad\blacksquare
\end{aligned}
$$

Remark 2.4.9. The variables $\mathbf{u}$ and t are merely auxiliary. (Cf. [123, §1].) The operators do not depend on the parameter $\boldsymbol{\alpha}$, rather, the subscripts indicate which indices they affect. Heuristically, the action of the operators (2.4.10) given in Lemma 2.4.8 may be deduced from that [121, Eqn. (1.5)] of operators in the dynamical symmetry algebra of ${}_kF_D$ by a formal contraction procedure [121, p. 1398], letting (in the notation of [121]) $\alpha=0$, $\boldsymbol{\beta}=\boldsymbol{\alpha}$, $\gamma=\boldsymbol{\alpha}_\bullet$ and dropping redundancies. Notice finally that the action of the operator E_{α_j} on $f_{\boldsymbol{\alpha}}$ corresponds to a differentiation in the variable s_j of the moment $\mu'_n[\mathbf{s},\boldsymbol{\alpha}]$, as in (2.3.6).

Remark 2.4.10. If $\boldsymbol{\alpha}_\bullet=1$, the action of the lowering operators $E_{-\alpha_i}$ vanishes on $\mathcal{O}_{\boldsymbol{\alpha}}$. This is natural when regarding $f_{\boldsymbol{\alpha}}$ as a formal power series, whereas it is conventional when regarding $f_{\boldsymbol{\alpha}}$ as a meromorphic function, for the functions $(1-\boldsymbol{\alpha}_\bullet)f_{\boldsymbol{\alpha}-\mathbf{e}_i}$ are in fact — after cancellations — well-defined, not identically vanishing, and holomorphic in $\mathbf{s}$ even for $\boldsymbol{\alpha}_\bullet=1$. The convention here reads $0\times\infty=0$, which is consistent with the usual convention in measure theory when we identify $\boldsymbol{\alpha}_\bullet-1$ with the quantity $(\sigma-\delta_y)X$ for any y in X; the reason for such identification will be apparent in §2.4.2 below.

Corollary 2.4.11. *The operators* (2.4.10) *fix* $\mathcal{O}_{\Lambda_{\boldsymbol{\alpha}}}$ *for any* $\boldsymbol{\alpha}\in\mathbb{C}^k$.

In the statement of the next Lemma and in the diagrams in Fig.s 2.2 and 2.3 we write for simplicity E_i in place of E_{α_i} and analogously for all other operators.

Lemma 2.4.12. *For $\boldsymbol{\alpha} \in \mathbb{C}^k$ consider the operators in $\mathfrak{g}_k$ as restricted to $\mathcal{O}_{\Lambda_{\boldsymbol{\alpha}}}$. The following commutation relations hold:*

$$[J_j - J_i, E_{p,-q}] = \begin{cases} 2E_{p,-q} & \text{if } j = p, i = q \\ -2E_{p,-q} & \text{if } j = q, i = p \\ E_{p,-q} & \text{if } j = p, i \neq q \\ & \quad \text{or } j \neq p, i = q \\ -E_{p,-q} & \text{if } j = q, i \neq p \\ & \quad \text{or } j \neq q, i = p \\ \mathbf{0} & \text{otherwise} \end{cases} , \quad \begin{aligned} [E_i, E_{-p}] &= \begin{cases} J_i & \text{if } i = p \\ E_{i,-p} & \text{otherwise} \end{cases} , \\ [E_i, E_{p,-q}] &= \begin{cases} -E_p & \text{if } i = q \\ \mathbf{0} & \text{otherwise} \end{cases} , \\ [E_{-i}, E_{p,-q}] &= \begin{cases} E_{-q} & \text{if } i = p \\ \mathbf{0} & \text{otherwise} \end{cases} , \end{aligned}$$

$$\begin{aligned} [J_i, E_{\pm p}] &= \begin{cases} \pm 2E_{\pm p} & \text{if } i = p \\ \pm E_{\pm p} & \text{otherwise} \end{cases} , \\ [J_i, J_p] &= \mathbf{0} , \\ [E_i, E_p] &= \mathbf{0} , \end{aligned} \qquad [E_{j,-i}, E_{p,-q}] = \begin{cases} J_j - J_i & \text{if } j = q, i = p \\ -E_{p,-i} & \text{if } j = q, i \neq p \\ E_{j,-q} & \text{if } j \neq q, i = p \\ \mathbf{0} & \text{otherwise} \end{cases} ,$$

where $i, j, p, q = 1, \dots, k$ with $i \neq j$, $p \neq q$.

Proof. Several applications of (2.4.11) yield

$$\begin{aligned}
[J_j - J_i, E_{p,-q}] =& \alpha_p (J_j - J_i) f_{\boldsymbol{\alpha}+\mathbf{e}_p-\mathbf{e}_q} - (\alpha_j - \alpha_i) E_{p,-q} f_{\boldsymbol{\alpha}} \\
=& \big(\alpha_p ((\boldsymbol{\alpha}+\mathbf{e}_p-\mathbf{e}_q)_j - (\boldsymbol{\alpha}+\mathbf{e}_p-\mathbf{e}_q)_i) - \alpha_p(\alpha_j - \alpha_i)\big) f_{\boldsymbol{\alpha}+\mathbf{e}_p-\mathbf{e}_q} \\
=& \big(\alpha_p (\alpha_j + \delta_{jp} - \delta_{jq} - \alpha_i - \delta_{ip} + \delta_{iq}) - \alpha_p(\alpha_j - \alpha_i)\big) f_{\boldsymbol{\alpha}+\mathbf{e}_p-\mathbf{e}_q} \\
=& \alpha_p (\delta_{jp} - \delta_{jq} - \delta_{ip} + \delta_{iq}) f_{\boldsymbol{\alpha}+\mathbf{e}_p-\mathbf{e}_q} \\
[E_{j,-i}, E_{p,-q}] f_{\boldsymbol{\alpha}} =& \alpha_p E_{j,-i} f_{\boldsymbol{\alpha}+\mathbf{e}_p-\mathbf{e}_q} - \alpha_j E_{p,-q} f_{\boldsymbol{\alpha}+\mathbf{e}_j-\mathbf{e}_i} \\
=& \big(\alpha_p (\boldsymbol{\alpha}+\mathbf{e}_p-\mathbf{e}_q)_j - \alpha_j (\boldsymbol{\alpha}+\mathbf{e}_j-\mathbf{e}_i)_p\big) f_{\boldsymbol{\alpha}+\mathbf{e}_j-\mathbf{e}_i+\mathbf{e}_p-\mathbf{e}_q} \\
=& \big(\alpha_p (\alpha_j + \delta_{jp} - \delta_{jq}) - \alpha_j (\alpha_p + \delta_{jp} - \delta_{ip})\big) f_{\boldsymbol{\alpha}+\mathbf{e}_j-\mathbf{e}_i+\mathbf{e}_p-\mathbf{e}_q} \\
=& \big(\alpha_p \delta_{jp} - \alpha_p \delta_{jq} - \alpha_j \delta_{jp} + \alpha_j \delta_{ip}\big) f_{\boldsymbol{\alpha}+\mathbf{e}_j-\mathbf{e}_i+\mathbf{e}_p-\mathbf{e}_q} \\
=& (-\alpha_p \delta_{jq} + \alpha_j \delta_{ip}) f_{\boldsymbol{\alpha}+\mathbf{e}_j-\mathbf{e}_i+\mathbf{e}_p-\mathbf{e}_q} \ , \\
[E_i, E_{p,-q}] f_{\boldsymbol{\alpha}} =& \alpha_p E_i f_{\boldsymbol{\alpha}+\mathbf{e}_p-\mathbf{e}_q} - \alpha_i E_{p,-q} f_{\boldsymbol{\alpha}+\mathbf{e}_i} \\
=& \big(\alpha_p (\boldsymbol{\alpha}+\mathbf{e}_p-\mathbf{e}_q)_i - \alpha_i (\boldsymbol{\alpha}+\mathbf{e}_i)_p\big) f_{\boldsymbol{\alpha}+\mathbf{e}_i+\mathbf{e}_p-\mathbf{e}_q} \\
=& \big(\alpha_p (\alpha_i + \delta_{ip} - \delta_{iq}) - \alpha_i (\alpha_p + \delta_{ip})\big) f_{\boldsymbol{\alpha}+\mathbf{e}_i+\mathbf{e}_p-\mathbf{e}_q} \\
=& - \alpha_p \delta_{iq} f_{\boldsymbol{\alpha}+\mathbf{e}_i+\mathbf{e}_p-\mathbf{e}_q} \\
[E_{-i}, E_{p,-q}] f_{\boldsymbol{\alpha}} =& \alpha_p E_{-i} f_{\boldsymbol{\alpha}+\mathbf{e}_p-\mathbf{e}_q} - (1 - \boldsymbol{\alpha}_\bullet) E_{p,-q} f_{\boldsymbol{\alpha}-\mathbf{e}_i} \\
=& \big(\alpha_p (1 - (\boldsymbol{\alpha}+\mathbf{e}_p-\mathbf{e}_q)_\bullet) - (1 - \boldsymbol{\alpha}_\bullet)(\boldsymbol{\alpha}-\mathbf{e}_i)_p\big) f_{\boldsymbol{\alpha}-\mathbf{e}_i+\mathbf{e}_p-\mathbf{e}_q} \\
=& \big(\alpha_p (1 - \boldsymbol{\alpha}_\bullet) - (1 - \boldsymbol{\alpha}_\bullet)(\alpha_p - \delta_{ip})\big) f_{\boldsymbol{\alpha}-\mathbf{e}_i+\mathbf{e}_p-\mathbf{e}_q} \\
=& (1 - \boldsymbol{\alpha}_\bullet) \delta_{ip} f_{\boldsymbol{\alpha}-\mathbf{e}_i+\mathbf{e}_p-\mathbf{e}_q} \\
[E_i, E_{-p}] f_{\boldsymbol{\alpha}} =& (1 - \boldsymbol{\alpha}_\bullet) E_i f_{\boldsymbol{\alpha}-\mathbf{e}_p} - \alpha_i E_{-p} f_{\boldsymbol{\alpha}+\mathbf{e}_i} \\
=& \big((\boldsymbol{\alpha}-\mathbf{e}_p)_i (1 - \boldsymbol{\alpha}_\bullet) - \alpha_i (1 - (\boldsymbol{\alpha}+\mathbf{e}_i)_\bullet)\big) f_{\boldsymbol{\alpha}+\mathbf{e}_i-\mathbf{e}_p}
\end{aligned}$$

$$
\begin{aligned}
&=\big((\alpha_i-\delta_{ip})(1-\boldsymbol{\alpha}_\bullet)+\alpha_i\boldsymbol{\alpha}_\bullet\big)f_{\boldsymbol{\alpha}+\mathbf{e}_i-\mathbf{e}_p}\\
&=(\delta_{ip}\boldsymbol{\alpha}_\bullet+\alpha_i-\delta_{ip})f_{\boldsymbol{\alpha}+\mathbf{e}_i-\mathbf{e}_p}\ ,\\
[J_i,E_p]f_{\boldsymbol{\alpha}} &=\alpha_pJ_if_{\boldsymbol{\alpha}+\mathbf{e}_p}-(\boldsymbol{\alpha}_\bullet+\alpha_i-1)E_pf_{\boldsymbol{\alpha}}\\
&=\big(\alpha_p((\boldsymbol{\alpha}+\mathbf{e}_p)_\bullet+(\boldsymbol{\alpha}+\mathbf{e}_p)_i-1)-\alpha_p(\boldsymbol{\alpha}_\bullet+\alpha_i-1)\big)f_{\boldsymbol{\alpha}+\mathbf{e}_p}\\
&=\big(\alpha_p(\boldsymbol{\alpha}_\bullet+1+\alpha_i+\delta_{ip}-1)-\alpha_p\boldsymbol{\alpha}_\bullet-\alpha_i\alpha_p+\alpha_p\big)f_{\boldsymbol{\alpha}+\mathbf{e}_p}\\
&=(1+\delta_{ip})\alpha_pf_{\boldsymbol{\alpha}+\mathbf{e}_p}\ ,\\
[J_i,E_{-p}]f_{\boldsymbol{\alpha}} &=(1-\boldsymbol{\alpha}_\bullet)J_if_{\boldsymbol{\alpha}-\mathbf{e}_p}-(\boldsymbol{\alpha}_\bullet+\alpha_i-1)E_{-p}f_{\boldsymbol{\alpha}}\\
&=\big((1-\boldsymbol{\alpha}_\bullet)((\boldsymbol{\alpha}-\mathbf{e}_p)_\bullet+(\boldsymbol{\alpha}-\mathbf{e}_p)_i-1)-(1-\boldsymbol{\alpha}_\bullet)(\boldsymbol{\alpha}_\bullet+\alpha_i-1)\big)f_{\boldsymbol{\alpha}-\mathbf{e}_p}\\
&=\big((1-\boldsymbol{\alpha}_\bullet)(\boldsymbol{\alpha}_\bullet-1+\alpha_i-\delta_{ip}-1)-(1-\boldsymbol{\alpha}_\bullet)(\boldsymbol{\alpha}_\bullet+\alpha_i-1)\big)f_{\boldsymbol{\alpha}-\mathbf{e}_p}\\
&=-(1+\delta_{ip})(1-\boldsymbol{\alpha}_\bullet)f_{\boldsymbol{\alpha}-\mathbf{e}_p}\\
[J_i,J_p]f_{\boldsymbol{\alpha}} &=(\boldsymbol{\alpha}_\bullet+\alpha_p-1)J_if_{\boldsymbol{\alpha}}-(\boldsymbol{\alpha}_\bullet+\alpha_i-1)J_pf_{\boldsymbol{\alpha}}\\
&=\big((\boldsymbol{\alpha}_\bullet+\alpha_p-1)(\boldsymbol{\alpha}_\bullet+\alpha_i-1)-(\boldsymbol{\alpha}_\bullet+\alpha_i-1)(\boldsymbol{\alpha}_\bullet+\alpha_p-1)\big)f_{\boldsymbol{\alpha}}\\
&=\mathbf{0}\ ,\\
[E_i,E_p]f_{\boldsymbol{\alpha}} &=\alpha_pE_if_{\boldsymbol{\alpha}+\mathbf{e}_p}-\alpha_iE_pf_{\boldsymbol{\alpha}+\mathbf{e}_i}\\
&=\big(\alpha_p(\boldsymbol{\alpha}+\mathbf{e}_p)_i-\alpha_i(\boldsymbol{\alpha}+\mathbf{e}_i)_p\big)f_{\boldsymbol{\alpha}+\mathbf{e}_i+\mathbf{e}_p}\\
&=\big(\alpha_p(\alpha_i+\delta_{ip})-\alpha_i(\alpha_p+\delta_{ip})\big)f_{\boldsymbol{\alpha}+\mathbf{e}_i+\mathbf{e}_p}\\
&=\mathbf{0}\ .
\end{aligned}
$$
■

Proposition 2.4.13. *Let $\rho\colon \mathfrak{l}_k\to\mathrm{End}(\mathcal{O})$ be the linear map defined by*

$$e_{0,i}\mapsto E_{\alpha_i}\ ,\ e_{i,j}\mapsto E_{\alpha_j,-\alpha_i}\ ,\ h_{0,i}\mapsto J_{\alpha_i}\ ,\ f_{i,0}\mapsto E_{-\alpha_i}\ ,\ f_{j,i}\mapsto E_{\alpha_i,-\alpha_j}$$

where $i,j\in[k]$ with $j>i$. Then, for any fixed $\boldsymbol{\alpha}\in\mathbb{C}^k$, the pair $\rho_{\boldsymbol{\alpha}}:=(\rho(\,\cdot\,)\big|_{\mathcal{O}_{\Lambda_{\boldsymbol{\alpha}}}},\mathcal{O}_{\Lambda_{\boldsymbol{\alpha}}})$ is a faithful Lie algebra representation of $\mathfrak{l}_k$ with image $\mathfrak{g}_k\big|_{\mathcal{O}_{\Lambda_{\boldsymbol{\alpha}}}}$. Furthermore, the functions $f_{\boldsymbol{\alpha}}$ transform as basis vectors for $\rho_{\boldsymbol{\alpha}}$, in the sense that for every v in the basis for $\mathfrak{l}_k$ and every $\boldsymbol{\varepsilon}$ in $\Lambda_{\boldsymbol{\alpha}}$ there exists a unique $\bar{\boldsymbol{\varepsilon}}=\bar{\boldsymbol{\varepsilon}}(\boldsymbol{\varepsilon},v)$ in $\Lambda_{\boldsymbol{\alpha}}$ such that $(\rho_{\boldsymbol{\alpha}}v)\,f_{\boldsymbol{\varepsilon}}\propto f_{\bar{\boldsymbol{\varepsilon}}}$.

Proof. By Corollary 2.4.11, $\rho_{\boldsymbol{\alpha}}$ is a well-defined linear morphism into $\mathrm{End}(\mathcal{O}_{\Lambda_{\boldsymbol{\alpha}}})$. The fact that $f_{\boldsymbol{\alpha}}$ transforms as a basis vector of $\mathcal{O}_{\Lambda_{\boldsymbol{\alpha}}}$ is an immediate consequence of Lemma 2.4.8. For $\boldsymbol{\varepsilon}\in\Lambda_{\boldsymbol{\alpha}}$ such that $\Re^\diamond\boldsymbol{\varepsilon}>\mathbf{1}$, the actions of operators in (2.4.10) on $\mathcal{O}_{\boldsymbol{\varepsilon}}$ are mutually different again by Lemma 2.4.8, hence $\rho_{\boldsymbol{\alpha}}$ is injective. In order to show that $\rho_{\boldsymbol{\alpha}}\mathfrak{l}=\mathfrak{g}\big|_{\mathcal{O}_{\Lambda_{\boldsymbol{\alpha}}}}$ is a Lie algebra of type A_k and that $\rho_{\boldsymbol{\alpha}}$ is a Lie algebra representation, it suffices to verify that the morphism $\rho\colon\mathfrak{l}_k\to\mathfrak{g}_k$ is a morphism of Lie algebras. Notice that $h_{i,i'}=h_{0,i'}-h_{0,i}$, hence $\rho(h_{i,i'})=J_{\alpha_{i'}}-J_{\alpha_i}$. Thus, the assertion follows by direct comparison of the commutators in Lemma 2.4.12 (for the choice $j>i$, $p>q$) with those of the presentation of $\mathfrak{l}_k$ given in Lemma 2.4.7. ■

Theorem 2.4.14. *For $\boldsymbol{\alpha}$ in $\operatorname{int}\Delta^{k-1}$ and $\mathbf{p}\in\mathbb{N}_0^k$ denote by $\mathsf{D}^{\mathbf{p}}_{\boldsymbol{\alpha}}$ the posterior distribution of $\mathsf{D}_{\boldsymbol{\alpha}}$ given atoms of mass p_i at point $i\in[k]$. Then,*

- *(i) the semi-lattice $\mathcal{O}_{\Lambda^+_{\boldsymbol{\alpha}}}$ is a weight $\mathfrak{l}$-module and $\mathfrak{U}(\mathfrak{l})$-module;*
- *(ii) the space $\mathcal{O}_{M_{\boldsymbol{\alpha},\ell}}$ in invariant under the action of the universal enveloping algebra $\mathfrak{U}(\mathfrak{f})<\mathfrak{U}(\mathfrak{l})$ for all $\ell\in\mathbb{N}_1$, while $\mathcal{O}_{H^+_{\boldsymbol{\alpha}}}$ is invariant under the action of $\mathfrak{U}(\mathfrak{h})\oplus\mathfrak{U}(\mathfrak{r}^+)$;*
- *(iii) for every $\mathbf{p}\in\mathbb{N}_0^k$ there exists a unique $v=v(\mathbf{p})\in\mathfrak{U}(\mathfrak{r}^+)$ such that $v.\mathcal{O}_{\boldsymbol{\alpha}}\cong\mathbb{C}\widehat{\mathsf{D}^{\mathbf{p}}_{\boldsymbol{\alpha}}}$;*
- *(iv) the canonical action of $\mathfrak{S}_k$ on $\mathscr{P}([k])$ corresponds to the natural action of the unique subgroup (isomorphic to) $\mathfrak{S}_k$ of the Weyl group of $\mathfrak{l}_k$ permuting roots corresponding to basis elements in $\mathfrak{r}^+_k$.*

Proof. By (2.4.11), the operators $\rho\,\mathfrak{sl}_{k+1}(\mathbb{C})$ fix $\mathcal{O}_{\Lambda^+_{\boldsymbol\alpha}} \subset \mathcal{O}$, thus $\rho_{\boldsymbol\alpha}$ is a (faithful) Lie algebra representation by Proposition 2.4.13, hence $\mathcal{O}_{\Lambda^+_{\boldsymbol\alpha}} \subset \mathcal{O}$ is an $\mathfrak{l}$-module for the linear extension of the action $v.f_{\boldsymbol\varepsilon} :=(\rho_{\boldsymbol\alpha} v)f_{\boldsymbol\varepsilon}$ varying v in the basis of $\mathfrak{l}$. The extension to a representation of $\mathfrak{U}(\mathfrak{l})$ is standard from the universal property of universal enveloping algebras. (See e.g., [77, §17.2].)

In order to prove (*i*)-(*ii*) it suffices to show that, for all $\boldsymbol\varepsilon \in \Lambda^+_{\boldsymbol\alpha}$ and $\ell \in \mathbb{N}_1$, one has

$$h_{0,i}.f_{\boldsymbol\varepsilon} = (\boldsymbol\varepsilon_\bullet - 1 + \varepsilon_i)f_{\boldsymbol\varepsilon}\ ,\ v.\mathcal{O}_{M_{\boldsymbol\varepsilon,\ell}} \subset \mathcal{O}_{M_{\boldsymbol\varepsilon,\ell}}\ ,\ w.\mathcal{O}_{H^+_{\boldsymbol\alpha}} \subset \mathcal{O}_{H^+_{\boldsymbol\alpha}}$$

for all $i \in [k]$, v in the basis of $\mathfrak{f}$, $\ell \in \mathbb{N}_1$ and w in the basis for $\mathfrak{h} \oplus \mathfrak{r}^+$. All of the above follow immediately from Lemma 2.4.8. Notably, since $\boldsymbol\alpha_\bullet = 1$, $\mathfrak{h}$ acts on $\mathcal{O}_{\boldsymbol\alpha}$ precisely by weight $\boldsymbol\alpha$.

Since $\boldsymbol\alpha \in \Delta^{k-1}$, then $f_{\boldsymbol\alpha+\mathbf{p}}(\,\cdot\,,\mathbf{1},1) = \widehat{\mathsf{D}_{\boldsymbol\alpha+\mathbf{p}}}(\,\cdot\,)$. By the Bayesian property of $\mathsf{D}_{\boldsymbol\alpha}$ the space $\mathcal{O}_{H^+_{\boldsymbol\alpha}}$ is spanned precisely by the Fourier transforms of the form $\widehat{\mathsf{D}^{\mathbf{p}}_{\boldsymbol\alpha}}$. It remains to show that $\mathfrak{U}(\mathfrak{r}^+).\mathcal{O}_{\boldsymbol\alpha} = \mathcal{O}_{H^+_{\boldsymbol\alpha}}$. Setting $v = e_1^{p_1} \cdots e_k^{p_k} \in \mathfrak{U}(\mathfrak{r}^+_k)$ yields $v.\mathcal{O}_{\boldsymbol\alpha} = \mathcal{O}_{\boldsymbol\alpha+\mathbf{p}}$ as required. The uniqueness of v follows by the fact that, since $\mathfrak{r}^+$ is Abelian, $\mathfrak{U}(\mathfrak{r}^+)$ coincides with the (Abelian) symmetric algebra generated by $\mathfrak{r}^+$. (See [77, §17.2].) This proves (*iii*).

In order to show (*iv*), recall (e.g., [77, §12.1]) that the Weyl group W_k of Ψ_k is isomorphic to $\mathfrak{S}_{k+1}$ and its action on Ψ_k may be canonically identified as dual to the action of $\mathfrak{S}_{k+1}$ on $\mathfrak{h}_k$ via conjugation by permutation matrices in $\mathfrak{P}_{k+1} \cong \mathfrak{S}_{k+1} < GL(\mathfrak{h}_k) \cong GL_{k+1}(\mathbb{C})$. Let $\mathfrak{P}_{2:k+1} < GL_{k+1}(\mathbb{C})$ denote the subgroup of permutations matrices whose action on $\mathrm{Mat}_{k+1}(\mathbb{C})$ fixes the first row and column. Clearly $\mathfrak{S}_k \cong \mathfrak{P}_{2:k+1} < \mathfrak{P}_{k+1}$. Composing the isomorphism $\rho_{\boldsymbol\alpha}$ with the identification of the action of $\mathfrak{P}_{k+1}$ above completes the proof. ■

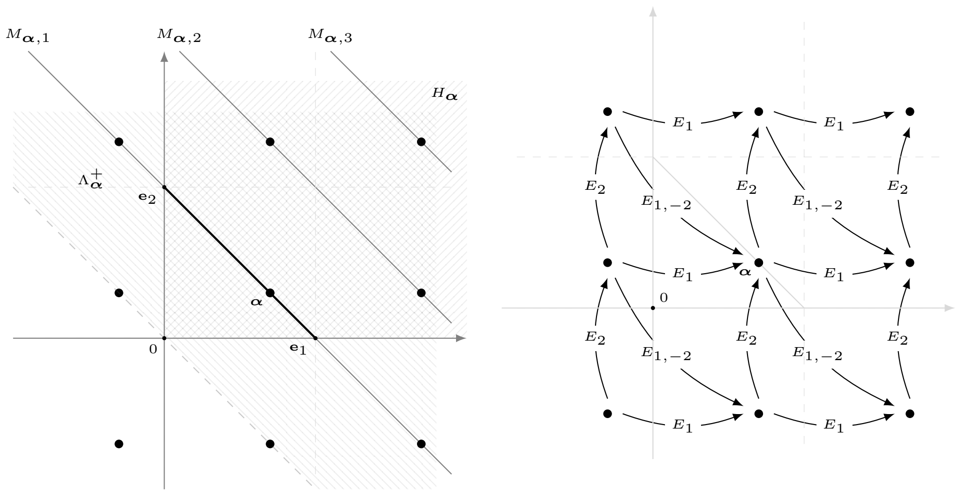

Figure 2.2: (both) *Each marked point corresponds to some $\boldsymbol\varepsilon \in \Lambda_{\boldsymbol\alpha}$ for fixed $\boldsymbol\alpha$, and is chosen to indicate the one-dimensional vector space $\mathcal{O}_{\boldsymbol\varepsilon}$.* (left) *The gray anti-diagonal lines denote the* isoplethic surfaces*: marked points $\boldsymbol\varepsilon$ lying on these surfaces belong to $M_{\boldsymbol\alpha,\ell}$, i.e. they have fixed length $\boldsymbol\varepsilon_\bullet = \ell \in \mathbb{N}_1$. The simplex Δ^1 is marked as a thick black segment. Analogously, marked points lying in the North-West dashed region delimited by the hyper-plane of equation $\mathbf{y}_\bullet = 0$ belong to the semi-lattice $\Lambda^+_{\boldsymbol\alpha}$, whereas marked points lying in the first hyper-octant (in the figure: the North-East dashed quadrant) belong to $H^+_{\boldsymbol\alpha}$.* (right) *The action of operators in $\rho_{\boldsymbol\alpha}(\mathfrak{n}^+_2)$ on the lattice $\mathcal{O}_{\Lambda_{\boldsymbol\alpha}}$ for $\boldsymbol\alpha = \left(\frac{2}{3}, \frac{1}{3}\right)$ is shown.*

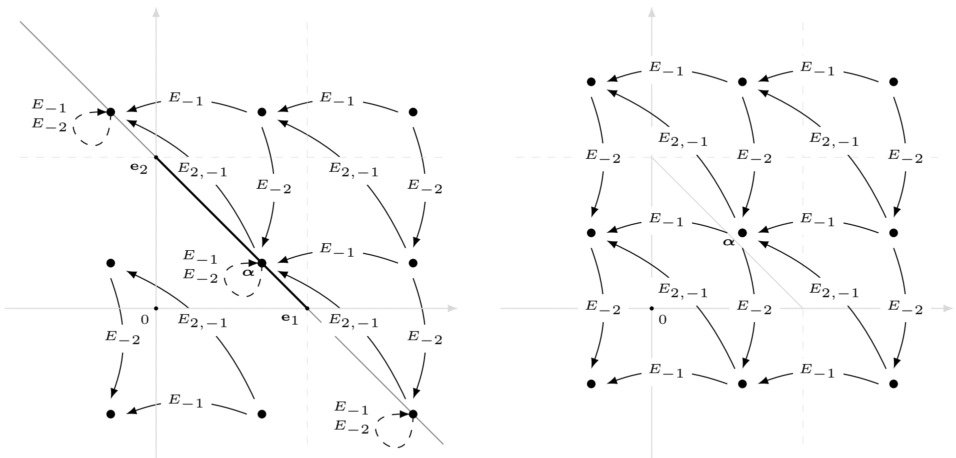

Figure 2.3: (both) *Each marked point corresponds to some $\boldsymbol{\varepsilon} \in \Lambda_{\boldsymbol{\alpha}}$ and is chosen to indicate the one-dimensional vector space $\mathcal{O}_{\boldsymbol{\varepsilon}}$.* (left) *The action of operators in $\rho_{\boldsymbol{\alpha}}(\mathfrak{n}_2^-)$ on the lattice $\mathcal{O}_{\Lambda_{\boldsymbol{\alpha}}}$ for $\boldsymbol{\alpha} = \left(\frac{2}{3}, \frac{1}{3}\right)$ is shown. Since $\boldsymbol{\alpha}_\bullet \in \mathbb{Z}$, the lowering operators $E_{-\alpha_1}, E_{-\alpha_2}$* (left) *vanish identically on the lowest positive isoplethic line (in gray), containing the standard simplex (the thick segment): their action is here represented by a dashed loop.* (right) *The action of operators in $\rho_{\boldsymbol{\alpha}}(\mathfrak{n}_2^-)$ on the lattice $\mathcal{O}_{\Lambda_{\boldsymbol{\alpha}}}$ for $\boldsymbol{\alpha} = \left(\frac{3}{5}, \frac{1}{2}\right)$ is shown. Since $\boldsymbol{\alpha}_\bullet \in \mathbb{R} \setminus \mathbb{Z}$, the lowering operators $E_{-\alpha_1}, E_{-\alpha_2}$ never vanish.*

The commutative action of $\mathfrak{h}_k$. It is the content of Theorem 2.4.14(i) that the characteristic functionals of the measures $\mathsf{D}_{\boldsymbol{\alpha}}$, varying $\boldsymbol{\alpha} \in \operatorname{int} \Delta^{k-1}$, are projectively invariant under the action of the maximal toral subalgebra $\mathfrak{h}_k < \mathfrak{l}_k$ in the representation $\rho_{\boldsymbol{\alpha}}$. Since $\mathfrak{h}_k$ acts on $\mathcal{O}_{\boldsymbol{\alpha}}$ by weight $\boldsymbol{\alpha}$ (See the proof of Thm. 2.4.14(i).), for arbitrary $J_{\mathbf{t}} := t_1 J_{\alpha_1} + \cdots + t_k J_{\alpha_k} \in \rho_{\boldsymbol{\alpha}} \mathfrak{h}_k$ one has

$$J_{\mathbf{t}} f_{\boldsymbol{\alpha}} = (\mathbf{t} \cdot \boldsymbol{\alpha}) f_{\boldsymbol{\alpha}} \ , \qquad \mathbf{t} \in \mathbb{R}^k \ . \tag{2.4.12}$$

This is to be compared with the case of $L_{\boldsymbol{\alpha}}$. Indeed, let $\mathbf{t} \in \mathbb{R}^k$ be such that $\mathbf{t}_\bullet = 0$ and set $\mathbf{s} := \exp^\diamond \mathbf{t} \in M_1^{k-1}$. Then, $\mathbf{s}^{-\boldsymbol{\alpha}} = \exp(-\boldsymbol{\alpha} \cdot \ln^\diamond \mathbf{s}) = \exp(-\mathbf{t} \cdot \boldsymbol{\alpha})$. Thus, by (2.1.4),

$$\mathrm{d}\big((\exp^\diamond \mathbf{t}).\big)_\sharp L_{\boldsymbol{\alpha}} = \exp(-\mathbf{t} \cdot \boldsymbol{\alpha}) \; \mathrm{d}L_{\boldsymbol{\alpha}} \ , \qquad \mathbf{t} \in \mathbb{R}^k \ , \ \mathbf{t}_\bullet = 0 \ .$$

Improper hyper-priors. Before commenting on the non-commutative action of $\mathfrak{l}_k$, let us introduce a family of distinguished (possibly improper) hyper-priors of the Dirichlet distribution.

Definition 2.4.15 (Dirichlet-categorical hyper-priors)**.** Let $\boldsymbol{\alpha}_0 \in \Delta^{k-1}$ and fix $\boldsymbol{\alpha} \in \Lambda^+_{\boldsymbol{\alpha}_0}$. For $\boldsymbol{\varepsilon} \in \Lambda^+_{\boldsymbol{\alpha}_0} \cap H^-_{\boldsymbol{\alpha}}$ we denote by $\tilde{\mathsf{D}}_{\boldsymbol{\varepsilon}}$ the (possibly non-finite) definite (i.e., positive or negative, *not* signed) measure with density

$$\mathbb{1}_{\Delta^{k-1}}(\mathbf{y}) \, \frac{\mathbf{y}^{\boldsymbol{\varepsilon} - \mathbf{1}}}{\mathrm{B}(\boldsymbol{\varepsilon})}$$

with respect to the k-dimensional Lebesgue measure on the hyperplane of equation $\mathbf{y}_\bullet = 1$ in $\mathbb{R}^k$, concentrated on (the interior of) Δ^{k-1}. We term this measure the *Dirichlet-categorical hyper-prior* of parameter $\boldsymbol{\varepsilon}$. The measure $\tilde{\mathsf{D}}_{\boldsymbol{\varepsilon}}$ has sign given by

$$\operatorname{sgn}(\mathrm{B}(\boldsymbol{\varepsilon})) = \operatorname{sgn}(\Gamma(\boldsymbol{\varepsilon})) = \begin{cases} 1 & \text{if } \boldsymbol{\varepsilon} \in H^+_{\boldsymbol{\alpha}_0} \\ (-1)^{\lceil \varepsilon_1 \rceil + \cdots + \lceil \varepsilon_k \rceil} & \text{otherwise} \end{cases} \ .$$

The non-commutative action of $\mathfrak{l}_k$. If $\boldsymbol{\alpha}_0 \in \operatorname{int}\Delta^{k-1}$ and $\boldsymbol{\alpha} \in H^+_{\boldsymbol{\alpha}_0}$, then

(a) the action of basis elements in $\mathfrak{r}_k^+$ amounts to take (characteristic functionals of) Dirichlet-categorical *posteriors*; it fixes the space $\mathcal{O}_{H^+_{\boldsymbol{\alpha}}}$ of (characteristic functionals of) such posteriors;

(b) the action of basis elements in $\mathfrak{r}_k^-$ amounts to take (characteristic functionals of) Dirichlet-categorical (hyper-)priors. The action of $\mathfrak{r}_k^-$ fixes the space $\mathcal{O}_{\Lambda^+_{\boldsymbol{\alpha}_0}\cap H^-_{\boldsymbol{\alpha}}}$ of (characteristic functionals of) all such (hyper-)priors and vanishes on the line $M_{\boldsymbol{\alpha}_0,1}$, since $M_{\boldsymbol{\alpha}_0,0}$ is the singular set of the normalization constant $\mathrm{B}[\boldsymbol{\varepsilon}]^{-1}$;

(c) the action of basis elements in $\mathfrak{f}_k$ contains every non-trivial combination of the actions (a) and (b), and fixes isoplethic hypersurfaces $M_{\boldsymbol{\alpha}_0,\ell}$, i.e. those where the intensity $\boldsymbol{\varepsilon}$ has constant total mass $\boldsymbol{\varepsilon}_\bullet = \ell$.

In this framework, the case $\boldsymbol{\alpha} \in \operatorname{bd}\Delta^{k-1}$ is spurious, since the intensity measure $\boldsymbol{\alpha}$ should always be assumed fully supported.

2.4.2 Infinite-dimensional statements. For $a \in \mathbb{R}$ we denote by $\mathscr{M}_b^{>a}(X)$ the space of finite *signed* measures ν in $\mathscr{M}_b(X)$ such that $\nu X > a$.

Theorem 2.4.16. *Let $(X, \tau(X), \mathcal{B})$ be a second countable locally compact Hausdorff space, ν a diffuse fully supported non-negative finite measure on X. Let further*

$$\Phi[\nu, f] := \sum_{n=0}^{\infty} \frac{\mathrm{i}^n}{\langle \nu X\rangle_n} Z_n(\nu f^1, \ldots, \nu f^n)$$

and

$$\begin{aligned} E_A\Phi[\nu, f] &:= \int_A \mathrm{d}\nu(y)\, \Phi[\nu + \delta_y, f]\ , \\ E_{A,-B}\Phi[\nu, f] &:= \int_{A\setminus B} \mathrm{d}\nu(y)\, \Phi[\nu + \delta_y, f] + \int_{B\setminus A} \mathrm{d}\nu(y)\, \Phi[\nu - \delta_y, f]\ . \end{aligned} \tag{2.4.13}$$

Then,

(i) *$\Phi[\nu, f]$ is a well-defined extension of the characteristic functional $\widehat{\mathcal{D}_\nu}(f^*)$ on $\mathscr{M}_b^{>0}(X) \times \mathcal{C}_c(X)$;*

(ii) *for every ν in $\mathscr{M}_b^{>1}(X)$, every f in $\mathcal{C}_c(X)$, every A, B in $\mathcal{B}$, and every good approximation $(f_h)_h$ of f locally constant on $\mathbf{X}_h$ with values $\mathbf{s}_h$ for some $(\mathbf{X}_h)_h \in \mathfrak{Na}(A, B \subset X)$, one has*

$$E_A\Phi[\nu, f] = \lim_h \left(\sum_{i \mid X_{h,i} \subset A} E_{\alpha_{h,i}} \right) {}_{k_h}\Phi[\nu^\diamond \mathbf{X}_h, \mathbf{s}_h]\ ,$$

$$E_{A,-B}\Phi[\nu, f] = \lim_h \left(\sum_{\substack{i \mid X_{h,i} \subset A\setminus B \\ j \mid X_{h,j} \subset B\setminus A}} E_{\alpha_{h,i}, -\alpha_{h,j}} \right) {}_{k_h}\Phi[\nu^\diamond \mathbf{X}_h, \mathbf{s}_h]\ ,$$

where $\boldsymbol{\alpha}_h := \nu^\diamond \mathbf{X}_h$ and $E_{\alpha_{h,i}}$, $E_{\alpha_{h,i}, -\alpha_{h,j}} \in \mathfrak{g}_{k_h}$.

(iii) *let σ be a diffuse fully supported probability on $(X, \tau(X))$ and $(\mathbf{X}_h)_h \in \mathfrak{Na}(X, \tau(X), \sigma)$. For σ-a.e. x, such that $X_{h,i_h} \downarrow_h \{x\}$, and for every good approximation $(f_h)_h$ of f, locally* *constant on $\mathbf{X}_h$ and uniformly convergent to f, there exist the pointwise limiting rescaled actions*

$$\begin{aligned}\lim_h \alpha_{h,i_h}^{-1} E_{\alpha_{i_h}} \widehat{\mathcal{D}_\sigma}(f_h^*) =&\widehat{\mathcal{D}_\sigma^x}(f^*) \ , \\ \lim_h \alpha_{h,i_h}^{-1} J_{\alpha_{i_h}} =&\mathrm{Id} \ , \\ \lim_h \alpha_{h,i_h}^{-1} E_{-\alpha_{i_h}} =&0 \ .\end{aligned} \tag{2.4.14}$$

Proof. The functional $\Phi[\nu, f]$ is well-defined in the first place since $\nu X > 0$. For $c, t > 0$ denote by $P_{c,t} \subset \mathbb{R}^n$ the polydisk $\{\mathbf{y} \in \mathbb{R}^n \mid |y_i| \leq c\, t^i\}$. By induction and (2.2.2) it is not difficult to show that $\max_{P_{c,t}} |Z_n| = Z_n[c(t\,\mathbf{1})^{\diamond \vec{\mathbf{n}}}]$; moreover, by (2.2.1) and Theorem 2.3.3 (Cf. also Rmk. 2.3.4), the latter equals $t^n \langle c\rangle_n /n!$. As a consequence, for arbitrary ν in $\mathscr{M}_b^{>0}(X)$ and $f \in \mathcal{C}_c(X)$, letting $y_i \coloneqq \nu f^i$ above,

$$|\Phi[\nu, f]| \leq \sum_{n=0}^\infty \langle \nu X\rangle_n^{-1} \max_{P_{\|\nu\|,\|f\|}} |Z_n| = \sum_{n=0}^\infty \frac{\langle \|\nu\|\rangle_n}{n!\,\langle \nu X\rangle_n} \|f\|^n = {}_1F_1\left[\|\nu\|\,; \nu X; \|f\|\right] \ ,$$

which is finite since $\nu X > 0$. This shows (i). Notably, if ν is positive, then $|\Phi[\nu, f]| \leq \exp \|f\|$ independently of $\|\nu\|$.

Let now A be in $\mathcal{B}$ and $(\mathbf{X}_h)_h$ as in (ii). Fix f in $\mathcal{C}_c(X)$, set $\boldsymbol{\alpha}_h \coloneqq \nu^\diamond \mathbf{X}_h$ and let $(f_h)_h$ be a good approximation of f, locally constant on $\mathbf{X}_h$ with values $\mathbf{s}_h$. Equation (2.4.11) yields by summation

$$\left(\sum_{i|X_{h,i}\subset A} E_{\alpha_i}\right) {}_{k_h}\Phi\left[\boldsymbol{\alpha}_h; \mathbf{s}_h\right] = \sum_{i|X_{h,i}\subset A} \alpha_{h,i}\ {}_{k_h}\Phi\left[\boldsymbol{\alpha}_h + \mathbf{e}_i; \mathbf{s}_h\right] \ . \tag{2.4.15}$$

More explicitly, since f_h is constant on each $X_{h,i}$ with value $s_{h,i}$, Proposition 2.3.5 yields

$$\begin{aligned}&\left(\sum_{i|X_{h,i}\subset A} E_{\alpha_i}\right) {}_{k_h}\Phi\left[\boldsymbol{\alpha}_h; \mathbf{s}_h\right] = \\ &= \sum_{i|X_{h,i}\subset A} \nu X_{h,i} \sum_{n=0}^\infty \frac{1}{\langle \nu X + 1\rangle_n} Z_n\left(\nu f_h + \frac{\nu(f_h\,\mathbb{1}_{X_{h,i}})}{\nu X_{h,i}}, \dots, \nu f_h^n + \frac{\nu(f_h^n\,\mathbb{1}_{X_{h,i}})}{\nu X_{h,i}}\right) \\ &= \sum_{n=0}^\infty \frac{\mathrm{i}^n}{\langle \nu X + 1\rangle_n} \times \\ &\quad\times \sum_{i|X_{h,i}\subset A} \int_{X_{h,i}} \mathrm{d}\nu(y)\, Z_n\left(\nu f_h + \frac{\nu(f_h\,\mathbb{1}_{X_{h,i}})}{\nu X_{h,i}} \mathbb{1}_{X_{h,i}}(y), \dots, \nu f_h^n + \frac{\nu(f_h^n\,\mathbb{1}_{X_{h,i}})}{\nu X_{h,i}} \mathbb{1}_{X_{h,i}}(y)\right) \\ &= \sum_{n=0}^\infty \frac{\mathrm{i}^n}{\langle \nu X + 1\rangle_n} \sum_{i|X_{h,i}\subset A} \int_{X_{h,i}} \mathrm{d}\nu(y)\, Z_n\left(\nu f_h + f_h(y), \dots, \nu f_h^n + f_h(y)^n\right) \\ &= \sum_{n=0}^\infty \frac{\mathrm{i}^n}{\langle \nu X + 1\rangle_n} \int_A \mathrm{d}\nu(y)\, Z_n\left(\nu f_h + f_h(y), \dots, \nu f_h^n + f_h(y)^n\right) \ . \\ &= \int_A \mathrm{d}\nu(y) \sum_{n=0}^\infty \frac{\mathrm{i}^n}{\langle (\nu + \delta_y) X\rangle_n} Z_n\left(\nu f_h + f_h(y), \dots, \nu f_h^n + f_h(y)^n\right) \ .\end{aligned} \tag{2.4.16}$$

Since $|f_h| \leq |f|$ pointwise, the sequence $\left(f_h^i\right)_h$ converges strongly in L^1_ν for every $i \leq n$ for every $n \in \mathbb{N}_1$, thus by continuity of Z_n, there exists the limit

$$\lim_h \int_A \mathrm{d}\nu(y) \sum_{n=0}^\infty \frac{\mathrm{i}^n}{\langle \nu X + 1\rangle_n} Z_n\left(\nu f_h + f_h(y), \dots, \nu f_h^n + f_h(y)^n\right) = E_A \Phi[\nu, f] \ .$$

The proof of the statement for $E_{A,-B}$ is analogous. This completes the proof of (*ii*). The requirement that $\nu X > 1$ is necessary to the convergence of $\Phi[\nu - \delta_y, f]$ for $y \in X$ in the definition of $E_{A,-B}$, whereas it may be relaxed to $\nu X > 0$ in the case of E_A. We will make use of this fact in the proof of (*iii*).

Fix now x in X and let $i_h \coloneqq i_h(x)$ be such that $X_{h,i_h} \downarrow_h \{x\}$. By Lemma 2.5.1, the sequence $(i_h)_h$ is unique for σ-a.e. x. With the same notation of (*ii*), let now $A = X_{h,i_h}$ in (2.4.16). Then,

$$\alpha_{i_h}^{-1} E_{\alpha_{i_h} k_h} \Phi[\boldsymbol{\alpha}_h; \mathbf{s}_h] = \alpha_{i_h}^{-1} E_{\alpha_{i_h}} \widehat{\mathcal{D}_\sigma}(f_h^*) = \frac{1}{\sigma X_{h,i_h}} \int_{X_{h,i_h}} \mathrm{d}\sigma(y)\, \widehat{\mathcal{D}_{\sigma+\delta_y}}(f_h^*) \ . \tag{2.4.17}$$

By (2.3.14) and uniform convergence of the approximation

$$\lim_h \left| \tfrac{1}{\sigma X_{h,i_h}} E_{X_{h,i_h}} \Phi[\sigma, f_h] - \tfrac{1}{\sigma X_{h,i_h}} E_{X_{h,i_h}} \Phi[\sigma, f] \right| \le \lim_h e^{\|f\|} \|f - f_h\| = 0 \ , \tag{2.4.18}$$

thus, (2.4.17) and (2.4.18) yield, together with the continuity of $y \mapsto \widehat{\mathcal{D}_{\sigma+\delta_y}}(f^*)$ for fixed f and σ,

$$\lim_h \alpha_{i_h}^{-1} E_{\alpha_{i_h}} \widehat{\mathcal{D}_\sigma}(f_h^*) = \lim_h \frac{1}{\sigma X_{h,i_h}} \int_{X_{h,i_h}} \mathrm{d}\sigma(y)\, \widehat{\mathcal{D}_{\sigma+\delta_y}}(f^*) = \widehat{\mathcal{D}_{\sigma+\delta_x}}(f^*) \ .$$

By the Bayesian property $\mathcal{D}_\sigma^x = \mathcal{D}_{\sigma+\delta_x}$, this yields the conclusion for the limiting raising action. Finally, since σ is a probability measure, $(\boldsymbol{\alpha}_h)_\bullet = 1$ for all h, thus by (2.4.11),

$$\begin{aligned} \lim_h \alpha_{i_h}^{-1} J_{\alpha_{i_h}} \widehat{\mathcal{D}_\sigma}(f_h^*) &= \lim_h \widehat{\mathcal{D}_\sigma}(f_h^*) = \widehat{\mathcal{D}_\sigma}(f^*) \ , \\ \lim_h \alpha_{i_h}^{-1} E_{-\alpha_{i_h}} \widehat{\mathcal{D}_\sigma}(f_h^*) &= \lim_h 0 = 0 \ , \end{aligned}$$

where the second equality for the first limiting action follows by (2.3.14). In all three cases, independence of the limits from the chosen (good) approximation is straightforward. ■

Remark 2.4.17. The existence of a dynamical symmetry algebra $\mathfrak{g}_\sigma$ for the characteristic functional of $\mathcal{D}_\sigma$ when σ is a *diffuse* measure should *not* be expected. Indeed, as a consequence of (2.4.14), the limiting action of the standard negative Borel subalgebra would be trivial and, in particular, the limiting action of the standard Cartan subalgebra would collapse to the identity. As a consequence, the limiting action of the standard positive Borel subalgebra would be Abelian. Assuming this is the case, we provide a conjectural statement for the structure of $\mathfrak{g}_\sigma$ in the general case.

Let $\sigma = \sigma^{\mathrm{a}} + \sigma^{\mathrm{d}}$, where σ^{a}, resp. σ^{d}, is the purely atomic, resp. diffuse, part of σ. By the Mapping Theorem 2.3.9, we main assume without loss of generality that the atoms of σ^{a} are isolated points in $X = \mathrm{supp}\sigma^{\mathrm{d}} \sqcup \mathrm{supp}\sigma^{\mathrm{a}}$. Together with the infinite-dimensional analogue of [145, Lem. 3.1], this suggests that one might in fact have $\mathfrak{g}_\sigma = \mathfrak{g}_{\sigma^{\mathrm{d}}} \oplus \mathfrak{g}_{\sigma^{\mathrm{a}}}$. If we assume further, as heuristically argued before, that the dynamical symmetry algebra $\mathfrak{g}_{\sigma^{\mathrm{d}}}$ is Abelian, then $\mathfrak{g}_\sigma = \mathfrak{g}_{\sigma^{\mathrm{d}}} \oplus \mathfrak{g}_{\sigma^{\mathrm{a}}}$ would be reductive with semisimple part $\mathfrak{g}_{\sigma^{\mathrm{a}}}$. Thus, we may consider, without loss of generality, the case of purely atomic intensity measures $\sigma = \sigma^{\mathrm{a}}$.

Finally, if σ^{a} has support $[k]$, then clearly $\mathfrak{g}_{\sigma^{\mathrm{a}}} = \mathfrak{g}_k$ by Theorem 2.3.10, since $\mathcal{D}_{\sigma^{\mathrm{a}}} = \mathsf{D}_{\boldsymbol{\alpha}}$ for some $\boldsymbol{\alpha}$. The case $\#\mathrm{supp}\sigma^{\mathrm{a}} = \infty$ remains open, although we expect that $\mathfrak{g}_{\sigma^{\mathrm{a}}} \cong \mathfrak{fsl}_\infty(\mathbb{C})$, the finitary special linear Lie algebra of traceless infinite matrices with finitely many non-zero entries.

2.5 Appendix

We collect here some results in topology and measure theory.

Lemma 2.5.1. *Let $(X, \tau(X), \mathcal{B}, \sigma)$ be a second countable locally compact Hausdorff Borel measure space of finite diffuse fully supported measure. Then, for every $(\mathbf{X}_h)_h \in \mathfrak{Na}(X, \tau(X), \sigma)$ for σ-a.e. x in X there exists a unique sequence $(X_{h,i_h})_h$, with $i_h := i_h(x)$, such that $\mathbf{X}_h \ni X_{h,i_h} \downarrow_h \{x\}$.*

Proof. Proposition 2.2.4 justifies well-posedness of the requirements in the definition of $(\mathbf{X}_h)_h$.

Without loss of generality, each $X_{h,i}$ may be chosen to be closed by replacing it with its closure $\mathrm{cl}X_{h,i} = X_{h,i} \cup \mathrm{bd}X_{h,i}$. Hence $\mathbf{X}_h$ may be chosen to be consisting of closed sets (disjoint up to a σ-negligible set) with non-empty interior. It follows by the finite intersection property that every decreasing sequence of sets $(X_{h,i_h})_h$ such that $X_{h,i_h} \in \mathbf{X}_h$ admits a non-empty limit, which is a singleton because of the vanishing of diameters. Vice versa, however chosen $(\mathbf{X}_h)_h$, for every point x in X it is not difficult to construct a (possibly non-unique) sequence X_{h,i_h} (with $i_h := i_h(x)$) convergent to x and such that $X_{h,i_h} \in \mathbf{X}_h$. Furthermore, letting x be a point for which there exists more than one such sequence, we see that for every h the point x belongs to some intersection $X_{h,i_1} \cap X_{h,i_2} \cap \dots$, hence, since every partition has disjoint interiors by construction, $x \in \mathrm{bd}X_{h,i_1} \cap \mathrm{bd}X_{h,i_2} \cap \dots$. Since for every h and $i \leq k_h$ each set $X_{h,i}$ is a continuity set for σ, the whole union $\cup_{h\geq 0} \cup_{i\in[k_h]} \mathrm{bd}X_{h,i}$ is σ-negligible, thus so is the set of points x considered above, so that for σ-a.e. x there exists a unique sequence $(X_{h,i_h})_h$ such that $X_{h,i_h} \in \mathbf{X}^h$ and $\lim_h X_{h,i_h} = \{x\}$ and x belongs to each X_{h,i_h} in the sequence. ∎

Finally, recall the following form of Lévy's Continuity Theorem.

Theorem 2.5.2 ([160, Thm. 3.1, p. 224])**.** *Let $(Y, \tau(Y))$ be a completely regular Hausdorff topological space, V be a linear subspace of $\mathcal{C}(Y)$ separating points in Y and χ be a complex-valued functional on V. If $(\mu_\gamma)_\gamma$ is a narrowly precompact net of Radon probability measures on $(Y, \mathcal{B}(Y))$ and $\lim_\gamma \widehat{\mu_\gamma}(v) = \chi(v)$ for every v in V, then $(\mu_\gamma)_\gamma$ converges narrowly to a Radon probability measure μ, the characteristic functional thereof coincides with χ.*

Corollary 2.5.3. *Let $(\mu_\gamma)_\gamma$ be a narrowly precompact net of random probabilities over $(X, \mathcal{B})$. If $\lim_\gamma \widehat{\mu_\gamma}(f^*) = \chi(f^*)$ for every f in $\mathcal{C}_c(X)$, then $(\mu_\gamma)_\gamma$ converges narrowly to a random probability μ, the characteristic functional thereof coincides with χ.*

Proof. By Proposition 2.2.4 the space $(X, \tau(X))$ is Polish, hence so is $\mathscr{M}_b^+(X)$ [83, 15.7.7], thus the space $\mathscr{M}_{\leq 1}^+(X) := \{\mu \in \mathscr{M}_b^+(X) \mid \mu X \leq 1\}$ is too, being closed, and $\mathscr{P}(X)$, being a G_δ-set in $\mathscr{M}_{\leq 1}^+(X)$. Since every finite measure on a Polish space is Radon [20, Thm. 7.1.7], each μ_γ is Radon. Consider $\mathscr{M}_b(X)$ endowed with the vague topology. The dense subset $\mathcal{C}_c(X)$ of the topological dual $(\mathscr{M}_b(X), \tau_v(\mathscr{M}_b(X)))' = \mathcal{C}_0(X)$ separates points in $\mathscr{M}_b(X)$, hence it separates points in $\mathscr{P}(X) \subset \mathscr{M}_b(X)$. The conclusion follows now by the Theorem choosing $Y = \mathscr{P}(X)$ and $V = \mathcal{C}_c(X)$. ∎

Chapter 3
A Mecke-type characterization of Dirichlet–Ferguson measures

In this Chapter, we characterize the Dirichlet–Ferguson measure over a locally compact Polish finite diffuse measure space as the unique random measure satisfying a Mecke-type identity.

3.1 Introduction

Let X be a *locally compact Polish space*[1], i.e., the topology on X is completely metrizable, separable, and locally compact. We denote by $\mathcal{B}$ the Borel σ-algebra on X.

We denote by $\mathscr{M}^+$ the cone of *positive* (not necessarily finite) *Radon measures*, endowed with the *vague topology*, i.e., the weakest topology on $\mathscr{M}^+$ with respect to which all maps $\mathscr{M}^+ \ni \eta \mapsto \eta f$ with $f \in \mathcal{C}_c$ are continuous. Here $\mathcal{C}_c$ is the space of continuous compactly supported functions on X, and for $f \in \mathcal{C}_c$ and $\eta \in \mathscr{M}^+$, we denote $\eta f := \int_X f \, \mathrm{d}\eta$, i.e., the usual duality pair between $\mathcal{C}_c$ and $\mathscr{M}^+$. By (the law of) a *random measure* over X we mean any Borel probability measure $\mathcal{Q}$ on $\mathscr{M}^+$.

We denote by Υ the *space of configurations* in X, i.e., the subset of $\mathscr{M}^+$ consisting of Radon measures of the form $\gamma = \sum_{i \geq 1} \delta_{x_i}$ with $x_i \neq x_j$ if $i \neq j$. Here, for $x \in X$, δ_x denotes the Dirac measure with mass at x, and we also require that the zero measure on X (i.e., an empty configuration) belongs to Υ. It holds that Υ is a Borel subset of $\mathscr{M}^+$. If a random measure $\mathcal{Q}$ is concentrated on Υ, we say that it is (the law of) a (*proper*, *simple*) *point process* in X.

Let σ be a *diffuse* (i.e., atomless) Radon measure on $(X, \mathcal{B})$, henceforth an *intensity.* Among all point processes, a remarkable and ubiquitous example is given by the *Poisson point process* or *Poisson measure* $\mathcal{P}_\sigma$ with intensity σ [88, 102], i.e., the point process in X whose Laplace transform satisfies:

$$\int_\Upsilon e^{\gamma f} \, \mathrm{d}\mathcal{P}_\sigma(\gamma) = \exp\left[\int_X \left(e^f - 1\right) \mathrm{d}\sigma\right] , \qquad f \in \mathcal{C}_c \; . \tag{3.1.1}$$

Recall the following characterization of $\mathcal{P}_\sigma$, usually known as the *Mecke identity.*

Theorem 3.1.1 (Mecke identity for $\mathcal{P}_\sigma$ [119, 3.1])**.** *Let X and σ be as above and let $\mathcal{Q}$ be a random measure over X. Then, the following statements are equivalent:*

(*i*) *$\mathcal{Q}$ is the Poisson measure $\mathcal{P}_\sigma$ with intensity σ;*

(*ii*) *for every non-negative measurable function $F \colon \mathscr{M}^+ \times X \to \mathbb{R}$,*

$$\int_{\mathscr{M}^+} \mathrm{d}\mathcal{Q}(\gamma) \int_X \mathrm{d}\gamma(x) \, F(\gamma, x) = \int_{\mathscr{M}^+} \mathrm{d}\mathcal{Q}(\gamma) \int_X \mathrm{d}\sigma(x) \, F(\gamma + \delta_x, x) \; . \tag{3.1.2}$$

[1] It is more common in the literature on random measures to refer to such spaces as *second countable locally compact Hausdorff spaces.* The two definitions are equivalent.

Remark 3.1.2. Note that, for $\mathcal{Q}=\mathcal{P}_\sigma$, in the Mecke identity (3.1.2), integration is over Υ rather than $\mathscr{M}^+$. Furthermore, since the measure σ is diffuse, for each $\gamma=\sum_{i\geq 1}\delta_{x_i}\in\Upsilon$, we have $\sigma\big(\{x_i\}_{i\geq 1}\big)=0$, hence $\gamma+\delta_x\in\Upsilon$ for σ-a.a. $x\in X$.

The Mecke identity (3.1.2) and its generalization to Gibbs measures, the *Georgii–Nguyen–Zessin formula* [62, 129], have important applications in the theory of point processes and stochastic dynamics of interacting particle systems. (See, e.g., [3, 4, 34, 113, 135].)

We denote by $\mathscr{P}$ the subset of *probability measures* in $\mathscr{M}^+$. It holds that $\mathscr{P}$ is a Borel subset of $\mathscr{M}^+$. If a random measure is concentrated on $\mathscr{P}$, we say that it is (the law of) a *random probability measure.*

The aim of this Chapter is to show how the Dirichlet–Ferguson measure $\mathcal{D}_\sigma$ (see §3.2 below) may be regarded as the natural analog of the Poisson measure when one replaces the configuration space Υ with $\mathscr{P}$.

Theorem 3.1.3 (A Mecke-type characterization of $\mathcal{D}_\sigma$)**.** *Let X be as above, let σ be a non-zero* finite *diffuse measure on $(X,\mathcal{B})$, and set $\beta:=\sigma X$. Then, for any random measure $\mathcal{Q}$ over X, the following statements are equivalent:*

(i) $\mathcal{Q}$ is the Dirichlet–Ferguson measure $\mathcal{D}_\sigma$ with intensity σ;

(ii) for every non-negative measurable function $G\colon\mathscr{M}^+\to\mathbb{R}$,

$$\int_{\mathscr{M}^+}\mathrm{d}\mathcal{Q}(\eta)\,\eta X\,G(\eta)=\int_{\mathscr{M}^+}\mathrm{d}\mathcal{Q}(\eta)\int_X\mathrm{d}\sigma(x)\int_0^1\mathrm{d}t\,(1-t)^{\beta-1}\,G\big((1-t)\eta+t\delta_x\big)\ . \tag{3.1.3}$$

Moreover, for every non-negative (or bounded) measurable function $F\colon\mathscr{P}\times X\to\mathbb{R}$,

$$\int_{\mathscr{P}}\mathrm{d}\mathcal{D}_\sigma(\eta)\int_X\mathrm{d}\eta(x)\,F(\eta,x)=\int_{\mathscr{P}}\mathrm{d}\mathcal{D}_\sigma(\eta)\int_X\mathrm{d}\sigma(x)\int_0^1\mathrm{d}t\,(1-t)^{\beta-1}\,F\big((1-t)\eta+t\delta_x,x\big)\ , \tag{3.1.4}$$

and for every non-negative (or bounded) measurable function $R\colon\mathscr{P}\times X\times[0,1]\to\mathbb{R}$,

$$\int_{\mathscr{P}}\mathrm{d}\mathcal{D}_\sigma(\eta)\int_X\mathrm{d}\eta(x)\,R(\eta,x,\eta_x)=\int_{\mathscr{P}}\mathrm{d}\mathcal{D}_\sigma(\eta)\int_X\mathrm{d}\sigma(x)\int_0^1\mathrm{d}t\,(1-t)^{\beta-1}\,R\big((1-t)\eta+t\delta_x,x,t\big)\ . \tag{3.1.5}$$

Remark 3.1.4. Note that, when σ is a probability measure on X, in formulas (3.1.3)–(3.1.5), the factor $(1-t)^{\beta-1}$ becomes 1.

Remark 3.1.5. Denote by B the Euler Beta function and by

$$\mathrm{d}\mathsf{B}_{\alpha,\beta}(t):=\frac{t^{\alpha-1}(1-t)^{\beta-1}\,\mathrm{d}t}{\mathrm{B}(\alpha,\beta)}$$

the Beta distribution on $[0,1]$ with shape parameters $\alpha>0$ and $\beta>0$. Set $\overline{\sigma}:=\sigma/\beta\in\mathscr{P}$. Then we have the following equality of the probability measures on $X\times[0,1]$:

$$(1-t)^{\beta-1}\,\mathrm{d}\sigma(x)\,\mathrm{d}t=\beta(1-t)^{\beta-1}\,\mathrm{d}\overline{\sigma}(x)\,\mathrm{d}t=\mathrm{d}\overline{\sigma}(x)\,\mathrm{d}\mathsf{B}_{1,\beta}(t)\ .$$

Remark 3.1.6. For $\mathcal{Q}=\mathcal{D}_\sigma$, in formula (3.1.3), integration is over $\mathscr{P}$ rather than $\mathscr{M}^+$, and the term ηX is equal to 1. Furthermore, for $\mathcal{Q}=\mathcal{D}_\sigma$, formula (3.1.3) is a special case of formula (3.1.4), while formulas (3.1.4) and (3.1.5) are in fact equivalent.

Let us provide some heuristics on the form of the Mecke-type identity for the Dirichlet–Ferguson measure. In the case of the Poisson measure, we seek a way to compute the $\mathcal{P}_\sigma$-average of the functional $\Upsilon \ni \gamma \mapsto \int_X F(\gamma, x)\,\mathrm{d}\gamma(x)$. It is the statement of the Mecke identity that the latter coincides with the $\mathcal{P}_\sigma$-average of some 'augmented' functional that we construct in the following way. Firstly, we augment $F(\gamma, x)$ by adding δ_x to γ, and secondly we take the σ-average in x. In the case where $\eta \in \mathscr{P}$, $\eta + \delta_x$ is not anymore a probability measure, so this operation makes no sense in $\mathscr{P}$. Nevertheless, δ_x belongs to $\mathscr{P}$. So, we may instead consider the convex hull of (actually a straight line between) η and δ_x, i.e., the set $\{(1-t)\eta + t\delta_x\}_{t\in[0,1]}$. Thus, the Mecke-type identity (3.1.4) states that the $\mathcal{D}_\sigma$-average of the functional $\mathscr{P} \ni \eta \mapsto \int_X F(\eta, x)\,\mathrm{d}\eta(x)$ coincides with the $\mathcal{D}_\sigma$-average of the augmented functional that we construct as follows. We consider $F\big((1-t)\eta + t\delta_x, x\big)$ and take its $\overline{\sigma}$-average in x and $\mathsf{B}_{1,\beta}$-average in t.

Our interest in a Mecke-type identity for the Dirichlet–Ferguson measure originated from the expected applications to the study of stochastic dynamics related to $\mathcal{D}_\sigma$, very much in the spirit of results of [31, 71, 96], which were obtained for measure-valued Lévy processes, in particular, for the gamma measure, see §3.2 below. Recall that, in those papers, a suitable analog of the Mecke identity (see formula (3.2.4) below regarding the gamma measure) plays a key rôle.

We also note that the Dirichlet–Ferguson measure is the unique stationary, reversible distribution for the Fleming–Viot process with parent-independent mutation. (See e.g., [54, Thm.s 5.3 and 5.4].)

Below, in §3.2, we discuss preliminary notions and facts, and in §3.3 we prove Theorem 3.1.3 and discuss several corollaries, including a characterization of the Dirichlet distribution. We expect that the results of this section remain true when X is a standard Borel space, that is, without the assumption that X is locally compact.

3.2 Preliminaries

The Dirichlet–Ferguson measure. For integer $k \geq 2$, let Δ^{k-1} denote the standard closed $(k-1)$-dimensional simplex in $\mathbb{R}^k$, i.e.,

$$\Delta^{k-1} \coloneqq \big\{(y_1, \ldots, y_k) \mid y_i \geq 0,\ y_1 + \cdots + y_k = 1\big\}\ .$$

Denote $\mathbb{R}_+ \coloneqq (0,\infty)$. For $\boldsymbol{\alpha} \coloneqq (\alpha_1, \ldots, \alpha_k)$ in $\mathbb{R}^k_+$, the *Dirichlet distribution* with parameter $\boldsymbol{\alpha}$ is the probability measure on Δ^{k-1} denoted by $\mathsf{D}_{\boldsymbol{\alpha}}$ and defined by

$$\mathsf{D}_{\boldsymbol{\alpha}}(A) \coloneqq \int_{\mathbb{R}^{k-1}} \mathbf{1}_A(y_1, \ldots, y_k)\, \frac{1}{\mathrm{B}(\boldsymbol{\alpha})} \left(\prod_{i=1}^k y_i^{\alpha_i - 1}\right) \mathrm{d}y_1 \cdots \mathrm{d}y_{k-1} \tag{3.2.1}$$

for each measurable subset A of Δ^{k-1}. In formula (3.2.1), B is the multivariate Beta function and $y_k \coloneqq 1 - y_1 - \cdots - y_{k-1}$.

For integer $k \geq 2$, we denote by $\mathfrak{P}_k(X)$ the set of ordered partitions $\mathbf{X} \coloneqq (X_1, \ldots, X_k)$ of X with $X_i \in \mathcal{B}$ and $\sigma X_i > 0$, $i \in [k]$. For each $\mathbf{X} \in \mathfrak{P}_k(X)$, we define the evaluation map $\mathrm{ev}_{\mathbf{X}} \colon \mathscr{P} \to \Delta^{k-1}$ by

$$\mathrm{ev}_{\mathbf{X}}(\eta) \coloneqq \big(\eta X_1, \ldots, \eta X_k\big)\ .$$

Note that the map $\mathrm{ev}_{\mathbf{X}}$ is measurable.

The *Dirichlet–Ferguson measure* $\mathcal{D}_\sigma$ with intensity σ [55] is the unique random probability measure over X satisfying the following two conditions:

(i) for each $B \in \mathcal{B}$ with $\sigma B = 0$, we have $\eta B = 0$ for $\mathcal{D}_\sigma$-a.a. $\eta \in \mathscr{P}$;

(ii) for each integer $k \geq 2$ and $\mathbf{X} \in \mathfrak{P}_k(X)$,

$$(\mathrm{ev}_{\mathbf{X}})_\sharp \mathcal{D}_\sigma = \mathsf{D}_{\mathrm{ev}_{\mathbf{X}}(\sigma)} \ , \tag{3.2.2}$$

i.e., the push-forward of $\mathcal{D}_\sigma$ under $\mathrm{ev}_{\mathbf{X}}$ is equal to $\mathsf{D}_{\mathrm{ev}_{\mathbf{X}}(\sigma)}$.

The gamma measure. Denote by $\mathscr{M}_b^{+,\mathrm{pa}}$ the subset of $\mathscr{M}^+$ that consists of discrete finite measures on X. Thus, each measure $\nu \in \mathscr{M}_b^{+,\mathrm{pa}}$ is finite and has a representation $\nu = \sum_{i\geq 1} s_i \delta_{x_i}$ with $s_i > 0$ and $x_i \neq x_j$ if $i \neq j$. As shown in [72], $\mathscr{M}_b^{+,\mathrm{pa}}$ is a Borel subset of $\mathscr{M}^+$. Consider the space $\mathbb{R}_+$ as endowed with the logarithmic distance $d_{\log}(s_1, s_2) := |\log(s_1/s_2)|$. Denote by $\hat{X}$ the (locally compact Polish) space $X \times \mathbb{R}_+$. Further let $\hat{\Upsilon}$ denote the space of configurations in $\hat{X}$. Consider the mapping $\mathbf{H} : \mathscr{M}_b^{+,\mathrm{pa}} \to \hat{\Upsilon}$ given by

$$\mathbf{H}\colon \sum_{i\geq 1} s_i \delta_{x_i} \longmapsto \sum_{i\geq 1} \delta_{(s_i, x_i)} \ ,$$

and let $\hat{\mathscr{M}}_b^{+,\mathrm{pa}}$ denote the range of $\mathbf{H}$. Clearly, the mapping $\mathbf{H} : \mathscr{M}_b^{+,\mathrm{pa}} \to \hat{\mathscr{M}}_b^{+,\mathrm{pa}}$ is a bijection. It was shown in [72, Thm. 6.2] that both $\mathbf{H}$ and $\mathbf{H}^{-1}$ are Borel-measurable.

Let λ be a Borel measure on $\mathbb{R}_+$ such that $\int_{\mathbb{R}_+} 1 \wedge s \,\mathrm{d}\lambda(s) < \infty$ (a Lévy measure). Consider the measure $\hat{\sigma} := \sigma \otimes \lambda$ on $(\hat{X}, \hat{\mathcal{B}})$. The Poisson measure $\mathcal{P}_{\hat{\sigma}}$ with intensity $\hat{\sigma}$ is concentrated on $\hat{\mathscr{M}}_b^{+,\mathrm{pa}}$, and we define a Borel probability measure $\mathcal{R}_{\sigma,\lambda}$ on $\mathscr{M}_b^{+,\mathrm{pa}}$ as the push-forward of $\mathcal{P}_{\hat{\sigma}}$ under $\mathbf{H}^{-1}$. The measure $\mathcal{R}_{\sigma,\lambda}$ is called a *measure-valued Lévy process* and it has the Laplace transform (See [96].)

$$\int_{\mathscr{M}_b^{+,\mathrm{pa}}} e^{\nu f} \,\mathrm{d}\mathcal{R}_{\sigma,\lambda}(\nu) = \exp\left[\int_{\hat{X}} \left(e^{s\, f(x)} - 1\right) \mathrm{d}\hat{\sigma}(x,s)\right] \ , \qquad f \in \mathcal{C}_c \ .$$

When $\mathrm{d}\lambda(s) := s^{-1}e^{-s}\,\mathrm{d}s$, we write $\mathcal{G}_\sigma := \mathcal{R}_{\sigma,\lambda}$ for the *gamma measure* with intensity σ. It has the Laplace transform (e.g., [157])

$$\int_{\mathscr{M}_b^+} e^{\nu f} \,\mathrm{d}\mathcal{G}_\sigma(\nu) = \exp\left[-\int_X \mathrm{d}\sigma \log(1-f)\right] \ , \qquad f \in \mathcal{C}_c \ , \ f < 1 \ . \tag{3.2.3}$$

Lemma 3.2.1 (Mecke identity for the gamma measure)**.** *The gamma measure with intensity σ satisfies*

$$\int_{\mathscr{M}_b^{+,\mathrm{pa}}} \mathrm{d}\mathcal{G}_\sigma(\nu) \int_X \mathrm{d}\nu(x)\, G(\nu, x) = \int_{\mathscr{M}_b^{+,\mathrm{pa}}} \mathrm{d}\mathcal{G}_\sigma(\nu) \int_X \mathrm{d}\sigma(x) \int_0^\infty \mathrm{d}\lambda'(s)\, G(\nu + s\delta_x, x) \tag{3.2.4}$$

for every non-negative measurable function $G\colon \mathscr{M}_b^{+,\mathrm{pa}} \times X \to \mathbb{R}$. Here $\mathrm{d}\lambda'(s) := s \ \mathrm{d}\lambda(s) = e^{-s}\ \mathrm{d}s$.

Proof. The statement is a straightforward corollary of the Mecke identity satisfied by the Poisson measure $\mathcal{P}_{\hat{\sigma}}$ and the construction of the gamma measure $\mathcal{G}_\sigma$. (See [72, Thm. 6.3] for the details in the case $X = \mathbb{R}^d$.) ■

Remark 3.2.2. A similar Mecke identity holds, of course, for each measure-valued Lévy process $\mathcal{R}_{\sigma,\lambda}$. (See [96, Eq. (29)].)

It was shown in [55] that the Dirichlet–Ferguson measure $\mathcal{D}_\sigma$ is the 'simplicial part' of $\mathcal{G}_\sigma$. More precisely, we denote by Γ the Euler gamma function and by

$$\mathrm{d}\mathsf{G}_{k,\theta}(r) := \frac{\theta^{-k}}{\Gamma(k)} r^{k-1} e^{-\frac{r}{\theta}}\,\mathrm{d}r$$

the gamma distribution on $\mathbb{R}_+$ with shape parameter $k > 0$ and scale parameter $\theta > 0$. Consider the measurable mapping $\mathbf{R} : \mathscr{M}_b^{+,\mathrm{pa}} \to \mathscr{P} \times \mathbb{R}_+$ given by

$$\mathbf{R} \colon \nu \longmapsto (\overline{\nu}, \nu X) \ .$$

Then,

$$\mathbf{R}_\sharp \, \mathcal{G}_\sigma = \mathcal{D}_\sigma \otimes \mathsf{G}_{\beta,1} \ , \tag{3.2.5}$$

i.e., the push-forward of $\mathcal{G}_\sigma$ under $\mathbf{R}$ is the product measure $\mathcal{D}_\sigma \otimes \mathsf{G}_{\beta,1}$. (Recall that $\beta = \sigma X$.) Note that $\mathsf{G}_{1,1} = \lambda'$. Note also that the Dirichlet–Ferguson measure $\mathcal{D}_\sigma$ is concentrated on the set of discrete probability measure, $\mathscr{P}^{\mathrm{pa}} := \mathscr{M}_b^{+,\mathrm{pa}} \cap \mathscr{P}$, and that the mapping $\mathbf{R} : \mathscr{M}_b^{+,\mathrm{pa}} \to \mathscr{P}^{\mathrm{pa}} \times \mathbb{R}_+$ is bijective.

3.3 Proof and corollaries

We start with the proof of the main result.

Proof of Theorem 3.1.3. We first prove that $\mathcal{D}_\sigma$ satisfies formula (3.1.4), hence (3.1.3). Using Lemma 3.2.1 and formula (3.2.5) yields

$$\begin{aligned}
&\int_{\mathscr{P}} \mathrm{d}\mathcal{D}_\sigma(\eta) \int_X \mathrm{d}\eta(x)\, F(\eta, x) \\
&= \frac{\Gamma(\beta)}{\Gamma(\beta+1)} \int_{\mathscr{P}} \mathrm{d}\mathcal{D}_\sigma(\eta) \int_0^\infty \mathrm{d}\mathsf{G}_{\beta,1}(r)\, r \int_X \mathrm{d}\eta(x)\, F(\eta, x) \\
&= \frac{\Gamma(\beta)}{\Gamma(\beta+1)} \int_{\mathscr{M}_b^{+,\mathrm{pa}}} \mathrm{d}\mathcal{G}_\sigma(\nu) \int_X \mathrm{d}\nu(x)\, F(\overline{\nu}, x) \\
&= \frac{\Gamma(\beta)}{\Gamma(\beta+1)} \int_{\mathscr{M}_b^{+,\mathrm{pa}}} \mathrm{d}\mathcal{G}_\sigma(\nu) \int_X \mathrm{d}\sigma(x) \int_0^\infty \mathrm{d}s\, e^{-s}\, F\left(\frac{\nu + s\delta_x}{\nu X + s}, x\right) \\
&= \frac{\Gamma(\beta)}{\Gamma(\beta+1)} \int_{\mathscr{P}} \mathrm{d}\mathcal{D}_\sigma(\eta) \int_0^\infty \mathrm{d}\mathsf{G}_{\beta,1}(r) \int_X \mathrm{d}\sigma(x) \int_0^\infty \mathrm{d}s\, e^{-s} F\left(\frac{r\eta + s\delta_x}{r+s}, x\right) \\
&= \frac{1}{\Gamma(\beta+1)} \int_{\mathscr{P}} \mathrm{d}\mathcal{D}_\sigma(\eta) \int_0^\infty \mathrm{d}s\, e^{-s} \int_X \mathrm{d}\sigma(x) \int_0^\infty \mathrm{d}r\, r^{\beta-1} e^{-r} F\left(\frac{r}{r+s}\,\eta + \frac{s}{r+s}\,\delta_x, x\right) \ ,
\end{aligned}$$

whence the change of variable $t = \frac{s}{r+s}$ (for a fixed s) yields

$$\begin{aligned}
&= \frac{1}{\Gamma(\beta+1)} \int_{\mathscr{P}} \mathrm{d}\mathcal{D}_\sigma(\eta) \int_0^\infty \mathrm{d}s\, e^{-s} \int_X \mathrm{d}\sigma(x) \int_0^1 \frac{s\ \mathrm{d}t}{t^2}\, s^{\beta-1} \frac{(1-t)^{\beta-1}}{t^{\beta-1}} e^{-s(1-t)/t} \cdot \\
&\qquad \cdot F\big((1-t)\eta + t\delta_x, x\big) \\
&= \frac{1}{\Gamma(\beta+1)} \int_{\mathscr{P}} \mathrm{d}\mathcal{D}_\sigma(\eta) \int_X \mathrm{d}\sigma(x) \int_0^1 \mathrm{d}t\, \frac{(1-t)^{\beta-1}}{t^{\beta+1}}\, F\big((1-t)\eta + t\delta_x, x\big) \int_0^\infty \mathrm{d}s\, e^{-s} s^\beta \cdot \\
&\qquad \cdot e^{-s(1-t)/t} \\
&= \int_{\mathscr{P}} \mathrm{d}\mathcal{D}_\sigma(\eta) \int_X \mathrm{d}\sigma(x) \int_0^1 \mathrm{d}t\, (1-t)^{\beta-1} F\big((1-t)\eta + t\delta_x, x\big) \ .
\end{aligned}$$

To prove formula (3.1.5), choose $F(\eta, x) = R(\eta, x, \eta_x)$ in (3.1.4), which gives

$$\begin{aligned}
&\int_{\mathscr{P}} \mathrm{d}\mathcal{D}_\sigma(\eta) \int_X \mathrm{d}\eta(x)\, R\big(\eta, x, \eta_x\big) \\
&\quad = \int_{\mathscr{P}} \mathrm{d}\mathcal{D}_\sigma(\eta) \int_0^1 \mathrm{d}t\, (1-t)^{\beta-1} \int_X \mathrm{d}\sigma(x)\, R\big((1-t)\eta + t\delta_x, x, (1-t)\eta_x + t\big) \\
&\quad = \int_{\mathscr{P}^{\mathrm{pa}}} \mathrm{d}\mathcal{D}_\sigma(\eta) \int_0^1 \mathrm{d}t\, (1-t)^{\beta-1} \int_X \mathrm{d}\sigma(x)\, R\big((1-t)\eta + t\delta_x, x, t\big) \ ,
\end{aligned}$$

where we used that, for a fixed $\eta = \sum_{i\geq 1} s_i \delta_{x_i} \in \mathscr{P}^{\mathrm{pa}}$, we have $\sigma\big(\{x_i\}_{i\geq 1}\big) = 0$, hence $\eta_x = 0$ for σ-a.a. $x \in X$.

For the reverse implication we consider the set $\mathfrak{M}$ of all random measures $\mathcal{Q}$ over X that satisfy (3.1.3). We know that $\mathcal{D}_\sigma \in \mathfrak{M}$ and we need to prove that $\mathcal{D}_\sigma$ is the unique element of $\mathfrak{M}$. So let $\mathcal{Q} \in \mathfrak{M}$ and let us first show that $\mathcal{Q}$ is concentrated on $\mathscr{P}$. Choosing $G = 1$ in (3.1.3), we get

$$\int_{\mathscr{M}^+} \eta X \,\mathrm{d}\mathcal{Q}(\eta) = 1 \ . \tag{3.3.1}$$

In particular, $\eta X < \infty$ $\mathcal{Q}$-a.s.. Next, choosing $G(\eta) = \eta X$ in (3.1.3) and using (3.3.1), we get

$$\begin{aligned} \int_{\mathscr{M}^+} \mathrm{d}\mathcal{Q}(\eta)\, (\eta X)^2 &= \int_{\mathscr{M}^+} \mathrm{d}\mathcal{Q}(\eta) \int_X \mathrm{d}\sigma(x) \int_0^1 \mathrm{d}t\, (1-t)^{\beta-1} \big((1-t)\eta X + t\big) \\ &= \int_X \mathrm{d}\sigma(x) \int_0^1 \mathrm{d}t\, (1-t)^{\beta-1} \big((1-t) + t\big) = 1 \ . \end{aligned} \tag{3.3.2}$$

By (3.3.1) and (3.3.2), the random variable ηX has zero variance under $\mathcal{Q}$, hence it is deterministic. Thus, $\eta X = 1$ $\mathcal{Q}$-a.s., so $\mathcal{Q}$ is concentrated on $\mathscr{P}$. Hence, formula (3.1.3) becomes

$$\int_{\mathscr{P}} \mathrm{d}\mathcal{Q}(\eta)\, G(\eta) = \int_{\mathscr{P}} \mathrm{d}\mathcal{Q}(\eta) \int_X \mathrm{d}\sigma(x) \int_0^1 \mathrm{d}t\, (1-t)^{\beta-1} G\big((1-t)\eta + t\delta_x\big) \ , \tag{3.3.3}$$

and it holds for every measurable bounded function $G : \mathscr{P} \to \mathbb{R}$.

Let $B \in \mathcal{B}$ be such that $\sigma B = 0$. By (3.3.3),

$$\begin{aligned} \int_{\mathscr{M}^+} \eta B \,\mathrm{d}\mathcal{Q}(\eta) &= \int_{\mathscr{M}^+} \mathrm{d}\mathcal{Q}(\eta) \int_X \mathrm{d}\sigma(x) \int_0^1 \mathrm{d}t\, (1-t)^{\beta-1} \big((1-t)\eta B + t\, \mathbb{1}_B(x)\big) \\ &= \beta \int_{\mathscr{M}^+} \eta B \,\mathrm{d}\mathcal{Q}(\eta) \int_0^1 \mathrm{d}t\, (1-t)^{\beta} + \sigma B \int_0^1 \mathrm{d}t\, t(1-t)^{\beta-1} \\ &= \frac{\beta}{\beta+1} \int_{\mathscr{M}^+} \eta B \,\mathrm{d}\mathcal{Q}(\eta) \ , \end{aligned}$$

which implies

$$\int_{\mathscr{M}^+} \eta B \,\mathrm{d}\mathcal{Q}(\eta) = 0 \ .$$

Hence, $\eta B = 0$ for $\mathcal{Q}$-a.e. $\eta \in \mathscr{M}^+$. Next, for an integer $k \geq 2$, consider an ordered partition $\mathbf{X} = (X_1, \ldots, X_k) \in \mathfrak{P}_k(X)$. To prove that $\mathcal{Q} = \mathcal{D}_\sigma$, it remains to show that the distribution of the random vector $\mathrm{ev}_{\mathbf{X}}(\eta) = (\eta X_1, \ldots, \eta X_k)$ in $\mathbb{R}^k$ (in fact, in Δ^{k-1}) under $\mathcal{Q}$ is uniquely determined by (3.3.3). We recall that the *Hadamard product* $\diamond : \mathbb{R}^k \times \mathbb{R}^k \to \mathbb{R}^k$ is defined by

$$\mathbf{s}_1 \diamond \mathbf{s}_2 := \big(s_{1,1}s_{2,1}, \ldots, s_{1,k}s_{2,k}\big) \ , \qquad \mathbf{s}_i := (s_{i,1}, \ldots, s_{i,k}) \in \mathbb{R}^k \ , \qquad i = 1, 2 \ .$$

This binary operation is obviously associative and commutative.

Denote by $\mathcal{B}_b(X)$ the linear space of bounded measurable functions $g\colon X \to \mathbb{R}$. For any $g \in \mathcal{B}_b(X)$ and a finite measure η on X, we let $\eta g := \int_X g \,\mathrm{d}\eta$. Set further $\boldsymbol{\alpha} := \mathrm{ev}_{\mathbf{X}}(\sigma)$. Fix any $\mathbf{s} = (s_1, \ldots, s_k) \in \mathbb{R}^k$, and let $g(x) := \sum_{i=1}^k s_i \mathbb{1}_{X_i}(x) \in \mathcal{B}_b(X)$. Then

$$\eta g = \mathbf{s} \cdot \mathrm{ev}_{\mathbf{X}}(\eta) \ , \qquad \eta \in \mathscr{P} \ , \tag{3.3.4}$$

and

$$\sigma g^n = \mathbf{s}^{\diamond n} \cdot \boldsymbol{\alpha} \ , \qquad n \in \mathbb{N}_0 \ . \tag{3.3.5}$$

For $n \in \mathbb{N}_0$, we get, by (3.3.3)–(3.3.5),

$$\int_{\mathscr{P}} \big(\mathbf{s} \cdot \mathrm{ev}_{\mathbf{X}}(\eta)\big)^n \,\mathrm{d}\mathcal{Q}(\eta)$$

$$
\begin{aligned}
&= \int_{\mathscr{P}} \mathrm{d}\mathcal{Q}(\eta) \int_X \mathrm{d}\sigma(x) \int_0^1 \mathrm{d}t\,(1-t)^{\beta-1}\big((1-t)\eta g + t g(x)\big)^n \\
&= \sum_{i=0}^n \tbinom{n}{i} \int_{\mathscr{P}} (\eta g)^i \,\mathrm{d}\mathcal{Q}(\eta) \int_X g^{n-i}\,\mathrm{d}\sigma \int_0^1 (1-t)^{\beta+i-1} t^{n-i}\,\mathrm{d}t \\
&= \sum_{i=0}^n \tbinom{n}{i} \mathrm{B}(\beta+i, n-i+1) \int_{\mathscr{P}} \big(\mathbf{s}\cdot \mathrm{ev}_{\mathbf{X}}(\eta)\big)^i \big(\mathbf{s}^{\diamond(n-i)}\cdot \boldsymbol{\alpha}\big)\,\mathrm{d}\mathcal{Q}(\eta) \\
&= \sum_{i=0}^n \frac{n!\,\Gamma(\beta+i)}{i!\,\Gamma(\beta+n+1)} \int_{\mathscr{P}} \big(\mathbf{s}\cdot \mathrm{ev}_{\mathbf{X}}(\eta)\big)^i \big(\mathbf{s}^{\diamond(n-i)}\cdot \boldsymbol{\alpha}\big)\,\mathrm{d}\mathcal{Q}(\eta) \\
&= \sum_{i=0}^n \frac{(n)_{n-i}}{(\beta+n)_{n+1-i}} \int_{\mathscr{P}} \big(\mathbf{s}\cdot \mathrm{ev}_{\mathbf{X}}(\eta)\big)^i \big(\mathbf{s}^{\diamond(n-i)}\cdot \boldsymbol{\alpha}\big)\,\mathrm{d}\mathcal{Q}(\eta) \\
&= \frac{\beta}{\beta+n} \int_{\mathscr{P}} \big(\mathbf{s}\cdot \mathrm{ev}_{\mathbf{X}}(\eta)\big)^n\,\mathrm{d}\mathcal{Q}(\eta) \\
&\quad + \sum_{i=0}^{n-1} \frac{(n)_{n-i}}{(\beta+n)_{n+1-i}} \int_{\mathscr{P}} \mathrm{d}\mathcal{Q}(\eta)\, \big(\mathbf{s}\cdot \mathrm{ev}_{\mathbf{X}}(\eta)\big)^i \big(\mathbf{s}^{\diamond(n-i)}\cdot \boldsymbol{\alpha}\big)\ ,
\end{aligned}
$$

where $(r)_k$ denotes the falling factorial: $(r)_0 \coloneqq 1$ and $(r)_k \coloneqq r(r-1)\cdots(r-k+1)$ for $k\in\mathbb{N}_1$. Therefore,

$$
\begin{aligned}
\int_{\mathscr{P}} \big(\mathbf{s}\cdot \mathrm{ev}_{\mathbf{X}}(\eta)\big)^n\,\mathrm{d}\mathcal{Q}(\eta) =& \frac{\beta+n}{n} \sum_{i=0}^{n-1} \frac{(n)_{n-i}}{(\beta+n)_{n+1-i}} \int_{\mathscr{P}} \mathrm{d}\mathcal{Q}(\eta)\, \big(\mathbf{s}\cdot \mathrm{ev}_{\mathbf{X}}(\eta)\big)^i \big(\mathbf{s}^{\diamond(n-i)}\cdot \boldsymbol{\alpha}\big) \\
=& \sum_{i=0}^{n-1} \frac{(n-1)_{n-1-i}}{(\beta+n-1)_{n-i}} \int_{\mathscr{P}} \big(\mathbf{s}\cdot \mathrm{ev}_{\mathbf{X}}(\eta)\big)^i \big(\mathbf{s}^{\diamond(n-i)}\cdot \boldsymbol{\alpha}\big)\,\mathrm{d}\mathcal{Q}(\eta)\ .
\end{aligned} \tag{3.3.6}
$$

The recurrence relation (3.3.6) uniquely determines the moments

$$
\int_{\mathscr{P}} \big(\mathbf{s}\cdot \mathrm{ev}_{\mathbf{X}}(\eta)\big)^n\,\mathrm{d}\mathcal{Q}(\eta)\ , \qquad \mathbf{s}\in\mathbb{R}^k\ ,\ n\in\mathbb{N}_0\ . \tag{3.3.7}
$$

Since the measure $(\mathrm{ev}_{\mathbf{X}})_\sharp\mathcal{Q}$ has a compact support in $\mathbb{R}^k$, it is uniquely determined by its moment sequence (3.3.7). (See e.g., [16, Ch. 8, §5].) Therefore, $(\mathrm{ev}_{\mathbf{X}})_\sharp\mathcal{Q} = (\mathrm{ev}_{\mathbf{X}})_\sharp\mathcal{D}_\sigma$. Thus, $\mathcal{Q}=\mathcal{D}_\sigma$. ■

Remark 3.3.1. We stress that our proof of the reverse implication in Theorem 3.1.3 is different from the proofs of the analogous characterizations for the Poisson and the gamma measure (resp. [119, Satz 3.1] and [72, Thm. 6.3]). Indeed, the latter proofs rely on a characterization of the Laplace transform of the random measure in question by some ordinary differential equation. This approach seems however not possible in the case of the Dirichlet–Ferguson measure, whose Laplace transform is a kind of an infinite-variable hypergeometric function. (See Thm. 2.1.1.) On the other hand, proper analogs of our proof (through the uniqueness of the solution of a multidimensional moment problem under an appropriate bound on the moments) allow one to prove the corresponding statements for both the Poisson and the gamma measure. (See [102, Thm. 4.1].)

Corollary 3.3.2 (Moments of the Dirichlet distribution)**.** *Let* $\boldsymbol{\alpha} \coloneqq (\alpha_1,\dots,\alpha_k)\in\mathbb{R}^k_+$ *and assume that* $\alpha_\bullet \coloneqq \alpha_1+\cdots+\alpha_k = 1$. *Then*

(*i*) *The moments of the Dirichlet distribution* $\mathsf{D}_{\boldsymbol{\alpha}}$ *satisfy the following recurrence relation:*

$$
\int_{\mathbb{R}^k} \prod_{i=1}^n (\mathbf{s}_i\cdot\mathbf{y})\,\mathrm{d}\mathsf{D}_{\boldsymbol{\alpha}}(\mathbf{y}) = \frac{1}{n} \sum_{\substack{\xi\subset[n]\\ \#\xi<n}} \tbinom{n}{\#\xi}^{-1} \Bigg(\Diamond_{j\in[n]\setminus\xi} \mathbf{s}_j\Bigg)\cdot\boldsymbol{\alpha} \int_{\mathbb{R}^k} \prod_{i\in\xi} (\mathbf{s}_i\cdot\mathbf{y})\,\mathrm{d}\mathsf{D}_{\boldsymbol{\alpha}}(\mathbf{y}) \tag{3.3.8}
$$

for all $n \in \mathbb{N}_0$ *and* $\mathbf{s}_1, \dots, \mathbf{s}_n \in \mathbb{R}^k$. *(Here,* $\#\xi$ *denotes the number of elements of the set* ξ.*) In particular, for all* $n \in \mathbb{N}_0$ *and* $\mathbf{s} \in \mathbb{R}^k$,

$$\int_{\mathbb{R}^k} (\mathbf{s} \cdot \mathbf{y})^n \, \mathrm{d}\mathsf{D}_{\boldsymbol{\alpha}}(\mathbf{y}) = \frac{1}{n} \sum_{i=0}^{n-1} \int_{\mathbb{R}^k} (\mathbf{s} \cdot \mathbf{y})^i \, \mathrm{d}\mathsf{D}_{\boldsymbol{\alpha}}(\mathbf{y}) \big(\mathbf{s}^{\diamond(n-i)} \cdot \boldsymbol{\alpha}\big) \; . \tag{3.3.9}$$

(*ii*) *For all* $n \in \mathbb{N}_0$ *and* $\mathbf{s} \in \mathbb{R}^k$,

$$\int_{\mathbb{R}^k} (\mathbf{s} \cdot \mathbf{y})^n \, \mathrm{d}\mathsf{D}_{\boldsymbol{\alpha}}(\mathbf{y}) = Z_n\big(\mathbf{s}^{\diamond 1} \cdot \boldsymbol{\alpha}, \dots, \mathbf{s}^{\diamond n} \cdot \boldsymbol{\alpha}\big) \; ,$$

where Z_n *denotes the cycle index polynomial of the symmetric group* $\mathfrak{S}_n$.

Proof. Choose $X = [0,1]$, $\mathrm{d}\sigma(x) = \mathrm{d}x$, and choose a partition $\mathbf{X}$ such that $\mathrm{ev}_{\mathbf{X}}(\sigma) = \boldsymbol{\alpha}$. Then formula (3.3.9) follows from (3.3.6) if we note that, for $\beta = 1$,

$$\frac{\langle n-1 \rangle_{n-1-i}}{\langle \beta + n - 1 \rangle_{n-i}} = \frac{\langle n-1 \rangle_{n-1-i}}{\langle n \rangle_{n-i}} = \frac{1}{n} \; .$$

Next, note that the right hand side of formula (3.3.8) is an n-linear symmetric form of $\mathbf{s}_1, \dots, \mathbf{s}_n \in \mathbb{R}^k$, and for $\mathbf{s} = \mathbf{s}_1 = \dots = \mathbf{s}_n$, the right-hand side of (3.3.8) is equal to the right-hand side of formula (3.3.9). Hence, (3.3.8) follows from (3.3.9) and the polarization identity. The second statement follows by noticing that the cycle index polynomials of $\mathfrak{S}_n$ satisfy the recurrence relation (3.3.9). (See e.g., [37, §2.1, Eq. (2.6)]). ■

Remark 3.3.3. Corollary 3.3.2(*ii*) is shown by different methods in [37, 4.2].

Remark 3.3.4. By using formula (3.3.6), one can immediately extend Corollary 3.3.2(*i*) to the case of a general $\boldsymbol{\alpha} \in \mathbb{R}^k_+$.

Corollary 3.3.5 (Moments of the Dirichlet–Ferguson measure)**.** *Let* $\sigma \in \mathscr{P}$ *(i.e.* $\beta = 1$*). Then, the moments of the Dirichlet–Ferguson measure* $\mathcal{D}_\sigma$ *satisfy the following recurrence relation:*

$$\int_{\mathscr{P}} \prod_{i=1}^n \eta g_i \, \mathrm{d}\mathcal{D}_\sigma(\eta) = \frac{1}{n} \sum_{\substack{\xi \subset [n] \\ \#\xi < n}} \binom{n}{\#\xi}^{-1} \int_{\mathscr{P}} \prod_{i \in \xi} \eta g_i \, \mathrm{d}\mathcal{D}_\sigma(\eta) \int_X \prod_{j \in [n] \setminus \xi} g_j \, \mathrm{d}\sigma \tag{3.3.10}$$

for all $n \in \mathbb{N}_1$ *and* $g_1, \dots, g_n \in \mathcal{B}_b(X)$. *In particular, for all* $n \in \mathbb{N}_1$ *and* $g \in \mathcal{B}_b(X)$,

$$\int_{\mathscr{P}} (\eta g)^n \, \mathrm{d}\mathcal{D}_\sigma(\eta) = \frac{1}{n} \sum_{i=0}^{n-1} \int_{\mathscr{P}} (\eta g)^i \, \mathrm{d}\mathcal{D}_\sigma(\eta) \int_X g^{n-i} \, \mathrm{d}\sigma \; .$$

Proof. In the case where the functions $g_1, \dots, g_n \in \mathcal{B}_b(X)$ take on a finite number of values, formula (3.3.10) follows from (3.3.4) and (3.3.8). In the general case, formula (3.3.10) follows by approximation and the dominated convergence theorem. ■

Remark 3.3.6. Similarly to Remark 3.3.4, one can easily extend Corollary 3.3.5 to the case of a general finite intensity measure σ.

Remark 3.3.7. A non-recursive formula for the moments of the Dirichlet–Ferguson measure, namely the full expansion of (3.3.10), may be found in [54, Lem. 5.2].

Corollary 3.3.8 (A characterization of the Dirichlet distribution)**.** *Let* $k \geq 2$. *Let* θ *be a probability measure on* $\mathbb{R}^k_+$. *Then, the following statements are equivalent:*

(*i*) θ *is the Dirichlet distribution* $\mathsf{D}_{\boldsymbol{\alpha}}$ *with parameter* $\boldsymbol{\alpha} \in \mathbb{R}^k_+$;

(ii) for every non-negative measurable function $g\colon \mathbb{R}_+^k \to \mathbb{R}$*,*

$$\int_{\mathbb{R}_+^k} \mathrm{d}\theta(\mathbf{y})\, \mathbf{y}_\bullet\, g(\mathbf{y}) = \int_{\mathbb{R}_+^k} \mathrm{d}\theta(\mathbf{y}) \int_0^1 \mathrm{d}t\, (1-t)^{\boldsymbol{\alpha}_\bullet - 1} \sum_{i=1}^k \alpha_i\, g\big((1-t)\mathbf{y} + t\mathbf{e}_i\big) \ . \tag{3.3.11}$$

Here $\mathbf{y}_\bullet \coloneqq y_1 + \cdots + y_k$ *for* $\mathbf{y} \in \mathbb{R}_+^k$ *and* $\{\mathbf{e}_i\}_{i\in[k]}$ *is the canonical basis of* $\mathbb{R}^k$*.*

Moreover, for every non-negative (or bounded) measurable function $f\colon \Delta^{k-1} \times [k] \to \mathbb{R}$*,*

$$\int_{\Delta^{k-1}} \sum_{i=1}^k y_i f(\mathbf{y}, i)\ \mathrm{d}\mathsf{D}_{\boldsymbol{\alpha}}(\mathbf{y}) = \int_{\Delta^{k-1}} \mathrm{d}\mathsf{D}_{\boldsymbol{\alpha}}(\mathbf{y}) \int_0^1 \mathrm{d}t\, (1-t)^{\boldsymbol{\alpha}_\bullet - 1} \sum_{i=1}^k \alpha_i\, f\big((1-t)\mathbf{y} + t\mathbf{e}_i, i\big) \ . \tag{3.3.12}$$

Proof. Assume (i) holds. Similarly to the proof of Corollary 3.3.2, choose $X = [0,1]$ and $\mathrm{d}\sigma(x) = \boldsymbol{\alpha}_\bullet\, \mathrm{d}x$, so that $\beta = \boldsymbol{\alpha}_\bullet$, and choose a partition $\mathbf{X}$ such that $\mathrm{ev}_{\mathbf{X}}(\sigma) = \boldsymbol{\alpha}$. Applying formula (3.3.3) to $G \coloneqq g \circ \mathrm{ev}_{\mathbf{X}}$ and recalling (3.2.2) gives

$$\begin{aligned}
&\int_{\Delta^{k-1}} g(\mathbf{y})\, \mathrm{d}\mathsf{D}_{\boldsymbol{\alpha}}(\mathbf{y}) = \int_{\mathscr{P}} g\big(\eta X_1, \ldots, \eta X_k\big)\, \mathrm{d}\mathcal{D}_\sigma(\eta) \\
&= \sum_{i=1}^k \int_{\mathscr{P}} \mathrm{d}\mathcal{D}_\sigma(\eta) \int_{X_i} \mathrm{d}x \int_0^1 \mathrm{d}t\, (1-t)^{\boldsymbol{\alpha}_\bullet - 1} g\big((1-t)\eta X_1, \ldots, (1-t)\eta X_i + t, \ldots, (1-t)\eta X_k\big) \\
&= \int_{\Delta^{k-1}} \mathrm{d}\mathsf{D}_{\boldsymbol{\alpha}}(\mathbf{y}) \int_0^1 \mathrm{d}t\, (1-t)^{\boldsymbol{\alpha}_\bullet - 1} \sum_i^k \alpha_i g\big((1-t)\mathbf{y} + t\mathbf{e}_i\big) \ .
\end{aligned}$$

Thus, (3.3.11) holds for $\theta = \mathsf{D}_{\boldsymbol{\alpha}}$. Formula (3.3.12) is proven analogously by applying (3.1.4) to

$$F(\eta, x) = \sum_{i=1}^k f\big(\mathrm{ev}_{\mathbf{X}}(\eta), i\big)\, \mathbb{1}_{X_i}(x) \ .$$

In order to prove that (3.3.11) uniquely identifies the measure θ, one uses essentially the same arguments as in the proof of Theorem 3.1.3. One first shows that

$$\int_{\mathbb{R}_+^k} (\mathbf{y}_\bullet)^n\, \mathrm{d}\theta(\mathbf{y}) = 1 \ , \qquad n = 1, 2 \ ,$$

which implies $\mathbf{y}_\bullet = 1$ θ-a.s., i.e., θ is concentrated on Δ^{k-1} in $\mathbb{R}^k$. Then one chooses $g(\mathbf{y}) \coloneqq (\mathbf{s} \cdot \mathbf{y})^n$ and finds the recurrence relation for the moments of θ. ■

Chapter 4
A Rademacher-type Theorem on Wasserstein spaces

Let $\mathbb{P}$ be any Borel probability measure on the L^2-Wasserstein space $(\mathscr{P}_2(M), W_2)$ over a closed Riemannian manifold M. In this Chapter, we consider the Dirichlet form $\mathcal{E}$ induced by $\mathbb{P}$ and by the Wasserstein gradient on $\mathscr{P}_2(M)$. Under natural assumptions on $\mathbb{P}$, we show that W_2-Lipschitz functions on $\mathscr{P}_2(M)$ are contained in the Dirichlet space $\mathscr{D}(\mathcal{E})$ and that W_2 is dominated by the intrinsic metric induced by $\mathcal{E}$. We detail several examples.

4.1 Introduction

We consider the L^2-Wasserstein space $\mathscr{P}_2 = (\mathscr{P}_2(M), W_2)$ associated to a closed Riemannian manifold (M, g). Since the seminal work of F. Otto [131], the geometry of $\mathscr{P}_2$ has been widely studied from several view points. Definitions have been proposed and thoroughly studied of a 'weak Riemannian structure' on $\mathscr{P}_2$ (e.g., Lott [111]), of a gradient for 'smooth' functions on $\mathscr{P}_2$, of tangent space to $\mathscr{P}_2$ at a point (See Gigli [63] for a detailed account of several such notions), of an exponential map [63], of a Levi-Civita connection [64], of differential forms [60]. This heuristic picture of $\mathscr{P}_2$ as an infinite-dimensional Riemannian manifold calls for the existence of a measure on $\mathscr{P}_2$ canonically and uniquely associated to the metric structure. As it is the case for a differentiable manifold, such a measure — if any — would deserve the name of *Riemannian volume measure* which we shall adopt in the following.

In this framework, the question of the existence of such a Riemannian volume measure on $\mathscr{P}_2$ has been insistently posed (e.g., [28, 63, 140, 151]). In the case of $M = \mathbb{S}^1$, M.-K. von Renesse and K.-T. Sturm [140] proposed as a candidate the *entropic measure* on $\mathscr{P}_2(\mathbb{S}^1)$ (Example 4.5.15). Whereas a suitable definition of entropic measure on $\mathscr{P}_2(M)$ for a closed Riemannian manifold M was given by K.-T. Sturm in [151], most of its properties in this general case remain unknown. Here, we rather address the question of discerning the properties of a *volume measure* $\mathbb{P}$ on $\mathscr{P}_2$. By 'volume measure' we shall mean any analogue on $\mathscr{P}_2$ of *a* measure on a differentiable manifold induced by a volume form via integration.

We do so by proving a Rademacher-type result on the $\mathbb{P}$-a.e. *Fréchet* differentiability of W_2-Lipschitz functions (Thm. 4.2.4). Namely, we consider a Dirichlet space $\mathscr{F}$ associated to $\mathbb{P}$ and to a natural gradient, with core being the algebra $\mathfrak{Z}^\infty$ of cylinder functions induced by smooth potential energies (Dfn. 4.2.1). Combining the strategy of [142] with the fine analysis of tangent plans performed by N. Gigli in [63], we study, for functions in $\mathscr{F}$, suitable concepts of *directional derivative* and *differential*, proving their consistency on $\mathfrak{Z}^\infty$. We show that, if $\mathbb{P}$ is quasi-invariant with respect to the family of shifts defining the gradient, then the space of W_2-Lipschitz functions is contained in $\mathscr{F}$.

The requirement of the Rademacher property is indeed a natural one for a volume measure. For instance, it was recently shown by G. De Philippis and F. Rindler [35, Thm. 1.14] that, if μ

is a positive Radon measure on $\mathbb{R}^d$ such that every Lipschitz function is μ-a.e. differentiable, then $\mu \ll \mathrm{Leb}^d$. In infinite dimensions, the problem has been addressed in linear spaces (e.g. Bogachev–Mayer-Wolf [22]), in particular on the abstract Wiener space (Enchev–Stroock [49]), and — in the 'non-flat', albeit finitary, case — on configuration spaces (Röckner–Schied [142]).

Finally, we detail some examples of measures satisfying, fully or in part, our assumptions. These are mainly taken from the theory of point processes and include *normalized mixed Poisson measures*, the *Dirichlet–Ferguson measure* [55], as well as the *entropic measure* [140] and an image on $\mathscr{P}_2(\mathbb{S}^1)$ of the *Malliavin–Shavgulidze measure* [116]. We show through these examples how the situation on $\mathscr{P}_2$ is opposite to the aforementioned result in [35]. In particular, there exist mutually singular fully supported measures on $\mathscr{P}_2$ satisfying the Rademacher property.

Auxiliary results are collected in the Appendix, together with a discussion of the notion of 'tangent bundle' to $\mathscr{P}_2$ from the point of view of global derivations of the algebra $\mathfrak{Z}^\infty$.

4.2 A Rademacher Theorem on $\mathscr{P}_2$

Let (X, d) be a complete and separable metric space and let further $\mathscr{P} = \mathscr{P}(X)$ be the space of all Borel probability measures on X. Given $\mu_1, \mu_2 \in \mathscr{P}$, we denote by $\mathrm{Cpl}(\mu_1, \mu_2)$ the set of *couplings* (or *transport plans*) between μ_1 and μ_2, that is, the set of Borel probability measures on $X^{\times 2}$ such that $\mathrm{pr}^i_\sharp \pi = \mu_i$ for $i = 1, 2$. We let further

$$\mathscr{P}_2 := \left\{ \mu \in \mathscr{P} \mid \int_X \mathsf{d}^2(x, x_0)\, \mathrm{d}\mu(x) < \infty \right\}$$

for some fixed $x_0 \in X$, and

$$W_2(\mu_1, \mu_2) := \inf_{\pi \in \mathrm{Cpl}(\mu_1, \mu_2)} \left(\int_{X^{\times 2}} \mathsf{d}^2(x, y)\, \mathrm{d}\pi(x, y) \right)^{1/2} . \tag{4.2.1}$$

The space $(\mathscr{P}_2, W_2)$ is a metric space, termed L^2-*Wasserstein space over* (X, d). (Notice that $\mathscr{P}_2$ does not depend on the choice of x_0, by triangle inequality.) We denote by $\mathrm{Opt}(\mu, \nu)$ the set of *optimal plans* $\pi \in \mathrm{Cpl}(\mu, \nu)$ attaining the infimum in (4.2.1). This set is always non-empty.

Everywhere in the following let (M, g) be a closed (i.e. *compact*, without boundary) connected smooth d-dimensional Riemannian manifold with Riemannian distance $\mathsf{d} = \mathsf{d}_\mathsf{g}$ and volume measure m. As a consequence of the compactness of M, the space $\mathscr{P}_2$ coincides, as a set, with the space $\mathscr{P}$. It is well-known that, under our assumptions on M, the space $(\mathscr{P}_2, W_2)$ is a compact (in particular: complete and separable) geodesic metric space. (See e.g., [10] or [165, Ch. 6].)

In order to perform computations for functions on $\mathscr{P}$ in the spirit of [111, 131], we recall the definition of *potential energy* — in the sense of [164, §5.2.2]. Namely, given a continuous function $f\colon M \to \mathbb{R}$, we define the potential energy $f^*\colon \mathscr{P} \to \mathbb{R}$ associated to f by setting

$$f^*\mu := \mu f = \int_M f\, \mathrm{d}\mu .$$

Definition 4.2.1 (Cylinder functions)**.** For $f_i \in \mathcal{C}^0(M)$, $i \leq k$, we set $\mathbf{f} := (f_1, \dots, f_k)$ and $\mathbf{f}^*\colon \mathscr{P} \ni \mu \mapsto (f_1^*\mu, \dots, f_k^*\mu) \in \mathbb{R}^k$, and define the algebra of *cylinder functions* on $\mathscr{P}$

$$\mathfrak{Z}^\infty := \left\{ u\colon \mathscr{P} \to \mathbb{R} \mid u = F \circ \mathbf{f}^* \text{ for some } k \in \mathbb{N}_0 \ , \ F \in \mathcal{C}^\infty(\mathbb{R}^k) \ , \ f_i \in \mathcal{C}^\infty(M) \right\} . \tag{4.2.2}$$

Remark 4.2.2. By compactness of $\mathscr{P}_2$, in the definition above one might equivalently take $F \in \mathcal{C}_c^\infty(\mathbb{R}^k)$. The given definition makes more apparent that $f^* \in \mathfrak{Z}^\infty$ for all $f \in \mathcal{C}^\infty(M)$. By continuity of f^*, cylinder functions are continuous and thus (Borel) measurable.

Motivated by a similar choice in the framework of configuration spaces, (Cf. [142, Eq. (1.1)], see §4.5.3 below.) we define the gradient of $u \in \mathfrak{Z}^\infty$ by

$$\boldsymbol{\nabla}\, u(\mu)(x) \coloneqq \sum_i^k (\partial_i F)(\mathbf{f}^*\mu)\, \nabla f_i(x) \ . \tag{4.2.3}$$

This choice is consistent, by chain rule, with the Fréchet differentiability of f^* with respect to a natural Riemannian structure on the space of absolutely continuous measures $\mu = \rho\mathsf{m} \in \mathscr{P}$ (Cf., e.g., [111] or [164, §9.1].) and more generally with the differentiability of functionals on probability measures (e.g., [11]); furthermore, it is also consistent with the definition of a Wasserstein gradient in the recent work [29]. (See in particular [29, Dfn. 2.3 and Rmk. 2.4].)

We will also need a concept of directional derivative for functions in $\mathfrak{Z}^\infty$ and thus a concept of 'direction' at a point μ in $\mathscr{P}$. It is not surprising that such a definition ought to be "inherited" from the differentiable structure of the manifold M, henceforth the *base space*. Indeed, let T_xM be the tangent space to M at the point x. We denote by $\mathfrak{X}^0$ the space of continuous vector fields, that is, sections of the tangent bundle TM, endowed with the supremum norm

$$\|w\|_{\mathfrak{X}^0} \coloneqq \sup_{x\in M} |w_x|_{\mathsf{g}} \ .$$

We let further $\mathfrak{X}^\infty \subset \mathfrak{X}^0$ be the algebra of smooth vector fields on M. For any $w \in \mathfrak{X}^\infty$ we denote by $\left(\psi^{w,t}\right)_{t\in\mathbb{R}}$ the flow generated by w, i.e. a map $\psi^{w,t} \colon M \to M$ such that

$$\forall x \in M \qquad \dot{\psi}^{w,t}(x) = w\big(\psi^{w,t}(x)\big) \quad \text{and} \quad \psi^{w,0}(x) = x \ ,$$

where by $\dot{\psi}^{w,t}(x)$ we mean the velocity of the curve $s \mapsto \psi^{w,s}(x)$ at time t. By compactness of M every $w \in \mathfrak{X}^\infty$ admits a unique flow, well-defined and a smooth orientation-preserving diffeomorphism in $\mathrm{Diff}^\infty_+(M)$ for all times $t \in \mathbb{R}$. (See e.g. [14, §1.3.7(ii)].) If we denote by

$$\Psi^{w,t} \coloneqq \psi^{w,t}_\sharp \colon \mathscr{P} \longrightarrow \mathscr{P}$$

the push-forward via $\psi^{w,t}$, then a straightforward computation (see Lem. 4.6.2 below) shows that

$$(\boldsymbol{\nabla}_w\, u)(\mu) \coloneqq \mathrm{d}_t\big|_{t=0}(u \circ \Psi^{w,t})(\mu) = \left\langle \boldsymbol{\nabla}\, u(\mu) \,\middle|\, w \right\rangle_{\mathfrak{X}_\mu} \ , \qquad u \in \mathfrak{Z}^\infty \ , \tag{4.2.4}$$

where

$$\left\langle w^0 \,\middle|\, w^1 \right\rangle_{\mathfrak{X}_\mu} \coloneqq \int_M \left\langle w^0_x \,\middle|\, w^1_x \right\rangle_{\mathsf{g}} \mathrm{d}\mu(x) \ , \qquad w^0, w^1 \in \mathfrak{X}^\infty \ . \tag{4.2.5}$$

This motivates (Cf. [142] for the case of configuration spaces.) to define the tangent space to $\mathscr{P}$ at a point μ as the space $\mathfrak{X}_\mu \coloneqq \mathrm{cl}_{L^2_\mu} \mathfrak{X}^\infty$, that is, the abstract linear completion of $\mathfrak{X}^\infty$ with respect to the norm $\|\cdot\|_{\mathfrak{X}_\mu}$ induced by the Hilbert scalar product $\langle \cdot \mid \cdot \rangle_{\mathfrak{X}_\mu}$, the (non-relabeled) extension to $\mathfrak{X}_\mu$ of the scalar product (4.2.5). We shall also write $T^{\mathrm{Der}}_\mu \mathscr{P}_2$ for $\mathfrak{X}_\mu$ and thus $T^{\mathrm{Der}}\mathscr{P}_2$ for the associated fiber-"bundle". Let us further set $T^\nabla_\mu \mathscr{P}_2 \coloneqq \mathrm{cl}_{\mathfrak{X}_\mu} \mathfrak{X}^\infty_\nabla$, where $\mathfrak{X}^\infty_\nabla \coloneqq \nabla \mathcal{C}^\infty(M)$ denotes the family of vector fields *of gradient type*; the associated fiber-"bundle" will be denoted by $T^\nabla \mathscr{P}_2$. As it is well-established in the optimal transport literature, (e.g., [10, 60, 63, 64]) the space $T^\nabla_\mu \mathscr{P}_2$ is the space of those geodesic directions at μ that are induced by optimal transport maps in the sense of Brenier–McCann Theorem. (See Thm. 4.3.8 below.) In the following we will make use of both *non-equivalent* definitions. An exhaustive discussion of this choice is postponed to §4.6.1. However, let us remark here that the two definitions are in fact equivalent on configuration spaces.

Assumption. We say that a Borel probability measure $\mathbb{P}$ on $\mathscr{P}_2$ satisfies Assumption 4.2 if 4.2 (i)–4.2 (iv) below hold for $\mathbb{P}$, namely if

4.2 (i) $\mathbb{P}$ is fully supported;

4.2 (ii) $\mathbb{P}$ is diffuse (i.e., it has no atoms);

4.2 (iii) $\mathbb{P}$ satisfies the following integration by parts formula. If $u, v \in \mathfrak{Z}^\infty$ and $w \in \mathfrak{X}^\infty$, then there exists a measurable function $\mu \mapsto \boldsymbol{\nabla}^*_w v \in \mathfrak{X}_\mu$ such that

$$\int_{\mathscr{P}} \boldsymbol{\nabla}_w u \cdot v \,\mathrm{d}\mathbb{P} = \int_{\mathscr{P}} u \cdot \boldsymbol{\nabla}^*_w v \,\mathrm{d}\mathbb{P} \; ; \tag{4.2.6}$$

4.2 (iv) $\mathbb{P}$ is quasi-invariant with respect to the action of the family of flows $\mathrm{Flow}(M)$ on $\mathscr{P}$, i.e. $\mathbb{P}$ and $\Psi^{w,t}_\sharp \mathbb{P}$ are mutually absolutely continuous for all $w \in \mathfrak{X}^\infty$ and $t \in \mathbb{R}$. Moreover, for all finite $s \leq t$ it holds that

$$\text{for } \mathbb{P}\text{-a.e. } \mu \in \mathscr{P} \qquad \mathrm{Leb}^1\text{-}\operatorname*{essinf}_{r\in[s,t]} R^w_r(\mu) > 0 \qquad \text{where} \quad R^w_r := \frac{\mathrm{d}(\Psi^{w,r}_\sharp \mathbb{P}) \otimes \mathrm{d}r}{\mathrm{d}\mathbb{P} \otimes \mathrm{d}r} \; . \tag{4.2.7}$$

The validity and necessity of these assumptions are widely illustrated through examples in §4.5.

In the following, we shall also need the stronger assumption

4.2 (v) $\mathbb{P}$ satisfies 4.2 (iv) and the Radon–Nikodým derivative R^w_r defined in (4.2.7) is such that for every $w \in \mathfrak{X}^\infty$

- $r \mapsto R^w_r(\mu)$ is differentiable in a neighborhood of 0 for $\mathbb{P}$-a.e. μ;
- $\mu \mapsto |\partial_r R^w_r(\mu)|$ is integrable w.r.t. $\mathbb{P}$ uniformly in r on a neighborhood of 0.

We shall comment on this latter assumption in Proposition 4.5.6.

Definition 4.2.3 (Cylinder vector fields)**.** Let $\mathcal{XC}^\infty := \mathfrak{Z}^\infty \otimes_{\mathbb{R}} \mathfrak{X}^\infty$ denote the vector space of *cylinder vector fields* on $\mathscr{P}$, i.e. the $\mathbb{R}$-vector space of sections W of $T^{\mathrm{Der}}\mathscr{P}$ of the form

$$W(\mu)(x) = \sum_j^n v_j(\mu) w_j(x) \tag{4.2.8}$$

with $n \in \mathbb{N}$, $v_j \in \mathfrak{Z}^\infty$ and $w_j \in \mathfrak{X}^\infty$. (By $\otimes_{\mathbb{R}}$ we denote the *algebraic* $\mathbb{R}$-tensor product.) By $\mathcal{XC}_{\mathbb{P}}$ we mean the abstract linear completion of the space $\mathcal{XC}^\infty$ endowed with the pre-Hilbert norm defined by setting

$$\|W\|^2_{\mathcal{XC}_{\mathbb{P}}} := \sum_j^n \int_{\mathscr{P}} |v_j(\mu)|^2 \, \|w_j\|^2_{\mathfrak{X}_\mu} \,\mathrm{d}\mathbb{P}(\mu) \; .$$

It follows by linearity from Assumption 4.2 (iii) that

$$\forall u \in \mathfrak{Z}^\infty \quad \forall W \in \mathcal{XC}^\infty \qquad \int_{\mathscr{P}} \left\langle \boldsymbol{\nabla} u \,\middle|\, W \right\rangle_{\mathfrak{X}_\cdot} \mathrm{d}\mathbb{P} = -\int_{\mathscr{P}} u \cdot \mathbf{div}_{\mathbb{P}} W \,\mathrm{d}\mathbb{P} \tag{4.2.9}$$

where, for any W as in (4.2.8),

$$\mathbf{div}_{\mathbb{P}} W(\mu) := -\sum_i^n \boldsymbol{\nabla}^*_{w_i} v_i(\mu) \; .$$

Then, $(\mathbf{div}_{\mathbb{P}}, \mathcal{XC}^\infty)$ is a densely defined linear operator from the space of sections $\Gamma_{L^2_{\mathbb{P}}} T^{\mathrm{Der}}\mathscr{P}_2$ to $L^2_{\mathbb{P}}(\mathscr{P})$ and we denote its adjoint by $(\mathbf{d}_{\mathbb{P}}, \mathbf{W}^{1,2})$. By definition, functions in $\mathbf{W}^{1,2}$ are weakly differentiable, in the sense that (4.2.9) holds for all $u \in \mathbf{W}^{1,2}$ with $\mathbf{d}_{\mathbb{P}} u$ in lieu of $\boldsymbol{\nabla} u$.

We denote by $\mathcal{F}$ the set of all bounded measurable functions u on $\mathscr{P}$ for which there exists a measurable section $\mathbf{D}u$ of $T^{\mathrm{Der}}\mathscr{P}_2$ such that

$$\mathcal{E}(u,u) := \int_{\mathscr{P}} \left\langle \mathbf{D}u(\mu) \,\middle|\, \mathbf{D}u(\mu)\right\rangle_{\mathfrak{X}_\mu} \mathrm{d}\mathbb{P}(\mu) < \infty \tag{4.2.10}$$

and such that for every $w \in \mathfrak{X}^\infty$ and $s \in \mathbb{R}$ there exists the directional derivative

$$\frac{u \circ \Psi^{w,t} - u}{t} \xrightarrow{t\to 0} \langle \mathbf{D}u \,|\, w\rangle_{\mathfrak{X}\,.} \quad \text{in} \quad L^2(\mathscr{P}, \Psi^{w,s}_\sharp \mathbb{P})\ . \tag{4.2.11}$$

Finally, set $\mathcal{F}_{\mathrm{cont}} := \mathcal{F} \cap \mathcal{C}(\mathscr{P})$ and observe that $\mathfrak{Z}^\infty \subset \mathcal{F}_{\mathrm{cont}} \subset \mathcal{F}$ and that, *a priori*, every inclusion may be a strict one.

Before stating the main result, we introduce the following — quite restrictive — assumption on the base space. We will comment extensively about this assumption, and about its connection with the Ma–Trudinger–Wang curvature condition, in §4.5.2.

Assumption (Smooth Transport Property). We say that M satisfies the *smooth transport property* (in short: STP) if, whenever $\mu, \nu \in \mathscr{P}$, $\mu, \nu \ll \mathsf{m}$ with smooth nowhere vanishing densities, then there exists a *smooth* optimal transport map $g\colon M \to M$ mapping μ to ν in the sense of Thm. 4.3.8 below.

Theorem 4.2.4. *Suppose that $\mathbb{P}$ satisfies Assumptions 4.2 (ii) and 4.2 (iii). Then,*

4.2.4 (1) *the bilinear forms $(\mathcal{E}, \mathfrak{Z}^\infty)$, $(\mathcal{E}, \mathcal{F}_{\mathrm{cont}})$ and $(\mathcal{E}, \mathcal{F})$ are closable and their closures, respectively denoted by $(\mathcal{E}, \mathscr{F}_0)$, $(\mathcal{E}, \mathscr{F}_{\mathrm{cont}})$ and $(\mathcal{E}, \mathscr{F})$ are strongly local Dirichlet forms. Clearly, $\mathscr{F}_0 \subset \mathscr{F}_{\mathrm{cont}} \subset \mathscr{F}$;*

4.2.4 (2) *for each $u \in \mathscr{F}$ there exists a measurable section $\mathbf{D}u$ of the tangent bundle $T^{\mathrm{Der}}\mathscr{P}_2$ such that*

$$\mathbf{D}u = \boldsymbol{\nabla} u\ , \qquad u \in \mathfrak{Z}^\infty\ , \tag{4.2.12}$$

and

$$\mathcal{E}(u,u) = \int_{\mathscr{P}} \|\mathbf{D}u(\mu)\|^2_{\mathfrak{X}_\mu} \mathrm{d}\mathbb{P}(\mu)\ , \tag{4.2.13}$$

i.e. the form $(\mathcal{E}, \mathscr{F})$ admits carré du champ $\boldsymbol{\Gamma}(u)(\mu) := \|\mathbf{D}u(\mu)\|^2_{\mathfrak{X}_\mu}$;

4.2.4 (3) (Rademacher property) *let $u\colon \mathscr{P} \to \mathbb{R}$ be W_2-Lipschitz continuous. Then $u \in \mathscr{F}_{\mathrm{cont}}$ and, if additionally STP holds, then $u \in \mathscr{F}_0$. Furthermore, there exist a measurable set $\Omega^u \subset \mathscr{P}$ of full $\mathbb{P}$-measure and a measurable section $\mathbf{D}u$ of $T^{\mathrm{Der}}\mathscr{P}_2$, satisfying (4.2.12) and (4.2.13), such that*

4.2.4 (3.*i*) *for all $\mu \in \Omega^u$ it holds that $\|\mathbf{D}u(\mu)\|_{\mathfrak{X}_\mu} \le \mathrm{Lip}[u]$;*

4.2.4 (3.*ii*) *if additionally Assumption 4.2 (iv) holds, then*

$$\forall \mu \in \Omega^u \qquad \|\mathbf{D}u(\mu)\|_{\mathfrak{X}_\mu} \le |Du|(\mu)\ , \tag{4.2.14}$$

where $|Du|$ is the slope *of u (see (4.3.2) below), and, for all $w \in \mathfrak{X}^\infty$*

$$\lim_{t\to 0} \frac{(u \circ \Psi^{w,t} - u)(\,\cdot\,)}{t} = \langle \mathbf{D}u(\,\cdot\,) \,|\, w\rangle_{\mathfrak{X}\,.} \tag{4.2.15}$$

pointwise on Ω^u and in $L^2_{\mathbb{P}}(\mathscr{P})$.

We now collect some remarks on the statement of our main theorem.

Remark 4.2.5. Let us notice that, in the definition of the pre-domain $\mathcal{F}$, we only ask for the existence of *a* measurable section $\mathbf{D}u$ of $T^{\mathrm{Der}}\mathscr{P}_2$ with finite L^2-norm in the sense of (4.2.10). In particular, we do not require (4.2.12) which is only used to uniquely identify the form $(\mathcal{E},\mathscr{F})$ in the theorem. We pay this arbitrariness with the Fréchet differentiability (4.2.11) of $u\in\mathcal{F}$ in $L^2(\mathscr{P},\Psi^{w,s}_\sharp\mathbb{P})$ for each real s, a seemingly stronger condition than the conclusion (4.2.15) of Theorem 4.2.4 (3.*ii*). In fact though, under Assumption 4.2 (*iv*), condition (4.2.11) is equivalent to (4.2.15) for every W_2-Lipschitz u, since the $L^2(\mathscr{P},\Psi^{w,s}_\sharp\mathbb{P})$-topology is equivalent to the $L^2(\mathscr{P},\mathbb{P})$-topology for every s.

Remark 4.2.6 (On the definition of 'volume measure' on $\mathscr{P}_2$). Assumptions 4.2 (*i*) and 4.2 (*ii*) are of a general kind. In fact, Assumption 4.2 (*i*) (i.e. $\mathbb{P}$ fully supported) is not necessary to the conclusions of Theorem 4.2.4. Rather, it rules out some trivial cases. (See Example 4.5.4.)

On the contrary, Assumptions 4.2 (*iii*) and 4.2 (*iv*) are — as already noticed in [142, Rmk. p. 329] for measures on configuration spaces — specifically proper of a volume measure (as discussed in the Introduction). In particular, Assumption 4.2 (*iii*) may be regarded as a form of 'gradient-divergence duality' for $\mathbb{P}$. Assumption 4.2 (*iv*) (and its stronger version 4.2 (*v*)) is also expected from a differential geometry point of view and it is equally important in light of Proposition 4.5.6 below.

Remark 4.2.7. As already noticed in the case of configuration spaces (Cf. [142, Prop. 1.4(iii)].), the Dirichlet forms $(\mathcal{E},\mathscr{F}_0)$, $(\mathcal{E},\mathscr{F}_{\mathrm{cont}})$ and $(\mathcal{E},\mathscr{F})$ do in principle differ. A sufficient condition for their coincidence is the essential self-adjointness of the generator of $(\mathcal{E},\mathscr{F})$ on the core $\mathfrak{Z}^\infty$.

It is readily seen that, by compactness of $\mathscr{P}_2$ and the Stone–Weierstraß Theorem, the spaces $\mathfrak{Z}^\infty$ and $\mathcal{F}_{\mathrm{cont}}$ are uniformly dense in $\mathcal{C}(\mathscr{P}_2)$. Together with the Theorem, this implies that the Dirichlet forms $(\mathcal{E},\mathscr{F}_0)$ and $(\mathcal{E},\mathscr{F}_{\mathrm{cont}})$ are regular strongly local Dirichlet forms on $\mathscr{P}_2$, thus properly associated to Markov diffusion processes by the theory of Dirichlet forms. (See e.g. [112].)

Remark 4.2.8 (On the definition of 'Rademacher-type' properties). Assume we have already shown that $u_\nu\colon\mu\mapsto W_2(\nu,\mu)$ belongs to $\mathscr{F}_{\mathrm{cont}}$, resp. $\mathscr{F}_0$, (cf. Lem.s 4.4.3 and 4.4.4 below) and $\mathbf{\Gamma}(u_\nu)\leq\mathbb{1}$. Then, 4.2.4 (3.*i*) may be deduced by the general results on (non-local) Dirichlet forms in [58]. On the contrary — even if it is proven that the Dirichlet form $(\mathcal{E},\mathscr{F})$ is strongly local and regular — the finer estimate (4.2.14) does not follow by [97, Thm. 2.1], where the reference measure (in our case $\mathbb{P}$) is assumed to be doubling. In fact it may be proved that *no* (fully supported) doubling measure exists on $\mathscr{P}_2$, since the latter is infinite-dimensional.

Both of the previous results may be considered as 'Rademacher-type' properties for the Dirichlet form(s) in question. Nonetheless, in the case of the Wasserstein space $\mathscr{P}_2$, we have — in addition to the general assumptions of [58] or [97] — a good notion of *directional* derivative for functions on $\mathscr{P}_2$. As a consequence, the statement of what we call a 'Rademacher Theorem on $(\mathscr{P}_2,W_2,\mathbb{P})$' comprises more properly assertion 4.2.4 (3.*ii*), where we check that each directional derivative of a "differentiable" function $u\in\mathscr{F}$ along a "smooth direction" $w\in\mathfrak{X}^\infty$ coincides with the scalar product of the "gradient" $\mathbf{D}u$ and "direction" w.

To conclude this preliminary section we anticipate that the statement of our main theorem is non-void, and that our assumptions pose no restriction to the subset of measures in $\mathscr{P}$ whereon $\mathbb{P}$ is concentrated. In particular

Remark 4.2.9 (See Rmk. 4.5.19 below). Define

- $\mathscr{A}_1$ the set of measures in $\mathscr{P}$ absolutely continuous w.r.t. the volume of M;
- $\mathscr{A}_2$ the set of measures in $\mathscr{P}$ singular continuous w.r.t. the volume of M;

- $\mathscr{A}_3$ the set of purely atomic measures in $\mathscr{P}$;
- $\mathscr{A}_4$ the set of transport regular measures in $\mathscr{P}$. (See Dfn. 4.3.6 below.)

Then, $M = \mathbb{S}^1$ has the STP (Ass. 4.2), and, for any choice of $a_1, a_2, a_3 \geq 0$ and such that $a_1 + a_2 + a_3 = 1$, there exists $\mathbb{P} \in \mathscr{P}(\mathscr{P}_2)$, satisfying Assumption 4.2 and such that $\mathbb{P}(\mathscr{A}_i) = a_i$ for every $i = 1, 2, 3$ and $\mathbb{P}(\mathscr{A}_4) = a_1 + a_2$.

4.3 Preliminaries

4.3.1 Setting and further notation. By a *measure* we always mean a *non-negative* measure. We denote by I, resp. I°, the unit interval $[0, 1]$, resp. $(0, 1)$, always endowed with the usual metric, σ-algebra and with the one-dimensional Lebesgue measure $\mathrm{dLeb}^1(r) = \mathrm{d}r$. Analogously, we denote by Leb^d the d-dimensional Lebesgue measure on (any subset of) $\mathbb{R}^d$.

Probability measures on M. We indicate by $\mathscr{P}^{\mathsf{m}} \subset \mathscr{P}$ the space of probability measures $\mu \ll \mathsf{m}$, by $\mathscr{P}^\infty$ the subset of probability measures $\mu \in \mathscr{P}^{\mathsf{m}}$ with smooth densities, by $\mathscr{P}^{\infty,\times}$ the subset of measures in $\mathscr{P}^\infty$ whose densities with respect to m are bounded away from 0 (the boundedness (from above) of such densities is rather a consequence of their continuity and of the compactness of M). We denote further by η any *purely atomic* measure in $\mathscr{P}$. Usually, we think of any such η as an infinite marked configuration and thus we write, with slight abuse of notation, η_x in place of $\eta\{x\}$ and $x \in \eta$ whenever $\eta_x > 0$. We denote further by $\operatorname{ptws} \eta$ the set of points $x \in M$ such that $\eta_x > 0$, termed the *pointwise support of* η. For r in I and any $\mu \in \mathscr{P}$ also set

$$\mu +_r \delta_x := (1 - r)\mu + r\delta_x \ . \tag{4.3.1}$$

4.3.2 Lipschitz functions. Everywhere in the following let (Y, τ) be a compact Polish (that is, it is separable and completely metrizable) space with Borel σ-algebra $\mathcal{B}$ and let n be a finite fully supported (Radon) measure on $(Y, \mathcal{B})$. Let ρ be any metric metrizing (Y, τ). In the rest of this section, the metric measure space (Y, ρ, n) will play the rôle of $(\mathscr{P}, W_2, \mathbb{P})$.

We say that a real-valued function $h \colon Y \to \mathbb{R}$ is *L-Lipschitz* (with respect to ρ) if there exists a constant $L > 0$ such that

$$\forall y_1, y_2 \in Y \qquad |h(y_1) - h(y_2)| \leq L\, \rho(y_1, y_2) \ ,$$

in which case we denote by $\mathrm{Lip}_\rho[h]$ the infimal such constant and by

$$|Dh|_\rho (y) := \limsup_{z \to y} \frac{|h(y) - h(z)|}{\rho(y, z)} \leq L \tag{4.3.2}$$

the *slope* (or *local Lipschitz constant*) of h at a point $y \in Y$. The metric ρ is omitted in the notation whenever apparent from context. We set $\rho_z(\,\cdot\,) := \rho(z, \,\cdot\,)$ and, for any $A := (a_i)_i^n \subset \mathbb{R}$ and $E := (z_i)_i^n \subset Y$, we let $\rho_{A,E,L}(\,\cdot\,) := \vee_{i \leq n}(a_i - L\rho_{z_i}(\,\cdot\,))$.

Lemma 4.3.1. *Let $Z \subset Y$ be a dense set and fix $h \in \mathrm{Lip}_\rho Y$. For $\varepsilon > 0$ let $E_\varepsilon := (z_{\varepsilon,i})_i^{n_\varepsilon} \subset Z$ be an ε-net for Y, that is, E_ε is such that $\rho(z_{\varepsilon,i}, z_{\varepsilon,j}) > \varepsilon/2$ for all $i \neq j$ and $\sup_{y \in Y} \rho(y, E_\varepsilon) \leq \varepsilon$. Set $A_\varepsilon := (h(z_{\varepsilon,i}))_i^{n_\varepsilon} \subset \mathbb{R}$. Then, the function $h_\varepsilon := \rho_{A_\varepsilon, E_\varepsilon, \mathrm{Lip}[h]}$ satisfies $\mathrm{Lip}[h_\varepsilon] \leq \mathrm{Lip}[h]$ and $\|h - h_\varepsilon\|_{\mathcal{C}^0} \leq C\varepsilon$ where C is a constant only depending on $\mathrm{Lip}[h]$.*

Proof. The existence of E_ε as above follows by density of Z in Y and compactness of Y. This shows that the statement is well-posed. The function h_ε is ρ-Lipschitz continuous with $\mathrm{Lip}[h_\varepsilon] \leq \mathrm{Lip}[h]$ for it is a maximum of ρ-Lipschitz continuous functions with Lipschitz constant $\mathrm{Lip}[h]$. Since h is Lipschitz continuous, it coincides with its lower McShane extension [118], i.e. $h(y) = \sup_{z\in Y} \{h(z) - \rho_y(z)\}$. Thus, $h_\varepsilon \leq h$. Furthermore, for all $y \in Y$ there exists $\bar z := \bar z(y)$ such that $h(y) \leq h(\bar z) - \rho(y, \bar z) + \varepsilon$ and, by definition of E_ε, there exists $\bar\imath := \bar\imath(y)$ such that $\rho(\bar z, z_{\varepsilon,\bar\imath}) \leq \varepsilon$. Hence,

$$\begin{aligned}
h_\varepsilon(y) \leq h(y) \leq& h(\bar z) - \rho(y, \bar z) + \varepsilon\\
\leq& h(\bar z) - h(z_{\varepsilon,\bar\imath}) + h(z_{\varepsilon,\bar\imath}) - \rho(y, \bar z) + \rho(y, z_{\varepsilon,\bar\imath}) - \rho(y, z_{\varepsilon,\bar\imath}) + \varepsilon\\
\leq& h(z_{\varepsilon,\bar\imath}) - \rho(y, z_{\varepsilon,\bar\imath}) + |h(\bar z) - h(z_{\varepsilon,\bar\imath})| + |\rho(y, z_{\varepsilon,\bar\imath}) - \rho(y, \bar z)| + \varepsilon\\
\leq& h_\varepsilon(y) + \mathrm{Lip}[h]\varepsilon + \varepsilon + \varepsilon
\end{aligned}$$

respectively by definition of h_ε, Lipschitz continuity of h and by reverse triangle inequality and definition of $z_{\varepsilon,\bar\imath}$. The conclusion follows by letting $C := \mathrm{Lip}[h] + 2$. ■

4.3.3 Dirichlet forms. We recall some facts on Dirichlet forms and prove some auxiliary results. Whenever $(Q, \mathscr{D}(Q))$ is a non-negative definite symmetric bilinear form, we denote by the same symbol the associated quadratic form, defined as $Q(u) := Q(u, u)$ if $u \in \mathscr{D}(Q)$ and $Q(u) := +\infty$ otherwise.

Definition 4.3.2 (Energy measure, carré du champ, intrinsic distance)**.** Let $(\mathcal{E}, \mathscr{D}(\mathcal{E}))$ be a regular strongly local (in the sense of [59, §1.1]) Dirichlet form on $L^2_{\mathsf{n}}(Y)$, additionally such that $\mathbb{1} \in \mathscr{D}(\mathcal{E})$ and $\mathcal{E}(\mathbb{1}) = 0$. Then, (e.g., [23]) the form $\mathcal{E}$ can be written as

$$\mathcal{E}(u, v) = \int_Y \mathrm{d}\mathbf{\Gamma}(u, v)$$

for all $u, v \in \mathscr{D}(\mathcal{E})$, where $\mathbf{\Gamma}$, termed the *energy measure of* $(\mathcal{E}, \mathscr{D}(\mathcal{E}))$, is an $\mathscr{M}(Y, \mathcal{B})$-valued non-negative definite symmetric bilinear form defined by the formula

$$\int_Y \phi \,\mathrm{d}\mathbf{\Gamma}(u, v) := \tfrac{1}{2}\big(\mathcal{E}(u, \phi v) + \mathcal{E}(v, \phi u) - \mathcal{E}(uv, \phi)\big)$$

for all $u, v \in \mathscr{D}(\mathbf{\Gamma}) := \mathscr{D}(\mathcal{E}) \cap L^\infty_{\mathsf{n}}(Y)$ and $\phi \in \mathscr{D}(\mathcal{E}) \cap \mathcal{C}(Y)$. (Notice that $\mathcal{C}(Y) = \mathcal{C}_c(Y)$ by compactness of Y.)

We say that $(\mathcal{E}, \mathscr{D}(\mathcal{E}))$ admits *carré du champ operator* if $\mathbf{\Gamma}(u, v) \ll \mathsf{n}$ for every $u, v \in \mathscr{D}(\mathbf{\Gamma})$, in which case, with usual abuse of notation, we indicate again by $(\mathbf{\Gamma}, \mathscr{D}(\mathbf{\Gamma}))$ the $L^1_{\mathsf{n}}(Y)$-valued non-negative definite symmetric bilinear form $\frac{\mathrm{d}\mathbf{\Gamma}(u,v)}{\mathrm{d}\mathsf{n}}$. By $\mathbf{\Gamma}(u) \leq \mathsf{n}$ we mean that $\mathbf{\Gamma}(u)$ is absolutely continuous with respect to n and $\mathbf{\Gamma}(u) \leq 1$ n-a.e..

A strongly local Dirichlet form $(\mathcal{E}, \mathscr{D}(\mathcal{E}))$ on $L^2_{\mathsf{n}}(Y)$ with carré du champ operator $\mathbf{\Gamma}$ induces an intrinsic extended pseudo-metric $\mathsf{d}_{\mathcal{E}}$ on Y, termed the *intrinsic metric of* $(\mathcal{E}, \mathscr{D}(\mathcal{E}))$ and defined by

$$\mathsf{d}_{\mathcal{E}}(y_1, y_2) := \sup\{u(y_1) - u(y_2) \mid u \in \mathscr{D}(\mathbf{\Gamma}) \cap \mathcal{C}(Y), \mathbf{\Gamma}(u) \leq \mathsf{n}\} \ . \tag{4.3.3}$$

By *extended* we mean that $\mathsf{d}_{\mathcal{E}}$ may attain the value $+\infty$, by the prefix "*pseudo-*" that it may vanish outside the diagonal in $Y^{\times 2}$.

We will make wide use of the following lemma, which is thus worth to state separately. A proof is standard. (See e.g. [112, Lem. I.2.12] for the first part.)

Lemma 4.3.3. *Let $(\mathcal{E},\mathscr{D}(\mathcal{E}))$ be a Dirichlet form on $L^2_{\mathsf{n}}(Y)$ with energy measure $(\mathbf{\Gamma},\mathscr{D}(\mathbf{\Gamma}))$ and let $(u_n)_n \subset \mathscr{D}(\mathcal{E})$ be such that $\sup_n \mathcal{E}(u_n) < \infty$. If there exists $u \in L^2_{\mathsf{n}}(Y)$ such that L^2_{n}-$\lim_n u_n = u$, then*

$$u \in \mathscr{D}(\mathcal{E}) \qquad \textit{and} \qquad \mathcal{E}(u) \le \liminf_n \mathcal{E}(u_n) \ .$$

If additionally $(u_n)_n \subset \mathscr{D}(\mathbf{\Gamma})$ and $\limsup_n \mathbf{\Gamma}(u_n) \le \mathsf{n}$, then, additionally, $u \in \mathscr{D}(\mathbf{\Gamma})$ and $\mathbf{\Gamma}(u) \le \mathsf{n}$.

Lemma 4.3.4. *Let $(\mathcal{E},\mathscr{D}(\mathcal{E}))$ be a (possibly not regular) strongly local Dirichlet form on $L^2_{\mathsf{n}}(Y)$ with energy measure $(\mathbf{\Gamma},\mathscr{D}(\mathbf{\Gamma}))$. Let ρ be any metric metrizing (Y,τ) and assume further that $\rho_z := \rho(z,\,\cdot\,) \in \mathscr{D}(\mathbf{\Gamma})$ and $\mathbf{\Gamma}(\rho_z) \le \mathsf{n}$ for every $z \in Z$ a dense subset of Y. Then, every ρ-Lipschitz function $u\colon Y \to \mathbb{R}$ satisfies $u \in \mathscr{D}(\mathbf{\Gamma})$ and $\mathbf{\Gamma}(u) \le \mathrm{Lip}[u]^2\,\mathsf{n}$.*

Proof. Without loss of generality, up to rescaling, we can restrict ourselves to the case when $\mathrm{Lip}[u] \le 1$, for which we claim $\mathbf{\Gamma}[u] \le \mathsf{n}$. Let u_ε be defined as in Lemma 4.3.1. Since Y is compact and $\mathbb{1} \in \mathscr{D}(\mathcal{E})$, functions locally in the domain of the form belong to $\mathscr{D}(\mathcal{E})$, thus we have $u_\varepsilon \in \mathscr{D}(\mathcal{E})$ and $\mathbf{\Gamma}(u_\varepsilon) \le \mathsf{n}$ by [97, Thm. 2.1] (where the regularity of $(\mathcal{E},\mathscr{D}(\mathcal{E}))$ is in fact not needed and the fact that $\mathbf{\Gamma}(\rho_{z_i}) \le \mathsf{n}$ is granted by assumption). Choose now $\varepsilon := \varepsilon_n \searrow 0$ as $n \to \infty$. Since u_{ε_n} converges to u uniformly as $n \to \infty$ by Lemma 4.3.1, the conclusion follows by Lemma 4.3.3. ■

4.3.4 Optimal transport. We collect here some known results in metric geometry based on optimal transport. The reader is referred to [10] for an expository treatment.

Everywhere in the following let $\exp_x\colon T_xM \to M$ be the exponential map of (M,g) at a point $x \in M$ and set $\mathsf{c} := \frac{1}{2}\mathsf{d}^2\colon M^{\times 2} \to \mathbb{R}$.

Definition 4.3.5 (c-transform, c-convexity, conjugate map)**.** For any $\varphi\colon M \to \mathbb{R}$, we define its c-*transform* (often termed c_--transform, e.g. [10]) by

$$\varphi^{\mathsf{c}}(x) := -\inf_{y\in M} \{\mathsf{c}(x,y) + \varphi(y)\} \ . \tag{4.3.4}$$

Any such φ is termed c-*convex* if there exists $\psi\colon M \to \mathbb{R}$ such that $\varphi = \psi^{\mathsf{c}}$, in which case it holds that $\varphi = \varphi^{\mathsf{cc}}$. (See e.g. [10, Dfn. 1.9].) Every c-convex function on M is Lipschitz. (See [10, Prop. 1.30], where the statement is proven for c-*concave* functions. It is equivalent to our claim by [10, Rmk. 1.12].) By the classical Rademacher Theorem on M, the set Σ_φ of singular points of φ has m-measure 0.

Definition 4.3.6 (Regular measures)**.** We say that $\mu \in \mathscr{P}$ is (*transport*) *regular* if $\mu\Sigma_\varphi = 0$ for every semi-convex function φ. We denote by $\mathscr{P}^{\mathrm{reg}}$ the set of regular measures in $\mathscr{P}$.

It is well-known that every finite measure on a Polish space is *regular* in the classical sense of measure theory. Thus we will henceforth refer to 'transport regular' measures simply as to 'regular' measures. Since we only consider finite measures on Polish spaces, no confusion may arise.

Remark 4.3.7. The above definition of a regular measure is rather intrinsic. Regularity is a local property. For an extrinsic definition in local charts we refer the reader to [63, Dfn. 2.8]. The equivalence of our definition to the one in [63] is shown in the proof of [63, Prop. 2.10].

Theorem 4.3.8 (Brenier–McCann, [10, Thm. 1.33], Gigli, [63, Prop. 2.10 and Thm. 7.4])**.** *The following are equivalent:*

(*i*) $\mu \in \mathscr{P}^{\mathrm{reg}}$;

(*ii*) *for each* $\nu \in \mathscr{P}$ *there exists a unique optimal transport plan* $\pi \in \mathrm{Opt}(\mu, \nu)$ *and* π *is induced by a map (say,* $g_{\mu\to\nu}$*).*

Furthermore, if any of the previous holds, then there exists a c*-convex* $\varphi_{\mu\to\nu}$*, unique up to additive constant, termed a* Kantorovich potential*, such that* $g_{\mu\to\nu} = \exp \nabla\varphi_{\mu\to\nu}$ μ*-a.e. on* M*.*

Proposition 4.3.9 (AC curves in $(\mathscr{P}_2, W_2)$, [10, Thm. 2.29])**.** *For every* $(\mu_t)_{t\in I} \in \mathrm{AC}^1(I; \mathscr{P}_2)$ *there exists a Borel measurable time-dependent family of vector fields* $(w_t)_{t\in I}$ *with* $\|w_t\|_{\mathfrak{X}_{\mu_t}} \leq |\dot\mu_t|$ *for* $\mathrm{d}t$*-a.e.* $t \in I$ *and additionally such that the* continuity equation

$$\partial_t\mu_t + \mathrm{div}(w_t\mu_t) = 0 \tag{4.3.5}$$

holds in the sense of distributions on $I \times M$*, that is*

$$\forall \varphi \in \mathcal{C}_c^\infty(I \times M) \qquad \int_0^1\int_M \left(\partial_t\varphi(t,x) + \langle\nabla\varphi(t,x) \,|\, w_t(x)\rangle_{\mathsf{g}}\right) \mathrm{d}\mu_t(x)\,\mathrm{d}t = 0\ . \tag{4.3.6}$$

Conversely, if $(\mu_t, w_t)_{t\in I}$ *satisfies* (4.3.5) *in the sense of distributions and* $\|w_t\|_{\mathfrak{X}_{\mu_t}} \in L^1(I)$*, then, up to redefining* $t \mapsto \mu_t$ *on a* $\mathrm{d}t$*-negligible set of times,* $(\mu_t)_t \in \mathrm{AC}^1(I; \mathscr{P}_2)$ *and* $|\dot\mu_t| \leq \|w_t\|_{\mathfrak{X}_{\mu_t}}$ *for* $\mathrm{d}t$*-a.e.* $t \in I$*.*

4.3.5 Geometry of $\mathscr{P}_2$. A detailed study of the Riemannian structure of $\mathscr{P}_2$ has been carried out by N. Gigli in [63, 64]. We shall need the following definitions and results from [63] to which we refer the reader for further references.

We consider the tangent bundle TM as endowed with the Sasaki metric g_* and the associated Riemannian distance $\mathsf{d}_* \coloneqq \mathsf{d}_{\mathsf{g}_*}$ which turn it into a (non-compact connected oriented) Riemannian manifold.

Definition 4.3.10 (Tangent plans)**.** For $\mu \in \mathscr{P}_2$ we let $\mathscr{P}_2(TM)_\mu \subset \mathscr{P}_2(TM)$ be the space of *tangent plans* $\boldsymbol{\gamma} \in \mathscr{P}(TM)$ such that

$$\mathrm{pr}^M_\sharp\boldsymbol{\gamma} = \mu \qquad \text{and} \qquad \int_{TM} |\mathrm{v}|^2_{\mathsf{g}_x}\,\mathrm{d}\boldsymbol{\gamma}(x,\mathrm{v}) < \infty\ . \tag{4.3.7}$$

Definition 4.3.11 (Exponential map)**.** We denote by $\mathbf{exp}_\mu\colon \mathscr{P}_2(TM)_\mu \to \mathscr{P}_2$ the *exponential map* $\mathbf{exp}_\mu(\boldsymbol{\gamma}) = \exp_\sharp\boldsymbol{\gamma}$, with right-inverse $\mathbf{exp}_\mu^{-1}\colon \mathscr{P}_2 \to \mathscr{P}_2(TM)_\mu$ defined by

$$\mathbf{exp}_\mu^{-1}(\nu) \coloneqq \left\{\boldsymbol{\gamma} \in \mathscr{P}_2(TM)_\mu \mid \mathbf{exp}_\mu(\boldsymbol{\gamma}) = \nu, \int_{TM} |\mathrm{v}|^2_{\mathsf{g}_x}\,\mathrm{d}\boldsymbol{\gamma}(x,\mathrm{v}) = W_2^2(\mu,\nu)\right\}\ .$$

Equivalently, $\mathbf{exp}_\mu^{-1}(\nu)$ is the set of all tangent plans $\boldsymbol{\gamma} \in \mathscr{P}_2(TM)$ such that

$$(\mathrm{pr}^M, \exp)_\sharp\boldsymbol{\gamma} \in \mathrm{Cpl}(\mu,\nu) \tag{4.3.8}$$

and

$$\int_{TM} |\mathrm{v}|^2_{\mathsf{g}_x}\,\mathrm{d}\boldsymbol{\gamma}(x,\mathrm{v}) = W_2^2(\mu,\nu)\ . \tag{4.3.9}$$

Remark 4.3.12 (Cf. [63, p. 131]). Notice that (4.3.9) may not be dropped even if (4.3.8) is strengthened to

$$(\mathrm{pr}^M, \exp)_\sharp\boldsymbol{\gamma} \in \mathrm{Opt}(\mu,\nu)\ . \tag{4.3.10}$$

The joint requirement of (4.3.8) *and* (4.3.9) is however equivalent to that of (4.3.10) *and* (4.3.9).

Remark 4.3.13. Notably, $\mathbf{exp}_\mu^{-1}(\nu)$ need not be a singleton even when $\mathrm{Opt}(\mu,\nu)$ is. Consider e.g. the case when $\mu=\delta_p$ and $\nu=\delta_q$ are Dirac masses at antipodal points $p,q\in\mathbb{S}^1$ and let $\mathrm{v}:=\frac{1}{2}\partial_p\in T_p\mathbb{S}^1$. Then $\mathrm{Cpl}(\mu,\nu)=\mathrm{Opt}(\mu,\nu)=\{\delta_{(p,q)}\}$, yet $\mathbf{exp}_\mu^{-1}(\nu)=\{\delta_{p,\mathrm{v}}+_r\delta_{p,-\mathrm{v}}\}_{r\in I}$. (Cf. (4.3.1).)

Definition 4.3.14 (Rescaling of tangent plans)**.** For $t\in\mathbb{R}$ we denote by $t\cdot\boldsymbol{\gamma}$ the *rescaling*

$$t\cdot\boldsymbol{\gamma}:=(\mathrm{pr}^M,t\,\mathrm{pr}^1)_\sharp\boldsymbol{\gamma}\ . \tag{4.3.11}$$

Definition 4.3.15 (Double tangent)**.** We denote by $T^2M:=\{(x,\mathrm{v}_1,\mathrm{v}_2)\mid \mathrm{v}_1,\mathrm{v}_2\in T_xM\}$ the *double tangent bundle* to M, with natural projections

$$\mathrm{pr}^M\colon(x,\mathrm{v}_1,\mathrm{v}_2)\mapsto x\in M\ ,\qquad \mathrm{pr}^i\colon(x,\mathrm{v}_1,\mathrm{v}_2)\mapsto \mathrm{v}_i\in T_xM\ ,\ i=1,2$$

and endowed with the distance

$$\mathsf{d}_{*2}:=\sqrt{\mathsf{d}_*^2\circ(\mathrm{pr}^1,\mathrm{pr}^1)+\mathsf{d}_*^2\circ(\mathrm{pr}^2,\mathrm{pr}^2)}\ .$$

All of the previous definitions are instrumental to the statement of the following result by N. Gigli, concerned with the one-sided differentiability of the squared L^2-Wasserstein distance along a family of nice curves including W_2-geodesic curves.

Theorem 4.3.16 (Directional derivatives of the squared Wasserstein distance, [63, Thm. 4.2])**.** *Fix $\mu_0\in\mathscr{P}$ and $\boldsymbol{\gamma}\in\mathscr{P}_2(TM)_{\mu_0}$ and set $\mu_t:=\mathbf{exp}_{\mu_0}(t\cdot\boldsymbol{\gamma})$. Then, for every $\nu\in\mathscr{P}$ there exists the right derivative*

$$\mathrm{d}_t^+\big|_{t=0}\tfrac{1}{2}W_2^2(\mu_t,\nu)=-\sup_{\boldsymbol{\alpha}}\int_{T^2M}\langle\mathrm{v}_1\mid\mathrm{v}_2\rangle_{\mathsf{g}_x}\,\mathrm{d}\boldsymbol{\alpha}(x,\mathrm{v}_1,\mathrm{v}_2) \tag{4.3.12}$$

where the supremum is taken over all $\boldsymbol{\alpha}\in\mathscr{P}_2(T^2M)$

$$(\mathrm{pr}^M,\mathrm{pr}^1)_\sharp\boldsymbol{\alpha}=\boldsymbol{\gamma}\ ,\qquad \textit{and}\qquad (\mathrm{pr}^M,\mathrm{pr}^2)_\sharp\boldsymbol{\alpha}\in\mathbf{exp}_{\mu_0}^{-1}(\nu)\ . \tag{4.3.13}$$

The following is a straightforward corollary. We provide a proof for the sake of completeness.

Corollary 4.3.17. *In the same notation of Theorem 4.3.16, there exists the left derivative*

$$\mathrm{d}_t^-\big|_{t=0}\tfrac{1}{2}W_2^2(\mu_t,\nu)=-\inf_{\boldsymbol{\alpha}}\int_{T^2M}\langle\mathrm{v}_1\mid\mathrm{v}_2\rangle_{\mathsf{g}_x}\,\mathrm{d}\boldsymbol{\alpha}(x,\mathrm{v}_1,\mathrm{v}_2) \tag{4.3.14}$$

where the infimum is taken over all $\boldsymbol{\alpha}\in\mathscr{P}_2(T^2M)$ satisfying (4.3.13)*.*

Proof. Given $\boldsymbol{\gamma}^+\in\mathscr{P}_2(TM)_{\mu_0}$ let $\boldsymbol{\gamma}^-:=(-1)\cdot\boldsymbol{\gamma}$ be defined by (4.3.11) and set $\mu_t^\pm:=\mathbf{exp}_\mu(t\cdot\boldsymbol{\gamma}^\pm)$ for $t\geq 0$. Notice that $\mu_{-t}^+=\mu_t^-$ for every $t\geq0$, hence, by definition,

$$\mathrm{d}_t^-\big|_{t=0}\tfrac{1}{2}W_2^2(\mu_t^+,\nu)=-\,\mathrm{d}_t^+\big|_{t=0}\tfrac{1}{2}W_2^2(\mu_t^-,\nu)$$

which exists by choosing $\boldsymbol{\gamma}=\boldsymbol{\gamma}^-$ in Theorem 4.3.16. Let $A^\pm$ be the set of plans $\boldsymbol{\alpha}\in\mathscr{P}_2(T^2M)$ satisfying (4.3.13) with $\boldsymbol{\gamma}^\pm$ in lieu of $\boldsymbol{\gamma}$ and define $\mathrm{re}^1:=(\mathrm{pr}^M,-\mathrm{pr}^1,\mathrm{pr}^2)\colon T^2M\to T^2M$. It is straightforward that $A^\pm=\mathrm{re}^1_\sharp A^\mp$, thus, by Theorem 4.3.16,

$$\begin{aligned}-\,\mathrm{d}_t^+\big|_{t=0}\tfrac{1}{2}W_2^2(\mu_t^-,\nu)&=\sup_{\boldsymbol{\alpha}\in A^-}\int_{T^2M}\langle\mathrm{v}_1\mid\mathrm{v}_2\rangle_{\mathsf{g}_x}\,\mathrm{d}\boldsymbol{\alpha}(x,\mathrm{v}_1,\mathrm{v}_2)\\&=\sup_{\boldsymbol{\alpha}\in A^+}\int_{T^2M}\langle-\mathrm{v}_1\mid\mathrm{v}_2\rangle_{\mathsf{g}_x}\,\mathrm{d}\boldsymbol{\alpha}(x,\mathrm{v}_1,\mathrm{v}_2)\\&=-\inf_{\boldsymbol{\alpha}\in A^+}\int_{T^2M}\langle\mathrm{v}_1\mid\mathrm{v}_2\rangle_{\mathsf{g}_x}\,\mathrm{d}\boldsymbol{\alpha}(x,\mathrm{v}_1,\mathrm{v}_2)\ ,\end{aligned}$$

whence the conclusion by combining the last two chains of equalities. ■

4.4 Proof of the main result

Let us briefly outline the proof of Theorem 4.2.4. The proofs of 4.2.4 (1) and 4.2.4 (2) are mainly a standard consequence of Assumption 4.2 (*iii*) and the abstract theory of Dirichlet forms. Therefore, we shall rather focus on the proofs of Theorem 4.2.4 (3.*i*) and 4.2.4 (3.*ii*). These are respectively reminiscent of the "global" proof of the Rademacher Theorem for strongly local regular Dirichlet forms in the work [97] by P. Koskela and Y. Zhou, and of the "local" proof [127] of the classical Rademacher Theorem on $\mathbb{R}^n$ by A. Nekvinda and L. Zajíček. Informally, the "global" method consists in showing that Lipschitz functions belong to the Sobolev space $W^{1,2}_{\mathrm{loc}}$ (Here: to the domain $\mathscr{F}$) and that $|\nabla\cdot| \leq \mathrm{Lip}[\,\cdot\,]$. The "local" method consists instead in showing that a Lipschitz function is Gâteaux differentiable (along every direction) at a.e. point and, subsequently, that a Lipschitz function Gâteaux differentiable at a point is in fact Fréchet differentiable at that point.

A "local" proof of 4.2.4 (3.*ii*) *under quasi-invariance.* Let $u\colon \mathscr{P}_2 \to \mathbb{R}$ be a W_2-Lipschitz function. In §4.4.2, we study the differentiability of any such u along the flow curves $t \mapsto \Psi^{w,t}\mu$. In Corollary 4.4.10 we show how we may trade the dt-a.e. differentiability of the curve $t \mapsto u(\Psi^{w,t}\mu)$ for every μ, with the differentiability of $t \mapsto u(\Psi^{w,t}\mu)$ at $t = 0$ (i.e. the Gâteaux differentiability of u along w) for $\mathbb{P}$-a.e. μ. This result is the only one relying on the quasi-invariance Assumption 4.2 (*iv*). Finally, assuming that u is Gâteaux differentiable along every w in a countable dense subspace of directions, we improve its differentiability to the Fréchet differentiability in 4.2.4 (3.*ii*). This is the content of Proposition 4.4.9, adapted from the proof of [142, Thm. 1.3].

A "global proof" of 4.2.4 (3.*i*) *without quasi-invariance.* Set $U_{\mu,\nu}(t) := W_2(\nu, \Psi^{w,t}\mu)$. In §4.4.1 we show that $t \mapsto U_{\mu,\nu}(t)$ is differentiable at $t = 0$ for every fixed ν in the dense set $\mathscr{P}^{\mathrm{reg}}$ and every fixed $\mu \neq \nu$. This shows the Gâteux differentiability of $u_\nu := W_2(\nu, \,\cdot\,)$ at every $\mu \neq \nu$ along every $w \in \mathfrak{X}^\infty$, hence we can apply Proposition 4.4.9 to conclude 4.2.4 (3.*i*) for all metric-cone functions u_ν with $\nu \in \mathscr{P}^{\mathrm{reg}}$. The extension to all Lipschitz functions is given by the general Lemma 4.3.4 with $Z = \mathscr{P}^{\mathrm{reg}}$.

Refined statements under STP. We start by considering truncated metric cones $u_\nu \vee \theta$ with u_ν as above and $\theta > 0$. In Lemma 4.4.4 we construct an approximating sequence $u_{\nu,\theta,n} \in \mathfrak{Z}^\infty$ and $\mathcal{E}_1$-convergent to $u_\nu \vee \theta$ as $n \to \infty$. This shows that $u_\nu \vee \theta \in \mathscr{F}_0$ for every $\nu \in \mathscr{P}_2$. By applying Proposition 4.4.9 to $u_\nu \vee \theta$ we improve the bound $\|\mathbf{D}(u_\nu \vee \theta)(\mu)\|_{\mathfrak{X}_\mu} \lesssim \theta^{-1}$ obtained in Lemma 4.4.4 to the sharp bound $\|\mathbf{D}(u_\nu \vee \theta)(\mu)\|_{\mathfrak{X}_\mu} \leq \mathrm{Lip}[u_\nu \vee \theta] = 1$. By Lemma 4.3.3 we may then remove the truncation by θ, thus showing that $u_\nu \in \mathscr{F}_0$ for every $\nu \in \mathscr{P}_2$. Once more, the extension to all Lipschitz functions follows by Lemma 4.3.4.

4.4.1 On the differentiability of W_2-cone functions. In this section we collect some results on the differentiability of the Wasserstein distance along (flow) curves. We aim to show (Lem. 4.4.3) that, for nice μ and $\nu \in \mathscr{P}_2$, the function $t \mapsto U_{\mu,\nu}(t) := W_2(\nu, \Psi^{w,t}\mu)$ is differentiable at $t = 0$. Since $U \geq 0$, with equality only if $\mu = \nu$, it suffices to study the differentiability of $t \mapsto U_{\mu,\nu}(t)^2$. In order to do so, we firstly need to trace back the computation of $\mathrm{d}_t\big|_{t=0} U_{\mu,\nu}(t)^2$ to the setting of Gigli's Theorem 4.3.16. Informally, we exploit the following fact: The curves $\psi^{w,t}(x)$ and $\exp_x(tw)$ are tangent to each other at every point $x \in M$ (Lem. 4.4.1). As a consequence, their lifts on $\mathscr{P}_2$ by push-forward, namely $\Psi^{w,t}\mu := \psi^{w\,t}_\sharp\mu$ and $\mathbf{exp}_\mu(t \cdot \boldsymbol{\gamma}) = \exp_{\,\cdot}(tw_{\,\cdot})_\sharp\mu$ (here, $\boldsymbol{\gamma}$ is some tangent plan depending on w) are themselves tangent to each other in a suitable sense. (See Step 2 in the proof of Lem. 4.4.3.) This shows that $\mathrm{d}_t\big|_{t=0} U_{\mu,\nu}(t) = \mathrm{d}_t\big|_{t=0} V_{\mu,\nu}(t)$, where $V_{\mu,\nu}(t) \coloneqq W_2(\nu, \mathbf{exp}_\mu(t\cdot\boldsymbol{\gamma}))$. Thus, as for $U_{\mu,\nu}$, it suffices to show the differentiability of $t \mapsto V_{\mu,\nu}(t)^2$.

While Theorem 4.3.16 and Corollary 4.3.17 provide the *one-sided* differentiability of $t \mapsto V_{\mu,\nu}(t)^2$ at $t=0$, the two-sided differentiability generally fails, since the left and right derivatives need not coincide. However, $t \mapsto V_{\mu,\nu}(t)^2$ is differentiable as soon as we show that, for given μ, ν, the set of plans $\boldsymbol{\alpha}$ over which we are extremizing in the right-hand sides of (4.3.12) and (4.3.14) is in fact a singleton. This is the case if either μ or ν is regular. (See Prop. 4.4.2.)

We denote by $\mathrm{inj}_M > 0$ the injectivity radius of M.

Lemma 4.4.1. *Let $w \in \mathfrak{X}^\infty$. Then,*

$$\mathsf{d}(\exp_x(tw_x), \psi^{w,t}(x)) \in o(t) \quad \textit{as } t \to 0$$

uniformly in $x \in M$.

Proof. Let $(\partial_i)_{i=1,\dots,d}$ be a g-orthonormal basis of T_xM, $(\mathrm{d}_i)_{i=1,\dots,d}$ be its g-dual basis in T_x^*M and recall the Lie series expansion of $\psi^{w,t}$ about $t=0$, viz.

$$\forall f \in \mathcal{C}^\infty(M) \qquad f(\psi^{w,t}(x)) = \sum_{k\geq 0} \frac{t^k}{k!} w^k(f)_x \ .$$

Set $c_0 \coloneqq \mathrm{inj}_M\big(1 \wedge \|w\|_{\mathfrak{X}^0}^{-1}\big)$ and let $0 < c_1 < c_0$ be such that $\psi^{w,t}(x) \in B_{c_0}(x)$ for all $t < c_1$. Letting $w_x = \mathrm{w}^j\partial_j$ and choosing $f = \mathrm{d}_i \circ \exp_x^{-1}$ (suitably restricted to a coördinate chart around x) above yields

$$\begin{aligned}(\mathrm{d}_i \circ \exp_x^{-1})(\psi^{w,t}(x)) =& (\mathrm{d}_i \circ \exp_x^{-1})(x) + tw(\mathrm{d}_i \circ \exp_x^{-1})_x + o(t)\\ =& t\mathrm{w}^j\partial_j(\mathrm{d}_i \circ \exp_x^{-1})_x + o(t) = t\mathrm{w}^i + o(t)\ ,\end{aligned}$$

whence $(\exp_x^{-1} \circ \psi^{w,t})(x) = tw + o(t)$. Since $\exp_x$ is a smooth diffeomorphism on $B_{c_1}(\mathbf{0}_{T_xM})$, there exists $L > 0$ such that

$$\forall y_1, y_2 \in B_{c_1}(x) \qquad \mathsf{d}(y_1, y_2) \leq L\left|\exp_x^{-1}(y_1) - \exp_x^{-1}(y_2)\right| \ .$$

Thus, finally

$$\mathsf{d}(\exp_x(tw), \psi^{w,t}(x)) \leq L\left|tw - tw - o(t)\right|_{\mathsf{g}_x} \in o(t)\ ,$$

which concludes the proof. ∎

Proposition 4.4.2. *Let either $\mu \in \mathscr{P}^{\mathrm{reg}}$ or $\nu \in \mathscr{P}^{\mathrm{reg}}$. Then, $\mathbf{exp}_\mu^{-1}(\nu)$ is a singleton.*

Proof. Assume first $\mu \in \mathscr{P}^{\mathrm{reg}}$. By Theorem 4.3.8 there exists a c-convex φ (unique up to additive constant) such that

$$\nu = (\exp_{\cdot} \nabla\varphi_{\cdot})_\sharp \mu \qquad \text{and} \qquad W_2^2(\mu,\nu) = \int_M \mathsf{d}^2(x, \exp_x \nabla\varphi_x)\,\mathrm{d}\mu(x)\ . \tag{4.4.1}$$

Moreover, for μ-a.e. $x \in M$ there exists a unique geodesic $(\alpha_r^x)_{r\in I}$ connecting x to $g_{\mu\to\nu}(x)$ given by $\alpha_r^x \coloneqq \exp_x(r\nabla\varphi_x)$. (Cf. [10, Rmk. 1.35].) We call this property the *geodesic uniqueness* property.

Claim: $\mathbf{exp}_\mu^{-1}(\nu) \neq \varnothing$. *Proof.* Set $\boldsymbol{\gamma}_0 \coloneqq (\mathrm{id}_M(\,\cdot\,), \nabla\varphi_{\,\cdot\,})_\sharp \mu \in \mathscr{P}(TM)$. It is straightforward that $\boldsymbol{\gamma}_0 \in \mathscr{P}_2(TM)_\mu$. Additionally,

$$\begin{aligned}\int_{TM} |\mathrm{v}|^2_{\mathsf{g}_x} \,\mathrm{d}\boldsymbol{\gamma}_0(x, \mathrm{v}) &= \int_M |\nabla\varphi_x|^2_{\mathsf{g}_x} \,\mathrm{d}\mu(x) \\ &= \int_M \mathsf{d}(x, \exp_x \nabla\varphi_x)^2 \,\mathrm{d}\mu(x) = W_2^2(\mu, \nu) \ ,\end{aligned} \tag{4.4.2}$$

where $|\nabla\varphi_x|_{\mathsf{g}_x} = \mathsf{d}(x, \exp_x \nabla\varphi_x)$ for μ-a.e. x by geodesic uniqueness. This shows (4.3.9), hence that $\boldsymbol{\gamma}_0 \in \mathbf{exp}_\mu^{-1}(\nu)$.

Claim: $\mathbf{exp}_\mu^{-1}(\nu) = \{\boldsymbol{\gamma}_0\}$. *Proof.* Let $\boldsymbol{\gamma} \in \mathbf{exp}_\mu^{-1}(\nu)$. By (4.3.7), $\mathrm{pr}^M_\sharp \boldsymbol{\gamma} = \mu$, thus there exists the Rokhlin disintegration $\{\boldsymbol{\gamma}^x\}_{x \in M}$ of $\boldsymbol{\gamma}$ along pr^M with respect to μ. By (4.3.10), $\mathbf{exp}_\mu \boldsymbol{\gamma} = \nu = (\exp_{\,\cdot\,} \nabla\varphi_{\,\cdot\,})_\sharp \mu$, thus, for μ-a.e. $x \in M$, $\boldsymbol{\gamma}^x$ is concentrated on the set $A_x \coloneqq \exp_x^{-1}(\exp_x \nabla\varphi_x)$. Moreover, (4.4.2) holds with $\boldsymbol{\gamma}$ in place of $\boldsymbol{\gamma}_0$ by (4.3.9), hence, by optimality, $\boldsymbol{\gamma}^x$ is in fact concentrated on the set

$$\mathrm{pr}^{A_x}(\mathbf{0}_{T_xM}) \coloneqq \operatorname*{argmin}_{\mathrm{v} \in T_xM} \mathrm{dist}_{\mathsf{g}_x}(A_x, \mathbf{0}_{T_xM}) \ . \tag{4.4.3}$$

By geodesic uniqueness, one has $\mathrm{pr}^{A_x}(\mathbf{0}_{T_xM}) = \{\nabla\varphi_x\}$ for μ-a.e. $x \in M$, hence $\boldsymbol{\gamma}^x = \boldsymbol{\delta}_{(x, \nabla\varphi_x)}$ for μ-a.e. $x \in M$. Thus finally (4.4.2) holds and $\boldsymbol{\gamma} = \boldsymbol{\gamma}_0$.

Assume now $\nu \in \mathscr{P}^{\mathrm{reg}}$. By Theorem 4.3.8 there exists a c-convex ψ (unique up to additive constant) such that (4.4.1) holds when exchanging ν with μ and replacing φ with ψ. Moreover, geodesic uniqueness holds too, for the geodesics defined by $\beta^y_r \coloneqq \exp_y(r\nabla\psi_y)$.

For a measurable vector field w, we denote by $\mathbf{T}^t_s((\alpha)) w_{\alpha_s}$ the parallel transport (of the Levi-Civita connection) from α_s to α_t of the vector w_{α_s} along the curve $(\alpha) \coloneqq (\alpha_r)_r$. We set further

$$\begin{aligned}\mathbf{R}\colon TM &\longrightarrow TM \\ (x, \mathrm{v}) &\longmapsto \big(\exp_x \mathrm{v}, -\mathbf{T}^1_0\big((\exp_x(r\mathrm{v}))_r\big)\mathrm{v}\big)\end{aligned}$$

Claim: $\mathbf{exp}_\mu^{-1}(\nu) \neq \varnothing$. *Proof.* Set $\boldsymbol{\gamma}_0 \coloneqq \mathbf{R}_\sharp(\mathrm{id}_M(\,\cdot\,), \nabla\psi_{\,\cdot\,})_\sharp \nu$. Since

$$\mathrm{pr}^M \circ \mathbf{R} \circ (\mathrm{id}_M(\,\cdot\,), \nabla\psi_{\,\cdot\,}) = \exp_{\,\cdot\,} \nabla\psi_{\,\cdot\,}$$

and $\mu = (\exp_{\,\cdot\,} \nabla\psi_{\,\cdot\,})_\sharp \nu$, then $\boldsymbol{\gamma}_0 \in \mathscr{P}_2(TM)_\mu$. Additionally,

$$\begin{aligned}\int_{TM} |\mathrm{v}|^2_{\mathsf{g}_x} \,\mathrm{d}\boldsymbol{\gamma}_0(x, \mathrm{v}) &= \int_M \big|-\mathbf{T}^1_0((\beta^y_r)_r)\nabla\psi_y\big|^2_{\mathsf{g}_{x(y)}} \,\mathrm{d}\nu(y) \qquad x(y) \coloneqq \beta^y_1 \\ &= \int_M |\nabla\psi_y|^2_{\mathsf{g}_y} \,\mathrm{d}\nu(y) \ ,\end{aligned}$$

where the last equality holds since, being $(\beta^y_r)_r$ a geodesic and the Levi-Civita connection being a metric connection, the parallel transport

$$\mathbf{T}^1_0((\beta^y_r)_r)\colon (T_{\beta^y_0}M, \mathsf{g}_{\beta^y_0}) \to (T_{\beta^y_1}M, \mathsf{g}_{\beta^y_1})$$

is an isometry. Thus, arguing as in the proof of the first claim, $\boldsymbol{\gamma}_0 \in \mathbf{exp}_\mu^{-1}(\nu)$.

Claim: $\mathbf{exp}_\mu^{-1}(\nu)=\{\boldsymbol{\gamma}_0\}$. *Proof.* Let $\boldsymbol{\gamma}\in\mathbf{exp}_\mu^{-1}(\nu)$. By definition $\mathbf{exp}_\mu\boldsymbol{\gamma}=\exp_\sharp\boldsymbol{\gamma}=\nu$, thus there exists the Rokhlin disintegration $\{\boldsymbol{\gamma}^y\}_{y\in M}$ of $\boldsymbol{\gamma}$ along exp with respect to ν. By (4.3.10),

$$\begin{aligned}(\mathrm{id}_M(\,\cdot\,),\exp_{\cdot}\,-)_\sharp\boldsymbol{\gamma}\in\mathrm{Opt}(\mu,\nu)=&(\mathrm{pr}^2,\mathrm{pr}^1)_\sharp\mathrm{Opt}(\nu,\mu)\\ =&\{(\exp_{\cdot}\,\nabla\psi_{\cdot},\mathrm{id}_M(\,\cdot\,))_\sharp\nu\}\ ,\end{aligned}$$

thus, for ν-a.e. $y\in M$, $\boldsymbol{\gamma}^y$ is concentrated on the set

$$C_y:=\exp^{-1}_{\beta^y_1}(y)\subset T_{\beta^y_1}M\ .$$

By a similar reasoning to that in the second claim, for ν-a.e. $y\in M$, $\boldsymbol{\gamma}^y$ is in fact concentrated on $\mathrm{pr}^{C_y}(\mathbf{0}_{T_{\beta^y_1}M})$, defined analogously to (4.4.3). By definition of parallel transport and since $(\beta^y_r)_r$ is a geodesic, the latter set is a singleton

$$\mathrm{pr}^{C_y}(\mathbf{0}_{T_{\beta^y_1}M})=\left\{-\mathbf{T}^1_0((\beta^y_r)_r)\nabla\psi_y\right\}\ .$$

This concludes the proof analogously to that of the second claim. ∎

Lemma 4.4.3 (Derivatives of the Wasserstein distance along flow curves)**.** *Fix $w\in\mathfrak{X}^\infty$, $\mu_0\in\mathscr{P}$ and set $\mu_t:=\Psi^{w,t}\mu_0$. Then, for every $\nu\in\mathscr{P}\setminus\{\mu_0\}$, there exists the right derivative*

$$\mathrm{d}_t^+\big|_{t=0}W_2(\mu_t,\nu)=-W_2^{-1}(\mu_0,\nu)\sup_{\boldsymbol{\gamma}}\int_{TM}\langle w_x\,|\,\mathrm{v}\rangle_{\mathsf{g}_x}\,\mathrm{d}\boldsymbol{\gamma}(x,\mathrm{v})\tag{4.4.4}$$

where the supremum is taken over all $\boldsymbol{\gamma}\in\mathbf{exp}_{\mu_0}^{-1}(\nu)$. Moreover, if additionally either $\mu_0\in\mathscr{P}^{\mathrm{reg}}$ or $\nu\in\mathscr{P}^{\mathrm{reg}}$, then there exists the two-sided derivative $\mathrm{d}_t\big|_{t=0}W_2(\mu_t,\nu)$.

Proof. The proof is divided into several steps. Firstly, we show that there exists

$$\lim_{t\downarrow 0}\frac{W_2(\mu_t',\nu)-W_2(\mu_0,\nu)}{t}=-W_2^{-1}(\mu_0,\nu)\sup_{\boldsymbol{\gamma}}\int_{TM}\langle w_x\,|\,\mathrm{v}\rangle_{\mathsf{g}_x}\,\mathrm{d}\boldsymbol{\gamma}(x,\mathrm{v})\tag{4.4.5}$$

where $\mu_t':=(\exp_{\cdot}(tw))_\sharp\mu_0$ and $\boldsymbol{\gamma}$ is as above. Next, profiting the fact that for small $t>0$ the flow $\exp_{\cdot}(tw_{\cdot})$ is tangent to the flow $\psi^{w,t}(\,\cdot\,)$ at each point in M, we show that the same holds for the corresponding lifted flows $(\exp_{\cdot}(tw_{\cdot}))_\sharp$ and $(\psi^{w,t}(\,\cdot\,))_\sharp$ at each point in $\mathscr{P}$, hence that the right derivative (4.4.4) exists and coincides with (4.4.5).

Step 1. Set $\iota^w:=(\mathrm{id}_M(\,\cdot\,),w_{\cdot})\colon M\to TM$, let $\boldsymbol{\gamma}_0:=\iota^w_\sharp\mu_0\in\mathscr{P}_2(TM)$ and notice that

$$\mathbf{exp}_{\mu_0}(t\cdot\boldsymbol{\gamma}_0)=\left(\exp_\sharp\circ(\mathrm{pr}^M,t\,\mathrm{pr}^1)_\sharp\circ\iota^w_\sharp\right)\mu_0=(\exp_{\cdot}(tw_{\cdot}))_\sharp\mu_0=:\mu_t'\ .$$

By Theorem 4.3.16, there exists the right derivative

$$\mathrm{d}_t^+\big|_{t=0}\tfrac{1}{2}W_2^2(\mu_t',\nu)=-\sup_{\boldsymbol{\alpha}}\int_{T^2M}\langle\mathrm{v}_1\,|\,\mathrm{v}_2\rangle_{\mathsf{g}_x}\,\mathrm{d}\boldsymbol{\alpha}(x,\mathrm{v}_1,\mathrm{v}_2)$$

where $\boldsymbol{\alpha}$ is as in (4.3.13). In particular, for every such $\boldsymbol{\alpha}$, it holds that $(\mathrm{pr}^M,\mathrm{pr}^1)_\sharp\boldsymbol{\alpha}=\boldsymbol{\gamma}_0=\iota^w_\sharp\mu_0$, that is $(\mathrm{pr}^M,\mathrm{pr}^1)_\sharp\boldsymbol{\alpha}$ is supported on the graph $\mathrm{Graph}(\iota^w)\subset TM$ of the map ι^w. As a consequence, $\boldsymbol{\alpha}$ is concentrated on the set

$$\{(x,\mathrm{v}_1,\mathrm{v}_2)\mid(x,\mathrm{v}_1)\in\mathrm{Graph}(\iota^w)\}=\left\{(x,w_x,\mathrm{v}_2)\in T^2M\right\}\subset T^2M\ ,$$

thus, in fact

$$\mathrm{d}_t^+\big|_{t=0}\tfrac{1}{2}W_2^2(\mu_t',\nu)=-\sup_{\boldsymbol{\gamma}}\int_{TM}\langle w_x\,|\,\mathrm{v}\rangle_{\mathsf{g}_x}\,\mathrm{d}\boldsymbol{\gamma}(x,\mathrm{v})$$

where the supremum is taken over all $\boldsymbol{\gamma}\in\mathbf{exp}_{\mu_0}^{-1}(\nu)$. The existence of $\mathrm{d}_t^+\big|_{t=0}W_2(\mu_t',\nu)$ and (4.4.5) follow from the existence of $\mathrm{d}_t^+\big|_{t=0}\frac{1}{2}W_2^2(\mu_t',\nu)$ by chain rule.

Step 2. By Lemma 4.4.1 there exists a constant $c_1 > 0$ such that

$$\forall t \in (0, c_1) \quad \forall x \in M \qquad \mathsf{d}^2\big(\exp_x(tw), \psi^{w,t}(x)\big) \in o(t^2) \ . \tag{4.4.6}$$

Furthermore, since $\big(\exp_{\cdot}(tw), \psi_t^w(\,\cdot\,)\big)_\sharp \mu_0$ is a coupling between μ_t' and μ_t, equation (4.4.6) yields

$$\forall t \in (0, c_1) \qquad W_2^2(\mu_t', \mu_t) \leq \int_M \mathsf{d}^2\big(\exp_x(tw), \psi_t^w(x)\big) \, \mathrm{d}\mu_0(x) \in o(t^2) \ ,$$

thus there exists

$$\mathrm{d}_t\big|_{t=0} W_2(\mu_t', \mu_t) = \lim_{t \to 0} \tfrac{1}{t} \left| W_2(\mu_t', \mu_t) - W_2(\mu_0, \mu_0) \right| = 0 \ .$$

Step 3. By triangle inequality

$$W_2(\mu_t, \nu) - W_2(\mu_0, \nu) \leq W_2(\mu_t, \mu_t') + W_2(\mu_t', \nu) - W_2(\mu_0, \nu) \ ,$$

while by reverse triangle inequality

$$\begin{aligned} W_2(\mu_t, \nu) - W_2(\mu_0, \nu) \geq & \left| W_2(\nu, \mu_t') - W_2(\mu_t', \mu_t) \right| - W_2(\mu_0, \nu) \\ \geq & W_2(\mu_t', \nu) - W_2(\mu_0, \nu) - W_2(\mu_t', \mu_t) \ . \end{aligned}$$

As a consequence, setting

$$\begin{aligned} \overline{\mathrm{d}}_t^+\big|_{t=0} W_2(\mu_t, \nu) \coloneqq & \limsup_{t \downarrow 0} \frac{W_2(\mu_t', \nu) - W_2(\mu_0, \nu)}{t} \ , \\ \underline{\mathrm{d}}_t^+\big|_{t=0} W_2(\mu_t, \nu) \coloneqq & \liminf_{t \downarrow 0} \frac{W_2(\mu_t', \nu) - W_2(\mu_0, \nu)}{t} \ , \end{aligned}$$

one has

$$\begin{aligned} & - \mathrm{d}_t\big|_{t=0} W_2(\mu_t, \mu_t') + \mathrm{d}_t^+\big|_{t=0} W_2(\mu_t', \nu) \leq \\ & \qquad \underline{\mathrm{d}}_t^+\big|_{t=0} W_2(\mu_t, \nu) \leq \overline{\mathrm{d}}_t^+\big|_{t=0} W_2(\mu_t, \nu) \\ & \qquad \qquad \leq \mathrm{d}_t\big|_{t=0} W_2(\mu_t, \mu_t') + \mathrm{d}_t^+\big|_{t=0} W_2(\mu_t', \nu) \end{aligned}$$

where the derivatives above exist by the previous steps. Since $\mathrm{d}_t\big|_{t=0} W_2(\mu_t, \mu_t') = 0$ by Step 2, the right derivative $\mathrm{d}_t^+\big|_{t=0} W_2(\mu_t, \nu)$ exists and coincides with (4.4.5).

The last assertion follows by Step 1 and Corollary 4.3.17 since $\mathbf{exp}_{\mu_0}^{-1}(\nu)$ is a singleton by Proposition 4.4.2. ∎

Lemma 4.4.4. *Let* (M, g) *be additionally satisfying STP (Ass. 4.2). Then, for every* $\nu \in \mathscr{P}$ *and every* $\theta > 0$*, the function* $u_{\nu,\theta} \colon \mu \mapsto W_2(\nu, \mu) \vee \theta$ *belongs to* $\mathscr{F}_0$*.*

Proof. We construct an approximation of $u_{\nu,\theta}$ by functions in $\mathfrak{Z}^\infty$.

Preliminaries. By Kantorovich duality (see e.g. [10, Thm. 1.17])

$$W_2^2(\nu, \mu) = 2 \cdot \sup \{\nu\psi + \mu\varphi\}$$

where the supremum is taken over all $(\psi, \varphi) \in L^1_\nu(M) \times L^1_\mu(M)$ satisfying $\psi(x) + \varphi(y) \leq \mathsf{c}(x, y)$ for ν-a.e. x and μ-a.e. y in M. An optimal pair (ψ, φ) always exists and satisfies $\psi = \varphi^{\mathsf{c}}$ ν-a.e. where φ^{c} is the c-conjugate (4.3.4) of φ.

Let $\mathscr{P}^{\infty,\times}$ be the set of measures in $\mathscr{P}^\infty$ with densities bounded away from 0 and fix a countable set $(\mu_i)_i \subset \mathscr{P}^{\infty,\times}$ and dense in $\mathscr{P}_2$.

Construction of the approximation. We start by showing that $W_2(\nu,\,\cdot\,)\vee\theta\in\mathscr{F}_0$ for fixed $\nu\in\mathscr{P}^{\infty,\times}$. Let (ψ_i,φ_i) be the optimal pair of Kantorovich potentials for the pair (ν,μ_i), so that

$$\tfrac{1}{2}W_2^2(\nu,\mu_i)=\nu\psi_i+\mu_i\varphi_i\ , \tag{4.4.7}$$

where φ_i and ψ_i are smooth maps by assumption for all i's. Let further $\mathbf{t}\coloneqq(t_1,\dots,t_n)\in\mathbb{R}^n$ and, for small $\varepsilon>0$, let $F_{n,\varepsilon}\colon\mathbb{R}^n\to[-\varepsilon,\infty)$ be a smooth regularization of the function $F_n(\mathbf{t})\coloneqq 2\cdot\max_{i\leq n}t_i$. Since F_n is 2-Lipschitz for every n, the functions $F_{n,\varepsilon}$ may be chosen in such a way that

$$\lim_{\varepsilon\downarrow 0}F_{n,\varepsilon}(\mathbf{t})=F_n(\mathbf{t})\ ,\qquad n\in\mathbb{N}\ ,\ \mathbf{t}\in\mathbb{R}^n\ ; \tag{4.4.8a}$$

$$F_{n,\varepsilon_1}(\mathbf{t})\geq F_{n,\varepsilon_2}(\mathbf{t})\ ,\qquad n\in\mathbb{N}\ ,\ \varepsilon_1<\varepsilon_2\ ,\ \mathbf{t}\in\mathbb{R}^n\ ; \tag{4.4.8b}$$

$$2\cdot\mathbb{1}_{B_{n,i}}\leq\partial_i F_{n,\varepsilon}\leq 2\cdot\mathbb{1}_{(B_{n,i})_\varepsilon}\ ,\qquad n\in\mathbb{N}\ ,\ i\leq n\ ,\ \varepsilon>0\ , \tag{4.4.8c}$$

where

$$B_{n,i}\coloneqq\left\{\mathbf{t}\in\mathbb{R}^n\ \middle|\ \begin{array}{l}t_i>t_j\ \text{for all}\ 1\leq j<i\\ t_i\geq t_j\ \text{for all}\ i\leq j\leq n\end{array}\right\} \tag{4.4.9}$$

and, for any $B\subset\mathbb{R}^n$, we put $B_\varepsilon\coloneqq\{\mathbf{t}\in\mathbb{R}^n\mid \operatorname{dist}(\mathbf{t},B)<\varepsilon\}$.

For small $0<\delta<\theta$, let $\varrho_{\theta,\delta}\colon\mathbb{R}\to[\theta-\delta,\infty)$ be a smooth regularization of $\varrho_\theta\colon t\mapsto\sqrt{t\vee\theta}$ such that

$$\lim_{\delta\downarrow 0}\varrho_{\theta,\delta}(t)=\varrho_\theta(t)\ ,\qquad 0<\delta<\theta\ ,\ t\in\mathbb{R}\ ; \tag{4.4.10a}$$

$$\varrho_{\theta,\delta_1}(t)\geq\varrho_{\theta,\delta_2}(t)\ ,\qquad 0<\delta_1<\delta_2<\theta\ ,\ t\in\mathbb{R}\ ; \tag{4.4.10b}$$

$$\mathbb{1}_{[\theta,\infty)}/(2\varrho_\theta)\leq\varrho'_{\theta,\delta}\leq\mathbb{1}_{[\theta-\delta,\infty)}/(2\varrho_\theta)\ . \tag{4.4.10c}$$

Now, by smoothness of all functions involved, the function $u_{\theta,n,\varepsilon,\delta}\colon\mathscr{P}\to\mathbb{R}$ defined by

$$u_{\theta,n,\varepsilon,\delta}(\mu)\coloneqq\varrho_{\theta,\delta}\left(F_{n,\varepsilon}(c_1+\varphi_1^*\mu,\dots,c_n+\varphi_n^*\mu)\right)\quad\text{where}\quad c_i\coloneqq\psi_i^*\nu\ .$$

belongs to $\mathfrak{Z}^\infty$ and one has

$$\begin{aligned}\boldsymbol{\nabla}\,u_{\theta,n,\varepsilon,\delta}(\mu)=\sum_i^n&\varrho'_{\theta,\delta}\left(F_{n,\varepsilon}(c_1+\varphi_1^*\mu,\dots,c_n+\varphi_n^*\mu)\right)\times\\&\times(\partial_i F_{n,\varepsilon})(c_1+\varphi_1^*\mu,\dots,c_n+\varphi_n^*\mu)\nabla\varphi_i\ .\end{aligned}$$

By (4.4.8a) and (4.4.8b), resp. (4.4.10a) and (4.4.10b), and Dini's Theorem,

$$\lim_{\varepsilon\downarrow 0}\lim_{\delta\downarrow 0}(\varrho_{\theta,\delta}\circ F_{n,\varepsilon})(\mathbf{t})=(\varrho_\theta\circ F_n)(\mathbf{t})$$

locally uniformly in $\mathbf{t}\in\mathbb{R}^n$ and for all n and $\theta>0$. As a consequence, for all n and uniformly in $\mu\in\mathscr{P}$

$$\lim_{\varepsilon\downarrow 0}\lim_{\delta\downarrow 0}u_{\theta,n,\varepsilon,\delta}(\mu)=u_{\theta,n}(\mu)\coloneqq\varrho_\theta\big(F_n(c_1+\varphi_1^*\mu,\dots,c_n+\varphi_n^*\mu)\big)\ . \tag{4.4.11}$$

Moreover, by (4.4.8c), resp. (4.4.10c), $\lim_{\varepsilon\downarrow 0}\partial_i F_{n,\varepsilon}=2\cdot\mathbb{1}_{B_{n,i}}$ pointwise on $\mathbb{R}^n$ for all $i\leq n$, for all n, resp. $\lim_{\delta\downarrow 0}\varrho'_{\theta,\delta}\colon t\mapsto\mathbb{1}_{[\theta,\infty)}/(2\varrho_\theta)$ pointwise on $\mathbb{R}$ for all $\theta>0$. Thus, for all n and for all $\mu\in\mathscr{P}$ one has

$$\lim_{\varepsilon\downarrow 0}\lim_{\delta\downarrow 0}\boldsymbol{\nabla}\,u_{\theta,n,\varepsilon,\delta}(\mu)=\sum_i^n\frac{\mathbb{1}_{A_{\theta,n,i}}(\mu)}{\varrho_\theta\big(F_n(c_1+\varphi_1^*\mu,\dots,c_n+\varphi_n^*\mu)\big)}\nabla\varphi_i\ , \tag{4.4.12}$$

where the sets

$$A_{\theta,n,i} := \left\{ \mu \in \mathscr{P} \;\middle|\; \begin{aligned} &c_i + \varphi_i^* \mu \geq \theta \\ &c_i + \varphi_i^* \mu > c_j + \varphi_j^* \mu \text{ for all } 1 \leq j < i \\ &c_i + \varphi_i^* \mu \geq c_j + \varphi_j^* \mu \text{ for all } i \leq j \leq n \end{aligned} \right\}$$

are, for all n, measurable by continuity of φ_i^* and pairwise disjoint for all $i \leq n$, since the same holds for $B_{n,i}$ in (4.4.9).

Finally, again by McCann Theorem, $|\nabla \varphi_i|_{\mathsf{g}} \leq \mathrm{diam} M$, hence

$$|\boldsymbol{\nabla}\, u_{\theta,n,\varepsilon,\delta}(\mu)(x)|_{\mathsf{g}} \leq n (\mathrm{diam} M)/\sqrt{\theta}$$

whence, by Dominated Convergence, (4.4.11) and (4.4.12),

$$\mathcal{E}_1^{1/2}\text{-}\lim_{\varepsilon \downarrow 0} \left(\mathcal{E}_1^{1/2}\text{-}\lim_{\delta \downarrow 0} u_{\theta,n,\varepsilon,\delta} \right) = u_{\theta,n} \in \mathscr{F}_0 \;,$$

$$\mathbf{D} u_{\theta,n}(\mu)(x) = \sum_i^n \frac{\mathbb{1}_{A_{\theta,n,i}}(\mu)}{\varrho_\theta \left(F_n(c_1 + \varphi_1^* \mu, \ldots, c_n + \varphi_n^* \mu) \right)} \nabla \varphi_i(x) \;.$$

Pre-compactness of the approximation. Since $L^2_{\mathbb{P}}\text{-}\lim_n u_{\theta,n} = u_{\nu,\theta}$ by Dominated Convergence and (4.4.11), by Lemma 4.3.3 it suffices to show that

$$\text{for } \mathbb{P}\text{-a.e. } \mu \qquad \limsup_n \boldsymbol{\Gamma}(u_{\theta,n})(\mu) \leq C_{\nu,\theta} \tag{4.4.13}$$

for some constant $C_{\nu,\theta}$ to get $u_{\nu,\theta} \in \mathscr{F}_0$ and $\boldsymbol{\Gamma}(u_{\nu,\theta}) \leq C_{\nu,\theta}$ $\mathbb{P}$-a.e.. Indeed,

$$\begin{aligned} \|\mathbf{D} u_{\theta,n}(\mu)\|_{\mathfrak{X}_\mu}^2 &= \int_M |\mathbf{D} u_{\theta,n}(\mu)(x)|_{\mathsf{g}}^2 \,\mathrm{d}\mu(x) \\ &= \sum_i^n \frac{\mathbb{1}_{A_{\theta,n,i}}(\mu)}{\varrho_\theta^2 \left(F_n(c_1 + \varphi_1^* \mu, \ldots, c_n + \varphi_n^* \mu) \right)} \int_M |\nabla \varphi_i|_{\mathsf{g}}^2 \,\mathrm{d}\mu \end{aligned} \tag{4.4.14}$$

since the sets $A_{\theta,n,i}$ are pairwise disjoint. Thus

$$\|\mathbf{D} u_{\theta,n}(\mu)\|_{\mathfrak{X}_\mu}^2 = \sum_i^n \frac{\mathbb{1}_{A_{\theta,n,i}}(\mu)}{\theta \vee 2(c_i + \varphi_i^* \mu)} \int_M |\nabla \varphi_i|_{\mathsf{g}}^2 \,\mathrm{d}\mu \leq \frac{(\mathrm{diam} M)^2}{\theta} =: C_\theta \;.$$

General case. Fix an arbitrary $\nu \in \mathscr{P}$ and let $(\nu_k)_k$ be a sequence in $\mathscr{P}^{\infty,\times}$ narrowly converging to ν. It is readily seen that $u_{\nu_k,\theta}$ converges to $u_{\nu,\theta}$ in $L^2_{\mathbb{P}}(\mathscr{P})$ and $\|\mathbf{D} u_{\nu_k,\theta}\|_{\mathfrak{X}_\cdot}^2 \leq C_\theta$ $\mathbb{P}$-a.e. by the previous step. Thus, $u_{\nu,\theta} \in \mathscr{F}_0$ and $\|\mathbf{D} u_{\nu,\theta}\|_{\mathfrak{X}_\cdot}^2 \leq C_\theta$ $\mathbb{P}$-a.e. by Lemma 4.3.3. ∎

4.4.2 On the differentiability of functions along flow curves.

Lemma 4.4.5. *Fix $w \in \mathfrak{X}^\infty$, $\mu_0 \in \mathscr{P}$ and set $\mu_t := \Psi^{w,t} \mu_0$. Then, the curve $(\mu_t)_{t \in \mathbb{R}}$ is Lipschitz continuous with Lipschitz constant $M \leq \|w\|_{\mathfrak{X}^0}$ and satisfies $|\dot{\mu}_t| = \|w\|_{\mathfrak{X}_{\mu_t}}$ for every $t \in \mathbb{R}$.*

Proof. Since constant functions are in particular Lipschitz, we can assume without loss of generality $w \neq 0$. Set $c_1 := \mathrm{inj}_M / \|w\|_{\mathfrak{X}^0}$ and let $\mu'_{t,\varepsilon} := (\exp_\cdot(\varepsilon w))_\sharp \mu_t$. For $\varepsilon \in (-c_1, c_1)$, the curve $\varepsilon \mapsto \exp_x(\varepsilon w)$ is a minimizing geodesic. Thus, $(\exp_\cdot(\varepsilon w))_\sharp \mu_t \in \mathrm{Opt}(\mu_t, \mu'_{t,\varepsilon})$ and, for every $t \in \mathbb{R}$,

$$\mathrm{d}_\varepsilon \big|_{\varepsilon=0} W_2(\mu_t, \mu'_{t,\varepsilon}) = \lim_{\varepsilon \to 0} \left(\frac{1}{\varepsilon^2} \int_M \mathsf{d}^2\big(x, \exp_x(\varepsilon w)\big) \,\mathrm{d}\mu_t(x) \right)^{1/2} = \|w\|_{\mathfrak{X}_{\mu_t}} \;.$$

Arguing as in Step 3 in the proof of Lemma 4.4.3 with $\mu'_{t,\varepsilon}$, $\mu_{t+\varepsilon}$ and μ_t in lieu of μ'_t, μ_t and ν respectively,

$$\left|\dot{\mu}_t\right| \coloneqq \mathrm{d}_\varepsilon\big|_{\varepsilon=0} W_2(\mu_t, \mu_{t+\varepsilon}) = \mathrm{d}_\varepsilon\big|_{\varepsilon=0} W_2(\mu_t, \mu'_{t,\varepsilon}) \ .$$

Combining the last two equalities yields the second assertion. Moreover, by [11, Thm. 1.1.2],

$$\forall s < t \qquad W_2(\mu_s, \mu_t) \leq \int_s^t \left|\dot{\mu}_r\right| \mathrm{d}r = \int_s^t \|w\|_{\mathfrak{X}_{\mu_r}} \mathrm{d}r \leq \|w\|_{\mathfrak{X}^0} \left|t - s\right| \ .$$

This concludes the proof. ∎

Lemma 4.4.6. *Fix $w \in \mathfrak{X}^\infty$, $\mu_0 \in \mathscr{P}$ and set $\mu_t \coloneqq \Psi^{w,t}\mu_0$. If u is L-Lipschitz continuous, then the map $U \colon t \mapsto u(\mu_t)$ is Lipschitz continuous with $\mathrm{Lip}[U] \leq L \|w\|_{\mathfrak{X}^0}$ for every choice of μ_0 and*

$$\forall t \in \mathbb{R} \quad \left|DU\right|(t) \leq \left|Du\right|(\mu_t) \|w\|_{\mathfrak{X}_{\mu_t}} \ . \tag{4.4.15}$$

Proof. The Lipschitz continuity of U follows from those of u and $t \mapsto \mu_t$ (Lem. 4.4.5). By definition of slope,

$$\left|DU\right|(t) \leq \limsup_{\nu \to \mu_t} \frac{\left|u(\mu_t) - u(\nu)\right|}{W_2(\mu_t, \nu)} \limsup_{s \to t} \frac{W_2(\mu_t, \mu_s)}{\left|t - s\right|} = \left|Du\right|(\mu_t) \left|\dot{\mu}_t\right|$$

for every $t \in \mathbb{R}$, whence (4.4.15) again by Lemma 4.4.5. ∎

Lemma 4.4.7. *Let $(\mu_t)_{t \in I}$ be an absolutely continuous curve in $\mathscr{P}_2$ connecting μ_0 to μ_1. Then, for every $u \in \mathfrak{Z}^\infty$ there exists for a.e. $t \in \mathbb{R}$ the derivative*

$$\mathrm{d}_t u(\mu_t) = \langle \boldsymbol{\nabla}\, u(\mu_t) \,|\, w_t \rangle_{\mathfrak{X}_{\mu_t}} \ ,$$

where (μ_t, w_t) is any distributional solution of the continuity equation (4.3.5), and one has

$$u(\mu_t) - u(\mu_0) = \int_0^t \langle \boldsymbol{\nabla}\, u(\mu_s) \,|\, w_s \rangle_{\mathfrak{X}_{\mu_s}} \mathrm{d}s \ . \tag{4.4.16}$$

Proof. Let f be in $\mathcal{C}^\infty(M)$, $\varphi \in \mathcal{C}_c^\infty(\mathbb{R})$ be an arbitrary test function and denote by $\langle \cdot \,|\, \cdot \rangle$ the canonical duality pair of distributions. Then,

$$\begin{aligned}
\langle \mathrm{d}_t f^*\mu_t \,|\, \varphi \rangle &= \int_{\mathbb{R}} \varphi'(t)\, f^*\mu_t \,\mathrm{d}t = \int_{\mathbb{R}} \varphi'(t) \int_M f(x) \,\mathrm{d}\mu_t(x) \,\mathrm{d}t \\
&= \int_{\mathbb{R}} \int_M \partial_t (f\varphi)(t,x) \,\mathrm{d}\mu_t(x) \,\mathrm{d}t \\
&= \int_{\mathbb{R}} \varphi(t) \int_M \langle \nabla f(x) \,|\, w_t(x) \rangle_{\mathsf{g}} \,\mathrm{d}\mu_t(x) \,\mathrm{d}t
\end{aligned}$$

for any time dependent vector field $(w_t)_t$ such that $(\mu_t, w_t)_t$ is a solution of (4.3.5). Thus the distributional derivative is representable by

$$\mathrm{d}_t f^*\mu_t = \int_M \langle \nabla f(x) \,|\, w_t(x) \rangle_{\mathsf{g}} \,\mathrm{d}\mu_t(x)$$

and

$$\left|\mathrm{d}_t f^*\mu_t\right| \leq \|\nabla f\|_{\mathcal{C}^0} \|w_t\|_{\mathfrak{X}_{\mu_t}} \ .$$

By Proposition 4.3.9 and absolute continuity of $(\mu_t)_t$ the function $t \mapsto \|w_t\|_{\mathfrak{X}_{\mu_t}}$ is in $L^1_{\mathrm{loc}}(\mathbb{R})$. Thus $t \mapsto \mathrm{d}_t f^* \mu_t$ is itself in $L^1_{\mathrm{loc}}(\mathbb{R})$. Let now $u := F \circ \mathbf{f}^* \in \mathfrak{Z}^\infty$. The above reasoning yields, in the sense of distributions,

$$\begin{aligned} \mathrm{d}_t u(\mu_t) &= \sum_i^k (\partial_i F)(\mathbf{f}^* \mu_t)\, \mathrm{d}_t (f_i^* \mu_t) = \sum_i^k (\partial_i F)(\mathbf{f}^* \mu_t) \int_M \langle \nabla f_i \,|\, w_t \rangle_{\mathsf{g}} \, \mathrm{d}\mu_t \\ &= \langle \boldsymbol{\nabla}\, u(\mu_t) \,|\, w_t \rangle_{\mathfrak{X}_{\mu_t}} \quad , \end{aligned}$$

where $(\mu_t, w_t)_t$ is a solution of (4.3.5) as above and we used (4.2.3). Since $t \mapsto \boldsymbol{\nabla}\, u(\mu_t)$ is continuous and bounded by definition of u, the distributional derivative of the function $t \mapsto u(\mu_t)$ is again representable by some function in $L^1_{\mathrm{loc}}(\mathbb{R})$. Thus, the Fundamental Theorem of Calculus applies and one has

$$u(\mu_t) - u(\mu_0) = \int_0^t \mathrm{d}_r\big|_{r=s} u(\mu_r)\, \mathrm{d}s = \int_0^t \langle \boldsymbol{\nabla}\, u(\mu_r) \,|\, w_r \rangle_{\mathfrak{X}_{\mu_r}} \, \mathrm{d}s \ .$$

This concludes the proof. ∎

The following Lemma is taken — almost verbatim — from [142].

Lemma 4.4.8 ([142, Lem. 6.1]). *Fix $w \in \mathfrak{X}^\infty$. Then, for every bounded measurable $u \colon \mathscr{P} \to \mathbb{R}$ and every $v \in \mathfrak{Z}^\infty$, for every $t \in \mathbb{R}$*

$$\int \big(u \circ \Psi^{w,t} - u\big)\, v \, \mathrm{d}\mathbb{P} = - \int_0^t \int u \circ \psi_\sharp^{w,s} \cdot \boldsymbol{\nabla}_w^* \, v \; \mathrm{d}\mathbb{P} \; \mathrm{d}s \ . \tag{4.4.17}$$

4.4.3 On the differentiability of Lipschitz functions. In the following let $u \in \mathrm{Lip}\mathscr{P}_2$, $w \in \mathfrak{X}^\infty$ and set

$$\Omega_w^u := \big\{ \mu \in \mathscr{P} \mid \exists\, G_w u(\mu) := \mathrm{d}_t\big|_{t=0} (u \circ \Psi^{w,t})(\mu) \big\} \ . \tag{4.4.18}$$

Since the function $u \circ \Psi^{w,t}$ is continuous, the existence of $G_w u$ coincides with that of the limit $\lim_{r \to 0} \frac{1}{r}(u(\psi_\sharp^r \mu) - u(\mu))$, $r \in \mathbb{Q}$. As a consequence the set Ω_w^u is measurable.

Proposition 4.4.9. *Fix $u \in \mathrm{Lip}\mathscr{P}_2$ and for any $w \in \mathfrak{X}^\infty$ let Ω_w^u be defined as in* (4.4.18)*. Let further $\mathscr{X} \subset \mathfrak{X}^\infty$ be a countable $\mathbb{Q}$-vector space dense in $\mathfrak{X}^0$ and assume $\mathbb{P}\,\Omega_w^u = 1$ for all $w \in \mathscr{X}$. Then, the assertions* 4.2.4 (3.*i*) *and* 4.2.4 (3.*ii*) *in Theorem 4.2.4 hold for u.*

Proof. Fix $w \in \mathscr{X}$. By assumption on $\mathscr{X}$, there exists

$$G_w u(\mu) = \lim_{t \to 0} \frac{u(\Psi^{w,t}\mu) - u(\mu)}{t}$$

for all μ in the set Ω_w^u of full $\mathbb{P}$-measure. Moreover, by (4.4.15),

$$\sup_{t \in [-1,1]} \left| \frac{u(\Psi^{w,t}\mu) - u(\mu)}{t} \right| \leq \sup_{t \in [-1,1]} \frac{\mathrm{Lip}[u]}{t} \int_0^t \|w\|_{\mathfrak{X}_{\Psi^{w,r}\mu}} \, \mathrm{d}r \leq \mathrm{Lip}[u]\, \|w\|_{\mathfrak{X}^0} \ ,$$

thus, by Dominated Convergence,

$$G_w u = L^2_{\mathbb{P}}\text{-}\lim_{t \to 0} \frac{u \circ \Psi^{w,t} - u}{t} \ . \tag{4.4.19}$$

By continuity of $t \mapsto \frac{1}{t}(u \circ \psi^{w,t} - u)$, combining Lemma 4.4.8 with (4.4.19) yields

$$\forall v \in \mathfrak{Z}^\infty \qquad \int G_w u \cdot v \, \mathrm{d}\mathbb{P} = \int u \cdot \boldsymbol{\nabla}_w^* \, v \, \mathrm{d}\mathbb{P}$$

Next, notice that the map $w \mapsto \boldsymbol{\nabla}^*_w v$ is linear for all $v \in \mathfrak{Z}^\infty$ by Assumption 4.2 (*iii*). Hence, if $w = s_1 w_1 + \cdots + s_k w_k$ for some $s_i \in \mathbb{R}$ and $w_i \in \mathscr{X}$, then

$$\int G_w u \cdot v \, \mathrm{d}\mathbb{P} = \sum_i^k s_i \int u \cdot \boldsymbol{\nabla}^*_{w_i} v \, \mathrm{d}\mathbb{P} = \sum_i^k s_i \int G_{w_i} u \cdot v \, \mathrm{d}\mathbb{P} \ ,$$

thus

$$G_w u = \sum_i^k s_i \, G_{w_i} u \quad \mathbb{P}\text{-a.e. .} \tag{4.4.20}$$

Since $\mathscr{X}$ is countable, the set $\Omega^u_0 := \bigcap_{w \in \mathscr{X}} \Omega^u_w$ has full $\mathbb{P}$-measure by assumption. Therefore, the set Ω^u of measures $\mu \in \Omega^u_0$ such that $w \mapsto G_w u(\mu)$ is a $\mathbb{Q}$-linear functional on $\mathscr{X}$ has itself full $\mathbb{P}$-measure by (4.4.20).

For fixed $\mu \in \Omega^u$ we have $|G_w u(\mu)| \leq |Du|(\mu) \, \|w\|_{\mathfrak{X}_\mu}$ for every $w \in \mathscr{X}$ by Lemma 4.4.6. Since $\mathscr{X}$ is $\mathfrak{X}^0$-dense in $\mathfrak{X}^\infty$, it is in particular $\mathfrak{X}_\mu$-dense in $\mathfrak{X}^\infty$ for every $\mu \in \mathscr{P}$. Hence the map $w \mapsto G_w u(\mu)$ is a $\mathfrak{X}_\mu$-continuous linear functional on the dense subset $\mathscr{X}$ and may thus be extended on the whole space $\mathfrak{X}^\infty$ (in fact: on $\mathfrak{X}_\mu$) to a continuous linear functional, again denoted by $w \mapsto G_w u(\mu)$ and again such that $|G_w u(\mu)| \leq |Du|(\mu) \, \|w\|_{\mathfrak{X}_\mu}$.

Thus, for every μ in the set of full $\mathbb{P}$-measure Ω^u there exists $\mathbf{D}u(\mu) \in T_\mu \mathscr{P}_2$ such that $G_w u(\mu) = \langle \mathbf{D}u(\mu) \,|\, w \rangle_{\mathfrak{X}_\mu}$ and $\|\mathbf{D}u(\mu)\|_{\mathfrak{X}_\mu} \leq |Du|(\mu)$. This concludes the proof of the first statement in 4.2.4 (3.*ii*), which in turn implies 4.2.4 (3.*i*) since $|Du|(\,\cdot\,) \leq \mathrm{Lip}[u]$.

By definition of Ω^u one has $\Omega^u \subset \Omega^u_w$ for all $w \in \mathscr{X}$, hence 4.2.4 (3.*ii*) is already proven for all $w \in \mathscr{X}$. In order to prove it for $w \in \mathfrak{X}^\infty \setminus \mathscr{X}$, fix $\varepsilon > 0$ and let $w' \in \mathscr{X}$ be such that $\|w - w'\|_{\mathfrak{X}^0} < \varepsilon$. Since M is compact, a straightforward modification of [142, Lem. 5.5] yields

$$\left| u(\Psi^{w,t}\mu) - u(\Psi^{w',t}\mu) \right| \leq \mathrm{Lip}[u] \, W_2(\Psi^{w,t}\mu, \Psi^{w',t}\mu) \leq t \, \mathrm{Lip}[u] \, c_0 \, e^{c_0 \, t} \, \varepsilon$$

for some constant $c_0 := c_0(M, w) < \infty$. As a consequence,

$$\begin{aligned} \forall \mu \in \Omega^u \qquad \left| \frac{u \circ \Psi^{w,t} - u}{t} - \langle \mathbf{D}u(\mu) \,|\, w \rangle_{\mathfrak{X}_\mu} \right| \leq & \varepsilon \, \mathrm{Lip}[u] \, c_0 \, e^{c_0 \, t} + \varepsilon \, \|\mathbf{D}u(\mu)\|_{\mathfrak{X}_\mu} \\ & + \left| \frac{u \circ \Psi^{w',t} - u}{t} - \left\langle \mathbf{D}u(\mu) \,\middle|\, w' \right\rangle_\mu \right| \end{aligned}$$

and letting $t \to 0$ yields the conclusion of 4.2.4 (3.*ii*) by arbitrariness of ε.

As consequence of 4.2.4 (3.*ii*) and the bound $\|\mathbf{D}u(\mu)\|_{\mathfrak{X}_\mu} \leq \mathrm{Lip}[u]$, by definition, $u \in \mathcal{F}_{\mathrm{cont}}$. ∎

Corollary 4.4.10. *Assume* $\mathbb{P}$ *additionally satisfies Assumption* 4.2 (*iv*) *and let* $u \in \mathrm{Lip}\mathscr{P}_2$. *Then, the assertions* 4.2.4 (3.*i*) *and* 4.2.4 (3.*ii*) *in Theorem 4.2.4 hold for* u.

Proof. Let $w \in \mathfrak{X}^\infty$ and denote its flow by $\left(\psi^{w,t}\right)_{t \in \mathbb{R}}$. It suffices to show that u satisfies the assumption on Ω^u_w in Proposition 4.4.9. By Lemma 4.4.6 the set $\left\{ r \in [s,t] \mid \Psi^{w,r}\mu \in \Omega^u_w \right\}$ has full Lebesgue measure for every $s < t$ in $\mathbb{R}$ and every $\mu \in \mathscr{P}$. Thus

$$0 = \int_0^1 \int \mathbb{1}_{(\Omega^u_w)^\mathrm{c}}(\Psi^{w,r}\mu) \, \mathrm{d}\mathbb{P}(\mu) \, \mathrm{d}r = \int_0^1 (\Psi^{w,r}_\sharp \mathbb{P})\big((\Omega^u_w)^\mathrm{c}\big) \, \mathrm{d}r = \int_0^1 \int R^w_r(\mu) \, \mathbb{1}_{(\Omega^u_w)^\mathrm{c}}(\mu) \, \mathrm{d}\mathbb{P}(\mu) \, \mathrm{d}r \ ,$$

whence $\mathbb{P}\big((\Omega^u_w)^\mathrm{c}\big) = 0$ by (4.2.7). ∎

Corollary 4.4.11. *Let (M, g) be additionally satisfying the STP (Ass. 4.2). Then, for every $\nu \in \mathscr{P}$ the function $u_\nu \colon \mu \mapsto W_2(\nu, \mu)$ belongs to $\mathscr{F}_0$ and $\|\mathbf{D}u_\nu\|_{\mathfrak{X}_\cdot} \leq 1$ $\mathbb{P}$-a.e..*

Proof. Assume first $\nu \in \mathscr{P}^{\mathrm{reg}}$ and set $S_\theta(\nu) \coloneqq \{\mu \in \mathscr{P} \mid u_\nu(\mu) = \theta\}$. Since $\mathbb{P}$ is a probability measure, there exists a sequence $\theta_n \to 0$ as $n \to \infty$ such that $\mathbb{P}\, S_{\theta_n}(\nu) = 0$. As a consequence of this fact and of Lemma 4.4.3, Proposition 4.4.9 applies to the map $u_{\nu,\theta_n} \colon \mu \mapsto W_2(\nu, \mu) \vee \theta_n$ with $\Omega^{u_{\nu,\theta_n}} \coloneqq \mathscr{P} \setminus S_{\theta_n}(\nu)$, yielding $\|\mathbf{D}u_{\nu,\theta_n}\|_{\mathfrak{X}_\cdot} \leq \mathrm{Lip}[u_{\nu,\theta_n}] = 1$ $\mathbb{P}$-a.e.. On the other hand, $u_{\nu,\theta_n} \in \mathscr{F}_0$ by Lemma 4.4.4 and it is clear by reverse triangle inequality that $\lim_n u_{\nu,\theta_n} = u_\nu$ uniformly, whence $u_\nu \in \mathscr{F}_0$ by Lemma 4.3.3.

If $\nu \in \mathscr{P} \setminus \mathscr{P}^{\mathrm{reg}}$, choose $\nu_n \in \mathscr{P}^{\mathrm{reg}}$ narrowly convergent to ν. Again by reverse triangle inequality $\lim_n u_{\nu_n} = u_\nu$ uniformly and $\|\mathbf{D}u_{\nu_n}\|_{\mathfrak{X}_\cdot} \leq 1$ $\mathbb{P}$-a.e. as above, hence the conclusion again by Lemma 4.3.3. ∎

4.4.4 Proof of Theorem 4.2.4.

Proof of 4.2.4 (1) *and* 4.2.4 (2). The proof of [142, Prop. 1.4(i) and (iv)], together with the auxiliary results [142, Lem.s 6.3, 6.4], carries over *verbatim* to our case. This proves the closability of the forms in assertion 4.2.4 (1) and assertion 4.2.4 (2). Since $\mathscr{F}_0 \subset \mathscr{F}_{\mathrm{cont}} \subset \mathscr{F}$, it suffices to prove the strong locality of $(\mathcal{E}, \mathscr{F})$. That is, by [23, Rmk. I.5.1.5] it suffices to show that if $u \in \mathcal{F}$, then $\varrho_1 \circ u, \varrho_2 \circ u \in \mathscr{F}$ and $\mathcal{E}(\varrho_1 \circ u, \varrho_2 \circ u) = 0$ for $\varrho_1, \varrho_2 \in \mathcal{C}_c^\infty(\mathbb{R})$ such that $\varrho_1(0) = \varrho_2(0) = 0$ and $\mathrm{supp}\varrho_1 \cap \mathrm{supp}\varrho_2 = \varnothing$.

Fix $w \in \mathfrak{X}^\infty$ and denote by $\left(\psi^{w,t}\right)_{t \in \mathbb{R}}$ its flow. Since $u \in \mathcal{F}$ is bounded, the map $U \colon t \mapsto u \circ \Psi^{w,t}$ satisfies $U(t) \in L^2_{\mathbb{P}}(\mathscr{P})$ for every $t \in \mathbb{R}$, hence, [142, Lem. 6.4] yields for $i = 1, 2$

$$\mathrm{d}_t\big|_{t=0}\, \varrho_i(U(t)) = \varrho_i'(U(0))\; \mathrm{d}_t\big|_{t=0} U(t) = (\varrho_i' \circ u)\, \langle \mathbf{D}u \,|\, w \rangle_{\mathfrak{X}_\cdot}$$

where all derivatives are taken in $L^2_{\mathbb{P}}(\mathscr{P})$. Hence, the map $\mu \mapsto \varrho_i'(u(\mu))\mathbf{D}u(\mu)$ is a measurable section of $T^{\mathrm{Der}}\mathscr{P}_2$, satisfies (4.2.11) and is such that

$$\mathcal{E}(\varrho_i \circ u, \varrho_i \circ u) = \int \varrho_i'(u(\mu))\, \|\mathbf{D}u(\mu)\|^2_{\mathfrak{X}_\mu}\, \mathrm{d}\mathbb{P}(\mu) \leq \left\|\varrho_i'\right\|^2_{\mathcal{C}^0} \mathcal{E}(u, u) < \infty\,. \tag{4.4.21}$$

As a consequence, $\varrho_i \circ u \in \mathcal{F}$ and the locality property follows now by (4.4.21) and polarization. ∎

Proof of 4.2.4 (3). For fixed $\nu \in \mathscr{P}^{\mathrm{reg}}$ let $u_\nu \colon \mathscr{P} \to \mathbb{R}$ be defined by $u_\nu \colon \mu \mapsto W_2(\nu, \mu)$. By Lemma 4.4.3, for every $\mu \in \Omega^\nu \coloneqq \mathscr{P} \setminus \{\nu\}$ and every $w \in \mathfrak{X}^\infty$ there exists the limit $G_w u_\nu(\mu)$ defined in (4.4.18). Since $\mathbb{P}$ is diffuse by Assumption 4.2 (*ii*), the set Ω^ν has full $\mathbb{P}$-measure, hence Proposition 4.4.9 applies to u_ν with $\Omega^{u_\nu} = \Omega^\nu$ and one has $\|\mathbf{D}u_\nu(\mu)\|_{\mathfrak{X}_\mu} \leq \mathrm{Lip}[u_\nu] = 1$.

Since additionally $u_\nu \in \mathcal{F}_{\mathrm{cont}}$ by Proposition 4.4.9, if u is W_2-Lipschitz continuous, then $u \in \mathscr{F}_{\mathrm{cont}}$ and $\|\mathbf{D}u\|_{\mathfrak{X}_\cdot} \leq \mathrm{Lip}[u]$ $\mathbb{P}$-a.e. by strong locality of $(\mathcal{E}, \mathscr{F})$ and Lemma 4.3.4 applied to the dense set $\mathscr{P}^{\mathrm{reg}}$, which proves 4.2.4 (3.*i*). If M additionally satisfies the STP, then we may replace $\mathscr{F}_{\mathrm{cont}}$ in the above reasoning with $\mathscr{F}_0$ thanks to Corollary 4.4.11.

If $\mathbb{P}$ additionally satisfies Assumption 4.2 (*iv*), then assertion 4.2.4 (3.*ii*) reduces to Corollary 4.4.10. ∎

Intrinsic distances. Given a family of functions $\mathscr{A} \subset \mathscr{F}$ set, for all $\mu, \nu \in \mathscr{P}$,

$$\mathsf{d}_{\mathscr{A}}(\mu, \nu) \coloneqq \sup\left\{u(\mu) - u(\nu) \mid u \in \mathscr{A} \cap \mathcal{C}(\mathscr{P}), \mathbf{\Gamma}(u) \leq 1\ \mathbb{P}\text{-a.e. on } \mathscr{P}\right\}\,.$$

Corollary 4.4.12 (Intrinsic distances)**.** *Suppose that* $\mathbb{P}$ *satisfies Assumption 4.2 and let*

$$\mathsf{d}_{\mathscr{F}_0} \leq \mathsf{d}_{\mathscr{F}_{\mathrm{cont}}} \leq \mathsf{d}_{\mathscr{F}}$$

be the intrinsic distances (4.3.3) *of the Dirichlet forms* $(\mathcal{E}, \mathscr{F}_0)$, $(\mathcal{E}, \mathscr{F}_{\mathrm{cont}})$ *and* $(\mathcal{E}, \mathscr{F})$ *respectively. Then,*

$$\mathsf{d}_{\mathfrak{Z}^\infty} \leq W_2 \leq \mathsf{d}_{\mathscr{F}_{\mathrm{cont}}} \ .$$

If additionally STP holds, then the above statement holds with $\mathsf{d}_{\mathscr{F}_0}$ *in lieu of* $\mathsf{d}_{\mathscr{F}_{\mathrm{cont}}}$.

Proof. Let $\mathscr{A} = \mathscr{F}_0, \mathscr{F}_{\mathrm{cont}}, \mathscr{F}$. If $u_\nu \in \mathscr{A}$ then

$$\mathsf{d}_{\mathscr{A}}(\mu, \nu) \geq u_\nu(\mu) - u_\nu(\nu) = W_2(\mu, \nu) \ ,$$

hence it suffices to keep track of the assumptions under which $u_\nu \in \mathscr{F}_0, \mathscr{F}_{\mathrm{cont}}, \mathscr{F}$ respectively in order to show $W_2 \leq \mathsf{d}_{\mathscr{A}}$. One has $u_\nu \in \mathscr{F}_{\mathrm{cont}} \subset \mathscr{F}$ by the proof of Theorem 4.2.4 (3) above, while $u_\nu \in \mathscr{F}_0$ under the STP by Corollary 4.4.11.

Let now $u \in \mathfrak{Z}^\infty$ with $\|\mathbf{D}u\|_{\mathfrak{X}_\cdot} \leq 1$ $\mathbb{P}$-a.e.. Since $\mathbf{D}u = \boldsymbol{\nabla} u$ is continuous, if $\mathbb{P}$ is fully supported (Assumption 4.2 (i)), then $\|\mathbf{D}u(\mu)\|_{\mathfrak{X}_\mu} \leq 1$ for all $\mu \in \mathscr{P}$. In the same notation of Lemma 4.4.7, it follows from (4.4.16) that

$$u(\mu_1) - u(\mu_0) = \int_0^1 \langle \boldsymbol{\nabla} u(\mu_s) \,|\, w_s \rangle_{\mathfrak{X}_{\mu_s}} \, \mathrm{d}s \leq \int_0^1 \|w_s\|_{\mathfrak{X}_{\mu_s}} \, \mathrm{d}s \ .$$

Taking the infimum of the above inequality over all distributional solutions $(\mu_s, w_s)_{s \in I}$ of (4.3.5) with fixed μ_0, μ_1 yields $u(\mu_1) - u(\mu_0) \leq W_2(\mu_0, \mu_1)$ by e.g. [10, Prop. 2.30].

This settles all the inequalities in the assertion. ■

4.5 Examples

We collect here some examples of measures satisfying our main Theorem 4.2.4. These include the family of normalized mixed Poisson measures §4.5.3 (for any M), the entropic measure §4.5.5 and an image on $\mathscr{P}_2$ of the Malliavin–Shavgulidze measure §4.5.6 (both in the case $M = \mathbb{S}^1$). We notice that a proof of Theorem 4.2.4 for the entropic measure was already sketched in [140, Prop. 7.26]. Again when M is arbitrary, we also provide an example of a measure *not* satisfying Assumption 4.2, namely the Dirichlet–Ferguson measure §4.5.4. However, relying on results in the present chapter, we will show in Chapter 5 that the assertions 4.2.4 (1)–4.2.4 (3.i) in Theorem 4.2.4 hold for this measure too. Finally, we show how to construct more examples from those listed above, by considering shifted measures, weighted measures and convex combinations.

Notation. Everywhere in this section let $\phi \in \mathrm{Diff}^\infty(M)$ and denote by $\Phi \colon \mathscr{M}_b^+ \to \mathscr{M}_b^+$ the shift by ϕ, by $\phi^* \colon L^0(M) \to L^0(M)$ the pullback by ϕ, and by J_ϕ^{m} the modulus of the Jacobian determinant of ϕ with respect to m.

Denote further by $\mathbf{N} \colon \mathscr{M}_b^+ \to \mathscr{P}$ the normalization map $\mathbf{N} \colon \nu \mapsto \overline{\nu} := \nu/\nu M$. It is straightforward that $\mathbf{N}$ is continuous with respect to the chosen topologies, hence measurable with respect to the chosen σ-algebras. Moreover, it is readily verified that $\mathbf{N}$ and $\Phi := \phi_\sharp$ commute, i.e.

$$\mathbf{N} \circ \Phi = \Phi \circ \mathbf{N} \colon \mathscr{M}_b^+ \longrightarrow \mathscr{P} \ . \tag{4.5.1}$$

4.5.1 On Assumption 4.2. We collect here some comments on Assumption 4.2. First of all, let us show how one can construct examples of measures satisfying Assumption 4.2 starting from a single one.

Lemma 4.5.1. *Let* $w \in \mathfrak{X}^\infty$ *and* $u \in \mathfrak{Z}^\infty$*. Then,*

$$\boldsymbol{\nabla}_w(u \circ \Phi) = \boldsymbol{\nabla}_{\phi_* w}\, u \circ \Phi \ .$$

Proof. Let $f \in \mathcal{C}^\infty(M)$. Then

$$\boldsymbol{\nabla}(f^* \circ \Phi) = \boldsymbol{\nabla}((f \circ \phi)^*) = \nabla(f \circ \phi) \ .$$

By (4.2.4), the proof reduces now to the following computation

$$\begin{aligned} \langle \boldsymbol{\nabla}(u \circ \Phi)(\mu) \,|\, w \rangle_{\mathfrak{X}_\mu} &= \sum_i^k (\partial_i F)(\mathbf{f}^*(\Phi\mu)) \int_M \mathrm{d}(f \circ \phi)_x(w_x)\, \mathrm{d}\mu(x) \\ &= \sum_i^k (\partial_i F)(\mathbf{f}^*(\Phi\mu)) \int_M \mathrm{d}f_{\phi(x)}(\mathrm{d}\phi_x w_x)\, \mathrm{d}\mu(x) \\ &= \sum_i^k (\partial_i F)(\mathbf{f}^*(\Phi\mu)) \int_M \mathrm{d}f_y(\mathrm{d}\phi_{\phi^{-1}(y)} w_{\phi^{-1}(y)})\, \mathrm{d}\Phi\mu(y) \\ &= \langle \boldsymbol{\nabla}\, u(\Phi\mu) \,|\, \phi_* w \rangle_{\mathfrak{X}_{\Phi\mu}} \ . \end{aligned}$$

■

Proposition 4.5.2. *Let* $\mathbb{P} \in \mathscr{P}(\mathscr{P})$*,* $\phi \in \mathrm{Diff}^\infty(M)$ *and* $\varphi \in \mathscr{F}$ *be such that* $\varphi > 0$ $\mathbb{P}$*-a.e. and* $\|\varphi\|_{L^2_\mathbb{P}} = 1$*. Set* $\mathbb{P}' := \Phi_\sharp \mathbb{P}$ *and* $\mathbb{P}^\varphi := \varphi^2 \cdot \mathbb{P}$*. Then,*

(*i*) *if* $\mathbb{P}$ *satisfies Assumption* 4.2 (*i*)*, then so do* $\mathbb{P}'$ *and* $\mathbb{P}^\varphi$*;*

(*ii*) *if* $\mathbb{P}$ *satisfies Assumption* 4.2 (*ii*)*, then so do* $\mathbb{P}'$ *and* $\mathbb{P}^\varphi$*;*

(*iii*) *if* $\mathbb{P}$ *satisfies Assumption* 4.2 (*iii*)*, then so do* $\mathbb{P}'$ *and* $\mathbb{P}^\varphi$*;*

(*iv*) *if* $\mathbb{P}$ *satisfies Assumption* 4.2 (*iv*)*, then so does* $\mathbb{P}^\varphi$*. If additionally* $\phi = \psi^{w,t}$ *for some* $w \in \mathfrak{X}^\infty$*,* $t \in \mathbb{R}$*, then, additionally,* $\mathbb{P}'$ *satisfies Assumption* 4.2 (*iv*) *too.*

Proof. Since ϕ is bijective, so are $\Phi := \phi_\sharp$ and $\Phi_\sharp$. This proves (*i*) and (*ii*) for $\mathbb{P}'$; they are also straightforward for $\mathbb{P}^\varphi$ since $\varphi^2 > 0$ $\mathbb{P}$-a.e.. In both cases, (*iv*) is straightforward by (4.2.7).

In order to show (*iii*) for $\mathbb{P}'$, we need to show that there exists an operator $\boldsymbol{\nabla}^{*'}_w : \mathfrak{Z}^\infty \to L^2_{\mathbb{P}'}(\mathscr{P})$ such that (4.2.6) holds with $\mathbb{P}'$ in lieu of $\mathbb{P}$ and $\boldsymbol{\nabla}^{*'}_w$ in lieu of $\boldsymbol{\nabla}^*_w$. Since ϕ is a diffeomorphism, the notations ϕ_*^{-1} and ${\phi^{-1}}_\sharp = {\phi_\sharp}^{-1} = \Phi^{-1}$ are unambiguous. Then, by Lemma 4.5.1,

$$\begin{aligned} \int \boldsymbol{\nabla}_w\, u \cdot v \,\mathrm{d}\mathbb{P}' &= \int \boldsymbol{\nabla}_w\, u \circ \Phi \cdot v \circ \Phi \,\mathrm{d}\mathbb{P} = \int \boldsymbol{\nabla}_{\phi_*^{-1} w}(u \circ \Phi) \cdot v \circ \Phi \,\mathrm{d}\mathbb{P} \\ &= \int u \circ \Phi \cdot \boldsymbol{\nabla}^*_{\phi_*^{-1} w}(v \circ \Phi) \,\mathrm{d}\mathbb{P} = \int u \cdot \boldsymbol{\nabla}^*_{\phi_*^{-1} w}(v \circ \Phi) \circ \Phi^{-1} \,\mathrm{d}\mathbb{P}' \ . \end{aligned}$$

Assertion (*iii*) follows by putting $\boldsymbol{\nabla}^{*'}_w\, v := \boldsymbol{\nabla}^*_{\phi_*^{-1} w}(v \circ \Phi) \circ \Phi^{-1}$.

In order to show (*iii*) for $\mathbb{P}^\varphi$ assume first that $\varphi \in \mathfrak{Z}^\infty$, whence φ is continuous and bounded (Rmk. 4.2.2). Then, by (4.2.3) and (4.2.4)

$$\int u\varphi^2 \cdot \boldsymbol{\nabla}^*_w\, v \,\mathrm{d}\mathbb{P} = \int \boldsymbol{\nabla}_w(u\varphi^2) \cdot v \,\mathrm{d}\mathbb{P} = \int \boldsymbol{\nabla}_w\, u \cdot \varphi^2 v \,\mathrm{d}\mathbb{P} + \int \varphi^2 uv \cdot (2\varphi^{-1}\, \boldsymbol{\nabla}_w\, \varphi) \,\mathrm{d}\mathbb{P}$$

and the assertion follows by setting $\boldsymbol{\nabla}^{*;\varphi}_w\, v := \boldsymbol{\nabla}^*\, v - (2\varphi^{-1}\, \boldsymbol{\nabla}_w\, \varphi) v$. The general case follows by approximation as soon as we show that the pre-Dirichlet form

$$\mathscr{F}^\varphi := \left\{ u \in \mathscr{F} \mid \int \left(u^2 + \|\mathbf{D}u\|^2_{\mathfrak{X}_\cdot}\right) \varphi^2 \,\mathrm{d}\mathbb{P} < \infty \right\}$$

$$\mathcal{E}^\varphi(u,v) := \int \varphi^2 \left\langle \mathbf{D}u \,|\, \mathbf{D}v \right\rangle_{\mathfrak{X}_\cdot} \, \mathrm{d}\mathbb{P}$$

is closable. Provided that $(\mathcal{E}, \mathscr{F})$ is a strongly local Dirichlet form by Theorem 4.2.4 (1), this last assertion is the content of [48, Thm. 1.1]. ■

Remark 4.5.3. While points (*i*)–(*iii*) of the Proposition suggest that Assumptions 4.2 (*i*)–4.2 (*iii*) are quite generic with respect to shifting $\mathbb{P}$ by (the lift of) a diffeomorphism, point (*iv*) is (by far) more restrictive, as the inclusion $\mathrm{Flow}(M) \subsetneq \mathrm{Diff}^\infty_+(M)$ is always strict, even on $\mathbb{S}^1$. (See e.g. [66].)

It is clear that the closability of the pre-Dirichlet forms $(\mathcal{E}, \mathfrak{Z}^\infty)$ and $(\mathcal{E}, \mathcal{F}_{\mathrm{cont}})$ associated to $\mathbb{P}$ is essential to our approach in discussing Rademacher-type theorems, which settles the necessity of Assumption 4.2 (*iii*). Assumption 4.2 (*i*) is instead motivated by the following trivial example.

Example 4.5.4. Denote by $\delta\colon M \mapsto \mathscr{P}$ the Dirac embedding $x \mapsto \delta_x$ and set $\mathbb{P} := \delta_\sharp \mathsf{m}$. Since $\mathbb{P}$ is supported on the family of Dirac masses, it does not satisfy 4.2 (*i*). On the other hand, since $W_2(\delta_{x_1}, \delta_{x_2}) = \mathsf{d}_\mathsf{g}(x_1, x_2)$ for every $x_1, x_2 \in M$, it is clear that $(\mathscr{P}, W_2, \mathbb{P})$ and $(M, \mathsf{d}_\mathsf{g}, \mathsf{m})$ are isomorphic as metric measure spaces, which shows 4.2 (*ii*). Moroever,

$$\Phi_\sharp \delta_\sharp \mathsf{m} = (\Phi \circ \delta)_\sharp \mathsf{m} = (\delta \circ \phi)_\sharp \mathsf{m} = \delta_\sharp(\Phi \mathsf{m}) = \delta_\sharp(J^\mathsf{m}_\phi \cdot \mathsf{m}) = (J^\mathsf{m}_\phi)^* \cdot \delta_\sharp \mathsf{m}$$

and 4.2 (*iii*) holds for $\mathbb{P}$ as well.

Remark 4.5.5. Incidentally, notice that Theorem 4.2.4 applied to Example 4.5.4 provides a non-local proof of the classical Rademacher Theorem on a closed Riemannian manifold. Indeed it suffices to notice that $T^{\mathrm{Der}}_{\delta_x} \cong T_x M$ as Hilbert spaces for every $x \in M$ and that every Lipschitz function $f \in \mathrm{Lip}(M)$ induces a Lipschitz function $\tilde{f} \in \mathrm{Lip}(\mathscr{P}_2)$, namely the (e.g. lower) McShane extension $\tilde{f}$ of the function $f \circ \delta^{-1}$ defined on the image of δ.

Proposition 4.5.6. *The following chain of implications holds true:*

$$4.2\,(v) \implies 4.2\,(iv) \land 4.2\,(iii) \implies 4.2\,(iv) \implies 4.2\,(ii)\ .$$

In particular: 4.2 (*i*) *and* 4.2 (*v*) *together imply Assumption 4.2.*

Proof. The implication 4.2 (*v*) $\implies$ 4.2 (*iv*) is trivial. It is readily seen that Assumption 4.2 (*ii*) is already implied by the first part of 4.2 (*iv*). It remains to show that 4.2 (*v*) $\implies$ 4.2 (*iii*). Indeed,

$$\begin{aligned}
\int \boldsymbol{\nabla}_w\, u \cdot v \,\mathrm{d}\mathbb{P} = & \int \lim_{t\to 0} \frac{u \circ \Psi^{w,t} - u}{t} \cdot v \,\mathrm{d}\mathbb{P} \\
= & \lim_{t\to 0} \frac{1}{t} \int \big(u \cdot v \circ \Psi^{w,-t} \cdot R^w_{-t} - uv\big)\,\mathrm{d}\mathbb{P} \\
= & \lim_{t\to 0} \frac{1}{t} \int u\big(v \circ \Psi^{w,-t} - v\big)\,\mathrm{d}\mathbb{P} \\
& + \lim_{t\to 0} \frac{1}{t} \int u\,\big(v \circ \Psi^{w,-t} - v\big)\,(R^w_{-t} - 1)\,\mathrm{d}\mathbb{P} \\
& + \lim_{t\to 0} \frac{1}{t} \int uv\,(R^w_{-t} - 1)\,\mathrm{d}\mathbb{P}\ .
\end{aligned}$$

The first limit in the last equality satisfies, by Dominated Convergence,

$$\lim_{t\to 0} \frac{1}{t} \int u\big(v \circ \Psi^{w,-t} - v\big)\,\mathrm{d}\mathbb{P} = -\int u \cdot \boldsymbol{\nabla}_w\, v \,\mathrm{d}\mathbb{P}\ .$$

The second limit vanishes, again by Dominated Convergence, since $t \mapsto R_t^w(\mu)$ is continuous (differentiable) at $t = 0$ for $\mathbb{P}$-a.e. μ. In light of Assumption 4.2 (v), differentiating under integral sign, the third limit satisfies

$$\lim_{t\to 0} \frac{1}{t} \int uv\, (R_{-t}^w - 1)\, \mathrm{d}\mathbb{P} = \int uv \cdot \partial_t\big|_{t=0} R_{-t}^w \, \mathrm{d}\mathbb{P} \ .$$

As a consequence, Assumption 4.2 (iii) is satisfied by letting

$$\boldsymbol{\nabla}_w^* \, v := - \boldsymbol{\nabla}_w \, v - \partial_t\big|_{t=0} R_{-t}^w \cdot v \ .$$

This concludes the proof. ■

4.5.2 On the Smooth Transport Property. The reader is referred to [56] and references therein for an expository treatment of regularity theory of optimal transport maps on Riemannian manifolds, whereof we make use in the present section. We denote by $\mathbb{S}T_xM := \{\mathrm{w} \in T_xM \mid |\mathrm{w}|_{\mathsf{g}_x} = 1\}$ the unit tangent space to (M, g) at x. Everywhere in the following also let $\mathsf{c} := \frac{1}{2}\mathsf{d}^2$.

Further geometrical assumptions. For $x \in M$ and $\mathrm{w} \in T_xM$ define the *cut*, resp. *focal*, *time* by

$$t_{\mathrm{C}}(x, \mathrm{w}) := \inf \{t > 0 \mid s \mapsto \exp_x(s\mathrm{w}) \text{ is not a } \mathsf{d}\text{-minimizing curve from } x \text{ to } \exp_x(t\mathrm{w})\}$$
$$t_{\mathrm{F}}(x, \mathrm{w}) := \inf \left\{t > 0 \mid \mathrm{d}_{t\mathrm{w}} \exp_x \colon T_xM \to T_{\exp_x(t\mathrm{w})}M \text{ is not invertible}\right\}$$

and the (*tangent*), resp. (*tangent*) *focal*, *cut locus* and *injectivity domain* by

$$\mathrm{TCL}(x) := \{t_{\mathrm{C}}(x, \mathrm{w})\mathrm{w} \mid \mathrm{w} \in \mathbb{S}T_xM\} \ , \qquad \mathrm{cut}(x) := \exp_x(\mathrm{TCL}(x)) \ ,$$
$$\mathrm{TFL}(x) := \{t_{\mathrm{F}}(x, \mathrm{w})\mathrm{w} \mid \mathrm{w} \in \mathbb{S}T_xM\} \ , \qquad \mathrm{fcut}(x) := \exp_x(\mathrm{TCL}(x) \cap \mathrm{TFL}(x)) \ ,$$
$$I(x) := \{t\mathrm{w} \mid \mathrm{w} \in \mathbb{S}T_xM, 0 \le t < t_{\mathrm{C}}(x, \mathrm{w})\} \ .$$

Finally, recall the definition of the Ma–Trudinger–Wang tensor

$$\mathfrak{S}_{(x,y)}(\mathrm{w}, \mathrm{w}') := -\tfrac{3}{2}\, \mathrm{d}_s^2\big|_{s=0} \mathrm{d}_t^2\big|_{t=0} \mathsf{c}\big(\exp_x(t\mathrm{w}), \exp_x(\mathrm{v} + s\mathrm{w}')\big)$$

where $x \in M$, $y \in I(x)$, $\mathrm{w}, \mathrm{w}' \in T_xM$ and $\mathrm{v} := \exp_x^{-1}(y)$.

The following definitions are taken from [56].

Definition 4.5.7 (Non-focality of cut loci)**.** We say that (M, g) is *non-focal* if it additionally satisfies $\mathrm{fcut}(x) = \varnothing$ for all $x \in M$.

Definition 4.5.8 (Strong Ma–Trudinger–Wang condition $\mathrm{MTW}(K)$)**.** We say that (M, g) satisfies the *strong Ma–Trudinger–Wang condition with constant* K (in short: M is $\mathrm{MTW}(K)$) if there exists a constant $K > 0$ such that

$$\forall x \in M \ , \ y \in \exp_x(I(x)) \qquad \mathfrak{S}_{(x,y)}(\mathrm{w}, \mathrm{w}') \ge K \big|\mathrm{w}\big|_{\mathsf{g}_x}^2 \big|\mathrm{w}'\big|_{\mathsf{g}_x}^2 \quad \text{whenever} \quad \mathrm{w}^\top [\mathsf{c}_{\cdot,\cdot}]\mathrm{w}' = 0 \ ,$$

where $[\mathsf{c}_{\cdot,\cdot}]$ denotes the matrix of derivatives $\mathsf{c}_{i,j} := \partial^2_{x_i, y_j} \mathsf{c}$.

Our main interest in the previous definitions is the following regularity result.

Theorem 4.5.9 (Loeper–Villani (See e.g. [56, Cor. 3.13].))**.** *Let* (M, g) *be additionally non-focal and satisfying* $\mathrm{MTW}(K)$*. Then* M *satisfies the STP (Ass. 4.2).*

Remark 4.5.10. The strong MTW condition is sufficient, whereas not necessary, to establish the above result. A discussion of optimal assumptions is here beyond our purposes. It will suffice to say that the proof strategy of Lemma 4.4.4 fails as soon as $\mathrm{MTW}(0)$ is negated, which in turn implies that c-convex $\mathcal{C}^1$ functions are *not* uniformly dense in (Lipschitz) c-convex functions. (See [56, Thm. 3.4].)

4.5.3 Normalized mixed Poisson measures. We denote by $\ddot{\Upsilon}$ the *space of integer-valued Radon measures* over (M, g) with arbitrary *finite* number of atoms, always regarded as a subspace of $\mathscr{M}_b^+$, endowed with the vague topology (which coincides with the narrow topology by compactness of M) and with the associated Borel σ-algebra. Similarly to [7, 142], we let $\rho \in \mathcal{C}^1(M; \mathbb{R}^+)$ and denote by $\mathcal{P}_\sigma$ the Poisson measure of intensity $\sigma \coloneqq \rho\mathsf{m}$ on $\ddot{\Upsilon}$. Given $\lambda \in \mathscr{P}(\mathbb{R}^+)$ such that $\lambda(1 \wedge \mathrm{id}_{\mathbb{R}^+}) < \infty$, henceforth a *Lévy measure*, we denote by $\mathcal{Q}_{\lambda,\sigma}$ the *mixed Poisson measure*

$$\mathcal{Q}_{\lambda,\sigma}(\,\cdot\,) = \int_{\mathbb{R}^+} \mathcal{P}_{s\cdot\sigma}(\,\cdot\,)\,\mathrm{d}\lambda(s)\ .$$

Recall that $\mathcal{P}_\sigma$, hence $\mathcal{Q}_{\lambda,\sigma}$, is concentrated on the *configuration space*

$$\Upsilon \coloneqq \left\{\gamma \in \ddot{\Upsilon} \mid \gamma\{x\} \in \{0,1\} \text{ for all } x \in M\right\}\ .$$

Moreover (see [7, Prop. 2.2]), for all $\gamma \in \Upsilon$

$$\frac{\mathrm{d}\big(\Phi_\sharp \mathcal{P}_\sigma\big)}{\mathrm{d}\mathcal{P}_\sigma}(\gamma) = \exp\big(\sigma(\mathbb{1} - p^\sigma_\Phi)\big)\prod_{x\in\gamma} p^\sigma_\Phi(x)\ , \qquad \text{where} \qquad p^\sigma_\Phi(x) \coloneqq \frac{\Phi^*\rho(x)}{\rho(x)} J^{\mathsf{m}}_\Phi(x) \tag{4.5.2}$$

and by $x \in \gamma$ we mean $\gamma\{x\} > 0$. Since we chose $\rho \in L^1_{\mathsf{m}}(M)$, the measure σ is finite, hence $\gamma M < \infty$ for $\mathcal{P}_\sigma$-a.e. γ, i.e. $\mathcal{P}_\sigma$-a.e. γ is concentrated on a finite number of points. As a consequence, the same statement holds for $\mathcal{Q}_{\lambda,\sigma}$ in lieu of $\mathcal{P}_\sigma$ and one has

$$\text{for } \mathcal{Q}_{\sigma,\lambda}\text{-a.e. } \gamma \qquad R^\sigma_\Phi(\gamma) \coloneqq \prod_{x\in\gamma} p^\sigma_\Phi(x) = \exp\int_M \ln\big(p^\sigma_\Phi(x)\big)\,\mathrm{d}\gamma(x)\ . \tag{4.5.3}$$

Example 4.5.11 (Normalized mixed Poisson measures). Let $\lambda \in \mathscr{P}(\mathbb{R}^+)$ be a Lévy measure with *compact* support and set $\mathbb{P} \coloneqq \mathbf{N}_\sharp \mathcal{Q}_{\lambda,\sigma}$. Assumption 4.2 (*ii*) is satisfied because of the diffuseness of σ, whence that of $\mathcal{P}_\sigma$ and, in turn, that of $\mathcal{Q}_{\lambda,\sigma}$. Assumptions 4.2 (*iv*) and 4.2 (*iii*) are respectively verified in Lemmas 4.6.5 and 4.6.6 below. In particular, the closability of the pre-Dirichlet form in (4.2.13) is obtained as a consequence of the quasi-invariance of $\mathbb{P}$. Assumption 4.2 (*i*) is verified in Lemma 4.6.7 below.

Denote now by $M^{\odot n} \coloneqq M^{\times n}/\mathfrak{S}_n$ the quotient of the n-fold cartesian product $M^{\times n}$ by the symmetric group $\mathfrak{S}_n$ acting by permutation of coördinates. Let further $M^{\times n}_\circ$ denote the set of points $\mathbf{x} \coloneqq (x_1, \dots, x_n)$ in $M^{\times n}$ such that $x_i \neq x_j$ for $i \neq j$, and set

$$M^{(n)} \coloneqq M^{\times n}_\circ/\mathfrak{S}_n\ .$$

Denote by $\mathrm{pr}^{\mathfrak{S}_n}\colon M^{\times n}_\circ \to M^{(n)}$ the quotient projection, and set $\sigma^{(n)} \coloneqq \mathrm{pr}^{\mathfrak{S}_n}_\sharp \sigma^{\otimes n}$. It is well-known that, when (M, σ) is a finite Radon measure space, then $(\Upsilon, \mathcal{P}_\sigma)$ is isomorphic, as a measure space, to the space

$$\bigoplus_{n\in\mathbb{N}_1} \big(M^{(n)}, e^{-\sigma M}\sigma^{(n)}/n!\big)\ . \tag{4.5.4}$$

More explicitly, the isomorphism is given by identifying $M^{(n)}$ with $\Upsilon^{(n)}$, the space of configurations $\gamma \in \Upsilon$ such that $\gamma M = n$. Finally, define the following subsets of $\mathscr{P}$

$$\begin{aligned} \mathbf{N}\big(\Upsilon^{(n)}\big) \subset \mathring{\Delta}^n &\coloneqq \left\{\sum_i^n s_i\delta_{x_i} \mid \mathbf{x} \in M^{(n)}, s_i \in \mathbb{R}^+\right\} \subset \mathring{\Delta}^{\mathrm{fin}} \coloneqq \bigcup_n \mathring{\Delta}^n\ , \\ \mathbf{N}\big(\ddot{\Upsilon}^{(n)}\big) \subset \Delta^n &\coloneqq \left\{\sum_i^n s_i\delta_{x_i} \mid \mathbf{x} \in M^{\times n}, s_i \in \mathbb{R}^+\right\} \subset \Delta^{\mathrm{fin}} \coloneqq \bigcup_n \Delta^n\ . \end{aligned} \tag{4.5.5}$$

Remark 4.5.12. While the support $\mathring{\Delta}^1 = \Delta^1 \cong M$ of the measure constructed in Example 4.5.4 is "small" in various senses — e.g., it is a closed nowhere dense subset of $\mathscr{P}$ —, the normalized (mixed) Poisson measures in Example 4.5.11 are fully supported. On the other hand though, even these measures are concentrated on $\mathring{\Delta}^{\mathrm{fin}}$, which may itself be still regarded as "small" — e.g., since the measure space $(\Delta^{\mathrm{fin}}, \mathbf{N}_\sharp \mathcal{Q}_{\lambda,\sigma})$ may be approximated in many senses via the sequence of compact *finite-dimensional* measure spaces $(\Delta^n, \mathbf{N}_\sharp \mathcal{Q}_{\lambda,\sigma}\big|_{\Delta^n})$.

4.5.4 The Dirichlet–Ferguson measure. Example 4.5.11 shows that the laws of (normalized) point processes on M may be examples of measures on $\mathscr{P}$ satisfying Assumption 4.2. In light of Remark 4.5.12, the question arises, whether such laws may be chosen to be concentrated on sets richer than $\mathring{\Delta}^{\mathrm{fin}}$, and in particular on the whole set of purely atomic measures.

In this section we introduce for further purposes a negative example, the *Dirichlet–Ferguson measure* over M (see below), satisfying Assumptions 4.2 (i)–4.2 (ii) and the closability of the form $(\mathcal{E}, \mathscr{D}(\mathcal{E}))$, whereas not 4.2 (iii) nor 4.2 (iv). These properties are verified in Chapter 5, basing on the characterization of the measure in Theorem 4.5.13 below.

Preliminaries. Denote by $\overline{\mathsf{m}}$ the *normalized* volume measure of M. Everywhere in the following let $\beta \in (0,\infty)$ be defined by $\mathsf{m} = \beta\overline{\mathsf{m}}$. Set further $\hat{M} := M \times I$, always endowed with the product topology, σ-algebra and with the measure $\hat{\mathsf{m}}_\beta := \overline{\mathsf{m}} \otimes \mathsf{B}_\beta$, where

$$\mathrm{d}\mathsf{B}_\beta(r) := \beta(1-r)^{\beta-1}\,\mathrm{d}r$$

is the Beta distribution on I with parameters 1 and β.

The Dirichlet–Ferguson measure. We denote by $\mathcal{D}_\mathsf{m}$ the Dirichlet–Ferguson measure [55] over $(M, \mathcal{B})$ with intensity m. The measure is also known as: *Dirichlet*, *Poisson–Dirichlet* [157], (law of the) *Fleming–Viot process with parent-independent mutation* [132]. The characteristic functional of $\mathcal{D}_\mathsf{m}$ may be found in Chapter 2, together with further properties of the measure. The following characterization is originally found, in the form of a distributional equation, in [145, Eqn. (3.2)].

Theorem 4.5.13 (Mecke-type identity for $\mathcal{D}_\mathsf{m}$ [145], see Chapter 3)**.** *Let $u\colon \mathscr{P} \times \hat{M} \to \mathbb{R}$ be measurable semi-bounded. Then, there exists a unique measure $\mathcal{D}_\mathsf{m}$ on $\mathscr{P}$ satisfying*

$$\iint_M u(\eta, x, \eta_x)\,\mathrm{d}\eta(x)\,\mathrm{d}\mathcal{D}_\mathsf{m}(\eta) = \iint_{\hat{M}} u(\eta +_r \delta_x, x, r)\,\mathrm{d}\hat{\mathsf{m}}_\beta(x,r)\,\mathrm{d}\mathcal{D}_\mathsf{m}(\eta)\ . \tag{4.5.6}$$

The unacquainted reader may take this result as a definition of $\mathcal{D}_\mathsf{m}$.

4.5.5 The entropic measure. In this section we recall an example showing that there exist measures on $\mathscr{P}$ — other than normalized mixed Poisson measures — satisfying Assumption 4.2.

Preliminaries. Similarly to [140, §2.2], define

$$\mathscr{G}(\mathbb{R}) := \{g\colon \mathbb{R} \to \mathbb{R},\ \text{right-cont. non-decr., s.t. } \forall x \in \mathbb{R} \quad g(x+1) = g(x) + 1\}$$

In light of the equi-variance property, each $g \in \mathscr{G}(\mathbb{R})$ uniquely induces a Borel function $\mathrm{pr}^{\mathscr{G}}(g)\colon \mathbb{S}^1 \to \mathbb{S}^1$ and we set $\mathscr{G} := \mathrm{pr}^{\mathscr{G}}(\mathscr{G}(\mathbb{R}))$, endowed with the L^2-distance

$$\|g_1 - g_2\|_{\mathscr{G}} := \left(\int_{\mathbb{S}^1} |g_1(t) - g_2(t)|^2\,\mathrm{d}\mathsf{m}(t)\right)^{1/2} .$$

Letting $\mathbb{S}^1 \cong \mathbb{R}/\mathbb{Z}$, define further for every $a \in \mathbb{S}^1$ the *translation* $\tau_a\colon \mathbb{S}^1 \to \mathbb{S}^1$ by

$$\tau_a\colon t \mapsto t + a \pmod 1\ ,$$

and define an equivalence relation $\sim$ on $\mathscr{G}$ by setting $g \sim h$ for $g, h \in \mathscr{G}$ if and only if $g = h \circ \tau_a$ for some $a \in \mathbb{S}^1$. Denote by $\mathrm{pr}^{\mathscr{G}_1}$ the quotient map of $\mathscr{G}$ modulo this equivalence relation, with values in the quotient space $\mathscr{G}_1 := \mathrm{pr}^{\mathscr{G}_1}(\mathscr{G}) = \mathscr{G}/\mathbb{S}^1$ endowed with the quotient L^2-distance

$$\|g_1 - g_2\|_{\mathscr{G}_1} := \left(\inf_{s \in \mathbb{S}^1} \int_{\mathbb{S}^1} |g_1(s) - g_2(t+s)|^2 \, \mathrm{dm}(t)\right)^{1/2} .$$

Equivalently, $\mathscr{G}_1$ is the semi-group of right-continuous non-decreasing functions on $\mathbb{S}^1 \cong [0,1)$ fixing $0 \in \mathbb{S}^1$. Finally, the space $\big(\mathscr{G}_1, \|\cdot\|_{\mathscr{G}_1}\big)$ is *isometric* ([140, Prop. 2.2]) to $\mathscr{P}_2 := \big(\mathscr{P}_2(\mathbb{S}^1), W_2\big)$ via the map

$$\chi \colon g \mapsto g_\sharp \mathsf{m} . \tag{4.5.7}$$

The conjugation map $\mathfrak{C}^{\overline{\mathsf{m}}}$ *(cf. [151, §3]).* For $\mu \in \mathscr{P}$ let $\varphi_\mu := \varphi_{\overline{\mathsf{m}} \to \mu}$ be given by Theorem 4.3.8 (recall that $\overline{\mathsf{m}} \in \mathscr{P}^{\mathrm{reg}}$). The *conjugation* map $\mathfrak{C}^{\overline{\mathsf{m}}} \colon \mathscr{P} \to \mathscr{P}$ is defined by

$$\mathfrak{C}^{\overline{\mathsf{m}}} \colon \mu \mapsto \big(\exp \nabla(\varphi_\mu^{\mathsf{c}})\big)_\sharp \overline{\mathsf{m}} .$$

It was shown in [151, Thm. 3.6] that $\mathfrak{C}^{\overline{\mathsf{m}}}$ is an involutive homeomorphism of $\mathscr{P}_2$. If $M = \mathbb{S}^1$, then the conjugation map may be alternatively defined in the following equivalent way. Let

$$g_\mu(t) := \inf \{s \in I \mid \mu[0,s] > t\}$$

(Here: conventionally, $\inf \varnothing := 1$.) denote the cumulative distribution function of $\mu \in \mathscr{P}(\mathbb{S}^1)$. Observe that $g_\mu \in \mathscr{G}_1$, hence it admits a left inverse g_μ^{-1} in $\mathscr{G}_1$ given by

$$g_\mu^{-1}(t) := \inf \{s \geq 0 \mid g(s) > t\} .$$

Then, $\mathfrak{C}^{\overline{\mathsf{m}}}(\mu) = \mathrm{d}g_\mu^{-1}$ where, for any $g \in \mathscr{G}_1$, we denoted by $\mathrm{d}g$ the Lebesgue–Stieltjes measure associated to φ (see [140] for the detailed construction).

Definition 4.5.14 (entropic measure over M [151, Dfn. 6.1])**.** The *entropic measure* $\mathbb{P}_{\mathsf{m}}$ is the Borel probability measure on $\mathscr{P}_2$ defined by $\mathbb{P}_{\mathsf{m}} := \mathfrak{C}^{\overline{\mathsf{m}}}_\sharp \mathcal{D}_{\mathsf{m}}$, where $\mathcal{D}_{\mathsf{m}}$ is the Dirichlet–Ferguson measure of §4.5.4.

Since $\mathfrak{C}^{\overline{\mathsf{m}}}$ is a homeomorphism, $\mathbb{P}_{\mathsf{m}}$ satisfies Assumptions 4.2 (*i*), 4.2 (*ii*) because so does $\mathcal{D}_{\mathsf{m}}$. The quasi-invariance of $\mathbb{P}_{\mathsf{m}}$ as in Assumption 4.2 (*iv*) and Assumption 4.2 (*iii*) (hence the closability of the Dirichlet form (4.2.13)) are a challenging problem. They have been proven in the seminal work [140] for the case $M = \mathbb{S}^1$, which leads us to the following example.

Example 4.5.15 (The entropic measure over $\mathbb{S}^1$ [140, Dfn. 3.3]). Let $\beta > 0$ be a fixed constant and let $M = \mathbb{S}^1$ be endowed with the rescaled volume measure $\mathsf{m} := \beta \mathrm{Leb}^1$. The quasi-invariance of $\mathbb{P}_{\mathsf{m}}$ — as in Assumption 4.2 (*iv*) — was proven in [140, Cor. 4.2] (in fact, it was proven for the action of the whole of $\mathrm{Diff}^2(M)$ rather than only for $\mathrm{Flow}(M)$, cf. Rmk. 4.5.3). Although not apparent, the bound (4.2.7) for the Radon–Nikodým derivative R_r^w may be deduced from the explicit computations in [140, Lem. 4.8]. In fact, Assumption 4.2 (*v*) holds too, because of [140, Lem. 5.1(ii)]. Assumption 4.2 (*iii*) holds as a consequence of 4.2 (*v*) by Proposition 4.5.6. Together with the previous discussion, this shows that $\mathbb{P}_{\mathsf{m}}$ satisfies Assumption 4.2.

The closability of the form $(\mathcal{E}, \mathscr{F}_0)$ is proven in [140, Thm. 7.25], where the family of cylinder functions $\mathfrak{Z}^\infty$ is introduced in [140, Dfn. 7.24] and denoted by $\mathfrak{Z}^\infty(\mathscr{P})$. A proof of the Rademacher property in the form of our Theorem 4.2.4 (3.*i*) is sketched in [140, Prop. 7.26].

Remark 4.5.16. Finally, let us notice that $\mathbb{P}_{\mathsf{m}}$-a.e. μ is concentrated on an m-negligible set [140, Cor. 3.11]. In fact, it is not difficult to show that $\mathbb{P}_{\mathsf{m}}$-a.e. μ is concentrated on the set of irrational points of a Cantor space, i.e. any (non-empty) totally disconnected perfect metrizable compact space.

4.5.6 An image on $\mathscr{P}$ of the Malliavin–Shavgulidze measure. As a final example, we introduce here an image on $\mathscr{P}(\mathbb{S}^1)$ of the Malliavin–Shavgulidze measure on $\mathrm{Diff}^1_+(\mathbb{S}^1)$.

Preliminaries. See e.g. [100] for a detailed exposition and further references. Let $M = \mathbb{S}^1$ with volume measure $\mathsf{m} := \mathrm{Leb}^1$ and denote by $\mathcal{C}_0(I^\circ)$ the space of continuous functions on I vanishing at both 0 and 1, endowed with the trace topology of $\mathcal{C}(I)$. Consider the space $\mathrm{Diff}^1_+(\mathbb{S}^1)$ of orientation preserving $\mathcal{C}^1$-diffeomorphisms of $\mathbb{S}^1$, endowed with the topology of uniform convergence, and let $\xi\colon \mathrm{Diff}^1_+(\mathbb{S}^1) \to \mathbb{S}^1 \times \mathcal{C}_0(I^\circ)$ be the homeomorphism defined by

$$\xi\colon g(t) \mapsto \big(g(0), \ln g'(t) - \ln g'(0)\big) \ .$$

Definition 4.5.17 (The Malliavin–Shavgulidze measure)**.** Let $\mathcal{W}_0$ be the Borel probability on $\mathcal{C}(I)$ defined as the law of the Brownian Bridge connecting 0 to 0 in time 1, concentrated on $\mathcal{C}_0(I^\circ)$. The *Malliavin–Shavgulidze measure* $\mathcal{M}$ on $\mathrm{Diff}^1_+(\mathbb{S}^1)$ [116] is the Borel probability measure defined by $\mathcal{M} := (\xi^{-1})_\sharp(\mathsf{m} \otimes \mathcal{W}_0)$.

Denote further by S the *Schwarzian derivative* operator

$$S\colon \phi \mapsto \frac{\phi'''}{\phi'} - \frac{3}{2}\left(\frac{\phi''}{\phi'}\right)^2 \ ,$$

and consider the left action $L_\phi\colon g \mapsto \phi \circ g$ of the subgroup $\mathrm{Diff}^3_+(\mathbb{S}^1)$. The measure $\mathcal{M}$ is quasi-invariant with respect to L_ϕ and the following quasi-invariance formula holds true (see e.g. [116]) for every Borel $A \subset \mathrm{Diff}^1_+(\mathbb{S}^1)$

$$\mathcal{M}(L_\phi(A)) = \int_A \exp\left[\int_{\mathbb{S}^1} S(\phi)(g(t)) \cdot g'(t)^2 \,\mathrm{d}\mathsf{m}(t)\right] \mathrm{d}\mathcal{M}(g) \ . \tag{4.5.8}$$

The Malliavin–Shavgulidze image measure. Every $\mathcal{C}^1$-function in $\mathscr{G}$ is a $\mathcal{C}^1$-diffeomorphism of $\mathbb{S}^1$, orientation-preserving since induced by a non-decreasing function, and every such diffeomorphism arises in this way. Furthermore, $\mathrm{Diff}^1_+(\mathbb{S}^1)$ embeds continuously into $\mathscr{G}$. It follows that $\mathcal{M}$ may be regarded as a (non-relabeled) measure on $\mathscr{G}$.

Example 4.5.18 (The Malliavin–Shavgulidze image measure). Consider the Borel probability measure $\mathcal{M}$ on $\mathscr{G}$. The measure $\mathcal{M}_1 := \mathrm{pr}^{\mathscr{G}_1}_\sharp \mathcal{M}$ is a well-defined Borel probability measure on $\mathscr{G}_1$ by measurability (continuity) of $\mathrm{pr}^{\mathscr{G}_1}$. The *Malliavin–Shavgulidze image measure* $\mathcal{S}$ is the Borel probability measure on $\mathscr{P}$ defined by

$$\mathcal{S} := \chi_\sharp(\mathrm{pr}^{\mathscr{G}_1}_\sharp \mathcal{M}) \ .$$

Assumptions 4.2 (i) for $\mathcal{S}$ is readily verified from the properties of the Malliavian–Shavgulidze measure $\mathcal{M}$. In fact, $\mathcal{S}$ is concentrated on the set

$$\big(\mathrm{Diff}^1_+(\mathbb{S}^1)/\mathrm{Isom}_+(\mathbb{S}^1)\big)_\sharp \mathsf{m} \subset \mathscr{P}^{\mathsf{m}}(\mathbb{S}^1) \ .$$

Assumption 4.2 (v) is verified in Lemma 4.6.8 below, which suffices to establish Assumption 4.2 by Proposition 4.5.6.

Remark 4.5.19. Examples 4.5.11, 4.5.15 and 4.5.18 clarify that Assumption 4.2 poses no restriction to the subset of $\mathscr{P}$ where $\mathbb{P}$ is concentrated. Indeed, as argued above

- $\mathbf{N}_\sharp \mathcal{Q}_{\lambda,\sigma}$-a.e. $\mu \in \mathscr{P}_2(\mathbb{S}^1)$ is *purely atomic*;
- $\mathbb{P}_{\mathsf{m}}$-a.e. $\mu \in \mathscr{P}_2(\mathbb{S}^1)$ is *singular continuous* (w.r.t. the volume measure of $\mathbb{S}^1$);
- $\mathcal{S}$-a.e. $\mu \in \mathscr{P}_2(\mathbb{S}^1)$ is *absolutely continuous* (w.r.t. the volume measure of $\mathbb{S}^1$).

Furthermore, it is readily seen that, if $\mathbb{P}$ and $\mathbb{P}'$ both satisfy Assumption 4.2, then so does any convex combination thereof. Thus, it is possible to construct a measure $\mathbb{P}$ on $\mathscr{P}_2(\mathbb{S}^1)$ such that $\mathbb{P}$-a.e. μ has Lebesgue decomposition consisting of both a singular, a singular continuous and an absolutely continuous part.

4.6 Appendix

4.6.1 On the notion of tangent bundle to $\mathscr{P}_2$. The concept of '*tangent space*' to $\mathscr{P}_2$ at a point μ or '*space of directions*' through μ has been widely investigated. (See [11, 60, 63, 64] and, especially, the bibliographical notes [10, §6.4].) At least the following three different notions are available

- the *tangent space* $T_\mu^\nabla \mathscr{P}_2 \coloneqq \mathrm{cl}_{\mathfrak{X}_\mu} \mathfrak{X}_\nabla^\infty$;
- the *geometric tangent space*, denoted here by $\mathbf{T}_\mu \mathscr{P}_2$, defined in [63, Dfn. 5.4];
- the *pseudo-tangent space*, denoted here by $T_\mu^{\mathrm{Der}} \mathscr{P}_2 \coloneqq \mathfrak{X}_\mu$, considered as auxiliary space in [29, 60].

It was proven in [63, Prop.s 6.1, 6.3] (cf. [10, Thm. 6.1]) that $T_\mu^\nabla \mathscr{P}_2 \cong \mathbf{T}_\mu \mathscr{P}_2$ if and only if $\mu \in \mathscr{P}^{\mathrm{reg}}$; if otherwise, then $T_\mu^\nabla \mathscr{P}_2$ embeds canonically non-surjectively in $\mathbf{T}_\mu \mathscr{P}_2$ and the latter is not a Hilbert space. The relation between $T_\mu^\nabla \mathscr{P}_2$ and $T_\mu^{\mathrm{Der}} \mathscr{P}_2$ is made explicit in the following.

Preliminaries. By a *Fréchet space* we mean a locally convex completely metrizable topological vector space. In this section, we endow $\mathcal{C}^\infty(M)$ with its usual Fréchet topology $\tau_{\mathcal{C}^\infty(M)}$ and denote by $\mathcal{C}^\infty(M)^*$ the topological dual $\left(\mathcal{C}^\infty(M), \tau_{\mathcal{C}^\infty(M)}\right)^*$ endowed with the weak* topology (see e.g. [155, §1.9]). Analogously, we endow $\mathfrak{Z}^\infty$ with the locally convex metrizable linear topology $\tau_{\mathfrak{Z}^\infty}$ induced by the countable family of semi-norms

$$\left|u\right|_k \coloneqq \sup_{w_1,\dots,w_k \in \mathfrak{X}^\infty} \left\|(\boldsymbol{\nabla}_{w_1} \circ \cdots \circ \boldsymbol{\nabla}_{w_k})\, u\right\|_{\mathcal{C}(\mathscr{P}_2)} \ , \qquad k \in \mathbb{N}_0 \ ,$$

where it is understood that $\left|u\right|_0$ is but the uniform norm on $\mathcal{C}(\mathscr{P}_2)$. We denote by $\mathfrak{Z}^{\infty *}$ the topological dual of $(\mathfrak{Z}^\infty, \tau_{\mathfrak{Z}^\infty})$, endowed with the weak* topology.

Divergence operator (cf. [60, §2.3]). The *divergence operator* $\mathrm{div}_\mu \colon \mathfrak{X}^\infty \to \mathcal{C}^\infty(M)^*$ mapping

$$w \mapsto \left(\langle \mathrm{div}_\mu w \,|\, \cdot \,\rangle : f \mapsto -\int_M (\mathrm{d}f(w))(x) \,\mathrm{d}\mu(x) \right)$$

satisfies

$$\langle \mathrm{div}_\mu w \,|\, f \rangle \leq \left\|\nabla f\right\|_{\mathfrak{X}_\mu} \left\|w\right\|_{\mathfrak{X}_\mu} \ , \tag{4.6.1}$$

hence it extends by continuity to a (non-relabeled) operator $\mathrm{div}_\mu \colon T_\mu^{\mathrm{Der}} \mathscr{P}_2 \to \mathcal{C}^\infty(M)^*$ and one has (e.g. [60, Rmk. 2.7])

$$T_\mu^{\mathrm{Der}} \mathscr{P}_2 = T_\mu^\nabla \mathscr{P}_2 \oplus \ker \mathrm{div}_\mu \ , \tag{4.6.2}$$

where the symbol $\oplus$ denotes the orthogonal direct sum of Hilbert spaces.

On the one hand, it is clear that, if $\mu \in \mathscr{P}^{\infty,\times}$, then $\ker \mathrm{div}_\mu$ is non-trivial as soon as $\mathfrak{X}^\infty \neq \mathfrak{X}_\nabla^\infty$. This holds in particular if (M, g) has non-trivial de Rham cohomology group $H^1_{\mathrm{dR}}(M;\mathbb{R})$. On the other hand (cf. [60, Example 2.8]), if $\eta \in \mathscr{P}$ has finite support, then

$$T_\eta^\nabla \mathscr{P}_2 = T_\eta^{\mathrm{Der}} \mathscr{P}_2 = \bigoplus_{x \in \mathrm{ptws}\,\eta} \left(T_x M, \eta_x \cdot \mathsf{g}_x\right) \ . \tag{4.6.3}$$

Local derivations. Motivated by the definition, for finite-dimensional differentiable manifolds, of *space of derivatives at a point* (or *pointwise derivations*) (see e.g. [32, Cor. 2.2.22]), we define for fixed $\mu \in \mathscr{P}$ the linear functional $\partial^\mu_w \colon \mathfrak{Z}^\infty \to \mathbb{R}$ by

$$\partial^\mu_w \colon u \mapsto \langle \boldsymbol{\nabla}\, u(\mu) \,|\, w \rangle_{\mathfrak{X}_\mu} \quad .$$

Letting $\mathrm{ev}_\mu \colon \mathfrak{Z}^\infty \to \mathbb{R}$ be defined by $\mathrm{ev}_\mu(u) := u(\mu)$, it is readily verified (cf. Lem 4.6.2 below) that ∂^μ_w satisfies Leibniz rule in the form

$$\partial^\mu_w(uv) = \mathrm{ev}_\mu(v)\, \partial^\mu_w u + \mathrm{ev}_\mu(u)\, \partial^\mu_w v \ . \tag{4.6.4}$$

We denote by $\mathrm{Der}(\mathfrak{Z}^\infty)_\mu \subset \mathfrak{Z}^{\infty *}$ the space of continuous linear functionals on $\mathfrak{Z}^\infty$ satisfying (4.6.4), endowed with the trace topology. Since $\mathfrak{Z}^{\infty *}$ is Hausdorff and complete, the (uniformly) continuous linear operator $\partial^\mu_\cdot \colon \mathfrak{X}^\infty \to \mathfrak{Z}^{\infty *}$ extends to a uniquely-defined non-relabeled operator $\partial^\mu_\cdot \colon \mathfrak{X}_\mu \to \mathfrak{Z}^{\infty *}$ by [156, §I.5, Thm. 5.1, p. 39]. Moreover,

$$\partial^\mu_w(u) \le \|\boldsymbol{\nabla}\, u(\mu)\|_{\mathfrak{X}_\mu} \|w\|_{\mathfrak{X}_\mu} \ , \tag{4.6.5}$$

hence one has in fact $\partial^\mu_w \in \mathrm{Der}(\mathfrak{Z}^\infty)_\mu$ for every $w \in \mathfrak{X}_\mu$.

Proposition 4.6.1. *Denote by* $\mathsf{j} \colon f \mapsto f^*$ *the canonical injection* $\mathcal{C}(M) \to \mathcal{C}(M)^*$.
Then, $\mathrm{div}_\mu(\,\cdot\,) = -\partial^\mu_\cdot \circ \mathsf{j} \colon \mathfrak{X}_\mu \to \mathcal{C}^\infty(M)^*$ *and* $\ker \mathrm{div}_\mu \cong \ker \partial^\mu_\cdot \subset \mathfrak{X}_\mu$ *as Hilbert spaces.*

Proof. For any $f \in \mathcal{C}^\infty(M)$ and $w \in \mathfrak{X}^\infty$ it holds that

$$(\partial^\mu_w \circ \mathsf{j})(f) = \partial^\mu_w(f^*) = \int_M \langle \nabla f_x \,|\, w_x \rangle_{\mathsf{g}} \,\mathrm{d}\mu(x) = \mu(\mathrm{d}f(w)) = -\langle \mathrm{div}_\mu w \,|\, f \rangle \ , \tag{4.6.6}$$

that is $\mathrm{div}_\mu(\,\cdot\,)(\cdot) = -\partial^\mu_\cdot(\mathsf{j}(\cdot))$ on $\mathfrak{X}^\infty \otimes \mathcal{C}^\infty(M)$. By (4.6.5) applied to $u = f^* \in \mathfrak{Z}^\infty$, the operator $\partial^\mu_\cdot \circ \mathsf{j} \colon \mathfrak{X}^\infty \to \mathcal{C}^\infty(M)^*$ may be extended to a uniquely defined non-relabeled operator $\partial^\mu_\cdot \circ \mathsf{j} \colon \mathfrak{X}_\mu \to \mathcal{C}^\infty(M)^*$ and the notation is consistent in the sense that this operator coincides with the previously defined extension of $\partial^\mu_\cdot$ applied to j. Since both $\partial^\mu_\cdot \circ \mathsf{j}$ and $-\mathrm{div}_\mu(\,\cdot\,)$ are linear and $\|\cdot\|_{\mathfrak{X}_\mu}$-continuous and coincide on the dense set $\mathfrak{X}^\infty \subset \mathfrak{X}_\mu$, they coincide on the whole space $\mathfrak{X}_\mu$. It remains to show that $\ker \partial^\mu_\cdot = \ker \partial^\mu_\cdot \circ \mathsf{j}$, which follows immediately by noticing that for any $u = F \circ \mathbf{f}^* \in \mathfrak{Z}^\infty$ and $w \in \mathfrak{X}_\mu$

$$\begin{aligned}\partial^\mu_w(u) &= \sum_i^k (\partial_i F)(\mathbf{f}^*\mu) \int_M \langle \nabla_x f_i \,|\, w_x \rangle_{\mathsf{g}} \,\mathrm{d}\mu(x) = -\sum_i^k (\partial_i F)(\mathbf{f}^*\mu)\, \langle \mathrm{div}_\mu w \,|\, f_i \rangle \\ &= \sum_i^k (\partial_i F)(\mathbf{f}^*\mu)(\partial^\mu_w \circ \mathsf{j})(f_i) \ .\end{aligned}$$

This concludes the proof. ■

Tangent bundles. Let us denote by $T^\nabla \mathscr{P}_2$ the *tangent bundle* to $\mathscr{P}_2$, set-wise defined as the disjoint union of $T^\nabla_\mu \mathscr{P}_2$ varying $\mu \in \mathscr{P}_2$. The *pseudo-tangent bundle* $T^{\mathrm{Der}} \mathscr{P}_2$ is analogously defined. Whereas this terminology is well-established, it is clear that $T^\nabla \mathscr{P}_2$ is *not* a vector *bundle* in the standard sense — nor in any reasonable sense —, since it admits *no* local trivialization by reasons of the dimension of $T^\nabla_\mu \mathscr{P}_2$. Indeed, for any $x_0 \in M$ and every $\varepsilon > 0$ one can find a smooth function $\rho_\varepsilon \in \mathcal{C}^\infty(M)$ such that $\mu_\varepsilon := \rho_\varepsilon \overline{\mathsf{m}} \in \mathscr{P}^{\mathsf{m}}$ and $W_2(\delta_{x_0}, \mu_\varepsilon) < \varepsilon$, yet $T^\nabla_{\delta_{x_0}} \mathscr{P}_2 \cong T_{x_0} M$ while $T^\nabla_\mu \mathscr{P}_2$ is infinite-dimensional. The same is true for $T^{\mathrm{Der}} \mathscr{P}_2$.

Despite this fact, the gradient $\boldsymbol{\nabla}\, u$ of a cylinder function $u \in \mathfrak{Z}^\infty$ may well be regarded as a 'smooth *section*' of $T^\nabla \mathscr{P}_2$ since $\boldsymbol{\nabla}\, u(\mu) \in T^\nabla_\mu \mathscr{P}_2$ by (4.2.3). Again by (4.2.3) the space of all such gradients is a subspace of the space $\mathfrak{Z}^\infty \otimes_{\mathbb{R}} \mathfrak{X}^\infty_\nabla$ of $\mathfrak{Z}^\infty$-linear combinations of gradient-type vector fields. This motivates the Definition 4.2.3 of *cylinder vector fields* $\mathcal{XC}^\infty := \mathfrak{Z}^\infty \otimes_{\mathbb{R}} \mathfrak{X}^\infty$, henceforth regarded — in analogy to the case of finite-dimensional manifolds — as (a subspace of) the space of 'smooth sections' of the tangent bundle $T^{\mathrm{Der}}\mathscr{P}_2$. In spite of Proposition 4.6.1, the fiber-bundle $T^{\mathrm{Der}}\mathscr{P}_2$ does in fact convey more information than the fiber-bundle $T^\nabla \mathscr{P}_2$.

Global derivations. Consider the space $\mathrm{Der}(\mathfrak{Z}^\infty)$ of abstract $\mathbb{R}$-derivations of $\mathfrak{Z}^\infty$.

Lemma 4.6.2. *Let $w \in \mathfrak{X}^\infty$. Then, the map*

$$\partial_w \colon u \mapsto \mathrm{d}_t\big|_{t=0}(u \circ \Psi^{w,t}) = \langle \boldsymbol{\nabla}\, u \,|\, w \rangle_{\mathfrak{X}}\,.$$

is an element of $\mathrm{Der}(\mathfrak{Z}^\infty)$.

Proof. One has

$$\begin{aligned}
\mathrm{d}_t\big|_{t=0} u(\Psi^{w,t}\mu) &= \sum_i^k (\partial_i F)(\mathbf{f}^* \psi_\sharp^{w,0}\mu) \times \mathrm{d}_t\big|_{t=0} f_i^* \Psi^{w,t}\mu = \sum_i^k (\partial_i F)(\mathbf{f}^*\mu) \times \mathrm{d}_t\big|_{t=0} \mu(f_i \circ \psi^{w,t}) \\
&= \sum_i^k (\partial_i F)(\mathbf{f}^*\mu) \times \mu\big(\mathrm{d}_t\big|_{t=0}(f_i \circ \psi^{w,t})\big) = \sum_i^k (\partial_i F)(\mathbf{f}^*\mu) \times \mu \,\langle \nabla f_i \,|\, w\rangle_{\mathsf{g}} \\
&= \sum_i^k (\partial_i F)(\mathbf{f}^*\mu) \times \langle \nabla f_i \,|\, w\rangle_{\mathsf{g}}^* \,\mu \qquad (4.6.7) \\
&= \langle \boldsymbol{\nabla}\, u(\mu) \,|\, w\rangle_{\mathfrak{X}_\mu} \quad . \qquad (4.6.8)
\end{aligned}$$

Since $\langle \nabla f_i \,|\, w\rangle_{\mathsf{g}} \in \mathcal{C}^\infty(M)$ by the choice of f_i and w, and since $\mathfrak{Z}^\infty$ is an algebra, (4.6.7) shows that $\partial_w \colon \mathfrak{Z}^\infty \to \mathfrak{Z}^\infty$. The Leibniz rule is straightforward from the same property of d_t, while $\mathfrak{Z}^\infty$-linearity is a consequence of the representation in (4.6.8). ∎

Proposition 4.6.3. *Let $W \in \mathcal{XC}^\infty$ be as in* (4.2.8)*. Then, the map*

$$\partial \colon W \mapsto \partial_W := \sum_j^n v_j \partial_{w_j}$$

is a linear injection $\partial \colon \mathcal{XC}^\infty \to \mathrm{Der}(\mathfrak{Z}^\infty)$.

Proof. The fact that $\partial_W \in \mathrm{Der}(\mathfrak{Z}^\infty)$ is a consequence of Lemma 4.6.2 and of the choice of the v_j's. The $\mathfrak{Z}^\infty$-linearity is immediate, while the $\mathfrak{X}^\infty$-linearity follows from (4.6.8).

Let now $W \neq \mathbf{0}_{\mathcal{XC}^\infty}$, that is, there exists $\mu_0 \in \mathscr{P}$ and $x_0 \in M$ such that $W(\mu_0)(x_0) \neq \mathbf{0}_{T_{x_0}M}$. Since $W(\,\cdot\,)(x_0)$ is continuous and the set of purely atomic finitely supported probability measures is dense in $\mathscr{P}_2$, (see e.g. the proof of [165, Thm. 6.18]) we can find a purely atomic finitely supported $\eta \in \mathscr{P}$ such that $W(\eta)(x_0) \neq \mathbf{0}_{T_{x_0}M}$. Without loss of generality, up to choosing $\eta' := \eta +_\varepsilon \delta_{x_0}$ for some small $\varepsilon > 0$, we can assume $\eta_{x_0} > 0$ (for the notation see (4.3.1)). By standard arguments, there exists $f \in \mathcal{C}^\infty(M)$ such that $\nabla f_{x_0} = W(\eta)(x_0)$. Moreover, since $\mathrm{ptws}\,\eta$ is discrete (finite), we can find $g \in \mathcal{C}^\infty(M)$ such that $g \equiv \mathbb{1}$ on an open neighborhood of x_0 and $g \equiv \mathbf{0}$ on an open neighborhood of every point in $\mathrm{ptws}\,\eta$ other than x_0. Set $h = fg$ and notice that $\nabla h_{x_0} = W(\eta)(x_0)$ while $\nabla h = \mathbf{0}$ for every point in $\mathrm{ptws}\,\eta$ other than x_0. Now,

$$\partial_W(h^*)(\eta) = \langle \boldsymbol{\nabla}\, h^* \,|\, W(\eta)\rangle_{\mathfrak{X}_\eta} = \int_M \langle \nabla h_x \,|\, W(\eta)(x)\rangle_{\mathsf{g}} \,\mathrm{d}\eta(x) = \eta_{x_0}\, |W(\eta)(x_0)|_{\mathsf{g}}^2 > 0 \;.$$

Since ∂ is linear, this shows that it is also injective, which concludes the proof. ∎

Remark 4.6.4. Proposition 4.6.3 is motivated by the analogy (e.g. [32, Prop. 3.5.3]) with finite-dimensional compact differentiable manifolds, where the map

$$\partial\colon \mathfrak{X}^\infty \ni w \longmapsto \big(\partial_w\colon f \mapsto \mathrm{d}f(w)\big) \in \mathrm{Der}(\mathcal{C}^\infty(M))$$

is straightforwardly injective, and surjective because of the classical Hadamard Lemma. In the case of $\mathscr{P}_2$, I do not know whether ∂ is surjective, however, it should be noted that, in the case of infinite-dimensional smooth manifolds, this is not necessarily the case, again already at the pointwise scale. (Cf. [98, Thm. 28.7], for a proof of surjectivity under additional assumptions.)

Throughout all computations in Section 4.4, vector fields $w \in \mathfrak{X}^\infty$ ought to be interpreted as 'smooth directions' at every point $\mu \in \mathscr{P}$. This is the right notion to be compared with the definition of directional derivative given in (4.2.4) in light of Proposition 4.6.3.

4.6.2 Auxiliary results on normalized mixed Poisson measures.

Lemma 4.6.5. *The measure $\mathbb{P}$ defined in Example 4.5.11 satisfies Assumption* 4.2 (*iv*).

Proof. Retain the notation in §4.5. By (4.5.1) and combining (4.5.2) with (4.5.3),

$$\begin{aligned}\mathrm{d}\big(\Phi_\sharp\mathbb{P}\big)(\mu) =& \,\mathrm{d}\left(\int_{\mathbb{R}^+} \mathbf{N}_\sharp \Phi_\sharp \mathcal{P}_{s\cdot\sigma}(\,\cdot\,)\,\mathrm{d}\lambda(s)\right)(\mu)\\ =&\,\mathrm{d}\left(\int_{\mathbb{R}^+} \exp\big(s\cdot\sigma(\mathbb{1}-p_\Phi^{s\cdot\sigma})\big)\cdot \mathbf{N}_\sharp\big(R_\Phi^{s\cdot\sigma}\cdot\mathcal{P}_{s\cdot\sigma}\big)(\,\cdot\,)\,\mathrm{d}\lambda(s)\right)(\mu)\ .\end{aligned}$$

Noticing further that $\mathbf{N}$ is injective on $\ddot{\Upsilon}$ and denoting by $\mathbf{N}^{-1}$ its right-inverse, the function $R_\Phi^{s\cdot\sigma}\circ\mathbf{N}^{-1}$ is well-defined on $\ddot{\Upsilon}$, hence $\mathcal{P}_{s\cdot\sigma}$-a.e. on Υ. It follows that

$$\mathrm{d}\big(\Phi_\sharp\mathbb{P}\big)(\mu) = \int_{\mathbb{R}^+}\Big(\exp\big(s\cdot\sigma(\mathbb{1}-p_\Phi^{s\cdot\sigma})\big)\cdot\big(R_\Phi^{s\cdot\sigma}\circ\mathbf{N}^{-1}\big)(\mu)\cdot\mathrm{d}\left(\mathbf{N}_\sharp\mathcal{P}_{s\cdot\sigma}\right)(\mu)\Big)\,\mathrm{d}\lambda(s)\ .$$

Moreover, $p_\Phi^\sigma = p_\Phi^{s\cdot\sigma}$ for every $s>0$ by definition (cf. (4.5.2)), thus $R_\Phi^\sigma = R_\Phi^{s\cdot\sigma}$ and

$$\begin{aligned}\mathrm{d}\big(\Phi_\sharp\mathbb{P}\big)(\mu) =& \big(R_\Phi^\sigma\circ\mathbf{N}^{-1}\big)(\mu)\int_{\mathbb{R}^+}\Big(\exp\big(s\cdot\sigma(\mathbb{1}-p_\Phi^\sigma)\big)\cdot\mathrm{d}\left(\mathbf{N}_\sharp\mathcal{P}_{s\cdot\sigma}\right)(\mu)\Big)\,\mathrm{d}\lambda(s)\\ =&\big(R_\Phi^\sigma\circ\mathbf{N}^{-1}\big)(\mu)\cdot\mathrm{d}\mathbf{N}_\sharp\left(\int_{\mathbb{R}^+}\exp\big(s\cdot\sigma(\mathbb{1}-p_\Phi^\sigma)\big)\cdot\mathcal{P}_{s\cdot\sigma}(\,\cdot\,)\,\mathrm{d}\lambda(s)\right)(\mu)\ ,\end{aligned}\tag{4.6.9}$$

where it is possible to pull $\mathbf{N}_\sharp$ outside the integral sign since the integrand does not depend on μ.

Finally, for every measurable $A\subset\mathscr{P}$,

$$e^{-c_{\lambda,\sigma,\Phi}}\big(R_\Phi^\sigma\circ\mathbf{N}^{-1}\cdot\mathbb{P}\big)A \le \big(\Phi_\sharp\mathbb{P}\big)A \le e^{c_{\lambda,\sigma,\Phi}}\big(R_\Phi^\sigma\circ\mathbf{N}^{-1}\cdot\mathbb{P}\big)A\ ,\tag{4.6.10}$$

where $c_{\lambda,\sigma,\Phi} :=(\sup\operatorname{supp}\lambda)\,\big|\sigma(\mathbb{1}-p_\Phi^\sigma)\big|$. Since $R_\Phi^\sigma(\gamma)>0$ for $\mathcal{Q}_{\lambda,\sigma}$-a.e. $\gamma\in\Upsilon$, it follows from (4.6.10) that $\mathbb{P}$ and $\Phi_\sharp\mathbb{P}$ are mutually absolutely continuous, hence the quasi-invariance in 4.2 (*iv*) holds. Letting $w\in\mathfrak{X}^\infty$, equation (4.2.7) is similarly verified since $\#\operatorname{supp}\mu<\infty$ for $\mathbb{P}$-a.e. μ, hence, for all $t\in\mathbb{R}$,

$$\big(R_{\psi^{w,t}}^\sigma\circ\mathbf{N}^{-1}\big)(\mu)\ge\prod_{x\in\mu}\frac{(\psi^{w,t})^*\rho(x)}{\rho(x)}J_{\psi^{w,t}}^{\mathsf{m}}(x)\ge\left(\min_M\frac{(\psi^{w,t})^*\rho}{\rho}J_{\psi^{w,t}}^{\mathsf{m}}\right)^{\#\operatorname{supp}\mu}>0\ .$$

This concludes the proof. ■

Lemma 4.6.6. *The measure $\mathbb{P}$ defined in Example 4.5.11 satisfies Assumption* 4.2 (*iii*).

Proof. We show the assertion when $\lambda=\delta_1$, i.e. when $\mathbb{P}=\mathbf{N}_\sharp\mathcal{P}_\sigma$, similarly to [7, Thm. 3.1]. The general case is readily proved by integration w.r.t. λ in light of the mutual absolute continuity of $\mathcal{P}_{s\cdot\sigma}$ w.r.t. $\mathcal{P}_\sigma$ (hence of their normalizations) for every choice of $s>0$ (hence λ-a.e., cf. (4.5.4)).

Preliminaries. Retain the notation established in §4.5 and denote by $\beta^\sigma := \nabla\rho/\rho$ the logarithmic derivative of σ, which is well-defined on M since $\rho \in \mathcal{C}^1(M;\mathbb{R}^+)$. Let further $w \in \mathfrak{X}^\infty$ and set

$$\beta^\sigma_w(x) := \langle \beta^\sigma_x \mid w_x\rangle_{\mathsf{g}_x} + \mathrm{div}_{\mathsf{m}} w_x \ ,$$

where $\mathrm{div}_{\mathsf{m}}$ denotes the divergence on M with respect to the volume measure m. By integration by parts (cf. e.g. [7, Eq. (3.11)]) one can readily show that

$$\nabla^*_w = -\nabla_w - \beta^\sigma_w \ ,$$

where ∇^*_w denotes the adjoint of ∇_w in $L^2_\sigma(M)$ and we denote the closure of ∇_w again by the same symbol.

Claim. Letting $B^\sigma_w := (\beta^\sigma_w)^*$, we claim that $\boldsymbol{\nabla}^*_w := -\boldsymbol{\nabla}_w - B^\sigma_w$ satisfies (4.2.6) *for our choice of $\mathbb{P}$.*

Some differentiation. For all $u \in \mathfrak{Z}^\infty$ denote by the same symbol the natural extension of u to $\mathscr{M}^+_b$. By [7, Prop. 2.1] and several applications of (4.5.1) we have

$$\int u \circ \Psi^{w,t} \cdot v \,\mathrm{d}\mathbf{N}_\sharp \mathcal{P}_\sigma = \int u \cdot v \circ \Psi^{w,-t} \,\mathrm{d}\mathbf{N}_\sharp \mathcal{P}_{\Psi^{w,t}\sigma} \ . \tag{4.6.11}$$

Differentiating the l.h.s. of (4.6.11) under the sign of integral with respect to t yields the l.h.s. of (4.2.6) by (4.2.4). Moreover, letting $\lambda = \delta_1$ in (4.6.9) yields

$$\int u \cdot v \circ \Psi^{w,-t} \,\mathrm{d}\mathbf{N}_\sharp \mathcal{P}_{\Psi^{w,t}\sigma} = \int \exp\left(\sigma(\mathbb{1} - p^\sigma_{\psi^{w,t}})\right) \cdot R^\sigma_{\psi^{w,t}} \circ \mathbf{N}^{-1} \cdot u \cdot v \circ \Psi^{w,-t} \,\mathrm{d}\mathbf{N}_\sharp \mathcal{P}_\sigma \ , \tag{4.6.12}$$

where $R^\sigma_{\psi^{w,t}} \circ \mathbf{N}^{-1}$ is well-defined as in Lemma 4.6.5. Also, with obvious meaning of the notation $x \in \mu$ for $\mu \in \mathbf{N}(\ddot{\Upsilon})$,

$$\begin{aligned}\mathrm{d}_t\big|_{t=0} &\left(\exp\left(\sigma(\mathbb{1} - p^\sigma_{\psi^{w,t}})\right) \cdot R^\sigma_{\psi^{w,t}} \circ \mathbf{N}^{-1}\right)(\mu) = \\ &= \mathrm{d}_t\big|_{t=0} \prod_{x\in\mu} \frac{(\psi^{w,t})^*\rho(x)}{\rho(x)} J^{\mathsf{m}}_{\psi^{w,t}}(x) + \mathrm{d}_t\big|_{t=0} \exp\left[\int_M \left(\mathbb{1} - \frac{(\psi^{w,t})^*\rho(x)}{\rho(x)} J^{\mathsf{m}}_{\psi^{w,t}}(x)\right) \mathrm{d}\sigma(x)\right] \ .\end{aligned}$$

Now the arguments in the proof of [7, Thm. 3.1] apply verbatim, yielding

$$\mathrm{d}_t\big|_{t=0} \left(\exp\left(\sigma(\mathbb{1} - p^\sigma_{\psi^{w,t}})\right) \cdot R^\sigma_{\psi^{w,t}} \circ \mathbf{N}^{-1}\right)(\mu) = -B^\sigma_w \ . \tag{4.6.13}$$

Proof of the claim. Finally, differentiating $v \circ \Psi^{w,-t}$ with respect to t yields $-\boldsymbol{\nabla}_w v$, again by (4.2.4). Combing this fact with (4.6.13), the derivative under integral sign with respect to t of the r.h.s. of (4.6.12) reads $\int u \cdot (-\boldsymbol{\nabla}_w v - B^\sigma_w) \,\mathrm{d}\mathbb{P}$, which proves the claim. ■

Lemma 4.6.7. *The measure $\mathbb{P}$ defined in Example 4.5.11 satisfies Assumption* 4.2 (*i*).

Proof. By definition, $\mathbb{P}$ is concentrated on the set $\mathbf{N}(\Upsilon)$, which is dense in $\mathscr{P}$. (See e.g. [165, Thm. 6.18].) Let $U \neq \varnothing$ be open in $\mathscr{P}_2$. Then $U \cap \mathbf{N}(\Upsilon) \neq \varnothing$ by density of $\mathbf{N}(\Upsilon)$ in $\mathscr{P}_2$. By continuity of $\mathbf{N}$ the set $\widetilde{U} := \mathbf{N}^{-1}(U) \cap \Upsilon = \mathbf{N}^{-1}(U \cap \mathbf{N}(\Upsilon)) \neq \varnothing$ is open in Υ. Since $\mathcal{Q}_{\sigma,\lambda}$ is fully supported on Υ (cf. [142, Prop. 5.6]), then $\mathbb{P}U = \mathcal{Q}_{\sigma,\lambda}\widetilde{U} > 0$. ■

4.6.3 Auxiliary results on the Malliavin–Shavgulidze image measure.

Lemma 4.6.8. *The measure $\mathcal{S}$ defined in Example 4.5.18 satisfies Assumption* 4.2 (*v*)*.*

Proof. Retain all notation from §4.5.6. It follows from (4.5.8) that $(L_{\tau_a})_\sharp\mathcal{M} = \mathcal{M}$ for every $a \in \mathbb{S}^1$, hence $\mathcal{M}_1$ is quasi-invariant with respect to the left action of $\mathrm{Diff}^\infty_+(\mathbb{S}^1)$ (in fact: of $\mathrm{Diff}^3_+(\mathbb{S}^1)$) on $\mathscr{G}_1$ given by post-composition, i.e. (4.5.8) holds true with $\mathcal{M}_1$ in place of $\mathcal{M}$ for every Borel $A \subset \mathscr{G}_1$ and every ϕ in $\mathrm{Diff}^\infty_+(\mathbb{S}^1)$.

By definition of χ (eq. (4.5.7)), for every ϕ and Φ as in the beginning of §4.5, it holds that

$$\chi(L_{\phi^{-1}}(g)) = (\phi^{-1} \circ g)_\sharp \mathsf{m} = \phi_\sharp^{-1}(g_\sharp \mathsf{m}) = \phi_\sharp^{-1}\chi(g) = \Phi^{-1}(\chi(g)) \ .$$

As a consequence, for $\mu = \chi(g)$,

$$\begin{aligned}\mathrm{d}\Phi_\sharp\mathcal{S}(\mu) =&\, \mathrm{d}\mathcal{S}(\Phi^{-1}(\chi(g))) = \mathrm{d}\mathcal{S}(\chi(L_{\phi^{-1}}(g))) = \mathrm{d}\mathcal{M}_1(L_{\phi^{-1}}(g)) = R_\phi(g)\cdot \mathrm{d}\mathcal{M}_1(g)\\ =&(R_\phi \circ \chi^{-1})(\mu)\cdot \mathrm{d}\mathcal{S}(\mu)\end{aligned}$$

where $R_\phi(g)$ is the Radon–Nikodým derivative

$$R_\phi(g) \coloneqq \exp\left[\int_{\mathbb{S}^1} S(\phi^{-1})(g(r))\cdot g'(r)^2\,\mathrm{d}\mathsf{m}(r)\right] \ .$$

The conclusion straightforwardly follows from the form of R_ϕ. ∎

Chapter 5
The Dirichlet–Ferguson Diffusion

In this Chapter construct a recurrent diffusion process with values in the space of probability measures over a closed Riemannian manifold of arbitrary dimension. The process is associated with the Dirichlet energy integral defined by integration of the L^2-Wasserstein gradient w.r.t. the Dirichlet–Ferguson measure. Together with two different constructions of the process, we discuss its ergodicity, invariant sets and finite-dimensional approximations.

5.1 Introduction

We provide two constructions of a Markov diffusion $\eta_\bullet$ with values in the space of probability measures $\mathscr{P}$ over a closed Riemannian manifold M of dimension $d \geq 2$.

On the one hand, combining results by Bendikov–Saloff-Coste [15] and Albeverio–Daletskii–Kondratiev [5, 6] about elliptic diffusions on infinite products, we characterize $\eta_\bullet$ as the superprocess constituted by any number of independent massive Brownian particles with volatility equal to their inverse mass. Thus, we may regard $\eta_\bullet$ as a possible counterpart over M of Konarovskyi's Modified Massive Arratia Flow [91] over the unit interval. Here, no coalescence occurs by reasons of the dimension of M.

On the other hand, we show that $\eta_\bullet$ is associated with a symmetric Dirichlet form $\mathcal{E}$ on the space of real-valued functions on $\mathscr{P}$ square-integrable with respect to the Dirichlet–Ferguson random measure $\mathcal{D}$ [55]. The form $\mathcal{E}$ is defined as the closure of the Dirichlet integral induced by $\mathcal{D}$ and by the natural gradient of the L^2-Wasserstein geometry of $\mathscr{P}$, on the algebra of cylinder functions induced by smooth potential energies in the sense of Otto calculus. Thus, we may regard $\eta_\bullet$ as a possible candidate for a "Brownian motion" — that is, a canonical diffusion process — on the L^2-Wasserstein space $\mathscr{P}_2(M)$.

Among other results, we prove the following.

Main Theorem. *Let (M, g) be a closed Riemannian manifold of dimension $d \geq 2$ with volume measure m, Riemannian distance d and Laplace–Beltrami operator Δ. Let $(\mathscr{P}_2, W_2)$ be the L^2-Wasserstein space over (M, d), endowed with Otto's metric $\langle \cdot \mid \cdot \rangle_{T_\mu \mathscr{P}_2}$ and with the Dirichlet–Ferguson (probability) measure $\mathcal{D}_\mathsf{m}$ [55] with intensity measure m. Let $\widehat{\mathfrak{Z}}_0^\infty$ be the algebra of functions $u\colon \mathscr{P}_2 \to \mathbb{R}$ defined in Definition 5.4.1.*

Then, the symmetric bilinear form $(\mathcal{E}, \widehat{\mathfrak{Z}}_0^\infty)$ given by

$$\mathcal{E}(u, v) := \tfrac{1}{2} \int_{\mathscr{P}_2} \mathrm{d}\mathcal{D}_\mathsf{m}(\eta) \ \langle \boldsymbol{\nabla} u(\mu) \mid \boldsymbol{\nabla} v(\mu) \rangle_{T_\mu \mathscr{P}_2} \ , \qquad u, v \in \widehat{\mathfrak{Z}}_0^\infty$$

is closable. Its closure $(\mathcal{E}, \mathscr{D}(\mathcal{E}))$ is a regular strongly local recurrent (conservative) Dirichlet form with generator the (Friedrichs) extension of the essentially self-adjoint operator $(\mathbf{L}, \widehat{\mathfrak{Z}}_0^\infty)$ given by

$$\mathbf{L}u(\eta) := \tfrac{1}{2} \int_M \mathrm{d}\eta(x) \, \frac{\Delta^z \big|_{z=x} u(\eta + \eta\{x\}\,\delta_z - \eta\{x\}\,\delta_x)}{(\eta\{x\})^2} \qquad \text{for } \mathcal{D}_\mathsf{m}\text{-a.e. } \eta \ , \quad u \in \widehat{\mathfrak{Z}}_0^\infty \ .$$

Additionally:

- *For any W_2-Lipschitz function $u\colon \mathscr{P}_2 \to \mathbb{R}$ it holds that $u \in \mathscr{D}(\mathcal{E})$ and $\langle \boldsymbol{\nabla} u \,|\, \boldsymbol{\nabla} u\rangle_{T_\mu \mathscr{P}_2} \leq \mathrm{Lip}[u]^2$;*
- *The associated Markov kernel*

$$p_t(A_1, A_2) := \int_{A_1} \mathrm{d}\mathcal{D}_{\mathsf{m}}\, e^{-t\,\mathbf{L}}\, \mathbb{1}_{A_2} \ , \qquad A_1, A_2 \subset \mathscr{P}_2$$

satisfies the one-sided Varadhan-type upper estimate

$$\lim_{t\downarrow 0} t \,\log p_t(A_1, A_2) \leq -\tfrac{1}{2} \inf_{\mu_i \in A_i} W_2(\mu_1, \mu_2)^2 \ .$$

- *The properly associated Markov diffusion $\eta_\bullet$ is a $\mathscr{P}_2$-valued martingale solution to the stochastic partial differential equation (Cf.* (5.2.24) *and Prop. 5.2.4 below)*

$$\mathrm{d}\eta_t = \mathrm{div}\big(\sqrt{\eta_t}\ \mathrm{d}W_t\big) + \left(\tfrac{1}{2} \sum_{x \mid \eta_t\{x\}>0} \Delta\delta_x\right) \mathrm{d}t \ , \qquad t > 0$$

tested on functions in $\widehat{\mathfrak{Z}}^\infty_0$. (Here $W_\bullet$ is a cylindrical Brownian motion.)

5.2 Motivations, main results and literature comparison

Wasserstein geometry. In the last two decades, the space $\mathscr{P}$ of probability measures over a Riemannian manifold (M, g), endowed with the L^2-Kantorovich–Rubinshtein distance W_2 (5.3.13), has proven both a powerful tool and an interesting geometric object in its own right. Since the fundamental works of Y. Brenier, R. J. McCann, F. Otto, C. Villani and many others (see, e.g., [24, 117, 131, 164]), several geometric notions have been introduced, including those of geodesics, tangent space $T_\mu\mathscr{P}$ at a point $\mu \in \mathscr{P}$ and gradient $\boldsymbol{\nabla} u(\mu)$ of a scalar-valued function u at μ (see, e.g., [63, 64, 111]). Indeed, the metric space $\mathscr{P}_2 := (\mathscr{P}, W_2)$ may — to some extent — be regarded as a kind of infinite-dimensional Riemannian manifold. Furthermore, provided that (M, g) be a closed manifold with non-negative sectional curvature, $\mathscr{P}_2$ has non-negative lower curvature bound in the sense of Alexandrov [10, Thm. 2.20].

Volume measures on $\mathscr{P}_2$. The question of the existence of a *Riemannian* volume measure on $\mathscr{P}_2$, say $\mathrm{dvol}_{\mathscr{P}_2}$, has been insistently posed and remains to date not fully answered. A first natural requirement that one might ask of such a measure — if any — is an *integration-by-parts* formula for the gradient, which would imply the closability of the form

$$\mathcal{E}(u, v) := \tfrac{1}{2} \int_{\mathscr{P}} \langle \boldsymbol{\nabla} u(\mu) \,|\, \boldsymbol{\nabla} v(\mu)\rangle_{T_\mu \mathscr{P}_2} \,\mathrm{dvol}_{\mathscr{P}_2}(\mu) \ . \tag{5.2.1}$$

In turn, the theory of Dirichlet forms would then grant the existence of a diffusion process associated to $\mathcal{E}$ and thus deserving the name of *Brownian motion* on $\mathscr{P}_2$.

Further requirements are the validity of a Rademacher-type Theorem, i.e. the $\mathrm{dvol}_{\mathscr{P}_2}$-a.e. differentiability of W_2-Lipschitz functions, which motivated the work in Chapter 4, and of its converse, the Sobolev-to-Lipschitz property. Together, these properties would grant the identification of W_2 with the intrinsic distance induced by $\mathcal{E}$.

Diffusions processes on $\mathscr{P}$. In the case when $M = \mathbb{S}^1$, the unit sphere, or $M = I$, the (closed) unit interval, M.-K. von Renesse and K.-T. Sturm proposed the *entropic measure* $\mathbb{P}^\beta$ [140, Dfn. 3.3] as a candidate for $\mathrm{dvol}_{\mathscr{P}_2}$ and constructed the associated *Wasserstein diffusion* $\mu_\bullet^{\mathrm{WD}}$. Whereas the construction of the entropic measure in the case when M is an arbitrary closed Riemannian manifold was subsequently achieved by K.-T. Sturm in [151], many of its properties and in particular the closability of the associated form (5.2.1) remain unknown.

Similar constructions to the Wasserstein diffusion — up to now confined to one-dimensional base spaces — include J. Shao's *Dirichlet–Wasserstein diffusion* [147] (when $M = \mathbb{S}^1$ or I), and V. Konarovskyi *modified massive Arratia flow* $\mu_\bullet^{\mathrm{AF}}$ [91, 93] (when $M = I$) and V. Konarovskyi and M.-K. von Renesse's *coalescing-fragmentating Wasserstein dynamics* $\mu_\bullet^{\mathrm{CF}}$ [92] (when $M = \mathbb{R}$). Finally, it is worth mentioning two constructions in the case $M = \mathbb{R}^d$, namely the superprocesses of stochastic flows introduced by Z.-M. Ma and K.-N. Xiang in [114] and the recent work [29] by Y. T. Chow and W. Gangbo, concerned with a stochastic process $\mu_\bullet^{\mathrm{CG}}$ on $\mathscr{P}_2$ ≪modeled after Brownian motion≫ and generated by a ≪partial Laplacian≫.

A canonical process. If not otherwise stated, we shall assume the following.

Assumption (Riemannian manifolds). By a *Riemannian manifold* we shall mean any closed (i.e. compact, without boundary) connected oriented smooth Riemannian manifold (M, g) with (smooth) Riemannian metric g, intrinsic distance d_g, volume measure m, *normalized* volume measure $\overline{\mathsf{m}}$ and heat kernel $\mathsf{h}_t(x, \mathrm{d}y)$. If not otherwise stated, we shall assume that $d \coloneqq \dim M \geq 2$.

In the following, we shall construct a stochastic diffusion process

$$\eta_\bullet \coloneqq \left(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, (\eta_t)_{t\geq 0}, (P_\eta)_{\eta\in\mathscr{P}_2}\right) \tag{5.2.2}$$

with state space $\mathscr{P}_2$, modeled after the Brownian motion on M. By this we mean that $\eta_\bullet$ enjoys the following property: Let $(\eta_t^{\eta_0})_t$ denote the stochastic path of $\eta_\bullet$ starting at η_0. If

$$\eta_0 \coloneqq (1-r)\delta_{x_0} + r\delta_{y_0}\ , \qquad x_0, y_0 \in M\ ,\ r \in I\ ,$$

then

$$\eta_t^{\eta_0}(\omega) = (1-r)\delta_{x_{t/(1-r)}(\omega)} + r\delta_{y_{t/r}(\omega)}\ , \tag{5.2.3}$$

where $x_\bullet$ and $y_\bullet$ are *independent* Brownian motions on (M, g) respectively starting at x_0, y_0. If $r = 0$ or 1, then (5.2.3) entails that $\eta_\bullet$ respects the Dirac embedding, that is

$$\eta_0 = \delta_{x_0} \qquad \Longrightarrow \qquad \eta_t(\omega) = \delta_{x_t(\omega)}$$

for some Brownian motion $x_\bullet$ starting at x_0. This is a natural requirement, since $\delta\colon x \mapsto \delta_x$ is an isometric embedding $(M, \mathsf{d}_\mathsf{g}) \to (\mathscr{P}_2, W_2)$. If $r \in (0, 1)$, then (5.2.3) and its straightforward n-points generalizations may be easily interpreted in terms of particle systems. Indeed, $\eta_\bullet$ as in (5.2.3) describes the evolution of the two massive particles $(x_0, 1-r)$ and (y_0, r), and translates into the requirement that the evolution of their positions be independent up to the choice of suitable volatilities, namely the inverse of the mass carried by each atom.

We will provide two different constructions of $\eta_\bullet$.

Construction via semigroups. In the following let $\mathbf{I} \coloneqq \prod^\infty I$ with the product topology and set

$$\begin{aligned}\mathbf{\Delta} &\coloneqq \left\{ \mathbf{s} \coloneqq (s_i)_{i\in\mathbb{N}} \in \mathbf{I} \mid \sum_i^\infty s_i = 1 \right\}\ ,\\ \mathbf{T} &\coloneqq \left\{ \mathbf{s} \coloneqq (s_i)_{i\in\mathbb{N}} \in \mathbf{\Delta} \mid s_i \geq s_{i+1} \geq 0 \right\}\ .\end{aligned}$$

Let $\mathbf{T}_\circ \subset \mathbf{T}$ be defined similarly to $\mathbf{T}$ with $>$ in place of $\geq$. For $\mathbf{s} \in \mathbf{T}_\circ$ we put $M_i := (M, s_i \mathsf{g})$ and consider the infinite product $\mathbf{M} = \prod_i^\infty M_i$, endowed with the product measure

$$\overline{\mathsf{m}} := \bigotimes^\infty \overline{\mathsf{m}} \ .$$

Letting $\mathbf{x} = (x_i)_{i\in\mathbb{N}}$, $\mathbf{y} := (y_i)_{i\in\mathbb{N}}$ and defining the family of measures

$$\mathsf{h}^{\mathbf{s}}_t(\mathbf{x}, \mathrm{d}\mathbf{y}) := \bigotimes_i^\infty \mathsf{h}_{t/s_i}(x_i, \mathrm{d}y_i) \ , \qquad \mathbf{x} \in \mathbf{M} \ , \tag{5.2.4}$$

the resulting product semigroup $(\mathsf{H}^{\mathbf{s}}_t)_{t\geq 0}$ given by

$$(\mathsf{H}^{\mathbf{s}}_t u)(\mathbf{x}) := \int_{\mathbf{M}} u(\mathbf{y})\, \mathsf{h}^{\mathbf{s}}_t(\mathbf{x}, \mathrm{d}\mathbf{y}) \ , \qquad u \in L^2_{\overline{\mathsf{m}}}(\mathbf{M}) \ , \qquad t > 0 \ , \tag{5.2.5}$$

is an ergodic Markov semigroup with invariant measure $\overline{\mathsf{m}}$. By the general results of A. Bendikov and L. Saloff-Coste [15] about infinite-dimensional elliptic diffusions, $\mathsf{H}^{\mathbf{s}}_t$ admits a density, i.e. $\mathsf{h}^{\mathbf{s}}_t(\mathbf{x}, \mathrm{d}\mathbf{y}) = \mathsf{h}^{\mathbf{s}}_t(\mathbf{x}, \mathbf{y})\, \mathrm{d}\overline{\mathsf{m}}(\mathbf{y})$ for every $\mathbf{x}, \mathbf{y} \in \mathbf{M}$ and every $t > 0$, which is additionally continuous and bounded on $(0, \infty) \times \mathbf{M}^{\times 2}$ for every $\mathbf{s} \in \mathbf{T}_\circ$. We denote by

$$\mathsf{W}^{\mathbf{s}}_\bullet := \big(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, (\mathsf{W}^{\mathbf{s}}_t)_{t\geq 0}, (P^{\mathbf{s}}_{\mathbf{x}})_{\mathbf{x}\in\mathbf{M}}\big) \tag{5.2.6}$$

the associated time-homogeneous recurrent ergodic Markov process with state space $\mathbf{M}$ and transition kernels $(\mathsf{h}^{\mathbf{s}}_\bullet(\mathbf{x}, \cdot\,))_{\mathbf{x}\in\mathbf{M}}$. Let now $\mathbf{P}$ be any probability on $\mathbf{T}_\circ$ such that $(\mathbf{T}_\circ, \mathbf{P})$ be a standard Borel probability space. The semigroup defined on $\widehat{\mathbf{M}} := \mathbf{T} \times \mathbf{M}$ as

$$\begin{aligned} (\widehat{\mathsf{H}}_t v)(\mathbf{s}, \mathbf{x}) :=& \big((\mathrm{id} \otimes \mathsf{H}^{\mathbf{s}}_t)\, v\big)(\mathbf{s}, \mathbf{x}) \ , \qquad && t > 0 \ , \\ =& \big(\mathsf{H}^{\mathbf{s}}_t\, v(\mathbf{s}, \cdot\,)\big)(\mathbf{x}) \ , && v \in L^2(\widehat{\mathbf{M}}, \mathbf{P} \otimes \overline{\mathsf{m}}) \end{aligned} \tag{5.2.7}$$

is itself a Markov semigroup on $L^2(\widehat{\mathbf{M}}, \mathbf{P} \otimes \overline{\mathsf{m}})$. Setting

$$\mathbf{M}_\circ := \{\mathbf{x} \in \mathbf{M} \mid x_i \neq x_j \text{ for } i \neq j\} \ ,$$

the map

$$\mathbf{\Phi} \colon \mathbf{\Delta} \times \mathbf{M} \longrightarrow \mathscr{P}_2 \ , \qquad \mathbf{\Phi}(\mathbf{s}, \mathbf{x}) := \sum_i^\infty s_i \delta_{x_i} \tag{5.2.8}$$

is injective when restricted to $\widehat{\mathbf{M}}_\circ := \mathbf{T}_\circ \times \mathbf{M}_\circ$. We say that $\mathbf{M}_\circ$ is $\mathsf{W}^{\mathbf{s}}_\bullet$-coexceptional for every $\mathbf{s} \in \mathbf{T}_\circ$ if the process $\mathsf{W}^{\mathbf{s}}_\bullet$ never leaves $\mathbf{M}_\circ$. Since this turns out to be the case, (See Lem. 5.3.9) then $\widehat{\mathbf{M}}_\circ$ is coexceptional for the process $\widehat{\mathsf{W}}_\bullet$ associated to $\widehat{\mathsf{H}}_t$. Thus, provided that $\mathbf{\Phi}$ is suitably measurable (Prop. 5.3.14), we may consider the induced stochastic process on $\mathscr{P}$ pathwise defined as

$$\eta^{\eta_0}_t := \mathbf{\Phi} \circ \widehat{\mathsf{W}}^{\mathbf{s},\mathbf{x}_0}_t = \mathbf{\Phi}\big(\mathbf{s}, \mathsf{W}^{\mathbf{s};\mathbf{x}_0}_t\big) \ , \qquad \eta_0 := \mathbf{\Phi}\big(\mathbf{s}, \mathbf{x}_0\big) \ , \qquad t > 0 \ , \tag{5.2.9}$$

where by $\mathsf{W}^{\mathbf{s};\mathbf{x}_0}_t$ we mean $\mathsf{W}^{\mathbf{s}}_t$ starting at $\mathbf{x}_0$.

By construction, $\eta_\bullet$ is a time-homogeneous Markov process with state space $\mathscr{P}_2$. However, since $\mathbf{\Phi}$ is not continuous, it is not clear at this stage whether $\eta_\bullet$ has continuous paths, and its properties may vary wildly depending on the choice of the law $\mathbf{P}$ for the starting point $\mathbf{s}$.

A choice for $\mathrm{dvol}_{\mathscr{P}_2}$. Everywhere in the following we let $\beta > 0$ be fixed. For the moment, we shall think of β as the total volume of M, so that $\mathsf{m} = \beta\overline{\mathsf{m}}$. We denote by

$$\mathrm{dB}_\beta(r) := \beta(1-r)^{\beta-1}\,\mathrm{d}r$$

the *Beta distribution* on I with shape parameters 1 and β; by $\mathbf{B}_\beta := \otimes^\infty \mathrm{B}_\beta$ the corresponding product measure on $\mathbf{I}$. For every measure μ and every measurable map T, we denote the induced push-forward measure by

$$T_\sharp\mu := \mu \circ T^{-1} .$$

We set $\mathscr{P}^{\mathrm{pa}}_{\mathrm{iso}} = \mathbf{\Phi}(\widehat{\mathbf{M}}_\circ)$, the space of *purely atomic* probability measures with *infinite strictly ordered* masses. By standard results (e.g., [165, Thm. 6.18, Rmk. 6.19]), the latter space is dense in the compact space $\mathscr{P}_2$. Thus, if we assume $\mathbf{P}$ to be fully supported on $\mathbf{T}$, then

$$\mathbf{Q} := \mathbf{\Phi}_\sharp(\mathbf{P} \otimes \overline{\mathsf{m}})$$

is fully supported on $\mathscr{P}_2$. Under such assumption, the property of $\mathbf{Q}$ being a 'canonical' measure — in any suitable sense — on $\mathscr{P}$ is equivalent to that of $\mathbf{P}$ being 'canonical' on $\mathbf{T}$. As a candidate for $\mathbf{P}$ we choose the *Poisson–Dirichlet measure* Π_β introduced by J. F. C. Kingman in [87]. We recall its definition following the neat exposition of P. Donnelly and G. Grimmet [44].

Definition 5.2.1 (Poisson–Dirichlet measure)**.** For $\mathbf{r} := (r_i)_{i\in\mathbb{N}} \in \mathbf{I}$ we denote by $\mathbf{\Upsilon}(\mathbf{r})$ the vector of its entries in non-increasing order, by $\mathbf{\Upsilon}\colon \mathbf{I} \to \mathbf{T} \subset \mathbf{I}$ the reordering map, measurable by e.g. [45, p. 91]. Further, we let $\mathbf{\Lambda}\colon \mathbf{I} \to \mathbf{\Delta}$ be defined by

$$\Lambda_1(r_1) := r_1 \ , \qquad \Lambda_k(r_1,\dots,r_k) := r_k \prod_i^{k-1}(1-r_i) \ , \qquad \mathbf{\Lambda}(\mathbf{r}) := (\Lambda_1(r_1), \Lambda_2(r_1,r_2), \dots) \ . \tag{5.2.10}$$

The *Poisson–Dirichlet measure* Π_β *with parameter* β on $\mathbf{T}$, concentrated on $\mathbf{T}_\circ$, is defined as

$$\Pi_\beta := (\mathbf{\Upsilon} \circ \mathbf{\Lambda})_\sharp \mathbf{B}_\beta \ .$$

For such a choice of $\mathbf{P}$, the measure $\mathbf{Q}$ is the *Dirichlet–Ferguson measure* $\mathcal{D}_\mathsf{m}$ introduced by T. S. Ferguson in his seminal work [55]. The dependence of $\mathcal{D}_\mathsf{m}$ on β is implicit in the constraint $\mathsf{m} = \beta\overline{\mathsf{m}}$. Since in the following $\mathcal{D}_\mathsf{m}$ will play the rôle of $\mathrm{dvol}_{\mathscr{P}_2}$, we state here one of its several characterizations.

Theorem 5.2.2 (A characterization of $\mathcal{D}_\mathsf{m}$)**.** *Let* $\mathbf{Q}$ *be a probability measure on* $\mathscr{P}$*. For* $\eta \in \mathscr{P}$*,* $x \in M$ *and* $r \in I$ *set* $\eta_x := \eta\{x\} \in I$ *and*

$$\eta^x_r := (1-r)\eta + r\delta_x \in \mathscr{P} \ . \tag{5.2.11}$$

Then, the following are equivalent:

- $\mathbf{Q}$ *is the Dirichlet–Ferguson measure* $\mathcal{D}_\mathsf{m} := \mathbf{\Phi}_\sharp(\Pi_\beta \otimes \overline{\mathsf{m}})$*;*
- *if* η *is a* $\mathbf{Q}$*-distributed* $\mathscr{P}$*-valued random field,* x *is* $\overline{\mathsf{m}}$*-distributed and* r *is* B_β*-distributed, then* $\mathbf{Q}$ *satisfies* Sethuraman's fixed-point[1] characterization *(See [145, Eqn. (3.2)])*

$$\eta \overset{\mathrm{d}}{=} \eta^x_r \ , \tag{5.2.12}$$

where $\overset{\mathrm{d}}{=}$ *denotes equality in law;*

[1] I am grateful to Prof. F. Bassetti for having suggested me this interpretation of (5.2.12).

- $\mathbf{Q}$ *satisfies the* Mecke-type identity *or* Georgii–Nguyen–Zessin formula *(See Chapter 3.)*
$$\int_{\mathscr{P}} \mathrm{d}\mathbf{Q}(\eta) \int_M \mathrm{d}\eta(x)\, u(\eta, x, \eta_x) = \int_{\mathscr{P}} \mathrm{d}\mathbf{Q}(\eta) \int_M \mathrm{d}\overline{\mathsf{m}}(x) \int_I \mathrm{dB}_\beta(r)\, u(\eta_r^x, x, r) \qquad (5.2.13)$$
for any semi-bounded measurable $u\colon \mathscr{P}_2 \times M \times I \to \mathbb{R}$*.*

We will mostly dwell upon the characterization (5.2.13), obtained with E. W. Lytvynov in Chapter 3 and which is in fact but the integral version of (5.2.12), originally proven by J. Sethuraman in [145]. (See also G. Last [101] for a similar characterization on more general spaces, Chapter 2 for a characterization via Fourier transform and T. J. Jiang–J. M. Dickey–K.-L. Kuo's work [79] for a characterization via c-transform.)

Construction via Dirichlet forms theory. By construction, the measure $\mathbf{\Phi}_\sharp^{-1}\mathcal{D}_{\mathsf{m}} = \Pi_\beta \otimes \overline{\mathsf{m}}$ is an invariant measure of $\widehat{\mathbf{W}}_\bullet$. Choosing $\mathcal{D}_{\mathsf{m}}$ as $\mathrm{dvol}_{\mathscr{P}_2}$ in (5.2.1), we will show that the process $\eta_\bullet$ in (5.2.9) is the Markov diffusion (i.e. special Hunt, sample-continuous) associated with the Dirichlet form $\mathcal{E}$. This requires however some preparations.

We shall follow a similar strategy to the one adopted in Yu. G. Kondratiev, E. W. Lytvynov and A. M. Vershik [96], where analogous results are presented for Gibbs measures on the space of non-negative Radon measures over $\mathbb{R}^d$. Firstly, let $\hat f\colon M \times I \to \mathbb{R}$ be of the form $\hat f \coloneqq f \otimes \varrho$, where $f \in \mathcal{C}^\infty(M)$ and $\varrho \in \mathcal{C}^\infty(I)$ is supported in the *open* interval $(0,1)$. Recalling the notation $\eta_x \coloneqq \eta\{x\}$, we let further

$$\hat f^\star(\eta) \coloneqq \int_M \mathrm{d}\eta(x)\, f(x) \cdot \varrho(\eta_x) \qquad (5.2.14)$$

and consider

- the algebra $\widehat{\mathfrak{Z}}_0$ of cylinder functions $u\colon \mathscr{P} \to \mathbb{R}$ of the form
$$u(\eta) = F\big(\hat f_1^\star(\eta), \dots, \hat f_k^\star(\eta)\big) \ ,$$
where $F \in \mathcal{C}_b^\infty(\mathbb{R}^k)$ and $\hat f_i$ is as above for $i \le k$.
- the algebra $\mathfrak{B}$ of cylinder functions induced by measurable potential energies, i.e. such that
$$u(\eta) = F\big(\eta f_1, \dots, \eta f_k\big) \ ,$$
where $F \in \mathcal{C}_b^\infty(\mathbb{R}^k)$ and $f_i \in \mathcal{B}_b(M;\mathbb{R})$ for $i \le k$;
- the algebra $\mathfrak{Z}$ of cylinder functions induced by smooth potential energies, defined as $\mathfrak{B}$, with the additional requirement that $f_i \in \mathcal{C}^\infty(M)$ for $i \le k$.

Let now w be a smooth vector field and $\big(\psi^{w,t}\big)_{t\ge 0}$ be its flow (5.3.3). For $\mu \in \mathscr{P}$ we denote by $T_\mu^{\mathrm{Der}}\mathscr{P}_2$ the completion of the space of all smooth vector fields w with respect to the pre-Hilbertian norm $w \mapsto \big\| \left|w\right|_{\mathsf{g}} \big\|_{L^2_\mu}$. (The superscript 'Der' stands for *derivation.* See §4.6.1.) It is well-established in the optimal transport theory (e.g., [10, 2.31 and §7.2], cf. also [60, 63, 64]) that the *tangent space* $T_\mu\mathscr{P}_2$ to the 'Riemannian manifold' $\mathscr{P}_2$ at μ is

$$T_\mu\mathscr{P}_2 \coloneqq \mathrm{cl}_{T_\mu^{\mathrm{Der}}\mathscr{P}_2}\left\{\nabla f \mid f \in \mathcal{C}^\infty(M)\right\} \ .$$

The inclusion $T_\mu\mathscr{P}_2 \subset T_\mu^{\mathrm{Der}}\mathscr{P}_2$ is generally a strict one. We shall make use of both definitions, the interplay of which was detailed in Chapter 4. The 'directional derivative' of functions $u \in \widehat{\mathfrak{Z}}_0$ or $\mathfrak{Z}$ in the smooth 'direction' w is given by (Lem. 5.4.7)

$$\boldsymbol{\nabla}_w\, u(\mu) \coloneqq \mathrm{d}_t\big|_{t=0} u\big(\psi_\sharp^{w,t}\mu\big) \ . \qquad (5.2.15)$$

For $u,v\in\widehat{\mathfrak{Z}}_0$ and any smooth w, we show that there exists some small $\varepsilon=\varepsilon_{u,v}>0$ such that we have the *integration by parts formula* (Thm. 5.4.9)

$$\mathbf{E}_{\mathcal{D}_\mathsf{m}}\big[\boldsymbol{\nabla}_w\, u\cdot v\big]=-\mathbf{E}_{\mathcal{D}_\mathsf{m}}\big[u\cdot\boldsymbol{\nabla}_w\, v\big]-\mathbf{E}_{\mathcal{D}_\mathsf{m}}\big[u\cdot v\cdot\mathbf{B}_\varepsilon[w]\big]\ ,$$

where

$$\mathbf{B}_\varepsilon[w](\eta)\coloneqq\sum_{x\,|\,\eta_x>\varepsilon}\mathrm{div}^\mathsf{m}_x w\ . \tag{5.2.16}$$

Provided that $w\mapsto\boldsymbol{\nabla}_w\, u(\mu)$ be a $T^{\mathrm{Der}}_\mu\mathscr{P}_2$-continuous linear functional, a gradient

$$\mu\longmapsto\boldsymbol{\nabla}\, u(\mu)\in T^{\mathrm{Der}}_\mu\mathscr{P}_2 \tag{5.2.17}$$

is induced by Riesz Representation Theorem. The latter integration-by-parts formula is then a main tool in establishing the following theorem.

Theorem 5.2.3 (See Thm. 5.4.11 and Cor. 5.4.19)**.** *The quadratic form $(\mathcal{E},\mathfrak{Z})$ defined by*

$$\mathcal{E}(u,v)\coloneqq\int_{\mathscr{P}}\mathrm{d}\mathcal{D}_\mathsf{m}(\eta)\big\langle\,\boldsymbol{\nabla}\, u(\eta)\,\big|\,\boldsymbol{\nabla}\, v(\eta)\big\rangle_{T_\eta\mathscr{P}_2}\ ,\qquad u,v\in\mathfrak{Z}$$

is closable. Its closure $(\mathcal{E},\mathscr{D}(\mathcal{E}))$ is a regular strongly local recurrent non-ergodic Dirichlet form on $L^2_{\mathcal{D}_\mathsf{m}}(\mathscr{P}_2)$ with carré du champ operator

$$\boldsymbol{\Gamma}(u,v)(\eta)\coloneqq\big\langle\,\boldsymbol{\nabla}\, u(\eta)\,\big|\,\boldsymbol{\nabla}\, v(\eta)\big\rangle_{T_\eta\mathscr{P}_2}\ ,\qquad u,v\in\mathfrak{Z}\ . \tag{5.2.18}$$

A comparison with the Fleming–Viot process. Before exploring any further the properties of $\mathcal{E}$ and its relation to $\eta_\bullet$, it is worth comparing its carré du champ operator (5.2.18) with the carré du champ operators of other processes on $\mathscr{P}$. In [132] L. Overbeck, M. Röckner and B. Schmuland showed that, letting[2] (cf. [132, p. 2])

$$\big(\tfrac{\partial}{\partial\delta_x}u\big)(\mu)(x)\coloneqq\mathrm{d}_t\big|_{t=0}u(\mu+t\delta_x)\ , \tag{5.2.19}$$

the Dirichlet form $(\mathcal{E}^{\mathrm{FV}},\mathscr{D}(\mathcal{E}^{\mathrm{FV}}))$ with carré du champ operator

$$\boldsymbol{\Gamma}^{\mathrm{FV}}(u)(\mu)\coloneqq\mathrm{Var}_\mu\big(\tfrac{\partial}{\partial\delta_\cdot}u(\mu)\big)\qquad u\in\mathfrak{B}$$

and invariant measure $\mathcal{D}_\mathsf{m}$ is properly associated with the *Fleming–Viot process* [57] with parent independent mutation. J. Shao observed in [147] that the increment in (5.2.19) is not internal to $\mathscr{P}$. To overcome the issue, he considered the map S_f [147, Eqn. (2.7)] (there termed 'exponential map'. See Remark 5.4.6 below.) originally introduced by K. Handa in [73]

$$S_f(\mu)\coloneqq\frac{e^f\cdot\mu}{\mu(e^f)}\qquad f\in\mathcal{C}(M)\ . \tag{5.2.20}$$

For $\mu\in\mathscr{P}$ we recall the notation (5.2.11) and set[3] (also cf. [143, Eqn. (1.1)])

$$\big(\widetilde{\tfrac{\partial}{\partial\delta_x}}\big)u(\mu)\coloneqq\mathrm{d}_t\big|_{t=0}u(\mu^x_t)\ .$$

Then,

$$\mathrm{d}_t\big|_{t=0}S_{tf}(\mu)=\Big\langle\big(\widetilde{\tfrac{\partial}{\partial\delta_\cdot}}u\big)(\mu)\,\Big|\,f\Big\rangle_{L^2_\mu}\qquad u\in\mathfrak{Z}\ ,$$

[2]Here, we are forced to change the notation for the gradient in [132], since it conflicts with ours.
[3]Here, we are forced to change the notation for the gradient in [147], since it conflicts with ours.

and

$$\mathbf{\Gamma}^{\mathrm{FV}}(u)(\mu) = \left\| \widetilde{\tfrac{\partial}{\partial \delta_{\cdot}} u} \right\|^2_{L^2_\mu} \quad .$$

As noted by M. Döring and W. Stannat in [46, Rmk. 1.5], the carré du champ operator (5.2.18) is strictly stronger than $\mathbf{\Gamma}^{\mathrm{FV}}$ and one has in fact

$$\mathbf{\Gamma}(u)(\mu) = \left\| \nabla \cdot \tfrac{\partial}{\partial \delta_{\cdot}} u \right\|^2_{L^2_\mu} = \left\| \nabla \cdot \widetilde{\tfrac{\partial}{\partial \delta_{\cdot}} u} \right\|^2_{L^2_\mu} \quad .$$

In the case $M = \mathbb{S}^1$ (whence $T_\mu \mathscr{P}_2 = T^{\mathrm{Der}}_\mu \mathscr{P}_2 = L^2_\mu(\mathbb{S}^1;\mathbb{R})$), one has

$$\mathbf{\Gamma} = \mathbf{\Gamma}^{\mathrm{WD}} \ , \tag{5.2.21}$$

the carré du champ [140, Dfn. 7.24] of the Wasserstein diffusion [140]. Letting $(\mathcal{E}^{\mathrm{WD}}, \mathscr{D}(\mathcal{E}^{\mathrm{WD}}))$ be the Dirichlet form [140, Thm. 7.25] of Wasserstein diffusion, equality (5.2.21) is interpreted in the following sense: By definition $u \in \mathfrak{Z} \subset \mathscr{D}(\mathcal{E}^{\mathrm{WD}})$. Then, for each $u \in \mathfrak{Z}$ there exist a continuous $\mathbb{P}^\beta$-representative $\widetilde{\mathbf{\Gamma}^{\mathrm{WD}}(u)}$ of $\mathbf{\Gamma}^{\mathrm{WD}}(u)$ and a continuous $\mathcal{D}_{\mathsf{m}}$-representative $\widetilde{\mathbf{\Gamma}(u)}$ of $\mathbf{\Gamma}(u)$ such that $\widetilde{\mathbf{\Gamma}^{\mathrm{WD}}(u)} = \widetilde{\mathbf{\Gamma}(u)}$ (everywhere) on $\mathscr{P}(\mathbb{S}^1)$.

A comparison with the Wasserstein diffusion. In addition to the carré du champ operator, the generator of $(\mathcal{E}, \mathscr{D}(\mathcal{E}))$ entails further geometrical information. Up to Friedrichs extension,

$$\mathbf{L}u = \mathbf{L}_1 u + \mathbf{L}_2 u \ , \qquad u \in \widehat{\mathfrak{Z}}_0 \ ,$$

where

$$\begin{aligned} \mathbf{L}_1 u(\eta) &:= \tfrac{1}{2} \sum_{i,j}^k (\partial^2_{ij} F)\big(\hat{f}^\star_1(\eta), \dots, \hat{f}^\star_k(\eta)\big) \cdot \int_M \mathrm{d}\eta(x)\, \varrho_i(\eta_x) \varrho_j(\eta_x)\ \langle \nabla_x f_i \,|\, \nabla_x f_j \rangle_{\mathsf{g}} \ , \\ \mathbf{L}_2 u(\eta) &:= \tfrac{1}{2} \sum_{i}^k (\partial_i F)\big(\hat{f}^\star_1(\eta), \dots, \hat{f}^\star_k(\eta)\big) \cdot \sum_{x \,|\, \eta_x > \varepsilon} \varrho_i(\eta_x)\, \Delta_x f_i \ . \end{aligned} \tag{5.2.22}$$

For functions in the core $\mathfrak{Z}$ of $(\mathcal{E}, \mathscr{D}(\mathcal{E}))$, the first operator takes the form

$$\mathbf{L}_1 u(\eta) = \tfrac{1}{2} \sum_{i,j}^k (\partial^2_{ij} F)\big(\eta f_1, \dots, \eta f_k\big) \cdot \langle \nabla f_i \,|\, \nabla f_j \rangle_{T_\eta \mathscr{P}_2} \ ,$$

the *diffusion part* of the generator. Indeed, in the case when $M = \mathbb{S}^1$, we have

$$\mathbf{L}_1 = \mathbf{L}_1^{\mathrm{WD}} \ ,$$

where $\mathbf{L}_1^{\mathrm{WD}}$ is the diffusion part of the generator $\mathbf{L}^{\mathrm{WD}}$ of the Wasserstein diffusion in the decomposition [140, Thm. 7.25] and equality is interpreted as in (5.2.21). As noted in [140, Rmk. 7.18], $\mathbf{L}_1^{\mathrm{WD}}$ ≪describes the [Wasserstein] diffusion [...] in all directions of the respective tangent spaces≫. Thus, the process $\mu^{\mathrm{WD}}_\bullet$ associated with $(\mathcal{E}^{\mathrm{WD}}, \mathscr{D}(\mathcal{E}^{\mathrm{WD}}))$ ≪experiences [...] the full tangential noise≫. In the present case, the same statement may be formulated rigorously, in terms of Hino's *index* [74] of the form (Prop. 5.4.22).

The first order operator $\mathbf{L}_2$ represents instead the *drift part* of the generator, constraining the process $\eta_\bullet$ on $\mathscr{P}^{\mathrm{pa}}_{\mathrm{iso}}$. We notice here that the expression of $\mathbf{L}_2$ in (5.2.22) does *not* converge for functions in $\mathfrak{Z}$ (i.e., in the pointwise limit $\varrho_i \to \mathbb{1}_I$). This is consistent with the heuristic observation of N. Gigli (see [64, Rmk. 5.6]) that the Laplacian of potential energies on $\mathscr{P}_2$ should *not* exist. On the other hand though, this does not prevent the closability of $(\mathcal{E}, \mathfrak{Z})$ above.

This seeming contradiction is resolved in the understanding that the operator $\mathbf{L}_2$ is in fact a *boundary term*, and, as such, it was not accounted for in [64, *ibid.*]. Indeed — in the present framework — the set $\mathscr{P}^{\mathrm{pa}}_{\mathrm{iso}}$ where $\mathcal{D}_{\mathsf{m}}$ is concentrated ought to be thought of as part of the *geodesic boundary* of $\mathscr{P}_2$. Here, we say that a point $\mu_0 \in \mathscr{P}$ is a geodesic boundary point if there exists some W_2-geodesic $(\mu_t)_t$ for which μ_0 is extremal, that is, $(\mu_t)_t$ may not be further prolonged *through* μ_0. The fact that measures with atoms satisfy this property is a consequence of the same result for Dirac masses, originally proved by J. Bertrand and B. R. Kloeckner (see [19, Lem. 2.2]) and of the known fact that transport optimality is inherited by restrictions (see [165, Thm. 4.6]).

When $M = \mathbb{S}^1$ the operator $\mathbf{L}_2^{\mathrm{WD}}$ (see [140, Rmk. 7.18, Thm. 7.25]) may be given the same interpretation of $\mathbf{L}_2$, analogously to the case of $\mathbf{L}_1$ and $\mathbf{L}_1^{\mathrm{WD}}$. Finally, we notice that the operator $\mathbf{L}_3^{\mathrm{WD}}$ in [140, Rmk. 7.18] has no counterpart in our case (which should rather be compared with [140, Thm. 7.25]), since it is an artifact of the boundary of I.

A comparison with the Modified Massive Arratia Flow. In [91] V. V. Konarovskyi introduced the *Modified Massive Arratia Flow*, a random element $y(\,\cdot\,,\,\cdot\,)$ in the Skorokhod space $D\big(I;\mathcal{C}([0,T])\big)$ whose corresponding measure-valued process $\mu^{\mathrm{AF}}_\bullet$ defined by $\mu^{\mathrm{AF}}_t := y(\,\cdot\,,t)_\sharp \mathrm{Leb}^1$ is a solution to (5.2.23 AF) below. For the purpose of comparison, let us recall the stochastic partial differential equations solved by all the processes mentioned so far. Namely, for $t \geq 0$

$$\begin{aligned}
\mathrm{d}\mu_t^{\mathrm{AF}} &= \operatorname{div}\big(\sqrt{\mu_t^{\mathrm{AF}}}\,\mathrm{d}W_t\big) + \mathbf{L}_2^{\mathrm{AF}}(\mu_t^{\mathrm{AF}})\,\mathrm{d}t && \text{on } M=\mathbb{R}\ , && (5.2.23\,\mathrm{AF})\\
\mathrm{d}\mu_t^{\mathrm{WD}} &= \operatorname{div}\big(\sqrt{\mu_t^{\mathrm{WD}}}\,\mathrm{d}W_t\big) + \mathbf{L}_2^{\mathrm{WD}}(\mu_t^{\mathrm{WD}})\,\mathrm{d}t + \beta\Delta\mu_t^{\mathrm{WD}}\,\mathrm{d}t && \text{on } M=I\ , && (5.2.23\,\mathrm{WD}_0)\\
\mathrm{d}\mu_t^{\mathrm{WD}} &= \operatorname{div}\big(\sqrt{\mu_t^{\mathrm{WD}}}\,\mathrm{d}W_t\big) + \mathbf{L}_2^{\mathrm{WD}}(\mu_t^{\mathrm{WD}})\,\mathrm{d}t && \text{on } M=\mathbb{S}^1\ , && (5.2.23\,\mathrm{WD}_1)\\
-\frac{1}{\sqrt{2\gamma}}\,\mathrm{d}\mu_t^{\mathrm{CG}} &= \operatorname{div}\big(\mu_t^{\mathrm{CG}}\,\mathrm{d}W_t\big) - \sqrt{\gamma/2}\,\Delta\mu_t^{\mathrm{CG}}\,\mathrm{d}t && \text{on } M=\mathbb{R}^d\ , && (5.2.23\,\mathrm{CG})
\end{aligned}$$

where $W_\bullet$ is a standard Brownian motion and the equations are tested on functions of the form $f^\star$ for $f \in \mathcal{C}^\infty_c(M)$ (hence $f^\star \in \mathcal{C}_b(\mathscr{P}_2)$ by Rmk. 5.4.3(***d***)). Then, by e.g. (5.2.23 AF) we mean

$$\mathrm{d}\big(f^\star\mu_t^{\mathrm{AF}}\big) = \Big(\big(|\nabla f|^2_{\mathsf{g}}\big)^\star \mu_t^{\mathrm{AF}}\Big)\,\mathrm{d}W_t + \big(f^\star(\mathbf{L}_2^{\mathrm{AF}}\mu_t^{\mathrm{AF}})\big)\,\mathrm{d}t\ , \qquad f\in\mathcal{C}^\infty_c(\mathbb{R})\ ,$$

or, equivalently,

$$\mathrm{d}\int_M f\,\mathrm{d}\mu_t^{\mathrm{AF}} = \bigg(\int_M |\nabla f|^2_{\mathsf{g}}\,\mathrm{d}\mu_t^{\mathrm{AF}}\bigg)\,\mathrm{d}W_t + \big((\mathbf{L}_2^{\mathrm{AF}}\mu_t^{\mathrm{AF}})f\big)\,\mathrm{d}t\ , \qquad f\in\mathcal{C}^\infty_c(\mathbb{R})\ .$$

Finally, for μ such that $|\mathrm{ptws}\,(\mu)| < \infty$, the operator $\mathbf{L}_2^{\mathrm{AF}}\mu$ is the distribution

$$\mathbf{L}_2^{\mathrm{AF}}\mu = \mathbf{L}_2\mu = \sum_{x \mid \mu_x>0} \delta''_x\ , \tag{5.2.24}$$

whereas, for μ singular continuous w.r.t. Leb^1, the operator $\mathbf{L}_2^{\mathrm{WD}}\mu$ is instead given by

$$(\mathbf{L}_2^{\mathrm{WD}}\mu)f := \sum_{J\in\mathrm{gaps}(\mu)} \left[\frac{f''(J_+)+f''(J_-)}{2} - \frac{f'(J_+)-f'(J_-)}{|J|}\right]\ ,$$

where $\mathrm{gaps}(\mu)$ is the set of maximal intervals $J := (J_-, J_+)$ such that $\mu J = 0$, and $|J| := J_+ - J_-$.

Equation (5.2.23 AF) is [93, Eqn. (1.2)]. It was subsequently shown by V. V. Konarovskyi and M.-K. von Renesse (See [92, Rmk. 1.2]) that (5.2.23 AF) admits multiple solutions, among which the *coalescing-fragmentating Wasserstein dynamics* [92]. Equation (5.2.23 WD_0), describing the Wasserstein diffusion $\mu^{\mathrm{WD}}_\bullet$ on $\mathscr{P}(I)$, is also found shortly after [93, Eqn. (1.2)]. The corresponding Equation (5.2.23 WD_1), describing $\mu^{\mathrm{WD}}_\bullet$ on $\mathscr{P}(\mathbb{S}^1)$, is readily deduced from [140, Rmk. 7.18].

Equation (5.2.23 CG), describing Y. T. Chow and W. Gangbo's process $\mu_\bullet^{\mathrm{CG}}$, is a reformulation of [29, Eqn. (1.7)]. (Cf. also [25, Eqn. (8)] for a formulation analogous to the one given here.) We are forced to change the notation of [29] in that we denote here the *noise intensity* by $\gamma > 0$ (instead of β, as originally in [25, 29]) since γ is opposite in meaning to $\beta > 0$ in (5.2.23 WD$_0$) which is rather an *inverse temperature*. (Cf. Rmk. 2.3.16.) For the relation of (5.2.23 WD$_0$) to the Dean–Kawasaki dynamics for supercooled liquid models, cf. [92, §1.1].

In the case when $d \geq 2$, our process $\eta_\bullet$ may be regarded as a counterpart on multidimensional base spaces to the Modified Massive Arratia Flow, in the following weak sense:

Proposition 5.2.4. *Assume $d \geq 2$. Then, the process $\eta_\bullet$ is a $\mathscr{P}$-valued martingale solution to* (5.2.23 AF) *tested on functions of the form $\hat{f}^\star$ as in* (5.2.14).

In the case when $d = 1$ the form $(\mathcal{E}, \mathscr{D}(\mathcal{E}))$ above is properly associated with a Markov diffusion, again denoted by $\eta_\bullet$. In this case however, the identification of $\eta_\bullet$ with $\mathbf{\Phi} \circ \widehat{\mathbf{W}}_\bullet$ does not hold. We collect some remarks on the relations between $\eta_\bullet$ and the Modified Massive Arratia Flow in §5.5.3, postponing a thorough analysis to future studies.

Quasi-invariance, representations and Helmoltz decomposition. If G is a group acting measurably on a probability space $(\Omega, \mathcal{F}, \mathbf{P})$ (write $G \circlearrowright \Omega$) we say that $\mathbf{P}$ is *quasi-invariant* with respect to the action of $h \in G$ (write $h.\omega$) if

$$\mathbf{P}^h := (h.)_\sharp \mathbf{P} = \mathbf{R}[h] \cdot \mathbf{P}$$

for some $\mathcal{F}$-measurable Radon–Nikodým derivative $\mathbf{R}[h] \colon \Omega \to [0, \infty]$. It is *invariant* if $\mathbf{P}^h = \mathbf{P}$.

In the case when $M = \mathbb{S}^1$ and G is the Virasoro group $\mathrm{Diff}^\infty_+(\mathbb{S}^1)$ of smooth orientation-preserving diffeomorphisms of $\mathbb{S}^1$, the quasi-invariance of the entropic measure $\mathbb{P}^\beta$ and of the Dirichlet–Ferguson measure $\mathcal{D}$ has been a key tool in establishing the closability of the form (5.2.1).

Let us briefly recall the definition of the actions in [140, 141, 147]. Following [140, §2.2] we set

$$\mathscr{G}(\mathbb{R}) := \{g \colon \mathbb{R} \to \mathbb{R} \ , \quad \text{right-continuous, non-decreasing, s.t. } g(x+1) = g(x) + 1\} \ .$$

Let further $\mathrm{pr}^{\mathbb{S}^1} \colon \mathbb{R} \to \mathbb{S}^1 \cong \mathbb{R}/\mathbb{Z}$ denote the quotient projection and set $\mathscr{G}(\mathbb{S}^1) := \mathrm{pr}^{\mathbb{S}^1}(\mathscr{G}(\mathbb{R}))$. By equi-variance, $g \colon \mathbb{S}^1 \to \mathbb{S}^1$ for every $g \in \mathscr{G}(\mathbb{S}^1)$ and the set $\mathscr{G}(\mathbb{S}^1)$ endowed with the usual composition of functions, $\circ$, is a semi-group with identity $\mathrm{id}_{\mathbb{S}^1}$. In particular, the group $\mathrm{Diff}^\infty_+(\mathbb{S}^1)$ injects into $\mathscr{G}(\mathbb{S}^1)$. (See, e.g., [140] or Chapter 4.) Again following [140], we set

$$\mathscr{G}_1 := \mathscr{G}(\mathbb{S}^1)/\mathbb{S}^1 \quad , \tag{5.2.25}$$

where $g, h \in \mathscr{G}(\mathbb{S}^1)$ are identified if $g(\,\cdot\,) = h(\,\cdot + a)$ for some $a \in \mathbb{S}^1$, and define the maps

$$\begin{aligned} \zeta \colon \mathscr{G}(\mathbb{S}^1) &\longrightarrow \mathscr{P}(\mathbb{S}^1) \\ \zeta \colon g &\longmapsto \mathrm{d}g \end{aligned} \qquad \text{and} \qquad \begin{aligned} \chi \colon \mathscr{G}_1 &\longrightarrow \mathscr{P}(\mathbb{S}^1) \\ \chi \colon g &\longmapsto g_\sharp \overline{\mathsf{m}} \end{aligned} \quad , \tag{5.2.26}$$

where $\mathrm{d}g$ is the Lebesgue–Stieltjes measure induced by g and $\overline{\mathsf{m}}$ denotes here the normalized Lebesgue measure on $\mathbb{S}^1$. Both maps are invertible. Namely, the inverse ζ^{-1} assigns to μ its cumulative distribution function, while χ^{-1} assigns to μ its generalized inverse distribution function (see [140, Eqn. (2.2)]). In particular, up to passing from $\mathbb{S}^1$ to I,

$$\chi^{-1} = \,\cdot^{-1} \circ \zeta^{-1} \ , \tag{5.2.27}$$

where $\cdot^{-1} \colon g \mapsto g^{-1}$ is the right-inversion map defined by $g^{-1}(t_0) := \inf\{t \in I \mid g(t) > t_0\}$.

For $h \in \mathrm{Diff}^\infty_+(\mathbb{S}^1)$ we consider the left and right action on $\mathscr{G}(\mathbb{S}^1)$ defined by

$$\ell_h\colon g \longmapsto h \circ g \ , \tag{5.2.28 ℓ}$$

$$r_h\colon g \longmapsto g \circ h \ . \tag{5.2.28 r}$$

It is then the content of [140, Thm. 4.1] (See also [141, Thm. 4.1]) that the measure on $\mathscr{G}(\mathbb{S}^1)$ defined as $\mathbb{Q}^\beta := \zeta_\sharp^{-1}\mathcal{D}_{\beta\overline{\mathsf{m}}}$ is quasi-invariant with respect to the *left* action (5.2.28 ℓ). This has two consequences. On the one hand (See [147, Thm. 3.4]), the measure $\mathcal{D}_\mathsf{m}$ is quasi-invariant with respect to the "left" action $L_h := \zeta \circ \ell_h \circ \zeta^{-1}$ of $\mathrm{Diff}^\infty_+(\mathbb{S}^1)$ on $\mathscr{P}$ corresponding to (5.2.28 ℓ) on $\mathscr{G}(\mathbb{S}^1)$ via ζ. On the other hand (see [140, Cor. 4.2]), the entropic measure $\mathbb{P}^\beta = \chi_\sharp\mathbb{Q}^\beta$ is quasi-invariant with respect to the "right" (because of (5.2.27)) action $R_h := \chi \circ r_h \circ \chi^{-1}$ of $\mathrm{Diff}^\infty_+(\mathbb{S}^1)$ on $\mathscr{P}$ corresponding to (5.2.28 r) on $\mathscr{G}(\mathbb{S}^1)$ via χ.

The action (5.2.28 ℓ) is meaningful only for one-dimensional base spaces, where the representation of μ via its cumulative distribution function makes sense. As a consequence, it is not possible to generalize the results of [147] to base spaces of arbitrary dimension. Analogously, since the R_h-quasi-invariance of the entropic measure $\mathbb{P}^\beta$ is a consequence of the ℓ_h-quasi-invariance of $\mathbb{Q}^\beta$, it is bound to hold only in the case of one-dimensional base spaces.

Notwithstanding this fact, let us notice that

$$g_{h_\sharp\mu} := \zeta^{-1}(h_\sharp\mu) = g_\mu \circ h =: r_h(g_\mu) \ ,$$

thus, the action $K_h := \zeta \circ r_h \circ \zeta^{-1}$ of $\mathrm{Diff}^\infty_+(\mathbb{S}^1)$ on $\mathscr{P}(\mathbb{S}^1)$ is meaningful in the general case, as we detail now. Indeed, let $\mathfrak{G} := \mathrm{Diff}^\infty_+(M)$ be the Lie group of orientation-preserving smooth diffeomorphisms of M. The natural action of $\mathfrak{G}$ on M lifts to an action of $\mathfrak{G}$ on $\mathscr{P}$, given by

$$\begin{aligned} .\colon \mathfrak{G} \times \mathscr{P} &\longrightarrow \mathscr{P} \\ (\psi \ , \ \mu) &\longmapsto \psi_\sharp\mu \end{aligned} \ . \tag{5.2.29}$$

The quasi-invariance of $\mathrm{dvol}_{\mathscr{P}_2}$ with respect to the action $\mathfrak{G} \circlearrowright \mathscr{P}$ is a natural question within representation theory (cf. e.g., [2, 96]), where it corresponds to the action above defining a quasi-regular representation of the infinite-dimensional Lie group $\mathfrak{G}$ on $L^2(\mathscr{P}_2)$. In turn, this relates to the closability of the gradient (5.2.17) on $\mathscr{P}_2$. Indeed, the Lie algebra of $\mathfrak{G}$ is the algebra $\mathfrak{X}^\infty := \Gamma^\infty(TM)$ of smooth vector fields on M and its exponential curves based at $\mathrm{id}_\mathfrak{G} = \mathrm{id}_M$ are precisely the shifts $\psi^{w,t}$ defining the directional derivative (5.2.15).

It turns out that the Dirichlet–Ferguson measure $\mathcal{D}_\mathsf{m}$ is *not* quasi-invariant with respect to the action of $\mathfrak{G}$: Were this the case, then the Gamma measure $\mathcal{G}_\mathsf{m} = \mathcal{D}_\mathsf{m} \otimes \mathsf{G}_[1, \beta]$ too would be quasi-invariant with respect to the analogous action $\mathfrak{G} \circlearrowright \mathscr{M}_b^+(M) = \mathscr{P}(M) \times \mathbb{R}_+$. However, this does not hold (see the introduction to §2.4 in [96]).

In order to address this issue, we recall the following definition from [96, Dfn. 9]. (See also [78, §5.2].)

Definition 5.2.5 (Partial quasi-invariance)**.** $\mathbf{P}$ is termed *partially quasi-invariant* with respect to $G \circlearrowright \Omega$ if there exists a filtration $\mathcal{F}_\bullet := (\mathcal{F}_n)_{n\in\mathbb{N}}$ such that:

(i) $\mathcal{F} = \mathcal{F}_\infty$ the σ-algebra generated by $\mathcal{F}_\bullet$;

(ii) for each $h \in G$ and $n \in \mathbb{N}$ there exists $n' \in \mathbb{N}$ such that $h.\mathcal{F}_n = \mathcal{F}_{n'}$;

(iii) for each $h \in G$ and $n \in \mathbb{N}$ there exists an $\mathcal{F}_n$-measurable $\mathbf{R}_n[h]\colon \Omega \to [0,\infty]$ such that

$$\int_\Omega \mathrm{d}\mathbf{P}^h(\omega)\, u(\omega) = \int_\Omega \mathrm{d}\mathbf{P}(\omega)\, u(\omega)\mathbf{R}_n[h](\omega)$$

for each $\mathcal{F}_n$-measurable semi-bounded $u\colon \Omega \to [-\infty,\infty]$.

If $\mathbf{P}$ is quasi-invariant with respect to $G \circlearrowleft \Omega$, then it is partially quasi-invariant (choose $\mathcal{F}_n = \mathcal{F}$). Finally, $\mathbf{R}_n[h]$ is $\mathbf{P}$-a.e. uniquely defined (see [96, Rmk. 10]).

We let $\mathcal{B}_\bullet(\mathscr{P}_2) := \big(\mathcal{B}_\varepsilon(\mathscr{P}_2)\big)_{\varepsilon\in I}$ be the filtration of σ-algebras on $\mathscr{P}_2$ generated by the functions

$$\mathbf{R}_\varepsilon[\psi]\colon \eta \longmapsto \prod_{x|\eta_x>\varepsilon} \tfrac{\mathrm{d}\psi_\sharp \mathsf{m}}{\mathrm{d}\mathsf{m}}(x) \ , \qquad \psi \in \mathrm{Diff}^\infty_+(M) \ . \tag{5.2.30}$$

Then, $\mathcal{B}_1(\mathscr{P}_2)$ is the trivial σ-algebra and the restriction $\mathcal{B}_0(\mathscr{P}_2)_{\mathscr{P}^{\mathrm{pa}}}$ of $\mathcal{B}_0(\mathscr{P}_2)$ to $\mathscr{P}^{\mathrm{pa}}$ coincides with the Borel σ-algebra $\mathcal{B}(\mathscr{P}_2)_{\mathscr{P}^{\mathrm{pa}}}$ (Lem. 5.4.4). We shall prove the following

Theorem 5.2.6 (See Prop. 5.4.20 and Cor. 5.4.21)**.** *Let $\psi \in \mathrm{Diff}^\infty_+(M)$. Then, **(i)** $\mathcal{D}_\mathsf{m}$ is partially quasi-invariant w.r.t. the action of ψ on the filtration $\big(\mathcal{B}_{1/n}(\mathscr{P}_2)\big)_{n\in\mathbb{N}}$; **(ii)** $\mathcal{D}_\mathsf{m}$ is quasi-invariant w.r.t. the action of ψ if and only if $\psi_\sharp\mathsf{m} = \mathsf{m}$, in which case it is in fact invariant; **(iii)** if $\psi^{w,t}$ is the flow of a smooth vector field w, then $\mathbf{B}_\bullet$, defined in* (5.2.16)*, satisfying*

$$\mathbf{B}_\bullet[w] = \mathrm{d}_t\big|_{t=0} \mathbf{R}_\bullet[\psi^{w,t}] \ ,$$

is a centered square-integrable $\mathcal{D}_\mathsf{m}$-martingale adapted to $\mathcal{B}_\bullet(\mathscr{P}_2)$.

By the theorem, the algebra $\mathfrak{X}$ is decomposed, as a vector space, into a direct sum $\mathfrak{X}^{\mathrm{inv}} \oplus \mathfrak{X}^{\mathrm{pqi}}$, where $\mathfrak{X}^{\mathrm{inv}}$, resp. $\mathfrak{X}^{\mathrm{pqi}}$, denotes the space of vectors such that $\mathcal{D}_\mathsf{m}$ is invariant, resp. partially quasi-invariant not quasi-invariant, with respect to the action of $\psi^{w,t}$. Now, it is readily checked (see, e.g., [10, Rmk. 1.29]) that, if $\psi = \psi^{w,1}$ for some $w \in \mathfrak{X}$, then ψ is m-measure-preserving (i.e. $\psi_\sharp\mathsf{m} = \mathsf{m}$) if and only if w is divergence-free. Thus, $\mathfrak{X}^{\mathrm{inv}} = \mathfrak{X}^{\mathrm{div}}$ the space of divergence-free vector fields, whereas $\mathfrak{X}^{\mathrm{pqi}} = \mathfrak{X}^{\nabla}$ the space of gradient-type vector fields. This is but an instance of the classical *Helmholtz decomposition*, and extends for every η to an orthogonal decomposition of the tangent space $T^{\mathrm{Der}}_\eta\mathscr{P}_2$ into the subspaces $T_\eta\mathscr{P}_2 = \mathrm{cl}_{T^{\mathrm{Der}}_\eta\mathscr{P}_2}(\mathfrak{X}^{\nabla})$ and $\mathrm{cl}_{T^{\mathrm{Der}}_\eta\mathscr{P}_2}(\mathfrak{X}^{\mathrm{div}})$. (Also cf. [10, Prop. 1.28].)

Properties of the process. By the standard theory of Dirichlet forms there is a Markov process $\eta_\bullet$ with state space $\mathscr{P}$ properly associated to $(\mathcal{E}, \mathscr{D}(\mathcal{E}))$ in the sense of [112, Dfn. IV.2.5(i)]. In order to show, as anticipated, that $\eta_\bullet = \mathbf{\Phi} \circ \widehat{\mathsf{W}}_\bullet$, we shall construct finite-dimensional approximations of $\eta_\bullet$ and $\widehat{\mathsf{W}}_\bullet$ and prove their coincidence up to a suitable restriction of the map $\mathbf{\Phi}$. Namely, we construct

- a sequence of Dirichlet forms $(\mathcal{E}^n, \mathscr{D}(\mathcal{E}^n))$ defined as a martingale-type approximation of $(\mathcal{E}, \mathscr{D}(\mathcal{E}))$ w.r.t. the filtration $\big(\mathcal{B}_{1/n}(\mathscr{P}_2)\big)_{n\in\mathbb{N}}$ given by (5.2.30);
- a sequence of Dirichlet form $(\hat{\mathsf{E}}^n, \mathscr{D}(\hat{\mathsf{E}}^n))$ (See Prop. 5.5.3) associated to the processes $\widehat{\mathsf{W}}^n_\bullet$ obtained by truncation of $\widehat{\mathsf{W}}_\bullet$ onto the first n components of the product space $\mathbf{M}$ and onto the first n elements of $\mathbf{s} \in \mathbf{T}$.

We show their coincidence and their generalized Mosco convergence to $(\mathcal{E}, \mathscr{D}(\mathcal{E}))$ in the sense of Kuwae–Shioya [99] (see Prop. 5.5.6). As already noticed in [93, §1] for the Modified Massive Arratia Flow in the case $d = 1$, also in the case $d \geq 2$ we do not expect the family $\big(\widehat{\mathsf{W}}^n_\bullet\big)_n$ to be a compatible family of Feller semigroups in the sense of Le Jan–Raimond [105, Dfn. 1.1]; thus the process $\eta_\bullet$ would not be induced by a stochastic flow.

The previous approximation allows to identify, up to quasi-homeomorphism, the Dirichlet form $\mathcal{E}$ with the Dirichlet form $\widehat{\mathsf{E}}$ associated to $\widehat{\mathsf{W}}_\bullet$, hence to further specify $\eta_\bullet$'s sample-continuity properties and to classify its invariant sets and invariant measures (see Thm. 5.5.17).

Finally, profiting the essential self-adjointness of the generator $\mathbf{L}$ on $\widehat{\mathfrak{Z}}_0$ (Prop. 5.5.7), we are able to show the $\mathcal{D}_\mathsf{m}$-a.e. differentiability of W_2-Lipschitz functions (Prop. 5.5.8) and to provide a one-sided Varadhan-type estimate of the short-time asymptotics for the heat kernel of $\mathcal{E}$ (Cor. 5.5.9).

5.3 Preliminaries

Everywhere in the following let S be any *Hausdorff* topological space with topology $\tau(S)$ and Borel σ-algebra $\mathcal{B}(S)$. We denote by $\mathcal{C}(S)$, resp. $\mathcal{C}_c(S)$, $\mathcal{C}_0(S)$, $\mathcal{C}_b(S)$, $\mathcal{B}_b(S)$, the space of (real-valued) continuous, resp. continuous compactly supported, continuous vanishing at infinity, continuous bounded, bounded (Borel) measurable, functions on $(S, \tau(S))$. Whenever $U \in \tau(S)$, we always regard the spaces $\mathcal{C}_c(U)$ and $\mathcal{C}_0(U)$ as embedded into $\mathcal{C}(S)$ by taking the trivial extension of $f \in \mathcal{C}_c(U)$ identically vanishing on $U^{\mathrm{c}} := S \setminus U$.

For $n \in \mathbb{N}$ and $\mathbf{h}\colon S \to \mathbb{R}^k$, $\mathbf{h} := (h_1, \dots, h_k)$, we set $\|\mathbf{h}\|_\infty := \sup_{s\in S} \max_{i\le k} |h_i(s)|$.

5.3.1 Dirichlet forms. By a Dirichlet form we shall mean either a *symmetric* bilinear Dirichlet form $E(u, v)$ with domain $\mathscr{D}(E) \subset L^2_{\mathsf{n}}(S)$ or the associated functional $E(u) := E(u, u)$, denoted by the same symbol. We adhere to the terminology of [112].

Definition 5.3.1. In the following, we let $(Y, \tau(Y))$ be a Lusin space, (i.e. Y is homeomorphic to a Borel subset of a compact metric space) and n be a fully supported non-negative finite measure on $(Y, \mathcal{B}(Y))$. Every such measure is Radon by [20, Thm. 7.4.3 (Vol. II)].

Definition 5.3.2 (Capacities)**.** Let $(E, \mathscr{D}(E))$ be a strongly local Dirichlet form on Y. Let $K \subset Y$ be compact and $U \subset Y$ be open and such that $K \subset U$. We define the *capacities*

$$\begin{aligned} \operatorname{cap}^{(0)}(K, U) &:= \inf\{E(u) \mid u \in \mathscr{D}(E), \mathbb{1}_K \le u \le \mathbb{1}_U \ \mathsf{n}\text{-a.e.}\} \ , \\ \operatorname{cap}(K, U) &:= \inf\{E_1(u) \mid u \in \mathscr{D}(E), \mathbb{1}_K \le u \le \mathbb{1}_U \ \mathsf{n}\text{-a.e.}\} \ . \end{aligned}$$

If $A \subset B \subset Y$ and A is relatively compact, denote by $\overline{A}$ the closure of A in Y. We set further

$$\operatorname{cap}^{(0)}(A, B) := \inf_{B\subset U\in\tau(Y)} \operatorname{cap}^{(0)}(\overline{A}, U) \ , \qquad \operatorname{cap}(A, B) := \inf_{B\subset U\in\tau(Y)} \operatorname{cap}(\overline{A}, U) \ .$$

Write $\operatorname{cap}^{(0)}(A) := \operatorname{cap}^{(0)}(A, Y)$ and analogously for $\operatorname{cap}(A)$. Every infimum above is always achieved by some $u_{A,B} \in \mathscr{D}(E)$, termed the *equilibrium potential* of the pair (A, B). (See e.g. [59, Thm. 2.1.5] for the case $B = Y$.) We refer the reader to [59, §2.1] for additional properties of capacities which we shall use in the following without explicit mention. Finally, we say that a set $A \subset Y$ is *E-capacitable* if $\operatorname{cap}(A) < \infty$.

Definition 5.3.3 (E-invariance)**.** Let $(E, \mathscr{D}(E))$ be a conservative Dirichlet form on $L^2_{\mathsf{n}}(Y)$. A Borel set $A \subset Y$ is termed E-invariant if

$$\forall u, v \in \mathscr{D}(E) \qquad \mathbb{1}_A\, u \in \mathscr{D}(E) \qquad \text{and} \qquad E(u) = E(\mathbb{1}_A\, u) + E(\mathbb{1}_{A^{\mathrm{c}}}\, u) \ . \tag{5.3.1}$$

A Borel set A is E-invariant if and only if so is A^{c}. Since $\mathbb{1} \in \mathscr{D}(E)$ and $E(\mathbb{1}) = 0$, choosing $u = \mathbb{1}$ in (5.3.1) yields $E(\mathbb{1}_A) = 0$ for every E-invariant A. Finally, recall that if $(E, \mathscr{D}(E))$ is additionally (quasi-)regular with properly associated Markov diffusion process $M_\bullet$, then A is E-invariant iff it is $M_\bullet$-invariant.

5.3.2 Group actions. Let G be a group acting on Y, write $G \circlearrowright Y$ and $g.y \in Y$ for any $g \in G$ and $y \in Y$. We denote by Y/G the quotient of Y by the action of G, always endowed with the quotient topology and the induced Borel σ-algebra, and by $\operatorname{pr}^G\colon Y \to Y/G$ the projection to the quotient.

We say that $A \subset Y$ is G-invariant if $G.A := \{g.y \mid g \in G, y \in A\} \subset A$ (equivalently $G.A = A$) and that $f\colon Y \to \mathbb{R}$ is G-invariant if it is constant on G-orbits, i.e. $f(y) = f(g.y)$ for every $g \in$

G, $y \in Y$. We say that $A \subset Y$ is $(G, \mathfrak{n})$-invariant if there exists a G-invariant $A_1 \in \mathcal{B}(Y)$ such that $A \triangle A_1$ is $\mathfrak{n}$-negligible. If $A \in \mathcal{B}(Y)$ and $(Y, \mathcal{B}(Y), G, \mathfrak{n})$ is a continuous dynamical system with invariant measure $\mathfrak{n}$, then $(G, \mathfrak{n})$-invariance coincides with the classical definition (e.g. [33, Eqn. (1.2.13)]). For any $n \in \mathbb{N}$ let ***(a)*** $Y^{\times n} := \prod_{i \leq n} Y$, resp. $\mathbf{Y} := \prod_{i \in \mathbb{N}} Y$, always endowed with the product topology; and ***(b)***

$$Y_\circ^{\times n} := \left\{ (y_i)_{i \leq n} \mid y_i \neq y_j \text{ for } i \neq j \right\} , \tag{5.3.2}$$

resp. $\mathbf{Y}_\circ$, defined analogously, always endowed with the trace topology of $Y^{\times n}$, resp. $\mathbf{Y}$. Additionally, let $\mathrm{pr}_n \colon \mathbf{Y} \to Y^{\times n}$ be defined by $\mathrm{pr}_n \colon \mathbf{y} := (y_i)_i^\infty \mapsto (y_i)_i^n$.

If $G \circlearrowright Y$, then $G \circlearrowright Y^{\times n}$ and $G \circlearrowright Y_\circ^{\times n}$ coördinate-wise. We say that $G \circlearrowright Y$ is ***(a)*** *transitive* if for every $y_1, y_2 \in Y$ there exists $g \in G$ such that $g.y_1 = y_2$; ***(b)*** *n-transitive* if $G \circlearrowright Y_\circ^{\times i}$ is transitive for every $i \leq n$; ***(c)*** *finitely transitive* if $G \circlearrowright Y_\circ^{\times n}$ is transitive for every finite n; ***(d)*** *σ-transitive* if $G \circlearrowright \mathbf{Y}_\circ$ is transitive. Finally, for $p \in [1, \infty]$, we denote by $L^p_{\mathfrak{n},G}(Y)$ the family of classes $u \in L^p_{\mathfrak{n}}(Y)$ such that u has a G-invariant representative.

Proposition 5.3.4. *Let G be a group acting on Y. Then, a G-invariant subset $A \subset Y$ is Borel measurable if and only if so is $\mathrm{pr}^G(A)$. Furthermore, for every $p \in [1, \infty]$, the space $L^p_{\mathfrak{n},G}$ is isomorphic to the space $L^p_{\mathrm{pr}^G_\sharp \mathfrak{n}}(Y/G)$.*

Proof. Let $\mathrm{pr} := \mathrm{pr}^G$. If $\mathrm{pr}(A)$ is Borel, then so is $A = \mathrm{pr}^{-1}(\mathrm{pr}(A))$ by measurability (continuity) of pr. Vice versa, if A is Borel G-invariant, then so is A^{c}. Moreover, $\mathrm{pr}(G.\{y\})^{\mathrm{c}} = \mathrm{pr}((G.\{y\})^{\mathrm{c}})$, hence, by G-invariance of A, A^{c},

$$\begin{aligned}\mathrm{pr}(A)^{\mathrm{c}} =&\mathrm{pr}(G.A)^{\mathrm{c}} = \mathrm{pr}\left(\cup_{y \in A} G.\{y\}\right)^{\mathrm{c}} = \cap_{y \in A} \mathrm{pr}(G.\{y\})^{\mathrm{c}} \\ =& \cap_{y \in A} \mathrm{pr}((G.\{y\})^{\mathrm{c}}) = \mathrm{pr}\left((\cup_{y \in A} G.\{y\})^{\mathrm{c}}\right) = \mathrm{pr}((G.A)^{\mathrm{c}}) = \mathrm{pr}(A^{\mathrm{c}}) \ .\end{aligned}$$

By continuity of pr, both $\mathrm{pr}(A)$ and $\mathrm{pr}(A)^{\mathrm{c}}$ are analytic (Suslin), thus Borel by [20, Cor. 6.6.10 (Vol. II)]. The second assertion is a straightforward consequence. ∎

5.3.3 Riemannian manifolds. The main object of our analysis are Riemannian manifolds satisfying Assumption 5.2. We refer the reader to the monograph [69] for a detailed account of (stochastic) analysis on manifolds. We state here without proof the main results we shall assume in the following.

For $w \in \mathfrak{X}^\infty$, the algebra of smooth vector fields on M, we denote by $\psi^{w,t}$ its *flow*, satisfying

$$\begin{aligned} \mathrm{d}_t \psi^{w,t}(x) =& w(\psi^{w,t}(x)) \\ \psi^{w,0}(x) =& x \end{aligned} \quad , \qquad x \in M \ , \ t \in \mathbb{R} \ . \tag{5.3.3}$$

As a consequence of its compactness, M enjoys the following additional properties: ***(a)*** for every $w \in \mathfrak{X}^\infty$ the flow $\psi^{w,t}$ is well-defined and a smooth diffeomorphism for every $t \in \mathbb{R}$, with inverse $(\psi^{w,t})^{-1} = \psi^{w,-t}$; ***(b)*** the manifold M is geodesically complete, that is, every geodesic curve is infinitely prolongable to a locally length-minimizing curve; ***(c)*** the Laplace—Beltrami operator Δ^{g} is a densely defined linear operator on $L^2_{\mathsf{m}}(M)$, essentially self-adjoint on $\mathcal{C}^\infty(M)$ and with discrete spectrum; ***(d)*** the manifold M is stochastically complete, that is the heat semigroup $\mathsf{H}_t := e^{-t\Delta^{\mathsf{g}}} \colon L^2_{\mathsf{m}}(M) \to L^2_{\mathsf{m}}(M)$ has (absolutely continuous) kernel with density $y \mapsto \mathsf{h}_t(\,\cdot\,, y)$, satisfying

$$(\mathsf{H}_t \, \mathbb{1}_M)(x) = \int_M \mathrm{d}\overline{\mathsf{m}}(y) \, \mathsf{h}_t(x,y) = 1 \ , \qquad x \in M \ , \ t > 0 \ .$$

For $f \in \mathcal{C}^1(M)$ and $w \in \mathfrak{X}^\infty$ we denote further *(a)* by $\nabla^{\mathsf{g}}_w f = (\mathrm{d}f)w$ the directional derivative of f in the direction w; *(b)* by $\nabla^{\mathsf{g}} f$ the gradient of f; *(c)* by $\mathrm{div}^{\mathsf{m}} w$ the divergence of w induced by the volume measure m, satisfying the integration by parts formula

$$\int_M \mathrm{d}\mathsf{m}\, (\nabla^{\mathsf{g}}_w f_1) \cdot f_2 = -\int_M \mathrm{d}\mathsf{m}\, f_1 \cdot (\nabla^{\mathsf{g}}_w f_2) - \int_M \mathrm{d}\mathsf{m}\, f_1 \cdot f_2 \cdot \mathrm{div}^{\mathsf{m}} w \ .$$

Whenever no confusion may arise, *we drop the superscript* g *from the notation.* We denote variables by a superscript: e.g. $\nabla^z\big|_{z=x} f$ denotes the gradient of f in the variable z computed at the point $x \in M$.

Canonical Dirichlet forms. We endow (M, g) with the *canonical Dirichlet form* $(\mathsf{E}^{\mathsf{g}}, \mathscr{D}(\mathsf{E}^{\mathsf{g}}))$, defined as the closure of the pre-Dirichlet form

$$\mathsf{E}^{\mathsf{g}}(f_1, f_2) := \int_M \mathrm{d}\overline{\mathsf{m}}\, \Gamma^{\mathsf{g}}(f_1, f_2) \ , \qquad f_i \in \mathcal{C}^\infty(M) \ , \tag{5.3.4}$$

where Γ^{g} is the carré du champ operator $\Gamma^{\mathsf{g}}(f_1, f_2) := \frac{1}{2} \langle f_1 \,|\, f_2 \rangle_{\mathsf{g}}$. We stress that the reference measure here is the *normalized* volume $\overline{\mathsf{m}}$ (as opposed to the volume) and that we adhere to the stochastic convention, taking $\frac{1}{2}\Delta^{\mathsf{g}}$ as generator of E^{g} (as opposed to Δ^{g}). If not otherwise stated, by a *Brownian motion on* M we shall mean the diffusion process associated to E^{g}; due to the normalization in the measure, this differs from the usual Brownian motion by a linear deterministic time change.

Conformal rescaling. We will make extensive use of conformal rescaling for metric objects on M, some of which are listed below. Let $a > 0$. Then,

$$\begin{aligned} a\nabla^{a\mathsf{g}} &= \nabla^{\mathsf{g}} \ , & a\Delta^{a\mathsf{g}} &= \Delta^{\mathsf{g}} \ , & \Gamma^{\mathsf{g}}(\,\cdot\,) &:= |\nabla^{\mathsf{g}} \cdot|^2_{\mathsf{g}} = a\Gamma^{a\mathsf{g}}(\,\cdot\,) \ , \\ \mathsf{m}^{a\mathsf{g}} &= a^{d/2}\mathsf{m}^{\mathsf{g}} \ , & \mathsf{h}^{a\mathsf{g}}_t &= \mathsf{h}^{\mathsf{g}}_{t/a} \ , & \mathsf{H}^{a\mathsf{g}}_t &:= e^{-t\Delta^{a\mathsf{g}}} = \mathsf{H}^{\mathsf{g}}_{t/a} \ . \end{aligned} \tag{5.3.5}$$

Product manifolds. Everywhere in the following let $\mathbf{M}$, resp. $\mathbf{I}$, be the infinite-product manifold of M, resp. I, endowed with the respective product topologies, and $\mathbf{T}_\circ \subset \mathbf{T} \subset \boldsymbol{\Delta} \subset \mathbf{I}$ be endowed with the trace topology. Define the topology τ_{u} on $\widehat{\mathbf{M}} := \mathbf{T} \times \mathbf{M}$ as the product topology of the spaces $\mathbf{M}$ and $\mathbf{T}$ and denote by the same symbol the trace topology on any subset of $\widehat{\mathbf{M}}$. For the sake of notational simplicity we let further $\widehat{\mathbf{M}}_\circ := \mathbf{T}_\circ \times \mathbf{M}_\circ$.We always endow $\widehat{\mathbf{M}}$ with the fully supported measure $\widehat{\mathbf{m}}_\beta := \Pi_\beta \otimes \overline{\mathbf{m}}$, concentrated on $\widehat{\mathbf{M}}_\circ$.

Definition 5.3.5. For $n \in \mathbb{N}$ and $\mathbf{s} \in \mathbf{T}_\circ$, we denote by $M^{n,\mathbf{s}}$ the product manifold $M^{\times n}$ endowed with Riemannian metric $\mathsf{g}^{n,\mathbf{s}} := \oplus_\ell^n s_\ell\, \mathsf{g}$, normalized volume measure $\overline{\mathsf{m}}^n := \overline{\mathsf{m}}^{\otimes n}$, canonical form $(\mathsf{E}^{n,\mathbf{s}}, \mathscr{D}(\mathsf{E}^{n,\mathbf{s}}))$, heat semigroup $\mathsf{H}^{n,\mathbf{s}}_t$ with kernel $\mathsf{h}^{n,\mathbf{s}}_t$, and Brownian motion

$$\mathsf{W}^{n,\mathbf{s}}_\bullet := \Big(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}\,, (\mathsf{W}^{n,\mathbf{s}}_t)_{t\geq 0}\,, (P^{n,\mathbf{s}}_{\mathbf{x}})_{\mathbf{x}\in M^{n,\mathbf{s}}}\Big) \ , \tag{5.3.6}$$

where $\mathcal{F}_\bullet$ is the natural filtration in the sense of [112, Dfn. IV.1.8]. For every $\mathbf{x}_0 := \big(x_0^\ell\big)_{\ell\leq n} \in M^{\times n}$ and $t > 0$ one has $\mathsf{W}^{n,\mathbf{s};\mathbf{x}_0}_t = \big(x^1_{t/s_1}, \dots, x^n_{t/s_n}\big)$, where $\big(x^\ell_\bullet\big)_{\ell\leq n}$ are independent Brownian motions on M respectively starting at x_0^ℓ.

Denote further by $\mathbf{M}^{\mathbf{s}}$ the infinite-product manifold $\mathbf{M}$ endowed with the symmetric tensor field $\mathsf{g}^{\mathbf{s}} := \oplus_\ell^\infty s_\ell\, \mathsf{g}$ and normalized volume measure $\overline{\mathbf{m}} := \overline{\mathsf{m}}^{\otimes\infty}$. Each of the above objects is well-defined since $\mathbf{s} \in \mathbf{T}_\circ$, as opposed to $\mathbf{T}$.

Lemma 5.3.6. *The set* $M_\circ^{\times n}$ *is* $\mathsf{W}^{n,\mathbf{s}}_\bullet$*-coexceptional for every* $\mathbf{s} \in \mathbf{T}_\circ$.

Proof. By the standard theory of Dirichlet forms (see, e.g. [59, Thm. 4.1.2(i)] or [112, Thm. 5.29(i)]), the statement is equivalent to the set $M_\circ^{\times n}$ being $\mathsf{E}^{n,\mathbf{s}}$-coexceptional. Let

$$M^n_{i,j} \coloneqq \left\{ \mathbf{x} \in M^{\times n} \mid x_i = x_j \text{ for } i \neq j \right\} \ .$$

Since $M^{\times n} \setminus M_\circ^{\times n} \subset \bigcup_{i,j \mid i \neq j} M^n_{i,j}$, it suffices to show that $\mathrm{cap}_{n,\mathbf{s}}(M^n_{i,j}) = 0$, where $\mathrm{cap}_{n,\mathbf{s}}$ denotes the capacity associated to $(\mathsf{E}^{n,\mathbf{s}}, \mathscr{D}(\mathsf{E}^{n,\mathbf{s}}))$. Without loss of generality, we can assume $i = 1$, $j = 2$. Set $\mathsf{g}_i \coloneqq s_i\, \mathsf{g}$, $\mathsf{d}_{1,2} \coloneqq \mathsf{d}_{\mathsf{g}_1 \oplus \mathsf{g}_2}$ and $B^{1,2}_\varepsilon(A) \coloneqq B^{\mathsf{d}_{1,2}}_\varepsilon(A) \subset M^{\times 2}$ be the ε-neighborhood of $A \subset M^{\times 2}$. Denote by $\mathrm{cap}_{1,2}$, resp. $\mathrm{cap}_{3,\dots,n}$, the capacity of the canonical form $\big(\mathsf{E}^{2,(s_1,s_2)}, \mathscr{D}(\mathsf{E}^{2,(s_1,s_2)})\big)$, resp. $\big(\mathsf{E}^{n-2,(s_3,\dots,s_n)}, \mathscr{D}(\mathsf{E}^{n-2,(s_3,\dots,s_n)})\big)$. For $0 < \varepsilon < r$, let now $u_{1,2,\varepsilon}$ be the equilibrium potential of $B^{1,2}_\varepsilon(\Delta M)$ for the $\mathrm{cap}_{1,2}$. By Lemma 5.6.23,

$$\begin{aligned}
\mathrm{cap}_{n,\mathbf{s}}(M^n_{1,2}) \leq& \mathrm{cap}_{n,\mathbf{s}}\big(B^{1,2}_\varepsilon(\Delta M) \times M^{\times n-2}, M^{\times 2} \times M^{\times n-2}\big) \\
\leq& \mathrm{cap}_{1,2}\big(B^{1,2}_\varepsilon(\Delta M)\big) \left\|\mathbb{1}\right\|^2_{L^2_{\overline{\mathfrak{m}}^{n-2}}} + \mathrm{cap}_{3,\dots,n}(M^{n-2}, M^{n-2}) \left\|u_{1,2,\varepsilon}\right\|^2_{L^2_{\overline{\mathfrak{m}}^2}} \\
\leq& \mathrm{cap}_{1,2}\big(B^{1,2}_\varepsilon(\Delta M)\big) \cdot 1 + 1 \cdot \left\|u_{1,2,\varepsilon}\right\|^2_{L^2_{\overline{\mathfrak{m}}^2}} \\
\leq& 2\, \mathrm{cap}_{1,2}\big(B^{1,2}_\varepsilon(\Delta M)\big) \ .
\end{aligned}$$

The conclusion follows by Proposition 5.6.24 letting $\varepsilon \to 0$. ■

We summarize several results about the canonical form and heat semigroup (kernel) on $\mathbf{M}^{\mathbf{s}}$. For the sake of notational simplicity, $A \coloneqq \{\ell_1, \dots, \ell_k\}$ shall always denote a *finite* subset of $\mathbb{N}$, and we set $\mathbf{x}_A \coloneqq (x_{\ell_1}, \dots, x_{\ell_k})$ and analogously for $\mathbf{s}$. For $m \in \mathbb{N}$, define the algebra of cylinder functions (cf. [6, Eqn. (3)])

$$\mathcal{FC}^m \coloneqq \left\{ u \colon \mathbf{M} \longrightarrow \mathbb{R} \mid u(\mathbf{x}) = F(\mathbf{x}_A) \ , \ F \in \mathcal{C}^m(M^{|A|}) \right\} \ . \tag{5.3.7}$$

Theorem 5.3.7 (Albeverio–Daletskii–Kondratiev, Bendikov–Saloff-Coste)**.** *Fix $\mathbf{s} \in \mathbf{T}_\circ$. Then, the following holds: **(i)** $\mathbf{M}^{\mathbf{s}}$ is a Banach manifold modelled on the space $\ell^\infty(\mathbb{N}; \mathbb{R}^d)$ with norm $\left\|\mathbf{a}\right\|_{\mathbf{s}} \coloneqq \sup_\ell s_\ell \left|a_\ell\right|_{\mathbb{R}^d}$. **(ii)** The form $(\mathsf{E}^{\mathbf{s}}, \mathcal{FC}^1)$ given by (cf. [6, Eqn. (25)])*

$$\mathsf{E}^{\mathbf{s}}(u) \coloneqq \tfrac{1}{2} \int_{\mathbf{M}} \mathrm{d}\overline{\mathbf{m}}(\mathbf{x}) \sum_{\ell \in A} s_\ell^{-1} \left|\nabla^{\mathsf{g}, x_\ell} F(\mathbf{x}_A)\right|^2_{\mathsf{g}_{x_\ell}} \ , \qquad u(\mathbf{x}) = F(\mathbf{x}_A) \in \mathcal{FC}^1 \ , \tag{5.3.8}$$

*is closable. **(iii)** its closure is a regular strongly local Dirichlet form on $L^2_{\overline{\mathbf{m}}}(\mathbf{M})$ with standard core $\mathcal{FC}^1$. Furthermore, it has **(iv)** generator $(\mathbf{\Delta}^{\mathbf{s}}, \mathscr{D}(\mathbf{\Delta}^{\mathbf{s}}))$, essentially self-adjoint on $\mathcal{FC}^2$, with*

$$\mathbf{\Delta}^{\mathbf{s}} u(\mathbf{x}) = \tfrac{1}{2} \sum_{\ell \in A} s_\ell^{-1} \Delta^{\mathsf{g}, x_\ell} F(\mathbf{x}_A) \ , \qquad u(\mathbf{x}) = F(\mathbf{x}_A) \in \mathcal{FC}^2 \ ;$$

*and **(v)** heat kernel $\mathsf{h}^{\mathbf{s}}_\bullet$, defined as in (5.2.4), absolutely continuous w.r.t. $\overline{\mathbf{m}}$ with density in $\mathcal{C}_b(\mathbf{M})$; **(vi)** properly associated Brownian motion $\mathbf{W}^{\mathbf{s}}_\bullet$, defined as in (5.2.6), satisfying*

$$\mathrm{pr}^n \circ \mathbf{W}^{\mathbf{s};\mathbf{x}_0}_t = \mathsf{W}^{\mathbf{s},n;\mathbf{x}_0^{(n)}}_t \ , \qquad \mathbf{x}_0 \in \mathbf{M} \ , \quad t > 0 \ ,$$

where $\mathbf{x}_0^{(n)} \coloneqq \mathrm{pr}^n(\mathbf{x}_0)$ and $\mathsf{W}^{n,\mathbf{s};\mathbf{x}_0^{(n)}}_t$ is defined as in Definition 5.3.5.

Proof. Throughout the proof we shall refer to results in [5, 6] concerned with the infinite-product manifold $\mathbf{M} = \mathbf{M}^{\mathbf{1}}$, rather than with $\mathbf{M}^{\mathbf{s}}$. However, as noted in [6, Rmk. 2.1], this construction is possible and nearly identical for arbitrary $\mathbf{s} \in \mathbf{T}$. Again throughout the proof, we refer to the form (5.3.8) as coinciding with the one in [6, Eqn. (25)].

Assertion ***(i)*** is claimed in [6, p. 284]. Assertion ***(ii)***: The closability of the form is claimed in [6, p. 289]; it is a consequence of the integration by parts formula [5, Eqn. (44)] with $\Lambda_k \equiv \mathbf{0}$ for all k, in the notation of [5]. Assertion ***(iii)***: The fact that $\mathcal{FC}^1$ is a core is straightforward; its standardness is immediate. By [23, Cor. I.5.1.4, Rmk. I.5.1.5], it is sufficient to check strong locality on the core $\mathcal{FC}^1$; by finiteness of the set A in (5.3.7) this is in turn a standard finite-dimensional fact. Assertion ***(iv)*** is claimed in [5, Thm. 4] and [6, Thm. 4.1]. Provided we can identify the semigroup $\mathsf{T}^{\mathbf{s}}_\bullet$ of $(\mathsf{E}^{\mathbf{s}}, \mathscr{D}(\mathsf{E}^{\mathbf{s}}))$ with $\mathsf{H}^{\mathbf{s}}_\bullet$ as in (5.2.5), ***(v)*** is the content of [15, Thm. 1.1] since

$$\sum_n^\infty e^{-2\lambda_1 t/s_n} < \infty \ , \qquad \mathbf{s} := (s_n)_n^\infty \in \mathbf{T}_\circ \ , \ t > 0 \ , \tag{5.3.9}$$

where λ_1 denotes the spectral gap of the Laplace–Beltrami operator of $(M, \mathsf{g}, \mathsf{m})$. In order to prove (5.3.9) it is sufficient to show that $\limsup_n e^{-2\lambda_1 t/(n\, s_n)} < 1$, by the root test. In turn, this is equivalent to $\liminf_n n\, s_n < \infty$. In fact, since $\mathbf{s} \in \mathbf{T}_\circ$, there exists $\lim_n n\, s_n = 0$ by the Abel–Olivier–Pringsheim criterion. The rest of the proof is devoted to the identification of the semigroup $\mathsf{T}^{\mathbf{s}}_\bullet$ with $\mathsf{H}^{\mathbf{s}}_\bullet$. Since $\mathsf{h}^{\mathbf{s}}_t(\mathbf{x}, \mathrm{d}\mathbf{y}) \ll \overline{\mathsf{m}}$, by [82, Lem. 6] we have in fact

$$\mathsf{h}^{\mathbf{s}}_t(\mathbf{x}, \mathrm{d}\mathbf{y}) = \left(\prod_\ell^\infty \mathsf{h}_{t/s_\ell}(x_\ell, y_\ell) \right) \mathrm{d}\overline{\mathsf{m}}(\mathbf{y}) \ , \qquad \mathbf{x}, \mathbf{y} \in \mathbf{M} \ .$$

(In particular, the product of the densities converges). For $u \in \mathcal{FC}^2$ one has (Also cf. [18, §4.1])

$$\begin{aligned}\lim_{t\downarrow 0} \tfrac{1}{t}\big(\mathsf{H}^{\mathbf{s}}_t u - u\big)(\mathbf{x}) &= \lim_{t\downarrow 0} \frac{1}{t} \left(\int_{\mathbf{M}} \mathrm{d}\overline{\mathsf{m}}(\mathbf{y}) \left(\prod_\ell^\infty \mathsf{h}_{t/s_\ell}(x_\ell, y_\ell) \right) F(\mathbf{y}_A) - F(\mathbf{x}_A) \right) \\ &= \lim_{t\downarrow 0} \frac{1}{t} \left(\int_{M^{\times |A|}} \mathrm{d}\overline{\mathsf{m}}^{|A|}(\mathbf{y}) \left(\prod_{\ell \in A} \mathsf{h}_{t/s_\ell}(x_\ell, y_\ell) \right) F(\mathbf{y}) - F(\mathbf{x}) \right) \ .\end{aligned}$$

The standard finite-dimensional computation now shows that the generator, say $(\mathsf{L}^{\mathbf{s}}, \mathscr{D}(\mathsf{L}^{\mathbf{s}}))$, of $\mathsf{H}^{\mathbf{s}}_\bullet$ satisfies $\mathsf{L}^{\mathbf{s}} = \mathbf{\Delta}^{\mathbf{s}}$ on $\mathcal{FC}^2$. This concludes the proof of ***(v)*** by essential self-adjointness ***(iv)*** of $\mathbf{\Delta}^{\mathbf{s}}$ on $\mathcal{FC}^2$. ***(vi)*** is a direct consequence of ***(v)***. ∎

Remark 5.3.8. If $d = 1$, i.e. $M = \mathbb{S}^1$, Theorem 5.3.7***(iv)*** and ***(v)*** are [18, §4.1, Thm.s 4.3, 4.6].

Lemma 5.3.9. *Let* $\mathbf{s} \in \mathbf{T}_\circ$ *and* $\mathsf{W}^{\mathbf{s}}_\bullet$ *be defined as in* (5.2.6). *Then,* $\mathbf{M}_\circ$ *is* $\mathsf{W}^{\mathbf{s}}_\bullet$*-coexceptional.*

Proof. Denote by $\tau^{\mathbf{s}}$ the *first touching time* of $\mathbf{M}_\circ^{\mathrm{c}}$ for $\mathsf{W}^{\mathbf{s}}_\bullet$ in the sense of [112, §IV.5, Eqn. (5.14)]. Since $\mathbf{M}_\circ^{\mathrm{c}}$ is measurable and $\mathsf{W}^{\mathbf{s}}_\bullet$ has infinite life-time, it suffices to show

$$P^{\mathbf{s}}_{\overline{\mathsf{m}}}\{\tau^{\mathbf{s}} < \infty\} = 0 \ . \tag{5.3.10}$$

(Here, $P^{\mathbf{s}}_{\overline{\mathsf{m}}}$ is defined analogously to [112, §IV.1, Eq. (1.4)].) With slight abuse of notation, for every $\mathbf{x}_0 \in \mathbf{M}$ we denote both $\mathbf{x}_0$ and $\mathrm{pr}^n(\mathbf{x}_0)$ by $\mathbf{x}_0$, the distinction being apparent from the contextual index n. Notice that $\mathbf{M} \setminus \mathbf{M}_\circ = \bigcap_n \mathrm{pr}_n^{-1}\big(M^{\times n} \setminus M_\circ^{\times n}\big)$ and $\mathrm{pr}^n \circ \mathsf{W}^{\mathbf{s};\mathbf{x}_0}_\bullet = \mathsf{W}^{n,\mathbf{s};\mathbf{x}_0}_\bullet$ by Theorem 5.3.7***(vi)***. Moreover, since $M^{\times n}$ is compact, pr^n is a closed map, hence $\mathrm{pr}^n\big(\overline{\mathsf{W}^{\mathbf{s}}_{[0,t]}}\big) = \overline{\mathsf{W}^{n,\mathbf{s}}_{[0,t]}}$. (Cf. [50, Cor. 3.1.11].) As a consequence, letting $\tau^{n,\mathbf{s}}$ be the first touching time of $M^{\times n} \setminus M_\circ^{\times n}$ for $\mathsf{W}^{n,\mathbf{s}}_\bullet$,

$$\{\tau^{n,\mathbf{s}} < \infty\} \subset \left\{\tau^{n+1,\mathbf{s}} < \infty\right\} \subset \{\tau^{\mathbf{s}} < \infty\} = \lim_n \{\tau^{n,\mathbf{s}} < \infty\} \ . \tag{5.3.11}$$

Furthermore,

$$P^{\mathbf{s}}_{\overline{\mathsf{m}}}\{\tau^{n,\mathbf{s}} < \infty\} = P^{n,\mathbf{s}}_{\overline{\mathsf{m}}^n}\{\tau^{n,\mathbf{s}} < \infty\} = 0 \tag{5.3.12}$$

since $M^{\times n} \setminus M_\circ^{\times n}$ is $\mathsf{W}^{n,\mathbf{s}}_\bullet$-exceptional for every $\mathbf{s} \in \mathbf{T}_\circ$ by Lemma 5.3.6. Finally, (5.3.11) and (5.3.12) yield (5.3.10) by the Borel–Cantelli Lemma. ∎

5.3.4 Spaces of measures. Let $\mathscr{P} := \mathscr{P}(M)$, resp. $\mathscr{M}_1^+ := \mathscr{M}_1^+(M)$, denote the space of Borel probability, resp. subprobability, measures on M. On $\mathscr{P}$ we consider different topologies, namely ***(a)*** the *narrow* (or *weak*) *topology* τ_n, induced by duality with $\mathcal{C}_b(M)$; ***(b)*** the *weak atomic topology* τ_a [52, §2]; ***(c)*** the *strong* (or *norm*) *topology* τ_s, induced by the total variation.

For $i = 1, 2$ let $\mu_i \in \mathscr{P}$. We denote by $\mathrm{Cpl}(\mu_1, \mu_2)$ the set of couplings of μ_1, μ_2, i.e. the set of probability measures π on $M^{\times 2}$ such that $\mathrm{pr}^i_\sharp \pi = \mu_i$, where $\mathrm{pr}^i \colon M^{\times 2} \to M$ is the projection on the i^{th} component of the product. For $p \in [1, \infty)$, the L^p-Kantorovich–Rubinshtein distance W_p on $\mathscr{P}$ is defined by

$$W_p(\mu_1, \mu_2)^p := \inf_{\pi \in \mathrm{Cpl}(\mu_1, \mu_2)} \int_{M^{\times 2}} \mathrm{d}\pi(x, y)\, \mathsf{d}_\mathsf{g}(x, y)^p \ . \tag{5.3.13}$$

Since M is compact, the narrow topology coincides with both the *vague topology* (induced by duality with $\mathcal{C}_c(M)$) and with the topology induced by W_p for any p. (See, e.g., [165, Cor. 6.13].) For the reader's convenience, we collect here the main properties of the weak atomic topology which we shall dwell upon in the following, taken, almost verbatim, from [52].

Remark 5.3.10. On $\mathscr{P}$ we only consider the Borel σ-algebra $\mathcal{B}_\mathrm{n}(\mathscr{P}) := \mathcal{B}(\mathscr{P}, \tau_\mathrm{n})$; in fact, it holds that $\mathcal{B}_\mathrm{n}(\mathscr{P}) = \mathcal{B}(\mathscr{P}, \tau_\mathrm{a})$. (See [52, p. 5].)

Proposition 5.3.11 (Ethier–Kurtz [52])**.** *The following holds: **(i)** τ_a is strictly finer than τ_n; **(ii)** $(\mathscr{P}, \tau_\mathrm{a})$ is a Polish space; **(iii)** suppose τ_n-$\lim_n \mu_n = \mu_\infty$. For $N \in \overline{\mathbb{N}}$ let $\big(s_{N,i}\delta_{x_{N,i}}\big)_{i \le m_N}$ be the set of atoms of μ_N, ordered so that $s_{N,i-1} \ge s_{N,i}$ for all $i \le m_N \in \overline{\mathbb{N}_0}$. Then,*

$$\tau_\mathrm{a}\text{-}\lim_n \mu_n = \mu_\infty \quad \textit{if and only if} \quad s_{n,i} \longrightarrow s_{\infty,i} \quad \textit{for all } i \le m_\infty \ ;$$

***(iv)** suppose τ_n-$\lim_n \mu_n = \mu_\infty$ and that μ_∞ is purely atomic. Then,*

$$\tau_\mathrm{a}\text{-}\lim_n \mu_n = \mu_\infty \quad \textit{if and only if} \quad \lim_n \sum_i |s_{n,i} - s_{\infty,i}| = 0 \ .$$

*Conversely, if τ_a-$\lim_n \mu_n = \mu_\infty$ and $s_{\infty,k} > s_{\infty,k-1}$ for some $k \le m_\infty$, then the set of locations $\{x_{n,1}, \dots, x_{n,k}\}$ converges to $\{x_{\infty,1}, \dots, x_{\infty,k}\}$. In particular, if $s_{\infty,i} > s_{\infty,i+1}$ for all i, then $x_{n,i} \longrightarrow x_{\infty,i}$ for all i; **(v)** $(\mathscr{P}, \tau_\mathrm{a})$ is* not *compact, even if M is.*

Proof. For a metric metricizing $(\mathscr{P}, \tau_\mathrm{a})$ see [52, Eqn. (2.2)]. For separability and completeness see [52, Lem. 2.3]. The inclusion in ***(i)*** follows by comparison of [52, Eqn. (2.2)] with the Prohorov metric; it is strict by [52, Example 2.7]. For ***(iii)***–***(iv)*** see [52, Lem. 2.5(b)]. Assume M compact. We sketch a proof of ***(v)***. By e.g. [165, Rmk. 6.19], $(\mathscr{P}, \tau_\mathrm{n})$ is compact. Argue by contradiction that $(\mathscr{P}, \tau_\mathrm{a})$ is compact. It is known that a continuous injection from a compact Hausdorff space is a homeomorphism onto its image. Applying this to $\mathrm{id} \colon (\mathscr{P}, \tau_\mathrm{a}) \to (\mathscr{P}, \tau_\mathrm{n})$ contradicts ***(i)***. ■

The Dirichlet–Ferguson measure. Everywhere in the following let $\beta \in (0, \infty)$ be defined by $\mathsf{m} = \beta\overline{\mathsf{m}}$. Let $I := [0, 1]$, resp. $I^\circ := (0, 1)$, be the closed, resp. open, unit interval and set $\widehat{M} := M \times I$ and $\widehat{M}^\circ := M \times I^\circ$, always endowed with the product topology, Borel σ-algebra $\mathcal{B}(\widehat{M})$ and with the measure $\widehat{\mathsf{m}}_\beta := \overline{\mathsf{m}} \otimes \mathrm{B}_\beta$. The next result may be regarded as a corollary of the Mecke identity (5.2.13).

Corollary 5.3.12 (cf. [55, Prop. 1])**.** *Let η be a $\mathcal{D}_\mathsf{m}$-distributed $\mathscr{P}$-valued random field. Then, for every semi-bounded measurable $f \colon M \to \mathbb{R}$,*

$$\mathbf{E}[\eta f] = \overline{\mathsf{m}} f \ . \tag{5.3.14}$$

In particular, for every measurable A it holds with $\mathcal{D}_\mathsf{m}$-probability 1 that $\eta A > 0$ iff $\mathsf{m}A > 0$.

Definition 5.3.13. Denote by $\mathscr{P}^{\mathrm{pa}}$ the set of *purely atomic* probability measures on M; by $\mathscr{P}^{\mathrm{pa}}_{\mathrm{iso}}$ the set of measures $\eta \in \mathscr{P}^{\mathrm{pa}}$ with infinitely many atoms and such that $\eta_{x_1} \neq \eta_{x_2}$ whenever $\eta_{x_1} > 0$ and $x_1 \neq x_2$; by $\mathscr{P}^{\mathrm{fs}}$ the set of *fully supported* Borel probability measures on M. Finally set $\mathscr{P}^{\mathrm{pa,fs}}_{\mathrm{so}} := \mathscr{P}^{\mathrm{fs}} \cap \mathscr{P}^{\mathrm{pa}}_{\mathrm{iso}}$.

Proposition 5.3.14. *The following holds: **(i)** $\mathscr{P}^{\mathrm{pa,fs}}_{\mathrm{so}}, \mathscr{P}^{\mathrm{pa}}_{\mathrm{iso}} \in \mathcal{B}_{\mathrm{n}}(\mathscr{P})$ and $\mathcal{D}_{\mathsf{m}}\mathscr{P}^{\mathrm{pa,fs}}_{\mathrm{so}} = 1$; **(ii)** let $\mathbf{\Phi}$ be defined as in* (5.2.8)*; then, its* non-relabeled *restriction*

$$\mathbf{\Phi}\colon \big(\widehat{\mathbf{M}}_\circ, \tau_{\mathrm{u}}, \widehat{\mathbf{m}}_\beta\big) \longrightarrow (\mathscr{P}^{\mathrm{pa}}_{\mathrm{iso}}, \tau_{\mathrm{a}}, \mathcal{D}_{\beta\overline{\mathsf{m}}})$$

*is a homeomorphism and an isomorphism of measure spaces; **(iii)** the set*

$$\mathscr{N} := \{(\eta, x) \in \mathscr{P} \times M \mid \eta_x > 0\} \times I \subset \mathscr{P} \times \widehat{M} \tag{5.3.15}$$

is $\mathcal{B}_{\mathrm{n}}(\mathscr{P}) \otimes \mathcal{B}(\widehat{M})$-measurable and $\mathcal{D}_{\mathsf{m}} \otimes \widehat{\mathsf{m}}_\beta$-negligible.

Proof. Assertion ***(i)*** is known, see e.g. [52] or [55, §4]. In order to prove ***(ii)*** notice that each of the topologies involved is metrizable (including τ_{a}, by Prop. 5.3.11***(ii)***), thus it suffices to show the continuity of $\mathbf{\Phi}$, resp. $\mathbf{\Phi}^{-1}$, along sequences. To this end, for $N \in \overline{\mathbb{N}}$ let

$$\mathbf{x}_N := (x_{N,i})_i^\infty \in \mathbf{M}_\circ \ , \qquad \mathbf{s}_N := (s_{N,i})_i^\infty \in \mathbf{T}_\circ \ , \qquad \mu_N := \mathbf{\Phi}(\mathbf{s}_N, \mathbf{x}_N) \in \mathscr{P}^{\mathrm{pa}} \ .$$

The latter association is unique, since $\mathbf{\Phi}$ is bijective.

Assume first $\tau_{\mathrm{u}}\text{-}\lim_n(\mathbf{x}_n, \mathbf{s}_n) = (\mathbf{x}_\infty, \mathbf{s}_\infty)$. In particular $\ell^1\text{-}\lim_n \mathbf{s}_n = \mathbf{s}_\infty$. For every $f \in \mathcal{C}_b(M)$,

$$\begin{aligned}\left|\int_M \mathrm{d}\mu_n\, f - \int_M \mathrm{d}\mu_\infty\, f\right| =& \left|\sum_i^\infty \big(s_{n,i} f(x_{n,i}) - s_{\infty,i} f(x_{\infty,i})\big)\right| \\ \leq& \sum_i^\infty s_{n,i}\, |f(x_{n,i}) - f(x_{\infty,i})| + \sum_i^\infty |s_{n,i} - s_{\infty,i}|\, |f(x_{\infty,i})| \\ \leq& \sum_i^\infty s_{n,i}\, |f(x_{n,i}) - f(x_{\infty,i})| + \|f\|_\infty \|\mathbf{s}_n - \mathbf{s}_\infty\|_{\ell^1} \ .\end{aligned}$$

The first term vanishes as $n \to \infty$ by Dominated Convergence Theorem with varying dominating functions $\|f\|_\infty \mathbf{s}_n \in \ell^1(\mathbb{N})$; the second term vanishes by assumption. By arbitrariness of f, $\tau_{\mathrm{n}}\text{-}\lim_n \mu_n = \mu_\infty$, whence $\tau_{\mathrm{a}}\text{-}\lim_n \mu_n = \mu_\infty$ by Proposition 5.3.11***(iv)***. This shows the continuity of $\mathbf{\Phi}$. The continuity of $\mathbf{\Phi}^{-1}$ is precisely the converse statement in Proposition 5.3.11***(iv)***. It follows that $\mathbf{\Phi}$ is bi-measurable. The measure isomorphism property is also known (*Sethuraman stick-breaking representation*). ***(iii)*** The set $\mathscr{N}$ is measurable by Lemma 5.6.2. Moreover, its sections $\mathscr{N}_x := \{\eta \in \mathscr{P} \mid (\eta, x, r) \in \mathscr{N}\}$ are $\mathcal{D}_{\mathsf{m}}$-negligible for every $x \in M$ by Corollary 5.3.12, hence $\mathscr{N}$ is $\mathcal{D}_{\mathsf{m}} \otimes \widehat{\mathsf{m}}_\beta$-negligible itself. ■

Remark 5.3.15. It is not possible to extend the homeomorphism in Proposition 5.3.14***(ii)***, in the sense that the spaces $(\mathscr{P}^{\mathrm{pa}}, \tau_{\mathrm{a}})$ and $\mathbf{T} \times \mathbf{M}_\circ$ are *not* homeomorphic. Clearly, the same holds for $(\mathscr{P}^{\mathrm{pa}}, \tau_{\mathrm{a}})$ and $\mathbf{T}_\circ \times \mathbf{M}$, for which $\mathbf{\Phi}$ is not even bijective.

Remark 5.3.16. It was noticed in [13, Prop. 3.1] that $\mathscr{P}^{\mathrm{pa}}$ is an $F_{\sigma\delta}$-set in $(\mathscr{P}, \tau_{\mathrm{n}})$, and thus so is $\mathscr{P}^{\mathrm{pa,fs}}$. The same holds in τ_{a}. Neither subspace is locally compact in τ_{n}, nor in τ_{a}.

5.4 The Dirichlet form

In this section, assume $d \geq 1$ whenever not stated otherwise.

5.4.1 Cylinder functions. Here, we introduce some spaces of suitably differentiable functions.

Definition 5.4.1 (Cylinder functions)**.** Let $k, \ell, m, n \in \mathbb{N}_0$ and $\mathcal{C}_b^m(\mathbb{R}^k)$ be the space of real-valued bounded m-differentiable functions on $\mathbb{R}^k$ with bounded derivatives of any order up to m.

For $\hat{f} \in \mathcal{B}_b(\widehat{M})$ and $\eta \in \mathscr{P}$ set

$$\hat{f}^\star(\eta) \coloneqq \sum_{x \in \eta} \eta_x \, \hat{f}(x, \eta_x) = \int_M \mathrm{d}\eta(x) \, \hat{f}(x, \eta_x) \ . \tag{5.4.1}$$

For $\hat{f}_i \in \mathcal{B}_b(\widehat{M})$ for $i \leq k$, set $\hat{\mathbf{f}} \coloneqq \big(\hat{f}_1, \dots, \hat{f}_k\big)$ and $\hat{\mathbf{f}}^\star(\eta) \coloneqq \big(\hat{f}_1^\star(\eta), \dots, \hat{f}_k^\star(\eta)\big)$.

For $\varepsilon \in I$ set further $\widehat{M}_\varepsilon \coloneqq M \times (\varepsilon, 1]$. We always regard $\mathcal{C}_c^m(\widehat{M}_\varepsilon)$ as the subspace of $\mathcal{C}^m(\widehat{M})$ obtained by extension by 0. Consistently with this identification, we put $\mathcal{C}^0(\widehat{M}_1) \coloneqq \{0\}$ by convention. Notice that $\mathcal{C}_0^0(\widehat{M}_0) \subsetneq \mathcal{C}^0(\widehat{M})$. We define the following families of *cylinder functions*

$$\begin{aligned}
\widehat{\mathfrak{Z}}^m &\coloneqq \left\{ \begin{matrix} u\colon \mathscr{P} \longrightarrow \mathbb{R} \mid u = F \circ \hat{\mathbf{f}}^\star, \\ F \in \mathcal{C}_b^m(\mathbb{R}^k), \hat{\mathbf{f}} \in \mathcal{C}^m(\widehat{M})^{\otimes k} \end{matrix} \right\} , &\qquad \widehat{\mathfrak{Z}}_-^m &\coloneqq \left\{ \begin{matrix} u\colon \mathscr{P} \to \mathbb{R} \mid u = F \circ \hat{\mathbf{f}}^\star, F \in \mathcal{C}_b^m(\mathbb{R}^k), \\ \hat{f}_i = \mathbb{1}_M \otimes \varrho_i, \varrho_i \in \mathcal{C}^m(I) \quad i \leq k \end{matrix} \right\} , \\
\widehat{\mathfrak{Z}}_\varepsilon^m &\coloneqq \left\{ \begin{matrix} u\colon \mathscr{P} \longrightarrow \mathbb{R} \mid u = F \circ \hat{\mathbf{f}}^\star, \\ F \in \mathcal{C}_b^m(\mathbb{R}^k), \hat{\mathbf{f}} \in \mathcal{C}_c^m(\widehat{M}_\varepsilon)^{\otimes k} \end{matrix} \right\} , &\qquad \mathfrak{Z}^m &\coloneqq \left\{ \begin{matrix} u\colon \mathscr{P} \to \mathbb{R} \mid u = F \circ \hat{\mathbf{f}}^\star, F \in \mathcal{C}_b^m(\mathbb{R}^k), \\ \hat{f}_i = f_i \otimes \mathbb{1}_I, f_i \in \mathcal{C}^m(M) \quad i \leq k \end{matrix} \right\} , \\
\widehat{\mathfrak{Z}}_{-,\varepsilon}^m &\coloneqq \widehat{\mathfrak{Z}}_-^m \cap \widehat{\mathfrak{Z}}_\varepsilon^m \ .
\end{aligned} \tag{5.4.2}$$

For $u \in \widehat{\mathfrak{Z}}^m$ we define the *vanishing threshold* ε_u *of* u by

$$\varepsilon_u \coloneqq \sup \left\{ \varepsilon \in I \mid u \in \widehat{\mathfrak{Z}}_\varepsilon^m \right\} \ .$$

Notice that $\varepsilon_u > 0$ for all $u \in \widehat{\mathfrak{Z}}_0^0$. Finally, for $\varepsilon \in I$, define the family of σ-algebras

$$\mathcal{B}_\varepsilon(\mathscr{P}) \coloneqq \sigma_0\big(\widehat{\mathfrak{Z}}_\varepsilon^\infty\big) \ . \tag{5.4.3}$$

Remark 5.4.2 (Representation of cylinder functions)*.* The representation of u by F and $\hat{\mathbf{f}}$ is never unique. Indeed, assume $u \in \widehat{\mathfrak{Z}}^m$ may be written as $u = F \circ \hat{\mathbf{f}}^\star$ for appropriate F and $\hat{\mathbf{f}}$. By compactness of $\mathscr{P}$ and $\widehat{M}$ and by our definition of the test functions $\hat{f}^\star$, if $G\colon \mathbb{R}^k \to \mathbb{R}$ satisfies $G \equiv F$ identically on $\prod_i^k \mathrm{im} \hat{f}_i$, then $u = G \circ \hat{\mathbf{f}}^\star$. As a consequence: ***(a)*** the families in (5.4.2) remain unchanged if we replace $\mathcal{C}_b^m(\mathbb{R}^k)$ with $\mathcal{C}_c^m(\mathbb{R}^k)$ or $\mathcal{C}^m(\mathbb{R}^k)$; ***(b)*** in particular, if $\hat{f} \in \mathcal{C}^m(\widehat{M})$, then the induced test function $\hat{f}^\star$ belongs to $\widehat{\mathfrak{Z}}^m$ (and analogously for the other families of functions in (5.4.2)); ***(c)*** if additionally F is constant in the direction $\mathbf{e}_j$ on $\mathrm{im} \hat{f}_j$ for some $j \leq k$, then $u = G \circ \hat{\mathbf{g}}^\star$, where $\hat{\mathbf{g}} \coloneqq \big(\hat{f}_1, \dots, \hat{f}_{j-1}, \hat{f}_{j+1}, \dots, \hat{f}_k\big)$ and $G \in \mathcal{C}_b^m(\mathbb{R}^{k-1})$ is such that, for some, hence any, $\bar{t} \in \mathrm{im} \hat{f}_j$,

$$\forall \mathbf{s} \coloneqq (s_1, \dots, s_{k-1}) \in \prod_{i \neq j}^k \mathrm{im} \hat{f}_i \qquad G(\mathbf{s}) = F(s_1, \dots, s_j, \bar{t}, s_{j+1}, \dots, s_{k-1}) \ ;$$

(d) if $u \in \widehat{\mathfrak{Z}}^m$ there exists a minimal k such that $u = F \circ \hat{\mathbf{f}}^\star$ for $F \in \mathcal{C}^m(\mathbb{R}^k)$ and appropriate $\hat{\mathbf{f}}^\star$. If this is the case, we say that u is written in minimal form. In the following, we shall always assume every cylinder function to be written in minimal form.

Remark 5.4.3 (Measurability and continuity of cylinder functions)*.* ***(a)*** Every function $u \in \widehat{\mathfrak{Z}}^0$ is measurable (consequence of Lemma 5.6.1); ***(b)*** every non-constant function $u \in \widehat{\mathfrak{Z}}_0^0$ is τ_n-discontinuous at $\mathcal{D}_\mathsf{m}$-a.e. μ, even for $u \in \widehat{\mathfrak{Z}}_\varepsilon^\infty$, for every $\varepsilon \in I$; ***(c)*** every function $u \in \widehat{\mathfrak{Z}}_0^0$ is τ_a-continuous (consequence of [52, Rmk. 2.6]); ***(d)*** every function in $\mathfrak{Z}^0$ is τ_n-continuous (by definition of τ_n); ***(e)*** $\mathfrak{Z}^m$, $\widehat{\mathfrak{Z}}_-^m$, $\widehat{\mathfrak{Z}}_\varepsilon^m$ and $\widehat{\mathfrak{Z}}^m$ are algebras with respect to the pointwise multiplication of real-valued functions on $\mathscr{P}$ and are closed with respect to pre-composition with m-differentiable functions, i.e. if e.g. $u \in \mathfrak{Z}^m$ and $\psi \in \mathcal{C}^m(\mathbb{R};\mathbb{R})$, then $\psi \circ u \in \mathfrak{Z}^m$; ***(f)*** the sequences $m \mapsto \mathfrak{Z}^m$, $\widehat{\mathfrak{Z}}^m_-$, $\widehat{\mathfrak{Z}}^m_\varepsilon$, $\widehat{\mathfrak{Z}}^m$ are decreasing; ***(g)*** $\mathfrak{Z}^m$, $\widehat{\mathfrak{Z}}^m_-$, $\widehat{\mathfrak{Z}}^m_\varepsilon \subsetneq \widehat{\mathfrak{Z}}^m$ (*strict* inclusion) for every $m \in \overline{\mathbb{N}_0}$ and $\varepsilon \in I$; ***(h)*** $\varepsilon \mapsto \widehat{\mathfrak{Z}}^m_\varepsilon$ is decreasing and left-continuous, in the sense that $\widehat{\mathfrak{Z}}^m_\varepsilon = \bigcup_{\delta>\varepsilon} \widehat{\mathfrak{Z}}^m_\delta$ for every $m \in \overline{\mathbb{N}_0}$ and $\varepsilon \in I$; ***(i)*** $\widehat{\mathfrak{Z}}^0_0 \cap \mathfrak{Z}^0 = \mathbb{R}$ (constant functions); ***(j)*** $\mathcal{B}_\varepsilon(\mathscr{P})$ does not separate points in $\mathscr{P}$ for any $\varepsilon \geq 0$.

Lemma 5.4.4. *It holds that **(i)** $\mathcal{B}_1(\mathscr{P}) = \{\varnothing, \mathscr{P}\}$; **(ii)** $\mathcal{B}_\mathrm{n}(\mathscr{P}^\mathrm{pa}) = \mathcal{B}_0(\mathscr{P})_{\mathscr{P}^\mathrm{pa}}$; and **(iii)** $\mathrm{cl}_{L^2_{\mathcal{D}_\mathsf{m}}(\mathscr{P},\mathcal{B}_\mathrm{n})}(\widehat{\mathfrak{Z}}^\infty_\varepsilon) = \mathrm{cl}_{L^2_{\mathcal{D}_\mathsf{m}}(\mathscr{P},\mathcal{B}_\varepsilon)}(\widehat{\mathfrak{Z}}^\infty_\varepsilon) = L^2_{\mathcal{D}_\mathsf{m}}(\mathscr{P},\mathcal{B}_\varepsilon)$.*

Proof. ***(i)*** is immediate, since $\widehat{\mathfrak{Z}}^\infty_1 = \mathbb{R}$. As for ***(ii)***, notice that the family of pointwise limits of sequences in $\widehat{\mathfrak{Z}}^\infty_0$ contains the algebra

$$\mathfrak{B}^0_0 := \left\{ \begin{matrix} u\colon \mathscr{P} \to \mathbb{R} \mid u := F \circ \hat{\mathbf{f}}^\star, F \in \mathcal{C}^0_b(\mathbb{R}^k), \\ \hat{f}_i = f_i \otimes \mathbb{1}_{(0,1]}, f_i \in \mathcal{C}^0_b(M) \quad i \leq k \end{matrix} \right\} ,$$

and, for $u = F \circ \big((f_i \otimes \mathbb{1}_{(0,1]})_{i\leq k} \big)^\star \in \mathfrak{B}^0_0$ let $\tilde{u} := F \circ \mathbf{f}^\star \in \mathfrak{Z}^0$. Clearly $u(\eta) = \tilde{u}(\eta)$ for every $\eta \in \mathscr{P}^\mathrm{pa}$, hence $\mathcal{B}_0(\mathscr{P})_{\mathscr{P}^\mathrm{pa}} := \sigma_0(\widehat{\mathfrak{Z}}^\infty_0)_{\mathscr{P}^\mathrm{pa}} \supset \sigma_0(\mathfrak{B}^0_0)_{\mathscr{P}^\mathrm{pa}} = \sigma_0(\mathfrak{Z}^0)_{\mathscr{P}^\mathrm{pa}}$. Since τ_n on $\mathscr{P}$ is generated by the linear functionals $f^\star \in \mathfrak{Z}^0$ varying $f \in \mathcal{C}_b(M)$, one has $\sigma_0(\mathfrak{Z}^0)_{\mathscr{P}^\mathrm{pa}} = \mathcal{B}_\mathrm{n}(\mathscr{P}^\mathrm{pa})$. Thus, $\mathcal{B}_0(\mathscr{P})_{\mathscr{P}^\mathrm{pa}} \supset \mathcal{B}_\mathrm{n}(\mathscr{P}^\mathrm{pa})$. On the other hand, $\mathcal{B}_\mathrm{n}(\mathscr{P}) = \mathcal{B}_\mathrm{a}(\mathscr{P}) \supset \mathcal{B}_0(\mathscr{P})$, (by Rmk.s 5.3.10 and 5.4.3***(c)*** respectively) hence $\mathcal{B}_\mathrm{n}(\mathscr{P}^\mathrm{pa}) \supset \mathcal{B}_0(\mathscr{P})_{\mathscr{P}^\mathrm{pa}}$ and the conclusion follows. The first equality in ***(iii)*** is immediate, since $\widehat{\mathfrak{Z}}^\infty_\varepsilon \subset L^2_{\mathcal{D}_\mathsf{m}}(\mathscr{P},\mathcal{B}_\varepsilon)$ by boundedness of functions in $\widehat{\mathfrak{Z}}^\infty_\varepsilon$ and finiteness of $\mathcal{D}_\mathsf{m}$. The second equality is not entirely straightforward (cf. Rmk. 5.4.3***(j)***). It is however a consequence of Proposition 5.6.3 which we postpone to the Appendix. ■

As a consequence of the proof of Lemma 5.4.4***(iii)***, we have that $\mathcal{B}_\varepsilon(\mathscr{P})_{\mathscr{P}^\mathrm{pa}} = \sigma_0(\widehat{\mathfrak{Z}}^m_\varepsilon)_{\mathscr{P}^\mathrm{pa}}$ and that we may replace $\widehat{\mathfrak{Z}}^\infty_\varepsilon$ with $\widehat{\mathfrak{Z}}^m_\varepsilon$ in the statement of Lemma 5.4.4***(iii)*** for any $m \in \mathbb{N}_0$.

Directional derivatives of cylinder functions. In the following, if $\phi\colon M \to M$ is measurable, set $\Phi := \phi_\sharp\colon \mathscr{P} \to \mathscr{P}$. In particular,

$$\Psi^{w,t} := \psi^{w,t}_\sharp .$$

Definition 5.4.5. For $w \in \mathfrak{X}^\infty$ and $u \in \widehat{\mathfrak{Z}}^1$ we define the derivative of u in the direction of w

$$\boldsymbol{\nabla}_w\, u(\eta) := \mathrm{d}_t\big|_{t=0} u\big(\Psi^{w,t}\eta\big) \tag{5.4.4}$$

whenever it exists.

Remark 5.4.6 (Geometries of $\mathscr{P}$). It is important to notice that the *shift* S_{tf} in (5.2.20) (considered in [73, 144, 147]. See [73, p. 546] for the terminology.) is *not* the 'exponential map' of $\mathscr{P}_2$ (i.e. in the sense of the L^2-Wasserstein geometry of $\mathscr{P}$). Rather, it is associated to $\mathscr{P}_1$, where the convex combination $\mu \mapsto \mu^x_t$ *is* a geodesic curve. In fact, the map $\Psi^{w,t}$ (one might suggestively write $(e^{tw})_\sharp$) is also not the exponential map $\mathbf{exp}$ of $\mathscr{P}_2$, studied in [63]. However, $\Psi^{w,t}$ is tangent to $\mathbf{exp}\, t \cdot \boldsymbol{\gamma}$ for some appropriately chosen 'tangent plan' $\boldsymbol{\gamma} \in \mathscr{P}_2(TM)$ depending on w, as shown in the proof of Lemma 4.4.3.

Lemma 5.4.7 (Directional derivative)**.** *Let $u \in \widehat{\mathfrak{Z}}^1$, $w \in \mathfrak{X}^\infty$ and $\eta \in \mathscr{P}$. Then, there exists*

$$\boldsymbol{\nabla}_w\, u(\eta) = \sum_i^k (\partial_i F)(\hat{\mathbf{f}}^\star \eta) \cdot \int_M \mathrm{d}\eta(x) \left\langle \nabla \hat{f}_i(x, \eta_x) \,\middle|\, w(x) \right\rangle_\mathsf{g} . \tag{5.4.5}$$

Furthermore $\boldsymbol{\nabla}_w\colon \mathfrak{Z} \longrightarrow \mathfrak{Z}$ for $\mathfrak{Z} = \widehat{\mathfrak{Z}}^\infty, \widehat{\mathfrak{Z}}^\infty_\varepsilon, \mathfrak{Z}^\infty$ while $\boldsymbol{\nabla}_w\colon \widehat{\mathfrak{Z}}^1_- \longrightarrow \{0\}$, and

$$\big\| \boldsymbol{\nabla}_w\, u \big\|_{L^2_{\mathcal{D}_\mathsf{m}}} \leq \sqrt{k}\, \|\nabla F\|_\infty \max_i \big\|\nabla \hat{f}_i\big\|_{\mathfrak{X}^0} \|w\|_{\mathfrak{X}_{\overline{m}}} .$$

Proof. We show that the curve $t \mapsto u\big(\Psi^{w,t}\eta\big)$ is differentiable for every t. Indeed

$$\begin{aligned}\mathrm{d}_t u\left(\Psi^{w,t}\eta\right) =&\sum_i^k (\partial_i F)(\hat{\mathbf{f}}^\star\eta)\cdot \mathrm{d}_t \int_M \mathrm{d}\left(\Psi^{w,t}\eta\right)(y)\ \hat f_i\left(y, \left(\Psi^{w,t}\eta\right)_y\right)\\ =&\sum_i^k (\partial_i F)(\hat{\mathbf{f}}^\star\eta)\cdot \mathrm{d}_t \int_M \mathrm{d}\eta(y)\ \hat f_i\left(\psi^{w,t}(y),\eta_y\right)\ .\end{aligned}$$

Since $\hat f\in\mathcal{C}^1(\widehat{M})$, differentiation under integral sign yields

$$\mathrm{d}_t u\left(\Psi^{w,t}\eta\right) =\sum_i^k (\partial_i F)(\hat{\mathbf{f}}^\star\eta)\cdot \int_M \mathrm{d}\eta(y)\ \left\langle \nabla \hat f_i\left(\psi^{w,t}(y),\eta_y\right) \,\middle|\, \dot\psi^{w,t}(y)\right\rangle_{\mathsf{g}}$$

Computing at $t=0$ yields (5.4.5). For the second claim, notice that, by smoothness of w, $\big\langle\nabla \hat f_i(\,\cdot\,,\,\cdot\,)\,\big|\, w(\,\cdot\,)\big\rangle_{\mathsf{g}}\in\mathcal{C}^\infty(\widehat{M})$ as soon as $\hat f_i$ is. One can estimate

$$\begin{aligned}\left\|\,\boldsymbol{\nabla}_w\, u\right\|^2_{L^2_{\mathcal{D}_{\mathsf{m}}}} \leq& \|\nabla F\|^2_\infty\cdot k\max_i \int_{\mathscr{P}} \mathrm{d}\mathcal{D}_{\mathsf{m}}(\eta) \left|\int_M \mathrm{d}\eta(x)\ \left\langle \nabla \hat f_i(x,\eta_x)\,\middle|\, w(x)\right\rangle_{\mathsf{g}}\right|^2\\ \leq& k\,\|\nabla F\|^2_\infty\, \max_i \left\|\nabla \hat f_i\right\|^2_{\mathfrak{X}^0} \int_{\mathscr{P}} \mathcal{D}_{\mathsf{m}}(\eta)\,\|w\|^2_{\mathfrak{X}_\eta}\\ =& k\,\|\nabla F\|^2_\infty\, \max_i \left\|\nabla \hat f_i\right\|^2_{\mathfrak{X}^0} \|w\|^2_{\mathfrak{X}_{\overline{\mathsf{m}}}}\end{aligned}$$

by (5.3.14), which concludes the proof. ■

Integration by parts formula. We discuss integration by parts for cylinder functions.

Lemma 5.4.8 (Local derivative and Laplacian)**.** *Let $u:=F\circ\hat{\mathbf{f}}^\star\in\widehat{\mathfrak{Z}}^0_0$. Then, the function $U\colon(\eta,z,r)\mapsto u(\eta^z_r)$ is $\mathcal{B}_{\mathrm{n}}(\mathscr{P})\otimes\mathcal{B}(\widehat{M})$-measurable. Furthermore,*

(i) *if u is in $\widehat{\mathfrak{Z}}^1_0$, then for $\mathcal{D}_{\mathsf{m}}\otimes\widehat{\mathsf{m}}_\beta$-a.e. (η,x,r) the map $z\mapsto U(\eta,z,r)$ is differentiable in a neighborhood of $z=x$ and*

$$\nabla^z_w\big|_{z=x} U(\eta,z,r) = r\sum_i^k (\partial_i F)\big(\hat{\mathbf{f}}^\star(\eta^x_r)\big)\cdot\nabla_w \hat f_i(x,r)\ . \tag{5.4.6}$$

(ii) *if u is in $\widehat{\mathfrak{Z}}^2_0$, then for $\mathcal{D}_{\mathsf{m}}\otimes\widehat{\mathsf{m}}_\beta$-a.e. (η,x,r) the map $z\mapsto U(\eta,z,r)$ is twice differentiable in a neighborhood of $z=x$ and*

$$\begin{aligned}\Delta^z\big|_{z=x} U(\eta,z,r) =& r^2\sum_{i,j}^k (\partial^2_{ij} F)\big(\hat{\mathbf{f}}^\star(\eta^x_r)\big)\cdot\big\langle\nabla \hat f_i(x,r)\,\big|\,\nabla \hat f_j(x,r)\big\rangle_{\mathsf{g}}\\ &+ r\sum_i^k (\partial_i F)\big(\hat{\mathbf{f}}^\star(\eta^x_r)\big)\cdot\Delta \hat f_i(x,r)\ .\end{aligned} \tag{5.4.7}$$

Furthermore, the right-hand sides of (5.4.6) *and* (5.4.7) *are $\mathcal{B}_{\mathrm{n}}(\mathscr{P})\otimes\mathcal{B}(\widehat{M})$-measurable.*

Proof. By continuity of the Dirac embedding $x\mapsto\delta_x$ and Lem. 5.6.1 the function $(\eta,x,r)\mapsto\eta^x_r$ is continuous. As a consequence, U is measurable by Remark 5.4.3. Let $\mathscr{N}$ be as in (5.3.15). For $(\eta,z,r)\notin\mathscr{N}$ and every $\hat f\in\mathcal{C}^0_0(\widehat{M})$ one has

$$\hat f^\star(\eta^z_r) = \sum_{y\in\eta^z_r}(\eta^z_r)_y \hat f\big(y,(\eta^z_r)_y\big) = r\hat f(z,r)+\sum_{y\in\eta}(1-r)\eta_y\hat f\big(y,(1-r)\eta_y\big)\ . \tag{5.4.8}$$

Thus,

$$\nabla^z_w\big|_{z=x} F(\hat{\mathbf{f}}^\star\eta^z_r) =$$

$$
\begin{aligned}
&= \sum_i^k (\partial_i F)(\hat{\mathbf{f}}^\star \eta_r^x) \cdot \nabla_w^z\big|_{z=x} \hat{f}_i^\star(\eta_r^z) \\
&= \sum_i^k (\partial_i F)(\hat{\mathbf{f}}^\star \eta_r^x) \cdot \nabla_w^z\big|_{z=x} \sum_{y\in\eta_r^z} (\eta_r^z)_y\, \hat{f}_i\big(y, (\eta_r^z)_y\big) \\
&= \sum_i^k (\partial_i F)(\hat{\mathbf{f}}^\star \eta_r^x) \cdot \left(\nabla_w^z\big|_{z=x} (\eta_r^z)_z\, \hat{f}_i\big(z, (\eta_r^z)_z\big) + \sum_{y\in\eta} \nabla_w^z\big|_{z=x} (\eta_r^z)_y\, \hat{f}_i\big(y, (\eta_r^z)_y\big) \right) ,
\end{aligned}
$$

where the gradient may be exchanged with the sum, since the latter is always over a finite number of points by the choice of $\hat{f}_i$. In light of (5.4.8),

$$
\begin{aligned}
\nabla_w^z\big|_{z=x} F\big(\hat{\mathbf{f}}^\star(\eta_r^z)\big) =& \sum_i^k (\partial_i F)\big(\hat{\mathbf{f}}^\star(\eta_r^x)\big) \\
&\cdot \left(\nabla_w^z\big|_{z=x} r \hat{f}_i(z,r) + \sum_{y\in\eta} \nabla_w^z\big|_{z=x} (1-r)\eta_y\, \hat{f}_i\big(y, (1-r)\eta_y\big) \right) , \\
=& \sum_i^k (\partial_i F)\big(\hat{\mathbf{f}}^\star(\eta_r^x)\big) \cdot r \nabla_w \hat{f}_i(x,r) \ .
\end{aligned}
$$

By (5.4.6) and arbitrariness of w one has

$$
\nabla^z\big|_{z=x} u(\eta_r^z) = r \sum_i^k (\partial_i F)\big(\hat{\mathbf{f}}^\star(\eta_r^x)\big) \cdot \nabla \hat{f}_i(x,r) \ ,
$$

hence, if u is sufficiently regular,

$$
\begin{aligned}
\Delta^z\big|_{z=x} u(\eta_r^z) =& (\operatorname{div}^{\mathsf{m},z} \circ \nabla^z)\big|_{z=x} u(\eta_r^z) \\
=& r \sum_i^k \big\langle \nabla^z\big|_{z=x} (\partial_i F)\big(\hat{\mathbf{f}}^\star(\eta_r^z)\big) \,\big|\, \nabla \hat{f}_i(x,r) \big\rangle_{\mathsf{g}} + r \sum_i^k (\partial_i F)\big(\hat{\mathbf{f}}^\star(\eta_r^x)\big) \cdot \Delta^z\big|_{z=x} \hat{f}_i(z,r) \\
=& r^2 \sum_{i,j}^k (\partial^2_{ji} F)\big(\hat{\mathbf{f}}^\star(\eta_r^x)\big) \cdot \big\langle \nabla \hat{f}_j(x,r) \,\big|\, \nabla \hat{f}_i(x,r) \big\rangle_{\mathsf{g}} + r \sum_i^k (\partial_i F)\big(\hat{\mathbf{f}}^\star(\eta_r^x)\big) \cdot \Delta \hat{f}_i(x,r) \ .
\end{aligned}
$$

This shows (5.4.6) and (5.4.7) outside the $\mathcal{D}_{\mathsf{m}} \otimes \widehat{\mathsf{m}}_\beta$-negligible set $\mathscr{N}$. ∎

Theorem 5.4.9 (Integration by parts)**.** *Let $w \in \mathfrak{X}^\infty$ and $u := F \circ \hat{\mathbf{f}}^\star$, $v := G \circ \hat{\mathbf{g}}^\star$ be cylinder functions in $\widehat{\mathfrak{Z}}_0^1$. Set $\varepsilon := \varepsilon_u \wedge \varepsilon_v > 0$. Then, the following integration by parts formula holds:*

$$
\int_{\mathscr{P}} \mathrm{d}\mathcal{D}_{\mathsf{m}}\ \boldsymbol{\nabla}_w\, u \cdot v = - \int_{\mathscr{P}} \mathrm{d}\mathcal{D}_{\mathsf{m}}\, u \cdot \boldsymbol{\nabla}_w\, v - \int_{\mathscr{P}} \mathrm{d}\mathcal{D}_{\mathsf{m}}\, u \cdot v \cdot \mathbf{B}_\varepsilon[w] \ , \tag{5.4.9}
$$

where

$$
\mathbf{B}_\varepsilon[w](\eta) := \sum_{x \mid \eta_x > \varepsilon} \operatorname{div}_x^{\mathsf{m}} w \ . \tag{5.4.10}
$$

Proof. We can compute

$$
\begin{aligned}
\int_{\mathscr{P}} \mathrm{d}\mathcal{D}_{\mathsf{m}}\ \boldsymbol{\nabla}_w\, u \cdot v &= \int_{\mathscr{P}} \mathrm{d}\mathcal{D}_{\mathsf{m}}(\eta)\, v(\eta) \cdot \sum_i^k (\partial_i F)\big(\hat{\mathbf{f}}^\star(\eta)\big) \int_M \mathrm{d}\eta(x)\, \big\langle \nabla \hat{f}_i(x,\eta_x) \,\big|\, w(x) \big\rangle_{\mathsf{g}} \\
&= \int_{\mathscr{P}} \mathrm{d}\mathcal{D}_{\mathsf{m}}(\eta) \int_M \mathrm{d}\eta(x)\, v(\eta) \cdot \sum_i^k (\partial_i F)\big(\hat{\mathbf{f}}^\star(\eta)\big) \cdot \big\langle \nabla \hat{f}_i(x,\eta_x) \,\big|\, w(x) \big\rangle_{\mathsf{g}} ,
\end{aligned}
$$

whence, by the Mecke identity (5.2.13) and by (5.4.6),

$$\begin{aligned}&= \int_{\mathscr{P}} \mathrm{d}\mathcal{D}_{\mathsf{m}}(\eta) \int_{\widehat{M}} \mathrm{d}\widehat{\mathsf{m}}_\beta(x,r)\, v(\eta_r^x) \cdot \sum_i^k (\partial_i F)\big(\hat{\mathbf{f}}^\star(\eta_r^x)\big) \big\langle \nabla \hat{f}_i(x,r) \,\big|\, w(x)\big\rangle_{\mathsf{g}} \,, \\ &= \int_{\mathscr{P}} \mathrm{d}\mathcal{D}_{\mathsf{m}}(\eta) \int_0^1 \frac{\mathrm{d}\mathrm{B}_\beta(r)}{\beta}\, \mathbb{1}_{(\varepsilon,1]}(r) \int_M \mathrm{d}\mathsf{m}(x)\, v(\eta_r^x) \cdot \tfrac{1}{r} \nabla_w^z\big|_{z=x} u(\eta_r^z) \,.\end{aligned}$$

Since $\mathrm{bd} M = \varnothing$, integration by parts on M now yields

$$\begin{aligned}\int_{\mathscr{P}} \mathrm{d}\mathcal{D}_{\mathsf{m}}\ \boldsymbol{\nabla}_w\, u \cdot v =& - \int_{\mathscr{P}} \mathrm{d}\mathcal{D}_{\mathsf{m}}(\eta) \int_{\widehat{M}} \mathrm{d}\widehat{\mathsf{m}}_\beta(x,r)\, \nabla_w^z\big|_{z=x} v(\eta_r^z) \cdot u(\eta_r^x) \\ &- \int_{\mathscr{P}} \mathrm{d}\mathcal{D}_{\mathsf{m}}(\eta) \int_{\widehat{M}} \mathrm{d}\widehat{\mathsf{m}}_\beta(x,r)\, \frac{\mathbb{1}_{(\varepsilon,1]}(r)}{r} (uv)(\eta_r^x) \cdot \mathrm{div}_x^{\mathsf{m}} w \\ =& - \int_{\mathscr{P}} \mathrm{d}\mathcal{D}_{\mathsf{m}}(\eta) \int_{\widehat{M}} \mathrm{d}\widehat{\mathsf{m}}_\beta(x,r)\, u(\eta_r^x) \cdot \sum_j^h (\partial_j G)(\hat{\mathbf{g}}^\star \eta_r^x)\ \langle \nabla \hat{g}_j(x,r) \,|\, w(x)\rangle_{\mathsf{g}} \\ &- \int_{\mathscr{P}} \mathrm{d}\mathcal{D}_{\mathsf{m}}(\eta) \int_{\widehat{M}} \mathrm{d}\widehat{\mathsf{m}}_\beta(x,r)\, \frac{\mathbb{1}_{(\varepsilon,1]}(r)}{r} (uv)(\eta_r^x) \cdot \mathrm{div}_x^{\mathsf{m}} w \,.\end{aligned}$$

Applying the Mecke identity (5.2.13) to the first integral yields

$$\begin{aligned}&\int_{\mathscr{P}} \mathrm{d}\mathcal{D}_{\mathsf{m}}(\eta) \int_{\widehat{M}} \mathrm{d}\widehat{\mathsf{m}}_\beta(x,r)\, u(\eta_r^x) \cdot \sum_j^h (\partial_j G)\big(\hat{\mathbf{g}}^\star(\eta_r^x)\big)\ \langle \nabla \hat{g}_j(x,r) \,|\, w(x)\rangle_{\mathsf{g}} \\ &\quad = \int_{\mathscr{P}} \mathrm{d}\mathcal{D}_{\mathsf{m}}(\eta)\, u(\eta) \cdot \boldsymbol{\nabla}_w\, v(\eta) \,.\end{aligned}$$

Applying the Mecke identity (5.2.13) to the second integral instead yields

$$\begin{aligned}&\int_{\mathscr{P}} \mathrm{d}\mathcal{D}_{\mathsf{m}}(\eta) \int_{\widehat{M}} \mathrm{d}\widehat{\mathsf{m}}_\beta(x,r)\, \frac{\mathbb{1}_{(\varepsilon,1]}(r)}{r} (uv)(\eta_r^x) \cdot \mathrm{div}_x^{\mathsf{m}} w \\ &\quad = \int_{\mathscr{P}} \mathrm{d}\mathcal{D}_{\mathsf{m}}(\eta) \int_M \mathrm{d}\eta(x)\, \frac{\mathbb{1}_{(\varepsilon,1]}(\eta_x)}{\eta_x} (uv)(\eta) \cdot \mathrm{div}_x^{\mathsf{m}} w \\ &\quad = \int_{\mathscr{P}} \mathrm{d}\mathcal{D}_{\mathsf{m}}(\eta)\, (uv)(\eta) \cdot \sum_{\eta_x > \varepsilon} \mathrm{div}_x^{\mathsf{m}} w \,.\end{aligned}$$

■

5.4.2 Gradient and Dirichlet form on $\mathscr{P}$. At each point μ in $\mathscr{P}$, the directional derivative $\boldsymbol{\nabla}_w\, u$ of any $u \in \mathscr{D}(\boldsymbol{\nabla}_w)$ defines a linear form $w \mapsto \boldsymbol{\nabla}_w\, u(\mu)$ on $\mathfrak{X}^\infty$. Let $\|\cdot\|_\mu$ be a pre-Hilbert norm on $\mathfrak{X}^\infty$ such that this linear form is continuous, and let $T_\mu\mathscr{P}$ denote the completion of $\mathfrak{X}^\infty$ with respect to the said norm. By Riesz Representation Theorem there exists a unique element $\boldsymbol{\nabla}\, u(\mu)$ in $T_\mu\mathscr{P}$ such that $\boldsymbol{\nabla}_w\, u(\mu) = \big\langle \boldsymbol{\nabla}\, u(\mu) \,\big|\, w\big\rangle_\mu$, where $\langle \cdot \,|\, \cdot \rangle_\mu$ denotes the scalar product of the Hilbert space $T_\mu\mathscr{P}$. Different choices of $\|\cdot\|_\mu$, hence of $T_\mu\mathscr{P}$, yield different gradient maps $\boldsymbol{\nabla}\, u$, namely, as suggested by Lemma 5.4.7, the closures of the operator

$$\boldsymbol{\nabla}\, u(\mu)(x) := \sum_i^k (\partial_i F)\big(\hat{\mathbf{f}}^\star(\mu)\big) \cdot \nabla \hat{f}_i(x, \mu_x) \,, \qquad u := F \circ \hat{\mathbf{f}}^\star \in \widehat{\mathfrak{Z}}^1 \,. \tag{5.4.11}$$

Remark 5.4.10 (Measurability of gradients). The function $x \mapsto \boldsymbol{\nabla}\, u(\mu)(x)$ is measurable for every $u \in \widehat{\mathfrak{Z}}_0^1$ and μ in $\mathscr{P}$ by measurability of $x \mapsto \mu_x$, whereas it is generally discontinuous at $\mathcal{D}_{\mathsf{m}}$-a.e. μ, even for $u \in \widehat{\mathfrak{Z}}^\infty$.

The Dirichlet form $\mathcal{E}$. Throughout this section we fix $T_\mu\mathscr{P} = T^{\mathrm{Der}}_\mu\mathscr{P}_2 := \mathfrak{X}_\mu$. The $\|\cdot\|_{\mathfrak{X}_\mu}$-continuity of $w \mapsto \boldsymbol{\nabla} u(\mu)$, granting that our choice is admissible, readily follows from (5.4.5). We refer the reader to §4.6.1 for the geometrical meaning of this choice.

For $u := F \circ \hat{\mathbf{f}}^\star \in \widehat{\mathfrak{Z}}^0$ where $\hat{\mathbf{f}} := \big(\hat{f}_1, \dots, \hat{f}_k\big)$, denote by $\tilde{u}$ the extension of u to $\mathscr{M}_b(M)$ defined by extending $\hat{f}_i^\star \colon \mathscr{P} \to \mathbb{R}$ to $\hat{f}_i^\star \colon \mathscr{M}_b(M) \to \mathbb{R}$ in the obvious way for all i's. For the purpose of clarity, in the *statement* of the following theorem we distinguish u from $\tilde{u}$. Everywhere else, with slight abuse of notation, we will denote both u and $\tilde{u}$ simply by u.

Theorem 5.4.11. *Assume $d \geq 1$. For u, v in $\widehat{\mathfrak{Z}}^2_0$ set*

$$\begin{aligned}
\mathcal{E}(u,v) &:= \tfrac{1}{2} \int_{\mathscr{P}} \mathrm{d}\mathcal{D}_{\beta\overline{\mathsf{m}}}(\eta) \left\langle \boldsymbol{\nabla} u(\eta) \,\middle|\, \boldsymbol{\nabla} v(\eta) \right\rangle_{\mathfrak{X}_\eta} , \\
\mathbf{L}u(\eta) &:= \tfrac{1}{2} \int_M \mathrm{d}\eta(x)\, \frac{\Delta^z\big|_{z=x} \tilde{u}(\eta + \eta_x\delta_z - \eta_x\delta_x)}{(\eta_x)^2} , \qquad \eta \in \mathscr{P}^{\mathrm{pa}} , && (5.4.12) \\
\boldsymbol{\Gamma}(u,v)(\eta) &:= \tfrac{1}{2} \left\langle \boldsymbol{\nabla} u(\eta) \,\middle|\, \boldsymbol{\nabla} v(\eta) \right\rangle_{\mathfrak{X}_\eta} , \qquad \eta \in \mathscr{P} . && (5.4.13)
\end{aligned}$$

Then, $(\mathbf{L}, \widehat{\mathfrak{Z}}^2_0)$ is a symmetric operator on $L^2_{\mathcal{D}_\mathsf{m}}(\mathscr{P})$ satisfying

$$\forall u, v \in \widehat{\mathfrak{Z}}^2_0 \qquad \mathcal{E}(u,v) = \left\langle u \,\middle|\, -\mathbf{L}v \right\rangle_{L^2_{\mathcal{D}_\mathsf{m}}} .$$

The bilinear form $(\mathcal{E}, \widehat{\mathfrak{Z}}^2_0)$ is a closable symmetric form on $L^2_{\mathcal{D}_\mathsf{m}}(\mathscr{P})$. Its closure $(\mathcal{E}, \mathscr{D}(\mathcal{E}))$ is a strongly local recurrent (in particular: conservative) Dirichlet form with generator the Friedrichs extension $(\mathbf{L}_{\mathrm{F}}, \mathscr{D}(\mathbf{L}_{\mathrm{F}}))$ of $(\mathbf{L}, \widehat{\mathfrak{Z}}^2_0)$. Moreover, $(\mathcal{E}, \mathscr{D}(\mathcal{E}))$ has carré du champ operator $(\boldsymbol{\Gamma}, \mathscr{D}(\boldsymbol{\Gamma}))$ where $\mathscr{D}(\boldsymbol{\Gamma}) := \mathscr{D}(\mathcal{E}) \cap L^\infty_{\mathcal{D}_\mathsf{m}}(\mathscr{P})$, that is, for all $u, v, z \in \mathscr{D}(\boldsymbol{\Gamma})$,

$$2 \int_{\mathscr{P}} \mathrm{d}\mathcal{D}_\mathsf{m}\, z\, \boldsymbol{\Gamma}(u,v) = \mathcal{E}(u, vz) + \mathcal{E}(uz, v) - \mathcal{E}(uv, z) . \tag{5.4.14}$$

Proof. By definition of $\boldsymbol{\nabla} u$,

$$2\,\mathcal{E}(u,v) = \int_{\mathscr{P}} \mathrm{d}\mathcal{D}_\mathsf{m}(\eta) \int_M \mathrm{d}\eta(x) \sum_{i,j}^{k,h} \Big\langle (\partial_i F)\big(\hat{\mathbf{f}}^\star(\eta)\big) \nabla \hat{f}_i(x, \eta_x) \,\Big|\, (\partial_j G)\big(\hat{\mathbf{g}}^\star(\eta)\big) \nabla \hat{g}_j(x, \eta_x) \Big\rangle_{\mathsf{g}} , \tag{5.4.15}$$

whence, by the Mecke identity (5.2.13) and integrating by parts on M,

$$\begin{aligned}
&= \int_{\mathscr{P}} \mathrm{d}\mathcal{D}_\mathsf{m}(\eta) \int_I \mathrm{dB}_\beta(r) \int_M \mathrm{d}\overline{\mathsf{m}}(x) \left\langle \tfrac{1}{r} \nabla^z\big|_{z=x} u(\eta_r^z) \,\middle|\, \tfrac{1}{r} \nabla^z\big|_{z=x} v(\eta_r^z) \right\rangle_{\mathsf{g}} \\
&= - \int_{\mathscr{P}} \mathrm{d}\mathcal{D}_\mathsf{m}(\eta) \int_I \frac{\mathrm{dB}_\beta(r)}{\beta r^2} \int_M \mathrm{d}\mathsf{m}(x)\, u(\eta_r^x) \cdot \Delta^z\big|_{z=x} v(\eta_r^z) \\
&= - \int_{\mathscr{P}} \mathrm{d}\mathcal{D}_\mathsf{m}(\eta) \int_{\widehat{M}} \mathrm{d}\widehat{\mathsf{m}}_\beta(x,r)\, u(\eta_r^x) \cdot \frac{\Delta^z\big|_{z=x} v(\eta_r^x + r\delta_z - r\delta_x)}{r^2} ,
\end{aligned}$$

thus, again by the Mecke identity (5.2.13),

$$= -\,2 \int_{\mathscr{P}} \mathrm{d}\mathcal{D}_\mathsf{m}(\eta)\, u(\eta) \cdot \mathbf{L}v(\eta) .$$

Let $\mathcal{H} := \mathrm{cl}_{L^2_{\mathcal{D}_\mathsf{m}}} \widehat{\mathfrak{Z}}^2_0$, thought of as a Hilbert subspace of $L^2_{\mathcal{D}_\mathsf{m}}(\mathscr{P})$. Clearly $\widehat{\mathfrak{Z}}^2 \subset \mathcal{H}$, hence in particular $\mathfrak{Z}^2 \subset \mathcal{H}$ and the family $\mathfrak{Z}^2$ is a unital nowhere-vanishing algebra of continuous functions (cf. Rem. 5.4.3) separating points in $\mathscr{P}$, thus it is uniformly dense in $\mathcal{C}(\mathscr{P})$ by compactness of $\mathscr{P}$ and the Stone–Weierstraß Theorem. Since $(\mathscr{P}, \mathcal{B}_{\mathrm{n}}(\mathscr{P}), \mathcal{D}_\mathsf{m})$ is a compact Polish probability space, one has $\mathrm{cl}_{L^2_{\mathcal{D}_\mathsf{m}}} \mathcal{C}(\mathscr{P}) = L^2_{\mathcal{D}_\mathsf{m}}(\mathscr{P})$, thus finally $\mathcal{H} = L^2_{\mathcal{D}_\mathsf{m}}(\mathscr{P})$. It is straightforward that (5.4.12) defines a linear operator $\mathbf{L}\colon \widehat{\mathfrak{Z}}_0^2 \to L^2_{\mathcal{D}_\mathsf{m}}(\mathscr{P})$. The symmetry (and coercivity) of the bilinear form $(\mathcal{E}, \widehat{\mathfrak{Z}}_0^2)$ is obvious. Its closability on $L^2_{\mathcal{D}_\mathsf{m}}(\mathscr{P})$ and the existence of the Friedrichs extension $(\mathbf{L}, \mathscr{D}(\mathbf{L}))$ follow from [136, Thm. X.23]. The Markov and strong local properties are also straightforward since $\widehat{\mathfrak{Z}}_0^2$ is closed w.r.t. post-composition with smooth real functions.

By the Leibniz rule for $\boldsymbol{\nabla}$, (5.4.14) holds for all $u, v, z \in \widehat{\mathfrak{Z}}_0^1$. Arbitrary $u, v, z \in \mathscr{D}(\mathcal{E}) \cap L^\infty_{\mathcal{D}_\mathsf{m}}(\mathscr{P})$ may be respectively approximated both in $\mathcal{E}_1^{1/2}$ and $\mathcal{D}_\mathsf{m}$-a.e. by uniformly bounded sequences $u_n, v_n, z_n \in \widehat{\mathfrak{Z}}_0^1$. Thus $\lim_n u_n v_n = uv$, $\lim_n u_n z_n = uz$ and $\lim_n v_n z_n = vz$ in $\mathcal{E}_1^{1/2}$ and

$$\begin{aligned}\lim_n \int_{\mathscr{P}} &\mathrm{d}\mathcal{D}_\mathsf{m} \big| z\,\boldsymbol{\Gamma}(u,v) - z_n \boldsymbol{\Gamma}(u_n, v_n)\big| \\ &\leq \lim_n \int_{\mathscr{P}} \mathrm{d}\mathcal{D}_\mathsf{m}\, |z - z_n|\, \boldsymbol{\Gamma}(u,v) + \lim_n \int_{\mathscr{P}} \mathrm{d}\mathcal{D}_\mathsf{m}\, |z_n| \,\big|\boldsymbol{\Gamma}(u,v) - \boldsymbol{\Gamma}(u_n,v_n)\big| = 0 \ ,\end{aligned}$$

whence (5.4.14) carries over from $\widehat{\mathfrak{Z}}_0^1$ to $\mathscr{D}(\mathcal{E}) \cap L^\infty_{\mathcal{D}_\mathsf{m}}(\mathscr{P})$. Since $\mathbb{1} \in \mathscr{D}(\mathcal{E})$ and $\mathcal{E}(\mathbb{1}) = 0$, the form is recurrent (e.g. [59, Thm. 1.6.3]), thus conservative (e.g. [59, Lem. 1.6.5]). ■

Remark 5.4.12. Notice that $(\mathcal{E}, \mathscr{D}(\mathcal{E}))$, $(\mathbf{L}_\mathrm{F}, \mathscr{D}(\mathbf{L}_\mathrm{F}))$ and $(\boldsymbol{\Gamma}, \mathscr{D}(\boldsymbol{\Gamma}))$ all depend on β. Rigorously we ought to write $\mathcal{E}_{(\beta)}$ for the form $\mathcal{E}$ defined on $L^2_{\mathcal{D}_{\beta\overline{\mathsf{m}}}}(\mathscr{P})$ and analogously for $\boldsymbol{\Gamma}$ and $\mathbf{L}$. We assume $\beta > 0$ to be fixed and drop it from the notation.

Remark 5.4.13. For $u = F \circ \hat{\mathbf{f}}^\star \in \widehat{\mathfrak{Z}}_0^2$ with vanishing threshold ε_u, set, consistently with (5.4.10),

$$\mathbf{B}[\nabla \hat{f}_i](\eta) = \sum_{x \in \eta} \Delta \hat{f}_i(x, \eta_x) = \sum_{x \mid \eta_x > \varepsilon_u} \Delta \hat{f}_i(x, \eta_x) \ , \qquad i \leq k \ ,$$

and

$$\mathbf{L}_1 u := \tfrac{1}{2} \sum_{i,p}^{k} (\partial^2_{ip} F \circ \hat{\mathbf{f}}^\star) \cdot \boldsymbol{\Gamma}(\hat{f}_i^\star, \hat{f}_p^\star) \ , \qquad \mathbf{L}_2 u := \tfrac{1}{2} \sum_{i}^{k} (\partial_i F \circ \hat{\mathbf{f}}^\star) \cdot \mathbf{B}[\nabla \hat{f}_i] \ . \tag{5.4.16}$$

Also notice that

$$\boldsymbol{\Gamma}(\hat{f}_i^\star, \hat{f}_p^\star) = \Gamma(\hat{f}_i, \hat{f}_p)^\star \ , \qquad i, p \leq k \ , \tag{5.4.17}$$

where $\Gamma(\hat{f}_i, \hat{f}_p)(x) := \frac{1}{2} \big\langle \nabla_x \hat{f}_i \,\big|\, \nabla_x \hat{f}_p \big\rangle_\mathsf{g}$ is but the carré du champ operator of the Laplace–Beltrami operator on (M, g). Then, consistently with (5.4.10) and (5.4.9),

$$\mathbf{L} = \mathbf{L}_1 + \mathbf{L}_2 \ , \tag{5.4.18}$$

which makes apparent that $\mathbf{L}u$ is defined everywhere on $\mathscr{P}$ and is identically vanishing on the subspace of diffuse measures. for every $u \in \widehat{\mathfrak{Z}}_0^2$.

The τ_n-regularity of $\mathcal{E}$. In view of Remark 5.4.3, $\mathscr{D}(\mathcal{E})$ might appear unsuitable for the form to be regular, since we defined the latter on a core of *non-continuous* functions. The goal of this section is to show that, in fact, $\mathscr{D}(\mathcal{E})$ contains sufficiently many continuous functions.

Definition 5.4.14 (Sobolev functions of mixed regularity and Sobolev cylinder functions)**.** Denote by I_β the measure space (I, B_β) and consider the space

$$W_{\widehat{\mathsf{m}}_\beta} := L^2\big(I_\beta; W^{1,2}_{\overline{\mathsf{m}}}(M)\big) \cong L^2(I_\beta) \mathbin{\widehat{\otimes}} W^{1,2}_{\overline{\mathsf{m}}}(M) \ ,$$

where $\widehat{\otimes}$ denotes the tensor product of *Hilbert* spaces. It coincides with the completion of $\mathcal{C}^\infty(\widehat{M})$ with respect to the norm defined by

$$\|\hat{f}\|^2_{W_{\widehat{\mathsf{m}}_\beta}} := \int_{\widehat{M}} \mathrm{d}\widehat{\mathsf{m}}_\beta \left(\left|\nabla \hat{f}\right|^2_{\mathsf{g}} + |\hat{f}|^2 \right) .$$

To fix notation, let $\hat{f}\colon (x,s) \mapsto \hat{f}_s(x) := \hat{f}(x,s) \in W_{\widehat{\mathsf{m}}_\beta}$. We denote further by D the distributional gradient on M and by $D^{1,0} := (D \otimes \mathrm{id}_{I^\circ})$ the distributional differential operator given, locally on a chart of $\widehat{M}^\circ$, by differentiation along coördinate directions on M.

By [9, Prop. 3.105], for $\hat{f} \in W_{\widehat{\mathsf{m}}_\beta}$ and for a.e. $s \in I^\circ$ one has that, locally on $\widehat{M}^\circ$, differentiation in the M-directions commutes with restriction in the I°-direction, i.e. $D\hat{f}_s = (D^{1,0}\hat{f})_s$. Thus, for any such $\hat{f}$, the notation $D\hat{f}$ is unambiguous. For $\hat{f} \in W_{\widehat{\mathsf{m}}_\beta}$, we denote by $[\hat{f}, D\hat{f}]$ any of its Borel representatives. We write $[\hat{f}, D\hat{f}]_1$ when referring only to the representative of $\hat{f}$, and $[\hat{f}, D\hat{f}]_2$ when referring only to the representative of $D\hat{f}$. Finally set

$$\widehat{\mathfrak{W}}^{2,2}_b := \left\{ \begin{matrix} u\colon \mathscr{P} \to \mathbb{R} \mid u = F \circ \left([\hat{f}_1, D\hat{f}_1]^\star_1, \dots, [\hat{f}_k, D\hat{f}_k]^\star_1\right), \\ F \in \mathcal{C}^2_b(\mathbb{R}^k) \ , \ \hat{f}_i \in W_{\widehat{\mathsf{m}}_\beta} \ , \ [\hat{f}_i, D\hat{f}_i]_1 \in \mathcal{B}_b(\widehat{M}) \quad i \leq k \end{matrix} \right\} . \tag{5.4.19}$$

Remark 5.4.15. The specification of representatives for both $\hat{f}$ and $D\hat{f}$ in the definition of $\widehat{\mathfrak{W}}^{2,2}_b$ is instrumental to the statement of Lemma 5.4.16 below. It is then the content of the Lemma that such a specification is in fact immaterial.

Lemma 5.4.16. *Let $(\mathcal{E}, \mathscr{D}(\mathcal{E}))$ be defined as in Theorem 5.4.11. Then,* ***(i)*** $\widehat{\mathfrak{Z}}^2 \subset \mathscr{D}(\mathcal{E})$*; and* ***(ii)*** $\widehat{\mathfrak{W}}^{2,2}_b \subset \mathscr{D}(\mathcal{E})$ *and $u \in \mathscr{D}(\mathcal{E})$ of the form* (5.4.19) *does not depend on the choice of the representatives for $\hat{f}_i$. Moreover, for any such u, for $\mathcal{D}_{\mathsf{m}}$-a.e. $\eta \in \mathscr{P}$,*

$$\mathbf{\Gamma}(u)(\eta) = \sum_{i,p} (\partial_i F)\left([\hat{\mathbf{f}}, D\hat{\mathbf{f}}]^\star_1(\eta)\right) \cdot (\partial_p F)\left([\hat{\mathbf{f}}, D\hat{\mathbf{f}}]^\star_1(\eta)\right) \int_M \mathrm{d}\eta(x) \left\langle [\hat{f}_i, D\hat{f}_i]_2 \,\middle|\, [\hat{f}_p, D\hat{f}_p]_2 \right\rangle_{\mathsf{g}} (x, \eta_x)$$

(with usual meaning of the notation $\hat{\mathbf{f}}$) does not depend on the choice of representatives for $\hat{f}_i$.

Proof. ***(i)*** Let $u = F \circ \hat{\mathbf{f}} \in \widehat{\mathfrak{Z}}^2$ and $\varrho_n \in \mathcal{C}^\infty(I)$ be such that $\varrho_n \uparrow_n \mathbb{1}_I$ pointwise and $\mathrm{supp}\varrho_n \subset [1/n, 1]$. For $i \leq k$ set $\hat{f}_{n,i} := \varrho_n \cdot \hat{f}_i$ and notice that $\hat{f}_{n,i} \in \mathcal{C}^2(\widehat{M}_{1/n})$, hence $u_n := F \circ \hat{\mathbf{f}}_n \in \widehat{\mathfrak{Z}}^2_{1/n}$ for every $n \in \mathbb{N}$. It is straightforward that

$$\max_{i \leq k} \lim_n \left\|\hat{f}_{n,i} - \hat{f}_i\right\|_{W_{\widehat{\mathsf{m}}_\beta}} = 0 ,$$

hence there exists $C_u > 0$ such that $\max_{i\leq k} \sup_n \left\|\hat{f}_{n,i}\right\|_{W_{\widehat{\mathsf{m}}_\beta}} \leq C_u$. Thus,

$$\begin{aligned} 2\,\mathcal{E}(u_n - u_m) =& \mathbf{E}_{\mathcal{D}_{\mathsf{m}}} \left[\int_M \mathrm{d}\eta(x) \left| \sum_i^k \Big((\partial_i F)\big(\hat{\mathbf{f}}^\star_n(\eta)\big) \cdot \nabla \hat{f}_{n,i} - (\partial_i F)\big(\hat{\mathbf{f}}^\star_m(\eta) \cdot \nabla \hat{f}_{m,i}\big) \Big) \right|^2_{\mathsf{g}} (x, \eta_x) \right] \\ \leq& 2\mathbf{E}_{\mathcal{D}_{\mathsf{m}}} \left[\int_M \mathrm{d}\eta(x) \left| \sum_i^k (\partial_i F)\big(\hat{\mathbf{f}}^\star_n(\eta)\big) \cdot (\nabla \hat{f}_{n,i} - \nabla \hat{f}_{m,i})(x, \eta_x) \right|^2_{\mathsf{g}} \right] \\ &+ 2\mathbf{E}_{\mathcal{D}_{\mathsf{m}}} \left[\int_M \mathrm{d}\eta(x) \left| \sum_i^k \Big((\partial_i F)\big(\hat{\mathbf{f}}^\star_n(\eta)\big) - (\partial_i F)\big(\hat{\mathbf{f}}^\star_m(\eta)\big) \Big) \cdot \nabla \hat{f}_{m,i}(x, \eta_x) \right|^2_{\mathsf{g}} \right] \\ \leq& 2^k k \cdot \mathrm{Lip}(F)^2 \cdot \sum_i^k \mathbf{E}_{\mathcal{D}_{\mathsf{m}}} \left[\int_M \mathrm{d}\eta(x) \left| \nabla \hat{f}_{n,i} - \nabla \hat{f}_{m,i} \right|^2_{\mathsf{g}} (x, \eta_x) \right] \\ &+ 2^k k \cdot \max_i \mathrm{Lip}(\partial_i F)^2 \cdot C_u^2 \cdot \sum_i^k \mathbf{E}_{\mathcal{D}_{\mathsf{m}}} \left[\left| \int_M \mathrm{d}\eta(x) \left(\hat{f}_{n,i} - \hat{f}_{m,i}\right)(x, \eta_x) \right|^2 \right] \end{aligned}$$

$$\leq K_u \cdot \sum_i^k \mathcal{E}_1\big(\hat{f}^\star_{n,i} - \hat{f}^\star_{m,i}\big) \ , \tag{5.4.20}$$

for some appropriate constant K_u, independent of i, n, m. Now, by Jensen inequality

$$\begin{aligned} 2\,\mathcal{E}_1\big(\hat{f}^\star_{n,i} - \hat{f}^\star_{m,i}\big) =& \mathbf{E}_{\mathcal{D}_\mathsf{m}}\left[\Gamma(\hat{f}_{n,i} - \hat{f}_{m,i})^\star + \left|(\hat{f}_{n,i} - \hat{f}_{m,i})^\star\right|^2\right] \\ \leq& \mathbf{E}_{\mathcal{D}_\mathsf{m}}\left[\Gamma\big(\hat{f}_{n,i} - \hat{f}_{m,i}\big)^\star + \big(\big|\hat{f}_{n,i} - \hat{f}_{m,i}\big|^2\big)^\star\right] \ , \end{aligned}$$

thus, by the Mecke identity (5.2.13)

$$\begin{aligned} 2\,\mathcal{E}_1(\hat{f}^\star_{n,i} - \hat{f}^\star_{m,i}) \leq& \mathbf{E}_{\mathcal{D}_\mathsf{m}}\mathbf{E}_{\widehat{\mathsf{m}}_\beta}\left[\Gamma(\hat{f}_{n,i} - \hat{f}_{m,i}) + \left|\hat{f}_{n,i} - \hat{f}_{m,i}\right|^2\right] \\ =& \mathbf{E}_{\widehat{\mathsf{m}}_\beta}\left[\Gamma(\hat{f}_{n,i} - \hat{f}_{m,i}) + \left|\hat{f}_{n,i} - \hat{f}_{m,i}\right|^2\right] \\ =& \left\|\hat{f}_{n,i} - \hat{f}_{m,i}\right\|^2_{W_{\widehat{\mathsf{m}}_\beta}} \ . \end{aligned} \tag{5.4.21}$$

This shows that the sequence $\big(\mathcal{E}(u_n)\big)_{n\in\mathbb{N}}$ is fundamental (hence bounded). Analogously, one can show that the sequence u_n converges to u strongly in $L^2_{\mathcal{D}_\mathsf{m}}(\mathscr{P})$. Thus, $u \in \mathscr{D}(\mathcal{E})$ by [112, Lem. I.2.12]. Since $\big(\mathcal{E}_1(u_n)\big)_{n\in\mathbb{N}}$ is fundamental, letting $n \to \infty$ in (5.4.21) and combining it with (5.4.20) yields

$$\lim_n \mathcal{E}_1(u_n - u) = 0 \ , \qquad u \in \widehat{\mathfrak{Z}}^2 \ , \ u_n \in \widehat{\mathfrak{Z}}^2_{1/n} \ . \tag{5.4.22}$$

Notice that the condition $[\hat{f}_i, D\hat{f}_i]_1 \in \mathcal{B}_b(\widehat{M})$ grants that $[\hat{f}_i, D\hat{f}_i]^\star_1(\eta)$ is well-defined by (5.4.1). The measurability of u follows as in Remark 5.4.3. As a consequence, ***(ii)*** may be proven similarly to ***(i)*** by $\|\cdot\|_{W_{\widehat{\mathsf{m}}_\beta}}$-density of $\mathcal{C}^2(\widehat{M})$ in $W_{\widehat{\mathsf{m}}_\beta}$. ■

Remark 5.4.17. In view of Lemma 5.4.16***(ii)***, everywhere in the following we write $\hat{f}^\star$ in lieu of $[\hat{f}, D\hat{f}]^\star_1 \in \widehat{\mathfrak{W}}^{2,2}_b \setminus \widehat{\mathfrak{Z}}^2$. Analogously, for any such $\hat{f}$ we will write $\mathbf{\Gamma}(\hat{f}^\star) = \big(\big|D\hat{f}\big|^2_\mathsf{g}\big)^\star$ omitting any explicit indication of the representative of $\hat{f} \in W_{\widehat{\mathsf{m}}_\beta}$. We notice that, with a little more effort, one could show that $\hat{f}^\star$ is a well-defined element of $L^2_{\mathcal{D}_\mathsf{m}}$ for any $\hat{f} \in L^2(\widehat{M}, \widehat{\mathsf{m}}_\beta)$ and independent of the choice of representatives for $\hat{f}$. In a similar way, one can show that $\widehat{\mathfrak{W}}^{2,2} \subset \mathscr{D}(\mathcal{E})$, with obvious meaning of the notation $\widehat{\mathfrak{W}}^{2,2}$ (as opposed to $\widehat{\mathfrak{W}}^{2,2}_b$).

Corollary 5.4.18. *Let $u \in \mathscr{D}(\mathcal{E})$. Then, there exists $(u_n)_{n\in\mathbb{N}}$ such that **(a)** $u_n \in \widehat{\mathfrak{Z}}^2_{1/n}$ for all $n \in \mathbb{N}$; **(b)** $\mathcal{E}_1^{1/2}$-$\lim_n u_n = u$; **(c)** $\mathcal{D}_\mathsf{m}$-$\lim_n u_n = u$; **(d)** $\mathcal{D}_\mathsf{m}$-$\lim_n \left\|\boldsymbol{\nabla} u_n - \boldsymbol{\nabla} u\right\|_{\mathfrak{X}_\cdot} = 0$.*

Proof. ***(c)*** and ***(d)*** are a standard consequence of ***(b)*** up to passing to a suitable subsequence. Thus, it suffices to show ***(a)*** and ***(b)***, which in turn follow by Lemma 5.4.16***(i)*** and an $\varepsilon/3$-argument. ■

Corollary 5.4.19. *Assume $d \geq 2$. The Dirichlet form $(\mathcal{E}, \mathscr{D}(\mathcal{E}))$ on $L^2(\mathscr{P}, \tau_\mathrm{n}, \mathcal{D}_\mathsf{m})$ is a regular strongly local recurrent (in particular: conservative) Dirichlet form with standard core*

$$\mathfrak{L}^{2,1} := \left\{ \begin{matrix} u\colon \mathscr{P} \to \mathbb{R} \mid u := F \circ \hat{\mathbf{f}}^\star, F \in \mathcal{C}^2_b(\mathbb{R}^k), \\ \hat{f}_i = f_i \otimes \mathbb{1}_I, f_i \in \mathrm{Lip}(M) \quad i \leq k \end{matrix} \right\} \ .$$

Proof. The family $\mathfrak{Z}^\infty \subset \widehat{\mathfrak{W}}^{2,2}_b \subset \mathscr{D}(\mathcal{E})$ is uniformly dense in $\mathcal{C}(\mathscr{P})$ as in the proof of Thm. 5.4.11 *Closability*. A proof that $\mathfrak{Z}^\infty$ is also dense in $\mathscr{D}(\mathcal{E})$ if $d \geq 2$ is postponed to Lemma 5.6.26.

One has $\mathfrak{L}^{2,1} \subset \widehat{\mathfrak{W}}^{2,2}_b \subset \mathscr{D}(\mathcal{E})$ and in fact $\mathfrak{L}^{2,1} \subset \mathscr{D}(\mathcal{E})$ as in the proof of Lemma 5.4.16***(ii)***. In particular, for any $f \in \mathrm{Lip}(M)$, the section Df is defined on A^c, where the singular set A of f

satisfies $A \in \mathcal{B}(M)$ and $\mathsf{m}A = 0$ by the classical Rademacher Theorem. For $u = F \circ \mathbf{f}^\star \in \mathfrak{L}^{2,1}$ with $\mathbf{f} := (f_1, \dots, f_k)$, let A_i denote the singular set of f_i and set $A := \cup_{i \le k} A_i$. Then,

$$\forall u \in \mathfrak{L}^{2,1}\ , \qquad \text{for } \mathcal{D}_\mathsf{m}\text{-a.e. } \eta \qquad \mathbf{\Gamma}(u)(\eta) = \tfrac{1}{2} \int_M \mathrm{d}\eta(x) \sum_{i,p}^k (\partial_i F)(\mathbf{f}^\star \eta) \cdot (\partial_p F)(\mathbf{f}^\star \eta) \cdot \langle Df_i \,|\, Df_p \rangle_\mathsf{g}$$

is well-defined, since the set of measures $\eta \in \mathscr{P}^{\mathrm{pa,fs}}_{\mathrm{so}}$ charging A is $\mathcal{D}_\mathsf{m}$-negligible by Corollary 5.3.12. The fact that $\mathfrak{L}^{2,1}$ is a standard core is a straightforward consequence of the definition of $\mathfrak{L}^{2,1}$ and of the classical chain rule. ■

Partial quasi-invariance of $\mathcal{D}_\mathsf{m}$ and integration by parts formula. The following result is heuristically clear from the analogous result [96, Thm. 13] for the Gamma measure. However, it seems to us that it cannot be rigorously deduced from it. Thus, we provide an independent proof.

Proposition 5.4.20. *The measure $\mathcal{D}_\mathsf{m}$ is partially quasi-invariant with respect to the action $\mathfrak{G} \circlearrowleft \mathscr{P}$ as in (5.2.29) on the filtration $\mathcal{B}_\bullet(\mathscr{P}^{\mathrm{pa}}) := \big(\mathcal{B}_{1/n}(\mathscr{P}^{\mathrm{pa}})\big)_{n\in\mathbb{N}}$ as in (5.4.3), with Radon–Nikodým derivatives $\mathbf{R}_{1/n}[\psi]$ as in (5.2.30).*

Proof. It suffices to establish ***(iii)*** in Definition 5.2.5. Indeed ***(i)*** was noticed in Definition 5.4.1 and ***(ii)*** is straightforward with $n' = n$. In order to check ***(iii)*** it suffices to restrict ourselves to functions $u \in \widehat{\mathfrak{Z}}^\infty_{1/n}$, since they generate $\mathcal{B}_{1/n}(\mathscr{P}^{\mathrm{pa}})$ by definition. Now, for $\psi \in \mathrm{Diff}^\infty_+(M)$,

$$\int_{\mathscr{P}} \mathrm{d}(\psi.)_\sharp \mathcal{D}_\mathsf{m}(\eta)\, u(\eta) = \int_{\mathscr{P}^{\mathrm{pa}}_{\mathrm{iso}}} \mathrm{d}\big(\mathbf{\Phi}_\sharp \widehat{\mathbf{m}}_\beta\big)(\eta)\, u(\psi_\sharp \eta)$$

by Proposition 5.3.14***(ii)***. Let $\eta = \sum_i^\infty s_i \delta_{x_i}$. Since $s_1 > s_2 > \dots$, one has $s_{n+1} < 1/n$, hence, for every $i \le k$, every $x \in M$ and every $n' > n$ it holds that $\hat{f}_j(x, s_{n'}) = 0$ by definition of $\hat{f}_i$. Thus,

$$\begin{aligned}
&\int_{\mathscr{P}} \mathrm{d}(\psi.)_\sharp \mathcal{D}_\mathsf{m}(\eta)\, u(\eta) = \\
&= \int_{\mathbf{M}_\circ} \mathrm{d}\overline{\mathbf{m}}(\mathbf{x}) \int_{\mathbf{T}_\circ} \mathrm{d}\Pi_\beta(\mathbf{s})\, F\left(\sum_i^\infty s_i \hat{f}_1(\psi(x_i), s_i), \dots, \sum_i^\infty s_i \hat{f}_k(\psi(x_i), s_i)\right) \\
&= \int_{M^{\times n}} \mathrm{d}\overline{\mathsf{m}}^n(x_1, \dots, x_n) \int_{\mathbf{T}_\circ} \mathrm{d}\Pi_\beta(\mathbf{s})\, F\left(\sum_i^n s_i \hat{f}_1(\psi(x_i), s_i), \dots, \sum_i^n s_i \hat{f}_k(\psi(x_i), s_i)\right) \\
&= \int_{M^{\times n}} \mathrm{d}(\psi_\sharp \overline{\mathsf{m}})^{\otimes n}(x_1, \dots, x_n) \int_{\mathbf{T}_\circ} \mathrm{d}\Pi_\beta(\mathbf{s})\, F\left(\sum_i^n s_i \hat{f}_1(x_i, s_i), \dots, \sum_i^n s_i \hat{f}_k(x_i, s_i)\right) \\
&= \int_{M^{\times n}} \mathrm{d}\overline{\mathsf{m}}^n(x_1, \dots, x_n) \prod_i^n J^\mathsf{m}_\psi(x_i) \int_{\mathbf{T}_\circ} \mathrm{d}\Pi_\beta(\mathbf{s})\, F\left(\sum_i^n s_i \hat{f}_1(x_i, s_i), \dots, \sum_i^n s_i \hat{f}_k(x_i, s_i)\right) \\
&= \int_{\mathbf{M}_\circ} \mathrm{d}\overline{\mathbf{m}}(\mathbf{x}) \int_{\mathbf{T}_\circ} \mathrm{d}\Pi_\beta(\mathbf{s}) \left(\prod_{i \mid s_i > 1/n} J^\mathsf{m}_\psi(x_i)\right) F\left(\sum_i^n s_i \hat{f}_1(x_i, s_i), \dots, \sum_i^n s_i \hat{f}_k(x_i, s_i)\right) \\
&= \int_{\mathscr{P}} \mathrm{d}\mathcal{D}_\mathsf{m}(\eta)\, \mathbf{R}_{1/n}[\psi](\eta) \cdot u(\eta)\ .
\end{aligned}$$

■

Together with the definition of partial quasi-invariance, Proposition 5.4.20 suggests that some of the quantities we defined in terms of the σ-algebras $\mathcal{B}_\varepsilon(\mathscr{P})$ ought to be martingales with respect to the filtration $\big(\mathcal{B}_\varepsilon(\mathscr{P})\big)_{\varepsilon\in I}$. Indeed, this turns out to be the case. The following is a corollary of Theorem 5.4.9.

Corollary 5.4.21. *Let $w \in \mathfrak{X}^\infty$ and $\mathbf{B}_\varepsilon[w]$ be defined as in* (5.4.10)*. Then, with the same notation of Proposition 5.4.20, **(i)** the stochastic process $\mathbf{B}_\bullet[w] := \big(\mathbf{B}_\varepsilon[w]\big)_{\varepsilon\in I}$ is a centered square-integrable martingale on $\big(\mathscr{P}^{\mathrm{pa}}, \mathcal{B}_{\mathrm{n}}(\mathscr{P}^{\mathrm{pa}}), \mathcal{D}_{\mathsf{m}}\big)$ with respect to the filtration $\mathcal{B}_\bullet(\mathscr{P}^{\mathrm{pa}}) := (\mathcal{B}_\varepsilon(\mathscr{P}^{\mathrm{pa}}))_{\varepsilon\in I}$ (cf.* (5.4.3)*); **(ii)** it holds that*

$$\mathbf{B}_\bullet[w] = \mathrm{d}_t\big|_{t=0}\mathbf{R}_\bullet[\psi^{w,t}] \ ;$$

***(iii)** the quadratic form*

$$\mathcal{A}_0^w \colon \widehat{\mathfrak{Z}}_0^1 \times \widehat{\mathfrak{Z}}_0^1 \ni (u,v) \longmapsto \mathbf{E}_{\mathcal{D}_{\mathsf{m}}}\big[u\cdot v\cdot \mathbf{B}_\varepsilon[w]\big] \qquad \varepsilon := \varepsilon_u \wedge \varepsilon_v$$

is $\mathcal{E}_1^{1/2}$-bounded, and uniquely extends to an $\mathcal{E}_1^{1/2}$-bounded quadratic form $\mathcal{A}^w$ on $\mathscr{D}(\mathcal{E})$;

Proof. **(*i*)** Let $\delta > \varepsilon > 0$ and $\varrho_{\varepsilon,\delta} \in \mathcal{C}_c^\infty([\varepsilon,1];\mathbb{R})$ be such that $\varrho_{\varepsilon,\delta}(r) = 1/r$ for every $r \geq \delta$. Set $\hat f_{w,\varepsilon,\delta} := \operatorname{div}^{\mathsf{m}} w \otimes \varrho_{\varepsilon,\delta} \in \widehat{\mathfrak{Z}}_\varepsilon^\infty$ and notice that $\mathbf{B}_\varepsilon[w] = \lim_{\delta\downarrow\varepsilon} \hat f^\star_{w,\varepsilon,\delta}$ pointwise on $\mathscr{P}^{\mathrm{pa}}$. Thus, $\mathbf{B}_\varepsilon[w]$ is $\mathcal{B}_\varepsilon(\mathscr{P}^{\mathrm{pa}})$-measurable. This shows that the process $\mathbf{B}_\bullet[w]$ is adapted to $\mathcal{B}_\bullet(\mathscr{P}^{\mathrm{pa}})$. Moreover,

$$\big|\mathbf{B}_\varepsilon[w](\eta)\big| \leq \sum_{x \mid \eta_x > \varepsilon} \|\operatorname{div}^{\mathsf{m}} w\|_{\mathcal{C}^0} \leq \lfloor \varepsilon^{-1} \rfloor \, \|\operatorname{div}_{\mathsf{m}} w\|_{\mathcal{C}^0} \ . \tag{5.4.23}$$

Choosing $v = \mathbb{1}$ in (5.4.9) yields

$$\int_{\mathscr{P}} \mathrm{d}\mathcal{D}_{\mathsf{m}}(\eta)\ \boldsymbol{\nabla}_w\, u(\eta) = \int_{\mathscr{P}} \mathrm{d}\mathcal{D}_{\mathsf{m}}(\eta)\, u(\eta)\, \mathbf{B}_\varepsilon[w](\eta) \ .$$

Since u is $\mathcal{B}_\varepsilon(\mathscr{P}^{\mathrm{pa}})$-measurable, it is also $\mathcal{B}_\varepsilon(\mathscr{P}^{\mathrm{pa}})$-measurable for all $\delta \leq \varepsilon$, hence

$$\forall \delta \leq \varepsilon \qquad \int_{\mathscr{P}} \mathrm{d}\mathcal{D}_{\mathsf{m}}(\eta)\, u(\eta)\, \mathbf{B}_\delta[w](\eta) = \int_{\mathscr{P}} \mathrm{d}\mathcal{D}_{\mathsf{m}}(\eta)\, u(\eta)\, \mathbf{B}_\varepsilon[w](\eta) \ .$$

By arbitrariness of $u \in \widehat{\mathfrak{Z}}_\varepsilon^1$ and Lemma 5.4.4**(*iii*)**,

$$\mathbf{E}_{\mathcal{D}_{\mathsf{m}}}\big[\mathbf{B}_\delta[w] \mid \mathcal{B}_\varepsilon(\mathscr{P}^{\mathrm{pa}})\big] = \mathbf{B}_\varepsilon[w] \ . \tag{5.4.24}$$

Then, $\mathbf{B}_\bullet[w]$ is a martingale by (5.4.23) and (5.4.24). Finally, choosing $u = v = \mathbb{1}$ in (5.4.9) yields $\mathbf{E}_{\mathcal{D}_{\mathsf{m}}}\big[\mathbf{B}_\varepsilon[w]\big] = 0$. **(*ii*)** For every $\varepsilon > 0$ one has

$$\begin{aligned}
\mathrm{d}_t\big|_{t=0}\mathbf{R}_\varepsilon[\psi^{w,t}](\eta) &= \mathrm{d}_t\big|_{t=0} \exp\left[\int_M \mathrm{d}\eta(x)\, \mathbb{1}_{(\varepsilon,1]}(\eta_x) \ln \frac{\mathrm{d}\psi^{w,t}_\sharp \mathsf{m}}{\mathrm{d}\mathsf{m}}(x)\right] \\
&= \mathrm{d}_t\big|_{t=0} \int_M \mathrm{d}\eta(x)\, \mathbb{1}_{(\varepsilon,1]}(\eta_x) \ln \frac{\mathrm{d}\psi^{w,t}_\sharp \mathsf{m}}{\mathrm{d}\mathsf{m}}(x) \\
&= \int_M \mathrm{d}\eta(x)\, \mathbb{1}_{(\varepsilon,1]}(\eta_x)\, \mathrm{d}_t\big|_{t=0} \ln \frac{\mathrm{d}\psi^{w,t}_\sharp \mathsf{m}}{\mathrm{d}\mathsf{m}}(x)
\end{aligned}$$

by Dominated Convergence Theorem. Finally, since $\psi^{w,t}$ is orientation-preserving, $\frac{\mathrm{d}\psi^{w,t}_\sharp \mathsf{m}}{\mathrm{d}\mathsf{m}} = \det \mathrm{d}\psi^{w,t}$, whence

$$\begin{aligned}
&\mathrm{d}_t\big|_{t=0}\mathbf{R}_\varepsilon[\psi^{w,t}](\eta) = \\
&= \int_M \mathrm{d}\eta(x)\, \mathbb{1}_{(\varepsilon,1]}(\eta_x)\, \mathrm{d}_t\big|_{t=0} \ln\det \mathrm{d}\psi^{w,t}(x) = \int_M \mathrm{d}\eta(x)\, \mathbb{1}_{(\varepsilon,1]}(\eta_x) \operatorname{tr}\big[\mathrm{d}_t\big|_{t=0} \mathrm{d}\psi^{w,t}(x)\big] \\
&= \int_M \mathrm{d}\eta(x)\, \mathbb{1}_{(\varepsilon,1]}(\eta_x) \operatorname{div}^{\mathsf{m}}_x w = \mathbf{B}_\varepsilon[w](\eta) \ .
\end{aligned}$$

(iii) By (5.4.9), Cauchy–Schwarz inequality, (5.4.5) and (5.4.11)

$$\begin{aligned}\left|\int_{\mathscr{P}} \mathrm{d}\mathcal{D}_{\mathsf{m}}\, u \cdot v \cdot \mathbf{B}_\varepsilon[w]\right| \leq& \int_{\mathscr{P}} \mathrm{d}\mathcal{D}_{\mathsf{m}} \left| \boldsymbol{\nabla}_w u \cdot v\right| + \int_{\mathscr{P}} \mathrm{d}\mathcal{D}_{\mathsf{m}} \left|u \cdot \boldsymbol{\nabla}_w v\right| \\ \leq& \left\| \boldsymbol{\nabla}_w u\right\|_{L^2_{\mathcal{D}_{\mathsf{m}}}} \|v\|_{L^2_{\mathcal{D}_{\mathsf{m}}}} + \|u\|_{L^2_{\mathcal{D}_{\mathsf{m}}}} \left\| \boldsymbol{\nabla}_w v\right\|_{L^2_{\mathcal{D}_{\mathsf{m}}}} \\ \leq& \|w\|_{\mathfrak{X}_{\overline{\mathsf{m}}}} \|u\|_{\mathcal{E}_1^{1/2}} \|v\|_{\mathcal{E}_1^{1/2}} \ .\end{aligned}$$

The existence and uniqueness of $\mathcal{A}^w$ are a standard consequence. ∎

Next, we show that $(\mathcal{E}, \mathscr{D}(\mathcal{E})$ describes a truly infinite-dimensional diffusion. We refer to [74, Dfn. 2.9] for the concept of *index* of a Dirichlet form.

Proposition 5.4.22. *The form $(\mathcal{E}, \mathscr{D}(\mathcal{E})$ has pointwise index $p(\eta) = \infty$ $\mathcal{D}_{\mathsf{m}}$-a.e.. Moreover, the index is 'full' in the following sense: For $\mathcal{D}_{\mathsf{m}}$-a.e. $\eta \in \mathscr{P}$ there exists an orthonormal basis $(e_i)_i$ of $T^{\mathrm{Der}}_\eta \mathscr{P}$ and a function $u = u_i \in \widehat{\mathfrak{Z}}^2_0 \subset \mathscr{D}(\mathcal{E})$ such that $\boldsymbol{\nabla} u = e_i$ for any choice of i.*

Proof. Since $\mathcal{D}_{\mathsf{m}}(\mathscr{P}^{\mathrm{pa,fs}}_{\mathrm{so}}) = 1$, we can restrict our attention to $\eta = \sum_i^\infty s_i \delta_{x_i} \in \mathscr{P}^{\mathrm{pa,fs}}_{\mathrm{so}}$. For all such η one has $T^{\mathrm{Der}}_\eta \mathscr{P} \cong \oplus^\perp_i (T_{x_i} M, s_i \mathsf{g})$, where $\oplus^\perp$ denotes the orthogonal direct sum. For the rest of the proof we tacitly assume this identification.

As a basis for $T^{\mathrm{Der}}_\eta \mathscr{P}$ we fix $(e_{i,\ell})_{i \in \mathbb{N}, \ell \leq d}$, where $(e_{i,\ell})_{\ell \leq d}$ is a g-orthonormal basis for $T_{x_i} M$ for every i. In order to show the second assertion, let $f = f_{i,\ell} \in \mathcal{C}^\infty(M)$ be such that $e_{i,\ell}(f)_{x_i} = 1$ and $e_{i,\ell'}(f)_{x_i} = 0$ for every $\ell' \neq \ell$ and $\varrho = \varrho_i \in \mathcal{C}^\infty(I)$ be such that $\varrho(s_i) = 1$ and $\varrho(s_{i'}) = 0$ for every $i \neq i'$. The existence of f is standard, while the existence of ϱ follows from the fact that $\eta \in \mathscr{P}^{\mathrm{pa}}_{\mathrm{iso}}$, hence $(\mathrm{pr}^{\mathbf{T}} \circ \boldsymbol{\Phi}^{-1})(\eta) \in \mathbf{T}_\circ$. Letting $u = u_{i,\ell} := (f \otimes \varrho)^\star$, one has $\boldsymbol{\nabla} u(\eta) = e_{i,\ell}$.

By definition of $(e_{i,\ell})_{i,\ell}$, one has $\boldsymbol{\Gamma}(u_{i,\ell}, u_{i',\ell'})(\eta) = \delta_{ii'} \langle e_{i,\ell} \,|\, e_{i,\ell'}\rangle_{\mathsf{g}} = 0$ for every $(i,\ell) \neq (i',\ell')$. As a consequence, setting

$$\mathbf{A}_{ii'}(\eta) := \left[\boldsymbol{\Gamma}(u_{i,\ell}, u_{i',\ell'})(\eta)\right]^{\ell' \leq d}_{\ell \leq d} \in \mathbb{R}^{d \times d} \qquad \text{and} \qquad \mathbf{A}(\eta) := \left[\mathbf{A}_{ii'}(\eta)\right]^{i' \leq n}_{i \leq n} \in \mathbb{R}^{d^2 \times n^2} \ ,$$

one has $\mathbf{A}(\eta) = \mathrm{id}_{\mathbb{R}^{nd}}$ for every $\eta \in \mathscr{P}^{\mathrm{pa,fs}}_{\mathrm{so}}$. Thus $p(\eta) \geq \mathrm{rk}(\mathbf{A}(\eta)) = nd$ for every n, which shows the first assertion. ∎

Remark 5.4.23 (A comparison with the Cheeger energy). A known object in metric measure space's analysis is the *Cheeger energy* of a (complete and separable) metric measure space $(Y, \mathsf{d}, \mathsf{n})$

$$\begin{aligned}\mathsf{Ch}_{\mathsf{d},\mathsf{n}}(f) :=& \inf\left\{\liminf_n \tfrac{1}{2} \int_Y \mathrm{d}\mathsf{n} \left|Df_n\right|^2 \ \middle|\ f_n \in \mathrm{Lip}(Y, \mathsf{d}) \ ,\ f_n \to f \text{ in } L^2_{\mathsf{n}}(Y)\right\} \ , \\ \left|Df\right|(y) :=& \limsup_{z \to y} \frac{|f(y) - f(z)|}{\mathsf{d}(y,z)} \ , \qquad f \in \mathrm{Lip}(Y, \mathsf{d}) \ .\end{aligned}$$

A comparison of the Cheeger energy $\mathsf{Ch}_{W_2, \mathcal{D}_{\mathsf{m}}}$ of $(\mathscr{P}, W_2, \mathcal{D}_{\mathsf{m}})$ with the form $(\mathcal{E}, \mathscr{D}(\mathcal{E}))$ constructed in Theorem 5.4.11 is here beyond our scope. However, let us notice that, at a merely heuristic level, we do not expect $\mathsf{Ch}_{W_2, \mathcal{D}_{\mathsf{m}}}$ to be a quadratic form. Indeed, $\mathcal{D}_{\mathsf{m}}$-a.e. $\eta \in \mathscr{P}$ is *not* a regular measure in the sense of optimal transport (e.g., [63]), hence the tangent space at η accessed by Lipschitz functions is the full 'abstract tangent space' $\mathrm{AbstrTan}_\eta$ [63, Dfn. 3.7]. By the results in [63, §6], $\mathfrak{X}^\nabla_\eta$ embeds canonically, non-surjectively into $\mathrm{AbstrTan}_\eta$ and the latter is $\mathcal{D}_{\mathsf{m}}$-a.e. *not* a Hilbert space (rather, it is merely a Banach space). Additionally, it is not clear to me whether $\mathsf{Ch}_{W_2, \mathcal{D}_{\mathsf{m}}}$ is non-trivial (that is, not identically vanishing).

5.5 The associated process

In the case $d \geq 2$, by e.g., [112, Thm. IV.5.1], the form $(\mathcal{E}, \mathscr{D}(\mathcal{E}))$ is properly associated with a $\mathcal{D}_{\mathsf{m}}$-symmetric recurrent Markov diffusion process

$$\eta_\bullet := \left(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, (\eta_t)_{t\geq 0}, \{P_\eta\}_{\eta\in\mathscr{P}}\right) \tag{5.5.1}$$

which we now characterize.

5.5.1 Finite-dimensional approximations. Everywhere in this section, assume $d \geq 1$ whenever not explicit stated otherwise. We construct a sequence of forms $(\hat{\mathsf{E}}^n, \mathscr{D}(\hat{\mathsf{E}}^n))$ enjoying the following properties: ***(a)*** $\hat{\mathsf{E}}^n$ is defined on $L^2(Y_n)$ for some finite-dimensional compact manifold Y_n and ***(b)*** $(\hat{\mathsf{E}}^n, \mathscr{D}(\hat{\mathsf{E}}^n))$ Mosco converges to $(\mathcal{E}, \mathscr{D}(\mathcal{E}))$ (in the generalized sense).

We start with the following definition of a family of Dirichlet forms approximating $(\mathcal{E}, \mathscr{D}(\mathcal{E}))$.

Definition 5.5.1 (Approximating forms)**.** For $\varepsilon \in (0,1)$ we consider the form $(\mathcal{E}, \widehat{\mathfrak{Z}}^2_\varepsilon)$. Notice that ***(a)*** $(\mathcal{E}, \widehat{\mathfrak{Z}}^2_\varepsilon)$ is a closable, strongly local Dirichlet form, with closure $(\mathcal{E}^\varepsilon, \mathscr{D}(\mathcal{E}^\varepsilon))$, closability and strong locality being inherited by $(\mathcal{E}, \widehat{\mathfrak{Z}}^2_0)$; ***(b)*** $(\mathcal{E}^\varepsilon, \mathscr{D}(\mathcal{E}^\varepsilon))$ is not densely defined on $L^2_{\mathcal{D}_{\mathsf{m}}}(\mathscr{P})$, yet it is densely defined on $L^2_{\mathcal{D}_{\mathsf{m}}}(\mathscr{P}, \mathcal{B}_\varepsilon(\mathscr{P}))$ by Lemma 5.4.4***(iii)***.

Definition 5.5.2 (Simplices and projections)**.** For $n \in \overline{\mathbb{N}}$, $\varepsilon \in I$ and $\beta > 0$ set

$$\begin{aligned} \Sigma^n &:= \mathrm{pr}^n(\mathbf{T}) \;, & \widehat{M}^n &:= \mathrm{pr}^n(\widehat{\mathbf{M}}) \;, \\ \Sigma^n_\varepsilon &:= \{\mathbf{s} \in \Sigma^n \mid s_1 \geq \varepsilon\} \;, & \widehat{M}^n_\varepsilon &:= \Sigma^n_\varepsilon \times M^{\times n} \;, \end{aligned}$$

each endowed with the usual topology and σ-algebra. We endow Σ^n (resp. Σ^n_ε) with (the restriction of) the probability measure

$$\pi^n_\beta := \mathrm{pr}^n_\sharp \Pi_\beta \;, \tag{5.5.2}$$

and $\widehat{M}^n$ (resp. $\widehat{M}^n_\varepsilon$) with (the restriction of) the measure $\widehat{\mathsf{m}}^n_\beta := \pi^n_\beta \otimes \overline{\mathsf{m}}^n$. Finally, we set

$$\begin{aligned} \Phi_n \colon \widehat{M}^n &\longrightarrow \mathscr{M}_1^+ \\ (\mathbf{s}, \mathbf{x}) &\longmapsto \sum_\ell s_\ell \delta_{x_\ell} \end{aligned} \;. \tag{5.5.3}$$

Everywhere in the following, for fixed $n \in \mathbb{N}$ and $\varepsilon > 0$ let

$$\begin{aligned} H^n_\varepsilon &:= \mathrm{cl}_{L^2_{\widehat{\mathsf{m}}^n_\beta}} \mathcal{C}^\infty(\widehat{M}^n_\varepsilon) = L^2_{\widehat{\mathsf{m}}^n_\beta}(\widehat{M}^n_\varepsilon) \;, & H_n &:= H^n_{1/n} \;, \\ H_{1/n} &:= L^2_{\mathcal{D}_{\mathsf{m}}}(\mathscr{P}, \mathcal{B}_{1/n}(\mathscr{P})) \;, & H &:= L^2_{\mathcal{D}_{\mathsf{m}}}(\mathscr{P}) \;. \end{aligned} \tag{5.5.4}$$

The following is a particular case of the *direct integral of Dirichlet forms* constructed in §5.6.4.

Proposition 5.5.3 (Randomization of Dirichlet forms)**.** *For fixed $n \in \mathbb{N}$ and $\varepsilon > 0$ let*

$$\hat{\mathsf{E}}^{n,\varepsilon}(h) = \int_{\Sigma^n_\varepsilon} \mathrm{d}\pi^n_\beta(\mathbf{s}) \int_{M^{n,\mathbf{s}}} \mathrm{d}\overline{\mathsf{m}}^n(\mathbf{x}, \mathbf{s}) \left|\nabla^{\mathsf{g}^{n,\mathbf{s}},\mathbf{z}}\big|_{\mathbf{z}=\mathbf{x}} h(\mathbf{z}, \mathbf{s})\right|^2_{(\mathsf{g}^{n,\mathbf{s}})_\mathbf{x}} \;, \qquad h \in \mathcal{C}^1(\widehat{M}^n_\varepsilon) \;.$$

For every $i \leq n$ let further $x^i_\bullet := (x^i_t)_{t>0}$ be independent Brownian motions on (M, g) starting at x^i_0 and defined on a common probability space $(\Omega, \mathcal{F}, \mathbf{P})$. We define a stochastic process $\widehat{\mathsf{W}}^{n,\varepsilon}$ on $(\Omega, \mathcal{F}, \mathbf{P})$ with state space $\widehat{M}^n_\varepsilon$ by

$$\widehat{\mathsf{W}}^{n,\varepsilon,\mathbf{x}_0,\mathbf{s}}_t(\omega) := \left(x^1_{t/s_1}(\omega), \dots, x^n_{t/s_n}(\omega)\right) \;, \qquad \mathbf{s} := (s_1, \dots, s_n) \in \Sigma^n_\varepsilon \;, \qquad \omega \in \Omega \;,$$

where $\widehat{\mathsf{W}}^{n,\varepsilon,\mathbf{x}_0,\mathbf{s}}_\bullet(\omega)$ *is any stochastic path of* $\widehat{\mathsf{W}}^{n,\varepsilon}_\bullet$ *starting at* $(\mathbf{x}_0,\mathbf{s})\in\widehat{M^n_\varepsilon}$*.*

Then, ***(i)*** *the form* $\big(\hat{\mathsf{E}}^{n,\varepsilon},\mathcal{C}^1(\widehat{M^n_\varepsilon})\big)$ *is closable and its closure* $\big(\hat{\mathsf{E}}^{n,\varepsilon},\mathscr{D}(\hat{\mathsf{E}}^{n,\varepsilon})\big)$ *is a regular strongly local Dirichlet form on* H^n_ε *with special core* $\mathcal{C}^1(\widehat{M^n_\varepsilon})$*; with* ***(ii)*** *semigroup*

$$\begin{aligned}(\mathsf{H}^{n,\varepsilon}_t h)(\mathbf{x},\mathbf{s}) =&\big((\mathsf{H}^{\mathbf{s}}_t\otimes \mathrm{id}_{H^n_\varepsilon})\, h\big)(\mathbf{x},\mathbf{s})\\ =&c_{n,\varepsilon}\int_{M^{\times n}} \mathrm{d}\overline{\mathsf{m}}^n(\mathbf{y}) \prod_i^n \mathsf{h}_{t/s_i}(x_i,y_i)\, h(y_1,\dots,y_n,s_1,\dots,s_n)\ ,\end{aligned}$$

where $c_{n,\varepsilon}\coloneqq \pi^n_\beta\Sigma^n_\varepsilon\uparrow_{\varepsilon\downarrow 0} 1$*;* ***(iii)*** $(\hat{\mathsf{E}}^{n,\varepsilon},\mathscr{D}(\hat{\mathsf{E}}^{n,\varepsilon}))$ *is properly associated to the process* $\widehat{\mathsf{W}}^{n,\varepsilon}_\bullet$*;* ***(iv)*** $M^{\times n}_\circ\times\Sigma^n_\varepsilon$ *is* $\widehat{\mathsf{W}}^{n,\varepsilon}_\bullet$*-coexceptional.*

Proof. ***(i)***–***(iii)*** are a direct consequence of Proposition 5.6.18. We omit the details. ***(iv)*** follows by showing that $M^{\times n}_\circ$ is $\mathsf{W}^{n,\mathbf{s}}_\bullet$-coexceptional for π^n_β-a.e. $\mathbf{s}\in\Sigma^n_\varepsilon$, which is Lemma 5.3.6. ■

Lemma 5.5.4. *Let* $D_n\coloneqq\Phi^*_n\widehat{\mathfrak{Z}}^1_{1/n}$ *and* $(T^n_m)_m$ *be a family of closed sets in* $\Sigma^n_{1/n}\cap\mathrm{pr}^n(\mathbf{T}_\circ)$ *such that* $T^n_m\uparrow T^n$ *with* T^n *of full* π^n_β*-measure in* $\Sigma^n_{1/n}$*. Let further* $(F^n_m)_m$ *be a nest for* $\mathsf{E}^{n,\mathbf{s}}$ *for every* $\mathbf{s}\in T^n$ *and such that* $F^n_m\subset\mathrm{int}\,F^n_{m+1}$*. Set* $\hat F^n_m\coloneqq T^n\times F^n_m\subset\widehat{M^n_{1/n}}$ *and let*

$$D_{n,m}=(D_n)_{\hat F^n_m}\coloneqq\Big\{u\in D_n \mid u\equiv 0\ \ \pi^n_\beta\text{-a.e. on } (\hat F^n_m)^{\mathrm{c}}\Big\}\ .$$

Then, $\bigcup_m D_{n,m}$ *is both dense in* H_n *and dense in* $\mathscr{D}(\hat{\mathsf{E}}^n)$*.*

Proof. It suffices to show the second density statement. Let $(D_{n,m})_{\mathbf{s}}\coloneqq\big\{h(\,\cdot\,,\mathbf{s})\mid h\in D_{n,m}\big\}$. In order to show that $\bigcup_m D_{n,m}$ is dense in $\mathscr{D}(\hat{\mathsf{E}}^n)$, it suffices to show that $\bigcup_m(D_{n,m})_{\mathbf{s}}$ is dense in $\mathscr{D}(\mathsf{E}^{n,\mathbf{s}})$ for π^n_β-a.e. $\mathbf{s}\in\Sigma^n_{1/n}$ (cf. Prop. 5.6.18***(iii)***). Since $(F^n_m)_m$ is a nest for $(\mathsf{E}^{n,\mathbf{s}},\mathscr{D}(\mathsf{E}^{n,\mathbf{s}}))$ for all $\mathbf{s}$ in the set of full π^n_β-measure T^n, we have that $\bigcup_m\mathscr{D}(\mathsf{E}^{n,\mathbf{s}})_{F^n_m}$ is dense in $\mathscr{D}(\mathsf{E}^{n,\mathbf{s}})$ for all $\mathbf{s}\in T^n$. Thus, it suffices to show that

$$\mathrm{cl}_{(\mathsf{E}^{n,\mathbf{s}})^{1/2}_1}\big((D_{n,m+1})_{\mathbf{s}}\big)\supset\mathscr{D}(\mathsf{E}^{n,\mathbf{s}})_{F^n_m}\ ,\qquad m\in\mathbb{N}\ . \tag{5.5.5}$$

To this end, we firstly show that $(D_{n,m+1})_{\mathbf{s}}\supset\mathcal{A}_{n,m+1}\coloneqq\big(\mathcal{C}^1(M)^{\otimes n}\big)_{F^n_{m+1}}$. Indeed, since, in particular, $\mathbf{s}\in\mathrm{pr}^n(\mathbf{T}_\circ)$, for $\ell\leq n$ there exists $\varrho_\ell\in\mathcal{C}^\infty_c((1/n,1))$ (depending on $\mathbf{s}$) such that $\varrho_{\ell_1}(s_{\ell_2})=s_{\ell_1}^{-1}\delta_{\ell_1\,\ell_2}\in(0,\infty)$. Thus, for any choice of $(f_\ell)^n_\ell\subset\mathcal{C}^1(M)$, one has that

$$\bigotimes_\ell^n f_\ell=\big(\Phi^*\big(\prod_\ell^n(f_\ell\,\varrho_\ell)^\star\big)\big)(\,\cdot\,,\mathbf{s})\ ,\qquad \mathbf{s}\in T^n\ .$$

Finally, it is clear that

$$\mathrm{cl}_{(\mathsf{E}^{n,\mathbf{s}})^{1/2}_1}(\mathcal{A}_{n,m+1})=\mathrm{cl}_{(\mathsf{E}^{n,\mathbf{s}})^{1/2}_1}\big(\mathcal{C}^1(M^{\times n})_{F^n_{m+1}}\big)\supset\mathscr{D}(\mathsf{E}^{n,\mathbf{s}})_{F^n_m}$$

where the latter inclusion follows by a localization argument with smooth partitions of unity and regularization by convolution since $F^n_m\subset\mathrm{int}\,F^n_{m+1}$. (We omit the details.) This concludes the proof of (5.5.5). ■

Lemma 5.5.5. *Let* $(\hat{\mathsf{E}}^n,\mathscr{D}(\hat{\mathsf{E}}^n))\coloneqq(\hat{\mathsf{E}}^{n,1/n},\mathscr{D}(\hat{\mathsf{E}}^{n,1/n}))$ *be defined as in Proposition 5.5.3. Then, the forms* $(\hat{\mathsf{E}}^n,\mathscr{D}(\hat{\mathsf{E}}^n))$ *and* $(\mathcal{E}^{1/n},\mathscr{D}(\mathcal{E}^{1/n}))$ *are intertwined via* Φ^*_n *as in* (5.5.3)*.*

Proof. Let $\eta\in\mathscr{P}^{\mathrm{pa}}$ be of the form $\eta=\sum_\ell^N s_\ell\delta_{x_\ell}$ for some $N\in\overline{\mathbb{N}}$. For $u=F\circ\hat{\mathbf{f}}\in\widehat{\mathfrak{Z}}^0_\varepsilon$ define

$$\sum_\ell^N s_\ell\,\hat{\mathbf{f}}(x_\ell,s_\ell)\coloneqq\left(\sum_\ell^N s_\ell\hat f_1(x_\ell,s_\ell),\cdots,\sum_\ell^N s_\ell\hat f_k(x_\ell,s_\ell)\right)\ ,\qquad N\in\overline{\mathbb{N}} \tag{5.5.6}$$

$$U_n\colon u\longmapsto \left(\Phi_n^* u\colon (\mathbf{s},\mathbf{x})\mapsto F\left(\sum_\ell^n s_\ell\, \hat{\mathbf{f}}(x_\ell,s_\ell)\right)\right)\ , \qquad n\in\mathbb{N}\ , \quad (\mathbf{s},\mathbf{x})\in\Sigma^n\ . \tag{5.5.7}$$

By definition of u one has

$$\hat f_i(x,s)=0\ , \qquad x\in M\ ,\ s\leq 1/n\ ,\ i\leq k\ , \tag{5.5.8}$$

hence

$$\boldsymbol{\Phi}^* u=\Phi_n^* u\circ \mathrm{pr}_n\ , \qquad u\in \widehat{\mathfrak{Z}}^0_{1/n}\ . \tag{5.5.9}$$

Now, by Proposition 5.3.14***(ii)*** and the fact that $\mathcal{E}^{1/n}=\mathcal{E}$ on $\widehat{\mathfrak{Z}}^1_{1/n}$ one has for all $u\in\widehat{\mathfrak{Z}}^1_{1/n}$

$$\mathcal{E}^{1/n}(u)=\int_{\widehat{\mathbf{M}}_\circ}\mathrm{d}\widehat{\mathbf{m}}_\beta(\mathbf{s},\mathbf{x})\sum_{i,p}^k(\partial_i F\cdot\partial_p F)\left(\sum_\ell^\infty s_\ell\,\hat{\mathbf{f}}(x_\ell,s_\ell)\right)\cdot\sum_\ell^\infty s_\ell\,\Gamma^{\mathsf{g}}(\hat f_i,\hat f_p)(x_\ell,s_\ell)\ . \tag{5.5.10}$$

If $\ell>n$, then $s_\ell\leq 1/n$ because $s_1\geq s_2\geq\dots$. Thus, by (5.5.10) and (5.3.5),

$$\begin{aligned}\mathcal{E}^{1/n}(u)=&\int_{\widehat{M}^n}\mathrm{d}\widehat{\mathsf{m}}^n_\beta(\mathbf{s},\mathbf{x})\sum_{i,p}^k(\partial_i F\cdot\partial_p F)\left(\sum_\ell^n s_\ell\,\hat{\mathbf{f}}(x_\ell,s_\ell)\right)\cdot\sum_\ell^n s_\ell^2\,\Gamma^{s_\ell\mathsf{g}}(\hat f_i,\hat f_p)(x_\ell,s_\ell)\\ =&\int_{\widehat{M}^n}\mathrm{d}\widehat{\mathsf{m}}^n_\beta(\mathbf{s},\mathbf{x})\left|\nabla^{\mathsf{g}^{n,\mathbf{s}},\mathbf{z}}\big|_{\mathbf{z}=\mathbf{x}}F\left(\sum_\ell^n s_\ell\,\hat{\mathbf{f}}(z_\ell,s_\ell)\right)\right|^2_{\mathsf{g}^{n,\mathbf{s}}_{\mathbf{x}}} &(5.5.11)\\ =&\int_{\widehat{M}^n_{1/n}}\mathrm{d}\widehat{\mathsf{m}}^n_\beta(\mathbf{s},\mathbf{x})\left|\nabla^{\mathsf{g}^{n,\mathbf{s}},\mathbf{z}}\big|_{\mathbf{z}=\mathbf{x}}F\left(\sum_\ell^n s_\ell\,\hat{\mathbf{f}}(z_\ell,s_\ell)\right)\right|^2_{\mathsf{g}^{n,\mathbf{s}}_{\mathbf{x}}} &(5.5.12)\\ =&\hat{\mathsf{E}}^n(\Phi_n^* u)\ ,\end{aligned}$$

where we may reduce the domain of integration in (5.5.11) to the one in (5.5.12) since the integrand vanishes identically on $\widehat{M}^n\setminus\widehat{M}^n_{1/n}$ for all $u\in\widehat{\mathfrak{Z}}^1_{1/n}$, again as a consequence of (5.5.8). In particular, $D_n\coloneqq U_n\,\widehat{\mathfrak{Z}}^1_\varepsilon\subset\mathscr{D}(\hat{\mathsf{E}}^n)$. An analogous computation shows that $\Phi_n^* u$ satisfies

$$\|u\|_{H_{1/n}}=\|\Phi_n^* u\|_{H_n}\ , \qquad u\in\widehat{\mathfrak{Z}}^0_{1/n}\ .$$

The family $\widehat{\mathfrak{Z}}^1_{1/n}$ is dense in $H_{1/n}$ by Lemma 5.4.4***(iii)*** and dense in $\mathscr{D}(\mathcal{E}^{1/n})$ by definition of the latter. As a consequence, the operator U_n defined in (5.5.7) uniquely extends to a non-relabeled isometric operator $U_n\colon\mathscr{D}(\mathcal{E}^{1/n})_1\to\mathscr{D}(\hat{\mathsf{E}}^n)_1$, and, subsequently to an isometric operator $U_n\colon H_{1/n}\to H_n$. It suffices to show the intertwining property on dense subsets. Thus, the conclusion follows by showing that $D_n\coloneqq U_n\,\widehat{\mathfrak{Z}}^1_{1/n}=\Phi_n^*\widehat{\mathfrak{Z}}^1_{1/n}$ is both dense in H_n and dense in $\mathscr{D}(\hat{\mathsf{E}}^n)$. This follows by Lemma 5.5.4 with $F_m\coloneqq M$ for every m. ■

Proposition 5.5.6. *Let* $\big(\hat{\mathsf{E}}^n,\mathscr{D}(\hat{\mathsf{E}}^n)\big)\coloneqq\big(\hat{\mathsf{E}}^{n,1/n},\mathscr{D}(\hat{\mathsf{E}}^{n,1/n})\big)$ *be defined as in Proposition 5.5.3. Then, the sequence* $\big(\hat{\mathsf{E}}^n,\mathscr{D}(\hat{\mathsf{E}}^n)\big)_{n\in\mathbb{N}}$ *Mosco converges to* $(\mathcal{E},\mathscr{D}(\mathcal{E}))$ *in the sense of Definition 5.6.7.*

Proof. Recall the notation in (5.5.4). We claim that $(\mathcal{E}^{1/n},\mathscr{D}(\mathcal{E}^{1/n}))$ converges to $(\mathcal{E},\mathscr{D}(\mathcal{E}))$ as $n\to\infty$ in the generalized Mosco sense. Indeed let $P_n\colon H\to H$ be the projection operator $P_n\coloneqq\mathbf{E}_{\mathcal{D}_{\mathsf{m}}}[\,\cdot\mid\mathcal{B}_{1/n}(\mathscr{P}^{\mathrm{pa}})]$ given by the conditional expectation w.r.t. $\mathcal{B}_{1/n}(\mathscr{P}^{\mathrm{pa}})$. By definition, $H_{1/n}=P_n(H)$. Since $\mathcal{B}_0(\mathscr{P})_{\mathscr{P}^{\mathrm{pa}}}=\mathcal{B}_{\mathrm{n}}(\mathscr{P}^{\mathrm{pa}})$ by Lemma 5.4.4***(ii)***, the sequence $(P_n)_{n\in\mathbb{N}}$ converges strongly to id_H. Regard $(\mathcal{E}^{1/n},\mathscr{D}(\mathcal{E}^{1/n}))$ as a (*not* densely defined) quadratic form on H. By Lemma 5.6.10 applied to the family $(P_n)_{n\in\mathbb{N}}$, it suffices to check the Mosco convergence of $(\mathcal{E}^{1/n},\mathscr{D}(\mathcal{E}^{1/n}))$ to $(\mathcal{E},\mathscr{D}(\mathcal{E}))$ in the classical sense. The strong Γ-lim sup condition (5.6.5) is the content of Corollary 5.4.18. Since $\mathcal{E}(u) = \mathcal{E}^{1/n}(u)$ for every $u \in \mathscr{D}(\mathcal{E}^{1/n})$, the weak Γ-lim inf condition (5.6.4) is a consequence of the weak lower semi-continuity of $\mathcal{E}$ (Lem. 5.6.5).

Now, Lemma 5.6.9 applies to $Q = \mathcal{E}^{1/n}$ and $Q^\sharp = \hat{\mathsf{E}}^n$ with $U_n = \Phi_n^*$ as in (the proof of) Lemma 5.5.5. Therefore, $(\hat{\mathsf{E}}^n, \mathscr{D}(\hat{\mathsf{E}}^n))$ Mosco converges to $(\mathcal{E}, \mathscr{D}(\mathcal{E}))$ as $n \to \infty$. ■

Proposition 5.5.7. *The non-negative operator* $(-\mathbf{L}, \widehat{\mathfrak{Z}}_0^\infty)$ *is essentially self-adjoint.*

Proof. We show that for every $u_0 \in \widehat{\mathfrak{Z}}_0^\infty$ and every $T \in [0, \infty)$ there exists ***(a)*** a sequence $(u_{0,n})_{n\in\mathbb{N}} \subset \widehat{\mathfrak{Z}}_0^\infty$ such that $L^2_{\mathcal{D}_\mathsf{m}}(\mathscr{P})\text{-}\lim_n u_{0,n} = u_0$ and ***(b)*** strong solutions u_n of the Cauchy problems

$$\begin{aligned} (\mathrm{d}_t u_n)(t) - (-\mathbf{L}u_n)(t) =&0 \ , \\ u_n(T) =&u_{0,n} \ , \qquad u_n(t) \in \widehat{\mathfrak{Z}}_0^2 \subset \mathscr{D}(\mathbf{L}) \ , \end{aligned} \qquad t \in [0,T] \ . \tag{5.5.13}$$

Then, the assertion follows by [17, §II.5, Thm. 1.10, p. 30]. (Condition (*ii*) there is trivially satisfied since we chose, in the notation of [17, *ibid.*], $A_n = A$.)

Solutions to the heat equation. For $n \leq N \in \overline{\mathbb{N}}$ let $\eta = \sum_i^N s_i \delta_{x_i}$ and $u \in \widehat{\mathfrak{Z}}_{1/n}^\infty$, and recall the notation in (5.5.6). Analogously to the proof of Lemma 5.5.5, by (5.4.16) and (5.4.17) we have

$$\begin{aligned} 2\,(\mathbf{L}u_0)(\eta) =& \sum_{i,p}^k (\partial^2_{ip} F)\left(\sum_\ell^n s_\ell \hat{\mathbf{f}}(x_\ell, s_\ell)\right) \sum_\ell^n s_\ell\, \Gamma^{\mathsf{g}}(\hat{f}_i, \hat{f}_p)(x_\ell, s_\ell) \\ &+ \sum_i^k (\partial_i F)\left(\sum_\ell^n s_\ell \hat{\mathbf{f}}(x_\ell, s_\ell)\right) \sum_\ell^n \Delta^z\big|_{z=x_\ell} \hat{f}_i(z, s_\ell) \\ =& \sum_{i,p}^k (\partial^2_{ip} F)\left(\sum_\ell^n s_\ell \hat{\mathbf{f}}(x_\ell, s_\ell)\right) \sum_\ell^n s_\ell^2\, \Gamma^{s_\ell \mathsf{g}}(\hat{f}_i, \hat{f}_p)(x_\ell, s_\ell) \\ &+ \sum_i^k (\partial_i F)\left(\sum_\ell^n s_\ell \hat{\mathbf{f}}(x_\ell, s_\ell)\right) \sum_\ell^n s_\ell\, \Delta^{s_\ell \mathsf{g}, z}\big|_{z=x_\ell} \hat{f}_i(z, s_\ell) \end{aligned} \tag{5.5.14}$$

$$= (\Delta^{\mathsf{g}^{n,\mathbf{s}},\mathbf{z}}\big|_{\mathbf{z}=\mathbf{x}} \Phi_n^* u_0)(\mathbf{z}, \mathbf{s}) \ , \tag{5.5.15}$$

By (5.5.15) together with the time-reversal $t \mapsto T - t$, the Cauchy problem (5.5.13) with u_0 in place of $u_{0,n}$ transforms into the Cauchy problem

$$\begin{aligned} \partial_t h - \tfrac{1}{2}\Delta^{\mathsf{g}^{n,\mathbf{s}}} h =&0 \ , \\ h(0) =&\Phi_n^* u_0 \ , \end{aligned} \qquad t \in [0,T] \ . \tag{5.5.16}$$

Since $M^{n,\mathbf{s}}$ is a closed manifold, by standard results the latter Cauchy problem has a unique solution, say $t \mapsto h(t)$, additionally satisfying $h(t) \in \mathcal{C}^\infty(M^{n,\mathbf{s}})$ for all $t \in [0,T]$. Finally, notice that every function $h \in \mathcal{C}^\infty(M^{n,\mathbf{s}})$ may be written as

$$h(\mathbf{x}) = (U_n v)(\mathbf{x}, \mathbf{s})$$

for some $v = G \circ \hat{\mathbf{g}}^\star \in \widehat{\mathfrak{Z}}_{1/n}^\infty$ (not necessarily in minimal form) with $G \in \mathcal{C}_c^\infty(\mathbb{R}^{nd}; \mathbb{R})$. As a consequence, there exist functions $t \mapsto G(t) \in \mathcal{C}_c^\infty(\mathbb{R}^{nd}; \mathbb{R})$ and $t \mapsto \hat{g}_i(t) \in \mathcal{C}_c^\infty(\widehat{M}_{1/n})$ for $i \leq nd$ such that $h(t) = \Phi_n^* u(t)$, where $u(t) \coloneqq G(t) \circ \hat{\mathbf{g}}(t)^\star$. We have thus constructed the unique solution $t \mapsto u(t)$ of the Cauchy problem (5.5.13) with initial data $u_0 \in \widehat{\mathfrak{Z}}_{1/n}^\infty$, additionally satisfying $u(t) \in \widehat{\mathfrak{Z}}_{1/n}^\infty$. As usual, the representation of u by G and $\hat{\mathbf{g}}$ is not unique (cf. Rmk. 5.4.2). Notice that the strong solution $h_{u(t)}$ to (5.5.16) is smooth, hence the corresponding function $u(t) \in \widehat{\mathfrak{Z}}_{1/n}^\infty$ is a strong solution to (5.5.13) in the sense of the strong topology of $L^2_{\mathcal{D}_\mathsf{m}}(\mathscr{P})$ and therefore satisfies ***(b)***.

Approximations. Let $u_{0,n} \in \widehat{\mathfrak{Z}}^\infty_{1/n}$ be given by Corollary 5.4.18, thus satisfying **(a)**. Constructing solutions $t \mapsto u_n(t)$ to the Cauchy problems (5.5.13) as above concludes the proof. ∎

Next, we show a weak form of the Rademacher property for $(\mathcal{E}, \mathscr{D}(\mathcal{E}))$. We assume the reader to be familiar with the setting of Chapter 4, from which a proof is adapted.

Proposition 5.5.8. *Assume $d \geq 2$. If $u \in \mathrm{Lip}(\mathscr{P}_2)$, then $u \in \mathscr{D}(\mathcal{E})$ and $\mathbf{\Gamma}(u) \leq \mathrm{Lip}[u]^2$ $\mathcal{D}_{\mathsf{m}}$-a.e..*

Proof. By e.g. Theorem 2.3.9, $\Psi^{w,t}_\sharp \mathcal{D}_{\mathsf{m}} = \mathcal{D}_{\Psi^{w,t}_\sharp \mathsf{m}}$. Let $\mathcal{F}$ be the set of all bounded measurable functions u on $\mathscr{P}$ for which there exists a measurable section $\mathbf{D}u$ of $T^{\mathrm{Der}}\mathscr{P}_2$ such that

$$\int_{\mathscr{P}} \mathrm{d}\mathcal{D}_{\mathsf{m}}(\eta)\ \langle \mathbf{D}u(\eta) \,|\, \mathbf{D}u(\eta)\rangle_{\mathfrak{X}_\eta} < \infty$$

and, for all $s \in \mathbb{R}$ and $w \in \mathfrak{X}^\infty$,

$$\frac{u \circ \Psi^{w,t} - u}{t} \xrightarrow{t \to 0} \langle \boldsymbol{\nabla} u \,|\, w\rangle_{\mathfrak{X}} \quad \text{in} \quad L^2(\mathscr{P}, \mathcal{D}_{\Psi^{w,s}_\sharp \mathsf{m}}) \ .$$

By Lemma 5.4.7, $\widehat{\mathfrak{Z}}^\infty_0 \subset \mathcal{F}$. Since the generator of $(\mathcal{E}, \mathscr{D}(\mathcal{E}))$ is essentially self-adjoint on $\widehat{\mathfrak{Z}}^\infty_0$ (Prop. 5.5.7), the form $(\mathcal{E}, \mathscr{D}(\mathcal{E}))$ coincides with the form $(\mathcal{E}, \mathscr{F})$ defined in Chapter 4 with $\mathbb{P} = \mathcal{D}_{\mathsf{m}}$. Moreover, $(\mathcal{E}, \mathscr{D}(\mathcal{E}))$ coincides with the closure of $(\mathcal{E}, \mathfrak{Z}^\infty)$ (in the notation of Chapter 4, $\mathfrak{Z}^\infty = \mathcal{FC}^\infty$) by Lemma 5.6.26. Thus, for $\mathbb{P} = \mathcal{D}_{\mathsf{m}}$, the forms $(\mathcal{E}, \mathscr{F}_0)$, $(\mathcal{E}, \mathscr{F}_{\mathrm{cont}})$ and $(\mathcal{E}, \mathscr{F})$ defined in Thm. 4.2.4 (1) all coincide with $(\mathcal{E}, \mathscr{D}(\mathcal{E}))$ by Remark 4.2.7.

As already noticed in §4.5.4, $\mathcal{D}_{\mathsf{m}}$ satisfies assumptions $(\mathbf{P}_1) - (\mathbf{P}_2)$ there. Since we have the closability of $(\mathcal{E}, \mathfrak{Z}^\infty)$ independently of $(\mathbf{P}_3)$ there, the strategy of Chapter 4 applies verbatim, except for Lemma 4.4.8 Proposition 4.4.9. We show how to replace both of them in Lemma 5.6.28 and Proposition 5.6.29 below. ∎

For $A_1, A_2 \in \mathcal{B}_{\mathrm{n}}(\mathscr{P})$ of positive $\mathcal{D}_{\mathsf{m}}$-measure set $\mathsf{d}_{W_2}(A_1, A_2) := \mathcal{D}_{\mathsf{m}}\text{-}\operatorname{essinf}_{\mu_i \in A_i} W_2(\mu_1, \mu_2)$ and $p_t(A_1, A_2) := \int_{A_1} \mathrm{d}\mathcal{D}_{\mathsf{m}}(\mu_1) \int_{A_2} p_t(\mu_1, \mathrm{d}\mu_2)$ and let

$$\mathsf{d}_{\mathcal{E}}(\mu, \nu) := \sup\{u(\mu) - u(\nu) \mid u \in \mathscr{D}(\mathbf{\Gamma}) \cap \mathcal{C}(\mathscr{P})\ ,\ \mathbf{\Gamma}(u) \leq 1 \quad \mathcal{D}_{\mathsf{m}}\text{-a.e.}\} \tag{5.5.17}$$

be the intrinsic distance of $(\mathcal{E}, \mathscr{D}(\mathcal{E}))$. Then,

Corollary 5.5.9 (Gaussian short-time asymptotics lower bound)**.** *It holds that $W_2 \leq \mathsf{d}_{\mathcal{E}}$ and*

$$\lim_{t \downarrow 0} t \log p_t(A_1, A_2) \leq -\tfrac{1}{2} \mathsf{d}_{W_2}(A_1, A_2)^2 \ . \tag{5.5.18}$$

Proof. The first statement is an immediate consequence of Proposition 5.5.8. Since, $\mathsf{d}_{\mathcal{E}} \geq \mathsf{d}_{W_2}$, the Varadhan-type estimate (5.5.18) follows by the general result [75, Thm. 1.1]. ∎

5.5.2 A quasi-homeomorphic Dirichlet form. In this section, assume $d \geq 2$ whenever not explicitly stated otherwise. We construct a Dirichlet form on $L^2(\widehat{\mathbf{M}}, \widehat{\mathsf{m}}_\beta)$ quasi-homeomorphic to $\mathcal{E}$. Namely, the Π_β-randomization $(\widehat{\mathsf{E}}, \mathscr{D}(\widehat{\mathsf{E}}))$ (Dfn. 5.6.19) of the forms $(\mathsf{E}^{\mathbf{s}}, \mathscr{D}(\mathsf{E}^{\mathbf{s}}))$, varying $\mathbf{s} \in \mathbf{T}$.

Definition 5.5.10. We denote by $(\widehat{\mathsf{E}}, \mathscr{D}(\widehat{\mathsf{E}}))$ the Dirichlet form on $L^2(\widehat{\mathbf{M}}, \widehat{\mathsf{m}}_\beta)$ with semigroup $\widehat{\mathsf{H}}_\bullet$ defined as in (5.2.7).

Theorem 5.5.11. *The forms $(\widehat{\mathsf{E}}, \mathscr{D}(\widehat{\mathsf{E}}))$ on $L^2(\widehat{\mathbf{M}}, \tau_{\mathrm{u}}, \widehat{\mathsf{m}}_\beta)$ and $(\mathcal{E}, \mathscr{D}(\mathcal{E}))$ on $L^2(\mathscr{P}^{\mathrm{pa}}_{\mathrm{iso}}, \tau_{\mathrm{a}}, \mathcal{D}_{\mathsf{m}})$ are quasi-homeomorphic (in the sense of [27, Dfn. 3.1]) via the map $\mathbf{\Phi}$ defined in* (5.2.8).

Proof. For $i<j\in\mathbb{N}$ set $\mathbf{U}_{i,j,\delta}\coloneqq\{\mathbf{x}\in\mathbf{M}\mid \mathsf{d_g}(x_i,x_j)<\delta\}$. Notice that $\mathbf{U}_{i,j,\delta}$ is open and that for every $\varepsilon>0$ there exists $\delta(\varepsilon)>0$ such that $\overline{\mathbf{m}}\,\mathbf{U}_{i,j,\delta(\varepsilon)}<\varepsilon$. As a consequence, the set

$$\mathbf{U}_m\coloneqq\bigcup_{\substack{i,j\in\mathbb{N}\\ i<j}}\mathbf{U}_{i,j,\delta(2^{-i-j}/m)}\subset\mathbf{M}\ ,\qquad m\in\mathbb{N}\ ,$$

is open, relatively compact, and satisfies $\overline{\mathbf{m}}\,\mathbf{U}_m\leq 1/m$, $\overline{\mathbf{U}_{m+1}}\subset\mathbf{U}_m$ and $\mathbf{U}_m\downarrow_m\mathbf{M}_\circ^{\mathrm{c}}$. Finally, set $\mathbf{F}_m\coloneqq\mathbf{U}_m^{\mathrm{c}}$ and notice that $\mathbf{F}_m$ is compact (closed) and satisfies $\mathbf{F}_m\uparrow_m\mathbf{M}_\circ$.

A nest for $\widehat{\mathsf{E}}$. The set $\mathbf{M}_\circ$ is $\mathsf{E}^{\mathbf{s}}$-coexceptional by Lemma 5.3.9. Since $\mathrm{cap}_{\mathbf{s}}\coloneqq\mathrm{cap}_{\mathsf{E}^{\mathbf{s}}}$ is a Choquet capacity,

$$\begin{aligned}\lim_m \mathrm{cap}_{\mathbf{s}}(\mathbf{F}_m^{\mathrm{c}})&\leq\lim_m\mathrm{cap}_{\mathbf{s}}\big(\overline{\mathbf{F}_m^{\mathrm{c}}}\big)=\inf_m\mathrm{cap}_{\mathbf{s}}\big(\overline{\mathbf{U}_m}\big)\\ &=\mathrm{cap}_{\mathbf{s}}\Big(\bigcap_m\overline{\mathbf{U}_m}\Big)=\mathrm{cap}_{\mathbf{s}}\Big(\bigcap_m\mathbf{U}_m\Big)=\mathrm{cap}_{\mathbf{s}}(\mathbf{M}_\circ^{\mathrm{c}})=0\ ,\end{aligned}$$

hence $(\mathbf{F}_m)_m$ is a nest for $(\mathsf{E}^{\mathbf{s}},\mathscr{D}(\mathsf{E}^{\mathbf{s}}))$ for every $\mathbf{s}\in\mathbf{T}$. Set now

$$\mathbf{T}_m\coloneqq\big\{\mathbf{s}\in\mathbf{T}\mid s_\ell-s_{\ell+1}\geq 2^{-\ell-1}/m\big\}$$

and notice that $\mathbf{T}_m$ is compact (closed) and satisfies $\mathbf{T}_m\uparrow_m\mathbf{T}_\circ$. Then, $\widehat{\mathbf{F}}_m\coloneqq\mathbf{T}_m\times\mathbf{F}_m$ is compact (closed) in $\widehat{\mathbf{M}}_\circ$ (in $\widehat{\mathbf{M}}$) and satisfies $\widehat{\mathbf{F}}_m\uparrow_m\widehat{\mathbf{M}}_\circ$. By Proposition 5.6.18***(vi)***, $\big(\widehat{\mathbf{F}}_m\big)_m$ is a nest for $(\widehat{\mathsf{E}},\mathscr{D}(\widehat{\mathsf{E}}))$.

A nest for $\mathcal{E}$. Since $\mathbf{\Phi}\colon(\widehat{\mathbf{M}}_\circ,\tau_{\mathrm{u}})\to(\mathscr{P},\tau_{\mathrm{a}})$ is a homeomorphism onto its image $\mathscr{P}^{\mathrm{pa}}_{\mathrm{iso}}$ (Prop. 5.3.14***(ii)***), then $G_m\coloneqq\mathbf{\Phi}(\widehat{\mathbf{F}}_m)\subset\mathscr{P}$ is itself compact in $(\mathscr{P}^{\mathrm{pa}}_{\mathrm{iso}},\tau_{\mathrm{a}})$, hence compact in $(\mathscr{P},\tau_{\mathrm{a}})$ and, in turn, compact (closed) in $(\mathscr{P},\tau_{\mathrm{n}})$ by Proposition 5.3.11***(i)***. (Also cf. [52, Lem. 2.4].) Set

$$\big(\widehat{\mathfrak{Z}}^2_\varepsilon\big)_{G_m}\coloneqq\big\{u\in\widehat{\mathfrak{Z}}^2_\varepsilon\mid u\equiv\mathbf{0}\ \mathcal{D}_{\mathsf{m}}\text{-a.e. on }G_m^{\mathrm{c}}\big\}\subset\mathscr{D}(\mathcal{E})_{G_m}\ ,\qquad\varepsilon\in I\ ,$$

and notice that $\big(\widehat{\mathfrak{Z}}^2_0\big)_{G_m}\subset\mathcal{C}(\mathscr{P},\tau_{\mathrm{a}})$ for every $m\in\mathbb{N}$ by Remark 5.4.3***(c)***. Then, in order to prove that $(G_m)_m$ is a nest for $(\mathcal{E},\mathscr{D}(\mathcal{E}))$ we need to show that $\bigcup_m\big(\widehat{\mathfrak{Z}}^2_0\big)_{G_m}$ is dense in $\mathscr{D}(\mathcal{E})$.

We start by reducing the statement to a finite-dimensional case. In fact, by Corollary 5.4.18, it suffices to show that $\mathcal{C}_n\coloneqq\bigcup_m\big(\widehat{\mathfrak{Z}}^2_{1/n}\big)_{G_m}$ is dense in $\widehat{\mathfrak{Z}}^2_{1/n}$ for $n\in\mathbb{N}$. To this end, fix $n\in\mathbb{N}$ and let $u\coloneqq F\circ\hat{\mathbf{f}}^\star$ be arbitrary in $\widehat{\mathfrak{Z}}^2_{1/n}$. By definition of $\widehat{\mathfrak{Z}}^0_{1/n}$, it holds that $\mathbf{\Phi}^*u=(\Phi_n^*u)\circ\mathrm{pr}^n$ $\widehat{\mathbf{m}}_\beta$-a.e.. Therefore, by Lemma 5.5.5 it suffices to establish that $D_n'\coloneqq\Phi^*\mathcal{C}_n$ is dense in $\mathscr{D}(\hat{\mathsf{E}}^n)$.

To this end, set $T_m^n\coloneqq\mathrm{pr}^n(\mathbf{T}_m)$, $F_m^n\coloneqq\mathrm{pr}^n(\mathbf{F}_m)$ and $\hat{F}_m^n\coloneqq T_m^n\times F_m^n$. Since $\mathbf{\Phi}$ is a homeomorphism onto G_m for every m, one has $\Phi_n^*\big(\big(\widehat{\mathfrak{Z}}^1_{1/n}\big)_{G_m}\big)=\big(\Phi_n^*\big(\widehat{\mathfrak{Z}}^1_{1/n}\big)\big)_{\hat{F}_m^n}$. Thus, the conclusion follows by Lemma 5.5.4 with T_m^n and F_m^n as above.

Intertwining. It suffices to prove the intertwining property $\widehat{\mathsf{E}}\circ\mathbf{\Phi}^*=\mathcal{E}$ for all $u\in\mathcal{C}$ with $\mathcal{C}$ dense in $\mathscr{D}(\mathcal{E})$ and $\mathbf{\Phi}^*\mathcal{C}$ dense in $\mathscr{D}(\widehat{\mathsf{E}})$. We choose $\mathcal{C}\coloneqq\widehat{\mathfrak{Z}}^1_0=\bigcup_n\widehat{\mathfrak{Z}}^1_{1/n}$ (cf. Rmk. 5.4.3***(h)***). The first density requirement follows by definition of $(\mathcal{E},\mathscr{D}(\mathcal{E}))$.

By standard topological facts and Lemma 5.5.5, one has

$$\begin{aligned}\mathrm{cl}_{\widehat{\mathsf{E}}_1^{1/2}}\left(\mathbf{\Phi}^*\bigcup_n\widehat{\mathfrak{Z}}^1_{1/n}\right)&\supset\bigcup_n\mathrm{cl}_{\widehat{\mathsf{E}}_1^{1/2}}\big(\mathbf{\Phi}^*\widehat{\mathfrak{Z}}^1_{1/n}\big)=\bigcup_n\mathrm{cl}_{\widehat{\mathsf{E}}_1^{1/2}}\big(\Phi_n^*\widehat{\mathfrak{Z}}^1_{1/n}\circ\mathrm{pr}^n\big)\\ &=\bigcup_n\mathrm{cl}_{(\hat{\mathsf{E}}^n)_1^{1/2}}\big(\Phi_n^*\widehat{\mathfrak{Z}}^1_{1/n}\big)\circ\mathrm{pr}^n=\bigcup_n\mathscr{D}(\hat{\mathsf{E}}^n)\circ\mathrm{pr}^n\ .\end{aligned}$$

As a consequence, it suffices to show that $\bigcup_n\mathscr{D}(\hat{\mathsf{E}}^n)\circ\mathrm{pr}^n$ is dense in $\mathscr{D}(\widehat{\mathsf{E}})$. By our usual reduction argument, it suffices to show that $\bigcup_n\mathscr{D}(\mathsf{E}^{n,\mathbf{s}})\circ\mathrm{pr}^n$ is dense in $\mathscr{D}(\mathsf{E}^{\mathbf{s}})$ for Π_β-a.e. $\mathbf{s}\in\mathbf{T}$. This is however immediate by definition of $(\mathsf{E}^{\mathbf{s}},\mathscr{D}(\mathsf{E}^{\mathbf{s}}))$, since $\bigcup_n\mathscr{D}(\mathsf{E}^{n,\mathbf{s}})\circ\mathrm{pr}^n\supset\mathcal{FC}^\infty$. (Cf. (5.3.7).)

As for the intertwining, for all $u \in \widehat{\mathfrak{Z}}^1_{1/n}$ it holds by (5.5.9) that

$$\widehat{\mathsf{E}}(\mathbf{\Phi}^* u) = \widehat{\mathsf{E}}(\Phi_n^* u \circ \mathrm{pr}_n) \ . \tag{5.5.19}$$

Noticing that $\Phi_n^* u \circ \mathrm{pr}_n \in \mathcal{FC}^1$ by definition of $\widehat{\mathfrak{Z}}^1_{1/n}$, it follows by definition of $\widehat{\mathsf{E}}$ and (5.3.8) that

$$\widehat{\mathsf{E}}(\Phi_n^* u \circ \mathrm{pr}_n) = \hat{\mathsf{E}}^n(\Phi_n^* u) \ . \tag{5.5.20}$$

Respectively by Lemma 5.5.5 and definition of $(\mathcal{E}^{1/n}, \mathscr{D}(\mathcal{E}^{1/n}))$,

$$\hat{\mathsf{E}}^n(\Phi_n^* u) = \mathcal{E}^{1/n}(u) = \mathcal{E}(u) \ . \tag{5.5.21}$$

Finally, combining (5.5.19)–(5.5.21) concludes the proof of the intertwining property. ■

As a consequence of the regularity of $(\widehat{\mathsf{E}}, \mathscr{D}(\widehat{\mathsf{E}}))$ on $L^2(\widehat{\mathbf{M}}, \tau_{\mathrm{u}}, \widehat{\mathsf{m}}_\beta)$ and of [27, Thm. 3.7] we have

Corollary 5.5.12. *The Dirichlet form $(\mathcal{E}, \mathscr{D}(\mathcal{E}))$ on $L^2(\mathscr{P}^{\mathrm{pa}}_{\mathrm{iso}}, \tau_{\mathrm{a}}, \mathcal{D}_{\mathsf{m}})$ is quasi-regular.*

Remark 5.5.13. Although $\mathbf{\Phi}^* \colon L^2_{\mathcal{D}_{\mathsf{m}}}(\mathscr{P}) \to L^2_{\widehat{\mathsf{m}}_\beta}(\widehat{\mathbf{M}})$ is an order isomorphism, Theorem 5.5.11 does not follow from the general result [106, Thm. 3.12], where the intertwined quasi-regular Dirichlet forms are additionally assumed irreducible. We postpone a study of the $\mathcal{E}$-invariant sets to Theorem 5.5.17 below.

5.5.3 Properties of $\eta_\bullet$. Recall that $\mathfrak{G} := \mathrm{Diff}^\infty_+(M)$ and let $\mathfrak{F}^w := \left(\psi^{w,t}\right)_{t\in\mathbb{R}}$ be the one-parameter subgroup of $\mathfrak{G}$ generated by $w \in \mathfrak{X}^\infty$, and $\mathfrak{I} := \mathrm{Iso}(M, \mathcal{B}(M))$ be the group of bijective bi-measurable transformations of $(M, \mathcal{B}(M))$. The natural action $G \circlearrowleft M$ of any $G \subset \mathfrak{I}$ lifts to an action on $\mathscr{P}$ as in (5.2.29), denoted by $G_\sharp$.

Proposition 5.5.14. *Assume $d \geq 1$. Then, **(i)** $\mathfrak{I} \circlearrowleft M$ is σ-transitive. Assume $d \geq 2$. Then, **(ii)** for every $n \in \mathbb{N}$ and every $\mathbf{x}, \mathbf{x}' \in M_\circ^{\times n}$ there exists $w \in \mathfrak{X}^\infty$ such that*

$$\mathbf{x}' = (\psi^{w,1})^{\times n}(\mathbf{x}) := \left(\psi^{w,1}(x_1), \ldots, \psi^{w,1}(x_n)\right) \ ; \tag{5.5.22}$$

*and **(iii)** $\mathfrak{G} \circlearrowleft M$ is finitely transitive.*

Proof. ***(i)*** Let $\mathbf{x}, \mathbf{x}' \in \mathbf{M}_\circ$. The map $g \colon M \to M$ defined by $g(x_i) := x'_i$ for all $i \in \mathbb{N}$ and $g(x) := x$ if $x \neq x_i$ for all $i \in \mathbb{N}$ is bijective since $x_i \neq x_j$ and $x'_i \neq x'_j$ for every $i \neq j$ and it is straightforwardly bi-measurable. ***(iii)*** is an immediate consequence of ***(ii)***. Since $\mathbf{x}, \mathbf{x}' \in M_\circ^{\times n}$ there exist $\varepsilon > 0$ and smooth arcs $\gamma^i_\cdot \colon I \to M$ satisfying ***(a)*** $\gamma^i_0 = x_i$ and $\gamma^i_1 = x'_i$ for all $i \leq n$; ***(b)*** $\gamma^i_\cdot$ is a simple open arc and $B_\varepsilon(\mathrm{im}\gamma^i_\cdot)$ is contractible for all $i \leq n$; and ***(c)*** $B_\varepsilon(\mathrm{im}\gamma^i_\cdot) \cap B_\varepsilon(\mathrm{im}\gamma^j_\cdot) = \varnothing$ for $i \neq j \leq n$. For each $i \leq n$ define a vector field w^i on $\mathrm{im}\gamma^i_\cdot$ by $w^i_{\gamma^i_t} := \dot\gamma^i_t$ for every $t \in I$ (this definition is well-posed by (*c*)). By standard techniques involving partitions of unity, each w^i may be extended to a (non-relabeled) globally defined smooth vector field vanishing outside $B_\varepsilon(\mathrm{im}\gamma^i_\cdot)$. Let $w := \sum_i w^i$. By construction, one has $\psi^{w,t}(x_i) = \gamma^i_t$ for every $i \leq n$ and $t \in I$, thus (5.5.22) holds. ■

As a consequence of Propositions 5.3.4, 5.3.14***(ii)*** and 5.5.14***(i)*** we have

Corollary 5.5.15. *The (Borel) spaces $\left((\mathscr{P}^{\mathrm{pa}}_{\mathrm{iso}}, \tau_{\mathrm{a}})/\mathfrak{I}_\sharp\ , \mathrm{pr}_\sharp^{\mathfrak{I}_\sharp} \mathcal{D}_{\beta\overline{\mathsf{m}}}\right)$ and $(\mathbf{T}_\circ, \Pi_\beta)$ are homeomorphic, isomorphic measure spaces.*

We say that $A\subset\mathscr{P}^{\mathrm{pa}}_{\mathrm{iso}}$ is $\mathfrak{F}^{n,w}_\sharp$-*invariant* if

$$\eta=\sum_i s_i\delta_{x_i}\in A\ , \qquad \mathrm{orb}_{\mathfrak{F}^w_n}(\eta):=\bigcup_{t\in\mathbb{R}}\left\{\sum_{i\leq n} s_i\delta_{\psi^{w,t}(x_i)}+\sum_{i>n}s_i\delta_{x_i}\right\}\subset A\ .$$

By definition, $\mathfrak{F}^{\infty,w}_\sharp=\mathfrak{F}^w_\sharp$. For $N\in\overline{\mathbb{N}}$ we say that A is $\mathfrak{F}^N_\sharp$-invariant if it is $\mathfrak{F}^{N,w}_\sharp$-invariant for each $w\in\mathfrak{X}^\infty$. Consistently, for any $\eta\in\mathscr{P}^{\mathrm{pa}}_{\mathrm{iso}}$ we set $\mathrm{orb}_{\mathfrak{F}^n}(\eta):=\bigcup_{w\in\mathfrak{X}^\infty}\mathrm{orb}_{\mathfrak{F}^{n,w}}(\eta)$ and analogously for $\mathrm{orb}_{\mathfrak{F}}(\eta)$. In light of the fact that the natural action of $\mathrm{Diff}^\infty(M)$ on M is finitely transitive but not σ-transitive, $\mathfrak{F}^n_\sharp$-invariance and $\mathfrak{F}_\sharp$-invariance are not comparable notions.

Proposition 5.5.16. *Let* $A\subset\mathscr{P}^{\mathrm{pa}}_{\mathrm{iso}}$. *Then,* ***(i)*** A *is* $\mathfrak{I}_\sharp$*-invariant if and only if it is* $\mathfrak{F}^n_\sharp$*-invariant for all* $n\in\mathbb{N}$*;* ***(ii)*** *if* A *is* $\mathfrak{F}^n_\sharp$*-invariant for some* $n\in\mathbb{N}$*, then it is also* $\mathfrak{F}^k_\sharp$*-invariant for all* $k\leq n\in\mathbb{N}$*;* ***(iii)*** *there exists* $A\subset\mathscr{P}^{\mathrm{pa,fs}}_{\mathrm{so}}$ *such that* A *is* $\mathfrak{F}_\sharp$*-invariant but not* $\mathfrak{I}_\sharp$*-invariant.*

Proof. The forward implication in ***(i)*** is straightforward (cf. the proof of Prop. 5.5.14***(i)***). For the reverse implication let A be $\mathfrak{F}^n_\sharp$-invariant for all $n\in\mathbb{N}$. Set $\mathrm{pr}^\times_n:=\mathrm{pr}_n\circ\mathbf{\Phi}^{-1}$. By Prop. 5.5.14***(ii)***

$$\mathrm{pr}^\times_n(A)=\mathrm{pr}^\times_n(\mathfrak{F}^n.A)=\mathrm{pr}_{\Sigma^n}\circ\mathbf{\Phi}^{-1}(A)\times M^{\times n}_\circ=\mathrm{pr}^\times_n(\mathfrak{I}.A)\ .$$

As a consequence,

$$A=\bigcap_n(\mathrm{pr}^\times_n)^{-1}\big(\mathrm{pr}^\times_n(A)\big)=\bigcap_n(\mathrm{pr}^\times_n)^{-1}\big(\mathrm{pr}^\times_n(\mathfrak{I}.A)\big)=\mathfrak{I}.A\ ,$$

that is, A is $\mathfrak{I}$-invariant. ***(ii)*** is straightforward since $\mathfrak{F}^k.A\subset\mathfrak{F}^n.A$ for all $A\subset\mathscr{P}^{\mathrm{pa}}_{\mathrm{iso}}$ and all $k\leq n$.

In order to show ***(iii)*** let $\eta=\sum_i s_i\delta_{x_i}\in\mathscr{P}^{\mathrm{pa,fs}}_{\mathrm{so}}$ be arbitrary. By ***(i)*** $\bigcup_n\mathrm{orb}_{\mathfrak{F}_n}(\eta)=\mathrm{orb}_{\mathfrak{I}}(\eta)$. Let $\mathfrak{H}:=\mathrm{Homeo}(M)$. We show the possibly stronger statement that $\mathrm{orb}_{\mathfrak{I}}(\eta)\supsetneq\mathrm{orb}_{\mathfrak{H}}(\eta)\supset\mathrm{orb}_{\mathfrak{F}}(\eta)$. Indeed, for fixed $n\in\mathbb{N}$ let $y\neq x_n$. On the one hand, the measure $\tilde\eta:=\sum_{i\in\mathbb{N}\setminus\{n\}}s_i\delta_{x_i}+s_n\delta_y$ satisfies $\tilde\eta\in\mathrm{orb}_{\mathfrak{I}}(\eta)$ and $\tilde\eta\neq\eta$ (cf. the proof of Prop. 5.5.14***(i)***). On the other hand, argue by contradiction that there exists $h\in\mathrm{Homeo}(M)$ such that $h_\sharp\eta=\tilde\eta$. Since for $i\neq j$ one has $s_i\neq s_j$ by definition of $\mathscr{P}^{\mathrm{pa,fs}}_{\mathrm{so}}$, then $h(x_i)=x_i$ for all $i\neq n$. Again since $\eta\in\mathscr{P}^{\mathrm{pa,fs}}_{\mathrm{so}}$, the set $\{x_i\}_{i\in\mathbb{N}\setminus\{n\}}$ is dense in M, hence, by continuity of h it must be $h=\mathrm{id}_M$, and $\tilde\eta=\eta$, a contradiction. ∎

Theorem 5.5.17. *Assume* $d\geq 2$ *and let* $\eta_\bullet$ *be defined as in* (5.5.1)*. Then,* ***(i)*** $\eta_\bullet$ *satisfies* (5.2.9)*;* ***(ii)*** $\eta_\bullet$ *is not irreducible: a measurable set* $A\subset\mathscr{P}$ *is* $\eta_\bullet$*-invariant if and only if it is* $(\mathfrak{I}_\sharp,\mathcal{D}_{\mathsf{m}})$*-invariant;* ***(iii)*** $\eta_\bullet$ *is not ergodic;* ***(iv)*** $\eta_\bullet$ *has a (non-relabeled) distinguished extension to all starting points in* $\mathscr{P}^{\mathrm{pa}}$ *and satisfying* (5.2.3)*;* ***(v)*** $\eta_\bullet$ *has* τ_{a}*-continuous sample paths;* ***(vi)*** *let the initial distribution of* η_0 *be satisfying* $\mathrm{law}(\eta_0)\ll\mathcal{D}_{\mathsf{m}}$*. Then, for each* $u\in\widehat{\mathfrak{Z}}^2_0$*, the process*

$$M^u_t:=u(\eta_t)-u(\eta_0)-\int_0^t \mathrm{d}s\,\mathbf{L}u(\eta_s)$$

is a martingale with quadratic variation process

$$[M^u]_t=\int_0^t\mathrm{d}s\,\mathbf{\Gamma}(u)(\eta_s)\ .$$

Proof. Since $\mathbf{\Phi}$ is bijective between an $\widehat{\mathsf{E}}$-coexceptional and an $\mathcal{E}$-coexceptional set, Equation (5.2.9) is satisfied as a consequence of Theorem 5.5.11.

Invariant sets. Assume first that $A\subset\mathscr{P}$ is $(\mathfrak{I}_\sharp,\mathcal{D}_\mathsf{m})$-invariant. Without loss of generality, $A\subset\mathscr{P}^{\mathrm{pa}}_{\mathrm{iso}}$, since $\mathcal{D}_\mathsf{m}\mathscr{P}^{\mathrm{pa}}_{\mathrm{iso}}=1$ and $\mathscr{P}^{\mathrm{pa}}_{\mathrm{iso}}$ is $\mathfrak{I}_\sharp$-, hence $(\mathfrak{I}_\sharp,\mathcal{D}_\mathsf{m})$-, invariant. By a straightforward density argument and Corollary 5.5.15, (cf. (5.4.2))

$$L^2(\mathbf{T}_\circ,\Pi_\beta)\cong L^2\left(\mathscr{P}^{\mathrm{pa}}_{\mathrm{iso}}/\mathfrak{I}_\sharp\,,\mathrm{pr}^{\mathfrak{I}_\sharp}_\sharp\mathcal{D}_\mathsf{m}\right)\cong L^2_{\mathcal{D}_\mathsf{m},\mathfrak{I}_\sharp}(\mathscr{P})=\mathrm{cl}_{L^2_{\mathcal{D}_\mathsf{m}}}\widehat{\mathfrak{Z}}^1_{-,0}\ .$$

Since $\boldsymbol{\nabla}\equiv\mathbf{0}$ on $\widehat{\mathfrak{Z}}^1_{-,0}$, one has $\mathcal{E}\equiv\mathbf{0}$ on $\mathrm{cl}_{\mathcal{E}_1^{1/2}}(\widehat{\mathfrak{Z}}^1_{-,0})=\mathrm{cl}_{L^2_{\mathcal{D}_\mathsf{m}}}\widehat{\mathfrak{Z}}^1_{-,0}$. By strong locality, $\boldsymbol{\Gamma}$ satisfies the Leibniz rule, hence

$$\forall u,v\in\mathscr{D}(\mathcal{E})\qquad \mathbb{1}_A\,u\in\mathscr{D}(\mathcal{E})\qquad\text{and}\qquad \mathcal{E}(\mathbb{1}_A\,u,v)=\mathcal{E}(\mathbb{1}_A\,u,\mathbb{1}_A\,v)=\mathcal{E}(u,\mathbb{1}_A\,v)\tag{5.5.23}$$

as soon as $\mathbb{1}_A\in L^2_{\mathcal{D}_\mathsf{m},\mathfrak{I}_\sharp}(\mathscr{P})$ or, equivalently, A is $(\mathfrak{I}_\sharp,\mathcal{D}_\mathsf{m})$-invariant. Thus, (5.3.1) follows by (5.5.23) since $\boldsymbol{\Gamma}(\mathbb{1}_A,u)=0$ for every $u\in\mathscr{D}(\mathcal{E})$. Viceversa, assume that A is $\mathcal{E}$-invariant. If $\mathcal{D}_\mathsf{m}A=0$, resp. 1, then A is $(\mathfrak{I}_\sharp,\mathcal{D}_\mathsf{m})$-invariant since $\varnothing$, resp. $\widehat{\mathbf{M}}_\circ$, is. Assume then $\mathcal{D}_\mathsf{m}A\in(0,1)$. Without loss of generality, $A\subset\mathscr{P}^{\mathrm{pa}}_{\mathrm{iso}}$, since $\mathscr{P}^{\mathrm{pa}}_{\mathrm{iso}}$ is $\mathcal{E}$-coexceptional. Thus, $\boldsymbol{\Phi}^{-1}$ is well-defined on A and $B:=\boldsymbol{\Phi}^{-1}(A)\subset\widehat{\mathbf{M}}_\circ$ is $\widehat{\mathsf{E}}$-invariant by Theorem 5.5.11. Since $(\mathsf{E}^{\mathbf{s}},\mathscr{D}(\mathsf{E}^{\mathbf{s}}))$ is ergodic for every $\mathbf{s}\in\mathbf{T}$ by the discussion in [15, §1], it follows by Proposition 5.6.18***(v)*** that the only $\widehat{\mathsf{E}}$-invariant sets are of the form $C\times\prod_i U_i$ where $C\in\mathcal{B}(\mathbf{T}_\circ)$ is any measurable set and U_i satisfies either $U_i=M_i$ or $U_i=\varnothing$ for all $i\in\mathbb{N}$. As a consequence of the fact that $\widehat{\mathsf{m}}_\beta B\in(0,1)$, it must be $U_i=M$ for every $i\in\mathbb{N}$ and $\Pi_\beta C\in(0,1)$, that is $B=C\times\mathbf{M}$. The $(\mathfrak{I}_\sharp,\mathcal{D}_\mathsf{m})$-invariance of A follows by the $\mathfrak{I}\circlearrowleft\mathbf{M}$-invariance of $\mathbf{M}$ since

$$\iota_\sharp\boldsymbol{\Phi}(\mathbf{s},\mathbf{x})=\boldsymbol{\Phi}(\mathbf{s},\iota^{\times\infty}(\mathbf{x}))\ ,\qquad \iota\in\mathfrak{I}\ .$$

Lack of ergodicity. Let $\varepsilon\in(0,1)$. Since Π_β is diffuse, by the main result in [148] there exists $A''\in\mathcal{B}(\mathbf{T}_\circ)$ with $\Pi_\beta A''=\varepsilon$. Let A' be the corresponding subset of $\mathscr{P}^{\mathrm{pa}}_{\mathrm{iso}}/\mathfrak{I}_\sharp$ via the homeomorphism of Corollary 5.5.15. Then, $A:=(\mathrm{pr}^{\mathfrak{I}_\sharp})^{-1}(A')$ is $\mathfrak{I}_\sharp$-invariant by definition, $\mathcal{B}_{\mathrm{n}}(\mathscr{P})$ measurable by Proposition 5.3.4, satisfying $\mathcal{D}_\mathsf{m}A=\varepsilon$ by Corollary 5.5.15 and $\mathcal{E}$-invariant by the previous step. As a consequence, $(\mathcal{E},\mathscr{D}(\mathcal{E}))$ is *not* ergodic.

Continuity of paths and extension. For every $\eta_0\in\mathscr{P}^{\mathrm{pa}}_{\mathrm{iso}}$ the path $t\mapsto\eta^{\eta_0}_t$ is τ_{a}-continuous as a consequence of Corollary 5.5.12 and the standard theory of Dirichlet forms.

Consistently with the definition of $\mathsf{W}^{\mathbf{s}}_\bullet$ for $\mathbf{s}\in\mathbf{T}_\circ$, for $\mathbf{s}\in\mathbf{T}\setminus\mathbf{T}_\circ$ we set

$$\mathsf{W}^{\mathbf{s};\mathbf{x}_0}_t(\omega):=\left(x^1_{t/s_1},x^2_{t/s_2},\dots\right)\ ,$$

where $\left(x^i_t\right)_{t\geq0}$ are independent Brownian motions on M and, conventionally, $x^i_{t/s_i}=x^i_0$ for all $t\geq0$ whenever $s_i=0$. Then, letting

$$\widehat{\mathsf{W}}^{\mathbf{s},\mathbf{x}_0}_\bullet:=\mathsf{W}^{\mathbf{s};\mathbf{x}_0}_\bullet\ ,\qquad\text{and}\qquad \eta_0:=\boldsymbol{\Phi}(\mathbf{s},\mathbf{x}_0)\ ,\quad \eta^{\eta_0}_\bullet:=\boldsymbol{\Phi}\circ\widehat{\mathsf{W}}^{\mathbf{s},\mathbf{x}_0}_\bullet$$

yields the desired extension to all starting points in $\boldsymbol{\Phi}(\mathbf{T}\times\mathbf{M}_\circ)=\mathscr{P}^{\mathrm{pa}}$ satisfying (5.2.3).

A proof of ***(vi)*** is standard and it is therefore omitted. ■

As a consequence of Theorem 5.5.17***(v)***–***(vi)***, the process $\eta_\bullet$ defined as in (5.5.1) is, equivalently, a solution of the following martingale problem.

Corollary 5.5.18 (Martingale problem)**.** *For every $\hat f\in\mathcal{C}^2(\widehat{M}_0)$ (Def. 5.4.1) the process*

$$M^{\hat f}_t:=\hat f^\star(\eta_t)-\hat f^\star(\eta_0)-\int_0^t\mathrm{d}s\,\mathbf{B}_0\big[\nabla\hat f\big](\eta_s)$$

is a continuous martingale with quadratic variation process

$$\left[M^{\hat f}\right]_t=\int_0^t\mathrm{d}s\,\Gamma(\hat f)^\star(\eta_s)\ .$$

Remark 5.5.19 (Distinguished invariant measures). By Theorem 5.5.11, it follows from the extremality of $\widehat{\mathbf{W}}_\bullet$-ergodic measures that $\mathbf{Q} \in \mathscr{P}(\mathscr{P})$ is $\eta_\bullet$-ergodic if and only if it is of the form $\mathbf{Q}_\mathbf{s} \coloneqq \mathbf{\Phi}_\sharp(\delta_\mathbf{s} \otimes \overline{\mathsf{m}})$ for some $\mathbf{s} \in \mathbf{T}_\circ$. It is straightforward that every such measure satisfies

$$\mathbf{E}_{\mathbf{Q}_\mathbf{s}} \mathbf{E}_{P_\cdot} [\eta_t^\cdot A] = \overline{\mathsf{m}} A \ , \qquad A \in \mathcal{B}(A) \ , \ t \geq 0 \ . \tag{5.5.24}$$

More generally, (5.5.24) holds for any $\eta_\bullet$-invariant measure $\mathbf{Q}$, since $\mathbf{Q} \in \overline{\mathrm{conv}}\,\{\mathbf{Q}_\mathbf{s} \mid \mathbf{s} \in \mathbf{T}_\circ\}$.

Remark 5.5.20. We notice that the requirement in (5.2.3) is not met by the process defined by Y. Chow and W. Gangbo in [29], which satisfies instead for each fixed starting point $\mu_0 \in \mathscr{P}_2$

$$\mu_t^{\mathrm{CG},\mu_0}(\omega) = \big(\mathrm{id}_M + \sqrt{2}\, b_t(\omega)\big)_\sharp \mu_0 \ ,$$

where $M = \mathbb{R}^d$, and $b_\bullet$ is a standard d-dimensional Brownian motion. The process $\mu_\bullet^{\mathrm{CG}}$ satisfies

$$\mathbf{E}_{P_{\mu_0}} [\mu_t^{\mathrm{CG},\mu_0} A] = \mu_0 A \ , \qquad A \in \mathcal{B}(A) \ , \ t \geq 0 \ ,$$

and it may then be made to satisfy (5.5.24) if the initial distribution of μ_0 is chosen according to some suitable randomness and independently of $b_\bullet$.

Some remarks on the case of one-dimensional base space. Although most of the previous results only hold when $d \geq 2$, we are able to construct a regular strongly local Dirichlet form on $\mathscr{P}$ also in the case when $d = 1$, i.e. when $M = \mathbb{S}^1$.

Definition 5.5.21 (Reduced form)**.** We denote by $(\mathcal{E}^{\mathrm{red}}, \mathscr{D}(\mathcal{E}^{\mathrm{red}}))$ the τ_n-regular strongly local Dirichlet form on $L^2(\mathscr{P}_2(\mathbb{S}^1), \mathcal{D}_\mathsf{m})$ defined as the closure of the form $(\mathcal{E}, \mathfrak{Z}^1)$.

Proof. By Lemma 5.4.16 we have $\mathfrak{Z}^1 \subset \mathscr{D}(\mathcal{E})$, hence the statement is well-posed. The closability of $(\mathcal{E}, \mathfrak{Z}^1)$ follows since $(\mathcal{E}, \mathscr{D}(\mathcal{E}))$ is closed. The Markov property and strong locality are inherited from $(\mathcal{E}, \mathscr{D}(\mathcal{E}))$. The density of $\mathfrak{Z}^1$ in $\mathcal{C}(\mathscr{P}_2)$ holds as in the proof of Corollary 5.4.19. ■

Remark 5.5.22. If $d \geq 2$ we have $(\mathcal{E}^{\mathrm{red}}, \mathscr{D}(\mathcal{E}^{\mathrm{red}})) = (\mathcal{E}, \mathscr{D}(\mathcal{E}))$ by Lemma 5.6.26. However, if $d = 1$, our proof of Lemma 5.6.26 fails and the form $(\mathcal{E}, \mathscr{D}(\mathcal{E}))$ might be *not* τ_n-regular.

5.6 Appendix

5.6.1 Measurability properties.

Lemma 5.6.1. *For any $A \in \mathcal{B}(X)$ and $\mu \in \mathscr{P}$ define the evaluation map $\mathrm{ev}_A \colon \mu \mapsto \mu A$. Then,* ***(i)*** *the map ev_A is $\mathcal{B}_\mathrm{n}(\mathscr{P})$-measurable;* ***(ii)*** *$\mathcal{B}_\mathrm{n}(\mathscr{P})$ is generated by the maps $\{\mathrm{ev}_A\}_{A \in \mathcal{B}(X)}$;* ***(iii)*** *the r-parametric convex combination $(r, \mu, \nu) \mapsto (1-r)\mu + r\nu$ is jointly $\mathcal{B}(I) \otimes \mathcal{B}_\mathrm{n}(\mathscr{P})^{\otimes 2}$-measurable.*

Proof. ***(i)*** is a consequence of ***(ii)*** which is in turn [84, Thm. 1.5]. Since $\mathscr{M}_b(X) \supset \mathscr{P}$, endowed with the weak* topology, is a measurable vector space ***(iii)*** follows by [160, Prop. I.2.3, p. 16]. ■

Lemma 5.6.2. *The map $\mathrm{ev} \colon (\mu, x) \mapsto \mu_x \coloneqq \mu\{x\}$ is $\mathcal{B}_\mathrm{n}(\mathscr{P}) \otimes \mathcal{B}(X)$-measurable.*

Proof. Denote by $\mathsf{h}_t^*\colon \mathscr{P}\to\mathscr{P}$ the heat flow on measures. Then, **(a)** $\mathsf{h}_t^*\colon \mathscr{P}\to\mathscr{P}$ is narrowly continuous for every $t>0$; **(b)** $t\mapsto \mu_t:=\mathsf{h}_t^*\mu$ is narrowly continuous for every $\mu\in\mathscr{P}$; **(c)** $\mu_t\ll\mathsf{m}$ for every $t>0$ and every $\mu\in\mathscr{P}$. For each $\varepsilon>0$, $t>0$ and $x\in X$ the map $\mu\mapsto\mu_t B_\varepsilon(x)$ is measurable, since it is the composition of the continuous map $\mu\mapsto\mu_t$ with the measurable map $\mathrm{ev}_{B_\varepsilon(x)}$ (see Lem. 5.6.1). Moreover, it is readily seen by Dominated Convergence that for each $\varepsilon>0$, $t>0$ and $\mu\in\mathscr{P}$ the map $x\mapsto\mu_t B_\varepsilon(x)$ is continuous, since $\mathrm{d}\mu_t(y)=f_t(y)\,\mathrm{d}\mathsf{m}(y)$ for some $f_t\in L^1_{\mathsf{m}}(X)$. That is, $\mathrm{ev}_{\varepsilon,t}\colon(\mu,x)\mapsto\mu_t B_\varepsilon(x)$ is a Carathéodory map between Polish spaces, hence it is jointly measurable. Since the pointwise limit of (jointly-)measurable maps is (jointly-)measurable, it suffices to show the existence of $\lim_{\varepsilon\downarrow 0}\lim_{t\downarrow 0}\mathrm{ev}_{\varepsilon,t}=\mathrm{ev}$. To this end,

$$\begin{aligned}\liminf_{\varepsilon\downarrow 0}\liminf_{t\downarrow 0}\mu_t B_\varepsilon(x)&\geq\liminf_{\varepsilon\downarrow 0}\mu B_\varepsilon(x)=\mu_x\ ,\\ \limsup_{\varepsilon\downarrow 0}\limsup_{t\downarrow 0}\mu_t B_\varepsilon(x)&\leq\limsup_{\varepsilon\downarrow 0}\limsup_{t\downarrow 0}\mu_t\left(\overline{B_\varepsilon(x)}\right)\leq\limsup_{\varepsilon\downarrow 0}\mu\left(\overline{B_\varepsilon(x)}\right)\\ &\leq\limsup_{\varepsilon\downarrow 0}\mu B_{2\varepsilon}(x)=\mu_x\end{aligned}$$

by the Portmanteau Theorem and the outer regularity of μ. ■

Proposition 5.6.3. *Let Ω be any non-empty set, $\mathcal{A}$ be a multiplicative system of bounded real-valued functions on Ω. Let $\mathcal{B}$ be the σ-algebra generated by $\mathcal{A}$ and denote by $\mathcal{B}_b$ the space of bounded $\mathcal{B}$-measurable real-valued functions. Then, for any non-negative finite measure μ on $(\Omega,\mathcal{B})$, the system $\mathcal{A}$ is dense in $L^2_\mu(\Omega)$.*

Proof. Since μ is finite and functions in $\mathcal{A}$ are bounded, then $\mathcal{A}\subset L^2_\mu(\Omega)$. Let now $v\in\mathcal{A}^\perp\subset L^2_\mu(\Omega)$ and $H\subset\mathcal{B}_b$ be maximal such that $\int_\Omega\mathrm{d}\mu\,v\,h=0$ for every $h\in H$. It suffices to show that $v=0$ μ-a.e.. We show that $\mathcal{B}_b\subset H$, from which the previous assertion readily follows. Observe that H is a vector space, uniformly closed in $\mathbb{R}^\Omega$ and closed under monotone convergence of non-negative uniformly bounded sequences by Dominated Convergence. Since $\mathcal{A}\subset H$ is multiplicative, $\mathcal{B}_b\subset H$ by Dynkin's Multiplicative System Theorem [20, Thm. 2.12.9(i) (Vol. I)]. ■

5.6.2 Quadratic forms. Let $(H,\|\cdot\|_H)$ be a real *separable* Hilbert space.

Definition 5.6.4. By a *quadratic form* (Q,D) on H we shall always mean a symmetric positive semi-definite — if not otherwise stated, densely defined — bilinear form. To (Q,D) we associate the non-relabeled functional $Q\colon H\to\mathbb{R}\cup\{+\infty\}$ defined by

$$Q(u):=\begin{cases}Q(u,u) & \text{if } u\in D\\ +\infty & \text{otherwise}\end{cases}\ ,\qquad u\in H\ .$$

Additionally, we set for every $\alpha>0$

$$\begin{aligned}Q_\alpha(u,v):=&Q(u,v)+\alpha\langle u\,|\,v\rangle_H\ ,\qquad u,v\in D\ ,\\ Q_\alpha(u):=&Q(u)+\alpha\|u\|_H^2\ ,\qquad u\in H\ .\end{aligned}$$

For $\alpha>0$, we let $\mathscr{D}(Q)_\alpha$ be the completion of D, endowed with the Hilbert norm $Q_\alpha^{1/2}$.

The following result is well-known.

Lemma 5.6.5. *Let (Q,D) be a quadratic form on H. The following are equivalent:* ***(a)*** *(Q,D) is closable, say, with closure $(Q,\mathscr{D}(Q))$;* ***(b)*** *the canonical inclusion $\iota\colon D\to H$ extends to a continuous injection $\iota_\alpha\colon\mathscr{D}(Q)_\alpha\to H$ satisfying $\|\iota_\alpha\|\leq\alpha^{-1}$;* ***(c)*** *$Q$ is lower semi-continuous w.r.t. the strong topology of H;* ***(d)*** *Q is lower semi-continuous w.r.t. the weak topology of H.*

To every closed quadratic form $(Q, \mathscr{D}(Q))$ we associate a non-negative self-adjoint operator $-L$, with domain defined by the equality $\mathscr{D}(\sqrt{-L}) = \mathscr{D}(Q)$, such that $Q(u,v) = \langle -Lu \,|\, v\rangle_H$ for all $u, v \in \mathscr{D}(Q)$. We denote the associated semigroup by $T_t \coloneqq e^{tL}$, $t > 0$, and the associated resolvent by $G_\alpha \coloneqq (\alpha - L)^{-1}$, $\alpha > 0$. By Hille–Yosida Theorem (See e.g. [112, p. 27])

$$Q_\alpha(G_\alpha u, v) = \langle u \,|\, v\rangle_H \ , \qquad v \in \mathscr{D}(Q), u \in H \ , \tag{5.6.1a}$$

$$T_t = H\text{-}\lim_{\alpha\to\infty} e^{t\alpha(\alpha G_\alpha - 1)} \ . \tag{5.6.1b}$$

5.6.3 Generalized Mosco convergence of quadratic forms. We shall need K. Kuwae and T. Shioya's *generalized Mosco convergence* (see [99]). We start by recalling the simplified setting introduced by A. Kolesnikov in [90, §2].

Definition 5.6.6 (Convergences of Hilbert spaces, vectors, operators)**.** Let $(H_n)_{n\in\mathbb{N}}$ and H be Hilbert spaces and set $\mathcal{H} \coloneqq H \sqcup \bigsqcup_{n\in\mathbb{N}} H_n$. Let further $D \subset H$ be a dense subspace and $(\Phi_n)_{n\in\mathbb{N}}$ be densely defined linear operators

$$\Phi_n \colon D \to H_n \ . \tag{5.6.2}$$

We say that H_n $\mathcal{H}$*-converges* to H if

$$\forall u \in D \qquad \lim_n \|\Phi_n u\|_{H_n} = \|u\|_H \ , \tag{5.6.3}$$

in which case we say further that a sequence $(u_n)_{n\in\mathbb{N}}$, $u_n \in H_n$, ***(a)*** $\mathcal{H}$*-strongly converges* to $u \in H$ if there exists a sequence $(\tilde{u}_m)_{m\in\mathbb{N}} \subset D$ such that

$$\lim_m \|\tilde{u}_m - u\|_H = 0 \qquad \text{and} \qquad \lim_m \limsup_n \|\Phi_n \tilde{u}_m - u_n\|_{H_n} = 0 \ ;$$

(b) $\mathcal{H}$*-weakly converges* to $u \in H$ if, for every sequence $(v_n)_{n\in\mathbb{N}}$, $v_n \in H_n$, $\mathcal{H}$-strongly converging to $v \in H$,

$$\lim_n \langle u_n \,|\, v_n\rangle_{H_n} = \langle u \,|\, v\rangle_H \ .$$

Let further $(B_n)_{n\in\mathbb{N}}$ be a sequence of bounded operators $B_n \in \mathcal{B}(H_n)$. We say that B_n converges $\mathcal{H}$-strongly to $B \in \mathcal{B}(H)$ if $B_n u_n$ converges $\mathcal{H}$-strongly to Bu for all sequences $(u_n)_{n\in\mathbb{N}}$, $u_n \in H_n$, such that u_n $\mathcal{H}$-strongly converges to $u \in H$.

Definition 5.6.7 (Kuwae–Shioya's Mosco convergence)**.** Let $\big((Q_n, \mathscr{D}(Q_n))\big)_{n\in\mathbb{N}}$ be a sequence of closed quadratic forms, Q_n on H_n, and $(Q, \mathscr{D}(Q))$ be a quadratic form on H. We say that Q_n *Mosco converges* to Q if the following conditions hold: ***(a)*** H_n $\mathcal{H}$-converges to H; ***(b)*** (*weak* Γ-$\liminf$) if $(u_n)_{n\in\mathbb{N}}$, $u_n \in H_n$, $\mathcal{H}$-weakly converges to $u \in H$, then

$$Q(u) \le \liminf_n Q_n(u_n) \ ; \tag{5.6.4}$$

(c) (*strong* Γ-$\limsup$) for every $u \in H$ there exists a sequence $(u_n)_{n\in\mathbb{N}}$, $u_n \in H_n$, $\mathcal{H}$-strongly convergent to u and such that

$$Q(u) = \lim_n Q_n(u_n) \tag{5.6.5}$$

Clearly, in condition ***(b)*** we can additionally assume $u_n \in \mathscr{D}(Q_n)$.

Remark 5.6.8. In all the above definitions, the notion of convergence *does depend* on the family of linear operators $(\Phi_n)_{n\in\mathbb{N}}$. The latter is however omitted from the notation, for it will be apparent from the context.

Lemma 5.6.9. *Let $\big((Q_n,\mathscr{D}(Q_n))\big)_{n\in\mathbb{N}}$, resp. $\big((Q^\sharp_n,\mathscr{D}(Q^\sharp_n))\big)_{n\in\mathbb{N}}$, be a sequence of closed quadratic forms Q_n on H_n, resp. $Q^\sharp_n$ on $H^\sharp_n$, and $(Q,\mathscr{D}(Q))$ be a quadratic form on H. Assume that there exist unitary operators $(U_n)_{n\in\mathbb{N}}$ such that $Q^\sharp_n\circ U_n=Q_n$. Then, $\big((Q_n,\mathscr{D}(Q_n))\big)_{n\in\mathbb{N}}$ Mosco converges to $(Q,\mathscr{D}(Q))$ if and only if $\big((Q^\sharp_n,\mathscr{D}(Q^\sharp_n))\big)_{n\in\mathbb{N}}$ Mosco converges to $(Q,\mathscr{D}(Q))$.*

Proof. Assume $\big((Q_n,\mathscr{D}(Q_n))\big)_{n\in\mathbb{N}}$ Mosco converges to $(Q,\mathscr{D}(Q))$. Let $D\subset H$ and $\Phi_n\colon D\to H_n$ be as in Definition 5.6.6 for $n\in\mathbb{N}$. Set $\Phi^\sharp_n:=U_n\circ\Phi_n\colon D\to H^\sharp_n$. Then, since $U_n\colon H_n\to H^\sharp_n$ is unitary and by (5.6.3),

$$\forall u\in D \qquad \lim_n\left\|\Phi^\sharp_n u\right\|_{H^\sharp_n}=\lim_n\|\Phi_n u\|_{H_n}=\|u\|_H \ ,$$

hence $H^\sharp_n$ $\mathcal{H}$-converges to H. Analogously, one can show that $(u_n)_{n\in N}$, $u_n\in H_n$, $\mathcal{H}$-strongly, resp. $\mathcal{H}$-weakly, converges to $u\in H$ if and only if $\big(u^\sharp_n\big)_{n\in\mathbb{N}}$, $u^\sharp_n:=U_n(u_n)\in H^\sharp_n$, $\mathcal{H}$-strongly, resp. $\mathcal{H}$-weakly converges to $u\in H$. Thus, let $u\in H$ and $(u_n)_{n\in N}$, $u_n\in H_n$, be as in (5.6.5) and notice that $\big(u^\sharp_n\big)_{n\in\mathbb{N}}$ defined as above $\mathcal{H}$-strongly converges to $u\in H$ and

$$Q(u)=\lim_n Q_n(u_n)=\lim_n Q^\sharp_n(U_nu_n)=\lim_n Q^\sharp_n(u^\sharp_n) \ ,$$

which proves the Γ-lim sup condition (5.6.5) for $Q^\sharp_n$. Finally, let $\big(u^\sharp_n\big)_{n\in\mathbb{N}}$, $u^\sharp_n\in H^\sharp_n$, be $\mathcal{H}$-weakly converging to $u\in H$ and set $u_n:=U_n^{-1}(u^\sharp_n)\in H_n$. Then, $(u_n)_{n\in\mathbb{N}}$ $\mathcal{H}$-weakly converges to $u\in H$, hence, by assumption on Q_n,

$$Q(u)\le\liminf_n Q_n(u_n)=\liminf_n Q^\sharp_n(U_nu_n)=\liminf_n Q^\sharp_n(u^\sharp_n) \ ,$$

which proves the Γ-lim inf condition (5.6.4) for $Q^\sharp_n$ and concludes the proof. ■

Lemma 5.6.10. *Let H be a Hilbert space and $(P_n)_{n\in\mathbb{N}}$ be an increasing sequence of orthogonal projectors $P_n\colon H\to H$ strongly converging to id_H. Set $H_n:=\mathrm{ran}P_n$ and let further $u\in H$ and $(u_n)_{n\in\mathbb{N}}$ be a sequence such that $u_n\in H_n$. Then,* ***(i)*** *H_n $\mathcal{H}$-converges to H;* ***(ii)*** *$(u_n)_{n\in\mathbb{N}}$ $\mathcal{H}$-strongly converges to $u\in H$ if and only if it strongly converges to u in H;* ***(iii)*** *$(u_n)_{n\in\mathbb{N}}$ $\mathcal{H}$-weakly converges to $u\in H$ if and only if it weakly converges to u in H.*

Proof. ***(i)*** is an immediate consequence of the strong convergence of P_n to id_H.

(ii) Assume u_n strongly converges to u and choose $D=H$, $\Phi_n:=P_n$ and $\tilde u_m=u_m$ in Definition 5.6.6. By strong convergence of P_n to id_H one has $\lim_m\|u_m-u\|_H=0$. Furthermore,

$$\begin{aligned}\lim_m\limsup_n\|P_nu_m-u_n\|_{H_n}&=\lim_m\limsup_n\|P_n(u_m-u_n)\|_{H_n}\\&\le\lim_m\limsup_n\|P_n\|\,\|u_m-u_n\|_H=0\end{aligned}$$

and the conclusion follows. Viceversa, assume that u_n $\mathcal{H}$-strongly converges to u. Then,

$$\begin{aligned}\|u_n-u\|_H&\le\|u_n-P_n\tilde u_m\|_H+\|P_n\tilde u_m-\tilde u_m\|_H+\|\tilde u_m-u\|_H\\&=\|u_n-P_n\tilde u_m\|_{H_n}+\|P_n\tilde u_m-\tilde u_m\|_H+\|\tilde u_m-u\|_H \ .\end{aligned}$$

Taking first the limit superior in n and, subsequently, the limit in m, the above inequality readily yields the conclusion. A proof of ***(iii)*** follows similarly to ***(ii)*** (by definition of $\mathcal{H}$-weak convergence) and thus it is omitted. ■

The main result concerning generalized Mosco convergence is the following

Theorem 5.6.11 (Kuwae–Shioya [99, Thm. 2.4])**.** *Let $\big((Q_n,\mathscr{D}(Q_n))\big)_{n\in\mathbb{N}}$ be a sequence of closed quadratic forms, Q_n on H_n, and $(Q,\mathscr{D}(Q))$ be a closed quadratic form on H. Then, the following are equivalent:* ***(a)*** *Q_n Mosco converges to Q;* ***(b)*** *$G_{n,\alpha}$ $\mathcal{H}$-strongly converges to G_α for every $\alpha>0$;* ***(c)*** *$T_{n,t}$ $\mathcal{H}$-strongly converges to T_t for every $t>0$.*

5.6.4 Direct integrals of quadratic forms. In the following, we shall need the notion of a *direct integral of quadratic forms*. We provide here a minimal background for the reader's convenience, referring to [43, §§II.1, II.2] for the general theory of direct integrals of Hilbert spaces. We adhere to the notation in [43] except for minor modifications.

Everywhere in the following let $(Z, \mathcal{B}, \nu)$ be a measure space with σ-algebra $\mathcal{B}$, endowed with a σ-finite measure ν. Denote by $(Z, \mathcal{B}^\nu, \hat{\nu})$ its completion. Sets in $\mathcal{B}^\nu$ are termed *ν-measurable.* A real-valued function is termed *ν-measurable* if it is measurable w.r.t. $\mathcal{B}^\nu$.

Definition 5.6.12 (Direct integrals (cf. [43, §II.1.3, Def. 1, p. 164, II.1.5, Prop. 5, p. 169]))**.** Let $(H_\zeta)_{\zeta \in Z}$ be a family of Hilbert spaces and let F be the linear space $F := \prod_{\zeta\in Z} H_\zeta$. We say that $\zeta \mapsto H_\zeta$ is a *ν-measurable field of Hilbert spaces* (*with underlying space* S) if there exists a linear subspace S of F such that ***(a)*** for every $u \in S$, the function $\zeta \mapsto \|u_\zeta\|_\zeta$ is ν-measurable; ***(b)*** if $v \in F$ is such that $\zeta \mapsto \langle u_\zeta \,|\, v_\zeta \rangle_\zeta$ is ν-measurable for every $u \in S$, then $v \in S$; ***(c)*** there exists a sequence $(u_n)_{n\in\mathbb{N}} \subset S$ such that $(u_{n,\zeta})_{n\in\mathbb{N}}$ is a total sequence[4] in H_ζ for every $\zeta \in Z$.

Any such S is termed a *space of ν-measurable vector fields*. Any sequence in S possessing property ***(c)*** is termed a *fundamental* sequence. A ν-measurable vector field u is termed *square-integrable* if

$$\|u\| := \left(\int_Z \mathrm{d}\nu(\zeta)\ \|u_\zeta\|_\zeta^2 \right)^{1/2} < \infty\ .$$

Two square-integrable vector fields are termed *equivalent* if $\|u - v\| = 0$. The space H of equivalence classes of square-integrable vector fields, endowed with the norm $\|\cdot\|$, is a separable Hilbert space, termed the *direct integral of* $\zeta \mapsto H_\zeta$ (*with underlying space* S) and denoted by

$$H = {}^{S}\!\!\int_Z^\oplus \mathrm{d}\nu(\zeta)\, H_\zeta\ . \tag{5.6.6}$$

The superscript S is omitted whenever the chosen space S is apparent from context or its specification is unnecessary.

Definition 5.6.13 (Measurable fields of bounded operators, decomposable operators)**.** Let H be defined as in (5.6.6). A field of bounded operators $\zeta \mapsto B_\zeta \in \mathcal{B}(H_\zeta)$ is termed *ν-measurable* (*with underlying space* S) if $\zeta \mapsto B_\zeta u_\zeta \in H_\zeta$ is a ν-measurable vector field for every ν-measurable vector field u. A ν-measurable vector field of bounded operators is termed *ν-essentially bounded* if $\nu\text{-esssup}_{\zeta\in Z} \|B_\zeta\|_{\mathrm{op},\zeta} < \infty$. A bounded operator $B \in \mathcal{B}(H)$ is termed *decomposable* if Bu is represented by a ν-essentially bounded ν-measurable field of bounded operators $\zeta \mapsto B_\zeta$, in which case we write

$$B = \int_Z^\oplus \mathrm{d}\nu(\zeta)\, B_\zeta\ .$$

Lemma 5.6.14. *Let H be defined as in* (5.6.6)*, $B \in \mathcal{B}(H)$ be decomposable and $\varphi \in \mathcal{C}(\sigma(B))$. Then, the continuous functional calculus $\varphi(B)$ of B is decomposable and*

$$\varphi(B) = \int_Z^\oplus \mathrm{d}\nu(\zeta)\, \varphi(B_\zeta)\ .$$

Proof. Well-posedness follows by [43, §II.2.3, Prop. 2, p. 181]. The proof is then a straightforward application of [43, §II.2.3, Prop. 3, p. 182] and [43, §II.2.3, Prop. 4(ii), p. 183] (by approximation of φ with suitable polynomials, since $\sigma(B)$ is compact). ∎

[4] A sequence $(u_n)_{n\in\mathbb{N}}$ in a Banach space B is termed *total* if the strong closure of its linear span coincides with B.

Definition 5.6.15 (Direct integral of quadratic forms)**.** For $\zeta \in Z$ let (Q_ζ, D_ζ) be a closable quadratic form on a Hilbert space H_ζ. We say that $\zeta \mapsto (Q_\zeta, D_\zeta)$ is a *ν-measurable field of quadratic forms on Z* if ***(a)*** $\zeta \mapsto H_\zeta$ is a ν-measurable field of Hilbert spaces on Z, and ***(b)*** $\zeta \mapsto \mathscr{D}(Q_\zeta)_1$ is a ν-measurable field of Hilbert spaces on Z, both with common underlying space S (under the identification of $\mathscr{D}(Q_\zeta)$ with a subspace of H_ζ granted by Lemma 5.6.5). We denote by

$$Q = {}^{S}\!\!\int_Z^{\oplus} \mathrm{d}\nu(\zeta)\, Q_\zeta$$

the *direct integral* of $\zeta \mapsto (Q_\zeta, \mathscr{D}(Q_\zeta))$, i.e. the quadratic form defined on H as in (5.6.6) given by

$$\mathscr{D}(Q) := \left\{ u \in H \mid \int_Z \mathrm{d}\nu(\zeta)\, Q_{\zeta,1}(u_\zeta) < \infty \right\} , \tag{5.6.7}$$

$$Q(u,v) := \int_Z \mathrm{d}\nu(\zeta)\, Q_\zeta(u_\zeta, v_\zeta) , \qquad u, v \in \mathscr{D}(Q) . \tag{5.6.8}$$

Lemma 5.6.16. *Let $(Q, \mathscr{D}(Q))$ be defined as above. Then, **(i)** $(Q, \mathscr{D}(Q))$ is a densely defined closed quadratic form on H; **(ii)** $\zeta \mapsto G_{\zeta,\alpha}$, $\zeta \mapsto T_{\zeta,t}$ are ν-measurable fields of bounded operators for every $\alpha, t > 0$ **(iii)** Q has resolvent and semigroup respectively defined by*

$$\begin{aligned} G_\alpha &:= {}^{S}\!\!\int_Z^{\oplus} \mathrm{d}\nu(\zeta)\, G_{\zeta,\alpha} , \qquad \alpha > 0 ; \\ T_t &:= {}^{S}\!\!\int_Z^{\oplus} \mathrm{d}\nu(\zeta)\, T_{\zeta,t} , \qquad t > 0 . \end{aligned} \tag{5.6.9}$$

Proof. ***(i)*** Let $u \in \mathscr{D}(Q)$. Since $\zeta \mapsto H_\zeta$ is a ν-measurable family of Hilbert spaces by Definition 5.6.15***(a)***, the map $\zeta \mapsto \|u_\zeta\|_\zeta$ is ν-measurable for every $u \in H$ by Definition 5.6.12***(a)***. Analogously, the map $\zeta \mapsto Q_{\zeta,1}^{1/2}(u_\zeta)$ is ν-measurable for every $u \in \mathscr{D}(Q)$. Together with the polarization identity for $\mathscr{D}(Q)_1$, this yields the measurability of the maps

$$\zeta \mapsto Q_{\zeta,\alpha}(u_\zeta, v_\zeta) , \qquad u, v \in \mathscr{D}(Q) , \qquad \alpha > 0 .$$

As a consequence $\zeta \mapsto \mathscr{D}(Q_\zeta)_\alpha$ is a ν-measurable field of Hilbert spaces (on Z, with underlying space S) for every $\alpha > 0$. Thus, it admits a direct integral of Hilbert spaces

$$D_\alpha := \int_Z^{\oplus} \mathrm{d}\nu(\zeta)\, \mathscr{D}(Q_\zeta)_\alpha , \qquad \alpha > 0 .$$

For $\alpha > 0$ let $(u_n^\alpha)_{n\in\mathbb{N}}$ be a fundamental sequence of ν-measurable vector fields for D_α and $(u_n)_{n\in\mathbb{N}}$ be a fundamental sequence of ν-measurable vector fields for H. Since Q_ζ is closable for every $\zeta \in Z$, the (extension of the) canonical inclusion $\iota_\zeta \colon \mathscr{D}(Q_\zeta)_1 \to H_\zeta$ is injective and contractive for every $\zeta \in Z$ by Lemma 5.6.5. Since D_α and H are defined on the same underlying space S by Definition 5.6.15, the maps

$$\zeta \mapsto \langle \iota_{\zeta,\alpha} u_i^\alpha \mid u_j \rangle_\zeta = \langle u_i^\alpha \mid u_j \rangle_\zeta , \qquad i, j \in \mathbb{N} , \qquad \alpha > 0$$

are ν-measurable. Together with the uniform boundedness of $\iota_{\zeta,\alpha}$ in $\zeta \in Z$, this yields the decomposability of the operator $\iota_\alpha \colon D_\alpha \to H$, defined by

$$\iota_\alpha := \int_Z^{\oplus} \mathrm{d}\nu(\zeta)\, \iota_{\zeta,\alpha} .$$

By [43, §II.2.3, Example, p. 182] and the injectivity of $\iota_{\zeta,\alpha}$ for every $\zeta \in Z$ and every $\alpha > 0$, the map $\iota_\alpha \colon D_\alpha \to \mathscr{D}(Q)_\alpha$ is an isomorphism. In particular, the composition of ι_1 with the canonical inclusion of $\mathscr{D}(Q)$ into H is injective, thus Q is closed.

Finally, since $\left(u^\alpha_{n,\zeta}\right)_{n\in\mathbb{N}}$ is $Q^{1/2}_{\zeta,\alpha}$-total in $\mathscr{D}(Q_\zeta)_\alpha$ for every $\zeta \in Z$ by Definition 5.6.12***(c)***, it is additionally H_ζ-total for every $\zeta \in Z$ by H_ζ-density of $\mathscr{D}(Q_\zeta)$ in H_ζ. As a consequence, $\left(u^\alpha_n\right)_{n\in\mathbb{N}}$ is fundamental also for H, thus $\mathscr{D}(Q)$ is H-dense in H.

(ii) For fixed $\alpha > 0$ consider the field of linear operators $\zeta \mapsto G_{\zeta,\alpha}$. The map (cf. (5.6.1a))

$$\zeta \mapsto Q_{\zeta,\alpha}(G_{\zeta,\alpha} u^\alpha_{i,\zeta}, u^\alpha_{j,\zeta}) = \left\langle u^\alpha_{i,\zeta} \,\middle|\, u^\alpha_{j,\zeta} \right\rangle_\zeta$$

is ν-measurable for every $i, j \in \mathbb{N}$ since u^α_n is a ν-measurable vector field. Since $\left\|G_{\alpha,\zeta}\right\|_\zeta \leq \alpha^{-1}$ and $(u^\alpha_n)_{n\in\mathbb{N}}$ is a fundamental sequence of ν-measurable vector fields for H, then $\zeta \mapsto G_{\zeta,\alpha}$ is a ν-measurable field of bounded operators by [43, §II.2.1, Prop. 1, p. 179] and the operator G_α defined in (5.6.9) is decomposable for every $\alpha > 0$.

By Lemma 5.6.14 any image of G_α via its continuous functional calculus is itself decomposable.

For every $\zeta \in Z$ one has $T_{\zeta,t} = \lim_{\alpha\to\infty} e^{t\alpha(\alpha G_{\zeta,\alpha}-1)}$ strongly in H_ζ by (5.6.1a), hence

$$\zeta \mapsto \left\langle T_{\zeta,t} u^\alpha_{i,\zeta} \,\middle|\, u^\alpha_{j,\zeta} \right\rangle_\zeta = \lim_{\alpha\to\infty} \left\langle e^{t\alpha(\alpha G_{\zeta,\alpha}-1)} u^\alpha_{i,\zeta} \,\middle|\, u^\alpha_{j,\zeta} \right\rangle_\zeta$$

is a pointwise limit of ν-measurable functions, hence ν-measurable, for every $i, j \in \mathbb{N}$ and every $t > 0$. As a consequence, $\zeta \mapsto T_{\zeta,t}$ is a ν-measurable field of bounded operators for every $t > 0$, again by [43, §II.2.1, Prop. 1, p. 179]. Since $\|T_{\zeta,t}\| \leq 1$, the operator T_t defined in (5.6.9) is decomposable too, for every $t > 0$.

(iii) It suffices to show (5.6.1) for $(Q, \mathscr{D}(Q))$, G_α and T_t defined in (5.6.9). Now, by definition of $(Q, \mathscr{D}(Q))$ one has for every $\alpha > 0$

$$\begin{aligned}
Q_\alpha(G_\alpha u, v) &= \int_Z \mathrm{d}\nu(\zeta)\, Q_\zeta\big((G_\alpha u)_\zeta, v_\zeta\big) + \alpha \int_Z \mathrm{d}\nu(\zeta)\ \langle (G_\alpha u)_\zeta \,|\, v_\zeta \rangle_\zeta \\
&= \int_Z \mathrm{d}\nu(\zeta)\, Q_{\zeta,\alpha}\big((G_\alpha u)_\zeta, v_\zeta\big) \\
&= \int_Z \mathrm{d}\nu(\zeta)\, Q_{\zeta,\alpha}\big((G_{\alpha,\zeta} u_\zeta)_\zeta, v_\zeta\big) \ .
\end{aligned}$$

By [43, §II.2.3, Cor., p. 182] and decomposability of G_α, one has $G_{\alpha,\zeta} = G_{\zeta,\alpha}$ for ν-a.e. $\zeta \in Z$, whence, by (5.6.1a) applied to $(Q_\zeta, \mathscr{D}(Q_\zeta))$ and $G_{\zeta,\alpha}$,

$$= \int_Z \mathrm{d}\nu(\zeta)\, \langle u_\zeta \,|\, v_\zeta \rangle_\zeta = \langle u \,|\, v \rangle \ ,$$

which is the desired conclusion. The proof of (5.6.1b) for T_t is a consequence of (5.6.1a) and the approximation given in ***(ii)*** and is therefore omitted. ■

Remark 5.6.17 (cf. [43, p. 168, Rmk.]). Each of the above statements holds with identical proof if one substitutes 'ν-measurable' with 'measurable'.

Proposition 5.6.18 (Direct integrals of Dirichlet forms)**.** *Let $(Z, \mathcal{B}, \nu)$ be satisfying Definition 5.3.1. For $\zeta \in Z$ let further **(a)** n_ζ be a fully supported finite measure on Y such that **(a$_1$)** $\zeta \mapsto \mathsf{n}_\zeta f$ is measurable for every $f \in \mathcal{C}_0(Y)$; and **(a$_2$)** ν-$\operatorname{esssup}_{\zeta\in Z} \mathsf{n}_\zeta Y < \infty$; **(b)** $(E_\zeta, \mathscr{D}(E_\zeta))$ be a family of strongly local regular Dirichlet forms on $L^2_{\mathsf{n}_\zeta}(Y)$ with common core $\mathcal{C} \subset \mathcal{C}_0(Y)$ and such that $\zeta \mapsto E_\zeta(u, v)$ is measurable for every $u, v \in \mathcal{C}$.*

*Then, **(i)** $\zeta \mapsto (E_\zeta, \mathscr{D}(E_\zeta))$ is a measurable field of quadratic forms (Def. 5.6.15); **(ii)** its direct integral $(E, \mathscr{D}(E))$ is a regular strongly local Dirichlet form on*

$$H := \int_Z^\oplus \mathrm{d}\nu(\zeta)\, L^2_{\mathsf{n}_\zeta}(Y) \cong L^2_\nu(Z; L^2_{\mathsf{n}_\cdot}(Y)); \tag{5.6.10}$$

(iii) *$\mathbb{1}_{Y\times Z}\in\mathscr{D}(E)$ if and only if $\mathbb{1}_Y\in\mathscr{D}(E_\zeta)$ for ν-a.e. $\zeta\in Z$;* ***(iv)*** *$(E,\mathscr{D}(E))$ has core $\mathcal{C}_0(Z)\otimes\mathcal{C}$;* ***(v)*** *$(E,\mathscr{D}(E))$ has semigroup given on its core by*

$$(T_t\,u)(y,\zeta)=\big((T_{\zeta,t}\otimes\mathrm{id})u\big)(y,\zeta):=\big(T_{\zeta,t}\,u(\,\cdot\,,\zeta)\big)(y)$$

for ν-a.e. $\zeta\in Z$ and n_ζ-a.e. $y\in Y$; ***(vi)*** *whenever $A\subset Z$ is ν-measurable and $U\subset Y$ is E_ζ-capacitable for every $\zeta\in Z$, then $A\times U\subset Z\times Y$ satisfies*

$$\mathrm{cap}_E(A\times U)\leq\int_A\mathrm{d}\nu(\zeta)\,\mathrm{cap}_{E_\zeta}(U)\ .$$

Proof. ***(i)*** Let $(u_n)_{n\in\mathbb{N}}\subset\mathcal{C}$ be a $\|\cdot\|_\infty$-total system in $\mathcal{C}_0(Y)$. Since $\mathcal{C}$ is a core of $(E_\zeta,\mathscr{D}(E_\zeta))$, then $\mathcal{C}$ is $\|\cdot\|_\zeta$-total in $H_\zeta:=L^2_{\mathsf{n}_\zeta}(Y)$ and $E^{1/2}_{\zeta,1}$-total in $\mathscr{D}(E_\zeta)_1$ for every $\zeta\in Z$. By ***(a***$_1$***)***, the function $\zeta\mapsto\langle u_i\,|\,u_j\rangle_\zeta$ is measurable for every $i,j\in\mathbb{N}$. By ***(b)*** the same holds for the function $\zeta\mapsto E_\zeta(u_i,u_j)$, therefore, additionally, for the function $\zeta\mapsto E_{\zeta,1}(u_i,u_j)$. This verifies properties ***(a)*** and ***(c)*** in Definition 5.6.12 for both $\zeta\mapsto L^2_{\mathsf{n}_\zeta}(Y)$ and $\zeta\mapsto\mathscr{D}(E_\zeta)_1$. In particular, $(u_n)_{n\in\mathbb{N}}$ is a fundamental sequence (Def. 5.6.12). By [43, §II.1.4, Prop. 4, p. 167] there exists a unique family of functions S such that $\zeta\mapsto(E_\zeta,\mathscr{D}(E_\zeta))$ satisfies Definition 5.6.15 with underlying space S. Let now $u\in\mathcal{C}_0(Y)$ be arbitrary; for every $n\in\mathbb{N}$ one has $u\cdot u_n\in\mathcal{C}_0(Y)$, whence the measurability of $\zeta\mapsto\langle u\,|\,u_n\rangle=\mathsf{n}_\zeta(u\cdot u_n)$ by ***(a***$_2$***)***. By [43, §II.1.4, Prop. 2, p. 166], the measurability of $\zeta\mapsto\langle u\,|\,u_n\rangle_\zeta$ for every u_n in a fundamental sequence is sufficient to establish the conclusion in Definition 5.6.12***(b)***, that is, the linear map

$$\begin{aligned}\iota\colon\mathcal{C}_0(Y)&\longrightarrow\prod_{\zeta\in Z}H_\zeta\\ u&\longmapsto\big(\zeta\mapsto u=u\cdot\mathbb{1}_Z(\zeta)\big)\end{aligned}$$

has range in S. This shows that $\mathcal{C}_0(Y)\subset S$ and that S does not depend on the total system $(u_n)_{n\in\mathbb{N}}$. Furthermore, $\iota(u)\in H$ by ***(a***$_2$***)*** since

$$\|\iota(u)\|^2_H=\nu\big(\|\iota(u)\|^2_\cdot\big)\leq\|\iota(u)\|_\infty\,\nu\text{-}\operatorname*{esssup}_{\zeta\in Z}\mathsf{n}_\zeta Y\ .$$

Finally, since n_ζ is fully supported for ν-a.e. $\zeta\in Z$ by ***(a)***, one has $\|\iota(u)\|=0$ if and only if $u\equiv\mathbf{0}$, that is $\iota\colon\mathcal{C}_0(Y)\to H$ is an injection. Let now $v\in\mathcal{C}_0(Z)$ be arbitrary and notice that it is measurable by assumption. Thus $\zeta\mapsto v(\zeta)\iota(u)$ is an element in S, and in fact in H, exactly as before. This shows that $\mathcal{C}_0(Z)\otimes\mathcal{C}_0(Y)$ injects into H. Since $\mathcal{C}_0(Z)\otimes\mathcal{C}_0(Y)$ is norm-dense in $L^2_\nu(Z;L^2_{\mathsf{n}_\cdot}(Y))$, a proof of the isomorphism in (5.6.10) and that it respects fibers is now standard.

(ii) The Markov property for the quadratic form $(E,\mathscr{D}(E))$ is a straightforward consequence of the Markov property for $(E_\zeta,\mathscr{D}(E_\zeta))$. The strong locality is also straightforward, due to our topological assumptions on Y and Z. The regularity is a consequence of ***(iv)*** below.

(iii) is immediate. ***(iv)*** The $\|\cdot\|_\infty$-density of $\mathcal{C}_0(Z)\otimes\mathcal{C}$ in $\mathcal{C}_0(Z)\otimes\mathcal{C}_0(Y)$ is a consequence of that of $\mathcal{C}$ in $\mathcal{C}_0(Y)$. The $\|\cdot\|_\infty$-density of $\mathcal{C}_0(Z)\otimes\mathcal{C}_0(Y)$ into $\mathcal{C}_0(Z\times Y)$ is in turn standard. The $E^{1/2}_1$-density in $\mathscr{D}(E)$ is a straightforward consequence of the $E^{1/2}_{\zeta,1}$-density of $\mathcal{C}$ in $\mathscr{D}(E_\zeta)$ for ν-a.e. $\zeta\in Z$. ***(v)*** is but a rephrasing of Lemma 5.6.16***(iii)***. ***(vi)*** is straightforward. ∎

Definition 5.6.19 (Randomization of Dirichlet forms)**.** Let n be as in §5.3 and assume that $\mathsf{n}_\zeta=\mathsf{n}$ for ν-a.e. $\zeta\in Z$ in Proposition 5.6.18***(a)***. In this case, we term the direct integral of Dirichlet forms constructed in Proposition 5.6.18 the *ν-randomization* of the family $\big((E_\zeta,\mathscr{D}(E_\zeta))\big)_{\zeta\in Z}$, a Dirichlet form on the concrete Hilbert space $L^2_{\nu\otimes\mathsf{n}}(Z\times Y)$. We denote it by $(\widehat{E}^\nu,\mathscr{D}(\widehat{E}^\nu))$, dropping ν from the notation whenever apparent from the context.

5.6.5 Capacity estimates. In order to simplify the statement of the next results, set

$$c_{k,d} := \sqrt{\frac{k}{d-1}} \ , \qquad s_{k,d}(r) := \begin{cases} \sin(c_{k,d}\, r) & \text{if } k > 0 \\ r & \text{if } k = 0 \\ \sinh(c_{k,d}\, r) & \text{if } k < 0 \end{cases} \ , \qquad V_{k,d}(r) := \int_0^r \mathrm{d}u\, s_{k,d}(u)^{d-1}$$

and

$$v_r(x) := \mathsf{m} B^{\mathsf{g}}_r(x) \ , \quad x \in M \ , \ r > 0 \ , \qquad D_{\mathsf{g}} := \operatorname{diam}_{\mathsf{d}_{\mathsf{g}}} M \ . \tag{5.6.11}$$

The following is well-known.

Proposition 5.6.20 (Bishop–Gromov volume comparison). *Let* (M, g) *be satisfying Assumption 5.2. Then,* $(M, \mathsf{d}_{\mathsf{g}}, \mathsf{m})$ *satisfies*

$$\frac{v_R(x)}{v_r(x)} \leq \frac{V_{k,d}(R)}{V_{k,d}(r)} \ , \qquad x \in M \ , \ 0 < r < R \tag{5.6.12}$$

where $k := (d-1) \inf_{x \in M} \operatorname{Ric}_x$.

Definition 5.6.21 (Packings and coverings). For fixed $r > 0$, we say that $(x_j)_j^n \subset M$ is ***(a)*** an *r-packing* of M if $\left(B^{\mathsf{g}}_r(x_j)\right)_{j \leq n}$ is a disjoint family and $B^{\mathsf{g}}_r(x) \cap \bigcup_j^n B^{\mathsf{g}}_r(x_j) \neq \varnothing$ for every $x \in M$; or ***(b)*** an *r-covering* of M if $M \subset \bigcup_j^n B^{\mathsf{g}}_r(x_j)$. The *covering number* of M is defined by

$$c_{M,\mathsf{g}}(r) := \min \left\{ n \in \mathbb{N} \mid \exists\, (x_j)_j^n \quad r\text{-covering of } M \right\} \ .$$

We say that an r-covering $(x_j)_j^n$ is (r-)*optimal* if $n = c_{M,\mathsf{g}}(r)$.

The following is an exercise in [70, E_+, 5.31 Ex. (b)]. We provide a proof for completeness.

Lemma 5.6.22 (Covering number of M). *Let $r > 0$. Then,* $c_{M,\mathsf{g}}(r) \leq V_{k,d}(D_{\mathsf{g}}/2)/V_{k,d}(r/2)$.

Proof. Alternatively letting $r \to 0$ or $R \to D_{\mathsf{g}}/2$ in (5.6.12) we have

$$r^d \lesssim \beta \frac{V_{k,d}(r)}{V_{k,d}(D_{\mathsf{g}}/2)} \leq v_r(x) \leq V_{k,d}(r) \lesssim r^d \ , \qquad r > 0 \ , \ x \in M \ . \tag{5.6.13}$$

Let $(x_j)_j^n$ be an $r/2$-packing of M. Notice that it is an r-covering. By disjointness,

$$\mathsf{m} \bigcup_j^n B^{\mathsf{g}}_{r/2}(x_j) = \sum_j^n \mathsf{m} B^{\mathsf{g}}_{r/2}(x_j) \leq \mathsf{m} M = \beta \ ,$$

hence $n \leq \beta / \inf_{x \in M} v_{r/2}(x)$ and the conclusion follows by (5.6.13). ■

Lemma 5.6.23. *For $i = 1, 2$ let (M_i, g_i) be satisfying Assumption 5.2, with canonical form* $(\mathsf{E}^i, \mathscr{D}(\mathsf{E}^i))$. *Denote by (M, g) the product manifold $(M_1, \mathsf{g}_1) \times (M_2, \mathsf{g}_2)$, with canonical form* $(\mathsf{E}^{\mathsf{g}}, \mathscr{D}(\mathsf{E}^{\mathsf{g}}))$. *For every E^i-capacitable A_i, B_i with $A_i \subset B_i$ let further $u_i := u_{A_i, B_i} \in \mathscr{D}(\mathsf{E}^i)$ be the equilibrium potential of (A_i, B_i). Then, the set $A_1 \times A_2$ is E^{g}-capacitable and*

$$\operatorname{cap}_{\mathsf{g}}(A_1 \times A_2, B_1 \times B_2) \leq \operatorname{cap}_1(A_1, B_1) \left\| u_2 \right\|^2_{L^2_{\mathsf{m}_2}} + \operatorname{cap}_2(A_2, B_2) \left\| u_1 \right\|^2_{L^2_{\mathsf{m}_1}} \ .$$

Proof. Straightforward. ■

Proposition 5.6.24. *For $i = 1, 2$ let (M, g_i) be satisfying Assumption 5.2, with canonical form $(E^i, \mathscr{D}(E^i))$ and* same *underlying differential manifold M. Denote by $(M^{\times 2}, \mathsf{g})$ the product manifold $(M, \mathsf{g}_1) \times (M, \mathsf{g}_2)$, with canonical form $(E^{\mathsf{g}}, \mathscr{D}(E^{\mathsf{g}}))$. Then,* $\operatorname{cap}_{\mathsf{g}}(\Delta M) = 0$.

Proof. Fix $0<\varepsilon<\delta<1$ and let $\big(x^i_j\big)_j^{n_i}$ be an optimal ε-covering for (M,g_i). Then, their union, relabeled $(y_j)_j^n$, is an ε-covering of both (M,g_i) and $n\coloneqq n_1+n_2\lesssim\varepsilon^{-d}$ by Lemma 5.6.22. For simplicity of notation, for any $r>0$ write $B^i_r(x)\coloneqq B^{\mathsf{g}_i}_r(x)$, $B^i_{r,j}\coloneqq B^{\mathsf{g}_i}_r(y_j)$ and $v_{i,r}(x)\coloneqq v^{\mathsf{g}_i}_r(x)$. Let $u_{i,j,\varepsilon,\delta}$ be the equilibrium potential of the pair $(B^i_{\varepsilon,j},B^i_{\delta,j})$ for the form $(\mathsf{E}^i,\mathscr{D}(\mathsf{E}^i))$. Since $(y_j)_j^n$ is a covering, $\Delta M\subset\bigcup_j^n B^1_{\varepsilon,j}\times B^2_{\varepsilon,j}$, thus

$$\begin{aligned}\operatorname{cap}_{\mathsf{g}}(\Delta M)\leq&\operatorname{cap}_{\mathsf{g}}\left(\bigcup_j^n B^1_{\varepsilon,j}\times B^2_{\varepsilon,j}\right)\leq\sum_j^n\operatorname{cap}_{\mathsf{g}}\big(B^1_{\varepsilon,j}\times B^2_{\varepsilon,j}\big)\\ \leq&\sum_j^n\operatorname{cap}_{\mathsf{g}}\big(B^1_{\varepsilon,j}\times B^2_{\varepsilon,j},B^1_{\delta,j}\times B^2_{\delta,j}\big)\\ \leq&\sum_j^n\operatorname{cap}_1\big(B^1_{\varepsilon,j},B^1_{\delta,j}\big)\left\|u_{2,j,\varepsilon,\delta}\right\|^2_{L^2_{\mathfrak{m}_2}}+\operatorname{cap}_2\big(B^2_{\varepsilon,j},B^2_{\delta,j}\big)\left\|u_{1,j,\varepsilon,\delta}\right\|^2_{L^2_{\mathfrak{m}_1}}\end{aligned}$$

by Lemma 5.6.23. As a consequence, since $0\leq u_{i,j,\varepsilon,\delta}\leq\mathbb{1}_{B^i_{\delta,j}}$,

$$\operatorname{cap}_{\mathsf{g}}(\Delta M)\lesssim\varepsilon^{-d}\sup_{x\in M}\Big(\operatorname{cap}_1\big(B^1_\varepsilon(x),B^1_\delta(x)\big)\cdot v_{2,\delta}(x)+\operatorname{cap}_2\big(B^2_\varepsilon(x),B^2_\delta(x)\big)\cdot v_{1,\delta}(x)\Big)\ . \tag{5.6.14}$$

Now, if $i,j=1,2$, $i\neq j$,

$$\operatorname{cap}_i\big(B^i_\varepsilon(x),B^i_\delta(x)\big)\cdot v_{j,\delta}(x)\leq\operatorname{cap}^{(0)}_i\big(B^i_\varepsilon(x),B^i_\delta(x)\big)\cdot v_{j,\delta}(x)+v_{1,\delta}(x)\cdot v_{2,\delta}(x)\ . \tag{5.6.15}$$

By [150, Eqn. (2.2)] (also cf. [68, Eqn. (2.2)]) and (5.6.13), one has

$$\begin{aligned}\sup_{x\in M}\operatorname{cap}^{(0)}_i\big(B^i_\varepsilon(x),B^i_\delta(x)\big)\leq&\sup_{x\in M}\left(\int_\varepsilon^\delta\mathrm{d}r\,\frac{r-\varepsilon}{v_{i,r}(x)-v_{i,\varepsilon}(x)}\right)^{-1}\\ \leq&\left(\int_\varepsilon^\delta\mathrm{d}r\,\frac{r-\varepsilon}{\sup_{x\in M}v_{i,r}(x)-\inf_{x\in M}v_{i,\varepsilon}(x)}\right)^{-1}\\ \leq&\left(\int_\varepsilon^\delta\mathrm{d}r\,\frac{r-\varepsilon}{V_{k,d}(r)-V_{k,d}(\varepsilon)\,\beta_i^{-1}V_{k,d}(D_i/2)^{-1}}\right)^{-1}\\ \lesssim& c_\varepsilon\coloneqq\begin{cases}\varepsilon^{d-2}&\text{if }\ d\geq3\ \text{ and }\ \delta\coloneqq2\varepsilon\\ \big(\ln\frac{\delta+\varepsilon}{2\varepsilon}\big)^{-1}&\text{if }\ d=2\ \text{ and }\ \delta\coloneqq1\wedge D_1/2\wedge D_2/2\end{cases}\ .\end{aligned}\tag{5.6.16}$$

Finally, combining Equations (5.6.13)–(5.6.16) yields

$$\operatorname{cap}_{\mathsf{g}}(\Delta M)\lesssim\varepsilon^{-d}(c_\varepsilon\,\varepsilon^d+\varepsilon^d\,\varepsilon^d)$$

and letting ε tend to 0 yields the desired conclusion since $c_\varepsilon\to0$. ■

5.6.6 Operators and domains.

Lemma 5.6.25. *Let $\varrho\in\mathcal{C}^\infty(I)$, $f\in\mathcal{C}^\infty(M)$ and $g\in\mathcal{C}^\infty(M^{\times2})$. Then,*

$$u_{g,\varrho}\colon\mu\mapsto\int_M\mathrm{d}\mu(x)\,f(x)\cdot\varrho\left(\int_M\mathrm{d}\mu(y)\,g(x,y)\right)$$

satisfies $u_{g,\varrho}\in\operatorname{cl}_{\mathcal{E}_1^{1/2}}(\mathfrak{Z}^\infty)$ and

$$\begin{aligned}\big(\mathrm{g}^\flat\,\boldsymbol{\nabla}\,u_{g,\varrho}(\eta)(x)\big)(\,\cdot\,)=&(\mathrm{d}f)_x(\,\cdot_x)\cdot\varrho\big(g(x,\,\cdot\,)^\star\eta\big)\\ &+f(x)\cdot\varrho'\big(g(x,\,\cdot\,)^\star\eta\big)\int_M\mathrm{d}\eta(y)\,(\mathrm{d}^{\otimes2}g)_{x,y}(\,\cdot_x,\,\cdot_y)\ .\end{aligned}\tag{5.6.17}$$

Proof. Let $u := u_{g,\varrho}$. Notice that

$$\begin{aligned}\boldsymbol{\nabla}_w\, u(\eta) &= \mathrm{d}_t\big|_{t=0} \int_M \mathrm{d}\eta(x)\, (f\circ \psi^{w,t})(x)\cdot \varrho\left(\int_M \mathrm{d}\eta(y)\, g\big(\psi^{w,t}(x), \psi^{w,t}(y)\big)\right)\\ &= \int_M \mathrm{d}\eta(x) \left\langle \nabla_x f \,\middle|\, w_x\right\rangle_{\mathsf{g}} \cdot \varrho\big(g(x,\,\cdot\,)^\star\eta\big)\\ &\quad + \int_M \mathrm{d}\eta(x)\, f(x)\cdot \varrho'\big(g(x,\,\cdot\,)^\star\eta\big)\cdot \int_M \mathrm{d}\eta(y) \left\langle \nabla^{\otimes 2}_{x,y} g \,\middle|\, (w_x, w_y)\right\rangle_{\mathsf{g}_x\oplus \mathsf{g}_y} ,\end{aligned}$$

whence (5.6.17) follows. By a straightforward approximation argument in the appropriate $\mathcal{C}^1$-topologies, it suffices to show $u_{g,\varrho} \in \mathrm{cl}_{\mathcal{E}_1^{1/2}}(\mathfrak{Z}^\infty)$ when $\varrho \in I[r]$, the space of real-valued polynomials on I, and g is of the form $g = \sum_k^n a_k \otimes b_k$, where $k \leq n \in \mathbb{N}$ and $a_k, b_k \in \mathcal{C}^\infty(M)$. Finally, since $\varrho \mapsto u_{g,\varrho}$ is linear and $\boldsymbol{\nabla}$ is a linear operator, it suffices to show the statement when $\varrho(r) := r^N$ for $N \in \mathbb{N}$. For such a choice of g and ϱ, one has in fact

$$u_{g,\varrho}(\eta) = \sum_{\substack{\mathbf{j}\in\mathbb{N}_0^n\\ |\mathbf{j}|=N}} \tbinom{N}{\mathbf{j}} (f\cdot \mathbf{a}^{\mathbf{j}})^\star\eta \cdot (\mathbf{b}^\star\eta)^{\mathbf{j}} \in \mathfrak{Z}^\infty , \qquad \mathbf{a} := (a_k)_k^n , \ \mathbf{b} := (b_k)_k^n . \qquad \blacksquare$$

Lemma 5.6.26. *The set $\mathfrak{Z}^\infty$ is dense in $\mathscr{D}(\mathcal{E})$.*

Proof. In order to prove the statement, it suffices to show that $u := (f\otimes\varrho)^\star \in \mathrm{cl}_{\mathcal{E}_1^{1/2}}(\mathfrak{Z}^\infty)$ for all $f\in\mathcal{C}^\infty(M)$ and $\varrho\in\mathcal{C}^\infty(I)$. Denote by cap the (first order) capacity associated to the canonical form $(\mathsf{E}, \mathscr{D}(\mathsf{E}))$ of $(M,\mathsf{g})^{\times 2}$. Let $(g_n)_n \in \mathscr{D}(\mathsf{E})$ be a minimizing sequence for $\mathrm{cap}(\Delta M) = 0$ (by Prop. 5.6.24). By standard arguments, we may assume that $g_n \in \mathcal{C}^\infty(M^{\times 2})$ additionally satisfies

$$0\leq g_n\leq 1 , \qquad g_n(x,x) = 1 , \qquad \nabla^z\big|_{z=x} g_n(z,z) = 0 , \qquad \mathsf{E}_1(g_n)\leq 2^{-n} \qquad x\in M ,\ n\in\mathbb{N} ,$$

and further that $g_n(x,y) = g_n(y,x)$, thus we may write $g_n^x := g_n(x,\,\cdot\,)$, unambiguously.

By Lemma 5.6.25, for every $f\in\mathcal{C}^\infty(M)$, $\varrho\in I[r]$ and $n\in\mathbb{N}$,

$$u_n(\eta) := \int_M \mathrm{d}\eta(x)\, f(x)\cdot\varrho\big((g_n^x)^\star\eta\big) \in \mathrm{cl}_{\mathcal{E}_1^{1/2}}(\mathfrak{Z}^\infty) .$$

Thus, it suffices to show that $\mathcal{E}_1^{1/2}$-$\lim_n u_n = u$. By (5.6.17),

$$\begin{aligned}\mathcal{E}(u-u_n) &\leq \||\nabla f|\|_\infty^2 \underbrace{\int_{\mathscr{P}} \mathrm{d}\mathcal{D}_{\mathsf{m}}(\eta) \int_M \mathrm{d}\eta(x)\, \left|\varrho(\eta_x) - \varrho\big((g_n^x)^\star\eta\big)\right|^2}_{I_{1,n}}\\ &\quad + \|f\|_\infty^2 \underbrace{\int_{\mathscr{P}} \mathrm{d}\mathcal{D}_{\mathsf{m}}(\eta) \int_M \mathrm{d}\eta(x) \left|\varrho'\big((g_n^x)^\star\eta\big)\right|^2 \int_M \mathrm{d}\eta(y)\, \left|\nabla^{\otimes 2}_{x,y} g_n\right|^2}_{I_{2,n}} .\end{aligned}$$

Concerning $I_{1,n}$, one has, by the Mecke identity (5.2.13) and properties of g_n^x, that

$$\begin{aligned} I_{1,n} &\leq \left\|\varrho'\right\|_\infty^2 \int_{\mathscr{P}} \mathrm{d}\mathcal{D}_{\mathsf{m}}(\eta) \int_M \mathrm{d}\overline{\mathsf{m}}(x) \int_I \mathrm{dB}(r) \left| r - r g_n^x(x) - (1-r)\int_M \mathrm{d}\eta(y)\, g_n^x(y)\right|^2\\ &\leq \left\|\varrho'\right\|_\infty^2 \int_{\mathscr{P}} \mathrm{d}\mathcal{D}_{\mathsf{m}}(\eta) \int_M \mathrm{d}\overline{\mathsf{m}}(x) \int_M \mathrm{d}\eta(y)\, |g_n^x(y)|^2\\ &= \left\|\varrho'\right\|_\infty^2 \int_{M^{\times 2}} \mathrm{d}\overline{\mathsf{m}}^{\otimes 2}(x,y)\, |g_n^x(y)|^2 \leq 2^{-n} \left\|\varrho'\right\|_\infty^2 \end{aligned}$$

and therefore vanishing as $n\to\infty$. A proof of the convergence $\lim_n u_n = u$ pointwise on $\mathscr{P}^{\mathrm{pa}}$ and in $L^2_{\mathcal{D}_{\mathsf{m}}}(\mathscr{P})$ is analogous to that for $I_{1,n}$ and therefore it is omitted.

Concerning $I_{2,n}$, by Cauchy–Schwarz inequality, properties of g_n and the Mecke identity (5.2.13),

$$\begin{aligned}I_{2,n} &\leq \left\|\varrho'\right\|_\infty^2 \int_{\mathscr{P}} \mathrm{d}\mathcal{D}_{\mathsf{m}}(\eta) \int_M \mathrm{d}\overline{\mathsf{m}}(x) \int_I \mathrm{dB}_\beta(r) \left[(1-r)\int_M \mathrm{d}\eta(y)\, \left|\nabla^{\otimes 2}_{x,y} g_n\right|^2 + r\left|\nabla^{\otimes 2}_{x,x} g_n\right|^2\right] \\ &\leq \left\|\varrho'\right\|_\infty^2 \int_M \mathrm{d}\overline{\mathsf{m}}^{\otimes 2}(x,y)\, \left|\nabla^{\otimes 2}_{x,y} g_n\right|^2 \leq 2^{-n} \left\|\varrho'\right\|_\infty^2 \ ,\end{aligned}$$

which concludes the proof by letting $n \to \infty$. ■

Lemma 5.6.27 (Iterated carré du champ operator)**.** *Denote by* $\mathbf{\Gamma}_2$ *the iterated carré du champ*

$$\mathbf{\Gamma}_2(u,v) \coloneqq \tfrac{1}{2}\big(\mathbf{L}\mathbf{\Gamma}(u,v) - \mathbf{\Gamma}(u,\mathbf{L}v) - \mathbf{\Gamma}(\mathbf{L}u,v)\big) \ ,$$

by Γ *(resp.* Γ_2*) the (iterated) carré du champ operator of the Laplace–Beltrami operator on* (M,g)*. For* $m \in \mathbb{N}_0$ *and* $\hat{f} \in \mathcal{C}_0^m(\widehat{M})$*, set further*

$$\mathcal{S} \colon \hat{f} \mapsto \mathcal{S}(\hat{f})(x,s) \coloneqq s^{-1}\hat{f}(x,s) \ .$$

Then, for $u = F \circ \hat{\mathbf{f}}^\star, v = G \circ \hat{\mathbf{g}}^\star \in \widehat{\mathfrak{Z}}_0^3$ *one has*

$$\begin{aligned}\mathbf{\Gamma}_2(u,v) = &\sum_{i,p,j,q}^{k,k,h,h} (\partial^2_{ip}F)\circ\hat{\mathbf{f}}^\star \cdot (\partial^2_{jq}G)\circ\hat{\mathbf{g}}^\star \cdot \Gamma(\hat{f}_i,\hat{g}_j)^\star \cdot \Gamma(\hat{f}_p,\hat{g}_q)^\star \\ &+ \sum_{i,j}^{k,h} (\partial_i F)\circ\hat{\mathbf{f}}^\star \cdot (\partial_j G)\circ\hat{\mathbf{g}}^\star \cdot \mathcal{S}\big(\Gamma_2(\hat{f}_i,\hat{g}_j)\big)^\star \\ &+ \tfrac{1}{2}\sum_{i,p,j}^{k,k,h} (\partial^2_{ip}F)\circ\hat{\mathbf{f}}^\star \cdot (\partial_j G)\circ\hat{\mathbf{g}}^\star \cdot \big(\Gamma(\hat{f}_p,\Gamma(\hat{f}_i,\hat{g}_j)) - \Gamma(\Gamma(\hat{f}_i,\hat{f}_p),\hat{g}_j)\big)^\star \\ &+ \tfrac{1}{2}\sum_{i,j,q}^{k,h,h} (\partial_i F)\circ\hat{\mathbf{f}}^\star \cdot (\partial^2_{jq}G)\circ\hat{\mathbf{g}}^\star \cdot \big(\Gamma(\Gamma(\hat{f}_i,\hat{g}_j),\hat{g}_q) - \Gamma(\hat{f}_i,\Gamma(\hat{g}_j,\hat{g}_q))\big)^\star \ .\end{aligned}$$

In particular,

$$\mathbf{\Gamma}_2(u) = (\mathbf{L}_1 u)^2 + \sum_{i,p}^{k,k} (\partial_i F)\circ\hat{\mathbf{f}}^\star \cdot (\partial_p F)\circ\hat{\mathbf{f}}^\star \cdot \mathcal{S}\big(\Gamma_2(\hat{f}_i,\hat{f}_p)\big)^\star \ . \tag{5.6.18}$$

Proof. For u, v as above one has

$$\mathbf{\Gamma}(u,v) = \sum_{i,j}^{h,k} H^{(ij)} \circ \hat{\mathbf{h}}^\star_{(ij)}$$

where $H^{(ij)} \coloneqq \partial_i F \otimes \partial_j G \otimes \mathrm{id}_{\mathbb{R}} \colon \mathbb{R}^{k+h+1} \to \mathbb{R}$ and $\hat{\mathbf{h}}_{(ij)} \coloneqq \hat{\mathbf{f}} \oplus \hat{\mathbf{g}} \oplus \Gamma(\hat{f}_i,\hat{g}_j)$.

Then (recall the definition of $\mathbf{L}_1, \mathbf{L}_2$ from (5.4.16)),

$$\begin{aligned}\mathbf{L}_1\mathbf{\Gamma}(u,v) = &\sum_{i,j}^{k,h} \sum_{p_1,p_2}^{k,k} (\partial^3_{p_1p_2i}F)\circ\hat{\mathbf{f}}^\star \cdot (\partial_j G)\circ\hat{\mathbf{g}}^\star \cdot \Gamma(\hat{f}_{p_1},\hat{f}_{p_2})^\star \cdot \Gamma(\hat{f}_i,\hat{g}_j)^\star \\ &+ \sum_{i,j}^{k,h} \sum_{q_1,q_2}^{h,h} (\partial_i F)\circ\hat{\mathbf{f}}^\star \cdot (\partial^3_{q_1q_2j}G)\circ\hat{\mathbf{g}}^\star \cdot \Gamma(\hat{g}_{q_1},\hat{g}_{q_2})^\star \cdot \Gamma(\hat{f}_i,\hat{g}_j)^\star \\ &+ 2\sum_{i,j}^{k,h} \sum_{p,q}^{k,h} (\partial^2_{pi}F)\circ\hat{\mathbf{f}}^\star \cdot (\partial^2_{qj}G)\circ\hat{\mathbf{g}}^\star \cdot \Gamma(\hat{f}_p,\hat{g}_q)^\star \cdot \Gamma(\hat{f}_i,\hat{g}_j)^\star\end{aligned}$$

$$
\begin{aligned}
&+\sum_{i,j}^{k,h}\sum_{p}^{k}(\partial^2_{pi}F)\circ\hat{\mathbf{f}}^\star\cdot(\partial_j G)\circ\hat{\mathbf{g}}^\star\cdot\Gamma(\hat f_p,\Gamma(\hat f_i,\hat g_j))^\star\\
&+\sum_{i,j}^{k,h}\sum_{q}^{h}(\partial_i F)\circ\hat{\mathbf{f}}^\star\cdot(\partial^2_{qj} G)\circ\hat{\mathbf{g}}^\star\cdot\Gamma(\Gamma(\hat f_i,\hat g_j),\hat g_q)^\star\\
&+\sum_{i,j}^{k,h}0
\end{aligned}
$$

and

$$
\begin{aligned}
\mathbf{L}_2\mathbf{\Gamma}(u,v)=&\sum_{i,j}^{k,h}\sum_{p}^{k}(\partial^2_{pi}F)\circ\hat{\mathbf{f}}^\star\cdot\mathbf{B}[\nabla\hat f_p]\cdot(\partial_j G)\circ\hat{\mathbf{g}}^\star\cdot\Gamma(\hat f_i,\hat g_j)\\
&+\sum_{i,j}^{k,h}\sum_{q}^{h}(\partial_i F)\circ\hat{\mathbf{f}}^\star\cdot(\partial^2_{qj} G)\circ\hat{\mathbf{g}}^\star\cdot\mathbf{B}[\nabla\hat g_q]\cdot\Gamma(\hat f_i,\hat g_j)\\
&+\sum_{i,j}^{k,h}(\partial_i F)\circ\hat{\mathbf{f}}^\star\cdot(\partial_j G)\circ\hat{\mathbf{g}}^\star\cdot\mathbf{B}[\nabla\Gamma(\hat f_i,\hat g_j)]\ .
\end{aligned}
$$

By bilinearity, $\mathbf{\Gamma}(\mathbf{L}u,v)=\mathbf{\Gamma}(\mathbf{L}_1u,v)+\mathbf{\Gamma}(\mathbf{L}_2u,v)$. Moreover,

$$
\mathbf{L}_1u=\sum_{p_1,p_2}^{k,k}L_1^{(p_1p_2)}\circ\hat{\mathbf{a}}^\star_{p_1p_2}
$$

where $L_1^{(p_1p_2)}\coloneqq\partial^2_{p_1p_2}F\otimes\mathrm{id}_{\mathbb{R}}\colon\mathbb{R}^{k+1}\to\mathbb{R}$ and $\hat{\mathbf{a}}_{p_1p_2}\coloneqq\hat{\mathbf{f}}\oplus\Gamma(\hat f_{p_1},\hat f_{p_2})$, and

$$
\mathbf{L}_2u=\sum_{p}^{k}L_2^{(p)}\circ\hat{\mathbf{b}}^\star_p
$$

where $L_2^{(p)}\coloneqq\partial_pF\otimes\mathrm{id}_{\mathbb{R}}\colon\mathbb{R}^{k+1}\to\mathbb{R}$, $\hat{\mathbf{b}}_p\coloneqq\hat{\mathbf{f}}\oplus\Delta\mathcal{S}(\hat f_p)$. Thus,

$$
\begin{aligned}
\mathbf{\Gamma}(\mathbf{L}_1u,v)=&\sum_{i,j}^{k,h}\sum_{p_1,p_2}^{k,k}(\partial^3_{ip_1p_2}F)\circ\hat{\mathbf{f}}^\star\cdot\Gamma(\hat f_{p_1},\hat f_{p_2})^\star\cdot(\partial_j G)\circ\hat{\mathbf{g}}^\star\cdot\Gamma(\hat f_i,\hat g_j)^\star\\
&+\sum_{j}^{h}\sum_{p_1,p_2}^{k,k}(\partial^2_{p_1p_2}F)\circ\hat{\mathbf{f}}^\star\cdot(\partial_j G)\circ\hat{\mathbf{g}}^\star\cdot\Gamma(\Gamma(\hat f_{p_1},\hat f_{p_2}),\hat g_j)^\star\\
\mathbf{\Gamma}(\mathbf{L}_2u,v)=&\sum_{i,j}^{k,h}\sum_{p}^{k}(\partial^2_{ip}F)\circ\hat{\mathbf{f}}^\star\cdot\mathbf{B}[\nabla\hat f_p]\cdot(\partial_j G)\circ\hat{\mathbf{g}}^\star\cdot\Gamma(\hat f_i,\hat g_j)^\star\\
&+\sum_{j}^{h}\sum_{p}^{k}(\partial_p F)\circ\hat{\mathbf{f}}^\star\cdot(\partial_j G)\circ\hat{\mathbf{g}}^\star\cdot\Gamma(\Delta\mathcal{S}(\hat f_p),\hat g_j)^\star
\end{aligned}
$$

It follows from the previous computations that

$$
\begin{aligned}
\mathbf{\Gamma}_2(u,v)=&\sum_{i,p,j,q}^{k,k,h,h}(\partial^2_{ip}F)\circ\hat{\mathbf{f}}^\star\cdot(\partial^2_{jq}G)\circ\hat{\mathbf{g}}^\star\cdot\Gamma(\hat f_i,\hat g_j)^\star\cdot\Gamma(\hat f_p,\hat g_q)^\star\\
&+\tfrac12\sum_{i,p,j}^{k,k,h}(\partial^2_{ip}F)\circ\hat{\mathbf{f}}^\star\cdot(\partial_j G)\circ\hat{\mathbf{g}}^\star\cdot\big(\Gamma(\hat f_p,\Gamma(\hat f_i,\hat g_j))-\Gamma(\Gamma(\hat f_i,\hat f_p),\hat g_j)\big)^\star\\
&+\tfrac12\sum_{i,j,q}^{k,h,h}(\partial_i F)\circ\hat{\mathbf{f}}^\star\cdot(\partial^2_{jq}G)\circ\hat{\mathbf{g}}^\star\cdot\big(\Gamma(\Gamma(\hat f_i,\hat g_j),\hat g_q)-\Gamma(\hat f_i,\Gamma(\hat g_j,\hat g_q))\big)^\star
\end{aligned}
$$

$$+\sum_{i,j}(\partial_i F)\circ \hat{\mathbf{f}}^\star \cdot (\partial_j G)\circ \hat{\mathbf{g}}^\star \cdot \tfrac{1}{2}\mathcal{S}\left(\left(\Delta\Gamma(\hat f_i,\hat g_j)\right)-\Gamma(\Delta\mathcal{S}(\hat f_i),\hat g_j)-\Gamma(\hat f_i,\Delta\mathcal{S}(\hat g_j))\right)^\star \ .$$

Since the map $\mathcal{S}$ does not affect evaluation in the space variable, it commutes with any *linear* differential operator affecting only the space variable. Thus, the conclusion follows by definition of Γ_2. ■

Lemma 5.6.28. *For $w \in \mathfrak{X}^\infty$ let $\mathcal{A}^w$ be the form on $\mathscr{D}(\mathcal{E})$ defined in Corollary 5.4.21**(iii)**. Then, for all bounded measurable $u\colon \mathscr{P}\to\mathbb{R}$ and all $v\in\mathfrak{Z}^\infty$,*

$$\int_{\mathscr{P}} \mathrm{d}\mathcal{D}_{\mathsf{m}}\,(u\circ\Psi^{w,t}-u)v = -\int_0^1 \mathrm{d}s\left[\int_{\mathscr{P}}\mathrm{d}\mathcal{D}_{\mathsf{m}}\,u\circ\Psi^{w,s}\cdot\boldsymbol{\nabla}_w\,v+\mathcal{A}^w(u\circ\Psi^{w,s},v)\right] \ . \tag{5.6.19}$$

Proof. We follow [142, Lem. 6.1]. By a monotone class argument, it suffices to show (5.6.19) for $u\in\mathfrak{Z}^\infty$. Then, $u\circ\Psi^{w,t}\in\mathfrak{Z}^\infty$ too. By Lemma 4.4.7,

$$u\circ\Psi^{w,t}-u=\int_0^t \mathrm{d}s\ \boldsymbol{\nabla}_w(u\circ\Psi^{w,s}) \ ,$$

whence, integrating and applying Fubini's Theorem,

$$\int_{\mathscr{P}}\mathrm{d}\mathcal{D}_{\mathsf{m}}\,(u\circ\Psi^{w,t}-u)v=\int_0^t\mathrm{d}s\int_{\mathscr{P}}\mathrm{d}\mathcal{D}_{\mathsf{m}}\ \boldsymbol{\nabla}_w(u\circ\Psi^{w,s})\cdot v \ ,$$

hence the conclusion by properties of $\mathcal{A}^w$. ■

Proposition 5.6.29. *For $u\in\mathrm{Lip}(\mathscr{P}_2)$ and $w\in\mathfrak{X}^\infty$ set*

$$\Omega^u_w \coloneqq \left\{\mu\in\mathscr{P}\mid \exists G_w u(\mu)\coloneqq \mathrm{d}_t\big|_{t=0}(u\circ\Psi^{w,t})(\mu)\right\} \ .$$

Let further $\mathscr{X}\subset\mathfrak{X}^\infty$ be a countable $\mathcal{Q}$-vector space dense in $\mathfrak{X}^0$ and assume $\mathcal{D}_{\mathsf{m}}\Omega^u_w=1$ for all $w\in\mathscr{X}$. Then $u\in\mathscr{D}(\mathcal{E})$ and $\boldsymbol{\Gamma}(u)\le\mathrm{Lip}[u]$ $\mathcal{D}_{\mathsf{m}}$-a.e..

Proof. It suffices to show Equation (4.4.20): the rest of the proof is identical to Proposition 4.4.9. By continuity of $t\mapsto\mathcal{A}^w(u\circ\Psi^{w,t},v)$ for $u,v\in\mathfrak{Z}^\infty$, Equation (4.4.19) and Lemma 5.6.28 yield

$$\int_{\mathscr{P}}\mathrm{d}\mathcal{D}_{\mathsf{m}}\,G_w u\cdot v=-\int_{\mathscr{P}}\mathrm{d}\mathcal{D}_{\mathsf{m}}\,u\,\boldsymbol{\nabla}_w\,v-\mathcal{A}^w(u,v) \ , \qquad u,v\in\mathfrak{Z}^\infty$$

Next, notice that the map $w\mapsto\mathcal{A}^w(u,v)$ is linear for every $u,v\in\mathfrak{Z}^\infty$, since it is the limit of the linear maps $w\mapsto\mathcal{A}^w_0(u_n,v_n)$, where $u_n,v_n\in\widehat{\mathfrak{Z}}^\infty_{1/n}$ are the approximation of u, v constructed in Corollary 5.4.18. As a consequence, if $w=s_1w_1+\cdots+s_kw_k$ for some $s_i\in\mathbb{R}$ and $w_i\in\mathscr{X}$, then

$$\int_{\mathscr{P}}\mathrm{d}\mathcal{D}_{\mathsf{m}}\,G_w u\cdot v=-\sum_i^k s_i\left[\int_{\mathscr{P}}\mathrm{d}\mathcal{D}_{\mathsf{m}}\,u\,\boldsymbol{\nabla}_{w_i}\,v+\mathcal{A}^{w_i}(u,v)\right]=\sum_i^k\int_{\mathscr{P}}\mathrm{d}\mathcal{D}_{\mathsf{m}}\,G_{w_i}u\cdot v$$

and Equation (4.4.20) follows. ■

Bibliography

[1] Aarnes, J. F. A Large Bi-Invariant Nuclear Function Space on a Locally Compact Group. *Math. Scand.*, 29(2):281–297, 1972.

[2] Airault, H. and Malliavin, P. Quasi-invariance of Brownian measures on the group of circle homeomorphisms and infinite-dimensional Riemannian geometry. *J. Funct. Anal.*, 241:99–142, 2006.

[3] Albeverio, S., Daletskii, A., Kondratiev, Yu., and Lytvynov, E. Laplace operators in deRham complexes associated with measures on configuration spaces. *J. Geom. Phys.*, 47(2-3):259–302, 2003.

[4] Albeverio, S., Daletskii, A., and Lytvynov, E. De Rham Cohomology of Configuration Spaces with Poisson Measure. *J. Funct. Anal.*, 185:240–273, 2001.

[5] Albeverio, S., Daletskii, A. Yu., and Kondrat'ev, Yu. G. Infinite Systems of Stochastic Differential Equations and Some Lattice Models on Compact Riemannian Manifolds. *Ukr. Math. J.*, 49(3):360–372, 1997.

[6] Albeverio, S., Daletskii, A. Yu., and Kondrat'ev, Yu. G. Stochastic Analysis on Product Manifolds: Dirichlet Operators on Differential Forms. *J. Funct. Anal.*, 176:280–316, 2000.

[7] Albeverio, S., Kondratiev, Yu. G., and Röckner, M. Analysis and Geometry on Configuration Spaces. *J. Funct. Anal.*, 154:444–500, 1998.

[8] Ambrosio, L., Erbar, M., and Savaré, G. Optimal transport, Cheeger energies and contractivity of dynamic transport distances in extended spaces. *Nonlinear Anal.*, 137:77–134, 2016.

[9] Ambrosio, L., Fusco, N., and Pallara, D. *Functions of Bounded Variation and Free Discontinuity Problems.* Oxford Mathematical Monographs. Oxford Science Publications, 2000.

[10] Ambrosio, L. and Gigli, N. A User's Guide to Optimal Transport. In Ambrosio, L., Bressan, A., Helbing, D., Klar, A., and Zuazua, E., editors, *Modelling and Optimisation of Flows on Networks – Cetraro, Italy 2009, Editors: Benedetto Piccoli, Michel Rascle*, volume 2062 of *Lecture Notes in Mathematics*, pages 1–155. Springer, 2013. Throughout the present work, we refer to (the numbering of) results in the extended version, available at `http://cvgmt.sns.it/media/doc/paper/195/`.

[11] Ambrosio, L., Gigli, N., and Savaré, G. *Gradient Flows in Metric Spaces and in the Space of Probability Measures.* Lectures in Mathematics - ETH Zürich. Birkhäuser, 2nd edition, 2008.

[12] Andrews, G. E. *The theory of partitions.* Addison-Wesley Pub. Co., Reading, Mass., 1976.

[13] Banakh, T. O. and Radul, T. N. Topology of spaces of probability measures. *Mat. Sb.*, 188(7):23–46, 1997.

[14] Banyaga, A. *The Structure of Classical Diffeomorphism Groups*, volume 400 of *Mathematics and Its Applications.* Springer, 1997.

[15] Bendikov, A. and Saloff-Coste, L. Elliptic diffusions on infinite products. *J. reine angew. Math.*, 493:171–220, 1997.

[16] Berezanskiĭ, Ju. M. *Expansions in Eigenfunctions of Selfadjoint Operators*, volume 17 of *Translations of Mathematical Monographs.* Amer. Math. Soc., Providence, RI, 1968.

[17] Berezansky, Yu. M. and Kondratiev, Yu. G. *Spectral Methods in Infinite-Dimensional Analysis.* Springer Netherlands, Dordrecht, 1995.

[18] Berg, Ch. Potential Theory on the Infinite Dimensional Torus. *Invent. Math.*, 32:49–100, 1976.

[19] Bertrand, J. and Kloeckner, B. R. A Geometric Study of Wasserstein Spaces: Isometric Rigidity in Negative Curvature. *International Mathematics Research Notices*, 2015.

[20] Bogachev, V. I. *Measure Theory.* Springer, 2007.

[21] Bogachev, V. I. and Kolesnikov, A. V. The Monge–Kantorovich problem: achievements, connections, and perspectives. *Russ. Math. Surv.+*, 67(5):785 – 890, 2012.

[22] Bogachev, V. I. and Mayer-Wolf, E. Some Remarks on Rademacher's Theorem in Infinite Dimensions. *Potential Anal.*, 5:23–30, 1996.

[23] Bouleau, N. and Hirsch, F. *Dirichlet forms and analysis on Wiener space.* De Gruyter, 1991.

[24] Brenier, Y. Polar Factorization and Monotone Rearrangement of Vector-Valued Functions. *Comm. Pur. Appl. Math.*, 44:375–417, 1991.

[25] Cardaliaguet, P., Delarue, F., Lasry, J.-M., and Lions, P.-L. The master equation and the convergence problem in mean field games. *arXiv:1509.02505*, 2015.

[26] Carlson, B. C. *Special Functions of Applied Mathematics.* Academic Press, New York, 1977.

[27] Chen, Z.-Q., Ma, Z.-M., and Röckner, M. Quasi-homeomorphisms of Dirichlet forms. *Nagoya Math. J.*, 136:1–15, 1994.

[28] Chodosh, O. A lack of Ricci bounds for the entropic measure on Wasserstein space over the interval. *J. Funct. Anal.*, 262(10):4570–4581, 2012.

[29] Chow, Y. T. and Gangbo, W. A partial Laplacian as an infinitesimal generator on the Wasserstein space. *arXiv:1710.10536*, Oct. 2017.

[30] Ciftan, M. On the Combinatorial Structure of State Vectors in $U(n)$. II. The Generalization of Hypergeometric Functions on $U(n)$ States. *J. Math. Phys.*, 10(9):1635–1646, 1969.

[31] Conache, D., Kondratiev, Yu. G., and Lytvynov, E. Equilibrium Diffusion on the Cone of Discrete Radon Measures. *Potential Anal.*, 44:71–90, 2016.

[32] Conlon, L. *Differentiable Manifolds.* Birkhäuser Advanced Texts. Birkhäuser, 2^{nd} edition, 2001.

[33] Da Prato, G. and Zabczyk, J. *Ergodicity for infinite-dimensional systems*, volume 229 of *London Mathematical Society Lecture Note Series.* Cambridge University Press, 1996.

[34] da Silva, J., Kondratiev, Yu., and Röckner, M. On a Relation between Intrinsic and Extrinsic Dirichlet Forms for Interacting Particle Systems. *Math. Nachr.*, 222:141–157, 2001.

[35] De Philippis, G. and Rindler, F. On the Structure of $\mathscr{A}$-Free Measures and Applications. *Ann. Math.*, 184:1017–1039, 2016.

[36] Decreusefond, L. Wasserstein Distance on Configuration Space. *Potential Anal.*, 28:283–300, 2008.

[37] Dello Schiavo, L. Moments and Cumulants in Infinite Dimensions with Applications to Poisson, Gamma and Dirichlet–Ferguson Random Measures. *arXiv:1611.02208*, 2016.

[38] Dello Schiavo, L. A Rademacher-type theorem on L^2-Wasserstein spaces over closed Riemannian manifolds. *arXiv:1810.10227*, 2018.

[39] Dello Schiavo, L. Characteristic Functionals of Dirichlet Measures. *arXiv:1810.09790*, 2018.

[40] Dello Schiavo, L. The Dirichlet–Ferguson Diffusion on the Space of Probability Measures over a Closed Riemannian Manifold. *arXiv:1811.11598*, 2018.

[41] Dello Schiavo, L. and Lytvynov, E. A Mecke-type Characterization of the Dirichlet–Ferguson Measure. *arXiv 1706.07602*, 2017.

[42] DiBenedetto, E. *Real Analysis.* Birkhäuser advanced texts. Birkhäuser, 2002.

[43] Dixmier, J. *Von Neumann Algebras.* North-Holland, 1981.

[44] Donnelly, P. and Grimmett, G. On the asymptotic distribution of large prime factors. *J. London Math. Soc.*, 47(3):395–404, 1993.

[45] Donnelly, P. and Joyce, P. Continuity and weak convergence of ranked and size-biased permutations on the infinite simplex. *Stoch. Proc. Appl.*, 31:89–103, 1989.

[46] Döring, M. and Stannat, W. The logarithmic sobolev inequality for the wasserstein diffusion. *Probab. Theory Relat. Fields*, 145:189–209, 2009.

[47] Driver, B. K. and Gordina, M. Heat kernel analysis on infinite-dimensional Heisenberg groups. *J. Funct. Anal.*, 255:2395–2461, 2008.

[48] Eberle, A. Girsanov-type transformations of local Dirichlet forms: An analytic approach. *Osaka J. Math.*, 33(2):497–531, 1996.

[49] Enchev, O. and Stroock, D. W. Rademacher's theorem for Wiener functionals. *Ann. Probab.*, 21(1):25–33, 1993.

[50] Engelking, R. *General Topology*, volume 6 of *Sigma series in pure mathematics*. Heldermann, Berlin, 1989.

[51] Erbar, M. and Huesmann, M. Curvature bounds for configuration spaces. *Calc. Var.*, 54:307–430, 2015.

[52] Ethier, S. N. and Kurtz, T. G. Convergence to Fleming–Viot process in the weak atomic topology. *Stoch. Proc. Appl.*, 54:1–27, 1994.

[53] Exton, H. Ch. *Multiple hypergeometric functions and applications.* Whiley, 1976.

[54] Feng, S. *The Poisson-Dirichlet Distribution and Related Topics.* Probability and its Applications. Springer-Verlag, Berlin, Heidelberg, 2010.

[55] Ferguson, T. S. A Bayesian analysis of some nonparametric problems. *Ann. Statist.*, pages 209–230, 1973.

[56] Figalli, A. Regularity of optimal transport maps (After Ma–Trudinger–Wang and Loeper). *Asterisque J.*, 2010. Séminaire BOURBAKI 61ème année, 2008-2009, no 1009, Juin 2009.

[57] Fleming, V. H. and Viot, M. Some Measure-Valued Markov Processes in Population Genetics Theory. *Indiana J. Math.*, 28(5):817–843, 1979.

[58] Frank, R. L., Lenz, D., and Wingert, D. Intrinsic metrics for non-local symmetric Dirichlet forms and applications to spectral theory. *J. Funct. Anal.*, 266(8):4765–4808, 2014.

[59] Fukushima, M., Oshima, Y., and Takeda, M. *Dirichlet forms and symmetric Markov processes*, volume 19 of *De Gruyter Studies in Mathematics.* de Gruyter, extended edition, 2011.

[60] Gangbo, W., Kim, H. K., and Pacini, T. Differential Forms on Wasserstein Space and Infinite-dimensional Hamiltonian Systems. *Mem. Am. Math. Soc.*, 211(995), 2010.

[61] Gel'fand, I. M. and Vilenkin, N. Ya. *Generalized Functions vol. 4: Applications of harmonic analysis.* Academic Press, 1964.

[62] Georgii, H.-O. Canonical and Grand Canonical Gibbs States for Continuum Systems. *Comm. Math. Phys.*, 48:31–51, 1976.

[63] Gigli, N. On the inverse implication of Brenier–McCann theorems and the structure of $(\mathscr{P}_2(M), W_2)$. *Methods and Applications of Analysis*, 18(2):127–158, 2011.

[64] Gigli, N. Second Order Analysis on $(\mathscr{P}_2(M), W_2)$. *Mem. Am. Math. Soc.*, 216(1018), 2012.

[65] Gordina, M. An Application of a Functional Inequality to Quasi-Invariance in Infinite Dimensions. In Carlen E., Madiman, M., and Werner, E., editors, *Convexity and Concentration*, number 161 in The IMA Volumes in Mathematics and its Applications. Springer, New York, NY, 2017.

[66] Grabowski, J. Free subgroups of diffeomorphism groups. *Fundam. Math.*, 131(2):103–121, 1988.

[67] Griffiths, R. C. On the Distribution of Points in a Poisson–Dirichlet Process. *J. Appl. Probab.*, 25(2):336–345, 1988.

[68] Grigor'yan, A. Isoperimetric inequalities and capacities on Riemannian manifolds. In Rossmann, J., Takáč, P., and Wildenhain, G., editors, *The Maz'ya Anniversary Collection – Volume 1: On Maz'ya's work in functional analysis, partial differential equations and applications*, volume 109 of *Operator Theory – Advances and Applications*, pages 139–153. Springer Basel AG, 1999.

[69] Grigor'yan, A. *Heat Kernel and Analysis on Manifolds*, volume 47 of *Advanced Studies in Mathematics.* AMS–IP, 2009.

[70] Gromov, M. *Metric structures for Riemannian and non-Riemannian spaces*, volume 152 of *Progress in Mathematics.* Birkhäuser, 3 edition, 2006.

[71] Hagedorn, D., Kondratiev, Y. G., Lytvynov, E. W., and Vershik, A. M. Laplace operators in gamma analysis. In Bernido C., Carpio-Bernido, M., Grothaus M., Kuna T., Oliveira, M., and da Silva, J., editors, *Stochastic and Infinite Dimensional Analysis*, Trends in Mathematics, pages 119–147. Birkhäuser, 2016.

[72] Hagedorn, D., Kondratiev, Yu. G., Pasurek, T., and Röckner, M. Gibbs states over the cone of discrete measures. *J. Funct. Anal.*, 264(11):2550–2583, June 2013.

[73] Handa, K. Quasi-invariance and reversibility in the Fleming–Viot process. *Probab. Theory Relat. Fields*, 122:545–566, 2002.

[74] Hino, M. Energy measures and indices of Dirichlet forms, with applications to derivatives on some fractals. *Proc. London Math. Soc.*, 100:269–302, 2009.

[75] Hino, M. and Ramírez, J. A. Small-Time Gaussian Behavior of Symmetric Diffusion Semigroups. *Ann. Probab.*, 31(3):1254–1295, 2003.

[76] Hollander, M. A Conversation with Jayaram Sethuraman. *Statist. Sci.*, 23(2):272–285, 2008.

[77] Humphreys, J. E. *Introduction to Lie Algebras and Representation Theory.* Springer-Verlag, 1972.

[78] Ibraheem, H. O. and Lytvynov, E. W. Quasi-invariance of completely random measure. *Methods of Functional Analysis and Topology*, 2018.

[79] Jiang, T. J., Dickey, J. M., and Kuo, K.-L. A new multivariate transform and the distribution of a random functional of a Ferguson–Dirichlet process. *Stoch. Proc. Appl.*, 111(1):77–95, 2004.

[80] Jiang, T. J. and Kuo, K.-L. Distribution of a Random Functional of a Ferguson-Dirichlet Process over the Unit Sphere. *Elect. Comm. in Probab.*, 13:518–525, 2008.

[81] Jordan, R., Kinderlehrer, D., and Otto, F. The Variational Formulation of the Fokker-Planck Equation. *SIAM J. Math. Anal.*, 29:1–17, 1998.

[82] Kakutani, S. On Equivalence of Infinite Product Measures. *Ann. Math.*, 49(1):214–224, Jan 1948.

[83] Kallenberg, O. *Random measures.* Academic Press, London, 3rd edition, 1983.

[84] Kallenberg, O. *Random Measures, Theory and Applications.* Springer, 2017.

[85] Kechris, Alexander S. *Classical Descriptive Set Theory*, volume 156 of *Graduate Texts in Mathematics.* Springer, Berlin, 1995.

[86] Kerov, S.V. and Tsilevich, N. V. The Markov–Krein correspondence in several dimensions. *Zap. Nauchn. Sem. POMI*, 283:98–122, 2001. In Russian.

[87] Kingman, J. F. C. Random Discrete Distributions. *J. Roy. Stat. Soc. B Met.*, 37(1):1–22, 1975.

[88] Kingman, J. F. C. *Poisson Processes*, volume 3 of *Oxford Studies in Probability.* Oxford Science Publications, 1993.

[89] Kingman, J. F. C. Poisson processes revisited. *Probab. Math. Statist.*, 26(1):77–95, 2006.

[90] Kolesnikov, A. V. Mosco convergence of Dirichlet forms in infinite dimensions with changing reference measures. *J. Funct. Anal.*, 230(2):382–418, 2006.

[91] Konarovskyi, V. V. A System of Coalescing Heavy Diffusion Particles on the Real Line. *Ann. Probab.*, 45(5):3293–3335, 2017.

[92] Konarovskyi, V. V. and Renesse, M.-K. von. Reversible Coalescing-Fragmentating Wasserstein Dynamics on the Real Line. *arXiv:1709.02839*, page 59, 2017.

[93] Konarovskyi, V. V. and Renesse, M.-K. von. Modified Massive Arratia flow and Wasserstein diffusion. *Comm. Pur. Appl. Math.*, 2018. (to appear).

[94] Kondratiev, Yu. G., da Silva, J. L., Streit, L., and Us, G. F. Analysis on Poisson and Gamma Spaces. *Infin. Dimens. Anal. Quantum Probab. Relat. Top.*, 01(01):91–117, 1998.

[95] Kondratiev, Yu. G. and Lytvynov, E. W. Operators of Gamma White Noise Calculus. *Infin. Dimens. Anal. Quantum Probab. Relat. Top.*, 3, 2000.

[96] Kondratiev, Yu. G., Lytvynov, E. W., and Vershik, A. M. Laplace operators on the cone of Radon measures. *J. Funct. Anal.*, 269(9):2947–2976, 2015.

[97] Koskela, P. and Zhou, Y. Geometry and analysis of Dirichlet forms. *Adv. Math.*, 231:2755–2801, 2012.

[98] Kriegl, A. and Michor, P. W. *The Convenient Setting of Global Analysis*, volume 53 of *Mathematical Surveys and Monographs.* American Mathematical Society, 1997.

[99] Kuwae, K. and Shioya, T. Convergence of spectral structures: a functional analytic theory and its applications to spectral geometry. *Comm. Anal. Geom.*, 11(4):599–673, 2003.

[100] Kuzmin, P. A. On circle diffeomorphisms with discontinuous derivatives and quasi-invariance subgroups of Malliavin–Shavgulidze measures. *J. Math. Anal. appl.*, 330:744–750, 2007.

[101] Last, G. An integral characterization of the Dirichlet process. *arXiv:1803.11465*, 2018.

[102] Last, G. and Penrose, M. *Lectures on the Poisson Process.* IMS Textbook. Cambridge University Press (In press)., 2018.

[103] Lauricella, G. Sulle funzioni ipergeometriche a più variabili. *Rendiconti Circolo Mat. Palermo*, 7:111–158, 1893.

[104] Lautensack, C. *Random Laguerre Tessellations*. PhD thesis, Universität Karlsruhe, 2007.

[105] Le Jan, Y. and Raimond, O. Flows, Coalescence and Noise. *Ann. Probab.*, 32(2):1247–1315, 2004.

[106] Lenz, D., Schmidt, M., and Wirth, M. Geometric Properties of Dirichlet Forms under Order Isomorphism. *arXiv:1801.08326*, 2018.

[107] Letac, G. and Piccioni, M. Dirichlet curves, convex order and Cauchy distribution. *Bernoulli*, 24(1):1–29, 2018.

[108] Lijoi, A. and Prünster, I. Models beyond the Dirichlet process. In *Bayesian nonparametrics*, pages 80–136. Cambridge Univ. Press, Cambridge, 2010.

[109] Lijoi, A. and Regazzini, E. Means of a Dirichlet process and multiple hypergeometric functions. *Ann. Probab.*, 32(2):1469–1495, 2004.

[110] Lisini, Stefano. Characterization of absolutely continuous curves in Wasserstein spaces. *Calc. Var.*, 28:85–120, 2007.

[111] Lott, J. Some Geometric Calculations on Wasserstein Space. *Commun. Math. Phys.*, 277(2):423–437, 2007.

[112] Ma, Z.-M. and Röckner, M. *Introduction to the Theory of (Non-Symmetric) Dirichlet Forms*. Graduate Studies in Mathematics. Springer, 1992.

[113] Ma, Z.-M. and Röckner, M. Construction of Diffusions on Configuration Spaces. *Osaka J. Math.*, 37:273–314, 2000.

[114] Ma, Z.-M. and Xiang, K.-N. Superprocesses of Stochastic Flows. *Ann. Probab.*, 29(1):317–343, 2001.

[115] Majumdar, S. On topological support of Dirichlet prior. *Stat. Probabil. Lett.*, 15:385–388, 1992.

[116] Malliavin, P. M. and Malliavin, P. An Infinitesimally Quasi Invariant Measure on the Group of Diffeomorphisms of the Circle. In Kashiwara, M. and Miwa, T., editors, *ICM-90 Satellite Conference Proceedings – Special Functions*, pages 234–244. Springer, 1990.

[117] McCann, R. J. A Convexity Principle for Interacting Gases. *Adv. Math.*, 128, 1997.

[118] McShane, E. J. Extension of Range of Functions. *Bull. Amer. Math. Soc.*, 40(12):837–842, 1934.

[119] Mecke, J. Stationäre zufällige Maße auf lokalkompakten Abelschen Gruppen. *Z. Wahrscheinlichkeitstheorie*, 9:36–58, 1967.

[120] Melleray, J., Petrov, F. V., and Vershik, A. M. Linearly rigid metric spaces and the embedding problem. *Fund. Math.*, 199:177–194, 2008.

[121] Miller, W. Jr. Lie Theory and the Lauricella Functions F_D. *J. Math. Phys.*, 13(9):1393–1399, 1972.

[122] Miller, W. Jr. Lie Theory and Generalizations of the Hypergeometric Functions. *SIAM J. Appl. Math.*, 25(2):226–235, 1973.

[123] Miller, W. Jr. Lie theory and the Appell functions F_1. *SIAM J. Math. Anal.*, 4:638–655, 1973.

[124] Miller, W. Jr. *Symmetry and Separation of Variables.* Addison–Wesley, Reading, Massachusetts, 1977.

[125] Milnor, J. Remarks on Infinite-dimensional Lie groups. In De Witt, B. S. and Stora, R., editors, *Les Houches 1983 Proceedings: Relativity, Groups and Topology II.* North-Holland, 1984.

[126] Myers, S. B. and Steenrod, N. E. The Group of Isometries of a Riemannian Manifold. *Ann. Math.*, 40(2):400–416, 1939.

[127] Nekvinda, A. and Zajíček, L. A simple proof of the Rademacher theorem. *Časopis pro Pěstování Matematiky*, 113(4):337–341, 1988.

[128] Ng, K. W., Tian, G.-L., and Tang, M.-L. *Dirichlet and Related Distributions – Theory, Methods and Applications.* Wiley Series in Probability and Statistics. John Wiley & Sons, 2011.

[129] Nguyen, X. X. and Zessin, H. Integral and Differential Characterizations of the Gibbs Process. *Math. Nachr.*, 88:105–115, 1979.

[130] Okounkov, A. Yu. On the Representations of the Infinite Symmetric Group. *arXiv:math/9803037v1*, 1998.

[131] Otto, F. The Geometry of Dissipative Evolution Equations: The Porous Medium Equation. *Comm. Part. Diff. Eq.*, 26(1-2):101–174, 2001.

[132] Overbeck, L., Röckner, M., and Schmuland, B. An analytic approach to Fleming–Viot processes with interactive selection. *Ann. Probab.*, 23(1):1–36, 1995.

[133] Palais, Richard S. On the Differentiability of Isometries. *Proc. Amer. Math. Soc.*, 8(4):805–807, 1957.

[134] Pólya, G. Kombinatorische Anzahlbestimmungen für Gruppen, Graphen und chemische Verbindungen. *Acta Math.*, 68:145–254, 1937.

[135] Privault, N. Moments of Poisson stochastic integrals with random integrands. *Probab. Math. Statist.*, 32:227–239, 2012.

[136] Reed, M. and Simon, B. *Methods of Modern Mathematical Physics II – Fourier Analysis, Self-Adjointness.* Academic Press, New York, London, 1975.

[137] Regazzini, E. An example of the interplay between statistics and special functions. In *Tricomi's Ideas and Contemporary Applied Mathematics*, volume 147 of *Atti dei Convegni Lincei*, pages 303–320. Accademia Nazionale dei Lincei, Rome, 1998.

[138] Regazzini, E., Guglielmi, A., and Di Nunno, G. Theory and numerical analysis for exact distributions of functionals of a Dirichlet process. *Ann. Statist.*, 30:1376–1411, 2002.

[139] Regazzini, E. and Sazonov, V.V. Approximation of laws of random probabilities by mixtures of Dirichlet distributions with applications to nonparametric Bayesian inference. *Teor. Veroyatnost. i Primenen.*, 45:103–124, 2000. In Russian.

[140] Renesse, M.-K. von and Sturm, K.-T. Entropic measure and Wasserstein diffusion. *Ann. Probab.*, 37(3):1114–1191, 2009.

[141] Renesse, M.-K. von, Yor, M., and Zambotti, L. Quasi-invariance properties of a class of subordinators. *Stoch. Proc. Appl.*, 118(11):2038–2057, 2008.

[142] Röckner, M. and Schied, A. Rademacher's Theorem on Configuration Spaces and Applications. *J. Funct. Anal.*, 169:325–356, 1999.

[143] Schied, A. Geometric Aspects of Fleming–Viot and Dawson–Watanabe Processes. *Ann. Probab.*, 25(3):1160–1179, 1997.

[144] Schied, A. Geometric Analysis for Symmetric Fleming–Viot Operators: Rademacher's Theorem and Exponential Families. *Potential Anal.*, 17:351–374, 2002.

[145] Sethuraman, J. A constructive definition of Dirichlet priors. *Stat. Sinica*, 4(2):639–650, 1994.

[146] Sethuraman, J. and Tiwari, R. C. Convergence of Dirichlet Measures and the interpretation of their parameter. In Gupta, S. S. and Berger, J. O., editors, *Statistical Decision Theory and Related Topics III*, pages 305–315, 1981.

[147] Shao, J. A New Probability Measure-Valued Stochastic Process with Ferguson-Dirichlet Process as Reversible Measure. *Electron. J. Probab.*, 16(9):271–292, 2011.

[148] Sierpiński, W. F. Sur les fonctions d'ensemble additives et continues. *Fund. Math.*, 3(1):240–246, 1922.

[149] Stanley, R. P. *Enumerative Combinatorics*. Cambridge University Press, 2001.

[150] Sturm, K.-T. Sharp estimates for capacities and applications to symmetric diffusions. *Probab. Theory Relat. Fields*, 103:73–89, 1995.

[151] Sturm, K.-T. Entropic Measure on Multidimensional Spaces. In Dalang, R., Dozzi, M., and Russo, F., editors, *Seminar on Stochastic Analysis, Random Fields and Applications VI*, volume 63 of *Progress in Probability*, pages 261–277. Springer, 2011.

[152] Surgailis, D. On multiple Poisson stochastic integrals and associated Markov semigroups. *Probab. Math. Statist.*, 3(2):217–239, 1984.

[153] Teh, Y. W., Jordan, M. I., Beal, M. J., and Blei, D. M. Hierarchical Dirichlet Processes. *J. Am. Stat. Assoc.*, 101(476):1566–1581, 2006.

[154] Thoma, E. Die unzerlegbaren, positiv-definiten Klassenfunktionen der abzählbar unendlichen, symmetrischen Gruppe. *Math. Z.*, 85:40–61, 1964.

[155] Trèves, F. *Linear Partial Differential Equations with Constant Coefficients – Existence, Approximation and Regularity of Solutions*, volume 6 of *Mathematics and its Applications*. Gordon and Breach – Science Publishers, London, 1966.

[156] Trèves, F. *Topological Vector Spaces, Distributions and Kernels*, volume 25 of *Pure and Applied Mathematics*. Academic Press, New York, 1967.

[157] Tsilevich, N., Vershik, A. M., and Yor, M. An Infinite-Dimensional Analogue of the Lebesgue Measure and Distinguished Properties of the Gamma Process. *J. Funct. Anal.*, 185:274–296, 2001.

[158] Tsilevich, N. V. and Vershik, A. M. Quasi-invariance of the gamma process and multiplicative properties of the Poisson–Dirichlet measures. *C. R. Acad. Sci. Paris*, 329:163–168, 1999.

[159] Tsilevich, N. V., Vershik, A. M., and Yor, M. Distinguished properties of the gamma process and related topics. arXiv:math/0005287, 2000.

[160] Vakhania, N., Tarieladze, V., and Chobanyan, S. *Probability Distributions on Banach Spaces*. Mathematics and its Applications. D. Reidel Publishing Company, Dordrecht, 1987.

[161] Vershik, A. M. Does there exist the Lebesgue measure in the infinite-dimensional space? *Proc. Steklov Insitute*, 259:248–272, 2007.

[162] Vershik, A. M., Gel'fand, I. M., and Graev, M. I. Representations of the Group of Diffeomorphisms. *Russ. Math. Surv.+*, 30(6):1–50, 1975.

[163] Vershik, A. M. and Tsilevich, N. V. Fock factorizations, and decompositions of the L^2-spaces over general Lévy processes. *Russ. Math. Surv.+*, 58(3):427–472, 2003.

[164] Villani, C. *Topics in Optimal Transportation*, volume 58 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2003.

[165] Villani, C. *Optimal transport, old and new*, volume 338 of *Grundlehren der mathematischen Wissenschaften*. Springer-Verlag, 2009.