

SEDIMENTATION OF PARTICLE SUSPENSIONS IN STOKES FLOWS

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Summary

In this thesis, we consider problems arising from the physical phenomenon of particle sedimentation. We focus on non-Brownian particles in fluids at zero Reynolds number. Microscopically, the particle system is described by a system of ordinary differential equations that determine the particle trajectories which is coupled to a partial differential equation for the fluid flow. The objective of this thesis is the rigorous derivation of macroscopic equations from microscopic particle dynamics as well as the analysis of such macroscopic equations.

For a suitable limit of many small spherical inertialess particles, we prove convergence to a nonlinear system of partial differential equations that couples a transport equation for the particles with Stokes equations for the fluid. This system has been used as a sedimentation model in the physics literature. The limit behavior occurs in the regime where the volume fraction of particles is very small but there are still enough particles such that the interaction of the particles through the fluid is relevant. In the considered limit, we prove that the microscopic dynamics is well-posed until time T which tends to infinity, and we prove well-posedness and convergence to the macroscopic system globally in time. The result has been published in *Communications in Mathematical Physics*, [Höf18a].

The proof of the result in [Höf18a] uses a technique to represent the solution of the fluid equations which is known as the method of reflections. We systematically study this technique and show how to use it in order to derive classical homogenization results. This part is based on the author's master's thesis [Höf15]. The results presented in this thesis are an improved version of [HV18], published in *Archive for Rational Mechanics and Analysis*. In comparison with [HV18], the assumptions under which convergence of the method of reflections is proved have been relaxed significantly.

For aerosols, the particle inertia becomes relevant in appropriate scaling limits and it has been suggested that the microscopic dynamics converge to a Vlasov equation coupled with Navier-Stokes/Brinkman equations. A rigorous proof of this result is presently completely out of scope, even in the case when the fluid inertia is neglected. There are several papers in the literature, though, that derive the Brinkman equations in the quasi-stationary case as a first step of the proof of the full problem. Since the particle distances are very difficult to control for the full problem, it is important to minimize the restrictions on these distances in the derivation of the Brinkman equations. Based on an idea that we first applied to a similar problem for the Poisson instead of the Stokes equations, we generalize these results to stochastic distributions of particles with very mild constraints on the sizes and distances of the particles. The results have been obtained together with Arianna Giunti and Juan Velázquez, [GHV18; GH18] (arXiv preprints).

The Vlasov-Navier-Stokes equations involve several dimensionless physical parameters, such as the Reynolds number and the Stokes number, which account for the fluid inertia and particle inertia respectively. We consider the equations at zero Reynolds number, which are then called the Vlasov-Stokes equations, and study the limit of the Stokes number tending to zero in such a way that the interaction strength of the particles – determined by another physical parameter – is held fixed. In this limit, we prove convergence to the coupled transport-Stokes system which has been derived from the microscopic picture in [Höf18a]. This result has been published in *SIAM Journal on Mathematical Analysis*, [Höf18b].

If the sedimenting particles are non-spherical, the orientation of the particles influences the macroscopic behavior. We study a macroscopic model for the sedimentation of inertialess rod-like particles. We investigate the well-posedness of this system, which is more delicate than in the case of the transport-Stokes equations and the Vlasov-Stokes equations. Short-time existence for the full system and global well-posedness for cylindrically symmetric initial data is established.

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Chapter 1

Introduction

Small particles moving in a fluid are encountered in various situations in nature and technology. In many cases, gravitation is the driving force for the movement of the particles. Although the force acting on each particle due to the gravity is directly proportional to its mass, and there is no direct (e.g. electromagnetic) interaction between the particles themselves, the motion of the particles will be quite complex in many situations. The complexity arises from the interaction of the particles through the fluid. Indeed, the presence of each particle induces a disturbance in the fluid flow which again influences all the other particles. In viscous fluids, this disturbance is of a long range nature, decaying like one over the distance. Therefore, the motion of a cloud of particles will in general show a significantly different qualitative behavior than the motion of a single particle. In particular, while the latter problem might be studied by explicit analytic solution, this is beyond hope for a system of many particles. Moreover, if the number of particles exceeds a certain value, the full problem becomes inaccessible by numerical simulations. On the other hand, from a practical point of view, the exact particle trajectory of every single particle is not relevant in most applications, but rather macroscopic quantities like the local average density and velocity of particles. Therefore, it is desirable to derive effective macroscopic equations that govern the motion of the particles in the limit of many small particles.

Various physical parameters determine the effective sedimentation dynamics. On the one hand, there are the properties of the fluid surrounding the particles, namely its mass density, viscosity, and temperature, as well as the size and shape of the container bounding the fluid. On the other hand, there are the properties of the particles, their mass density, volume fraction in the fluid, and size and shape of the cloud of particles under consideration. Most prominently, though, it is the size of the particles that determine the dynamics. Indeed, sedimentation processes occur at various length-scales of particles ranging from macromolecules and viruses ($\sim 10\text{ nm} - 100\text{ nm}$) to suspensions of dust and sand to large macroscopic objects like rocks (see e.g. [Dho96; Van06; Jul10]).

Larger particles settle faster than smaller particles, and the fluid flow tends to become turbulent then. On the other hand, for very small particles, the effect of Brownian motion of the particles becomes important. This thesis focuses on the intermediate regime of particles where both Brownian motion and fluid inertia is negligible. The fluid flow can then be modeled by the Stokes equations.

This regime is described by two dimensionless quantities, low Reynolds number Re such that the fluid inertia is small, and high Péclet number Pe such that the motion due to gravity (and the induced fluid flow) is large compared to Brownian motion. We will elaborate more on the identification of these regimes in Section 2.2.1, where we consider the settling of a single sphere, and in Section 2.3, where we formally derive macroscopic equations for sedimentation. In Section 2.3, we will also discuss two more dimensionless quantities that determine the macroscopic behavior the dynamics; the Stokes number St which accounts for the inertia of the particles and the parameter γ for the interaction

strength between the particles.

Another important factor for the sedimentation is the shape of the particles. In the simplest case the particles are spherical such that their orientation is irrelevant. In contrast, if one considers a single particle of elongated shape, then it will fall faster oriented in the direction of gravity than with a transverse orientation. For the sake of simplicity of the model and the feasibility of the mathematical analysis, we will mostly restrict our study to spherical particles. However, the sedimentation of very elongated particles such as fibers and polymers occurs in various applications, and it shows some strikingly different phenomena compared to the sedimentation of spherical particles. We therefore also study a macroscopic model for such rod-like particles.

As another simplification of our models, we will consider monodisperse suspensions, i.e., all particles have identical size (and shape). We will only include polydispersity in the formal derivation of macroscopic equations in Section 2.3. Finally, we consider all the sedimentation processes in the idealized situation of a fluid that fills the whole space \mathbb{R}^3 , and at very small volume fraction of the particles. We briefly discuss effects of finite volume fraction and sedimentation in bounded containers in Section 2.5.

1.1 Macroscopic sedimentation models

The main focus of this thesis is directed towards the following three macroscopic sedimentation models which we give here in dimensionless form.

The first one models the dynamics of inertialess spherical particles:

$$\begin{aligned} \partial_t \rho + \left(u + \frac{2}{9} \gamma^{-1} g \right) \cdot \nabla \rho &= 0, \\ -\Delta u + \nabla p &= \rho g, \quad \operatorname{div} u = 0 \quad \text{in } \mathbb{R}^3, \quad u(x) \rightarrow 0 \quad \text{as } |x| \rightarrow \infty. \end{aligned} \tag{1.1.1}$$

Here, $\rho(t, x)$ is the density of particles at time t and position $x \in \mathbb{R}^3$, and $u(t, x)$ and $p(t, x)$ are the fluid velocity and pressure, respectively. Moreover, g is the (dimensionless) gravitational acceleration, and γ is the dimensionless quantity accounting for the interaction strength between the particles mentioned above.

These equations, which we call the transport-Stokes equations, have the following interpretation: Since the particles are inertialess, their velocity is determined by the fluid velocity at the position of the particle and the strength of the gravity as $u + 2/(9\gamma)g$. Moreover, the gravitational force acting on each particle must be balanced by the drag force exerted by the fluid. By Newton's third law, this implies that the particles are exerting the same force on the fluid which leads to the source term ρg in the Stokes equations above.

The second model describes sedimentation of spherical particles when the inertia of the particles becomes relevant:

$$\begin{aligned} \partial_t f + v \cdot \nabla_x f + \operatorname{St}^{-1} \operatorname{div}_v \left(\gamma^{-1} g f + \frac{9}{2} (u - v) f \right) &= 0, \\ -\Delta u + \nabla p &= 6\pi\gamma \int_{\mathbb{R}^3} (v - u) f dv, \quad \operatorname{div} u = 0 \quad \text{in } \mathbb{R}^3, \quad u(x) \rightarrow 0 \quad \text{as } |x| \rightarrow \infty. \end{aligned} \tag{1.1.2}$$

Here, $f(t, x, v)$ is the density of particles at time t and position $x \in \mathbb{R}^3$ with velocity $v \in \mathbb{R}^3$, and St is the Stokes number that accounts for the effect of the particle inertia.

This system of equations is called the Vlasov-Stokes equations. The particles are accelerated by gravity and by the fluid drag force which is proportional to the difference between the fluid and particle velocity. On the other hand, again by Newton's third law, the particles exert the same force on the fluid, leading to the fluid equations above, which are known as Brinkman equations.

Finally, for inertialess rod-like particles, we consider the system

$$\begin{aligned} \partial_t f + \left(u + \frac{1}{8\gamma^{-1}}(\text{Id} + \xi \otimes \xi)g \right) \cdot \nabla_x f + \text{div}_\xi (P_{\xi^\perp}(\xi \cdot \nabla)uf) &= 0, \\ -\Delta u + \nabla p &= \int_{S^2} fg \, d\xi, \quad \text{div } u = 0 \quad \text{in } \mathbb{R}^3, \quad u(x) \rightarrow 0 \quad \text{as } |x| \rightarrow \infty. \end{aligned} \tag{1.1.3}$$

Here $f(t, x, \xi)$ denotes the density of particles at time t and position x that have an orientation ξ considered as an element in the two-dimensional unit sphere¹ S^2 . Moreover, Id denotes the identity matrix, and P_{ξ^\perp} denotes the orthogonal projection to the orthogonal complement of ξ .

In contrast to the inertialess system for spherical particles (1.1.1), the velocity of the particles now depends also on their orientation. This orientation undergoes changes due to the fluid flow, which satisfies the analogous of the Stokes equations in (1.1.1) since $\int_{S^2} f \, d\xi$ to the spatial particle density corresponds to ρ in (1.1.1).

1.2 Outline

The systems (1.1.1), (1.1.2), and (1.1.3) and related equations are used in the physics and engineering literature for the macroscopic modeling of sedimentation (see e.g. [Wil18; Koc90; O'R81; CP83; DE88]). So far, very little is known about these models from the mathematical perspective. The ultimate goal would be a comprehensive analysis of these equations as well as their rigorous derivation as limits of particle systems. The only one of these models which has been rigorously derived so far is the transport-Stokes system (1.1.1). This derivation is the content of Chapter 4. Regarding the Vlasov-Stokes equations (1.1.2), there are only results concerning the derivation of the fluid equations, the Brinkman equations, without taking into account the time evolution of the particles. We extended the existing results in the literature in [GHV18] and [GH18], which we will present in Chapter 5 and 6. We study the full macroscopic Vlasov-Stokes equations (1.1.2) in Chapter 7 and prove convergence of the system to the transport-Stokes equations (1.1.1) in the asymptotic limit of small particle inertia $\text{St} \rightarrow 0$. In Chapter 8, we prove local well-posedness for the rod model 1.1.3 and existence of global solutions under the assumption of cylinder symmetry. The remaining sections of the introduction are devoted to a more detailed outline and discussion of these main results of the thesis. For the precise statements, though, we will refer to the respective chapters.

In Chapter 2 we will give a more detailed introduction and overview of the microscopic and macroscopic aspects of sedimentation and suspension dynamics without proving original mathematical results. Starting from a discussion of the settling of a single body in infinite Stokes fluids, we will give a formal derivation of the three macroscopic sedimentation models (1.1.1), (1.1.2), and (1.1.3) that we introduced above. We also discuss the various physical regimes in which these equations can be expected to hold. Furthermore, we give an overview of some phenomena regarding the macroscopic behavior of sedimentation dynamics which have been observed experimentally and numerically. It can be hoped that some of these phenomena could be understood in the future by analyzing the underlying macroscopic equations. Finally, we briefly discuss two effects on particle sedimentation that we are neglecting for the rest of this thesis: the effects of finite volume fraction of the particles and the effects of container walls.

¹Strictly speaking, the orientation is of course undirected, and thus an element of the projective space. For all our purposes, though, it is more convenient to work with $\xi \in S^2$.

1.3 The method of reflections

In Chapter 3, we study the method of reflections and some applications to homogenization problems. We will discuss below how this method is used in and linked to the study of sedimentation problems. The method of reflections is a method to represent the solution of a (linear) boundary value problem where the boundary consists of several connected components. One example is the Dirichlet problem of the Poisson equation in perforated domains. More precisely, we consider the space \mathbb{R}^3 which is perforated by pairwise disjoint particles $\{\Omega_i\}_{i \in I}$ where I is a finite or countable index set. Then, given a source term $f \in \dot{H}^{-1}(\mathbb{R}^3)$, the dual space of the homogeneous Sobolev space $\dot{H}^1(\mathbb{R}^3) := \{v \in L^6(\mathbb{R}^3) : \nabla v \in L^2(\mathbb{R}^3)\}$, we study the problem of finding the solution to

$$\begin{aligned} -\Delta u &= f \quad \text{in } \mathbb{R}^3 \setminus \bigcup_{i \in I} \overline{\Omega_i}, \\ u &= 0 \quad \text{in } \bigcup_{i \in I} \overline{\Omega_i}. \end{aligned} \tag{1.3.1}$$

By the standard theory, this problem has a unique weak solution $u \in \dot{H}^1(\mathbb{R}^3)$. The Method of Reflection formally gives this solution in terms of a series expansion

$$u = \sum_{k \in \mathbb{N}} \Phi_k. \tag{1.3.2}$$

Here, $\Phi_0 \in \dot{H}_0^1(\mathbb{R}^3)$ is the solution to the problem neglecting the effect of the particles, i.e., Φ_0 solves

$$-\Delta \Phi_0 = f \quad \text{in } \mathbb{R}^3.$$

Then, one defines $\Phi_1 := \sum_{i \in I} \Phi_{1,i}$, where the function $\Phi_{1,i}$ is the reflection of Φ_0 at the i -th particle, which means that

$$\begin{aligned} -\Delta \Phi_{1,i} &= 0 \quad \text{in } \mathbb{R}^3 \setminus \overline{\Omega_i}, \\ \Phi_{1,i} &= -\Phi_0 \quad \text{in } \overline{\Omega_i}. \end{aligned}$$

In other words $\Phi_{1,i}$ is chosen such that $\Phi_0 + \Phi_{1,i}$ would solve (1.3.1) if the i -th particle was the only particle. If there is more than one particle, $\Psi_1 := \Phi_0 + \Phi_1$ will in general not solve (1.3.1), since $\Phi_{1,i} \neq 0$ in Ω_j for $j \neq i$. Therefore, next order corrections are needed. More precisely, one inductively defines the functions $\Phi_{k,i}$ as the solutions to

$$\begin{aligned} -\Delta \Phi_{k,i} &= 0 \quad \text{in } \mathbb{R}^3 \setminus \overline{\Omega_i}, \\ \Phi_{k,i} &= -\sum_{j \neq i} \Phi_{k-1,j} \quad \text{in } \overline{\Omega_i}. \end{aligned}$$

Then, with $\Phi_k = \sum_{i \in I} \Phi_{k,i}$, one obtains the formal series expansion (1.3.2). If this series is convergent, then the error terms $\Phi_{k,i}$ that we are adding to fulfill the boundary conditions become smaller and smaller. Thus, convergence of this series should imply that the thereby defined function u indeed solves (1.3.1). Sometimes the method explained above is referred to as the parallel method of reflections to distinguish from its not so widespread sequential variant, where an ordering of the set I is chosen and analogous correctors $\Phi_{k,i}$ are computed considering one particle after the other and already taking into account the correction on the particles considered before.

The method of reflections was introduced by Smoluchowski in 1911 in [Smo11] in order to calculate the interaction of spherical particles through a Stokes fluid. The importance of the method

of reflections for similar problems in the physics and engineering community is evident from its detailed presentation in several standard textbooks (see for instance [HB12; Dho96; KK13]). However, the method appears to have been seldom studied from the mathematical perspective.

Luke considered in [Luk89] the sequential method of reflections for the Stokes equations with sedimentation boundary conditions (cf. equation (1.4.1) below) for any configuration of finitely many particles in a bounded domain. The key to his proof was to observe that the method of reflections can be formulated in terms of orthogonal projections. Later, Traytak [Tra06] studied the parallel method of reflections in the case of the Poisson equations with Dirichlet boundary conditions and finitely many spherical particles in the whole space \mathbb{R}^3 and proved necessary and sufficient conditions for the convergence of the method.

Jabin and Otto used a related method in [JO04], where they identified the regime of inertialess particle sedimentation that is so dilute that the interaction between the particles becomes negligible. However, the estimates used in [JO04] seem to be too rough in order to directly apply to the regime where the interaction between the particles becomes relevant.

Recently, the method of reflections also attracted the attention in numerical analysis. In [LLS17], the authors give an overview of both variants of the method of reflections and analyze the convergence rate numerically. In [CGHS18], which also contains a nice historical introduction to the method of reflections, the authors study relations between the method of reflections and classical numerical methods, such as Schwarz methods.

In Chapter 3, we study the parallel version of the method of reflections in the whole space \mathbb{R}^3 perforated by finitely or infinitely many particles of arbitrary shape for the Poisson and the Stokes equations. We formulate the method in terms of orthogonal projections as

$$\sum_{k=0}^N \Phi_k = \left(1 - \sum_{i \in I} Q_i\right)^N \Phi_0. \quad (1.3.3)$$

If the Poisson equation is replaced by the so called screened Poisson equation $(-\Delta u + \ell^{-2})u = f$ for some $\ell > 0$, we prove that the convergence of the method of reflections is related to the (harmonic/electrostatic) capacity density of the particles μ : If $\ell^2 \mu$ is sufficiently small, the method converges linearly. For higher capacity densities, the method in general yields a divergent series. However, in the framework of orthogonal projections, replacing the term on the right-hand side of (1.3.3) by

$$\left(1 - \gamma \sum_{i \in I} Q_i\right)^N \Phi_0, \quad (1.3.4)$$

yields a sequence which converges to the solution u of the screened Poisson equation if γ is chosen such that $\gamma \ell^2 \mu$ is sufficiently small.

For the Poisson and Stokes equations, a similar result applies for finitely many particles. The method when (1.3.3) is replaced by (1.3.4) has been studied independently of the author's work in [LLS17] for $\gamma = N^{-1}$, where N is the number of particles. In [LLS17], this variant of the method is then called the averaged (parallel) method of reflections, and it is proved that it always leads to a convergent sequence.

For infinitely many particles, the method yields a divergent series due to the long range structure of these equations. However, it is possible to consider instead of (1.3.4) the sequence given by

$$\left(1 - \sum_{i \in I} \gamma_i Q_i\right)^N \Phi_0. \quad (1.3.5)$$

We show in Chapter 3 how to choose γ_i such that this defines a sequence which converges to the solution u of (1.3.1).

Chapter 3 contains the results of [HV18], which is based on the author's master's thesis [Höf15]. However, the results we are able to prove convergence of the (modified) method of reflections in Chapter 3 under much milder assumption on the particle configuration compared to [HV18] and [Höf15]. In [Höf15], the assumption is made, that all the particles are spherical with identical radii and centers on the lattice $(d\mathbb{Z})^3$. In [HV18], the particles are allowed to have different size and shape. Instead of particles on a lattice, particle configurations are treated where the particles satisfy the following minimal distance condition: Each particle Ω_i is contained in a ball $B_{r_i}(x_i)$, such that

$$\sup_{i \in I} \sup_{j \neq i} \frac{r_i}{|x_i - x_j|^3} \leq \mu_0 < \infty.$$

The quantity μ_0 provides an upper bound for the capacity density of the particles (see Section 1.5). In [HV18], we also simplified some of the arguments and extended the results of [Höf15] to the Stokes equations. In Chapter 3, we prove the same convergence results for the method of reflections under the much less restrictive assumption

$$\sum_{i \in I} \sum_{j \neq i} \frac{r_i r_j e^{\frac{2|x_i - x_j|}{\ell}}}{|x_i - x_j|^2} < \infty$$

for some $\ell > 0$.

In Chapter 3, we will also show how to use the method of reflections in order to give a new proof of homogenization results in domains perforated by many small particles. In particular, the method allows us to prove the homogenization of the Stokes equations to the Brinkman equations in the limit of many small particles. The Brinkman equations are the fluid equations in the macroscopic sedimentation model (1.1.2). We will comment more on these homogenization results in Section 1.5. Moreover, as we will explain in Section 1.4, the method of reflections in its framework of orthogonal projections is one of the key ingredients in proving the convergence of the microscopic inertialess particle sedimentation to the macroscopic equations (1.1.1), which is proved in Chapter 4.

1.4 The derivation of the transport-Stokes equations

In Chapter 4, we rigorously derive the transport-Stokes equations (1.1.1) as a mean field limit of microscopic inertialess particle sedimentation in Stokes flows. More precisely, we consider N spherical particles $B_i := B_R(X_i)$, $1 \leq i \leq N$ and study the following dimensionless system for inertialess non-rotating particles

$$\begin{aligned} -\Delta v + \nabla p &= 0, \quad \operatorname{div} v = 0 \quad \text{in } \mathbb{R}^3 \setminus \bigcup_{i=1}^N \overline{B_i}, \\ v &= V_i \quad \text{in } \overline{B_i}, \quad v(x) \rightarrow 0 \quad \text{as } |x| \rightarrow \infty, \\ \dot{X}_i &= V_i, \quad \frac{4\pi}{3N} g = - \int_{\partial B_i} \sigma[v] n \, d\mathcal{H}^2, \text{ for } 1 \leq i \leq N. \end{aligned} \tag{1.4.1}$$

Here, $\sigma[v] = \nabla v + (\nabla v)^T - p \operatorname{Id}$ is the fluid stress. The boundary conditions imposed on the fluid are the so called sedimentation boundary conditions (or boundary conditions of the fourth kind). They consist in prescribing the total force each particle acts on the fluid (last equation) and the no-slip condition that the fluid flow has to equal a constant at the boundary of each of the particles

(since they are not allowed to rotate). Since the particles are inertialess, these constants, which are the particle velocities V_i , are not prescribed but determined by solving the fluid equation. For convenience, the fluid velocity v is extended by V_i inside of the particle B_i . The rotations of particles are neglected for the sake of simplicity of the model only. They are negligible in the homogenization limit of many small particles (cf. [Mec18]).

More details on the approximations and nondimensionalization leading to the microscopic model (1.4.1) as well as a discussion about the physical regime of its validity can be found in Chapter 2.3.

The system (1.4.1) has been widely used as a model for particle sedimentation, in particular as a starting point for formal calculations of sedimentation velocities and numerical simulations (see for instance [Has59; Bat72; Hin77; Nic+95; GM12b]).

There are a number of papers (e.g. [CP83; Koc90]) about the derivations of the transport-Stokes equations (1.1.1). However, the first preliminary step towards a rigorous derivation of a macroscopic system from the microscopic particle system (1.5.1) has only been made in [JO04], where the regime is identified where the particle interaction is negligible, and thus all the particles settle like isolated particles. In particular, they found that if the minimal distance between the particles $d_{N,\min}$ satisfies $d_{N,\min} \gtrsim N^{-1/3}$, the quantity $\gamma := NR$ determines the interaction strength between the particles (note that due to the nondimensionalization, the system size, which affects the interaction strength, has been rescaled to 1).

In Chapter 4, which contains the results of [Höf18a], we prove the convergence of the particle model (1.4.1) towards the macroscopic equations (1.1.1) in the limit $N \rightarrow \infty$ with $R \rightarrow 0$ and $\gamma \rightarrow \gamma_* \in (0, \infty]$. We postpone the precise formulation of the convergence result to Chapter 4. Roughly speaking, we prove uniform convergence in $[0, T) \times \mathbb{R}^3$ for any time $T > 0$ for an averaged particle density ρ_N to the solution ρ of the (1.1.1) with initial datum ρ_0 under the three conditions, first, that the initial average particle density $\rho_N(0, \cdot)$ converges to ρ_0 , second, that the initial particle configurations satisfies the estimate $d_{N,\min} \gtrsim N^{-1/3}$ uniformly in N , and third, that $NR^3 \log N \rightarrow 0$.

This last condition is slightly more restrictive than imposing that the volume fraction $\phi_N := NR^3$ converges to zero, which is very natural since a non-vanishing limit volume fraction ϕ would (at least formally) change the limit viscosity in (1.1.1) according to the Einstein law $\mu_{\text{eff}} = (1 + 5/2\phi)\mu$ (with $\mu = 1$ in our case due to the nondimensionalization). We will discuss this effect in more details (but on a formal level) in Chapter 2.5.2.

The constraint $d_{N,\min} \gtrsim N^{-1/3}$ on the initial particle configurations is imposed to rule out the presence of clusters of too close particles that would change the effective behavior of the dynamics. However, this assumption is the main drawback of the result, as it is not satisfied with a probability that tends to 1 as $R \rightarrow 0$ if the initial particle distribution is random according to any reasonable probability distribution. It would be highly desirable to be able to consider such stochastic initial data, since from the physical point of view it is practically impossible to have an exact control on the initial configuration.

As a first step in this direction, Mecherbet [Mec18] has been able to relax the condition $d_{N,\min} \gtrsim N^{-1/3}$ to allow more initial particle configurations. However, her conditions are still not general enough in order to allow for a probabilistic result. In [Mec18], the microscopic model includes particle rotations, and it is shown that the rotations indeed do not affect the macroscopic behavior. Moreover, a quantitative convergence rate of the microscopic particle density towards the macroscopic particle in terms of Wasserstein distances is proved. As a drawback, the results in [Mec18] are only proved for short times.

The rigorous derivation of mean field limits similar to the systems (1.1.1), (1.1.2), and (1.1.3) from particle models are in general very difficult. Although such a mean field limit can be derived in

a quite straightforward manner if the particle interaction is given by

$$\dot{V}_i = \frac{1}{N} \sum_{j \neq i} F(X_i - X_j), \quad (1.4.2)$$

for a smooth force F (see [BH77]), it has long been recognized to be a very challenging task when the interactions are singular, which is the case in many physical relevant problems, most prominently, $F(x) \sim \pm 1/|x|^2$ in the case of gravitational or Coulombian force. Here, the mean field limit has only been accomplished if the singularity is cut off or the singularity is sufficiently mild (see [HJ07; HJ15]).

For the particle systems considered in this thesis, each particle changes the fluid velocity at each point in a manner that behaves like the inverse distance to that particle. Hence, all the problems that we study here correspond to singular interactions. However, for the system of inertialess particles (1.4.1), the interaction does not determine the acceleration but the velocity of the particles. This has certain advantages, as it can be hoped that very close particles have almost the same velocities leading to estimates on their relative distance over time. We will comment on the difficulties arising from inertial particles in the next section. Another system, where the interaction determines the velocity and not the acceleration is the vortex model, where the vortex formulation of the Euler equations has been obtained in the mean field limit in [GHL90; HL90]

On the other hand, there is an additional obstruction to the derivation of sedimentation models: the particle interaction is not given as in (1.4.2) by an explicitly given force which only depends on the distance between the particles. In contrast, in sedimentation problems, the particles are interacting with each other by modifying the fluid flow. Therefore, the interaction has a more complex structure, since one has to solve a boundary value Stokes problem.

A way to overcome this problem is to find sufficiently good approximations of the fluid velocities which are more explicit. This is exactly what we will do in Chapter 4, by considering the fluid flow u_N that one gets if one assumes that the force which every particle exerts on the fluid is distributed uniformly on the boundary of that particle. Then, since the total force is given by the last equation in (1.4.1), the only difference between the exact solution v_N to the fluid equations of (1.4.1) and the approximation u_N is how these forces are distributed at each particles. Since the particles are very small, one is therefore led to think that the approximation of v_N to u_N is very accurate.

However, in order to control each of the particle trajectories, we effectively need L^∞ -estimates for $u_N - v_N$ which are usually difficult to obtain. This is where we employ the method of reflections yielding an explicit series representation for v_N . It is possible to apply the method in such a way that the first term in this series is given by the approximation u_N , and the k -th order terms can be shown to be of order ϕ_N^k , which proves the desired estimate for $u_N - v_N$, since the volume fraction ϕ_N is assumed to tend to zero in the limit $N \rightarrow \infty$.

The reason why the condition $d_{N,\min} \gtrsim N^{-1/3}$ on the initial particle configuration is imposed is that the Method of Reflection breaks down when some of the particles are too close to each other, and it is indeed to be suspected that the approximation of v_N by u_N is actually not accurate at *every* particle if they are not sufficiently well separated.

Thus, another difficulty in the proof is to show that the condition $d_{N,\min} \gtrsim N^{-1/3}$, which can only be imposed for the initial distribution of particles, persists globally in time under the dynamical behavior of the particle system. As pointed out above this can be hoped to be true because the particle velocities only depend on their positions and it can actually be proven again with the help of the method of reflections.

1.5 The Brinkman equations as the homogenized Stokes equations

The Vlasov-Stokes equations (1.1.2) as a sedimentation model has been mathematically studied by Hamdache [Ham98], who proved global well-posedness of a similar system where the stationary Stokes equations are replaced by the instationary Stokes equations. Since then, these and related models for inertial particles in fluids have attracted a lot of attention: In [AMB97; BDGM09; Yu13; WY15; CK15], global well-posedness has been proven for the Vlasov-Navier-Stokes equations. Global existence of solution to the Vlasov-Navier-Stokes-Fokker-Planck equations (thus taking into account particle diffusion) has been established in [GHMZ10; CKL11]. In [MV07] existence of weak solutions to the compressible version of this system has been proved. Finally, local existence of weak solutions to the compressible Vlasov-Euler equations have been proved in [BD06]. Several asymptotic limits of some of these equations have been studied in [GJV04a; GJV04b; CG06; MV08].

All these results study the macroscopic systems. In contrast to the transport-Stokes equations (1.1.1), the rigorous derivation of the Vlasov-Stokes equations (1.1.2) from a particle system is still a widely open problem. Partial results on the derivation of the Vlasov-Navier-Stokes-Fokker-Planck have been recently obtained in [FLR18]. For completeness, we also mention the papers [BDGR17; BDGR18], where the Vlasov-Navier-Stokes equations are obtained as a scaling limit of a multiphase Boltzmann system.

The dimensionless microscopic model for inertial spherical particles (corresponding to the inertialess model (1.4.1)) reads

$$\begin{aligned} -\Delta u + \nabla p &= 0, \quad \operatorname{div} u = 0 \quad \text{in } \mathbb{R}^3 \setminus \bigcup_{i=1}^N \overline{B_i}, \\ u &= V_i \quad \text{in } \overline{B_i}, \quad u(x) \rightarrow 0 \quad \text{as } |x| \rightarrow \infty, \\ \dot{X}_i &= V_i, \quad \dot{V}_i = \lambda \left(g + \frac{3N}{4\pi} \int_{\partial B_i} \sigma[u] n \, d\mathcal{H}^2 \right). \end{aligned} \tag{1.5.1}$$

The main obstacle which makes the analysis of this model even more challenging than the one for inertialess particles is caused by the fact that inertial particles are moving significantly relative to each other even if they are already quite close together, while for inertialess particles we are able to rule out such behavior since their velocities only depend on their positions.

Due to the singular nature of the particle interactions, close pairs of particles move significantly different than well separated particles. If the number of such pairs or clusters is sufficiently small, it can be hoped that they do not affect the macroscopic behavior of the system. However, even when we restrict ourselves to initial conditions of well separated particles, it seems plausible that one can construct pathological initial configurations where many close pairs of particles occur after short times, for example by putting a slower particle in front of each faster particle. In those pathological situations, convergence to the macroscopic system (1.1.2) cannot be expected to hold.

Consequently, one can only hope for the convergence of the microscopic dynamics to the macroscopic equations for a large set of initial data, and the exceptional set seems very hard to characterize. This problem is quite standard in the derivation of kinetic equations, in particular for the Boltzmann equations (see e.g. [CIP94]) and it is therefore natural both physically and mathematically to consider stochastic initial data. Thus, as a first step towards the derivation of the Vlasov-Stokes equations (1.1.2), it is necessary to study the quasi-static problem of the convergence of the fluid equations to the Brinkman equations, the fluid equations in (1.1.2), for random distributions of particles (without considering their time-evolution), and to investigate the fluctuations of the fluid velocity.

Before we further discuss the derivation of the Brinkman equations, we remark that the issue of particles approaching each other also occurs in certain inertialess sedimentation models: For

non-spherical inertialess particles, the particle velocities do not only depend on their positions but also on their orientations, such as for rod-like particles, for which the macroscopic model (1.1.3) is proposed. Moreover, even for inertialess spherical but polydisperse particle distributions, i.e., systems with particles of different size, the same problem occurs since larger particles settle faster than smaller particles. Thus, the convergence proof for inertialess spherical particles which we will give in Chapter 4 is not easily adapted to the case of polydisperse particle distributions since the condition $d_{\min} \lesssim N^{-1/3}$ does not persist in time.

For the study of the homogenization problem of the Stokes to the Brinkman equations, we consider the simplified problem, where the particle velocities are set to zero. More precisely, we study the homogenization limit of the problem

$$\begin{aligned} -\Delta u_\varepsilon + \nabla p_\varepsilon &= f, & \operatorname{div} u_\varepsilon &= 0 & \text{in } D^\varepsilon, \\ u_\varepsilon &= 0 & \text{on } \partial D^\varepsilon \end{aligned} \quad (1.5.2)$$

in a perforated domain D^ε , that is obtained from $D \subset \mathbb{R}^3$ by removing a number of small particles indexed by I_ε . We also call those particles holes in the following, in order to make clear that they are fixed. As $\varepsilon \rightarrow 0$, the number of holes tends to infinity and their sizes tend to zero.

To simplify the problem even more, we first study the corresponding problem for the Poisson instead of the Stokes equations, i.e., set to zero. More precisely, we consider the homogenization limit of the problem

$$\begin{aligned} -\Delta u_\varepsilon &= f, & \text{in } D^\varepsilon, \\ u_\varepsilon &= 0 & \text{on } \partial D^\varepsilon. \end{aligned} \quad (1.5.3)$$

This problem can be interpreted as studying the electrostatic potential u_ε in the presence of many small grounded holes. The order of magnitude of the effect of each hole on the potential is given by the electrostatic capacity of the holes. For spherical particles $\{B_{R_i}(X_i)\}_{i \in I_\varepsilon}$, which we restrict our attention to, their individual capacity is given by $4\pi R_i$. We therefore introduce the capacity density

$$\mu_\varepsilon := 4\pi \sum_{i \in I} R_i \delta_{X_i}.$$

A nontrivial effect of the holes can be expected in the limit $\varepsilon \rightarrow 0$, if the capacity density of the holes is of order one. On the other hand, if the capacity density tends to zero, we expect $u_\varepsilon \rightarrow u$, where u solves the Poisson equation without holes, and, if the capacity density tends to infinity, we expect $u_\varepsilon \rightarrow 0$.

In Chapter 3, we consider the case $D = \mathbb{R}^3$ and use the method of reflections to study the limit as $\varepsilon \rightarrow 0$. Under the assumption that the holes are sufficiently well separated and that the capacity density μ_ε converges in a suitable way to some $\mu \in L^\infty(\mathbb{R}^3)$ which is uniformly bounded below, we prove that $u_\varepsilon \rightarrow u$ in $H^1(\mathbb{R}^3)$, where u is the solution of the screened Poisson equation

$$-\Delta u + \mu u = f, \quad \text{in } \mathbb{R}^3.$$

We also prove the analogous result for the Stokes equation. Here the role of the electrostatic capacity is replaced by the Stokes resistance of the holes (see Section 2.2). In the case of a spherical hole, this resistance is given by $6\pi R_i$. With the corresponding definition of μ_ε , we obtain under the same assumptions as for the Poisson equation that the sequence of solutions u_ε to (1.5.2) weakly converges in $H^1(\mathbb{R}^3)$ to the solution u of the Brinkman equations

$$-\Delta u + \mu u + \nabla p = f, \quad \operatorname{div} u = 0 \quad \text{in } \mathbb{R}^3.$$

Very similar results have been obtained before (see e.g. [CM82a; CM82b; All90a; DGR08]; a detailed discussion of the existing literature will follow). We revisit these problems in Chapter 3 in order to investigate and illustrate the use of the method of reflections, which allows us to give a completely new proof for homogenization results in perforated domains. This new proof avoids the classical usage of cleverly chosen test-functions in order to pass to the limit equation. Instead, the method of reflections yields a series expansion of the solutions u_ε , which can be shown to converge to u by some Riemann-sum argument. It can be hoped that this approach could lead to a better quantitative understanding of the convergence. A variant of the method of reflections has actually been used in [FOT85] and [Rub86] where the fluctuations u_ε around the limit solution u is determined when the holes are randomly distributed. However, [FOT85] and [Rub86] are only able to treat the case where $(-\Delta)$ in (1.5.3), (1.5.2) is replaced by $(-\Delta + \lambda)$ for $\lambda \in \mathbb{R}$ has to be sufficiently large. As mentioned in Section 1.3, this ensures the convergence of the method of reflections. In Chapter 3, we are able to use the method of reflections without introducing λ by relying on the modified Method of Reflection (cf. equation (1.3.5)). It can be hoped that with similar methods, also the fluctuation results from [FOT85] and [Rub86] can be proved for the case $\lambda = 0$.

Although the Method of Reflection seems to provide a very effective tool in order to get quantitative estimates, it becomes very difficult to apply as soon as clustering of holes occurs. In Chapter 5 and 6 we analyze under which minimal assumptions on the configuration of holes the homogenization results for the Poisson and Stokes equations hold revisiting the classical methods that have been used by Cioranescu and Murat [CM82a; CM82b] for the Poisson equation and by Allaire [All90a] for the Stokes equations. There is a huge literature on the homogenization of the Poisson and Stokes equations in perforated domains. Most results, though, in particular in the case of the Stokes equations, apply to a deterministic setting, where situations of very close particles are explicitly ruled out. A detailed overview of the existing literature on those homogenization problems can be found in Stokes 5.1 and 6.1. The results in Chapter 5 and 6 are published in [GHV18] and in the preprint [GH18].

We consider perforated domains $D^\varepsilon \subset \mathbb{R}^d$, $d > 2$ which are obtained by removing a randomly generated configuration of spherical holes. More precisely, in Chapter 5

$$D^\varepsilon := D \setminus \bigcup_{z_i \in \Phi \cap \frac{1}{\varepsilon} D} B_{\varepsilon^{\frac{d}{d-2}} \rho_i}(\varepsilon z_i),$$

where $\frac{1}{\varepsilon} D := \{x \in \mathbb{R}^d : \varepsilon x \in D\}$. We assume that the collection Φ of the hole centers z_i is generated according to a stationary point process on \mathbb{R}^d and that the radii $\{\rho_i\}_{z_i \in \Phi}$ are unbounded random variables with identical 1-correlation functions and short-range correlations. We impose the minimal constraint that the expectation of the d -dimensional capacity of each hole is finite, i.e.,

$$\langle \rho^{d-2} \rangle < \infty. \quad (1.5.4)$$

(Note that the rescaling $B_{\varepsilon^{\frac{d}{d-2}} \rho_i}(\varepsilon z_i)$ is chosen according to this capacity density.) Moreover, we require Φ to satisfy a strong mixing condition and that the variance of the number of holes in bounded sets is finite.

These assumptions are very mild (in particular they are satisfied if Φ is a Poisson point process) and allow for big clusters of close and even overlapping holes $B_{\varepsilon^{\frac{d}{d-2}} \rho_i}(\varepsilon z_i)$. More precisely, clusters of arbitrary numbers of overlapping holes appear with probability one in the limit $\varepsilon \rightarrow 0$. Nevertheless, we prove that \mathbb{P} -almost surely, when $\varepsilon \rightarrow 0$, the solutions u_ε of (1.5.3) weakly converge in $H_0^1(D)$ to the solution of

$$\begin{aligned} -\Delta u + \mu u &= f && \text{in } D \\ u &= 0 && \text{on } \partial D, \end{aligned}$$

where μ is the average capacity density of the holes. Due to stationarity, this average capacity density is constant and given by

$$\mu = (d-2)\mathcal{H}^{d-1}(S^{d-1})\langle\rho^{d-2}\rangle\langle\#(\Phi\cap Q)\rangle,$$

where S^{d-1} is the $(d-1)$ -dimensional unit sphere in \mathbb{R}^d , and $\langle\#(\Phi\cap Q)\rangle$ is the expected number of holes in a unit cube Q .

The main challenge in the proof of this result is to deal with the clusters of close holes. We treat these particles by a careful study of the geometry of the perforation leading to a smallness estimate of the capacity of these “bad” holes. Due to subadditivity of the capacity, we are then able to prove that the effect of these “bad” holes is negligible in the limit $\varepsilon \rightarrow 0$.

In Chapter 6, we present corresponding results for the Stokes equations. Due to the incompressibility, the analysis is significantly more difficult in this case, since for the Stokes resistance, that corresponds to the electrostatic capacity for the Poisson equation, subadditivity fails. However, we are nevertheless able to prove the homogenization to the Brinkman equations under the slightly strengthened condition when (1.5.4) is replaced by

$$\langle\rho^{d-2+\beta}\rangle < \infty \quad \text{for some } \beta > 0. \quad (1.5.5)$$

For simplicity, we only consider the case, when Φ is a Poisson point process and the radii ρ_i are identically and independently distributed. Condition (1.5.5) is used in a even more detailed analysis of the geometry of the random perforation, where we prove that \mathbb{P} -almost surely, as $\varepsilon \rightarrow 0$, there are no clusters of more than M close holes of *similar* size, where M is a number that depends on β and d . It must be emphasized, though, that clusters of arbitrary number of overlapping holes which are composed of different sizes of holes still appear with probability one in the limit $\varepsilon \rightarrow 0$.

1.6 The inertialess limit of the Vlasov-Stokes equations

In Chapter 7, we prove the convergence of the solutions to the Vlasov-Stokes equations (1.1.2) to the solutions of the transport-Stokes equations (1.1.1) in the limit $\text{St} \rightarrow 0$ with $\gamma \in \mathbb{R}_+$ fixed. This result is expected, since the Stokes number St in the Vlasov-Stokes equations (1.1.2) determines the strength of the particle inertia and the transport-Stokes equations 1.1.1 models inertialess particles.

We again postpone the precise formulation of the convergence to Chapter 7. We remark, however, that the solution to the Vlasov-Stokes equations (1.1.2) with parameter St is a function f_{St} depending on time t , position x and velocity v . On the other hand, since the transport-Stokes equations model inertialess particles, the solution is a function ρ depending only on time t and position x . Thus, the convergence takes place for the velocity averages of f_{St} , i.e., for $\rho_{\text{St}} := \int_{\mathbb{R}^3} f_{\text{St}} dv$. The main aspect of the convergence proof is to establish that in the limit $\text{St} \rightarrow 0$, f_{St} concentrates in the space of velocities around the transport velocity $u + 2/9\gamma^{-1}g$ of (1.1.1). This concentration happens on the timescale St , inducing a boundary layer at time zero, where the particles adapt their velocities without moving. Consequently, the convergence of the fluid velocities only happens for positive times and not at the initial time.

As already mentioned at the beginning of Section 1.5, asymptotic limits of the related equations have been studied before in [GJV04a; GJV04b; CG06; MV08]. However, all these papers study the case where the diffusion of the particles is significant, thus modeling the particle evolution by a Vlasov-Fokker-Planck equation coupled to some equations for the fluid flow. The diffusive term completely changes the nature of the system. In particular, the analysis in [GJV04a; GJV04b; CG06; MV08] is based on a relation between the dissipated energy of the system and a relative entropy, which is not available for the study of the Vlasov-Stokes equations. Therefore, the proof of the

convergence result in Chapter 7 is based on the method of characteristics. Together with an estimate for the energy

$$E := \int_{\mathbb{R}^3 \times \mathbb{R}^3} |v|^2 f_{\text{St}} dv dx, \quad (1.6.1)$$

this also allows for a global well-posedness proof of the Vlasov-Stokes equations.

The main obstacle in the proof is that the energy estimate depends on St , which makes it hard to achieve uniform estimates in the limit of $\text{St} \rightarrow 0$. Therefore a very careful analysis of the characteristics is needed in order to show that the particle velocities adapt sufficiently fast to the corresponding inertialess velocities.

1.7 Well-posedness results for a model of sedimentation of rod-like particles

In Chapter 8 we consider the macroscopic equations (1.1.3) that models the sedimentation of inertialess rod-like particles.

The mathematical literature on suspensions of rigid rod-like particles is even more limited than for spherical particles. There are only some results concerning the so called Doi model which goes back to [Doi81] (see also [DE88]). In this model, Brownian motion of the particles is taken into account leading to an additional diffusion terms in the system (1.1.3). In [OT08] stationary solutions and perturbations around those solutions have been studied. Existence of weak solution to the same model has been proved in [BT13] and it has been studied numerically by [HO06].

In Chapter, we will prove short time existence of solutions to the rod model (1.1.3). The proof is based on the method of characteristics and a fixed point argument. It is very similar to the existence proof to solutions of the Vlasov-Stokes equations (1.1.2) that we give in Chapter 4.3. However, in contrast to the Vlasov-Stokes equations, we are not able to prove the existence of global solutions for the rod model (1.1.3). In comparison with the Vlasov-Stokes equations, there are two aspects that make the existence proof more challenging. First, in the rod-model (1.1.3), not only the fluid velocity u appears but also its gradient, which is more singular. Second, in the Vlasov-Stokes equations (1.1.2), we are able to use energy estimates are available for the kinetic energy of the particles (1.6.1). A similar energy for the rod model (1.1.3) does not seem to be available.

Regarding the first issue, the presence of ∇u and its regularity, the rod-model resembles more the well-known Vlasov-Poisson equations than the Vlasov-Stokes system. The Vlasov-Poisson equations are

$$\begin{aligned} \partial_t f + v \cdot \nabla_x f + \nabla u \cdot \nabla_v f &= 0 \quad \text{in } (0, T) \times \mathbb{R}^3 \times \mathbb{R}^3, \\ -\Delta u &= \int_{\mathbb{R}^3} f dv \quad \text{in } (0, T) \times \mathbb{R}^3, \quad u(x) \rightarrow 0 \quad \text{as } |x| \rightarrow \infty. \end{aligned} \quad (1.7.1)$$

For the Vlasov-Poisson equations global well-posedness has first been proved in [Pfa92] (see also [Sch91] for an improved version of the proof of [Pfa92]), also based on the method of characteristics. However, even if from the regularity point of view the Vlasov-Poisson equations (1.7.1) look very similar to the rod model (1.1.3), there are important differences: as for the Vlasov-Stokes equations, the Vlasov-Poisson system comes with a natural kinetic energy. The main difference, though, lies in the (a priori) possible blowup scenarios for solutions to the Vlasov-Poisson equations on the one hand and for solutions to the rod model on the other hand.

In both cases, the possible blowup lies in the L^∞ -norm of the spatial particle density ρ . For the Vlasov-Poisson equations, we have $\rho = \int_{\mathbb{R}^3} f dv$, for the rod-model $\rho = \int_{S^2} f d\xi$. For the Vlasov-Poisson equations, the L^∞ -norm of f is conserved. Hence, blowup of ρ can only occur if the velocity

of the particles blow up. Therefore, the main technical effort in [Pfa92] goes into estimates on the support of $f(t, x \cdot)$ in the space of velocities (uniformly in x) under the assumption of compactly supported initial data f_0 . On the other hand, for the rod-model (1.1.3), this is no issue since S^2 is compact. However, the dynamics does not preserve the L^∞ -norm of f . Therefore, in order to prove global well-posedness of the rod model (1.1.3), estimates for $\|f(t, \cdot)\|_\infty$ have to be proved.

In order to get insights on whether blowup in finite time of solutions to the rod-model (1.1.3) occurs or how to rule out such behavior, we also study in Chapter 8 a simplified version of (1.1.3). This simplified version is derived from the full system (1.1.3) in the case of cylindrically initial data f_0 . This symmetry assumption considerably reduces the complexity of the system, but it preserves the general structure which a priori could produce blowup phenomena. However, we will prove in Chapter 8 that this rod model for cylinder symmetrical solutions is well-posed globally in time.

Chapter 2

Microscopic and macroscopic sedimentation models

This chapter introduces the theory of sedimentation. This introduction is not intended to be comprehensive but rather focuses on problems treated later in this thesis and related problems. This chapter does not contain original rigorous mathematical results.

The classical problem of a single body in an infinite fluid modeled by Stokes equations is the topic of Section 2.2. This is the starting point for almost any sedimentation model. We also discuss in the context of this section the assumption of neglecting the fluid (and particle) inertia as well as particle diffusion. In Section 2.3 we formally derive the macroscopic sedimentation models (1.1.1), (1.1.2), and (1.1.3) and discuss their physical range of application. In Section 2.4, we formally take several scaling limits of the Vlasov-Stokes equations (1.1.2), including the inertialess limit, which is rigorously treated in Chapter 7. Finally, in Section 2.5, we discuss some further phenomena on particle sedimentation that have attracted a lot of attention in the physics community but are not treated in this thesis. It seems worthwhile to consider these problems in future research.

For a general introduction to physical hydrodynamics, we refer the reader to [LL87; BB67; GHPM01] and to [HB12] for hydrodynamics at small Reynolds numbers.

2.1 List of Symbols

Throughout this chapter, a lot of symbols are used to denote various physical quantities. Therefore a list of symbols used in this chapter is given below. We will give a precise definition of these symbols when they first appear.

The usage of these symbols in the other chapters might deviate in some cases.

u, v	fluid velocity
p, q	fluid pressure
R	particle radius
X	particle position
V	particle velocity
Ω	particle angular velocity
Ξ	particle orientation for a rod-like particle
ρ_f	fluid mass density
ρ_p	particle mass density
μ	dynamic viscosity
$\sigma[v]$	fluid stress related to a fluid velocity v

g	gravitational acceleration
\hat{g}	normalized gravitational acceleration ($\hat{g} := g/ g $)
m_p	particle mass
I_p	particle moment of inertia
F	force acting on a particle
\mathcal{T}	torque acting on a particle
V_{St}	“Stokes velocity”, settling velocity of a single particle in infinite unperturbed fluid
M	Stokes resistance matrix
Θ	temperature
T	typical time
L	typical length
U	typical velocity
D	diffusion constant
ϕ	volume fraction of the particles
Re	Reynolds number
St	Stokes number
Pe	Péclet number
k_B	Boltzmann constant
γ	interaction strength
λ	$(\text{St}\gamma)^{-1}$
S^2	two dimensional unit sphere
n	outer unit normal
\mathcal{H}^2	two dimensional Hausdorff measure
$\delta_{\partial B_r(x)}$	normalized Hausdorff measure on $\partial B_r(x)$
Id	identity matrix

2.2 The settling of a single particle

In this section, we summarize results on the settling of a single particle in the idealized situation of an otherwise unperturbed infinite fluid. As throughout the thesis, we mostly restrict the discussion to the idealized situation of zero Reynolds numbers and infinite Péclet number, where inertia of the fluid and Brownian diffusion of the particle is negligible. A discussion about the regime where this is a valid approximation is included in the following subsection. In this regime, there is a linear dependence between the particle velocity and the drag force exerted on it by the fluid, and this relation completely describes the settling of the particle. We will not give the detailed computations leading to the results that we summarize in this section. They can be found in standard textbooks, e.g. in [HB12].

The first systematic study of the diffusive motion (low Péclet number) of particles in fluids has been undertaken by Einstein in a series of papers that can be found in [Ein56]. For an introduction to this topic, we also refer the reader to [Dho96]. The motion of bodies in turbulent flows (high Reynolds number) is an extremely complicated issue. In particular, the relation between the velocity of the body and the drag force acting on it is nonlinear for nonzero Reynolds numbers. Experiments suggest that for sufficiently large Reynolds numbers the drag force is proportional to the square of the velocity with a constant independent of the Reynolds number. At present, the mathematical analysis of this topic is very much limited to numerical simulations. An introduction to turbulent flows can be found in [Bat53; LL87].

2.2.1 The settling of a single spherical particle

The mathematical study of sedimentation goes back to Stokes [Sto51] who calculated the drag force acting on a solid body moving in a creeping flow. More precisely, let us consider a spherical particle of radius R and mass density ρ_p settling in a fluid of (dynamic) viscosity μ and density ρ_f extending to all of \mathbb{R}^3 and which is at rest at infinity. Then, modeling the fluid flow by the incompressible Navier-Stokes equations, the problem is to find the solution to

$$\begin{aligned} \rho_f(\partial_t v + (v \cdot \nabla)v) - \mu \Delta v + \nabla p &= \rho_f g, \quad \operatorname{div} v = 0 \quad \text{in } \mathbb{R}^3 \setminus B_R(X(t)), \\ v(t, x) &= V(t) + \Omega(t) \times (x - X(t)) \quad \text{in } B_R(X(t)), \quad v(t, x) \rightarrow 0 \quad \text{as } |x| \rightarrow \infty, \\ \dot{X} &= V, \quad m_p \dot{V} = m_p g + \int_{\partial B_R(X(t))} \sigma[v] n \, d\mathcal{H}^2, \\ I_p \dot{\Omega} &= \int_{\partial B_R(X(t))} (x - X(t)) \times (\sigma[v] n) \, d\mathcal{H}^2. \end{aligned}$$

Here, $v(t, x)$ and $p(t, x)$ are the fluid velocity and pressure at time t and position x , and $\sigma[v] := \mu(\nabla v + (\nabla v)^T) - p \operatorname{Id}$ denotes the fluid stress. Moreover, $g \in \mathbb{R}^3$ is the gravitational acceleration, X, V , and Ω denote the position, the velocity, and the angular velocity of the particle respectively, and m_p and I_p are the mass and the moment of inertia of the particle. Furthermore, n denotes the outer unit normal and \mathcal{H}^2 the two-dimensional Hausdorff measure.

Stokes simplified this problem by making the assumption that the Reynolds number

$$\operatorname{Re} = \frac{\rho_f L U}{\mu}$$

is small compared to unity, where U is the ‘typical’ value of $|v|$, and L is the ‘typical’ length-scale over which v changes. If $\operatorname{Re} \ll 1$, one might justify (at least formally) to drop the convective/inertial term $\rho_f(\partial_t v + (v \cdot \nabla)v)$ yielding the Stokes equations instead of the Navier-Stokes equations. Then, neglecting also the rotation of the particle (which can actually be shown to converge to zero as $t \rightarrow \infty$ for a spherical particle) one can determine the terminal velocity V_{St} of the particle, by solving the stationary problem

$$\begin{aligned} -\mu \Delta v + \nabla p &= \rho_f g, \quad \operatorname{div} v = 0 \quad \text{in } \mathbb{R}^3 \setminus B_R(0), \quad v(x) \rightarrow 0 \quad \text{as } |x| \rightarrow \infty, \\ v &= V_{\text{St}} \quad \text{in } B_R(0), \quad m_p g = - \int_{\partial B_R(0)} \sigma[v] n \, d\mathcal{H}^2. \end{aligned}$$

Note the special type of boundary conditions that are imposed here: The total force that the fluid acts on the particle is prescribed to equal $m_p g$. On the other hand, the fluid velocity v needs to be constant in $B_R(0)$, but this constant, which we denote by V_{St} , is not given but part of the problem. These boundary conditions are sometimes referred to as boundary conditions of the fourth type, or, more intuitively for our purpose, as sedimentation boundary conditions.

After changing the pressure by defining $p(x) =: q(x) + \rho_f g \cdot x$, and using $m_p = (4\pi/3)R^3 \rho_p$ this problem becomes

$$\begin{aligned} -\mu \Delta v + \nabla q &= 0, \quad \operatorname{div} v = 0 \quad \text{in } \mathbb{R}^3 \setminus B_R(0), \quad v(x) \rightarrow 0 \quad \text{as } |x| \rightarrow \infty, \\ v &= V_{\text{St}} \quad \text{in } B_R(0), \quad F := \frac{4\pi}{3} R^3 (\rho_p - \rho_f) g = - \int_{\partial B_R(0)} \sigma[v] n \, d\mathcal{H}^2, \end{aligned} \tag{2.2.1}$$

where now (abusing the notation) $\sigma[v] := \mu(\nabla v + (\nabla v)^T) - q \operatorname{Id}$. Stokes [Sto51] explicitly solved this problem (by using the symmetry of the problem and introducing the so called stream function) and

computed

$$V_{\text{St}} = \frac{F}{6\pi\mu R} = \frac{2(\rho_p - \rho_f)R^2 g}{9\mu}. \quad (2.2.2)$$

Moreover, the explicit computation reveals that the force $F = 6\pi\mu R V_{\text{St}}$, which the particle is dragging the fluid with, is uniformly distributed on the boundary of the spherical particle. In particular, there is no torque exerted on the particle (as one can also deduce from a simple symmetry argument). This is different for non-spherical particles as we will discuss in the next subsection. Since the force F is uniformly distributed on the boundary of the particle, u solves the problem

$$-\mu\Delta v + \nabla q = F\delta_{\partial B_R(0)}, \quad \operatorname{div} v = 0 \quad \text{in } \mathbb{R}^3, \quad v(x) \rightarrow 0 \quad \text{as } |x| \rightarrow \infty, \quad (2.2.3)$$

where $\delta_{\partial B_R(0)}$ denotes the normalized uniform measure on the sphere, i.e.,

$$\delta_{\partial B_R(0)} := \frac{\mathcal{H}^2|_{\partial B_R(0)}}{\mathcal{H}^2(\partial B_R(0))}. \quad (2.2.4)$$

We will now check a posteriori the assumption $\operatorname{Re} \ll 1$: As typical length-scale and typical velocity we identify R and $|V_{\text{St}}|$. Thus,

$$\operatorname{Re} = \frac{\rho_f R |V_{\text{St}}|}{\mu} = \frac{2\rho_f(\rho_p - \rho_f)R^3 |g|}{9\mu^2}. \quad (2.2.5)$$

To get a feeling for the regime $\operatorname{Re} \ll 1$, we insert some typical values for ρ_f, ρ_p, μ and g . We consider water with $\rho_f \approx 10^3 \text{ kg m}^{-3}$ and $\mu \approx 1 \text{ kg m}^{-1} \text{ s}^{-1}$, and a grain of sand with $\rho_p \approx 2 \times 10^3 \text{ kg m}^{-3}$. Moreover, $g \approx 10 \text{ m s}^{-2}$. Inserting into (2.2.5) yields

$$\operatorname{Re} \approx \frac{2}{9} 10^7 R^3 \text{ m}^{-3}.$$

Hence, $\operatorname{Re} \ll 1$ if $R \ll 1 \text{ cm}$. Thus, for a typical sand grain sedimenting in water, $\operatorname{Re} \ll 1$ holds.

We can also consider the instationary problem when the particle initially has a velocity V_0 different from V_{St} . We again neglect the rotation of the particle. (If the spherical particle was not rotating initially, it will also not rotate at any later time since the torque vanishes as we observed above, and if it was rotating initially, the rotation will decay exponentially in time.) Solving only for the velocity $V(t)$ we use translational invariance to reduce the problem to

$$\begin{aligned} -\mu\Delta v + \nabla q &= 0 \quad \operatorname{div} u = 0 \quad \text{in } \mathbb{R}^3 \setminus B_R(0), \quad v(x) \rightarrow 0 \quad \text{as } |x| \rightarrow \infty, \\ v &= V(t) \quad \text{in } B_R(0), \quad \dot{V} = \frac{\rho_p - \rho_f}{\rho_p} g + \frac{3}{4\pi R^3 \rho_p} \int_{\partial B_R(0)} \sigma[v] n d\mathcal{H}^2. \end{aligned}$$

From (2.2.1) and (2.2.2), we already know that

$$\int_{\partial B_R(X(t))} \sigma[v] n d\mathcal{H}^2 = -6\pi R \mu V(t).$$

Thus,

$$\dot{V} = \frac{\rho_p - \rho_f}{\rho_p} g - \frac{9\mu}{2R^2 \rho_p} V(t),$$

which solution is given by

$$V(t) = V_{\text{St}} + \exp\left(-\frac{9\mu t}{2R^2\rho_p}\right) (V(0) - V_{\text{St}}).$$

In particular, the particle velocity adapts to the terminal settling velocity in times of order

$$T_t = \frac{2}{9} \frac{R^2 \rho_p}{\mu}. \quad (2.2.6)$$

On the other hand, the time in which the body, having attained its terminal velocity, falls down the distance given by its radius is

$$T = \frac{R}{|V_{\text{St}}|} = \frac{9}{2} \frac{\mu}{(\rho_p - \rho_f)R|g|}.$$

Comparing those times, this yields for the so called Stokes number

$$\text{St} := \frac{T_t}{T} = \left(\frac{2}{9}\right)^2 \frac{(\rho_p - \rho_f)\rho_p R^3 |g|}{\mu^2}.$$

The Stokes number St determines the importance of the particle inertia. Comparing the Stokes number St to the Reynolds number Re in (2.2.5), we see that their values are quite similar in this situation, if the fluid density ρ_f and the particle density ρ_p are of the same order. Therefore, (with the possible exception of dilute gases when $\rho_f \ll \rho_p$) if the fluid inertia is negligible for the settling of a single body, so is the particle inertia.

In the above computation, we completely ignored the phenomenon that a particle suspended in a fluid undergoes Brownian diffusion due to random collision with fluid molecules. Throughout this thesis, we restrict our study to non-Brownian particles. However, in order to identify the regime, where Brownian motion is negligible, we will briefly address this issue in the next paragraphs. For a more comprehensive introduction to the diffusive motion of particles in fluids, we refer the reader to [Dho96].

The diffusion constant for a particle is given by the Einstein relation

$$D = k_B \Theta M^{-1} \quad (2.2.7)$$

where Θ is the absolute temperature, k_B the Boltzmann constant, and M^{-1} is the mobility, the inverse of the resistance to translations of the particle. More precisely, if the particle translates with velocity V in an infinite fluid at rest, then the force the particle exerts on the fluid is given by $F = MV$. For a spherical particle with radius R , equation (2.2.2) implies $M = 6\pi R\mu$. (For a particle of arbitrary shape, M will generally be a matrix (see Section 2.2.2).)

The variance of the particle's position X after time t due to Brownian motion is given by¹

$$\mathbb{E}(|X|^2) = 6Dt.$$

Thus, the typical distance the particle travels due to diffusion in time t is given by $\sqrt{|D|t}$. In order to find out, whether Brownian motion may be neglected, we need to compare this distance to the distance the particle travels neglecting the Brownian motion. Since this distance depends linearly on time in contrast to the square-root dependence of the diffusion distance, the result will depend on the considered time-scale. The typical time is related to the typical length L by $T = L/U$, where U

¹For a general particle, both M and D are matrices, $6D$ in this identity has to be replaced $2\text{tr } D$

is the absolute value of the velocity of the particle. Hence, the relevant quantity for the importance of Brownian motion is the so-called Péclet number

$$\left(\frac{UT}{\sqrt{DT}}\right)^2 = \frac{UL}{|D|} =: \text{Pe}. \quad (2.2.8)$$

For the problem of a sedimenting sphere in infinite fluid, one might choose as the typical length the radius of the particle R . Then, since the settling velocity is given by the Stokes velocity (2.2.2), we have

$$\text{Pe} = \frac{|V_{\text{St}}|R}{D} = \frac{2}{9} \frac{(\rho_p - \rho_f)R^3 g}{\mu} \frac{6\pi R\mu}{k_B\Theta} = \frac{4}{3}\pi \frac{(\rho_p - \rho_f)R^4 g}{k_B\Theta} \quad (2.2.9)$$

If we consider again a grain of sand in water with the same specifications as above and at temperature $\Theta \approx 300 \text{ K}$, we find

$$\text{Pe} \approx 10^{25} R^4 \text{ m}^4.$$

Consequently, to have a large Péclet number, and thus negligible Brownian motion, the sand grain needs to have a radius of at least 10^{-6} m .

We emphasize that for a sedimenting cloud of particles, the value of both the Reynolds number and the Péclet number might change, since both the particle velocity and the relevant length-scale might differ. We will address this question in Section 2.3.5.

2.2.2 The settling of a single particle of arbitrary shape

The trajectory of a settling body of arbitrary shape is in general more complicated than for a spherical particle. Due to the lack of symmetry, the solution cannot be computed explicitly not even in the small Reynolds number regime when the Navier-Stokes equations are replaced by the Stokes equations. In this case, however, due to the linearity of the Stokes equations, there is a linear dependence between the force F and the torque T exerted on the particle on the one hand, and the velocity V and the angular velocity Ω of the particle on the other hand. More precisely, let us consider a particle occupying a closed, bounded (and sufficiently regular) set $K \subset \mathbb{R}^3$ with center of mass at the origin. Then, for any $V \in \mathbb{R}^3$ and $\Omega \in \mathbb{R}^3$ there exists a unique solution to the problem

$$\begin{aligned} -\mu\Delta v + \nabla p &= 0, \quad \text{div } v = 0 \quad \text{in } \mathbb{R}^3 \setminus K, \\ v &= V + \Omega \times x \quad \text{in } K, \quad v(x) \rightarrow 0 \quad \text{as } |x| \rightarrow \infty, \end{aligned}$$

and we find the force and torque acting on the fluid by

$$F := \int_{\partial K} \sigma[v]n \, d\mathcal{H}^2, \quad \mathcal{T} := \int_{\partial K} x \times (\sigma[v]n) \, d\mathcal{H}^2.$$

Then, there exists a matrix $M \in \mathbb{R}^{6 \times 6}$ such that for all $V \in \mathbb{R}^3$ and $\Omega \in \mathbb{R}^3$

$$(F, \mathcal{T}) = -M(V, \Omega),$$

where $(F, \mathcal{T}) \in \mathbb{R}^6$ denotes the vector composed of F and \mathcal{T} . For a spherical particle studied in the previous subsection, M is a diagonal matrix, and, by isotropy, the first three diagonal entries of M are identical as are the last three. In general, however, this is not true, meaning, that a purely translating object might experience a force which is not parallel to its velocity, and might even experience a torque. In particular, a sedimenting particle starting from a resting position will in general not move parallel to the direction of gravity and it will start to rotate.

The matrix M is called the resistance matrix², and it can be shown to be symmetric. If the particle possesses symmetries, the complexity of the resistance matrix M reduces. In particular, if the particle is symmetric with respect to all three coordinate axes, M is a diagonal matrix. If, in addition, the particle is “spherically isotropic”, i.e., it is symmetric under any exchange of the coordinate axes, then, the first three diagonal entries of M are identical as are the last three. Examples of a spherical isotropic particle are spheres and cubes.

2.2.3 The settling of a single rod-like particle

In many applications involving sedimentation, the particles are very elongated in one direction in comparison to the other two directions. We call such particles rod-like particles, or simply rods.

We consider a rod which is symmetric with respect to all three coordinate axes and which is oriented with its elongated part in the direction of the x_3 -coordinate. For definiteness, we may think of a particle given by

$$K := \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_3 \in [-l, l], |(x_1, x_2)| \leq r(|x_3|)\}, \quad (2.2.10)$$

with $r: [0, l] \rightarrow \mathbb{R}_+$ and $r_0 := \sup_{s \in [0, l]} r(s) \ll l$. As discussed in the previous subsection, the resistance matrix M is diagonal. In particular, a purely translational motion of the rod will not cause any torque on it. Our intuition tells us that the resistance of the rod to any movement perpendicular to its orientation is larger than its resistance to moving parallel to its orientation.

For a rod-like particle, one can approximately calculate the entries of the resistance matrix using a so called slender body theory. This approach goes back to Burgers [Bur38] and was later used by Batchelor [Bat70] to calculate the resistance for slender bodies of the form (2.2.10). Although these are formal computations, they can probably rigorously justified without much difficulty. As a result, (under the assumption that r does not in a sense vary too much over $[0, l]$) with $\varepsilon = \log(2l/r_0)^{-1}$

$$M_{11} = M_{22} = 8\pi\mu l\varepsilon(1 + O(\varepsilon)), \quad M_{33} = 4\pi\mu l\varepsilon(1 + O(\varepsilon)). \quad (2.2.11)$$

Strikingly, the force caused by translations perpendicular to the orientation of the rod is twice as high as for translations parallel to the rod, independently of the exact shape of the rod. Hence, the terminal settling velocity of a rod subject to a constant force F depends on its orientation $\xi \in S^2$ and is approximately given by

$$V = (\text{Id} + \xi \otimes \xi) \frac{F}{8\pi\mu l\varepsilon}. \quad (2.2.12)$$

The slender body theory also yields the resistance of the particle to rotations perpendicular to its orientation as

$$M_{44} = M_{55} = \frac{8\pi}{3}\mu l^3\varepsilon(1 + O(\varepsilon)). \quad (2.2.13)$$

2.3 Formal derivation of macroscopic sedimentation models

We consider a cloud of N spherical particles located at $(\bar{X}_i)_{1 \leq i \leq N}$ with radii $(\bar{R}_i)_{1 \leq i \leq N}$ and velocities $(\bar{V}_i)_{1 \leq i \leq N}$. The fluid surrounding the particles is assumed to satisfy the Navier-Stokes

²Some authors only call the upper left 3×3 block of M the resistance matrix.

equations with no-slip boundary conditions at the particles, neglecting particle rotations. Thus, analogously to Section 2.2, we study the problem

$$\rho_f(\partial_t \bar{v} + (\bar{v} \cdot \nabla) \bar{v}) - \mu \Delta \bar{v} + \nabla p = 0, \quad \operatorname{div} \bar{v} = 0 \quad \text{in } \mathbb{R}^3 \setminus \bigcup_{i=1}^N \bar{B}_i, \quad (2.3.1)$$

$$\bar{v} = \bar{V}_i \quad \text{in } \bar{B}_i, \quad \bar{v}(x) \rightarrow 0 \quad \text{as } |x| \rightarrow \infty, \quad (2.3.2)$$

$$\dot{\bar{X}}_i = \bar{V}_i, \quad \rho_p \frac{4\pi}{3} \bar{R}_i^3 \dot{\bar{V}}_i = \frac{4\pi}{3} \bar{R}_i^3 (\rho_p - \rho_l) g + \int_{\partial B_i} \bar{\sigma}[\bar{v}] n d\mathcal{H}^2, \quad (2.3.3)$$

Here, $B_i := B_{\bar{R}_i}(\bar{X}_i)$, and $\bar{\sigma}[\bar{v}] = \mu(\nabla \bar{v} + (\nabla \bar{v})^T) - p \operatorname{Id}$ is the fluid stress. Notice that we have already absorbed the gravitational force acting on the fluid into the pressure as explained in Section 2.2.1. Therefore, we separately added the buoyancy to the force acting on particle by the fluid.

In the following subsections, we formally derive macroscopic equations for these dynamics in different regimes leading to the transport-Stokes equations (1.1.1) and the Vlasov-Stokes equation (1.1.2). The derivation of the system (1.1.3) modeling the sedimentation of rod-like particles will be given in Section 2.3.4. We will always consider the regime, where the interaction between the particles through the fluid is important. We will always assume that the particles do not collide. If there are too many collisions (or close pairs of particles), the macroscopic dynamics is expected to change. It can be hoped that this assumption is satisfied, if the volume fraction ϕ of the particles is small. In Section 2.3.5, we will discuss the physical regimes in which these macroscopic equations can be expected to be valid.

In contrast to the rigorous results of this thesis outlined in Chapter 1, we allow the particles to have different radii, i.e., we consider a polydisperse distribution of particles. On a formal level, in particular ignoring possible particle collisions, this does not cause any difficulties, and we therefore include this aspect for the sake of a more general picture. We have argued in Chapter 1.5, why polydispersity causes problems for the rigorous derivation of these models.

On the other hand, we do not include the rotation of particles for the sake of simplicity of the computations. The rotations do not affect the macroscopic equations (see [Mec18] in the case of inertialess particles).

Formal derivation of variants of the models considered here have also been formally derived in the physics literature. The transport-Stokes equations (1.1.1) for the sedimentation of inertialess particles are for instance considered in [Feu84]. The Vlasov-Stokes equations are formally derived in [Koc90], where also the effect of particle collisions is considered. In [DE88], various models for suspensions of rod-like particles are considered, including the study of elastic rods and the Brownian motion.

2.3.1 Inertialess particles in Stokes flows

If we neglect both the inertia of the particle and the inertia of the fluid, problem (2.3.1) – (2.3.3) becomes

$$-\mu \Delta \bar{v} + \nabla p = 0, \quad \operatorname{div} \bar{v} = 0 \quad \text{in } \mathbb{R}^3 \setminus \bigcup_{i=1}^N \bar{B}_i, \quad (2.3.4)$$

$$\bar{v} = \bar{V}_i \quad \text{in } \bar{B}_i, \quad \bar{v}(x) \rightarrow 0 \quad \text{as } |x| \rightarrow \infty,$$

$$\dot{\bar{X}}_i = \bar{V}_i, \quad \bar{F}_i := \frac{4\pi}{3} \bar{R}_i^3 (\rho_p - \rho_f) g = - \int_{\partial B_i} \bar{\sigma}[\bar{v}] n d\mathcal{H}^2. \quad (2.3.5)$$

We emphasize that the boundary conditions imposed on the fluid are the sedimentation boundary conditions introduced in Section 2.2.1. The second equation in (2.3.3), which determines the

acceleration of the particles, has turned into the second equation in (2.3.5), which is a constraint on the fluid flow. On the other hand, the (extended) fluid velocity \bar{u} only has to be constant in all the particles B_i and these constants, denoted by \bar{V}_i , are then determined by the constraints coming from the second equation in (2.3.5).

Clearly, the fluid flow satisfies an equation of the form

$$-\mu\Delta\bar{v} + \nabla p = \sum_{i=1}^N f_i \quad \text{div } \bar{v} = 0 \quad \text{in } \mathbb{R}^3, \quad \bar{v}(x) \rightarrow 0 \quad \text{as } |x| \rightarrow \infty,$$

where the force distributions f_i are supported on ∂B_i and satisfy $\int_{\partial B_i} f_i = \bar{F}_i$. If the particles are very small, one could expect that for each particle velocity $\bar{V}_i = \bar{v}(X_i)$ it does not matter how exactly the forces f_j are distributed on ∂B_j for $j \neq i$. Moreover, we already know from (2.2.3) that for a single sedimenting sphere, the distribution of the force \bar{F}_i is uniform on ∂B_i . Therefore, a good approximation of \bar{v} should be the solution to the problem

$$-\mu\Delta\bar{u} + \nabla p = \sum_{i=1}^N \bar{F}_i \delta_{\partial B_i} \quad \text{div } \bar{u} = 0 \quad \text{in } \mathbb{R}^3, \quad \bar{u}(x) \rightarrow 0 \quad \text{as } |x| \rightarrow \infty,$$

where $\delta_{\partial B_i}$ is defined as in (2.2.4). (Note that we still denote the pressure by p although it is different from the one in (2.3.4). We will repeatedly abuse the notation for the pressure in this way. No confusion will arise from this abuse of notation since the pressure will never appear without the corresponding velocity field.)

Using the linearity of the Stokes equations and the solution for a single moving sphere, one can write down \bar{u} explicitly as

$$\bar{u}(x) = \sum_{i=1}^N w_{\bar{R}_i}(x - \bar{X}_i),$$

where $w_{\bar{R}_i}$ denotes the solution to the problem of a single sphere of radius \bar{R}_i centered at the origin with a force \bar{F}_i (which only depends on the radius of the particle \bar{R}_i) acting on it. In particular, by (2.2.2),

$$\bar{V}(X_i) \approx \bar{u}(\bar{X}_i) = \frac{\bar{F}_i}{6\pi\mu\bar{R}_i} + \sum_{j \neq i} w_{\bar{R}_i}(\bar{X}_i - \bar{X}_j). \quad (2.3.6)$$

The functions $w_{\bar{R}_i}$ decay like $1/|x|$. Moreover, since all the forces \bar{F}_i point in the direction of g ,

$$w_{\bar{R}_i}(x - \bar{X}_i) \cdot g \sim \frac{|\bar{F}_i|}{\mu|x - \bar{X}_i|}$$

for $|x - \bar{X}_i| \geq \bar{R}_i$. In particular, there are no cancellation effects in the sum on the right-hand side of (2.3.6) in the direction of g . Let us assume that the particles are distributed in a region Ω of diameter L and have typical radii \bar{R} . Then, the order of magnitude of the contribution to $V(X_i)$ by the collective effect due to the sum on the right-hand side of (2.3.6) is given by

$$\left| \sum_{j \neq i} w_{\bar{R}_i}(X_i - X_j) \right| \sim \frac{N|\bar{F}_{\bar{R}}|}{\mu L^3} \int_{\Omega} \frac{1}{|\bar{X}_i - x|} dx \sim \frac{N|\bar{F}_{\bar{R}}|}{\mu L},$$

where $\bar{F}_{\bar{R}} = \frac{4\pi}{3}\bar{R}_i^3(\rho_p - \rho_l)g$. Comparing this collective effect, with the settling velocity V_{St} of an isolated sphere, which is given by the first term on the right hand side of (2.3.6), yields (assuming $\bar{F}_i \sim \bar{F}_{\bar{R}}$)

$$\left| \sum_{j \neq i} w_{\bar{R}_i}(X_i - X_j) \right| \sim \frac{N\bar{R}}{L}|V_{\text{St}}| =: \gamma|V_{\text{St}}| =: \frac{9}{2}U_c.$$

Thus, for small values of γ , we expect all the particles to behave similar to isolated spheres (this has been rigorously proved in [JO04] for monodisperse suspensions). For γ of order one or larger, we expect the interactions between the particles to become important. Therefore, we focus on the latter case in the following.

We now non-dimensionalize the dynamics. We define

$$\begin{aligned} T &:= \frac{L}{U_c}, \\ v(s, y) &:= \frac{\bar{v}(Ts, Ly)}{U_c}, \quad u(s, y) := \frac{\bar{u}(Ts, Ly)}{U_c}, \\ V_i(s) &:= \frac{\bar{V}_i(Ts)}{U_c}, \quad X_i(s) := \frac{\bar{X}_i(Ts)}{L}, \\ R_i &:= \frac{\bar{R}_i}{L}, \quad R := \frac{\bar{R}}{L}, \quad r_i := \frac{R_i}{R}, \end{aligned} \tag{2.3.7}$$

and we redefine $B_i := B_{R_i}(X_i)$. Then,

$$\begin{aligned} -\Delta v + \nabla p &= 0, \quad \text{div } v = 0 \quad \text{in } \mathbb{R}^3 \setminus \bigcup_{i=1}^N \bar{B}_i, \\ v &= V_i \quad \text{in } \bar{B}_i, \quad v(x) \rightarrow 0 \quad \text{as } |x| \rightarrow \infty, \\ \dot{X}_i &= V_i, \quad F_i := \frac{4\pi}{3N}r_i^3\hat{g} = - \int_{\partial B_i} \sigma[v]n \, d\mathcal{H}^2, \end{aligned}$$

where $\hat{g} = g/|g|$ and $\sigma[v] = (\nabla v + (\nabla v)^T) - p\text{Id}$ are the dimensionless gravity and fluid stress respectively. Analogously, u is the solution to the problem

$$-\Delta u + \nabla p = \frac{4\pi}{3N} \sum_{i=1}^N r_i^3 \delta_{\partial B_i} \hat{g}, \quad \text{div } u = 0 \quad \text{in } \mathbb{R}^3, \quad u(x) \rightarrow 0 \quad \text{as } |x| \rightarrow \infty.$$

We introduce the particle density

$$\rho_N(t, x, r) := \frac{4\pi}{3N} \sum_{i=1}^N \delta_{\partial B_i}(x) \delta_{r_i}(r).$$

Here $\delta_{\partial B_i}$ is defined as in (2.2.4), which means that the spacial support of ρ consists of the boundaries of the particles. This is convenient since, by this definition,

$$-\Delta u + \nabla p = \int_0^\infty r^3 \rho_N \hat{g} \, dr, \quad \text{div } u = 0 \quad \text{in } \mathbb{R}^3, \quad u(x) \rightarrow 0 \quad \text{as } |x| \rightarrow \infty.$$

Notice that ρ_N satisfies the transport equation

$$\partial_t \rho + v \cdot \nabla \rho = 0.$$

We assume that in the limit $N \rightarrow \infty$ with $R \rightarrow 0$ and $\gamma \rightarrow \gamma_* \in (0, \infty]$, we have $\rho_N \rightarrow \rho$ in a suitable (weak) sense. Then, we formally deduce $u \rightarrow v_*$, and since $v \approx u$ also $v \rightarrow v_*$, which solves

$$-\Delta v_* + \nabla p = \int_0^\infty r^3 \rho \hat{g}, \quad \operatorname{div} v_* = 0 \quad \text{in } \mathbb{R}^3, \quad v_*(x) \rightarrow 0 \quad \text{as } |x| \rightarrow \infty. \quad (2.3.8)$$

However, reconsidering equation (2.3.6), v_* will only contain the information of the second term on the right hand side of that equation, which is the collective term. Therefore, we expect that ρ solves

$$\partial_t \rho + (v_* + \frac{2}{9} \gamma_*^{-1} r_i^2 \hat{g}) \cdot \nabla \rho = 0, \quad (2.3.9)$$

where the term $\frac{2}{9} \gamma_*^{-1} r_i^2 \hat{g}$ exactly corresponds to the settling velocity of an isolated particle. The coupled system (2.3.9), (2.3.8) is the polydisperse version of the system (1.1.1).

2.3.2 The Brinkman equations

In this and the next subsection, we revisit the system (2.3.1), (2.3.2), (2.3.3). In contrast to the previous subsection, we keep the inertia of the particles in order to derive the Vlasov-Stokes equations (1.1.2). Moreover, in contrast to the rest of this thesis, we also allow for nonzero Reynolds numbers Re . Formally the fluid inertia does not cause any difficulties in the derivation of the macroscopic dynamics.

Using the dimensionless quantities defined in (2.3.7), the full dynamics including inertial effects given by (2.3.1), (2.3.2), (2.3.3) becomes

$$\operatorname{Re}(\partial_t u + (u \cdot \nabla)u) - \Delta u + \nabla p = 0, \quad \operatorname{div} u = 0 \quad \text{in } \mathbb{R}^3 \setminus \bigcup_{i=1}^N \overline{B_i}, \quad (2.3.10)$$

$$u = V_i \quad \text{in } \overline{B_i}, \quad u(x) \rightarrow 0 \quad \text{as } |x| \rightarrow \infty. \quad (2.3.11)$$

$$\dot{X}_i = V_i, \quad \dot{V}_i = \lambda \left(g + \frac{3}{4\pi} \frac{N}{r_i^3} \int_{\partial B_i} \sigma[u] n d\mathcal{H}^2 \right), \quad (2.3.12)$$

where

$$\operatorname{Re} = \frac{N \bar{R}^3 \rho_f (\rho_p - \rho_f) |g|}{\mu^2}, \quad \lambda = \frac{\mu^2 L^3}{\rho_p (\rho_p - \rho_f) N^2 \bar{R}^6 |g|}.$$

We will first consider only the system (2.3.10), (2.3.11) with the particle positions and velocities held fixed and derive the Brinkman equations. To this end, we notice that on the length-scale of a single particle, the relevant Reynolds number is

$$\operatorname{Re}_p = \frac{\rho_f U_c \bar{R}}{\mu} = R \operatorname{Re}.$$

Thus, even if Re is not small, we might assume that Re_p is small and therefore approximate the Navier-Stokes equations by the Stokes equations near each particle. We assume that there exists a macroscopic fluid velocity v_* close to u in an averaged sense (and thus in the weak topology). We expect v_* to be close to u away from the particles. Consider the ball $B_d(X_i)$, where d is chosen such that $R \ll d \ll L$ and $\operatorname{dist}(B_d(X_i), B_j) \gg R$ for all $j \neq i$. Then, in $B_d(X_i)$, we expect u to be approximately given as the solution to

$$\begin{aligned} -\Delta u + \nabla p &= 0, \quad \operatorname{div} u = 0 \quad \text{in } B_d(X_i) \setminus \overline{B_i}, \\ u &= v_* \quad \text{on } \partial B_d(X_i), \quad u = V_i \quad \text{on } \partial B_i. \end{aligned}$$

Since v_* changes on length-scales of order L , we can replace the value v_* on $\partial B_d(X_i)$ by $v_*(X_i)$. Then, the solution to the above Dirichlet problem can be computed explicitly, and since $R \ll d$ it is very close to the solution to

$$\begin{aligned} -\Delta u + \nabla p &= 0, \quad \operatorname{div} u = 0 \quad \text{in } \mathbb{R}^3 \setminus \overline{B_i}, \\ u(x) &\rightarrow v_*(X_i) \quad \text{as } |x| \rightarrow \infty, \quad u = V_i \quad \text{on } \partial B_i. \end{aligned}$$

This is the problem of a translating sphere in an infinite fluid. Thus, (pretending that all the approximations for u are exact) we know from (2.2.2) and (2.2.3) that

$$\int_{\partial B_i} \sigma[u] n d\mathcal{H}^2 = 6\pi R_i(v_*(X_i) - V_i), \quad (2.3.13)$$

and

$$-\Delta u + \nabla p = 6\pi R_i(v_*(X_i) - V_i)\delta_{\partial B_i}, \quad \operatorname{div} u = 0 \quad \text{in } B_d(0).$$

Thus, we formally deduce

$$\operatorname{Re}(\partial_t u + (u \cdot \nabla)u) - \Delta u + \nabla p = \sum_{i=1}^N 6\pi R_i(v_i - v_*)\delta_{\partial B_i}, \quad \operatorname{div} u = 0 \quad \text{in } B_d(0).$$

We assume that the empirical particle density $f_N(x, v, r) = \sum_{i=1}^N \delta_{X_i}(x)\delta_{V_i}(v)\delta_{r_i}(r)$ (weakly) converges to some f in the limit $N \rightarrow \infty$ with $\operatorname{Re} \rightarrow 0$ such that $NR \rightarrow \gamma_* \in (0, \infty)$, $\operatorname{Re} \rightarrow \operatorname{Re}_* \in [0, \infty)$. Then, taking formally the homogenization limit, yields for $u \rightarrow v_*$

$$\begin{aligned} \operatorname{Re}_*(\partial_t v_* + (v_* \cdot \nabla)v_*) - \Delta v_* + \nabla p &= 6\pi\gamma_* \int_0^\infty \int_{\mathbb{R}^3} r(v - v_*)f dv dr \quad \text{in } \mathbb{R}^3, \\ \operatorname{div} v_* &= 0 \quad \text{in } \mathbb{R}^3, \quad v_*(x) \rightarrow 0 \quad \text{as } |x| \rightarrow \infty. \end{aligned}$$

Clearly, in the case of a monodisperse distribution $f(x, v, r) = h(x, v)\delta_1(r)$, and $\operatorname{Re}_* = 0$, these are the Brinkman equations which appear in the Vlasov-Stokes equations (1.1.2).

2.3.3 The Vlasov-Navier-Stokes equations

We now consider again the full dynamics (2.3.10) - (2.3.12). Having derived the Brinkman equation for the fluid flow in the previous subsection, we only have to take the limit in (2.3.12). We observe, that we can use (2.3.13) to simplify that equation yielding (approximately)

$$\dot{X}_i = V_i \quad \dot{V}_i = \lambda \left(g + \frac{9}{2} \frac{\gamma}{r_i^2} (v_*(X_i) - V_i) \right). \quad (2.3.14)$$

Thus, the particle density $f_N(x, v, r) = \sum_{i=1}^N \delta_{X_i}(x)\delta_{V_i}(v)\delta_{r_i}(r)$ satisfies (approximately)

$$\partial_t f_N + v \cdot \nabla_x f_N + \lambda \operatorname{div}_v \left(\left(\hat{g} + \frac{9}{2} \frac{\gamma}{r^2} (v_* - v) \right) f_N \right) = 0.$$

Taking the limit $N \rightarrow \infty$ assuming $\lambda \rightarrow \lambda_* \in (0, \infty)$, we thus deduce the polydisperse version of the so called Vlasov-Navier-Stokes equations

$$\begin{aligned} \partial_t f + v \cdot \nabla_x f + \lambda_* \operatorname{div}_v \left(\hat{g} f + \frac{9}{2} \frac{\gamma_*}{r^2} (v_* - v) f \right) &= 0, \\ \operatorname{Re}_*(\partial_t v_* + (v_* \cdot \nabla)v_*) - \Delta v_* + \nabla p &= 6\pi\gamma_* \int_0^\infty \int_{\mathbb{R}^3} r(v - v_*)f dv dr \quad \text{in } \mathbb{R}^3, \\ \operatorname{div} v_* &= 0 \quad \text{in } \mathbb{R}^3, \quad v_*(x) \rightarrow 0 \quad \text{as } |x| \rightarrow \infty. \end{aligned} \quad (2.3.15)$$

For $\operatorname{Re}_* = 0$ and monodisperse particle distribution, these equations turn into the Vlasov-Stokes equations (1.1.2) with $\operatorname{St} = (\gamma_* \lambda_*)^{-1}$.

2.3.4 Sedimentation of inertialess rod-like particles

Instead of spherical particles, we now investigate the sedimentation of a cloud of rods. For simplicity, we restrict ourselves to monodisperse suspensions of rods. As we have seen in Section 2.2.3, the settling velocity of rods depends on their orientation. Therefore, it is not longer possible to ignore particle rotations in the model. On the other hand, we have also seen that the exact shape of the rods does not matter too much as long as they are very elongated. We therefore consider particles of the form

$$K := \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1 \in [-\bar{l}, \bar{l}], |(x_2, x_3)| \leq \bar{R}(|x_1|)\},$$

where we assume $\bar{l} \gg \bar{R}$. Then, every particle is prescribed by the position of its center \bar{X}_i and its orientation $\bar{\Xi}_i \in S^2$. We denote the set occupied by the i -th particle by $K_i := \bar{X}_i + O(\bar{\Xi}_i)K$, where $O(\bar{\Xi}_i) \in SO(3)$ is chosen such that $O(\bar{\Xi}_i)e_1 = \bar{\Xi}_i$.

Then, our starting point for the sedimentation of rods are the inertialess microscopic dynamics

$$\begin{aligned} -\mu\Delta\bar{v} + \nabla p &= 0, \quad \operatorname{div} \bar{v} = 0 \quad \text{in } \mathbb{R}^3 \setminus \bigcup_{i=1}^N K_i, \\ \bar{v} &= \bar{V}_i + \Omega_i \times (x - \bar{X}_i) \quad \text{in } \bar{B}_i, \quad \bar{v}(x) \rightarrow 0 \quad \text{as } |x| \rightarrow \infty, \\ \dot{\bar{X}}_i &= V_i, \quad \bar{F} := \pi\bar{R}^2\bar{l}(\rho_p - \rho_f)g = - \int_{\partial K_i} \bar{\sigma}[v]n \, d\mathcal{H}^2, \\ \dot{\bar{\Xi}}_i &= P_{\bar{\Xi}_i^\perp} \Omega_i, \quad 0 = \int_{\partial K_i} (x - \bar{X}_i) \times (\sigma[v]n) \, d\mathcal{H}^2. \end{aligned} \tag{2.3.16}$$

Here $P_{\bar{\Xi}_i^\perp} = (\operatorname{Id} - \bar{\Xi}_i \otimes \bar{\Xi}_i)$ denotes the orthogonal projection to $\bar{\Xi}_i^\perp = T_{\bar{\Xi}_i}S^2$, the tangential space of S^2 at $\bar{\Xi}_i$. As for the problem (2.3.4) – (2.3.5), the constants V_i and Ω_i are determined by the equations for the fluid.

By the same heuristics as in Section 2.3.1, we approximate the fluid velocity by

$$\bar{u}(x) = \sum_{i=1}^N w_{\bar{\Xi}_i}(x - \bar{X}_i),$$

where $w_{\bar{\Xi}_i}$ is the solution to the problem of a single rod with orientation $\bar{\Xi}_i$ centered at the origin with a force \bar{F} acting on it. Note that this approximation of the fluid velocity does not account for the rotations of the particles. However, at least formally, the effect of the rotation of the particles on the fluid velocity is negligible as in the case of spherical particles. It is only important to take into account the rotations of the particles because of the direct effect of the orientation of each particle on its velocity.

Indeed, from (2.2.12), we have

$$\bar{V}_i \approx \bar{u}(\bar{X}_i) = (\operatorname{Id} + \bar{\Xi} \otimes \bar{\Xi}) \frac{\bar{F}}{8\pi\mu\bar{l}\varepsilon} + \sum_{j \neq i} w_{\bar{\Xi}_j}(\bar{X}_j - \bar{X}_i),$$

where $\varepsilon = \log^{-1}(2\bar{l}/\bar{R})$.

For the rotational motion, we observe that for all $0 < \delta < \bar{l}$

$$\bar{v}(\bar{X}_i + \delta\bar{\Xi}_i) = \bar{V}_i + \delta\Omega_i \times \bar{\Xi}_i = \bar{V}_i + \delta\dot{\bar{\Xi}}_i.$$

Thus,

$$\dot{\Xi}_i = P_{\Xi_i^\perp}(\Xi_i \cdot \nabla) \bar{v}(\bar{X}_i) \approx P_{\Xi_i^\perp}(\Xi_i \cdot \nabla) \bar{u}(\bar{X}_i) = P_{\Xi_i^\perp}(\Xi_i \cdot \nabla) \sum_{j \neq i} w_{\Xi_j}(\bar{X}_j - \bar{X}_i). \quad (2.3.17)$$

As in Section 2.3.1, we compare the order of magnitude of the self-interaction term to the collective. The self interactive term scales like

$$V_{\text{St}} := \frac{\bar{F}}{8\pi\mu\bar{\ell}\varepsilon} = \frac{\bar{R}^2(\rho_p - \rho_f)g}{8\varepsilon\mu}.$$

On the other hand, the collective term is of order

$$\left| \sum_{j \neq i} w_{\Xi_i}(X_i - X_j) \right| \sim \frac{N\bar{F}}{\mu L^3} \int_{\Omega} \frac{1}{|X_i - x|} dx \sim \frac{N\bar{\ell}\varepsilon}{L} |V_{\text{St}}| =: \gamma |V_{\text{St}}| =: 8U_c. \quad (2.3.18)$$

We rescale the dynamics with the typical length L and the velocity U_c analogously as in Section 2.3.1. Then, formally in the limit of $N \rightarrow \infty$, $R \rightarrow 0$ with $\gamma \rightarrow \gamma_* \in (0, \infty]$

$$\begin{aligned} \partial_t f + \left(u + \frac{1}{8\gamma_*^{-1}} (\text{Id} + \xi \otimes \xi) \hat{g} \right) \cdot \nabla_x f + \text{div}_\xi (P_{\xi^\perp}(\xi \cdot \nabla) u f) &= 0, \\ -\Delta u + \nabla p &= \int_{S^2} f \hat{g} d\xi, \quad \text{div } u = 0. \end{aligned} \quad (2.3.19)$$

It is interesting to notice that the rotations of particles are important only when $\gamma \sim 1$. Indeed, if the interactions are very strong ($\gamma \gg 1$), then the particles are transported by the fluid velocity u . Their orientation, although it does change in times of order one, does not have a significant influence on the sedimentation velocity. Hence, in this case, there is no difference between spherical and non-spherical particles. On the other hand, if the interactions are very weak ($\gamma \ll 1$), the velocity of the particles due to the self-interaction is much faster than their rotation. Hence, in the relevant time-scale, their orientation is fixed.

2.3.5 The regime of validity of these equations

For single particles it was argued in Section 2.2.1 that neglecting fluid and particle inertia as well as Brownian motion is justified for a wide range of relevant particle sizes – typically for $1 \mu\text{m} \ll R \ll 1 \text{cm}$ for solids sedimenting in liquids. We now address in which regimes those effects are negligible in the case of a sedimenting cloud of particles and thus in which regimes the macroscopic equations derived in the previous subsections can be expected to be appropriate to model the sedimentation dynamics. For clouds of particles, these regimes are in general different from the ones for isolated particles due to the possibility of much higher fluid and particle velocities caused by the interaction between the particles.

We recall that the interaction strength is determined by the parameter γ . For $\gamma \ll 1$, the particles do not interact significantly, but behave like isolated particles studied in Section 2.2. In some sense, this is the case of very dilute suspensions. This is not the case which we are interested in. We are interested in the regime $\gamma \sim 1$ or $\gamma \gg 1$, which will also be characterized below. On the other hand, we need to consider suspensions that are sufficiently dilute in the sense that the volume fraction of the particles ϕ is small. If this is not the case, one expects that collisions of particles or at least the effect of very close pairs of particles might become relevant. For positive volume fraction of the particles, there is also the effect of the change of viscosity of the fluid according to Einstein's law. We briefly comment on this phenomenon in Section 2.5.2.

The regime of negligible fluid inertia

This regime is characterized by small Reynolds numbers. If γ is of order one or large, a macroscopic fluid velocity arises from the collective effect of the particles. This fluid velocity is expected to change on the length scale on which the macroscopic particle density changes. We assume that this length scale is L . Thus, the Reynolds number is given as

$$\text{Re} = \frac{\rho_f L U_c}{\mu} = \frac{\rho_f(\rho_p - \rho_f) N \bar{R}^3 |g|}{\mu^2} = \frac{\rho_f(\rho_p - \rho_f) L^3 \phi |g|}{\mu^2}$$

where we introduced the volume fraction of the particles

$$\phi := \frac{N \bar{R}^3}{L^3} = N R^3.$$

We calculate the Reynolds number for an explicit example. Let us consider as in Section 2.2.1 the example of sand grains in water ($\rho_f \approx 10^3 \text{ kg m}^{-3}$, $\mu \approx 1 \text{ kg m}^{-1} \text{ s}^{-1}$, $\rho_p \approx 2 \times 10^3 \text{ kg m}^{-3}$, $g \approx 10 \text{ m s}^{-2}$). This yields

$$\text{Re} \approx 10^7 N \bar{R}^3 \text{ m}^{-3}.$$

Typical sand grains have a size of $\bar{R} \sim 10^{-3} \text{ m}$. Then, $\text{Re} \approx 10^{-2} N$. Thus, the assumption is violated as soon as $N \sim 10^2$. However, there are examples of much smaller particles of size $\bar{R} \sim 10^{-6} \text{ m}$ such as clay, which allows to consider clouds of significantly more particles ($N \sim 10^{10}$) with $\text{Re} \ll 1$.

Concerning the interaction strength $\gamma = N \bar{R} L^{-1}$, we see that in the case $\bar{R} \sim 10^{-6} \text{ m}$, $N \sim 10^{10}$, there is wide range of γ depending on the diameter of the cloud L . For $L \sim 10^4 \text{ m}$, we have $\gamma \sim 1$, and for smaller clouds with the particle number and radius held fixed, the interaction becomes dominant.

In order to have small volume fraction $\phi = N \bar{R}^3 L^{-3}$, the cloud should not become too small. Considering again $\bar{R} \sim 10^{-6} \text{ m}$, $N \sim 10^{10}$, we find that the diameter L should not fall below about 10^{-2} m to fulfil this requirement.

The regime of negligible particle inertia

In the inertialess regime of the particles, the transport-Stokes equations (1.1.1) are expected to be valid. In order to identify this regime from the microscopic point of view we compute the so called Stokes number, which is the ratio between the characteristic time of the particle and the characteristic time of the fluid. The former is the relaxation time for a particle needed to adjust its velocity to the quasi-stationary, “inertialess” velocity, where gravity is balanced by the fluid drag force. The latter is the time over which the fluid velocity changes along the particle trajectory. From (2.3.14), we deduce, analogously as we have obtained (2.2.6), that the relaxation time for particle i is given by

$$T_p = \frac{2}{9} \frac{r_i^2}{\lambda \gamma} T \sim \frac{1}{\lambda \gamma} T.$$

where T is defined in (2.3.7) (thus, T_p is the relaxation time for the unrescaled dynamics). On the other hand, the characteristic time of the fluid is $T = L/U_c$ because the fluid velocity changes over the length-scale L and these changes are transported with the fluid velocity with typical value U_c . Thus, we find for the Stokes number

$$\text{St} \sim \frac{1}{\lambda \gamma} = \frac{\rho_p(\rho_p - \rho_f) |g| \phi R^2 L^3}{\mu^2}.$$

We compare the Stokes number with the Reynolds number

$$\frac{\text{St}}{\text{Re}} = \frac{\rho_p}{\rho_f} \frac{\bar{R}^2}{L^2} = \frac{\rho_p}{\rho_f} R^2 = . \quad (2.3.20)$$

Since we consider $R \ll 1$, we typically have $\text{St} \ll \text{Re}$. Thus, the inertia of the particles is typically negligible if the inertia of the fluid is negligible. In particular, the inertialess dynamics considered in Section 2.3.1 seem to have a wide range of validity.

The regime of negligible fluid inertia but significant particle inertia

In the regime of the Vlasov-Stokes equations (1.1.2) where the inertia of the particles count but the inertia of the fluid does not, (2.3.20) implies that we need to have $\rho_p \gg \rho_f$. Let us check, whether this regime is physically relevant. For very heavy particles like gold, we have $\rho_p \sim 2 \times 10^4 \text{ kg m}^{-3}$. On the other hand, all (common) liquids have a density of at least $\rho_f \sim 10^3 \text{ kg m}^{-3}$. Thus, whenever $\text{Re} \ll 1$ and $R \ll 1$ we have $\text{St} \ll 1$.

However, this regime might be physical relevant for (rarefied) gases: air at standard conditions has a density of $\rho_f \sim 1 \text{ kg m}^{-3}$ and the density can be reduced further by reducing the pressure.

Furthermore, if we consider the physical regime of the Vlasov-Navier-Stokes equations (2.3.15), i.e., we allow $\text{Re} \sim 1$ or even large, it is possible to have both relevant fluid and particle inertia.

The assumption of neglecting the Brownian motion of the particles

We recall from Section 2.2.1 that this is formally justified if the Péclet number Pe given by (2.2.8) is large compared to unity. Applying this to sedimentation of a cloud of particles, the typical distance and velocity are again given by L and $\gamma|V_{\text{St}}|$, respectively. Thus, recalling from (2.2.7) the diffusion constant, we find

$$\text{Pe} = \frac{\gamma|V_{\text{St}}|L}{D} = \frac{2}{9} \frac{(\rho_p - \rho_f)N\bar{R}^3|g|}{\mu} \frac{6\pi\bar{R}\mu}{k_B\Theta} = \frac{4}{3}\pi \frac{(\rho_p - \rho_f)N\bar{R}^4|g|}{k_B\Theta}.$$

Comparing with (2.2.9), this is just N times the value of the Péclet number of a single sedimenting sphere. Hence, on the considered length-scale of the cloud diameter, the Brownian motion is negligible even for smaller particles provided the number of particles is large.

The regime of validity of the macroscopic rod model

The effect of particle and fluid inertia for a cloud of rod-like particles is described very similarly as for spherical particles. However, when it comes to Brownian motion, there is an important additional aspect: rotational diffusion could also become relevant as it changes the orientation of the particles. This effect is described analogously to translational diffusion (see (2.2.7)) by the rotational diffusion constant

$$D_r = k_B\Theta M_r^{-1},$$

where M_r is the resistance of a particle to rotations. We only consider rotations that change the orientation of the particles. Then the resistance is given by (2.2.13), i.e.,

$$M_r \approx \frac{8\pi}{3} \mu \bar{l}^3 \varepsilon.$$

The variance in the orientation after time t satisfies

$$\mathbb{E}[\bar{\Xi}^2] \sim D_r t,$$

for $t \leq D_r$. In particular, the orientation changes due to Brownian motion in times of order

$$T_r = \frac{1}{D_r} = \frac{M_r}{k_B \Theta}. \quad (2.3.21)$$

On the other hand, the particle orientations change due to the gradient of the fluid velocity. By (2.3.17) and (2.3.18), we have

$$|\dot{\bar{\Xi}}| \sim \frac{U_c}{L}.$$

Thus the relevant time scale for this effect is described by the same typical time that is the time scale for the macroscopic system (2.3.19),

$$T = \frac{L}{U_c}. \quad (2.3.22)$$

Comparing those time scales given by (2.3.21) and (2.3.22), we find that the rotational diffusion is negligible if the “rotational Péclet number” Pe_r is large, where Pe_r is given by

$$\text{Pe}_r := \frac{T_r}{T} = \frac{U_c M_r}{L k_B \Theta} = \frac{\pi (\rho_p - \rho_f) g \varepsilon N \bar{R}^2 \bar{l}^4}{k_B \Theta L^2}.$$

We compare this quantity with the (translational) Péclet number, given analogously as for spherical particles by

$$\text{Pe} = \frac{U_c L}{D_t} = \frac{U_c L M_t}{k_B \Theta},$$

where we denoted the translational diffusion constant and resistance by D_t and M_t , respectively. The order of M_t is given by (2.2.11). In particular, we find

$$\text{Pe}_r = \frac{M_r}{M_t L^2} \text{Pe} \sim \frac{\bar{l}^2}{L^2} \text{Pe}.$$

Since $\bar{l} \ll L$, this implies that the rotational Péclet number is much smaller than the translational Péclet number.

We want to identify how small the rods can be such that Pe_r is still large. We impose one more constraint, namely that the particle suspension is very dilute in the sense that

$$\eta := \frac{N \bar{l}^3}{L^2} \ll 1. \quad (2.3.23)$$

This constraint is stronger than to require that the particle volume fraction is small. However, it is reasonable to make this restriction because one could expect that, if (2.3.23) is not satisfied, the rotation of close rods lead to additional interaction. We also introduce $A = \bar{l} \bar{R}^{-1}$, the aspect ratio of the rods, which needs to be large. Moreover, we consider fluid and particle densities such that $\rho_p - \rho_f \sim 10^3 \text{ kg m}^{-3}$, and standard conditions for the temperature and gravity. This yields for the rotational Péclet number

$$\text{Pe}_r \sim 10^{25} \frac{\varepsilon \eta L \bar{l}^3}{A^2} \text{ m}^{-4}.$$

To find the smallest possible value for \bar{l} , for which Pe_r is still large, we assume $\eta \sim 10^{-1}$, $A \sim 10$. We can then neglect $\varepsilon = (\log A)^{-1}$ and arrive at

$$\text{Pe}_r \sim 10^{22} L \bar{l}^3 \text{m}^{-4}.$$

We cannot choose L too large. Indeed, following the same computations as for spherical particles above, it follows that for the specified quantities, we have $\text{Re} \sim 10^4 L^3 \text{m}^{-3}$. Thus, we have to choose at least $L \sim 1 \text{cm}$. Thus, the rotational Péclet number can only be small for rods with at least $l \sim 1 \mu\text{m}$.

Since this length is quite small, we conclude that the macroscopic rod-model (2.3.19) has a wide range of applicability. On the other hand, the case of very small rods which are much more elongated than in the example considered above are very important, for example in the study of polymers. Thus it is reasonable also to study the macroscopic models for Brownian rods that are proposed in [DE88]. As mentioned in Section 1.7, there are also some mathematical results for these models [OT08; BT13; HO06].

2.4 Formal limits of the Vlasov-Navier-Stokes equations

There are three parameters involved in the Vlasov-Navier-Stokes equations (2.3.15): Re_* , $\text{St}_* = (\gamma_* \lambda_*)^{-1}$, and γ_* . We will drop the index $*$ in this section. Since we are only concerned with the macroscopic dynamics, no confusion will arise with the corresponding microscopic quantities considered in the previous section.

We briefly summarize the role of these parameters discussed in the previous section: The Reynolds number Re determines the strength of the fluid inertia, and the Stokes number determines the strength of the particle inertia. The parameter γ determines the interaction strength between the particles, or, more precisely, the ratio between the interaction strength of the particles and the “self-interaction” due to the direct influence of the gravity on each single particle. The fact that the parameter γ is involved both in the equations for the fluid velocity and for the particle density is just a consequence of Newton’s third law.

Various asymptotic limits of the Vlasov-Navier-Stokes equations (2.3.15) can be considered in the case of different combinations of limits of those parameters. In the next subsections we will formally consider three of these asymptotic limits:

First, the inertialess limit (with not too strong interactions) which is characterized by $\text{St} \rightarrow 0$ and $\text{St}\gamma \rightarrow 0$. In this case, we recover the transport-Stokes equations. This is the only one of the asymptotic limits of the Vlasov-Stokes equations which has been rigorously proved so far. This proof is the content of Chapter 7 of this thesis.

Second, the case of very large interactions, characterized by $\gamma \rightarrow 0$, $\text{St} \sim 1$. In this case, we formally obtain a Vlasov equation, which is coupled to a variant of the Darcy’s law.

The third limit that we consider is intermediate between the first two. It is characterized by small inertia and high interaction strength, $\text{St} \rightarrow 0$ and $\text{St}\gamma \sim 1$. As in the inertialess limit with not too strong interactions, the Vlasov equations turns into a transport equation. However, the particle inertia does not vanish from the system but reappears as a convective term in the fluid equations.

In the main part of this thesis where rigorous results are derived, we restrict our study to sedimentation of monodisperse particles in the regimes of zero Reynolds numbers. We recall that in

this case, the macroscopic dynamics is described by the system (1.1.2), which we repeat here:

$$\begin{aligned} St(\partial_t f + v \cdot \nabla_x f) + \operatorname{div}_v \left(\gamma^{-1} \hat{g} f + \frac{9}{2} (u - v) f \right) &= 0, \\ -\Delta u + \nabla p &= 6\pi\gamma \int_{\mathbb{R}^3} (v - u) f \, dv, \quad \operatorname{div} u = 0 \quad \text{in } \mathbb{R}^3, \quad u(x) \rightarrow 0 \quad \text{as } |x| \rightarrow \infty. \end{aligned} \quad (2.4.1)$$

For the ease of notation, also in this section, instead of the Vlasov-Navier-Stokes equations (2.3.15) for polydisperse particles, we consider this simplified model even though from the physical point of view the case of non-zero Reynolds number might be more relevant as discussed in Section 2.3.5. Formally however, it is formally straightforward to adapt to the case of non-vanishing Reynolds number and polydisperse particles.

The physical relevance of the considered limits can be studied similarly to Section 2.3.5, but we omit this discussion here. From the mathematical point of view, these limits appear to be very interesting even if they might only provide toy models for more complicated systems which are physically more relevant.

2.4.1 Inertialess limit with not too strong interactions: $St \rightarrow 0$ with $St\gamma \rightarrow 0$

In this limit, we will recover the inertialess system derived in Section 2.3.1: Since $\gamma^{-1} \gg St \rightarrow 0$ we can neglect the first term in the first equation of (2.4.1). Hence,

$$\operatorname{div}_v \left(\frac{2}{9} \gamma_*^{-1} \hat{g} f + (u - v) f \right) = 0,$$

where $\gamma_* \in [0, \infty]$ denotes the limit of γ . We will first assume $\gamma_* > 0$.

Multiplying by v and integrating yields with $\rho(t, x) = \int_{\mathbb{R}^3} f \, dv$

$$\int_{\mathbb{R}^3} (u - v) f \, dv = \frac{2}{9} \gamma_*^{-1} \rho \hat{g}. \quad (2.4.2)$$

Thus, the second equation in (2.4.1) becomes

$$-\Delta u + \nabla p = \frac{4\pi}{3} \rho \hat{g}, \quad \operatorname{div} u = 0 \quad \text{in } \mathbb{R}^3, \quad u(x) \rightarrow 0 \quad \text{as } |x| \rightarrow \infty.$$

Moreover, we can integrate the first equation in (2.4.1) over v , and use (2.4.2) and the fact that u is divergence free to get

$$\partial_t \rho + \left(\frac{2}{9} \gamma_*^{-1} \hat{g} + u \right) \cdot \nabla \rho = 0. \quad (2.4.3)$$

Up to the factor $4\pi/3$, this is exactly the system (2.3.8), (2.3.9) we obtained in Section 2.3.1. We can get rid of this factor by considering $\tilde{\rho} = (4\pi/3)\rho$.

If $\gamma_* = 0$, after changing the timescale $t' = \gamma^{-1}t$, one arrives at

$$\partial_{t'} \rho + \frac{2}{9} \hat{g} \cdot \nabla \rho = 0.$$

In this case, since the interactions are negligible, the particles just fall down like single inertialess particles.

2.4.2 Limit of strong interactions: $\gamma \rightarrow \infty$ with $\text{St} \sim 1$

In this limit, the first equation in (2.4.1) simply becomes

$$\text{St}(\partial_t f + v \cdot \nabla_x f) + \text{div}((u - v)f) = 0. \quad (2.4.4)$$

Furthermore, the Brinkman equations turn into Darcy's law,

$$\int f(u - v) dv + \nabla p = 0, \quad \text{div } u = 0. \quad (2.4.5)$$

This is a variant of Darcy's law, which usually appears in the form

$$u = M_0^{-1}(h - \nabla p), \quad \text{div } u = 0,$$

where h is some given external force and M_0 is the (rescaled) particle resistance. The particle positions are fixed in this system and the particle resistance density is assumed to be constant. These equations go back to the experimental study of fluid flows through porous media by Darcy [Dar56]. The Brinkman equations have been later suggested as a intermediate system between the Stokes equations and Darcy's law [Bri47]. Both the Brinkman equations and Darcy's law have been rigorously derived from the Stokes equations in domains perforated by many small particles (see e.g. [All90a], [All90b]). The convergence to the Brinkman equations occurs when the density of the Stokes resistance of the particles is of order one. Darcy's law is found when this density is very large. For the problems considered here, the resistance density of the particles corresponds to γ .

It seems that Darcy's law in the form of (2.4.5) has never been studied. Thus, one of the mathematical challenges in the derivation and the study of the Vlasov-Darcy system (2.4.4), (2.4.5) is the analysis of Darcy's law for non-constant particle density. In particular, Darcy's law (2.4.5) can only expected to hold inside the cloud of particles. Outside the cloud, the Stokes equations should still be valid. Therefore, suitable boundary conditions at the boundary of the particle cloud must be determined that account for a boundary layer for the transition between Darcy's law and the Stokes equations.

2.4.3 Inertialess limit with strong interactions: $\text{St} \rightarrow 0$ with $\text{St}\gamma \rightarrow \lambda_*^{-1} \in (0, \infty)$

In this case, the inertial term in the first equation of (2.4.1) is of the same order as the self-interaction term involving γ^{-1} . Thus, it is now more subtle to derive the limit equation for the fluid than in the cases considered in the previous subsections. It is convenient to work with the parameter $\lambda = (\text{St}\gamma)^{-1}$ instead of St in this limit.

We expect $u - v \sim \gamma^{-1}$. Therefore, we introduce as a new variable

$$w = \gamma(u - v),$$

and we define

$$h(t, x, w) = \gamma^{-3} f\left(t, x, u(x, t) + \frac{w}{\gamma}\right).$$

Then,

$$-\Delta u + \nabla p = 6\pi \int_{\mathbb{R}^3} w h dw, \quad \text{div } u = 0 \quad \text{in } \mathbb{R}^3, \quad u(x) \rightarrow 0 \quad \text{as } |x| \rightarrow \infty. \quad (2.4.6)$$

and

$$\partial_t h - \gamma \nabla_w h \partial_t u + \left(u + \frac{w}{\gamma}\right) (\nabla_x h - \gamma \nabla_w h \nabla_x u) + \lambda \gamma \text{div}_w \left(\left(\hat{g} - \frac{9}{2} w\right) h \right) = 0.$$

This is equivalent to

$$\partial_t h + \left(u + \frac{w}{\gamma}\right) \nabla_x h + \gamma \operatorname{div}_w \left(\left(\lambda \left(\hat{g} - \frac{9}{2} w \right) - \partial_t u - \left(u + \frac{w}{\gamma}\right) \nabla_x u \right) h \right) = 0, \quad (2.4.7)$$

where we used that u is divergence free. We use the notation

$$\rho(t, x) = \int_{\mathbb{R}^3} h \, dw, \quad j(t, x) = \int_{\mathbb{R}^3} w h \, dw.$$

Integrating equation (2.4.7) and taking the limit $\gamma \rightarrow \infty$ yields

$$\partial_t \rho + u \cdot \nabla \rho = 0.$$

It is not surprising that this is the same transport equation as (2.4.3) for $\gamma_* = \infty$; the particles are transported by the fluid since both the inertia of the particles and the self-interactive term do not have a direct effect on the evolution of the particles. The particle inertia only has an effect on the fluid velocity. Taking the limit $\gamma \rightarrow \infty$ in (2.4.7), we find

$$\operatorname{div}_w \left(\left(\lambda_* \left(\hat{g} - \frac{9}{2} w \right) - \partial_t u - u \nabla_x u \right) h \right) = 0.$$

Multiplying this equation by w and integrating yields

$$\frac{9}{2} \lambda_* j = (\lambda_* \hat{g} - \partial_t u - u \nabla_x u) \rho.$$

Inserting this identity in (2.4.6), yields the fluid equations

$$-\Delta u + \nabla p = 6\pi j = \frac{4\pi}{3} (\hat{g} - \lambda_*^{-1} (\partial_t u + u \nabla_x u)) \rho, \quad \operatorname{div} u = 0 \quad \text{in } \mathbb{R}^3.$$

Equivalently, we can write the whole system as

$$\begin{aligned} \partial_t \rho + u \cdot \nabla \rho &= 0, \\ \frac{4\pi}{3} \lambda_*^{-1} \rho (\partial_t u + u \nabla_x u) - \Delta u + \nabla p &= \frac{4\pi}{3} \hat{g} \rho, \quad \operatorname{div} u = 0 \quad \text{in } \mathbb{R}^3. \end{aligned} \quad (2.4.8)$$

This is a very interesting limit equation. The inertia of the particles is negligible for the equation of the particles. Because of the Brinkman equations for the fluid in (2.4.1), the deviation between the particle and the fluid velocity is still the driving force for the fluid velocity. Since this deviation involves the particle inertia which is small but compensated by the high interaction strength $\gamma \sim \text{St}^{-1}$, it endows the fluid with inertia, yielding Navier-Stokes equations for the fluid but with the particle density in place of the fluid density.

Since the *stationary* Stokes equations turned into *instationary* Navier-Stokes equations, we clearly need an initial datum for u in order to solve (2.4.8), which was not needed for the original problem (2.4.1), where only an initial datum for the particle density f needs to be prescribed.

As we shall see, this initial datum for u is determined by the initial datum of the particles and a boundary layer in time: Consider the rescaled time $\tau = \lambda \gamma t = (\text{St})^{-1} t$. This is the time-scale in which the particle velocities adapt to the fluid velocity. With $\tilde{f}(\tau, \cdot) := f(\tau/(\lambda \gamma), \cdot)$ and $\tilde{u}(\tau, \cdot) := u(\tau/(\lambda \gamma), \cdot)$, we find

$$\partial_\tau \tilde{f} + \frac{v}{\lambda \gamma} \nabla \tilde{f} + \operatorname{div}_v \left(\left(\frac{1}{\gamma} \hat{g} + \tilde{u} - v \right) \tilde{f} \right) = 0.$$

Hence, in the limit $\gamma \rightarrow \infty$

$$\partial_\tau \tilde{f} + \operatorname{div}_v((\tilde{u} - v)\tilde{f}) = 0. \quad (2.4.9)$$

This is coupled with the fluid equations which becomes Darcy's law in the limit $\gamma \rightarrow \infty$,

$$\int \tilde{f}(\tilde{u} - v) dv + \nabla p = 0. \quad (2.4.10)$$

With the notation $\tilde{\rho} = \int \tilde{f} dv$ and $\tilde{\rho}V = \int v\tilde{f} dv$, we have

$$V - \tilde{u} = \frac{1}{\tilde{\rho}} \nabla p.$$

Clearly, if V is divergence free, then $u = V$ is a solution. From equation (2.4.9) we find by integrating

$$\partial_\tau \tilde{\rho} = 0, \quad \tilde{\rho} \partial_\tau V - \tilde{\rho}(\tilde{u} - V) = 0.$$

Dividing by ρ and taking the divergence yields

$$\partial_\tau \operatorname{div} V = -\operatorname{div} V.$$

Thus,

$$\operatorname{div} V = \operatorname{div} V_0 e^{-\tau},$$

if V_0 corresponds to the given initial datum for the particles. Hence, the particle velocities will rearrange in such a way that V becomes divergence free, and exponential fast convergence of \tilde{u} in the time-scale of τ can be expected. This happens in times of order $1/(\lambda\gamma)$ in the original time-scale and therefore instantaneously as $\gamma \rightarrow \infty$. Thus, the desired initial datum for u in the Navier-Stokes equations in (2.4.8) is given as the limit of the solution \tilde{u} to (2.4.9), (2.4.10) as $\tau \rightarrow \infty$.

2.5 Further sedimentation phenomena and open problems

In this section, we discuss some intriguing phenomena related to the sedimentation models studied before, which have been observed experimentally and numerically, leading to interesting mathematical problems for future study.

2.5.1 The analogy between a sedimenting cloud of inertialess spherical particles and a fluid drop

The qualitative behavior of the system (1.1.1) describing the evolution of a cloud of inertialess spherical particles is independent of the parameter $\gamma \in (0, \infty]$. Indeed, since the Stokes-equations are invariant under translations, the value of γ only changes the particle velocity by a constant.

An important observation is that, for $\gamma = \infty$, equation (1.1.1) can be interpreted as modeling the evolution of a fluid with variable density but fixed viscosity. In this case, ρ is the difference of density of the fluid to the density at infinity. In particular, (1.1.1) models the settling of a fluid drop surrounded by a fluid of lower density ρ_f . Recall from Section 2.3.1 that we absorbed the term $\rho_f g$ in the fluid equation into the pressure. Therefore, the fluid equation in (1.1.1) is equivalent to

$$-\Delta v + \nabla q = (\rho_f + \rho)g, \quad \operatorname{div} v = 0 \quad \text{in } \mathbb{R}^3.$$

This equation shows that ρ really resembles an increase in the fluid density.

The analogy between a suspension of particles and a fluid drop has been observed in experiments and numerical simulations (see e.g. [AKY78; GM12b; PM82; KHA84; NB97; MMNS01; MNG07]).

Moreover, the macroscopic equation (1.1.1), has been obtained by formal computations in the physics literature and referred to as a “continuum model for sedimentation” (see e.g. [Feu84; Luk00; MMNS01]).

There is an interesting quasi-stationary solution to (1.1.1), which has been independently computed by Hadamard and Rybczynski [Had11; Ryb11]. They showed that $\rho(t, x) = \mathbf{1}_{B_r(X_0 + \alpha t g)}(x)$ solves the equation where α depends only on r . Thus, according to this solution, a spherical cloud of particles (or fluid drop) will maintain its shape and settle with constant velocity αg . In contrast to a rigid particle, though, the fluid velocity inside this spherical drop is not constant but performs a circular motion.

However, both numerical and experimental studies suggest that this solution is instable both for particle clouds and fluid drops (see e.g. [AKY78; KHA84; Poz90; MMNS01; MNG07]). It has been observed that an initially spherically cloud will transform into a toroidal shape by developing a hole in the axis parallel to the gravity. Then, this torus will expand in the directions perpendicular to the gravity and finally beak into several pieces. Although different mechanisms have been suggested to cause this instability, it remains unclear to the present time.

2.5.2 The Einstein law for the effective viscosity of a suspension

One of the most well-known features of particle-laden flows is that the suspension of particles effectively increases the viscosity of the fluid. Einstein [Ein06] calculated the effective viscosity to first order in ϕ as $\mu_{\text{eff}} = (1 + 5/2\phi)\mu$.

Mathematically rigorous proofs of this formula have been obtained only recently. In [HM12], the Einstein law is proved on the level of the effective energy dissipation for homogeneous particle densities. In [Sch19], it is shown that the convergence also takes place on the level of the fluid equations. The result in [Sch19] uses the Method of Reflections in the framework described in Chapters 3 and 4 of this thesis. As a drawback, the result is so far restricted to the static case, where the particle evolution is not taken into account, and a condition on the minimal distance between the particles is imposed similar as in Chapter 4.

In Section 2.5.1, we discussed the analogy between the sedimentation of a cloud of particles and a fluid drop of larger mass density within a surrounding fluid. As a consequence of the increase of the effective viscosity due to the particles, it has been suggested (see. e.g [GM12b]) that the cloud of particles resemble a fluid drop of larger density *and viscosity*. Moreover, both the increase in density and the increase in viscosity are supposed to be at first order proportional to ϕ . At first glance, this seems to contradict the validity of the macroscopic equations (1.1.1), since there is no change in viscosity. (Recall that the macroscopic equations are rescaled in time with a factor including the volume fraction ϕ .) It turns out, however, although the viscosity increases at first order linearly with the particle volume fraction, the effect on the fluid velocity is quadratic, and therefore negligible in the limit $\phi \rightarrow 0$.

More precisely, taking into account this increase in viscosity, the macroscopic fluid equations for non-vanishing ϕ (and without rescaling with ϕ) differs from the fluid equation in (1.1.1) in the following way:

$$\begin{aligned} -\operatorname{div} \left(\left(1 + \frac{5}{2}\phi\rho \right) (\nabla u + \nabla u^T) \right) + \nabla p &= \phi\rho g \quad \text{in } \mathbb{R}^3, \\ \operatorname{div} u &= 0 \quad \text{in } \mathbb{R}^3, \quad u(x) \rightarrow 0 \quad \text{as } |x| \rightarrow \infty. \end{aligned} \tag{2.5.1}$$

The effect of the right hand side (increase of density) on the fluid velocity is of order ϕ , whereas the effect of the change of viscosity is of order ϕ^2 . Indeed, replacing again u by $v = \phi^{-1}u$ (and

adjusting the pressure in order to avoid a prefactor in front of it), the first equation in (2.5.1) becomes

$$-\Delta v - \frac{5}{2}\phi \operatorname{div}(\rho(\nabla v + \nabla v^T)) + \nabla p = \rho g, \operatorname{div} v = 0 \quad \text{in } \mathbb{R}^3, \quad v(x) \rightarrow 0 \quad \text{as } |x| \rightarrow \infty$$

Therefore, in the limit $\phi \rightarrow 0$, we recover the fluid equation in (1.1.1).

2.5.3 Fluid backflow in finite containers – the Batchelor correction

At least formally, the analogous equations that describe sedimentation in the whole space \mathbb{R}^3 are also valid when considering instead of a bounded domain E , where a no-slip boundary condition is imposed for the fluid velocity at ∂E . In particular, for inertialess spherical particles, the system (1.1.1) is replaced by

$$\begin{aligned} \partial_t \rho + \left(u + \frac{2}{9}\gamma^{-1}g\right) \cdot \nabla \rho &= 0, \\ -\Delta u + \nabla p &= \rho g, \quad \operatorname{div} u = 0 \quad \text{in } E, \quad u = 0 \quad \text{on } \partial E. \end{aligned} \tag{2.5.2}$$

This system makes sense at least when ρ is compactly supported in E and for times until the first particle reaches the boundary of E . (One can then extend the dynamics by imposing suitable constraints at the boundary, e.g. by imposing that mass that hits the boundary of E is taken out of the system, and that no mass can enter through the boundary.)

Moreover, if we fix an initial distribution of particles ρ_0 and take the limit $E \rightarrow \mathbb{R}^3$, we expect the dynamics to converge to the dynamics of the whole space. However, there is one fundamental difference between particle sedimentation in finite domains and in the whole space. The fluid velocity in the whole space given as the solution of the Stokes equation with right-hand side ρg satisfies

$$g \cdot u \geq 0 \quad \text{in } \mathbb{R}^3. \tag{2.5.3}$$

Indeed, using the fundamental solution of the Stokes equations, u can be explicitly computed as

$$u(x) = \frac{1}{8\pi} \int_{\mathbb{R}^3} \rho(y) \left(\frac{g}{|x-y|} + \frac{(g \cdot (x-y))(x-y)}{|x-y|^3} \right) dy.$$

This implies (2.5.3) since $\rho \geq 0$ (in fact, the inequality in (2.5.3) is strict provided $\rho \neq 0$).

On the other hand, if we solve the Stokes equations in a bounded domain E with no-slip conditions, we find for any $a \in \mathbb{R}^3$

$$\int_E a \cdot u(y) dy = \int_E \operatorname{div}((a \cdot y)u(y)) = 0. \tag{2.5.4}$$

In particular, u has mean 0, and (2.5.3) does not hold. Heuristically speaking, if the fluid is pushed down in some region in E , there has to be a backflow somewhere else because it is incompressible. Indeed, analogously to (2.5.4), the total fluid flow through each surface $\Sigma \subset E$ with $\partial \Sigma \subset \partial E$ vanishes. This is not true in the whole space, though: roughly speaking, fluid can enter from infinity.

As a consequence of the backflow, the behavior of solutions in finite containers is expected to differ significantly from the behavior in the whole space as soon as the particles are filling a large portion of the container. This can be easily verified by considering the extreme case of a homogeneous distribution of particles in the whole container, i.e., $\rho = \mathbf{1}_E$. Then, the solution to (2.5.2) is given by $u = 0$ and the particles simply settle with the Stokes velocity of a single particle $2/(9\gamma)g$.

We only used that u is divergence-free and satisfies the no-slip condition on ∂E to show that $\int_E u = 0$. Therefore, this also holds for the microscopic fluid velocity, when we extend u into the

particles. Since the particles are moving in the direction of gravity (at least on average), this implies that the fluid has to flow in the opposite direction in between the particles. Therefore, when the particles are distributed homogeneously in the whole container, it is expected that the average particle velocity is even smaller than the Stokes settling velocity of a single particle, where the correction depends on the particle volume fraction ϕ . This phenomenon is known as hindered settling, and it has attracted a lot of attention in the physics community (see e.g. [Has59; Bat72; Hin77; Nic+95; GM12b]). The exact value of the average velocity seem to be highly sensitive to the particle distribution. Hasimoto [Has59] showed that the correction to the particle velocity due to the fluid backflow is of order $\phi^{1/3}$ times the Stokes velocity V_{St} if the particles are periodically distributed in the whole space. To mimic the situation of a bounded container, he considered the solution to the Stokes equations in a cell Q with periodic boundary conditions imposing the constraint $\int_Q u = 0$. On the other hand, Batchelor [Bat72] suggested a settling velocity $V = (1 - 6.55\phi)V_{St}$ to first order in ϕ in the case when the particles are placed according to a hard spheres distribution. It is an interesting problem to prove rigorous results in that direction, in particular, when taking the time-evolution into account.

2.5.4 The Caffisch-Luke paradox

Caffisch and Luke [CL85] pointed out that although the average particle velocity in a suspension homogeneously distributed in a bounded container is of order V_{St} , the variance of this velocity formally increases with the size of the container at fixed particle concentration. In fact, their computation suggest that this variance is as large as in the case of a cloud of particles sedimenting in infinite fluid. This computation has been supported by early numerical simulations [Koc94; Lad97]. However experiments did not observe this divergence but found that the variance is independent of the container size for sufficiently large containers [NG95; SHC97]. Therefore, this issue became known as the Caffisch-Luke paradox, and several mechanisms were suggested for its resolution [KS91; Bre99; Luk00], proposing that the variance would decay in time due to the onset of some structure in the particle distribution. Although this decay has been qualitatively confirmed by simulations [BGGH03; NL05], there is still no convincing theory explaining a suitable rearrangement of the particles leading to this behavior. For a review on the Caffisch-Luke paradox, we refer the reader to [GH11].

2.5.5 The instability of sedimentation of rod-like particles

As in the case of inertialess spherical particles, we can consider the corresponding macroscopic equations for inertialess rod-like particles (1.1.3) in bounded domains $E \subset \mathbb{R}^3$,

$$\begin{aligned} \partial_t f + \left(u + \frac{1}{8\gamma^{-1}} (\text{Id} + \xi \otimes \xi) g \right) \cdot \nabla_x f + \text{div}_\xi (P_{\xi^\perp}(\xi \cdot \nabla) u f) &= 0, \\ -\Delta u + \nabla p &= \int_{S^2} f g \, d\xi, \quad \text{div } u = 0 \quad \text{in } E, \quad u(x) = 0 \quad \text{on } \partial E. \end{aligned} \tag{2.5.5}$$

Analogously as for spherical particles, if $\int_{S^2} f \, d\xi = \mathbf{1}_E$, then $u = 0$. In particular, ignoring boundary conditions of f at ∂E , any function $f(t, x, \xi) = h(\xi)$ is a stationary solution to (2.5.5).

However, these solutions are experimentally and numerically found to be instable, whereas the corresponding constant density $\rho = \mathbf{1}_E$ is observed to be a stable solution of the transport-Stokes equations (2.5.2) for spherical particles. Experiments and numerical simulations (based on the microscopic model (2.3.16)) of the sedimentation of rods at small volume fraction found that the particles tend to orient towards the direction of gravity and form packets that settle fast with surrounding regions of little particles rising up (see e.g. [HGMS96; MS98; HG99; SDS05; MBG07]). In particular, the average settling velocity is not found to be decreased by the presence of other

particles due to the fluid backflow as for spherical particles, but it is larger than the sedimentation velocity of a single vertically oriented rod in unperturbed infinite fluid.

In [KS89], the macroscopic model (2.5.5) was studied by a linear stability analysis. It was found that indeed the stationary solutions $f(t, x, \xi) = h(\xi)$ are unstable. However, in contradiction to the mentioned experiments and simulations, it was predicted by [KS89] that the horizontal width of the developed particle clusters are of the order of the container width. Some mechanisms have been suggested for this different wave number selection in the instability in experiments and simulations [SSD06a; SSD06b].

A review over the instability of the sedimentation of rods can be found in [GH11].

Chapter 3

The method of reflections

In this chapter, we study the method of reflections which is used to obtain series representations for the solutions of certain boundary value problems in perforated domains. As discussed in Section 1.3, the method of reflections has been widely used in the physics and engineering literature, but only few mathematical papers have considered the method.

In this chapter, we give a precise mathematical meaning of the formal series obtained by means of the method of reflections and explain how these series can be used to obtain the asymptotic behavior of the solutions of the Poisson and Stokes equations in the limit of domains perforated by many small particles.

The method of reflections is also used in Chapter 4 for the rigorous derivation of the transport-Stokes equations (4.1.9) from the microscopic particle system.

The content of this chapter has been published in *Archive for Rational Mechanics and Analysis*, [HV18]. In comparison with [HV18], the assumptions on the configurations of particles have been relaxed considerably. However, the structure of the proof is unaltered. In fact, the improvement has only been achieved by application of the Cauchy-Schwarz inequality in the proof of Lemma 3.2.8 in a way that has been overlooked in [HV18].

3.1 Introduction

We consider Poisson and Stokes equations in perforated domains

$$-\Delta u = f \text{ in } \mathbb{R}^3 \setminus K, \quad u = 0 \text{ in } K, \quad (3.1.1)$$

and

$$-\Delta v + \nabla p = f, \quad \nabla \cdot v = 0 \text{ in } \mathbb{R}^3 \setminus K, \quad v = 0 \text{ in } K. \quad (3.1.2)$$

where u is a scalar function, and v is a vector field with values in \mathbb{R}^3 . Here, the set K consists of mutually disjoint sets,

$$K = \bigcup_{i \in I} \overline{\Omega_i}, \quad (3.1.3)$$

for some $\Omega_i \subset \mathbb{R}^3$ open, where I is a finite or countable index set.

Problems analogous to (3.1.1) and (3.1.2) have been often studied in the physics literature using the so-called method of reflections. This method allows to obtain some formal series for the solutions of these equations which eventually should approximate them.

However, the series obtained by means of the method of reflections are divergent for problems like (3.1.1) and (3.1.2) where K extends to the whole space. This divergence takes place even if the source term f is compactly supported or decays very fast at infinity.

3.1.1 The method of reflections

The method of reflections in hydrodynamic equations was introduced by Smoluchowski (cf. [Smo11]). This method allows to approximate the solutions of boundary value problems for the Poisson or Stokes equations in domains with complex boundaries consisting of many connected components. We write any of those equations as

$$\mathcal{L}\phi = f \quad \text{in } \Omega \quad (3.1.4)$$

where ϕ is the solution to be computed and f is a suitable source term, and where \mathcal{L} could be in principle any linear elliptic operator. We will assume by definiteness that we wish to solve these equations in the domain $\Omega = \mathbb{R}^d \setminus \bigcup_j C_j$, where the sets C_j , which from now on will be denoted as particles, are compact sets and $C_j \cap C_k = \emptyset$ if $j \neq k$. The boundary conditions might be Dirichlet, Neumann or Robin or any other type as long as they are linear. We will write the boundary condition at each set C_j as

$$\mathcal{B}\phi = g_j \quad \text{on } \partial C_j. \quad (3.1.5)$$

Suppose that the exterior boundary value problem outside each of the sets C_j can be solved explicitly, i.e., we have explicit formulas (typically in terms of integrals) for the problems

$$\mathcal{L}\psi_j = 0 \quad \text{in } \mathbb{R}^d \setminus C_j, \quad \mathcal{B}\psi_j = h_j \quad \text{on } \partial C_j. \quad (3.1.6)$$

It is then possible to compute iteratively a solution for the boundary value problem (3.1.4), (3.1.5) in Ω as follows. We write as zero order approximation Φ_0 to the solution of (3.1.4), (3.1.5) just as the solution of

$$\mathcal{L}\Phi_0 = f \quad \text{in } \mathbb{R}^d. \quad (3.1.7)$$

This solution cannot be expected to satisfy the boundary condition (3.1.5). We then define a first order approximation to ϕ adding to Φ_0 the solutions of the problems (3.1.6) where h_j is chosen as the difference between the desired boundary condition and the one given by Φ_0 . More precisely we define $\Phi_{1,j}$ as the solution of

$$\mathcal{L}\Phi_{1,j} = 0 \quad \text{in } \mathbb{R}^d \setminus C_j, \quad \mathcal{B}\Phi_{1,j} = g_j - \mathcal{B}\Phi_0 \quad \text{on } \partial C_j. \quad (3.1.8)$$

We then define $\Phi_1 = \sum_j \Phi_{1,j}$. Then $\Phi_0 + \Phi_1$ yields a new approximation to ϕ . This new approximation does not satisfy the boundary conditions on $\bigcup_j \partial C_j$. We can then define a new correction Φ_2 , defining functions $\Phi_{2,j}$ in a manner analogous to (3.1.8). More precisely we define inductively functions $\Phi_{k,j}$ as

$$\mathcal{L}\Phi_{k,j} = 0 \quad \text{in } \mathbb{R}^d \setminus C_j, \quad \mathcal{B}\Phi_{k,j} = -\mathcal{B} \left(\sum_{l \neq j} \Phi_{k-1,l} \right) \quad \text{on } \partial C_j \quad \text{for } k = 2, 3, \dots, \quad (3.1.9)$$

$$\Phi_k = \sum_j \Phi_{k,j} \quad (3.1.10)$$

Iterating the method, we obtain a series $\Psi_N = \Phi_0 + \Phi_1 + \Phi_2 + \dots + \Phi_N$. The reason, why this sequence can be hoped to converge to the solution of the boundary value problem (3.1.4), (3.1.5) is that Ψ_N satisfies (3.1.4) and, by induction,

$$\mathcal{B}\Psi_N = g_j - \mathcal{B}\Phi_{N+1,j} \quad \text{on } \partial C_j.$$

There are several clear difficulties that one encounters when trying to prove the convergence of the method described above to the solution ϕ . If there are infinitely many particles C_j , it is not clear

whether the functions Φ_k would be defined since they are given by a series with infinitely many terms. Actually, the divergence of these series might be expected in this situation because the solutions of Poisson and Stokes equations yield long range interactions which decay as power laws with a too slow decay. Even if the functions Φ_k are well defined, the convergence of $\{\Psi_N\}_N$ as $N \rightarrow \infty$ is not clear. Divergence of this series might happen if the particles C_j are too close and their mutual interactions do not tend to zero sufficiently fast. More precisely, divergence is expected if

$$\left| \sum_{l \neq j} \Phi_{k,l}(x_j) \right| > |\Phi_{k,j}(x_j)|$$

for most of the particles j . Indeed this condition implies, that adding Φ_k does not bring the function closer to the right boundary conditions at those particles j .

For the analysis of the convergence of the method of reflections applied to problem (3.1.1), it turns out that some characteristic length is of great importance, namely the screening length. This concept was introduced in the physics literature in [MR84]. A precise mathematical discussion of this length and its relevance in phase transition problems driven by diffusive effects can be found in [NO01], [NV06]. In the following, we illustrate the effect of the screening length for simplicity in the special case of spherical particles with equal radii distributed on a lattice. (Later we will go back to more general particle configurations.) More precisely, in (3.1.3) we choose $\Omega_i = B_r(x_i)$ and $\{x_i\}_{i \in I} = (d\mathbb{Z})^3$ for some $r, d > 0$. In the following, we will use the notation $B_i := B_r(x_i)$. We consider equal charges on all particles that are contained in a ball of radius ρ . Then, we look at the potential at the particle which is at the center of this ball. This potential is the sum of the potential that is induced by the charge on that particular particle and the potential due to all the other particles. Then, the screening length is the critical radius ρ at which those two portions are equal. More precisely, we define u_j to be the unique solution with $u_j(x) \rightarrow 0$ as $|x| \rightarrow \infty$ of the problem

$$\begin{aligned} -\Delta u_j &= 0 \quad \text{in } \mathbb{R}^3 \setminus B_j, \\ u_j &= 1 \quad \text{on } \partial B_j. \end{aligned}$$

Then, the screening length is defined as

$$\Lambda := \sup \left\{ \rho > 0 : \sup_{\partial B_j} \left(\sum_{l \neq j, x_l \in B_\rho(x_j)} u_l(x) \right) < 1 \right\}.$$

If we now apply the method of reflections for Poisson equation to the system containing only the particles in a cloud of radius R , i.e., for $K_R = K \cap B_R(0)$, a sufficient condition for convergence would be

$$R < \Lambda.$$

Indeed, adding Φ_k would then really bring the function closer to the right boundary conditions for most of the particles, leading to the estimate

$$\|\Phi_{k+1}\| \leq \theta \|\Phi_k\|$$

in a suitable norm, where

$$\theta := \sup_{\partial B_k} \left(\sum_{j \neq k, x_j \in B_R} u_k(x) \right) < 1. \quad (3.1.11)$$

This condition is similar to the sufficient condition obtained in [Tra06] for the convergence of the method of reflections for the Laplace equation in exterior domains with Dirichlet boundary conditions. The condition there reads

$$\max_i \sum_{k \neq i} \frac{\frac{r_k}{|x_i - x_k|}}{1 - \frac{r_i}{|x_i - x_k|}} < 1. \quad (3.1.12)$$

The proof in [Tra06] relies on the maximum principle for the Poisson equation. Therefore, it is not clear how to generalize it to other problems like the Stokes equations.

In systems with many particles of small radii and typical distance between particles d , the conditions (3.1.11) and (3.1.12) are roughly equivalent to

$$d^{-3} r \max_i \sum_{k \neq i} \frac{d^3}{|x_i - x_k|} < C$$

with C of order one. Approximating the sum by an integral and assuming that the particles are contained in ball with radius R , this would be equivalent to

$$d^{-3} r \int_{B_R(0)} \frac{dy}{|y|} < C.$$

Thus, the screening length Λ is of order $\sqrt{r^{-1}d^3}$.

3.1.2 Assumptions on the particle configuration

For a general configuration of particles Ω_i , we assume that there exist $x_i \in \mathbb{R}^3$ and $r_i > 0$ such that $\Omega_i \subset B_{x_i}(r_i) =: B_i$ and the balls B_i are pairwise disjoint. For each particle $i \in I$ we define the distance to the nearest other particle

$$d_i := \inf_{j \neq i} |x_i - x_j|.$$

Then the sets $B_{d_i/2}(x_i)$ are disjoint.

In the following, we will always assume that the following two conditions are satisfied.

Condition 3.1.1. *There exists a constant ℓ such that*

$$C_\ell^2 := \sum_i \sum_{j \neq i} \frac{r_i r_j e^{\frac{2|x_i - x_j|}{\ell}}}{|x_i - x_j|^2} < \infty \quad (3.1.13)$$

Condition 3.1.2.

There exists a constant $\kappa > 1$ such that

$$\frac{d_i}{2} > \kappa^2 r_i \quad \text{for all } i \in I.$$

Condition 3.1.1 appears to be rather complicated. However, it can be viewed as a relaxed version of (3.1.12): due to the exponential cutoff, it is a localized version. Moreover, due to the second sum on the left-hand side of (3.1.13) instead of the maximum in (3.1.12), Condition 3.1.1 allows clusters of close particles as long as there are not too many of them. The following minimal distance condition implies Condition 3.1.1.

Condition 3.1.1*. There exists $\mu_0 < \infty$ such that

$$r_i d_i^{-3} \leq \mu_0 \quad \text{for all } i \in I \quad (3.1.14)$$

Indeed, if Condition 3.1.1* is satisfied for some $\mu_0 < \infty$, then also Condition 3.1.1 holds with

$$C_\ell \leq C\mu_0\ell^2, \quad (3.1.15)$$

where C is a universal constant. Clearly, Condition 3.1.1 is much less restrictive than Condition 3.1.1*. In view of the previous subsection, Condition 3.1.1* implies a lower bound for the screening length Λ of the particles. Equivalently, μ_0 provides an upper bound for the *capacity density* of the particles (see also Section 3.1.5).

Condition 3.1.1 is also satisfied for certain random distribution of particles: Consider spherical particles that are distributed according to a Poisson point process with intensity $n(x)g(r)$. We define the capacity density

$$\mu(x) := n(x) \int_0^\infty rg(r) dr.$$

Then, if $\mu \in L^\infty(\mathbb{R}^3)$, the expectation of C_ℓ is estimated as

$$\mathbb{E}(C_\ell) = \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{\mu(x)\mu(y)e^{\frac{2|x-y|}{\ell}}}{|x-y|^2} \leq C\|\mu\|_\infty\ell^2.$$

In particular, Condition 3.1.1 is satisfied with probability one.

Condition 3.1.2 rules out overlapping or touching particles by requiring that their distance is controlled by their radius. This condition is not very restrictive. Indeed, we are interested in the case of many small particles with screening length $\sqrt{r^{-1}d^3}$ of order one. Thus the typical distance d needs to be much larger than the typical radius r . In particular, we observe that Condition 3.1.1* implies Condition 3.1.2 if all the radii r_i are sufficiently small.

3.1.3 Main results for the Screened Poisson equation

In order to avoid divergences but still allow for infinitely many particles, instead of the Poisson equation, we will consider first a modified version of the problem (3.1.1), namely the screened Poisson equation

$$-\Delta u + \ell^{-2}u = f \text{ in } \mathbb{R}^3 \setminus K, \quad u = 0 \text{ in } K \quad (3.1.16)$$

for some $\ell > 0$. The basic difference between (3.1.1) and (3.1.16) is that the Green's function associated to the second problem decreases exponentially in distances of order ℓ , which can be thought of as the effective system size. Thus, the series defining the functions Φ_k are well defined. Moreover, the series in the Method of Reflections converges provided the particle configuration satisfies the assumptions of the previous subsection with C_ℓ sufficiently small. This condition implies that the screening length of the system is sufficiently large compared to the effective system size, which guarantees that the interaction does not exceed a certain value.

We also have to impose that $(r_i)_{i \in I}$ is bounded.

Condition 3.1.3. *There exists a constant α such that*

$$r_i \leq \alpha\ell \text{ for all } i \in I.$$

Theorem 3.1.4. *Assume Conditions 3.1.1, 3.1.2, and 3.1.3 are satisfied for some $\ell > 0$. Let $\mathcal{L} = -\Delta + \ell^{-2}$, $\mathcal{B} = I$, $\Omega = \mathbb{R}^3 \setminus K$ with K as in (3.1.3), and let $g_j = 0$ on ∂C_j . Suppose also that $f \in H^{-1}(\mathbb{R}^3)$ and define Φ_0 as in (3.1.7) and inductively the functions Φ_k by means of (3.1.9),*

(3.1.10). Let u be the unique solution of (3.1.16) in $H^1(\mathbb{R}^3 \setminus K)$. Then, there exists a constant $\theta < 1$ depending only on κ and a constant C_1 depending only on κ and α such that

$$\left\| \sum_{k=0}^N \Phi_k - u \right\|_{H^1(\mathbb{R}^3 \setminus K)} \leq \max\{1, \ell^2\} \max\{\theta, C_1 C_\ell\}^N \|f\|_{H^{-1}(\mathbb{R}^3)},$$

where C_ℓ is defined as in (3.1.13).

Remark 3.1.5. In particular, if (3.1.14) is satisfied, the above theorem together with (3.1.15) implies that $\sum_{k=0}^N \Phi_k$ converges to u for all particle configurations with $\ell^2 \mu_0$ sufficiently small.

As indicated in the theorem above, if C_ℓ is large, the series $\sum_{k=0}^\infty \Phi_k$ is in general divergent and the method of reflections cannot be applied, at least not in the form stated in Theorem 3.1.4. However, it turns out that it is possible to give a meaning to the formal series arising in the method of reflections in order to obtain a modified series which converges to the solution of (3.1.16).

Indeed, instead of $\sum_{k=0}^\infty \Phi_k$, we prove (see Theorem 3.2.12) that for arbitrary values of C_ℓ there exists a double sequence $q(k, N)$ defined for $k, N \in \mathbb{N}$ and $0 \leq k \leq N$, such that

$$\Psi_N = \sum_{k=0}^N q(k, N) \Phi_k \tag{3.1.17}$$

converges as $N \rightarrow \infty$ to the unique solution u of (3.1.16) in $H^1(\mathbb{R}^3 \setminus K)$.

This result can be thought as a summation method for the original series $\sum_{k=0}^\infty \Phi_k$. The precise construction of the sequence $q(k, N)$ will be given in Section 3.2.

3.1.4 The summation procedure and the main result for the Poisson equation

Theorems 3.1.4 and 3.2.12 refer to the Dirichlet problem for the screened Poisson equation (3.1.16) containing a parameter ℓ which restricts the range of interaction between particles to the finite value ℓ . It is natural to ask if the result can be generalized to the Dirichlet problem for the Poisson equation (3.1.1) which corresponds to $\ell = \infty$.

In this case, the series (3.1.10) defining the functions Φ_k does not converge if the particles extend to the whole space \mathbb{R}^3 and then the method of reflections as formulated in Theorem 3.1.4 becomes meaningless.

Nevertheless, using the formal series $\sum_{k=0}^\infty \Phi_k$, it is possible to construct an alternative series which converges to the solution of (3.1.1). However, the relation between the original (divergent) series and the modified one, is much more involved than in the case of the screened Poisson equation Theorem 3.2.12. Therefore, we will first give an idea of the summation method.

The summation method is based on an interpretation of the method of reflections using an abstract idea of functional analysis in Hilbert spaces. It is well known that by means of convenient choices of Hilbert spaces H , the solutions of many boundary value problems for a large class of equations with the form (3.1.4) is equivalent to the orthogonal projection of $\mathcal{L}^{-1}f$ to the subspace of the Hilbert space for which the boundary conditions hold. We denote here by \mathcal{L}^{-1} the operator solving (3.1.4) in the whole space, which can be easily computed using the Green's function associated to (3.1.4). We will denote this orthogonal projection operator providing the solution of the boundary value problem (3.1.4) by P . This projection maps the Hilbert space H into the subspace satisfying the boundary conditions, which will be denoted by V . On the other hand, we can associate another orthogonal projection operator P_j to the solution of the boundary value problem for a single particle j . This projection maps H in a subspace V_j for which the boundary conditions are satisfied at the

particle j . We have $V = \cap_j V_j$. Let Q_j denote the orthogonal projection from H in the orthogonal of V_j in H .

It turns out that the partial sums for the Method of Projections $\sum_{k=0}^N \Phi_k$ can be written as

$$\left(1 - \sum_j Q_j\right)^N \mathcal{L}^{-1} f.$$

Thus, the Method of Projections converges to the solution of (3.1.4) if

$$P = \lim_{N \rightarrow \infty} \left(1 - \sum_j Q_j\right)^N \quad (3.1.18)$$

in some suitable way. This result would hold trivially if the subspaces $\{V_j\}$ were mutually orthogonal. However, if the angles between some of these subspaces are too small, a geometrical argument shows that (3.1.18) will fail. It is precisely the condition of smallness of C_ℓ that ensures that the convergence (3.1.18) takes place for the Dirichlet problem of the screened Poisson equation (3.1.16). This is the main idea in the proof of Theorem 3.1.4.

A related geometrical interpretation of the method of reflections has been analyzed in [Luk89]. The method used in [Luk89] can be applied to systems with finitely many particles, and the convergence of the method of reflections used there, which does not treat all the particles simultaneously but sequentially, leads to showing that

$$\lim_{N \rightarrow \infty} \left(\prod_j P_j\right)^N = P,$$

where the product is taken over the finite number of particles chosen in any order. Actually, the method of reflections used in [Luk89] is not applied in the case of Dirichlet boundary conditions. Instead, it is applied to the Stokes system imposing the set of mixed boundary conditions at the particles satisfied by sedimenting inertialess particles, and to the Poisson equation with analogous boundary conditions.

As indicated above, the convergence stated in (3.1.18) cannot be expected if C_ℓ is large. However, a geometrical argument shows that, as long as the sum $\sum_j Q_j$ is convergent, the following convergence takes place.

$$P = \lim_{N \rightarrow \infty} \left(I - \gamma \sum_j Q_j\right)^N,$$

if $\gamma > 0$ is small enough. Actually the right hand side can be written as the series (3.1.17) which is directly related to the original series $\sum_{k=0}^N \Phi_k$.

For the Poisson equation (3.1.1) with particles extending to the whole space, the series $\sum_j Q_j$ is in general divergent. However, a similar idea can be applied by including in γ an additional dependence on the particle position.

Theorem 3.1.6. *Suppose Conditions 3.1.1 and 3.1.2 are satisfied with some $\ell > 0$. Let $f \in \dot{H}^{-1}(\mathbb{R}^3)$. There exists a $\gamma_0 > 0$ depending only on C_ℓ from Condition 3.1.1 and κ from Condition 3.1.2 such that the sequence*

$$\lim_{N \rightarrow \infty} \left(1 - \gamma \sum_j e^{-\ell|x_j|} Q_j\right)^N (-\Delta)^{-1} f$$

converges to the solution of (3.1.1) in $\dot{H}^1(\mathbb{R}^3)$ for all $\gamma < \gamma_0$.

Remark 3.1.7. *We denote by $\dot{H}^1(\mathbb{R}^3) := \{v \in L^6(\mathbb{R}^3) : \nabla v \in L^2(\mathbb{R}^3)\}$ the homogeneous Sobolev space and by $\dot{H}^{-1}(\mathbb{R}^3)$ its dual space.*

3.1.5 Homogenization results

To illustrate the possible use of the method of reflections, we will give a proof of classical homogenization results in perforated domains using only the tools developed in this chapter. We assume that for $0 < \delta < 1$ we have configurations of spherical particles with radii $r_{i,\delta}$ and centers $x_{i,\delta}$ for $i \in I_\delta$.

In [CM82a; CM82b] (see [CM97] for an English version), the question considered is the homogenization problem

$$-\Delta u_\delta = f \text{ in } \Omega \setminus K_\delta, \quad u_\delta = 0 \text{ in } K_\delta \cup \partial\Omega, \quad (3.1.19)$$

where Ω is an open bounded subset of \mathbb{R}^n and K_δ is a sequence of sets occupied by particles. It was proved in [CM97] that under some abstract conditions on the particle configurations, there is $\mu \in W^{-1,\infty}(\Omega)$ such that for all $f \in L^2(\Omega)$ the sequence of solutions u_δ converges weakly in $H^1(\Omega)$ as $\delta \rightarrow 0$ to the solution of

$$-\Delta u + \mu u = f \text{ in } \Omega, \quad u = 0 \text{ in } \partial\Omega, \quad (3.1.20)$$

In [CM97], also specific particle configurations have been studied. In particular, they considered particles of radius r periodically distributed on the lattice $(d\mathbb{Z})^3$. Then, in the limit $r, d \rightarrow 0$ such that rd^{-3} is fixed, the sequence of solutions to (3.1.19) converges to the solution of (3.1.20) with $\mu = 4\pi rd^{-3}$.

We have already explained the importance of the quantity rd^{-3} when we introduced the screening length Λ . Furthermore, we can draw the following analogy to the theory of electrostatics. The electrostatic capacity of a conductor is the charge induced on it by a difference of potential. We recall that the electrostatic capacity of a sphere of radius r is $4\pi r$ (cf. [Jac75]). Thus, $\mu 4\pi rd^{-3}$ is the capacity density of the system.

Therefore, for more complicated particle configuration, it is natural to prove convergence to the homogenized equation (3.1.20) given that the capacity density converges in a suitable sense. For $\delta > 0$ and $x \in (\delta\mathbb{Z})^3$, we define q_x^δ to be half open cubes with edges of length δ such that \mathbb{R}^3 is the disjoint union of those cubes. Having fixed those cubes, for any $y \in \mathbb{R}^3$, we use the notation q_y^δ for the unique cube q_x^δ containing y , where $x \in (\delta\mathbb{Z})^3$. We define the averaged capacity density

$$\mu_\delta(x) := \frac{4\pi}{\delta^3} \sum_{x_{i,\delta} \in q_x^\delta} r_{i,\delta}. \quad (3.1.21)$$

Then, under the following assumption, we can prove the homogenization result below.

Assumption 3.1.8. *For all $0 < \delta < 1$, the particle configurations satisfy Conditions 3.1.1* and 3.1.2 uniformly in δ , i.e. with the same constants ℓ and κ and with C_ℓ uniformly bounded in δ . Moreover, there exists $\mu \in L^\infty(\mathbb{R}^3)$ and $\mu_1 > 0$ such that $\mu_\delta \geq \mu_1$ for all $0 < \delta < 1$ and*

$$\lim_{\delta \rightarrow 0} \mu_\delta \rightarrow \mu \text{ in } L^\infty(\mathbb{R}^3).$$

Note that instead of Condition 3.1.1 we require the stronger Condition 3.1.1*. This is used for a Riemann sum argument in the proof of the homogenization result. It seems to be possible to weaken Condition 3.1.1* but it avoids certain technicalities to require it.

Note that

$$r_{\delta,\max} := \sup_{i \in I_\delta} r_i < C\delta \leq C. \quad (3.1.22)$$

Indeed, by Assumption 3.1.8 $\mu_\delta \geq \mu_1 > 0$, and therefore every cube q_x^δ contains at least one particle, and those cubes are of length $\delta < 1$.

Theorem 3.1.9. *Suppose that $f \in H^{-1}(\mathbb{R}^3)$. Then, under Assumption, 3.1.8, the problems (3.1.1) with $K = K_\delta$ have unique solutions $u_\delta \in H^1(\mathbb{R}^3)$. In the limit $\delta \rightarrow 0$, u_δ converges weakly in $H^1(\mathbb{R}^3)$ to the unique solution $u \in H^1(\mathbb{R}^3)$ of the problem*

$$-\Delta u + \mu u = f \text{ in } \mathbb{R}^3.$$

An analogous result can also be proved for the solutions of the equation (3.1.16). In that case, the limit equation reads

$$-\Delta u + (\ell^2 + \mu)u = f.$$

Generalizations of this result have been developed, including more general elliptic operators, in particular Stokes equations [All90a; DGR08; MK74] (see [MK08] for a English version). Most of the homogenization results for elliptic problems have been obtained in bounded domains. The homogenization problem associated to (3.1.19) with $\Omega = \mathbb{R}^3$ has been considered in [NV04a; NV06] with assumptions on the particle configurations similar to Assumption 3.1.8. In particular, it was proved in those papers that assuming that $f \in L^\infty(\mathbb{R}^3)$, the unique bounded solutions of (3.1.19) converge weakly in $H_{loc}^1(\mathbb{R}^3)$ as $\delta \rightarrow 0$ to the solution of (3.1.20) (with $\Omega = \mathbb{R}^3$). The proof of the homogenization results in [NV04a; NV06] relies heavily in the derivation of the so-called screening estimate, which states that the fundamental solution for the Laplace equation in a perforated domain with homogeneous Dirichlet boundary conditions decreases exponentially over distances of the order of the screening length $\Lambda = \frac{1}{\sqrt{\mu}}$. The proof of this estimate given in [NV06] uses the maximum principle for second order elliptic operators and therefore the proof cannot be easily generalized to higher order operators. We want to emphasize that our proof of Theorem 3.1.9 does not use the maximum principle.

Analogous theorems as Theorem 3.1.6 and Theorem 3.1.9 can be obtained also for Stokes equations (3.1.2), see Theorem 3.5.8 and Theorem 3.5.12. The homogenized equations in the case of Stokes equations are

$$-\Delta u + \nabla p + \mu u = f \text{ in } \mathbb{R}^3, \quad \nabla \cdot u = 0, \quad (3.1.23)$$

and the factor 4π in equation (3.1.21) has to be replaced by 6π .

Related results have been obtained in [All90a; DGR08; MK08]. The system of equations (3.1.23) is known as Brinkman equations, which is a well established model in the theory of filtration. It can be viewed as intermediate equations between the Stokes equations and Darcy's law in porous media (see [SP82; All90b]). All the results in those papers have been obtained in bounded domains and for particles of identical radii. Theorem 3.5.12 above provides a new proof of this type of homogenization results by means of the method of reflections. Note that the homogenization result in Theorem 3.5.12 is valid for particle distributions in the whole space and for particles with different radii. However, we do not think that the method of reflections is really needed for this generalization, because seemingly the methods of [DGR08] might be easily adapted to prove Theorem 3.5.12. We just want to emphasize that the convergence result in Theorem 3.5.8 based on the method of reflections is strong enough to allow the derivation of the homogenization limit.

3.1.6 Organisation of this chapter

The rest of the chapter is organized as follows.

In Section 3.2, we will prove Theorem 3.1.4 and 3.2.12. To do so, after repeating a basic lemma from functional analysis, we will give the precise formulation of the method of reflections in terms of orthogonal projections in Section 3.2.2, which will directly lead to necessary and sufficient conditions for convergence of the series obtained by the method of reflections. In Section 3.2.3, we will provide the necessary estimate to prove Theorem 3.1.4. In Section 3.2.4, we will explain in detail the geometrical

idea leading to the summation method yielding Theorem 3.2.12. In Section 3.2.5, we will analyze the summation method on the level of the original series obtained by the method of reflections.

In Section 3.3, we will explain the modifications needed to adapt the method derived in Section 3.2 to the Poisson equation. These modifications basically consist in a spatial cutoff in order to solve the problem of divergent series due to the long range structure of the Poisson equation. This leads to the proof of Theorem 3.1.6.

In Section 3.4, we prove the homogenization result, Theorem 3.1.9. In Section 3.4.1, we show that, under Assumption 3.1.8, problem (3.1.1) is well posed in $H^1(\mathbb{R}^3)$ due to the existence of a Poincaré inequality in $H_0^1(\mathbb{R}^3 \setminus K_\delta)$. Thereafter, we give a formal derivation of the homogenization result based on the original formal series obtained by the method of reflections. Finally, we give the rigorous proof of Theorem 3.1.9 using the tools and results from the previous sections.

In Section 3.5, we apply the method to the Stokes equations (3.1.2) in order to prove the analogous results as for the Poisson equation. Since most parts work exactly the same way as for the Poisson equation, we refrain from going through all the details again, but rather point out the necessary modifications.

3.2 The Screened Poisson equation

Throughout this section, we will always assume that a particle configuration $(\Omega_i)_{i \in I}$ with corresponding balls $B_i \supset \Omega_i$ and a number $\ell > 0$ is given which satisfy Conditions 3.1.1, 3.1.2 and 3.1.3 for some κ , ℓ and α .

3.2.1 Preliminaries of functional analysis

In the following $G_0 := (-\Delta + \ell^{-2})^{-1}$ denotes the solution operator for the screened Poisson equation in the whole space \mathbb{R}^3 . Then, $G_0 f = W_\ell * f$, where

$$W_\ell(x) = \frac{e^{-\frac{|x|}{\ell}}}{4\pi|x|}. \quad (3.2.1)$$

Moreover, G_0 is an isometric isomorphism from $H^{-1}(\mathbb{R}^3)$ to $H^1(\mathbb{R}^3)$ if we modify the standard scalar product in $H^1(\mathbb{R}^3)$ according to

$$(u, v)_{H_\ell^1} := (\nabla u, \nabla v)_{L^2} + \ell^{-2}(u, v)_{L^2}.$$

We will always consider $H^1(\mathbb{R}^3)$ endowed with this scalar product.

Furthermore, we will denote the dual pairing between $H^{-1}(\mathbb{R}^3)$ and $H^1(\mathbb{R}^3)$ by $\langle \cdot, \cdot \rangle$.

Moreover, we will use the following notation that differs slightly from the usual terminology. Given any closed set $K \subset \mathbb{R}^3$ we will denote as $H_0^1(\mathbb{R}^3 \setminus K)$ the closure in the $H^1(\mathbb{R}^3)$ topology of the set of functions $u \in C_c^\infty(\mathbb{R}^3)$ such that $u = 0$ in K . Notice that with this convention the elements of $H_0^1(\mathbb{R}^3 \setminus K)$ are also elements of $H^1(\mathbb{R}^3)$.

We now recall a classical Functional Analysis result which allows to interpret the solutions of the Dirichlet problem for elliptic equations using projections. These projection operators will be an essential tool for the analysis of the method of reflections.

Lemma 3.2.1. *Let $\Omega \subset \mathbb{R}^3$ be open. Then, for every $f \in H^{-1}(\mathbb{R}^3)$, the problem*

$$\begin{aligned} -\Delta u + \ell^{-2}u &= f & \text{in } \mathbb{R}^3 \setminus \overline{\Omega}, \\ u &= 0 & \text{in } \overline{\Omega} \end{aligned} \quad (3.2.2)$$

has a unique weak solution $u \in H^1(\mathbb{R}^3)$. Moreover, the solution for problem (3.2.2), is given by

$$P_\Omega G_0 f,$$

where P_Ω is the orthogonal projection from $H^1(\mathbb{R}^3)$ to the subspace $H_0^1(\mathbb{R}^3 \setminus \overline{\Omega})$.

Proof. Existence and uniqueness follow directly from the Riesz Representation Theorem since the weak formulation reads

$$(u, v)_{H_\ell^1(\mathbb{R}^3)} = \langle v, f \rangle \quad \text{for all } v \in H_0^1(\mathbb{R}^3 \setminus \overline{\Omega}).$$

Furthermore, denoting by u the solution to problem (3.2.2), we have for $v \in H_0^1(\mathbb{R}^3 \setminus \overline{\Omega})$

$$(G_0 f - u, v)_{H_\ell^1(\mathbb{R}^3)} = \langle v, f \rangle - \langle v, f \rangle = 0.$$

Hence, $u = P_\Omega G_0$. □

3.2.2 Formulation of the method of reflections using orthogonal projections

We now recall the method of reflections and give directly an interpretation involving the projection operators mentioned in the introduction. These projection operators are defined by

$$Q_i = 1 - P_i, \tag{3.2.3}$$

where $P_i := P_{\Omega_i}$ are the projection operators from Lemma 3.2.1. Thus, Q_i is the orthogonal projection in $H^1(\mathbb{R}^3)$ to the subspace $H_0^1(\mathbb{R}^3 \setminus \overline{\Omega_i})^\perp$. Equivalently, for $u \in H^1(\mathbb{R}^3)$, $Q_i u$ solves

$$\begin{aligned} -\Delta Q_i u + \ell^{-2} Q_i u &= 0 \quad \text{in } \mathbb{R}^3 \setminus \overline{\Omega_i}, \\ Q_i u &= u \quad \text{in } \overline{\Omega_i}. \end{aligned} \tag{3.2.4}$$

This also yields the characterization

$$H_0^1(\mathbb{R}^3 \setminus \overline{\Omega_i})^\perp = \{v \in H^1(\mathbb{R}^3) : -\Delta v + \ell^{-2} v = 0 \text{ in } \mathbb{R}^3 \setminus \overline{\Omega_i}\}. \tag{3.2.5}$$

For $f \in H^{-1}(\mathbb{R}^3)$, we define $\Phi_0 := G_0 f$. Then, the first order correction for a particle i is given by $\Phi_{1,i} := -Q_i \Phi_0$, and the first order approximation for the solution is obtained by subtracting from Φ_0 the correctors $\Phi_{1,i}$ for all the particles, i.e.,

$$\Psi_1 = \Phi_0 + \sum_{i \in I} \Phi_{1,i}.$$

Similarly, the k -th order correction for a particle i is given by

$$\Phi_{k,i} = -Q_i \sum_{j \neq i} \Phi_{k-1,j}.$$

Then, we define

$$\Phi_k = \sum_i \Phi_{k,i}. \tag{3.2.6}$$

and the k -th order approximation $\Psi_k = \Phi_0 + \dots + \Phi_k$. Therefore, the method of reflections yields the series

$$G_0 f - \sum_{i_1} Q_{i_1} G_0 f + \sum_{i_1} \sum_{i_2 \neq i_1} Q_{i_1} Q_{i_2} G_0 f - \sum_{i_1} \sum_{i_2 \neq i_1} \sum_{i_3 \neq i_2} Q_{i_1} Q_{i_2} Q_{i_3} G_0 f + \dots \tag{3.2.7}$$

As mentioned in the introduction, we want to rewrite this series in terms of powers of a certain operator. To do so, the key observation is that

$$\Phi_{k,i} = -Q_i \sum_{j \neq i} \Phi_{k-1,j} = -Q_i \Psi_{k-1}.$$

This is due to the fact that

$$\Psi_{k-1} = \Phi_{k,i} \quad \text{in } \Omega_i, \quad (3.2.8)$$

which follows inductively from the definition of Ψ_k and $\Phi_{k,i}$.

Therefore, we have

$$\Psi_{k+1} = \left(1 - \sum_i Q_i\right) \Psi_k,$$

and thus, the partial sums of the scattering series are given by

$$\left(1 - \sum_i Q_i\right)^n G_0 f. \quad (3.2.9)$$

Definition 3.2.2. *The operator $L: H^1(\mathbb{R}^3) \supset \mathcal{D}(L) \rightarrow H^1(\mathbb{R}^3)$ is defined as*

$$L = \sum_i Q_i.$$

The domain $\mathcal{D}(L)$ of this operator consists of all function $u \in H^1(\mathbb{R}^3)$ such that the series $\sum_i Q_i$ exists.

Remark 3.2.3. *We will show below (cf. Proposition 3.2.7) that L is a bounded operator in the whole of $H^1(\mathbb{R}^3)$. As mentioned in the introduction, this is due to the exponential decay in the fundamental solution of the screened Poisson equation and fails for the Poisson equation.*

Remark 3.2.4. *We note that $\mathcal{D}(L) = H^1(\mathbb{R}^3)$ implies that L is a nonnegative self-adjoint operator, since the operators Q_i are orthogonal projections*

Theorem 3.2.5. *(i) If the series (3.2.7) obtained by the method of reflections is absolutely convergent, then it yields a solution to the Dirichlet problem (3.1.16).*

(ii) The series (3.2.7) is absolutely convergent for every $f \in H^{-1}(\mathbb{R}^3)$ if the operator L from Definition 3.2.2 is a bounded operator on $H^1(\mathbb{R}^3)$ with $\|L\| < 2$. The series (3.2.7) is convergent for every $f \in H^{-1}(\mathbb{R}^3)$, then L defines a bounded operator on $H^1(\mathbb{R}^3)$ with $\|L\| \leq 2$.

(iii) Assume L is a bounded operator on $H^1(\mathbb{R}^3)$ with $\|L\| < 2$, and L has a spectral gap, i.e.,

$$\inf\{\lambda \in \sigma(L) \setminus \{0\}\} = c > 0,$$

where $\sigma(L)$ denotes the spectrum of L . Then,

$$\|(1-L)^n G_0 f - u\|_{H_\ell^1(\mathbb{R}^3)} \leq \max\{1-c, \|L\|-1\}^n \|f\|_{H_\ell^{-1}(\mathbb{R}^3)} \quad \text{for all } f \in H^{-1}(\mathbb{R}^3), \quad (3.2.10)$$

where u denotes the solution to the Dirichlet problem (3.1.16).

Proof. As above, we denote the partial sums of the series (3.2.7) by Ψ_n . Since $(-\Delta + \ell^{-2})Q_i v = 0$ in $\mathbb{R}^3 \setminus K$ for all $v \in H^1(\mathbb{R}^3)$ (cf. (3.2.4)), it follows

$$(-\Delta + \ell^{-2})\Psi_n = f \quad \text{in } \mathbb{R}^3 \setminus K.$$

Thus, this equation is also satisfied by the limit. By (3.2.8) we have $\Psi_n = \Phi_{n+1,i} \rightarrow 0$ in Ω_i since $\Phi_{n+1,i}$ appears in the series (3.2.7) which we assumed to be absolutely convergent. This implies that the limit indeed solves (3.1.16).

To prove the second statement, we observe that by (3.2.9), the partial sums of the series (3.2.7) can be written as $(1 - L)^n G_0 f$. Since G_0 is an isometry, these partial sums only exist if $\mathcal{D}(L) = H^1(\mathbb{R}^3)$. Then, by Remark 3.2.4, L is a nonnegative self-adjoint operator. Thus, by the spectral theorem (for unbounded self-adjoint operators), up to an isometry, L is a multiplication operator T on $H := L^2_\nu(X)$ for some measure space (X, \mathcal{A}, ν) , i.e., there exists a function $f \in L^\infty_\nu(X)$ such that $T\varphi = f\varphi$ for all $\varphi \in L^2_\nu(X)$. Thus, $(1 - L)^n G_0 f$ corresponds to

$$(1 - f)^n \varphi$$

which converges iff

$$-1 < (1 - f) \leq 1 \quad \nu\text{-a.e.}$$

Since L is nonnegative, this is equivalent to $f < 2$, ν -a.e., and hence, a sufficient condition for convergence is $\|L\| < 2$, and a necessary condition is $\|L\| \leq 2$.

If, in addition, L has a spectral gap, then for ν -a.e. x , $f(x) = 0$ or $f(x) \geq c$ and (3.2.10) follows. \square

Remark 3.2.6. *It is essential to observe the following. If the operator L from Definition 3.2.2 defines a bounded operator on $H^1(\mathbb{R}^3)$ with $\|L\| < 2$, then $(1 - L)^n$ converges to the orthogonal projection to the kernel of L . Indeed, by decomposing any $u \in H^1(\mathbb{R}^3)$ into $u = u_1 + u_2$, where $u_1 \in \ker L$ and $u_2 \in (\ker L)^\perp$, we see that $(1 - L)^n u_2 = u_2$ and $(1 - L)^n u_1 \rightarrow 0$ using the spectral theorem as in the proof above.*

We recall that $L = \sum_i Q_i$, where Q_i are orthogonal projections to $H_0^1(\mathbb{R}^3 \setminus \overline{\Omega_i})^\perp$. Therefore,

$$\ker L = \bigcap_i H_0^1(\mathbb{R}^3 \setminus \overline{\Omega_i}) = H_0^1(\mathbb{R}^3 \setminus K) =: V.$$

Hence, the series (3.2.7) written as $(1 - L)^n G_0 f$ converges to $PG_0 f$, where P denotes orthogonal projection to V . However, this is just a different way to see that the series indeed converges to the solution of problem (3.1.16). Indeed, the fact that $PG_0 f$ solves problem (3.1.16) follows directly from Lemma 3.2.1.

3.2.3 Estimates for the operator L

Proposition 3.2.7. *There exists a constant $C_1 > 0$ depending only on α and κ from Condition 3.1.2 and 3.1.3 such that*

$$\|L\| \leq 1 + C_1 C_\ell,$$

where C_ℓ is the defined in Condition 3.1.1

The key estimate for the proof of the above proposition is the following lemma. Roughly speaking, it states that correlations between H^{-1} functions which are supported in the particles are controlled by the capacity density times the norms of the functions themselves.

Lemma 3.2.8. Assume $(f_i)_{i \in I} \subset H^{-1}(\mathbb{R}^3)$ satisfies $\text{supp } f_i \subset \overline{B_i}$ for all $i \in I$. Then,

$$c \sum_i \|f_i\|_{H_\ell^{-1}(\mathbb{R}^3)}^2 \leq \left\| \sum_i f_i \right\|_{H_\ell^{-1}(\mathbb{R}^3)}^2 \leq (1 + C_1 C_\ell) \sum_i \|f_i\|_{H_\ell^{-1}(\mathbb{R}^3)}^2, \quad (3.2.11)$$

where $c > 0$ is a universal constant and C_1 depends only on α and κ from Condition 3.1.2 and 3.1.3.

For the proof we need the following lemma.

Lemma 3.2.9. Let $i, j \in I$. Assume $f \in H^{-1}(\mathbb{R}^3)$ is supported in $\overline{B_j}$. Then, there exists a function $v \in H_0^1(B_{\kappa r_i}(x_i))$ such that $v = G_0 f$ in B_i , and

$$\|v\|_{H_\ell^1(\mathbb{R}^3)} \leq C \sqrt{r_i r_j} \frac{e^{-\frac{|x_i - x_j|}{\ell}}}{|x_i - x_j|} \|f\|_{H_\ell^{-1}(\mathbb{R}^3)},$$

for a constant C that depends only on α and κ from Condition 3.1.2 and 3.1.3.

Proof. For $z \in B_{\kappa r_i}(x_i)$, we define $\theta \in C_c^\infty(B_{\kappa r_j}(z - x_j))$ such that $\theta = 1$ in $B_{r_j}(z - x_j)$ and $|\nabla \theta| \leq \frac{C}{r_j}$, (where the constant depends on κ). We use that f is supported in $\overline{B_j}$. Therefore, using the fundamental solution (3.2.1),

$$\begin{aligned} |(G_0 f)(z)| &= |(W_\ell * f)(z)| = |((\theta W_\ell) * f)(z)| \\ &= |\langle (\theta W_\ell)(z - \cdot), f \rangle| \leq \|f\|_{H_\ell^{-1}(\mathbb{R}^3)} \|\theta W_\ell\|_{H_\ell^1(\mathbb{R}^3)}, \end{aligned} \quad (3.2.12)$$

and

$$|\nabla(G_0 f)(z)| \leq \|f\|_{H_\ell^{-1}(\mathbb{R}^3)} \|\theta \nabla W_\ell\|_{H_\ell^1(\mathbb{R}^3)}. \quad (3.2.13)$$

We observe that Condition 3.1.2 and $z \in B_{\kappa r_i}(x_i)$ implies for $y \in B_{\kappa r_j}(z - x_j)$

$$|y| \geq |x_i - x_j| - \kappa(r_j + r_i) \geq \left(1 - \frac{1}{\kappa}\right) |x_i - x_j| \geq c |x_i - x_j|.$$

Using also Condition 3.1.3, we estimate

$$\begin{aligned} \|\theta W_\ell\|_{H_\ell^1(\mathbb{R}^3)} &\leq \|W_\ell\|_{H_\ell^1(B_{\kappa r_j}(z - x_j))} + \frac{C}{r_j} \|W_\ell\|_{L^2(B_{\kappa r_j}(z - x_j))} \\ &\leq C r_j^{3/2} e^{-\frac{|x_i - x_j| - \kappa(r_j + r_i)}{\ell}} \left(\frac{1}{c |x_i - x_j|^2} + \frac{1}{r_j c |x_i - x_j|} + \frac{1}{\ell c |x_i - x_j|} \right) \\ &\leq C r_j^{1/2} \frac{e^{-\frac{|x_i - x_j|}{\ell}}}{|x_i - x_j|}, \end{aligned}$$

and

$$\|\theta \nabla W_\ell\|_{H_\ell^1(\mathbb{R}^3)} \leq C r_j^{1/2} \frac{e^{-\frac{|x_i - x_j|}{\ell}}}{|x_i - x_j|^2}.$$

Now, we use another cutoff function $\eta \in C_c^\infty(B_{\kappa r_i}(x_i))$ such that $\eta = 1$ in B_i and $|\nabla \eta| \leq \frac{C}{r_i}$ to define $v := \eta(G_0 f)$. Then, we get from the pointwise estimates on $G_0 f$, (3.2.12) and (3.2.13),

$$\begin{aligned} \|v\|_{H_\ell^1(\mathbb{R}^3)} &= \|\eta(G_0 f)\|_{H_\ell^1(\mathbb{R}^3)} \leq \|G_0 f\|_{H_\ell^1(B_{\kappa r_i}(x_i))} + \frac{C}{r_i} \|G_0 f\|_{L^2(B_{\kappa r_i}(x_i))} \\ &\leq C \sqrt{r_i r_j} \frac{e^{-\frac{|x_i - x_j|}{\ell}}}{|x_i - x_j|} \|f\|_{H^{-1}(\mathbb{R}^3)}. \end{aligned} \quad \square$$

Proof of Lemma 3.2.8. Let $\eta_i \in C_c^\infty(B_{\kappa r_i}(x_i))$ such that $\eta_i = 1$ in B_i and $|\nabla \eta_i| \leq \frac{C}{r_i}$. Now, we observe that for all $u \in H^1(\mathbb{R}^3)$

$$\|u\|_{L^2(B_{\kappa r_i}(x_i))} \leq \|u\|_{L^6(B_{\kappa r_i}(x_i))} \|1\|_{L^3(B_{\kappa r_i}(x_i))} \leq C r_i \|\nabla u\|_{L^2(\mathbb{R}^3)},$$

where we have used the Gagliardo-Nirenberg-Sobolev inequality $\|u\|_{L^6(\mathbb{R}^3)} \leq C \|\nabla u\|_{L^2(\mathbb{R}^3)}$. Hence,

$$\|\eta_i u\|_{H_\ell^1(\mathbb{R}^3)} \leq \|u\|_{H_\ell^1(\mathbb{R}^3)} + \frac{C}{r_i} \|u\|_{L^2(B_{\kappa r_i}(x_i))} \leq C \|u\|_{H_\ell^1(\mathbb{R}^3)}. \quad (3.2.14)$$

On the other hand, denoting $f = \sum_i f_i$,

$$\begin{aligned} \sum_i \|f_i\|_{H_\ell^{-1}(\mathbb{R}^3)}^2 &= \sum_i \langle G_0 f_i, f_i \rangle = \sum_i \langle \eta_i G_0 f_i, f_i \rangle \\ &= \sum_i \langle \eta_i G_0 f_i, f \rangle \leq \|f\|_{H_\ell^{-1}(\mathbb{R}^3)} \left\| \sum_i \eta_i G_0 f_i \right\|_{H_\ell^1(\mathbb{R}^3)}. \end{aligned}$$

By taking squares on both sides and using the fact that the balls $B_{\kappa r_i}(x_i)$ are disjoint together with the preliminary estimate (3.2.14), we deduce

$$\left(\sum_i \|f_i\|_{H_\ell^{-1}(\mathbb{R}^3)}^2 \right)^2 \leq C \|f\|_{H_\ell^{-1}(\mathbb{R}^3)}^2 \sum_i \|G_0 f_i\|_{H_\ell^1(\mathbb{R}^3)}^2.$$

Since G_0 is an isometry, this yields the first inequality in (3.2.11).

For the second inequality, we use again that G_0 is an isometry to get

$$\begin{aligned} \left\| \sum_i f_i \right\|_{H_\ell^{-1}(\mathbb{R}^3)}^2 &= \left\| \sum_i G_0 f_i \right\|_{H_\ell^1(\mathbb{R}^3)}^2 \\ &= \sum_i \|G_0 f_i\|_{H_\ell^1(\mathbb{R}^3)}^2 + \sum_i \sum_{j \neq i} \langle G_0 f_i, G_0 f_j \rangle_{H_\ell^1(\mathbb{R}^3)} \\ &= \sum_i \|f_i\|_{H_\ell^{-1}(\mathbb{R}^3)}^2 + \sum_i \sum_{j \neq i} \langle G_0 f_j, f_i \rangle. \end{aligned}$$

Let $i \neq j$. Since f_i is supported in $\overline{B_i}$, we have

$$\langle G_0 f_j, f_i \rangle = \langle v, f_i \rangle,$$

for any $v \in H^1(\mathbb{R}^3)$ such that $v = G_0 f_j$ in B_i . Therefore, application of Lemma 3.2.9 yields

$$|\langle G_0 f_j, f_i \rangle| \leq C \sqrt{r_i r_j} \frac{e^{-\frac{|x_i - x_j|}{\ell}}}{|x_i - x_j|} \|f_i\|_{H_\ell^{-1}(\mathbb{R}^3)} \|f_j\|_{H_\ell^{-1}(\mathbb{R}^3)}.$$

Hence, by the Cuachy-Schwarz inequality

$$\begin{aligned} \sum_i \sum_{j \neq i} \langle G_0 f_j, f_i \rangle &\leq C \sum_i \sum_{j \neq i} \sqrt{r_i r_j} \frac{e^{-\frac{|x_i - x_j|}{\ell}}}{|x_i - x_j|} \|f_i\|_{H_\ell^{-1}(\mathbb{R}^3)} \|f_j\|_{H_\ell^{-1}(\mathbb{R}^3)} \\ &\leq C \sum_{i \neq j} \frac{r_i r_j e^{-\frac{2|x_i - x_j|}{\ell}}}{|x_i - x_j|^2} \sum_i \|f_i\|_{H_\ell^{-1}(\mathbb{R}^3)}^2 \end{aligned}$$

By definition of C_ℓ in Condition 3.1.1 this yields the desired estimate. \square

Proof of Proposition 3.2.7. We choose an enumeration of the index set I and define

$$L_N := \sum_{i=1}^N Q_i,$$

where Q_i was defined in (3.2.3). From (3.2.5) we see that every function in the image of $G_0^{-1}Q_i$ is supported in $\overline{B_i}$. Using that G_0 is an isometry, Lemma 3.2.8 implies

$$\begin{aligned} \|L^N u\|_{H_\ell^1(\mathbb{R}^3)}^2 &\leq (1 + C_1 C_\ell) \sum_{i=1}^N \|Q_i u\|_{H_\ell^1(\mathbb{R}^3)}^2 = (1 + C_1 C_\ell) \sum_{i=1}^N (Q_i u, u)_{H_\ell^1(\mathbb{R}^3)} \\ &= (1 + C_1 C_\ell) (L^N u, u)_{H_\ell^1(\mathbb{R}^3)} \\ &\leq (1 + C_1 C_\ell) \|L^N u\|_{H_\ell^1(\mathbb{R}^3)} \|u\|_{H_\ell^1(\mathbb{R}^3)}. \end{aligned}$$

Thus, $\|L^N\| \leq 1 + C_1 C_\ell$.

On the other hand, convergence of $L^N u$ holds for any $u \in H^1(\mathbb{R}^3)$ that is compactly supported, because particles lying outside of the support of u do not play any role. Thus, $Lu = \sum_{i=1}^\infty Q_i u$ is well defined for all $u \in H^1(\mathbb{R}^3)$ and $\|L\| \leq 1 + C_1 C_\ell$. Indeed, let $\varepsilon > 0$ and $u \in H^1(\mathbb{R}^3)$. Then, there exists a $v \in H^1(\mathbb{R}^3)$ with compact support such that $\|u - v\|_{H_\ell^1(\mathbb{R}^3)} \leq \varepsilon(1 + C_1 C_\ell)^{-1}$. Let $N_0 \in \mathbb{N}$ be such that all particles Ω_i lie outside of the support of v for $i > N_0$. Then, for $N, M > N_0$

$$\|L^N u - L^M u\| \leq \|L^N u - L^N v\| + \|L^N v - L^M v\| + \|L^M u - L^M v\| \leq 2\varepsilon. \quad \square$$

Remark 3.2.10. The second estimate in (3.2.11) is sharp in the following sense. For all particle configurations, $\|L\| \geq 1$. Moreover, there exists a constant c such that for all $\ell > 0$ and for all C_ℓ , there exist particle configurations satisfying Conditions 3.1.1 with prescribed C_ℓ , 3.1.2, and 3.1.3 such that

$$\|L\| \geq c C_\ell.$$

In particular, Theorem 3.2.5 implies that the series (3.2.7) is in general not convergent if μ_0 is sufficiently large.

Proof. Consider any particle configuration and a function supported in one particle, i.e., $u \in H_0^1(\Omega_i)$ for some $i \in I$. Then u is a fixed point of the operator $L = \sum_i Q_i$, because $Q_i u = u$ and $Q_j u = 0$ for all $j \neq i$. Hence $\|L\| \geq 1$.

To see that $\|L\| \gtrsim C_\ell$ we consider particles with equal radii r distributed on a lattice, i.e., $\{x_i\}_{i \in I} = (d\mathbb{Z})^3$, $\Omega_i = B_i = B_r(x_i)$, and we denote by $\mu_0 = r/d^{-3}$ the capacity density. Then, as mentioned in Section 3.1.2, approximating the sums in (3.1.13) by integrals, leads to

$$c\mu_0 \ell^2 \leq C_\ell \leq C\mu_0 \ell^2,$$

for universal constants c, C . The fact that the capacity density μ_0 has to appear on the right hand side of an estimate for $\|L\|$ follows more or less directly from the definition of the electrostatic capacity: The capacity of a set K is defined as

$$\|\nabla v\|_{L^2(\mathbb{R}^3 \setminus K)}^2,$$

where v is the solution to

$$\begin{aligned} -\Delta v &= 0 \quad \text{in } \mathbb{R}^3 \setminus K, \\ v &= 1 \quad \text{in } K. \end{aligned}$$

Now, we choose $d \ll 1$ and consider $u \in H^1(\mathbb{R}^3)$ such that $u = 1$ in $B := B_1(0)$. Then, for each $B_i \subset B$, we have for $y \in \mathbb{R}^3 \setminus B_i$

$$(Q_i u)(y) = r e^{\frac{r}{\ell}} \frac{e^{-\frac{|y-x_i|}{\ell}}}{|y-x_i|},$$

and thus,

$$\|Q_i u\|_{H_\ell^1(\mathbb{R}^3)}^2 \geq \|\nabla Q_i u\|_{L^2(\mathbb{R}^3)}^2 \geq r^2 \int_r^\infty \frac{e^{-\frac{s}{\ell}}}{s^2} ds \geq C \ell^2 r.$$

Therefore, using again that Q_i is an orthogonal projection,

$$\|L\| \geq c(Lu, u)_{H_\ell^1(\mathbb{R}^3)} = c \sum_i (Q_i u, u)_{H_\ell^1(\mathbb{R}^3)} \geq c \sum_{i: B_i \subset B} \|Q_i u\|_{H_\ell^1(\mathbb{R}^3)}^2 \geq c \ell^2 \sum_{i: B_i \subset B} r,$$

where we put the norm of u into the constant because u has been chosen independently of the particle distribution. Since the number of x_i in $(d\mathbb{Z})^3 \cap B$ is of order $d^{-3} = \mu_0 r^{-1}$, we conclude $\|L\| \geq c \ell^2 \mu_0$. \square

Using the bound on the norm of L that we proved in Proposition 3.2.7 it follows from Theorem 3.2.5 that the series (3.2.7) obtained by the method of reflections converges to the solution of problem (3.1.16). Uniform convergence also follows from Theorem 3.2.5 and the following Lemma.

Lemma 3.2.11. *There exists a constant $c_1 > 0$ depending only on κ from Condition 3.1.2 such that*

$$(Lu, u)_{H^1(\mathbb{R}^3)} \geq c_1 \|u\|_{H^1(\mathbb{R}^3)}^2,$$

for all $u \in H_0^1(\mathbb{R}^3 \setminus K)^\perp$.

Proof. Let $\eta_i \in C_c^\infty(B_{\kappa r_i}(x_i))$ such that $\eta_i = 1$ in B_i and $|\nabla \eta_i| \leq \frac{C}{r_i}$. As shown at the beginning of the proof of Lemma 3.2.8, we have

$$\|\eta_i v\|_{H_\ell^1(\mathbb{R}^3)} \leq C \|v\|_{H_\ell^1(\mathbb{R}^3)}.$$

On the other hand, we know that every $u \in H_0^1(\mathbb{R}^3 \setminus K)^\perp$ satisfies $-\Delta u + \ell^{-2}u = 0$ in $\mathbb{R}^3 \setminus K$ (cf. equation (3.2.5)). Thus, the variational form of this equation implies that u is the function of minimal norm in the set $X_u := \{v \in H^1(\mathbb{R}^3) : v = u \text{ in } K\}$. Clearly, $\sum_i \eta_i Q_i u \in X_u$, and hence,

$$\begin{aligned} (Lu, u)_{H_\ell^1(\mathbb{R}^3)} &= \sum_i (Q_i u, u)_{H_\ell^1(\mathbb{R}^3)} = \sum_i \|Q_i u\|_{H_\ell^1(\mathbb{R}^3)}^2 \\ &\geq c \sum_i \|\eta_i Q_i u\|_{H_\ell^1(\mathbb{R}^3)}^2 = c \left\| \sum_i \eta_i Q_i u \right\|_{H_\ell^1(\mathbb{R}^3)}^2 \geq c \|u\|_{H_\ell^1(\mathbb{R}^3)}^2. \end{aligned} \quad \square$$

Proof of Theorem 3.1.4. By Proposition 3.2.7, we have $\|L\| \leq 1 + C_1 C_\ell$. Furthermore, Lemma 3.2.11 implies

$$\|Lu\| \geq c_1 \|u\| \quad (3.2.15)$$

for all $u \in H_0^1(\mathbb{R}^3 \setminus K)^\perp$. By Remark 3.2.6, we have $\ker L = H_0^1(\mathbb{R}^3 \setminus K)$. Thus, Estimate (3.2.15) implies that L has a spectral gap. Therefore, Theorem 3.2.5 implies the exponential convergence

$$\|(1-L)^n G_0 f - u\|_{H_\ell^1(\mathbb{R}^3)} \leq \max\{1 - c_1, C_1 C_\ell\}^N \|f\|_{H_\ell^{-1}(\mathbb{R}^3)} \quad \text{for all } f \in H^{-1}(\mathbb{R}^3),$$

Using equivalence of the norm $\|\cdot\|_{H_\ell^1(\mathbb{R}^3)}$ to the standard H^1 -norm concludes the proof. \square

3.2.4 Convergence of a modified method of reflections

In the previous subsection, we proved that the series (3.2.7) obtained by the method of reflections converges for small capacities.

Our aim is now to prove the following theorem.

Theorem 3.2.12. *Suppose that Conditions 3.1.1, 3.1.2 and 3.1.3 hold with some constants ℓ , α and κ . Then, there exists $c_1 > 0$ depending only on κ and C_1 depending only on α and κ such that for all $\gamma > 0$*

$$\|(1 - \gamma L)^M G_0 f - u\|_{H_\ell^1} \leq \max\{1 - \gamma c_1, \gamma(1 + C_1 C_\ell) - 1\}^M \|f\|_{H_\ell^{-1}}.$$

In particular, there exists γ_0 depending only on α , κ , and C_ℓ such that for all $0 < \gamma < \gamma_0$ $(1 - \gamma L)^N G_0 f$ converges to u .

Moreover, there exists a double sequence $q(n, M, \gamma)$ such that

$$(1 - \gamma L)^M G_0 f = \sum_{n=0}^M q(n, M, \gamma) \Phi_n,$$

where Φ_n as in (3.2.6) are the n -th order correction obtained by the method of reflections.

Recall that the series is given by

$$\lim_{n \rightarrow \infty} (1 - L)^n G_0 f. \quad (3.2.16)$$

First of all, we note that the series is indeed divergent if the capacity is sufficiently large. Indeed, as shown in Remark 3.2.10 the operator norm of L diverges as the capacity tends to infinity and we have already observed in Theorem 3.2.5 that the series is divergent if the operator norm of L is larger than 2.

Now we want to give the series a meaning for arbitrary capacities. As seen in Remark 3.2.6, the solution to problem (3.1.16), which we want to obtain by the method of reflections, is given by $PG_0 f$, where P is the orthogonal projection to the kernel of L . Therefore, the modification simply consists in replacing (3.2.16) by

$$\lim_{n \rightarrow \infty} (1 - \gamma L)^n G_0 f,$$

with $\gamma := 1/\|L\|$. Using again the spectral theorem, we will show in Proposition 3.2.13 below that this ensures convergence to the solution to problem (3.1.16). However, let us first give a heuristic explanation why this can be expected.

We can give the following interpretation of the method of reflections using the representation (3.2.16). To the solution of the equation without boundary conditions $G_0 f$, we add the sum of all the correctors, which is $-L$. Doing this, we expect to push the function towards zero boundary conditions. By iterating this, we hope to obtain a sequence converging to the solution to the Dirichlet problem (3.1.16). However, if $G_0 f$ has the same sign in several particles that are close to each other and sufficiently large (i.e., large capacity), then, the effect of L is too large: The boundary conditions in each of those particles are not only corrected by the corresponding projection operator, but they also undergo a push in the same direction by the effect of all the other particles. In other words, we push in the right direction but too far. Therefore, reducing the push by multiplying with γ might solve this problem.

We can also give a purely geometrical interpretation. Let P denote the orthogonal projection to $\ker L$, and Q the projection to its orthogonal complement. We recall that L is the sum of the operators Q_i which are the orthogonal projections. Let us denote their kernel by V_i . Then

$$\ker L = \bigcap_i V_i =: V.$$

If the subspaces V_i were orthogonal to each other, then, we would have

$$1 - L = 1 - \sum_i Q_i = 1 - Q = P,$$

and the convergence of $(1 - L)^n$ to P would trivially hold.

However, they are not orthogonal to each other. Indeed, the closer two particles are, the more they interact with each other. Interaction of the particles, however, means lack of orthogonality. Therefore, the series diverges if there is too much interaction between particles – corresponding to too large values of C_ℓ .

In Figure 3.1, we see what happens in the orthogonal complement V^\perp if the angles between the subspaces V_i are small. We consider the simplest non-trivial case in which only two particles are present.

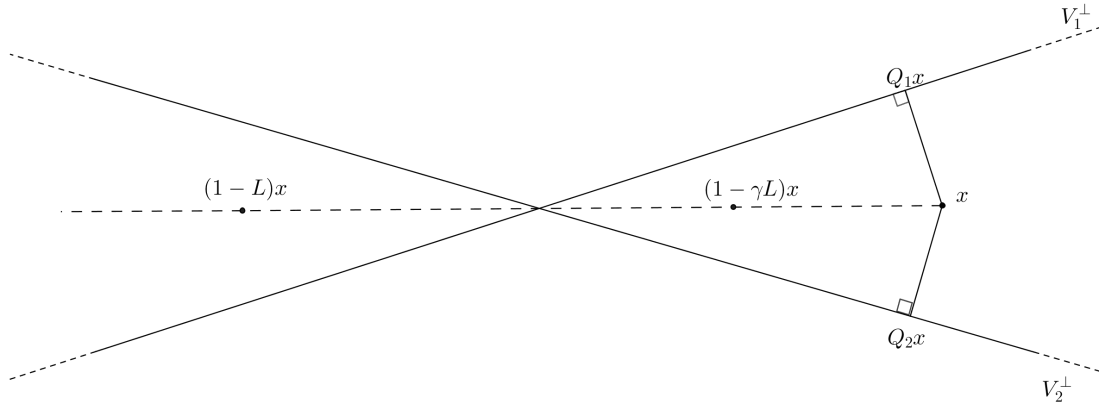


Figure 3.1: For sufficiently small angles between the subspaces as in this example, $(1 - L)x$ might end up on the other side of the origin than x . In this example, $(1 - L)x$ is still closer to the origin than x . This is a feature of the case of only two subspaces since $\|L\| < 2$ as long as the subspaces V_i have trivial intersection. Therefore, the method of reflections always yields a convergent sequence if there are only two particles and they do not intersect. However, if more particles are present and the angles between the subspaces are sufficiently small, $(1 - L)x$ will be larger than x . In that case, adding a small parameter γ in front of L will solve this problem. Indeed, in this way, we can ensure that $(1 - \gamma L)x$ lies on the same side of the origin as x by choosing $\gamma < 1/\|L\|$.

Proposition 3.2.13. *Assume H is a Hilbert space and $V_k \subset H$ are closed subspaces for $k \in J$, where J is a finite or countable index set. Define Q_k to be the orthogonal projections from H to V_k^\perp . Let $V = \cap_{k \in J} V_k$ and define P to be the orthogonal projection from H to V . If $S := \sum_{k \in J} Q_k$ defines a bounded operator, then, for all $0 < \gamma < \frac{2}{\|S\|}$,*

$$\lim_{M \rightarrow \infty} (1 - \gamma S)^M = P,$$

pointwise in H . Moreover, if S is strictly positive in V^\perp , i.e., there exists $c > 0$ such that

$$(Sx, x)_H \geq c\|x\|_H^2 \quad \text{for all } x \in V^\perp, \quad (3.2.17)$$

then,

$$\|(1 - \gamma S)^M - P\| \leq \max\{1 - \gamma c, \gamma\|S\| - 1\}^M. \quad (3.2.18)$$

Remark 3.2.14. To optimize the exponential convergence in (3.2.18), one can choose

$$\gamma = \frac{2}{\|S\| + c}.$$

Proof. By definition of S , we have $\ker S = V$. Thus, $(1 - \gamma S)^M x = x$ for all $x \in V$. On the other hand, as S is self-adjoint, we have $\mathcal{R}(S) \subset (\ker S)^\perp = V^\perp$.

We define T as the restriction of S to V^\perp (in both the domain and the range) satisfies $\|1 - \gamma T\| \leq \max\{1 - \gamma c, \gamma\|S\| - 1\}$. Thus, it suffices to show that $(1 - T)^n \rightarrow 0$ pointwise in H .

Being a sum of orthogonal projections, S and also T are self-adjoint operators. Hence, by the spectral theorem, we can assume that T is a multiplication operator on $H = L_\nu^2(X)$ for some measure space (X, \mathcal{A}, ν) , i.e., there exists a function $f \in L_\nu^\infty(X)$ such that $T\varphi = f\varphi$ for all $\varphi \in L_\nu^2(X)$. Since T is positive and bounded by $\|S\|$, we have $0 < f \leq \|S\|$. Therefore,

$$\|(1 - \gamma T)^M \varphi\|_H^2 = \int_X |\varphi|^2 (1 - \gamma f)^{2M} d\nu \rightarrow 0.$$

If in addition, (3.2.17) holds, then $c < f \leq \|S\|$. Thus,

$$\begin{aligned} \|(1 - \gamma T)\varphi\|_H^2 &= \int_X |\varphi|^2 (1 - \gamma f)^2 d\nu \\ &\leq \|1 - \gamma f\|_{L_\nu^\infty(X)}^2 \|\varphi\|_H^2 \\ &\leq \max\{1 - \gamma c, \gamma\|S\| - 1\}^2 \|\varphi\|_H^2. \end{aligned} \quad \square$$

Corollary 3.2.15. Let C_1 be the constant from Proposition 3.2.7. Then, for all particle configuration satisfying

$$C_1 C_\ell \leq C_2,$$

for some $C_2 < \infty$, there exists a constant γ_0 , which depends only on C_2 , with the following property. For all $\gamma \leq \gamma_0$,

$$(1 - \gamma L)^M \rightarrow P \quad \text{in } \mathcal{L}(H^1(\mathbb{R}^3)) \quad \text{as } M \rightarrow \infty,$$

where P is the orthogonal projection from $H^1(\mathbb{R}^3)$ to $H_0^1(\mathbb{R}^3 \setminus K)$.

Moreover, there exists $\varepsilon < 1$ depending only on κ , and C_2 such that

$$\|(1 - \gamma_0 L)^M - P\|_{\mathcal{L}(H^1(\mathbb{R}^3))} \leq C\varepsilon^n,$$

where C depends only on ℓ .

Proof. We define $\gamma_0 = 1/(1 + C_2)$. Proposition 3.2.7 implies $\gamma_0 \leq 1/\|L\|$. Then, the assertion follows directly from Proposition 3.2.13 and Lemma 3.2.11. \square

3.2.5 The modified method of reflections as a summation method

Lemma 3.2.16. Let $f \in H^{-1}(\mathbb{R}^3)$. Let Φ_n as in (3.2.6) be the n -th order correction obtained by the method of reflections. Then, for all $\gamma > 0$

$$(1 - \gamma L)^M G_0 f = \sum_{n=0}^M q(n, M, \gamma) \Phi_n,$$

where $q(0, M, \gamma) := 0$, $q(n, M, \gamma) = 0$ for $n > M$, and

$$q(n, M, \gamma) = \frac{M!}{(M-n)!(n-1)!} \int_0^\gamma t^{n-1} (1-t)^{M-n} dt = \frac{M!}{(M-n)!(n-1)!} B(\gamma; n, M-n+1),$$

for $0 < n \leq M$. Here, B denotes the incomplete Beta function. In particular, for all $\gamma > 0$, and $n \in \mathbb{N}$ it holds

$$\lim_{M \rightarrow \infty} q(n, M, \gamma) = 1.$$

Proof. As we have seen in (3.2.9), it holds

$$\sum_{n=0}^M \Phi_n = (-L_r)^M G_0 f.$$

By induction, this leads to the following identity

$$(-L_r)^M G_0 f = \sum_{n=1}^M (-1)^{M-n} \binom{M-1}{n-1} \Phi_n. \quad (3.2.19)$$

Expanding $(1 - \gamma L)^M$ and using (3.2.19) leads to $q(0, M, \gamma) = 1$, $q(n, M, \gamma) = 0$ for $n > M$, and, for $0 < n \leq M$,

$$\begin{aligned} q(n, M, \gamma) &= \sum_{l=n}^M \binom{M}{l} \gamma^l (-1)^{l-n} \binom{l-1}{n-1} \\ &= (-1)^n \sum_{l=n}^M \frac{M!}{l(M-l)!} \frac{(-\gamma)^l}{(n-1)!(l-n)!} \\ &= (-1)^n \frac{M!}{(n-1)!} \sum_{k=0}^{M-n} \frac{1}{k+l} \frac{(-\gamma)^{k+l}}{(M-n-k)!k!}. \end{aligned}$$

Defining

$$\psi(z) := \sum_{k=0}^{M-n} \frac{1}{k+n} \frac{z^{k+l}}{(M-n-k)!k!},$$

we find

$$\begin{aligned} \frac{d}{dz} \psi(z) &= \sum_{k=0}^{M-n} \frac{z^{k+n-1}}{(M-n-k)!k!} \\ &= \frac{z^{n-1}}{(M-n)!} (1+z)^{M-n}, \end{aligned}$$

and hence,

$$\psi(z) = \frac{1}{(M-n)!} \int_0^z t^{n-1} (1+t)^{M-n} dt.$$

Inserting this in the above equation, we finally get

$$\begin{aligned} \sum_{l=n}^M \binom{M}{l} \gamma^l (-1)^{l-n} \binom{l-1}{n-1} &= (-1)^n \frac{M!}{(M-n)!(n-1)!} \int_0^{-\gamma} t^{n-1} (1+t)^{M-n} dt \\ &= \frac{M!}{(M-n)!(n-1)!} \int_0^{\gamma} t^{n-1} (1-t)^{M-n} dt \\ &= \frac{M!}{(M-n)!(n-1)!} B(\gamma; n, M-n+1). \end{aligned} \quad \square$$

Proof of Theorem 3.2.12. The result is a direct consequence of Corollary 3.2.15 and Lemma 3.2.16. \square

3.3 The Poisson equation

Throughout this section, we will again assume that a particle configuration $(\Omega_i)_{i \in I}$ with corresponding balls $B_i \supset \Omega_i$ is given which satisfy Conditions 3.1.1 and 3.1.2 for some κ and ℓ . For the ease of notation, we assume $\ell = 2$.

In order to directly apply the method to the Poisson equation, we need to change the spaces that we work in to make it possible to solve the Poisson equation in the whole space.

Definition 3.3.1. We define the homogeneous Sobolev space $\dot{H}^1(\mathbb{R}^3)$ as the closure of $C_c^\infty(\mathbb{R}^3)$ with respect to the L^2 -norm of the gradient and denote its dual by $\dot{H}^{-1}(\mathbb{R}^3)$. Moreover, for an open set $\Omega \subset \mathbb{R}^3$, we define the space $\dot{H}_0^1(\Omega)$ to be $\{u \in \dot{H}^1 : u = 0 \text{ in } \mathbb{R}^3 \setminus \Omega\}$.

Note that, with these definitions, the Laplacian is an isometry from \dot{H}^1 into $\dot{H}^{-1}(\mathbb{R}^3)$. Correspondingly to the previous section, we denote $G_0 = (-\Delta)^{-1}$. Then, $G_0 f = \Phi * f$, where $\Phi(x) = (4\pi|x|)^{-1}$.

The following lemma corresponds to Lemma 3.2.1.

Lemma 3.3.2. Let $\Omega \subset \mathbb{R}^3$ be open. Then, for every $f \in \dot{H}^{-1}(\mathbb{R}^3)$, the problem

$$\begin{aligned} -\Delta u &= f & \text{in } \mathbb{R}^3 \setminus \overline{\Omega}, \\ u &= 0 & \text{in } \overline{\Omega} \end{aligned} \tag{3.3.1}$$

has a unique weak solution $u \in \dot{H}^1(\mathbb{R}^3)$. Moreover, the solution for problem (3.3.1) is given by

$$P_\Omega G_0 f,$$

where P_Ω is the orthogonal projection from $\dot{H}^1(\mathbb{R}^3)$ to the subspace $\dot{H}_0^1(\mathbb{R}^3 \setminus \overline{\Omega})$.

As before, we define

$$Q_i = 1 - P_i,$$

where $P_i := P_{\Omega_i}$ are the projection operators from Lemma 3.3.2. Moreover, we note as in (3.2.5) that Q_i is the orthogonal projection to

$$\dot{H}_0^1(\mathbb{R}^3 \setminus \overline{\Omega_i})^\perp = \{v \in \dot{H}^1(\mathbb{R}^3) : -\Delta v = 0 \text{ in } \mathbb{R}^3 \setminus \overline{\Omega_i}\}. \tag{3.3.2}$$

As mentioned before, the operator $\sum_i Q_i$, which we have denoted L for the screened Poisson equation, will in general not be a bounded operator for infinitely particles. This is due to the long range interactions of the Laplacian. Therefore, we use a spatial cutoff to define the operator L for the Poisson equation.

Definition 3.3.3. We define

$$L := \sum_i e^{-|x_i|} Q_i.$$

Remark 3.3.4. The choice of the specific exponential cutoff has only been made for definiteness and to make the proof of the estimate for L (cf. Lemma 3.3.5) as analogous to the screened Poisson equation as possible. However, any cutoff η would work which maps $\dot{H}^1(\mathbb{R}^3)$ to $\dot{H}^{-1}(\mathbb{R}^3)$ in the sense that $\int \eta u v dx$ is well defined for all $u, v \in \dot{H}^1(\mathbb{R}^3)$. (Note that $\int u v dx$ is not well defined for $u, v \in \dot{H}^1(\mathbb{R}^3)$, and thus $\dot{H}^1(\mathbb{R}^3)$ is not contained in $\dot{H}^{-1}(\mathbb{R}^3)$ in this sense.) In particular, we could choose a polynomial cutoff with sufficiently fast decay.

3.3.1 Convergence of the modified method of reflections

Lemma 3.3.5. *The operator L from Definition 3.3.3 is a well defined, bounded, nonnegative, self-adjoint operator on $\dot{H}^1(\mathbb{R}^3)$ with*

$$\|L\| \leq (1 + CC_2),$$

where C_2 is the constant defined in Condition 3.1.1 and C depends only on κ from Condition 3.1.2.

The proof follows the lines of the proof of the corresponding result for the screened Poisson equation, Proposition 3.2.7. The only difference is that the exponential cutoff in the definition of L replaces the the exponential decay of the fundamental solution of the screened Poisson equation (3.2.1). We omit the details of the proof. However, we state the lemma corresponding to Lemma 3.2.8 for further reference.

Lemma 3.3.6. *Assume $(f_i)_{i \in I} \subset \dot{H}^{-1}(\mathbb{R}^3)$ satisfy $\text{supp } f_i \subset \overline{B_i}$. Then,*

$$\left\| \sum_i e^{-|x_i|} f_i \right\|_{\dot{H}^{-1}(\mathbb{R}^3)}^2 \leq (1 + CC_2) \sum_i e^{-|x_i|} \|f_i\|_{\dot{H}^{-1}(\mathbb{R}^3)}^2,$$

where the constant C depends only on κ from Condition 3.1.2.

As in Proposition 3.2.13 we would like to prove convergence for

$$(1 - \gamma L)^n G_0 f = (1 - \sum_i \gamma e^{-|x_i|} Q_i)^n G_0 f.$$

for sufficiently small $\gamma > 0$. The only difference is that, instead of putting the same small factor γ in front of all the operators Q_i , we now have factors depending on the particle position due to the spatial cutoff $e^{-|x_i|}$ in Definition 3.3.3. Thus, we will see in Proposition 3.3.7 below, that convergence to the desired solution still holds for sufficiently small γ . However, due to the spatial cutoff, L lacks the coercivity on $\dot{H}_0^1(\mathbb{R}^3 \setminus K)^\perp$ the analogous of which we had in the case of the screened Poisson equation (cf. Lemma 3.2.11): Clearly, if $u \in \dot{H}_0^1(\mathbb{R}^3 \setminus K)^\perp$ is only non-zero in particles very far away from the origin, then, $\|Lu\|_{\dot{H}^1}$ is very small compared to $\|u\|_{\dot{H}^1}$. Hence, we cannot expect any result about uniform convergence of $(1 - \gamma L)^n G_0$ from a purely abstract argument as in Proposition 3.2.13. Indeed, the farther the mass of the source term f is away from the origin, the slower we expect the convergence to take place.

Proposition 3.3.7. *Let H be a Hilbert space and $V_k \subset H$ closed subspaces for $k \in J$, where J is a finite or countable index set. Define Q_k to be the orthogonal projections from H to V_k^\perp . Let $V = \cap_{k \in J} V_k$ and define P to be the orthogonal projection from H to V . Assume $\gamma_k > 0$, $k \in J$, are chosen such that $S := \sum_{k \in J} \gamma_k Q_k$ defines a bounded operator with $\|S\| < 2$. Then,*

$$\lim_{M \rightarrow \infty} (1 - S)^M = P,$$

pointwise in H .

If $\|S\| \leq 1$, then for all $x \in H$,

$$(Sx, x)_H \geq \|Sx\|_H^2, \tag{3.3.3}$$

and

$$(S(1 - S)x, (1 - S)x)_H \leq (Sx, x)_H. \tag{3.3.4}$$

Proof. The statement about convergence is proven in the same way as in Proposition 3.2.13.

Observe that estimates (3.3.3) and (3.3.4) are trivially satisfied in V . We define again T as the restriction of S to V^\perp (in both the domain and the range). Using the spectral theorem, we can assume T to be a multiplication operator on $H = L^2_\nu(X)$ for some measure space (X, \mathcal{A}, ν) , i.e., there exists a function $f \in L^\infty_\nu(X)$ such that $T\varphi = f\varphi$ for all $\varphi \in L^2_\nu(X)$. By assumption, we know $0 < f \leq 1$. Therefore,

$$(T\varphi, \varphi)_H = \int_X f\varphi^2 d\nu \geq \int_X f^2\varphi^2 d\nu = \|T\varphi\|_H^2,$$

and

$$(T(1-T)\varphi, (1-T)\varphi)_H = \int_X f(1-f)^2\varphi^2 d\nu \leq \int_X f\varphi^2 d\nu = (T\varphi, \varphi)_H. \quad \square$$

Proof of Theorem 3.1.6. We define $\gamma_0 \leq 1/\|L_r\|$. Proposition 3.2.7 ensures that this is possible in such a way that γ_0 depends only on C_2 and κ . Then, the assertion follows directly from Proposition 3.3.7 and Lemma 3.3.2. \square

3.3.2 The modified method of reflections on the level of the original series

In this subsection, we will show how to compute the expansion of the term $(1 - \gamma L)^n$ in order to obtain a series similar to the original series obtained by the method of reflections (3.2.7). This is not only interesting in itself, but will be used to derive the homogenization results Theorem 3.1.9 and 3.5.12 in Section 3.4.

This leads to the following definition and lemma.

Definition 3.3.8. Let $n \in \mathbb{N}_*$ and $\beta \in \mathbb{N}_*^n$, where we denote $\mathbb{N}_* := \mathbb{N} \setminus \{0\}$. Then, we define the operator $A_\beta: \dot{H}^1(\mathbb{R}^3) \rightarrow \dot{H}^1(\mathbb{R}^3)$ by

$$A_\beta = \sum_{i_1} e^{-\beta_1|x_{i_1}|} Q_{i_1} \sum_{i_2 \neq i_1} e^{-\beta_2|x_{i_2}|} Q_{i_2} \cdots \sum_{i_n \neq i_{n-1}} e^{-\beta_n|x_{i_n}|} Q_{i_n}.$$

Lemma 3.3.9. For all $n \in \mathbb{N}_*$, the following identity holds

$$(L)^n = \sum_{l=1}^n \sum_{\substack{\beta \in \mathbb{N}_*^l \\ |\beta|=n}} A_\beta.$$

In particular, for all $\beta \in \mathbb{N}_*^n$, A_β is a bounded operator with

$$\|A_\beta\| \leq (1 + CC_2)^n,$$

where C_2 is the constant defined in Condition 3.1.1 and C depends only on κ from Condition 3.1.2.

Proof. For $n = 1$, the assertion is trivial. Let $n \geq 2$ and $\beta \in \mathbb{N}_*^n$. We write $\beta = (\beta_1, \beta')$ for some $\beta' \in \mathbb{N}_*^{n-1}$. Using $Q_x^2 = Q_x$, it is easy to see that

$$LA_\beta = A_{(1, \beta)} + A_{(\beta_1+1, \beta')}.$$

Observe that for every $1 \leq l \leq n+1$ and every $\gamma \in \mathbb{N}_*^l$ with $|\gamma| = n+1$, either $\gamma_1 = 1$, then, there exists a unique $\beta \in \mathbb{N}_*^{l-1}$ with $|\beta| = n$ such that $\gamma = (1, \beta)$, or $\gamma_1 > 1$, then, $l \leq n$, and there exists a unique $\beta \in \mathbb{N}_*^l$ with $|\beta| = n$ such that $\gamma = (\beta_1 + 1, \beta')$. Therefore, the assertion for n follows from the one for $n-1$.

For $\beta \in \mathbb{N}_*^n$ with $\beta_j = 1$ for all $1 \leq j \leq n$, the estimate for the operators A_β follows directly from the bound on L (see Lemma 3.3.5) and the identity that we just have proven, since all the operators Q_i are positive. For general $\gamma \in \mathbb{N}_*^n$, we clearly have $\|A_\gamma\| \leq \|A_\beta\|$ if β is chosen as above. This concludes the proof. \square

3.4 Homogenization

In the following, we will always consider particle configurations indexed by δ that satisfy Assumption 3.1.8. We will sometimes put an index δ on objects defined in the previous sections to emphasize the dependence on δ . On the other hand, we will often omit the index δ for the ease of notation. In particular, we will often write x_i , r_i , and B_i instead of $x_{i,\delta}$, $r_{i,\delta}$ and $B_{i,\delta}$. The same applies to objects that are going to be defined in this section.

3.4.1 A Poincaré inequality for perforated domains

A consequence of the lower bound $\mu_\delta \geq \mu_1$ from Assumption 3.1.8 is that problem (3.1.1) admits a unique solution in $H^1(\mathbb{R}^3)$ for sources $f \in H^{-1}(\mathbb{R}^3)$, instead of solutions only in $\dot{H}^1(\mathbb{R}^3)$ for sources in $\dot{H}^{-1}(\mathbb{R}^3)$. This is due to the existence of a Poincaré inequality in the space $H_0^1(\mathbb{R}^3 \setminus K)$. A related inequality has been proven in [All90b]. Since we allow for more general particle configurations, the proof needs to be modified.

We first notice the following local Poincaré inequality.

Lemma 3.4.1. *Assume $z \in \mathbb{R}^3$, $R > \rho > 0$ and $u \in H^1(B_R(z))$. Then,*

$$\|u\|_{L^2(B_R(z) \setminus B_\rho(z))}^2 \leq C \frac{R^3}{\rho} \|\nabla u\|_{L^2(B_R(z))}^2 + C \frac{R^3}{\rho^2} \|u\|_{L^2(\partial B_\rho(z))}^2,$$

for a universal constant C . Furthermore, one can replace $B_R(z)$ by any $\Omega \subset B_R(z)$ which is star shaped with respect to z (i.e., the line segment connecting z and x is contained in Ω for every $x \in \Omega$).

Proof. It suffices to prove the estimate for $z = 0$ and for smooth functions. Let $\varphi \in C^1(B_R(0))$. Then, denoting the unit sphere in \mathbb{R}^3 by S^2 we have for every $x \in S^2$ and every $t \in (\rho, R)$

$$|\varphi(tx)| \leq |\varphi(\rho x)| + \int_\rho^R |\nabla \varphi(sx)| ds.$$

Thus,

$$\begin{aligned} \int_{B_R(0) \setminus B_\rho(0)} |\varphi|^2 dy &\leq C \int_{S^2} \int_\rho^R t^2 \left(\int_\rho^R |\nabla \varphi(sx)| ds \right)^2 dt dx + C \int_{S^2} \int_\rho^R t^2 |\varphi(\rho x)|^2 dt dx \\ &\leq CR^3 \int_{S^2} \int_\rho^R \frac{1}{s^2} ds \int_\rho^R s^2 |\nabla \varphi(sx)|^2 ds dx + C \frac{R^3}{\rho^2} \int_{S^2} \rho^2 |\varphi(\rho x)|^2 dt dx \\ &\leq C \frac{R^3}{\rho} \int_{B_R(0) \setminus B_\rho(0)} |\nabla \varphi|^2 dy + C \frac{R^3}{\rho^2} \int_{\partial B_\rho(0)} |u|^2 dy. \quad \square \end{aligned}$$

Lemma 3.4.2. *Let $\Omega \subset \mathbb{R}^3$ be open and bounded and $c > 0$. Then there exists a constant C such that for all measurable $U \subset \Omega$ with $|U| \geq c$ and all $u \in H^1(\Omega)$*

$$\|u\|_{L^2(\Omega)} \leq C (\|u\|_{L^2(U)} + \|\nabla u\|_{L^2(\Omega)}).$$

Proof. We prove the statement by contradiction. Assume there is no such constant C . Then there exist sequences $U_k \subset \Omega$ with $|U_k| \geq c$ and $u_k \in H^1(\Omega)$ such that $\|u_k\|_{L^2(\Omega)} = 1$ and

$$\|u\|_{L^2(U_k)} + \|\nabla u\|_{L^2(\Omega)} \leq \frac{1}{k}. \quad (3.4.1)$$

Then, (3.4.1) implies $\nabla u_k \rightarrow 0$ in $L^2(\Omega)$. Thus, by Rellich embedding and $\|u_k\|_{L^2(\Omega)} = 1$ we find $u_k \rightarrow |\Omega|^{-1/2}$ in $L^2(\Omega)$. However, (3.4.1) implies

$$\|u_k - |\Omega|^{-1/2}\|_{L^2(\Omega)} \geq \|u_k - |\Omega|^{-1/2}\|_{L^2(U_k)} \geq \| |\Omega|^{-1/2} \|_{L^2(U_k)} - \|u_k\|_{L^2(U_k)} \geq c^{1/2} |\Omega|^{-1/2} - \frac{1}{k}.$$

This yields the desired contradiction. \square

By a standard scaling argument, one proves the following result.

Lemma 3.4.3. *For all $c > 0$ there exists a constant $C > 0$ with the following property. Let $0 < \delta < 1$, $q \subset \mathbb{R}^3$ be a cube of length δ and assume $U \subset q$ satisfies $|U| \geq c\delta^3$. Then, for all $u \in H^1(q)$,*

$$\|u\|_{L^2(q)} \leq C \left(\|u\|_{L^2(U)} + \|\nabla u\|_{L^2(q)} \right).$$

Lemma 3.4.4. *There exists a constant C_1 depending only on C_ℓ and μ_0 from Condition 3.1.1* and Assumption 3.1.8 such that for all $0 < \delta < 1$ all $x \in (\delta\mathbb{Z})^3$, and all $u \in H^1(\mathbb{R}^3)$ such that $u = 0$ in B_i for all $i \in I_\delta$ with $x_i \in q_x^\delta$*

$$\|u\|_{L^2(q_x^\delta)} \leq C_1 \|\nabla u\|_{L^2(q_x^\delta)}.$$

Proof. Let $x \in (\delta\mathbb{Z})^3$. Define $\bar{d}_i = (\mu_0^{-1}r_i)^{1/3}/2$ and notice that by Condition 3.1.1* the balls $B_{\bar{d}_i}(x_i)$ are disjoint. We further denote

$$U := q_x^\delta \cap \bigcup_{x_i \in q_x^\delta} B_{\bar{d}_i}(x_i).$$

If there $x_i \in q_x^\delta$ with $\bar{d}_i \geq \delta/2$, then $|U| \geq \pi\delta^3/6$. Otherwise, for all $x_i \in q_x^\delta$ at least a sector of volume of $1/8$ of the ball $B_{\bar{d}_i}(x_i)$ is contained in q_x^δ . Then,

$$|U| \geq C \sum_{x_i \in q_x^\delta} \mu_0^{-1}r_i \geq C\delta^3 \frac{\mu_1}{\mu_0},$$

where we used $\mu_\delta \geq \mu_1$ by Assumption 3.1.8. Let $u \in H^1(\mathbb{R}^3)$ such that $u = 0$ in B_i for all $i \in I_\delta$ with $x_i \in q_x^\delta$. Then, Lemma 3.4.3 implies

$$\|u\|_{L^2(q_x^\delta)} \leq C \left(\|u\|_{L^2(U)} + \|\nabla u\|_{L^2(q_x^\delta)} \right), \quad (3.4.2)$$

where the constant C depends only on μ_1/μ_0 . By Lemma 3.4.1, applied to the sets $B_{\bar{d}_i}(x_i) \cap q_x^\delta$, we have

$$\|u\|_{L^2(U)} \leq \mu_0^{-1/2} \|\nabla u\|_{L^2(U)}.$$

Inserting this in equation (3.4.2) finishes the proof. \square

As a direct consequence of Lemma 3.4.4, we have the following Lemma and Corollary.

Lemma 3.4.5. *For all $0 < \delta < 1$ and all $u \in H_0^1(\mathbb{R}^3 \setminus K_\delta)$*

$$\|u\|_{L^2(\mathbb{R}^3)}^2 \leq C_1 \|\nabla u\|_{L^2(\mathbb{R}^3)}^2$$

for a constant C_1 which depends only on μ_0 and μ_1 from Condition 3.1.1 and Assumption 3.1.8.*

Corollary 3.4.6. *For all $0 < \delta < 1$ and all $f \in H^{-1}(\mathbb{R}^3)$, there exists a unique weak solution $u \in H^1(\mathbb{R}^3)$ to the problem*

$$\begin{aligned} -\Delta u &= f & \text{in } \mathbb{R}^3 \setminus K_\delta, \\ u &= 0 & \text{in } K_\delta, \end{aligned} \quad (3.4.3)$$

which satisfies

$$\|u\|_{H^1(\mathbb{R}^3)}^2 \leq C_1 \|f\|_{H^{-1}(\mathbb{R}^3)}^2$$

for a constant C_1 which depends only on μ_0 and μ_1 from Condition 3.1.1* and Assumption 3.1.8.

Lemma 3.4.7. *For all $\lambda > 0$ there exists a constant C_1 depending only on λ , μ_0 , and μ_1 from Condition 3.1.1* and Assumption 3.1.8 with the following property. For all $0 < \delta < 1$, all $x \in (\delta\mathbb{Z})^3$, all $q_x^\delta \subset \Omega \subset B_{\lambda\delta}(x)$ which are star shaped with respect to x , and all $u \in H^1(\mathbb{R}^3)$ such that $u = 0$ in B_i for all $i \in I_\delta$ with $x_i \in q_x^\delta$*

$$\|u\|_{L^2(\Omega)} \leq C_1 \|\nabla u\|_{L^2(\Omega)}.$$

Proof. By Lemma 3.4.4, it suffices to prove

$$\|u\|_{L^2(\Omega \setminus q_x^\delta)}^2 \leq C_1 \left(\|\nabla u\|_{L^2(\Omega)}^2 + \|u\|_{H^1(q_x^\delta)}^2 \right).$$

We have $B_{\delta/2}(x) \subset q_x^\delta$. Using the trace theorem and scaling yields

$$\|u\|_{L^2(\partial B_{\delta/2}(x))}^2 \leq C\delta \|\nabla u\|_{L^2(B_{\delta/2}(x))}^2 + \frac{C}{\delta} \|u\|_{L^2(B_{\delta/2}(x))}^2 \leq \frac{C}{\delta} \|u\|_{H^1(q_x^\delta)}^2.$$

Thus, applying Lemma 3.4.1 yields

$$\begin{aligned} \|u\|_{L^2(\Omega \setminus B_{\delta/2}(x))}^2 &\leq C \frac{(\lambda\delta)^3}{\delta} \|\nabla u\|_{L^2(\Omega \setminus B_{\delta/2}(x))}^2 + C \frac{(\lambda\delta)^3}{\delta^2} \|u\|_{L^2(\partial B_{\delta/2}(x))}^2 \\ &\leq C\lambda^3 \left(\|\nabla u\|_{L^2(\Omega)}^2 + \|u\|_{H^1(q_x^\delta)}^2 \right). \end{aligned} \quad \square$$

3.4.2 The main idea of the proof

In order to explain the idea how to prove the homogenization result, we need the definition below. Here and below, we will use the notation

$$(u)_{i,\delta} := \int_{B_{r_{i,\delta}}(x_{i,\delta})} u(y) dy.$$

Definition 3.4.8. *For a particle with radius $r_{i,\delta}$ at position $x_{i,\delta}$, we define the operator $T_{i,\delta}$ from $\dot{H}^1(\mathbb{R}^3)$ to $\dot{H}^{-1}(\mathbb{R}^3)$ by means of*

$$Q_{i,\delta} = G_0 T_{i,\delta}.$$

Moreover, we define $M_{i,\delta}: \dot{H}^1(\mathbb{R}^3) \rightarrow \dot{H}^{-1}(\mathbb{R}^3)$ to be the uniform charge density approximation of $T_{i,\delta}$,

$$(M_{i,\delta}u)(y) = \frac{(u)_{i,\delta}}{r} \mathcal{H}^2|_{\partial B_{i,\delta}},$$

where \mathcal{H}^2 denotes the two dimensional Hausdorff measure. Furthermore, we define $\tilde{Q}_{i,\delta} = G_0 M_{i,\delta}$ to be the induced approximation for $Q_{i,\delta}$.

The uniform charge density approximations of the operators $A_\beta^{(\delta)}$ from Definition 3.3.8 are defined by

$$M_\beta^{(\delta)} := \sum_{i_1 \in I_\delta} e^{-\beta_1 |x_{i_1}|} \tilde{Q}_{i_1} \sum_{i_2 \neq i_1} e^{-\beta_2 |x_{i_2}|} \tilde{Q}_{i_2} \cdots \sum_{i_n \neq i_{n-1}} e^{-\beta_n |x_{i_n}|} \tilde{Q}_{i_n}.$$

Remark 3.4.9. For $u \in H^1(\mathbb{R}^3)$, $T_i u$ is supported in $\overline{B_i}$. Since $T_i = G_0^{-1} Q_i$, and Q_i is the orthogonal projection to $H_0^1(\mathbb{R}^3 \setminus \overline{B_i})^\perp$, this follows directly from the characterization (3.3.2).

To understand the meaning of the operator T_i , we take any potential $u \in \dot{H}^1(\mathbb{R}^3)$ and denote by $f := G_0^{-1} u$ the source corresponding to u . Moreover, we denote $g = T_i u$. Then, subtracting g from f , gives a source $f - g$, which corresponds to a potential $v := G_0(f - g)$ that solves

$$\begin{aligned} -\Delta v &= f & \text{in } \mathbb{R}^3 \setminus \overline{B_i}, \\ v &= 0 & \text{in } \overline{B_i}. \end{aligned}$$

We can also draw the following analogy to electrostatics. In this context, $-g = -T_i G_0 f$ gives the charge density that is induced by f in B_i if B_i represents a grounded conductor (surrounded by vacuum).

With this definition the original series obtained by the Method of Reflection (3.2.7) becomes,

$$G_0 - \sum_{i_1} G_0 T_{i_1} G_0 + \sum_{i_1} \sum_{i_2 \neq i_1} G_0 T_{i_1} G_0 T_{i_2} G_0 - \dots, \quad (3.4.4)$$

This is how the series appears in [Kir82], where T_i is called a scattering operator. In this paper, the Method of Reflection is interpreted as a scattering process. Viewing G_0 as some kind of propagator, (3.4.4) inherits the interpretation of the potential due to a source which propagates according to G_0 and scattered at the particles by T_i .

We want to give an heuristic explanation for the homogenization result Theorem 3.1.9. To do so, let us pretend for the moment that the series (3.4.4) exists, and that all the operators are well defined on $H^1(\mathbb{R}^3)$ (instead of $\dot{H}^1(\mathbb{R}^3)$). Moreover, let us assume that we already know that in the limit $\delta \rightarrow 0$, we can replace the operator T_i by M_i in Definition 3.4.8. Using the definition of M_i Assumption 3.1.8 guarantees that $\langle \varphi, \sum_{i \in I_\delta} M_i u \rangle$ can be interpreted as a Riemann sum for $\langle \varphi, J\mu u \rangle$, where J is the inclusion from $H^1(\mathbb{R}^3)$ to $H^{-1}(\mathbb{R}^3)$. This leads to

$$\sum_{i \in I_\delta} T_i u \approx \sum_{i \in I_\delta} M_i u \rightharpoonup J\mu u \quad \text{in } H^{-1}(\mathbb{R}^3),$$

as $\delta \rightarrow 0$

Therefore, the first order term in the series (3.4.4) converges to $(-G_0 J\mu) G_0 f$. It seems plausible that the higher order terms converge weakly to $(-G_0 J\mu)^k G_0 f$. Thus, the weak limit of the sequence of solutions is formally given by

$$\sum_{k=0}^{\infty} (-G_0 J\mu)^k G_0 = (1 + G_0 J\mu)^{-1} G_0 = (-\Delta + J\mu)^{-1},$$

which is the desired result.

Since the series (3.4.4) is in reality divergent, we use the modified version

$$(1 - \gamma L_\delta)^n G_0 f, \quad (3.4.5)$$

which we already know to converge to the solution of (3.1.1). We want to expand (3.4.5) in powers of L_δ and then to take the weak limit as δ tends to zero in each of the resulting terms separately. However, one has to take into account that the weak limit is not interchangeable with taking powers. Therefore it turns out, that it is convenient to use Lemma 3.3.9 in order to write $(L_\delta)^n$ as a sum of terms such that no particle appears back to back with itself like in the formal series in (3.4.4).

Somewhat surprisingly, the exponential cutoff in the definition of the operator L_δ does not cause much trouble when computing the weak limit. The only difference to the heuristic reasoning above is that some additional combinatorial identities are needed.

3.4.3 Weak limits of powers of L

Since the inclusion map from $\dot{H}^1(\mathbb{R}^3)$ to $\dot{H}^{-1}(\mathbb{R}^3)$ is not well defined (cf. Remark 3.3.4), we need the following replacement.

Definition 3.4.10. We define X to be the following subspace of $\dot{H}^1(\mathbb{R}^3)$ endowed with the $\dot{H}(\mathbb{R}^3)$ -topology.

$$X := \{u \in \dot{H}^1(\mathbb{R}^3) : u = -\Delta v \text{ for some } v \in \dot{H}^1(\mathbb{R}^3)\}.$$

Moreover, we define $J: X \rightarrow \dot{H}^{-1}(\mathbb{R}^3)$ by means of

$$\langle Ju, w \rangle = (\nabla v, \nabla w)_{L^2(\mathbb{R}^3)} \quad \text{for all } w \in \dot{H}^1,$$

where $v \in \dot{H}^1(\mathbb{R}^3)$ is the solution to $-\Delta v = u$.

Remark 3.4.11. Note that X is a dense subspace of $\dot{H}^1(\mathbb{R}^3)$ as it contains $C_c^\infty(\mathbb{R}^3)$. Moreover, J can be viewed as the inclusion map, since $\langle Ju, w \rangle = \int_{\mathbb{R}^3} uw \, dx$, whenever the latter is well defined.

Lemma 3.4.12. The operator $A: \dot{H}^1(\mathbb{R}^3) \rightarrow \dot{H}^1(\mathbb{R}^3)$,

$$(Au)(x) = e^{-|x|}u(x),$$

is a bounded linear operator with range $\mathcal{R}(A) \subset X$. Moreover, the composition JA , where J is the inclusion operator from Definition 3.4.10, is a bounded operator from $\dot{H}^1(\mathbb{R}^3)$ to $\dot{H}^{-1}(\mathbb{R}^3)$.

Proof. We observe that the range of A satisfies $\mathcal{R}(A) \subset \dot{H}^1(\mathbb{R}^3) \cap L^{6/5}(\mathbb{R}^3) \subset X$. The first inclusion follows from the Gagliardo-Nirenberg-Sobolev inequality $\|w\|_{L^6(\mathbb{R}^3)} \leq C\|\nabla w\|_{L^2(\mathbb{R}^3)}$ and Hölder's inequality. The second one is deduced by the Gagliardo-Nirenberg-Sobolev inequality, too, since this implies boundedness of the functional $F(w) := \int_{\mathbb{R}^3} uw \, dx$ in \dot{H}^1 if $u \in \dot{H}^1(\mathbb{R}^3) \cap L^{6/5}(\mathbb{R}^3)$, providing in turn a solution $v \in \dot{H}^1(\mathbb{R}^3)$ to $-\Delta v = u$.

The second assertion follows from $\|Ju\|_{\dot{H}^{-1}(\mathbb{R}^3)} = \|v\|_{\dot{H}^1(\mathbb{R}^3)}$ and the reasoning above. \square

Proposition 3.4.13. Let $u \in \dot{H}^1(\mathbb{R}^3)$ and $n \in \mathbb{N}_*$. Then, in the limit $\delta \rightarrow 0$ with μ as in Assumption 3.1.8,

$$L_\delta^n u \rightharpoonup \sum_{l=1}^n \sum_{\substack{\beta \in \mathbb{N}_*^l \\ |\beta|=n}} \left(\prod_{j=1}^l \mu G_0 J A^{\beta_j} \right) u = G_0 J \mu A (G_0 J \mu A + A)^{n-1} u =: R_n u \quad \text{in } \dot{H}^1(\mathbb{R}^3).$$

In particular, for all $\gamma > 0$ and all $M \in \mathbb{N}$

$$(1 - \gamma L_\delta)^M u \rightharpoonup \left(1 + \sum_{n=1}^M \binom{M}{n} (-\gamma)^n R_n \right) u =: S_M u \quad \text{in } \dot{H}^1(\mathbb{R}^3)$$

The fact that the complicated looking weak limit of L_δ^n equals R_n follows from the combinatorial consideration that, expanding the power in the definition of R_n , each term in the sum on the right hand side will appear exactly once.

As mentioned above, the proof of Proposition 3.4.13 is based on a Riemann sum argument using the operators T_i and M_i from Definition 3.4.8. This is not very difficult but technical. Therefore, we first show how to derive the homogenization result from Proposition 3.4.13 and the results from Section 3.3.

Proposition 3.4.14. *Let $M \in \mathbb{N}$ and S_M be the pointwise weak limit of $(1 - \gamma L_\delta)^M$ from Proposition 3.4.13. Then, there exists $\gamma_0 > 0$ such that for all $\gamma \leq \gamma_0$ and all $f \in \dot{H}^{-1}(\mathbb{R}^3)$,*

$$\lim_{M \rightarrow \infty} S_M G_0 f = u,$$

where u is the unique weak solution to

$$-\Delta u + \mu u = f \quad \text{in } \mathbb{R}^3. \quad (3.4.6)$$

Proof. We first show that $G_0 J\mu + 1$ as an operator from X to $\dot{H}^1(\mathbb{R}^3)$ is invertible and that the inverse mapping is bounded. Since G_0 is an isometry, it suffices to prove that $J\mu + G_0^{-1}$ is invertible from X to $\dot{H}^{-1}(\mathbb{R}^3)$.

Indeed, we know that for any $f \in \dot{H}^{-1}(\mathbb{R}^3) \subset H^{-1}(\mathbb{R}^3)$, problem (3.4.6) has a unique weak solution $u \in H^1(\mathbb{R}^3) \subset \dot{H}^1(\mathbb{R}^3)$. Moreover, $u = -\mu^{-1} \Delta(v - u)$, where $v \in \dot{H}^1(\mathbb{R}^3)$ is the solution to $-\Delta v = f$. Thus, $u \in X$, and therefore, the solution operator $E: \dot{H}^{-1}(\mathbb{R}^3) \rightarrow X$ for problem (3.4.6) is well defined. Hence, we have $(J\mu + G_0^{-1})^{-1} = E$.

Thus, $(G_0 J\mu + 1)^{-1} = E G_0^{-1}$. Additionally, we see that $(G_0 J\mu + 1)^{-1}$ is a bounded operator since E is bounded because for u and f as above we have $\|Ef\|_{\dot{H}^1(\mathbb{R}^3)} = \|\nabla u\|_{L^2(\mathbb{R}^3)} \leq \|f\|_{\dot{H}^{-1}(\mathbb{R}^3)}$.

Therefore, inserting the definitions of S_M and R_n from the previous theorem, we deduce

$$\begin{aligned} S_M &= 1 + \sum_{n=1}^M \binom{M}{n} (-\gamma)^n R_n = 1 + \sum_{n=1}^M \binom{M}{n} (-\gamma)^n G_0 J\mu A (G_0 J\mu A + A)^{n-1} \\ &= 1 + G_0 J\mu (G_0 J\mu + 1)^{-1} \sum_{n=1}^M \binom{M}{n} (-\gamma)^n ((G_0 J\mu + 1)A)^n \\ &= 1 + G_0 J\mu (G_0 J\mu + 1)^{-1} ((1 - \gamma(G_0 J\mu + 1)A)^M - 1). \end{aligned}$$

Next, we show that $(1 - \gamma(G_0 J\mu + 1)A)^M \rightarrow 0$ pointwise in $\dot{H}^1(\mathbb{R}^3)$ as $M \rightarrow \infty$. First, by Lemma 3.4.12, we know that $G_0 J\mu A$ is a bounded operator. Second, $G_0 J\mu A$ is also a positive operator since

$$(G_0 J\mu A u, u)_{\dot{H}^1(\mathbb{R}^3)} = \langle J\mu A u, u \rangle = \int \mu A u \cdot u \, dx = \int e^{-|x|} \mu(x) |u(x)|^2 \, dx.$$

Finally, $G_0 J\mu A$ is clearly self-adjoint since

$$(G_0 J\mu A u, v)_{\dot{H}^1(\mathbb{R}^3)} = \int \mu A u \cdot v \, dx = \int \mu A v \cdot u \, dx.$$

Therefore, using the spectral theorem for bounded self-adjoint operators as in the proof of Proposition 3.2.13, we conclude $(1 - \gamma(G_0 J\mu + 1)A)^M \rightarrow 0$ pointwise in $\dot{H}^1(\mathbb{R}^3)$ for small enough γ .

Furthermore,

$$G_0 J\mu (G_0 J\mu + 1)^{-1} = 1 - (G_0 J\mu + 1)^{-1},$$

and hence, this is a bounded operator, as well. Therefore, multiplying by G_0 from the right and taking the limit $M \rightarrow \infty$ yields

$$(1 - (1 - (G_0 J\mu + 1)^{-1}))G_0 = (1 + G_0 J\mu)^{-1}G_0 = (G_0^{-1} + J\mu)^{-1} = (-\Delta + \mu)^{-1},$$

which is the desired result. \square

3.4.4 Uniform estimates and proof of Theorem 3.1.9

Combining Proposition 3.4.13 and 3.4.14 we see that $(1 - \gamma L_\delta)^M G_0 f$ converges weakly to the solution of (3.4.6) if we take the limits in the order $\delta \rightarrow 0$ followed by $M \rightarrow \infty$. In order to prove Theorem 3.1.9, it remains interchange the order of taking the limits. For this purpose, we will prove that the speed of convergence of $(1 - \gamma L_\delta)^M G_0 f$ to u_δ in $\dot{H}_{\text{loc}}^1(\mathbb{R}^3)$ as M tends to infinity is uniform in δ .

Corresponding to Lemma 3.2.11, we have the following lemma. It implies that the sequence $(1 - \gamma L_\delta)^M G_0 f$ is close to zero boundary conditions in the particles in any fixed bounded region uniformly in δ as $M \rightarrow \infty$.

Lemma 3.4.15. *Let $u \in \dot{H}_0^1(\mathbb{R}^3 \setminus K_\delta)^\perp$ and $R > 0$, we define $v \in \dot{H}^1(\mathbb{R}^3)$ to be the solution to*

$$\begin{aligned} -\Delta v &= 0 & \text{in } \mathbb{R}^3 \setminus (K_\delta \cap \overline{B_R(0)}), \\ v &= u & \text{in } K_\delta \cap \overline{B_R(0)}. \end{aligned}$$

Then,

$$(L_\delta u, u)_{\dot{H}^1(\mathbb{R}^3)} \geq c e^{-R} \|v\|_{\dot{H}^1(\mathbb{R}^3)}^2,$$

where $c > 0$ is a constant that depends only on κ from Condition 3.1.2.

Proof. The proof follows the same argument as the proof of Lemma 3.2.11. Let $\eta_i \in C_c^\infty(B_{\kappa r_i}(x_i))$ such that $\eta_i = 1$ in B_i and $|\nabla \eta_i| \leq \frac{C}{r_i}$. By the same calculation as in the beginning of the proof of Lemma 3.2.8, we have

$$\|\eta_i w\|_{\dot{H}^1(\mathbb{R}^3)} \leq C \|w\|_{\dot{H}^1(\mathbb{R}^3)}.$$

On the other hand, by the variational form of the equation for v , we know that v is the function of minimal norm in the set $X_v := \{w \in \dot{H}^1(\mathbb{R}^3) : w = v \text{ in } K_\delta \cap \overline{B_R}\}$. Recall from equation (3.1.22) that

$$r_0 := \sup_{\delta} \sup_{i \in I_\delta} r_i < \infty.$$

Clearly, $\sum_{x_i \in B_{R+\kappa r_0}} \eta_i Q_i v \in X_u$, and hence,

$$\begin{aligned} \langle L_\delta v, v \rangle &= \sum_{i \in I_\delta} e^{-|x_i|} \|Q_i v\|_{\dot{H}^1(\mathbb{R}^3)}^2 \\ &\geq c e^{-R} \sum_{x_i \in B_{R+\kappa r_0}} \|\eta_i Q_i v\|_{\dot{H}^1(\mathbb{R}^3)}^2 \\ &= c e^{-R} \left\| \sum_{x_i \in B_{R+\kappa r_0}} \eta_i Q_i v \right\|_{\dot{H}^1(\mathbb{R}^3)}^2 \\ &\geq c e^{-R} \|v\|_{\dot{H}^1(\mathbb{R}^3)}^2. \end{aligned}$$

□

The next Lemma is needed to ensure that the values of $(1 - \gamma L_\delta)^M G_0 f$ in a fixed bounded region is very little affected by particles far away from this region.

For simplicity, we will write B_s instead of $B_s(0)$ in the following.

Lemma 3.4.16. *There exists a nonincreasing function $e: \mathbb{R}_+ \rightarrow \mathbb{R}_+$ with $\lim_{s \rightarrow \infty} e(s) = 0$ with the following property. For all $0 < \delta < 1$, all $0 \leq \rho \leq R$, all $w \in \dot{H}_0^1(\mathbb{R}^3 \setminus K_\delta)^\perp$ with $w = 0$ in $K_\delta \cap B_R$ satisfy*

$$\|\nabla w\|_{L^2(B_\rho)} \leq e(R - \rho) \|\nabla w\|_{L^2(\mathbb{R}^3)}.$$

The proof uses the classical Widman's hole filling technique (see e.g. [GM12a]). However, it is not possible to use the Poincaré estimate for mean zero functions, since the equation that w as in the Lemma satisfies is not invariant under adding constants. Therefore, we need the following Poincaré inequality.

Lemma 3.4.17. *There exist universal constant s_0 and a constant C which depends only on μ_0 and μ_1 from Condition 3.1.1* and Assumption 3.1.8 such that for all $0 < \delta < 1$, all $s_1, s \geq s_0$, and all $u \in \dot{H}_0^1(\mathbb{R}^3 \setminus K_\delta)$,*

$$\|u\|_{L^2(B_{s_1+s} \setminus B_{s_1})}^2 \leq C_1 \|\nabla u\|_{L^2(B_{s_1+s} \setminus B_{s_1})}^2.$$

Proof. Let $s_1, s > 0$. Let J denote the set of all $x \in (\delta\mathbb{Z})^3$ such that $q_x^\delta \subset B_{(s_1+s)} \setminus B_{s_1}$. Then, it is possible to choose s_0 large enough (independently of δ) such that for all $s_1, s > s_0$ the balls $B_{4\delta}(x)$, $x \in J$, are a covering of $B_{s_1+s} \setminus B_{s_1}$, and the sets $B_{4\delta}(x) \cap (B_{s_1+s} \setminus B_{s_1})$ are star shaped with respect to x . Therefore, the assertion follows from Lemma 3.4.7. \square

Proof of Lemma 3.4.16. Fix δ, R, ρ , and w according to the assumptions. Let s_0 be as in Lemma 3.4.17. Assume $R - s_0 \geq \rho + s_0$ (otherwise set $e(R - \rho) = 1$). For $\rho_0 + s_0 \leq s \leq R - s_0$, we define $\eta_s \in C_c^\infty(B_{s+s_0})$ such that $\eta_s = 1$ in B_s , $|\eta_s| \leq 1$, and $|\nabla \eta_s| \leq C$. We use $\eta^2 w$ as a test function in the weak form of the equation w satisfies, namely,

$$\begin{aligned} -\Delta w &= 0 & \text{in } \mathbb{R}^3 \setminus K_\delta \\ w &= 0 & \text{in } K_\delta \cap B_R. \end{aligned}$$

This yields

$$0 = \int_{B_{s+s_0}} \nabla w \nabla (\eta^2 w) dx = \int_{B_{s+s_0}} (\eta \nabla w)^2 + 2\eta \nabla w \nabla \eta w dx.$$

Using the Cauchy-Schwartz inequality and the Poincaré inequality in the annulus $B_{s+s_0} \setminus B_s$, provided by Lemma 3.4.17, we deduce

$$\|\nabla w\|_{L^2(B_s)}^2 \leq \|\eta \nabla w\|_{L^2(B_s)}^2 \leq C \|w\|_{L^2(B_{s+s_0} \setminus B_s)}^2 \leq C_1 \|\nabla w\|_{L^2(B_{s+s_0} \setminus B_s)}^2.$$

Let us denote $a_k := \|\nabla w\|_{L^2(B_{\rho+(k+1)s_0})}^2$ and

$$n := \max\{k \in \mathbb{N} : \rho + (k+2)s_1 \leq R\},$$

which depends only on $R - \rho$. Then, the above estimate implies for all $0 \leq k \leq n$

$$a_k \leq C_1(a_{k+1} - a_k).$$

Therefore,

$$a_k \leq \frac{C_1}{C_1 + 1} a_{k+1} =: \lambda a_{k+1},$$

and $\lambda < 1$. By iterating up to n , we conclude

$$\|\nabla w\|_{L^2(B_\rho)}^2 \leq \lambda^n \|\nabla w\|_{L^2(\mathbb{R}^3)}^2,$$

which is the desired estimate. \square

Remark 3.4.18. *As seen in the proof, the decay of e is exponential. This can be interpreted as a screening effect due to the presence of the particles. This effect can be exploited to prove homogenization results also for sources $f \in L^\infty(\mathbb{R}^3)$ (cf. [NV04a; NV06]).*

Proposition 3.4.19. *There exists $\gamma > 0$ depending only on μ_1 , μ_2 and κ such that for all $f \in \dot{H}^{-1}(\mathbb{R}^3)$ the sequence*

$$\lim_{N \rightarrow \infty} (1 - \gamma L_\delta)^N G_0 f \rightarrow v$$

uniformly in $\dot{H}_{\text{loc}}^1(\mathbb{R}^3)$ for all $0 < \delta < 1$, where v is the solution of (3.1.1).

Proof. As in the proof of Theorem 3.1.6, we choose $\gamma \leq 1/\|L_\delta\|$. Lemma 3.3.5 ensures that this is possible such that γ depends only on μ_0 and κ .

Let $\rho > 0$, $\varepsilon > 0$, and $u := G_0 f \in \dot{H}^1(\mathbb{R}^3)$. Since $\ker(L_\delta) = \dot{H}_0^1(\mathbb{R}^3 \setminus K_\delta)$, it suffices to consider $u \in \dot{H}_0^1(\mathbb{R}^3 \setminus K_\delta)^\perp$. Define $u_M := (1 - \gamma L_\delta)^M u$.

Then, we know from Proposition 3.3.7

$$\begin{aligned} \|(1 - \gamma L_\delta)u\|_{\dot{H}^1(\mathbb{R}^3)}^2 &= \|u\|_{\dot{H}^1(\mathbb{R}^3)}^2 - 2(\gamma L_\delta u, u)_{\dot{H}^1(\mathbb{R}^3)} + \|\gamma L_\delta u\|_{\dot{H}^1(\mathbb{R}^3)}^2 \\ &\leq \|u\|_{\dot{H}^1(\mathbb{R}^3)}^2 - \gamma(L_\delta u, u)_{\dot{H}^1(\mathbb{R}^3)}. \end{aligned}$$

Iterating and using monotonicity of $(L_\delta u_M, u_M)_{\dot{H}^1(\mathbb{R}^3)}$, which follows from the estimate (3.3.4) in Proposition 3.3.7, yields

$$0 \leq \|u_{M+1}\|_{\dot{H}^1(\mathbb{R}^3)}^2 \leq \|u\|_{\dot{H}^1(\mathbb{R}^3)}^2 - (M+1)\gamma(L_\delta u_M, u_M)_{\dot{H}^1(\mathbb{R}^3)}.$$

Thus,

$$(L_\delta u_M, u_M)_{\dot{H}^1(\mathbb{R}^3)} \leq \frac{1}{(M+1)\gamma} \|u\|_{\dot{H}^1(\mathbb{R}^3)}^2.$$

Define $v_M \in \dot{H}^1(\mathbb{R}^3)$ to be the solution to

$$\begin{aligned} -\Delta v_M &= 0 \quad \text{in } \mathbb{R}^3 \setminus (K_\delta \cap \overline{B_R}), \\ v_M &= u_M \quad \text{in } K_\delta \cap \overline{B_R}, \end{aligned}$$

and $w_M := u_M - v_M$. Then, Lemma 3.4.16 implies for all $R > \rho$

$$\begin{aligned} \|\nabla w_M\|_{L^2(B_\rho(0))} &\leq e(R - \rho) \|w_M\|_{\dot{H}^1(\mathbb{R}^3)} \leq e(R - \rho) \left(\|u_M\|_{\dot{H}^1(\mathbb{R}^3)} + \|v_M\|_{\dot{H}^1(\mathbb{R}^3)} \right) \\ &\leq e(R - \rho) \left(\|u\|_{\dot{H}^1(\mathbb{R}^3)} + \|v_M\|_{\dot{H}^1(\mathbb{R}^3)} \right), \end{aligned}$$

and it is possible to choose R large enough such that $e(R - \rho) < \frac{\varepsilon}{3}$. On the other hand, by Lemma 3.4.15, we have

$$ce^{-R} \|v_M\|_{\dot{H}^1(\mathbb{R}^3)}^2 \leq (L_\delta u_M, u_M)_{\dot{H}^1(\mathbb{R}^3)} \leq \frac{1}{(M+1)\gamma} \|u\|_{\dot{H}^1(\mathbb{R}^3)}^2.$$

Therefore, choosing M_0 large enough yields for all $M \geq M_0$

$$\|v_M\|_{\dot{H}^1(\mathbb{R}^3)} < \frac{\varepsilon}{3} \|u\|_{\dot{H}^1(\mathbb{R}^3)}.$$

By combining the estimates for v_M and w_M , we conclude (assuming without restriction $\varepsilon \leq 3$)

$$\|\nabla u_M\|_{L^2(B_\rho(0))} < \varepsilon \|u\|_{\dot{H}^1(\mathbb{R}^3)} = \varepsilon \|f\|_{\dot{H}^{-1}(\mathbb{R}^3)}.$$

□

Proof of Theorem 3.1.9. We first prove the statement for all sources $f \in \dot{H}^{-1}(\mathbb{R}^3)$. Let u and u_δ be as in the statement of the theorem. The functions u_δ are well defined and bounded in $\dot{H}^1(\mathbb{R}^3)$ by Corollary 3.4.6. Therefore, it suffices to consider test functions in $C_c^\infty(\mathbb{R}^3)$. Let $\varphi \in C_c^\infty(\mathbb{R}^3)$ and choose $R > 0$ such that $\text{supp } \varphi \subset B_R(0)$. Further, let $\gamma < \gamma_0$ from Proposition 3.4.19 and denote by S_M the corresponding pointwise weak limit of $(1 - \gamma L_\delta)^M$ from Proposition 3.4.13. Then, for all $M > 0$,

$$\begin{aligned} |(u_\delta - u, \varphi)_{\dot{H}^1}| &\leq |(S_M G_0 f - u, \varphi)_{\dot{H}^1}| + |(1 - \gamma L_\delta)^M G_0 f - S_M G_0 f, \varphi)_{\dot{H}^1}| \\ &\quad + |(u_\delta - (1 - \gamma L_\delta)^M G_0 f, \varphi)_{\dot{H}^1}|. \end{aligned}$$

The third term on the right hand side is estimated by

$$\|\nabla(u_\delta - (1 - \gamma L_\delta)^M G_0 f)\|_{L^2(B_R)} \|\varphi\|_{\dot{H}^1}.$$

Choosing M sufficiently large, Proposition 3.4.19 ensures that this term becomes small independently of r . On the other hand, also the first term becomes small by choosing M large, and the second term vanishes in the limit $\delta \rightarrow 0$.

Weak convergence in $\dot{H}^1(\mathbb{R}^3)$ is equivalent to weak convergence in $L^2(\mathbb{R}^3)$ of the gradients. However, due to Corollary 3.4.6, the sequence u_r is uniformly bounded in $H^1(\mathbb{R}^3)$. Therefore, we can extract subsequences that converge weakly in $H^1(\mathbb{R}^3)$. Since their weak limit is uniquely determined by the weak limit of their gradients, the whole sequence converges weakly in $H^1(\mathbb{R}^3)$.

The result for $f \in H^{-1}(\mathbb{R}^3)$ follows from density of $\dot{H}^{-1}(\mathbb{R}^3)$ in $H^{-1}(\mathbb{R}^3)$ using again that the solution operators for problem (3.4.3) are uniformly bounded. \square

3.4.5 Proof of Proposition 3.4.13

Lemma 3.4.20. *The following holds for the operators defined in Definition 3.4.8 and 3.3.8.*

(i) *There exists a constant C such that, for all $0 < \delta < 1$, all $i \in I_\delta$, and all $u \in \dot{H}^1(\mathbb{R}^3)$,*

$$\|(T_i - M_i)u\|_{\dot{H}^{-1}(\mathbb{R}^3)} \leq C \|\nabla u\|_{L^2(B_i)}.$$

(ii) *For all $u \in \dot{H}^1(\mathbb{R}^3)$, all $n \in \mathbb{N}$, and all $\beta \in \mathbb{N}_*^n$,*

$$\left\| \left(M_\beta^{(\delta)} - A_\beta^{(\delta)} \right) u \right\|_{\dot{H}^1(\mathbb{R}^3)} \rightarrow 0 \quad \text{as } \delta \rightarrow 0. \quad (3.4.7)$$

For the proof we need the following lemma.

Lemma 3.4.21. *For $r > 0$ and $x \in \mathbb{R}^3$, let $H_r := \{u \in H^1(B_r(x)) : \int_{B_r(x)} u = 0\}$. Then, for all $r > 0$, there exists an extension operator $E_r : H_r \rightarrow H_0^1(B_{2r}(x))$ such that*

$$\|\nabla E_r u\|_{L^2(B_{2r}(x))} \leq C \|\nabla u\|_{L^2(B_r(x))} \quad \text{for all } u \in H_r,$$

where the constant C is independent of r .

Proof. For $r = 1$ let $E_1 : H^1(B_1(x)) \rightarrow H_0^1(B_2(x))$ be a continuous extension operator. Then, by the Poincaré inequality on H_1 , we get for all $u \in H_1$

$$\|\nabla E_1 u\|_{L^2(B_2(x))} \leq \|E_1 u\|_{H^1(B_2(x))} \leq C \|u\|_{H^1(B_1(x))} \leq C \|\nabla u\|_{L^2(B_1(x))}.$$

The assertion for general $r > 0$ follows from scaling by defining $(E_r u)(x) := (E_1 u_r)(\frac{x}{r})$, where $u_s(x) := u(sx)$. \square

Proof of Lemma 3.4.20. Let $u \in \dot{H}^1(\mathbb{R}^3)$. First, we observe by a straightforward calculation that

$$(\tilde{Q}_i u)(y) = \begin{cases} (u)_i, & \text{if } x \in B_i, \\ (u)_i \frac{r_i}{|y - x_i|}, & \text{otherwise.} \end{cases}$$

Now, we use that G_0 is an isometry and that $Q_i = G_0 T_i$ is the orthogonal projection to the subspace

$$\dot{H}_0^1(\mathbb{R}^3 \setminus \overline{B_i})^\perp = \{u \in H^1(\mathbb{R}^3) : -\Delta u = 0 \text{ in } \mathbb{R}^3 \setminus (\overline{B_i})\}.$$

Therefore, we can characterize $Q_i u$ as the function $v \in \dot{H}^1(\mathbb{R}^3)$ that solves

$$\begin{aligned} -\Delta v &= 0 & \text{in } \mathbb{R}^3 \setminus \overline{B_i}, \\ v &= u & \text{in } \overline{B_i}. \end{aligned}$$

Hence, v is the function of minimal norm that coincides with u inside the ball B_i . Clearly, $\tilde{Q}_i u \in \dot{H}_0^1(\mathbb{R}^3 \setminus \overline{B_i})^\perp$, and thus, $Q_i \tilde{Q}_i = \tilde{Q}_i$. Therefore,

$$(Q_i - \tilde{Q}_i)u = Q_i(u - \tilde{Q}_i u).$$

Since $\tilde{Q}_i u = (u)_i$ in B_i , we can use the extension operator E_{r_i} from Lemma 3.4.21 (since, by the Rellich embedding theorem, the restriction of a \dot{H}^1 function to a ball is a H^1 function in that ball) and estimate

$$\begin{aligned} \|(Q_i - \tilde{Q}_i)u\|_{\dot{H}^1(\mathbb{R}^3)} &\leq \|E_{r_i}((u - \tilde{Q}_i u)|_{B_i})\|_{\dot{H}^1(\mathbb{R}^3)} \\ &= \|\nabla E_{r_i}((u - (u)_i)|_{B_i})\|_{L^2(\mathbb{R}^3)} \\ &\leq C \|\nabla u\|_{L^2(B_i)}. \end{aligned}$$

This concludes the proof of assertion (i).

Observe that $M_i u$ satisfies $\text{supp}(M_i u) \subset \partial B_i$. It can easily be seen that Lemma 3.3.6 still holds true when replacing the cutoff $e^{-|x|}$ by $e^{-j|x|}$ for any $j \in \mathbb{N}_*$. Therefore, we get for $n = 1$

$$\begin{aligned} \|(M_\beta^{(\delta)} - A_\beta^{(\delta)})u\|_{\dot{H}^1(\mathbb{R}^3)}^2 &\leq (1 + C\mu_0) \sum_{i \in I_\delta} e^{-\beta|x_i|} \|(Q_i - \tilde{Q}_i)u\|_{\dot{H}^1(\mathbb{R}^3)}^2 \\ &\leq C(1 + \mu_0) \sum_{i \in I_\delta} e^{-\beta|x_i|} \|\nabla u\|_{L^2(B_i)}. \end{aligned}$$

Hence, the convergence (3.4.7) for $n = 1$ follows provided the volume of the particles inside a fixed bounded domain tends to zero as $\delta \rightarrow 0$. Indeed, we have by definition of μ_δ (see equation (3.1.21)) and Assumption 3.1.8

$$\sum_{x_i \in q_x^\delta} r_i^3 \leq \left(\sum_{x_i \in q_x^\delta} r_i \right)^3 \leq C\delta^9,$$

which implies that the volume of the particles in a fixed bounded domain is of order δ^6 .

The general assertion now follows by induction. For $n = 2$, we have

$$\begin{aligned} \|(M_\beta^{(\delta)} - A_\beta^{(\delta)})u\|_{\dot{H}^1(\mathbb{R}^3)}^2 &\leq \left\| \sum_{i \in I_\delta} \sum_{j \neq i} e^{-\beta_1|x_i|} e^{-\beta_2|x_j|} Q_i(Q_j - \tilde{Q}_j)u \right\|_{\dot{H}^1(\mathbb{R}^3)}^2 \\ &\quad + \left\| \sum_{i \in I_\delta} \sum_{j \neq i} e^{-\beta_1|x_i|} e^{-\beta_2|x_j|} (\tilde{Q}_i - Q_i)\tilde{Q}_j u \right\|_{\dot{H}^1(\mathbb{R}^3)}^2. \end{aligned} \tag{3.4.8}$$

To further estimate the first term on the right hand side, we use that $\sum_{i \in I_\delta} e^{-\beta_1|x_i|} Q_i$ is a bounded operator. Together with part (i) and using again $Q_i Q_i = Q_i$ and $Q_i \tilde{Q}_i = \tilde{Q}_i$, we get (with a constant that depends on μ_0)

$$\begin{aligned} & \left\| \sum_{i \in I_\delta} \sum_{j \neq i} e^{-\beta_1|x_i|} e^{-\beta_2|x_j|} Q_i (Q_j - \tilde{Q}_j) u \right\|_{\dot{H}^1(\mathbb{R}^3)} \\ & \leq \left\| \sum_{i \in I_\delta} e^{-\beta_1|x_i|} Q_i \sum_{j \in I_\delta} e^{-\beta_2|x_j|} (Q_j - \tilde{Q}_j) u \right\|_{\dot{H}^1(\mathbb{R}^3)} + \left\| \sum_{i \in I_\delta} e^{-\beta_1|x_i|} Q_i (Q_i - \tilde{Q}_i) u \right\|_{\dot{H}^1(\mathbb{R}^3)} \\ & \leq C \left\| \sum_{j \in I_\delta} e^{-\beta_2|x_j|} (Q_j - \tilde{Q}_j) u \right\|_{\dot{H}^1(\mathbb{R}^3)} + C \left\| \sum_{i \in I_\delta} e^{-\beta_1|x_i|} (Q_{x_i} - \tilde{Q}_{x_i}) u \right\|_{\dot{H}^1(\mathbb{R}^3)} \\ & \leq C \left\| \sum_{j \in I_\delta} e^{-\beta_2|x_j|} (Q_j - \tilde{Q}_j) u \right\|_{\dot{H}^1(\mathbb{R}^3)} \rightarrow 0. \end{aligned}$$

For the second term on the right hand side of (3.4.8), recall

$$(M_i - T_i)v = 0 \quad \text{in } \mathbb{R}^3 \setminus \overline{B_i}$$

for all $v \in \dot{H}^1(\mathbb{R}^3)$. Hence, for $u \in H^1(\mathbb{R}^3)$, we can use Lemma 3.3.6 to take out the sum in i , and we use the estimate for the uniform charge density approximation from part (i),

$$\left\| \sum_{i \in I_\delta} \sum_{j \neq i} e^{-\beta_1|x_i|} e^{-\beta_2|x_j|} (\tilde{Q}_i - Q_i) \tilde{Q}_j u \right\|_{\dot{H}^1(\mathbb{R}^3)}^2 \leq \sum_{i \in I_\delta} e^{-\beta_1|x_i|} \left\| \sum_{j \neq i} e^{-\beta_2|x_j|} \nabla \tilde{Q}_j u \right\|_{L^2(B_i)}^2. \quad (3.4.9)$$

Inserting the definition of \tilde{Q}_j , expanding the square of the sum over j , and estimating the integral yields

$$\begin{aligned} & \sum_{i \in I_\delta} e^{-\beta_1|x_i|} \left\| \sum_{j \neq i} e^{-\beta_2|x_j|} \nabla \tilde{Q}_j u \right\|_{L^2(B_{x_i})}^2 \\ & \leq C \sum_{i \in I_\delta} \sum_{j \neq i} \sum_{k \neq i} e^{-\beta_1|x_i|} r_i r_j^2 r_k^2 \frac{e^{-\beta_2|x_j|} |(u)_j|}{|x_i - x_j|^2} \frac{e^{-\beta_2|x_k|} |(u)_k|}{|x_i - x_k|^2}. \end{aligned} \quad (3.4.10)$$

Consider the off-diagonal terms first, i.e., $j \neq k$. We estimate

$$e^{-\beta_1|x_i|} e^{-\beta_2|x_j|} e^{-\beta_2|x_k|} \leq e^{-\frac{|x_i - x_j|}{2}} e^{-\frac{|x_i - x_k|}{2}} e^{-\frac{|x_j|}{2}} e^{-\frac{|x_k|}{2}}$$

and use $r_i \leq \mu_0 d_i^3$ from Condition 3.1.1* to bound the sum over i by an integral,

$$\sum_{i \neq j, k} r_i \frac{e^{-\frac{|x_i - x_j|}{2}}}{|x_i - x_j|^2} \frac{e^{-\frac{|x_i - x_k|}{2}}}{|x_i - x_k|^2} \leq C \mu_0 \int_{\mathbb{R}^3} \frac{e^{-\frac{|y - x_j|}{2}}}{|y - x_j|^2} \frac{e^{-\frac{|y - x_k|}{2}}}{|y - x_k|^2} dy.$$

To estimate the integral for $j \neq k$, we denote $z = x_j - x_k$ for the moment and split the integral to get

$$\int_{\mathbb{R}^3} \frac{e^{-\frac{|y|}{2}}}{|y|^2} \frac{e^{-\frac{|y - z|}{2}}}{|y - z|^2} dy \leq \int_{\mathbb{R}^3 \setminus B_{|z|/2}(0)} \frac{4e^{-\frac{|z|}{4}}}{|z|^2} \frac{e^{-\frac{|y - z|}{2}}}{|y - z|^2} dy + \int_{B_{|z|/2}(0)} \frac{e^{-|y|}}{|y|^2} \frac{4e^{-\frac{|z|}{4}}}{|z|^2} dy \leq C \frac{e^{-\frac{|z|}{4}}}{|z|^2}.$$

Hence, using

$$|(u)_j| |(u)_k| r_j^2 r_k^2 \leq \frac{1}{2} ((u)_j^2 r_j^3 r_k + (u)_k^2 r_k^3 r_j)$$

and symmetry, we deduce

$$\begin{aligned}
& \sum_{i \in I_\delta} \sum_{j \neq i} \sum_{k \neq i, j} r_i r_j^2 r_k^2 e^{-\frac{|x_j|}{2}} e^{-\frac{|x_k|}{2}} (u)_j (u)_k \frac{e^{-\frac{|x_i - x_j|}{2}}}{|x_i - x_j|^2} \frac{e^{-\frac{|x_i - x_k|}{2}}}{|x_i - x_k|^2} \\
& \leq \sum_{j \in I_\delta} \sum_{k \neq j} C \mu_0 r_j^3 r_k e^{-|x_j|} (u)_j^2 \frac{e^{-\frac{|x_j - x_k|}{4}}}{|x_j - x_k|^2} \\
& \leq \sum_{j \in I_\delta} C \mu_0^2 r_j^3 e^{-|x_j|} (u)_{x_j}^2 \int_{\mathbb{R}^3} \frac{e^{-\frac{|x_j - y|}{4}}}{|x_j - y|^2} dy \\
& \leq C \mu_0^2 \sum_{j \in I_\delta} r_j^3 e^{-|x_j|} \left(\int_{B_{x_j}} u(y) dy \right)^2 \\
& \leq C \mu_0^2 \sum_{j \in I_\delta} e^{-|x_j|} \|u\|_{L^2(B_j)}^2 \\
& \leq C \mu_0^2 \int_{\mathbb{R}^3} e^{-|y|} |u(y)|^2 \chi_{\cup_{j \in I_\delta} B_j} dy,
\end{aligned}$$

where we used that the radii r_i are uniformly bounded in δ by (3.1.22). Recall from the first part of the proof that the volume of the particles inside a fixed bounded domain converges to zero. Thus,

$$\int_{\mathbb{R}^3} e^{-|y|} |u(y)|^2 \chi_{\cup_{j \in I_\delta} B_j} dy \rightarrow 0$$

It remains to bound the diagonal terms $j = k$ in (3.4.10). For those, we use the estimate

$$\sum_{i \neq j} r_i \frac{e^{-|x_i - x_j|}}{|x_i - x_j|^4} \leq C \mu_0 \int_{\mathbb{R}^3 \setminus B_{d_j}(0)} \frac{e^{-\frac{|y|}{2}}}{|y|^4} dy \leq C \mu_0 d_j^{-1} \leq C \mu_0^{4/3} r_j^{1/3}.$$

Hence,

$$\sum_i \sum_{j \neq i} r_i r_j^4 e^{-|x_j|} (u)_j^2 \frac{e^{-|x_i - x_j|}}{|x_i - x_j|^4} \leq C \mu_0^{4/3} \sum_{j \in I_\delta} e^{-|x_j|} \|u\|_{L^2(B_j)}^2 \rightarrow 0,$$

For $n \geq 3$, one does same thing as for $n = 2$. We sketch the proof for $n = 3$. After an analogous splitting as in estimate (3.4.8), convergence of the first term is shown using the result for $n = 2$. For the second term we follow the estimates in the $n = 2$ case, replacing $(u)_j$ by $(\sum_{l \neq j} e^{-\beta_3 |x_l|} \tilde{Q}_l)_j$. Therefore, we are left to show

$$\sum_j e^{-|x_j|} \left\| \sum_{l \neq j} e^{-\beta_3 |x_l|} \tilde{Q}_l u \right\|_{L^2(B_j)}^2 \rightarrow 0. \quad (3.4.11)$$

and this can be estimated in the same way as the right hand side of equation (3.4.9). The only difference is that the gradient in (3.4.9) is not present in (3.4.11). However, this only means that the squares in the denominators in estimate (3.4.10) are missing, but these squares have not been important for the subsequent estimates due to the presence of the exponentials. Therefore, (3.4.11) holds. \square

Proof of Proposition 3.4.13. By Lemma 3.3.9 it suffices to prove

$$A_\beta^{(\delta)} u \rightharpoonup \left(\prod_{j=1}^n G_0 J \mu A^{\beta_j} \right) u \quad \text{in } \dot{H}^1(\mathbb{R}^3),$$

for all $u \in \dot{H}^1(\mathbb{R}^3)$, all $n \in \mathbb{N}_*$, and all $\beta \in \mathbb{N}_*^n$.

Since G_0 is an isometry, for $n = 1$, it suffices to show

$$\sum_{i \in I_\delta} e^{-\beta_1 |x_i|} T_i u \rightharpoonup J\mu A^{\beta_1} u \quad \text{in } \dot{H}^{-1}(\mathbb{R}^3)$$

for all $u \in \dot{H}^1(\mathbb{R}^3)$ and analogously for $n \geq 2$. Since by Lemma 3.3.9, we have a uniform bound on $A_\beta^{(\delta)}$, it suffices to prove the assertion for the dense subset $\dot{H}^1(\mathbb{R}^3) \cap C^1(\mathbb{R}^3)$. Lemma 3.4.20 implies that we can replace all the operators T_i by M_i . Moreover, it suffices to consider test function from the dense set $C_c^\infty(\mathbb{R}^3)$. Let $u \in \dot{H}^1(\mathbb{R}^3) \cap C^1(\mathbb{R}^3)$ and $\varphi \in C_c^\infty(\mathbb{R}^3)$. Let $x \in (\delta\mathbb{Z})^3$. Then, we estimate for $x_i \in q_x^\delta$

$$\begin{aligned} |\langle \varphi, e^{-\beta_1 |x_i|} M_i u \rangle - 4\pi r_i e^{-\beta_1 |x|} u(x) \varphi(x) | &\leq \frac{1}{r_i} \int_{\partial B_i} |e^{-\beta_1 |x_i|} (u)_i \varphi(y) - e^{-\beta_1 |x|} u(x) \varphi(x)| dy \\ &\leq C r_i \delta e^{-\beta_1 |x|} \|u\|_{C^1(\mathbb{R}^3)} \|\varphi\|_{C^1(\mathbb{R}^3)}. \end{aligned}$$

Thus, with μ_δ as in (3.1.21),

$$\begin{aligned} |\langle \varphi, \sum_{x_i \in q_x^\delta} e^{-\beta_1 |x_i|} M_i u \rangle - \delta^3 \mu_\delta(x) e^{-\beta_1 |x|} u(x) \varphi(x) | &\leq C \delta e^{-\beta_1 |x|} \|u\|_{C^1(\mathbb{R}^3)} \|\varphi\|_{C^1(\mathbb{R}^3)} \sum_{x_i \in q_x^\delta} r_i \\ &\leq C \delta e^{-\beta_1 |x|} \|u\|_{C^1(\mathbb{R}^3)} \|\varphi\|_{C^1(\mathbb{R}^3)} \sum_{x_i \in q_x^\delta} \mu_0 d_i^3 \\ &\leq C \delta^4 e^{-\beta_1 |x|} \|u\|_{C^1(\mathbb{R}^3)} \|\varphi\|_{C^1(\mathbb{R}^3)}, \end{aligned}$$

where we used that the balls $B_{d_i}(x_i)$ are disjoint and $|B_{d_i}(x_i) \cap q_x^\delta| \geq C d_i^3$. On the other hand,

$$\begin{aligned} &\left| \int_{q_x^\delta} \mu(y) e^{-\beta_1 |y|} u(y) \varphi(y) dy - \delta^3 \mu_\delta(x) e^{-\beta_1 |x|} u(x) \varphi(x) \right| \\ &\leq \int_{q_x^\delta} |\mu(y) e^{-\beta_1 |y|} u(y) \varphi(y) - \mu_\delta(y) e^{-\beta_1 |x|} u(x) \varphi(x)| dy \\ &\leq C \delta^3 (\|\mu - \mu_\delta\|_{L^\infty(\mathbb{R}^3)} + \delta) e^{-\beta_1 |x|} \|u\|_{C^1(\mathbb{R}^3)} \|\varphi\|_{C^1(\mathbb{R}^3)}. \end{aligned}$$

Now, we take the sum in $x \in (d\mathbb{Z})^3$ and use that $\cup_x q_x^\delta = \mathbb{R}^3$. Therefore, combining the above estimates leads to

$$|\langle \varphi, \sum_{i \in I_\delta} e^{-\beta_1 |x_i|} M_i u - J\mu A^{\beta_1} u \rangle| \leq C (\|\mu - \mu_\delta\|_{L^\infty(\mathbb{R}^3)} + \delta) \|u\|_{C^1(\mathbb{R}^3)} \|\varphi\|_{C^1(\mathbb{R}^3)}.$$

This proves the convergence for $n = 1$.

For $n = 2$, we write

$$\begin{aligned} &\sum_{i \in I_\delta} \sum_{j \neq i} e^{-\beta_1 |x_i|} e^{-\beta_2 |x_j|} M_i G_0 M_j u - J\mu A^{\beta_1} G_0 J\mu A^{\beta_2} u \\ &= \left(\sum_{i \in I_\delta} e^{-\beta_1 |x_i|} M_i - J\mu A^{\beta_1} \right) G_0 J\mu A^{\beta_2} u \\ &\quad + \sum_{i \in I_\delta} e^{-\beta_1 |x_i|} M_i G_0 \left(\sum_{j \neq i} e^{-\beta_2 |x_j|} M_j - J\mu A^{\beta_2} \right) u. \end{aligned} \tag{3.4.12}$$

The first term converges to zero weakly in $H^{-1}(\mathbb{R}^3)$ by the assertion for $n = 1$.

To control the second term we show for all $i \in I_\delta$ and all $z \in B_i$

$$\left| \left(G_0 \left(\sum_{j \neq i} e^{-\beta_2 |x_j|} M_j - J\mu A^{\beta_2} \right) u \right) (z) \right| \leq C(1 + \mu_0) (\|\mu - \mu_\delta\|_{L^\infty(\mathbb{R}^3)} + \delta) \|u\|_{C^1(\mathbb{R}^3)}. \quad (3.4.13)$$

Using this estimate and inserting the definition of M_i yields (with a constant depending on μ_0)

$$\left| \langle \varphi, M_i G_0 \left(\sum_{j \neq i} e^{-\beta_2 |x_j|} M_j - J\mu A^{\beta_2} \right) u \rangle \right| \leq C r_i (\|\mu - \mu_\delta\|_{L^\infty(\mathbb{R}^3)} + \delta) \|u\|_{C^1(\mathbb{R}^3)} \|\varphi\|_{L^\infty(\mathbb{R}^3)},$$

which converges to zero after multiplying with $e^{-\beta_1 |x_i|}$ and taking the sum over i . In order to prove (3.4.13), we write

$$G_0 \left(\sum_{j \neq i} e^{-\beta_2 |x_j|} M_j - J\mu A^{\beta_2} \right) u = \sum_{x \in (\delta\mathbb{Z})^3} G_0 \left(\sum_{\substack{j \neq i \\ x_j \in q_x^\delta}} e^{-\beta_2 |x_j|} M_j - J\mu A^{\beta_2} \chi_{q_x^\delta} \right) u$$

with $\chi_{q_x^\delta}$ denoting the indicator function of q_x^δ . We first consider $x \in (\delta\mathbb{Z})^3$ such that the cube q_x contains x_i or is adjacent to the cube containing x_i . Then, for all $z \in B_i$ and all $j \neq i$

$$|(G_0 M_j u)(z)| = \frac{|(u)_j|}{r_j} \int_{\partial B_j} |\Phi(z - y)| dy \leq C \|u\|_{L^\infty(\mathbb{R}^3)} r_i \frac{1}{|x_j - x_i|},$$

where the constant C depends only on κ from Condition 3.1.2. Using Condition 3.1.1*, we get

$$\sum_{\substack{j \neq i \\ x_j \in q_x^\delta}} |e^{-\beta_2 |x_j|} (G_0 M_j u)(z)| \leq C \delta^2 \mu_0 \|u\|_{L^\infty(\mathbb{R}^3)}.$$

Moreover,

$$\left| \left(G_0 J\mu A^{\beta_2} \chi_{q_x^\delta} u \right) (z) \right| \leq \|\mu\|_{L^\infty(\mathbb{R}^3)} \int_{q_x^\delta} |u(z)| |\Phi(y - z)| dz \leq C \mu_0 \delta^2 \|u\|_{L^\infty}. \quad (3.4.14)$$

Now, for $x \in (\delta\mathbb{Z})^3$ such that q_x^δ neither contains x_i nor is adjacent to the cube containing x_i . Then, $|x_i - x_j| \geq C|x - x_i|$ for all $x_j \in q_x^\delta$. Following the same argument as in the first part of the proof (the case $n = 1$) with $\Phi(z - \cdot)$ instead of φ , we have for $z \in B_i$

$$\begin{aligned} & \left| \left(G_0 \left(\sum_{x_j \in q_x^\delta} e^{-\beta_2 |x_j|} M_j - J\mu A^{\beta_2} \chi_{q_x^\delta} \right) u \right) (z) \right| \\ & \leq C \delta^3 (\|\mu - \mu_\delta\|_{L^\infty(\mathbb{R}^3)} + \delta) e^{-\beta_2 |x|} \|u\|_{C^1(\mathbb{R}^3)} \left(\frac{1}{|x - x_i|} + \frac{1}{|x - x_i|^2} \right). \end{aligned} \quad (3.4.15)$$

Combining (3.4.14) and (3.4.15) yields (3.4.13).

This finishes the proof for $n = 2$. Convergence of the higher order terms is proven by induction. One first splits the difference analogously as in equation (3.4.12), and uses the induction hypothesis on the first term. The second term is given by

$$\sum_{i_1 \in I_\delta} e^{-\beta_1 |x_{i_1}|} M_{i_1} G_0 \cdots \sum_{i_n \neq i_{n-1}} e^{-\beta_n |x_{i_n}|} M_{i_n} G_0 \left(\sum_{i_{n+1} \neq i_n} e^{-\beta_{n+1} |x_{i_{n+1}}|} M_{i_{n+1}} - J\mu A^{\beta_{n+1}} \right) u \quad (3.4.16)$$

Using estimate (3.4.13) and the definition of M_i yields convergence of (3.4.16) to zero in $H^{-1}(\mathbb{R}^3)$. \square

3.5 Adaptation to Stokes equations

In this section, we will adapt the previous results for the Poisson equation to the case of Stokes equations. We will not repeat everything from the previous sections but rather point out the necessary modifications.

3.5.1 The method of reflections applied to Stokes equations

The aim of this subsection is to prove Theorem 3.5.8 stated below after introducing the necessary notation.

In the following, we will work in spaces of divergence free functions, since this basically allows to ignore the presence of the pressure in the Stokes equations.

Definition 3.5.1. We define $\dot{H}_\sigma^1(\mathbb{R}^3; \mathbb{R}^3) \subset \dot{H}^1(\mathbb{R}^3; \mathbb{R}^3)$ to be the closed subspace of divergence free functions, and $\dot{H}_\sigma^{-1}(\mathbb{R}^3; \mathbb{R}^3)$ its dual space.

Notation 3.5.2. To improve readability, we will from now on write $\dot{H}^1(\mathbb{R}^3)$ instead of $\dot{H}^1(\mathbb{R}^3; \mathbb{R}^3)$ and similarly for $\dot{H}^{-1}(\mathbb{R}^3; \mathbb{R}^3)$, $\dot{H}_\sigma^1(\mathbb{R}^3; \mathbb{R}^3)$, etc.

Remark 3.5.3. Note that $\dot{H}_\sigma^{-1}(\mathbb{R}^3) \subset \dot{H}^{-1}(\mathbb{R}^3)$. Here, the inclusion for $f \in \dot{H}_\sigma^{-1}(\mathbb{R}^3)$ to a function in $\dot{H}^{-1}(\mathbb{R}^3)$ is given by $\langle u, f \rangle := \langle P_\sigma u, f \rangle$ for all $u \in \dot{H}^1(\mathbb{R}^3)$, where P_σ is the orthogonal projection from $\dot{H}^1(\mathbb{R}^3)$ to $\dot{H}_\sigma^1(\mathbb{R}^3)$.

Lemma 3.5.4. Let $f \in \dot{H}^{-1}(\mathbb{R}^3)$. Then, the Stokes equations

$$-\Delta u = -\nabla p + f, \quad \operatorname{div} u = 0$$

have a unique weak solution $(u, p) \in \dot{H}_\sigma^1(\mathbb{R}^3) \times L^2(\mathbb{R}^3)$. The solution operator \bar{G}_0 for the velocity field is given by

$$\bar{G}_0 f = \Phi * f,$$

where

$$\Phi(x) := \frac{1}{8\pi} \left(\frac{1}{|x|} + \frac{x \otimes x}{|x|^3} \right). \quad (3.5.1)$$

Moreover, the restriction of the solution operator to \dot{H}_σ^{-1} , which we denote by G_0 , is an isometric isomorphism.

Lemma 3.5.5. Let $\Omega \subset \mathbb{R}^3$ be open. Then, for every $f \in \dot{H}^{-1}(\mathbb{R}^3)$, the problem

$$\begin{aligned} -\Delta u &= -\nabla p + f, & \operatorname{div} u &= 0 & \text{in } \mathbb{R}^3 \setminus \bar{\Omega}, \\ u &= 0, & p &= 0 & \text{in } \bar{\Omega}, \end{aligned} \quad (3.5.2)$$

has a unique weak solution $(u, p) \in \dot{H}_\sigma^1(\mathbb{R}^3) \times L^2(\mathbb{R}^3)$. Moreover,

$$u = P_\Omega \bar{G}_0,$$

where P_Ω is the orthogonal projection from $\dot{H}_\sigma^1(\mathbb{R}^3)$ to the subspace $\dot{H}_{0,\sigma}^1(\mathbb{R}^3 \setminus \bar{\Omega})$.

Remark 3.5.6. Analogous to $H_0^1(\mathbb{R}^3 \setminus \bar{\Omega})$, we use the convention

$$\dot{H}_{0,\sigma}^1(\mathbb{R}^3 \setminus \bar{\Omega}) := \{u \in \dot{H}_\sigma^1(\mathbb{R}^3) : u = 0 \text{ in } \Omega\}.$$

Remark 3.5.7. The condition $p = 0$ in $\overline{\Omega}$ in equation (3.5.2) ensures uniqueness. Indeed, dropping this condition, p can be chosen equal to any constant in every bounded connected component of Ω . In $\mathbb{R}^3 \setminus \overline{\Omega}$ the pressure is normalized by the condition $p \in L^2(\mathbb{R}^3)$.

Again, for a particle i , we define the orthogonal projection $Q_i = 1 - P_i$, where $P_i = P_{\Omega_i}$. The main theorem regarding the method of reflections for the Stokes equations is the following.

Theorem 3.5.8. Assume Conditions 3.1.1 and 3.1.2 are satisfied. Let $f \in \dot{H}^{-1}(\mathbb{R}^3)$. There exists a $\gamma_0 > 0$ depending only on C_2 from Condition 3.1.1 and κ from Condition 3.1.2 such that the sequence

$$\lim_{N \rightarrow \infty} \left(1 - \gamma \sum_j e^{-|x_j|} Q_j \right)^N \bar{G}_0 f$$

converges to the solution of (3.1.2) in $\dot{H}^1(\mathbb{R}^3)$ for all $\gamma < \gamma_0$.

Notice that

$$\dot{H}_{0,\sigma}^1(\mathbb{R}^3 \setminus \overline{B_i})^{\perp_\sigma} = \{u \in \dot{H}_\sigma^1(\mathbb{R}^3) : -\Delta u = -\nabla p \text{ in } \mathbb{R}^3 \setminus \overline{B_i} \text{ for some } p \in L^2(\mathbb{R}^3 \setminus \overline{B_i})\},$$

where \perp_σ indicates that we take the orthogonal complement with respect to $\dot{H}_\sigma^1(\mathbb{R}^3)$.

Notice that $G_0^{-1} Q_i u \in \dot{H}_\sigma^{-1}(\mathbb{R}^3)$ is supported in $\overline{B_i}$, i.e., $\langle v, G_0^{-1} Q_i u \rangle = 0$ for every v in $\dot{H}_{0,\sigma}^1(\mathbb{R}^3 \setminus B_x)$. This, however, does not mean that $G_0^{-1} Q_i u$, viewed as an element of $\dot{H}^{-1}(\mathbb{R}^3)$, is supported in $\overline{B_i}$.

In the case of Poisson equation, we often used cutoff functions to exploit that a function $f \in \dot{H}_\sigma^{-1}(\mathbb{R}^3)$ is supported in $\overline{B_i}$. However, multiplication with a cutoff function destroys the property of a function to be divergence free. Therefore, the following Lemma is needed.

Lemma 3.5.9. Assume $f \in \dot{H}_\sigma^{-1}(\mathbb{R}^3)$ is supported in $\overline{B_i}$. Then, there exists a unique $p \in L^2(\mathbb{R}^3)$ with $p = 0$ in B_i such that $\tilde{f} := f + \nabla p$ is supported in $\overline{B_i}$ as a function in $\dot{H}^{-1}(\mathbb{R}^3)$. Moreover, $\|\tilde{f}\|_{\dot{H}^{-1}(\mathbb{R}^3)} \leq C \|f\|_{\dot{H}^{-1}(\mathbb{R}^3)}$ for a universal constant C . We denote by S the operator that maps f to \tilde{f} .

Proof. Since $f \in \dot{H}_\sigma^{-1}(\mathbb{R}^3)$ is supported in $\overline{B_i}$, we have $\langle f, v \rangle = 0$ for all $v \in \dot{H}_{0,\sigma}^1(\mathbb{R}^3 \setminus B_i)$. Hence, there exists a unique $p \in L^2(\mathbb{R}^3 \setminus \overline{B_i})$ such that $f = -\nabla p$ in $\mathbb{R}^3 \setminus \overline{B_i}$ and we can set $p = 0$ in B_i . By Lemma 3.5.10 below, we can find $u \in \dot{H}_0^1(\mathbb{R}^3 \setminus B_i)$ such that $\operatorname{div} u = p$ and $\|u\|_{\dot{H}^1(\mathbb{R}^3)} \leq C \|p\|_{L^2(\mathbb{R}^3)}$. Hence,

$$\|u\|_{\dot{H}^1(\mathbb{R}^3)} \|f\|_{\dot{H}^{-1}(\mathbb{R}^3)} \geq \langle u, f \rangle = \langle u, -\nabla p \rangle = \|p\|_{L^2(\mathbb{R}^3)}^2,$$

and thus $\|p\|_{L^2(\mathbb{R}^3)} \leq C \|f\|_{\dot{H}^{-1}(\mathbb{R}^3)}$. Hence, $\tilde{f} := f + \nabla p$ is supported in $\overline{B_i}$ as a function in $\dot{H}^{-1}(\mathbb{R}^3)$, and $\|\tilde{f}\|_{\dot{H}^{-1}(\mathbb{R}^3)} \leq C \|f\|_{\dot{H}^{-1}(\mathbb{R}^3)}$. \square

The following Lemma can be found in every standard textbook on Stokes equations, e.g., in [Gal11].

Lemma 3.5.10. Let $\Omega \subset \mathbb{R}^3$ be a locally Lipschitzian bounded or exterior domain. Then there exists a constant C with the following property. For all $f \in L^2(\Omega)$, that satisfies

$$\int_{\Omega} f \, dx = 0$$

if Ω is a bounded domain, there exists $u \in H_0^1(\Omega)$ such that

$$\operatorname{div} u = f$$

and

$$\|\nabla u\|_{L^2(\Omega)} \leq C \|f\|_{L^2(\Omega)}.$$

Remark 3.5.11. *The constant C is invariant under scaling of Ω .*

Now one can define the operator L analogously to the corresponding operator for the Poisson equation from Definition 3.3.3. Using Lemma 3.5.9, the estimate for L (cf. Lemma 3.3.5) follows in the same manner as before. Then, Theorem 3.5.8 follows immediately from Proposition 3.3.7 and Lemma 3.5.5.

3.5.2 Homogenization

For the homogenization of the Stokes equations, one has to replace the factor 4π in the definition of the averaged capacity density (3.1.21) by 6π . This is directly related to the fact that the absolute value of the Stokes drag force on a ball of radius r is $6\pi r$ if it is moving with unit speed in a fluid which is at rest at infinity (see Chapter 2.2.1).

Theorem 3.5.12. *Suppose that $f \in H^{-1}(\mathbb{R}^3)$. Then, under Assumption 3.1.8, the problems (3.1.2) with $K = K_\delta$ have unique solutions $u_\delta \in H^1(\mathbb{R}^3)$. In the limit $\delta \rightarrow 0$, u_δ converges weakly in $H^1(\mathbb{R}^3)$ to the unique solution $u \in H^1(\mathbb{R}^3)$ of the problem*

$$-\Delta u + \nabla p + \mu u = f, \quad \operatorname{div} u = 0 \quad \text{in } \mathbb{R}^3.$$

Corresponding to Definition 3.4.8, we introduce the following operators.

Definition 3.5.13. *We define $T_{i,\delta} : \dot{H}_\sigma^1(\mathbb{R}^3) \rightarrow \dot{H}^{-1}(\mathbb{R}^3)$ by $T_{i,\delta} = SG_0^{-1}Q_{i,\delta}$, where S is the operator from Lemma 3.5.9.*

Moreover, we define the uniform force density approximation of the operator T to be the operator $M_{i,\delta} : \dot{H}_\sigma^1(\mathbb{R}^3) \rightarrow \dot{H}^{-1}(\mathbb{R}^3)$,

$$(M_{i,\delta}u)(y) = \frac{3(u)_{x_i,\delta}}{2r_i} \mathcal{H}^2|_{\partial B_{i,\delta}}$$

Lemma 3.4.21 used in the proof Lemma 3.4.20 has to be replaced by the following Lemma.

Lemma 3.5.14. *For $r > 0$ and $x \in \mathbb{R}^3$, let $H_r := \left\{ u \in H_\sigma^1(B_r(x)) : \int_{B_r(x)} u = 0 \right\}$. Then, for all $r > 0$, there exists an extension operator $E_r : H_r \rightarrow H_{\sigma,0}^1(B_{2r}(x))$ such that*

$$\|\nabla E_r u\|_{L^2(B_{2r}(x))} \leq C \|\nabla u\|_{L^2(B_r(x))} \quad \text{for all } u \in H_r,$$

where the constant C is independent of r .

Remark 3.5.15. *An analogous statement holds for H_r replaced by $\left\{ u \in H_\sigma^1(B_r(x)) : \int_{\partial B_r(x)} u = 0 \right\}$.*

Proof. For $r = 1$, let $E_1 : H_\sigma^1(B_1(x)) \rightarrow H_{\sigma,0}^1(B_2(x))$ be a continuous extension operator. Then, by the Poincaré inequality in H_1 , we get for all $u \in H_1$

$$\|\nabla E_1 u\|_{L^2(B_2(x))} \leq \|E_1 u\|_{H^1(B_2(x))} \leq C \|u\|_{H^1(B_1(x))} \leq C \|\nabla u\|_{L^2(B_1(x))}.$$

The assertion for general $r > 0$ follows from scaling by defining $(E_r)u(x) := (E_1 u_r)(\frac{x}{r})$ where $u_s(x) := u(sx)$. \square

These are the only things that change in the proof of the homogenization result, Theorem 3.5.12, except for the result about locally uniform convergence in the particle configuration. For the Poisson equation, this result was stated in Proposition 3.4.19. The analogous statement for the Stokes equations remains valid.

However, the proof of Lemma 3.4.15 and 3.4.16 needed in the proof of Proposition 3.4.19 have to be modified due to the use of cutoff functions. Corresponding to Lemma 3.4.15 and 3.4.16, we will prove Lemma 3.5.17 and 3.5.19. For the proof of Lemma 3.5.17, we need the following lemma.

Lemma 3.5.16. *Let $\Omega \subset \mathbb{R}^3$ be a bounded and locally Lipschitzian domain and assume $v \in H^1(\Omega)$ satisfies*

$$\int_{\Omega} v \cdot \nu = 0.$$

Then, for any $R > 0$ and $x \in \mathbb{R}^3$ such that $\Omega \subset\subset B_R(x)$, there exists $u \in H_0^1(\Omega)$ such that

$$\begin{aligned} u &= v \text{ in } \Omega \\ \operatorname{div} u &= 0 \text{ in } B_R(x) \setminus \overline{\Omega} \end{aligned}$$

and

$$\|u\|_{H^1(B_R)} \leq C\|v\|_{H^1(\Omega)},$$

where the constant depends only on the domains Ω and $B_R(x)$.

In particular, for any $v \in H^1(B_r)$ with $\int_{B_r} v \cdot \nu = 0$, we can find $u \in H_0^1(B_{\kappa r}(x))$ such that

$$\begin{aligned} u &= v \text{ in } B_r(x) \\ \operatorname{div} u &= 0 \text{ in } B_{\kappa r}(x) \setminus B_r(x) \end{aligned}$$

and

$$\|\nabla u\|_{H^1(B_{\kappa r}(x))}^2 \leq \frac{C}{r^2} \|v\|_{L^2(B_r(x))}^2 + C \|\nabla v\|_{L^2(B_r(x))}^2 \leq C \|\nabla v\|_{L^2(\mathbb{R}^3)}^2,$$

where the constant is independent of r and v .

Proof. We take any (not necessarily divergence free) extension u_1 of v to $B_R(x)$ that satisfies the estimate, and take a solution $u_2 \in H^1(B_R \setminus \overline{\Omega})$ of $\operatorname{div} u_2 = -\operatorname{div} u_1$ provided by Lemma 3.5.10 and define $u = u_1 + u_2$.

The second assertion follows from scaling, and the last inequality is a consequence of Hölder's inequality and the Gagliardo-Nirenberg-Sobolev inequality. \square

Lemma 3.5.17. *Let $u \in \dot{H}_{0,\sigma}^1(\mathbb{R}^3 \setminus K_\delta)^{\perp_\sigma}$ and $R > 0$. We define $v \in \dot{H}_\sigma^1(\mathbb{R}^3)$ to be the solution to*

$$\begin{aligned} -\Delta v &= -\nabla p \quad \text{in } \mathbb{R}^3 \setminus (K_\delta \cap \overline{B_R(0)}), \\ \operatorname{div} v &= 0, \\ v &= u \quad \text{in } K_\delta \cap \overline{B_R(0)}. \end{aligned}$$

Then,

$$(L_\delta u, u)_{\dot{H}^1(\mathbb{R}^3)} \geq c e^{-R} \|v\|_{\dot{H}^1(\mathbb{R}^3)}^2,$$

where $c > 0$ is a universal constant.

Proof. By the variational form of the equation for v , we know that v is the function of minimal norm in the set $X_v := \{w \in \dot{H}_\sigma^1(\mathbb{R}^3) : w = v \text{ in } K_\delta \cap \overline{B_R}\}$. Denote

$$r_0 := \sup_{\delta} \sup_{i \in I_\delta} r_i < \infty.$$

For every $x_i \in B_{R+\kappa r_0}$, Corollary 3.5.16 provides functions $v_i \in H_0^1(B_{\kappa r_i}(x))$ with $\|v_i\|_{\dot{H}^1(\mathbb{R}^3)} \leq C\|Q_i v\|_{\dot{H}^1(\mathbb{R}^3)}$ such that $v_i = Q_i v = v$ in B_i . Clearly, $\sum_{x_i \in B_{R+\kappa r_0}} v_i \in X_u$, and hence,

$$\begin{aligned} \langle Lv, v \rangle &= \sum_i e^{-|x_i|} \|Q_i v\|_{\dot{H}^1(\mathbb{R}^3)}^2 \\ &\geq ce^{-R} \sum_{x_i \in B_{R+\kappa r_0}} \|v_i\|_{\dot{H}^1(\mathbb{R}^3)}^2 \\ &= ce^{-R} \left\| \sum_{x_i \in B_{R+\kappa r_0}} v_i \right\|_{\dot{H}^1(\mathbb{R}^3)}^2 \\ &\geq ce^{-R} \|v\|_{\dot{H}^1(\mathbb{R}^3)}^2. \end{aligned} \quad \square$$

For the proof of Lemma 3.5.19 below, we need the following lemma.

Lemma 3.5.18. *Let $u \in H^1(\mathbb{R}^3)$ and $x \in \mathbb{R}^3$. Assume $0 < \rho < R$. Then*

$$\|u\|_{L^2(B_\rho(x))}^2 \leq C \left(\frac{\rho^3}{R^3} \|u\|_{L^2(B_R(x))}^2 + \rho^2 \|\nabla u\|_{L^2(B_R(x))}^2 \right),$$

where C is a universal constant.

In particular, under condition 3.1.1* we have

$$\sum_{i \in I_\delta} \frac{1}{r_i^2} \|u\|_{L^2(B_i)}^2 \leq C\mu_0 \|u\|_{L^2(\mathbb{R}^3)}^2 + C\|\nabla u\|_{L^2(\mathbb{R}^3)}^2.$$

Proof. Define $(u)_{R,x} = f_{B_R(x)} u$. Then, using Lemma 3.4.21 we get

$$\begin{aligned} \|u - (u)_{R,x}\|_{L^2(B_\rho(x))} &\leq \|u - (u)_{R,x}\|_{L^6(B_\rho(x))} \|1\|_{L^3(B_\rho(x))} \\ &\leq C\rho \|\nabla E_R(u - (u)_{R,x})\|_{L^2(B_d(x))} \leq C\rho \|\nabla u\|_{L^2(B_R(x))}. \end{aligned}$$

Furthermore,

$$\|(u)_{R,x}\|_{L^2(B_\rho(x))}^2 = C\rho^3 \left(\int_{B_R(x)} u \, dx \right)^2 \leq C\rho^3 \int_{B_R(x)} u^2 \, dx = C \frac{\rho^3}{R^3} \|u\|_{L^2(B_R(x))}^2.$$

Combining these two estimates yields the assertion. \square

Lemma 3.5.19. *For all $\rho > 0$ there exists a nonincreasing function $e_\rho: \mathbb{R}_+ \rightarrow \mathbb{R}_+$ which satisfies $\lim_{s \rightarrow \infty} e_\rho(s) = 0$ and the following property. For all $0 < \delta < 1$, all $R > \rho$, all $w \in \dot{H}_{0,\sigma}^1(\mathbb{R}^3 \setminus K_\delta)^{\perp_\sigma}$ with $w = 0$ in $K_\delta \cap B_R$ satisfy*

$$\|\nabla w\|_{L^2(B_\rho)} \leq e_\rho(R) \|\nabla w\|_{L^2(\mathbb{R}^3)}.$$

Proof. First note that Assumption 3.1.8 implies

$$d_0 := \sup_{\delta} \sup_{i \in I_\delta} d_i < \infty,$$

because every cube q_x^δ contains at least one particle and those cubes are of length smaller than one.

Fix δ , R , ρ , and w according to the assumptions. Let s_0 be as in Lemma 3.4.17 and define $s_1 = \max\{s_0, \rho, 8d_0\}$. Assume $R \geq 2s_1$ and let $s \geq s_1$ such that $2s \leq R$. Note that w is the function of minimal norm in the set

$$X_w := \{v \in \dot{H}_\sigma^1 : v = 0 \text{ in } K_\delta \cap B_{2s}, v = w \text{ on } \partial B_{2s}\}$$

Define $\eta \in C^1(\mathbb{R}^3)$ to be a cut-off function with $\eta = 1$ in $\mathbb{R}^3 \setminus B_{2s-2d_0}$, $\eta = 0$ in B_{s+2d_0} , and $|\nabla \eta| \leq C/s$. Then, $v_1 := \eta w$ has the right boundary condition to be in the set X_w but fails to be divergence free. Indeed, $\operatorname{div} v_1 = \nabla \eta \cdot w$. Therefore, we use Lemma 3.5.10 to find a function $v_2 \in \dot{H}_0^1(B_{2s} \setminus B_s)$ with $\operatorname{div} v_2 = -\operatorname{div} v_1$ and

$$\|\nabla v_2\|_{L^2(B_{2s} \setminus B_s)} \leq C \|\operatorname{div} v_1\|_{L^2(B_{2s} \setminus B_s)} \leq \frac{C}{s} \|w\|_{L^2(B_{2s} \setminus B_s)}.$$

Now $v_1 + v_2$ is divergence free and equals w on ∂B_{2s} . To match the boundary conditions in $K_\delta \cap B_{2s}$, we use Lemma 3.5.16. For $x_i \in A := B_{2s-d_0} \setminus B_{s+d_0}$ it provides a function $v_i \in H_{0,\sigma}^1(B_{\kappa r_i}(x_i))$ with $v_i = -v_2$ in B_i and

$$\begin{aligned} \left\| \sum_{x_i \in A} v_i \right\|_{\dot{H}^1(\mathbb{R}^3)}^2 &\leq \sum_{x_i \in A} \frac{C}{r_i^2} \|v_2\|_{L^2(B_i)}^2 + C \|\nabla v_2\|_{L^2(B_i)}^2 \\ &\leq \sum_{x_i \in A} C \mu_0 \|v_2\|_{L^2(B_{d_i}(x_i))}^2 + C \|\nabla v_2\|_{L^2(B_i)}^2 \\ &\leq C(s^2 \mu_0 + 1) \|\nabla v_2\|_{L^2(B_{2s} \setminus B_s)}^2, \end{aligned}$$

where we used Lemma 3.5.18 for the second estimate and the Poincaré inequality in $\dot{H}_0^1(B_{2s} \setminus B_s)$ for the last one. By construction, $v := v_1 + v_2 + \sum_{x_i \in A} v_i$ is an element of X_w . Therefore,

$$\begin{aligned} 0 &\leq \|\nabla v\|_{L^2(\mathbb{R}^3)}^2 - \|\nabla w\|_{L^2(\mathbb{R}^3)}^2 \\ &\leq C \|\nabla w\|_{L^2(B_{2s} \setminus B_s)}^2 + C\left(\frac{1}{s^2} + \mu_0\right) \|w\|_{L^2(B_{2s} \setminus B_s)}^2 - \|\nabla w\|_{L^2(B_s)}^2. \end{aligned}$$

Since $s \geq s_0$ by assumption, the factor s^{-2} can be dropped. Using the Poincaré inequality in the annulus $B_{2s} \setminus B_s$, Lemma 3.4.17, we deduce

$$\|\nabla w\|_{L^2(B_s)}^2 \leq C_1 \|\nabla w\|_{L^2(B_{2s} \setminus B_s)}^2,$$

where C_1 depends on μ_0 , μ_1 and κ . Using again the hole filling technique as in the proof of Lemma 3.4.16 and iterating from $s = s_1$ until $2^k s \geq R/2$ concludes the proof. \square

Chapter 4

Sedimentation of inertialess particles in Stokes flows

In this chapter, we give a rigorous derivation of the transport-Stokes system (1.1.1) as the macroscopic model for the sedimentation dynamics of inertialess particles at zero Reynolds number. A formal derivation of this system has already been given in Chapter 2.3, where we also studied the physical relevance of these equations. In Chapter 2.5.1, we discussed the analogy between the transport-Stokes equations and the Stokes equations of variable density, as well as some phenomena regarding the solutions to these equations. An important device for the proof in this chapter is the method of reflections in the framework of orthogonal projections studied in Chapter 3.

The content of this chapter has been published in *Communications in Mathematical Physics*, [Höf18a]

4.1 Introduction

We recall from Chapter 2.3.1 the microscopic dynamics for inertialess particles in a Stokes fluid in dimensionless form. We consider N particles at centers X_i and with identical radii R and we denote $B_i := B_R(X_i)$ and the initial particle centers by X_i^0 .

$$\dot{X}_i = V_i, \quad X_i(0) = X_i^0, \quad (4.1.1)$$

$$-\Delta v + \nabla p = 0 \quad \text{in } \mathbb{R}^3 \setminus \bigcup_{i=1}^N \overline{B_i}, \quad (4.1.2)$$

$$\operatorname{div} v = 0 \quad \text{in } \mathbb{R}^3, \quad v = V_i \quad \text{in } \overline{B_i}, \quad v(x) \rightarrow 0 \quad \text{as } |x| \rightarrow \infty,$$

$$\int_{\partial B_i} \sigma n d\mathcal{H}^2 = \frac{4\pi}{3N} g =: F, \quad (4.1.3)$$

where the (dimensionless) gravitational acceleration g is given, and we recall $\sigma = \nabla v + (\nabla v)^T - pI$.

The particle velocities V_i are determined by the positions X_j of all the particles through the solution to the Stokes equations v . This dependence can be shown to be locally Lipschitz continuous, which leads to well-posedness until the first collision of particles occurs. There are several results on the well-posedness of rigid bodies in viscous fluids even allowing for collisions of the bodies, e.g. [DE99], [SST02], and [Fei03]. In those paper, Navier-Stokes equations are considered, and inertial effects of the bodies are taken into account. Both the setting and the well-posedness result needed here is simpler and is proven in Section 4.3.

Since the empirical measure for the particles solving (4.1.2) and (4.1.3) only depends on the particle positions, it is given by

$$\nu_N(t, x) := \frac{1}{N} \sum_i \delta(x - X_i). \quad (4.1.4)$$

It turns out that instead of the empirical measure it is more convenient to deal with regularized version of it, namely,

$$\rho_N(t, x) := \frac{1}{N} \sum_i \frac{4\pi}{3} \frac{\mathcal{H}^2|_{\partial B_i}}{|\partial B_i|}, \quad (4.1.5)$$

(where we included the factor $4\pi/3$ to avoid it from appearing in the limit equation). Since the fluid velocity v that solves (4.1.2) takes the values V_i in $\overline{B_i}$, the time evolution of ρ_N is given by

$$\partial_t \rho_N + v \cdot \nabla \rho_N = 0,$$

It is very useful to work with the measure ρ_N instead of ν_N because $\rho_N g$ gives an approximation for the force which the particles exert on the fluid. To see this, we observe that by equation (4.1.3) a priori only the total force at each particle is known. The quantity ρg therefore approximates those forces by uniformly distributing them on the surface of the particles. More precisely, the fluid velocity v can be approximated by u which is defined as the solution to

$$\begin{aligned} -\Delta u + \nabla p &= \rho_N g, \\ \operatorname{div} u &= 0, \quad u(x) \rightarrow 0 \quad \text{as } |x| \rightarrow \infty. \end{aligned} \quad (4.1.6)$$

Such an approximation for v cannot be hoped to be valid (at least not in L^∞) if we replace ρ_N by ν_N as this would lead to singularities of u . We expect u to be a good approximation for v in $L^\infty(\mathbb{R}^3)$. We observe that u can be written as $u = \sum_i u_i$ with u_i being the solution to

$$\begin{aligned} -\Delta u_i + \nabla p &= \frac{4\pi}{3N} g \frac{\mathcal{H}^2|_{\partial \overline{B_i}}}{|\partial \overline{B_i}|} =: f_i, \\ \operatorname{div} u_i &= 0, \quad u_i(x) \rightarrow 0 \quad \text{as } |x| \rightarrow \infty. \end{aligned} \quad (4.1.7)$$

We have encountered this fluid equation in Chapter 2.2.1, when we studied the settling of a single spherical particle. In particular, from (2.2.2) we know

$$u_i = \frac{4\pi}{3N} \frac{1}{6\pi R} g = \frac{2}{9} \gamma^{-1} g \quad \text{in } \overline{B_i}. \quad (4.1.8)$$

Hence, each particle velocity V_i is expected to be approximately given by

$$V_i = v(X_i) \approx u(X_i) = \sum_j u_j(X_i) = \frac{2}{9} \gamma^{-1} g + \sum_{j \neq i} u_j(X_i).$$

The first term on the right hand side is the velocity of a single inertialess particle, the second term is the collective effect due to the presence of all the other particles. In particular, the second term corresponds to the macroscopic fluid velocity around B_i , whereas the fluid velocity that produces the first term decays fast away from B_i .

The quantity γ determines the ratio between these two contributions, and therefore determines the interaction strength of the particles. It is proportional to the resistance density of the particles, more precisely, $\gamma = L^2 \mu$ if μ denotes the resistance density of the particles and L is the size of the particle cloud before rescaling it to one (see Chapter 2.3.1). The resistance density is the

analog of the capacity density in the electrostatic framework. In Chapter 3.1, we discussed the relation between the capacity density and the screening length Λ , which is given by $\Lambda^2 = \mu^{-1}$. Thus, $\gamma = L^2/\Lambda^2$.

One can interpret the screening length Λ in the following way. The presence of the particles screens the fluid velocity at infinity (which is zero) over distances of order Λ . In particular, if the screening length Λ is much larger than the system size L , the collective effect of the particles is small, which results in a small macroscopic fluid velocity. Hence, for small values of γ , all the particles approximately sink like isolated spheres. This is the result of [JO04]. We emphasize however that this screening effect is different from the one that is observed when the sedimentation boundary conditions in the Stokes equations are replaced by Dirichlet boundary conditions. In that case, as we discussed in Chapter 3.1, the screening effect results in a faster decay of the fundamental solution over distances of the screening length. This faster decay does not occur for the sedimentation boundary conditions.

In the limit $N \rightarrow \infty$, $R \rightarrow 0$ assuming $\gamma \rightarrow \gamma_* \in (0, \infty]$ and $\rho_N \rightarrow \rho$ in a suitable sense for the initial data, we expect convergence of the microscopic inertialess dynamics to the transport Stokes equations, which we write again:

$$\begin{aligned} \partial_t \rho + (v_* + \frac{2}{9} \gamma_*^{-1} g) \cdot \nabla_x \rho &= 0, \\ -\Delta v_* + \nabla p &= \rho g, \\ \operatorname{div} v_* &= 0, \quad v_*(x) \rightarrow 0 \quad \text{as } |x| \rightarrow \infty. \end{aligned} \tag{4.1.9}$$

We will prove the convergence (4.1.2) and (4.1.3) to the macroscopic equation (4.1.9). The precise assumptions on the particle configurations will be given in Section 4.2, and the main result will be stated in Theorem 4.2.9. Well-posedness of the transport-Stokes equations (4.1.9) is shown by standard arguments based on the Banach fixed point theorem in Section 4.4.

In the limit $\gamma_* \rightarrow 0$, we get after rescaling time, that the cloud of particles falls down with constant velocity that is determined by the velocity of a single particle. This is in accordance with the result of [JO04].

On the other hand, in the case $\gamma_* = \infty$, we see that the self-interaction term $\frac{2}{9} \gamma_*^{-1} g$ becomes negligible and the particles are transported by the macroscopic fluid velocity v_* . Finally, for positive but finite γ_* , the behavior of solutions to (4.1.9) is very similar as in the case $\gamma_* = \infty$. Indeed, although equation (4.1.9) is nonlinear, the only effect of the term $\frac{2}{9} \gamma_*^{-1} g$ is a translation velocity of the cloud (cf. Proposition 4.4.1), because the Stokes equations are invariant under translation.

4.2 Assumptions and main result

We will now specify the assumptions on the initial particle configurations, give a precise statement of the convergence result to the macroscopic equation (4.1.9) and an outline of the proof of the main result.

4.2.1 Assumptions on the initial particle configurations

We consider a sequence of initial particle configurations $\{X_{\varepsilon,i}^0\}_{1 \leq i \leq N_\varepsilon}$ indexed by ε and we assume $N_\varepsilon \rightarrow \infty$ and $R_\varepsilon \rightarrow 0$ in the limit $\varepsilon \rightarrow 0$.

Notation 4.2.1. *For the ease of notation, we write X_i^0 instead of $X_{\varepsilon,i}^0$ in the remainder of this chapter. We will also sometimes drop the index ε on other quantities, in particular when ε is fixed.*

We impose the following constraints on the initial particle distributions.

- (A1) We require the distance between every pair of particles to be at least of the order of the typical distance between particles. More precisely, the minimal distance

$$d_{\varepsilon,\min}(0) := \min_{i \neq j} |X_i^0 - X_j^0|$$

has to satisfy

$$N_\varepsilon (d_{\varepsilon,\min}(0))^3 \geq c_0$$

for some constant $c_0 > 0$ independent of ε .

- (A2) We require the volume fraction of the particles $\phi_\varepsilon = N_\varepsilon R_\varepsilon^3$ to satisfy

$$\phi_\varepsilon \log N_\varepsilon \rightarrow 0 \quad \text{as } \varepsilon \rightarrow 0.$$

- (A3) We require the quantity γ_ε which is the ratio between the screening length ξ_ε and the system size L_ε to converge to some limit $\gamma_* \in (0, \infty]$. More precisely,

$$\lim_{\varepsilon \rightarrow 0} \gamma_\varepsilon = \lim_{\varepsilon \rightarrow 0} N_\varepsilon R_\varepsilon = \gamma_* \in (0, \infty].$$

For fixed $\varepsilon > 0$, the minimal distance between the particles $d_{\varepsilon,\min}$ might change over time. Thus, an important issue for the analysis of the time evolution will be to examine whether condition (A1) is conserved over time, possibly with a smaller constant but uniformly in ε . Therefore we introduce the following quantity.

Definition 4.2.2. For $\varepsilon > 0$ we define

$$Y_\varepsilon(t) := \sup_{0 \leq s \leq t} \sup_{i \neq j} \frac{|X_i^0 - X_j^0|}{|X_i(t) - X_j(t)|},$$

where the particle positions implicitly depend on ε .

Remark 4.2.3. Implicitly, we also assume the particles to be disjoint. More precisely, we assume $d_{\varepsilon,\min}(0) \geq 4R_\varepsilon$. Indeed, for sufficiently small ε this is ensured for the initial configuration by (A1) and (A2).

Moreover, this is preserved up to time t , provided $Y_\varepsilon(t)$ satisfies a uniform bound for small ε :

$$d_{\varepsilon,\min}(t) \geq \frac{d_{\varepsilon,\min}(0)}{Y_\varepsilon(t)} \geq \frac{c_0^{1/3}}{N_\varepsilon^{1/3} Y_\varepsilon(t)} = \frac{c_0^{1/3}}{\phi_\varepsilon^{1/3} Y_\varepsilon(t)} R_\varepsilon \geq 4R_\varepsilon,$$

for all $\varepsilon < \varepsilon_0(t)$ small enough. We will always assume that ε is chosen small enough, such that this is the case.

Notice that in the case of a finite limit γ_* , assumption (A2) is automatically satisfied. Indeed, in this case

$$\phi_\varepsilon \log N_\varepsilon = N_\varepsilon R_\varepsilon^3 \log N_\varepsilon \leq R_\varepsilon (N_\varepsilon R_\varepsilon)^2 = R_\varepsilon \gamma_\varepsilon^2 \rightarrow 0.$$

4.2.2 Statement of the main result

Finally, we also impose that the initial particle configurations converge in a certain averaged sense. We recall the definition of the particle density ρ_ε from (4.1.5), and we write $\rho_{\varepsilon,0}$ for the corresponding initial particle density. We now define a “coarse grained” density function ρ_ε^δ by taking averages on small cubes, but much larger than the typical particle distance. Such an averaged density has been used in [NV04a] for a related homogenization problem. It has the advantage, in contrast to ρ_ε , that it is bounded in $L^\infty(\mathbb{R}^3)$, and therefore pointwise convergence of ρ_ε^δ can be expected. On the other hand, the averages are taken in such a way, that the total masses of ρ_ε and ρ_ε^δ in those small cubes coincide.

Definition 4.2.4. For $\delta > 0$ we decompose \mathbb{R}^3 into half-open disjoint cubes Q_δ^i with edge length δ . Then, we define ρ_ε^δ by

$$\rho_\varepsilon^\delta(t, x) = \int_{Q_\delta^i} \rho_\varepsilon(t, y) dy \quad \text{for } x \in Q_\delta^i.$$

We will denote the unique cube containing x by Q_δ^x .

We also specify the function space, where we require convergence of the averaged initial data $\rho_{\varepsilon,0}^\delta$.

Definition 4.2.5. Let $\beta \geq 0$. We define the norm

$$\|h\|_{X_\beta} := \sup_x (1 + |x|^\beta) |h(x)|,$$

and the space

$$X_\beta := \{h \in L^\infty(\mathbb{R}^3) : \|h\|_{X_\beta} < \infty\}.$$

It is convenient to work in the space X_β for two reasons. First, dealing with a supremum-norm enhances working with particle trajectories and the characteristics of the limit equation (4.1.9). Second, for $\beta > 2$, we have $\|Sh\|_{W^{1,\infty}(\mathbb{R}^3)} \leq C\|h\|_{X_\beta}$, where S is the solution operator to Stokes equations (see Lemma 4.4.3 and Remark 4.4.4).

Remark 4.2.6. We note that for all times $\varepsilon > 0$ and all T such that the dynamics (4.1.1), (4.1.2), (4.1.3) has a solution up to time T with $v_\varepsilon \in L^\infty((0, T) \times \mathbb{R}^3)$, we have $\rho_\varepsilon^\delta \in C^0([0, T]; X_\beta)$. Indeed, consider the function $\tau(x) = \mathcal{H}^2|_{\partial B_R(X(t))}$, for some Lipschitz curve X . Then, it is easy to check that $\tau^\delta \in C^0([0, T]; X_\beta)$ and ρ_ε^δ is the sum of such functions τ .

Assumption 4.2.7. There exists a sequence $d_{\varepsilon,\min}(0) \ll \delta_\varepsilon \rightarrow 0$ and a function $\rho_0 \in X_\beta$ with $\nabla \rho_0 \in X_\beta$ for some $\beta > 2$ such that

$$\lim_{\varepsilon \rightarrow 0} \|\rho_{\varepsilon,0}^\delta - \rho_0\|_{X_\beta} = 0.$$

Remark 4.2.8. Let $\tilde{\delta}_\varepsilon \rightarrow 0$, such that $\tilde{\delta}_\varepsilon = n_\varepsilon \delta_\varepsilon$ for some $n_\varepsilon \in \mathbb{N} \setminus \{0\}$. Then, since $\nabla \rho_0 \in X_\beta$,

$$\|\rho_{\varepsilon,0}^{\tilde{\delta}_\varepsilon} - \rho_0\|_{X_\beta} \rightarrow 0.$$

Thus, we can assume $\delta_\varepsilon \gg \phi_\varepsilon$.

The following theorem is the main result of this chapter.

Theorem 4.2.9. *Assume that conditions (A1) - (A3) are satisfied and that the initial data $\rho_{\varepsilon,0}$ converge to some ρ_0 in the sense of Assumption 4.2.7 with some $\delta_\varepsilon \rightarrow 0$ and $\beta > 2$. Then for all $T > 0$, there exists $\varepsilon_0 > 0$ and C_1 such that for all $\varepsilon < \varepsilon_0$, there exists a unique solution to the dynamics (4.1.1), (4.1.2), (4.1.3) up to time T with*

$$Y_\varepsilon(T) \leq e^{C_1 T}, \quad \text{for all } \varepsilon < \varepsilon_0.$$

In particular, there are no collisions up to time T . Moreover, for all $\tilde{\delta}_\varepsilon \rightarrow 0$ such that $\tilde{\delta}_\varepsilon = n_\varepsilon \delta_\varepsilon$ for some $n_\varepsilon \in \mathbb{N}^$ with $n_\varepsilon \rightarrow \infty$,*

$$\rho_{\varepsilon}^{\tilde{\delta}_\varepsilon} \rightarrow \rho \quad \text{in } C^0([0, T]; X_\beta),$$

where ρ is the unique classical solution to problem (4.1.9).

Note that we do not impose a lower bound on the rate of convergence of $n_\varepsilon \rightarrow \infty$, whereas the constraint $\tilde{\delta}_\varepsilon \rightarrow 0$ provides an upper bound. The reason for introducing n_ε is discussed in Section 4.6.1.

It is possible to relate the averaged convergence $\rho_{\varepsilon}^{\tilde{\delta}_\varepsilon} \rightarrow \rho$ to weak L^1 -convergence of ρ_ε and to weak convergence in the sense of measures of the empirical measures ν_ε defined in (4.1.4). Indeed, assuming the particles are all contained in a bounded region independently of ε , the assumption $\rho \in W^{1,\infty}(\mathbb{R}^3)$ implies that both weak L^1 -convergence $\rho_\varepsilon \rightharpoonup \rho$ and weak convergence in the sense of measures of the empirical measures $\nu_\varepsilon \rightharpoonup \rho$ are equivalent to averaged convergence in the sense of Assumption 4.2.7 for some δ_ε . In particular, we have the following corollary.

Corollary 4.2.10. *Assume that conditions (A1) - (A3) are satisfied and that there exists a cube $Q_0 \subset \mathbb{R}^3$ such that $X_{\varepsilon,i}^0 \subset Q_0$ for all $\varepsilon > 0$ and all $1 \leq i \leq N_\varepsilon$. Moreover, assume that the empirical measures $\nu_{\varepsilon,0}$ converge to some $\rho_0 \in X_\beta$ weakly in the sense of measures with $\nabla \rho_0 \in X_\beta$. Then for all $T > 0$, there exists $\varepsilon_0 > 0$ such that for all $\varepsilon < \varepsilon_0$, there exists a unique solution to the dynamics (4.1.1), (4.1.2), (4.1.3) without collisions up to time T . Moreover, $\nu_\varepsilon(t) \rightharpoonup \rho(t, \cdot)$ weakly in the sense of measures for almost every $t \geq 0$ where ρ is the unique classical solution to problem (4.1.9).*

Proof. We claim that the assumptions of the corollary imply Assumption 4.2.7, namely that there exists a sequence δ_ε such that $\rho_{\varepsilon,0}^{\delta_\varepsilon} \rightarrow \rho_0$ in X_β . First, we note that since the limit ρ_0 of the sequence $\nu_{\varepsilon,0}$ is continuous with respect to the Lebesgue measure, we have

$$\nu_{\varepsilon,0}(Q) \rightarrow \int_Q \rho_0(x) dx,$$

for all cubes $Q \subset \mathbb{R}^3$. In particular, we observe that $\rho_0 = 0$ in $\mathbb{R}^3 \setminus Q_0$. Moreover, by definition of $\rho_{\varepsilon,0}$ and $\nu_{\varepsilon,0}$, for any cube of length $\delta \geq d_{\varepsilon,\min}(0)$

$$\left| \frac{1}{|Q|} \nu_{\varepsilon,0}(Q) - \int_Q \rho_{\varepsilon,0}(y) dy \right| \leq C \frac{d_{\varepsilon,\min}(0)}{\delta},$$

where the error term comes from particles close to the boundary of the cube. Let $k \in \mathbb{N}$, and let $Q_{j,k}$, $1 \leq j \leq M_k$, be cubes of length $1/k$ according to Definition 4.2.4 that cover Q_0 up to a nullset. Then, there exists $\varepsilon_k > 0$ such that for all $\varepsilon < \varepsilon_k$

$$\left| \int_Q \rho_{\varepsilon,0}(y) - \rho_0(y) dx \right| < 1/k.$$

We define $\delta_\varepsilon = 1/k$ for $\varepsilon_k > \varepsilon \geq \varepsilon_{k+1}$. Then, for all $\varepsilon_k > \varepsilon \geq \varepsilon_{k+1}$, and all cubes $Q_{\delta_\varepsilon}^i$ as in Definition 4.2.4, and all $x \in Q_{\delta_\varepsilon}^i$

$$\left| \int_{Q_{\delta_\varepsilon}^i} \rho_{\varepsilon,0}(y) dy - \rho_0(x) \right| \leq \left| \int_{Q_{\delta_\varepsilon}^i} \rho_{\varepsilon,0}(y) - \rho_0(y) dy \right| + \frac{C}{k} \|\nabla \rho_0\|_{L^\infty(\mathbb{R}^3)} \leq \frac{C}{k}.$$

This proves the claim.

Hence, we can apply Theorem 4.2.9. Thus, it remains to prove that $\rho_{\varepsilon}^{\tilde{\delta}_{\varepsilon}}(t, \cdot) \rightarrow \rho(t, \cdot)$ in X_{β} implies weak convergence $\nu_{\varepsilon}(t) \rightharpoonup \rho(t, \cdot)$. First note that combining the estimate on Y_{ε} from Theorem 4.2.9 with Lemma 4.6.3 implies that the fluid velocity v_{ε} that transports the particles is uniformly bounded for $\varepsilon < \varepsilon_0$ up to time T . Thus, all the particles remain in a bounded region independently of ε . Moreover, also the limit fluid velocity v_* is uniformly bounded by Lemma 4.4.3 such that $\rho(t, \cdot)$ is uniformly compactly supported up to time T . Let $f \in W^{1,\infty}(\mathbb{R}^3)$, let $Q_{\tilde{\delta}_{\varepsilon}}^i$, $i \in I_{\varepsilon}$ with $|I_{\varepsilon}| \leq C\tilde{\delta}_{\varepsilon}^{-3}$ be as in Definition 4.2.4 such that $\cup_{i \in I_{\varepsilon}} Q_{\tilde{\delta}_{\varepsilon}}^i$ contains the support of $\rho(t, \cdot)$ and $\rho_{\varepsilon}(t, \cdot)$ for all $t \leq T$ and all $\varepsilon < \varepsilon_0$. Let x_i be the centers of those cubes. Then,

$$\begin{aligned} \left| \int_{\mathbb{R}^3} f d\nu_{\varepsilon}(t) - \int_{\mathbb{R}^3} f \rho(t, \cdot) dx \right| &\leq \sum_{i \in I_{\varepsilon}} \left| \frac{1}{N_{\varepsilon}} \sum_{X_k(t) \in Q_{\tilde{\delta}_{\varepsilon}}^i} f(X_k) - \int_{Q_{\tilde{\delta}_{\varepsilon}}^i} f \rho(t, \cdot) dx \right| \\ &\leq \sum_{i \in I_{\varepsilon}} \left| \nu(Q_{\tilde{\delta}_{\varepsilon}}^i) - \int_{Q_{\tilde{\delta}_{\varepsilon}}^i} \rho(t, \cdot) dx \right| \|f\|_{L^{\infty}(\mathbb{R}^3)} + 2\tilde{\delta}_{\varepsilon} \|\nabla f\|_{L^{\infty}(\mathbb{R}^3)} \\ &\leq C \frac{d_{\varepsilon, \min}(t)}{\tilde{\delta}_{\varepsilon}} + C \|\rho_{\varepsilon}^{\tilde{\delta}_{\varepsilon}}(t, \cdot) - \rho(t, \cdot)\|_{L^{\infty}(\mathbb{R}^3)} + C\tilde{\delta}_{\varepsilon} \\ &\leq C \frac{d_{\varepsilon, \min}(t)}{\tilde{\delta}_{\varepsilon}} + C \|\rho_{\varepsilon}^{\tilde{\delta}_{\varepsilon}}(t, \cdot) - \rho(t, \cdot)\|_{L^{\infty}(\mathbb{R}^3)} + C\tilde{\delta}_{\varepsilon}. \end{aligned}$$

Since Y_{ε} is uniformly bounded, the first term tends to zero, as well as the other two. This concludes the proof. \square

4.2.3 Outline of the proof of the main result

Well-posedness of both the microscopic dynamics (4.1.1), (4.1.2), and (4.1.3) away from collisions and the transport-Stokes equations (4.1.9) is done by standard arguments in the Sections 4.3 and 7.2.

Regarding the convergence result, the main difficulty of the analysis of the dynamics (4.1.1), (4.1.2), and (4.1.3) is that the fluid velocity v is only given implicitly. It satisfies Stokes equations but the source term is not given explicitly but only the total force on each particle (by (4.1.3)) and the constraint that the fluid velocity has to be constant at every particle. Those constants, however, which are a priori unknown, determine the velocity of the particles, and therefore are the relevant quantities in order to understand the dynamics.

As an approximation for the fluid velocity v_{ε} , we take the velocity u_{ε} defined in (4.1.6), which corresponds to a source term that consists of a sum of forces uniformly distributed on the boundary of the particles such that (4.1.3) holds. This approximated fluid velocity u_{ε} does not satisfy the constraint of constant velocity at the particles. Smallness of $\nabla(v_{\varepsilon} - u_{\varepsilon})$ in $L^2(\mathbb{R}^3)$ can be obtained by standard variational methods in the limit of small volume fraction ϕ_{ε} of the particles, and will be proven in Section 4.5.2. However, since we are interested in the values of v_{ε} at the particles, those estimates are not sufficient. We still give the proof of the L^2 -estimate because most of the ingredients will be used also in the proof of L^{∞} -estimates.

In Section 4.5.3, we prove L^{∞} -estimates of $v_{\varepsilon} - u_{\varepsilon}$ using a rigorous version of the so-called method of reflections, which gives a series representation for v_{ε} . The method of reflection is a method to express the solution operator for an elliptic problem with a boundary consisting of several connected components in terms of a series involving the solution operators for the individual components. This method is particularly useful if the solution operators for those individual components are well understood as in the case of spheres.

A version of this method has been used in [JO04]. We will use the formulation of the method of reflection in the framework of orthogonal projection that has been investigated in [HV18], where Stokes

equations with Dirichlet boundary conditions are considered. In that case, the series representation has been proven to converge if γ_ε (which is the capacity density of the particles) is sufficiently small. It turns out that in the case of the mixed boundary conditions given in (4.1.2), (4.1.3), the method is actually convergent under milder assumption. Indeed, we prove convergence of the series to v_ε in $L^\infty(\mathbb{R}^3)$ under the assumption that the particles are sufficiently separated and $\phi_\varepsilon \log N_\varepsilon$ is small (cf. (A2)). (This assumption, which is only slightly stronger than smallness of the particle volume ϕ_ε , seems to be unavoidable when using the method of reflections, at least without additional assumption on the distribution of particles. The L^2 -estimates derived in Section 4.5.2, however, suggest that smallness of the particle volume ϕ_ε should be sufficient.) Moreover, the zero order term of the representation, which is exactly the approximated fluid velocity u_ε that solves (4.1.6), is shown to be close to v_ε in $L^\infty(\mathbb{R}^3)$ for small ϕ_ε .

Replacing v_ε by u_ε is the most important step in the proof of the homogenization result. Indeed, u_ε is given explicitly in terms of the particle positions and is the solution of the Stokes equations with a source term proportional to the particle density. In Section 4.5.5, we prove that u_ε is close to $\frac{2}{9}\gamma_\varepsilon^{-1}g + w_\varepsilon$, where w_ε is the solution to the Stokes equations with source $\rho_\varepsilon^{\delta_\varepsilon}g$. The function w_ε can be viewed as the macroscopic fluid velocity. Indeed, since $\rho_\varepsilon^{\delta_\varepsilon}g$ is bounded in L^∞ the effect of every single particle on the fluid velocity w is negligible, whereas the effect of every single particle on the fluid velocity u_ε at the position of that particular particle is $\frac{2}{9}\gamma_\varepsilon^{-1}g$. The proof of smallness of $u_\varepsilon - \frac{2}{9}\gamma_\varepsilon^{-1}g - w_\varepsilon$ is straightforward based on three ingredients. First, the explicit form of u_ε as the sum of u_i as in (4.1.7) with the value of u_i in B_i being $\frac{2}{9}\gamma_\varepsilon^{-1}g$ by equation (4.1.8). Second, the fact that the total masses of ρ_ε and $\rho_\varepsilon^{\delta_\varepsilon}$ in the cubes Q_{δ_ε} coincide, where Q_{δ_ε} are the cubes from the definition of $\rho_\varepsilon^{\delta_\varepsilon}$ (Definition 4.2.4). Finally, the explicit formula of the solution operator of the Stokes equations as a convolution operator.

As mentioned before, an important issue is whether aggregation of particles takes place. Not only is this important to investigate in order to rule out particle collisions, but also since the estimates based on the method of reflections are only proved for particles that are sufficiently well separated, in the sense that

$$\alpha_\varepsilon := \sup_j \frac{1}{N_\varepsilon} \sum_{i \neq j} \frac{1}{|X_i - X_j|^2} \quad (4.2.1)$$

is not too large. In Section 4.5.4, we will prove Lipschitz type estimates for the fluid velocity v_ε , which enable us to control the quantity $Y_\varepsilon(t)$ from Definition 4.2.2. Unfortunately, these Lipschitz type estimates are again based on the method of reflections, and therefore dependent on the value of α_ε from above. It is not difficult to control α_ε in terms of Y_ε^2 , but due to this mutual dependence, we are a priori only able to show that particle aggregation cannot take place in short times uniformly for small enough volume fractions ϕ_ε .

As long as all the particles remain well separated in the sense that $Y_\varepsilon(t)$ does not blow up as $\varepsilon \rightarrow 0$, the homogenization result is then proved in Section 4.6.1. With the ingredients from the previous sections, the proof is not difficult but somewhat tedious based on comparing the particle trajectories to the characteristics of the limit problem (4.1.9).

Finally, as the last step of the proof of Theorem 4.2.9, we show in Section 4.6.2 that particle aggregation does not take place for arbitrary finite times in the sense that, for all times $t > 0$, $Y_\varepsilon(t)$ does not blow up as $\varepsilon \rightarrow 0$. For this we use estimates for the solution to the macroscopic equation (4.1.9) together with the convergence result for times up to which $Y_\varepsilon(t)$ does not blow up in the limit $\varepsilon \rightarrow 0$. Since pointwise convergence of the particle density to the solution ρ of the macroscopic equation only holds for the averaged density ρ_ε , and the averages are taken over distances much larger than the typical particle distance, this convergence does not directly imply information on the quantity $Y_\varepsilon(t)$. However, the convergence result turns out to be strong enough to yield good a posteriori estimates on the quantity α_ε defined in (4.2.1). Finally, by the Lipschitz type estimates

from Section 4.5.4 on the microscopic fluid velocity v_ε , we control Y_ε in terms of α_ε .

4.3 Well-posedness of the microscopic dynamics away from collisions

In this section, we prove well-posedness of the dynamics (4.1.1), (4.1.2), (4.1.3) away from particle collisions. We first show that for fixed particle positions X_i at a fixed time, (4.1.2), (4.1.3) admits a unique weak solution in v in the space of divergence free functions $\dot{H}_\sigma(\mathbb{R}^3)$ (see Definition 3.5.1 for the precise definition), which determines the values of V_i . Then, by standard ODE-theory, it suffices to show that the mapping $(X_i)_i \mapsto (V_i)_i$ is Lipschitz continuous away from particle collisions.

For a fixed time and given, it is well known (see e.g. [Gal11]) that problem (4.1.2) has a unique weak solution (v, p) in $\dot{H}_\sigma^1(\mathbb{R}^3) \times L^2(\mathbb{R}^3)$ for given data of the particles, $(X_i)_i$, $(V_i)_i$, provided the particles are not touching each other.

Moreover, for a fixed time, problem (4.1.2), (4.1.3) has a unique weak solution (v, p) in $\dot{H}_\sigma^1(\mathbb{R}^3) \times L^2(\mathbb{R}^3)$ in terms of $(X_i)_i$ provided the particles are non-touching. To see this, we fix the space positions of the particles $(X_i)_i$ and observe that, for given velocities V_i , the forces $G_i = -\int_{\partial B_i} \sigma n d\mathcal{H}^2$ are given by $G = AV$, where $A \in \mathbb{R}^{N \times N}$ is a linear map. Furthermore, A is coercive, because

$$V \cdot AV = V \cdot G = -\sum_{i=1}^N \int_{\partial B_i} V_i \cdot \sigma n d\mathcal{H}^2 = \int_{\mathbb{R}^3 \setminus \bigcup_{i=1}^N \overline{B_i}} |\nabla v|^2 = \|v\|_{\dot{H}^1(\mathbb{R}^3)}^2 \geq C\|V\|^2, \quad (4.3.1)$$

with a constant C that only depends on R and N . Hence, A is invertible, which yields V for prescribed G and X .

Theorem 4.3.1. *For any initial configurations of particles $(X_i^0)_{1 \leq i \leq N}$ such that the closed balls $\overline{B_i^0}$ are pairwise disjoint, there exists a time $T_* > 0$ such that the problem (4.1.1), (4.1.2), (4.1.3) has a unique solution in $[0, T_*)$. Moreover, $T_* = \infty$, or*

$$\liminf_{t \rightarrow T_*} d_{\min}(t) = 2R.$$

Proof. We fix N and R . We have seen that the velocities $V_i(t)$ are uniquely determined by the particle positions. Hence we can write $V_i(t) = W_i(X(t))$. Then, it suffices to prove that the functions W_i are Lipschitz continuous away from particle collisions. More precisely, we need to prove that for all $\varepsilon > 0$, there exists a constant C such that

$$|W_i(X) - W_i(\tilde{X})| \leq C|X - \tilde{X}|$$

for all particle configurations $(X_i)_i$ with $d_{\min} > 2R + \varepsilon$.

In the following, a constant C might depend on ε (and N and R). We can estimate the \dot{H}^1 -norm of the solution v to problem (4.1.2), (4.1.3) brutally, using (4.3.1) and the definition of F in (4.1.3),

$$\|v\|_{\dot{H}^1(\mathbb{R}^3)}^2 = \sum_i FV_i \leq C \sup_i |V_i| \leq C\|v\|_{\dot{H}^1(\mathbb{R}^3)}.$$

Thus,

$$\|v\|_{\dot{H}^1(\mathbb{R}^3)} \leq C. \quad (4.3.2)$$

Fix particle positions with non-touching particles $(X_i)_{1 \leq i \leq N}$. Then, there exists $\theta > 1$ depending only on ε and R such that the closed balls $\overline{B_{2\theta R}(X_i)}$ are pairwise disjoint. Let $(\tilde{X}_i)_{1 \leq i \leq N}$ be another particle configuration with

$$\sup_i |X_i - \tilde{X}_i| \leq \frac{(\theta - 1)R}{4}.$$

We define a deformation φ by

$$\varphi(x) := x + \sum_i (\tilde{X}_i - X_i) \eta_i(x),$$

where $\eta_i \in C_c^\infty(B_{\theta R}(X_i))$ are chosen such that $0 \leq \eta_i \leq 1$, $\eta_i = 1$ in $\overline{B_i}$, and

$$|\nabla \eta_i| \leq \frac{2}{(\theta - 1)R}.$$

Then, φ is a diffeomorphism and $|\nabla \varphi|, |\nabla \varphi^{-1}| \leq C$.

Consider now the solutions v and \tilde{v} of problem (4.1.2), (4.1.3) with particle positions X_i and \tilde{X}_i , respectively. We denote the velocities in the balls B_i and \tilde{B}_i by V_i and \tilde{V}_i , respectively. We define $u_1 := \tilde{v} \circ \varphi$. Then,

$$|\operatorname{div} u_1| \leq C \sum_i |\tilde{X}_i - X_i| |\nabla \tilde{v}(\varphi(x))| \chi_{B_{\theta R}(X_i) \setminus \overline{B_i}}.$$

By Lemma 3.5.10, there exists a function $u_2 \in H_0^1(\cup_i B_{\theta R_i}(X_i) \setminus \overline{B_i})$ such that $\operatorname{div} u_2 = \operatorname{div} u_1$ and

$$\|u_2\|_{\dot{H}^1(\mathbb{R}^3)} \leq C \|\operatorname{div} u_1\|_{L^2(\mathbb{R}^3)} \leq C \|X - \tilde{X}\| \|\tilde{v}\|_{\dot{H}^1(\mathbb{R}^3)}.$$

Finally, we define $u = u_1 - u_2$. Then, $u = \tilde{V}_i$ in B_i . Moreover, using the equation, that \tilde{v} satisfies, we observe

$$\begin{aligned} -\Delta u + \nabla p &= -\operatorname{div} g \quad \text{in } \mathbb{R}^3 \setminus \bigcup_{i=1}^N \overline{B_i}, \\ \operatorname{div} u &= 0 \quad \text{in } \mathbb{R}^3, \end{aligned}$$

where

$$g(x) = - \sum_i ((\tilde{X}_i - X_i) \otimes \nabla \eta_i(x)) \nabla \tilde{v}(\varphi(x)) - \nabla u_2.$$

Thus,

$$\|g\|_{L^2(\mathbb{R}^3)} \leq C \|X - \tilde{X}\| \|\tilde{v}\|_{\dot{H}^1(\mathbb{R}^3)}. \quad (4.3.3)$$

Moreover, with σ_u denoting the stress corresponding to u ,

$$\int_{\partial B_i} \sigma_u n \, d\mathcal{H}^2 = F,$$

where F is as in (4.1.3). Defining $w := u - v$, we then deduce that w satisfies the following equation in its weak formulation

$$(\nabla w, \nabla \psi) = (g, \nabla \psi) \quad \text{for all } \psi \in \dot{H}_\sigma^1(\mathbb{R}^3) \text{ with } \psi = \text{const in } B_i, \ 1 \leq i \leq N.$$

Testing with $\psi = w$ and using the bound for g from (4.3.3), and (4.3.2) for the norm of \tilde{v} , we deduce

$$\|w\|_{\dot{H}^1(\mathbb{R}^3)} \leq C \|X - \tilde{X}\|.$$

Since $w = V_i - \tilde{V}_i$ in B_i , this yields

$$\|V - \tilde{V}\| \leq C \|X - \tilde{X}\|,$$

which concludes the proof. \square

4.4 Well-posedness of the macroscopic equations

In this section, we prove well-posedness of the macroscopic equation (4.1.9), which we write here as

$$\begin{aligned} \partial_t \rho + (u + v_0) \cdot \nabla \rho &= 0, \\ \rho(0, \cdot) &= \rho_0, \\ -\Delta u + \nabla p &= \rho g, \\ \operatorname{div} u &= 0, \end{aligned} \tag{4.4.1}$$

where $v_0 \in \mathbb{R}^3$ and $g \in \mathbb{R}^3$ are some given constants.

We are interested in Lipschitz continuous solutions of this problem. More precisely, for a given initial datum $\rho_0 \in X_\beta$ with $\nabla \rho_0 \in X_\beta$, we look for a classical solution $(\rho, u) \in W^{1,\infty}(0, T; X_\beta) \times L^\infty(0, T; W^{1,\infty})$ with $\nabla \rho \in L^\infty(0, T; X_\beta)$ for any positive time T . Here, X_β is the space from Definition 4.2.5.

Proposition 4.4.1. *Let $u_0 \in \mathbb{R}^3$ and assume $(\rho, u) \in L^\infty(0, T; X_\beta) \times L^\infty(0, T; W^{1,\infty})$ is a solution to problem (4.4.1) with $v_0 = u_0$. Let*

$$\sigma(t, x) := \rho(t, x - tu_0)$$

and

$$v(t, x) := u(t, x - tu_0).$$

Then (σ, v) solves (4.4.1) with $v_0 = 0$.

Proof. The solution operator of the Stokes equations S is a convolution operator, and convolution commutes with translation. Therefore, $(S(\rho(t, \cdot)g))(x - tu_0) = (S(\sigma(t, \cdot)g))(x)$. \square

Theorem 4.4.2. *Assume $\rho_0 \in X_\beta$ with $\nabla \rho_0 \in X_\beta$ for some $\beta > 2$. Then, Problem (4.4.1) admits a unique solution $\rho \in W^{1,\infty}(0, T; X_\beta)$ for all $T > 0$. Moreover, $\nabla \rho \in L^\infty(0, T; X_\beta)$.*

For the proof, we need the following lemma.

Lemma 4.4.3. *For all $\beta > 2$ and all $h \in X_\beta$,*

$$\|Sh\|_{W^{1,\infty}(\mathbb{R}^3)} \leq C\|h\|_{X_\beta}.$$

Remark 4.4.4. *It is worth noticing that also an estimate of the form*

$$\|Sh\|_{L^\infty(\mathbb{R}^3)} \leq C\|h\|_{L^\infty(\mathbb{R}^3)}^\theta \|h\|_{L^1(\mathbb{R}^3)}^{1-\theta}$$

and similar for ∇Sh holds with $\theta = 1/3$ and $\theta = 2/3$ respectively. Moreover, since the functions ρ_ε^δ are normalized in L^1 , this yields estimates of the relevant fluid velocities in terms of the L^∞ -norms only. However, for the proof of the convergence result (see the proof of Theorem 4.6.1), it is crucial, that the exponent on the right hand side of these estimates is (at least) equal to 1. This is the reason, why we chose to work in the space X_β instead of L^∞ .

Proof. We recall that the solution operator S can be represented by the convolution with the Oseen tensor Φ from (3.5.1). Hence, by definition of X_β ,

$$\begin{aligned} |(Sh)(x)| &\leq C\|h\|_{X_\beta} \int_{\mathbb{R}^3} \frac{1}{|x-y|} \frac{1}{|1+|y|^\beta|} dy \\ &\leq C\|h\|_{X_\beta} \int_{B_{\frac{|x|}{2}}(x)} \frac{1}{|x-y|} \frac{1}{|1+|x|^\beta|} + C\|h\|_{X_\beta} \int_{\mathbb{R}^3 \setminus B_{\frac{|x|}{2}}(x)} \frac{1}{|y|} \frac{1}{|1+|y|^\beta|} \\ &\leq C\|h\|_{X_\beta} \frac{|x|^2}{1+|x|^\beta} + C\|h\|_{X_\beta}, \end{aligned}$$

since $\frac{1}{|y|} \frac{1}{|1+|y|^\beta|} \in L^1(\mathbb{R}^3)$ as $1 + \beta > 3$.

The estimate for $\nabla(Sh)(x)$ works analogously. \square

Proof. By Proposition 4.4.1, we only have to consider the case $v_0 = 0$.

We prove the statement using the Banach fixed point theorem. We can write problem (4.4.1) in a more compressed way as

$$\begin{aligned} \partial_t \rho + S(\rho g) \cdot \nabla \rho &= 0, \\ \rho(0, \cdot) &= \rho_0. \end{aligned} \quad (4.4.2)$$

The strategy of the proof is the following. In the first part, we derive estimates for the linear equation

$$\begin{aligned} \partial_t \rho + S(\tau g) \cdot \nabla \rho &= 0, \\ \rho(0, \cdot) &= \rho_0. \end{aligned} \quad (4.4.3)$$

In the second part, we show that the solution operator for this equation is a contraction on a suitable metric space for small times. In order to get a global in time solution, we finally derive estimates for this solution that show that no blow-up in finite time is possible.

Step 1. Estimates for the linear equation. We recall from Lemma 4.4.3

$$\|S(\tau g)\|_{W^{1,\infty}} \leq C \|\tau\|_{X_\beta}, \quad (4.4.4)$$

where C depends only on g .

We claim that the solution operator A for problem (4.4.3) maps $\tau \in L^\infty(0, T; X_\beta)$ to a function $\rho \in L^\infty(0, T; X_\beta)$. To this end, we denote $v := S(\tau g)$. Then, the solution to the transport equation (4.4.3) is given by

$$\rho(t, x) = \rho_0(\varphi(t, 0, x)),$$

where $\varphi(t, \cdot, \cdot)$ is the flow of v starting at time t . More precisely, φ is the solution to

$$\begin{aligned} \partial_s \varphi(t, s, x) &= v(s, \varphi(t, s, x)), \\ \varphi(t, t, x) &= x, \end{aligned} \quad (4.4.5)$$

which is well defined due to (4.4.4). We observe that,

$$|\varphi(t, 0, x) - x| \leq \int_0^t |v(s, \varphi(t, s, x))| ds \leq CT \|\tau\|_{L^\infty(0, T; X_\beta)}.$$

Thus,

$$(1 + |x|^\beta) \leq C_1 \left(1 + T^\beta \|\tau\|_{L^\infty(0, T; X_\beta)}^\beta\right) (1 + |\varphi(t, 0, x)|^\beta), \quad (4.4.6)$$

where we denote the generic constant by C_1 for future reference. In particular,

$$\|\rho\|_{L^\infty(0, T; X_\beta)} \leq C_1 \left(1 + T^\beta \|\tau\|_{L^\infty(0, T; X_\beta)}^\beta\right) \|\rho_0\|_{X_\beta}. \quad (4.4.7)$$

Step 2. Contraction for small times. We want to prove that A is a contraction in

$$Y := \overline{B_{2C_1 \|\rho_0\|_{X_\beta}}(0)} \subset L^\infty(0, T; X_\beta) \quad (4.4.8)$$

for sufficiently small times T , where C_1 is the constant from (4.4.6). Choosing $T \leq (2C_1 \|\rho_0\|_{X_\beta})^{-1}$, we have seen in (4.4.7) that the solution operator A for problem (4.4.3) maps Y to itself.

Let $\tau_1, \tau_2 \in Y$, and for $i = 1, 2$, define $v_i = S(\tau_i g)$ the solutions to the Stokes equations, φ_i the corresponding flows as in (4.4.5), and $\rho_i = A\tau_i$ the solutions to the linear transport equation (4.4.3). Then, for $t \leq T \leq (2C_1\|\rho_0\|_{X_\beta})^{-1}$ we estimate, using (4.4.6) and writing $L := 2C_1\|\rho_0\|_{X_\beta}\|\nabla\rho_0\|_{X_\beta}$,

$$\begin{aligned} (1 + |x|^\beta)|\rho_1(t, x) - \rho_2(t, x)| &= (1 + |x|^\beta)|\rho_0(\varphi_1(t, 0, x)) - \rho_0(\varphi_2(t, 0, x))| \\ &\leq L|\varphi_1(t, 0, x) - \varphi_2(t, 0, x)| \\ &\leq L \int_0^t |v_1(s, \varphi_1(t, s, x)) - v_2(s, \varphi_2(t, s, x))| ds \\ &\leq L \int_0^t |v_1(s, \varphi_1(t, s, x)) - v_1(s, \varphi_2(t, s, x))| ds \\ &\quad + L \int_0^t |v_1(s, \varphi_2(t, s, x)) - v_2(s, \varphi_2(t, s, x))| ds \\ &\leq L\|\nabla v_1\|_{L^\infty((0,t)\times\mathbb{R}^3)} \int_0^t |\varphi_1(t, s, x) - \varphi_2(t, s, x)| ds \\ &\quad + L\|v_1 - v_2\|_{L^\infty((0,t)\times\mathbb{R}^3)} t. \end{aligned}$$

Using again Gronwall, we deduce

$$\|\rho_1 - \rho_2\|_{L^\infty(0,T;X_\beta)} \leq LT\|v_1 - v_2\|_{L^\infty((0,T)\times\mathbb{R}^3)} e^{L\|\nabla v_1\|_{L^\infty((0,T)\times\mathbb{R}^3)} T}.$$

Hence, using $\|\tau_1\|_{L^\infty(0,T;X_\beta)} \leq 2C_1\|\rho_0\|_{X_\beta}$ from (4.4.8) together with the estimates for the Stokes equation (4.4.4), we conclude for all $t \leq T \leq (2C_1\|\rho_0\|_{X_\beta})^{-1}$

$$\|\rho_1 - \rho_2\|_{L^\infty(0,T;X_\beta)} \leq CLT\|\tau_1 - \tau_2\|_{L^\infty(0,T;X_\beta)} e^{2CC_1L\|\rho_0\|_{X_\beta} T}.$$

This proves that A is indeed a contraction if we choose T sufficiently small. Therefore, the Banach fixed point theorem provides a unique solution ρ up to this time T .

Step 3. Global solution. In order to get a global solution in time, we need to show that $\rho(t, \cdot)$ and $\nabla\rho(t, \cdot)$ do not blow up in X_β in finite time, if ρ is the solution to (4.4.1). Define $v = S(\rho e)$ and φ the flow of v as before. We observe that

$$\|\rho(t, \cdot)\|_{L^1(\mathbb{R}^3)} \leq C\|\rho(t, \cdot)\|_{X_\beta},$$

and

$$\|\rho(t, \cdot)\|_{L^\infty(\mathbb{R}^3)} \leq \|\rho(t, \cdot)\|_{X_\beta}.$$

Clearly, the spatial L^∞ -norm of ρ is conserved over time. Since v is divergence free, also the spatial L^1 -norm is conserved. Using the explicit convolution formula for the solution operator S yields

$$\|v(t, \cdot)\|_{W^{1,\infty}(\mathbb{R}^3)} \leq C(\|\rho(t, \cdot)\|_{L^1(\mathbb{R}^3)} + \|\rho(t, \cdot)\|_{L^\infty(\mathbb{R}^3)}) \leq C\|\rho_0\|_{X_\beta}. \quad (4.4.9)$$

Therefore, we estimate analogously as we have obtained (4.4.6)

$$(1 + |x|^\beta)|\rho(t, x)| = (1 + |x|^\beta)|\rho_0(\varphi(t, 0, x))| \leq C \left(1 + t^\beta\|\rho_0\|_{X_\beta}^\beta\right) \|\rho_0\|_{X_\beta},$$

and we conclude

$$\|\rho\|_{L^\infty(0,T;X_\beta)} \leq C \left(1 + T^\beta\|\rho_0\|_{X_\beta}^\beta\right) \|\rho_0\|_{X_\beta}.$$

In order to get estimates for the gradient of ρ , we differentiate equation (4.4.2) and obtain

$$\partial_t \partial_{x_i} \rho = v \nabla \cdot \partial_{x_i} \rho + \partial_{x_i} v \cdot \rho.$$

Hence,

$$\partial_{x_i} \rho(t, x) = \partial_{x_i} \rho_0(\varphi(t, 0, x)) + \int_0^t \partial_{x_i} v(s, \varphi(t, s, x)) \cdot \rho(s, \varphi(t, s, x)) ds.$$

Using (4.4.9) leads to

$$\|\nabla \rho\|_{L^\infty(0, T; X_\beta)} \leq C \left(1 + T^\beta \|\rho_0\|_{X_\beta}^\beta\right) \left(\|\nabla \rho_0\|_{X_\beta} + T \|\rho_0\|_{X_\beta} \|\rho\|_{L^\infty(0, T; X_\beta)}\right).$$

Therefore, both ρ and $\nabla \rho$ do not blow up in finite time. Thus, by a standard contradiction argument using Step 2, solutions to (4.4.1) exist and are unique for arbitrary times T . \square

4.5 Estimates for the fluid velocity

4.5.1 Preliminary estimates on the particle configuration

As mentioned in Section 4.2.3, quantities like α_ε from equation (4.2.1) play an important role for the estimates on the fluid velocity. More precisely, for the estimates in the following subsections, it is important to control

$$\alpha_{\varepsilon, k}(t) := \sup_j \frac{1}{N_\varepsilon} \sum_{i \neq j} \frac{1}{|X_i(t) - X_j(t)|^k}, \quad (4.5.1)$$

for $k = 1, 2$. Moreover, the method of reflections used in Section 4.5.3 only works provided $\phi_\varepsilon \alpha_{\varepsilon, 3}(t)$ is sufficiently small. In the next lemma, we prove estimates for $\alpha_{\varepsilon, k}$ in terms of c_0 from assumption (A1) and Y from Definition 4.2.2 by approximating the sum in (4.5.1) by an integral.

Lemma 4.5.1. *There exists a constant C with the following property. Let $\varepsilon > 0$ and assume that the dynamics (4.1.1), (4.1.2), (4.1.3) have a solution up to time T . Then, for $t < T$ and $k = 1, 2$*

$$\alpha_{\varepsilon, k}(t) \leq C c_0^{-1} Y_\varepsilon(t)^3, \quad (4.5.2)$$

where c_0 is the constant from Assumption (A1) and Y_ε as in Definition 4.2.2. Moreover,

$$\alpha_{\varepsilon, 3} \leq C c_0^{-1} Y_\varepsilon^3(t) \log \left(\frac{N_\varepsilon Y_\varepsilon(t)}{c_0} \right). \quad (4.5.3)$$

Proof. Since ε and t is fixed, we omit them in the following. We define $\psi: \mathbb{R}^3 \rightarrow \mathbb{R}$ by

$$\psi = d_{\min}^{-3} \sum_i \chi_{B_{d_{\min}/2}(X_i)},$$

where d_{\min} denotes the minimal particle distance. By Definition 4.2.2 and assumption (A1), we have

$$\|\psi\|_{L^\infty(\mathbb{R}^3)} \leq d_{\min}^{-3} \leq c_0^{-1} N Y^3,$$

and

$$\|\psi\|_{L^1(\mathbb{R}^3)} = C N.$$

Thus, for all particles j ,

$$\begin{aligned} \frac{1}{N} \sum_{i \neq j} \frac{1}{|X_i - X_j|^k} &\leq \frac{C}{N} \int_{\mathbb{R}^3} \frac{\psi(y)}{|y - X_j|^k} dy \\ &\leq \frac{C}{N} \left(\int_{\mathbb{R}^3 \setminus B_1(X_j)} \frac{\psi(y)}{|y - X_j|^k} dy + \int_{B_1(X_j)} \frac{\psi(y)}{|y - X_j|^k} dy \right) \\ &\leq \frac{C}{N} (\|\psi\|_{L^1(\mathbb{R}^3)} + \|\psi\|_{L^\infty(\mathbb{R}^3)}) \\ &\leq C c_0^{-1} Y^3, \end{aligned} \quad (4.5.4)$$

where we used in the last step that $Y \geq 1$. This proves (4.5.2).

To show (4.5.3), we estimate for any j

$$\begin{aligned}
\frac{1}{N} \sum_{i \neq j} \frac{1}{|X_i - X_j|^3} &\leq \frac{C}{N} \int_{\mathbb{R}^3 \setminus B_{d_{\min}/2}(X_j)} \frac{\psi(y)}{|y - X_j|^3} dy \\
&\leq \frac{C}{N} \left(\int_{\mathbb{R}^3 \setminus B_1(X_j)} \frac{\psi(y)}{|y - X_j|^3} dy + \int_{B_1(X_j) \setminus B_{d_{\min}/2}(X_j)} \frac{\psi(y)}{|y - X_j|^3} dy \right) \\
&\leq \frac{C}{N} \left(\|\psi\|_{L^1(\mathbb{R}^3)} + \|\psi\|_{L^\infty(\mathbb{R}^3)} \log \left(\frac{1}{d_{\min}} \right) \right) \\
&\leq C c_0^{-1} Y^3 \log \left(\frac{Y}{d_{\min}(0)} \right) \\
&\leq C c_0^{-1} Y^3 \log \left(\frac{NY}{c_0} \right). \quad \square
\end{aligned}$$

Remark 4.5.2. By splitting the integral in (4.5.4) with $B_r(X_j)$ instead of $B_1(X_j)$, one can choose the optimal r to find

$$\sup_j \frac{1}{N} \sum_{i \neq j} \frac{1}{|X_i - X_j|^2} \leq C c_0^{-\frac{2}{3}} Y^2.$$

4.5.2 Estimates for $u_\varepsilon - v_\varepsilon$ in $\dot{H}^1(\mathbb{R}^3)$

For the remainder of this section, with the exception of Lemma 4.5.15, we consider an arbitrary given particle configuration without time evolution and derive estimates for the fluid velocity v that solves the stationary equations (4.1.2), (4.1.3). Therefore we omit the index ε . We will always assume a configuration of particles with $d_{\min} \geq 4R$. We will write C for any constant independent of the particle configuration, and C might change its value from line to line.

As explained in Section 4.2.3, we need L^∞ -estimates of $u - v$, where v is the solution to (4.1.2), and (4.1.3), and u is the solution to (4.1.6). These estimates will be shown using the method of reflections in Section 4.5.3. There, we will also rely on standard methods exploiting the structure of the linear PDEs that u and v solve. In this subsection 4.5.2, we will explain these methods in detail and prove an L^2 -estimate for $\nabla(u - v)$.

First, we notice that both u and v are solutions to variational problems. We define

$$E(w) := \frac{1}{2} \int_{\mathbb{R}^3} |\nabla w|^2 dx - \sum_i F \oint_{\partial B_i} w d\mathcal{H}^2,$$

with F as in (4.1.3). Then, u , which is defined as the solution to (4.1.6), is the minimizer of E in $\dot{H}_\sigma^1(\mathbb{R}^3)$. Moreover, v , which solves (4.1.2) and (4.1.3), is the minimizer of E in the subspace

$$W := \{w \in \dot{H}_\sigma^1(\mathbb{R}^3) : w = \text{const in } B_i, 1 \leq i \leq N\}.$$

In particular, v is the orthogonal projection of u from $\dot{H}_\sigma^1(\mathbb{R}^3)$ to W . Thus

$$\|u - v\|_{\dot{H}^1(\mathbb{R}^3)} \leq \|u - w\|_{\dot{H}^1(\mathbb{R}^3)} \quad \text{for all } w \in W. \quad (4.5.5)$$

We will exploit this by choosing w in a smart way in order to get an estimate for $u - v$. Following this approach, we need the following lemma which provides estimates on u based on its explicit form.

Lemma 4.5.3. *For all particle configuration with $d_{\min} \geq 4R$, the function u defined in (4.1.6) satisfies*

$$\|u\|_{L^\infty(\cup_i B_i)} \leq C\alpha_1, \quad (4.5.6)$$

and

$$\|\nabla u\|_{L^\infty(\cup_i B_i)} \leq C\alpha_2, \quad (4.5.7)$$

where α_k are defined in (4.5.1).

Proof. We have $u = \sum_i u_i$, where $u_i = Sf_i$ with f_i defined as in (4.1.7) and $S: \dot{H}^{-1}(\mathbb{R}^3) \rightarrow \dot{H}_\sigma^1(\mathbb{R}^3)$ the solution operator for the Stokes equations. Recall from (4.1.8) that $u_i = 2/9\gamma^2 g$ in B_i . In particular, $\nabla u_i = 0$ in B_i . Using the explicit formula for S as a convolution operator with kernel Φ as in (3.5.1), we observe for all particles $i \neq j$

$$\|\nabla u_j\|_{L^\infty(B_i)} \leq C|F| \frac{1}{(|X_i - X_j| - R)^2} \leq C|F| \frac{1}{|X_i - X_j|^2}.$$

Thus, recalling $|F| = \frac{C}{N}$ from (4.1.3), deduce

$$\|\nabla u\|_{L^\infty(B_i)} = \left\| \sum_{j \neq i} \nabla u_j \right\|_{L^\infty(B_i)} \leq \frac{C}{N} \sum_{j \neq i} \frac{1}{|X_i - X_j|^2}.$$

This shows (4.5.7). Estimate (4.5.6) is proved analogously. \square

Remark 4.5.4. *Estimate (4.5.6) actually also holds outside of the particles since the functions w_i are uniformly bounded.*

We now prove the main result of this subsection, which shows that the difference between u and v is of the order of the volume fraction of the particles ϕ .

Proposition 4.5.5. *For all particle configuration with $d_{\min} \geq 4R$,*

$$\|u - v\|_{\dot{H}^1(\mathbb{R}^3)}^2 \leq \sum_i C \|\nabla u\|_{L^2(B_i)}^2 \leq C\alpha_2\phi,$$

with α_2 as in (4.5.1).

Proof. By Lemma 3.5.14, for each $1 \leq i \leq N$, we can find $w_i \in H_{0,\sigma}^1(B_{2R}(X_i))$ with $\|\nabla w_i\|_{L^2(\mathbb{R}^3)} \leq C\|\nabla u\|_{L^2(B_i)}$ and $w_i = u - (u)_i$ in B_i , where $(u)_i := \int_{B_i} u \, dx$. Since the balls $B_{2R}(X_i)$ are disjoint, we obtain $w := u - \sum_i w_i \in W$. Hence, by (4.5.5),

$$\|u - v\|_{\dot{H}^1(\mathbb{R}^3)}^2 \leq \|u - w\|_{\dot{H}^1(\mathbb{R}^3)}^2 = \left\| \sum_i w_i \right\|_{\dot{H}^1(\mathbb{R}^3)}^2 \leq \sum_i C \|\nabla u\|_{L^2(B_i)}^2. \quad (4.5.8)$$

Thus, Lemma 4.5.3 yields

$$\|u - v\|_{\dot{H}^1(\mathbb{R}^3)}^2 \leq \sum_i C \|\nabla u\|_{L^2(B_i)}^2 \leq CNR^3\alpha_2^2 = C\alpha_2^2\phi. \quad \square$$

4.5.3 Estimates for $u_\varepsilon - v_\varepsilon$ in L^∞ by the method of reflections

In this subsection, we prove smallness of $v - u$ (again, we drop the index ε) in $L^\infty(\mathbb{R}^3)$ stated in Proposition 4.5.8. We use the method of reflections in the framework of orthogonal projections that has been investigated in Chapter 3. As we will see below, the method has better convergence properties for the problem (4.1.2), (4.1.3) that v solves than for the Stokes equations with Dirichlet boundary conditions, which we have studied in Chapter 3. Indeed, for the latter the method only converges provided that γ is sufficiently small. In Chapter 3, this problem has been overcome by a suitable resummation procedure.

In the case at hand, however, we will see below, that the higher order terms are associated to force densities that are “dipoles” in the particles (i.e. their integral vanishes). This makes the method convergent if $\phi\alpha_3$ is sufficiently small (see (4.5.1) for the definition of α_3). By Lemma 4.5.1 and the assumptions (A1) and (A2), this is the case for sufficiently small ε provided that Y_ε is uniformly bounded.

We will now introduce the necessary framework to apply the method of reflections in this setting of particle sedimentation using a notation analogous to the one in Chapter 3.

We define

$$W_i = \left\{ w \in \dot{H}_\sigma^1(\mathbb{R}^3) : w = \text{const in } B_i \right\}.$$

Let P_i be the orthogonal projection from $\dot{H}_\sigma^1(\mathbb{R}^3)$ to W_i and $Q_i = 1 - P_i$. We observe

$$W_i^\perp = \left\{ w \in \dot{H}_\sigma^1(\mathbb{R}^3) : \exists p \in L^2(\mathbb{R}^3) \quad -\Delta w + \nabla p = 0 \text{ in } \mathbb{R}^3 \setminus \overline{B_i}, \quad \int_{\mathbb{R}^3} -\Delta w + \nabla p = 0 \right\}. \quad (4.5.9)$$

Here, the first condition has to be interpreted in the weak sense. It is satisfied for every $w \in W_i^\perp$ since $\dot{H}_{\sigma,0}^1(\mathbb{R}^3 \setminus B_i) \subset W_i$. Using the first condition, the second condition simply means $\langle -\Delta w + \nabla p, \psi \rangle_{\dot{H}^{-1}, \dot{H}^1} = 0$ for all $\psi \in \dot{H}^1(\mathbb{R}^3)$ with $\psi = \text{const in } B_i$, and this follows directly from the definition of W_i .

The electrostatic analogy of the characterization of W_i^\perp by (4.5.9) is that the functions in W_i^\perp are dipole potentials.

The method of reflections can now be stated as follows. As a zero order approximation for v , one takes u . Recall from (4.1.2) and (4.1.3) that v is determined by being constant inside of the particles and satisfying the constraint of the total force acting on each particle being given by (4.1.3). Moreover, the function u defined in (4.1.6) satisfies this force constraint (4.1.3) but fails to be constant inside of the particles.

The idea is now to add functions w_i to u in such a way that $u + w_i$ is constant inside of the particle i , and still satisfies the Stokes equations outside of the particles and (4.1.3). Thus, $w_i = -Q_i u$. Indeed, by definition of the space W_i , $P_i u = (1 - Q_i)u$ is constant in B_i . Moreover, since $Q_i u$ is a “dipole potential”, $(1 - Q_i)u$ still satisfies (4.1.3). As a first order approximation for v , we define

$$v_1 = \left(1 - \sum_i Q_i \right) u.$$

Clearly, since w_i is not constant inside the particles $j \neq i$, the function v_1 is still not constant inside the particles. Therefore, higher order approximations for v are obtained by repeating this process.

$$v_k = \left(1 - \sum_i Q_i \right)^k u. \quad (4.5.10)$$

Then, we have to show the convergence $v_k \rightarrow v$.

This approach seems a bit awkward at first glance. Indeed, we already know that $v = Pu$, where P is the orthogonal projection to $W = \cap_i W_i$. Therefore, the method of reflections consists in writing this projection as $P = \lim_{k \rightarrow \infty} (1 - \sum_i Q_i)^k$. However, the advantage of this method is that it is much easier to study Q_i than P .

For the proof that v_k converges to v , we need the following lemmas. Lemma 4.5.6 ensures that $(1 - Q_i)v_k = P_i v_k$ does not differ too much from u_k inside particle i . Lemma 4.5.7 is used to exploit that $Q_i v_k$ is a “dipole potential”, and therefore decays quickly.

Lemma 4.5.6. *Let $w \in \dot{H}_\sigma^1(\mathbb{R}^3)$. Then, $P_i w = (w)_i$ in B_i , where $(w)_i = f_{\partial B_i} w$.*

Proof. Let $\psi_0 \in \mathbb{R}^3$ and define $\psi \in \dot{H}_\sigma^1(\mathbb{R}^3)$ to be the solution to

$$\begin{aligned} -\Delta \psi + \nabla p &= 0 \quad \text{in } \mathbb{R}^3 \setminus \overline{B_i}, \\ \psi &= \psi_0 \quad \text{in } B_i. \end{aligned}$$

In other words, ψ is the velocity field corresponding to a moving single sphere without external forces. Hence, by (2.2.2),

$$-\Delta \psi + \nabla p = \frac{3}{2R} \psi_0 \delta_{\partial B_i}.$$

Furthermore, $\psi \in W_i$, and hence,

$$0 = (w - P_i w, \psi)_{\dot{H}^1(\mathbb{R}^3)} = \langle w - P_i w, -\Delta \psi \rangle = \frac{3}{2R} \psi_0 \cdot \int_{\partial B_i} w - P_i w \, d\mathcal{H}^2.$$

Since ψ_0 was arbitrary, we deduce

$$\int_{\partial B_i} w - P_i w \, d\mathcal{H}^2 = 0,$$

and the assertion follows. \square

Lemma 4.5.7. *Assume $f \in \dot{H}_\sigma^{-1}(\mathbb{R}^3)$ is supported in $\overline{B_i}$ and $\int_{\mathbb{R}^3} f = 0$, i.e., $Sf \in W_i^\perp$, where S is the solution operator for the Stokes equations. Then, for all $x \in \mathbb{R}^3 \setminus B_{2R}(X_i)$,*

$$|(Sf)(x)| \leq C \frac{R^{\frac{3}{2}}}{|x - X_i|^2} \|f\|_{\dot{H}_\sigma^{-1}(\mathbb{R}^3)}, \quad (4.5.11)$$

and

$$|\nabla(Sf)(x)| \leq C \frac{R^{\frac{3}{2}}}{|x - X_i|^3} \|f\|_{\dot{H}_\sigma^{-1}(\mathbb{R}^3)}. \quad (4.5.12)$$

Proof. We denote again by Φ the Oseen tensor (3.5.1). Then,

$$\begin{aligned} |(Sf)(x)| &= |(\Phi * f)(x)| = |((\Phi - (\Phi)_{x-X_i, 2R}) * f)(z)| \\ &= |(E(\Phi - (\Phi)_{x-X_i, 2R}) * f)(z)| \\ &\leq \|f\|_{\dot{H}_\sigma^{-1}(\mathbb{R}^3)} \|E(\Phi - (\Phi)_{x-X_i, 2R})\|_{\dot{H}^1(\mathbb{R}^3)}, \end{aligned}$$

where

$$(\Phi)_{x-X_i, R} = \int_{B_R(x-X_i)} \Phi(y) \, dy,$$

and $E(\Phi - (\Phi)_{x-X_i, R})$ is a divergence free extension of the restriction of $\Phi - (\Phi)_{x-X_i, R}$ to $B_R(x-X_i)$. By Lemma 3.5.14, we can choose this extension in such a way that

$$\|E(\Phi - (\Phi)_{x-X_i, R})\|_{\dot{H}^1(\mathbb{R}^3)} \leq C \|\nabla \Phi\|_{L^2 B_R(x-X_i)} = C \frac{R^{\frac{3}{2}}}{|x - X_i|^2}.$$

This establishes estimate (4.5.11). Estimate (4.5.12) is proven analogously. \square

Proposition 4.5.8. *Assume a particle configuration is given with $d_{\min} \geq 4R$ and let α_k be as in (4.5.1). Let v_k be defined as in (4.5.10). Then, for all particles j and all $y \notin B_{2R}(X_j)$,*

$$|Q_j v_k(y)| \leq C \frac{R^3}{|X_j - y|^2} \|\nabla v_k\|_{L^\infty(B_j)}, \quad (4.5.13)$$

and

$$|\nabla Q_j v_k(y)| \leq C \frac{R^3}{|X_j - y|^3} \|\nabla v_k\|_{L^\infty(B_j)}. \quad (4.5.14)$$

Furthermore, there exists a constant $\delta > 0$ with the following property. Assume that $\phi\alpha_3 < \delta$. Then,

$$\|\nabla v_k\|_{L^\infty(B_i)} \leq \alpha_2(C\delta)^k, \quad (4.5.15)$$

and

$$v_k \rightarrow v \quad \text{in } \dot{H}^1(\mathbb{R}^3).$$

Moreover, the convergence also holds in $L^\infty(\mathbb{R}^3)$, and we have

$$\|u - v\|_{L^\infty(\mathbb{R}^3)} \leq C\alpha_2(\alpha_2\phi + R). \quad (4.5.16)$$

Proof. Recall that the solution operator S for the Stokes equations is an isometry from $\dot{H}_\sigma^{-1}(\mathbb{R}^3)$ to $\dot{H}_\sigma^1(\mathbb{R}^3)$. Thus, by (4.5.9) and Lemma 4.5.7, we have for all particles j and all $y \notin B_{2R}(X_j)$

$$|Q_j v_k(y)| \leq C \frac{R^{\frac{3}{2}}}{|X_j - y|^2} \|Q_j v_k\|_{\dot{H}^1(\mathbb{R}^3)}. \quad (4.5.17)$$

By (4.5.9), $\text{supp } \Delta Q_j v_k \subset \overline{B_i}$ as a function in $\dot{H}_\sigma^{-1}(\mathbb{R}^3)$. Therefore, $Q_j v_k$ is the function of minimal norm in $\dot{H}_\sigma^1(\mathbb{R}^3)$ that coincides with $Q_j v_k$ in $\overline{B_i}$. By Lemma 4.5.6, we have

$$Q_j v_k = v_k - \oint_{\partial B_j} v_k d\mathcal{H}^2 \quad \text{in } B_j.$$

Hence, Lemma 3.5.14 yields

$$\|Q_j v_k\|_{\dot{H}^1(\mathbb{R}^3)} \leq C \|\nabla v_k\|_{L^2(B_j)} \leq CR^{\frac{3}{2}} \|\nabla v_k\|_{L^\infty(B_j)}. \quad (4.5.18)$$

Combining (4.5.17) and (4.5.18) yields (4.5.13). Estimate (4.5.14) is proven analogously.

We claim that v is the orthogonal projection of v_k to W for all $k \in \mathbb{N}$. In Section 4.5.2, we have seen that v is the orthogonal projection of u to W . Therefore, it suffices to observe that $Q_i w$ lies in the orthogonal complement of W for any $w \in \dot{H}_\sigma^1(\mathbb{R}^3)$. By definition, $Q_i w$ lies in the orthogonal complement of W_i . Since $W \subset W_i$ this implies $Q_i w \in W^\perp$.

Now it follows analogously as we have obtained (4.5.8)

$$\|v - v_k\|_{\dot{H}^1(\mathbb{R}^3)}^2 \leq \sum_i C \|\nabla v_k\|_{L^2(B_i)}^2 \leq \sum_i CR^3 \|\nabla v_k\|_{L^\infty(B_i)}^2. \quad (4.5.19)$$

In B_i , we have for $k \geq 1$

$$\nabla v_k = \nabla(v_{k-1} - \sum_j Q_j v_{k-1}) = \sum_{j \neq i} \nabla Q_j v_{k-1}$$

since $v_{k-1} - Q_i v_{k-1} = P_i v_{k-1} \in W_i$ is constant in B_i .

Thus,

$$\|\nabla v_k\|_{L^\infty(B_i)} \leq C \sum_{j \neq i} \frac{R^3}{|X_i - X_j|^3} \|\nabla v_{k-1}\|_{L^\infty(B_j)} \leq C\delta \|\nabla v_{k-1}\|_{L^\infty(\cup B_j)}. \quad (4.5.20)$$

By Lemma 4.5.3,

$$\|\nabla u\|_{L^\infty(\cup B_j)} \leq C\alpha_2.$$

Combining this with (4.5.20) yields (4.5.15). Hence, by the estimate (4.5.19), the series v_k converges to v in $\dot{H}^1(\mathbb{R}^3)$, provided $\delta < 1/C$.

To prove convergence in $L^\infty(\mathbb{R}^3)$ we choose for any fixed $x \in \mathbb{R}^3$ a particle X_i which has minimal distance to x . We note that Lemma 4.5.11 implies

$$\frac{1}{N} \sum_{j \neq i} \frac{1}{|x - X_j|^2} \leq C\alpha_2.$$

Application of Lemma 4.5.6 and Lemma 4.5.9 below for particle i , and Lemma 4.5.7 for the others, using also (4.5.18), yields

$$\begin{aligned} |v_{k+1}(x) - v_k(x)| &= \left| \sum_j Q_j v_k(x) \right| \leq \sum_{j \neq i} |Q_j v_k(x)| + |Q_i v_k(x)| \\ &\leq \sum_{j \neq i} C \frac{R^{\frac{3}{2}}}{(x - X_i)^2} \|Q_j v_k\|_{\dot{H}^1(\mathbb{R}^3)} + C \|v_k - (v_k)_i\|_{L^\infty(B_i)} \\ &\leq C\alpha_2 R^3 N \|\nabla v_k\|_{L^\infty(\cup B_j)} + CR \|\nabla v_k\|_{L^\infty(B_i)} \\ &= C\alpha_2 (\alpha\phi + R)(C\delta)^k. \end{aligned}$$

Therefore, $v_k - v$ converges to zero in $L^\infty(\mathbb{R}^3)$, and (4.5.16) holds. □

We need the following maximum modulus estimate for solutions to Dirichlet problems of the Stokes equations. A proof of this result can be found in [MRS99].

Lemma 4.5.9. *Let $\Omega \subset \mathbb{R}^3$ be an exterior domain and assume that $g \in C(\partial\Omega)$ satisfies*

$$\int_{\partial\Omega} g \nu \, d\mathcal{H}^2 = 0.$$

Then, the unique solution $u \in \dot{H}^1(\mathbb{R}^3)$ of the Dirichlet problem

$$\begin{aligned} -\Delta u + \nabla p &= 0 & \text{in } \Omega, \\ u &= g & \text{on } \partial\Omega \end{aligned}$$

satisfies

$$\|u\|_{L^\infty(\Omega)} \leq C \|g\|_{L^\infty(\partial\Omega)},$$

where the constant C depends only on Ω .

Remark 4.5.10. *Clearly, the constant C in the above statement is invariant under scaling of the domain. In fact, we will only apply the above lemma for Ω being the exterior of a ball.*

Lemma 4.5.11. *Let $y \in \mathbb{R}^3$ and let X_i be a particle that has minimal distance to y , i.e.,*

$$|y - X_i| \leq |y - X_j| \quad \text{for all } 1 \leq j \leq N.$$

Then,

$$|y - X_j| \geq \frac{1}{2}|X_i - X_j| \quad \text{for all } 1 \leq j \leq N.$$

In particular, for $k = 1, 2$,

$$\sum_{j \neq i} \frac{1}{|y - X_j|^k} \leq C \sum_{j \neq i} \frac{1}{|X_i - X_j|^k}.$$

Proof. We consider two cases.

Case 1.

$$|X_j - X_i| \leq 2|y - X_i|.$$

Then,

$$|y - X_j| \geq |y - X_i| \geq \frac{1}{2}|X_i - X_j|.$$

Case 2.

$$|X_j - X_i| \geq 2|y - X_i|.$$

Then,

$$|X_j - X_i| \leq |y - X_i| + |y - X_j| \leq \frac{1}{2}|X_j - X_i| + |y - X_j|,$$

and the assertion follows. \square

4.5.4 Lipschitz type estimates for v_ε and short time estimates on the particle distances

In order to apply the method of reflections, we have seen in Proposition 4.5.8 that we need to control the quantities α_k (see (4.5.1)) for $k = 2, 3$. By Lemma 4.5.1, we know how to control α_k in terms of Y_ε from Definition 4.2.2. The purpose of this subsection is to prove short time estimates for Y_ε . More precisely, we show in Proposition 4.5.15 that, starting at time T_0 , the time θ needed for two particles to halve their distance is bounded from below. For sufficiently small ε , this bound on θ depends only on $Y_\varepsilon(T_0)$. Thus, if we have estimates for $Y_\varepsilon(T_0)$ uniformly in ε , we can also bound $Y_\varepsilon(T_0 + \theta)$ uniformly in ε .

This a priori estimate enables us to prove the main theorem for small times (cf. Theorem 4.6.1). However, it does not rule out that Y_ε blows up in finite time. Therefore, we prove an a posteriori estimate on Y_ε in Section 4.6.2.

In order to control the particle distances, we need to estimate their relative velocities, which is provided by Lipschitz type estimates for the fluid velocity v_ε . Since v_ε solves the stationary equations (4.1.2), (4.1.3), we will again simplify the notation in the following by considering a fixed particle configuration neither depending on ε nor on time. First, we observe that we cannot expect a uniform estimate on the L^∞ -norm of the gradient of v . Indeed, as discussed in the formal derivation of the limit equation (4.1.9), we expect the value of v at a particle and the value of v at a distance of order d_{\min} from that particle to differ by $2/9\gamma^{-1}g$. Hence, ∇v is large near the particles, in particular the L^∞ -norm of ∇v tends to infinity as $R \rightarrow 0$.

Therefore, we have to directly prove estimates for $v(X_i) - v(X_j)$. First, in Lemma 4.5.12, we prove such estimates for the approximated fluid velocity u . Then, in Lemma 4.5.14, we again use the method of reflections to get the estimates for v as well.

Lemma 4.5.12. *Assume a particle configuration is given with $d_{\min} \geq 4R$ and let α_k be as in (4.5.1). Then, for all particles i, j and all $h \in B_R(0) \subset \mathbb{R}^3$,*

$$|u(X_i + h) - u(X_j + h)| \leq C\alpha_2|X_i - X_j|.$$

Proof. We recall from (4.1.6) that $u = \sum_i u_i$, where $u_i = Sf_i$ with f_i defined as in (4.1.7) and $S: \dot{H}^{-1}(\mathbb{R}^3) \rightarrow \dot{H}_o^1(\mathbb{R}^3)$ the solution operator for the Stokes equations. In particular, for $x \notin B_R(X_i)$,

$$|\nabla u_i(x)| \leq C \frac{1}{N|x - X_i|^2}.$$

By definition of u_i , we have $u_i(x) = w(x - X_i)$ for some $w \in \dot{H}_o^1(\mathbb{R}^3)$ with $w(-x) = w(x)$. In particular, for particles i and j

$$u_i(X_i + h) = w(h) = u_j(X_j + h). \quad (4.5.21)$$

Moreover, using symmetry of w ,

$$\begin{aligned} |u_j(X_i + h) - u_i(X_j + h)| &= |w(X_i - X_j + h) - w(X_j - X_i + h)| \\ &= |w(X_i - X_j + h) - w(X_i - X_j - h)| \\ &\leq C|\nabla w(X_i - X_j)||h| \\ &\leq C \frac{R}{N|X_i - X_j|^2} \\ &\leq C \frac{|X_i - X_j|}{N|X_i - X_j|^2}. \end{aligned} \quad (4.5.22)$$

Let us denote $x_i = X_i + h$ and $x_j = X_j + h$. Then, (4.5.21) and (4.5.22) imply

$$|u(x_i) - u(x_j)| \leq \sum_{k \neq i, j} |u_k(x_i) - u_k(x_j)| + C \frac{|X_i - X_j|}{N|X_i - X_j|^2}.$$

For all $k \neq i, j$ we use Lemma 4.5.13 below, which provides curves $s_k \in C^1([0, 1]; \mathbb{R}^3)$ from x_i to x_j such that

$$|s_k(t) - X_k| \geq \min\{|x_i - X_k|, |x_j - X_k|\},$$

and

$$|\dot{s}_k| \leq C|x_i - x_j|.$$

We deduce

$$\begin{aligned} |u_k(x_i) - u_k(x_j)| &\leq C \int_0^1 |\nabla u_k(s_k(t))| |x_i - x_j| dt \\ &\leq C \frac{|X_i - X_j|}{N} \left(\frac{1}{|x_i - X_k|^2} + \frac{1}{|x_j - X_k|^2} \right). \end{aligned}$$

Thus, using Lemma 4.5.11, we conclude

$$\begin{aligned} |u(x) - u(y)| &\leq \sum_{k \neq i, j} |u_k(x_i) - u_k(x_j)| + C \frac{|X_i - X_j|}{N|X_i - X_j|^2} \\ &\leq C \sum_{i \neq j, k} \frac{|X_i - X_j|}{N} \left(\frac{1}{|x_i - X_k|^2} + \frac{1}{|x_j - X_k|^2} \right) + C \frac{|X_i - X_j|}{N|X_i - X_j|^2} \\ &\leq C\alpha_2|X_i - X_j|. \end{aligned} \quad \square$$

Lemma 4.5.13. *Let $x, y, z \in \mathbb{R}^3$ be distinct. Then, there exists a curve $s \in C^1([0, 1]; \mathbb{R}^3)$ with $s(0) = x$, $s(1) = y$,*

$$|s(t) - z| \geq \min\{|x - z|, |y - z|\},$$

and

$$|\dot{s}| \leq C|x - y|.$$

Proof. It suffices to construct a semicircle γ with endpoints x and y such that

$$\text{dist}\{\gamma, z\} \geq \min\{|x - z|, |y - z|\}. \quad (4.5.23)$$

If x, y, z lie on a line, then we can choose any semicircle with endpoints x and y .

Otherwise, let E be the plane that x, y, z lie in. Then, there are exactly two semicircles in E with endpoints x and y , and it is easy to check, that (at least) one of them satisfies (4.5.23). \square

Using Proposition 4.5.8, we can deduce from Lemma 4.5.12

$$|v_\varepsilon(X_i) - v_\varepsilon(X_j)| \leq C\alpha_2|X_i - X_j| + C\alpha_2(\alpha_2\phi_\varepsilon + R_\varepsilon), \quad (4.5.24)$$

provided that the assumptions of both Proposition 4.5.8 and Lemma 4.5.12 are satisfied.

The particle volume ϕ_ε on the right hand side of (4.5.24) poses a problem, since it could be much larger than the minimal particle distance $d_{\varepsilon, \min}$. Thus, not even for small times t , does estimate (4.5.24) imply any lower estimate for $d_{\varepsilon, \min}(t)$ which is uniform in ε .

In order to get rid of ϕ_ε in (4.5.24), we will prove that the functions v_k from Proposition 4.5.8 all satisfy

$$\sup_k |v_k(X_i) - v_k(X_j)| \leq C\alpha_2|X_i - X_j|.$$

Lemma 4.5.14. *Assume a particle configuration is given with $d_{\min} \geq 4R$ and let α_k be as in (4.5.1). Then, there exists a constant $\delta > 0$ with the following property. Assume that $\phi\alpha_3 < \delta$. Then, the functions v_k defined in Proposition 4.5.8 satisfy for all particles i and j*

$$|v_k(X_i) - v_k(X_j)| \leq C\alpha_2|X_i - X_j|. \quad (4.5.25)$$

In particular,

$$|v(X_i) - v(X_j)| \leq C\alpha_2|X_i - X_j|. \quad (4.5.26)$$

Proof. The assertion follows from the following estimate which we will prove by induction in k .

$$|v_k(X_i + h) - v_k(X_j + h)| \leq C\delta^{-1}\alpha_2 \sum_{n=0}^k (C\delta)^n |X_i - X_j|, \quad (4.5.27)$$

for all particles i, j and all $h \in \overline{B_R(0)}$. For $k = 0$, this is the second part of Lemma 4.5.12.

Let us denote $x_i = X_i + h$ and $x_j = X_j + h$. Using the definition of v_k from Proposition 4.5.8, we observe

$$v_{k+1}(x_i) = v_k(x_i) - \sum_l Q_l v_k(x_i) = (v_k)_i - \sum_{l \neq i} Q_l v_k(x_i).$$

Here we used that by Lemma 4.5.6, $Q_i v_k = v_k - (v_k)_i$ in $\overline{B_i}$, where $(v_k)_i = \int_{\partial B_i} v_k$. Therefore,

$$\begin{aligned} & |v_{k+1}(x_i) - v_{k+1}(x_j)| \\ & \leq |(v_k)_i - (v_k)_j| + \sum_{l \neq i, j} |Q_l v_k(x_i) - Q_l v_k(x_j)| + |Q_j v_k(x_i)| + |Q_i v_k(x_j)|. \end{aligned} \quad (4.5.28)$$

For the first term on the right hand side, we use the induction hypothesis. Regarding the second term, for all $l \neq i, j$, we use Lemma 4.5.13, which provides curves $s_l \in C^1([0, 1]; \mathbb{R}^3)$ from X_i to X_j such that

$$|s(t) - X_l| \geq \min\{|X_l - x_i|, |X_l - x_j|\},$$

and

$$|\dot{s}| \leq C|x_i - x_j|.$$

Using in addition estimates (4.5.13) and (4.5.15) from Proposition 4.5.8, we deduce

$$\begin{aligned} |Q_l v_k(x_i) - Q_l v_k(x_j)| &\leq C|x_i - x_j| \int_0^1 |\nabla Q_l v_k(s_l(t))| dt \\ &\leq C|X_i - X_j| R^3 \left(\frac{1}{|x_i - X_l|^3} + \frac{1}{|x_j - X_l|^3} \right) \|\nabla v_k\|_{L^\infty(B_l)} \\ &\leq C\alpha_2 |X_i - X_j| R^3 \left(\frac{1}{|X_i - X_l|^3} + \frac{1}{|X_j - X_l|^3} \right) (C\delta)^k. \end{aligned}$$

Thus, for the second term on the right hand side of (4.5.28), we deduce

$$\sum_{l \neq i, j} |Q_l v_k(x_i) - Q_l v_k(x_j)| \leq C\alpha_2 |X_i - X_j| (C\delta)^{k+1}.$$

For the third term on the right hand side of (4.5.28) we observe that estimates (4.5.14) and (4.5.15) from Proposition 4.5.8 yield

$$|Q_j v_k(x_i)| \leq C \frac{R^3}{|X_i - X_j|^2} \|\nabla v_k\|_{L^\infty(B_l)} \leq C\alpha_2 \frac{R^3}{|x_i - X_j|^2} (C\delta)^k \leq C\delta^{-1} \alpha_2 (C\delta)^{k+1} |X_i - X_j|,$$

where we used $R < |X_i - X_j|$.

Since we get the same estimate for the fourth term, this finishes the proof of the induction step. Thus, estimate (4.5.27) holds true for all $k \in \mathbb{N}$ which implies (4.5.25). Since v_k converges to v in $L^\infty(\mathbb{R}^3)$ by Proposition 4.5.8, this also proves (4.5.26). \square

We now combine the previous lemma and Lemma 4.5.1 which provides bounds for $\alpha_{\varepsilon, k}(t)$ in terms of $Y_\varepsilon(t)$. As a result, we are able to prove a lower bound on the time θ that it takes for any two particles to halve their distance to each other. This lower bound is uniform in ε for sufficiently small values of ε .

Proposition 4.5.15. *Let $T_0 \geq 0$, and assume there exist $\varepsilon_0 > 0$ and $Y_0 < \infty$ such that $Y_\varepsilon(T_0) \leq Y_0$ for all $\varepsilon < \varepsilon_0$. Then, there exists $\varepsilon_1 > 0$ and $\theta > 0$ which depends only on Y_0 and c_0 from Assumption (A1) such that*

$$Y_\varepsilon(T_0 + \theta) \leq 2Y_\varepsilon(0) \quad \text{for all } \varepsilon < \varepsilon_1.$$

Proof. By Lemma 4.5.1,

$$\phi_\varepsilon \alpha_{\varepsilon, 3} \leq C_1 \phi_\varepsilon Y_\varepsilon^3 \log \left(\frac{N_\varepsilon Y_\varepsilon}{c_0} \right),$$

where we denote the constant by C_1 for definiteness. We choose $\varepsilon_1 < \varepsilon_0$ such that

$$C_1 \phi_\varepsilon (2Y_0^3) \log \left(\frac{2N_\varepsilon Y_\varepsilon}{c_0} \right) < \delta \quad \text{for all } \varepsilon < \varepsilon_1.$$

where δ is the constant from Lemma 4.5.14. This is possible, since $\phi_\varepsilon \log(N_\varepsilon) \rightarrow 0$ as $\varepsilon \rightarrow 0$ by Assumption (A2). Let

$$\theta_\varepsilon := \sup\{t \geq 0: Y_\varepsilon(T_0 + t) \leq 2Y_0\}.$$

Since Y_ε is continuous in time, $\theta_\varepsilon > 0$. Then, Lemma 4.5.14 and Lemma 4.5.1 yield

$$|v(X_i) - v(X_j)| \leq C_2 c_0^{-1} Y_0^3 |X_i - X_j| \quad \text{for all } \varepsilon < \varepsilon_1 \quad \text{and all } t \leq T_0 + \theta_\varepsilon.$$

Since the particles are transported by v , this implies

$$\frac{d}{dt} d_{\varepsilon, \min}(t) \geq C_2 c_0^{-1} Y_0^3 d_{\varepsilon, \min}(t) \quad \text{for all } \varepsilon < \varepsilon_1 \quad \text{and all } t \leq T_0 + \theta_\varepsilon.$$

Hence,

$$Y_\varepsilon(T_0 + t) \leq Y_0 e^{C_2 c_0^{-1} Y_0^3 t} \quad \text{for all } \varepsilon < \varepsilon_1 \quad \text{and all } t \leq \theta_\varepsilon.$$

By definition of θ_ε , this implies

$$\theta_\varepsilon \geq \frac{\log 2}{C_2 c_0^{-1} Y_0^3} =: \theta,$$

which finishes the proof. \square

4.5.5 Approximations for the macroscopic fluid velocity

In order to prove the convergence result Theorem 4.2.9, we need to relate the microscopic fluid velocity v_ε to the macroscopic fluid velocity v_* of the limit equation (4.1.9). More precisely, we have to prove that inside the particles, v_ε is close to $v_* + \frac{2}{9}\gamma_\varepsilon^2 g$. Recall that $v_* = S(\rho g)$, where S is the solution operator to Stokes equations, and we want to prove $\rho_\varepsilon^{\delta_\varepsilon} \rightarrow \rho$ in the space X_β from Definition 4.2.5, where $\rho_\varepsilon^{\delta_\varepsilon}$ is the averaged (or macroscopic) particle density introduced in Definition 4.2.4. Therefore, it is convenient to consider

$$w_{\delta, \varepsilon} := S(\rho_\varepsilon^\delta g) + \frac{2}{9}\gamma_\varepsilon^{-1} g. \quad (4.5.29)$$

We have already proved boundedness of the operator S in the space X_β (see Lemma 4.4.3) which implies estimates of $w_{\delta, \varepsilon} - v_* - \frac{2}{9}\gamma_\varepsilon^{-1} g$ in terms of $\rho_\varepsilon^\delta - \rho$. In addition, we need to estimate $w_{\delta, \varepsilon} - v_\varepsilon$. By Proposition 4.5.8, we already know that we can replace v_ε by u_ε . Therefore, we prove smallness of $w_{\delta, \varepsilon} - u_\varepsilon$ in Lemma 4.5.16.

Again, the time evolution and the dependency of the particle configuration on ε is not relevant for the estimates in this subsection, since we only consider the fluid velocity at a fixed time, which solves the stationary equations (4.1.2), (4.1.3). Therefore, to simplify the notation, we will formulate the estimates for a fixed particle configuration neither depending on time nor on ε .

Lemma 4.5.16. *Assume a particle configuration is given with $d_{\min} \geq 4R$, and let w_δ be defined as in (4.5.29). Then, we have for all $\delta \geq d_{\min}$*

$$\|u - w_\delta\|_{L^\infty(\cup_i B_i)} \leq C\delta \left(1 + \frac{1}{Nd_{\min}^3}\right). \quad (4.5.30)$$

Moreover,

$$\|w_\delta\|_{W^{1, \infty}(\mathbb{R}^3)} \leq C \left(1 + \gamma^{-1} + \frac{1}{Nd_{\min}^3}\right). \quad (4.5.31)$$

Furthermore, for all $n \in \mathbb{N}$ and $\tilde{\delta} = n\delta$,

$$\|w_{\tilde{\delta}} - w_\delta\|_{L^\infty(\mathbb{R}^3)} \leq C\tilde{\delta} \left(1 + \frac{1}{Nd_{\min}^3}\right). \quad (4.5.32)$$

Proof. Recall from (4.1.7) that $u = \sum_i u_i$, where $u_i = S f_i$. Since by (4.1.8) $u_i(x) = \frac{2}{9}\gamma^2 g$ in B_i , we have for all $x \in B_i$

$$u(x) - \left(S(\rho^\delta g)(x) + \frac{2}{9}\gamma^2 g \right) = u(x) - u_i(x) - (S(\rho^\delta g)(x)).$$

We denote by I_δ the set of centers of the cubes from Definition 4.2.4. We define $I_1 \subset I_\delta$ to contain the center of the cube Q_δ^x as well as the centers of all cubes adjacent to Q_δ^x . Then $|I_1| = 27$. Let $I_2 = I_\delta \setminus I_1$. We observe that for all $z \in \mathbb{R}^3$

$$\int_{Q_\delta^z} \Phi(x-y)g \left(\rho(y) - \oint_{Q_\delta^z} \rho(z') dz' \right) dy = \int_{Q_\delta^z} \left(\Phi(x-y) - \oint_{Q_\delta^z} \Phi(x-z') dz' \right) \rho(y)g dy. \quad (4.5.33)$$

Thus,

$$\begin{aligned} |S(\rho g - f_i)(x) - S(\rho^\delta g)(x)| &= \left| \int_{\mathbb{R}^3} \Phi(x-y)g \left(\rho(y) - f_i(y) - \oint_{Q_\delta^y} \rho(z) dz \right) dy \right| \\ &\leq C \sum_{z \in I_1} \int_{Q_\delta^z} |\Phi(x-y)| \left(|\rho(y)g - f_i(y)| + \rho^\delta(y) \right) dy \\ &\quad + C \sum_{z \in I_2} \int_{Q_\delta^z} \left| \Phi(x-y) - \oint_{Q_\delta^z} \Phi(x-z') dz' \right| \rho(y) dy \\ &=: A + B. \end{aligned}$$

Recalling the definition of ρ from (4.1.5), we have

$$\|\rho^\delta\|_{L^1(\mathbb{R}^3)} = C \quad (4.5.34)$$

$$\|\rho^\delta\|_{L^\infty(\mathbb{R}^3)} \leq \frac{C}{Nd_{\min}^3}, \quad (4.5.35)$$

where we used $\delta \geq d_{\min}$. Using (4.5.35) as well as $|\Phi(x)| \leq C/|x|$, we deduce

$$\begin{aligned} A &\leq C \sum_{\substack{X_j \in B_{C\delta}(x) \\ j \neq i}} \frac{1}{N|x - X_j|} + \frac{C}{Nd_{\min}^3} \int_{B_{C\delta}(x)} \frac{1}{|x-y|} dy \\ &\leq C \frac{1}{Nd_{\min}^3} \int_{B_{C\delta}(x)} \frac{1}{|y-x|} dy \leq C \frac{1}{Nd_{\min}^3} \delta^2. \end{aligned}$$

From the explicit expression of Φ in (3.5.1), it follows for all $z \in I_2$ and all $y \in Q_\delta^z$

$$\left| \Phi(x-y) - \oint_{Q_\delta^z} \Phi(x-z') dz' \right| \leq C \frac{\delta}{|x-z|^2}. \quad (4.5.36)$$

Hence, by (4.5.34), (4.5.35), and (4.5.36),

$$\begin{aligned} B &\leq C\delta \sum_{z \in I_2} \int_{Q_\delta^z} \frac{\rho(y)}{|x-z|} dy \leq C\delta \int_{\mathbb{R}^3} \frac{\rho^\delta(y)}{|x-y|} dy \\ &\leq C\delta \left(\|\rho^\delta(y)\|_{L^1(\mathbb{R}^3)} + \|\rho^\delta(y)\|_{L^\infty(\mathbb{R}^3)} \right) \leq C\delta \left(1 + \frac{1}{Nd_{\min}^3} \right). \end{aligned}$$

Combining the error estimates for A and B proves (4.5.30).

The proof of (4.5.32) is almost completely analogous. The only difference is that, due to the averaging, there is no problem with a particle that is close to the point where we estimate. Therefore, we only have to deal with an error term analogous to B , and the estimate holds true in the whole of \mathbb{R}^3 . Indeed, we have for all $x \in Q_{\delta}^z$

$$\rho^{\delta}(y) = \oint_{Q_{\delta}^y} \rho^{\delta}(z) dz.$$

Hence, using again (4.5.33), we find

$$\begin{aligned} |w_{\delta}(x) - w_{\tilde{\delta}}(x)| &= \left| \int_{\mathbb{R}^3} \Phi(x-y) g \left(\rho^{\delta}(y) - \oint_{Q_{\delta}^y} \rho^{\delta}(z) dz \right) dy \right| \\ &\leq C \sum_{z \in I_{\tilde{\delta}}} \int_{Q_{\delta}^z} \left| \Phi(x-y) - \oint_{Q_{\delta}^z} \Phi(x-z') dz' \right| \rho^{\delta}(y) dy. \end{aligned}$$

Proceeding as in the proof of (4.5.30) yields (4.5.32).

By using again the decay of the Oseen tensor Φ , estimate (4.5.31) is a direct consequence of the estimates (4.5.34) and (4.5.35). Indeed,

$$\begin{aligned} \|w_{\delta}\|_{L^{\infty}(\mathbb{R}^3)} &\leq C\gamma^{-1} + C \int_{\mathbb{R}^3} \frac{\rho^{\delta}}{|y-x|} dy \\ &\leq C \left(\gamma^{-1} + \|\rho^{\delta}\|_{L^1(\mathbb{R}^3)} + \|\rho^{\delta}\|_{L^{\infty}(\mathbb{R}^3)} \right) \\ &\leq C\delta \left(1 + \gamma^{-1} + \frac{1}{Nd_{\min}^3} \right), \end{aligned}$$

and analogously for the gradient. □

4.6 Convergence to the macroscopic equation

4.6.1 Convergence for small times

In this subsection, we prove the main result, Theorem 4.2.9, up to times for which the particles are well separated in the sense that the quantity Y_{ε} from Definition 4.2.2 is uniformly bounded for small ε . We already know from Proposition 4.5.15 that there exists such a time $T_0 > 0$.

In Section 4.6.2, we will prove Theorem 4.2.9 by showing that Y_{ε} is actually uniformly bounded for small ε for every finite time interval.

We first state the main result of this subsection.

Theorem 4.6.1. *Assume conditions (A1)-(A3) are satisfied. Moreover, assume that for $T_0 > 0$ there exists an $\varepsilon_0 > 0$ and $C_1 < \infty$ such that*

$$Y_{\varepsilon}(T_0) \leq C_1, \quad \text{for all } \varepsilon < \varepsilon_0.$$

Let $\tilde{\delta}_{\varepsilon} \rightarrow 0$, such that $\tilde{\delta}_{\varepsilon} = n_{\varepsilon}\delta_{\varepsilon}$ for some $n_{\varepsilon} \in \mathbb{N}^$ with $n_{\varepsilon} \rightarrow \infty$ as $\varepsilon \rightarrow 0$. Then, if Assumption 4.2.7 is satisfied with some $\beta > 2$,*

$$\rho_{\varepsilon}^{\tilde{\delta}_{\varepsilon}} \rightarrow \rho \quad \text{in } C^0([0, T_0]; X_{\beta}),$$

where ρ is the unique solution to problem (4.1.9).

The main ingredients in the proof of Theorem 4.6.1 are Proposition 4.5.8 and Lemma 4.5.16. Indeed, Proposition 4.5.8 allows to replace the fluid velocity v_ε , which transports the particles, by u_ε . Furthermore, by Lemma 4.5.16, we can replace u_ε in turn by $w_{\delta_\varepsilon, \varepsilon}$ defined in (4.5.29). The velocity field $w_{\delta_\varepsilon, \varepsilon}$ resembles $v_* + 2/9\gamma_*^{-1}g$ from the limit equation (4.1.9). The bounds on Y_ε in the assumptions of Theorem 4.6.1 ensure, that the estimates in Proposition 4.5.8 and Lemma 4.5.16 are uniform in ε as summarized in Lemma 4.6.3 below.

Throughout this subsection, we use the following notation and convention.

Notation 4.6.2. *In this subsection, we impose the assumptions of Theorem 4.6.1, and any constant \tilde{C} might depend on c_0 from Assumption (A1), γ_* from Assumption (A3), the fixed time T_0 , C_1 , $\|\rho_0\|_{X_\beta}$, and $\|\nabla\rho_0\|_{X_\beta}$. Moreover, we will implicitly consider times $t \leq T_0$. Moreover, we will implicitly consider $\varepsilon < \varepsilon_*$ such that $\delta_\varepsilon \geq d_{\min, \varepsilon}(t)$ for all $t \leq T_0$ and $\varepsilon < \varepsilon_*$. This is possible because of $\delta_\varepsilon \gg d_{\min, \varepsilon}(0)$ by Assumption 4.2.7 and the bound on $Y_0(T_0)$.*

Lemma 4.6.3. *Under the assumptions of Theorem 4.6.1, there exists $\varepsilon_1 > 0$ such that for all $\varepsilon < \varepsilon_1$*

$$\|w_{\delta_\varepsilon, \varepsilon} - v_\varepsilon\|_{L^\infty(\cup_i B_i(t))} \leq \tilde{C}\delta_\varepsilon, \quad \text{for all } t \leq T_0 \quad (4.6.1)$$

$$\|w_{\tilde{\delta}_\varepsilon, \varepsilon} - w_{\delta_\varepsilon, \varepsilon}\|_{L^\infty((0, T_0) \times \mathbb{R}^3)} \leq \tilde{C}\tilde{\delta}_\varepsilon, \quad (4.6.2)$$

$$\|w_{\delta_\varepsilon, \varepsilon}\|_{L^\infty((0, T_0); W^{1, \infty}(\mathbb{R}^3))} \leq \tilde{C}, \quad (4.6.3)$$

$$\|v_\varepsilon\|_{L^\infty((0, T_0) \times \mathbb{R}^3)} \leq \tilde{C}. \quad (4.6.4)$$

Proof. By Lemma 4.5.16, we have

$$\|w_{\delta_\varepsilon, \varepsilon}(t, \cdot) - v_\varepsilon(t, \cdot)\|_{L^\infty(\cup_i B_i(t))} \leq C\delta \left(1 + \frac{1}{Nd_{\min, \varepsilon}^3(t)}\right) \leq C\delta (1 + Y_\varepsilon(T_0)^3 c_0) \leq \tilde{C}\delta_\varepsilon.$$

This proves (4.6.1). Estimates (4.6.2) and (4.6.3) are proven analogously, and estimate (4.6.4) directly follows from (4.6.1) and (4.6.3). \square

For the ease of notation, we denote in the following

$$w_\varepsilon := w_{\delta_\varepsilon, \varepsilon} = S(\rho_\varepsilon^{\delta_\varepsilon} e) + \frac{2}{9}\gamma_\varepsilon g.$$

With the estimates from Lemma 4.6.3, the proof of Theorem 4.6.1 is not difficult but a bit tedious. It is based on comparing the particle trajectories with the characteristics of the limit equation (4.1.9). Roughly speaking, Lemma 4.6.3 implies that the velocity difference between the particle trajectories and the characteristics is of order δ_ε . This is why we prove the convergence for $\rho_\varepsilon^{\tilde{\delta}_\varepsilon}$ with $\tilde{\delta}_\varepsilon \gg \delta_\varepsilon$ instead of $\rho_\varepsilon^{\delta_\varepsilon}$. Indeed, consider a cube $Q_{\delta_\varepsilon}^z$ at time 0 and a particle $X_i(0) \in Q_{\delta_\varepsilon}^z$. Let the cube $Q_{\delta_\varepsilon}^z$ be transported by w_ε and the particle by v_ε . Then, after some time $t < T_0$, Lemma 4.6.3 suggests that the distance between the particle $X_i(t)$ and the cube $Q_{\delta_\varepsilon}^z(t)$ is of order $\tilde{C}\delta$. In particular, we do not know whether any of the particles that initially have lain in the cube $Q_{\delta_\varepsilon}^z$ are still in the transported cube after time t . If we consider $Q_{\tilde{\delta}_\varepsilon}^z$, however, we are able to show this for most of the particles.

This is one cause of some technicalities in the proof of Theorem 4.6.1. The other one is that it is difficult not to prove smallness of $\rho_\varepsilon^{\tilde{\delta}_\varepsilon} - \rho$ directly. Instead, we introduce “intermediate” particle densities τ_ε and σ_ε .

We define τ_ε to be the solution to

$$\begin{aligned} \tau_\varepsilon(0, \cdot) &= \rho_{\varepsilon, 0}, \\ \partial_t \tau_\varepsilon + w_\varepsilon \cdot \nabla \tau_\varepsilon &= 0, \end{aligned} \quad (4.6.5)$$

and σ_ε to be the solution to

$$\begin{aligned}\sigma_\varepsilon(0, \cdot) &= \rho_{0,\varepsilon}^{\tilde{\delta}_\varepsilon}, \\ \partial_t \sigma_\varepsilon + w_\varepsilon \cdot \nabla \sigma_\varepsilon &= 0.\end{aligned}\tag{4.6.6}$$

Then, the difference between ρ_ε and τ_ε lies only in the transport velocity, and the difference between τ_ε and σ_ε lies only in the initial datum. More precisely, σ_ε is the function we get from transporting the averaged initial datum, whereas $\tau_\varepsilon^{\tilde{\delta}_\varepsilon}$ is the average of the transported initial datum.

It is important first to replace the fluid velocity v_ε by w_ε – which is how we defined τ_ε – and afterwards to replace the averaging one cubes $Q_{\delta_\varepsilon}^z$ at every time by averaging at the beginning and considering the transported averages – which is how we defined σ_ε . The reason for that is that the estimate on $v_\varepsilon - w_\varepsilon$ in Lemma 4.6.3 only holds inside the particles.

In Lemma 4.6.4, we prove smallness of $\tau_\varepsilon^{\tilde{\delta}_\varepsilon} - \rho_\varepsilon^{\tilde{\delta}_\varepsilon}$, in Lemma 4.6.5, we prove smallness of $\tau_\varepsilon^{\tilde{\delta}_\varepsilon} - \sigma_\varepsilon$. Then, the proof of Theorem 4.6.1 reduces to proving smallness of $\sigma_\varepsilon - \rho$.

Lemma 4.6.4. *Let τ_ε be defined to be the solution to (4.6.5). Then, under the assumptions of Theorem 4.6.1,*

$$\|\tau_\varepsilon^{\tilde{\delta}_\varepsilon} - \rho_\varepsilon^{\tilde{\delta}_\varepsilon}\|_{L^\infty([0, T_0]; X_\beta)} \rightarrow 0,$$

where $\tau_\varepsilon^{\tilde{\delta}_\varepsilon}$ and $\rho_\varepsilon^{\tilde{\delta}_\varepsilon}$ are averages on cubes as in Definition 4.2.4.

Proof. We denote by ψ_ε the flow of w_ε . More precisely, $\psi_\varepsilon : [0, T_0] \times [0, T_0] \times \mathbb{R}^3 \rightarrow \mathbb{R}^3$ is the solution to

$$\begin{aligned}\partial_s \psi_\varepsilon(t, s, x) &= w_\varepsilon(s, \psi_\varepsilon(t, s, x)), \\ \psi_\varepsilon(t, t, x) &= x.\end{aligned}$$

This is well defined due to Lemma 4.6.3. Fix a particle B_i and consider a point x in that particle at time 0, i.e., $x \in B_i(0)$. Let $t < T_0$. Then, Lemma 4.6.3 implies

$$\begin{aligned}|X_i(t) - \psi_\varepsilon(0, t, x)| &\leq R + \int_0^t |v_\varepsilon(s, X_i(s)) - w_\varepsilon(s, \psi_\varepsilon(0, s, y))| ds \\ &\leq R + \int_0^t |v_\varepsilon(s, X_i(s)) - w_\varepsilon(s, \psi_\varepsilon(0, s, y))| ds \\ &\quad + \int_0^t |w_\varepsilon(s, X_i(s)) - w_\varepsilon(s, \psi_\varepsilon(0, s, y))| ds \\ &\leq \tilde{C}\delta_\varepsilon + \tilde{C} \int_0^t |X_i(s) - \psi_\varepsilon(0, s, x)| ds.\end{aligned}$$

Gronwall's inequality implies

$$|X_i(s) - \psi_\varepsilon(0, t, x)| \leq \tilde{C}\delta_\varepsilon e^{\tilde{C}t} \leq \tilde{C}\delta_\varepsilon =: \eta_\varepsilon.\tag{4.6.7}$$

This estimate implies that if B_i was transported by w_ε instead of v_ε , it would lie in $B_{\eta_\varepsilon}(X_i(t))$ at time t . For $x \in \mathbb{R}^3$, we define

$$q^x = \{y \in Q_{\delta_\varepsilon}^x : \text{dist}\{y, \partial Q_{\delta_\varepsilon}^x\} > \tilde{C}\eta_\varepsilon\},$$

and

$$\bar{q}^x = \{y \in \mathbb{R}^3 : \text{dist}\{y, Q_{\delta_\varepsilon}^x\} < \tilde{C}\eta_\varepsilon\}.$$

Then, (4.6.7) yields

$$\frac{1}{|Q_{\delta_\varepsilon}^x|} \int_{q^x} \rho_\varepsilon(t, y) dy \leq \int_{Q_{\delta_\varepsilon}^x} \tau_\varepsilon(t, y) dy = \tau_\varepsilon^{\tilde{\delta}_\varepsilon}(t, x) \leq \frac{1}{|Q_{\delta_\varepsilon}^x|} \int_{\bar{q}^x} \rho_\varepsilon(t, y) dy.$$

Clearly, we also have

$$\frac{1}{|Q_{\tilde{\delta}_\varepsilon}^x|} \int_{q^x} \rho_\varepsilon(t, y) dy \leq \int_{Q_{\tilde{\delta}_\varepsilon}^x} \rho_\varepsilon(t, y) dy = \rho_{\tilde{\delta}_\varepsilon}^\varepsilon(t, x) \leq \frac{1}{|Q_{\tilde{\delta}_\varepsilon}^x|} \int_{\bar{q}^x} \rho_\varepsilon(t, y) dy.$$

Hence, in order to prove the assertion, it suffices to prove smallness of

$$(1 + |x|^\beta) \left(\frac{1}{|Q_{\tilde{\delta}_\varepsilon}^x|} \int_{q^x} \rho_\varepsilon(t, y) dy - \frac{1}{|Q_{\tilde{\delta}_\varepsilon}^x|} \int_{\bar{q}^x} \rho_\varepsilon(t, y) dy \right) = \frac{1 + |x|^\beta}{|Q_{\tilde{\delta}_\varepsilon}^x|} \int_{q^x \setminus \bar{q}^x} \rho_\varepsilon(t, y) dy. \quad (4.6.8)$$

Fix a particle i such that $X_i(t) \in q^x \setminus \bar{q}^x$ and consider $B_{\delta_\varepsilon}(X_i(t))$. Then, by definition of $Y_\varepsilon(T_0)$, we know that $X_j(t) \in B_{\delta_\varepsilon}(X_i(t))$ implies $X_j(0) \in B_{Y(T_0)\delta_\varepsilon}(X_i(0))$. Thus,

$$\int_{B_{\delta_\varepsilon}(X_i(t))} \rho_\varepsilon(t, y) dy \leq \int_{B_{Y(T_0)\delta_\varepsilon}(X_i(0))} \rho_{\varepsilon,0}(y) dy.$$

Let I denote the set of centers z of the cubes $Q_{\tilde{\delta}_\varepsilon}^z$ from Definition 4.2.4 with $Q_{\tilde{\delta}_\varepsilon}^z \cap B_{Y_\varepsilon(T_0)\delta_\varepsilon}(X_i(0)) \neq \emptyset$. Then,

$$\int_{B_{\delta_\varepsilon}(X_i(t))} \rho_\varepsilon(t, y) dy \leq \sum_{z \in I} \int_{Q_{\tilde{\delta}_\varepsilon}^z} \rho_{\varepsilon,0}(y) dy \leq C \sum_{z \in I} \delta_\varepsilon^3 \frac{1}{1 + |z|^\beta} \|\rho_{\varepsilon,0}\|_{X_\beta}.$$

Using the bound on $Y(T_0)$, we have $|I| \leq \tilde{C}$. Furthermore, from Assumption 4.2.7, we know that $\rho_{\varepsilon,0}$ is uniformly bounded in X_β . Moreover, since v_ε is uniformly bounded in $L^\infty(\mathbb{R}^3)$ by Lemma 4.6.3, $X_i(t) \in q^x \setminus \bar{q}^x$ at time $t \leq T_0$ implies $X_i(0) \in B_{\tilde{C}}(x)$ and also $z \in B_{\tilde{C}}(x)$ for all $z \in I$. Therefore,

$$\int_{B_{\delta_\varepsilon}(X_i(t))} \rho_\varepsilon(t, y) dy \leq \tilde{C} \delta_\varepsilon^3 \frac{1}{1 + |x|^\beta}. \quad (4.6.9)$$

Finally, we note that the number M of balls $B_{\delta_\varepsilon}(X_i(t))$ with $X_i(t) \in q^x \setminus \bar{q}^x$ that are needed to cover all the particles $B_i(t)$ in $q^x \setminus \bar{q}^x$ is bounded by

$$M \leq C \frac{|q^x \setminus \bar{q}^x|}{\delta_\varepsilon^3} \leq C \frac{\eta_\varepsilon \tilde{\delta}_\varepsilon^2}{\delta_\varepsilon^3} \leq \tilde{C} \frac{\delta_\varepsilon \tilde{\delta}_\varepsilon^2}{\delta_\varepsilon^3} \leq \tilde{C} \frac{\tilde{\delta}_\varepsilon^2}{\delta_\varepsilon^2}.$$

Combining this with (4.6.9) yields

$$(1 + |x|^\beta) \frac{1}{|Q_{\tilde{\delta}_\varepsilon}^x|} \int_{q^x \setminus \bar{q}^x} \rho_\varepsilon(t, y) dy \leq M \tilde{C} \frac{\delta_\varepsilon^3}{|Q_{\tilde{\delta}_\varepsilon}^x|} \leq \tilde{C} \frac{\delta_\varepsilon}{\tilde{\delta}_\varepsilon} \rightarrow 0.$$

This finishes the proof. \square

Lemma 4.6.5. *Let σ_ε be defined to be the solution to (4.6.6). Then, under the assumptions of Theorem 4.6.1*

$$\|\tau_{\tilde{\delta}_\varepsilon}^{\tilde{\delta}_\varepsilon} - \sigma_\varepsilon\|_{L^\infty([0, T_0]; X_\beta)} \rightarrow 0.$$

Proof. Denoting again by ψ_ε the flow of w_ε , we get

$$\sigma_\varepsilon(t, x) = \int_{Q_{\tilde{\delta}_\varepsilon}^{\psi_\varepsilon(t, 0, x)}} \rho_{\varepsilon,0}(y) dy.$$

On the other hand, we have

$$\tau_{\varepsilon}^{\tilde{\delta}_{\varepsilon}}(t, x) = \int_{Q_{\tilde{\delta}_{\varepsilon}}^x} \rho_{\varepsilon,0}(\psi(t, 0, y)) dy = \int_{\psi_{\varepsilon}(t, 0, Q_{\tilde{\delta}_{\varepsilon}}^x)} \rho_{\varepsilon,0}(y) dy,$$

where we used that $\det(D\psi_{\varepsilon}) = 1$ which follows from the fact that w_{ε} is divergence free.

We estimate using Lemma 4.6.3

$$\begin{aligned} |\psi_{\varepsilon}(0, t, x) - \psi_{\varepsilon}(0, t, y)| &\leq |x - y| + \int_0^t |w_{\varepsilon}(s, \psi_{\varepsilon}(0, s, x)) - w_{\varepsilon}(s, \psi_{\varepsilon}(0, s, y))| ds \\ &\leq |x - y| + \tilde{C} \int_0^t |\psi_{\varepsilon}(0, s, x) - \psi_{\varepsilon}(0, s, y)| ds. \end{aligned}$$

Gronwall's inequality implies

$$|\psi_{\varepsilon}(0, t, x) - \psi_{\varepsilon}(0, t, y)| \leq |x - y| e^{\tilde{C}t} \leq \tilde{C}|x - y|. \quad (4.6.10)$$

By an analogous argument, we also get the lower bound

$$|\psi_{\varepsilon}(0, t, x) - \psi_{\varepsilon}(0, t, y)| \geq \frac{1}{\tilde{C}}|x - y|. \quad (4.6.11)$$

Consider a point $y \in Q_{\tilde{\delta}_{\varepsilon}}^x$ at time t . We want to find η_{ε} such that $\text{dist}\{y, \partial Q_{\tilde{\delta}_{\varepsilon}}^x\} > \eta_{\varepsilon}$ implies

$$\psi_{\varepsilon}(0, t, Q_{\tilde{\delta}_{\varepsilon}}^{\psi_{\varepsilon}(t, 0, y)}) \subset Q_{\tilde{\delta}_{\varepsilon}}^x. \quad (4.6.12)$$

Estimate (4.6.10) implies that this is true with

$$\eta_{\varepsilon} = \tilde{C}\delta_{\varepsilon}, \quad (4.6.13)$$

for all $t \leq T_0$. Let

$$q_{\varepsilon}(x) = \{y \in Q_{\tilde{\delta}_{\varepsilon}}^x : \text{dist}\{y, \partial Q_{\tilde{\delta}_{\varepsilon}}^x\} > \eta_{\varepsilon}\},$$

and

$$\bar{q}_{\varepsilon}(t, x) = \bigcup_{y \in q_{\varepsilon}(t, x)} Q_{\tilde{\delta}_{\varepsilon}}^{\psi_{\varepsilon}(t, 0, y)}.$$

Then, by (4.6.12),

$$q_{\varepsilon}(x) \subset \psi_{\varepsilon}(0, t, \bar{q}_{\varepsilon}(t, x)) \subset Q_{\tilde{\delta}_{\varepsilon}}^x. \quad (4.6.14)$$

Therefore,

$$\begin{aligned} (1 + |x|^{\beta})|\sigma_{\varepsilon}(t, x) - \tau_{\varepsilon}^{\tilde{\delta}_{\varepsilon}}(t, x)| &= (1 + |x|^{\beta}) \left| \int_{Q_{\tilde{\delta}_{\varepsilon}}^{\psi_{\varepsilon}(t, 0, x)}} \rho_{\varepsilon,0}(y) dy - \int_{Q_{\tilde{\delta}_{\varepsilon}}^x} \rho_{\varepsilon,0}(\psi_{\varepsilon}(t, 0, y)) dy \right| \\ &\leq (1 + |x|^{\beta}) \left| \int_{Q_{\tilde{\delta}_{\varepsilon}}^{\psi_{\varepsilon}(t, 0, x)}} \rho_{\varepsilon,0}(y) dy - \rho_0(\psi_{\varepsilon}(t, 0, x)) \right| \\ &\quad + (1 + |x|^{\beta}) \left| \frac{1}{|Q_{\tilde{\delta}_{\varepsilon}}^x|} \int_{\bar{q}_{\varepsilon}(t, x)} \rho_{\varepsilon,0}(y) dy - \rho_0(\psi_{\varepsilon}(t, 0, x)) \right| \\ &\quad + (1 + |x|^{\beta}) \frac{1}{|Q_{\tilde{\delta}_{\varepsilon}}^x|} \int_{Q_{\tilde{\delta}_{\varepsilon}}^x \setminus q_{\varepsilon}(t, x)} \rho_{\varepsilon,0}(\psi_{\varepsilon}(t, 0, y)) dy \\ &=: A_1 + A_2 + A_3. \end{aligned}$$

By Lemma 4.6.3, w_ε is uniformly bounded in $L^\infty(\mathbb{R}^3)$. Thus, $|\psi_\varepsilon(t, 0, y)| \geq |y| - \tilde{C}$ and

$$\frac{1}{1 + |y|^\beta} \leq \frac{\tilde{C}}{1 + |x|^\beta}, \quad \text{for all } y \in Q_{\tilde{\delta}_\varepsilon}^{\psi_\varepsilon(t, 0, x)}.$$

We estimate A_1 using the convergence $\rho_{0,\varepsilon}^{\delta_\varepsilon} \rightarrow \rho_0$ and boundedness of $\|\nabla \rho_0\|_{X_\beta}$.

$$A_1 \leq \tilde{C} \|\rho_{\varepsilon,0}^{\delta_\varepsilon} - \rho_0\|_{X_\beta} + \tilde{C} \tilde{\delta}_\varepsilon \|\nabla \rho_0\|_{X_\beta} \rightarrow 0.$$

In order to estimate A_3 , we proceed as in the estimate of the term in (4.6.8) from Lemma 4.6.4. We have to control the number of deformed particles transported by w_ε in $Q_{\tilde{\delta}_\varepsilon}^x \setminus q_\varepsilon(t, x)$ at time t . To this end, we define the trajectories of the particles transported by w_ε

$$\tilde{X}_i(t) := \psi_\varepsilon(0, t, X_i(0))$$

and

$$\tilde{B}_i(t) := \psi_\varepsilon(0, t, B_i(0)).$$

Then, estimate (4.6.11) implies for all $i \neq j$

$$|\tilde{X}_i(t) - \tilde{X}_j(t)| \geq \frac{\tilde{C}}{|X_i(0) - X_j(0)|}.$$

and

$$\text{diam } \tilde{B}_i(t) \leq \tilde{C} R_\varepsilon.$$

Therefore, A_3 tends to zero by the same argument as we have proved smallness of (4.6.8).

For A_2 , let $(x_i)_{i=1}^n$ denote the centers of the disjoint cubes that $\bar{q}_\varepsilon(t, x)$ consists of. Note that (4.6.14) implies $|q_\varepsilon(x)| \leq |\bar{q}_\varepsilon(t, x)|$ due to conservation of volume. Using also (4.6.10), we deduce

$$\begin{aligned} A_2 &\leq (1 + |x|^\beta) \frac{|Q_{\tilde{\delta}_\varepsilon}^x|}{|Q_{\tilde{\delta}_\varepsilon}^x|} \sum_{i=1}^n \left| \int_{Q_{\tilde{\delta}_\varepsilon}^{x_i}} \rho_{\varepsilon,0}(y) dy - \rho_0(\psi_\varepsilon(t, 0, x)) \right| + \left(1 - \frac{|\bar{q}_\varepsilon(t, x)|}{|Q_{\tilde{\delta}_\varepsilon}^x|} \right) \rho_0(\psi_\varepsilon(t, 0, x)) \\ &\leq \tilde{C} \|\rho_{\varepsilon,0}^{\delta_\varepsilon} - \rho_0\|_{X_\beta} + \tilde{C} \tilde{\delta}_\varepsilon \|\nabla \rho_0\|_{X_\beta} + \tilde{C} \frac{|Q_{\tilde{\delta}_\varepsilon}^x \setminus q_\varepsilon(x)|}{|Q_{\tilde{\delta}_\varepsilon}^x|} \|\rho_0\|_{X_\beta} \\ &\leq \tilde{C} \|\rho_{\varepsilon,0}^{\delta_\varepsilon} - \rho_0\|_{X_\beta} + \tilde{C} \tilde{\delta}_\varepsilon \|\nabla \rho_0\|_{X_\beta} + \tilde{C} \frac{\eta_\varepsilon}{\tilde{\delta}_\varepsilon} \|\rho_0\|_{L^\infty(\mathbb{R}^3)}. \end{aligned}$$

By equation (4.6.13) this tends to 0 as $\varepsilon \rightarrow 0$ because $\eta_\varepsilon \leq \tilde{C} \tilde{\delta}_\varepsilon \ll \tilde{\delta}_\varepsilon$. □

Proof of Theorem 4.6.1. We again define ψ_ε to be the flow of w_ε . Moreover, we define

$$w = v_* + \frac{2}{9} \gamma_*^{-1} g = S(\rho g) + \frac{2}{9} \gamma_*^{-1} g,$$

and denote by $\tilde{\psi}$ the flow of w .

We recall from Lemma 4.6.3 that w_ε is bounded uniformly with respect to ε in $L^\infty((0, T_0) \times \mathbb{R}^3)$. Moreover

$$\|w\|_{W^{1,\infty}((0,T_0) \times \mathbb{R}^3)} \leq \tilde{C}. \quad (4.6.15)$$

This follows from boundedness of ρ in $L^\infty(0, T_0; X_\beta)$, which is stated in Theorem 4.4.2, and Lemma 4.4.3. From the L^∞ -bounds on w and w_ε , we deduce for all $x \in \mathbb{R}^3$

$$\frac{1}{1 + |\psi_\varepsilon(t, 0, x)|^\beta} \leq \frac{\tilde{C}}{1 + |x|^\beta},$$

and the same inequality with $\tilde{\psi}$ replacing ψ .

Let σ_ε be the solution to (4.6.6). Then,

$$\begin{aligned} |\rho(t, x) - \sigma_\varepsilon(t, x)| &= |\rho_0(\tilde{\psi}(t, 0, x)) - \rho_{0,\varepsilon}^{\tilde{\delta}_\varepsilon}(\psi_\varepsilon(t, 0, x))| \\ &\leq |\rho_0(\tilde{\psi}(t, 0, x)) - \rho_0(\psi_\varepsilon(t, 0, x))| + |\rho_0(\psi_\varepsilon(t, 0, x)) - \rho_{0,\varepsilon}^{\tilde{\delta}_\varepsilon}(\psi_\varepsilon(t, 0, x))| \\ &\leq \frac{1}{1 + |x|^\beta} \left(\|\nabla \rho_0\|_{X_\beta} |\tilde{\psi}(t, 0, x) - \psi_\varepsilon(t, 0, x)| + \|\rho_0 - \rho_{0,\varepsilon}^{\tilde{\delta}_\varepsilon}\|_{X_\beta} \right). \end{aligned} \quad (4.6.16)$$

Concerning the first term on the right hand side, we have

$$\begin{aligned} |\tilde{\psi}(t, 0, x) - \psi_\varepsilon(t, 0, x)| &\leq \int_0^t |w(s, \tilde{\psi}(t, s, x)) - w_\varepsilon(s, \psi_\varepsilon(t, s, x))| ds \\ &\leq \|\nabla w\|_{L^\infty} \int_0^t |\tilde{\psi}(t, s, x) - \psi_\varepsilon(t, s, x)| ds \\ &\quad + \int_0^t \|w(s, \cdot) - w_\varepsilon(s, \cdot)\|_{L^\infty} ds. \end{aligned}$$

Gronwall's inequality yields

$$\begin{aligned} \|\tilde{\psi}(t, 0, \cdot) - \psi_\varepsilon(t, 0, \cdot)\|_{L^\infty} &\leq \int_0^t \|w(s, \cdot) - w_\varepsilon(s, \cdot)\|_{L^\infty} ds \\ &\quad + \|\nabla w\|_{L^\infty} \int_0^t \int_0^s \|w(\tau, \cdot) - w_\varepsilon(\tau, \cdot)\|_{L^\infty} d\tau e^{(t-s)\|\nabla w\|_{L^\infty}} ds \\ &\leq \tilde{C} \int_0^t \|w(s, \cdot) - w_\varepsilon(s, \cdot)\|_{L^\infty} ds. \end{aligned} \quad (4.6.17)$$

Combining estimates (4.6.15), (4.6.16), and (4.6.17), we deduce for $t < T_0$

$$\|\rho(t, \cdot) - \sigma_\varepsilon(t, \cdot)\|_{X_\beta} \leq \|\rho_0 - \rho_{0,\varepsilon}^{\tilde{\delta}_\varepsilon}\|_{X_\beta} + \tilde{C} \|\nabla \rho_0\|_{X_\beta} \int_0^t \|w(s, \cdot) - w_\varepsilon(s, \cdot)\|_{L^\infty} ds. \quad (4.6.18)$$

Lemma 4.4.3 and Lemma 4.6.3 yield

$$\begin{aligned} \|w(s, \cdot) - w_\varepsilon(s, \cdot)\|_{L^\infty} &\leq \|w_{\tilde{\delta}_\varepsilon, \varepsilon}(s, \cdot) - w_\varepsilon(s, \cdot)\|_{L^\infty} + \|w(s, \cdot) - w_{\tilde{\delta}_\varepsilon, \varepsilon}(s, \cdot)\|_{L^\infty} \\ &\leq \tilde{C} \tilde{\delta}_\varepsilon + \|S(\rho(s, \cdot) - \rho_\varepsilon^{\tilde{\delta}_\varepsilon}(s, \cdot))\|_{L^\infty} \\ &\leq \tilde{C} \tilde{\delta}_\varepsilon + C \|\rho(s, \cdot) - \rho_\varepsilon^{\tilde{\delta}_\varepsilon}(s, \cdot)\|_{X_\beta} \\ &\leq \tilde{C} \tilde{\delta}_\varepsilon + C \|\sigma_\varepsilon(s, \cdot) - \rho_\varepsilon^{\tilde{\delta}_\varepsilon}(s, \cdot)\|_{X_\beta} + C \|\rho(s, \cdot) - \sigma_\varepsilon(s, \cdot)\|_{X_\beta} \\ &=: \theta_1 + \tilde{C} \|\rho(s, \cdot) - \sigma_\varepsilon(s, \cdot)\|_{X_\beta}. \end{aligned} \quad (4.6.19)$$

Note that $\theta_1 \rightarrow 0$ as $\varepsilon \rightarrow 0$ by Lemma 4.6.4 and Lemma 4.6.5. Using estimate (4.6.19) in (4.6.18), we deduce

$$\|\rho(t, \cdot) - \sigma_\varepsilon(t, \cdot)\|_{X_\beta} \leq \|\rho_0 - \rho_{0,\varepsilon}^{\tilde{\delta}_\varepsilon}\|_{X_\beta} + \tilde{C} \left(\theta_1 + \int_0^t \|\rho(s, \cdot) - \sigma_\varepsilon(s, \cdot)\|_{X_\beta} ds \right).$$

We apply Gronwall's inequality once more to conclude

$$\|\rho(t, \cdot) - \sigma_\varepsilon(t, \cdot)\|_{X_\beta} \leq \tilde{C} \left(\theta_1 + \|\rho_0 - \rho_{0,\varepsilon}^{\tilde{\delta}_\varepsilon}\|_{X_\beta} \right),$$

which converges to zero, uniformly for $t \leq T_0$. Combining this estimate with Lemma 4.6.4 and Lemma 4.6.5 finishes the proof. \square

4.6.2 Extension of the convergence to arbitrary times

Using the convergence result, Theorem 4.6.1, we are able to prove a posteriori $\limsup_{\varepsilon \rightarrow 0} Y_\varepsilon(t) < \infty$ for all $t > 0$. This is the crucial assumption of Theorem 4.6.1 and therefore the last step to prove the main result, Theorem 4.2.9.

The key estimate, which we prove in Lemma 4.6.6 below, is that the convergence result stated in Theorem 4.6.1 allows to show that $Y_\varepsilon(t)$ only grows exponentially in t for small enough ε . With this estimate and Proposition 4.5.15, it is easy to control Y_ε by a standard contradiction argument.

In order to prove Lemma 4.6.6, the Lipschitz type estimate on v_ε from Lemma 4.5.14 implies that it suffices to bound the quantity $\alpha_{\varepsilon,2}(t)$ defined in (4.5.1). We are able to achieve this using the convergence $\rho_\varepsilon^{\tilde{\delta}_\varepsilon} \rightarrow \rho$. Indeed, fixing a particle X_i , particles X_j that are close to X_i do not contribute much to

$$\sum_{j \neq i} \frac{1}{|X_i - X_j|^2}. \quad (4.6.20)$$

For particles far away, however, their exact position within a cube of size $\tilde{\delta}_\varepsilon$ is not important for the value of (4.6.20).

Lemma 4.6.6. *There exists a constant C_* which only depends on $\|\rho_0\|_{L^\infty(\mathbb{R}^3)}$ with the following property. Assume the assumptions of Theorem 4.6.1 are satisfied for some time $T_0 > 0$. Then, there exists $\varepsilon_1 > 0$ such that*

$$Y_\varepsilon(t) \leq e^{C_* t} \quad \text{for all } \varepsilon < \varepsilon_1, \quad \text{and } t \leq T_0.$$

Proof. Claim. There exists $\varepsilon_1 > 0$ such that for all $\varepsilon < \varepsilon_1$ and all $t \leq T_0$,

$$\sup_j \frac{1}{N_\varepsilon} \sum_{i \neq j} \frac{1}{|X_i(t) - X_j(t)|^2} \leq C_*,$$

for some constant C_* , which only depends on $\|\rho_0\|_{L^\infty(\mathbb{R}^3)}$.

Let I be the set of the centers of the cubes with side length $\tilde{\delta}_\varepsilon$ from Definition 4.2.4. At a fixed time $t < T_0$, we fix a particle X_j and define I_1 to consist of the center of the cube containing X_j , and the centers of the cubes that are adjacent to that cube. Furthermore, we denote $I_2 = I \setminus I_1$. Then, we estimate

$$\begin{aligned} \frac{1}{N_\varepsilon} \sum_{i \neq j} \frac{1}{|X_i - X_j|^2} &\leq \frac{1}{N_\varepsilon} \sum_{y \in I_1} \sum_{X_i \in Q_{\tilde{\delta}_\varepsilon}^y} \frac{1}{|X_i - X_j|^2} + \frac{1}{N_\varepsilon} \sum_{y \in I_2} \sum_{X_i \in Q_{\tilde{\delta}_\varepsilon}^y} \frac{1}{|X_i - X_j|^2} \\ &=: A_1 + A_2. \end{aligned}$$

The first term, A_1 , we estimate

$$A_1 \leq \frac{C}{(d_{\varepsilon, \min}(t))^3 N_\varepsilon} \int_{B_{C\tilde{\delta}_\varepsilon}(X_j)} \frac{1}{|y - X_j|^2} dy \leq C c_0 \tilde{\delta}_\varepsilon Y_\varepsilon(t)^3 \leq C \quad (4.6.21)$$

for ε sufficiently small, since Y_ε is uniformly bounded for small ε .

In order to estimate the second term, A_2 , we define

$$M(x) := |\{X_i \in Q_{\tilde{\delta}_\varepsilon}^x\}|.$$

Note that

$$\|M\|_{L^1(\mathbb{R}^3)} = \tilde{\delta}_\varepsilon^3 N_\varepsilon.$$

Moreover,

$$\|M\|_{L^\infty(\mathbb{R}^3)} \leq C\tilde{\delta}_\varepsilon^3 N_\varepsilon \|\rho_\varepsilon^{\tilde{\delta}_\varepsilon}\|_{L^\infty(\mathbb{R}^3)},$$

since all the balls B_i with centers $X_i \in Q_{\tilde{\delta}_\varepsilon}^x$ are contained in the union of $Q_{\tilde{\delta}_\varepsilon}^x$ and the adjacent cubes. Thus, Theorem 4.6.1 implies that we can choose ε_0 small enough such that for $\varepsilon < \varepsilon_0$

$$\|M\|_{L^\infty(\mathbb{R}^3)} \leq C\tilde{\delta}_\varepsilon^3 N_\varepsilon \|\rho(t)\|_{L^\infty(\mathbb{R}^3)} = C\tilde{\delta}_\varepsilon^3 N_\varepsilon \|\rho_0\|_{L^\infty(\mathbb{R}^3)},$$

where we used that the L^∞ -norm of ρ is conserved in time. Combining the L^∞ - and L^1 -estimates of M yields

$$A_2 \leq \frac{C}{N_\varepsilon} \sum_{y \in I_2} \frac{M(y)}{|y - X_j|^2} \leq \frac{C}{N_\varepsilon \tilde{\delta}_\varepsilon^3} \int_{\mathbb{R}^3} \frac{M(y)}{|y - X_j|^2} dy \leq C(1 + \|\rho_0\|_{L^\infty(\mathbb{R}^3)}). \quad (4.6.22)$$

Combining the estimates for A_1 and A_2 , (4.6.21) and (4.6.22) proves the claim.

Recall from Lemma 4.5.1

$$\phi_\varepsilon \alpha_{\varepsilon,3}(t) \leq Cc_0^{-1} \phi_\varepsilon Y_\varepsilon^3(t) \log \left(\frac{N_\varepsilon Y_\varepsilon(t)}{c_0} \right),$$

and this converges to zero for any fixed time $t < T_0$ due to Assumption (A2) since $Y_\varepsilon(t)$ is bounded by assumption.

Thus, Lemma 4.5.14 yields for all particles i and j

$$|v_\varepsilon(t, X_i) - v_\varepsilon(t, X_j)| \leq C_* |X_i - X_j|$$

for all $t \leq T_0$ and all $\varepsilon < \varepsilon_0$ for some ε_0 small enough.

Hence,

$$\begin{aligned} |X_i(t) - X_j(t)| &\geq |X_i(0) - X_j(0)| - \int_0^t |v_\varepsilon(s, X_i(s)) - v_\varepsilon(s, X_j(s))| ds \\ &\geq |X_i(0) - X_j(0)| - C_* \int_0^t |X_i - X_j| ds. \end{aligned}$$

Using Gronwall's inequality and the definition of Y_ε finishes the proof. \square

Proof of Theorem 4.2.9. Let $T_0 > 0$. By Theorem 4.6.1, it suffices to prove that there exists $\varepsilon_1 > 0$ and $C_1 < \infty$ such that

$$Y_\varepsilon(t) \leq C_1, \quad \text{for all } \varepsilon < \varepsilon_1, \quad \text{and } t \leq T_0. \quad (4.6.23)$$

We argue by contradiction. Define T_0 to be the infimum over all times for which there is no pair (ε_1, C_1) such that (4.6.23) holds, and assume $T_0 < \infty$. By Proposition 4.5.15, we know $T_0 > 0$.

Let $0 < \theta < T_0$. Then, at time $T_* := T_0 - \theta$, application of Lemma 4.6.6 yields

$$Y_\varepsilon(t) \leq e^{C_* T_0}, \quad \text{for all } \varepsilon < \varepsilon_0, \quad \text{and } t \leq T_*,$$

for some $\varepsilon_0 > 0$. Now, we can apply again Proposition 4.5.15, which yields

$$Y_\varepsilon(t) \leq 2e^{C_* T_0}, \quad \text{for all } \varepsilon < \varepsilon_1, \quad \text{and } t \leq T_* + \theta_1,$$

for $\varepsilon_1 > 0$ and some θ_1 which depends only on $e^{C_* T_0}$. Thus, choosing $\theta < \theta_1$, we get a contradiction to the definition of T_0 . \square

Chapter 5

Homogenization of the Poisson equation

In this Chapter, we study the homogenization of the Poisson equations in perforated domains under very mild assumptions on the distribution and size of the holes. In particular, we study distributions of holes where clusters and even overlapping of holes occur with high probability. We prove that, under minimal assumptions of the average capacity density of the holes, the classical homogenization results are still valid. The techniques developed for this problem are also used in Chapter 6 in order to study the more involved problem of the homogenization of the Stokes equations to the Brinkman equations, which are the fluid equations in the sedimentation model described by the Vlasov-Stokes equations (1.1.2).

The content of this chapter can be found in the preprint [GHV18] and has been accepted for publication in *Communications in Partial Differential Equations*.

5.1 Introduction

We consider the problem

$$\begin{cases} -\Delta u_\varepsilon = f & \text{in } D^\varepsilon \\ u_\varepsilon = 0 & \text{on } \partial D^\varepsilon, \end{cases} \quad (5.1.1)$$

where the domain D^ε is obtained by removing from a bounded set $D \subset \mathbb{R}^d$, $d > 2$, the union of properly rescaled spherical holes: Given a collection of points $\Phi = \{z_i\}_{i \in \mathbb{N}} \subset \mathbb{R}^d$ and associated radii $\{\rho_i\}_{z_i \in \Phi} \subset \mathbb{R}_+$, we define

$$D^\varepsilon := D \setminus \bigcup_{z_i \in \Phi \cap \frac{1}{\varepsilon} D} B_{\varepsilon \frac{d}{d-2} \rho_i}(\varepsilon z_i), \quad (5.1.2)$$

where $\frac{1}{\varepsilon} D := \{x \in \mathbb{R}^d : \varepsilon x \in D\}$.

In this chapter, we assume that D^ε is a random set. More precisely, we assume that the collection Φ of the centres is generated according to a stationary point process on \mathbb{R}^d and that the radii $\{\rho_i\}_{z_i \in \Phi}$ are unbounded random variables with short-range correlations. We show that, \mathbb{P} -almost surely, when $\varepsilon \downarrow 0^+$, the solutions of (5.1.1) weakly converge in $H_0^1(D)$ to the solution of

$$\begin{cases} (-\Delta + C_0)u_h = f & \text{in } D \\ u_h = 0 & \text{on } \partial D. \end{cases} \quad (5.1.3)$$

Here, the constant $C_0 > 0$ may be expressed in terms of an averaged density of capacity generated by the holes. We thus recover in the limit the analogue of the well-known “strange term” obtained by Cioranescu and Murat in the case of deterministic and periodic holes [CM97]. In this latter case, which is equivalent to taking $\Phi = \mathbb{Z}^d$ and $\rho_i \equiv r$ for $r > 0$, C_0 equals the capacity of a ball of radius r . More precisely,

$$C_0 = (d-2)\mathcal{H}^{d-1}(S^{d-1})r^{d-2}, \quad (5.1.4)$$

where S^{d-1} is the $(d-1)$ -dimensional unit sphere in \mathbb{R}^d .

We show in this chapter that the homogenization of u_ε to u_h of (5.1.3) takes place under minimal assumptions on the decay of the density functions for the radii $\{\rho_i\}_{z_i \in \Phi}$ of the holes. Namely, we only assume that the average capacity of each hole is finite. More precisely, we will show that the homogenization result holds assuming that the configurations of the holes in \mathbb{R}^d , $d > 2$, are assigned to a class of probability measures for which the expectation of each radius ρ_i satisfies only

$$\langle \rho_i^{d-2} \rangle < \infty. \quad (5.1.5)$$

In view of (5.1.4), this is the minimal assumption under which one can expect C_0 in (5.1.3) to be finite and therefore the limit problem (5.1.3) to be meaningful. On the other hand, to assume only (5.1.5) on the hole distributions poses some difficulties due to the fact that the presence of balls in (5.1.2) with large radii could allow the onset of clusters having large capacity. These clusters might prevent the convergence of u_ε to the solutions of the homogenized problem u_h . In fact, it is known that the onset of large clusters take place with probability one in systems of spherical holes filling a small volume fraction of the space \mathbb{R}^d , if the radii are distributed according to probability distributions with sufficiently fat tails (cf. [Gri99; MR96]). However, in [Gri99; MR96] the holes are not rescaled as in (5.1.2). In this chapter, we prove that with the rescaling of the balls as in (5.1.2), the assumption (5.1.5) is sufficient to ensure that no percolating-like structures appear in the limit $\varepsilon \rightarrow 0$.

Stochastic homogenization problems for (5.1.1) have been considered before in the literature. The earliest results were the ones in [PV80] and [MK08]. In [PV80], the evolution version of (5.1.1) (i.e. the linear heat equation) is considered, and it is shown that the corresponding solutions u_ε converge to the analogous evolution version of u_h assuming that ε^{-d} spherical holes are distributed independently according to some density function $V \in C_0^\infty(\mathbb{R}^d)$. All the holes are assumed to have constant and fixed radius $r_j = \varepsilon^{\frac{d}{d-2}}$.

In [MK08], the authors consider spherical hole configurations constituted of ε^{-3} balls in a bounded domain $D \subset \mathbb{R}^3$, selected according to some classes of probability measures in which the balls cannot overlap due to the presence of a hard-sphere potential. These probability measures allow also to have short range correlations between two holes. As in this chapter, also in [MK08], the balls have random size $\varepsilon^{\frac{d}{d-2}}\rho_i$, where the random variables ρ_i are assumed to satisfy

$$\langle \rho_i^{3+\beta} \rangle < \infty \text{ with } \beta > 0. \quad (5.1.6)$$

Under these assumptions, which we further discuss below, it is proved in [MK08] that the solutions of (5.1.1) converge to the solutions of (5.1.3).

Stochastic homogenization for equations related to (5.1.1) and domains D^ε perforated as in (5.1.2) has been considered also more recently in [CM09; CJCDLL16; CJCDLL15]. In [CM09], the authors consider the homogenization limit for the obstacle problem associated to a Dirichlet functional in $D \subset \mathbb{R}^d$, $d \geq 2$, in which the solutions must satisfy $u_\varepsilon \geq 0$ in the collection of small compact sets

$D \setminus D^\varepsilon$. Differently from our setting, the compact sets constituting $D \setminus D^\varepsilon$ are centred on a periodic lattice, but they can have random shapes which are uniformly bounded by $M\varepsilon^{\frac{d}{d-2}}$, for a fixed constant $M > 0$. Assuming an ergodicity condition of the probability measure on the shapes of the sets, the authors of [CM09] prove that the minimizers u_ε converge to the solution u_h of the semi-linear equation $-\Delta u_h + \alpha(u_h)_- = f$ in D .

In [CJCDLL16; CJCDLL15], the main focus is to study the stochastic homogenization of elliptic equations which not only include (5.1.1) but also singular elliptic operators like the p -Laplacian operator. The probability measures considered in these works allow to have hole configurations having random shapes which are uniformly bounded by $\varepsilon^{\frac{d}{d-2}}$. Moreover, a stringent condition is assumed on the probability measure for the positions of the holes to ensure that the minimal distance between the holes is of order ε with probability one.

In all the papers listed above with the exceptions of [MK08], it is assumed that the size of each hole is of order $\varepsilon^{\frac{d}{d-2}}$ with probability one. We emphasize that the main technical difficulty in this chapter is due to the fact that under the sole assumption (5.1.5), namely for distributions of the size of holes having fat tails decreasing slowly enough, there exist, with probability one, domains D^ε with the form (5.1.2) punctured by clusters of two or more overlapping holes. These clusters do not occur (with probability tending to one as $\varepsilon \rightarrow 0$) under the assumption (5.1.6) which is made in [MK08].

In order to prove the homogenization results mentioned above there are different methods in the literature. The first one, which was introduced by Cioranescu and Murat in [CM97], is related to the energy method of Tartar [Tar09]. It is based on the construction of some *oscillating test functions* w_ε . A related approach has been used in the analysis of several deterministic and stochastic homogenization problems (cf. [CM09; CJCDLL16; CJCDLL15; DMG94]) and this is also the approach that will be used in this chapter.

A second approach is based on the construction of suitable projection operators in Hilbert spaces which are defined using the geometry of the perforated domains. This approach was introduced by Marchenko and Khruslov (cf. [MK08] and the references therein). A related but different approach is by the method of reflections that we demonstrated in Chapter 3.

A third approach, used for instance in [PV80], employs the probabilistic interpretation of the solutions u_ε of (5.1.1) (and its evolution analogue), in terms of the properties of the Brownian motion. In particular, the solutions of (5.1.1) as well as the term C_0 arising in the limit equation can be obtained in terms of expectations of functions of the survival time of a Brownian walker among obstacles.

Finally, we also mention that for problems related to (5.1.1), a different approach has been introduced in [Nie99; NV04a; NV04b]. In this series of papers, the main goal is to study a dynamical version of (5.1.1), where the holes evolve according to the function u_ε itself. In this case, the main challenge is thus to obtain estimates for the solution in the space L^∞ instead of the Sobolev space H^1 . The starting point used in [NV04a; NV04b] is an ansatz for the structure of the solution of (5.1.1) which gives rise to an explicit expression for an approximate solution of (5.1.1). The difference between this approximate solution and the solution of (5.1.1) is then estimated using the maximum principle. Stochastic homogenization results have been obtained using this approach in [NV04b], and in the case of solutions of (5.1.1) in unbounded domains, they rely on the study of screening properties [NV06]. Concerning the introduction and the study of such screening phenomena for interacting particles, we also refer to [Nie99] and [NO01].

In the problems of stochastic homogenization, two different types of convergence results are obtained. One approach consists in introducing a probability measure \mathbb{P} on the space of hole

configurations Ω (positions and shapes) in \mathbb{R}^d . The Dirichlet problem (5.1.1) is then solved for each fixed realization $\omega \in \Omega$ in a bounded domain which is obtained by means of (5.1.2). It is then proved that u_ε converges for \mathbb{P} -a.s. as $\varepsilon \rightarrow 0$ to the solutions of (5.1.3). This is the type of results obtained in [CM09; CJCDLL16; CJCDLL15], and also in this chapter.

The second approach consists in creating configurations containing ε^{-d} holes in a bounded domain according to a family of probability measures \mathbb{P}_ε defined on a space of configurations Ω_ε . The homogenization results is thus expressed in terms of convergence in probability, namely that for any $\delta > 0$, $\lim_{\varepsilon \rightarrow 0} \mathbb{P}_\varepsilon(\{\|u_\varepsilon - u\| > \delta\}) = 0$, where $\|\cdot\|$ is a suitable norm. The results obtained in [MK08], [NV04b] and [PV80] are of this type.

5.1.1 Main ideas and organisation of this chapter

As already mentioned in the previous discussion, in this chapter, we focus on probability measures where the radii of the balls in (5.1.2) satisfy merely the minimal condition (5.1.5) on their moments, and the centres of the balls are distributed according to a stationary point process on the whole space. We allow that both the centres and the radii have short-range correlations. This class of measures includes the cases of balls having independent and identically distributed radii and centres either periodic or distributed according to an homogeneous Poisson point process (cf. settings (a) and (b) in the next section). We also give some explicit examples of short-range correlated measures which are constructed starting from clustering or repulsive point processes for the centres of the holes (cf. setting (c) in the next section).

In order to prove the main homogenization result for these measures, we adapt the argument of [CM97] to translate the conditions on the geometry of the holes of D^ε into properties of the associated *oscillating test function* w_ε . These functions account for the presence of the holes in the domain D^ε by correcting any admissible test function $\phi \in C_0^\infty(D)$ for (5.1.3) into an admissible test function $w_\varepsilon \phi \in H_0^1(D^\varepsilon)$ for (5.1.1). The main breakthrough of [CM97] is the formulation of sufficient conditions on w_ε which allow to treat the error terms generated by the presence of w_ε in the weak formulation for (5.1.1). In the limit $\varepsilon \downarrow 0^+$, these errors are the ones giving rise to the additional term $C_0 u_h$ in (5.1.3). In the case of periodic balls in \mathbb{R}^d , the authors in [CM97] explicitly construct the test functions w_ε and obtain (5.1.3) with the value for C_0 given by (5.1.4); from this construction, the link between the term C_0 and the density of capacity generated by the holes becomes apparent and motivates the necessary choice in (5.1.2) of the length-scales $\varepsilon^{\frac{d}{d-2}}$ for the radii and ε for the distance between the centres.

The main challenge in this chapter is that with the sole assumption (5.1.5) on the radii, we need to deal in (almost) all the configurations with the presence of large radii. In spite of the scaling of (5.1.2), the associated big balls may overlap and potentially break down the construction of the functions w_ε . The main idea of our proof is to show that, even though with probability one there are regions where the balls overlap, the moment assumption on the radii is sufficient to ensure that almost surely these regions have a capacity which vanishes in the limit $\varepsilon \downarrow 0^+$. This yields that the contribution of the functions w_ε to the new term in the limit equation is restricted to the region of the domain D^ε where the balls are small and well-separated.

The structure of this chapter is the following: In the next section, we give a precise definition of the processes generating the holes in (5.1.2) and introduce some examples which are included in our setting; we then state the main homogenization result (Theorem 5.2.1). Section 5.3 contains the proof of the theorems provided that the oscillating test functions exist, while Section 5.4 is devoted to the crucial arguments for the construction of such oscillating test functions. Section 5.5 provides some probabilistic results for *marked point processes* on which the previous section relies and which make the arguments of this chapter totally self-contained. To this purpose, we also include a proof of

a Strong Law of Large Numbers which is tailored to the processes that we consider. In the sake of what we think is a more comfortable reading, we do not prove our main result directly for a general probability measure, but we first give the argument in the case of holes with periodic centres and i.i.d. radii. By relying on the abstract results of Section 5.5, we then show how to adapt this proof to a general measure with short-range correlations. This, we believe, gives a more intuitive structure to the arguments of this chapter.

5.2 Setting and main result

Let $D \subset \mathbb{R}^d$, $d > 2$, be an open and bounded set that it is star-shaped with respect to the origin¹. For $\varepsilon > 0$, we denote by $D^\varepsilon \subset D$ the domain obtained by removing from D the closure of a set of “small” holes $H^\varepsilon \subset \mathbb{R}^d$ of the form:

$$H^\varepsilon := \bigcup_{z_j \in \Phi \cap \frac{1}{\varepsilon}D} B_{\varepsilon^{\frac{d}{d-2}} \rho_j}(\varepsilon z_j), \quad (5.2.1)$$

where $\frac{1}{\varepsilon}D := \{x \in \mathbb{R}^d : \varepsilon x \in D\}$, the set $\Phi \subset \mathbb{R}^d$ is a random collection of (countably many) points and the radii $\{\rho_i\}_{z_i \in \Phi} \subset \mathbb{R}^+$ are random variables. The set H^ε may thus be thought as being generated by a marked point process (Φ, \mathcal{R}) on $\mathbb{R}^d \times \mathbb{R}_+$, where Φ is a point process on \mathbb{R}^d for the centres of the balls, and the marks $\mathcal{R} = \{\rho_i\}_{z_i \in \Phi} \subset \mathbb{R}_+$ are the radii associated to each centre. We refer to [DVJ08, Chapter 9, Definitions 9.1.I - 9.1.IV] for a rigorous definition of marked point processes as a class of random measures on $\mathbb{R}^d \times \mathbb{R}_+$. We remark indeed that there is a one-to-one correspondence between representing each realisation of the process as a collection of points and radii $\{(z_i, \rho_i)\}_{i \in \mathbb{N}} \subset \mathbb{R}^d \times \mathbb{R}_+$ as we do in this chapter, and as the atomic measure $\mu := \sum_{i \in \mathbb{N}} \delta_{(z_i, \rho_i)}$ on $\mathbb{R}^d \times \mathbb{R}_+$. We also note that both the previous representations are invariant under permutation of the indices $i \in \mathbb{N}$ and thus that there is no preferred ordering of the centres of the balls generating H^ε .

We denote by $(\Omega, \mathcal{F}, \mathbb{P})$ the probability space associated to the process (Φ, \mathcal{R}) and, for every $\omega \in \Omega$, we write $H^\varepsilon(\omega)$ for the set defined in (5.2.1) with $(\Phi, \mathcal{R})(\omega)$. Throughout this chapter, we assume that (Φ, \mathcal{R}) satisfies the the following properties:

- The process Φ is stationary: For every $x \in \mathbb{R}^d$ we have $\tau_x \circ \Phi \stackrel{\mathcal{L}}{=} \Phi$, where for each $\{z_i\}_{i \in \mathbb{N}} \subset \mathbb{R}^d$ the translations are defined as

$$\tau_x(\{z_i\}_{i \in \mathbb{N}}) = \{z_i + x\}_{i \in \mathbb{N}}.$$

- There exists $\lambda < +\infty$ such that for any unitary cube $Q \subset \mathbb{R}^d$

$$\langle \#(\Phi \cap Q)^2 \rangle^{\frac{1}{2}} \leq \lambda, \quad (5.2.2)$$

where $\#(S) \in \mathbb{N} \cup \infty$ denotes the cardinality of a set S and $\langle \cdot \rangle$ is the integration over Ω with respect to the measure \mathbb{P} . Note that, by stationarity of Φ , the left-hand side of (5.2.2) does not depend on the position of Q .

¹This assumption ensures that the sets in the family $\{\frac{1}{\varepsilon}D\}_{\varepsilon>0}$ (see definition after formula (5.2.1)) are nested. This is not a necessary condition for our results to hold, but it avoids some technicalities in our proof and it keeps our arguments and our notation leaner.

- The point process Φ satisfies a *strong mixing condition*: For any bounded Borel set $A \subset \mathbb{R}^d$, let $\mathcal{F}(A)$ be the smallest σ -algebra with respect to which the random variables $N(B)(\omega) := \#(\Phi \cap B)$ are measurable for every Borel set $B \subset A$. Then, there exist $C_1 < +\infty$ and $\gamma > d$ such that for every $A \subset \mathbb{R}^d$ as above, every $x \in \mathbb{R}^d$ with $|x| > \text{diam}(A)$ and every ξ_1, ξ_2 measurable with respect to $\mathcal{F}(A)$ and $\mathcal{F}(\tau_x A)$, respectively, we have

$$|\langle \xi_1 \xi_2 \rangle - \langle \xi_1 \rangle \langle \xi_2 \rangle| \leq \frac{C_1}{1 + (|x| - \text{diam}(A))^\gamma} \langle \xi_1^2 \rangle^{\frac{1}{2}} \langle \xi_2^2 \rangle^{\frac{1}{2}}. \quad (5.2.3)$$

- The marginal $\mathbb{P}_{\mathcal{R}}$ of the marks with respect to the process Φ has 1- and 2- correlation functions

$$\begin{aligned} f_1((z, \rho)) &= h(\rho), \\ f_2(z_i, \rho_i, z_j, \rho_j) &= h(\rho_i)h(\rho_j) + g(|z_i - z_j|, \rho_i, \rho_j) \quad \forall i \neq j \end{aligned} \quad (5.2.4)$$

with

$$\int \rho^{d-2} h(\rho) d\rho < +\infty \quad |g(r, \rho_1, \rho_2)| \leq \frac{c}{(1 + r^\gamma)(1 + \rho_1^p)(1 + \rho_2^p)} \quad (5.2.5)$$

for $p > d - 1, \gamma > d$ and $c \in \mathbb{R}_+$.

The previous assumptions imply that the $(d - 2)$ -moment of the radii of the balls is finite and that, conditioned to the positions of the centres, the radii for two balls with centres in z_1, z_2 have correlations which vanish when the distance $|z_1 - z_2| \rightarrow +\infty$. These correlations are short-range in the sense that the function g above is integrable in the variable $r := |z_1 - z_2|$.

Throughout this chapter, we denote $D^\varepsilon(\omega) := D \setminus H^\varepsilon(\omega)$ with $H^\varepsilon(\omega)$ as in (5.2.1), and we identify any $v \in H_0^1(D^\varepsilon(\omega))$ with the function $\tilde{v} \in H_0^1(D)$ obtained by extending v as $v \equiv 0$ in $H^\varepsilon(\omega)$. Then we have:

Theorem 5.2.1. *Let the holes in (5.2.1) be generated by a marked point process (Φ, \mathcal{R}) . Let Φ satisfy (5.2.2) and (5.2.3), and let the marginal $\mathbb{P}_{\mathcal{R}}$ satisfy (5.2.4) and (5.2.5). For $f \in H^{-1}(D)$ and $\varepsilon > 0$, let $u_\varepsilon = u_\varepsilon(\omega, \cdot) \in H_0^1(D^\varepsilon(\omega))$ solve (5.1.1). Then, there exist a constant $C_0 > 0$ and $u_h \in H_0^1(D)$ solving (5.1.3) such that for \mathbb{P} -almost every $\omega \in \Omega$*

$$u_\varepsilon(\omega, \cdot) \rightharpoonup u_h \quad \text{in } H_0^1(D), \quad \text{for } \varepsilon \downarrow 0^+.$$

Moreover, we have that the constant C_0 in (5.1.3) is defined as

$$C_0 = (d - 2)\sigma_d \langle N(Q) \rangle \langle \rho^{d-2} \rangle, \quad (5.2.6)$$

where $\sigma_d = \mathcal{H}^{d-1}(S^{d-1})$ and $N(Q)$ is the number of centres falling into any fixed unitary cube Q .

5.2.1 Some examples of processes generating the holes H^ε .

Among the processes which satisfy the conditions required in the previous theorem, we mention the following three examples: For the first two examples, it is immediate that (5.2.2), (5.2.3), (5.2.4) and (5.2.5) are satisfied. Also in the case of the examples given in (c), the previous conditions are satisfied and this follows by easy calculations that we postpone to Section 5.5.3.

- (a) *The set H^ε is a collection of balls with periodic centres and i.i.d. radii.*
 Here, the centres have deterministic positions $\Phi = \mathbb{Z}^d$ and the marks \mathcal{R} for the radii are a family of independent and identically distributed random variables which satisfy (5.1.5). In this case, we have that the constant C_0 in (5.2.6) reads $C_0 = (d-2)\sigma_d \langle \rho^{d-2} \rangle$.
- (b) *The set H^ε is a collection of balls with centres generated by a Poisson point process and i.i.d. radii.* The process Φ is an homogeneous Poisson point process, and, conditioned to Φ , the marks \mathcal{R} are as in case (a). In this case, we have that $C_0 = (d-2)\sigma_d \lambda \langle \rho^{d-2} \rangle$ with $\lambda > 0$ being the intensity of the Poisson point process Φ .
- (c) *The balls of H^ε have correlated radii and centres generated by a clustering or repulsive point process.* The process Φ is an attractive or a repulsive point processes with short-range correlations, respectively:

- (c.1) Neymann-Scott cluster process on \mathbb{R}^d (see, e.g. [DVJ03, Example 6.3]): Let $(\Phi_1, \{r_i\}_{i \in \Phi_1})$ be a marked point process where Φ_1 is a homogeneous Poisson point process and the marks are i.i.d. and uniformly distributed on $(0, R_c)$, with $0 < R_c < +\infty$. For $\lambda_2 \in L^\infty(\mathbb{R}^d)$ and $x \in \mathbb{R}^d$, let Φ_2^x be the heterogeneous Poisson point process having intensity $\lambda_2(\cdot - x)$. Then, we define

$$\Phi := \bigcup_{z_i \in \Phi_1} \Phi_2^{z_i} \cap B_{r_i}(z_i). \quad (5.2.7)$$

- (c.2) Strauss process Φ on \mathbb{R}^d with parameters $\alpha > 0$, $\beta \in [0, 1]$ and interaction distance $r_c > 0$ [DVJ03, Example 7.1(c)], [KR76]. For each bounded Borel set $B \subset \mathbb{R}^d$, we define

$$\mathbb{P}(\#(\Phi \cap B) = n) = Z_B^{-1} \frac{\alpha^n}{n!} \int_{B \times \dots \times B} \beta^{R(\{x_1, \dots, x_n\})} dx_1 \dots dx_n,$$

with

$$R(\{x_1, \dots, x_n\}) := \frac{1}{2} \sum_{\substack{i,j \\ i \neq j}}^n \mathbf{1}_{[0, r_c]}(|x_i - x_j|),$$

and

$$Z_B = \sum_{n=0}^{+\infty} \frac{\alpha^n}{n!} \int_{B \times \dots \times B} \beta^{R(\{x_1, \dots, x_n\})} dx_1 \dots dx_n.$$

This probability measure is well-defined in the repulsive case $\beta \in [0, 1)$, while it requires further assumptions in the attractive case $\beta \geq 1$. For $\beta = 0$, we remark that Φ is the hard-core process with radius r_c and intensity α (see also [DVJ03, Example 5.3(c)] and [KR76]). We remark that this process is the same as the macrocanonical Gibbs ensemble of Statistical Physics at temperature $T = 1$ for the pair-interaction potential

$$\psi(r) = \begin{cases} -\log \beta & \text{if } r \leq r_c \\ 0 & \text{if } r > r_c \end{cases}$$

and the chemical potential $\mu = \log \alpha$.

For each one of the previous point processes Φ , we let the marginal $\mathbb{P}_{\mathcal{R}}$ be any probability measure satisfying (5.2.4) and (5.2.5).

5.3 Proof of Theorem 5.2.1

As already discussed in the introduction, our strategy is to adapt the method of [CM97] and to show that, in spite of the unboundedness of the radii of the holes in H^ε , we may almost surely construct a sequence of suitable oscillating test function. The crucial result of this chapter is indeed the following:

Lemma 5.3.1. *Let $H^\varepsilon = H^\varepsilon(\omega)$ be as in Theorem 5.2.1. Then, for \mathbb{P} -almost every $\omega \in \Omega$, there exists a sequence $\{w_\varepsilon(\omega, \cdot)\}_{\varepsilon>0} \subset H^1(D)$ which satisfies*

(H1) *For every $\varepsilon > 0$, $w_\varepsilon(\omega, \cdot) = 0$ in $H^\varepsilon(\omega)$;*

(H2) *$w_\varepsilon(\omega, \cdot) \rightharpoonup 1$ in $H^1(D)$ for $\varepsilon \downarrow 0^+$;*

(H3) *For every sequence $v_\varepsilon \rightharpoonup v$ in $H_0^1(D)$ such that $v_\varepsilon \in H_0^1(D^\varepsilon)$ it holds that*

$$(-\Delta w_\varepsilon(\omega, \cdot), v_\varepsilon)_{H^{-1}, H_0^1} \rightarrow C_0 \langle \rho^{d-2} \rangle \int_D v$$

for $\varepsilon \downarrow 0^+$ and where C_0 defined as in Theorem 5.2.1.

By relying on the previous lemma, the proof of Theorem 5.2.1 follows exactly as in [CM97]:

Proof of Theorem 5.2.1. Let $\omega \in \Omega$ be fixed, and let it belong to the full-probability set $\Omega' \subset \Omega$ made of configurations for which, according to Lemma 5.3.1, the functions $\{w_\varepsilon\}_{\varepsilon>0} := \{w_\varepsilon(\omega, \cdot)\}_{\varepsilon>0}$ exist and satisfy hypothesis (H1), (H2) and (H3).

Since $u_\varepsilon \in H_0^1(D^\varepsilon)$, we may test equation (5.1.1) with u_ε itself and get by the standard energy estimate

$$\|u_\varepsilon\|_{H^1} \leq C \|f\|_{H^{-1}},$$

with a constant C that depends only on the domain D . By weak-compactness of $H_0^1(D)$, we infer that, up to a subsequence which may depend on ω ,

$$u_\varepsilon \rightharpoonup u_h \quad \text{in } H_0^1(D) \text{ for } \varepsilon \downarrow 0^+. \quad (5.3.1)$$

We show that $u_h \in H_0^1(D)$ is the solution of (5.1.3); by uniqueness, this extends the weak convergence of the solutions u_ε to the continuum limit $\varepsilon \downarrow 0$ and concludes the proof of the theorem.

To prove that u_h solves (5.1.3), let us fix any function $\phi \in C_0^\infty(D)$. Since (H1) yields $w_\varepsilon \phi \in H_0^1(D^\varepsilon)$, we can test the equation (5.1.1) with $w_\varepsilon \phi$ and obtain

$$\int \nabla(w_\varepsilon \phi) \cdot \nabla u_\varepsilon = (f, w_\varepsilon \phi)_{H^{-1}, H_0^1}. \quad (5.3.2)$$

By (H2), the right-hand side above converges to

$$(f, w_\varepsilon \phi)_{H^{-1}, H_0^1} \rightarrow (f, \phi)_{H^{-1}, H_0^1}. \quad (5.3.3)$$

We now use the product-rule and an integration by parts to rewrite the left-hand side in (5.3.2) as

$$\begin{aligned} \int \nabla(w_\varepsilon \phi) \cdot \nabla u_\varepsilon &= \int \phi \nabla w_\varepsilon \cdot \nabla u_\varepsilon + \int w_\varepsilon \nabla \phi \cdot \nabla u_\varepsilon \\ &= (-\Delta w_\varepsilon, \phi u_\varepsilon)_{H^{-1}, H_0^1} - \int u_\varepsilon \nabla w_\varepsilon \cdot \nabla \phi + \int w_\varepsilon \nabla \phi \cdot \nabla u_\varepsilon. \end{aligned}$$

Since by (H2) of Lemma 5.3.1 and (5.3.1), both u_ε and w_ε converge strongly in $L^2_{\text{loc}}(D)$, the last two terms on the right-hand side above converge to $\int \nabla \phi \cdot \nabla u_h$. Furthermore, by (5.3.1) and the assumption on ϕ , we apply hypothesis (H3) of Lemma 5.3.1 to the first term on the right-hand side above and conclude that

$$\int \nabla(w_\varepsilon \phi) \cdot \nabla u_\varepsilon \rightarrow C_0 \int \phi u_h + \int \nabla \phi \cdot \nabla u_h.$$

This, together with (5.3.2), (5.3.3) and the arbitrariness of $\phi \in C_0^\infty(D)$, yields that u_h weakly solves (5.1.3). The proof of Theorem 5.2.1 is complete. \square

5.4 Existence of the oscillating test functions (Proof of Lemma 5.3.1)

As already mentioned in Subsection 5.1.1, we proceed to prove Lemma 5.3.1 in two steps: We first give an argument in the simplest case of random holes H^ε having periodic centres and i.i.d. radii (cf. example (a) of Section 5.2). In that case, the crucial role played by assumption (5.1.5) on the random geometry of the set H^ε becomes clear. We then generalize this argument to an arbitrary process (Φ, \mathcal{R}) that satisfies the assumptions of Theorem 5.2.1. We observe that, as it becomes apparent in the proofs of this section, the full-probability set of realizations for which the statement of Lemma 5.2.1 holds true is selected by countable repeated applications of Strong Laws of Large Numbers-type of results. The final set $\Omega' \subset \Omega$ in which we prove the existence of the oscillating test functions is thus a countable intersection of full-probability sets and remains of full probability.

Before giving the proof of Lemma 5.3.1, we fix the following notation: For any two open sets $A \subset B \subset \mathbb{R}^d$, we define

$$\text{Cap}(A, B) := \inf \left\{ \int |\nabla v|^2 : v \in C_0^\infty(B), v \geq \mathbf{1}_A \right\}. \quad (5.4.1)$$

For a point process Φ on \mathbb{R}^d and any bounded set $E \subset \mathbb{R}^d$, we define the random variables

$$\begin{aligned} \Phi(E) &:= \Phi \cap E, & \Phi^\varepsilon(E) &:= \Phi \cap \left(\frac{1}{\varepsilon} E \right), \\ N(E) &:= \#(\Phi(E)), & N^\varepsilon(E) &:= \#(\Phi^\varepsilon(E)). \end{aligned} \quad (5.4.2)$$

For $\delta > 0$, we denote by Φ_δ a thinning for the process Φ obtained as

$$\Phi_\delta(\omega) := \{x \in \Phi(\omega) : \min_{\substack{y \in \Phi(\omega), \\ y \neq x}} |x - y| \geq \delta\}, \quad (5.4.3)$$

i.e. the points of $\Phi(\omega)$ whose minimal distance from the other points is at least δ . Given the process Φ_δ , we set $\Phi_\delta(E)$, $\Phi_\delta^\varepsilon(E)$, $N_\delta(E)$ and $N_\delta^\varepsilon(E)$ for the analogues for Φ_δ of the random variables defined in (5.4.2).

For a fixed $M > 0$, we define the truncated marks

$$\mathcal{R}^M := \{\rho_{j,M}\}_{z_j \in \Phi}, \quad \rho_{j,M} := \rho_j \wedge M. \quad (5.4.4)$$

Furthermore, throughout the proofs, we write

$$a \lesssim b$$

whenever $a \leq Cb$ for a constant $C = C(d)$ depending only on the dimension d .

Finally, we remark that, under the assumptions of the process (Φ, \mathcal{R}) in Theorem 5.2.1, the process $(\Phi, \{\rho^{d-2}\}_{z_i \in \mathbb{Z}^d})$ satisfies the assumptions of Section 5.5, and we therefore may apply all the results stated in that section.

5.4.1 Case (a): Periodic centres

In this setting, the holes H^ε are generated by $\Phi = \mathbb{Z}^d$ and a collection of i.i.d. random variables $\{\rho_i\}_{i \in \mathbb{Z}^d}$ satisfying (5.1.5). It is immediate to check that the marked process

Since the centres of the holes are periodically distributed, the only challenge in the construction of the functions w_ε of Lemma 5.3.1 is due to the random variables $\{\rho_i\}_{i \in \mathbb{Z}^d}$ which might generate very large holes under the mere condition (5.1.5). In fact, in [CM97] the construction of w_ε relies on the assumption that each hole $B_{\varepsilon^{\frac{d}{d-2}}}(\varepsilon z_i)$, $z_i \in \mathbb{Z}^d$, is strictly contained in the concentric cube of size ε ; this allows to explicitly construct w^ε by locally solving a PDE on each of these cubes. In our case, the sole assumption (5.1.5) does not exclude that there are big holes which overlap and where the previous construction breaks down. The main auxiliary result on which Lemma 5.3.1 for the periodic case (a) relies is the following Lemma 5.4.1 on the asymptotic geometry of the set H^ε . Roughly speaking, this lemma ensures that H^ε may be almost surely partitioned into two subsets, a “good” and a “bad” set of holes which we denote by H_g^ε and H_b^ε , respectively. The set H_g^ε contains most of the holes of H^ε and is made of small balls where the construction of w_ε may be carried out similarly to [CM97]. The remaining holes, some of which overlap with full probability, are all included in H_b^ε . This set is well separated from H_g^ε and small with respect to the macroscopic size of the domain D : We may indeed enclose H_b^ε into a set $D_b^\varepsilon \subset D$ which is still separated from H_g^ε and such that the harmonic capacity of H_b^ε with respect to this “safety layer” D_b^ε vanishes in the limit $\varepsilon \downarrow 0^+$. This allows us to implicitly define w^ε in D_b^ε as the capacitary function of H_b^ε in D_b^ε ; this choice ensures that the H^1 -norm of w_ε on D_b^ε converges to zero. Hence, in the verification of (H2) and (H3) of Lemma 5.3.1, we only need to focus on the construction of w_ε on $D \setminus D_b^\varepsilon$.

Lemma 5.4.1. *Let $\delta \in (0, \frac{2}{d-2})$ be fixed. Then, there exists $\varepsilon_0 = \varepsilon_0(\delta) > 0$ such that for \mathbb{P} -almost every $\omega \in \Omega$ and for all $\varepsilon \leq \varepsilon_0$ there exist $H_g^\varepsilon(\omega), H_b^\varepsilon(\omega), D_b^\varepsilon(\omega) \subset \mathbb{R}^d$ such that*

$$\begin{aligned} H^\varepsilon(\omega) &= H_g^\varepsilon(\omega) \cup H_b^\varepsilon(\omega), & H_b^\varepsilon(\omega) &\subset D_b^\varepsilon(\omega), \\ \text{dist}(H_g^\varepsilon(\omega), D_b^\varepsilon(\omega)) &\geq \frac{\varepsilon}{2}, \end{aligned} \quad (5.4.5)$$

where

$$\lim_{\varepsilon \downarrow 0^+} \text{Cap}(H_b^\varepsilon(\omega), D_b^\varepsilon(\omega)) = 0. \quad (5.4.6)$$

Moreover, $H_g^\varepsilon(\omega)$ may be written as the following union of disjoint balls centred in $n^\varepsilon(\omega) \subset \mathbb{Z}^d \cap \frac{1}{\varepsilon}D$:

$$H_g^\varepsilon(\omega) := \bigcup_{z_j \in n^\varepsilon} B_{\varepsilon^{\frac{d}{d-2}} \rho_j}(\varepsilon z_j), \quad (5.4.7)$$

$$\varepsilon^{\frac{d}{d-2}} \rho_j \leq \varepsilon^{1+\delta} < \frac{\varepsilon}{2}, \quad \lim_{\varepsilon \downarrow 0^+} \varepsilon^d \#(n^\varepsilon) = |D|. \quad (5.4.8)$$

Proof of Lemma 5.4.1. The partition of the set $H^\varepsilon(\omega)$ in the statement of the lemma clearly depends on the realization $\omega \in \Omega$; in the sake of a leaner notation, though, in the rest of the proof we omit the argument ω and write $H^\varepsilon, H_b^\varepsilon, H_g^\varepsilon$ instead of $H^\varepsilon(\omega), H_b^\varepsilon(\omega), H_g^\varepsilon(\omega)$. For each $z_i \in \mathbb{Z}^d$, we denote by Q_i^ε the cube of length ε centered at εz_i .

We begin by constructing the set H_b^ε and its “safety layer” D_b^ε . We first include in H_b^ε the particles which are large compared to the size of the cubes Q_i^ε : For δ as in the statement of the lemma, we consider the subset of \mathbb{Z}^d given by

$$J_\varepsilon^b := \left\{ z_i \in \mathbb{Z}^d \cap \frac{1}{\varepsilon}D : \varepsilon^{\frac{d}{d-2}} \rho_j \geq \varepsilon^{1+\delta} \right\}, \quad (5.4.9)$$

and the corresponding union of balls

$$\tilde{H}_b^\varepsilon := \bigcup_{z_j \in J_b^\varepsilon} B_{2\varepsilon^{\frac{d}{d-2}}\rho_j}(\varepsilon z_j).$$

We now extend J_b^ε by including the centres of the balls for which, independently from the size of their radius, the corresponding cell Q_i^ε intersects \tilde{H}_b^ε : We define

$$\tilde{I}_b^\varepsilon := \{z_i \in \mathbb{Z}^d : Q_i^\varepsilon \cap \tilde{H}_b^\varepsilon \neq \emptyset\} \supset J_b^\varepsilon, \quad I_b^\varepsilon := \tilde{I}_b^\varepsilon \cap \frac{1}{\varepsilon}D. \quad (5.4.10)$$

We finally set

$$H_b^\varepsilon := \bigcup_{z_j \in I_b^\varepsilon} B_{\varepsilon^{\frac{d}{d-2}}\rho_j}(\varepsilon z_j), \quad D_b^\varepsilon := \bigcup_{z_j \in \tilde{I}_b^\varepsilon} Q_j^\varepsilon. \quad (5.4.11)$$

By (5.4.11) it is immediate that $H_b^\varepsilon \subset D_b^\varepsilon$. To show (5.4.6) we first argue that provided $\varepsilon \leq \varepsilon_0(\delta)$, with $\varepsilon_0(\delta)$ such that $2\varepsilon_0^{1+\delta} \leq \varepsilon_0$, for every $z_j \in I_b^\varepsilon$ it holds

$$B_{2\varepsilon^{\frac{d}{d-2}}}(\varepsilon z_j) \subset D_b^\varepsilon. \quad (5.4.12)$$

Indeed, since by definition $\tilde{H}_b^\varepsilon \subset D_b^\varepsilon$, if $z_j \in J_b^\varepsilon$, then (5.4.12) follows immediately. If, otherwise, $z_j \in I_b^\varepsilon \setminus J_b^\varepsilon$, then the assumption $\varepsilon \leq \varepsilon_0$ implies $B_{2\varepsilon^{\frac{d}{d-2}}}(\varepsilon z_j) \subset Q_j^\varepsilon \subset D_b^\varepsilon$. By the subadditivity of the capacity (see definition (5.4.1)) we estimate

$$\begin{aligned} \text{Cap}((H_b^\varepsilon(\omega), D_b^\varepsilon(\omega))) &\stackrel{(5.4.11)}{\leq} \sum_{j \in I_b^\varepsilon} \text{Cap}(B_{\varepsilon^{\frac{d}{d-2}}\rho_j}(x_j), D_b^\varepsilon(\omega)) \\ &\stackrel{(5.4.12)}{\leq} \sum_{j \in I_b^\varepsilon} \text{Cap}(B_{\varepsilon^{\frac{d}{d-2}}\rho_j}(x_j), B_{2\varepsilon^{\frac{d}{d-2}}\rho_j}(x_j)) \\ &\lesssim \sum_{j \in I_b^\varepsilon} \varepsilon^d \rho_j^{d-2}. \end{aligned}$$

To conclude (5.4.6), it remains to show that the right-hand side above vanishes almost surely in the limit $\varepsilon \downarrow 0^+$. This follows from Lemma 5.5.3 for the process $(\mathbb{Z}^d, \{\rho_i^{d-2}\}_{z_i \in \mathbb{Z}^d})$ provided

$$\lim_{\varepsilon \downarrow 0^+} \varepsilon^d \#(I_b^\varepsilon) = 0. \quad (5.4.13)$$

To show (5.4.13), we first bound by (5.4.10)

$$\varepsilon^d \#(I_b^\varepsilon) \leq \varepsilon^d \#(J_b^\varepsilon) + \sum_{z_i \in I_b^\varepsilon \setminus J_b^\varepsilon} |Q_i^\varepsilon|.$$

We note that by (5.4.10) and (5.4.9), there exists a constant $c = c(d)$ such that, provided $\varepsilon \leq \varepsilon_0(d)$ (with $\varepsilon_0(d)$ possibly smaller than the one above), for any cube Q_i^ε with $z_i \in I_b^\varepsilon$, there exists $z_j \in J_b^\varepsilon$ such that

$$Q_i^\varepsilon \subset B_{2c\varepsilon^{\frac{d}{d-2}}\rho_j}(\varepsilon z_j).$$

Since all the cubes Q_i^ε are (essentially) disjoint, we use the previous inclusion and the definition of \tilde{H}_b^ε to bound

$$\begin{aligned} \varepsilon^d \#(I_b^\varepsilon) &\lesssim \varepsilon^d \#(J_b^\varepsilon) + |\tilde{H}_b^\varepsilon| \lesssim \varepsilon^d \#(J_b^\varepsilon) + \sum_{z_j \in J_b^\varepsilon} \left(\varepsilon^{\frac{d}{d-2}} \rho_j \right)^d \\ &\lesssim \varepsilon^d \#(J_b^\varepsilon) + \left(\varepsilon^{\frac{d}{d-2}} \max_{z_j \in \frac{1}{\varepsilon} D \cap \mathbb{Z}^d} \rho_j \right)^2 \varepsilon^d \sum_{z_j \in J_b^\varepsilon} \rho_j^{d-2}. \end{aligned}$$

By Lemma 5.5.2, we have almost surely

$$\limsup_{\varepsilon \downarrow 0^+} \varepsilon^{\frac{d}{d-2}} \max_{z_j \in \frac{1}{\varepsilon} D \cap \mathbb{Z}^d} \rho_j \leq \lim_{\varepsilon \downarrow 0^+} \left(\varepsilon^d \sum_{z_j \in \frac{1}{\varepsilon} D \cap \mathbb{Z}^d} \rho_j^{d-2} \right)^{\frac{1}{d-2}} = \langle \rho^{d-2} \rangle^{\frac{1}{d-2}}, \quad (5.4.14)$$

and thus estimate for ε small enough (this time depending on ω)

$$\varepsilon^d \#(I_b^\varepsilon) \lesssim \varepsilon^d \#(J_b^\varepsilon) + \langle \rho^{d-2} \rangle^{\frac{2}{d-2}} \sum_{z_j \in J_b^\varepsilon} \left(\varepsilon^{\frac{d}{d-2}} \rho_j \right)^{d-2}. \quad (5.4.15)$$

The first term on right-hand side above tends to zero thanks to

$$\varepsilon^d \#(J_b^\varepsilon) = \varepsilon^d \sum_{z_j \in \frac{1}{\varepsilon} D \cap \mathbb{Z}^d} \mathbf{1}_{\varepsilon^{\frac{d}{d-2}} \rho_j \geq \varepsilon^{1+\delta}} \leq \varepsilon^{2-\delta(d-2)} \varepsilon^d \sum_{\frac{1}{\varepsilon} D \cap \mathbb{Z}^d} \rho_j^{d-2}$$

and the choice $\delta < \frac{2}{d-2}$ together with the right-hand side of (5.4.14). By this estimate and Lemma 5.5.3, also the second term on the right-hand side of (5.4.15) vanishes almost surely in the limit $\varepsilon \downarrow 0^+$. We thus established (5.4.13) and therefore also (5.4.6).

We now define $H_g^\varepsilon := H^\varepsilon \setminus H_b^\varepsilon$, which allows to write H_g^ε as in (5.4.7) with $n_\varepsilon = (\mathbb{Z}^d \cap \frac{1}{\varepsilon} D) \setminus I_b^\varepsilon$. The first property in (5.4.8) is immediately implied by $J_b^\varepsilon \subset I_b^\varepsilon$ and (5.4.9). The second property in (5.4.8) follows from (5.4.13).

It remains to prove the last inequality in (5.4.5): By the definition of H_g^ε itself, if $B_{\varepsilon^{\frac{d}{d-2}} \rho_j}(\varepsilon z_j) \subset H_g^\varepsilon$, then $\varepsilon^{\frac{d}{d-2}} \rho_j \leq \varepsilon^{1+\delta}$. We choose $\varepsilon \leq \varepsilon_0(\delta)$ as in (5.4.12), such that $B_{\varepsilon^{\frac{d}{d-2}} \rho_j}(\varepsilon z_j) \subset Q_j^\varepsilon$ and

$$\frac{\varepsilon}{2} \leq \text{dist}\left(B_{\varepsilon^{\frac{d}{d-2}} \rho_j}(\varepsilon z_j), \partial Q_j^\varepsilon\right) \stackrel{(5.4.10)}{\leq} \text{dist}\left(B_{\varepsilon^{\frac{d}{d-2}} \rho_j}(\varepsilon z_j), D_b^\varepsilon\right).$$

This concludes the proof of Lemma 5.4.1. □

Proof of Lemma 5.3.1, case (a). Let us fix δ and $\varepsilon_0(\delta)$ as in the statement of Lemma 5.4.1. Then, we know that we may fix \mathbb{P} -almost any event $\omega \in \Omega$ such that we find $H_b^\varepsilon(\omega)$, $H_g^\varepsilon(\omega)$ and $D_b^\varepsilon(\omega)$ as in Lemma 5.4.1. Also in this proof, to keep the notation leaner, we omit the argument ω in the oscillating test functions and in the set of holes and write, for instance, w_ε , H^ε instead of $w_\varepsilon(\omega, \cdot)$ and $H^\varepsilon(\omega)$.

Step 1. We begin by a reduction argument: We claim that we may separately treat the two regions D_b^ε and $D \setminus D_b^\varepsilon$, which contain H_b^ε and H_g^ε respectively, and give an explicit construction for

w^ε only in the set $D \setminus D_b^\varepsilon$. We indeed claim that, for \mathbb{P} -almost every $\omega \in \Omega$ we may set $w^\varepsilon = w_1^\varepsilon \wedge w_2^\varepsilon$ with $w_1, w_2 \in H^1(D)$ and such that

$$w_1^\varepsilon \equiv 1 \quad \text{in } D \setminus D_b^\varepsilon, \quad w_1^\varepsilon = 0 \quad \text{in } H_b^\varepsilon, \quad (5.4.16)$$

$$0 \leq w_2^\varepsilon \leq 1, \quad w_2^\varepsilon \equiv 1 \quad \text{in } D_b^\varepsilon, \quad w_2^\varepsilon = 0 \quad \text{in } H_g^\varepsilon, \quad (5.4.17)$$

with, in addition,

$$w_1^\varepsilon \rightarrow 1 \quad \text{in } H^1(D). \quad (5.4.18)$$

If true, this decomposition for w^ε yields that (H1) is satisfied and, by (5.4.18), that (H2) needs to be argued only for the sequence $\{w_2^\varepsilon\}_{\varepsilon>0}$. Finally, since ∇w_1^ε and ∇w_2^ε have disjoint support, for any sequence $\{v_\varepsilon\}_{\varepsilon>0} \subset H^1(D)$ as in (H3) we have that

$$(-\Delta w^\varepsilon, v_\varepsilon)_{H^{-1}, H_0^1} = \int \nabla w_1^\varepsilon \cdot \nabla v_\varepsilon + \int \nabla w_2^\varepsilon \cdot \nabla v_\varepsilon,$$

and, by (5.4.18), that the first term on the right-hand side vanishes in the limit $\varepsilon \downarrow 0^+$. Since by an integration by parts, the second term on the right-hand side may be rewritten as $(-\Delta w_2^\varepsilon, v_\varepsilon)_{H^{-1}, H_0^1}$, we deduce that with the previous decomposition we may verify (H3) only for the measures $\{-\Delta w_2^\varepsilon\}_{\varepsilon>0}$.

Step 2. Construction of w_1^ε and w_2^ε . We begin with w_1^ε : Thanks to (5.4.5) of Lemma 5.4.1 for $H_b^\varepsilon, H_g^\varepsilon$ and D_b^ε , together with (5.4.1) and (5.4.6), for every $\varepsilon \leq \varepsilon_0$ there exists a function $\tilde{w}_1^\varepsilon \in H_0^1(D_b^\varepsilon)$, such that $\tilde{w}_1^\varepsilon = 1$ in H_b^ε , which satisfies

$$\int_{D_b^\varepsilon} |\nabla \tilde{w}_1^\varepsilon|^2 \leq 2 \text{Cap}(H_b^\varepsilon, D_b^\varepsilon).$$

If we now set $w_1^\varepsilon = 1 - \tilde{w}_1^\varepsilon$, and trivially extend w_1^ε by 1 outside D_b^ε , we immediately have that (5.4.16) for w_1^ε is satisfied. In addition, thanks to (5.4.6) and our choice of \tilde{w}_1^ε , also (5.4.18) follows.

We now turn to the construction of w_2^ε : By the properties of $H_g^\varepsilon, H_b^\varepsilon$ and D_b^ε of Lemma 5.4.1, the set $D \setminus D_b^\varepsilon$ contains only the holes of H_g^ε , which are all disjoint balls, each strictly contained in the concentric cube Q_i^ε of size ε . We define $w_2^\varepsilon \equiv 1$ on D_b^ε , and explicitly construct w_2^ε on $D \setminus D_b^\varepsilon$ as done in [CM97]: For each $z_i \in n^\varepsilon$, with n^ε defined in the statement of Lemma 5.4.1, we write $T_i^\varepsilon = B_{\frac{\varepsilon}{\rho_i}}(\varepsilon z_i)$ and $B_i = B_{\frac{\varepsilon}{2}}(\varepsilon z_i)$ and define

$$w_2^\varepsilon = 1 - \sum_{z_i \in n^\varepsilon} w_2^{\varepsilon, i}, \quad (5.4.19)$$

with each $w_2^{\varepsilon, i}$ solving

$$\begin{cases} -\Delta w_2^{\varepsilon, i} = 0 & \text{in } B_i \setminus T_i \\ w_2^{\varepsilon, i} = 1 & \text{in } T_i \\ w_2^{\varepsilon, i} = 0 & \text{in } D \setminus B_i. \end{cases} \quad (5.4.20)$$

Since by Lemma 5.4.1 all the balls B_i are disjoint and contained in $D \setminus D_b^\varepsilon$, definitions (5.4.19) and (5.4.20) yield that w_2^ε satisfies (5.4.17) of Step 1. We thus constructed $w_1^\varepsilon, w_2^\varepsilon$ satisfying (5.4.17) and (5.4.18) of Step 1. We conclude this step by remarking that definition (5.4.20) also implies that

$$w_2^{\varepsilon, i} = 1 - \text{argmin}\{\text{Cap}(T_i^\varepsilon, B_i^\varepsilon)\},$$

and that each $w_2^{\varepsilon,i}$ may be written explicitly as

$$\begin{cases} w_2^{\varepsilon,i}(x) &= \frac{|x-\varepsilon z_i|^{-(d-2)-(\frac{\varepsilon}{2})-(d-2)}}{\varepsilon^{-d}\rho_i^{-(d-2)-(\frac{\varepsilon}{2})-(d-2)}} & \text{in } B_i \setminus T_i \\ w_2^{\varepsilon,i} &= 1 & \text{in } T_i \\ w_2^{\varepsilon,i} &= 0 & \text{in } D \setminus B_i. \end{cases} \quad (5.4.21)$$

Step 3. Equipped with w_1^ε and w_2^ε constructed above, we show that $w^\varepsilon = w_1^\varepsilon \wedge w_2^\varepsilon$ satisfies properties (H1)-(H3). As already discussed in Step 1, it suffices to prove that $\{w_2^\varepsilon\}_{\varepsilon>0}$ satisfies (H2) and (H3).

We begin with (H2): By (5.4.19), (5.4.21) and (5.4.8) of Lemma 5.4.1, a direct calculation leads to

$$\|\nabla w_2^\varepsilon\|_{L^2(D)}^2 \lesssim \varepsilon^d \sum_{z_i \in n^\varepsilon} \rho_i^{d-2} \leq \varepsilon^d \sum_{z_i \in \mathbb{Z}^d \cap \frac{1}{\varepsilon} D} \rho_i^{d-2}. \quad (5.4.22)$$

By Lemma 5.5.2 applied to the right hand side, we infer that, almost surely,

$$\limsup_{\varepsilon \downarrow 0^+} \|\nabla w_2^\varepsilon\|_{L^2(D)}^2 \leq C. \quad (5.4.23)$$

In addition, since $1 - w_2^\varepsilon = 0$ in $\mathbb{R}^d \setminus (\bigcup_{z_i \in n^\varepsilon} B_i)$, and the balls $\{B_i\}_{z_i \in n^\varepsilon}$ are essentially disjoint, by Poincaré's inequality we obtain also

$$\|1 - w_2^\varepsilon\|_{L^2(D)}^2 \leq \sum_{z_i \in n^\varepsilon} \|1 - w_2^\varepsilon\|_{L^2(B_i)}^2 \lesssim \varepsilon^2 \sum_{z_i \in n^\varepsilon} \|\nabla w_2^\varepsilon\|_{L^2(B_i)}^2.$$

This, together with (5.4.22) and (5.4.23), yields that almost surely $w_2^\varepsilon \rightharpoonup 1$ in $H^1(D)$ when $\varepsilon \downarrow 0^+$. We thus established (H2).

To prove (H3) for w_2^ε , we first use (5.4.19) and (5.4.21) to decompose

$$-\Delta w_2^\varepsilon = \sum_{i=1}^{n^\varepsilon} (\mu^{\varepsilon,i} - \gamma^{\varepsilon,i}), \quad \mu^{\varepsilon,i} = -\partial_\nu w_2^{\varepsilon,i} \delta_{\partial B_i}, \quad \gamma^{\varepsilon,i} = -\partial_\nu w_2^{\varepsilon,i} \delta_{\partial T_i},$$

with ν denoting the outer normal and $\delta_{\partial B_i}$ and $\delta_{\partial T_i}$ being the $(d-1)$ -dimensional Hausdorff measure restricted to ∂B_i and ∂T_i , respectively. We start by remarking (see (H5)' of [CM97]) that, since in (H3) the functions v_ε are always assumed to be vanishing on each T_i , we only need to focus on the convergence (H3) for the sequence of measures

$$\mu^\varepsilon := - \sum_{i \in n^\varepsilon} \partial_\nu w_2^{\varepsilon,i} \delta_{\partial B_i} := \sum_{i \in n^\varepsilon} \mu^{\varepsilon,i}.$$

More precisely, we claim that for every $v_\varepsilon \rightharpoonup v$ in $H_0^1(D)$ such that $v_\varepsilon \in H_0^1(D^\varepsilon)$, it holds

$$(\mu^\varepsilon, v_\varepsilon)_{H^{-1}, H_0^1(D)} \rightarrow C_0 \int_D v, \quad (5.4.24)$$

where $C_0 := (d-2)\sigma_d \langle \rho^{d-2} \rangle$ corresponds to the definition (5.2.6) for the case $\Phi = \mathbb{Z}^d$ under consideration.

We begin by arguing that it suffices to prove (5.4.24) above for any truncated process $(\mathbb{Z}^d, \mathcal{R}^M)$, with $M \in \mathbb{N}$ and \mathcal{R}^M defined in (5.4.4). From now on, we use the lower index M to distinguish the

objects constructed with the truncated marks \mathcal{R}^M and the ones coming from \mathcal{R} . For instance, we denote by $w_{2,M}^\varepsilon, \mu_M^\varepsilon$ the analogues of w^ε and μ^ε introduced above. Note that, for \mathcal{R}^M , the constant in (5.4.24) reads $C_{0,M} = (d-2)\sigma_d \langle \rho_M^{d-2} \rangle$.

For any $M \in \mathbb{N}$, since $|C_0 - C_{0,M}| \lesssim \langle \rho^{d-2} \mathbf{1}_{\rho \geq M} \rangle$, we bound by the triangular inequality, an integration by parts and Cauchy-Schwarz inequality

$$\begin{aligned} \left| (-\Delta w^\varepsilon, v_\varepsilon)_{H^{-1}, H_0^1(D)} - C_0 \int_D v \right| &\lesssim \|\nabla(w_M^\varepsilon - w^\varepsilon)\|_{L^2(D)} \|\nabla v_\varepsilon\|_{L^2(D)} \\ &+ \left| (\mu_M^\varepsilon, v_\varepsilon)_{H^{-1}, H_0^1(D)} - C_{0,M} \int_D v \right| + \langle \rho^{d-2} \mathbf{1}_{\rho \geq M} \rangle \|v\|_{L^1}, \end{aligned} \quad (5.4.25)$$

By an argument similar to the one in (5.4.22), we estimate

$$\limsup_{\varepsilon \downarrow 0^+} \|\nabla(w_M^\varepsilon - w^\varepsilon)\|_{L^2(D)} \lesssim \langle \rho^{d-2} \mathbf{1}_{\rho \geq M} \rangle,$$

so that by letting $\varepsilon \downarrow 0^+$ in (5.4.25), this and the boundedness of the sequence v_ε in H^1 yield

$$\begin{aligned} \limsup_{\varepsilon \downarrow 0^+} \left| (-\Delta w^\varepsilon, v_\varepsilon)_{H^{-1}, H_0^1(D)} - C_0 \int_D v \right| \\ \lesssim \limsup_{\varepsilon \downarrow 0^+} \left| (\mu_M^\varepsilon, v_\varepsilon)_{H^{-1}, H_0^1(D)} - C_{0,M} \int_D v \right| + \langle \rho^{d-2} \mathbf{1}_{\rho \geq M} \rangle (\|v\|_{L^1} + 1). \end{aligned}$$

Hence, provided that (5.4.24) holds for μ_M^ε and any fixed $M \in \mathbb{N}$, we may then send $M \uparrow +\infty$ and establish (H3) by assumption (5.1.5).

To argue that almost surely and for every $M \in \mathbb{N}$ the convergence in (5.4.24) holds for μ_M^ε , we follow [CM97]. First, by the definition of $w_{i,M}^\varepsilon$, we compute

$$\mu_M^\varepsilon = \sum_{z_i \in n_\varepsilon \cap \frac{1}{\varepsilon} D} \frac{2^{d-1}(d-2)(\rho_{i,M})^{d-2}}{1 - 2^{d-2}\varepsilon^2(\rho_{i,M})^{d-2}} \varepsilon \delta_{\partial B_i}.$$

Since $\rho_{i,M} \leq M$, to obtain (5.4.24) it suffices to prove

$$\tilde{\mu}_M^\varepsilon := \sum_{z_i \in n_\varepsilon} 2^{d-1}(d-2)\rho_{i,M}^{d-2} \varepsilon \delta_{B_i} \rightarrow C_{0,M} \quad \text{strongly in } W^{-1,\infty}(D).$$

To show this, we fix $M \in \mathbb{N}$ and split the convergence (5.4.24) into the two following steps: If we define

$$\eta_M^\varepsilon := \sum_{z_i \in \mathbb{Z}^d \cap \frac{1}{\varepsilon} D} 2^d(d-2)d\rho_{M,i}^{d-2} \mathbf{1}_{B_i},$$

then we argue that

$$\tilde{\mu}_M^\varepsilon - \eta_M^\varepsilon \rightarrow 0 \quad \text{strongly in } W^{-1,\infty}(D), \quad (5.4.26)$$

and

$$\eta_M^\varepsilon \rightarrow C_{0,M} \quad \text{strongly in } W^{-1,\infty}(D). \quad (5.4.27)$$

To show (5.4.26), we consider the auxiliary problems

$$\begin{cases} -\Delta q_{i,M}^\varepsilon &= 2^d(d-2)d\rho_{i,M}^{d-2} & \text{in } B_i^\varepsilon \\ \frac{\partial q_{i,M}^\varepsilon}{\partial \nu} &= 2^{d-1}(d-2)\rho_{i,M}^{d-2}\varepsilon & \text{on } \partial B_i^\varepsilon, \end{cases} \quad (5.4.28)$$

which are in particular satisfied by the functions

$$q_{i,M}^\varepsilon(x) = 2^{d-1}(d-2)\rho_{i,M}^{d-2} \left(|x - z_i|^2 - \left(\frac{\varepsilon}{2}\right)^2 \right).$$

As $q_{i,M}^\varepsilon = 0$ on ∂B_i^ε , we may extend $q_{i,M}^\varepsilon$ by zero outside of B_i^ε and estimate

$$\|\nabla q_{i,M}^\varepsilon\|_{L^\infty(B_i)} = 2^{d-1}(d-2)\rho_{i,M}^{d-2}\varepsilon \lesssim M^{d-2}\varepsilon.$$

Using Poincaré's inequality, and since the balls B_i^ε are disjoint, we infer that

$$q_M^\varepsilon := \sum_{z_i \in \mathbb{Z}^d \cap \frac{1}{\varepsilon}D} q_{i,M}^\varepsilon \rightarrow 0 \quad \text{in } W^{1,\infty}(\mathbb{R}^d). \quad (5.4.29)$$

We observe that by (5.4.28)

$$\eta_M^\varepsilon - \tilde{\mu}_M^\varepsilon = -\Delta q_M^\varepsilon + \sum_{z_i \in (\mathbb{Z}^d \cap \frac{1}{\varepsilon}D) \setminus n_\varepsilon} 2^d(d-2)d\rho_{M,i}^{d-2} \mathbf{1}_{B_i} =: -\Delta q_M^\varepsilon + R_\varepsilon^M.$$

We have by (5.4.29) that $-\Delta q_M^\varepsilon \rightarrow 0$ in $W^{-1,\infty}(\mathbb{R}^d)$. On the other hand, for the term R_ε^M above, we have that by Lemma 5.5.3 and (5.4.8)

$$\lim_{\varepsilon \downarrow 0^+} \|R_\varepsilon^\varepsilon\|_{L^1} \lesssim \lim_{\varepsilon \downarrow 0^+} \varepsilon^d \sum_{z_i \in (\mathbb{Z}^d \cap \frac{1}{\varepsilon}D) \setminus n_\varepsilon} \rho_{M,i}^{d-2} = 0,$$

almost surely. Since R^ε is bounded in L^∞ , we also have that $R^\varepsilon \xrightarrow{*} 0$ in $L^\infty(D)$ and $R^\varepsilon \rightarrow 0$ in $W^{-1,\infty}(D)$. This yields (5.4.26).

In order to show (5.4.27), we first remark that it suffices to argue that

$$\eta_M^\varepsilon \xrightarrow{*} C_{0,M} \quad \text{in } L^\infty(D).$$

Since the family of functions $\{\eta_M^\varepsilon\}_{\varepsilon>0}$ is uniformly bounded in $L^\infty(D)$, we identify the w^* -limit by testing η_M^ε with any function $\zeta \in C_0^1(D)$: Indeed, by Lemma 5.5.4 applied to $(\mathbb{Z}^d, \{\rho_{i,M}^{d-2}\}_{i \in \mathbb{Z}^d})$ in the domain $B = D$ we infer almost surely that

$$(\eta_M^\varepsilon, \zeta)_{H^{-1}, H_0^1(D)} \rightarrow C_{0,M} \int_D \zeta.$$

This establishes (5.4.27) and thus concludes the proof for (H3) and for the whole lemma. \square

5.4.2 Proof of Lemma 5.3.1 in the general case

Let (Φ, \mathcal{R}) be a marked point process satisfying the assumptions of Theorem 5.2.1. In contrast with the previous subsection, when the centres of the holes are distributed according to a general point process Φ , there is not a deterministic positive lower bound for the minimal distance between the points of Φ . This requires some technical changes in the arguments of Subsection 5.4.1 but still allows us to obtain a statement on the asymptotic geometry of H^ε similar to Lemma 5.4.1 and to prove Lemma 5.3.1.

Lemma 5.4.2. *There exist an $\varepsilon_0 = \varepsilon_0(d)$ and a family of random variables $\{r_\varepsilon\}_{\varepsilon>0} \subset \mathbb{R}_+$ such that for \mathbb{P} -almost every $\omega \in \Omega$*

$$\lim_{\varepsilon \downarrow 0^+} r_\varepsilon(\omega) = 0, \quad (5.4.30)$$

and for any $\varepsilon \leq \varepsilon_0$ there exist $H_g^\varepsilon(\omega), H_b^\varepsilon(\omega), D_b^\varepsilon(\omega) \subset \mathbb{R}^d$ such that

$$\begin{aligned} H^\varepsilon(\omega) &= H_g^\varepsilon(\omega) \cup H_b^\varepsilon(\omega), \quad H_b^\varepsilon(\omega) \subset D_b^\varepsilon(\omega), \\ \text{dist}(H_g^\varepsilon(\omega), D_b^\varepsilon(\omega)) &\geq \frac{\varepsilon r_\varepsilon(\omega)}{2}, \end{aligned} \quad (5.4.31)$$

where

$$\lim_{\varepsilon \downarrow 0^+} \text{Cap}(H_b^\varepsilon(\omega), D_b^\varepsilon(\omega)) = 0. \quad (5.4.32)$$

Moreover, $H_g^\varepsilon(\omega)$ may be written as the following union of disjoint balls centred in $n^\varepsilon(\omega) \subset \Phi(\frac{1}{\varepsilon}D)$:

$$\begin{aligned} H_g^\varepsilon(\omega) &:= \bigcup_{z_j \in n^\varepsilon} B_{\varepsilon^{\frac{d}{d-2}} \rho_j}(\varepsilon z_j), \\ \min_{z_i \neq z_j \in n^\varepsilon} \varepsilon |z_i - z_j| &\geq 2r_\varepsilon \varepsilon, \quad \varepsilon^{\frac{d}{d-2}} \rho_j \leq \frac{\varepsilon r_\varepsilon(\omega)}{2}, \quad \lim_{\varepsilon \downarrow 0^+} \varepsilon^d \#(n^\varepsilon) = \langle N(Q) \rangle |D|. \end{aligned} \quad (5.4.33)$$

Furthermore, if for $\delta > 0$ the process Φ_δ is defined as in (5.4.3), then

$$\lim_{\varepsilon \downarrow 0^+} \varepsilon^d \#(\{z_i \in \Phi_{2\delta}^\varepsilon(D)(\omega) : \text{dist}(z_i, D_b^\varepsilon) \leq \delta \varepsilon\}) = 0. \quad (5.4.34)$$

We remark that the lower bounds in (5.4.31) and (5.4.33) differ from the ones of Lemma 5.4.1 by the factor r_ε . This implies, by (5.4.30), that the minimal distance between the balls of the “good” set H_g^ε is only $o(\varepsilon)$ for $\varepsilon \downarrow 0^+$ and not of order ε as required in the construction of the functions $\{w_\varepsilon\}_{\varepsilon>0}$ carried out in the previous subsection. We overcome this technical issue by comparing w_ε again with the oscillating test functions w_M^ε obtained by approximating H_g^ε by a simpler set $H_g^{\varepsilon, M}$. Here, $H_g^{\varepsilon, M}$ is obtained not only by truncating the radii at size M as in the previous subsection, but also by considering in $H_g^{\varepsilon, M}$ only the balls whose centres (in $n^\varepsilon \subset \Phi(D)$) satisfy (5.4.31) and (5.4.33) with $M^{-1}\varepsilon$ instead of $r_\varepsilon \varepsilon$. As in the previous subsection, we show that the sets $H^{\varepsilon, M}$ are a good approximation of H^ε , in the sense that the associated functions w_M^ε and w^ε are close in H^1 . This follows from the fact that the centres removed from H_g^ε , which are either too close to each other or to the “safety layer” D_b^ε , are few and may be taken care of by studying the properties of the thinnings Φ_δ of a process Φ defined in (5.4.3). In fact, the last limit (5.4.34) of the previous lemma states that the main error in considering the approximate holes $H_g^{\varepsilon, M}$ is given by the points which are too close to each other.

Proof of Lemma 5.4.2. As in the previous subsection, we suppress the argument $\omega \in \Omega$ for the random sets involved in the argument below. Let us fix an $\alpha \in (0, \frac{2}{d-2})$ and let us define

$$r_\varepsilon := (\varepsilon^{\frac{d}{d-2}} \max_{z_j \in \Phi^\varepsilon(D)} \rho_j)^{\frac{1}{d}} \vee \varepsilon^{\frac{\alpha}{4}}. \quad (5.4.35)$$

With this choice, we prove that the decomposition of H^ε required by the lemma holds true. We begin by showing that with this definition r_ε satisfies (5.4.30): Let

$$F^\varepsilon := \{z_j \in \Phi^\varepsilon(D) \mid \varepsilon^{\frac{d}{d-2}} \rho_j \geq \varepsilon\}.$$

If $F^\varepsilon = \emptyset$, then $r_\varepsilon \leq \varepsilon^{\frac{1}{d}} \vee \varepsilon^{\frac{\alpha}{4}}$. If otherwise, we estimate

$$\varepsilon^d \max_{z_j \in \Phi^\varepsilon(D)} \rho_j^{d-2} = \varepsilon^d \max_{z_j \in F^\varepsilon} \rho_j^{d-2} \leq \varepsilon^d \sum_{z_j \in F^\varepsilon} \rho_j^{d-2}.$$

To get (5.4.30), it suffices to show that almost surely the right hand side above tends to zero in the limit $\varepsilon \downarrow 0^+$. By Lemma 5.5.3, this holds provided

$$\varepsilon^d \#(F^\varepsilon) \rightarrow 0 \quad \varepsilon \downarrow 0^+.$$

We show this by bounding

$$\varepsilon^d \#(F^\varepsilon) \lesssim \varepsilon^2 \varepsilon^d \sum_{z_j \in \Phi^\varepsilon(D)} \rho_j^{d-2},$$

and using Lemma 5.5.2. We thus established (5.4.30).

Equipped with (5.4.35), we now set $\eta_\varepsilon = r_\varepsilon \varepsilon$ and begin by constructing the sets H_b^ε and D_b^ε . As in Lemma 5.4.1, we denote by I_b^ε the set of points in $\Phi^\varepsilon(D)$ which generate the sets H_b^ε and D_b^ε . We start by requiring that I_b^ε contains the points in $\Phi^\varepsilon(D)$ whose associated radii are “too big”, namely the set

$$J_b^\varepsilon = \left\{ z_j \in \Phi^\varepsilon(D) : \varepsilon^{\frac{d}{d-2}} \rho_j \geq \frac{\eta_\varepsilon}{2} \right\}. \quad (5.4.36)$$

Similarly to the periodic case, we set

$$\tilde{H}_b^\varepsilon := \bigcup_{z_j \in J_b^\varepsilon} B_{2\varepsilon^{\frac{d}{d-2}} \rho_j}(\varepsilon z_j). \quad (5.4.37)$$

We now include in I_b^ε also the points in $\Phi^\varepsilon(D) \setminus J_b^\varepsilon$ which, in spite of having radii below the threshold set in definition (5.4.36), are too close to each other. We indeed define

$$K_b^\varepsilon := \Phi^\varepsilon(D) \setminus (\Phi_{2r_\varepsilon}^\varepsilon(D) \cup J_b^\varepsilon). \quad (5.4.38)$$

Finally, we include into I_b^ε also the set of points which are not in $J_b^\varepsilon \cup K_b^\varepsilon$, but which might be close to \tilde{H}_b^ε : We denote them by

$$\tilde{I}_b^\varepsilon := \left\{ z_j \in \Phi^\varepsilon(D) \setminus (J_b^\varepsilon \cup K_b^\varepsilon) : \tilde{H}_b^\varepsilon \cap B_{\eta_\varepsilon}(\varepsilon z_j) \neq \emptyset \right\}. \quad (5.4.39)$$

We thus set

$$I_b^\varepsilon = \tilde{I}_b^\varepsilon \cup J_b^\varepsilon \cup K_b^\varepsilon, \quad (5.4.40)$$

$$H_b^\varepsilon := \bigcup_{z_j \in I_b^\varepsilon} B_{\varepsilon^{\frac{d}{d-2}} \rho_j}(\varepsilon z_j), \quad H_g^\varepsilon := H^\varepsilon \setminus H_b^\varepsilon, \quad D_b^\varepsilon := \bigcup_{z_j \in I_b^\varepsilon} B_{2\varepsilon^{\frac{d}{d-2}} \rho_j}(\varepsilon z_j). \quad (5.4.41)$$

It remains to show that, with H_b^ε , H_g^ε and D_b^ε defined as in (5.4.41), properties (5.4.31), (5.4.32), (5.4.33) and (5.4.34) are satisfied. We start with (5.4.32). As in the proof of Lemma 5.4.1, the definition of D_b^ε allows us to estimate by the sub-additivity of the capacity

$$\text{Cap}(H_b^\varepsilon, D_b^\varepsilon) \lesssim \varepsilon^d \sum_{z_j \in I_b^\varepsilon} \rho_j^{d-2}.$$

Thanks to Lemma 5.5.3, we conclude that, almost surely, when $\varepsilon \downarrow 0^+$, the right-hand side above vanishes provided that almost surely

$$\lim_{\varepsilon \downarrow 0^+} \varepsilon^d \#(I_b^\varepsilon) = 0. \quad (5.4.42)$$

We show this by using definition (5.4.40) and proving that each of the sets which constitute I_b^ε satisfies the limit above. We begin with J_b^ε : Definitions (5.4.35) and (5.4.36) yield

$$\varepsilon^d \#(J_b^\varepsilon) \lesssim \varepsilon^2 r_\varepsilon^{-(d-2)} \varepsilon^d \sum_{z_j \in \Phi^\varepsilon(D)} \rho_j^{d-2} \leq \varepsilon^{2-\alpha(d-2)} \varepsilon^d \sum_{z_j \in \Phi^\varepsilon(D)} \rho_j^{d-2}.$$

By Lemma 5.5.2 and the assumption $\alpha < \frac{2}{d-2}$, the right-hand side almost surely vanishes in the limit $\varepsilon \downarrow 0^+$. We thus established

$$\lim_{\varepsilon \downarrow 0^+} \varepsilon^d \#(J_b^\varepsilon) = 0, \quad (5.4.43)$$

i.e. limit (5.4.42) for J_b^ε . We now turn to K_b^ε : Let $\{\delta_k\}_{k \in \mathbb{N}}$ be any sequence such that $\delta_k \downarrow 0^+$. Since r_ε satisfies (5.4.30), we estimate for any δ_k

$$\limsup_{\varepsilon \downarrow 0^+} \varepsilon^d \#(K_b^\varepsilon) \stackrel{(5.4.38)}{=} \limsup_{\varepsilon \downarrow 0^+} \varepsilon^d (N^\varepsilon(D) - N_{r_\varepsilon}^\varepsilon(D)) \stackrel{(5.4.3)}{\leq} \limsup_{\varepsilon \downarrow 0^+} \varepsilon^d (N^\varepsilon(D) - N_{\delta_k}^\varepsilon(D)).$$

We now apply Lemma 5.5.2 to Φ and each Φ_{δ_k} to deduce that almost surely and for every δ_k

$$\limsup_{\varepsilon \downarrow 0^+} \varepsilon^d \#(K_b^\varepsilon) \leq \langle N(Q) - N_{\delta_k}(Q) \rangle |D|,$$

where Q is a unit cube. By sending $\delta_k \downarrow 0^+$, Lemma 5.5.2 yields

$$\lim_{\varepsilon \downarrow 0^+} \varepsilon^d \#(K_b^\varepsilon) = 0. \quad (5.4.44)$$

To conclude the proof of (5.4.42), it remains to show that almost surely also

$$\varepsilon^d \#(\tilde{I}_b^\varepsilon) \rightarrow 0 \quad \varepsilon \downarrow 0^+. \quad (5.4.45)$$

By definitions (5.4.36), (5.4.38) and (5.4.39), for each $z_i \in \Phi^\varepsilon(D) \setminus (J_b^\varepsilon \cup K_b^\varepsilon)$, we have

$$\min_{z_j \in \Phi^\varepsilon(D) \setminus \{z_i\}} \varepsilon |z_j - z_i| \geq 2\eta_\varepsilon, \quad \varepsilon^{\frac{d}{d-2}} \rho_i < \frac{\eta_\varepsilon}{2}. \quad (5.4.46)$$

On the one hand, by the first inequality above, the balls $\{B_{\eta_\varepsilon}(\varepsilon z_i)\}_{z_i \in \tilde{I}_b^\varepsilon}$ are all disjoint and satisfy

$$\varepsilon^d \#(\tilde{I}_b^\varepsilon) \lesssim \varepsilon^d \sum_{z_i \in \tilde{I}_b^\varepsilon} \eta_\varepsilon^{-d} |B_{\eta_\varepsilon}(\varepsilon z_i)| = r_\varepsilon^{-d} \sum_{z_i \in \tilde{I}_b^\varepsilon} |B_{\eta_\varepsilon}(\varepsilon z_i)|.$$

In addition, the inequalities (5.4.46), definitions (5.4.36), (5.4.38) and (5.4.37) also imply that $|\bigcup_{z_i \in \tilde{I}_b^\varepsilon} B_{\eta_\varepsilon}(\varepsilon z_i)| \lesssim |\tilde{H}_b^\varepsilon|$. By wrapping up the previous two inequalities, we bound

$$\begin{aligned} \varepsilon^d \#(\tilde{I}_b^\varepsilon) &\lesssim r_\varepsilon^{-d} |\tilde{H}_b^\varepsilon| \lesssim r_\varepsilon^{-d} \sum_{z_j \in J_b^\varepsilon} \left(\varepsilon^{\frac{d}{d-2}} \rho_j \right)^d \\ &\lesssim r_\varepsilon^{-d} \left(\varepsilon^{\frac{d}{d-2}} \max_{z_j \in J_b^\varepsilon} \rho_j \right)^2 \sum_{z_j \in J_b^\varepsilon} \left(\varepsilon^{\frac{d}{d-2}} \rho_j \right)^{d-2}. \end{aligned}$$

By definition (5.4.35), the inequality above reduces to

$$\varepsilon^d \#(\tilde{I}_b^\varepsilon) \lesssim \varepsilon^d \sum_{j \in J_b^\varepsilon} \left(\varepsilon^{\frac{d}{d-2}} \rho_j \right)^{d-2}.$$

Thanks to (5.4.43), we apply Lemma 5.5.3 and deduce (5.4.45). This, together with (5.4.43) and (5.4.44), yields (5.4.42) and concludes the proof of (5.4.32).

To show (5.4.31), we recall the definitions of D_b^ε , H_b^ε and H_g^ε in (5.4.41) and set $n^\varepsilon := \Phi^\varepsilon(D) \setminus I_b^\varepsilon$. Since all $z_i \in n^\varepsilon$ satisfy (5.4.46) and thus also

$$\text{dist}\left(B_{\varepsilon^{\frac{d}{d-2}} \rho_i}(\varepsilon z_i), \partial B_{\eta_\varepsilon}(\varepsilon z_i)\right) \geq \frac{\eta_\varepsilon}{2},$$

by definition (5.4.41) of D_b^ε , it suffices to show that for all $z_i \in \Phi^\varepsilon(D) \setminus I_b^\varepsilon$ and all $z_j \in I_b^\varepsilon$ we have

$$B_{\eta_\varepsilon}(\varepsilon z_i) \cap B_{2\varepsilon^{\frac{d}{d-2}} \rho_j}(\varepsilon z_j) = \emptyset. \quad (5.4.47)$$

For all $z_j \in J_b^\varepsilon \subset I_b^\varepsilon$, this identity holds by (5.4.39) and the definition of n^ε . If $z_j \in I_b^\varepsilon \setminus J_b^\varepsilon$, then we know that $2\varepsilon^{\frac{d}{d-2}} \rho_j \leq \eta_\varepsilon$ and, by (5.4.46) for z_i , we obtain (5.4.47) also in this case. This establishes (5.4.47) and also (5.4.31).

Finally, the properties (5.4.33) of the set H_g^ε are a consequence of (5.4.46), definition (5.4.42) and (5.5.8) of Lemma 5.5.2.

To show (5.4.34), we resort to the definition of D_b^ε to estimate

$$\begin{aligned} &\{z_i \in \Phi_{2\delta}^\varepsilon(D)(\omega) : \text{dist}(z_i, D_b^\varepsilon) \leq \delta\varepsilon\} \\ &\subset I_b^\varepsilon \cup \left\{ z_i \in n^\varepsilon(\omega) : \text{dist}\left(z_i, \bigcup_{z_j \in J_b^\varepsilon} B_{2\varepsilon^{\frac{d}{d-2}} \rho_j}(\varepsilon z_j)\right) \leq \delta\varepsilon \right\} \\ &\cup \left\{ z_i \in n^\varepsilon(\omega) \cap \Phi_{2\delta}^\varepsilon(D)(\omega) : \text{dist}\left(z_i, \bigcup_{z_j \in \tilde{I}_b^\varepsilon \cup K_b^\varepsilon} B_{2\varepsilon^{\frac{d}{d-2}} \rho_j}(\varepsilon z_j)\right) \leq \delta\varepsilon \right\} \\ &:= I_b^\varepsilon \cup E^\varepsilon \cup C^\varepsilon. \end{aligned}$$

We already know $\varepsilon^d \#(I_b^\varepsilon) \rightarrow 0$. Next, we argue that

$$\varepsilon^d \#(E^\varepsilon) \rightarrow 0.$$

This follows by an argument similar to the one for (5.4.45): Then, we may choose $\varepsilon_0 = \varepsilon_0(d)$ such that for all $\varepsilon \leq \varepsilon_0$, property (5.4.30) yields $\eta_\varepsilon = \varepsilon r_\varepsilon \leq \delta\varepsilon$. By definition of J_b^ε in (5.4.36) and of E^ε above, we infer that for such $\varepsilon \leq \varepsilon_0$, for all $z_j \in E^\varepsilon$ there exists $z_i \in J_b^\varepsilon$ such that

$$B_{\eta_\varepsilon}(\varepsilon z_j) \subset B_{2\delta\varepsilon + 2\varepsilon^{\frac{d}{d-2}} \rho_i}(\varepsilon z_i) \subset B_{6\delta r_\varepsilon^{-1} \varepsilon^{\frac{d}{d-2}} \rho_i}(\varepsilon z_i), \quad (5.4.48)$$

where in the second inequality we use that $r_\varepsilon^{-1}\delta \geq 1$. We note that by (5.4.33) the balls $B_{\eta_\varepsilon}(\varepsilon z_j)$ with $z_j \in n^\varepsilon$ are all disjoint. Hence,

$$\begin{aligned} \varepsilon^d \#(E^\varepsilon) &= r_\varepsilon^{-d} \eta_\varepsilon^d \#(E^\varepsilon) \stackrel{(5.4.48)}{\lesssim} r_\varepsilon^{-d} \left| \bigcup_{z_i \in J_b^\varepsilon} B_{6\delta r_\varepsilon^{-1} \varepsilon^{\frac{d}{d-2}} \rho_i}(\varepsilon z_i) \right| \\ &\lesssim \delta^d r_\varepsilon^{-2d} \sum_{z_i \in J_b^\varepsilon} \left(\varepsilon^{\frac{d}{d-2}} \rho_i \right)^d \stackrel{(5.4.35)}{\leq} \delta^d \varepsilon^d \sum_{z_i \in J_b^\varepsilon} \rho_i^{d-2}. \end{aligned}$$

By Lemma 5.5.3 and (5.4.43), almost surely the right hand side tends to zero in the limit $\varepsilon \downarrow 0^+$.

We conclude the argument for (5.4.34) by showing that the set C^ε is empty when ε is small: In fact, by construction, if $z_i \in n_\varepsilon$ satisfies

$$\text{dist}\left(\varepsilon z_i, \bigcup_{z_j \in \tilde{I}_b^\varepsilon \cup K_b^\varepsilon} B_{2\varepsilon^{\frac{d}{d-2}} \rho_j}(\varepsilon z_j)\right) \leq \delta \varepsilon,$$

then there exists a $z_j \in \tilde{I}_b^\varepsilon \cup K_b^\varepsilon$ such that for $\varepsilon \leq \varepsilon_0$

$$\varepsilon |z_i - z_j| \leq \text{dist}\left(\varepsilon z_i, B_{2\varepsilon^{\frac{d}{d-2}} \rho_j}(\varepsilon z_j)\right) + \eta_\varepsilon \leq 2\delta \varepsilon.$$

This yields $C^\varepsilon \subset \Phi^\varepsilon(D) \setminus \Phi_{2\delta}(D)$ and thus that it is empty since by definition we also have $C^\varepsilon \subset \Phi_{2\delta}^\varepsilon(D)$. The proof of Lemma 5.4.2 is complete. \square

Proof of Lemma 5.3.1, general case. We split the proof of the lemma in the same steps as in the proof for the case of periodic centres (case (a)). Some of these steps may be proven exactly as in the previous subsection by relying on Lemma 5.4.2 instead of Lemma 5.4.1. We thus focus below only on the parts of the proof which differ from Subsection 5.4.1.

Step 1. Since by Lemma 5.4.2, the sets H_g^ε and D_b^ε are disjoint, the splitting $w^\varepsilon = w_1^\varepsilon \wedge w_2^\varepsilon$ with $w_1^\varepsilon, w_2^\varepsilon$ solving (5.4.16), (5.4.17) and (5.4.18) remains unchanged from the case of periodic centres.

Step 2. Again by Lemma 5.4.2, we construct the sequence $\{w_1^\varepsilon\}_{\varepsilon>0}$ satisfying (5.4.16) and (5.4.18) as in Subsection 5.4.1. We thus only need to focus on the construction of the functions $\{w_2^\varepsilon\}_{\varepsilon>0}$, which we set equal to 1 on D_b^ε . For each $z_j \in n^\varepsilon$, with n^ε being the set of centers of the particles in H_g^ε (see Lemma 5.4.2), we denote the random variables

$$d_j^\varepsilon := \min\left\{\text{dist}(\varepsilon z_j, D_b^\varepsilon), \frac{1}{2} \min_{i \neq j} \varepsilon |z_i - z_j|, \varepsilon\right\}. \quad (5.4.49)$$

We remark that, in contrast with case (a) where we had by Lemma 5.4.1 that $d_j^\varepsilon \geq \frac{\varepsilon}{2}$, here Lemma 5.4.2 only implies that, for ε small, $d_j^\varepsilon \geq r_\varepsilon \varepsilon$ with r_ε satisfying (5.4.30). By defining for each $z_j \in n^\varepsilon$ the sets

$$T_j^\varepsilon = B_{\varepsilon^{\frac{d}{d-2}} \rho_j}(\varepsilon z_j) \quad B_j = B_{d_j^\varepsilon}(\varepsilon z_j),$$

we consider the function $w_2^{\varepsilon,j}$ solving (5.4.20) in $B_j \setminus T_j^\varepsilon$ and hence defined as

$$w_2^{\varepsilon,j}(x) = \begin{cases} \frac{|x - \varepsilon z_j|^{-(d-2)} - (d_j^\varepsilon)^{-(d-2)}}{\varepsilon^{-d} \rho_i^{-(d-2)} - (d_j^\varepsilon)^{-(d-2)}} & \text{in } B_j \setminus T_j^\varepsilon \\ 1 & \text{in } T_j^\varepsilon \\ 0 & \text{in } D \setminus B_j. \end{cases} \quad (5.4.50)$$

Note, that by definition of d_j^ε , (5.4.49), the functions $\nabla w^{\varepsilon,j}$ have disjoint support. Moreover, for ε sufficiently small,

$$d_j^\varepsilon \geq 2\varepsilon^{\frac{d}{d-2}} \rho_j. \quad (5.4.51)$$

Indeed, by Lemma 5.4.2,

$$2\varepsilon^{\frac{d}{d-2}} \rho_j \leq \varepsilon r_\varepsilon \leq \min \left\{ \frac{1}{2} \min_{i \neq j} \varepsilon |z_i - z_j|, \varepsilon \right\},$$

and

$$2\varepsilon^{\frac{d}{d-2}} \rho_j \leq \varepsilon^{\frac{d}{d-2}} \rho_j + \frac{\varepsilon r_\varepsilon}{2} \leq \varepsilon^{\frac{d}{d-2}} \rho_j + \text{dist}(T_j, D_b^\varepsilon) = \text{dist}(\varepsilon z_j, D_b^\varepsilon).$$

We thus set

$$w_2^\varepsilon = 1 - \sum_{z_j \in n^\varepsilon} w_2^{\varepsilon,j}, \quad (5.4.52)$$

which immediately satisfies condition (5.4.17). Therefore, as discussed in Step 1, the function $w^\varepsilon = w_1^\varepsilon \wedge w_2^\varepsilon$ satisfies (H1) and it suffices to prove (H2)-(H3) only for w_2^ε .

Step 3. We begin by showing that w_2^ε satisfies (H2): By the triangular inequality and definitions (5.4.50) and (5.4.52), we estimate

$$\begin{aligned} \|\nabla w_2^\varepsilon\|_2^2 &= \sum_{z_i \in n^\varepsilon} \|\nabla w_2^{\varepsilon,i}\|_{L^2(B_i)}^2 \lesssim \sum_{z_i \in n^\varepsilon} \frac{\varepsilon^d \rho_i^{d-2}}{1 - \left(\frac{\varepsilon^{\frac{d}{d-2}} \rho_i}{d_\varepsilon} \right)^{d-2}} \\ &\stackrel{(5.4.51)}{\lesssim} \sum_{i \in n^\varepsilon} \varepsilon^d \rho_i^{d-2} \stackrel{n^\varepsilon \subset \Phi^\varepsilon(D)}{\lesssim} \sum_{i \in \Phi^\varepsilon(D)} \varepsilon^d \rho_i^{d-2}. \end{aligned} \quad (5.4.53)$$

By Lemma 5.5.2, the right-hand side above is almost surely bounded in the limit $\varepsilon \downarrow 0^+$. This, together with Poincaré's inequality for $1 - w_2^\varepsilon$ in D , yields that almost surely, up to a subsequence, we have $w_2^\varepsilon \rightharpoonup w$ in $H^1(D)$ when $\varepsilon \downarrow 0^+$.

We claim that $w \equiv 1$. To this purpose, it is useful to consider the following “truncated” processes $(n_M^\varepsilon, \{\rho_{j,M}\}_{j \in n^\varepsilon})$ which we construct in the following way: For any $M \in \mathbb{N}$, we set

$$n_M^\varepsilon := \left\{ z_i \in n^\varepsilon : d_j^\varepsilon \geq \frac{\varepsilon}{M} \right\}, \quad \rho_{j,M} = \rho_j \wedge M.$$

In addition, let

$$H_g^{\varepsilon,M} := \bigcup_{z_j \in n_M^\varepsilon} B_{\varepsilon^{\frac{d}{d-2}} \rho_{j,M}}(\varepsilon z_j), \quad D^{\varepsilon,M} := D \setminus (H_g^{\varepsilon,M} \cup H_b^\varepsilon),$$

and let $w_2^{\varepsilon,M}$ be the function constructed as in (5.4.52) and (5.4.50) for the set $H_g^{\varepsilon,M}$.

By the same argument above for w_2^ε , almost surely and up to a subsequence, it holds $w_2^{\varepsilon,M} \rightharpoonup w^M$ for every $M \in \mathbb{N}$. Moreover, since $1 - w_2^{\varepsilon,M} = 0$ on $\mathbb{R}^d \setminus (\bigcup_{z_i \in n_M^\varepsilon} B_i)$ and the balls B_i are pairwise disjoint and have radii in $[M^{-1}\varepsilon, \varepsilon]$ we may argue as in Subsection 5.4.1 and infer that almost surely, and for every $M \in \mathbb{N}$, $w_2^{\varepsilon,M} \rightharpoonup 1$ in $H^1(D)$, and therefore also strongly in $L^2(D)$. This implies by the triangular inequality

$$\limsup_{\varepsilon \downarrow 0^+} \|w_2^\varepsilon - 1\|_2^2 \leq \limsup_{M \uparrow \infty} \limsup_{\varepsilon \downarrow 0^+} \|w_2^\varepsilon - w_2^{\varepsilon,M}\|_2^2.$$

Condition (H2) holds for w_2^ε provided that the limit on the right-hand side above vanishes. By Poincaré's inequality in D , it suffices to prove that

$$\lim_{M \uparrow \infty} \limsup_{\varepsilon \downarrow 0^+} \|\nabla(w_2^\varepsilon - w_2^{\varepsilon, M})\|_2^2 = 0. \quad (5.4.54)$$

To show this, we argue as follows: By construction

$$\begin{aligned} w_2^\varepsilon &= w_2^{\varepsilon, M} && \text{in } B_i \subset D \setminus D_b^\varepsilon, \text{ whenever } \rho_i \leq M \text{ and } d_i \geq M^{-1}\varepsilon, \\ w_2^{\varepsilon, M} &\equiv 1 && \text{in } B_i, \text{ whenever } d_i \leq M^{-1}\varepsilon. \end{aligned}$$

This implies that the L^2 -norm on the left-hand side above reduces to

$$\begin{aligned} \|\nabla(w^\varepsilon - w_2^{\varepsilon, M})\|_2^2 &= \sum_{z_i \in n^\varepsilon} \|\nabla(w_2^{\varepsilon, i} - w_2^{\varepsilon, M, i})\|_2^2 \mathbf{1}_{\rho_i \geq M} \mathbf{1}_{d_i \geq M^{-1}\varepsilon} \\ &\quad + \sum_{z_i \in n^\varepsilon} \|\nabla w_2^\varepsilon\|_2^2 \mathbf{1}_{d_i \leq M^{-1}\varepsilon}. \end{aligned} \quad (5.4.55)$$

Similarly to (5.4.53), we use the explicit formulation for $w_2^\varepsilon, w_2^{\varepsilon, M}$ to control the first term on the right-hand side above by

$$\sum_{z_i \in n^\varepsilon} \|\nabla(w_2^{\varepsilon, i} - w_2^{\varepsilon, M, i})\|_2^2 \mathbf{1}_{\rho_i \geq M} \mathbf{1}_{d_i \geq M^{-1}\varepsilon} \lesssim \sum_{z_i \in n^\varepsilon} \varepsilon^d \rho_i^{d-2} \mathbf{1}_{\rho_i \geq M}.$$

By Lemma 5.5.2 applied to the process Φ with marks $\{\rho_i^{d-2} \mathbf{1}_{\rho_i \geq M}\}_{z_i \in \Phi}$, and the assumption (5.1.5), we obtain that almost surely

$$\lim_{M \uparrow +\infty} \limsup_{\varepsilon \downarrow 0^+} \sum_{z_i \in n^\varepsilon} \|\nabla(w_2^{\varepsilon, i} - w_2^{\varepsilon, M, i})\|_2^2 \mathbf{1}_{\rho_i \geq M} \mathbf{1}_{d_i \geq M^{-1}\varepsilon} = 0. \quad (5.4.56)$$

By using again the same estimate as in (5.4.53) for w_2^ε , we have that

$$\sum_{z_i \in n^\varepsilon} \|\nabla w_2^{\varepsilon, i}\|_2^2 \mathbf{1}_{d_i \leq M^{-1}\varepsilon} \lesssim \sum_{z_i \in n^\varepsilon} \varepsilon^d \rho_i^{d-2} \mathbf{1}_{d_i \leq M^{-1}\varepsilon}. \quad (5.4.57)$$

By Definition of d_i in (5.4.49), we have that if $d_i \leq M^{-1}\varepsilon$, then either $z_i \in \Phi^\varepsilon(D) \setminus \Phi_{2M^{-1}}^\varepsilon(D)$ or

$$z_i \in I_M^\varepsilon := \left\{ z_i \in n_\varepsilon \cap \Phi_{2M^{-1}}^\varepsilon(D) : \text{dist}(z_i, D_b^\varepsilon) \leq \frac{\varepsilon}{M} \right\}.$$

Hence, from (5.4.57) we obtain

$$\limsup_{\varepsilon \downarrow 0^+} \sum_{z_i \in n^\varepsilon} \|\nabla w_2^{\varepsilon, i}\|_2^2 \mathbf{1}_{d_i \leq M^{-1}\varepsilon} \leq \limsup_{\varepsilon \downarrow 0^+} \varepsilon^d \sum_{z_i \in \Phi^\varepsilon(D) \setminus \Phi_{2M^{-1}, \varepsilon}^\varepsilon(D)} \rho_i^{d-2} + \limsup_{\varepsilon \downarrow 0^+} \varepsilon^d \sum_{z_i \in I_M^\varepsilon} \rho_i^{d-2}$$

On the one hand, by (5.4.34) and Lemma 5.5.3, the second term on the right-hand side vanishes. On the other hand, Lemma 5.5.2 for both Φ and $\Phi^{2M^{-1}}$ imply that

$$\limsup_{\varepsilon \downarrow 0^+} \sum_{z_i \in n^\varepsilon} \|\nabla w_2^{\varepsilon, i}\|_2^2 \mathbf{1}_{d_i \leq M^{-1}\varepsilon} \leq \langle N(Q) - N_{2M^{-1}}(Q) \rangle \langle \rho^{d-2} \rangle,$$

where Q is a unit cube. Finally, the right-hand side in the above estimate converges to zero in the limit $M \rightarrow \infty$ again by Lemma 5.5.2. If we now wrap up the previous estimate with (5.4.56) and

(5.4.55), we conclude (5.4.54). We thus established (H1) and (H2) for the sequence w_2^ε (and thus also for w^ε).

It remains to prove (H3). We consider again the truncated sequences $\{w_2^{\varepsilon,M}\}_{\varepsilon>0}$ above and start by arguing that it is enough to show that, for every $M \in \mathbb{N}$ fixed, condition (H3) is satisfied by $w_2^{\varepsilon,M}$, namely

$$(-\Delta w_2^{\varepsilon,M}, v_\varepsilon)_{H^{-1}, H_0^1} \rightarrow C_{0,M} \int_D v, \quad (5.4.58)$$

where $C_{0,M} := (d-2)\sigma_d \langle N_{2M-1}(Q) \rangle \langle \rho_M^{d-2} \rangle$. In fact, Cauchy-Schwarz's inequality yields

$$|(-\Delta(w_2^\varepsilon - w_2^{\varepsilon,M}), v_\varepsilon)_{H^{-1}, H_0^1}| \leq \left(\int |\nabla v_\varepsilon|^2 \right)^{\frac{1}{2}} \left(\int |\nabla(w_2^\varepsilon - w_2^{\varepsilon,M})|^2 \right)^{\frac{1}{2}},$$

and, as the family $\{v^\varepsilon\}_{\varepsilon>0}$ is uniformly bounded in $H^1(D)$, by (5.4.54) we get

$$\lim_{M \rightarrow \infty} \limsup_{\varepsilon \downarrow 0^+} |(-\Delta(w_2^\varepsilon - w_2^{\varepsilon,M}), v_\varepsilon)_{H^{-1}, H_0^1}| = 0.$$

Since $C_{0,M} \rightarrow C_0$ when $M \uparrow +\infty$ by (5.5.10) of Lemma 5.5.2, the above limit and the triangular inequality yield that to prove (H3) it suffices to show (5.4.58).

The proof of (5.4.58) follows the same lines of Step 3. in Subsection 5.4.1 (in particular, (5.4.24) for $w_2^{\varepsilon,M}$), so we just point out the differences: By arguing as in that case, it suffices to prove that

$$\eta_M^\varepsilon := \sum_{z_i \in n_M^\varepsilon} d(d-2) \rho_{i,M}^{d-2} \frac{\varepsilon^d}{d_i^d} \mathbf{1}_{B_i} \xrightarrow{*} C_{0,M} \quad \text{in } L^\infty(D) \quad (5.4.59)$$

The factor $\varepsilon^d d_i^{-d}$ in the above expression is due to the fact that the balls B_i have now radii d_i instead of ε . Hence, by including this factor, we have

$$\|\varepsilon^d d_i^{-d} \mathbf{1}_{B_i}\|_{L^1} = \|\mathbf{1}_{B_\varepsilon(\varepsilon z_i)}\|_{L^1} = \varepsilon^d$$

as in the periodic case.

Since

$$\|\eta_M^\varepsilon\|_{L^\infty} \lesssim M^{d(d-2)},$$

to show (5.4.59) it suffices to test η_M^ε with functions $\zeta \in C_0^1(D)$. We observe that if we define

$$\tilde{\eta}_M^\varepsilon := \sum_{z_i \in \Phi_{2M-1}^\varepsilon(D)} d(d-2) \rho_{i,M}^{d-2} \frac{\varepsilon^d}{d_i^d} \mathbf{1}_{B_i},$$

then as in Step 3 of Subsection 5.4.1, we use Lemma 5.5.2 for Φ_{2M-1} and Lemma 5.5.4 applied to $(\Phi_{2M-1}, \{\rho_{i,M}^{d-2}\})$ to infer that, almost surely and for all $\zeta \in C_0^1(D)$,

$$\int_D \tilde{\eta}_M^\varepsilon \zeta \rightarrow C_{0,M} \int_D \zeta.$$

We conclude (5.4.59) by arguing that, almost surely and for all $\zeta \in C_0^1(D)$, we have

$$\left| \int_D (\eta_M^\varepsilon - \tilde{\eta}_M^\varepsilon) \zeta \right| \rightarrow 0.$$

Indeed,

$$\begin{aligned} \left| \int_D (\eta_M^\varepsilon - \tilde{\eta}_M^\varepsilon) \zeta \right| &\lesssim M^{d-2} \sum_{z_i \in \Phi_{2M-1}^\varepsilon(D) \setminus n_M^\varepsilon} \int_{B_\varepsilon(\varepsilon z_i)} |\zeta| \\ &\lesssim M^{d-2} \|\zeta\|_\infty \varepsilon^d \# \left(\left\{ z_i \in \Phi_{2M-1}^\varepsilon(D) : d_i \leq \frac{\varepsilon}{M} \right\} \right). \end{aligned}$$

By definition of d_i and (5.4.34), the right-hand side tends to zero almost surely in the limit $\varepsilon \rightarrow 0$. This concludes the proof of (5.4.59) and thus establishes (H3) for the sequence $\{w_2^\varepsilon\}_{\varepsilon>0}$. The proof of Lemma 5.3.1 is complete. \square

5.5 Auxiliary results

In this section, we give some auxiliary results which have been used in the previous Sections. In Section 5.5.1, we present a version of the Strong Law of Large Numbers adapted to our needs. We use this result in Section 5.5.2 where we prove variants of the Strong Law of Large Numbers for marked processes on which the proofs of Section 5.4 rely. Finally, in Section 5.5.3, we verify the conditions of Theorem 5.2.1 for the processes defined in case (c) of Section 5.2.

5.5.1 Strong Law of Large Numbers for sums of random variables with correlations

This result is an easy adaptation to our setting, and to our needs, of the standard argument for the Strong Law of Large Numbers.

Lemma 5.5.1. *Let $\{x_i\}_{i \in \mathbb{N}} = \mathbb{Z}^d$, and let $\{X_i\}_{i \in \mathbb{N}}$ be identically distributed random variables with $X_i \geq 0$ and $\langle X \rangle < +\infty$. Let us assume that for every $i, j \in \mathbb{N}$ with $i \neq j$*

$$|\langle X_i X_j \rangle - \langle X \rangle^2| < \frac{C}{|x_i - x_j|^\gamma} \quad \gamma > d. \quad (5.5.1)$$

Then for every bounded Borel set $B \subset \mathbb{R}^d$ which is star-shaped with respect to the origin, we have

$$\lim_{\varepsilon \downarrow 0^+} \varepsilon^d \sum_{x_i \in \mathbb{Z}^d \cap \frac{1}{\varepsilon} B} X_i = \langle X \rangle |B| \quad \text{almost surely.} \quad (5.5.2)$$

Proof. The proof of this lemma is an easy adaptation of the standard argument for independent and identically distributed random variables: In particular, we adapt to the case of correlated variables the argument of [Dur10, Subsection 2.4].

Without loss of generality, we assume that $B = Q$, where Q is the unitary cube centred at the origin and $Q_\varepsilon^{\frac{1}{\varepsilon}} = \frac{1}{\varepsilon} Q$. Moreover, we may assume that $|x_i|$ is monotone in $i \in \mathbb{N}$. Thus, there exists a constant $c = c(d)$ such that for all $i \in \mathbb{N}$

$$|x_i| \geq ci^{\frac{1}{d}}. \quad (5.5.3)$$

The first step is to reduce the study of (5.5.2) to the sum of the truncated random variables $Y_i := X_i \mathbf{1}_{X_i \leq i}$. Indeed,

$$\sum_{i=1}^{\infty} \mathbb{P}(X_i > i) \leq \int_0^{\infty} \mathbb{P}(X > t) dt = \langle X \rangle < \infty.$$

Thus, by Borel-Cantelli theorem applied to the events $E_i := \{X_i > i\}$ we have that almost surely in (5.5.2) we may substitute the variables X_i with their truncated versions Y_i . Clearly, also the sequence $\{Y_i\}_{i \in \mathbb{N}}$ satisfies (5.5.1).

We define $\tilde{Y}_i := Y_i - \langle Y_i \rangle$. The next step of the proof is the following estimate:

$$\sum_{i=1}^{\infty} \frac{\langle \tilde{Y}_i^2 \rangle}{i^2} \leq 4\langle X \rangle < \infty. \quad (5.5.4)$$

We estimate

$$\langle \tilde{Y}_i^2 \rangle \leq \langle Y_i^2 \rangle \int_0^\infty 2y \mathbb{P}(Y_i > y) dy \leq \int_0^i 2y \mathbb{P}(X > y) dy.$$

Using the monotone convergence theorem, this yields

$$\sum_{i=1}^{\infty} \frac{\langle \tilde{Y}_i^2 \rangle}{i^2} \leq \sum_{i=1}^{\infty} \frac{1}{i^2} \int_0^\infty \mathbf{1}_{(0,i)}(y) 2y \mathbb{P}(X > y) dy \leq \int_0^\infty \sum_{i>y} \frac{1}{i^2} 2y \mathbb{P}(X > y) dy.$$

Since $\int_0^\infty \mathbb{P}(X > y) dy = \langle X \rangle$, to prove (5.5.4) it suffices to show

$$y \sum_{i>y} \frac{1}{i^2} \leq 2.$$

If $y \geq 1$, then

$$y \sum_{i>y} \frac{1}{i^2} = y \sum_{i=\lfloor y \rfloor + 1}^{\infty} \frac{1}{i^2} \leq y \int_{\lfloor y \rfloor}^{\infty} \frac{1}{t^2} dt = \frac{y}{\lfloor y \rfloor} \leq 2.$$

If $0 < y < 1$,

$$y \sum_{i>y} \frac{1}{i^2} \leq 1 + \sum_{i=2}^{\infty} \frac{1}{i^2} \leq 1 + \int_1^{\infty} \frac{1}{t^2} dt = 2.$$

This concludes the proof of (5.5.4).

Next, we define

$$S_\varepsilon := \sum_{i \in \mathbb{Z}^d \cap Q^{\frac{1}{\varepsilon}}} Y_i, \quad \tilde{S}_\varepsilon := \sum_{i \in \mathbb{Z}^d \cap Q^{\frac{1}{\varepsilon}}} \tilde{Y}_i.$$

Then, for every $\delta > 0$, we estimate by Chebyshev's inequality

$$\mathbb{P}(\varepsilon^d \tilde{S}_\varepsilon > \delta) \leq \varepsilon^{2d} \frac{\langle \tilde{S}_\varepsilon^2 \rangle}{\delta^2} = \delta^{-2} \varepsilon^{2d} \left\langle \sum_{j, i \in \mathbb{Z}^d \cap Q^{\frac{1}{\varepsilon}}} \tilde{Y}_i \tilde{Y}_j \right\rangle.$$

By definition of Y_i and assumption (5.5.1) the last term is bounded by

$$\mathbb{P}(\varepsilon^d \tilde{S}_\varepsilon > \delta) \leq \delta^{-2} \varepsilon^{2d} \sum_{i \in \mathbb{Z}^d \cap Q^{\frac{1}{\varepsilon}}} \langle \tilde{Y}_i^2 \rangle + \delta^{-2} \varepsilon^{2d} \sum_{\substack{j, i \in \mathbb{Z}^d \cap Q^{\frac{1}{\varepsilon}} \\ i \neq j}} \frac{C}{|i - j|^\gamma}.$$

We now restrict ourselves to consider the sequence $\varepsilon_k := \alpha^k$, $k \in \mathbb{N}$ and $\alpha \in (0, 1)$ and use the previous inequality to estimate

$$\sum_{k=1}^{+\infty} \mathbb{P}(\varepsilon_k^d \tilde{S}_{\varepsilon_k} > \delta) \leq \delta^{-2} \sum_{k=1}^{+\infty} \varepsilon_k^{2d} \sum_{i \in \mathbb{Z}^d \cap Q_{\frac{1}{\varepsilon_k}}} \langle \tilde{Y}_i^2 \rangle + \delta^{-2} \sum_{k=1}^{+\infty} \varepsilon_k^{2d} \sum_{\substack{j, i \in \mathbb{Z}^d \cap Q_{\frac{1}{\varepsilon_k}} \\ i \neq j}} \frac{C}{|z_i - z_j|^\gamma}. \quad (5.5.5)$$

For the second term on the right-hand side above, thanks to assumption $\gamma > d$, we have

$$\sum_{k=1}^{+\infty} \varepsilon_k^{2d} \sum_{j \neq i \in \mathbb{Z}^d \cap Q_{\frac{1}{\varepsilon_k}}} \frac{C}{|z_i - z_j|^\gamma} \leq \sum_{k=1}^{+\infty} \varepsilon_k^d < +\infty.$$

To estimate the first term on the right-hand side in (5.5.5), we can interchange the order of the sums since all terms are nonnegative. Thus,

$$\begin{aligned} \sum_{k=1}^{+\infty} \varepsilon_k^{2d} \sum_{i \in \mathbb{Z}^d \cap Q_{\frac{1}{\varepsilon_k}}} \langle \tilde{Y}_i^2 \rangle &= \sum_{i \in \mathbb{Z}^d} \langle \tilde{Y}_i^2 \rangle \sum_{k=1}^{+\infty} \varepsilon_k^{2d} \mathbf{1}_{x_i \in Q_{\frac{1}{\varepsilon_k}}} \stackrel{(5.5.3)}{\leq} \sum_{i \in \mathbb{Z}^d} \langle \tilde{Y}_i^2 \rangle \sum_{k: \varepsilon_k^{-d} \leq C i} \varepsilon_k^{2d} \\ &\lesssim \sum_{i \in \mathbb{Z}^d} \langle \tilde{Y}_i^2 \rangle \frac{1}{i^{2d}} \frac{1}{1 - \alpha^{2d}} \stackrel{(5.5.4)}{\lesssim} \langle X \rangle < \infty \end{aligned}$$

Therefore, for every $\delta > 0$ we have that $\sum_{k=1}^{+\infty} \mathbb{P}(\varepsilon_k^d \tilde{S}_{\varepsilon_k} > \delta) < +\infty$ and by Borel-Cantelli's lemma and the Dominated Convergence theorem we get

$$\tilde{S}_{\varepsilon_k} \rightarrow 0 \quad \text{almost surely.}$$

Since $\lim_{i \rightarrow \infty} \langle Y_i \rangle = \langle X \rangle$, this implies also

$$S_{\varepsilon_k} \rightarrow \langle X \rangle \quad \text{almost surely.} \quad (5.5.6)$$

To pass to the continuum limit $\varepsilon \downarrow 0^+$ for the same full-probability set, we argue as in [Dur10] by monotonicity. Indeed, for $\varepsilon_{k+1} \leq \varepsilon \leq \varepsilon_k$, we have

$$\mathbb{Z}^d \cap Q_{\frac{1}{\varepsilon_k}} \subset \mathbb{Z}^d \cap Q_{\frac{1}{\varepsilon}} \subset \mathbb{Z}^d \cap Q_{\frac{1}{\varepsilon_{k+1}}}$$

Hence, since $Y_i \geq 0$, it holds

$$\frac{\#(Q_{\frac{1}{\varepsilon_{k+1}}})}{\#(Q_{\frac{1}{\varepsilon_k}})} S_{\varepsilon_k} \leq S_\varepsilon \leq \frac{\#(Q_{\frac{1}{\varepsilon_{k+1}}})}{\#(Q_{\frac{1}{\varepsilon_k}})} S_{\varepsilon_{k+1}}. \quad (5.5.7)$$

By the choice $\varepsilon_k = \alpha^k$, we obtain

$$\frac{\#(Q_{\frac{1}{\varepsilon_{k+1}}})}{\#(Q_{\frac{1}{\varepsilon_k}})} \rightarrow \alpha^{-d}.$$

We now combine (5.5.6) and (5.5.7) to infer

$$\liminf_{\varepsilon \rightarrow 0} S_\varepsilon \geq \alpha^d \langle X \rangle, \quad \limsup_{\varepsilon \rightarrow 0} S_\varepsilon \leq \alpha^{-d} \langle X \rangle.$$

Since $\alpha \in (0, 1)$ is arbitrary, we conclude the proof by sending $\alpha \rightarrow 1^-$ in the inequalities above. \square

5.5.2 Strong Law of Large Numbers for marked point processes

We give these results for a general marked point process (Φ, \mathcal{X}) with Φ satisfying (5.2.2) and (5.2.3) and with the marks $\mathcal{X} := \{X_i\}_{z_i \in \Phi}$ satisfying (5.2.4) with

$$\langle X \rangle = \int_0^{+\infty} xh(x)dx < +\infty$$

and with the function g being bounded as in (5.2.5) (with ρ substituted by x and with $p > 2$).

Lemma 5.5.2. *Let Q a unitary cube and let (Φ, \mathcal{X}) be a marked point process as introduced above. Then, for every bounded set $B \subset \mathbb{R}^d$ which is star-shaped with respect to the origin, we have*

$$\lim_{\varepsilon \downarrow 0^+} \varepsilon^d N^\varepsilon(B) = \langle N(Q) \rangle |B| \quad \text{almost surely,} \quad (5.5.8)$$

and

$$\lim_{\varepsilon \downarrow 0^+} \varepsilon^d \sum_{z_i \in \Phi^\varepsilon(B)} X_i = \langle N(Q) \rangle \langle X \rangle |B| \quad \text{almost surely.} \quad (5.5.9)$$

Furthermore, for every $\delta < 0$ the process Φ_δ obtained from Φ as in (5.4.3) satisfies the analogues of (5.5.9), (5.5.8) and

$$\lim_{\delta \downarrow 0^+} \langle N_\delta(A) \rangle = \langle N(A) \rangle \quad (5.5.10)$$

for every bounded set $A \subset \mathbb{R}^d$.

Lemma 5.5.3. *In the same setting of Lemma 5.5.2, let $\{I_\varepsilon\}_{\varepsilon > 0}$ be a family of collections of points such that $I_\varepsilon \subset \Phi^\varepsilon(B)$ and*

$$\lim_{\varepsilon \downarrow 0^+} \varepsilon^d \#I_\varepsilon = 0 \quad \text{almost surely.} \quad (5.5.11)$$

Then,

$$\lim_{\varepsilon \downarrow 0^+} \varepsilon^d \sum_{z_i \in I_\varepsilon} X_i \rightarrow 0 \quad \text{almost surely.}$$

Lemma 5.5.4. *In the same setting of Lemma 5.5.2, let us assume that in addition the marks satisfy $\langle X^2 \rangle < +\infty$. For $z_i \in \Phi$ and $\varepsilon > 0$, let $r_{i,\varepsilon} > 0$, and assume there exists a constant $C > 0$ such that for all $z_i \in \Phi$ and $\varepsilon > 0$*

$$r_{i,\varepsilon} \leq C\varepsilon.$$

Then, almost surely, we have

$$\lim_{\varepsilon \downarrow 0^+} \sum_{z_i \in \Phi^\varepsilon(B)} X_i \frac{\varepsilon^d}{r_{i,\varepsilon}^d} \int_{B_{r_{i,\varepsilon}}(\varepsilon z_i)} \zeta(x) dx = \frac{\sigma_d}{d} \langle N(Q) \rangle \langle X \rangle \int_B \zeta(x) dx,$$

for every $\zeta \in C_0^1(B)$.

Proof of Lemma 5.5.2. Without loss of generality we assume that $B = Q^R$, with Q^R the cube of size R centred at the origin and $Q^{\frac{R}{\varepsilon}} = \frac{1}{\varepsilon}B$. Moreover, we denote by $\{Q_i\}_{i \in \mathbb{Z}^d}$ the partition of \mathbb{R}^d made of (essentially) disjoint unit cubes centred in the points of the lattice $\mathbb{Z}^d = \{x_i\}_{i \in \mathbb{N}}$.

The limit (5.5.9) is an easy consequence of a Strong Law of Large Numbers for correlated random variables, Lemma 5.5.1: For all $\mu > 0$ and all ε small enough

$$\varepsilon^d \sum_{z_i \in \Phi^\varepsilon(Q^R)} X_i \leq \varepsilon^d \sum_{x_j \in \mathbb{Z}^d \cap Q^{\frac{R+\mu}{\varepsilon}}} Z_j, \quad (5.5.12)$$

where $Z_j := \sum_{z_i \in \Phi(Q_j)} X_i$ are identically distributed random variables by stationarity of (Φ, \mathcal{X}) . Moreover, they have finite average

$$\left\langle \sum_{z_i \in \Phi(Q)} X_i \right\rangle = \langle N(Q) \rangle \langle X \rangle < +\infty \quad (5.5.13)$$

and satisfy for every $i, j \in \mathbb{N}$ with $i \neq j$

$$\begin{aligned} |\langle Z_i Z_j \rangle - \langle Z \rangle^2| &\stackrel{(5.5.13)}{=} \left| \left\langle \sum_{\substack{z_k \in \Phi(Q_i) \\ z_l \in \Phi(Q_j)}} X_k X_l \right\rangle - \langle N(Q) \rangle^2 \langle X \rangle^2 \right| \\ &\stackrel{(5.2.5)}{\leq} |\langle X \rangle^2 \langle N(Q_i) N(Q_j) \rangle - \langle N(Q) \rangle^2 \langle X \rangle^2| + \frac{C}{|x_i - x_j|^\gamma} \langle N(Q_i) N(Q_j) \rangle, \end{aligned} \quad (5.5.14)$$

where the constant C depends on the constants in (5.2.5). We now appeal to condition (5.2.3): By the stationarity assumption on Φ , we have that for any $i, j \in \mathbb{N}$

$$\langle N(Q_i) N(Q_j) \rangle = \langle N(Q_{i-j}) N(Q) \rangle,$$

so that (5.2.3) applied to the random variables $N(Q_{i-j})$ and $N(Q)$ yields

$$|\langle N(Q_i) N(Q_j) \rangle - \langle N(Q) \rangle^2| \lesssim \frac{\langle N(Q)^2 \rangle}{|x_i - x_j|^\gamma}.$$

We thus insert this bound into (5.5.14) and get

$$|\langle Z_i Z_j \rangle - \langle Z \rangle^2| \leq \frac{C \langle N(Q)^2 \rangle}{|x_i - x_j|^\gamma}.$$

Hence, condition (5.5.1) is satisfied with constant $C \langle N(Q)^2 \rangle < +\infty$, where C depends on the constants in (5.2.3) and (5.2.4). We apply Lemma 5.5.1 to the sequence $\{Z_i\}_{i \in \mathbb{Z}^d}$ in (5.5.12) and conclude

$$\limsup_{\varepsilon \downarrow 0^+} \varepsilon^d \sum_{z_i \in \Phi^\varepsilon(Q^R)} X_i \leq \langle N(Q) \rangle \langle X \rangle |Q^{R+\mu}|.$$

Arguing analogously for the limit inferior and taking the limit $\mu \rightarrow 0$ yields (5.5.9). Limit (5.5.8) follows exactly as (5.5.9) by substituting the marks X_i with 1.

For $\delta > 0$ be fixed. We show that Φ_δ satisfies the analogues of (5.5.8) and (5.5.9) together with (5.5.10). Since by definition (5.4.3) we have that $N^\delta(B) \leq N(B)$, the limit in (5.5.10) follows from the Dominated Convergence Theorem. To show (5.5.9) and (5.5.8) we may argue exactly as above

for the original process Φ and apply Lemma 5.5.1 to the random variables $Z_i^\delta := \sum_{z_i \in \Phi^\delta(Q_j)} X_i$. Since for each $x_i \in \mathbb{Z}^d$ we have $0 \leq Z_i^\delta \leq Z_i$, the only condition that remains to be shown for the collection $\{Z_i^\delta\}_{x_i \in \mathbb{Z}^d}$ is (5.5.1). By arguing as in (5.5.14), we use again (5.2.5) to reduce ourselves to show (5.5.1) for the random variables $\{N^\delta(Q_i)\}_{i \in \mathbb{Z}^d}$. To do so, for any $x \in \mathbb{R}^d$ we define

$$d_x := \min_{\substack{y \in \Phi(\omega), \\ y \neq x}} |x - y|,$$

so that

$$N^\delta(Q) = \sum_{z_i \in \Phi \cap Q} \mathbf{1}_{d_x > \delta}(z_i), \quad N^\delta(Q_i) = \sum_{z_i \in \tau_{-x_i} \Phi \cap Q} \mathbf{1}_{d_x > \delta}(z_i).$$

Since $\mathbf{1}_{d_x > \delta} = \mathbf{1}_{N(B_\delta(x) \setminus \{x\})=0}$, each $N^\delta(Q_i)$ are measurable random variables with respect to $\mathcal{F}(B_\delta(Q_i))$ defined in (5.2.3), with

$$B_\delta(Q_i) := \{x \in \mathbb{R}^d : \text{dist}(x, Q_i) \leq \delta\}.$$

We thus apply (5.2.3) as above and conclude that, with a constant depending on δ , condition (5.5.1) is satisfied by the sequence $N^\delta(Q_i)$. This yields (5.5.2) for Φ^δ and, by the same argument, also (5.5.8). \square

Proof of Lemma 5.5.3. Let $M \in \mathbb{N}$. For every $z_i \in \Phi$, we define truncated marks $Y_i := X_i \mathbf{1}_{[M, \infty)}(X_i)$ which satisfy assumption (5.2.4) and (5.2.5) thanks to the corresponding assumptions for the original marks $\{X_i\}_{i \in \mathbb{N}}$. Since

$$\langle Y_i \rangle \leq \langle X \rangle < +\infty,$$

we apply Lemma 5.5.2 to the point process Φ with the truncated marks $\{Y_i\}_{z_i \in \Phi}$ to infer that almost surely

$$\varepsilon^d \sum_{z_i \in \Phi^\varepsilon(B)} Y_i \rightarrow \langle X \mathbf{1}_{[M, \infty)}(X) \rangle.$$

This yields

$$\begin{aligned} \limsup_{\varepsilon \downarrow 0^+} \varepsilon^d \sum_{z_i \in I_\varepsilon} X_i &\leq \limsup_{\varepsilon \downarrow 0^+} \varepsilon^d \sum_{z_i \in I_\varepsilon} X_i \mathbf{1}_{[0, M)}(X_i) + \langle X \mathbf{1}_{[M, \infty)}(X) \rangle \\ &\leq M \limsup_{\varepsilon \downarrow 0^+} \varepsilon^d \#I_\varepsilon + \langle X \mathbf{1}_{[M, \infty)}(X) \rangle \stackrel{(5.5.11)}{=} \langle X \mathbf{1}_{[M, \infty)}(X) \rangle. \end{aligned}$$

Since $\langle X \rangle < +\infty$, we may take the limit $M \rightarrow \infty$ and conclude the proof. \square

Proof of Lemma 5.5.4. First, we argue that it suffices to prove the case $r_{i, \varepsilon} = \varepsilon$ for all $z_i \in \Phi$ and $\varepsilon > 0$. Indeed, for $\zeta \in C_0^1(B)$ we use a change of coordinates to get almost surely,

$$\limsup_{\varepsilon \downarrow 0^+} \sum_{z_i \in \Phi^\varepsilon(B)} \left| \frac{\varepsilon^d}{r_{i, \varepsilon}^d} \int_{B_{r_{i, \varepsilon}}(\varepsilon z_i)} \zeta(x) dx - \int_{B_\varepsilon(\varepsilon z_i)} \zeta(x) dx \right| \leq \limsup_{\varepsilon \downarrow 0^+} C\varepsilon \|\nabla \zeta\|_{L^\infty} \varepsilon^d N^\varepsilon(B) = 0$$

since $\varepsilon^d N^\varepsilon(B)$ is bounded by Lemma 5.5.2.

Without loss of generality we therefore assume $r_{i, \varepsilon} = \varepsilon$ and $|B| = 1$.

Next we observe that it suffices to argue that the assertion holds for any fixed $\zeta \in W_0^{1, \infty}(B)$. Indeed, once we have shown this, the statement follows because there exists a countable subset of $W_0^{1, \infty}(B)$ which is dense in $C_0^1(B)$.

We fix $\zeta \in W_0^{1,\infty}(B)$ and we begin by rewriting the term in the limit as

$$\begin{aligned} \sum_{z_i \in \Phi^\varepsilon(B)} X_i \int_{B_\varepsilon(\varepsilon z_i)} \zeta(x) dx &= \sum_{z_i \in \Phi^\varepsilon(B)} (X_i - \langle X \rangle) \int_{B_\varepsilon(\varepsilon z_i)} \zeta(x) dx \\ &\quad + \langle X \rangle \sum_{z_i \in \Phi^\varepsilon(B)} \int_{B_\varepsilon(\varepsilon z_i)} \zeta(x) dx, \end{aligned}$$

so that

$$\begin{aligned} &\left| \sum_{z_i \in \Phi^\varepsilon(B)} X_i \int_{B_\varepsilon(\varepsilon z_i)} \zeta(x) dx - \frac{\sigma_d}{d} \langle N(Q) \rangle \langle X \rangle \int_B \zeta \right| \\ &\leq \left| \sum_{z_i \in \Phi^\varepsilon(B)} (X_i - \langle X \rangle) \int_{B_\varepsilon(\varepsilon z_i)} \zeta(x) dx \right| + \langle X \rangle \left| \sum_{z_i \in \Phi^\varepsilon(B)} \int_{B_\varepsilon(\varepsilon z_i)} \zeta(x) dx - \langle N(Q) \rangle \int_B \zeta \right|. \end{aligned} \quad (5.5.15)$$

Let $\{Q_i\}_{i \in \mathbb{N}}$ be a partition of \mathbb{R}^d into (essentially) disjoint unitary cubes and let $\{y_i\}_{i \in \mathbb{N}}$ be the collection of their centres. We claim that if

$$\begin{aligned} T_\varepsilon(\zeta) &:= \int_B \zeta, & \tilde{T}_\varepsilon(\zeta) &:= \varepsilon^d \sum_{Q_i \cap \frac{1}{\varepsilon} B \neq \emptyset} \zeta(\varepsilon y_i), \\ R_\varepsilon(\zeta) &:= \sum_{z \in \Phi^\varepsilon(B)} \int_{B_\varepsilon(\varepsilon z)} \zeta(x) dx, & \tilde{R}_\varepsilon(\zeta) &:= \varepsilon^d \frac{\sigma_d}{d} \sum_{Q_i \cap \frac{1}{\varepsilon} B \neq \emptyset} N(Q_i) \zeta(\varepsilon y_i), \end{aligned}$$

then

$$\lim_{\varepsilon \downarrow 0^+} |T_\varepsilon(\zeta) - \tilde{T}_\varepsilon(\zeta)| = 0, \quad \lim_{\varepsilon \downarrow 0^+} |R_\varepsilon(\zeta) - \tilde{R}_\varepsilon(\zeta)| = 0 \quad \text{almost surely.} \quad (5.5.16)$$

The first limit is a standard Riemann sum; for the second limit we argue in a similar way: Since $\zeta \in W_0^{1,\infty}(B)$, we have that

$$\begin{aligned} |R_\varepsilon(\zeta) - \tilde{R}_\varepsilon(\zeta)| &= \left| \sum_{Q_i \cap \frac{1}{\varepsilon} B \neq \emptyset} \left(\sum_{z_j \in \Phi(Q_i)} \int_{B_\varepsilon(\varepsilon z_j)} \zeta - \varepsilon^d \frac{\sigma_d}{d} N(Q_i) \zeta(\varepsilon y_i) \right) \right| \\ &= \left| \sum_{Q_i \cap \frac{1}{\varepsilon} B \neq \emptyset} \sum_{z_j \in \Phi(Q_i)} \int_{B_\varepsilon(\varepsilon z_j)} (\zeta(x) - \zeta(\varepsilon y_i)) \right| \leq 2 \|\nabla \zeta\|_\infty \varepsilon^{d+1} N^\varepsilon(B). \end{aligned}$$

We now apply (5.5.8) of Lemma 5.5.2 to infer (5.5.16). This, together with (5.5.15) and the triangular inequality implies that almost surely

$$\begin{aligned} &\limsup_{\varepsilon \downarrow 0^+} \left| \sum_{z_i \in \Phi^\varepsilon(B)} X_i \int_{B_\varepsilon(\varepsilon z_i)} \zeta(x) dx - \frac{\sigma_d}{d} \langle N(Q) \rangle \langle X \rangle \int_B \zeta \right| \\ &\leq \limsup_{\varepsilon \downarrow 0^+} \left| \sum_{z_i \in \Phi^\varepsilon(B)} (X_i - \langle X \rangle) \int_{B_\varepsilon(\varepsilon z_i)} \zeta(x) dx \right| \\ &\quad + \limsup_{\varepsilon \downarrow 0^+} \left| \varepsilon^d \langle X \rangle \frac{\sigma_d}{d} \sum_{Q_i \cap \frac{1}{\varepsilon} B \neq \emptyset} \zeta(\varepsilon y_i) (N(Q_i) - \langle N(Q) \rangle) \right|. \end{aligned} \quad (5.5.17)$$

It remains to show that also the previous two terms on the right-hand side above vanish almost surely. Since to do this we follow an argument very similar to the one of Lemma 5.5.1, we only give the details of the parts in which the proof differs. For $\varepsilon > 0$ let

$$a_{i,\varepsilon} := \int_{B_\varepsilon(\varepsilon z_i)} \zeta(x) dx, \quad \tilde{X}_i := X_i - \langle X_i \rangle$$

and

$$S_\varepsilon := \sum_{z_i \in \Phi^\varepsilon(B)} a_{i,\varepsilon} X_i, \quad \tilde{S}_\varepsilon := \sum_{z_i \in \Phi^\varepsilon(B)} a_{i,\varepsilon} \tilde{X}_i.$$

We start by proving that the first term on the right-hand side of (5.5.17), i.e. \tilde{S}_ε above, vanishes in the limit; we may argue analogously that also second term on the right-hand side of (5.5.17) vanishes.

As in the proof of Lemma 5.5.1, we may use Chebyshev's inequality to estimate for each $\delta > 0$

$$\mathbb{P}(\tilde{S}_\varepsilon > \delta) \leq \delta^{-2} \langle \tilde{S}_\varepsilon^2 \rangle \quad (5.5.18)$$

and rewrite

$$\langle \tilde{S}_\varepsilon^2 \rangle = \left\langle \sum_{z_i, z_k \in \Phi(\frac{1}{\varepsilon}B)} a_{i,\varepsilon} a_{k,\varepsilon} \tilde{X}_i \tilde{X}_k \right\rangle = \sum_{\substack{Q_j \cap \frac{1}{\varepsilon}B \neq \emptyset \\ Q_i \cap \frac{1}{\varepsilon}B \neq \emptyset}} \left\langle \left(\sum_{z_l \in \Phi(Q_j)} a_{j,\varepsilon} \tilde{X}_j \right) \left(\sum_{z_k \in \Phi(Q_i)} a_{k,\varepsilon} \tilde{X}_k \right) \right\rangle.$$

If we now set $Y_i := \sum_{z_l \in \Phi(Q_j)} a_{j,\varepsilon} \tilde{X}_j$, since all $|a_{\varepsilon,i}| \leq \|\zeta\|_{L^\infty} \varepsilon^d$, we argue as for the random variables $\{Z_i\}_{i \in \mathbb{N}}$ in the proof of Lemma 5.5.2 and infer that

$$\begin{aligned} \langle \tilde{S}_\varepsilon^2 \rangle &\leq \sum_{Q_i \cap \frac{1}{\varepsilon}B \neq \emptyset} \|\zeta\|_\infty^2 \varepsilon^{2d} \langle N(Q)^2 \rangle \text{Var}(X) + C \varepsilon^{2d} \|\zeta\|_\infty^2 \sum_{\substack{Q_j \cap \frac{1}{\varepsilon}B \neq \emptyset \\ Q_i \cap Q_j = \emptyset}} \frac{\langle N(Q)^2 \rangle \langle X \rangle^2}{|x_i - x_j|^\gamma} \\ &\lesssim^{\gamma > d} \varepsilon^d \|\zeta\|_\infty^2 \langle N(Q)^2 \rangle \langle X^2 \rangle. \end{aligned}$$

Therefore, if we plug this into (5.5.18) and apply Borel-Cantelli's lemma to the subsequence $\varepsilon_n = \frac{1}{n}$ with $n \in \mathbb{N}$, we get that

$$\lim_{n \uparrow +\infty} \tilde{S}_{\varepsilon_n} = 0 \quad \text{almost surely.}$$

We appeal to an estimate similar to this one also for the second term on the right hand side of (5.5.17) (this time using the assumption (5.2.3)) and conclude from (5.5.17) that for the sequence $\{\varepsilon_n\}_{n \in \mathbb{N}}$ we have almost surely that

$$\lim_{n \uparrow +\infty} \sum_{z_i \in \Phi(\frac{1}{\varepsilon_n}B)} X_i \int_{B_{\varepsilon_n}(\varepsilon_n z_i)} \zeta(x) dx = \frac{\sigma_d}{d} \langle N(Q) \rangle \langle X \rangle \int_B \zeta(x) dx. \quad (5.5.19)$$

To extend (5.5.19) to any sequence $\varepsilon_j \downarrow 0$ and for the same full-probability set, we argue again similarly to Lemma 5.5.1. We first fix the following notation: For $0 < \varepsilon < 1$, we define

$$\underline{\varepsilon} := \left(\left\lfloor \frac{1}{\varepsilon} \right\rfloor + 1 \right)^{-1}, \quad \bar{\varepsilon} := \left(\left\lfloor \frac{1}{\varepsilon} \right\rfloor \right)^{-1}.$$

Note that $\bar{\varepsilon}^{-1}, \underline{\varepsilon}^{-1} \in \mathbb{N}$ and $\underline{\varepsilon} \leq \varepsilon \leq \bar{\varepsilon}$. By writing $\zeta = \zeta_+ + \zeta_-$ and using linearity, we observe that it suffices to consider non-negative functions ζ and thus reduce ourselves to the case $a_{i,\varepsilon} \geq 0$.

For any $\varepsilon_j \downarrow 0^+$ we may use the triangle inequality and the assumptions on the sign of the weights and the X_i 's to bound

$$S_{\varepsilon_j} \leq S_{\underline{\varepsilon}_j} + \sum_{i=1}^{N^{\varepsilon_j}(B)} |a_{i,\varepsilon_j} - a_{i,\underline{\varepsilon}_j}| X_i \leq S_{\underline{\varepsilon}_j} + \max_{i=1, \dots, N^{\varepsilon_j}(B)} |a_{i,\varepsilon_j} - a_{i,\underline{\varepsilon}_j}| \sum_{i=1}^{N^{\varepsilon_j}(B)} X_i. \quad (5.5.20)$$

We now claim that the weights are uniformly continuous in the second index, uniformly in the first index: More precisely we have that almost surely

$$\lim_{\varepsilon \downarrow 0^+} \frac{\max_{i \leq \#N^{\varepsilon}(B)} |a_{i,\varepsilon} - a_{i,\bar{\varepsilon}}|}{\bar{\varepsilon}^d} = \lim_{\varepsilon \downarrow 0^+} \frac{\max_{i \leq \#N^{\varepsilon}(B)} |a_{i,\varepsilon} - a_{i,\underline{\varepsilon}}|}{\underline{\varepsilon}^d} = 0. \quad (5.5.21)$$

We first argue that, if this is true, the proof of the lemma is concluded: From (5.5.20) we indeed obtain

$$S_{\varepsilon_j} \leq S_{\underline{\varepsilon}_j} + \frac{\max_{i=1, \dots, N^{\varepsilon_j}(B)} |a_{i,\varepsilon_j} - a_{i,\underline{\varepsilon}_j}|}{\underline{\varepsilon}_j^d} \sum_{i=1}^{N^{\varepsilon_j}(B)} X_i.$$

Limit (5.5.19), Lemma 5.5.2 and the second limit in (5.5.21) yield

$$\limsup_{\varepsilon_j \downarrow 0} S_{\varepsilon_j} \leq \frac{\sigma_d}{d} \langle N(Q) \rangle \langle X \rangle \int \zeta.$$

We may argue similarly as in (5.5.20) for the bound from below and get that

$$\liminf_{\varepsilon_j \downarrow 0} S_{\varepsilon_j} \geq \frac{\sigma_d}{d} \langle N(Q) \rangle \langle X \rangle \int \zeta.$$

This yields the claim of Lemma 5.5.4.

It thus remains to establish (5.5.21): Since $\zeta W_0^{1,\infty}(B)$, for any choice of $z_i \in B$ and $\varepsilon_1 \leq \varepsilon_2$ we estimate

$$\begin{aligned} |a_{i,\varepsilon_1} - a_{i,\varepsilon_2}| &= \int_{B_{\varepsilon_1}(0)} |\zeta(x + \varepsilon_1 z_i) - \zeta(x + \varepsilon_2 z_i)| dx + \int_{B_{\varepsilon_2}(0) \setminus B_{\varepsilon_1}(0)} \zeta(x + \varepsilon_2 z_i) dx \\ &\leq \|\nabla \zeta\|_{\infty} |\varepsilon_2 - \varepsilon_1| |z_i| \varepsilon_1^d + \|\zeta\|_{\infty} \left(\left(\frac{\varepsilon_2}{\varepsilon_1} \right)^d - 1 \right) \varepsilon_1^d. \end{aligned}$$

Since $N^{\varepsilon_2}(B) \leq N^{\varepsilon_1}(B)$ and thus $i \leq N^{\varepsilon_2}(B)$, we have that $|z_i| \leq \varepsilon_2^{-1}$ and

$$|a_{i,\varepsilon_1} - a_{i,\varepsilon_2}| \leq \|\zeta\|_{W^{1,\infty}} \left(\left(1 - \frac{\varepsilon_1}{\varepsilon_2} \right) + \left(\left(\frac{\varepsilon_2}{\varepsilon_1} \right)^d - 1 \right) \right) \varepsilon_1^d.$$

Therefore, for the choice $\varepsilon_1 = \varepsilon$, $\varepsilon_2 = \bar{\varepsilon}$ this yields

$$|a_{i,\varepsilon} - a_{i,\bar{\varepsilon}}| \leq \|\zeta\|_{W^{1,\infty}} \left(\varepsilon + \left(\frac{1}{1-\varepsilon} \right)^d - 1 \right) \bar{\varepsilon}^d.$$

and hence also the first limit in (5.5.4) by (5.5.8) of Lemma 5.5.2. The second limit may be argued in a similar way. \square

5.5.3 Conditions of Theorem 5.2.1 for the processes defined in case (c) of Section 5.2.

By construction, the processes are stationary. Moreover, the marginal $\mathbb{P}_{\mathcal{R}}$ satisfies (5.2.4), (5.2.5). Therefore, it suffices to prove that the point process Φ as defined in either (c.1) or (c.2) satisfies (5.2.2) and (5.2.3).

We begin with case (c.1): For a bounded set $D \subset \mathbb{R}^d$, $r > 0$ and a point $x_i \in \mathbb{R}^d$, we define $N_1(D) := \#\Phi_1(D)$ and $N_r^i(D) = \#\Phi_r^{x_i} \cap (B_{r_i}(x_i) \cap D)$. For $R > 0$, let

$$B_R(D) := \{x \in \mathbb{R}^d : \text{dist}(x, D) \leq R\}.$$

Then, by (5.2.7), we estimate

$$\begin{aligned} \langle N(Q)^2 \rangle &= \left\langle \left(\sum_{z_i \in \Phi_1(\mathbb{R}^d)} N_{r_i}^i(B_{r_i}(z_i) \cap Q) \right)^2 \right\rangle \leq \left\langle N_1(B_{R_c}(Q)) \sum_{z_i \in \Phi_1(B_{R_c}(Q))} (N_{r_i}^i(B_{r_i}(z_i)))^2 \right\rangle \\ &= \langle N_1(B_{R_c}(Q))^2 \rangle \langle N_{R_c}^0(B_{R_c}(0))^2 \rangle \leq \lambda_1^2 \|\lambda_2\|_{\infty}^2 R_c^{2d} |B_{R_c}(Q)|^2. \end{aligned}$$

After taking the square-root of the above inequality, we conclude (5.2.2).

Condition (5.2.3) is an easy consequence of the fact that the process under consideration has finite range of dependence R_c , namely that if $\text{dist}(A, B) > R_c$, then the random variables $N(A)$ and $N(B)$ are independent. Thus, condition (5.2.3) is satisfied for any $\gamma > 0$ and with a constant depending on R_c .

We now turn to case (c.2). In this case, property (5.2.2) is an immediate consequence of the choice $\beta < 1$ and the fact that, if $\tilde{\Phi}$ is the Poisson point process on \mathbb{R}^d with intensity α , then for every $m \in \mathbb{N}$ and bounded set $B \subset \mathbb{R}^d$

$$\langle N(B)^m \rangle \leq \langle (\#\tilde{\Phi}(B))^m \rangle.$$

Furthermore, as in the previous case, the process considered in (c.2) has finite range of dependence given by r_c and thus satisfies (5.2.3) for any $\gamma > 0$.

Chapter 6

Homogenization of the Stokes equations

In this Chapter, we consider the steady incompressible Stokes equations in randomly perforated domains under very mild assumption on the distribution and size of the holes. In the homogenization limit, we obtain the Brinkman equations, which are the fluid equations in the Vlasov-Stokes equations (1.1.2), which model sedimentation of inertial particles in a fluid at zero Reynolds number. We have discussed the significance of the homogenization result that we obtain in this chapter as a first step in the derivation of the Vlasov-Stokes system in Chapter 1.5. The basic strategy of the proof in this chapter follows the one for the Poisson equation from the previous chapter. However, several new ideas are needed in order to deal with the incompressibility of the fluid. We also need to slightly strengthen the assumptions on the sizes of the holes in comparison to the Poisson equation.

The content of this chapter has appeared as a preprint, [GH18].

6.1 Introduction

We study the problem

$$\begin{cases} -\Delta u_\varepsilon + \nabla p_\varepsilon = f & \text{in } D^\varepsilon \\ \nabla \cdot u_\varepsilon = 0 & \text{in } D^\varepsilon \\ u_\varepsilon = 0 & \text{on } \partial D^\varepsilon \end{cases} \quad (6.1.1)$$

in a domain D^ε , that is obtained by removing from a bounded set $D \subset \mathbb{R}^d$, $d > 2$, a random number of small balls having random centres and radii. More precisely, for $\varepsilon > 0$, we define

$$D^\varepsilon = D \setminus H^\varepsilon, \quad H^\varepsilon := \bigcup_{z_i \in \Phi \cap \frac{1}{\varepsilon} D} B_{\varepsilon \frac{d}{d-2} \rho_i}(\varepsilon z_i), \quad (6.1.2)$$

where Φ is a Poisson point process on \mathbb{R}^d with homogeneous intensity rate $\lambda > 0$, and the radii $\{\rho_i\}_{z_i \in \Phi} \subset \mathbb{R}_+$ are identically and independently distributed unbounded random variables. We comment on the exact assumptions on the distribution of each ρ_i later in this introduction. Our main result states that, for almost every realization of H^ε in (6.1.2), the solution u_ε to (6.1.1) weakly converges in $H_0^1(D)$ to the solution u_h of the Brinkman equations

$$\begin{cases} -\Delta u_h + \mu u_h + \nabla p_h = f & \text{in } D \\ \nabla \cdot u_h = 0 & \text{in } D \\ u_h = 0 & \text{on } \partial D. \end{cases} \quad (6.1.3)$$

The constant matrix μ appearing in the equations above satisfies

$$\mu = \mu_0 \mathbf{I}, \quad \mu_0 = C_d \lambda \langle \rho^{d-2} \rangle, \quad (6.1.4)$$

where $\langle \cdot \rangle$ denotes the expectation under the probability measure on the radii ρ_i , and the constant $C_d > 0$ depends only on the dimension d . In the case $d = 3$, we have $C_d = 6\pi$.

From a physical point of view, the equations in (6.1.1) represent the motion of an incompressible viscous fluid among many small obstacles; the additional term μu_h appearing in (6.1.3) corresponds to the effective friction force of the obstacles acting on the fluid. In the physical literature, the term μ is the Stokes resistance (see Chapter 2.2; in this chapter though, we mostly adopt for μ the term “Stokes capacity density” to emphasize the analogy with the harmonic capacity density which appears in the analogue homogenization problem for the Poisson equation studied in the previous chapter. More precisely, for a smooth and bounded set $E \subset \mathbb{R}^d$, let us define its *Stokes capacity* as the symmetric and positive-definite matrix given by

$$\xi^t \cdot M \xi = \inf_{w \in E_\xi} \int_{\mathbb{R}^d \setminus E} |\nabla w|^2, \quad \text{for all } \xi \in \mathbb{R}^d. \quad (6.1.5)$$

Here,

$$E_\xi = \{w \in H_{loc}^1(\mathbb{R}^d; \mathbb{R}^d) : \nabla \cdot w = 0, \ w = \xi \text{ in } E, \ w \rightarrow 0 \text{ for } |x| \uparrow +\infty\}.$$

Then, in the case $E = B_r$, we obtain $M = C_d r^{d-2} \mathbf{I}$ (see e.g. [All90a]). The definition (6.1.4) of μ is thus an averaged version of the previous formula where we take into account the intensity rate of the Process Φ according to which the balls of H^ε are generated.

This work is an adaptation to the Stokes equations of the homogenization result obtained in the previous chapter for the Poisson equation. In particular, the class of random holes considered in this chapter is included in the class studied in the previous chapter. In the latter, it is assumed that the identically distributed radii ρ_i in (6.1.2) satisfy

$$\langle \rho^{d-2} \rangle < +\infty. \quad (6.1.6)$$

In the this chapter, we require the slightly stronger condition

$$\langle \rho^{(d-2)+\beta} \rangle < +\infty, \quad \text{for some } \beta > 0. \quad (6.1.7)$$

Before further commenting on (6.1.7) in the next paragraph, we recall that in the case of the Poisson problem, the analogue of the term μ appearing in the homogenized equation (6.1.3) is the asymptotic harmonic capacity density generated by the holes H^ε . Assumption (6.1.6) is minimal in order to have that this quantity is finite in average, but does not exclude that with overwhelming probability some balls generating H^ε overlap. .

The main challenge in proving the results of this chapter is related to the regions of H^ε where there are clustering effects. More precisely, the main goal is to estimate their contribution to the Stokes capacity density, and thus to the limit term μ appearing in (6.1.3). In the case of the Poisson equation in the previous chapter, the analogue is done by relying on the sub-additivity of the harmonic capacity, together with (6.1.6) and a Strong Law of Large Numbers. In the case of the Stokes capacity (6.1.4), though, sub-additivity fails due to the incompressibility of the fluid (i.e. the divergence-free condition). We thus need to cook up a different method to deal with the balls in H^ε which overlap or are too close. Heuristically speaking, the main challenge is that the incompressibility condition yields that big velocities are needed to squeeze a fixed volume of fluid through a possible narrow opening. The main reason for the strengthened assumption (6.1.7) is that it allows us to obtain a

certain degree of information on the geometry of the clusters of H^ε . In particular, (6.1.7) rules out the occurrence of clusters made of too many holes of *similar size*. We emphasize, however, that it neither prevents the balls generating H^ε from overlapping, nor it implies a uniform upper bound on the number of balls of very different size which combine into a cluster (see Section 6.6). The main technical effort of this chapter goes into developing a strategy to deal with these geometric considerations and succeed in controlling the term in (6.1.3). We refer to Subsection 6.2.3 for a more detailed discussion on our strategy.

We also mention that, to avoid further technicalities, we only treat the case where the centres of the balls in (6.1.2) are distributed according to a homogeneous Poisson point process. It is easy to check that our result applies both to the case of periodic centres and to any (short-range) correlated point process treated in the previous chapter for which the probabilistic results contained in Section 6.6 hold.

After Brinkman proposed the equations (6.1.3) in [Bri47] for the fluid flow in porous media, an extensive literature has been developed to obtain a rigorous derivation of (6.1.3) from (6.1.1) in the case of periodic configuration of holes [Bri86; L83; SP82; MK08]. We take inspiration in particular from [All90a], where the method used in [CM97] for the Poisson equations is adapted to treat the case of the Stokes equations in domains with periodic holes of arbitrary and identical shape. In [All90a], by a compactness argument, the same techniques used for the Stokes equations also provide the analogous result in the case of the stationary Navier-Stokes equations. The same is true also in our setting (see Remark 6.2.2 in Section 6.2).

In [DGR08], with methods similar to [All90a] and [CM97], the homogenization of stationary Stokes and Navier-Stokes equations has been extended also to the case of spherical holes where different and constant Dirichlet boundary conditions are prescribed at the boundary of each ball. This corresponds to the *quasi-static regime* of holes slowly moving in a fluid, and gives rise in (6.1.3) to an additional source term μj , with j being the limit flux of the holes. In [DGR08], the holes have all the same radius, are not necessarily periodic, but satisfy a uniform minimal distance condition of the same order of ε as in the periodic setting. In [Hil18], this last condition has been weakened but not completely removed. In particular it is still assumed that, asymptotically for $\varepsilon \downarrow 0$, the radius of each hole is much smaller than its distance to any other hole.

In [HMS17], the quasi-static Stokes equations are considered in perforated domains with holes of different shapes which are both translating and rotating. Due to the shapes of the holes, the problem becomes non-isotropic, i.e. the matrix μ in (6.1.3) is not a multiple of the identity. Moreover, since also the rotations of the holes are included into the model, a more complicated source term $\bar{\mathbb{F}}$ arises on the right hand side of the limit problem. The result in [HMS17] is proved under the same uniform minimal distance assumption as in [DGR08].

Finally, we also mention that the homogenization in the Brinkman regime for evolutionary Navier-Stokes in a bounded domain of \mathbb{R}^3 has been considered in [FNN16]. In this paper, the holes are assumed to be disjoint, have arbitrary shape and uniformly bounded diameter. A condition on the minimal distance between the holes is substituted by a weaker assumption implying that, for ε small enough, the diameter of the holes is much smaller than the distance between them.

There are fewer results in the literature concerning the case of randomly distributed holes: In [Rub86], the case of N randomly distributed spherical holes of size N^{-1} in \mathbb{R}^3 is considered. Starting from the Brinkman equation (6.1.3) with the term μ sufficiently large, it is shown that in the limit $N \rightarrow \infty$ an additional zero-order term appears in the limit equation. This result has been recently generalized in [CH18] to the case of the Stokes equations in the quasi-static regime.

We emphasize that the main novelty of this chapter is that we consider spherical holes whose

radii are not uniformly bounded and only satisfy (6.1.7). As already mentioned above, for small β in (6.1.7), with probability tending to one as $\varepsilon \rightarrow 0$, the perforated domain D^ε in (6.1.2) contains many holes that overlap. In all the deterministic results listed above, overlapping balls are either excluded or asymptotically ruled out for $\varepsilon \downarrow 0$. Similarly, in the random settings of [Rub86] and [CH18], the overlapping are negligible in probability: Since the radii of the holes are chosen to be identically N^{-1} , it is shown that, with probability tending to one as $N \rightarrow \infty$, the minimal distance between them is bounded below by $N^{-\alpha}$ for $\alpha < 1$.

We finally mention that in this chapter we also give a convergence result for the pressures $\{p_\varepsilon\}_{\varepsilon>0}$. In all the papers mentioned above except for [All90a], the convergence of the pressure is not considered. In fact, the problem may be reformulated so that the pressure only plays the role of a Lagrange multiplier for the incompressibility of the fluid. As a physical quantity, though, the pressure is important in itself and obtaining bounds may turn out to be a challenging problem. In [All90a] it is shown that for a suitable extension $P_\varepsilon(p_\varepsilon)$ for p_ε on the whole domain D , the functions $P_\varepsilon p_\varepsilon$ converge to p_h weakly in $L^2(D)$. Since u_ε converges weakly in H^1 , this is the optimal result that one could expect. In our work, we prove a sub-optimal convergence result for a suitable modification \tilde{p}_ε of the pressures p_ε . The main difficulty in our case is again given by the presence of the clusters of H^ε that prevents us from finding suitable bounds for p_ε close to those regions. Roughly speaking, the definition of \tilde{p}_ε allows us to cut-off a small neighborhood E^ε of H^ε and show that, away from it, the pressures converge to p_h in L^q , $q < \frac{d}{d-1}$. The neighborhood E^ε is small in the sense that the harmonic capacity of the difference $E^\varepsilon \setminus H^\varepsilon$ almost surely vanishes in the limit $\varepsilon \downarrow 0^+$.

This chapter is organized as follows: In Section 6.2 we state the two main theorems, namely the convergence of the fluid velocity u_ε and a partial convergence result for the pressure p_ε . In Subsection 6.2.4 we formulate Lemma 6.2.4 which provides a rich class of test-functions for (6.1.1) and characterizes their behaviour in the limit $\varepsilon \rightarrow 0$. We then show how the convergence of u_ε follows from this result. In Section 6.3, we give some geometric properties for the realization of the holes H^ε that are needed in order to prove Lemma 6.2.4. These properties are split into two lemmas. The first one is analogous to the corresponding lemma in the previous chapter, the other one gives more detailed informations on the geometry of the clusters of H^ε and is the result which requires the strengthened version (6.1.7) of (6.1.6). In subsection 6.3.2, we prove the results stated in Section 6.3. In Section 6.4, we prove Lemma 6.2.4. In Section 6.5, we prove the main result concerning the convergence of pressure. In Section 6.6, we prove some probabilistic result on the number of comparable balls which may combine into a cluster of H^ε . These are the key ingredients used in subsection 6.3.2 to show the geometric results of Section 6.3. Finally, in Section 6.7, we show how to extend the convergence result from the Stokes equations to the Stationary Navier-Stokes equations, and in Section 6.8, we give some standard estimates for the solutions of the Stokes equations in annuli and exterior domains.

6.2 Setting and main result

Let $D \subset \mathbb{R}^d$, $d > 2$, be an open and bounded set that is star-shaped with respect to the origin. For $\varepsilon > 0$, we denote by $D^\varepsilon \subset D$ the domain obtained as in (6.1.2), namely by setting $D^\varepsilon = D \setminus H^\varepsilon$ with

$$H^\varepsilon := \bigcup_{z_j \in \Phi \cap \frac{1}{\varepsilon} D} B_{\varepsilon^{\frac{d}{d-2}} \rho_j}(\varepsilon z_j). \quad (6.2.1)$$

Here, $\Phi \subset \mathbb{R}^d$ is a homogeneous Poisson point process having intensity $\lambda > 0$ and the radii $\mathcal{R} := \{\rho_i\}_{z_i \in \Phi}$ are i.i.d. random variables which satisfy condition (6.1.7) for a fixed $\beta > 0$. Since

assumption (6.1.7) with $\beta_1 > 0$ implies (6.1.7) for every other $0 < \beta \leq \beta_1$, with no loss of generality we assume that $\beta \leq 1$.

Throughout this chapter we denote by $(\Omega, \mathcal{F}, \mathbb{P})$ the probability space associated to the marked point process (Φ, \mathcal{R}) , i.e. the joint process of the centres and radii distributed as above. We refer to the previous chapter for a more detailed introduction of marked point processes.

6.2.1 Notation

For a point process Φ on \mathbb{R}^d and any bounded set $E \subset \mathbb{R}^d$, we define the random variables

$$\begin{aligned} \Phi(E) &:= \Phi \cap E, & \Phi^\varepsilon(E) &:= \Phi \cap \left(\frac{1}{\varepsilon}E\right), \\ N(E) &:= \#(\Phi(E)), & N^\varepsilon(E) &:= \#(\Phi^\varepsilon(E)). \end{aligned} \quad (6.2.2)$$

For $\eta > 0$, we denote by Φ_η a thinning for the process Φ obtained as

$$\Phi_\eta(\omega) := \{x \in \Phi(\omega) : \min_{\substack{y \in \Phi(\omega), \\ y \neq x}} |x - y| \geq \eta\}, \quad (6.2.3)$$

i.e. the points of $\Phi(\omega)$ whose minimal distance from the other points is at least η . Given the process Φ_η , we set $\Phi_\eta(E)$, $\Phi_\eta^\varepsilon(E)$, $N_\eta(E)$ and $N_\eta^\varepsilon(E)$ for the analogues for Φ_η of the random variables defined in (6.2.2).

For a bounded and measurable set $E \subset \mathbb{R}^d$ and any $1 \leq p < +\infty$, we denote

$$L_0^p(E) := \{f \in L^p(E) : \int_E f = 0\}.$$

As in the previous chapter, we identify any $v \in H_0^1(D^\varepsilon)$ with the function $\bar{v} \in H_0^1(D)$ obtained by trivially extending v in H^ε .

Throughout the proofs in this chapter, we write $a \lesssim b$ whenever $a \leq Cb$ for a constant $C = C(d, \beta)$ depending only on the dimension d and β from assumption (6.1.7). Moreover, when no ambiguity occurs, we use a scalar notation also for vector fields and vector-valued function spaces, i.e. we write for instance $C_0^\infty(D)$, $H^1(\mathbb{R}^d)$, $L^p(\mathbb{R}^d)$ instead of $C_0^\infty(D; \mathbb{R}^d)$, $H^1(\mathbb{R}^d; \mathbb{R}^d)$, $L^p(\mathbb{R}^d; \mathbb{R}^d)$.

6.2.2 Main results

Let (Φ, \mathcal{R}) be a marked point process as above, and let H^ε be defined as in (6.2.1). Then, we have:

Theorem 6.2.1. *For $f \in H^{-1}(D; \mathbb{R}^d)$ and $\varepsilon > 0$, let $(u_\varepsilon, p_\varepsilon) = (u_\varepsilon(\omega, \cdot), p_\varepsilon(\omega, \cdot)) \in H_0^1(D^\varepsilon; \mathbb{R}^d) \times L_0^2(D^\varepsilon; \mathbb{R})$ be the solution of*

$$\begin{cases} -\Delta u_\varepsilon + \nabla p_\varepsilon = f & \text{in } D^\varepsilon \\ \nabla \cdot u_\varepsilon = 0 & \text{in } D^\varepsilon \\ u_\varepsilon = 0 & \text{on } \partial D^\varepsilon. \end{cases} \quad (6.2.4)$$

Then, for \mathbb{P} -almost every $\omega \in \Omega$

$$u_\varepsilon(\omega, \cdot) \rightharpoonup u_h \quad \text{in } H_0^1(D; \mathbb{R}^d), \quad \text{for } \varepsilon \downarrow 0^+,$$

where $(u_h, p_h) \in H_0^1(D; \mathbb{R}^d) \times L_0^2(D; \mathbb{R})$ is the solution of

$$\begin{cases} -\Delta u_h + \nabla p_h + C_d \lambda \langle \rho^{d-2} \rangle u_h = f & \text{in } D \\ \nabla \cdot u_h = 0 & \text{in } D \\ u_h = 0 & \text{on } \partial D, \end{cases} \quad (6.2.5)$$

with C_d as in (6.1.4).

Remark 6.2.2 (Stationary Navier-Stokes equations). As in the case of periodic holes [All90a], we remark that the same result of Theorem 6.2.1 holds in dimension $d = 3, 4$ for the solutions u_ε to the stationary Navier-Stokes system

$$\begin{cases} u_\varepsilon \cdot \nabla u_\varepsilon - \Delta u_\varepsilon + \nabla p_\varepsilon = f & \text{in } D^\varepsilon \\ \nabla \cdot u_\varepsilon = 0 & \text{in } D^\varepsilon, \\ u_\varepsilon = 0 & \text{on } \partial D^\varepsilon \end{cases} \quad (6.2.6)$$

with homogenized equations

$$\begin{cases} u_h \cdot \nabla u_h - \Delta u_h + C_d \lambda \langle \rho^{d-2} \rangle u_h + \nabla p_h = f & \text{in } D \\ \nabla \cdot u_h = 0 & \text{in } D \\ u_h = 0 & \text{on } \partial D, \end{cases} \quad (6.2.7)$$

We argue in Section 6.7 how the same argument that we give in the next section for Theorem 6.2.1 allows also to treat the non-linear term in (6.2.6).

The previous theorem shows that the holes of H^ε which overlap do not destroy the homogenization process and that their effect on the value of the Brinkman term is negligible. On the other hand, the complicated geometries which may arise from the clustering effects in H^ε prevent us from obtaining a suitable extension of the pressure terms p_ε to the whole domain D which converges to p_h . Nonetheless, in the next theorem we prove a convergence result for p_ε to p_h , as long as we remove from D an exceptional set E^ε containing H^ε . This set mostly coincides with H^ε in the sense that the difference $E^\varepsilon \setminus H^\varepsilon$ has vanishing harmonic capacity.

Theorem 6.2.3. *For almost every $\omega \in \Omega$, there exists a set $E^\varepsilon \subset \mathbb{R}^d$ such that $E^\varepsilon \supset H^\varepsilon$ and for $\varepsilon \downarrow 0^+$*

$$\text{Cap}(E^\varepsilon \setminus H^\varepsilon) \rightarrow 0, \quad (6.2.8)$$

where Cap denotes the harmonic capacity in \mathbb{R}^d . Moreover, for every compact set $K \Subset D$, the modification of the pressure

$$\tilde{p}_\varepsilon = \begin{cases} p_\varepsilon - f_{K \setminus E^\varepsilon} p_\varepsilon & \text{in } K \setminus E^\varepsilon \\ 0 & \text{in } D \setminus K \cup E^\varepsilon \end{cases} \quad (6.2.9)$$

satisfies for all $q < \frac{d}{d-1}$

$$\tilde{p}_\varepsilon \rightharpoonup p_h \quad \text{in } L_0^q(K; \mathbb{R}).$$

Since this result relies on some of the tools which will be developed along the proof of Theorem 6.2.1, we give the argument for Theorem 6.2.3 in Section 6.5.

6.2.3 Main ideas in proving Theorem 6.2.1 and Theorem 6.2.3

As already mentioned above, the structure and many arguments of this chapter are an adaptation of the previous chapter to the case of the Stokes equations. In this subsection, we point out the main differences and the challenges that we encountered along the process.

In contrast to the previous chapter, we prove the convergence of the fluid velocities u_ε by using an implicit version of the method of oscillating test-functions, which is similar to the one of [DGR08]: We construct an operator R_ε which acts on divergence-free test-functions v such that $R_\varepsilon v \in H_0^1(D^\varepsilon)$ is an admissible test function for (6.2.4), $R_\varepsilon v \rightarrow v$ in $H_0^1(D)$ and $\nabla \cdot R_\varepsilon v = 0$ in D . This last condition in particular implies that we may test the equation (6.2.4) with $R_\varepsilon v$ and do not need any bounds on the pressure p_ε . We emphasize that, as done in [All90a], a convergence result on the pressure terms $\{p_\varepsilon\}_{\varepsilon>0}$ is required if one constructs divergence-free oscillating functions $w_\varepsilon \in H_0^1(D^\varepsilon)$ and tests the equation (6.1.1) for u_ε with the products ϕw_ε , for arbitrary $\phi \in C_0^\infty(D)$. We remark that, in principle, the partial result that we obtain on the convergence of the pressure is strong enough to allow us to follow also this last approach. However, as we show in Section 6.5, obtaining bounds on the pressure in our setting strongly relies on the geometric properties of the clusters and requires a fairly (and further) technical argument. We thus find easier to first give a proof for the homogenization of u_ε which does not rely on any bounds on the sequence $\{p_\varepsilon\}_{\varepsilon>0}$, and only afterwards show how to extract a convergence result also for p_ε .

As in the previous chapter with the construction of the oscillating test-functions w_ε , the construction of the operator R_ε relies on a lemma dealing with the geometric properties of the set of holes H^ε which perforate D in (6.1.2). This lemma allows us to split the set H^ε into a “good” set H_g^ε , which contains holes which are small and well-separated, and a “bad” set H_b^ε , which contains big and overlapping holes. On the one hand, we construct $R_\varepsilon v$ such that it vanishes on H_g^ε by closely following the ideas in [All90a] and [DGR08]. On the other hand, to define $R_\varepsilon v$ in such a way that it vanishes also on H_b^ε , we need to improve the arguments used in the previous chapter. In fact, as pointed out in the introduction, in contrast to the previous chapter, by the incompressibility condition it is not enough to prove that the harmonic capacity of H_b^ε vanishes in the limit $\varepsilon \downarrow 0^+$.

In order to overcome this problem, we use the following strategy to construct $R_\varepsilon v$ such that, for any divergence-free $v \in C_0^\infty(D, \mathbb{R}^d)$, the function $R_\varepsilon v$ vanishes on the “bad” set H_b^ε , remains divergence-free in D and converges to v in $H_0^1(D; \mathbb{R}^d)$. We recall that in the set H_b^ε the balls may overlap; the challenge is therefore to find a suitable truncation for v on this set, which preserves the divergence-free condition and which remains bounded in an H^1 -sense. A first approach to construct $R_\varepsilon v$ would then be to solve the Stokes problem in a large enough neighbourhood D_b^ε of H_b^ε

$$\begin{cases} -\Delta w_\varepsilon + \nabla \pi_\varepsilon = \Delta v & \text{in } D_b^\varepsilon \setminus \overline{H_b^\varepsilon} \\ \nabla \cdot w = 0 & \text{in } D^\varepsilon \setminus \overline{H_b^\varepsilon} \\ w = 0 & \text{on } \partial H_b^\varepsilon \\ w(x) = v & \text{on } \partial D_b^\varepsilon. \end{cases} \quad (6.2.10)$$

The connection with the concept of “Stokes capacity” generated by the set H_b^ε thus becomes apparent; namely, at least in the case of sets E regular enough, the minimizer in (6.1.5) solves

$$\begin{cases} -\Delta w + \nabla \pi = 0 & \text{in } \mathbb{R}^d \setminus \overline{E} \\ \nabla \cdot w = 0 & \text{in } \mathbb{R}^d \setminus \overline{E} \\ w = \xi & \text{on } \partial E \\ w(x) \rightarrow 0 & \text{as } |x| \rightarrow \infty. \end{cases}$$

However, getting H^1 -estimates on the solution w^ε of (6.2.10) which depend explicitly on ε , requires more informations than we have on the geometry of the set H_b^ε . In fact, condition (6.1.7) does not prevent the balls from overlapping nor provides an upper bound on the number of balls in each of the clusters (cf. Lemma 6.6.1). The approach that we adopt to construct $R_\varepsilon v$ is therefore different and is based on finding a suitable covering \bar{H}_b^ε of the set H_b^ε . The set \bar{H}_b^ε is obtained by selecting some of the balls that constitute H_b^ε and dilating them by a uniformly bounded factor $\lambda_\varepsilon \leq \Lambda$. The main, crucial, feature of this covering is that it allows us to construct $R_\varepsilon v$ vanishing on $H_b^\varepsilon \subset \bar{H}_b^\varepsilon$ by solving different Stokes problems in disjoint annuli of the form $B_{\theta\lambda_\varepsilon \varepsilon^{\frac{d}{d-2}} \rho_i}(\varepsilon z_i) \setminus B_{\lambda_\varepsilon \varepsilon^{\frac{d}{d-2}} \rho_i}(\varepsilon z_i)$, $\theta > 1$, and iterating this procedure a finite number of steps. The advantage in this is that we construct $R_\varepsilon v$ iteratively and obtain bounds by applying a finite number of times some standard and rescaled estimates for solutions to Stokes equations in the annulus $B_\theta \setminus B_1$.

More precisely, \bar{H}_b^ε is chosen to satisfy the following properties:

- (a) \bar{H}_b^ε is the union of $M < +\infty$ families of balls such that, inside the same family, the balls $B_{\lambda_\varepsilon \varepsilon^{\frac{d}{d-2}} \rho_i}(\varepsilon z_i)$ are disjoint even if dilated by a further factor $\theta^2 > 0$, i.e. by considering $B_{\theta^2 \lambda_\varepsilon \varepsilon^{\frac{d}{d-2}} \rho_i}(\varepsilon z_i)$;

By this property, if we want to construct $R_\varepsilon v$ vanishing only in the holes of the same family, it suffices to solve (6.2.10) in the disjoint annuli $B_{\theta\lambda_\varepsilon \varepsilon^{\frac{d}{d-2}} \rho_i}(\varepsilon z_i) \setminus B_{\lambda_\varepsilon \varepsilon^{\frac{d}{d-2}} \rho_i}(\varepsilon z_i)$ and stitch the solutions together. This suffices to construct $R_\varepsilon v$ vanishing on the balls $B_{\lambda_\varepsilon \varepsilon^{\frac{d}{d-2}} \rho_i}(\varepsilon z_i)$ of the same family, and thus on the subset of H_b^ε covered by them. In order to obtain $R_\varepsilon v$ vanishing on the whole set H_b^ε , one may try to iterate the previous procedure: Let the families of balls constituting \bar{H}_b^ε be ordered with an index $k = 1, \dots, M$. Then:

- We construct a first solution v_ε^1 which solves (6.2.10) in all the (disjoint) annuli generated by the first family;
- We construct v_ε^2 solving (6.2.10) with v substituted by v_ε^1 in the (disjoint) annuli of the second family;
- We iterate the procedure up to the M -th family and set $R_\varepsilon v = v_\varepsilon^M$.

However, property (a) alone does not ensure that the final solution constructed in this fashion vanishes on H_b^ε : Since annuli generated by different families may still intersect, at each step the zero-boundary conditions of the previous steps may be destroyed (as an example, see Figure 6.1). This is the reason why we need that the covering \bar{H}_b^ε satisfies an additional property. This property should ensure that, if at step k the function v^k vanishes on a certain subset of H_b^ε , then also v^{k+1} vanishes on that same subset. We thus construct \bar{H}_b^ε in such a way that

- (b) all the balls $B_{\theta\lambda_\varepsilon \varepsilon^{\frac{d}{d-2}} \rho_i}(\varepsilon z_i)$ belonging to the k -th family do not intersect the balls of H_b^ε contained in the previous families (cf. property (6.3.8) of the Lemma 6.3.2).¹

The construction of \bar{H}_b^ε satisfying (a)-(b) is given in Lemma 6.3.2 and constitutes the most technically challenging part of this chapter.

¹Strictly speaking, this is a simplification of the statement of Lemma 6.3.2 (cf. Remark 6.3.3 in Section 6.3).

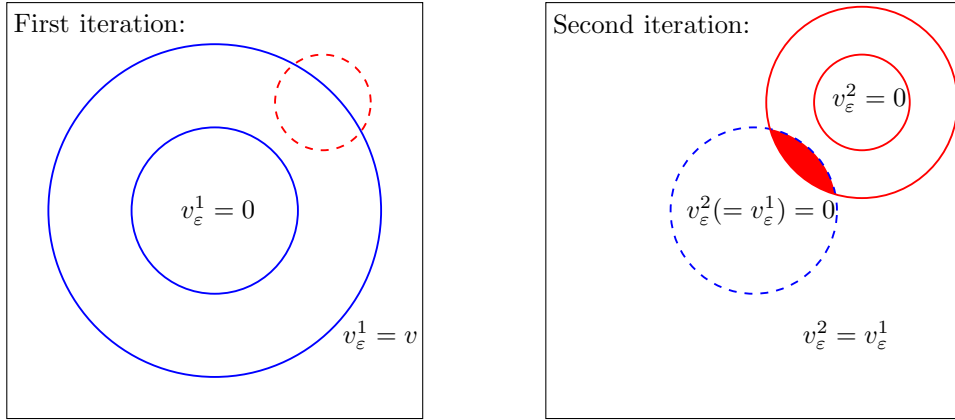


Figure 6.1: This is an example of a configuration which satisfies only (a) for which the algorithm to construct $R_\varepsilon v$ may not give a function vanishing on all the holes. The first picture on the left represents the first iteration step: The blue, full-lined, ball is the hole belonging to the first family generating \bar{H}_b^ε . We solve a Stokes problem in the blue annulus, with zero boundary conditions in the inner ball. The dashed, red ball represents a hole generated by another family of \bar{H}_b^ε , which is neglected in this step. The second picture represents the second iteration step: Given the solution v_ε^1 obtained in the first step, we solve another Stokes problem in the red, smaller, annulus with zero boundary conditions in the inner hole. Since this new annulus intersects the hole of the previous step, the function v_ε^2 may not vanish in the intersection in red.

6.2.4 Lemma 6.2.4 and proof of Theorem 6.2.1

The proof of Theorem 6.2.1 relies on the following lemma:

Lemma 6.2.4. *For almost every $\omega \in \Omega$ and for all $\varepsilon \leq \varepsilon_0(\omega)$ there exists a linear map*

$$R_\varepsilon: \{v \in C_0^\infty(D) : \nabla \cdot v = 0\} \rightarrow H^1(D)$$

with the following properties:

- (i) $R_\varepsilon v = 0$ in H^ε and, for ε small enough, also $R_\varepsilon v \in H_0^1(D)$;
- (ii) $\nabla \cdot R_\varepsilon v = 0$ in \mathbb{R}^d ;
- (iii) $R_\varepsilon v \rightharpoonup v$ in $H_0^1(D)$;
- (iv) $R_\varepsilon v \rightarrow v$ in $L^p(D)$ for all $1 \leq p < \infty$;
- (v) For all $u_\varepsilon \in H_0^1(D^\varepsilon)$ such that $\nabla \cdot u_\varepsilon = 0$ in D and $u_\varepsilon \rightharpoonup u$ in $H_0^1(D)$, we have

$$\int \nabla R_\varepsilon v : \nabla u_\varepsilon \rightarrow \int \nabla v : \nabla u + C_d \lambda \langle \rho^{d-2} \rangle \int v \cdot u,$$

with C_d as in Theorem 6.2.1.

Proof of Theorem 6.2.1. Let us fix $\omega \in \Omega$ such that the operator R_ε of Lemma 6.2.4 exists and satisfies all the properties (i) - (v). We trivially extend u^ε to the whole set D . Since by the standard energy estimate we have $\|u_\varepsilon\|_{H_0^1(D)} \leq \|f\|_{H^{-1}(D)}$, then up to a subsequence ε_j , we have $u_\varepsilon \rightharpoonup u^*$ in $H_0^1(D)$. Note that also $\nabla \cdot u^* = 0$ in D . We show that u^* solves (6.2.5) and, by uniqueness, that $u^* = u_h$ in $H_0^1(D)$. We thus may extend the convergences above to the whole limit $\varepsilon \downarrow 0^+$.

For any divergence-free $v \in C_0^\infty(D)$, we consider ε small enough such that the divergence-free vector field $R_\varepsilon v$ obtained by means of Lemma 6.2.4 is in $H_0^1(D)$. By testing (6.2.4) with this vector field, we obtain

$$\int \nabla R_\varepsilon v : \nabla u_\varepsilon = \langle R_\varepsilon v, f \rangle_{H^1, H^{-1}}.$$

We now apply (iii) and (v) of Lemma 6.2.4 to the left- and right-hand side of the above identity, respectively, and conclude that u^* satisfies

$$\int \nabla v : \nabla u^* + C_d \lambda \langle \rho^{d-2} \rangle \int v \cdot u^* = \langle v, f \rangle_{H^1, H^{-1}}.$$

Since $v \in C_0^\infty(D)$ is an arbitrary divergence-free test function, we conclude that u^* is the solution u_h of (6.2.5). \square

6.3 Geometric properties of the holes

This section is the core of the argument of Theorem 6.2.1 and Theorem 6.2.3 and provides some almost sure geometrical properties on H^ε . These allow us to construct the operator of Lemma 6.2.4.

The results contained in this section rely on assumption (6.1.7) and may be considered as an upgrade of Chapter 5.4. Since (6.1.7) is stronger than the one assumed in the previous chapter (see also (6.1.6)), the marked point process (Φ, \mathcal{R}) considered in this work is included in the class of processes studied in previous chapter. Therefore, all the results for H^ε contained in Chapter 5.4 hold also in our case. Bearing this in mind, we introduce the first main result of this section: This is almost a rephrasing of Lemma 5.4.2, where, thanks to (6.1.7), we are allowed to choose the sequence r_ε appearing in the statement of Lemma 5.4.2 as a power law $r_\varepsilon = \varepsilon^\delta$, for $\delta = \delta(d, \beta) > 0$.

Lemma 6.3.1. *There exists a $\delta = \delta(d, \beta) > 0$ such that for almost every $\omega \in \Omega$ and all $\varepsilon \leq \varepsilon_0 = \varepsilon_0(\omega)$, there exists a partition $H^\varepsilon = H_g^\varepsilon \cup H_b^\varepsilon$ and a set $D_b^\varepsilon \subset \mathbb{R}^d$ such that $H_b^\varepsilon \subset D_b^\varepsilon$ and*

$$\text{dist}(H_g^\varepsilon; D_b^\varepsilon) > \varepsilon^{1+\delta}, \quad |D_b^\varepsilon| \downarrow 0^+. \quad (6.3.1)$$

Furthermore, H_g^ε is a union of disjoint balls centred in $n^\varepsilon \subset \Phi^\varepsilon(D)$, namely

$$\begin{aligned} H_g^\varepsilon &= \bigcup_{z_i \in n^\varepsilon} B_{\varepsilon^{\frac{d}{d-2}} \rho_i}(\varepsilon z_i), \quad \varepsilon^d \# n^\varepsilon \rightarrow \lambda |D|, \\ \min_{z_i \neq z_j \in n^\varepsilon} \varepsilon |z_i - z_j| &\geq 2\varepsilon^{1+\frac{\delta}{2}}, \quad \varepsilon^{\frac{d}{d-2}} \rho_i \leq \varepsilon^{1+2\delta}. \end{aligned} \quad (6.3.2)$$

Finally, if for $\eta > 0$ the process $\Phi_{2\eta}^\varepsilon$ is defined as in (6.2.3), then

$$\lim_{\varepsilon \downarrow 0} \varepsilon^d \# (\{z_i \in \Phi_{2\eta}^\varepsilon(D) : \text{dist}(\varepsilon z_i, D_b^\varepsilon) \leq \eta \varepsilon\}) = 0. \quad (6.3.3)$$

The next result upgrades the previous lemma and is the key result on which relies the construction of the operator R_ε of Lemma 6.2.4. We introduce the following notation: We set $\mathcal{I}^\varepsilon := \Phi^\varepsilon(D) \setminus n^\varepsilon$, so that, by the previous lemma, we may write

$$H_b^\varepsilon := \bigcup_{z_i \in \mathcal{I}^\varepsilon} B_{\varepsilon^{\frac{d}{d-2}} \rho_i}(\varepsilon z_i). \quad (6.3.4)$$

As already discussed in Section 6.2.3, the main aim of the next result is to show that there exists a suitable covering for H_b^ε , which is of the form

$$\bar{H}_b^\varepsilon := \bigcup_{z_j \in J^\varepsilon} B_{\lambda_j^\varepsilon \varepsilon^{\frac{d}{d-2}} \rho_j}(\varepsilon z_j), \quad J^\varepsilon \subset \mathcal{I}^\varepsilon, \quad \sup_{z_j \in J^\varepsilon} \lambda_j^\varepsilon \leq \Lambda$$

and which satisfies (a) and (b) of Section 6.2.3. More precisely, we have:

Lemma 6.3.2. *Let $\theta > 1$ be fixed. Then for almost every $\omega \in \Omega$ and $\varepsilon \leq \varepsilon_0(\omega, \beta, d, \theta)$ we may choose $H_g^\varepsilon, H_b^\varepsilon$ of Lemma 6.3.1 in such a way that have the following:*

- There exist $\Lambda(d, \beta) > 0$, a sub-collection $J^\varepsilon \subset \mathcal{I}^\varepsilon$ and constants $\{\lambda_l^\varepsilon\}_{z_l \in J^\varepsilon} \subset [1, \Lambda]$ such that

$$H_b^\varepsilon \subset \bar{H}_b^\varepsilon := \bigcup_{z_j \in J^\varepsilon} B_{\lambda_j^\varepsilon \varepsilon^{\frac{d}{d-2}} \rho_j}(\varepsilon z_j), \quad \lambda_j^\varepsilon \varepsilon^{\frac{d}{d-2}} \rho_j \leq \Lambda \varepsilon^{2d\delta}. \quad (6.3.5)$$

- There exists $k_{\max} = k_{\max}(\beta, d) > 0$ such that we may partition

$$\mathcal{I}^\varepsilon = \bigcup_{k=-3}^{k_{\max}} \mathcal{I}_k^\varepsilon, \quad J^\varepsilon = \bigcup_{i=-3}^{k_{\max}} J_k^\varepsilon,$$

with $\mathcal{I}_k^\varepsilon \subset J_k^\varepsilon$ for all $k = 1, \dots, k_{\max}$ and

$$\bigcup_{z_i \in \mathcal{I}_k^\varepsilon} B_{\varepsilon^{\frac{d}{d-2}} \rho_i}(\varepsilon z_i) \subset \bigcup_{z_j \in J_k^\varepsilon} B_{\lambda_j^\varepsilon \varepsilon^{\frac{d}{d-2}} \rho_j}(\varepsilon z_j); \quad (6.3.6)$$

- For all $k = -3, \dots, k_{\max}$ and every $z_i, z_j \in J_k^\varepsilon$, $z_i \neq z_j$

$$B_{\theta^2 \lambda_i^\varepsilon \varepsilon^{\frac{d}{d-2}} \rho_i}(\varepsilon z_i) \cap B_{\theta^2 \lambda_j^\varepsilon \varepsilon^{\frac{d}{d-2}} \rho_j}(\varepsilon z_j) = \emptyset; \quad (6.3.7)$$

- For each $k = -3, \dots, k_{\max}$ and $z_i \in \mathcal{I}_k^\varepsilon$ and for all $z_j \in \bigcup_{l=-3}^{k-1} J_l^\varepsilon$ we have

$$B_{\varepsilon^{\frac{d}{d-2}} \rho_i}(\varepsilon z_i) \cap B_{\theta \lambda_j^\varepsilon \varepsilon^{\frac{d}{d-2}} \rho_j}(\varepsilon z_j) = \emptyset. \quad (6.3.8)$$

Finally, the set D_b^ε of Lemma 6.3.1 may be chosen as

$$D_b^\varepsilon = \bigcup_{z_i \in J^\varepsilon} B_{\theta \varepsilon^{\frac{d}{d-2}} \lambda_i^\varepsilon \rho_i}(\varepsilon z_i). \quad (6.3.9)$$

Remark 6.3.3. As explained in Section 6.2.3, property (6.3.8) is crucial for the construction of the operator R_ε of Lemma 6.2.4. However, it slightly differs from property (b) stated in that section. Namely, the balls $B_{\varepsilon^{\frac{d}{d-2}} \theta \lambda_j^\varepsilon \rho_j}(\varepsilon z_j)$, $z_j \in J_l^\varepsilon$ might intersect with some of the balls in H_b^ε that are contained in $B_{\varepsilon^{\frac{d}{d-2}} \lambda_i^\varepsilon \rho_i}(\varepsilon z_i)$ for $z_i \in J_k^\varepsilon$, $k > l$. This is why the additional index sets $\mathcal{I}_k^\varepsilon$ are introduced. In these index sets, the balls are not ordered by size, but in such a way that (6.3.8) holds. More precisely, if a ball in H_b^ε is contained in several of the dilated balls in J^ε , we will put it into the index set \mathcal{I}_k with k minimal such that it is contained in a dilated ball in J_k^ε .

6.3.1 Structure and main ideas in the proof of Lemma 6.3.1 and Lemma 6.3.2.

Since the proof of Lemma 6.3.2 requires different steps and technical constructions, we give a sketch of the ideas behind it. It is clear that Lemma 6.3.1 follows immediately from Lemma 6.3.2; we thus only need to focus on the proof of this last result.

To this end we introduce the following notation, which we will also use throughout the rigorous proof of Lemma 6.3.2 in the next subsection: Let

$$\delta := \frac{\beta}{2(d-2)(d-2+\beta)} \wedge \frac{\beta}{2d} \quad (6.3.10)$$

and

$$I_k^\varepsilon := \begin{cases} \{z_i \in \Phi^\varepsilon(D) : \varepsilon^{1-\delta k} \leq \varepsilon^{\frac{d}{d-2}} \rho_i < \varepsilon^{1-\delta(k+1)}\} & k \geq -2 \\ \{z_i \in \Phi^\varepsilon(D) : \varepsilon^{\frac{d}{d-2}} \rho_i < \varepsilon^{1+2\delta}\} & k = -3. \end{cases} \quad (6.3.11)$$

Note that $\Phi^\varepsilon(D) = \bigcup_{k \geq -3} I_k^\varepsilon$. We remark that the sets I_k^ε correspond to $I_{\delta,k}^\varepsilon$ in (6.6.1) of Section 6.6 with δ as in (6.3.10). Since we chose δ above such that $\delta < \frac{\beta}{2d}$, we may apply Lemma 6.6.1 with this choice of δ and infer that there exists $k_{\max} \in \mathbb{N}$ such that $I_k^\varepsilon = \emptyset$ for all $k > k_{\max}$. From now on, we assume that k_{\max} is chosen in this way and thus that

$$\Phi^\varepsilon(D) = \bigcup_{k=-3}^{k_{\max}} I_k^\varepsilon.$$

In addition, since we may bound

$$\varepsilon^{\frac{d}{d-2}} \max_{\Phi^\varepsilon(D)} \rho_i \leq \varepsilon^{\frac{d}{d-2} - \frac{d}{d-2+\beta}} \left(\varepsilon^d \sum_{z_i \in \Phi^\varepsilon(D)} \rho_i^{d-2+\beta} \right)^{\frac{1}{d-2+\beta}},$$

we use (6.1.7) and the Strong Law of Large Numbers, to infer that almost surely and for ε small enough

$$\varepsilon^{\frac{d}{d-2}} \max_{\Phi^\varepsilon(D)} \rho_i \lesssim \varepsilon^{\frac{d}{d-2} - \frac{d}{d-2+\beta}} \langle \rho^{d-2+\beta} \rangle^{\frac{1}{d-2+\beta}}.$$

This implies by (6.3.10) that

$$\max_{z_i \in \Phi^\varepsilon(D)} \varepsilon^{\frac{d}{d-2}} \rho_i \lesssim \varepsilon^{2d\delta}. \quad (6.3.12)$$

Step 1: Combining clusters of holes of similar size: We begin obtaining a first covering of H^ε made by a union of balls which, if of comparable size, are disjoint even if dilated by a constant factor $\alpha > 1$. Roughly speaking, we do this by merging the balls of H^ε generated each family $I_k^\varepsilon \cup I_{k-1}^\varepsilon$, in holes of similar size which are also disjoint. More precisely, we prove:

Claim: Let $\alpha > 1$. Then, there exists $\tilde{\Lambda} = \tilde{\Lambda}(d, \beta, \alpha) > 0$ such that for \mathbb{P} -almost every $\omega \in \Omega$ and all $\varepsilon < \varepsilon_0(\omega)$ and all $-3 \leq k \leq k_{\max}$ there are $\tilde{I}_k^\varepsilon \subset I_k^\varepsilon$ and $\{\tilde{\lambda}_j^\varepsilon\}_{z_j \in \tilde{I}_k^\varepsilon} \subset [1, \tilde{\Lambda}]$ with the following properties:

$$\forall z_i \in I_k^\varepsilon \exists z_j \in \bigcup_{l \geq k} \tilde{I}_l^\varepsilon : B_{\varepsilon^{\frac{d}{d-2}} \rho_i}(\varepsilon z_i) \subset B_{\varepsilon^{\frac{d}{d-2}} \tilde{\lambda}_j^\varepsilon \rho_j}(\varepsilon z_j). \quad (6.3.13)$$

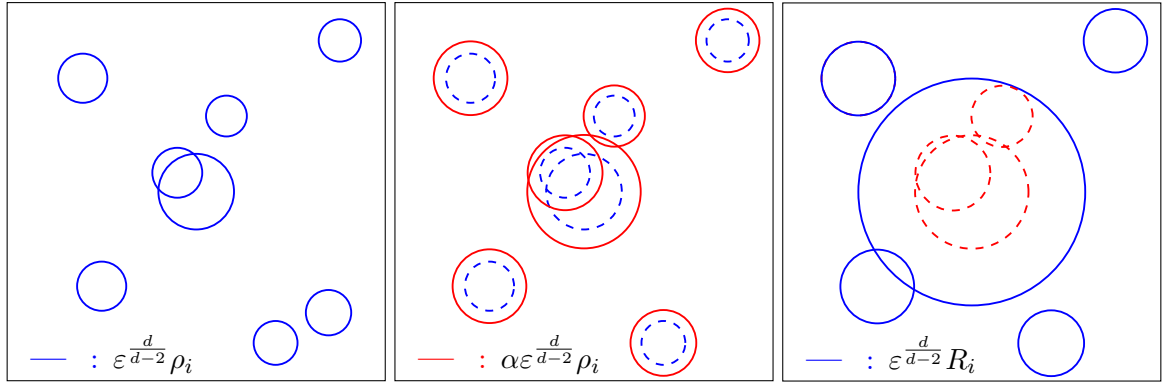


Figure 6.2: This sequence of pictures shows how to implement the algorithm of Step 1. From left to right: We begin with an initial configuration of comparable balls generated by centres in $I_{-3}^\varepsilon \cup I_{-2}^\varepsilon$ and with associated radii $\varepsilon^{\frac{d}{d-2}} \rho_i$. In the picture in the middle, the full line represents a dilation by a factor $\alpha = 1.5$ of this initial configuration (here drawn with a dashed line). In the last picture, the full line represents the new configuration obtained with the modified radii R_i which covers all the dilated balls of the previous figure (here drawn with a dashed line).

For each $-3 \leq k \leq k_{\max}$ the balls

$$\left\{ B_{\varepsilon^{\frac{d}{d-2}} \alpha \tilde{\lambda}_i^\varepsilon \rho_i}(\varepsilon z_i) \right\}_{z_i \in \tilde{I}_k^\varepsilon \cup \tilde{I}_{k-1}^\varepsilon} \quad \text{are pairwise disjoint.} \quad (6.3.14)$$

Note that “most” of the balls generated by the points in $I_{-2}^\varepsilon \cup I_{-3}^\varepsilon$ already satisfy (6.3.14) with $\lambda_i^\varepsilon = 1$. Hence, $\tilde{I}_{-3}^\varepsilon$ contains most of the points of I_{-3}^ε . The only elements of $I_{-2}^\varepsilon \cup I_{-3}^\varepsilon$ which might violate this conditions are the ones which are too close to each other. We will show that, since the collection $I_{-2}^\varepsilon \cup I_{-3}^\varepsilon$ is generated by a Poisson point process, these exceptional points are few for small values of $\varepsilon > 0$.

To construct the sets \tilde{I}_k above we adopt the following strategy (see Figure 6.2 for a sketch):

- Let $\alpha > 1$ and $-2 \leq k \leq k_{\max}$ be fixed. We multiply each one of the radii $\{\rho_i\}_{z_i \in I_k^\varepsilon \cup I_{k-1}^\varepsilon}$ by α and consider the set of balls

$$\left\{ B_{\alpha \varepsilon^{\frac{d}{d-2}} \rho_i}(\varepsilon z_i) \right\}_{z_i \in I_k^\varepsilon \cup I_{k-1}^\varepsilon}. \quad (6.3.15)$$

For each point $z_i \in I_k^\varepsilon \cup I_{k-1}^\varepsilon$ we now define a new radius R_i^ε in the following way: For each disjoint ball in the previous collection we set $R_i^\varepsilon := \rho_i$. We now consider the balls which are not disjoint: For each connected component C_k^ε of (6.3.15), we pick one of the largest balls belonging to C_k^ε , say $B_{\alpha \varepsilon^{\frac{d}{d-2}} \rho_l}(\varepsilon z_l)$, and set R_l^ε as the minimal one such that $C_k^\varepsilon \subset B_{\varepsilon^{\frac{d}{d-2}} R_l^\varepsilon}(\varepsilon z_l)$. We set $R_i^\varepsilon = 0$ for all the $z_i \neq z_l$ generating the balls contained in C_k^ε . We thus have a new collection of radii $\{R_i^\varepsilon\}_{z_i \in I_k^\varepsilon \cup I_{k-1}^\varepsilon}$.

- We multiply each R_i^ε above by the same factor α of the previous step and repeat the construction sketched above with ρ_i substituted by R_i^ε .

- We show that, almost surely, after a number $M = M(d, \beta) < +\infty$ of iterations of the previous two steps, all the radii R_i^ε obtained at the M^{th} -step do not change any further. This means that the balls $B_{\varepsilon^{\frac{d}{d-2}} R_i^\varepsilon}(\varepsilon z_i)$, for $R_i^\varepsilon \neq 0$, satisfy (6.3.13) and (6.3.14). Moreover, we may easily bound each ratio $\frac{R_i^\varepsilon}{\rho_i} =: \tilde{\lambda}_i^\varepsilon \leq \tilde{\Lambda}$.

The key idea to prove the existence of the threshold M is that the configurations $\omega \in \Omega$ for which the radii R_i 's obtained after M iterations continue to change is related to events of the form

“There exist $M + 1$ balls in $I_k^\varepsilon \cup I_{k-1}^\varepsilon$ which are connected when dilated by $C(\alpha, M)$ ”.

By Lemma 6.6.1, this event has zero probability for ε sufficiently small.

- The construction above can be expressed by a dynamical system (cf. (6.3.19)).
- We iterate this process for $I_k^\varepsilon \cap I_{k-1}^\varepsilon$, $-2 \leq k \leq k_{\max}$ starting from $k = -2$, each time working with the dilated radii that we got from the previous step.

Step 2: Construction of the sets \mathcal{I}^ε and J^ε : Let us set $\theta = \alpha^{\frac{1}{4}} \geq 1$, with $\alpha \geq 1$ as in Step 1 (see (6.3.14)). In the previous step we extracted from each family I_k^ε generating the whole $\Phi^\varepsilon(D)$ a sub-collection \tilde{I}_k^ε . These sub-collections provide a covering for the whole set H^ε and satisfy (6.3.14). The aim of this step is to use the previous result to find a way to extract from $\Phi^\varepsilon(D)$ the subset \mathcal{I}^ε generating the bad holes and to construct the covering \tilde{H}_b^ε .

We remark that, if we set $\lambda_i = \theta^2 \tilde{\lambda}_i$, the covering

$$\bigcup_{k=-3}^{k_{\max}} \bigcup_{z_j \in \tilde{I}_k^\varepsilon} B_{\varepsilon^{\frac{d}{d-2}} \tilde{\lambda}_j^\varepsilon \rho_j}(\varepsilon z_j) \supseteq H^\varepsilon$$

satisfies (6.3.7) thanks to (6.3.14).

The construction of this step is based on the following simple geometric fact: Let $z_1 \in \tilde{I}_{k_1}^\varepsilon$ and $z_2 \in \tilde{I}_{k_2}^\varepsilon$ with $k_1 < k_2 - 1$. Since by construction we had $\tilde{I}_k^\varepsilon \subset \mathcal{I}_k^\varepsilon$, this means by definition (6.3.11) of the sets $\mathcal{I}_k^\varepsilon$ that $\varepsilon^{\frac{d}{d-2}} \rho_1 \leq \varepsilon^\delta \varepsilon^{\frac{d}{d-2}} \rho_2$ and thus that the ball $B_{\varepsilon^{\frac{d}{d-2}} \rho_1}(\varepsilon z_1)$ is much smaller than $B_{\varepsilon^{\frac{d}{d-2}} \rho_2}(\varepsilon z_2)$. Therefore, for $\varepsilon \leq \varepsilon_0(d, \beta, \theta)$ we have that

$$B_{\varepsilon^{\frac{d}{d-2}} \theta^3 \tilde{\lambda}_1^\varepsilon \rho_1}(\varepsilon z_1) \cap B_{\varepsilon^{\frac{d}{d-2}} \tilde{\lambda}_2^\varepsilon \rho_2}(\varepsilon z_2) \neq \emptyset \Rightarrow B_{\varepsilon^{\frac{d}{d-2}} \theta \tilde{\lambda}_1^\varepsilon \rho_1}(\varepsilon z_1) \subseteq B_{\varepsilon^{\frac{d}{d-2}} \theta^2 \tilde{\lambda}_2^\varepsilon \rho_2}(\varepsilon z_2). \quad (6.3.16)$$

Indeed, if the inequality on the left-hand side above is true, for all $z \in B_{\varepsilon^{\frac{d}{d-2}} \theta \tilde{\lambda}_1^\varepsilon \rho_1}(\varepsilon z_1)$ we have

$$\varepsilon |z - z_2| \leq \varepsilon |z - z_1| + \varepsilon |z_1 - z_2| \leq \varepsilon^{\frac{d}{d-2}} \theta \tilde{\lambda}_1^\varepsilon \rho_1 + \varepsilon^{\frac{d}{d-2}} \theta^3 \tilde{\lambda}_1^\varepsilon \rho_1 + \varepsilon^{\frac{d}{d-2}} \tilde{\lambda}_2^\varepsilon \rho_2.$$

Since $\varepsilon^{\frac{d}{d-2}} \rho_1 \leq \varepsilon^\delta \varepsilon^{\frac{d}{d-2}} \rho_2$ and all $1 \leq \tilde{\lambda}_i^\varepsilon \leq \tilde{\Lambda}$, we may choose $\varepsilon^\delta < \frac{\theta^2 - 1}{\theta \tilde{\Lambda}(1 + \theta^2)}$ and obtain that

$$\varepsilon |z - z_2| \leq \varepsilon^{\frac{d}{d-2}} \theta^2 \tilde{\lambda}_2^\varepsilon \rho_2,$$

i.e. the right-hand side in (6.3.16).

By relying on (6.3.16), we construct the covering J^ε in the following way:

- We start with k_{max} and set $J_{k_{max}}^\varepsilon = \tilde{I}_{k_{max}}^\varepsilon$ and $J_{k_{max}-1}^\varepsilon = \tilde{I}_{k_{max}-1}^\varepsilon$. We know that all the balls of the form $B_{\varepsilon^{\frac{d}{d-2}} \tilde{\lambda}_i^\varepsilon \rho_i}(\varepsilon z_i)$ generated by $z_i \in \tilde{I}_{k_{max}}^\varepsilon \cup \tilde{I}_{k_{max}-1}^\varepsilon$ are disjoint in the sense of (6.3.14) (recall that $\theta^4 = \alpha$). The same holds for the balls $B_{\varepsilon^{\frac{d}{d-2}} \tilde{\lambda}_j^\varepsilon \rho_j}(\varepsilon z_j)$ generated by the centres in $\tilde{I}_{k_{max}-2}^\varepsilon \cup \tilde{I}_{k_{max}-1}^\varepsilon$. We thus focus on the intersections between the balls generated by $\tilde{I}_{k_{max}-2}^\varepsilon$ and $\tilde{I}_{k_{max}}^\varepsilon$.
- We show how to obtain the set $J_{k_{max}-2}^\varepsilon$ from $\tilde{I}_{k_{max}-2}^\varepsilon$ in such a way that (6.3.8) is satisfied by this family. We begin by dilating the balls generated by the centres in $J_{k_{max}}^\varepsilon$ of a factor θ^2 and thus obtain the set

$$E_{k_{max}}^\varepsilon = \bigcup_{z_j \in J_{k_{max}}^\varepsilon} B_{\varepsilon^{\frac{d}{d-2}} \lambda_j^\varepsilon \rho_j}(\varepsilon z_j)$$

(we recall that $\lambda_j^\varepsilon = \theta^2 \tilde{\lambda}_j^\varepsilon$). We define

$$J_{k_{max}-2}^\varepsilon := \{z_i \in \tilde{I}_{k_{max}-2}^\varepsilon : B_{\varepsilon^{\frac{d}{d-2}} \theta \tilde{\lambda}_i^\varepsilon \rho_i}(\varepsilon z_i) \not\subseteq E_{k_{max}}^\varepsilon\}.$$

Note that with this definition, for all $z_j \in J_{k_{max}-2}^\varepsilon$ and every $z_i \in J_{k_{max}}^\varepsilon$ we have that

$$B_{\varepsilon^{\frac{d}{d-2}} \theta \tilde{\lambda}_i^\varepsilon \rho_i}(\varepsilon z_i) \not\subseteq B_{\varepsilon^{\frac{d}{d-2}} \lambda_j^\varepsilon \rho_j}(\varepsilon z_j)$$

and thus by property (6.3.16) (with $z_i = z_1$ and $z_j = z_2$) that

$$B_{\varepsilon^{\frac{d}{d-2}} \theta \tilde{\lambda}_i^\varepsilon \rho_i}(\varepsilon z_i) \cap B_{\varepsilon^{\frac{d}{d-2}} \lambda_j^\varepsilon \rho_j}(\varepsilon z_j) = \emptyset.$$

Since $\tilde{\lambda}_j^\varepsilon \geq 1$, the previous equality implies that the collection $J_{k_{max}-2}^\varepsilon$ satisfies condition (6.3.8).

- We now iterate the previous construction: We define

$$E_{k_{max}-1}^\varepsilon = E_{k_{max}}^\varepsilon \cup \bigcup_{z_i \in J_{k_{max}-1}^\varepsilon} B_{\varepsilon^{\frac{d}{d-2}} \lambda_i^\varepsilon \rho_i}(\varepsilon z_i)$$

and

$$E_{k_{max}-2}^\varepsilon = (E_{k_{max}-1}^\varepsilon \setminus \bigcup_{z_i \in J_{k_{max}-2}^\varepsilon} B_{\varepsilon^{\frac{d}{d-2}} \theta \lambda_i^\varepsilon \rho_i}(\varepsilon z_i)) \cup \left(\bigcup_{z_i \in J_{k_{max}-2}^\varepsilon} B_{\varepsilon^{\frac{d}{d-2}} \lambda_i^\varepsilon \rho_i}(\varepsilon z_i) \right).$$

Note that in the definition of this last set we need to remove the annuli

$$B_{\varepsilon^{\frac{d}{d-2}} \theta \lambda_i^\varepsilon \rho_i}(\varepsilon z_i) \setminus B_{\varepsilon^{\frac{d}{d-2}} \lambda_i^\varepsilon \rho_i}(\varepsilon z_i)$$

in order to be able to iterate the argument of the previous step (see Figure 6.3 for an illustration of the construction of the set $E_{k_{max}-2}^\varepsilon$).

- We iterate the previous procedure and construct the sets J_k^ε , up to $-2 \leq k \leq k_{max}$. In the last step $k = -3$, we define J_{-3}^ε as the set of those elements which either intersect E_{-2}^ε or that are too close to each other. Thanks to this construction, some elements of $\tilde{I}_{-3}^\varepsilon$, i.e. the holes which are small and well-separated from the clusters and from each others, do not belong to any of the sets J_k^ε nor are covered by any of the dilated balls generated by these centres. We then show that the remaining elements in $\tilde{I}_{-3}^\varepsilon$ constitute the set n^ε generating the holes H_g^ε .

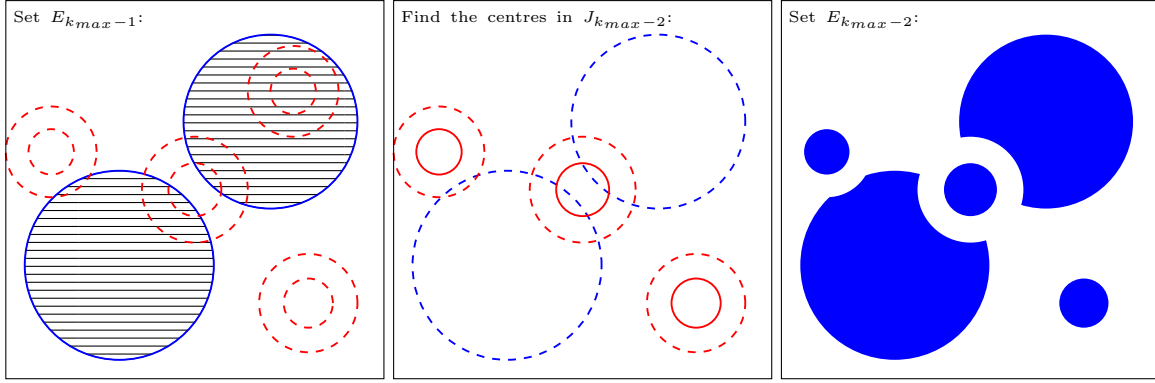


Figure 6.3: This sequence of pictures shows how to construct $E_{k_{max}-2}$ from $E_{k_{max}-1}$: In the first picture on the left, the set $E_{k_{max}-1}$ is the one filled with horizontal lines. Note that the balls are all disjoint and well-separated. The dashed annuli are the balls generated by centres in $\tilde{J}_{k_{max}-2}$ and dilated by the factor θ . The circles with the full line in the second picture represent the balls whose centres are in the set $J_{k_{max}-2}$. The third picture shows the set $E_{k_{max}-2}$.

- We finally define and partition the set \mathcal{I}^ε generating the holes of H_b^ε by using the sets $\{J_k^\varepsilon\}_{-3 \leq k \leq k_{max}}$: We insert in each $\mathcal{I}_k^\varepsilon$ the centres of the balls of H^ε such that k is the smallest integer for which J_k^ε provides a covering.

Step 3. Conclusion. We show that with these definitions of $J^\varepsilon, \mathcal{I}_k^\varepsilon$ and λ_j^ε , the covering obtained in the previous step satisfies all the properties of Lemma 6.3.1 and Lemma 6.3.2.

6.3.2 Proof of Lemma 6.3.1 and Lemma 6.3.2.

Proof of Lemma 6.3.2. In the sake of a leaner notation, when no ambiguity occurs we drop the index ε in the sets of points (e.g. $I_k^\varepsilon, J_k^\varepsilon, \dots$) and holes which are generated by them.

Proof of Step 1. We start by fixing a (total) ordering \leq of the points in $\Phi^\varepsilon(D)$ such that

$$z_i \leq z_j \Rightarrow \rho_i \leq \rho_j,$$

with ρ_i and ρ_j the radii of the balls in $H^\varepsilon(D)$ centred in z_i and z_j , respectively. We fix $\alpha > 1$ and set $C_0(\alpha, M) = (2\alpha M)^{M(k_{max}+3)} < +\infty$, where $M = M(\beta, d) \in \mathbb{N}$ is as in Lemma 6.6.1. We only consider $\omega \in \Omega$ belonging to the full-probability subset of Ω satisfying Lemma 6.6.1 with $\alpha = C_0$ and δ as in (6.3.10).

We introduce some more notation which is needed to implement the construction sketched in Step 1: Let $\Psi^\varepsilon \subset \Phi^\varepsilon(D)$ be any sub-collection of centres and let $\mathcal{R}^\varepsilon = \{R_i\}_{z_i \in \Psi^\varepsilon} \subset \mathbb{R}_+^{\#\Psi^\varepsilon}$ be their associated radii. Throughout this proof, unless there is danger of ambiguity, we forget about the dependence of both Ψ and \mathcal{R} on ε . For any two centres $z_i, z_j \in \Psi$ with radii R_i and R_j , respectively, we write

$$z_i \stackrel{\alpha}{-} z_j \Leftrightarrow B_{\alpha \varepsilon^{\frac{d}{d-2}} R_j}(\varepsilon z_j) \cap B_{\alpha \varepsilon^{\frac{d}{d-2}} R_i}(\varepsilon z_i) \neq \emptyset. \quad (6.3.17)$$

We define a notion of connection between points and associated radii in the following way: We say that (z_i, R_i) and (z_j, R_j) are connected, and we write that $z_i \sim_{(\Psi, \mathcal{R}), \alpha} z_j$ whenever

$$\exists z_1, \dots, z_m \in \Psi \text{ s.t. } z_i \stackrel{\alpha}{-} z_1 \stackrel{\alpha}{-} \dots \stackrel{\alpha}{-} z_m \stackrel{\alpha}{-} z_j.$$

This equivalence relation depends on ε , but we forget about it in the notation. We use the notation $[z_i](\Psi, \mathcal{R}, \alpha)$ for each equivalence class with respect to the previous equivalence relation $\sim_{(\Psi, \mathcal{R})\alpha}$. Each equivalence class constitutes a cluster of balls in the sense of (6.3.17).

By using this notation we may reformulate the result of Lemma 6.6.1: For almost every $\omega \in \Omega$, every $\varepsilon \leq \varepsilon_0(\omega, d, \beta)$ and any $k \geq -2$, if we choose $\Psi = I_k \cup I_{k-1}$, and $\mathcal{R} = \{\rho_i\}_{z_i \in \Psi}$, we have

$$\sup_{z \in \Psi} (\#[z](\Psi, \mathcal{R}, C_0)) \leq M, \quad (6.3.18)$$

i.e. every equivalence class contains at most M elements of Ψ . From now on, we thus fix $\omega \in \Omega$ and $\varepsilon \leq \varepsilon_0(\omega, d, \beta)$ satisfying this bound.

Given $\Psi \subset \Phi^\varepsilon(D)$, we introduce the map $T^{\Psi, \alpha} : \mathbb{R}_+^{\#\Psi} \rightarrow \mathbb{R}_+^{\#\Psi}$ which acts on $\mathcal{R} = \{R_i\}_{z_i \in \Psi}$ as

$$(T^{\Psi, \alpha}(\mathcal{R}))_j := \begin{cases} 0 & \text{if } \max\{z_i \in [z_j]_{\Psi, \mathcal{R}, \alpha}\} \neq z_j \\ \max_{z_i \in [z_j]_{\Psi, \mathcal{R}, \alpha}} (\varepsilon^{1-\frac{d}{d-2}} |z_j - z_i| + R_i) & \text{if } \max\{z_i \in [z_j]_{\Psi, \mathcal{R}, \alpha}\} = z_j \end{cases} \quad (6.3.19)$$

We recall that the maximum above is taken with respect to the ordering \leq between centres of $\Psi^\varepsilon(D)$. We observe that (6.3.19) implies that, if $[z_j](\Psi, \mathcal{R}, \alpha) = \{z_j\}$, then

$$T^{\Psi, \alpha}(\mathcal{R})_j = R_j.$$

By relying on (6.3.18), we use an iteration of the previous map to implement the construction sketched at Step 1. We begin by considering $k = -2$ and setting $\Psi = I_{-2} \cup I_{-3}$ and $\mathcal{R} = \{\rho_i\}_{z_i \in \Psi}$. We define the dynamical system

$$\begin{cases} \mathcal{R}(n) = T^{\Psi, \alpha}(\mathcal{R}(n-1)) & n \in \mathbb{N} \\ \mathcal{R}(0) = \mathcal{R} \end{cases} \quad (6.3.20)$$

and claim that

$$\mathcal{R}(n) = \mathcal{R}(M) \quad \forall n \geq M \quad (6.3.21)$$

$$(\mathcal{R}(n))_j \leq (2\alpha M)^n \rho_j \quad \forall z_j \in \Psi, \quad \forall n \leq M. \quad (6.3.22)$$

We start with (6.3.22) and prove it by induction over $n \leq M$. By definition (cf. (6.3.20)), the inequality trivially holds for $n = 0$. Let us now assume that (6.3.22) holds for some $0 \leq n < M$. We claim that at step $n+1$, each equivalence class $[z_i](\Psi, \mathcal{R}(n), \alpha)$ contains at most M elements: If otherwise, by the inductive hypothesis (6.3.22) for n and the choice of the constant $C_0(M, \alpha)$, also the equivalence class $[z_i](\Psi, \mathcal{R}(0), C_0)$ contains more than M elements. Since we chose $\mathcal{R}(0) = \{\rho_i\}_{z_i \in \Psi}$, by our choice of $\omega \in \Omega$ and $\varepsilon \leq \varepsilon(\omega, C_0)$, property (6.3.18) is contradicted. Thus, each equivalence class $[z_i](\Psi, \mathcal{R}(n), \alpha)$ contains at most M elements. This allows us to bound

$$(\mathcal{R}(n+1))_j \stackrel{(6.3.20)}{\leq} 2\alpha \sum_{z_i \in [z_j](\Psi, \mathcal{R}(n), \alpha)} R(n)_i \stackrel{(6.3.22)}{\leq} (2\alpha)^{n+1} M^n \sum_{z_i \in [z_j](\Psi, \mathcal{R}(n), \alpha)} \rho_i$$

We now observe that by construction (6.3.20) and definition (6.3.19), either $\mathcal{R}(n+1)_j = 0$, and thus the bound (6.3.22) holds trivially, or $\rho_j \geq \rho_i$ for all $z_i \in [z_j](\Psi, \mathcal{R}(n), \alpha)$. Thus, the previous inequality implies that

$$(\mathcal{R}(n+1))_j \leq (2\alpha M)^{n+1} \rho_j, \quad (6.3.23)$$

i.e. inequality (6.3.22) for $n + 1$. The induction proof for (6.3.22) is complete.

We now show (6.3.21): We begin by remarking that, by construction, if we have $\mathcal{R}(M) \neq \mathcal{R}(M+1)$, then there exist z_1, \dots, z_{M+1} such that

$$\bigcup_{k=1}^{M+1} B_{\varepsilon^{\frac{d}{d-2}} \rho_k}(\varepsilon z_k) \subset B_{\varepsilon^{\frac{d}{d-2}} \mathcal{R}(M+1)_1}(\varepsilon z_1).$$

This, together with estimate (6.3.22) for $n = M$, implies that the equivalence class $[z_i](\Psi, \mathcal{R}(0), C_0)$ contains more than M elements. As above, this contradicts our choice of the realization $\omega \in \Omega$ and ε . We established (6.3.21).

Equipped with properties (6.3.22) and (6.3.21) we may set for every $z_i \in \Phi^\varepsilon(D)$

$$\mathcal{R}_j^{(-2)} := \begin{cases} \mathcal{R}(M) & \text{if } z_i \in I_{-2} \cup I_{-3} \\ \rho_i & \text{otherwise} \end{cases}$$

and define

$$\tilde{I}_{-3} := \{z_i \in I_{-3} : \mathcal{R}_i^{(-2)} > 0\}.$$

Note that this definition of $\mathcal{R}^{(-2)}$ implies that the balls

$$\{B_{\alpha \varepsilon^{\frac{d}{d-2}} \mathcal{R}_i^{(-2)}}(\varepsilon z_i)\}_{z_i \in I_{-2} \cup \tilde{I}_{-3}}$$

are pairwise disjoint.

We now iterate the previous step up to $k = k_{max}$: For each $-1 \leq k \leq k_{max}$ we define recursively

$$\mathcal{R}_j^{(k)} := \begin{cases} \mathcal{R}(M) & \text{if } z_i \in I_k \cup I_{k-1} \\ \mathcal{R}^{(k-1)} & \text{otherwise,} \end{cases} \quad (6.3.24)$$

where $\mathcal{R}(M)$ is obtained by solving (6.3.19) with $\Psi = I_k \cup I_{k-1}$ and $\mathcal{R}(0) = \mathcal{R}^{(k-1)}$. We note that for a general $-1 \leq k \leq k_{max}$, (6.3.22) turns into

$$(\mathcal{R}^{(k)}(n))_j \leq (2\alpha M)^{(k+2)M+n} \rho_j \quad \forall z_j \in \Psi, \quad \forall n \leq M. \quad (6.3.25)$$

In fact, since for $n \leq M$ we have $(2\alpha M)^{(k+2)M+n} \leq C_0$, property (6.3.21) follows by this inequality exactly as in the case $k = -2$ shown above. We emphasize that, by definition (6.3.24), at each step k we have that the balls

$$\{B_{\alpha \varepsilon^{\frac{d}{d-2}} \mathcal{R}_i^{(k)}}(\varepsilon z_i)\}_{z_i \in I_k \cup \tilde{I}_{k-1}, \mathcal{R}_i^{(k)} > 0} \quad (6.3.26)$$

are pairwise disjoint.

From the previous construction we construct the sets \tilde{I}_k and the parameters $\{\tilde{\lambda}_i\}_{z_i \in \bigcup_{k=-3}^{k_{max}} \tilde{I}_k}$ of Step 1: For every $-3 \leq k \leq k_{max}$, let

$$\begin{aligned} \tilde{I}_k &:= \{z_i \in \mathcal{I}_k : (\mathcal{R}^{(k+1)}(M))_i > 0\}, \\ \tilde{\lambda}_i &= \frac{(\mathcal{R}^{(k+1)}(M))_i}{\rho_i} \quad \text{for } z_i \in \tilde{I}_k. \end{aligned} \quad (6.3.27)$$

By (6.3.25) and the definition of the sets \tilde{I}_k , we immediately have that each $\tilde{\lambda}_i \geq 1$ and is bounded by $\tilde{\Lambda} := (2\alpha M)^{(k_{max}+3)M}$. It remains to argue that \tilde{I}^k satisfy (6.3.13) and (6.3.14): Property (6.3.13) follows immediately from the construction and the definition of the operator $T^{\Psi, \alpha}$. To prove (6.3.14), we claim that is enough to show that for every $k = -2, \dots, k_{max}$ and $z_i \in \tilde{I}_k$,

$$\tilde{\lambda}_i = \frac{\mathcal{R}_i^{(k)}}{\rho_i}. \quad (6.3.28)$$

Indeed, if this is true, then (6.3.14) follows immediately from (6.3.26).

Let $-2 \leq k \leq k_{max}$ be fixed. By (6.3.24), to show (6.3.28) it enough to prove that

$$\mathcal{R}_i^{(k)} = \mathcal{R}_i^{(k+1)}, \quad \text{for all } z_i \in \tilde{I}_k.$$

Since by (6.3.24) we have for all $z_i \in \tilde{I}_k$ that $\mathcal{R}_i^{(k+1)} = \mathcal{R}(M)_i$, with $\mathcal{R}(M)$ solving

$$\begin{cases} \mathcal{R}(n) = T^{\Psi, \alpha}(\mathcal{R}(n-1)) & n \in \mathbb{N} \\ \mathcal{R}(0) = \mathcal{R}^{(k)}, \end{cases}$$

we need to make sure that $\mathcal{R}(n)_i = \mathcal{R}_i^{(k)}$ for each $1 \leq n \leq M$. By induction we show that for $z_i \in I_k$ we have

$$\mathcal{R}(n)_i \neq \mathcal{R}_i^{(k)} \Rightarrow \mathcal{R}(n+1)_i = \mathcal{R}^{(k+1)} = 0 \quad (6.3.29)$$

This implies (6.3.28) by definition (6.3.27).

For $n = 1$, property (6.3.29) is an easy consequence of (6.3.26) for the balls generated by points $z_i \in I_k$. Let us assume that (6.3.27) holds at step n . Then, again by (6.3.27), we have that for $z_i \in I_k$ either $\mathcal{R}(n)_i = 0$, or $\mathcal{R}(n)_i = \mathcal{R}_i^{(k)}$. Thus, if $\mathcal{R}(n+1)_i \neq \mathcal{R}(n)_i$, we necessarily have again by (6.3.26) that there exists $z_j \in I_{k+1}$ such that

$$B_{\alpha \varepsilon^{\frac{d}{d-2}} \mathcal{R}_j^{(n-1)}}(\varepsilon z_j) \cap B_{\alpha \varepsilon^{\frac{d}{d-2}} \mathcal{R}(n-1)_i}(\varepsilon z_i) \neq \emptyset.$$

This implies that $\rho_j \geq \rho_i$ and in turn that $z_j \geq z_i$. By definition of the map $T^{\Psi, \alpha}$, this yields necessarily that $\mathcal{R}(n+1)_i = 0$. The proof of (6.3.29) is complete. This establishes (6.3.28) and concludes the proof of (6.3.14).

We conclude this step with the following remark: Let $\Phi_{2\varepsilon\delta/2}^\varepsilon(D)$ be the thinned process (see (6.2.3)) with δ fixed as in (6.3.10). Moreover, let $S^\varepsilon := \Phi^\varepsilon(D) \setminus \Phi_{2\varepsilon\delta/2}^\varepsilon(D)$ and

$$I_{-3}^g = I_{-3} \cap \Phi_{2\varepsilon\delta/2}^\varepsilon(D), \quad I_{-3}^b = I_{-3} \setminus I_{-3}^g = I_{-3} \cap S^\varepsilon. \quad (6.3.30)$$

We claim that, up to taking $\varepsilon_0 = \varepsilon_0(d, \beta)$ smaller than above, we have

$$I_{-3}^g \subset \tilde{I}_{-3}, \quad \tilde{\lambda}_i = 1 \quad \text{for all } z_i \in I_{-3}^g. \quad (6.3.31)$$

As will be shown in the next step, the set I_{-3}^g contains the set n^ε generating H_g^ε .

To show (6.3.31), we observe that whenever $z_i, z_j \in I_{-3}^g \cup I_{-2}$ with $z_i \neq z_j$, then we may choose ε small enough to infer that

$$B_{\alpha \varepsilon^{\frac{d}{d-2}} \rho_i}(\varepsilon z_i) \cap B_{\alpha \tilde{\lambda}_i \varepsilon^{\frac{d}{d-2}} \rho_j}(\varepsilon z_j) = \emptyset.$$

Indeed, for $\varepsilon^{\frac{\delta}{2}} \leq (\alpha\tilde{\Lambda})^{-1}$, we bound

$$\varepsilon|z_i - z_j| \stackrel{(6.3.30)}{\geq} 2\varepsilon^{1+\frac{\delta}{2}} \geq 2\alpha\tilde{\Lambda}\varepsilon^{1+\delta} \stackrel{(6.3.11)}{\geq} \varepsilon^{\frac{d}{d-2}}(\alpha\rho_i + \tilde{\Lambda}\rho_j).$$

This implies that after M iterations of the dynamical system (6.3.23), we have $\mathcal{R}(M) = \rho_i$ for all $z_i \in I_{-3}^g$. Thanks to (6.3.27) we obtain (6.3.31).

Proof of Step 2. In this step we rigorously implement the method sketched in Step 2 and construct the sets J_k^ε as subsets of \tilde{I}_k^ε , $-3 \leq k \leq k_{max}$. We define $\lambda_j = \theta^2 \tilde{\lambda}_j$, with $\tilde{\lambda}_j \in [1, \tilde{\Lambda}]$ constructed in Claim 1 of Step 1, and $\theta^4 = \alpha$. Clearly, we may choose the upper bound Λ in the statement of Lemma 6.3.2 as $\Lambda := \theta\tilde{\Lambda}$. We start by setting

$$\begin{aligned} J_{k_{max}} &:= \tilde{I}_{k_{max}}^\varepsilon, \\ E_{k_{max}} &:= \bigcup_{z_j \in J_{k_{max}}} B_{\lambda_j \varepsilon^{\frac{d}{d-2}} \rho_j}(\varepsilon z_j), \end{aligned}$$

and inductively define for $-1 \leq l \leq k_{max}$

$$J_{l-1} := \left\{ z_j \in \tilde{I}_{l-1} : B_{\theta \tilde{\lambda}_j \varepsilon^{\frac{d}{d-2}} \rho_j}(\varepsilon z_j) \not\subset E_l \right\}, \quad (6.3.32)$$

$$E_{l-1} := \left(E_l \setminus \bigcup_{z_j \in J_{l-1}} B_{\theta \lambda_j \varepsilon^{\frac{d}{d-2}} \rho_j}(\varepsilon z_j) \right) \cup \bigcup_{z_j \in J_{l-1}} B_{\lambda_j \varepsilon^{\frac{d}{d-2}} \rho_j}(\varepsilon z_j). \quad (6.3.33)$$

To construct the remaining sets J_{-3} and E_{-3} , we need an additional step: We recall the definition of S^ε and I_{-3}^g from (6.2.3) and (6.3.30), respectively. We first set

$$\begin{aligned} \tilde{J}_{-3} &:= \left\{ z_j \in \tilde{I}_{-3} \cap S^\varepsilon : B_{\theta \lambda_j \varepsilon^{\frac{d}{d-2}} \rho_j}(\varepsilon z_j) \not\subset E_{-2} \right\}, \\ \tilde{E}_{-3} &:= \left(E_{-2} \setminus \bigcup_{z_j \in \tilde{J}_{-3}} B_{\theta \lambda_j \varepsilon^{\frac{d}{d-2}} \rho_j}(\varepsilon z_j) \right) \cup \bigcup_{z_j \in \tilde{J}_{-3}} B_{\lambda_j \varepsilon^{\frac{d}{d-2}} \rho_j}(\varepsilon z_j). \end{aligned} \quad (6.3.34)$$

Finally, for $z_i \in \Phi^\varepsilon(D)$ we define the set

$$K^\varepsilon := \left\{ z_j \in I_{-3}^g : B_{2\varepsilon^{1+\delta}}(\varepsilon z_j) \cap \bigcup_{z_i \in \bigcup_{k=-2}^{k_{max}} J_k \cup \tilde{J}_{-3}} B_{\theta \lambda_i \varepsilon^{\frac{d}{d-2}} \rho_i}(\varepsilon z_i) \neq \emptyset \right\}, \quad (6.3.35)$$

and finally consider

$$\begin{aligned} J_{-3} &:= \tilde{J}_{-3} \cup \left\{ z_j \in K^\varepsilon : B_{\theta \lambda_j \varepsilon^{\frac{d}{d-2}} \rho_j}(\varepsilon z_j) \not\subset \tilde{E}_{-3} \right\}, \\ \tilde{E}_{-3} &:= \left(E_{-2} \setminus \bigcup_{z_j \in J_{-3}} B_{\theta \lambda_j \varepsilon^{\frac{d}{d-2}} \rho_j}(\varepsilon z_j) \right) \cup \bigcup_{z_j \in J_{-3}} B_{\lambda_j \varepsilon^{\frac{d}{d-2}} \rho_j}(\varepsilon z_j). \end{aligned} \quad (6.3.36)$$

We remark that in the definitions of E_l , the annuli $B_{\theta \lambda_j \varepsilon^{\frac{d}{d-2}} \rho_j}(\varepsilon z_j) \setminus B_{\lambda_j \varepsilon^{\frac{d}{d-2}} \rho_j}(\varepsilon z_j)$ are cut out in order to satisfy (6.3.8). Moreover, we observe that each connected component of the set E_k is a subset of $B_{\lambda_j \varepsilon^{\frac{d}{d-2}} \rho_j}(\varepsilon z_j)$ for some $z_j \in J_l$, for $k \geq l$. This follows from the the definition of E_k and (6.3.14).

We finally denote

$$J := \bigcup_{k=-3}^{k_{max}} J_k. \quad (6.3.37)$$

and define the set \mathcal{I} of the centres generating H_b^ε as

$$\mathcal{I} := \left\{ z_i \in \Phi^\varepsilon(D) : B_{\varepsilon^{\frac{d}{d-2}} \rho_i}(\varepsilon z_i) \subset B_{\lambda_j \varepsilon^{\frac{d}{d-2}} \rho_j}(\varepsilon z_j) \text{ for some } z_j \in J \right\}, \quad (6.3.38)$$

$$\mathcal{I}_k := \left\{ z_i \in \mathcal{I} : k \text{ is minimal such that } B_{\varepsilon^{\frac{d}{d-2}} \rho_i}(\varepsilon z_i) \subset B_{\lambda_j \varepsilon^{\frac{d}{d-2}} \rho_j}(\varepsilon z_j) \text{ for } z_j \in J_k \right\}. \quad (6.3.39)$$

Equipped with the previous definition, we construct H_b^ε , \bar{H}_b^ε and D_b^ε as shown in (6.3.4), (6.3.5), and (6.3.9).

Proof of Step 3. We first argue that the sets H_b^ε , \bar{H}_b^ε and D_b^ε constructed in the previous step satisfy the conditions of Lemma 6.3.1.

We begin by claiming that

$$n_\varepsilon = I_{-3}^g \setminus K^\varepsilon, \quad (6.3.40)$$

with K^ε defined in (6.3.35). Since, by construction we set $H_g^\varepsilon = H^\varepsilon \setminus H_b^\varepsilon$, by (6.3.4) this also reads as

$$\Phi^\varepsilon(D) \setminus \mathcal{I} = I_{-3}^g \setminus K^\varepsilon. \quad (6.3.41)$$

The \supseteq -inclusion is a consequence of the fact that by (6.3.31) we have by construction $I_{-3}^g \cap \tilde{J}_{-3} = \emptyset$ (see (6.3.34), (6.2.3)). This yields that in the definition (6.3.36) of J_{-3} the only elements of I_{-3}^g in J_{-3} are the ones contained in K^ε . By (6.3.32) and (6.3.37), this yields that $(I_{-3}^g \setminus K) \cap J = \emptyset$. We now use (6.3.39) to infer that also $(I_{-3}^g \setminus K^\varepsilon) \cap \mathcal{I} = \emptyset$, i.e. the \supseteq -inclusion in (6.3.41).

For the \subset inclusion we argue the complementary statement which, by (6.3.30), also reads as

$$K^\varepsilon \cup \bigcup_{k \geq -2} I_k^\varepsilon \cup I_{-3}^b \subset \mathcal{I}. \quad (6.3.42)$$

We show how to argue that $I_k \subset \mathcal{I}$, for some $k \geq -2$. The argument for the other sets is analogous.

Let $z_i \in I_k$. Then, by (6.3.13), there exists $l \geq k$, $z_{j_1} \in \tilde{I}_l$ such that

$$B_{\varepsilon^{\frac{d}{d-2}} \rho_i}(\varepsilon z_i) \subset B_{\varepsilon^{\frac{d}{d-2}} \tilde{\lambda}_{j_1} \rho_{j_1}}(\varepsilon z_{j_1}).$$

By definition (6.3.32), this yields that either $z_{j_1} \in J_l$ or

$$B_{\varepsilon^{\frac{d}{d-2}} \theta \tilde{\lambda}_{j_1} \rho_{j_1}}(\varepsilon z_{j_1}) \subset E_{l+1}.$$

In the first case, it is immediate that $z_i \in \mathcal{I}$ (see (6.3.38)); in the second case, since each connected component of the set E_{l+1} is a subset of a ball $B_{\lambda_{j_2} \varepsilon^{\frac{d}{d-2}} \rho_{j_2}}(\varepsilon z_{j_2})$ for some $z_{j_2} \in J_{l_2}$ with $l_2 > l_1$, it follows that

$$B_{\varepsilon^{\frac{d}{d-2}} \rho_i}(\varepsilon z_i) \subset B_{\lambda_{j_2} \varepsilon^{\frac{d}{d-2}} \rho_{j_2}}(\varepsilon z_{j_2}).$$

Hence, also in this case $z_i \in \mathcal{I}$. We established $I_k \subset \mathcal{I}$. This concludes the proof of (6.3.42) and thus also of (6.3.41) and (6.3.40).

From identity (6.3.40), the second line of (6.3.2) immediately follows by (6.3.30) and definition (6.3.11) for the set I_{-3} . In addition, since K^ε is not contained in n^ε , also the first inequality in (6.3.1) holds. The remaining claims in (6.3.1), (6.3.2), and (6.3.3) may be obtained from (6.3.42) similarly to Lemma 5.4.1, thanks to the very explicit definition of the sets \bar{H}_b^ε and D_b^ε .

In the sake of completeness we give these arguments explicitly: We claim

$$\lim_{\varepsilon \downarrow 0} \varepsilon^d \#(\mathcal{I}) = 0. \quad (6.3.43)$$

By taking the complement with respect to $\Phi^\varepsilon(D)$ in (6.3.41), we have

$$\mathcal{I} = \bigcup_{k=-2}^{k_{max}} I_k \cup I_{-3}^b \cup K^\varepsilon.$$

We estimate the limit for $\varepsilon \downarrow 0^+$ for the first sets on the right-hand side by appealing to Lemma 5.5.2 and (6.3.10) (we recall that we assumed $\beta \leq 1$): Indeed, we have

$$\begin{aligned} \limsup_{\varepsilon \downarrow 0} \varepsilon^d \# \left(\bigcup_{k=-2}^{k_{max}} I_k \right) &= \limsup_{\varepsilon \downarrow 0} \varepsilon^d \# \{z_i \in \Phi^\varepsilon(D) : \varepsilon^{\frac{d}{d-2}} \rho_i \geq \varepsilon^{1+2\delta}\} \\ &\leq \limsup_{\varepsilon \downarrow 0} \varepsilon^{d-(d-2)(1+2\delta)} \varepsilon^d \sum_{z_i \in \Phi^\varepsilon(D)} \rho_i^{d-2} \rightarrow 0 \\ &\lesssim \limsup_{\varepsilon \downarrow 0} \varepsilon^{2(1-(d-2)\delta)} = 0. \end{aligned}$$

We now turn to I_{-3}^b : Let $\{\delta_k\}_{k \in \mathbb{N}}$ be any sequence such that $\delta_k \downarrow 0^+$. Since $2\varepsilon^{\delta/2} \rightarrow 0$, we estimate for any $\delta_k > 0$

$$\limsup_{\varepsilon \downarrow 0^+} \varepsilon^d \#(I_{-3}^b) \stackrel{(6.3.30)}{\leq} \limsup_{\varepsilon \downarrow 0^+} \varepsilon^d (N^\varepsilon(D) - N_{2\varepsilon^{\delta/2}}^\varepsilon(D)) \stackrel{(6.2.3)}{\leq} \lim_{\varepsilon \downarrow 0^+} \varepsilon^d (N^\varepsilon(D) - N_{\delta_k}^\varepsilon(D)).$$

We now apply Lemma 5.5.2 to Φ and each Φ_{δ_k} , $k \in \mathbb{N}$, to deduce that almost surely and for every $\delta_k > 0$

$$\limsup_{\varepsilon \downarrow 0^+} \varepsilon^d \#(I_{-3}^b) \leq \lambda|D| - \langle N_{\delta_k}(D) \rangle.$$

By sending $\delta_k \downarrow 0^+$, we use once more Lemma 5.5.2 on the last term on the right-hand side above and obtain

$$\lim_{\varepsilon \downarrow 0^+} \varepsilon^d \#(I_{-3}^b) = 0.$$

To conclude the proof of (6.3.43), it thus remains to show that almost surely also

$$\varepsilon^d \#(K^\varepsilon) \rightarrow 0 \quad \varepsilon \downarrow 0^+. \quad (6.3.44)$$

We have for all $z_i \in K^\varepsilon \subset I_{-3}^g$

$$\min_{z_j \in \Phi^\varepsilon(D) \setminus \{z_i\}} \varepsilon |z_j - z_i| \geq 2\varepsilon^{1+\delta/2}, \quad \varepsilon^{\frac{d}{d-2}} \rho_i < \varepsilon^{1+2\delta}. \quad (6.3.45)$$

In particular, by the first inequality above, the balls $\{B_{\varepsilon^{1+2\delta}}(\varepsilon z_i)\}_{z_i \in K^\varepsilon}$ are all disjoint, and therefore

$$\varepsilon^d \#(K^\varepsilon) \lesssim \varepsilon^d \sum_{z_i \in K^\varepsilon} \varepsilon^{-d(1+2\delta)} |B_{\varepsilon^{1+2\delta}}(\varepsilon z_i)| = \varepsilon^{-2d\delta} \sum_{z_i \in \tilde{I}_b^\varepsilon} |B_{\varepsilon^{1+2\delta}}(\varepsilon z_i)|. \quad (6.3.46)$$

In addition, we observe that by definition of K^ε , for any $z_i \in K^\varepsilon$ there exists $z_j \in \cup_{k=-2}^{k_{max}} J_k$ such that

$$B_{2\varepsilon^{1+\delta}}(\varepsilon z_i) \cap B_{\theta\lambda_j \varepsilon^{\frac{d}{d-2}} \rho_j}(\varepsilon z_j) \neq \emptyset. \quad (6.3.47)$$

Here we used $K_\varepsilon \subset \tilde{I}_{-3}$ and (6.3.14) to rule out that $z_j \in J_{-3} \subset \tilde{I}_{-3}$. In particular, (6.3.45) and (6.3.47) imply

$$2\varepsilon^{1+\delta/2} \leq \varepsilon |z_i - z_j| \leq 2\varepsilon^{1+\delta} + \theta\lambda_j \varepsilon^{\frac{d}{d-2}} \rho_j,$$

we obtain that $\theta\lambda_j \varepsilon^{\frac{d}{d-2}} \rho_j \geq 2\varepsilon^{1+\delta}$. We combine this inequality with condition (6.3.47) to infer that

$$B_{\varepsilon^{1+2\delta}}(\varepsilon z_i) \subset B_{2\theta\lambda_j \varepsilon^{\frac{d}{d-2}} \rho_j}(\varepsilon z_j)$$

and, by (6.3.46), to estimate

$$\begin{aligned} \varepsilon^d \#(K^\varepsilon) &\lesssim \varepsilon^{-2d\delta} \sum_{z_j \in \cup_{k=-2}^{k_{max}} J_k} |B_{2\theta\lambda_j \varepsilon^{\frac{d}{d-2}} \rho_j}(\varepsilon z_j)| \\ &\lesssim \varepsilon^{-2d\delta} \left(\varepsilon^{\frac{d}{d-2}} \max_{z_j \in \Phi^\varepsilon(D)} \varepsilon^{\frac{d}{d-2}} \rho_j \right)^2 \sum_{z_j \in \cup_{k=-2}^{k_{max}} J_k} (\varepsilon^{\frac{d}{d-2}} \rho_j)^{d-2} \\ &\stackrel{(6.3.12)}{\lesssim} \varepsilon^{2\delta d} \sum_{z_j \in \Phi^\varepsilon(D)} (\varepsilon^{\frac{d}{d-2}} \rho_j)^{d-2}. \end{aligned}$$

Thanks to Lemma 5.5.2, the right-hand side vanishes almost surely in the limit $\varepsilon \downarrow 0^+$. This concludes the proof of (6.3.43).

The limit in the first line of (6.3.2) is a direct consequence of (6.3.43). Moreover, the second inequality in (6.3.1) follows from (6.3.43) and Lemma 5.5.11.

To show (6.3.3), we resort to the definition of D_b^ε to estimate

$$\begin{aligned} &\{z_i \in \Phi_{2\eta}^\varepsilon(D)(\omega) : \text{dist}(z_i, D_b^\varepsilon) \leq \eta\varepsilon\} \\ &\subset \mathcal{I} \cup \left\{ z_i \in n^\varepsilon(\omega) : \text{dist}\left(z_i, \bigcup_{z_j \in \cup_{k=-2}^{k_{max}} J_k} B_{\Lambda\varepsilon^{\frac{d}{d-2}} \rho_j}(\varepsilon z_j)\right) \leq \eta\varepsilon \right\} \\ &\cup \left\{ z_i \in n^\varepsilon(\omega) \cap \Phi_{2\eta}^\varepsilon(D)(\omega) : \text{dist}\left(z_i, \bigcup_{z_j \in J_{-3}} B_{\Lambda\varepsilon^{\frac{d}{d-2}} \rho_j}(\varepsilon z_j)\right) \leq \eta\varepsilon \right\} \\ &:= I_b^\varepsilon \cup F^\varepsilon \cup C^\varepsilon. \end{aligned}$$

We already know $\varepsilon^d \#(I_b^\varepsilon) \rightarrow 0$. Next, we argue that

$$\varepsilon^d \#(F^\varepsilon) \rightarrow 0.$$

This follows by an argument similar to the one for (6.3.44): We may choose $\varepsilon_0 = \varepsilon_0(d)$ such that for all $\varepsilon \leq \varepsilon_0$, $\varepsilon^{\delta/2} \leq \eta$. By definition of J_k and of F^ε above, we infer that for such $\varepsilon \leq \varepsilon_0$, for all $z_j \in F^\varepsilon$ there exists $-2 \leq k \leq k_{max}$ and $z_i \in J_k$ such that

$$B_{\varepsilon^{1+\delta/2}}(\varepsilon z_j) \subset B_{2\eta\varepsilon + \Lambda\varepsilon^{\frac{d}{d-2}}\rho_i}(\varepsilon z_i) \subset B_{2\Lambda\eta\varepsilon^{-2\delta}\varepsilon^{\frac{d}{d-2}}\rho_i}(\varepsilon z_i), \quad (6.3.48)$$

where in the second inequality we use that $\varepsilon^{-2\delta}\eta \geq 1$ and $\varepsilon^{\frac{d}{d-2}}\rho_i \geq \varepsilon^{1+2\delta}$. We note that by (6.3.45) the balls $B_{\varepsilon^{1+\delta/2}}(\varepsilon z_j)$ with $z_j \in n^\varepsilon$ are all disjoint. Hence,

$$\begin{aligned} \varepsilon^d \#(F^\varepsilon) &\stackrel{(6.3.48)}{\lesssim} \varepsilon^{-d\delta} \left| \bigcup_{z_i \in \bigcup_{k=-2}^{k_{max}} J_k} B_{2\Lambda\eta\varepsilon^{-2\delta}\varepsilon^{\frac{d}{d-2}}\rho_i}(\varepsilon z_i) \right| \\ &\lesssim \eta^d \varepsilon^{-d(\delta+2\delta)} \left(\max_{z_j \in \Phi^\varepsilon(D)} \varepsilon^{\frac{d}{d-2}}\rho_j \right)^2 \sum_{z_j \in \Phi^\varepsilon(D)} (\varepsilon^{\frac{d}{d-2}}\rho_j)^{d-2} \\ &\stackrel{(6.3.12)}{\lesssim} \eta^d \varepsilon^{d\delta} \sum_{z_j \in \Phi^\varepsilon(D)} (\varepsilon^{\frac{d}{d-2}}\rho_j)^{d-2}. \end{aligned}$$

The right-hand side vanishes almost surely in the limit $\varepsilon \downarrow 0^+$ thanks to (6.1.7) and Lemma 5.5.2.

We conclude the argument for (6.3.3) by showing that the set C^ε is empty when ε is small: In fact, by construction, if $z_i \in n_\varepsilon$ satisfies

$$\text{dist}\left(\varepsilon z_i, \bigcup_{z_j \in J_{-3}} B_{\Lambda\varepsilon^{\frac{d}{d-2}}\rho_j}(\varepsilon z_j)\right) \leq \eta\varepsilon,$$

then there exists a $z_j \in J_{-3} \subset I_{-3}$ such that for $\varepsilon \leq \varepsilon_0$ with $\Lambda\varepsilon^{2\delta} \leq \eta$

$$\varepsilon|z_i - z_j| \leq \text{dist}\left(\varepsilon z_i, B_{\Lambda\varepsilon^{\frac{d}{d-2}}\rho_j}(\varepsilon z_j)\right) + \Lambda\varepsilon^{1+2\delta} \leq 2\eta\varepsilon.$$

This yields $C^\varepsilon \subset \Phi^\varepsilon(D) \setminus \Phi_{2\eta}^\varepsilon(D)$ and thus that it is empty since by definition we also have $C^\varepsilon \subset \Phi_{2\eta}^\varepsilon(D)$. This finishes the proof of (6.3.3).

We hence have shown that H_b^ε , \bar{H}_b^ε and D_b^ε in Lemma 6.3.1 may be chosen as in Step 2 (see (6.3.4), (6.3.5), and (6.3.9)). We also remark that it immediately follows by (6.3.12) and the bounds on $\lambda_i^\varepsilon \leq \Lambda$ obtained at the beginning of Step 2, that the radii $\lambda_i^\varepsilon \varepsilon^{\frac{d}{d-2}}\rho_i$ generating the balls of \bar{H}_b^ε satisfy the second inequality in (6.3.5).

It remains to argue (6.3.7) and (6.3.8). The first property follows directly from (6.3.14) for $J_k \subset \tilde{I}_k$ and the choice of the parameters $\lambda_i = \theta\tilde{\lambda}_i$ and $\theta^4 = \alpha$.

We now turn to (6.3.8) and begin by showing that it suffices to prove the following:

Claim: For all $-3 \leq k < l \leq k_{max}$ and every $z_k \in J_k$, $z_l \in \tilde{I}_l$ we have

$$B_{\tilde{\lambda}_l \varepsilon^{\frac{d}{d-2}}\rho_l}(\varepsilon z_l) \cap B_{\theta\lambda_k \varepsilon^{\frac{d}{d-2}}\rho_k}(\varepsilon z_k) = \emptyset. \quad (6.3.49)$$

We first prove (6.3.54) provided this claim holds. To do so, for any $k < l$ and $z_j \in J_l$ we begin by denoting by $E_k^{z_j}$ the set

$$E_k^{z_j} := B_{\varepsilon^{\frac{d}{d-2}}\lambda_j\rho_j}(\varepsilon z_j) \setminus \bigcup_{m=k}^{l-1} \bigcup_{z_i \in J_m} B_{\theta\lambda_i \varepsilon^{\frac{d}{d-2}}\rho_i}(\varepsilon z_i) \quad (6.3.50)$$

and arguing that

$$B_{\varepsilon^{\frac{d}{d-2}} \tilde{\lambda}_j \rho_j}(\varepsilon z_j) \subset E_k^{z_j} \subset E_k, \quad (6.3.51)$$

$$E_k = \bigcup_{l \geq k} \bigcup_{z_j \in J_l} E_k^{z_j}, \quad (6.3.52)$$

where each union above is between disjoint sets.

By (6.3.33) for E_{l-1} and (6.3.32) for J_l , we clearly have that

$$B_{\varepsilon^{\frac{d}{d-2}} \lambda_j \rho_j}(\varepsilon z_j) \subset E_{l-1}.$$

Note that, by construction, this ball is a connected component of the set E_{l-1} . From the previous inclusion, the second inclusion in (6.3.51) is an easy application of the recursive definition (6.3.33) of E_k . Similarly, (6.3.52) is an easy consequence of the definition (6.3.33) of the sets E_k . Furthermore, since each $J_m \subset \tilde{I}_m$, we apply claim (6.3.49) to z_j and all $z_k \in J_m$ with $m \leq l-1$, and conclude also the first inclusion in (6.3.51). We conclude that definition (6.3.50) immediately yields the monotonicity property $E_{k-1}^{z_j} \subset E_k^{z_j}$ for all $z_j \in J_l$ and $-3 \leq k \leq l$.

Equipped with (6.3.51)-(6.3.52), we now turn to (6.3.8): Let $z_0 \in \mathcal{I}_{k_0}$ for some $-2 \leq k_0 \leq k_{max}$. By definition (6.3.39), there exists $z_1 \in J_{k_0}$ such that

$$B_{\varepsilon^{\frac{d}{d-2}} \rho_0}(\varepsilon z_0) \subset B_{\lambda_1 \varepsilon^{\frac{d}{d-2}} \rho_1}(\varepsilon z_1). \quad (6.3.53)$$

By this, property (6.3.8) follows immediately if we prove that for any $l < k_0$ and all $z_3 \in J_l$ we have

$$B_{\varepsilon^{\frac{d}{d-2}} \rho_0}(\varepsilon z_0) \cap B_{\theta \lambda_3 \varepsilon^{\frac{d}{d-2}} \rho_3}(\varepsilon z_3) = \emptyset. \quad (6.3.54)$$

Let $-3 \leq k_2 \leq k_{max}$ be minimal such that there exists $z_2 \in \tilde{I}_{k_2}^\varepsilon$ with the property that

$$B_{\varepsilon^{\frac{d}{d-2}} \rho_0}(\varepsilon z_0) \subset B_{\tilde{\lambda}_2 \varepsilon^{\frac{d}{d-2}} \rho_2}(\varepsilon z_2). \quad (6.3.55)$$

Note that, by (6.3.13), we may always find such k_2 . If $k_0 \leq k_2$, we use the above claim (6.3.49) on $z_2 \in \tilde{I}_{k_2}$ and $z_3 \in J_l$ with $l < k_2$ and conclude (6.3.54). Let us now assume that $k_0 > k_2$: Since $z_0 \in \mathcal{I}_{k_0}$, by definition (6.3.39) we have that $z_2 \notin J_{k_2}$. This implies by (6.3.32) that

$$B_{\theta \tilde{\lambda}_2 \varepsilon^{\frac{d}{d-2}} \rho_2}(\varepsilon z_2) \subset E_{k_2+1}.$$

In particular, by (6.3.55) and (6.3.50) there exists a $\tilde{k}_0 > k_2$ and $\tilde{z}_1 \in J_{\tilde{k}_0}$ such that

$$B_{\varepsilon^{\frac{d}{d-2}} \rho_0}(\varepsilon z_0) \subset B_{\theta \tilde{\lambda}_2 \varepsilon^{\frac{d}{d-2}} \rho_2}(\varepsilon z_2) \subset E_{k_2+1}^{\tilde{z}_1}. \quad (6.3.56)$$

Moreover, by (6.3.50) and the assumption $k_2 < k_0$, we also have

$$B_{\varepsilon^{\frac{d}{d-2}} \rho_0}(\varepsilon z_0) \subset E_{k_2+1}^{\tilde{z}_1} \subset E_{k_0}^{\tilde{z}_1}.$$

On the other hand, by (6.3.53) also

$$B_{\varepsilon^{\frac{d}{d-2}} \rho_0}(\varepsilon z_0) \subset B_{\lambda_1 \varepsilon^{\frac{d}{d-2}} \rho_1}(\varepsilon z_1) = E_{k_0}^{z_1}.$$

By combining the previous two inequalities and using that the sets $E_k^{z_i}, E_k^{z_j}$ are whenever $z_i \neq z_j \in J$, we conclude that $\tilde{z}_1 = z_1$. Thus, since $z_1 \in J_{k_0}$, definition (6.3.50) applied to $E_{k_2+1}^{z_1}$ yields that for all $k_2 < l < k_0$ we have for all $z_i \in J_l$

$$E_{k_2+1}^{z_1} \cap B_{\theta \varepsilon^{\frac{d}{d-2}} \lambda_i \rho_i}(\varepsilon z_i) = \emptyset.$$

By using (6.3.56), the above inequality implies (6.3.54) with $z_i = z_3$ and for all $k_2 < l < k_0$. To extend (6.3.54) also to the indices $l \leq k_2$ it suffices to observe that for $l < k_2$ we may argue as above in the case $k_0 \leq k_2$. Finally, if $l = k_2$, we obtain (6.3.54) by applying (6.3.55) and (6.3.14) to $z_2 \in \tilde{I}_{k_2}$ and $z_3 \in J_{k_2} \subset \tilde{I}_{k_2}$.

It remains to prove claim (6.3.49). Let $z_l \in \tilde{I}_l^\varepsilon$, $-2 \leq l \leq k_{max}$. We begin by arguing that

$$B_{\theta \tilde{\lambda}_l \varepsilon^{\frac{d}{d-2}} \rho_l}(\varepsilon z_l) \subset E_l. \quad (6.3.57)$$

Indeed, if $z_l \in J_l$, this follows immediately from the definition of E_l . If $z_l \notin J_l$, then by (6.3.32) we have $B_{\lambda_l \varepsilon^{\frac{d}{d-2}} \rho_l}(\varepsilon z_l) \subset E_{l+1}$. We now use (6.3.14) on the family J_l and definition (6.3.33) of E_l to conclude (6.3.57). From (6.3.57) we may use again (6.3.14) to the families J_l, J_{l-1} and also obtain that

$$B_{\theta \tilde{\lambda}_l \varepsilon^{\frac{d}{d-2}} \rho_l}(\varepsilon z_l) \subset E_{l-1}. \quad (6.3.58)$$

We are now ready to argue (6.3.49) by contradiction: Let us assume that there exists a $k < l$ and $z_k \in J_k$ such that (6.3.49) fails, i.e.

$$B_{\tilde{\lambda}_l \varepsilon^{\frac{d}{d-2}} \rho_l}(\varepsilon z_l) \cap B_{\theta \lambda_k \varepsilon^{\frac{d}{d-2}} \rho_k}(\varepsilon z_k) \neq \emptyset. \quad (6.3.59)$$

Then, again by (6.3.14) applied to J_l and J_{l-1} , we necessarily have $k \leq l-2$. Let us now assume that $z_k \in J_{l-2}$: Then by (6.3.32) we have

$$B_{\varepsilon^{\frac{d}{d-2}} \theta \tilde{\lambda}_k \rho_k}(\varepsilon z_k) \subsetneq E_{l-1}. \quad (6.3.60)$$

This, together with (6.3.58) for z_l and (6.3.59) yields

$$B_{\theta \tilde{\lambda}_k \varepsilon^{\frac{d}{d-2}} \rho_k}(\varepsilon z_k) \cap \partial B_{\theta \tilde{\lambda}_l \varepsilon^{\frac{d}{d-2}} \rho_l}(\varepsilon z_l) \neq \emptyset. \quad (6.3.61)$$

For a general $k < l-2$, we claim that we may iterate the previous argument and obtain that (6.3.59) implies the existence of an integer $m \leq 1 + \lceil \frac{k_{max}}{2} \rceil$ and a collection $k_0, \dots, k_m \leq l-2$, such that $k = k_0$ and for all $0 \leq n \leq m-1$ we have $k_n \leq k_{n+1} - 2$ and there exist $z_{k_n} \in J_{k_n}$ and $z_m \in J_{k_m}$ satisfying (see Figure 6.4)

$$\begin{aligned} B_{\theta \tilde{\lambda}_{k_m} \varepsilon^{\frac{d}{d-2}} \rho_{k_m}}(\varepsilon z_{k_m}) \cap \partial B_{\theta \tilde{\lambda}_l \varepsilon^{\frac{d}{d-2}} \rho_l}(\varepsilon z_l) &\neq \emptyset, \\ B_{\theta \tilde{\lambda}_{k_n} \varepsilon^{\frac{d}{d-2}} \rho_{k_n}}(\varepsilon z_{k_n}) \cap B_{\theta \lambda_{k_{n+1}} \varepsilon^{\frac{d}{d-2}} \rho_{k_{n+1}}}(\varepsilon z_{k_{n+1}}) &\neq \emptyset. \end{aligned} \quad (6.3.62)$$

Indeed, for $z_k \in J_k$ with $k < l-2$, we know that by (6.3.32)

$$B_{\varepsilon^{\frac{d}{d-2}} \theta \tilde{\lambda}_k \rho_k}(\varepsilon z_k) \not\subset E_{k+1}. \quad (6.3.63)$$

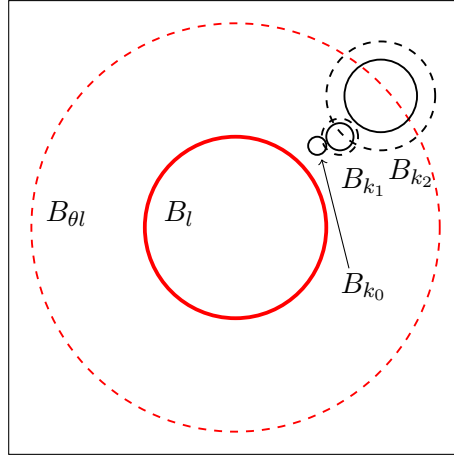


Figure 6.4: The thick ball B_l in the centre represents $B_{\theta \tilde{\lambda}_l \varepsilon^{\frac{d}{d-2}} \rho_l}(\varepsilon z_l)$, while the nested dashed ball $B_{\theta l}$ is its dilation by $\theta > 1$. The balls B_{k_0} , B_{k_1} and B_{k_2} correspond to $B_{\theta \tilde{\lambda}_{k_0} \varepsilon^{\frac{d}{d-2}} \rho_{k_0}}(\varepsilon z_{k_0})$, $B_{\theta \tilde{\lambda}_{k_1} \varepsilon^{\frac{d}{d-2}} \rho_{k_1}}(\varepsilon z_{k_1})$ and $B_{\theta \tilde{\lambda}_{k_2} \varepsilon^{\frac{d}{d-2}} \rho_{k_2}}(\varepsilon z_{k_2})$, respectively. The nested, dashed balls around B_{k_0} , B_{k_1} and B_{k_2} are the dilations by the factor θ^2 .

If also (6.3.61) is true, then we obtain (6.3.62) with $k_0 = k_m = k$. Let us assume, instead, that (6.3.61) does not hold and thus, by (6.3.59) that

$$B_{\varepsilon^{\frac{d}{d-2}} \theta \tilde{\lambda}_k \rho_k}(\varepsilon z_k) \subset B_{\varepsilon^{\frac{d}{d-2}} \theta \tilde{\lambda}_l \rho_l}(\varepsilon z_l) \stackrel{(6.3.58)}{\subset} E_{l-1}. \quad (6.3.64)$$

Then, by (6.3.63) and (6.3.33) there exists an index $k_1 \leq l-2$ and $z_{k_1} \in J_{k_1}$ such that

$$B_{\varepsilon^{\frac{d}{d-2}} \theta \tilde{\lambda}_k \rho_k}(\varepsilon z_k) \cap B_{\varepsilon^{\frac{d}{d-2}} \theta \tilde{\lambda}_{k_1} \rho_{k_1}}(\varepsilon z_{k_1}) \neq \emptyset. \quad (6.3.65)$$

Moreover, by (6.3.14), we necessarily have $k_1 \geq k+2$. We thus recovered the second line in (6.3.62). Since $z_{k_1} \in J_{k_1}$, we use again (6.3.32) to infer that

$$B_{\varepsilon^{\frac{d}{d-2}} \theta \tilde{\lambda}_{k_1} \rho_{k_1}}(\varepsilon z_{k_1}) \not\subset E_{k_1+1}.$$

Therefore, if $k_1 = l-2$, we argue as in (6.3.60) and conclude that (6.3.61) is satisfied with z_k substituted by z_{k_1} . This and (6.3.65) yield (6.3.62) with $m = 1$. Clearly, the same holds if $k_1 < l-2$ but (6.3.61) nonetheless satisfied by z_{k_1} . Let us now assume, instead, that z_{k_1} does not satisfy the first line in (6.3.62): By (6.3.65) and (6.3.64) this implies that

$$B_{\varepsilon^{\frac{d}{d-2}} \theta \tilde{\lambda}_{k_1} \rho_{k_1}}(\varepsilon z_{k_1}) \subset B_{\varepsilon^{\frac{d}{d-2}} \theta \tilde{\lambda}_l \rho_l}(\varepsilon z_l) \subset E_{l-1}.$$

We may now argue as for (6.3.63) above and obtain the existence of a new index $k_2 \geq k_1 + 2$ satisfying (6.3.65) with k and k_1 substituted by k_1 and k_2 respectively. By repeating the same argument above we iterate and conclude (6.3.62) for a general m . We remark that, since at each step n the index k_n increases of at least 2 this procedure necessarily stops whenever $k_n = l-2$. In other words, we obtain (6.3.62) after at most $1 + \lceil \frac{k_{max}}{2} \rceil$ iterations. We thus established (6.3.62).

Equipped with (6.3.62) we finally argue (6.3.49): Since for all $0 \leq n \leq m \leq 1 + \lceil \frac{k_{max}}{2} \rceil$, $1 \leq \lambda_{k_n} \leq \Lambda$ and $k_0 \leq \dots \leq k_m \leq l - 2$, we estimate

$$\begin{aligned} \varepsilon |z_l - z_k| &\geq \varepsilon |z_l - z_{k_m}| - \sum_{n=1}^m \varepsilon |z_{k_n} - z_{k_{n-1}}| \\ &\stackrel{(6.3.62)}{\geq} \theta \tilde{\lambda}_l \varepsilon^{\frac{d}{d-2}} \rho_l - (1 + 2m) \Lambda \varepsilon^{\frac{d}{d-2}} \rho_{k_m} \\ &\stackrel{\theta > 1}{\geq} \tilde{\lambda}_l \varepsilon^{\frac{d}{d-2}} \rho_l + (\theta - 1) \varepsilon^{\frac{d}{d-2}} \rho_l - (k_{max} + 4) \Lambda \varepsilon^{\frac{d}{d-2}} \rho_{k_m}. \end{aligned}$$

We now use the fact that since $z_l \in \tilde{I}_l$ and $z_{k_m} \in J_{k_m} \subset \tilde{I}_{k_m}$, we have by (6.3.11) and the assumptions on the indices k_n that $\frac{\rho_l}{\rho_{k_m}} \geq \varepsilon^{-\delta}$. From this inequality it follows that

$$\varepsilon |z_l - z_k| \geq \tilde{\lambda}_l \varepsilon^{\frac{d}{d-2}} \rho_l + ((\theta - 1) \varepsilon^{-\delta} - (k_{max} + 4) \Lambda) \varepsilon^{\frac{d}{d-2}} \rho_{k_m}$$

and for ε small enough we bound

$$\varepsilon |z_l - z_k| \geq \tilde{\lambda}_l \varepsilon^{\frac{d}{d-2}} \rho_l + 2\lambda_k \varepsilon^{\frac{d}{d-2}} \rho_{k_m},$$

where λ_k is the factor associated to z_k . We now observe that if $k_m = k_0 = k$, then the above inequality contradicts (6.3.59). If, otherwise $k = k_0 \neq k_m$, then by construction we have $k_0 \leq k_m - 2$ and thus by (6.3.11) that $\rho_k \leq \rho_{k_m}$. This and the above inequality contradict (6.3.59) also in this case. This proves claim (6.3.49) and establishes (6.3.8). The proof of Lemma 6.3.2 and Lemma 6.3.1 are complete. \square

6.4 Proof of Lemma 6.2.4

Proof of Lemma 6.2.4. For a $\theta > 1$ fixed, let $H^\varepsilon = H_b^\varepsilon \cup H_g^\varepsilon$ and the sets \bar{H}_b^ε , D_b^ε be as introduced in Lemma 6.3.1 and Lemma 6.3.2. Throughout this proof, we use the notation \lesssim for $\leq C$ with the constant depending on d, β, θ .

Step 1. We recall that the set D_b^ε is related to the partitioning of $H^\varepsilon = H_b^\varepsilon \cup H_g^\varepsilon$ and is such that $H_b^\varepsilon \subset \bar{H}_b^\varepsilon \subset D_b^\varepsilon$. We construct $R_\varepsilon v$ by distinguishing between the parts of domain D containing “small” holes (i.e. H_g^ε) and the ones containing the clusters of holes (i.e. H_b^ε). We set

$$R_\varepsilon v := \begin{cases} v_b^\varepsilon & \text{in } D_b^\varepsilon \\ v_g^\varepsilon & \text{in } D \setminus D_b^\varepsilon, \end{cases} \quad (6.4.1)$$

where the functions v_b^ε and v_g^ε satisfy

$$\begin{cases} v_b^\varepsilon = 0 & \text{in } H_b^\varepsilon, \quad v_b^\varepsilon = v & \text{in } D \setminus D_b^\varepsilon, \\ \nabla \cdot v_b^\varepsilon = 0 & \text{in } D, \\ v_b^\varepsilon \in H_0^1(D) & \text{for } \varepsilon \text{ small enough and } v_b^\varepsilon \rightarrow v & \text{in } H_0^1(D), \\ \|v_b^\varepsilon\|_{C^0} \lesssim \|v\|_{C^0(\bar{D})}. \end{cases} \quad (6.4.2)$$

and

$$\begin{cases} v_g^\varepsilon = v & \text{in } D_b^\varepsilon, \quad v_g^\varepsilon = 0 & \text{in } H_g^\varepsilon, \\ v_g^\varepsilon & \text{satisfies properties (i) - (v) with } H^\varepsilon \text{ substituted by } H_g^\varepsilon. \end{cases} \quad (6.4.3)$$

In particular, this means

$$R_\varepsilon v = v_b^\varepsilon + v_g^\varepsilon - v. \quad (6.4.4)$$

Before constructing the functions v_g^ε and v_b^ε , we argue that $R_\varepsilon v$ defined in (6.4.1) satisfies all the properties (i) - (v) enumerated in the lemma. Properties (i) and (ii) are immediately satisfied. We turn to properties (iii) and (iv). By (6.4.4), we rewrite

$$\|R_\varepsilon v - v\|_{L^p(\mathbb{R}^d)} = \|v_g^\varepsilon - v\|_{L^p(\mathbb{R}^d)} + \|v_b^\varepsilon - v\|_{L^p(D_b^\varepsilon)}.$$

The first term on the right-hand side vanishes almost surely in the limit thanks to the second line of (6.4.3) (property (iv) for v_g^ε). We bound the second term by using Hölder's inequality and the last estimate in (6.4.2):

$$\|v_b^\varepsilon - v\|_{L^p(D_b^\varepsilon)}^p \leq \|v - v_b^\varepsilon\|_{C^0(D)} |D_b^\varepsilon| \lesssim \|v\|_{C^0(D)} |D_b^\varepsilon|.$$

Thanks to (6.3.9), also this last line almost surely vanishes in the limit $\varepsilon \downarrow 0^+$. Thus, almost surely the whole norm $\|R_\varepsilon v - v\|_{L^p(\mathbb{R}^d)} \rightarrow 0$ when $\varepsilon \downarrow 0^+$. This yields property (iv) for $R_\varepsilon v$. To establish Property (iii) we use a similar argument to bound the L^2 -norm of $\nabla(R_\varepsilon v - v)$, this time using that by (6.4.2) the gradient $\nabla(v_b^\varepsilon - v)$ converges strongly to zero in $L^2(\mathbb{R}^d)$. Properties (i) - (iv) for $R_\varepsilon v$ are hence established.

It remains to argue property (v): Let $u_\varepsilon \in H_0^1(D_\varepsilon)$ be such that $u_\varepsilon \rightharpoonup u$ in $H^1(D)$ and $\nabla \cdot u_\varepsilon = 0$ in D . By (6.4.4), we have

$$\int \nabla R_\varepsilon v \cdot \nabla u_\varepsilon = \int \nabla v_g^\varepsilon \cdot \nabla u_\varepsilon + \int \nabla(v_b^\varepsilon - v) \cdot \nabla u_\varepsilon.$$

By (6.4.2) and the assumptions on u_ε , the second integral on the right-hand side almost surely converges to zero in the limit $\varepsilon \downarrow 0^+$. We treat the first integral term by observing that $H_0^1(D^\varepsilon) \subset H_0^1(D \setminus H_g^\varepsilon)$ and applying (6.4.3) (i.e. property (v) for v_g^ε). This implies property (v) for $R_\varepsilon v$ and concludes the proof of the lemma provided we construct v_g^ε and v_b^ε as above.

Step 2. Construction of v_b^ε satisfying (6.4.2).

To construct v_b^ε on D_b^ε , we exploit the construction of the covering \bar{H}_b^ε of Lemma 6.3.2, as sketched in Section 6.2.3. The main advantage in working with \bar{H}_b^ε instead of H_b^ε is related to the geometric properties satisfied by \bar{H}_b^ε which allow to define v_b^ε via a finite number of iterated Stokes problems on rescaled annuli.

Throughout this step, we skip the upper index ε and write v_b instead of v_b^ε . Let $J = \bigcup_{i=-3}^{k_{max}} J_i$ be the sub-collection of the centres of the balls generating \bar{H}_b^ε in the proof of Lemma 6.3.2. For each $z_j \in J$, we write

$$\begin{aligned} R_j^\varepsilon &:= \lambda_j^\varepsilon \rho_j, \quad B_j := B_{\varepsilon^{\frac{d}{d-2}} R_j}(\varepsilon z_j), \\ B_{\theta,j} &:= B_{\varepsilon^{\frac{d}{d-2}} \theta R_j}(\varepsilon z_j), \quad A_j := B_{\theta,j} \setminus B_j, \end{aligned} \quad (6.4.5)$$

with $\lambda_j^\varepsilon \in [1, \Lambda]$ the factors defined in Lemma 6.3.2.

As a first step, we consider the set $J_{k_{max}}$ and define the function v^0 on D as

$$\begin{cases} v^0 = v & \text{in } D \setminus \bigcup_{z_j \in J_{k_{max}}} B_{\theta,j} \\ v^0 = 0 & \text{in } B_j \text{ for all } z_j \in J_{k_{max}} \\ v^0 = v_j^0 & \text{in } A_j \text{ for all } z_j \in J_{k_{max}}, \end{cases} \quad (6.4.6)$$

where each v_j^0 solves

$$\begin{cases} -\Delta v_j^0 + \nabla p_j^0 = -\Delta v & \text{in } A_j \\ \nabla \cdot v_j^0 = 0 & \text{in } A_j \\ v_j^0 = 0 & \text{on } \partial B_j \\ v_j^0 = v & \text{on } \partial B_{\theta,j}. \end{cases} \quad (6.4.7)$$

This is well-defined since $\operatorname{div} v = 0$. In particular, each function $v_j^0 - v$ solves the first problem in (6.8.1) in A_i , and we apply to it the estimates (6.8.2) with the choice $R = \theta$ and after a rescaling by $\varepsilon^{\frac{d}{d-2}} R_j$ and a translation of εz_j . This yields

$$\begin{aligned} \|\nabla v_j^0\|_{L^2(A_j)}^2 &\lesssim \left(\|\nabla v\|_{L^2(B_{\theta,j})}^2 + \frac{1}{(\varepsilon^{\frac{d}{d-2}} R_j)^2} \|v\|_{L^2(B_{\theta,j})}^2 \right), \\ \|v_j^0\|_{C^0(\overline{B_{\theta,j}})} &\lesssim \|v\|_{C^0(\overline{B_{\theta,j}})}. \end{aligned}$$

We now use the definition (6.4.5) of R_j to obtain

$$\begin{aligned} \|\nabla v_j^0\|_{L^2(A_j)}^2 &\lesssim (\|\nabla v\|_{L^2(B_{\theta,j})}^2 + \varepsilon^d \lambda_j \rho_j^{d-2} \|v\|_{L^\infty}^2), \\ \|v_j^0\|_{C^0(\overline{B_{\theta,j}})} &\lesssim \|v\|_{C^0(\overline{B_{\theta,j}})}. \end{aligned} \quad (6.4.8)$$

Note that thanks to (6.3.7) of Lemma 6.3.2, we have that $B_{\theta,i} \cap B_{\theta,j} = \emptyset$ for all $z_i \neq z_j \in J_{k_{\max}}$ and $\lambda_i \leq \Lambda$ for all $z_i \in J$. Thus, this also implies by (6.4.6) that

$$\begin{aligned} \|\nabla v^0\|_{L^2(D)}^2 &\lesssim \|\nabla v\|_{L^2(D)}^2 + \varepsilon^d \sum_{z_j \in J_{k_{\max}}} \rho_j^{d-2} \|v\|_{L^\infty(D)}^2, \\ \|v^0\|_{C^0(D)} &\lesssim \|v\|_{C^0(D)}. \end{aligned} \quad (6.4.9)$$

Furthermore, since $v^0 - v$ is supported only in the balls $B_{\theta,j}$, the triangle inequality and (6.4.8) imply also that

$$\|\nabla(v^0 - v)\|_{L^2(D)}^2 \lesssim \sum_{z_j \in J_{k_{\max}}} \|\nabla v\|_{L^2(B_{\theta,j})}^2 + \varepsilon^d \sum_{z_j \in J_{k_{\max}}} \rho_j^{d-2} \|v\|_{L^\infty(D)}^2. \quad (6.4.10)$$

We observe also that, by using again the fact that by Lemma 6.3.2 all the balls B_j are disjoint, the function v^0 vanishes on

$$\bigcup_{z_j \in J_{k_{\max}}} B_j \stackrel{(6.3.6)}{\supseteq} \bigcup_{z_j \in \mathcal{I}_{k_{\max}}} B_{\varepsilon^{\frac{d}{d-2}} \rho_j}(\varepsilon z_j). \quad (6.4.11)$$

We now proceed iteratively and for $1 \leq i \leq k_{\max} + 3$ we consider the subsets $J_{k_{\max}-i} \subset J$. For each i in the range above, let v^i be defined as in (6.4.6) and (6.4.7), with v^{i-1} instead of v and the domains B_j and A_j generated by the elements $z_j \in J_{k_{\max}-i}$. We now argue that at each step i we have

$$\begin{aligned} \|\nabla v^i\|_{L^2(D)}^2 &\lesssim \|\nabla v\|_{L^2(D)}^2 + \varepsilon^d \sum_{z_j \in \bigcup_{k=0}^i J_{k_{\max}-k}} \rho_j^{d-2} \|v\|_{L^\infty(D)}^2, \\ \|v^i\|_{C^0(D)} &\lesssim \|v\|_{C^0(D)}, \end{aligned} \quad (6.4.12)$$

and

$$v^i = 0 \quad \text{in} \quad \bigcup_{z_j \in \bigcup_{k=0}^i \mathcal{I}_{k_{\max}-k}} B_{\varepsilon^{\frac{d}{d-2}} \rho_j}(\varepsilon z_j). \quad (6.4.13)$$

Moreover,

$$\begin{aligned} v^i - v &= 0 \quad \text{in} \quad D \setminus \left(\bigcup_{z_j \in \bigcup_{k=0}^i J_{k_{\max}-k}} B_{\theta,j} \right), \\ \|\nabla(v^i - v)\|_{L^2(D)}^2 &\lesssim \sum_{z_j \in \bigcup_{k=0}^i J_{k_{\max}-k}} \left(\|\nabla v\|_{L^2(B_{\theta,j})}^2 + \varepsilon^d \rho_j^{d-2} \|v\|_{L^\infty(D)}^2 \right). \end{aligned} \quad (6.4.14)$$

We prove the previous estimates by induction over $0 \leq i \leq k_{\max} + 3$.

It is easy to prove the estimates in (6.4.12) by induction: For $i = 0$, (6.4.9) is exactly (6.4.12). We now observe that at each step i we may argue as for v^0 and obtain (6.4.9) with v^0 , v and $J_{k_{\max}}$ substituted by v^i , v^{i-1} and $J_{k_{\max}-i}$, respectively. Therefore, if we now assume (6.4.12) holds at step $i - 1$, we only need to combine the analogue of (6.4.9) for v^i with (6.4.12) for v^{i-1} .

We now consider (6.4.13): For $i = 0$, this is implied immediately by (6.4.11). Let us now assume that (6.4.13) holds for $i - 1$. By definition of v^i (cf. (6.4.7)), the function vanishes on

$$\bigcup_{z_j \in J_{k_{\max}-i}} B_j \stackrel{(6.3.6)}{\supseteq} \bigcup_{z_j \in \mathcal{I}_{k_{\max}-i}} B_{\varepsilon^{\frac{d}{d-2}} \rho_j}(\varepsilon z_j)$$

and equals v^{i-1} on $D \setminus \bigcup_{z_j \in J_{k_{\max}-i}} B_{\theta,j}$. By the induction hypothesis (6.4.13) for $i - 1$, (6.4.13) for i follows provided

$$\left(\bigcup_{z_j \in J_{k_{\max}-i}} B_{\theta,j} \right) \cap \left(\bigcup_{z_j \in \bigcup_{k=0}^{i-1} \mathcal{I}_{k_{\max}-k}} B_{\varepsilon^{\frac{d}{d-2}} \rho_j}(\varepsilon z_j) \right) = \emptyset.$$

By recalling the definitions of the balls $B_{\theta,j}$, this identity is a consequence of property (6.3.8) of Lemma 6.3.2. We established (6.4.13) and (6.4.12) for each $0 \leq i \leq k_{\max} + 3$.

Finally, we turn to the claims in (6.4.14): For $i = 0$, both lines of (6.4.14) hold by construction and (6.4.10), respectively. If we now assume that (6.4.14) is true for $i - 1$, then v^i is by construction equal to v^{i-1} outside the set

$$\bigcup_{z_j \in J_{k_{\max}-i}} B_{\theta,j}.$$

It now suffices to apply the induction hypothesis for v^{i-1} to conclude the first statement in (6.4.14). In addition, by the triangle inequality we estimate

$$\|\nabla(v^i - v)\|_{L^2(D)}^2 \leq \|\nabla(v^i - v^{i-1})\|_{L^2(D)}^2 + \|\nabla(v^{i-1} - v)\|_{L^2(D)}^2.$$

We apply the induction hypothesis to the second term on the right-hand side above and get

$$\|\nabla(v^i - v)\|_{L^2(D)}^2 \leq \|\nabla(v^i - v^{i-1})\|_{L^2(D)}^2 + \sum_{z_j \in \bigcup_{k=0}^{i-1} J_{k_{\max}-k}} \left(\|\nabla v\|_{L^2(B_{\theta,j})}^2 + \varepsilon^d \rho_j^{d-2} \|v\|_{L^\infty(D)}^2 \right) \quad (6.4.15)$$

We now use the analogue of (6.4.8) with v^0 and v substituted by v^{i-1} and v^i to infer that

$$\|\nabla(v^i - v^{i-1})\|_{L^2(D)}^2 \lesssim \sum_{z_j \in J_{k_{\max}-i}} \left(\|\nabla v^{i-1}\|_{L^2(B_{\theta,j})}^2 + \varepsilon^d \lambda_j \rho_j^{d-2} \|v^{i-1}\|_{L^\infty(D)}^2 \right),$$

and, by (6.4.12) for v^{i-1} , that

$$\begin{aligned} \|\nabla(v^i - v^{i-1})\|_{L^2(D)}^2 &\lesssim \sum_{z_j \in J_{k_{max}-i}} \left(\|\nabla v^{i-1}\|_{L^2(B_{\theta,j})}^2 + \varepsilon^d \lambda_j \rho_j^{d-2} \|v\|_{L^\infty(D)}^2 \right) \\ &\lesssim \sum_{z_j \in J_{k_{max}-i}} \|\nabla(v^{i-1} - v)\|_{L^2(B_{\theta,j})}^2 \\ &\quad + \sum_{z_j \in J_{k_{max}-i}} \left(\|\nabla v\|_{L^2(B_{\theta,j})}^2 + \varepsilon^d \lambda_j \rho_j^{d-2} \|v\|_{L^\infty(D)}^2 \right). \end{aligned}$$

Since all $B_{\theta,j}$, $z_j \in J_{k_{max}-i}$, are disjoint, this implies that

$$\|\nabla(v^i - v^{i-1})\|_{L^2(D)}^2 \lesssim \|\nabla(v^{i-1} - v)\|_{L^2(D)}^2 + \sum_{z_j \in J_{k_{max}-i}} \left(\|\nabla v\|_{L^2(B_{\theta,j})}^2 + \varepsilon^d \lambda_j \rho_j^{d-2} \|v\|_{L^\infty(D)}^2 \right).$$

We may apply the induction hypothesis on v^{i-1} again and combine the above estimate with (6.4.15) to conclude (6.4.14) for v^i . The proof of (6.4.14) is complete.

Equipped with (6.4.12), (6.4.13) and (6.4.14), we finally set $v_b^\varepsilon := v^{k_{max}+3}$ and show that this choice fulfils all the conditions in (6.4.2): The first and the second line in (6.4.2) follow immediately by construction and the definition (6.3.9) of D_b^ε . The second estimate in (6.4.12) with $i = k_{max} + 3$ yields also the last inequality in (6.4.2). It thus only remain to show that, almost surely, $v_b^\varepsilon \in H_0^1(D)$ for ε small enough and $v_b^\varepsilon \rightarrow v$ in $H_0^1(D)$.

To do this, we begin by showing that $\nabla(v_b^\varepsilon - v) \rightarrow 0$ in $L^2(D)$: By (6.4.14) with $i = k_{max} + 3$ and the fact that $v \in C_0^\infty(D)$, we indeed obtain

$$\|\nabla(v_b^\varepsilon - v)\|_{L^2(D)} \lesssim \|v\|_{C^1(D)} \sum_{z_j \in J} ((\varepsilon^{\frac{d}{d-2}} \rho_j)^2 + 1) \varepsilon^d \rho_j^{d-2}.$$

We recall that the set J depends on ε , i.e. $J = J^\varepsilon$. In addition, since $J \subset \mathcal{I}$ (cf. Lemma 6.3.2) and $n^\varepsilon = \Phi^\varepsilon(D) \setminus I^\varepsilon$, the limit in (6.3.2) of Lemma 6.3.1 yields that almost surely $\varepsilon^d \# J^\varepsilon \rightarrow 0$ when $\varepsilon \downarrow 0^+$. This, together with (6.3.5), (6.1.7) and the Strong Law of Large numbers, Lemma (5.5.3) in the previous chapter, implies that the right-hand side above almost surely vanishes in the limit $\varepsilon \downarrow 0^+$. Hence, we showed that $\nabla(v_b^\varepsilon - v) \rightarrow 0$ in $L^2(\mathbb{R}^d)$. By Poincaré's inequality, it now suffices to argue that almost surely and for ε small enough $v_b^\varepsilon \in H_0^1(D)$ to infer that $v_b^\varepsilon \rightarrow v$ in $H_0^1(D)$ and thus conclude the proof of (6.4.2).

Let $K \Subset D$ be a compact set containing the support of v , and set $r = \text{dist}(K, D) > 0$. We show that, almost surely, $v_b^\varepsilon \in H_0^1(D)$ for all $\varepsilon \leq \bar{\varepsilon}$, with $\bar{\varepsilon} = \bar{\varepsilon}(r, \omega) > 0$. To do so, we fix any realization $\omega \in \Omega$ (which is independent from v) for which we have (6.3.12), and resort to the construction of v_b^ε via the solutions $v^0, v^1 \dots v^{k_{max}+3}$ obtained by iterating (6.4.7). We claim that for all $i = 0, \dots, k_{max} + 3$ we have

$$\text{supp}(v^i) =: K_i^\varepsilon \subset D, \quad \text{dist}(K_i^\varepsilon, D) \geq r - 2(i+1)\theta\Lambda\varepsilon^{2\delta d}, \quad (6.4.16)$$

for all ε such that the right-hand side in the last inequality is positive. Since $v_b^\varepsilon := v^{k_{max}+3}$, we may choose $\bar{\varepsilon}(r, \omega)$ such that $\varepsilon^{2\delta d} \leq \frac{r}{4(k_{max}+4)\theta\Lambda}$ and use the above estimate to infer that v_b^ε is compactly supported in D for all $\varepsilon \leq \bar{\varepsilon}(r, \omega)$.

We prove (6.4.16) iteratively and begin with $i = 0$: By (6.4.7) and the assumption on the support of v , it follows that, if for $z_i \in J_{k_{max}}$ the ball $B_{\theta,i}$ does not intersect the support K of v , then

$v^0 = v \equiv 0$ on $B_{\theta,i}$. This, together with property (6.3.7) of Lemma 6.3.2, implies that

$$\text{supp}(v^0) \subset K \bigcup_{\substack{z_i \in J_{kmax}, \\ B_{\theta,i} \cap K \neq \emptyset}} B_{\theta,i}. \quad (6.4.17)$$

By recalling that thanks to Lemma 6.3.2 each ball $B_{\theta,j}$ has radius

$$\theta \lambda_i \varepsilon^{\frac{d}{d-2}} \rho_i \leq \theta \Lambda \varepsilon^{\frac{d}{d-2}} \rho_i \stackrel{(6.3.12)}{\leq} \theta \Lambda \varepsilon^{2d\delta},$$

we observe that (6.4.17) yields estimate (6.4.16) for v^0 . Let us now assume (6.4.16) for v^i . Then, since v^{i+1} solves (6.4.7) with boundary datum v_i , we may argue as above to infer that

$$K_{i+1}^\varepsilon \subset K_i^\varepsilon \bigcup_{\substack{z_i \in J_{kmax}, \\ B_{\theta,i} \cap K_i^\varepsilon \neq \emptyset}} B_{\theta,i}$$

and thus that

$$\text{dist}(K_{i+1}^\varepsilon, D) \geq \text{dist}(K_i^\varepsilon, D) - 2\theta \Lambda \varepsilon^{2d\delta} \stackrel{(6.4.16)}{\geq} r - 2(i+1)\theta \Lambda \varepsilon^{2d\delta}.$$

This concludes the iterated estimate (6.4.16), which completes the proof of this step.

Step 3. Construction of v_g^ε satisfying (6.4.3). We now turn to the remaining set $D \setminus D_b^\varepsilon$ and construct the vector field v_g^ε in a way similar to [All90a][Subsection 2.3.2] and [DGR08].

For every $z_i \in n^\varepsilon$, we write

$$a_{\varepsilon,i} := \varepsilon^{\frac{d}{d-2}} \rho_i, \quad d_i := \min \left\{ \text{dist}(\varepsilon z_i, D_b^\varepsilon), \frac{1}{2} \min_{\substack{z_j \in n^\varepsilon, \\ z_j \neq z_i}} (\varepsilon |z_i - z_j|), \varepsilon \right\} \quad (6.4.18)$$

and

$$T_i = B_{a_{\varepsilon,i}}(\varepsilon z_i), \quad B_i := B_{\frac{d_i}{2}}(\varepsilon z_i), \quad B_{2,i} := B_{d_i}(\varepsilon z_i), \quad C_i := B_i \setminus T_i, \quad D_i := B_{2,i} \setminus B_i.$$

We remark that, since $z_i \in n^\varepsilon$, Lemma 6.3.1 implies that for $\delta > 0$

$$a_{\varepsilon,i} \leq \varepsilon^{1+2\delta}, \quad d_i \geq \varepsilon^{1+\delta}, \quad (6.4.19)$$

and that all the balls $B_{2,i}$ are pairwise disjoint.

For each $z_i \in n^\varepsilon$, we define the function v_g^ε in $B_{2,i}$ in the following way:

$$\begin{cases} v_g^\varepsilon = 0 & \text{in } T_i \\ v_g^\varepsilon = v - \tilde{v}_i^\varepsilon & \text{in } C_i, \end{cases}$$

where \tilde{v}_i^ε solves

$$\begin{cases} -\Delta \tilde{v}_i^\varepsilon + \nabla \pi_i^\varepsilon = 0 & \text{in } \mathbb{R}^d \setminus T_i \\ \nabla \cdot \tilde{v}_i^\varepsilon = 0 & \text{in } \mathbb{R}^d \setminus B_1 \\ \tilde{v}_i^\varepsilon = v & \text{on } \partial T_i \\ \tilde{v}_i^\varepsilon \rightarrow 0 & \text{for } |x| \rightarrow +\infty. \end{cases} \quad (6.4.20)$$

Finally, we require that on D_i , v_g^ε solves

$$\begin{cases} -\Delta v_g^\varepsilon + \nabla q_g^\varepsilon = \Delta v & \text{in } D_i \\ \nabla \cdot v_g^\varepsilon = 0 & \text{in } D_i \\ v_g^\varepsilon = v & \text{on } \partial B_{2,i} \\ v_g^\varepsilon = v - \tilde{v}_i^\varepsilon & \text{on } \partial B_i, \end{cases} \quad (6.4.21)$$

and we then extend v_g^ε by v on $\mathbb{R}^d \setminus \bigcup_{z_i \in n^\varepsilon} B_{2,i}$. By Lemma 6.3.1 and the definition (6.4.18) of d_i , we have that $D_b^\varepsilon \subset \mathbb{R}^d \setminus \bigcup_{z_i \in n^\varepsilon} B_{2,i}$. Therefore, this definition of v_g^ε satisfies the first line of (6.4.3) and property (i) with H^ε substituted by H_g^ε . It is immediate that by construction $\nabla \cdot v_g^\varepsilon = 0$ in D , i.e. v_g^ε satisfies also property (ii).

We observe that by uniqueness of the solution to (6.4.20), we may rescale the domains C_i and rewrite

$$v_g^\varepsilon = v - \phi_\infty^{\varepsilon,i} \left(\frac{\cdot - \varepsilon z_i}{a_{\varepsilon,i}} \right) \quad \text{in } C_i, \quad (6.4.22)$$

with $\phi_\infty^{\varepsilon,i}$ solving the second system in (6.8.1) in the annulus $\mathbb{R}^d \setminus B_1$ and with boundary datum $\psi(x) = v(a_{i,\varepsilon}x - \varepsilon z_i)$. Similarly, by uniqueness of the solutions to (6.4.21) we may rescale the domains D_i and write

$$v_g^\varepsilon = v - \phi_2^{\varepsilon,i} \left(\frac{\cdot - \varepsilon z_i}{d_i} \right) \quad \text{in } D_i, \quad (6.4.23)$$

with $\phi_2^{\varepsilon,i}$ solving the first system in (6.8.1) in the annulus $B_2 \setminus B_1$ and with boundary datum $\psi(x) = \phi_\infty^{\varepsilon,i} \left(\frac{d_i(x - \varepsilon z_i)}{a_{\varepsilon,i}} \right)$.

We now turn to properties (iii) and (iv) for v_g^ε : We write

$$\begin{aligned} \|v_g^\varepsilon - v\|_{L^p(\mathbb{R}^d)}^p &= \sum_{z_i \in n^\varepsilon} \|v_g^\varepsilon - v\|_{L^p(B_{2,i})}^p, \\ \|\nabla(v_g^\varepsilon - v)\|_{L^2(\mathbb{R}^d)}^2 &= \sum_{z_i \in n^\varepsilon} \|\nabla(v_g^\varepsilon - v)\|_{L^2(B_{2,i})}^2, \end{aligned} \quad (6.4.24)$$

and, since $B_{2,i} = D_i \cup C_i \cup T_i$, we may further split each norm on the right hand side into the contributions on each set D_i , C_i and T_i . We begin by focussing on the domains D_i : By (6.4.23), we apply (6.8.2) to $\phi_2^{\varepsilon,i}$ and infer that

$$\begin{aligned} \|\nabla(v_g^\varepsilon - v)\|_{L^2(D_i)}^2 &\lesssim \|\nabla \tilde{v}_i^\varepsilon\|_{L^2(D_i)}^2 + d_i^{-2} \|\tilde{v}_i^\varepsilon\|_{L^2(D_i)}^2, \\ \|v_g^\varepsilon - v\|_{C^0(D_i)} &\lesssim \|\tilde{v}_i^\varepsilon\|_{C^0(\partial B_{2,i})}. \end{aligned} \quad (6.4.25)$$

By using (6.4.22) and changing variables, we rewrite the second line above as

$$\|v_g^\varepsilon - v\|_{C^0(B_{2,i})} \lesssim \|\phi_\infty^{\varepsilon,i}\|_{C^0(\partial B_{\frac{a_{i,\varepsilon}}{d_i}})},$$

and use (6.8.4) on $\phi_\infty^{\varepsilon,i}$ to infer

$$\|v_g^\varepsilon - v\|_{C^0(B_i)} \lesssim \|v\|_{C^0} \left(\frac{a_{i,\varepsilon}}{d_i} \right)^{d-2} \lesssim \|v\|_{C^0} \varepsilon^{\delta(d-2)}.$$

In particular,

$$\|v_g^\varepsilon - v\|_{L^p(D_i)}^p \lesssim a_{i,\varepsilon}^d \|v\|_{C^0} \varepsilon^{\delta(d-2)} \lesssim \|v\|_{C^0} \varepsilon^{d+\delta(d-2)}. \quad (6.4.26)$$

We now turn to the first inequality in (6.4.25), use (6.4.22) on the right-hand side, and change variables to estimate

$$\begin{aligned} \|\nabla(v_g^\varepsilon - v)\|_{L^2(D_i)}^2 &\lesssim a_{\varepsilon,i}^{d-2} \|\nabla\phi_{\infty}^{\varepsilon,i}\|_{L^2(B_{d_i a_{i,\varepsilon}^{-1}} \setminus B_{\frac{1}{2}d_i a_{i,\varepsilon}^{-1}})}^2 + a_{\varepsilon,i}^d d_i^{-2} \|\phi_{\infty}^{\varepsilon,i}\|_{L^2(B_{d_i a_{i,\varepsilon}^{-1}} \setminus B_{\frac{1}{2}d_i a_{i,\varepsilon}^{-1}})}^2 \\ &\stackrel{(6.8.5)}{\lesssim} \|v\|_{C^1}^2 a_{\varepsilon,i}^{d-2} \left(\frac{a_{\varepsilon,i}}{d_i}\right)^{d-2} \stackrel{(6.4.19)}{\lesssim} \|v\|_{C^1}^2 \varepsilon^{d+\delta(d-2)} \rho_i^{d-2}. \end{aligned} \quad (6.4.27)$$

We consider the sets C_i : We use the definition (6.4.22) for v_g^ε on C_i and a change of variables to rewrite

$$\|\nabla(v_g^\varepsilon - v)\|_{L^2(C_i)}^2 = a_{\varepsilon,i}^{d-2} \|\nabla\phi_{\infty}^{\varepsilon,i}\|_{L^2(B_{\frac{1}{2}d_i a_{\varepsilon,i}^{-1}} \setminus B_1)}^2.$$

Hence, using (6.8.3) for $\phi_{\infty}^{\varepsilon,i}$, we obtain

$$\begin{aligned} \|\nabla(v_g^\varepsilon - v)\|_{L^2(C_i)}^2 &\lesssim \|\nabla v\|_{L^2(B_{2a_{\varepsilon,i}}(\varepsilon z_i) \setminus T_i)}^2 + a_{\varepsilon,i}^{-2} \|v\|_{L^2(B_{2a_{\varepsilon,i}}(\varepsilon z_i) \setminus T_i)}^2 \\ &\lesssim a_{\varepsilon,i}^{d-2} \|v\|_{C^1}^2 = \varepsilon^d \rho_i^{d-2} \|v\|_{C^1}^2. \end{aligned} \quad (6.4.28)$$

Similarly, by (6.4.22) and a change of variables, for each $2 \leq p < +\infty$ we have

$$\|v_g^{\varepsilon,i} - v\|_{L^p(C_i)}^p = a_{\varepsilon,i}^d \|\phi_{\infty}^{\varepsilon,i}\|_{L^p(B_{d_i a_{\varepsilon,i}^{-1}} \setminus B_1)}^p,$$

and, thanks to the pointwise estimate (6.8.4) for $\phi_{\infty}^{\varepsilon,i}$, we have that for all $p > \frac{d}{d-2}$

$$\|v_g^\varepsilon - v\|_{L^p(C_i)}^p \lesssim \|v\|_{C^0}^p a_{\varepsilon,i}^d \stackrel{(6.4.19)}{\lesssim} \|v\|_{C^0}^p \varepsilon^{2+4\delta} \varepsilon^d \rho_i^{d-2}. \quad (6.4.29)$$

We finally turn to T_i , on which we easily bound

$$\begin{aligned} \|\nabla(v_g^\varepsilon - v)\|_{L^2(T_i)}^2 &= \|\nabla v\|_{L^2(T_i)}^2 \leq \|v\|_{C^1}^2 a_{\varepsilon,i}^d \stackrel{(6.4.19)}{\lesssim} \|v\|_{C^1}^2 \varepsilon^{2(1+\delta)} \varepsilon^d \rho_i^{d-2}, \\ \|v_g^\varepsilon - v\|_{L^p(T_i)}^p &= \|v\|_{L^p(T_i)}^p \stackrel{(6.4.19)}{\lesssim} \|v\|_{C^0}^p \varepsilon^{2(1+2\delta)} \rho_i^{d-2}. \end{aligned} \quad (6.4.30)$$

By collecting all the estimates in (6.4.26), (6.4.27), (6.4.28), (6.4.29) and (6.4.30) we get

$$\|\nabla v_g^\varepsilon - v\|_{L^2(B_{2,i})}^2 \lesssim \|v\|_{C^1}^2 \varepsilon^d \rho_i^{d-2}, \quad (6.4.31)$$

and for all $p > \frac{d}{d-2}$

$$\|v_g^\varepsilon - v\|_{L^p(B_{2,i})}^p \lesssim \|v\|_{C^\infty} \varepsilon^d (\varepsilon^2 \rho_i^{d-2} + \varepsilon^{\delta p(d-2)}).$$

We insert these estimates in (6.4.24) and apply (6.1.7) and the Strong Law of Large Numbers on the right-hand sides to conclude that almost surely

$$\|\nabla v_g^\varepsilon\|_{L^2(D)} \lesssim 1$$

and that $v_g^\varepsilon \rightarrow v$ in $L^p(D)$ for $p > \frac{d}{d-2}$. Since v, v_g^ε are supported in the bounded domain D for ε small enough, we conclude properties (iii) and (iv) for v_g^ε .

We finally turn to property (v). We use an argument very similar to the one for Lemma 5.3.1. For any $N \in \mathbb{N}$ fixed and all $z_i \in n^\varepsilon$, let us define

$$n_N^\varepsilon := \left\{ z_i \in n^\varepsilon : d_i \geq \frac{\varepsilon}{N} \right\},$$

where $Q \subset \mathbb{R}^d$ is a unit cube. Moreover, let $\mathcal{R}^N := \{\rho_i^N\}_{z_i \in n^\varepsilon}$ be the truncated environment given by $\rho_i^N := \rho_i \wedge N$ and let $H_g^{\varepsilon, N}$ be the set of holes generated by n_N^ε with \mathcal{R}^N . Let $v_g^{\varepsilon, N}$ be the analogues of v_g^ε for $H_g^{\varepsilon, N}$. We begin by showing that $v_g^{\varepsilon, N}$ satisfy property (v) on $H_g^{\varepsilon, N}$ with

$$\mu^N = C_d \langle (\rho^N)^{d-2} \rangle \langle \#(N_{\frac{2}{N}}(Q)) \rangle,$$

where Q is a unit ball and $N_{\frac{2}{N}}$ is defined in Section 6.2.1.

Before showing this, we argue how to conclude also property (v) for v_g^ε : Let $u_\varepsilon \in H_0^1(D_\varepsilon)$ such that $u_\varepsilon \rightharpoonup u$ in $H^1(D)$. For each $N \in \mathbb{N}$ fixed we bound

$$\begin{aligned} \limsup_{\varepsilon \downarrow 0^+} \left| \int \nabla v_g^\varepsilon \cdot \nabla u_\varepsilon - \left(\int \nabla v \cdot \nabla u + \int v \cdot \mu u \right) \right| \\ \leq \limsup_{\varepsilon \downarrow 0^+} \left| \int \nabla v_g^{\varepsilon, N} \cdot \nabla u_\varepsilon - \left(\int \nabla v \cdot \nabla u + \int v \cdot \mu u \right) \right| + \limsup_{\varepsilon \downarrow 0^+} \left| \int \nabla (v_g^\varepsilon - v_g^{\varepsilon, N}) \cdot \nabla u_\varepsilon \right|. \end{aligned}$$

Since $H_g^{\varepsilon, N} \subset H_g^\varepsilon$, property (v) for $v_g^{\varepsilon, N}$ yields

$$\begin{aligned} \limsup_{\varepsilon \downarrow 0^+} \left| \int \nabla v_g^\varepsilon \cdot \nabla u_\varepsilon - \left(\int \nabla v \cdot \nabla u + \int v \cdot \mu u \right) \right| \\ \leq \left| \int v \cdot (\mu - \mu^N) u \right| + \limsup_{\varepsilon \downarrow 0^+} \left| \int \nabla (v_g^\varepsilon - v_g^{\varepsilon, N}) \cdot \nabla u_\varepsilon \right|. \end{aligned} \tag{6.4.32}$$

We now appeal to the explicit construction of the functions $v_g^\varepsilon, v_g^{\varepsilon, N}$ to observe that

$$\begin{aligned} \text{supp}(v_g^\varepsilon - v_g^{\varepsilon, N}) &\subset \bigcup_{\substack{z_i \in n_N^\varepsilon, \\ \rho_i \geq N}} B_{2,i} \cup \bigcup_{z_i \in n^\varepsilon \setminus n_N^\varepsilon} B_{2,i}, \\ v_g^\varepsilon - v_g^{\varepsilon, N} &= v_g^\varepsilon \quad \text{in} \quad \bigcup_{z_i \in n^\varepsilon \setminus n_N^\varepsilon} B_{2,i}. \end{aligned}$$

Therefore,

$$\|\nabla(v_g^\varepsilon - v_g^{\varepsilon, N})\|_{L^2(D)}^2 \lesssim \sum_{\substack{z_i \in n_N^\varepsilon, \\ \rho_i \geq N}} \|\nabla(v_g^\varepsilon - v_g^{\varepsilon, N})\|_{L^2(B_{2,i})}^2 + \sum_{z_i \in n^\varepsilon \setminus n_N^\varepsilon} \|\nabla v_g^\varepsilon\|_{L^2(B_{2,i})}^2.$$

We smuggle in the norms on the right-hand side the function v and appeal to (6.4.31) for v_g^ε (and the analogue for $v_g^{\varepsilon, N}$) to get that

$$\|\nabla(v_g^\varepsilon - v_g^{\varepsilon, N})\|_{L^2(D)}^2 \lesssim \|v\|_{C^1(D)} \varepsilon^d \left(\sum_{z_i \in n^\varepsilon} \rho_i^{d-2} \mathbf{1}_{\rho_i \geq N} + \sum_{z_i \in n^\varepsilon \setminus n_N^\varepsilon} (1 + \rho_i^{d-2}) \right).$$

Assumption (6.1.7) and the Strong Law of the Large Numbers yield that almost surely

$$\sum_{z_i \in n^\varepsilon} \rho_i^{d-2} \mathbf{1}_{\rho_i \geq N} \rightarrow \langle \rho \mathbf{1}_{\rho \geq N} \rangle.$$

Moreover, by (6.3.2) and (6.3.3) of Lemma 6.3.1, and (5.5.10) of Lemma 5.5.2, we have that almost surely

$$\lim_{N \uparrow +\infty} \lim_{\varepsilon \downarrow 0^+} \varepsilon^d \#(n^\varepsilon \setminus n_N^\varepsilon) = 0. \quad (6.4.33)$$

This yields by Lemma 5.5.3 that

$$\lim_{N \uparrow +\infty} \lim_{\varepsilon \downarrow 0^+} \|\nabla(v_g^\varepsilon - v_g^{\varepsilon, N})\|_{L^2(D)} = 0.$$

Since ∇u_ε is uniformly bounded in $L^2(D)$, we can insert this in (6.4.32) to conclude

$$\limsup_{\varepsilon \downarrow 0^+} \left| \int \nabla v_g^\varepsilon \cdot \nabla u_\varepsilon - \left(\int \nabla v \cdot \nabla u + \int v \cdot \mu u \right) \right| \lesssim \limsup_{N \uparrow +\infty} \left| \int v \cdot (\mu - \mu^N) u \right|.$$

By using again assumption (6.1.7) and (6.4.33) we infer that the right-hand side above vanishes almost surely and conclude property (v) for v_g^ε with μ as in Theorem 6.2.1.

We now turn to property (v) for $v_g^{\varepsilon, N}$. When no ambiguity occurs, we drop the upper index N . For every u_ε as above, we split the integral

$$\int \nabla v_g^\varepsilon \cdot \nabla u_\varepsilon = \int \nabla v \cdot \nabla u_\varepsilon - \int \nabla(v_g^\varepsilon - v) \cdot \nabla u_\varepsilon.$$

The first term converges to $\int \nabla v \cdot \nabla u$ by the assumption on the sequence u_ε . To conclude property (v) it thus remains to argue that

$$\int \nabla(v_g^\varepsilon - v) \cdot \nabla u_\varepsilon \rightarrow \int v \cdot \mu^N u. \quad (6.4.34)$$

To prove this, we recall the construction of v_g^ε , and we split the integral into

$$\int \nabla(v_g^\varepsilon - v) \cdot \nabla u_\varepsilon = \sum_{z_i \in n^\varepsilon} \int_{C_i} \nabla(v_g^\varepsilon - v) \cdot \nabla u_\varepsilon + \sum_{z_i \in n^\varepsilon} \int_{D_i} \nabla(v_g^\varepsilon - v) \cdot \nabla u_\varepsilon.$$

Note that the integral on each T_i vanishes by the assumption $u_\varepsilon \in H_0^1(D^\varepsilon)$. We first focus on the second sum on the right-hand side above and use Cauchy-Schwarz and (6.4.27) to bound

$$\sum_{z_i \in n^\varepsilon} \int_{D_i} \nabla(v_g^\varepsilon - v) \cdot \nabla u_\varepsilon \lesssim \|\nabla u_\varepsilon\|_{L^2(D)} \left(\varepsilon^{d+\delta(d-2)} \sum_{z_i \in n^\varepsilon} \rho_i^{d-2} \right)^{\frac{1}{2}} \|v\|_{C^\infty}.$$

By the assumption on the weak convergence for the sequence ∇u_ε and the Strong Law of Large Numbers, the right-hand side almost surely vanishes in the limit $\varepsilon \downarrow 0^+$. Thus,

$$\int \nabla(v_g^\varepsilon - v) \cdot \nabla u_\varepsilon = \sum_{z_i \in n^\varepsilon} \int_{C_i} \nabla(v_g^\varepsilon - v) \cdot \nabla u_\varepsilon + o(1). \quad (6.4.35)$$

We turn to the remaining term above: For each $z_i \in n^\varepsilon$, let $(\bar{\phi}_\infty^{\varepsilon,i}, \bar{\pi}_\infty^{\varepsilon,i})$ solve the Stokes problem (6.8.1) in the exterior domain $\mathbb{R}^d \setminus B_1$ and with constant boundary datum $v(\varepsilon z_i)$. We define

$$\bar{\phi}_\infty = \bar{\phi}_\infty\left(\frac{\cdot - \varepsilon z_i}{a_{\varepsilon,i}}\right), \quad \bar{\pi}_\infty := a_{\varepsilon,i}^{-1} \bar{\pi}_\infty\left(\frac{\cdot - \varepsilon z_i}{a_{\varepsilon,i}}\right), \quad (6.4.36)$$

and smuggle these functions in each one of the integrals over C_i . This yields

$$\sum_{z_i \in n^\varepsilon} \int_{C_i} \nabla(v_g^\varepsilon - v) \cdot \nabla u_\varepsilon = \sum_{z_i \in n^\varepsilon} \int_{C_i} \nabla(v_g^\varepsilon - v - \bar{\phi}_\infty^{\varepsilon,i}) \cdot \nabla u_\varepsilon + \sum_{z_i \in n^\varepsilon} \int_{C_i} \nabla(\bar{\phi}_\infty^{\varepsilon,i}) \cdot \nabla u_\varepsilon. \quad (6.4.37)$$

We claim that the first integral on the right-hand side vanishes in the limit $\varepsilon \downarrow 0^+$: By (6.4.22) and (6.4.36), each difference $v_g^\varepsilon - v - \bar{\phi}_\infty^{\varepsilon,i}$ solves the second system in (6.8.1) in $\mathbb{R}^d \setminus T_i$ with boundary datum $\psi = v - v(\varepsilon z_i)$. Therefore, by the first inequality in (6.8.3),

$$\|\nabla(v_g^\varepsilon - v - \bar{\phi}_\infty^{\varepsilon,i})\|_{L^2(C_i)}^2 \lesssim \|\nabla v\|_{L^2(B_{2a_{\varepsilon,i}}(\varepsilon z_i) \setminus T_i)}^2 + a_{\varepsilon,i}^{-2} \|v - v(\varepsilon z_i)\|_{L^2(B_{2a_{\varepsilon,i}}(\varepsilon z_i) \setminus T_i)}^2.$$

As the vector field v is smooth, we use a Lipschitz estimate on the last term, and conclude that

$$\|\nabla(v_g^\varepsilon - v - \bar{\phi}_\infty^{\varepsilon,i})\|_{L^2(C_i)}^2 \lesssim \|v\|_{C^1}^2 a_{\varepsilon,i}^d \stackrel{(6.4.19)}{\lesssim} \|v\|_{C^1}^2 \varepsilon^{2+4\delta} \varepsilon^d \rho_i^{d-2}.$$

By Cauchy-Schwarz inequality and this last estimate we find

$$\sum_{z_i \in n^\varepsilon} \int_{C_i} \nabla(v_g^\varepsilon - v - \bar{\phi}_\infty^{\varepsilon,i}) \cdot \nabla u_\varepsilon \leq \|\nabla u_\varepsilon\|_{L^2} \left(\varepsilon^{2+d} \sum_{z_i \in n^\varepsilon} \rho_i^{d-2} \right)^{\frac{1}{2}},$$

and use the the Strong Law of Large Numbers to conclude that almost surely the above right-hand side vanishes. This, together with (6.4.37) and (6.4.35), yields

$$\int \nabla(v_g^\varepsilon - v) \cdot \nabla u_\varepsilon = \sum_{z_i \in n^\varepsilon} \int_{C_i} \nabla \bar{\phi}_\infty^{\varepsilon,i} \cdot \nabla u_\varepsilon + o(1). \quad (6.4.38)$$

We now integrate the first integral on the right-hand side above by parts and, since u_ε vanishes in T_i , we obtain

$$\int_{C_i} \nabla \bar{\phi}_\infty^{\varepsilon,i} \cdot \nabla u_\varepsilon = - \sum_{z_i \in n^\varepsilon} \int_{C_i} \Delta \bar{\phi}_\infty^{\varepsilon,i} u_\varepsilon + \int_{\partial B_i} \partial_\nu \bar{\phi}_\infty^{\varepsilon,i} u_\varepsilon,$$

where ν denotes the outer unit normal. By using (6.4.36), the equation (6.8.1) for $(\bar{\phi}_\infty^{\varepsilon,i}, \bar{\pi}_\infty^{\varepsilon,i})$ and the fact that $\nabla \cdot u_\varepsilon = 0$ in D , we obtain

$$\int_{C_i} \nabla \bar{\phi}_\infty^{\varepsilon,i} \cdot \nabla u_\varepsilon = \sum_{z_i \in n^\varepsilon} \int_{\partial B_i} (\partial_n u \bar{\phi}_\infty^{\varepsilon,i} - \bar{\pi}_\infty^{\varepsilon,i} \nu) \cdot u_\varepsilon.$$

By wrapping this up with (6.4.38), we conclude that to show (6.4.34) it suffices to prove that

$$\sum_{z_i \in n^\varepsilon} \int_{\partial B_i} (\partial_\nu \bar{\phi}_\infty^{\varepsilon,i} - \bar{\pi}_\infty^{\varepsilon,i} \nu) \cdot u_\varepsilon \rightarrow \int v \cdot \mu^N u. \quad (6.4.39)$$

We establish (6.4.39) as in [All90a]: We remark, indeed, that by the uniqueness of the solutions in (6.8.1), for each $z_i \in n^\varepsilon$, we have

$$\bar{\phi}_\infty^{\varepsilon,i} = \sum_{k=1}^d v_k(\varepsilon z_i) w_k^\varepsilon, \quad \bar{\pi}^{\varepsilon,i} = \sum_{k=1}^d v_k(\varepsilon z_i) q_k^\varepsilon,$$

with $(w_k^\varepsilon, q_k^\varepsilon)$ the analogues of the oscillating test functions constructed in [All90a][Proposition 2.1.4]. We remark that the only difference is that in this setting, the scales $a_{\varepsilon,i}$ (i.e. the size of the holes T_i) depend on the index z_i and are not constant but bounded by N (we recall that we are considering the truncated environment \mathcal{R}^N). Therefore, by arguing as in Lemma 2.3.7 of [All90a] we use Lemma 2.3.5 of [All90a] and linearity to rewrite

$$\sum_{z_i \in n^\varepsilon} \int_{\partial B_i} (\partial_\nu \bar{\phi}_\infty^{\varepsilon,i} - \bar{\pi}^{\varepsilon,i} \nu) u_\varepsilon = (\mu_\varepsilon^N, u_\varepsilon)_{H^{-1}, H_0^1} + r_\varepsilon,$$

with

$$\mu_\varepsilon^N = \frac{C_d}{|B_1|} \sum_{z_i \in n^\varepsilon} v(\varepsilon z_i) (\rho_i^N)^{d-2} \frac{(2\varepsilon)^d}{d_i^d} \mathbf{1}_{B_i}, \quad r_\varepsilon \rightarrow 0 \quad \text{in } H^{-1}(D).$$

Since $v \in C_0^\infty(D)$ and the radii ρ_i^N are uniformly bounded, we can also replace μ_ε^N by

$$\tilde{\mu}_\varepsilon^N = \frac{C_d}{|B_1|} \sum_{z_i \in n^\varepsilon} (\rho_i^N)^{d-2} \frac{(2\varepsilon)^d}{d_i^d} \mathbf{1}_{B_i} v.$$

To establish (6.4.39), it remains to argue as in the proof of Lemma 5.3.1 in Chapter 5.4.2 (see from formula (5.4.59) on) and appeal to Lemma 5.5.4. This yields property (v) for v_g^ε and thus completes the proof of this step and of the whole lemma. \square

6.5 Estimates for the pressure (Proof of Theorem 6.2.3)

We begin this section by defining the set E^ε appearing in the statement of Theorem 6.2.3. In order to do so, we recall and introduce some notation. In order to keep the notation simpler we again often omit the index ε when no ambiguity occurs. From Lemma 6.3.1 and Lemma 6.3.2, we recall the definition of the index sets n^ε and J and the factors λ_j , $j \in J$. We use the notation

$$\begin{aligned} B_j &= B_{\varepsilon^{\frac{d}{d-2}} \rho_j}(\varepsilon z_j), & B_{j,\theta} &= B_{\theta \varepsilon^{\frac{d}{d-2}} \rho_j}(\varepsilon z_j) \quad \text{for } j \in n^\varepsilon \\ B_j &= B_{\lambda_j \varepsilon^{\frac{d}{d-2}} \rho_j}(\varepsilon z_j), & B_{j,\theta} &= B_{\theta \lambda_j \varepsilon^{\frac{d}{d-2}} \rho_j}(\varepsilon z_j) \quad \text{for } j \in J, \end{aligned}$$

and we denote $A_j = B_{j,\theta} \setminus B_j$.

Moreover, we recall the definition of the set E_l for $-3 \leq l \leq k_{\max} + 1$ from the proof of Lemma 6.3.2:

$$\begin{aligned} E_{k_{\max}+1} &:= \emptyset, \\ E_{l-1} &:= \left(E_l \setminus \bigcup_{z_j \in J_{l-1}} A_j \right) \cup \bigcup_{z_j \in J_{l-1}} B_j. \end{aligned} \tag{6.5.1}$$

We now define

$$E^\varepsilon := E_{-3} \cup H_g^\varepsilon, \tag{6.5.2}$$

where H_g^ε denotes the set of “good” holes as in Lemma 6.3.1. We remark that E^ε is precisely the set where the operator R_ε from Lemma 6.5.2 truncates to zero, i.e. $R_\varepsilon v = 0$ in E^ε for all $v \in C_0^\infty(D)$, and E^ε is the largest set with this property.

For the proof of Theorem 6.2.3, we will rely on some properties of the set E_{-3} that follow from the explicit construction in the proof of Lemma 6.3.2. We collect them in the following Lemma.

Lemma 6.5.1. *For $j \in J$, let E^{z_j} be the connected component of E_{-3} which contains εz_j . Then,*

$$E_{-3} = \bigcup_{j \in J} E^{z_j}. \quad (6.5.3)$$

Moreover, for $j \in J_k$, let $\tilde{E}^{z_j} = E^{z_j} \setminus E_{k+1}$. Then, $E^{z_j} \supset B_{\varepsilon \frac{d}{d-2} \rho_j}(\varepsilon z_j)$ and

$$|\tilde{E}^{z_j}| \gtrsim |B_{\varepsilon \frac{d}{d-2} \rho_j}(\varepsilon z_j)|. \quad (6.5.4)$$

Furthermore, there exists $N_1 \in \mathbb{N}_0$ and $z_{i_n} \in \cup_{l=-3}^{k-2} J_l$, $1 \leq n \leq N_1$ such that

$$E^{z_j} = B_j \setminus \left(\bigcup_{n=1}^{N_1} E^{i_n} \right), \quad (6.5.5)$$

and there exists $N_2 \in \mathbb{N}_0$ and $z_{j_n} \in \cup_{l=-3}^{k-2} J_l$, $1 \leq n \leq N_2$ such that

$$A_j \cap E_{k+1} \cap E = \bigcup_{n=1}^{N_2} E^{j_n} \cap A_j \cap E_{k+1}. \quad (6.5.6)$$

Proof. As mentioned above, the proof of this lemma follows from the construction in the proof of Lemma 6.3.2. First of all, the sets E^{z_j} have been defined in that proof after (6.3.49). Moreover, (6.5.3) is a direct consequence of (6.3.52), and (6.5.5) follows from (6.3.50).

We turn to the proof of (6.5.6): Since by construction of E_k and D_b^ε , $E_{k+1} \subset D_b^\varepsilon$, (6.3.1) implies $E_{k+1} \cap E = E_{k+1} \cap E_{-3}$. Moreover, by (6.5.1), $E_k \cap A_j = \emptyset$. Hence,

$$A_j \cap E_{-3} \subset A_j \cap \left(E_k \cup \bigcup_{l < k} \bigcup_{z_j \in J_l} E^{z_j} \right) = A_j \cap \bigcup_{l < k} \bigcup_{z_j \in J_l} E^{z_j}$$

This implies (6.5.6).

It remains to prove (6.5.4). To this end, we note that if $z_i \in J_k$ and $B_{\lambda_i \varepsilon \frac{d}{d-2} \rho_i}(\varepsilon z_i) \cap E_{k+1} \neq \emptyset$, then there are unique $l > k$ and $z_1 \in J_l$ such that

$$B_{\lambda_i \varepsilon \frac{d}{d-2} \rho_i}(\varepsilon z_i) \cap E_{k+1} = B_{\lambda_i \varepsilon \frac{d}{d-2} \rho_i}(\varepsilon z_i) \cap E_{k+1}^{z_1}. \quad (6.5.7)$$

Indeed, let $l_1 > k$ be minimal such that there is $z_1 \in J_{l_1}$ with

$$B_{\lambda_i \varepsilon \frac{d}{d-2} \rho_i}(\varepsilon z_i) \cap B_{\lambda_1 \varepsilon \frac{d}{d-2} \rho_1}(\varepsilon z_1) \neq \emptyset.$$

Then, since by (6.3.14) $l_1 \geq k+2$ we have $\rho_1 \ll \rho_i$,

$$B_{\lambda_i \varepsilon \frac{d}{d-2} \rho_i}(\varepsilon z_i) \subset B_{\theta \lambda_1 \varepsilon \frac{d}{d-2} \rho_1}(\varepsilon z_1).$$

Now assume there is $l_2 \geq l_1$ and $z_2 \in J_{l_2}$ such that

$$B_{\lambda_i \varepsilon^{\frac{d}{d-2}} \rho_i}(\varepsilon z_i) \cap E_{k+1}^{z_2} \neq \emptyset \quad (6.5.8)$$

Then, applying (6.3.14), $l_1 \leq l_2 - 2$. In particular

$$E_{k+1}^{z_2} \subset B_{\lambda_2 \varepsilon^{\frac{d}{d-2}} \rho_2}(\varepsilon z_2) \setminus B_{\theta \lambda_1 \varepsilon^{\frac{d}{d-2}} \rho_1}(\varepsilon z_1)$$

which contradicts (6.5.8) and thus proves (6.5.7). We remark, that this gives the set J the structure of a forest.

Furthermore, going through the proof of the claim (6.3.49) we see that actually for any $\gamma < \theta^2$ there exists ε sufficiently small such that for all $z_j \in J^\varepsilon$

$$B_{\varepsilon^{\frac{d}{d-2}} \gamma \tilde{\lambda} \rho_j}(\varepsilon z_j) \subset E^{z_j}.$$

Therefore, choosing $\theta < \gamma < \theta^2$, for $z_j \in J_k$,

$$|E^{z_j} \setminus E_{k+1}| \geq |B_{\varepsilon^{\frac{d}{d-2}} \gamma \tilde{\lambda} \rho_j}(\varepsilon z_j) \setminus E_{k+1}| \gtrsim |B_{\varepsilon^{\frac{d}{d-2}} \rho_j}(\varepsilon z_j)|$$

where the last inequality follows from (6.5.7) and the fact that $z_j \notin J_k$ if $B_{\varepsilon^{\frac{d}{d-2}} \theta \tilde{\lambda} \rho_j}(\varepsilon z_j) \subset E_{k+1}$. \square

The proof of Theorem 6.2.3 relies on the following two results. The first lemma below is an adaptation of Lemma 6.2.4 of Section 6.2.4 to the case of the reduction operator R_ε is applied to the function $v = e_k$, where e_k , $k = 1, \dots, d$ are the canonical vectors of \mathbb{R}^d . The second lemma below is a variant of the standard Bogovski lemma to the set $D \setminus E^\varepsilon$ which allows to obtain estimates for the pressure in the Stokes equations (6.1.1). The non-trivial aspect of that Lemma is that the estimate is uniform in ε for small ε . A priori, any such estimate highly depends on the exact geometry of the set of holes. To prove this result, we therefore again use an iteration scheme similar to the one in the construction of the operator R_ε .

Lemma 6.5.2. *Let $k = 1, \dots, d$ be fixed. Then, for almost every $\omega \in \Omega$ and any $\varepsilon \leq \varepsilon_0(\omega)$ and all $k = 1, \dots, d$, there exist $w_k^\varepsilon \in H^1(D; \mathbb{R}^d) \cap L^\infty(D; \mathbb{R}^d)$, $k = 1, \dots, d$, such that*

(H1) $w_k^\varepsilon = 0$ on E^ε and $\nabla \cdot w^\varepsilon = 0$ in D ;

(H2) $w_k^\varepsilon \rightharpoonup e_k$ in $H^1(D)$ and $w_k^\varepsilon \rightarrow e_k$ in $L^p(D)$ for any $1 \leq p < +\infty$;

(H3) For any $\phi \in C_0^\infty(D)$ and sequence $v_\varepsilon \rightharpoonup v$ in $H_0^1(D; \mathbb{R}^d)$ with $\nabla \cdot v_\varepsilon = 0$ on D we have

$$\lim_{\varepsilon \downarrow 0^+} \int \phi \nabla w_k^\varepsilon \cdot \nabla v_\varepsilon = \int \phi e_k \cdot \mu v,$$

with μ defined in Theorem 6.2.1.

Lemma 6.5.3. *Let $q > d$ and let $K \Subset D$. Then, almost surely, there exists $\varepsilon_0 > 0$ such that for all $\varepsilon < \varepsilon_0$ and all $g \in L_0^{d+}(K \setminus E^\varepsilon)$ there exists $v \in H_0^1(D \setminus E^\varepsilon)$ such that*

$$\begin{aligned} \operatorname{div} v &= g, \\ \|v\|_{H^1} &\leq C \|g\|_{L^q}, \end{aligned} \quad (6.5.9)$$

where $C = C(d, \beta, q)$.

Proof of Theorem 6.2.3. We first observe that (6.2.8) holds with the choice of E^ε as in (6.5.2). Indeed, $E^\varepsilon \setminus H_\varepsilon \subset D_b^\varepsilon$ and by (6.3.9), sub-additivity of the harmonic capacity, and Lemma 5.5.3

$$\text{Cap}(E^\varepsilon \setminus H_\varepsilon) \leq \sum_{z_j \in J^\varepsilon} \text{Cap}\left(B_{\Lambda \varepsilon^{\frac{d}{d-2}} \rho_j}(\varepsilon z_j)\right) \lesssim \varepsilon^d \sum_{z_j \in J^\varepsilon} \rho_j^{d-2} \rightarrow 0$$

almost surely as $\varepsilon \rightarrow 0$.

Let $K \Subset D$ and let $\varepsilon_0 > 0$ be as in Lemma 6.5.3. Let $g \in L_0^q(K \setminus E^\varepsilon)$ and let $v \in H_0^1(D \setminus E^\varepsilon)$ satisfy (6.5.9). Then, testing (6.2.4) with v yields

$$\begin{aligned} \int_{K \setminus E^\varepsilon} p_\varepsilon g &= \int_{D \setminus E^\varepsilon} p_\varepsilon \operatorname{div} v = (\nabla u_\varepsilon, \nabla v)_{L^2(D^\varepsilon)} + \langle f, v \rangle_{H^{-1}, H^1} \\ &\leq 2\|v\|_{H^1} \|f\|_{H^{-1}} \lesssim \|g\|_{L^q} \|f\|_{H^{-1}}. \end{aligned}$$

Since $g \in L^q(K \setminus E^\varepsilon)$ was arbitrary, this implies that, up to a subsequence, \tilde{p}_ε defined in (6.2.9) converges to p^* weakly in $L^{q'}(D)$, where q' is the Hölder conjugate of q . It remains to identify the limit p^* and extend the above convergence to the whole family $\varepsilon \downarrow 0^+$. To do so, it suffices to fix any smooth vector field $\phi \in C_0^\infty(\mathbb{R}^d)$ and test the equation (6.2.4) for u_ε with the admissible test function $\sum_{k=1}^d w_k^\varepsilon \phi_k$: The integral terms containing ∇u_ε and f may be treated as in the proof of Theorem 6.2.1 by relying on Lemma 6.5.2 instead of Lemma 6.2.4. It thus remains to show that also

$$\sum_{k=1}^d \int \nabla \cdot (w_k^\varepsilon \phi) p_\varepsilon \rightarrow \int \nabla \phi \cdot p^*. \quad (6.5.10)$$

This indeed yields that (u_h, p^*) solve (6.2.5) and, by uniqueness, that $p^* = p_h$ in $L^q(D)$.

Let $K \Subset D$ be the support of ϕ . Then, by (H1) of Lemma 6.5.2 each product $w_k^\varepsilon \phi_k$ is supported in $K \setminus E_\varepsilon$ and therefore

$$\sum_{k=1}^d \int \nabla \cdot (w_k^\varepsilon \phi) p_\varepsilon = \sum_{k=1}^d \int \nabla \cdot (w_k^\varepsilon \phi) \tilde{p}_\varepsilon = \sum_{k=1}^d \int w_k^\varepsilon \cdot \nabla \phi \tilde{p}_\varepsilon,$$

where in the last identity we used Leibniz rule and the divergence-free condition for w_k^ε in (H1) of Lemma 6.5.2. It now remains to combine the convergence of \tilde{p}_ε with (H2) of Lemma 6.5.2 and send $\varepsilon \downarrow 0^+$ in the right-hand side above. This establishes (6.5.10) and concludes the proof of Theorem 6.2.3. \square

Proof of Lemma 6.5.2. We construct w_k^ε as $R^\varepsilon e_k$ by mimicking the proof of Step 1 and Step 2 of Lemma 6.2.4, with the smooth vector field $v \in C_0^\infty(D, \mathbb{R}^d)$ substituted by e_k . We remark that the construction does not require that v is compactly supported in D . This yields from property (ii) of Lemma 6.2.4 that $\nabla \cdot w_k^\varepsilon = 0$ in D . Moreover, a careful look to the construction of Step 2 on the set D_b^ε shows that $R^\varepsilon e_k$ vanishes in the set $E^\varepsilon \cap D_b^\varepsilon \supset H_b^\varepsilon$ and, since $E^\varepsilon = H_g^\varepsilon$ on $D \setminus D_b^\varepsilon$, we may upgrade property (i) of Lemma 6.2.4 to obtain (H1) of Lemma 6.5.2. Property (H2) follows from (iii) and (iii) of Lemma 6.2.4. Similarly, we argue that (H3) for w_k^ε may be proven as (v) of Lemma 6.2.4, since the term on the left-hand side of (H3) may be rewritten as

$$\int \phi \nabla w_k^\varepsilon \cdot \nabla v_\varepsilon = - \int \nabla \phi \cdot \nabla w_k^\varepsilon v_\varepsilon - (\Delta w_k^\varepsilon, \phi v_\varepsilon)_{H^{-1}, H_0^1}.$$

Thanks to (H2) of Lemma 6.5.2 and the assumption on v_ε , the first term on the right-hand side vanishes almost surely in the limit $\varepsilon \downarrow 0^+$. The remaining term may be treated analogously to (6.4.39) in the proof of Lemma 6.2.4 (see also [All90a][Subsection 2.3.2]). \square

Proof of Lemma 6.5.3. Step 1: Strategy: Let $g_0 \in L_0^q(K \setminus E^\varepsilon)$ and extend it by zero to a function $g_0 \in L_0^q(D \setminus E^\varepsilon)$. The idea is to first solve the problem to find $v_0 \in H_0^1(K)$ such that

$$\begin{aligned} \operatorname{div} v_0 &= g_0, \\ \|v_0\|_{H^q} &\lesssim \|g_0\|_{L^q}. \end{aligned} \quad (6.5.11)$$

Clearly, since K does not depend on ε , this just follows from the classical estimates for the Bogovski operator (see e.g. [Gal11]). Then, we want to do corrections in order to have $v = 0$ in E . For $j \in n^\varepsilon$ the correction is straightforward by taking $v = v_0 + v_j$ in $B_{\theta,j}$, where v_j solves the problem

$$\begin{cases} -\Delta v_j + \nabla p_j = 0 & \text{in } A_j \\ \operatorname{div} v_j = 0 & \text{in } A_j \\ v_j = 0 & \text{on } \partial B_{j,\theta} \\ v_j = -v_0 & \text{in } B_j. \end{cases} \quad (6.5.12)$$

By (6.8.2), we have

$$\begin{aligned} \|v_j\|_{H^1(B_{\theta,j})} &\lesssim \|v_0\|_{H^1(B_{\theta,j})} + R_j^{\frac{d-2}{2}} \|v_0\|_{L^\infty}, \\ \|v_j\|_{C^0} &\lesssim \|v_0\|_{C^0}, \end{aligned} \quad (6.5.13)$$

where $R_j = \varepsilon^{\frac{d}{d-2}} \rho_\varepsilon$.

We would like to do this also for $z_j \in J$. We should start with $z_j \in J_{max}$. However, recall the complementary condition for existence of a solution to equation (6.5.12)

$$\int_{\partial B_j} v_0 \cdot \nu = 0.$$

This is in general not satisfied for those z_j since we have

$$\int_{\partial B_j} v_0 \cdot \nu = \int_{B_j} g_0,$$

and the latter integral might be nonzero if $B_j \not\subset E$ and we simply extended g_0 by zero inside E . (Clearly, $B_j \subset E$ holds for $z_j \in n^\varepsilon$.) Moreover, note that for $z_j \in J_k$, $-3 \leq k \leq k_{max}$, instead of the problem (6.5.12), we need to find a corrector v_j that solves

$$\begin{cases} \operatorname{div} v_j = g_0 & \text{in } A_j \cap E_{k+1} \\ \operatorname{div} v_j = 0 & \text{in } A_j \setminus E_{k+1} \\ v_j = 0 & \text{on } \partial B_{j,\theta} \\ v_j = -v^{(k+1)} & \text{in } B_j, \end{cases} \quad (6.5.14)$$

where $v^{(k)}$ is inductively defined by

$$\begin{aligned} v^{(k_{max}+1)} &:= v_0, \\ v^{(k)} &:= v^{(k+1)} + \sum_{z_j \in J_k} v_j. \end{aligned}$$

By Lemma 6.8.2, we can find a solution v_j to (6.5.14) with

$$\begin{aligned} \|v_j\|_{H^1} &\lesssim \|v^{(k+1)}\|_{H^1(B_{\theta,j})} + \|g\|_{L^2(B_{\theta,j})} + R_j^{\frac{d-2}{2}} \left(\|v^{(k+1)}\|_{C^0} + \|\operatorname{div} v^{(k+1)}\|_{L^q(B_r)} + \|g\|_{L^q} \right), \\ \|v_j\|_{C^0} &\lesssim \|v^{(k+1)}\|_{C^0} + \|\operatorname{div} v^{(k+1)}\|_{L^q(B_r)} + \|g\|_{L^q} \end{aligned} \quad (6.5.15)$$

with $R_j = \varepsilon^{\frac{d}{d-2}} \rho_j$, provided the complementary condition holds, namely

$$\int_{A_j \cap E_{k+1}} g_0 - \int_{\partial B_j} v^{(k+1)} \cdot \nu = 0. \quad (6.5.16)$$

Again, this is not satisfied in general, since

$$\int_{A_j \cap E_{k+1}} g_0 - \int_{\partial B_j} v^{(k+1)} \cdot \nu = \int_{A_j \cap E_{k+1}} g_0 - \int_{B_j \setminus E_{k+1}} g_0.$$

For this reason, instead of simply extending g_0 by zero, we need to extend it in a nontrivial way to a function $g \in L_0^q(D)$.

Step 2: Extension of the function g_0 : First, we extend g_0 by $g = 0$ to $\mathbb{R}^d \setminus E$. As seen above, for $z_j \in n^\varepsilon$, we can also simply choose $g = 0$ in B_j . For $z_j \in J$ let $N_1 \in \mathbb{N}_0$ and $z_{i_n} \in \cup_{l=-3}^{k-2} J_l$, $1 \leq n \leq N_1$ such (6.5.5) holds, and let $N_2 \in \mathbb{N}_0$ and $z_{j_n} \in \cup_{l=-3}^{k-2} J_l$, $1 \leq n \leq N_2$ such that (6.5.6) holds. We now choose $g = g_j = \text{const}$ in \tilde{E}^{z_j} and $g = 0$ in $E^{z_j} \setminus \tilde{E}^{z_j}$, where the constants g_j are uniquely determined by satisfying

$$\begin{aligned} 0 &= \int_{A_j \cap E_{k+1}} g - \int_{B_j \setminus E_{k+1}} g \\ &= \int_{A_j \cap E_{k+1} \setminus E} g_0 + \sum_{n=1}^{N_2} |\tilde{E}^{z_{j_n}} \cap A_j \cap E_{k+1}| g_{j_n} \\ &\quad - \int_{B_j \setminus (E_{k+1} \cup E)} g_0 - |\tilde{E}^{z_j}| g_j - \sum_{n=1}^{N_1} |\tilde{E}^{z_{i_n}} \cap B_j \setminus E_{k+1}| g_{i_n}. \end{aligned} \quad (6.5.17)$$

Indeed, since $z_{i_n}, z_{j_n} \in \cup_{l=-3}^{k-2} J_l$, this formula yields g_j for all $z_j \in J_k$, provided we already know g_i for $z_i \in \cup_{l=-3}^{k-2} J_l$. Therefore, all $z_j, j \in J$ are inductively defined by (6.5.17).

We observe that by this procedure we might extend the function g_0 non-trivially also in holes that are not contained in K , namely if they are within a cluster that intersects with K . Therefore, we fix some $K \Subset K' \Subset D$ and argue that for ε sufficiently small, $g = 0$ in $D \setminus K'$. Indeed, this follows by induction very similarly to the argument at the end of Step 2 in the proof of Lemma 6.2.4, only that here we start from the small holes towards the big holes. Indeed, $g_j = 0$ for all $j \in J_{-3}$ with $B_{\theta,j} \subset D \setminus K$, and $g_j = 0$ for $j \in J_k$ if $B_{\theta,j} \subset D \setminus K$ and $B_{\theta,j} \cap B_{\theta,i} = \emptyset$ for all $i \in \cup_{l=-3}^{k-1} J_l$ with $g_i \neq 0$.

Hence, instead of (6.5.11), we can find $v_0 \in H_0^1(K')$ with

$$\begin{aligned} \operatorname{div} v_0 &= g, \\ \|v_0\|_{H^q} &\lesssim \|g\|_{L^q}, \end{aligned} \quad (6.5.18)$$

and extend v_0 by zero to a function in D . In order to find such a v_0 , we need to check the complementary condition $\int g = 0$. By (6.5.17)

$$\begin{aligned} \int_{K'} g &= \int_K g_0 + \int_{E_{-3}} g = \int_{E_{-3}} g \\ &= \int_{E_{-2}} g + \sum_{j \in J_{-3}} \int_{B_j \setminus E_{-3}} g - \int_{A_j \cap E_{-3}} g = \int_{E_{-2}} g. \end{aligned}$$

By induction, this indeed yields $g = 0$ since $E_{k_{\max}+1} = \emptyset$.

Step 3: Solving $\operatorname{div} v = g$ and obtaining the desired estimates: We need to show that by the extension of g_0 to g , we did not increase its norm too much, i.e.,

$$\|g\|_{L^q(K')}^q \lesssim \|g_0\|_{L^q(K)}^q. \quad (6.5.19)$$

We claim that with the above definition of g_j , we have for all $z_j \in J_k$

$$|\tilde{E}^{z_j}|g_j| \leq (2k_{\max} + 3)^{k+2} \|g_0\|_{L^1(B_{\theta^2,j} \setminus E)}, \quad (6.5.20)$$

where $B_{\theta^2,j} := B_{\theta^2 \lambda_j \varepsilon^{\frac{d}{d-2}} \rho_j}(\varepsilon z_j)$. We prove (6.5.20) by induction over k . For $z_j \in J_{-3}$, we have

$$|\tilde{E}^{z_j}|g_j| = \int_{A_j \cap E_{k+1} \setminus E} g,$$

so (6.5.20) holds for $k = -3$. Assume that (6.5.20) holds for all $1 \leq l \leq k-1$ and consider $z_j \in J_k$. Let $N_1, N_2 \in \mathbb{N}_0$ and $z_{i_n}, z_{j_n} \in \cup_{l=-3}^{k-1} J_l$ such that (6.5.5) and (6.5.6) hold. Then,

$$\begin{aligned} |\tilde{E}^{z_j}|g_j| &\leq \int_{B_{\theta,j} \setminus E} |g_0| + \sum_{n=1}^{N_1} |B_{\theta,j} \cap \tilde{E}^{z_{i_n}}| |g_{i_n}| + \sum_{n=1}^{N_2} |B_{\theta,j} \cap \tilde{E}^{z_{j_n}}| |g_{j_n}| \\ &\leq \|g_0\|_{L^1(B_{\theta,j} \setminus E)} + \sum_{n=1}^{N_1} (2k_{\max} + 3)^{k+1} \|g_0\|_{L^1(B_{\theta^2,i_n} \setminus E)} \\ &\quad + \sum_{n=1}^{N_2} (2k_{\max} + 3)^{k+1} \|g_0\|_{L^1(B_{\theta^2,j_n} \setminus E)}. \end{aligned} \quad (6.5.21)$$

We observe that $B_{\theta^2,i_n} \subset B_{\theta^2,j}$ since $B_{i_n} \cap B_{\theta,j} \neq \emptyset$ and the radius of the ball B_{i_n} is much smaller than the one of B_j since $z_{i_n} \in J_l$ with $l \leq k-2$. Moreover, for every $x \in B_{\theta^2,j}$,

$$\#\{z_{i_n}, 1 \leq n \leq N : x \in B_{\theta^2,i_n}\} \leq k+1,$$

since, by (6.3.7), $B_{\theta^2,i_n} \cap B_{\theta^2,i_m} = \emptyset$ whenever $z_{i_m} \neq z_{i_n} \in J_l$ for some $1 \leq m, n \leq N$, $-3 \leq l \leq k-2$. Using this in (6.5.21) yields (6.5.20).

By definition of g , we have

$$\|g\|_{L^q(K')}^q = \|g_0\|_{L^q(K)}^q + \sum_{z_j \in J_{K'}} |\tilde{E}^{z_j}|g_j|^q.$$

We estimate for $z_j \in J$, using (6.5.20) and (6.5.4),

$$|\tilde{E}^{z_j}|g_j|^q \lesssim \frac{1}{|\tilde{E}^{z_j}|^{q-1}} \|g_0\|_{L^1(B_{\theta^2,j} \setminus E)}^q \lesssim \|g_0\|_{L^q(B_{\theta^2,j} \setminus E)}^q.$$

Using similar as above that for all $x \in K'$

$$\#\{z_j \in J : x \in B_{\theta,j}\} \leq k_{\max} + 1,$$

this yields (6.5.19).

Hence, the function v_0 solving (6.5.18) satisfies

$$\begin{aligned} \operatorname{div} v_0 &= g, \\ \|v_0\|_{H^q} &\lesssim \|g\|_{L^q} \lesssim \|g_0\|_{L^q}. \end{aligned}$$

Now we just proceed by adding correctors as sketched in Step 1: First, let v_j be the solutions to (6.5.12) for $z_j \in n^\varepsilon$ and define

$$v^{(k_{max}+1)} := v_0 + \sum_{z_j \in n^\varepsilon} v_j.$$

Then, $v^{(k_{max}+1)} \in H_0^1(D)$,

$$\begin{aligned} \operatorname{div} v^{(k_{max}+1)} &= g, \\ v^{(k_{max}+1)} &= 0 \quad \text{in } H_g^\varepsilon, \end{aligned} \tag{6.5.22}$$

and, since v_j have disjoint support, using (6.5.13)

$$\|v^{(k_{max}+1)}\|_{C^0} \lesssim \|g_0\|_{L^q}$$

and

$$\|v^{(k_{max}+1)}\|_{H^1}^2 = \sum_{z_j \in n^\varepsilon} \|v_j\|_{H^1}^2 \lesssim \sum_{z_j \in n^\varepsilon} \|v_0\|_{H^1(B_{\theta,j})}^2 + \varepsilon^{\frac{d}{d-2}} \rho_j \|v_0\|_{L^\infty} \lesssim \|g_0\|_{L^q},$$

almost surely, for ε small enough.

Then, inductively for $k = k_{max}, \dots, -3$, for all $z_j \in J_k$, we claim that we find solutions to v_j (6.5.14) that satisfy (6.5.15), and defining

$$v^{(k)} := v^{(k+1)} + \sum_{z_j \in J_k} v_j,$$

we have $v^{(k)} \in H_0^1(D)$ with

$$\begin{aligned} \operatorname{div} v^{(k)} &= g \quad \text{in } D \setminus E_k \\ v^{(k)} &= 0 \quad \text{in } H_g^\varepsilon \cup E_k, \end{aligned} \tag{6.5.23}$$

$$\|v^{(k)}\|_{H^1} + \|v^{(k)}\|_{C^0} \lesssim \|g_0\|_{L^q}.$$

It remains to prove this claim. Indeed, if (6.5.23) holds, then setting $v = v^{(-3)}$ yields the assertion.

The proof proceeds by induction in k . Indeed, for $k = k_{max} + 1$, (6.5.22) yields (6.5.23). Assume (6.5.23) holds for some $k + 1$. Then, we recall that the complementary condition for solving (6.5.14) is (6.5.16), which is equivalent to (6.5.17) since $\operatorname{div} v^{(k+1)} = g$ in $D \setminus E_{k+1}$. However, (6.5.17) holds, because this is exactly how we chose the values of g_i , $i \in J$. Therefore, v_j is well defined, and satisfies (6.5.15). In particular $v^{(k)}$ is well defined, and, using that $|\operatorname{div} v^{(k+1)}| \leq |g|$ pointwise together with the estimates for $v^{(k+1)}$, we get the estimate in (6.5.23) analogously as we obtained the estimates for $v^{(k_{max}+1)}$. Moreover, by construction, $\operatorname{div} v^{(k)} = g$ in $D \setminus E_k$.

Furthermore, $v^{(k)} \in H_0^1(D)$, since we only changed $v^{(k+1)}$ in $B_{\theta,j}$ for holes that are in certain cluster that overlaps with K' . These balls are contained in D by an argument analogous to the one at the end of Step 2 in the proof of Lemma 6.2.4. It remains remark that by construction $v^{(k)} = 0$ in $H_g^\varepsilon \cup E_k$, since $v^{(k)} = 0$ in $E_{k+1} \setminus \cup_{z_j \in J_k} B_{\theta,j}$ and in $\cup_{z_j \in J_k} B_{\theta,j}$. \square

6.6 Probabilistic results

The aim of this section is to give some probabilistic results on the random set H^ε , in terms of the size of the clusters generated by overlapping balls of comparable size; these results are used in Section 6.3 to obtain a good covering for H^ε and to estimate its size.

We introduce the following notation: For $\alpha \geq 1$, let

$$H_\alpha^\varepsilon = \bigcup_{z_i \in \Phi^\varepsilon(D)} B_{\varepsilon^{\frac{d}{d-2}} \alpha \rho_i}(\varepsilon z_i).$$

For a step-size $\delta > 0$, we partition the (random) collection of points $\Phi^\varepsilon(D)$ in terms of the order of magnitude of the associated radii: We define

$$\begin{aligned} I_{k,\delta}^\varepsilon &:= \{z_i \in \Phi^\varepsilon(D) : \varepsilon^{1-\delta k} < \varepsilon^{\frac{d}{d-2}} \rho_i \leq \varepsilon^{1-\delta(k+1)}\} \quad \text{for } k \geq -2, \\ I_{-3,\delta}^\varepsilon &:= \{z_i \in \Phi^\varepsilon(D) : \varepsilon^{\frac{d}{d-2}} \rho_i \leq \varepsilon^{1+2\delta}\}, \end{aligned} \quad (6.6.1)$$

and for every $k \geq -2$ also

$$\Psi_\delta^{k,\varepsilon} = I_k^\varepsilon \cup I_{k-1}^\varepsilon \subset \Phi^\varepsilon(D).$$

Each collection $\Psi_\delta^{k,\varepsilon}$ thus generates the set

$$H_{k,\alpha}^{\delta,\varepsilon} := \bigcup_{z_i \in \Psi_\delta^{k,\varepsilon}} B_{\varepsilon^{\frac{d}{d-2}} \alpha \rho_i}(\varepsilon z_i) \subset H_\alpha^\varepsilon \quad (6.6.2)$$

which is made of balls having radii which differ by at most two orders δ of magnitude.

Lemma 6.6.1. *Let $\alpha \geq 1$ and $0 < \delta < \frac{\beta}{2d}$ be fixed. Then, there exists $M(d, \beta), k_{\max}(\beta, d) \in \mathbb{N}$ such that for almost every $\omega \in \Omega$ and every $\varepsilon \leq \varepsilon_0(\omega)$*

(I) *For every $k > k_{\max}$ we have*

$$I_{\varepsilon,\delta}^k = \emptyset;$$

(II) *For every $-2 \leq k \leq k_{\max}$, each connected component of $H_{k,\alpha}^\varepsilon$ defined in (6.6.2) is made of at most M balls.*

Proof of Lemma 6.6.1. We begin with (I) and observe that assumption (6.1.7) and Chebyshev's inequality imply that for a constant $C < +\infty$

$$\langle \rho^{d-2+\beta} \rangle \leq C, \quad \mathbb{P}(\rho \geq r) \leq Cr^{-(d-2+\beta)}. \quad (6.6.3)$$

In addition, as already argued in Section 6.3.1 (see (6.3.12)), (6.1.7) and the Strong Law of Large Numbers (see Lemma 5.5.2) imply that for almost every $\omega \in \Omega$ and all ε sufficiently small

$$\max_{z_i \in \Phi^\varepsilon(D)} \varepsilon^{\frac{d}{d-2}} \rho_i \leq 2\varepsilon^{\frac{d}{d-2} - \frac{d}{d-2+\beta}} \langle \rho^{d-2+\beta} \rangle^{\frac{1}{d-2+\beta}}.$$

Hence, for the same choice of ω and ε we have $I^k = \emptyset$ whenever $k > k_{\max}$ with

$$\varepsilon^{1-\delta(k_{\max}+1)} < \varepsilon^{\frac{d}{d-2} - \frac{d}{d-2+\beta}},$$

namely if

$$1 - \delta(k_{\max} + 1) < \frac{d}{d-2} - \frac{d}{d-2+\beta}. \quad (6.6.4)$$

We may thus choose the minimal k_{max} satisfying the inequality above and conclude the proof for (II).

We now turn to (II) and fix $-2 \leq k \leq k_{max}$: For any $m \in \mathbb{N}$ we consider the event

$$A_{\varepsilon, \delta, k}^{\alpha, m} := \{\omega : \text{There exist } m \text{ intersecting balls in } H_{k, \alpha}^{\delta, \varepsilon}\}.$$

Then, (II) is equivalent to show that there exists an integer $M = M(\beta, d) \geq 2$ such that

$$\mathbb{P}\left(\bigcap_{\varepsilon_0 > 0} \bigcup_{\varepsilon \leq \varepsilon_0} \bigcup_{k \geq -2} A_{\varepsilon, \delta, k}^{\alpha, M}\right) = 0. \quad (6.6.5)$$

Furthermore, we begin by arguing that it suffices to prove that

$$\mathbb{P}\left(\bigcap_{l_0 \in \mathbb{N}} \bigcup_{l \geq l_0} \bigcup_{k \geq -2} A_{2^{-l}, 3\delta, k}^{\bar{\alpha}, M}\right) = 0, \quad (6.6.6)$$

i.e. statement (6.6.5) for the sequence $\varepsilon_l = 2^{-l}$ and α, δ substituted by $\bar{\alpha} = 2^{\frac{2}{d-2}}\alpha$ and 3δ .

Suppose, indeed, that (6.6.6) holds: For any $\varepsilon > 0$, let $l \in \mathbb{N}$ be such that $\varepsilon_{l+1} \leq \varepsilon \leq \varepsilon_l$. Then for every two $z_i, z_j \in \Psi^{k, \delta, \varepsilon}$ with $\rho_i \geq \rho_j$, definition (6.6.1) yields that

$$\rho_i - \rho_j \leq \rho_j \left(\frac{\rho_i}{\rho_j} - 1 \right) \leq \rho_j (\varepsilon_{l+1}^{-2\delta} - 1) \leq \rho_j \varepsilon_{l+1}^{-3\delta}.$$

This implies that if $\rho_j \in I_{\tilde{k}-1}^{\varepsilon_{l+1}, 3\delta}$ for some $\tilde{k} \in \mathbb{Z}$, then $\rho_i \in I_{\tilde{k}}^{\varepsilon_{l+1}, 3\delta}$. This is equivalent to

$$\Psi_k^{\delta, \varepsilon} \subset \Psi_{\tilde{k}}^{3\delta, \varepsilon_{l+1}}. \quad (6.6.7)$$

Equipped with this inclusion, we now show that

$$A_{\varepsilon, \delta, k}^{\alpha, m} \subset A_{\varepsilon_{l+1}, 3\delta, \tilde{k}}^{\bar{\alpha}, m}. \quad (6.6.8)$$

To do so, let us assume that $z_i, z_j \in \Psi_k^{\delta, \varepsilon}$ satisfy

$$B_{\alpha \varepsilon^{\frac{d}{d-2}} \rho_j}(\varepsilon z_j) \cap B_{\alpha \varepsilon^{\frac{d}{d-2}} \rho_i}(\varepsilon z_i) \neq \emptyset.$$

Then,

$$\varepsilon |z_i - z_j| \leq \alpha \varepsilon^{\frac{d}{d-2}} (\rho_i + \rho_j)$$

which yields

$$|z_i - z_j| \leq \alpha \varepsilon^{\frac{2}{d-2}} (\rho_i + \rho_j) \leq \alpha \varepsilon_l^{\frac{2}{d-2}} (\rho_i + \rho_j) = 2^{\frac{2}{d-2}} \alpha \varepsilon_{l+1}^{\frac{2}{d-2}} (\rho_i + \rho_j).$$

This is equivalent to

$$B_{\bar{\alpha} \varepsilon_{l+1}^{\frac{d}{d-2}} \rho_j}(\varepsilon_{l+1} z_j) \cap B_{\bar{\alpha} \varepsilon_{l+1}^{\frac{d}{d-2}} \rho_i}(\varepsilon_{l+1} z_i) \neq \emptyset.$$

Since the previous argument holds for any choice of two elements in $\Psi^{k, \delta, \varepsilon}$, this and (6.6.7) imply (6.6.8). This last statement allows also to conclude that for every $m \in \mathbb{Z}$

$$\bigcup_{k \geq -2} A_{\varepsilon, \delta, k}^{\alpha, m} \subset \bigcup_{k \geq -2} A_{\varepsilon_{l+1}, 2\delta, k}^{\bar{\alpha}, m}.$$

This establishes that (6.6.6) implies (6.6.5).

To conclude the proof of (II), it only remains to show (6.6.6): We begin by deriving a basic estimate for the probability of having a certain number of close points in a Poisson point process. We recall indeed that the centres $\Phi^\varepsilon(D)$ are distributed according to a Poisson point process in $1\varepsilon D$ with intensity λ . We also recall that, for a general set $A \subset \mathbb{R}^d$ we denote by $N(A)$ the random variable providing the number of points of the process which are in A .

For $0 < \eta < 1$, let

$$\mathcal{Q}_\eta := \{[-\eta 2, \eta 2]^d + y \mid y \in (\eta \mathbb{Z})^d\},$$

i.e. the set of cubes of length η centered at the points of the lattice $(\eta \mathbb{Z})^d$. Let S_η be the set containing the edges of the cube $[0, \eta 2]^d$, i.e.

$$S_\eta := \{z = (z_1, \dots, z_d) \in \mathbb{R}^d : z_k \in \{0, \frac{\eta}{2}\} \text{ for all } k = 1, \dots, d\}.$$

Then, for any $x \in \mathbb{R}^d$ there always exists $z \in S_\eta$ and $B_{\frac{\eta}{2}}(x) \subset Q$ for some $Q \in \mathcal{Q}_\eta + z$. Thus, if η is chosen such that $\lambda \eta^d \leq 1$, we use this geometric consideration to estimate

$$\mathbb{P}(\exists x \in \frac{1}{\varepsilon} D : N(B_{\frac{\eta}{2}}(x)) \geq m) \lesssim \mathbb{P}(\exists Q \in \mathcal{Q}_\eta, z \in S_\eta : (Q + z) \cap \frac{1}{\varepsilon} D \neq \emptyset, N(Q + z) \geq m),$$

and the distribution for $N(A)$ to conclude that

$$\mathbb{P}(\exists x \in \frac{1}{\varepsilon} D : N(B_{\frac{\eta}{2}}(x)) \geq m) \lesssim \varepsilon^{-d} \eta^{-d} e^{-\lambda \eta^d} \sum_{k=m}^{\infty} \frac{(\lambda \eta^d)^k}{k!} \lesssim (\eta \varepsilon)^{-d} (\lambda \eta^d)^m. \quad (6.6.9)$$

Equipped with (6.6.9), we estimate each $P(A_{\varepsilon,k}^{\alpha,m})$: Let us assume that $z_i, z_j \in \Psi^{k,\delta,\varepsilon}$ are such that

$$B_{\alpha \varepsilon^{\frac{d}{d-2}} \rho_j}(\varepsilon z_j) \cap B_{\alpha \varepsilon^{\frac{d}{d-2}} \rho_i}(\varepsilon z_i) \neq \emptyset.$$

Then,

$$\varepsilon |z_i - z_j| \leq \alpha \varepsilon^{\frac{d}{d-2}} (\rho_i + \rho_j) \leq 2\alpha \varepsilon^{1-\delta(k+1)}$$

and thus by setting

$$\kappa_k = -\delta(k+1), \quad (6.6.10)$$

we have

$$|z_i - z_j| \leq 2\alpha \varepsilon^{\kappa_k}, \quad A_{\varepsilon,k}^{\alpha,m} \subset \{\exists x \in \frac{1}{\varepsilon} D : \#(\Psi^{k,\delta,\varepsilon} \cap B_{m\alpha \varepsilon^{\kappa_k}}(x)) \geq m\}. \quad (6.6.11)$$

We now want to estimate the event in the right-hand side above by appealing to (6.6.9) for each ε and k fixed and with $\eta = \eta_k^\varepsilon$ given by

$$\eta_k^\varepsilon := m\alpha \varepsilon^{\kappa_k}. \quad (6.6.12)$$

We observe indeed that by definition (6.6.1), for every ε the processes $\Psi^{k,\delta,\varepsilon}$ are Poisson processes on $\frac{1}{\varepsilon} D$ with intensity given by

$$\lambda_k^\varepsilon = \lambda \mathbb{P}(\varepsilon^{-\frac{2}{d-2}-\delta(k-1)} \leq \rho \leq \varepsilon^{-\frac{2}{d-2}-\delta(k+1)}) \stackrel{(6.6.3)}{\lesssim} \varepsilon^{(d-2+\beta)\left(\frac{2}{d-2}+\delta(k-1)\right)} \quad (6.6.13)$$

for any $k \geq -1$, and

$$\lambda_{-2}^\varepsilon = \lambda \mathbb{P}(\rho \leq \varepsilon^{-\frac{2}{d-2}-\delta(-1)}) \leq \lambda \quad (6.6.14)$$

for $k = 2$.

We first argue that, provided that for every k and ε small enough, there exists $\mu_k > 0$ such that

$$\lambda_k^\varepsilon (\eta_k^\varepsilon)^d \leq \varepsilon^{\mu_k}, \quad (6.6.15)$$

then we conclude the proof of (6.6.6). Indeed, by the previous inequality we may apply (6.6.9) to the right-hand side of (6.6.11) and bound by (6.6.12) and (6.6.15)

$$\mathbb{P}(A_{\varepsilon,k}^{\alpha,m}) \lesssim \varepsilon^{m\mu_k - d(1+\kappa_k)}.$$

By choosing $m = M$, M sufficiently large, we thus get

$$\mathbb{P}(A_{\varepsilon,k}^{\alpha,m}) \lesssim \varepsilon^{\mu_k}.$$

Since by (I) we only have to consider finitely many values of $k = -3, \dots, k_{max}$, M can be chosen independently of k . Therefore, recalling that $\varepsilon_l = 2^{-l}$ in (6.6.6), we use the previous estimate and assumption (6.6.15) to infer

$$\sum_{l \in \mathbb{N}} \mathbb{P}\left(\bigcup_{k \geq -2} A_{\varepsilon_l, \delta, k}^{\alpha, M}\right) < \infty.$$

I thus remains to apply Borel-Cantelli's lemma to obtain (6.6.6) and thus (6.6.5) as well as (II).

To conclude the proof of the lemma, it thus remains to show (6.6.15). To do so, we recall the definitions (6.6.12) and (6.6.10) of η_k and κ_k and we also set for every $-1 \leq k \leq k_{max}$

$$\gamma_k := (d - 2 + \beta) \left(\frac{2}{d - 2} + \delta(k - 1) \right). \quad (6.6.16)$$

By (6.6.13), this definitions allows us to bound for each ε

$$\lambda_k^\varepsilon \leq \varepsilon^{\gamma_k}. \quad (6.6.17)$$

We first show (6.6.15) for $k = -2$: In this case, by (6.6.12), (6.6.10) and (6.6.14), we have

$$\lambda_{-2}^\varepsilon (\eta_{-2}^\varepsilon)^d \lesssim \varepsilon^{d\delta}$$

and we may thus simply choose $\mu_{-2} = d\delta > 0$. We now turn to the case $k > -2$: Again by (6.6.12) and, this time, by (6.6.17) we have

$$\lambda_k^\varepsilon (\eta_k^\varepsilon)^d \lesssim \varepsilon^{\gamma_k + d\kappa_k}.$$

Therefore we need

$$\mu_k = \gamma_k + d\kappa_k \stackrel{(6.6.16), (6.6.10)}{=} \frac{2(d - 2 + \beta)}{d - 2} - (2 - \beta)\delta(k - 1) - 2d\delta > 0.$$

Since we assumed that $\beta \leq 1$, we may use (6.6.4) on the second term in the right-hand side above and, after a short calculation, obtain that

$$\mu_k \geq 2 - (2 - \beta) - 2d\delta \geq \beta - 2d\delta.$$

Thanks to our assumption $\delta < \frac{\beta}{2d}$, we thus conclude that $\mu_k > 0$. This establishes (6.6.15) and completes the proof of the lemma. \square

6.7 Homogenization of the stationary Navier-Stokes equations (Proof of Remark 6.2.2)

The proof of the homogenization result in this case is analogous to the case of the Stokes equations, provided we prove the convergence of the non-linear term $u_\varepsilon \nabla \cdot u_\varepsilon$.

We recall the weak formulation of (6.2.6). We define the space $V_\varepsilon := \{w \in H_0^1(D_\varepsilon) : \operatorname{div} w = 0\}$ equipped with the norm $\|\nabla \cdot\|_{L^2}$. Then, we call $u_\varepsilon \in V$ a weak solution to (6.2.6) if

$$\mu \int \nabla u_\varepsilon \cdot \nabla \phi + \int u_\varepsilon \cdot \nabla u_\varepsilon \cdot \phi = \langle f, \phi \rangle \quad \forall \phi \in \tilde{V}_\varepsilon := \{w \in H_0^1(D_\varepsilon) \cap L^d : \operatorname{div} w = 0\},$$

where the space \tilde{V}_ε is chosen such that the nonlinear term makes sense. Furthermore, by Sobolev embedding we observe $\tilde{V}_\varepsilon = V_\varepsilon$ for $d \leq 4$. The weak formulation of (6.2.7) is analogous. Existence of solutions to (6.2.7) is well-known. However, the solution is only known to be unique if $d \leq 4$ and

$$\|f\|_{V'} < C(d, D). \quad (6.7.1)$$

If $d \leq 4$ testing with the solution u yields the energy estimate

$$\|\nabla u_i\|_{L^2} \leq \|f\|_{V'}. \quad (6.7.2)$$

For more details on the stationary Navier-Stokes equations see for example [Tem01] and [Gal11].

The proof of the convergence $u_\varepsilon \rightharpoonup u_h$ in $H^1(D)$ in the case $d = 3$ is now straightforward provided (6.7.1) holds. Indeed, thanks to (6.7.2), the sequence u_ε is bounded in H^1 , and by the uniqueness of the solutions to (6.2.7), it therefore suffices to prove that the weak limit u^* of any subsequence of u_ε satisfies (6.2.7). To this end, let $v \in C_0^\infty(D)$ with $\operatorname{div} v = 0$. Then, applying Lemma 6.2.4, we know

$$\begin{aligned} \int \nabla u_\varepsilon \cdot \nabla (R_\varepsilon v) &\rightarrow \int \nabla u^* \cdot \nabla v + \mu u^* \cdot v, \\ \langle f, R_\varepsilon v \rangle &\rightarrow \langle f, v \rangle. \end{aligned}$$

Therefore, it remains to show

$$\int u_\varepsilon \cdot \nabla u_\varepsilon \cdot (w_k^\varepsilon \phi) \rightarrow \int u^* \cdot \nabla u_k^* \phi.$$

However, since $2^* = 6 > 4$ both u_ε and $R_\varepsilon v$ converge strongly in L^4 and ∇u_ε converges weakly in L^2 . Thus, the convergence above follows immediately.

In the case $d = 4$ this argument just fails, since the embedding from H^1 to L^4 is not compact. However, since by Lemma 6.2.4 also $R_\varepsilon v \rightarrow v$ strongly in L^q , for any $4 < q < \infty$, the argument works again.

6.8 Estimates for the Stokes equations in annuli and in the exterior of balls

In this section we summarize some standard results for the solutions to the Stokes equation in annular and exterior domains (see, e.g. [Gal11; All90a]).

Lemma 6.8.1. *Let $R > 1$, denote $A_R := B_R \setminus B_1$, and let $\psi \in H^1(B_\theta) \cap C^0(\bar{B}_\theta)$ satisfy $\int_{\partial B_1} \psi \cdot \nu = 0$. Let (ϕ_R, π_R) and $(\phi_\infty, \pi_\infty)$ be the (weak) solutions of*

$$\begin{cases} \Delta \phi_R - \nabla \pi_R = 0 & \text{in } A_R \\ \nabla \cdot \phi_R = 0 & \text{in } A_R \\ \phi_R = \psi & \text{on } \partial B_1 \\ \phi_R = 0 & \text{on } \partial B_R, \end{cases} \quad \begin{cases} \Delta \phi_\infty - \nabla \pi_\infty = 0 & \text{in } \mathbb{R}^d \setminus B_1 \\ \nabla \cdot \phi_\infty = 0 & \text{in } \mathbb{R}^d \setminus B_1 \\ \phi_\infty = \psi & \text{on } \partial B_1 \\ \phi \rightarrow 0 & \text{for } |x| \rightarrow +\infty. \end{cases} \quad (6.8.1)$$

Then,

$$\begin{aligned} \|\pi_R\|_{L^2(A_R)/\mathbb{R}} + \|\nabla \phi_R\|_{L^2(A_R)} &\leq C_1 (\|\nabla \psi\|_{L^2(A_R)} + \|\psi\|_{L^2(A_R)}), \\ \|\phi_R\|_{C^0(\bar{A}_R)} &\leq C_1 \|\psi\|_{C^0(\partial B_1)}, \end{aligned} \quad (6.8.2)$$

with $C_1 = C_1(d, R)$. Moreover,

$$\begin{aligned} \|\pi_\infty\|_{L^2(\mathbb{R}^d \setminus B_1)} + \|\nabla \phi_\infty\|_{L^2(\mathbb{R}^d \setminus B_1)} &\leq C_2 (\|\nabla \psi\|_{L^2(A_2)} + \|\psi\|_{L^2(A_2)}), \\ \|\phi_\infty\|_{C^0} &\leq C_2 \|\psi\|_{C^0(\partial B_1)}, \end{aligned} \quad (6.8.3)$$

with $C_2 = C_2(d)$. Furthermore,

$$|\phi_\infty(x)| \leq C_2 \|\psi\|_{C^0(\partial B_1)} |x|^{2-d}, \quad (6.8.4)$$

and, if $\nabla \cdot \psi = 0$ in B_1 ,²

$$|\nabla \phi_\infty(x)| \leq C_2 \|\psi\|_{H^1(B_2)} |x|^{1-d} \quad \text{for all } |x| \geq 3. \quad (6.8.5)$$

Proof. The existence and uniqueness of solutions to both problems in (6.8.1) together with the first estimate in both (6.8.2) and (6.8.3) is a standard result [Gal11][Section IV and V]. The second estimate in both (6.8.2) and (6.8.3) can be found in [MRS99][Theorem 5.1 and Theorem 6.1]. Estimate (6.8.4) can be found in [MRS99][Theorem 6.1], too.

To prove (6.8.5), we extend ϕ_∞ by ψ inside B_1 and π_∞ by 0 inside B_1 . Then, by (6.8.3)

$$\begin{cases} -\Delta \phi_\infty + \nabla \pi_\infty = f & \text{in } \mathbb{R}^d \\ \nabla \cdot \phi_\infty = 0 & \text{in } \mathbb{R}^d \end{cases}$$

for some $f \in \dot{H}^{-1}(\mathbb{R}^d)$, with

$$\begin{aligned} \text{supp } f &\subset \bar{B}_1, \\ \|f\|_{\dot{H}^{-1}(\mathbb{R}^d)} &\lesssim \|\psi\|_{H^1(B_2)}. \end{aligned}$$

Here, $\dot{H}^{-1}(\mathbb{R}^d)$ is the dual of the homogeneous Sobolev space

$$\dot{H}^1(\mathbb{R}^d) := \left\{ v \in L^{\frac{2d}{d-2}}(\mathbb{R}^d) : \nabla v \in L^2(\mathbb{R}^d) \right\}, \quad \|\cdot\|_{\dot{H}^1(\mathbb{R}^d)} := \|\nabla \cdot\|_{L^2(\mathbb{R}^d)}.$$

Hence, with U being the fundamental solution of the Stokes equations we have

$$\phi_\infty(x) = (U * f)(x).$$

²This assumption is not needed, but makes the proof slightly simpler.

The fundamental solution satisfies

$$|D^\alpha U(x)| \lesssim C(d, |\alpha|) |x|^{2-d-|\alpha|}.$$

Using the compact support of f , and letting $\eta \in C_c^\infty(B_2)$ be a cut-off function with $\eta = 1$ in B_1 , we deduce for all $|x| > 3$

$$\begin{aligned} |\nabla \phi_\infty(x)| &= |\langle \eta \nabla U(x - \cdot), f \rangle_{H^1, \dot{H}^{-1}}| \\ &\leq \|\eta \nabla U(x - \cdot)\|_{\dot{H}^1(\mathbb{R}^d)} \|f\|_{\dot{H}^{-1}(\mathbb{R}^d)} \\ &\lesssim C_3 \|\psi\|_{H^1(B_2)} |x|^{1-d}. \end{aligned}$$

This proves (6.8.5). \square

Lemma 6.8.2. *Let $q > d$ and let $0 < r < 1$, $\theta > 1$, $B_r := B_r(0)$, $B_{r\theta} := B_{r\theta}(0)$, $A_{r,\theta} := B_{r\theta} \setminus B_r$. Assume $g \in L^q(B_{r\theta})$ and $v \in H^1(B_{r\theta}) \cap C^0(\overline{B_{r\theta}})$ with $\operatorname{div} v \in L^q(B_r)$ satisfy*

$$\int_{A_{r,\theta}} g + \int_{\partial B_r} v \cdot \nu = 0.$$

Then, there exists $u \in H_0^1(B_\theta) \cap C^0(\overline{B_\theta})$ solving

$$\begin{cases} \operatorname{div} u = g & \text{in } A_{r,\theta} \\ u = 0 & \text{on } \partial B_{r\theta} \\ u = v & \text{in } B_r, \end{cases}$$

with

$$\begin{aligned} \|u\|_{H^1} &\leq C \|v\|_{H^1} + \|g\|_{L^2} + r^{\frac{d-2}{2}} (\|v\|_{C^0} + \|\operatorname{div} v\|_{L^q(B_r)} + \|g\|_{L^q}), \\ \|u\|_{C^0} &\leq C \|v\|_{C^0} + \|\operatorname{div} v\|_{L^q(B_r)} + \|g\|_{L^q}. \end{aligned}$$

with $C = C(\theta, d, q)$.

Proof. We will define $u = u_1 + u_2$, where u_1 solves

$$\begin{cases} \operatorname{div} u_1 = g & \text{in } A_{r,\theta} \\ \operatorname{div} u_1 = \operatorname{div} v & \text{in } B_r \\ u_1 = 0 & \text{on } \partial B_{r\theta}, \end{cases}$$

and u_2 is the solution to

$$\begin{cases} -\Delta u_2 + \nabla p = 0 & \text{in } A_{r,\theta} \\ \operatorname{div} u_2 = 0 & \text{in } A_{r,\theta} \\ u_2 = 0 & \text{on } \partial B_{r\theta} \\ u_2 = v - u_1 & \text{in } B_r, \end{cases} \quad (6.8.6)$$

As it is well known (see e.g. [Gal11][Theorem 3.1]), the first problem has a solution with

$$\begin{aligned} \|u_1\|_{H^1} &\lesssim \|\operatorname{div} v\|_{L^2(B_r)} + \|g\|_{L^2}, \\ \|u_1\|_{W^{1,q}} &\lesssim \|\operatorname{div} v\|_{L^q(B_r)} + \|g\|_{L^q}. \end{aligned}$$

By Sobolev inequality,

$$\|u_1\|_{C^0} \lesssim \|\operatorname{div} v\|_{L^q(B_1)} + \|g\|_{L^q}.$$

Using estimate (6.8.2) rescaled with r for the solution to (6.8.6), we find

$$\begin{aligned} \|\nabla u_2\|_{L^2} &\lesssim \|\nabla(v - u_1)\|_{L^2} + \frac{1}{r}\|v - u_1\|_{L^2} \lesssim \|\nabla v\|_{L^2} + \|\nabla u_1\|_{L^2} + r^{\frac{d-2}{2}}\|v - u_1\|_{C^0} \\ &\lesssim \|\nabla v\|_{L^2} + \|g\|_{L^2} + r^{\frac{d-2}{2}}\left(\|v\|_{C^0} + \|\operatorname{div} v\|_{L^q(B_1)} + \|g\|_{L^q}\right), \end{aligned}$$

and

$$\|u_2\|_{C^0} \lesssim \|v - u_1\|_{C^0} \lesssim \|v\|_{C^0} + \|\operatorname{div} v\|_{L^q(B_1)} + \|g\|_{L^q}.$$

Combining these inequalities for u_1 and u_2 (and the Poincaré inequality) yields the desired estimate for u . \square

Chapter 7

The inertialess limit of the Vlasov-Stokes equations

In this Chapter, we study the Vlasov-Stokes equation (1.1.2), the sedimentation model for inertial particles at zero Reynolds number. We study the regime of small Stokes numbers St , in which, as explained in detail in Chapter 2, the effects of the particle inertia becomes small. We rigorously prove that in the limit $St \rightarrow 0$ the solution to the Vlasov-Stokes system converge to the solutions of the transport-Stokes system (1.1.1) which models the sedimentation of inertialess particles, as we proved in Chapter 7. A formal argument for this result has been given in Chapter 2.4.1.

The content of this chapter has been published in *SIAM Journal on Mathematical Analysis*, [Höf18b].

7.1 Introduction

We consider the Vlasov-Stokes equations (1.1.2). We denote $\lambda = (\gamma St^{-1})$ as in Chapter 2. For the ease of notation we set $\gamma = 1$ and also drop the numerical constants in the Vlasov-Stokes system, which then becomes

$$\begin{aligned} \partial_t f + v \cdot \nabla_x f + \lambda \operatorname{div}_v (gf + (u - v)f) &= 0, & f(0, \cdot, \cdot) &= f_0, \\ -\Delta u + \nabla p + \rho(u - \bar{V}) &= 0, & \operatorname{div} u &= 0. \end{aligned} \quad (7.1.1)$$

Here ρ and j denote the spatial particle density and current, i.e.,

$$\begin{aligned} \rho(t, x) &:= \int_{\mathbb{R}^3} f(t, x, v) dv, \\ j(t, x) &:= \rho(t, x) \bar{V}(t, x) := \int_{\mathbb{R}^3} f(t, x, v) v dv. \end{aligned} \quad (7.1.2)$$

Omitting these constants, the constant $\frac{2}{9}\gamma^{-1}$ also in transport-Stokes system (1.1.1) yields

$$\begin{aligned} \partial_t \rho_* + (g + u_*) \cdot \nabla \rho_* &= 0, & \rho_*(0, \cdot) &= \rho_0 := \int_{\mathbb{R}^3} f_0 dv, \\ -\Delta u_* + \nabla p &= g\rho_*, & \operatorname{div} u_* &= 0. \end{aligned} \quad (7.1.3)$$

We recall that we proved well-posedness of this system in Chapter 7.

7.1.1 Main result

The main result of this Chapter is the following theorem.

Theorem 7.1.1. *Assume $f_0 \in W^{1,\infty}(\mathbb{R}^3 \times \mathbb{R}^3)$ is compactly supported. Then, for $\lambda > 0$, there exists a unique solution (f_λ, u_λ) to (7.1.1). Let (ρ_*, u_*) be the unique solution to (7.1.3). Then, for all $0 < t < T$, and all $\alpha < 1$*

$$\begin{aligned} \rho_\lambda &\rightarrow \rho_* \quad \text{in } C^{0,\alpha}((0, T) \times \mathbb{R}^3), \\ u_\lambda &\rightarrow u_* \quad \text{in } L^\infty((t, T); W^{1,\infty}(\mathbb{R}^3)) \text{ and in } L^1((0, T); W^{1,\infty}(\mathbb{R}^3)). \end{aligned}$$

Formally, for large values of λ , the first equation in (7.1.1) forces the particle to attain the velocity $g + u(t, x)$, i.e., the density $f(t, x, v)$ concentrates around $g + u(t, x)$. Using that and integrating the first equation in (7.1.1) in v leads to the first equation in (7.1.3). Moreover, \bar{V} in the fluid equation in (7.1.1) can formally be replaced by $g + u(t, x)$, which leads to the fluid equation in (7.1.3).

Formally, the adjustment of the particle velocities described above happens in times of order $1/\lambda$. In fact, the process is more complicated as the fluid velocity changes very fast in this time scale as well. In other words, there is a boundary layer of width $1/\lambda$ at time zero for the convergence of the fluid (and particle) velocity. This is the reason, why the convergence $u_\lambda \rightarrow u_*$ can only hold uniformly on time intervals (t, T) for $t > 0$ as stated in the theorem. The particles, however, do not move significantly in times of order $1/\lambda$. Thus, there is no boundary layer in the convergence $\rho_\lambda \rightarrow \rho_*$.

7.1.2 Idea of the proof

We introduce the kinetic energy of the particles

$$E(t) := \int_{\mathbb{R}^3 \times \mathbb{R}^3} |v|^2 f \, dx \, dv.$$

Using the Vlasov-Stokes equations (7.1.1) yields the following energy identities for the fluid velocity and the particle energy (cf. Lemma 7.2.1 and Lemma 7.2.2).

$$\|\nabla u\|_{L^2(\mathbb{R}^3)}^2 + \|u\|_{L^2(\rho)}^2 = (u, j)_{L^2(\mathbb{R}^3)} \leq \|\bar{V}\|_{L_\rho^2}^2 \leq E, \quad (7.1.4)$$

$$\frac{1}{2} \frac{d}{dt} E = \lambda \left(g \cdot \int_{\mathbb{R}^3 \times \mathbb{R}^3} j \, dx - \int_{\mathbb{R}^3 \times \mathbb{R}^3} (u - v)^2 f \, dx \, dv - \|\nabla u\|_{L^2(\mathbb{R}^3)}^2 \right). \quad (7.1.5)$$

Here and in the following, the weighted L^p -norm is defined by

$$\|h\|_{L_\rho^p}^p := \int_{\mathbb{R}^3} |h|^p \rho \, dx.$$

As expected, equation (7.1.5) shows that there is loss of energy due to friction (friction between the particles and the fluid as well as friction inside of the fluid), but the gravity pumps energy into the system (if we assume $g \cdot \int_{\mathbb{R}^3 \times \mathbb{R}^3} j \, dx > 0$, which at least after some time should be the case). Note that the Vlasov-Stokes equations (7.1.1) also imply that the mass of the particles $\|\rho\|_{L^1(\mathbb{R}^3)}$ is conserved.

To analyze solutions to the Vlasov equation in (7.1.1), we look at the characteristic curves $(X, V, Z)(s, t, x, v)$ starting at time t at position $(x, v) \in \mathbb{R}^3 \times \mathbb{R}^3$, where denote the value of the

solution f along the characteristic curve by $Z(s, t, x, v) = f(s, X(s, t, x, v), V(s, t, x, v))$:

$$\begin{aligned} \partial_s X &= V, & X(t, t, x, v) &= x, \\ \partial_s V &= \lambda(g + u(s, X) - V(s, t, x, v)), & V(t, t, x, v) &= v, \\ \partial_s Z &= 3\lambda Z, & Z(t, t, x, v) &= f(t, x, v). \end{aligned} \quad (7.1.6)$$

The last equation has the explicit solution $Z(s, t, x, v) = e^{3\lambda(s-t)} f(t, x, v)$. By the standard theory for characteristics, any solution $f \in W^{1,\infty}((0, T) \times \mathbb{R}^3 \times \mathbb{R}^3)$ with $u \in L^\infty((0, T); W^{1,\infty}(\mathbb{R}^3))$ is of the form

$$f(t, x, v) = e^{3\lambda t} f_0(X(0, t, x, v), V(0, t, x, v)). \quad (7.1.7)$$

Using the characteristics as well as estimates based on the energy identities (7.1.4) and (7.1.5) and regularity theory of Stokes equations, we prove global well-posedness of the Vlasov-Stokes equations (7.1.1) for compactly supported initial data $f_0 \in W^{1,\infty}(\mathbb{R}^3 \times \mathbb{R}^3)$. A similar approach based on an analysis of the characteristics has been used to prove existence of solutions to the Vlasov-Poisson equations in [BD85], [Pfa92], and [Sch91] (see also [Gla96]). From the PDE point of view, the electrostatic potential appearing in the Vlasov-Poisson equation is similar to the fluid velocity in the Vlasov-Stokes equations. However, in the Vlasov-Poisson equations, the force acting on the particles is the gradient of the electrostatic potential, whereas in the Vlasov-Stokes equations, only the fluid velocity itself contributes. This makes it possible to prove existence (and also uniqueness) in a much simpler way for the Vlasov-Stokes equations.

In order to prove the convergence in Theorem 7.1.1, the starting point is integrating the characteristics which yields

$$\begin{aligned} V(t, 0, x, v) - V(0, 0, x, v) \\ = \lambda \left(\int_0^t (u_\lambda(s, X(s, 0, x, v)) + g) ds + X(0, 0, x, v) - X(t, 0, x, v) \right). \end{aligned} \quad (7.1.8)$$

Thus,

$$\left| X(t, 0, x, v) - x - \int_0^t (u_\lambda(s, X(s, 0, x, v)) + g) ds \right| \leq \frac{|V(t, 0, x, v) - v|}{\lambda}. \quad (7.1.9)$$

Therefore, provided the speed of the particles does not blow up, we see that for large values of λ the particles are almost transported by the fluid plus the gravity. Clearly, this is also what happens for solutions to the limit inertialess equations (7.1.3).

In order to show that u_λ is close to u , we introduce a fluid velocity \tilde{u}_λ which can be viewed as intermediate between u_λ and u_* by

$$-\Delta \tilde{u}_\lambda + \nabla p_\lambda = g\rho_\lambda, \quad \operatorname{div} \tilde{u}_\lambda = 0. \quad (7.1.10)$$

In order to prove smallness of $u_\lambda - \tilde{u}_\lambda$, one needs estimates on ρ_λ and u_λ that are uniform in λ , which are more difficult to obtain than those that we use in the proof of well-posedness. Indeed, in view of the energy identity for the particles (7.1.5), any naive estimate based on that equation will blow up as $\lambda \rightarrow \infty$. However, as the first term is linear in the velocity and the other terms (which have a good sign) are quadratic, the energy E cannot exceed a certain value as long as the particle density ρ is not too concentrated (cf. Lemma 7.3.2). In other words, if the energy is high enough, the quadratic friction terms will prevail over the linear gravitation terms and therefore prevent the energy from increasing further. However, if concentrations of the particle density occur, the particles essentially fall down like one small and heavy particle, leading to large velocities. Indeed, the terminal velocity of a spherical particle of radius R in a Stokes fluid at rest is

$$V = \frac{2}{9} \frac{\rho_p - \rho_f}{\mu} g R^2.$$

In order to rule out such concentration effects, we use again the representation of f in (7.1.7) obtained from the characteristics. Indeed, computing ρ by taking the integral over v in (7.1.7), we can show that the prefactor $e^{3\lambda t}$ in that formula is canceled due to concentration of f in velocity space in regions of size $e^{-\lambda t}$ as long as we control ∇u in a suitable way (cf. Lemma 7.3.4). As ∇u is controlled by E due to the energy identity (7.1.4), this enables us to get uniform estimates for both u , ∇u , and ρ for small times.

It turns out that also estimates on derivatives of ρ are needed to prove smallness of $u_\lambda - \tilde{u}_\lambda$. These are provided by a more detailed analysis of the characteristics.

7.1.3 Outline of the chapter

The rest of this chapter is organized as follows. In Section 7.2, we prove global well-posedness of the Vlasov Stokes equations (7.1.1), based on energy estimates, analysis of the characteristics, and a fixed point argument. In Section 7.3, we derive a priori estimates that are uniform in λ for small times by analyzing the characteristics more carefully. In particular we prove and use that the supports of the solutions concentrate in the space of velocities. In Section 7.4.1, we use those a priori estimates proven in Section 7.3 to show that the fluid velocity u_λ is close to the intermediate fluid velocity \tilde{u}_λ defined in (7.1.10) as $\lambda \rightarrow \infty$. In Section 7.4.2, we prove the assertion of the main result, Theorem 7.1.1, up to times where we have uniform a priori estimates. This follows from compactness due to the a priori estimates and convergence of averages of ρ_λ on small cubes, which we prove using again the characteristic equations. In Section 7.4.3, we finish the proof of the main result, Theorem 7.1.1, by extending the a priori estimates from Section 7.3 to arbitrary times. This is done by using both the a priori estimates and the convergence for small times.

7.2 Global well-posedness of the Vlasov-Stokes equations

In this section, we write C for any constant that depends only on the initial datum. Any additional dependencies are denoted by arguments of C , e.g. $C(\lambda t)$ is a constant that depends only on λt and the initial datum. We use the convention that C is monotone in all its arguments.

Throughout the chapter, to denote spatial norms of functions that depend on space and time we will often write $\|u\|_{L^p(\mathbb{R}^3)}$ instead of $\|u(t, \cdot)\|_{L^p(\mathbb{R}^3)}$ when there is no ambiguity on the dependence on time.

7.2.1 Estimates for the fluid velocity

Lemma 7.2.1. *Let $h \in L^\infty(\mathbb{R}^3 \times \mathbb{R}^3)$ be nonnegative, and assume $Q > 0$ is such that $\text{supp } h \subset B_Q(0) \subset \mathbb{R}^3 \times \mathbb{R}^3$. Let*

$$\begin{aligned}\rho(x) &:= \int_{\mathbb{R}^3} h(x, v) dv, \\ j(x) &:= \rho \bar{V} := \int_{\mathbb{R}^3} h(x, v) v dv, \\ E &:= \int_{\mathbb{R}^3 \times \mathbb{R}^3} h(x, v) |v|^2 dx dv.\end{aligned}$$

Then there exists a unique weak solution $u \in W^{1,\infty}(\mathbb{R}^3)$ to the Brinkman equation

$$-\Delta u + \nabla p + \rho u = j.$$

Moreover,

$$\|\nabla u\|_{L^2(\mathbb{R}^3)}^2 + \|u\|_{L_\rho^2(\mathbb{R}^3)} = (u, j)_{L^2(\mathbb{R}^3)} \leq \|\bar{V}\|_{L_\rho^2(\mathbb{R}^3)}^2 \leq E, \quad (7.2.1)$$

$$\|u\|_{L^\infty(\mathbb{R}^3)} \leq C(\|h\|_{L^\infty(\mathbb{R}^3 \times \mathbb{R}^3)}, \|h\|_{L^1(\mathbb{R}^3 \times \mathbb{R}^3)}, E)(1 + Q), \quad (7.2.2)$$

$$\|u\|_{W^{1,\infty}(\mathbb{R}^3)} \leq C(Q, E)\|h\|_{L^\infty(\mathbb{R}^3 \times \mathbb{R}^3)}. \quad (7.2.3)$$

Proof. Existence and uniqueness of weak solutions in the homogeneous Sobolev space $\dot{H}^1(\mathbb{R}^3) := \{w \in L^6(\mathbb{R}^3) : \nabla w \in L^2(\mathbb{R}^3)\}$ follows from the Lax-Milgram theorem.

In the following, we write $\|\cdot\|_q$ instead of $\|\cdot\|_{L^q(\mathbb{R}^3)}$ and $\|\cdot\|_{L^q(\mathbb{R}^3 \times \mathbb{R}^3)}$ and $\|\cdot\|_{L_\rho^q}$ instead of $\|\cdot\|_{L_\rho^q(\mathbb{R}^3)}$. Testing the Brinkman equation with u itself yields

$$\|\nabla u\|_2^2 + \|u\|_{L_\rho^2}^2 = (j, u)_{L^2(\mathbb{R}^3)} \leq \|u\|_{L_\rho^2} \|\bar{V}\|_{L_\rho^2}. \quad (7.2.4)$$

By the Cauchy-Schwarz inequality

$$\bar{V}^2 \rho = \frac{(\int_{\mathbb{R}^3} h(x, v) v \, dv)^2}{\int_{\mathbb{R}^3} h(x, v) \, dv} \leq \int_{\mathbb{R}^3} h(x, v) v^2 \, dv.$$

Hence,

$$\|u\|_{L^2(\rho)}^2 \leq \|\bar{V}\|_{L^2(\rho)}^2 \leq E.$$

Using again (7.2.4) yields (7.2.1). Using the critical Sobolev embedding, we have

$$\|u\|_6^2 \leq C\|\nabla u\|_2^2 \leq CE. \quad (7.2.5)$$

Moreover, we can use this Sobolev inequality in (7.2.1) to get

$$\|u\|_6^2 \leq C\|u\|_6\|j\|_{6/5}.$$

Using the definition of Q yields $\|j\|_{6/5} \leq C(Q)\|h\|_\infty$ and therefore

$$\|\nabla u\|_2 + \|u\|_6 \leq C(Q)\|h\|_\infty \quad (7.2.6)$$

Standard regularity theory for the Stokes equation (see [Gal11]) implies

$$\|\nabla^2 u\|_q \leq C\|\rho u\|_q + C\|j\|_q. \quad (7.2.7)$$

for all $1 < q < \infty$. In order to prove (7.2.3), we use (7.2.7) and (7.2.5) to get

$$\|\nabla^2 u\|_6 \leq C\|\rho u\|_6 + C\|j\|_6 \leq C\|\rho\|_\infty\|u\|_6 + C\|j\|_6 \leq C(E, Q)\|h\|_\infty.$$

Hence, by Sobolev embedding and (7.2.6)

$$\|\nabla u\|_\infty \leq C\|\nabla^2 u\|_6 + C\|\nabla u\|_2 \leq C(E, Q)\|h\|_\infty,$$

and similarly for $\|u\|_\infty$.

It remains to prove (7.2.2). Let $R > 0$. Then,

$$\begin{aligned} \rho &= \int_{\mathbb{R}^3} h \, dv \leq \int_{\{|v| < R\}} h \, dv + R^{-2} \int_{\{|v| > R\}} |v|^2 h \, dv \\ &\leq CR^3\|h\|_\infty + CR^{-2} \int_{\{|v| > R\}} |v|^2 h \, dv. \end{aligned}$$

We choose

$$R = \left(\int_{\mathbb{R}^3} |v|^2 f \, dv \right)^{1/5} \|h\|_{\infty}^{-1/5}.$$

Thus,

$$\rho \leq \|h\|_{\infty}^{2/5} \left(\int_{\mathbb{R}^3} |v|^2 h \, dv \right)^{3/5},$$

and therefore,

$$\|\rho\|_{5/3} \leq \|h\|_{\infty}^{2/5} E^{3/5}. \quad (7.2.8)$$

Moreover, by definition of Q , (7.2.8) implies for all $1 \leq p \leq 5/3$,

$$\|j\|_p \leq Q \|\rho\|_p \leq C(\|h\|_{\infty}, \|h\|_1, E)Q. \quad (7.2.9)$$

Sobolev and Hölder's inequalities imply

$$\|u\|_{10} \leq C \|\nabla^2 u\|_{30/23} \leq C \|\rho\|_{5/3} \|u\|_6 + C \|j\|_{30/23} \leq C(\|h\|_{\infty}, \|h\|_1, E)(1 + Q),$$

where we used (7.2.5), (7.2.8), and (7.2.9). Now, we can repeat the argument, using this improved estimate for u in (7.2.7). This yields

$$\|u\|_{30} \leq C(\|h\|_{\infty}, \|h\|_1, E)(1 + Q).$$

Using again (7.2.7) yields

$$\|\nabla^2 u\|_{30/19} \leq C(\|h\|_{\infty}, \|h\|_1, E)(1 + Q).$$

As $30/19 > 3/2$, we can apply Sobolev embedding to get

$$\|u\|_{\infty} \leq C \|\nabla^2 u\|_{30/19} + C \|u\|_6 \leq C(\|h\|_{\infty}, \|h\|_1, E)(1 + Q),$$

which finishes the proof of (7.2.2). \square

7.2.2 A priori estimates for the particle density

Lemma 7.2.2. *Let $T > 0$ and $f_0 \in W^{1,\infty}(\mathbb{R}^3 \times \mathbb{R}^3)$ and let $Q_0 > 0$ be minimal such that $\text{supp } f_0 \subset B_{Q_0}(0)$. Assume $f \in W^{1,\infty}((0, T) \times \mathbb{R}^3 \times \mathbb{R}^3)$ is a solution to (7.1.1) with $u \in L^{\infty}((0, T); W^{1,\infty}(\mathbb{R}^3))$. Then, f is compactly supported on $[0, T] \times \mathbb{R}^3 \times \mathbb{R}^3$. Let $Q(t)$ be minimal such that $\text{supp } f(t, \cdot, \cdot) \subset B_{Q(t)}(0)$. Furthermore, define*

$$E(t) := \int_{\mathbb{R}^3 \times \mathbb{R}^3} |v|^2 f \, dx \, dv.$$

Then,

$$\|f(t, \cdot, \cdot)\|_{L^{\infty}(\mathbb{R}^3 \times \mathbb{R}^3)} = e^{3\lambda t}, \quad (7.2.10)$$

$$\|\rho\|_1 = 1, \quad (7.2.11)$$

$$\frac{d}{dt} E = 2\lambda \left(g \cdot \int_{\mathbb{R}^3} j \, dx - \int_{\mathbb{R}^3 \times \mathbb{R}^3} (u - v)^2 f \, dx \, dv - \|\nabla u\|_{L^2(\mathbb{R}^3)}^2 \right) \quad (7.2.12)$$

$$\leq 2\lambda \left(CE^{\frac{1}{2}} - \int_{\mathbb{R}^3 \times \mathbb{R}^3} (v - \bar{V})^2 f \, dx \, dv - \|u - \bar{V}\|_{L^2_{\rho}(\mathbb{R}^3)}^2 - \|\nabla u\|_{L^2(\mathbb{R}^3)}^2 \right), \quad (7.2.13)$$

$$E(t) \leq C(1 + (\lambda t)^2), \quad (7.2.14)$$

$$Q(t) \leq C(t, \lambda). \quad (7.2.15)$$

Proof. By the regularity assumptions on f and u , the characteristics in (7.1.6) are well defined and (7.1.7) holds. This shows that the support of f remains uniformly bounded on compact time intervals.

The exponential growth of the L^∞ -norm of f (7.2.10) follows from the characteristic equations as we have seen in (7.1.7).

Mass conservation (7.2.11) follows directly from integrating the Vlasov equation (7.1.1).

We multiply the Vlasov equation by $|v|^2$ and integrate to find

$$\begin{aligned} \frac{d}{dt} E &= 2 \int_{\mathbb{R}^3 \times \mathbb{R}^3} v \cdot \lambda(g + u - v) f \, dx \, dv \\ &= 2\lambda \left(g \cdot \int_{\mathbb{R}^3 \times \mathbb{R}^3} v f \, dx \, dv - \int_{\mathbb{R}^3 \times \mathbb{R}^3} (u - v)^2 f \, dx \, dv + \int_{\mathbb{R}^3 \times \mathbb{R}^3} u \cdot (u - v) f \, dx \, dv \right) \\ &= 2\lambda \left(g \cdot \int_{\mathbb{R}^3 \times \mathbb{R}^3} j \, dx - \int_{\mathbb{R}^3 \times \mathbb{R}^3} (u - v)^2 f \, dx \, dv - \|\nabla u\|_{L^2(\mathbb{R}^3)}^2 \right). \end{aligned}$$

This yields the identity (7.2.12). By the Cauchy-Schwarz inequality

$$\int_{\mathbb{R}^3} |j| \, dx \leq \int_{\mathbb{R}^3 \times \mathbb{R}^3} |v| f \, dv \, dx \leq \|\rho\|_{L^1(\mathbb{R}^3)}^{1/2} E^{1/2}. \quad (7.2.16)$$

Moreover, by definition of \bar{V} in (7.1.2)

$$\begin{aligned} \int_{\mathbb{R}^3 \times \mathbb{R}^3} (u - v)^2 f \, dx \, dv &= \int_{\mathbb{R}^3 \times \mathbb{R}^3} ((v - \bar{V})^2 + (\bar{V} - u)^2 - 2(v - \bar{V})(\bar{V} - u)) f \, dx \, dv \\ &= \int_{\mathbb{R}^3 \times \mathbb{R}^3} (v - \bar{V})^2 f \, dx \, dv + \|u - \bar{V}\|_{L^2_\rho(\mathbb{R}^3)}^2. \end{aligned} \quad (7.2.17)$$

Using (7.2.16) and (7.2.17) shows (7.2.13).

In particular

$$\frac{d}{dt} E \leq C\lambda E^{1/2}.$$

This proves (7.2.14) by a comparison principle for ODEs.

The characteristic equation for V in (7.1.6) implies

$$\begin{aligned} |V(t, 0, x, v)| &= \left| e^{-\lambda t} \left(v + \lambda \int_0^t e^{\lambda s} (g + u(s, X(s, 0, x, v))) \, ds \right) \right| \\ &\leq e^{-\lambda t} v + |g| + \int_0^t \|u(s, \cdot)\|_{L^\infty(\mathbb{R}^3)} \, ds. \end{aligned}$$

Thus, for all $(x, v) \in \text{supp } f_0$, we get by Lemma 7.2.1, (7.2.10), (7.2.11), and (7.2.14)

$$\begin{aligned} |V(t, 0, x, v)| &\leq Q_0 + 1 + C(\|f\|_{L^\infty((0,t) \times \mathbb{R}^3 \times \mathbb{R}^3)}, \|E\|_{L^\infty(0,t)}) \int_0^t (1 + Q(s)) \, ds \\ &\leq C + C(\lambda t) \int_0^t (1 + Q(s)) \, ds. \end{aligned}$$

By the equation for X , we get for all $(x, v) \in \text{supp } f_0$

$$|X(t, 0, x, v)| \leq Q_0 + \int_0^t |V(s, 0, x, v)| \, ds \leq Q_0 + tC(\lambda t) \int_0^t (1 + Q(s)) \, ds.$$

Hence,

$$Q(t) \leq \sup_{(x,v) \in \text{supp } f_0} |(X(t, 0, x, v), V(t, 0, x, v))| \leq C + (1 + t)C(\lambda t) \int_0^t (1 + Q(s)) \, ds.$$

Grönwall's lemma yields (7.2.15). \square

7.2.3 Well-posedness by the Banach fixed point theorem

Proposition 7.2.3. *Let $f_0 \in W^{1,\infty}(\mathbb{R}^3 \times \mathbb{R}^3)$ with compact support. Then, for all $T > 0$, there exists a unique solution $f \in W^{1,\infty}((0, T) \times \mathbb{R}^3 \times \mathbb{R}^3)$ to (7.1.1) with $u \in L^\infty((0, T); W^{2,\infty}(\mathbb{R}^3)) \cap W^{1,\infty}((0, T) \times \mathbb{R}^3)$.*

Proof. We want to prove existence of solutions using the Banach fixed point theorem. Let $Q_1, E_1 > 0$. We define the metric space, where we want to prove contractiveness. We write $\Omega_T = (0, T) \times \mathbb{R}^3 \times \mathbb{R}^3$

$$Y := \left\{ h \in \Omega_T : h \geq 0, \|h(t, \cdot)\|_{L^1(\mathbb{R}^3 \times \mathbb{R}^3)} = \|f_0\|_{L^1(\mathbb{R}^3)}, \right. \\ \left. \int_{\mathbb{R}^3 \times \mathbb{R}^3} (1 + |v|^2) h \, dx \, dv \leq E_1, \text{supp } h \subset [0, T] \times \overline{B_{Q_1}(0)} \right\}.$$

Then, Y is a complete metric space. Let $T > 0$ and $h_1, h_2 \in Y$. For $i = 1, 2$, we define u_i to be the solution to

$$-\Delta u_i + \nabla p = \int_0^\infty \int_{\mathbb{R}^3} (v - u_i) h_i \, dv.$$

We define the characteristics $(X_i, V_i)(s, t, x, v)$ analogously to (7.1.6) by

$$\begin{aligned} \partial_s(X_i, V_i)(s, t, x, v) &= (V_i(s, t, x, v), g + u_i(s, X_i(s, t, x, v)) - V_i(s, t, x, v)), \\ (X_i, V_i)(t, t, x, v) &= (x, v). \end{aligned}$$

Then, the solutions to the equation

$$\partial_t f_i + v \cdot \nabla_x f_i + \lambda \operatorname{div}_v (g f_i + (u_i - v) f_i) = 0,$$

with initial datum f_0 is given by

$$f_i(t, x, v) = e^{3\lambda t} f_0((X_i, V_i)(0, t, x, v)), \quad (7.2.18)$$

and $f_i \in W^{1,\infty}(\Omega_T)$. In this way, we defined a map $S : Y \rightarrow W^{1,\infty}(\Omega_T)$ that maps h_i to f_i . A fixed point of S solves (7.1.1). In order to apply the Banach fixed point theorem, we show that S is contractive. We estimate

$$|f_1(t, x, v) - f_2(t, x, v)| \leq e^{3\lambda t} \|\nabla f_0\|_{L^\infty(\mathbb{R}^3 \times \mathbb{R}^3)} |(X_1, V_1)(0, t, x, v) - (X_2, V_2)(0, t, x, v)|. \quad (7.2.19)$$

Furthermore, writing $X_i(s)$ instead of $X_i(s, t, x, v)$ and similar for V_i , we have for all $0 \leq s \leq t$

$$\begin{aligned} & |(X_1, V_1)(s) - (X_2, V_2)(s)| \\ & \leq \int_s^t |(V_1(\tau) - V_2(\tau), \lambda(u_1(\tau, X_1(\tau)) - u_2(\tau, X_2(\tau)) - V_1(\tau) + V_2(\tau)))| \, d\tau \\ & \leq \int_s^t |V_1(\tau) - V_2(\tau)| + \|\nabla u_1(\tau, \cdot)\|_{L^\infty} |X_1(\tau) - X_2(\tau)| + \|u_1(\tau, \cdot) - u_2(\tau, \cdot)\|_{L^\infty} \, d\tau \\ & \leq C(Q_1, E_1) \|h_1\|_{L^\infty((s,t) \times \mathbb{R}^3 \times \mathbb{R}^3)} \int_s^t |(X_1, V_1)(\tau) - (X_2, V_2)(\tau)| \, d\tau \\ & \quad + C(Q_1, E_1)(t - s) \|h_1 - h_2\|_{L^\infty((s,t) \times \mathbb{R}^3 \times \mathbb{R}^3)}, \end{aligned}$$

where we used Lemma 7.2.1. Grönwall's lemma implies

$$|(X_1, V_1)(t) - (X_2, V_2)(t)| \leq C(Q_1, E_1) t \|h_1 - h_2\|_{L^\infty(\Omega_t)} \exp(C(Q_1, E_1) \|h_1\|_{L^\infty(\Omega_t)} t).$$

Inserting this in (7.2.19) yields

$$\begin{aligned} & \|f_1 - f_2\|_{L^\infty(\Omega_T)} \\ & \leq T e^{3T} C(Q_1, E_1) \|\nabla f_0\|_{L^\infty(\mathbb{R}^3 \times \mathbb{R}^3)} \|h_1 - h_2\|_{L^\infty(\Omega_T)} \exp(C(Q_1, E_1)T) \|h_1\|_{L^\infty(\Omega_T)} \end{aligned} \quad (7.2.20)$$

For $L > 0$, consider $B_L(0) \subset Y$. Then, for all L , equation (7.2.20) implies that there exists $T > 0$ such that the mapping $h \mapsto f$ is contractive. We have to check that $h \in B_L(0)$ implies $f \in B_L(0)$. First,

$$\|f(t, \cdot, \cdot)\|_{L^1(\mathbb{R}^3)} = \|f_0\|_{L^1(\mathbb{R}^3)} \quad (7.2.21)$$

follows from the equation. Moreover, for any $L > \|f_0\|_{L^\infty(\mathbb{R}^3 \times \mathbb{R}^3)}$, equation (7.2.18) implies that we can choose T sufficiently small such that

$$\|f\|_{L^\infty(\Omega_T)} = \|f_0\|_{L^\infty(\mathbb{R}^3 \times \mathbb{R}^3)} e^{3\lambda T} \leq L.$$

Furthermore, we have

$$\begin{aligned} \frac{d}{dt} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} |v|^2 f \, dx \, dv &= 2 \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} v \cdot (g + u - v) f \, dx \, dv \\ &\leq 2(|g| + \|u\|_{L^\infty(\mathbb{R}^3)}) \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} (1 + |v|^2) f \, dx \, dv. \end{aligned}$$

Hence, using mass conservation, equation (7.2.21),

$$\frac{d}{dt} \int_{\mathbb{R}^3 \times \mathbb{R}^3} (1 + |v|^2) f \, dx \, dv \leq (|g| + \|u\|_{L^\infty(\mathbb{R}^3)}) \int_{\mathbb{R}^3 \times \mathbb{R}^3} (1 + |v|^2) f \, dx \, dv.$$

Therefore, Lemma 7.2.1 and Grönwall's lemma imply

$$\int_{\mathbb{R}^3 \times \mathbb{R}^3} (1 + |v|^2) f \, dx \, dv \leq \int_{\mathbb{R}^3 \times \mathbb{R}^3} (1 + |v|^2) f_0 \, dv \, dx \exp(C(Q_1, E_1)Lt).$$

Thus, for any $E_1 > \int_{\mathbb{R}^3 \times \mathbb{R}^3} (1 + |v|^2) f_0 \, dv \, dx$, we can choose T small enough such that $\int_{\mathbb{R}^3 \times \mathbb{R}^3} (1 + |v|^2) f \, dx \, dv \leq E_1$ for all $t \leq T$.

Finally, we need to control the support of f . To do this, we follow the same argument as in the last part of the proof of Lemma 7.2.2 to get

$$Q(t) \leq Q_0 + (1+t) \int_0^t C(L, E_1, Q_1) \, ds \leq Q_0 + (1+t)tC(L, E_1, Q_1).$$

Again, for any $Q_1 > Q_0$, we can choose T small enough such that $Q(t) \leq Q_1$ for all $t \leq T$.

Therefore, by the Banach fixed point theorem, we get local in time existence of solutions to (7.1.1). Moreover, choosing for example $L = 2\|f_0\|_{L^\infty(\mathbb{R}^3 \times \mathbb{R}^3)}$, $E_1 = 2E_0$, $Q_1 = Q_0 + 1$, the time T for which we get existence in this way is a continuous and monotonically decreasing function of $\|f_0\|_{L^\infty(\mathbb{R}^3 \times \mathbb{R}^3)}$, Q_0 and E_0 . Thus, by a standard contradiction argument, global existence follows from the a priori estimates in Lemma 7.2.2 since these ensure that $\|f(t, \cdot)\|_{L^\infty(\mathbb{R}^3 \times \mathbb{R}^3)}$, Q and E do not blow up in finite time.

Since $f \in W^{1,\infty}(\Omega_T)$ with uniform compact support, higher regularity of u follows from taking derivatives in the Brinkman equations in (7.1.1) and using regularity theory for Stokes equations similarly to the proof of Lemma 7.2.1. \square

7.3 Uniform estimates on ρ_λ and u_λ

In the following, we assume that (f, u) is the solution to the Vlasov-Stokes equations (7.1.1) for some $\lambda > 0$ and some compactly supported initial datum $f_0 \in W^{1,\infty}(\mathbb{R}^3 \times \mathbb{R}^3)$. In this section we want to derive a priori estimates for these solutions that do not depend on λ . This is why we cannot use the a priori estimates derived in Lemma 7.2.2. However, the drawback of the estimates that we prove in this section is that they allow for blow-up in finite time. This is also why they are not suitable in the proof of global well-posedness, that we showed in the previous section. Later, we will use the limit equation in order to show that the estimates derived here allow for uniform estimates for arbitrary times.

Again, we denote by C any constant, which only depends on f_0 and may change from line to line.

7.3.1 Estimates for the fluid velocity

In this subsection we show that the fluid velocity as well as the particle velocity is controlled by $\|\rho\|_{L^\infty(\mathbb{R}^3)}$, uniformly in λ , which means that high velocities can only occur if particles concentrate in position space. This also implies control on the particle positions and velocities

The proof is based on the energy identity from Lemma 7.2.2, equation (7.2.12), and the subsequent estimate (7.2.13). The idea is to estimate the sum of the quadratic terms in that expression, which have a negative sign, by $E(t)$ from below. The following Lemma, which is a general observation on weighted L^2 -spaces, shows why such an estimate is true if $\|\rho\|_{L^{3/2}(\mathbb{R}^3)}$ is not too large.

Lemma 7.3.1. *There exists a constant c_0 , such that for all nonnegative functions $\sigma \in L^{3/2}(\mathbb{R}^3)$, and all $h \in L^2(\sigma)$ and $w \in H^1(\mathbb{R}^3)$,*

$$\|\nabla w\|_{L^2(\mathbb{R}^3)}^2 + \|w - h\|_{L_\sigma^2(\mathbb{R}^3)}^2 \geq c_0 \min\{\|\sigma\|_{L^{3/2}(\mathbb{R}^3)}^{-1}, 1\} \|h\|_{L_\sigma^2(\mathbb{R}^3)}^2.$$

Proof. We estimate using the critical Sobolev inequality

$$\|w\|_{L_\sigma^2(\mathbb{R}^3)}^2 \leq \|w\|_{L^6(\mathbb{R}^3)}^2 \|\sigma\|_{L^{3/2}(\mathbb{R}^3)} \leq C \|\nabla w\|_{L^2(\mathbb{R}^3)}^2 \|\sigma\|_{L^{3/2}(\mathbb{R}^3)}. \quad (7.3.1)$$

We have for any $\theta > 0$ and any $a, b \in H$ for some Hilbert space H

$$\|a - b\|^2 = \|a\|^2 + \|b\|^2 - 2(a, b) \geq (1 - \theta)\|a\|^2 + (1 - \frac{1}{\theta})\|b\|^2.$$

Applying this with $1 - \theta := -C^{-1}\|\sigma\|_{L^{3/2}(\mathbb{R}^3)}^{-1}$, where C is the constant from equation (7.3.1), we find

$$\|\nabla w\|_{L^2(\mathbb{R}^3)}^2 + \|w - h\|_{L_\sigma^2(\mathbb{R}^3)}^2 \geq \frac{\theta - 1}{\theta} \|h\|_{L_\sigma^2(\mathbb{R}^3)}^2.$$

To conclude, we notice that

$$\frac{\theta - 1}{\theta} = \frac{C^{-1}\|\sigma\|_{L^{3/2}(\mathbb{R}^3)}^{-1}}{1 + C^{-1}\|\sigma\|_{L^{3/2}(\mathbb{R}^3)}^{-1}} \geq c_0 \min\{\|\sigma\|_{L^{3/2}(\mathbb{R}^3)}^{-1}, 1\}.$$

□

Lemma 7.3.2. *There exists a constant C that depends only on f_0 such that for all $\lambda > 0$ and all $t > 0$, we have*

$$E(t) \leq C \sup_{s \leq t} \|\rho\|_{L^\infty(\mathbb{R}^3)}^{\frac{2}{3}}, \quad (7.3.2)$$

$$\|u(t, \cdot)\|_{L^\infty(\mathbb{R}^3)} \leq C \sup_{s \leq t} \|\rho(s, \cdot)\|_{L^\infty(\mathbb{R}^3)}, \quad (7.3.3)$$

$$\|\nabla u(t, \cdot)\|_{L^\infty(\mathbb{R}^3)} \leq C \sup_{s \leq t} \|\rho(s, \cdot)\|_{L^\infty(\mathbb{R}^3)}^2, \quad (7.3.4)$$

$$\|\bar{V}(t, \cdot)\|_{L^\infty(\mathbb{R}^3)} \leq C \sup_{s \leq t} \|\rho(s, \cdot)\|_{L^\infty(\mathbb{R}^3)}, \quad (7.3.5)$$

where \bar{V} is the average particle velocity defined in (7.1.2).

Moreover, for all $(x, v) \in \text{supp } f_0$,

$$|V(t, 0, x, v)| \leq C \sup_{s \leq t} \|\rho(s, \cdot)\|_{L^\infty(\mathbb{R}^3)}, \quad (7.3.6)$$

$$|X(t, 0, x, v)| \leq Ct \sup_{s \leq t} \|\rho(s, \cdot)\|_{L^\infty(\mathbb{R}^3)}. \quad (7.3.7)$$

Proof. For the sake of a leaner notation, we will again often omit the dependence of t of the norms.

By the energy estimate (7.2.13) from Lemma 7.2.2 and Lemma 7.3.1, we have for the energy of the particles

$$\begin{aligned} \frac{d}{dt} E &\leq 2\lambda \left(CE^{\frac{1}{2}} - \int_{\mathbb{R}^3 \times \mathbb{R}^3} (v - \bar{V})^2 f \, dx \, dv - \|u - \bar{V}\|_{L_\rho^2(\mathbb{R}^3)}^2 - \|\nabla u\|_{L^2(\mathbb{R}^3)}^2 \right) \\ &\leq 2\lambda \left(CE^{\frac{1}{2}} - \int_{\mathbb{R}^3 \times \mathbb{R}^3} (v - \bar{V})^2 f \, dx \, dv - c_0 \min\{\|\rho\|_{L^{3/2}(\mathbb{R}^3)}^{-1}, 1\} \|\bar{V}\|_{L_\rho^2(\mathbb{R}^3)}^2 \right) \\ &\leq 2\lambda \left(CE^{\frac{1}{2}} - c_0 \min\{\|\rho\|_{L^{3/2}(\mathbb{R}^3)}^{-1}, 1\} E \right). \end{aligned}$$

A comparison principle for ODEs implies

$$\begin{aligned} E^{\frac{1}{2}}(t) &\leq E(0)^{\frac{1}{2}} e^{-2\lambda t} + \frac{C}{c_0} \sup_{s \leq t} \max\{\|\rho(s, \cdot)\|_{L^{3/2}(\mathbb{R}^3)}, 1\} \\ &\leq C \sup_{s \leq t} \|\rho(s, \cdot)\|_{L^{3/2}(\mathbb{R}^3)} \leq C \sup_{s \leq t} \|\rho(s, \cdot)\|_{L^\infty(\mathbb{R}^3)}^{\frac{1}{3}}, \end{aligned} \quad (7.3.8)$$

where we used that the L^1 -norm of ρ is constant in time by (7.2.11). Note that here and in the following we also use that C might depend on f_0 in order to get rid of lower order terms (using that if $f_0 = 0$, the solution f is also trivial). This proves (7.3.2).

Recall from (7.2.1) that $\|\bar{V}\|_{L_\rho^2(\mathbb{R}^3)} \leq E^{\frac{1}{2}}$. Thus, (7.3.8) yields

$$\|\bar{V}(t, \cdot)\|_{L_\rho^2(\mathbb{R}^3)} \leq E^{\frac{1}{2}}(t) \leq C \sup_{s \leq t} \|\rho(s, \cdot)\|_{L^\infty(\mathbb{R}^3)}^{\frac{1}{3}}. \quad (7.3.9)$$

Using regularity theory for the Stokes equations (see [Gal11]) together with (7.2.1) and (7.3.9) yields

$$\begin{aligned} \|\nabla^2 u\|_{L^2(\mathbb{R}^3)} &\leq C \|\rho u\|_{L^2(\mathbb{R}^3)} + C \|\rho \bar{V}\|_{L^2(\mathbb{R}^3)} \leq C \|u\|_{L^6(\mathbb{R}^3)} \|\rho\|_{L^3(\mathbb{R}^3)} + C \|\rho \bar{V}\|_{L^2(\mathbb{R}^3)} \\ &\leq C \|\bar{V}\|_{L_\rho^2(\mathbb{R}^3)} \|\rho\|_{L^\infty(\mathbb{R}^3)}^{\frac{2}{3}} + \|\bar{V}\|_{L_\rho^2(\mathbb{R}^3)} \|\rho\|_{L^\infty(\mathbb{R}^3)}^{\frac{1}{2}} \\ &\leq C \|\bar{V}\|_{L_\rho^2(\mathbb{R}^3)} \|\rho\|_{L^\infty(\mathbb{R}^3)}^{\frac{2}{3}} \leq C \sup_{s \leq t} \|\rho(s, \cdot)\|_{L^\infty(\mathbb{R}^3)}. \end{aligned}$$

Sobolev inequality and (7.2.1) yield

$$\|u\|_{L^\infty(\mathbb{R}^3)} \leq \|u\|_{C^{0,\frac{1}{2}}(\mathbb{R}^3)} \leq C\|u\|_{W^{1,6}(\mathbb{R}^3)} \leq C\|\nabla u\|_{W^{1,2}(\mathbb{R}^3)} \leq C \sup_{s \leq t} \|\rho(s, \cdot)\|_{L^\infty(\mathbb{R}^3)}.$$

This proves (7.3.3).

Using the characteristic equations (7.1.6), we find for all $(x, v) \in \text{supp } f_0$

$$|V(t, 0, x, v)| \leq e^{-\lambda t}|v| + |g| + \sup_{s \leq t} \|u(s, \cdot)\|_{L^\infty(\mathbb{R}^3)} \leq C \sup_{s \leq t} \|\rho(s, \cdot)\|_{L^\infty(\mathbb{R}^3)},$$

with a constant C that depends on the support of f_0 . This proves (7.3.6). Moreover, using the equation for X , (7.3.6) implies (7.3.7).

Furthermore, by (7.3.6)

$$\|\bar{V}(t, \cdot)\|_{L^\infty(\mathbb{R}^3)} \leq C \sup_{s \leq t} \|\rho(s, \cdot)\|_{L^\infty(\mathbb{R}^3)},$$

which proves (7.3.5). This can be used again to derive a bound for $\nabla^2 u$ in $L^p(\mathbb{R}^3)$ to get (7.3.4). More precisely,

$$\begin{aligned} \|\nabla^2 u\|_{L^6(\mathbb{R}^3)} &\leq \|u\|_{L^6(\mathbb{R}^3)} \|\rho\|_{L^\infty(\mathbb{R}^3)} + \|\rho \bar{V}\|_{L^6(\mathbb{R}^3)} \\ &\leq C \|\bar{V}\|_{L^2_\rho(\mathbb{R}^3)} \|\rho\|_{L^\infty(\mathbb{R}^3)} + \|\bar{V}\|_{L^2_\rho(\mathbb{R}^3)}^{\frac{1}{3}} \|\bar{V}\|_{L^\infty(\mathbb{R}^3)}^{\frac{2}{3}} \|\rho\|_{L^\infty(\mathbb{R}^3)}^{\frac{5}{6}} \\ &\leq C \sup_{s \leq t} \|\rho(s, \cdot)\|_{L^\infty(\mathbb{R}^3)}^2. \end{aligned}$$

Thus,

$$\|\nabla u(t, \cdot)\|_{L^\infty(\mathbb{R}^3)} \leq C \sup_{s \leq t} \|\rho(s, \cdot)\|_{L^\infty(\mathbb{R}^3)}^2.$$

□

7.3.2 Estimates for the particle density

In this subsection we prove estimates on ρ that are uniform in λ for λ sufficiently large. However, these estimates will depend on u . Then, we will combine these estimates with the ones from Lemma 7.3.2 in order to get estimates on ρ independent of λ and u but only for small times.

We first prove a small lemma on estimates for ODEs that will be used several times analyzing the characteristics.

Lemma 7.3.3. *Let $T > 0$ and $a, b : [0, T] \rightarrow \mathbb{R}_+$ be Lipschitz continuous. Let $\alpha : [0, T] \rightarrow \mathbb{R}_+$ be continuous and $\lambda \geq 4 \max\{1, \|\alpha\|_{L^\infty(0, T)}\}$. Let $\beta \geq 0$ be some constant and assume that on $(0, T)$*

$$\begin{aligned} |\dot{a}| &\leq b, \\ \dot{b} &\leq \lambda(\alpha a - b) + \beta e^{-\lambda s}. \end{aligned}$$

(i) *If $a(T) = 0$, then for all $s, t \in [0, T]$ with $s \leq t$*

$$a(t) \leq \frac{2}{\lambda} b(t) + \frac{4}{\lambda^2} \beta e^{-\lambda t}, \tag{7.3.10}$$

$$b(t) \leq \exp\left(\int_s^t -\lambda + 2\alpha(\tau) d\tau\right) \left(b(s) + \frac{2\beta}{\lambda} e^{-\lambda s}\right). \tag{7.3.11}$$

(ii) If $\beta = 0$ and $b(0) = 0$, then for all $t \in [0, T]$

$$b(t) \leq 2\|\alpha\|_{L^\infty(0,T)}a.$$

Proof. We define

$$z(s) := b(s) - \frac{\lambda}{2}a(s) + \frac{2}{\lambda}\beta e^{-\lambda s}.$$

Then, if $a(T) = 0$,

$$z(T) = b(T) + \frac{2}{\lambda}\beta e^{-\lambda T} \geq 0,$$

and

$$\dot{z} \leq \lambda \left(\alpha a - \frac{b}{2} \right) - \beta e^{-\lambda s} = \lambda \left(\alpha a - \frac{\lambda}{4}a - \frac{z}{2} + \frac{\beta}{\lambda}e^{-\lambda s} \right) + \beta e^{-\lambda s} \leq -\frac{\lambda}{2}z.$$

Hence, (applying Grönwall's lemma to $-z(T-t)$) we find $z \geq 0$ in $[0, T]$. This proves (7.3.10). Moreover, (7.3.10) implies

$$\dot{b} \leq (2\alpha - \lambda)b + \left(1 + \frac{4}{\lambda}\right)\beta e^{-\lambda s} \leq (2\alpha - \lambda)b + 2\beta e^{-\lambda s}.$$

Thus, using the comparison principle for ODEs yields (7.3.11).

In order to prove (ii), we define $z := 2\|\alpha\|_{L^\infty(0,T)}a - b$. Then, $b(0) = 0$ implies $z(0) \geq 0$. Using the equations for a and b , one obtains $\dot{z} \geq -(\lambda/2)z$ as in the proof of part (i). This implies $z \geq 0$ in $[0, T]$, and the assertion follows. \square

Using the previous Lemma, we are able to prove that the particle velocities concentrate in regions of size $e^{-\lambda t}$ with an error due to fluctuations of the fluid velocity. Based on this result and equation (7.1.7), we also prove an estimate for ρ .

Lemma 7.3.4. *Let $T > 0$ and assume $\lambda \geq 4(1 + \|\nabla u\|_{L^\infty((0,T) \times \mathbb{R}^3)})$. Then, for all $t < T$ and all $x \in \mathbb{R}^3$, the map*

$$v \mapsto V(0, t, x, v)$$

is bi-Lipschitz. In particular its inverse $W(t, x, w)$ is well defined, and

$$\rho(t, x) = \int_{\mathbb{R}^3} e^{3\lambda t} f_0(X(0, t, x, W(t, x, w)), w) \det \nabla_w W(t, x, w) dw. \quad (7.3.12)$$

Moreover, denoting

$$M(t) := \exp \left(\int_0^t 2\|\nabla u(s, \cdot)\|_{L^\infty(\mathbb{R}^3)} ds \right), \quad (7.3.13)$$

we have

$$|\nabla_v V(0, t, x, v)| \leq M(t)e^{\lambda t}, \quad (7.3.14)$$

$$|\nabla_w W(t, x, w)| \leq M(t)e^{-\lambda t}, \quad (7.3.15)$$

$$0 \leq \det \nabla_w W(t, x, w) \leq M(t)^3 e^{-3\lambda t}. \quad (7.3.16)$$

Furthermore,

$$\|\rho(t, \cdot)\|_{L^\infty(\mathbb{R}^3)} \leq C_0 M(t)^3, \quad (7.3.17)$$

where the constant depends only on f_0 .

Proof. We fix t, x, v_1 , and v_2 and define

$$\begin{aligned} a(s) &= |X(s, t, x, v_1) - X(s, t, x, v_2)|, \\ b(s) &= |V(s, t, x, v_1) - V(s, t, x, v_2)|. \end{aligned}$$

Then,

$$\begin{aligned} |\dot{a}| &\leq b, & a(t) &= 0, \\ \dot{b} &\leq \lambda(\|\nabla u(s, \cdot)\|_{L^\infty(\mathbb{R}^3)} a - b), & b(t) &= |v_1 - v_2|. \end{aligned}$$

Then, with $\alpha(s) := \|\nabla u(s, \cdot)\|_{L^\infty(\mathbb{R}^3)}$ and $\beta = 0$, we can apply Lemma 7.3.3(i) to deduce

$$b(t) \leq b(0)M(t)e^{-\lambda t},$$

which implies

$$b(0) \geq M(t)^{-1}e^{\lambda t}|v_1 - v_2|. \quad (7.3.18)$$

Note that the first inequality in (7.3.10) also implies

$$a(t) \leq \frac{2}{\lambda}b(t).$$

Hence,

$$\dot{b} \geq \lambda(-\|\nabla u(s, \cdot)\|_{L^\infty(\mathbb{R}^3)} a - b) \geq (-\lambda - 2\|\nabla u(s, \cdot)\|_{L^\infty(\mathbb{R}^3)})b.$$

Thus

$$b(0) \leq e^{\lambda t}M(t)|v_1 - v_2|. \quad (7.3.19)$$

Estimates (7.3.18) and (7.3.19) imply that the map $v \mapsto V(0, t, x, v)$ is bi-Lipschitz and yield the bounds (7.3.14), (7.3.15), and (7.3.16). The Jacobian of W is positive since $W(0, x, v) = w$ and the Jacobian is continuous in t , which follows from the definition of V and regularity of u proven in Proposition 7.2.3.

Moreover, recalling (7.1.7), these estimates imply

$$\begin{aligned} \rho(t, x) &= \int_{\mathbb{R}^3} f(t, x, v) dv = \int_{\mathbb{R}^3} e^{3\lambda t} f_0(X(0, t, x, v), V(0, t, x, v)) dv \\ &= \int_{\mathbb{R}^3} e^{3\lambda t} f_0(X(0, t, x, W(t, x, w)), w) \det \nabla_w W(t, x, w) dw \\ &\leq \int_{\mathbb{R}^3} M(t)^3 f_0(X(0, t, x, W(t, x, w)), w) dw \\ &\leq C_0 M(t)^3, \end{aligned}$$

which finishes the proof. \square

We define

$$T_* := \sup \left\{ t \geq 0 : \limsup_{\lambda \rightarrow \infty} \|\rho_\lambda\|_{L^\infty((0, t) \times \mathbb{R}^3)} < \infty \right\} \quad (7.3.20)$$

In the lemma below, we prove that $T_* > 0$. Later we will show the convergence to the limit equation (7.1.3) first only up to times $T < T_*$ and finally, we will show $T_* = \infty$ using the convergence result for times $T < T_*$.

Lemma 7.3.5. *Let T_* be defined as in (7.3.20). Then,*

$$T_* > 0.$$

Proof. By Lemma 7.3.2, we have for all $t > 0$

$$\|\nabla u_\lambda\|_{L^\infty((0,t)\times\mathbb{R}^3)} \leq C \sup_{s \leq t} \|\rho_\lambda(s, \cdot)\|_{L^\infty(\mathbb{R}^3)}^2.$$

Moreover, by Lemma 7.3.4, if $\lambda \geq 4(\|\nabla u_\lambda\|_{L^\infty(0,t)\times\mathbb{R}^3} + 1)$, then

$$\sup_{s \leq t} \|\rho_\lambda(s, \cdot)\|_{L^\infty(\mathbb{R}^3)}^2 \leq C_0 \exp\left(2 \int_0^t \|\nabla u_\lambda(s, \cdot)\|_{L^\infty(\mathbb{R}^3)} ds\right).$$

Combining these two estimates, we see that $\lambda \geq C \sup_{s \leq t} \|\rho_\lambda(s, \cdot)\|_{L^\infty(\mathbb{R}^3)}^2$ implies

$$\sup_{s \leq t} \|\rho_\lambda(s, \cdot)\|_{L^\infty(\mathbb{R}^3)} \leq C_0 \exp\left(Ct \sup_{s \leq t} \|\rho_\lambda(s, \cdot)\|_{L^\infty(\mathbb{R}^3)}^2\right). \quad (7.3.21)$$

We define

$$T_\lambda := \sup\{t \geq 0: \sup_{s \leq t} \|\rho_\lambda(s, \cdot)\|_{L^\infty(\mathbb{R}^3)} \leq 2C_0\}.$$

Then, $T_\lambda > 0$ as ρ_λ is continuous (and C_0 is chosen such that $\|\rho(0, \cdot)\|_{L^\infty(\mathbb{R}^3)} \leq C_0$). Moreover, (7.3.21) implies for all $\lambda \geq 4(CC_0^2 + 1)$ and all $t < T_\lambda$

$$\sup_{s \leq t} \|\rho_\lambda(s, \cdot)\|_{L^\infty(\mathbb{R}^3)} \leq C_0 \exp(CC_0^2 t).$$

As ρ_λ is continuous, this yields for all $\lambda \geq 4(CC_0^2 + 1)$

$$T_\lambda \geq \frac{\log(2)}{CC_0^2},$$

which is independent of λ . Thus,

$$T_* \geq \inf_{\lambda \geq 4(CC_0^2 + 1)} T_\lambda > 0.$$

□

7.3.3 Higher order estimates

In this subsection, we prove estimates on $\partial_t \rho$ and $\nabla \rho$ which are uniform in λ for times $T < T_*$. On the one hand, this yields compactness of ρ_λ in Hölder spaces. On the other hand, we will also need these estimates in order to prove that the functions \tilde{u}_λ defined in (7.1.10) are close to u_λ for large values of λ .

From now on, any constant C might depend on T but not on λ . In particular, for $T < T_*$, C might depend on $\limsup_{\lambda \rightarrow \infty} \|\rho_\lambda\|_{L^\infty((0,T)\times\mathbb{R}^3)}$.

Lemma 7.3.6. *Let $T < T_*$. Then, there exist λ_0 and C depending on T and f_0 such that for all $\lambda \geq \lambda_0$ and all multiindices $\beta \in \mathbb{N}^3$,*

$$\|\rho\|_{W^{1,\infty}((0,T)\times\mathbb{R}^3)} \leq C, \quad (7.3.22)$$

$$\|u\|_{L^\infty((0,T_0);W^{2,\infty}(\mathbb{R}^3))} \leq C, \quad (7.3.23)$$

$$\|\bar{V}\|_{L^\infty((0,T_0)\times\mathbb{R}^3)} \leq C, \quad (7.3.24)$$

$$\left\| \nabla_x \int_{\mathbb{R}^3} v^\beta f dv \right\|_{L^\infty((0,T_0)\times\mathbb{R}^3)} \leq C. \quad (7.3.25)$$

Moreover, the support of f is uniformly bounded in λ up to time T .

Proof. By definition of T_* , there is some λ_0 such that for all $\lambda \geq \lambda_0$

$$\|\rho\|_{L^\infty((0,T)\times\mathbb{R}^3)} \leq C. \quad (7.3.26)$$

Thus, Lemma 7.3.2 yields (7.3.24) and

$$\|u\|_{L^\infty((0,T_0);W^{1,\infty}(\mathbb{R}^3))} \leq C. \quad (7.3.27)$$

Using this, we have $M(t) \leq C$ for all $t \leq T$, where M is the quantity from (7.3.13) in Lemma 7.3.4. Moreover, we can assume that λ_0 has been chosen such that for all $\lambda \geq \lambda_0$

$$\lambda \geq 4(1 + \|\nabla u\|_{L^\infty((0,T)\times\mathbb{R}^3)}). \quad (7.3.28)$$

In the following, we only consider $\lambda \geq \lambda_0$.

By Lemma 7.3.4, $V(0, t, x, v)$ is invertible with inverse $W(t, x, v)$, and we define

$$\begin{aligned} Y(s, t, x, w) &:= X(s, t, x, W(t, x, w)), \\ U(s, t, x, w) &:= V(s, t, x, W(t, x, w)). \end{aligned} \quad (7.3.29)$$

Then,

$$\begin{aligned} \partial_s Y &= U, & Y(t, t, x, w) &= x, \\ \partial_s U &= \lambda(g + u(Y, s) - U), & U(0, t, x, w) &= w, & U(t, t, x, w) &= W(t, x, w). \end{aligned}$$

Note that by (7.3.12)

$$\int_{\mathbb{R}^3} f(t, x, v) dv = e^{3\lambda t} \int_{\mathbb{R}^3} f_0(Y, w) \det \nabla_w W dw.$$

We compute

$$\partial_{x_i} \det \nabla_w W = \text{tr}(\text{adj } \nabla_w W \nabla_w \partial_{x_i} W) = \det \nabla_w W \text{tr}((\nabla_w W)^{-1} \nabla_w \partial_{x_i} W).$$

Thus, for any multiindex β ,

$$\begin{aligned} \partial_{x_i} \int_{\mathbb{R}^3} v^\beta f dv &= e^{3\lambda t} \int_{\mathbb{R}^3} \partial_{x_i} (W^\beta) f_0(Y, w) \det \nabla_w W dw \\ &\quad + e^{3\lambda t} \int_{\mathbb{R}^3} W^\beta \nabla_x f_0(Y, w) \cdot \partial_{x_i} Y \det \nabla_w W dw \\ &\quad + e^{3\lambda t} \int_{\mathbb{R}^3} W^\beta f_0(Y, w) \det \nabla_w W \text{tr}((\nabla_w W)^{-1} \nabla_w \partial_{x_i} W) dw \\ &=: A_1 + A_2 + A_3. \end{aligned} \quad (7.3.30)$$

We notice that

$$\begin{aligned} W(t, x, w) &= V(t, 0, X(0, t, x, W(t, x, w)), V(0, t, x, W(t, x, w))) \\ &= V(t, 0, Y(0, t, x, w), w). \end{aligned}$$

Hence, for all $(Y(0, t, x, w), w) \in \text{supp } f_0$, estimate (7.3.6) implies

$$|W(t, x, w)| \leq C. \quad (7.3.31)$$

Integrating the equation for U yields (analogously to (7.1.8))

$$Y(s, t, x, w) = x - \int_s^t g + u(\tau, Y) d\tau + \lambda^{-1}(U(t, t, x, w) - U(s, t, x, w)).$$

Therefore,

$$\nabla_x Y(s, t, x, w) = \text{Id} - \int_s^t \nabla u(\tau, Y) \nabla_x Y d\tau + \lambda^{-1}(\nabla_x U(t, t, x, w) - \nabla_x U(s, t, x, w)). \quad (7.3.32)$$

We claim that

$$|\nabla_x U(s, t, x, w)| \leq 2\|\nabla u\|_{L^\infty((0, T_0) \times \mathbb{R}^3)} |\nabla_x Y(s, t, x, w)|. \quad (7.3.33)$$

Indeed, with

$$\begin{aligned} a(s) &:= |\nabla_x Y(s, t, x, w)|, \\ b(s) &:= |\nabla_x \partial_s Y(s, t, x, w)|, \\ \alpha(s) &:= \|\nabla u(s, \cdot)\|_{L^\infty(\mathbb{R}^3)}, \end{aligned}$$

this follows from Lemma 7.3.3(ii) using (7.3.28).

We use estimate (7.3.33) in equation (7.3.32) to get

$$a(s) \leq 1 + \int_s^t \alpha(\tau) a(\tau) d\tau + \frac{2\|\alpha\|_{L^\infty(0, T)}}{\lambda} (a(t) + a(s)).$$

Since $a(t) = 0$ and equation (7.3.28) implies $4\|\alpha\|_{L^\infty(0, T)} \leq \lambda$, we have

$$a(s) \leq 2 + 2 \int_s^t \alpha(\tau) a(\tau) d\tau.$$

Therefore, (7.3.27) implies for all $0 \leq s \leq t \leq T$

$$|\nabla_x Y(s, t, x, w)| = a(s) \leq C. \quad (7.3.34)$$

Moreover, by (7.3.33), (7.3.27), and (7.3.34)

$$|\nabla_x W(t, x, w)| = |\nabla_x U(t, t, x, w)| \leq C. \quad (7.3.35)$$

We want to estimate $\nabla_x \det \nabla_w W$. We compute

$$\partial_{x_i} \det \nabla_w W = \text{tr}(\text{adj } \nabla_w W \nabla_w \partial_{x_i} W) = \det \nabla_w W \text{tr}((\nabla_w W)^{-1} \nabla_w \partial_{x_i} W).$$

By (7.3.14), we have

$$|(\nabla_w W(t, x, w))^{-1}| = |(\nabla_v V(0, t, x, W(0, t, x, w)))| \leq C e^{\lambda t}.$$

Thus, using also (7.3.16), we find

$$|\partial_{x_i} \det \nabla_w W| \leq \det \nabla_w W |(\nabla_w W)^{-1}| |\nabla_w \partial_{x_i} W| \leq C e^{-3\lambda t} e^{\lambda t} |\nabla_w \partial_{x_i} W|. \quad (7.3.36)$$

In order to estimate $|\nabla_w \partial_{x_i} W|$ we further analyze the characteristics Y and U defined in (7.3.29). Fix t , x , and w and denote

$$\begin{aligned} a(s) &:= |\nabla_w Y(s, t, x, w)|, \\ b(s) &:= |\nabla_w U(s, t, x, w)|, \\ \alpha(s) &:= \|\nabla u(s, \cdot)\|_{L^\infty(\mathbb{R}^3)}. \end{aligned}$$

Then, the assumptions of Lemma 7.3.3(i) are satisfied with $\beta = 0$.

Thus,

$$b(t) \leq \exp \left(\int_0^t -\lambda + 2\|\nabla u(s, \cdot)\|_{L^\infty(\mathbb{R}^3)} ds \right),$$

and

$$|\nabla_w Y(s, t, x, w)| = a(s) \leq \frac{2}{\lambda} b(s) \leq \frac{C}{\lambda} e^{-\lambda s}. \quad (7.3.37)$$

Next, we consider the second derivative. We denote

$$\begin{aligned} a(s) &:= |\nabla_w \nabla_x Y(s, t, x, w)|, \\ b(s) &:= |\nabla_w \nabla_x U(s, t, x, w)|, \\ \alpha(s) &:= \|\nabla u(s, \cdot)\|_{L^\infty(\mathbb{R}^3)}, \\ \beta &:= 4M(T)^3 \|\nabla^2 u\|_{L^\infty([0, T] \times \mathbb{R}^3)}, \end{aligned}$$

with M as in (7.3.17). Then, using the estimates for $|\nabla_x Y|$ and $|\nabla_w Y|$ from (7.3.34) and (7.3.37) respectively,

$$\begin{aligned} \dot{a} &\geq -b, \\ \dot{b} &\leq \lambda(\|\nabla^2 u\|_\infty |\nabla_x Y| |\nabla_w Y| + \|\nabla u\|_\infty a - b) \leq \lambda(\alpha a - b) + \beta e^{-\lambda s}. \end{aligned}$$

Hence, the assumptions of Lemma 7.3.3(i) are satisfied. Since $b(0) = 0$, Lemma 7.3.3(i) yields

$$|\nabla_w \nabla_x W(0, t, x, w)| = b(t) \leq C \frac{2\beta}{\lambda} e^{-\lambda t} \leq \frac{C}{\lambda} e^{-\lambda t} \|\nabla^2 u\|_{L^\infty((0, T) \times \mathbb{R}^3)}.$$

Inserting this in (7.3.36), we find

$$|\partial_{x_i} \det \nabla_w W| \leq \frac{C}{\lambda} e^{-3\lambda t} \|\nabla^2 u\|_{L^\infty((0, T) \times \mathbb{R}^3)}. \quad (7.3.38)$$

We recall the definition of A_1 , A_2 , and A_3 from equation (7.3.30). Using (7.3.16) and (7.3.35) yields

$$A_1 \leq C(\beta).$$

Estimates (7.3.16), (7.3.31), and (7.3.34) imply

$$A_2 \leq C(\beta).$$

Finally, (7.3.31) and (7.3.38) yield

$$A_3 \leq \frac{C(\beta)}{\lambda} e^{-3\lambda t} \|\nabla^2 u\|_{L^\infty((0, T) \times \mathbb{R}^3)}.$$

Inserting these bounds on A_i in (7.3.30) we have.

$$\left\| \nabla_x \int_{\mathbb{R}^3} v^\beta f(t, x, v) dv \right\|_{L^\infty(\mathbb{R}^3)} \leq C(\beta) \left(1 + \frac{1}{\lambda} \|\nabla^2 u\|_{L^\infty((0, T) \times \mathbb{R}^3)} \right). \quad (7.3.39)$$

Since the support of f in x is controlled by (7.3.7), we also have for any $1 \leq p \leq \infty$

$$\left\| \nabla_x \int_{\mathbb{R}^3} v^\beta f(t, x, v) dv \right\|_{L^p(\mathbb{R}^3)} \leq C(\beta)(1 + T) \left(1 + \frac{1}{\lambda} \|\nabla^2 u\|_{L^\infty((0, T) \times \mathbb{R}^3)} \right). \quad (7.3.40)$$

In order to control $\|\nabla^2 u\|_{L^\infty((0,T)\times\mathbb{R}^3)}$, the Brinkman equations in (7.1.1) and regularity theory for the Stokes equations yield

$$\begin{aligned}\|\nabla^3 u\|_{L^p(\mathbb{R}^3)} &\leq \|\nabla(\rho(u - \bar{V}))\|_{L^p(\mathbb{R}^3)} \\ &\leq \|\rho\|_{L^p(\mathbb{R}^3)} \|\nabla u\|_{L^\infty(\mathbb{R}^3)} + \|\nabla \rho\|_{L^p(\mathbb{R}^3)} \|u\|_{L^\infty(\mathbb{R}^3)} + \|\nabla(\rho \bar{V})\|_{L^p(\mathbb{R}^3)}.\end{aligned}$$

Note that both $\nabla \rho$ and $\nabla(\rho \bar{V})$ are of the form of the left hand side in (7.3.40). Therefore, using also Sobolev embedding together with (7.3.27) and (7.3.26) yields

$$\|\nabla^2 u\|_{L^\infty(0,T;C^{2,\alpha})} \leq C \left(1 + \frac{1}{\lambda} \|\nabla^2 u\|_{L^\infty((0,T)\times\mathbb{R}^3)} \right).$$

This implies (7.3.23) for λ sufficiently large.

Inserting (7.3.23) in (7.3.39) proves (7.3.25). The missing estimate for the time-derivative in (7.3.22) follows from the Vlasov-Stokes equations (7.1.1) and (7.3.25). \square

Remark 7.3.7. *One might wonder, whether the complicated splitting in (7.3.30) is really needed. Indeed, we also have*

$$\begin{aligned}\partial_{x_i} \int_{\mathbb{R}^3} v^\beta f \, dv &= \partial_{x_i} e^{3\lambda t} \int_{\mathbb{R}^3} v^\beta f_0(X, V) \, dv \\ &= e^{3\lambda t} \int_{\mathbb{R}^3} v^\beta \nabla_x f_0(X, V) \partial_{x_i} X + \nabla_v f_0(X, V) \partial_{x_i} V \, dv,\end{aligned}$$

an expression that involves only two terms and in particular does not involve any second derivatives. However, it turns out, that both $\partial_{x_i} X$ and $\partial_{x_i} V$ blow up as $\lambda \rightarrow \infty$. Therefore, estimating both terms individually in the above expression cannot lead to the assertion.

7.4 Proof of the main result

7.4.1 Error estimates for the particle and fluid velocities

Recall the definition of \tilde{u}_λ from (7.1.10), which can be viewed as intermediate between u_λ and u_* defined by (7.1.1) and (7.1.3) respectively. As a first step to show smallness of $u_\lambda - u_*$ (and also $\rho_\lambda - \rho_*$), we will show smallness of $u_\lambda - \tilde{u}_\lambda$. Comparing the PDEs that u_λ and \tilde{u}_λ fulfill, we observe that we have to prove smallness of $\rho(\bar{V} - u_\lambda - g)$. This is almost what we do in the proof of the lemma below. Indeed, it turns out that it is more convenient to consider the error term $\Phi = \bar{V} - \tilde{u}_\lambda - g$ instead of $\bar{V} - u_\lambda - g$ because we control the time derivative of \tilde{u} . Then, we are able to prove smallness of $u_\lambda - \tilde{u}_\lambda$ using energy identities for Φ and $u_\lambda - \tilde{u}_\lambda$ analogous to (7.1.4) and (7.1.5).

Lemma 7.4.1. *Assume $T < T_*$ and let \tilde{u}_λ be defined as in (7.1.10). Then, there exist λ_0 such that for all $\lambda \geq \lambda_0$*

$$\|\tilde{u}\|_{W^{1,\infty}((0,T_0)\times\mathbb{R}^3)} \leq C, \tag{7.4.1}$$

$$\|\tilde{u}(t, \cdot) - u(t, \cdot)\|_{W^{1,\infty}(\mathbb{R}^3)}^2 \leq C \left(e^{-c\lambda t} + \frac{1}{\lambda} \right). \tag{7.4.2}$$

Proof. Again, we consider only $\lambda > \lambda_0$ with λ_0 as in Lemma 7.3.6. Then, Lemma 7.3.6 implies that we control the L^∞ -norms of ρ and $\partial_t \rho$ and the support of ρ . Thus, (7.4.1) follows from regularity theory for the Stokes equations.

We define

$$\begin{aligned}\Phi &:= \bar{V} - \tilde{u} - g, \\ Z &:= u - \tilde{u}.\end{aligned}$$

Then,

$$-\Delta Z + \nabla p + (Z - \Phi)\rho = 0, \quad \operatorname{div} Z = 0.$$

Therefore

$$\|\nabla Z\|_{L^2(\mathbb{R}^3)}^2 = (Z, \Phi - Z)_{L^2_\rho(\mathbb{R}^3)}. \quad (7.4.3)$$

We compute

$$\begin{aligned}\partial_t(\rho\bar{V}) &= - \int_{\mathbb{R}^3} v \cdot \nabla_x f v \, dv + \lambda\rho(g + u - \bar{V}) = - \int_{\mathbb{R}^3} v \cdot \nabla_x f v \, dv + \lambda\rho(Z - \Phi), \\ \partial_t(\rho\Phi) &= \partial_t(\rho\bar{V}) - \partial_t(\rho\tilde{u}) = \lambda\rho(Z - \Phi) - \int_{\mathbb{R}^3} v \cdot \nabla_x f v \, dv - \partial_t(\rho\tilde{u}).\end{aligned} \quad (7.4.4)$$

Note that (7.4.1) and the bound on \bar{V} from Lemma 7.3.6 imply that $\Phi(t, \cdot)$ is uniformly bounded in $L^\infty(\mathbb{R}^3)$ up to time T . Thus, we use (7.4.4), (7.4.3), and the estimates from Lemma 7.3.6, (7.4.1), and Lemma 7.3.1 to obtain

$$\begin{aligned}\partial_t \frac{1}{2} \|\Phi\|_{L^2(\rho)}^2 &= \int_{\mathbb{R}^3} \partial_t(\rho\Phi) \cdot \Phi - \frac{1}{2} \partial_t \rho |\Phi|^2 \, dx \\ &= \lambda \int_{\mathbb{R}^3} \rho\Phi \cdot (Z - \Phi) \, dx - \int_{\mathbb{R}^3 \times \mathbb{R}^3} v \cdot \nabla_x f v \cdot \Phi \, dv \, dx \\ &\quad - \int_{\mathbb{R}^3} \partial_t(\rho\tilde{u}) \cdot \Phi \, dx - \frac{1}{2} \int_{\mathbb{R}^3} \partial_t \rho |\Phi|^2 \, dx \\ &\leq -\lambda \|\nabla Z\|_{L^2(\mathbb{R}^3)}^2 - \lambda \|Z - \Phi\|_{L^2_\rho(\mathbb{R}^3)}^2 + C \\ &\leq -c\lambda \|\Phi\|_{L^2_\rho(\mathbb{R}^3)}^2 + C.\end{aligned}$$

Therefore, we have

$$\|\Phi\|_{L^2_\rho(\mathbb{R}^3)}^2 \leq C \left(e^{-c\lambda t} + \frac{1}{\lambda} \right).$$

By the energy identity for the Brinkman equations (7.4.3), it follows

$$\|\nabla Z\|_{L^2(\mathbb{R}^3)}^2 + \|Z\|_{L^2_\rho(\mathbb{R}^3)}^2 \leq C \left(e^{-c\lambda t} + \frac{1}{\lambda} \right).$$

Regularity theory for Stokes equations implies

$$\begin{aligned}\|\nabla^2 Z\|_{L^2(\mathbb{R}^3)}^2 &\leq 2\|\rho Z\|_{L^2(\mathbb{R}^3)}^2 + 2\|\rho\Phi\|_{L^2(\mathbb{R}^3)}^2 \\ &\leq 2\|\rho\|_{L^3(\mathbb{R}^3)}^2 \|Z\|_{L^6(\mathbb{R}^3)}^2 + 2\|\Phi\|_{L^2_\rho(\mathbb{R}^3)}^2 \|\rho\|_{L^\infty(\mathbb{R}^3)} \\ &\leq C \left(e^{-c\lambda t} + \frac{1}{\lambda} \right).\end{aligned}$$

Thus, using Sobolev embedding,

$$\|Z\|_{L^\infty(\mathbb{R}^3)}^2 \leq C \left(e^{-c\lambda t} + \frac{1}{\lambda} \right).$$

Taking $\lambda_0 \geq 1$ and using again (7.4.4) yields

$$\partial_t(\rho\Phi) \leq -\rho\Phi + C(\lambda e^{-c\lambda t} + \sqrt{\lambda}).$$

Thus,

$$\|\rho\Phi\|_{L^\infty(\mathbb{R}^3)}^2 \leq C \left(e^{-c\lambda t} + \frac{1}{\lambda} \right),$$

which again yields smallness of Z in even better norms. More precisely, for $p \geq 2$

$$\begin{aligned} \|\nabla^2 Z\|_{L^p(\mathbb{R}^3)}^2 &\leq C\|\rho Z\|_{L^p(\mathbb{R}^3)}^2 + C\|\rho\Phi\|_{L^p(\mathbb{R}^3)}^2 \\ &\leq C\|\rho\|_{L^p(\mathbb{R}^3)}^2 \|Z\|_{L^\infty(\mathbb{R}^3)}^2 + C\|\rho\Phi\|_{L^\infty(\mathbb{R}^3)}^2 \\ &\leq C \left(e^{-c\lambda t} + \frac{1}{\lambda} \right). \end{aligned}$$

In particular,

$$\|Z\|_{W^{1,\infty}}^2 \leq C \left(e^{-c\lambda t} + \frac{1}{\lambda} \right).$$

By definition of Z , this proves (7.4.2). \square

7.4.2 Convergence for small times

We want to prove $\rho_\lambda \rightarrow \rho_*$ as $\lambda \rightarrow \infty$, where ρ_* is the solution to (7.1.3). By the a priori estimate from Lemma 7.3.6, we have that ρ_λ is uniformly bounded in $W^{1,\infty}((0, T_0) \times \mathbb{R}^3)$ for times $T_0 < T_*$ defined in (7.3.20). Hence, we can extract strongly convergent subsequences in $C^{0,\alpha}((0, T_0) \times \mathbb{R}^3)$ for all $\alpha < 1$. It remains to prove that any limit of these subsequences is ρ_* . To this end we will show that ρ_λ converges to ρ_* in a weaker sense by using again the characteristics.

We note that

$$\rho_*(t, x) = \rho_0(X_*(0, t, x)) = \int_{\mathbb{R}^3} f_0(X_*(0, t, x), v) dv, \quad (7.4.5)$$

where $X_*(s, t, x)$ is defined as the solution to

$$\begin{aligned} \partial_s X_*(s, t, x) &= g + u_*(s, X_*(s, t, x)), \\ X_*(t, t, x) &= x. \end{aligned}$$

We have seen in (7.1.9) that for large values of λ , the particles are almost transported by $u_\lambda + g$. Moreover, in Lemma 7.4.1, we have seen that the fluid velocity u_λ is close to \tilde{u}_λ , which roughly speaking is the fluid velocity corresponding to the limit equation (7.1.3).

In order to compare ρ_λ to ρ_* , we want to use the formula for ρ_λ from Lemma (7.3.4),

$$\rho_\lambda(t, x) = \int_{\mathbb{R}^3} e^{3\lambda t} f_0(X_\lambda(0, t, x, W_\lambda(t, x, w)), w) \det \nabla_w W_\lambda(t, x, w) dw. \quad (7.4.6)$$

Provided $X_\lambda(0, t, x, W_\lambda(t, x, w))$ is close to $X_*(0, t, x)$ independently of w , the right hand sides of (7.4.5) and (7.4.6) look very similar. However, we lack information on the Jacobian $\det \nabla_w W_\lambda(t, x, w)$. We know that $e^{3\lambda t} \det \nabla_w W_\lambda(t, x, w)$ is uniformly bounded (for small times t and large values of λ , cf. Lemma 7.3.4 and Lemma 7.3.6), but we do not know whether it tends to 1 in the limit $\lambda \rightarrow \infty$.

To avoid dealing with this Jacobian, we also integrate over a small set in position space. To this end, let $\Psi_\lambda(t, \xi) := (X_\lambda(t, 0, \xi), V_\lambda(t, 0, \xi))$ with $\xi = (x, v)$. Then, using the characteristic equations (7.1.6),

$$\partial_t \nabla \Psi_\lambda = \nabla \Psi_\lambda \begin{pmatrix} 0 & \text{Id} \\ \lambda \nabla u & -\lambda \text{Id} \end{pmatrix}.$$

Hence,

$$\partial_t \det \nabla \Psi_\lambda = \det \nabla \Psi_\lambda \operatorname{tr} \left((\nabla \Psi_\lambda)^{-1} \nabla \Psi_\lambda \begin{pmatrix} 0 & \operatorname{Id} \\ \lambda \nabla u & -\lambda \operatorname{Id} \end{pmatrix} \right) = -3\lambda \det \nabla \Psi_\lambda.$$

Thus,

$$\det \nabla \Psi_\lambda(t, \xi) = e^{-3\lambda t}.$$

Therefore, for $\Omega \subset \mathbb{R}^3$ measurable,

$$\int_\Omega \rho_\lambda(t, y) dy = \int_\Omega \int_{\mathbb{R}^3} e^{3\lambda t} f_0(\Psi_\lambda^{-1}(y, v)) dv dy = \int_{\Psi_\lambda^{-1}(\Omega \times \mathbb{R}^3)} f_0(y, v) dv dy. \quad (7.4.7)$$

On the other hand, since u_* is divergence-free, we observe that

$$\int_\Omega \rho_*(t, y) dy = \int_\Omega \rho_0(X_*(0, t, y)) dy = \int_{X(0, t, \Omega)} \rho_0(y) dy = \int_{X(0, t, \Omega) \times \mathbb{R}^3} f_0(y, v) dy dv. \quad (7.4.8)$$

Now, we have to compare the right hand sides of (7.4.8) and (7.4.7).

It is convenient to choose Ω to be a cube. We denote by \mathcal{Q}_δ the set of all cubes $Q \subset \mathbb{R}^3$ of length δ . We define

$$d_{\lambda, \delta}(t) := \sup_{Q \in \mathcal{Q}_\delta} \left| \int_Q \rho_\lambda(t, y) - \rho_*(t, y) dy \right|.$$

We will show that

$$\lim_{\lambda \rightarrow \infty} \lim_{\delta \rightarrow 0} d_{\lambda, \delta}(t) = 0 \quad \text{for all } t < T_*. \quad (7.4.9)$$

This implies $\rho_\lambda(t, \cdot) \rightarrow \rho_*(t, \cdot)$ weakly-* in $L^\infty(\mathbb{R}^3)$ as we will prove in Proposition 7.4.5. The idea is that by the uniform boundedness shown in Lemma 7.3.6, we already have weak convergence of subsequences to some limit, and (7.4.9) is enough to characterize this weak limit.

We will prove (7.4.9) in Proposition 7.4.4. To do so, we essentially need three ingredients. First, we will show in Lemma 7.4.2 that $d_{\lambda, \delta}$ is controlled by $|X_\lambda - X_*|$. Second, we will show in Lemma 7.4.3 that $\tilde{u}_\lambda - u_*$ is controlled by $d_{\lambda, \delta}$. Finally, we use that the particle trajectories X_λ are almost the ones, which one get from a transport velocity $\tilde{u}_\lambda + g$. This last ingredient is due to (7.1.9) and Lemma 7.4.1.

Lemma 7.4.2. *Let $T_0 < T_*$. Then, there exist constants C_1 and λ_0 such that for all $\lambda > \lambda_0$ and all $t < T_0$*

$$d_{\lambda, \delta}(t) \leq C \left(\sup_{(x, v) \in \operatorname{supp} f_0} |X_\lambda(t, 0, x, v) - X_*(t, 0, x)| + \delta + \frac{1}{\delta \lambda} \right).$$

Proof. Let $Q \in \mathcal{Q}_\delta$. Let $\Psi_\lambda(t, y, v) := (X_\lambda(t, 0, y, v), V_\lambda(t, 0, y, v))$. Recall from (7.4.8) and (7.4.7)

$$\int_Q \rho_*(t, y) dy = \int_{X(0, t, Q)} \rho_0(y) dy, \quad (7.4.10)$$

$$\int_Q \rho_\lambda(t, y) dy = \int_{\Psi_\lambda^{-1}(Q \times \mathbb{R}^3)} f_0(y, v) dy dv. \quad (7.4.11)$$

We want to replace the right hand side of (7.4.11) by an integral of ρ_0 to compare its value to the right hand side of (7.4.10). To this end, we have to replace the set $\Psi_\lambda^{-1}(Q \times \mathbb{R}^3)$ by a set of the form $\Omega \times \mathbb{R}^3$. We define

$$\begin{aligned} \Omega &:= \{X(0, t, z, w) : (z, w) \in \Psi(\operatorname{supp} f_0) \cap (Q \times \mathbb{R}^3)\} \\ &= \{y \in \mathbb{R}^3 : \text{there is } v \in \mathbb{R}^3 \text{ with } (y, v) \in \operatorname{supp} f_0, X_\lambda(t, 0, y, v) \in Q\}. \end{aligned} \quad (7.4.12)$$

Then, we claim

$$\Psi_\lambda^{-1}(Q \times \mathbb{R}^3) \cap \text{supp } f_0 \subset (\Omega \times \mathbb{R}^3) \cap \text{supp } f_0 \subset \Psi_\lambda^{-1}(Q_{C\lambda^{-1}} \times \mathbb{R}^3), \quad (7.4.13)$$

where C is a constant independent of δ (and λ), and

$$Q_{C\lambda^{-1}} := \bigcup_{y \in Q} B_{C\lambda^{-1}}(y).$$

The first inclusion in (7.4.13) follows from the definition of Ω . To prove the second inclusion, let $(y, v) \in \text{supp } f_0 \cap (\Omega \times \mathbb{R}^3)$. Then, by definition of Ω , there exists $\tilde{v} \in \mathbb{R}^3$ such that $(y, \tilde{v}) \in \text{supp } f_0$ and $X_\lambda(t, 0, y, \tilde{v}) \in Q$. From (7.1.9) and the fact that the support of f_λ is uniformly bounded up to time T_0 by Lemma 7.3.6, we know that

$$\begin{aligned} |X_\lambda(t, 0, y, v) - X_\lambda(t, 0, y, \tilde{v})| &\leq \frac{C}{\lambda} + \int_0^t |u(s, X_\lambda(s, 0, y, v)) - u(s, X_\lambda(s, 0, y, \tilde{v}))| ds \\ &\leq \frac{C}{\lambda} + \int_0^t \|\nabla u\|_{L^\infty} |X_\lambda(s, 0, y, v) - X_\lambda(s, 0, y, \tilde{v})| ds. \end{aligned}$$

Using the estimate for ∇u from Lemma 7.3.6 yields

$$|X_\lambda(t, 0, y, v) - X_\lambda(t, 0, y, \tilde{v})| \leq \frac{C}{\lambda} e^{Ct}.$$

Therefore, $X_\lambda(t, 0, y, v) \in Q_{C\lambda^{-1}}$ and thus $(y, v) \in \Psi_\lambda^{-1}(Q_{C\lambda^{-1}} \times \mathbb{R}^3)$. From (7.4.13) it follows

$$\begin{aligned} &\left| \int_{\Psi_\lambda^{-1}(Q \times \mathbb{R}^3)} f_0(y, v) dy dv - \int_{\Omega \times \mathbb{R}^3} f_0(y, v) dy dv \right| \\ &\leq \int_{\Psi_\lambda^{-1}((Q_{C\lambda^{-1}} \setminus Q) \times \mathbb{R}^3)} f_0(y, v) dy dv \\ &= \int_{Q_{C\lambda^{-1}} \setminus Q} \rho_\lambda(t, y) dy \\ &\leq \|\rho_\lambda(t, \cdot)\|_{L^\infty(\mathbb{R}^3)} |Q_{C\lambda^{-1}} \setminus Q| \\ &\leq C \frac{\delta^2}{\lambda}. \end{aligned} \quad (7.4.14)$$

Combining (7.4.10), (7.4.11), and (7.4.14) yields

$$\left| \int_Q \rho_\lambda(t, y) - \rho_*(t, y) dy \right| \leq \left| \int_{X_*(0, t, Q)} \rho_0(y) dy - \int_\Omega \rho_0(y) dy \right| + C \frac{\delta^2}{\lambda}. \quad (7.4.15)$$

To estimate the right hand side, we note that

$$|X_*(0, t, Q)| = |Q| = \delta^3, \quad (7.4.16)$$

since $\text{div } u_* = 0$. We want to show that $|\Omega| \approx |Q|$. To this end, we define \tilde{X}_λ to be the solution to

$$\begin{aligned} \partial_s \tilde{X}_\lambda(s, t, x) &= u_\lambda(s, \tilde{X}_\lambda(s, t, x)), \\ \tilde{X}_\lambda(t, t, x) &= x. \end{aligned}$$

Then, using (7.1.9), we have for all $(x, v) \in \text{supp } f_0$

$$\begin{aligned} |\tilde{X}_\lambda(t, 0, x) - X_\lambda(t, 0, x, v)| &\leq \frac{C}{\lambda} + \int_0^t |u_\lambda(s, \tilde{X}_\lambda(s, t, x)) - u_\lambda(s, X_\lambda(s, t, x, v))| ds \\ &\leq \frac{C}{\lambda} + \int_0^t \|\nabla u_\lambda\|_{L^\infty(\mathbb{R}^3)} |\tilde{X}_\lambda(s, t, x) - X_\lambda(s, t, x, v)| ds. \end{aligned}$$

Grönwall's lemma implies

$$|\tilde{X}_\lambda(t, 0, x) - X_\lambda(t, 0, x, v)| \leq \frac{C}{\lambda}.$$

Thus,

$$\tilde{X}_\lambda^{-1}(t, 0, I_{C\lambda^{-1}}(Q)) \subset \Omega \subset \tilde{X}_\lambda^{-1}(t, 0, Q_{C\lambda^{-1}}),$$

where

$$I_{C\lambda^{-1}}(Q) := \{y \in Q : B_{C\lambda^{-1}}(y) \subset Q\}.$$

Since $\text{div } u_\lambda = 0$, we have that \tilde{X}_λ is volume preserving as well. Therefore, using also (7.4.16)

$$||\Omega| - |X_*(0, t, Q)|| \leq |Q_{C\lambda^{-1}} \setminus I_{C\lambda^{-1}}(Q)| \leq C \frac{\delta^2}{\lambda}. \quad (7.4.17)$$

We observe that for any function $g \in W^{1,\infty}(\mathbb{R}^3)$ and measurable sets $E, F \subset \mathbb{R}^3$

$$\begin{aligned} &\left| \int_E g \, dx - \int_F g \, dx \right| \\ &\leq ||E| - |F|| \|g\|_{L^\infty} + \min\{|E|, |F|\} \|\nabla g\|_{L^\infty} \sup\{|x - y| : x \in E, y \in F\}. \end{aligned} \quad (7.4.18)$$

Indeed, using the first term on the right hand side, we may assume without loss of generality that E and F are of equal measure. Approximating E and F by equisized cubes further reduces the situation to the estimate for two of these cubes. For these cubes, the statement obviously holds.

Applying (7.4.18) together with (7.4.16) and (7.4.17) yields

$$\begin{aligned} &\left| \int_{X_*(0,t,Q)} \rho_0(y) \, dy - \int_\Omega \rho_0(y) \, dy \right| \\ &\leq ||\Omega| - |X_*(0, t, Q)|| \|\rho_0\|_{L^\infty} + \delta^3 \sup\{|y - z| : y \in \Omega, z \in X_*(0, t, Q)\} \|\nabla \rho_0\|_{L^\infty} \\ &\leq C \frac{\delta^2}{\lambda} + C \delta^3 \left(\sup_{y \in \Omega} \text{dist}(y, X_*(0, t, Q)) + \text{diam}(X_*(0, t, Q)) \right). \end{aligned} \quad (7.4.19)$$

We need to estimate the second term on the right hand side. To this end, recall the definition of the set Ω from (7.4.12). For any $y \in \Omega$, we find $(x, v) \in \text{supp } f_0$ such that $p := X_\lambda(t, 0, y, v) \in Q$. Define $z = X_*(0, t, p) \in X_*(0, t, Q)$. Then

$$\begin{aligned} |z - y| &= |X_*(0, t, p) - X_*(0, t, X_*(t, 0, y))| \\ &\leq \|\nabla X_*(0, t, \cdot)\|_{L^\infty(\mathbb{R}^3)} |X_\lambda(t, 0, y, v) - X_*(t, 0, y)| \\ &\leq \|\nabla X_*(0, t, \cdot)\|_{L^\infty(\mathbb{R}^3)} \sup_{(x,v) \in \text{supp } f_0} |X_\lambda(t, 0, x, v) - X_*(t, 0, x)|. \end{aligned} \quad (7.4.20)$$

Observe that

$$\|\nabla X_*(0, t, \cdot)\|_{L^\infty(\mathbb{R}^3)} \leq e^{\int_0^t \|\nabla u_*(s, \cdot)\| \, ds} \leq C. \quad (7.4.21)$$

Thus, (7.4.20) and (7.4.21) imply

$$\sup_{y \in \Omega} \text{dist}(y, X_*(0, t, Q)) \leq C \sup_{(x, v) \in \text{supp } f_0} |X_\lambda(t, 0, x, v) - X_*(t, 0, x)|. \quad (7.4.22)$$

Note that (7.4.21) also yields

$$\text{diam}(X_*(0, t, Q)) \leq \delta \|\nabla X_*(0, t, \cdot)\|_{L^\infty(\mathbb{R}^3)} \leq C\delta. \quad (7.4.23)$$

Finally, estimates (7.4.19), (7.4.23), and (7.4.22) yield

$$\begin{aligned} & \left| \int_{X_*(0, t, Q)} \rho_0(y) dy - \int_{\Omega} \rho_0(y) dy \right| \\ & \leq C \frac{\delta^2}{\lambda} + C\delta^3 \left(\sup_{(x, v) \in \text{supp } f_0} |X_\lambda(t, 0, x, v) - X_*(t, 0, x)| + \delta \right). \end{aligned}$$

Combining this estimate with (7.4.15) finishes the proof. \square

Lemma 7.4.3. *Let $T_0 < T_*$. For u_* and \tilde{u}_λ as in (7.1.3) and (7.1.10), we have for all $\delta \leq 1$ and for all $t < T_0$*

$$\|\tilde{u}_\lambda(t, \cdot) - u_*(t, \cdot)\|_{L^\infty(\mathbb{R}^3)} \leq C(d_{\lambda, \delta}(t) + \delta).$$

Proof. We choose disjoint cubes $(Q_i)_{i \in \mathbb{N}} \subset \mathcal{Q}_\delta$ that cover \mathbb{R}^3 up to a nullset. Define $I \subset \mathbb{N}$ to be the index set for those cubes that intersect with the support of either $\rho_\lambda(t, \cdot)$ or $\rho_*(t, \cdot)$ and let $(z_i)_{i \in I}$ be the centers of those cubes. Let $x \in \mathbb{R}^3$. Then,

$$\begin{aligned} |\tilde{u}_\lambda(t, x) - u_*(t, x)| &= \left| \int_{\mathbb{R}^3} \Phi(x - y)(\rho_\lambda(t, y) - \rho_*(t, y)) dy \right| \\ &\leq \sum_{j \in I} \left| \int_{Q_j} \Phi(x - y)(\rho_\lambda(t, y) - \rho_*(t, y)) dy \right|, \end{aligned}$$

where Φ is the fundamental solution of the Stokes equations,

$$\Phi(y) = \frac{1}{8\pi} \left(\frac{\text{Id}}{|y|} + \frac{y \otimes y}{|y|^3} \right).$$

Let $I_1 \subset I$ be the index set of those cubes Q_j which contain x or are adjacent to that cube. Then, $|I_1| \leq 27$ and for $j \in I_1$ we estimate

$$\begin{aligned} & \left| \int_{Q_j} \Phi(x - y)(\rho_\lambda(t, y) - \rho_*(t, y)) dy \right| \\ & \leq (\|\rho_\lambda(t, \cdot)\|_{L^\infty(\mathbb{R}^3)} + \|\rho_*(t, \cdot)\|_{L^\infty(\mathbb{R}^3)}) \left| \int_{Q_j} \Phi(x - y) dy \right| \\ & \leq C\delta^2. \end{aligned}$$

Let $I_2 = I \setminus I_1$. For $h \in L^1(\mathbb{R}^n)$ and $\Omega \subset \mathbb{R}^n$ measurable, we use the notation

$$(h)_\Omega := \int_{\Omega} h dx := \frac{1}{|\Omega|} \int_{\Omega} h dx.$$

Then, for $j \in I_2$,

$$\begin{aligned} & \left| \int_{Q_j} \Phi(x-y)(\rho_\lambda(t, y) - \rho_*(t, y)) dy \right| \\ & \leq |(\Phi(x-\cdot))_{Q_j}| \left| \int_{Q_j} (\rho_\lambda(t, y) - \rho_*(t, y)) dy \right| \\ & + \int_{Q_j} |\Phi(x-y) - (\Phi(x-\cdot))_{Q_j}| |\rho_\lambda(t, y) - \rho_*(t, y)| dy \\ & \leq \frac{\delta^3}{|x-z_j|} d_{\lambda, \delta}(t) + \frac{\delta^4}{|x-z_j|^2}, \end{aligned}$$

where we used that we control $\rho_\lambda(t, \cdot)$ and $\rho_*(t, \cdot)$ in $L^\infty(\mathbb{R}^3)$ by Lemma 7.3.6. Summing over all $j \in I$ yields

$$\begin{aligned} |\tilde{u}_\lambda(t, x) - u_*(t, x)| & \leq C\delta^2 + \sum_{j \in I^2} \frac{\delta^3}{|x-z_j|} d_{\lambda, \delta}(t) + \frac{\delta^4}{|x-z_j|^2} \\ & \leq C(\delta^2 + \delta + d_{\lambda, \delta}(t)), \end{aligned}$$

where the constant C depends on the spatial support of ρ_λ and ρ_* which we control uniformly up to time T_0 by Lemma 7.3.6. Using $\delta \leq 1$ finishes the proof. \square

Proposition 7.4.4. *Let $t < T_*$. Then*

$$\lim_{\delta \rightarrow 0} \lim_{\lambda \rightarrow \infty} d_{\lambda, \delta}(t) = 0.$$

Proof. We define

$$\eta(t) := \sup_{(x, v) \in \text{supp } f_0} |X_\lambda(t, 0, x, v) - X_*(t, 0, x)|.$$

Let $(x, v) \in \text{supp } f_0$. We write again $X_\lambda(t)$ instead of $X_\lambda(t, 0, x, v)$ and similar for X_* . We estimate using first (7.1.9) together with the fact that the support of f_λ remains uniformly bounded up to time T_0 , and then applying Lemma 7.4.1, Lemma 7.4.3, and Lemma 7.4.2

$$\begin{aligned} & |X_\lambda(t) - X_*(t)| \\ & \leq \int_0^t |u_\lambda(s, X_\lambda(s)) - u_*(s, X_*(s))| ds + \frac{C}{\lambda} \\ & \leq \int_0^t |\tilde{u}_\lambda(s, X_\lambda(s)) - u_*(s, X_*(s))| + |\tilde{u}_\lambda(s, X_\lambda(s)) - u_\lambda(s, X_*(s))| ds + \frac{C}{\lambda} \\ & \leq \int_0^t \|\tilde{u}_\lambda(s, \cdot) - u_*(s, \cdot)\|_{L^\infty(\mathbb{R}^3)} + \|\nabla u_*(s, \cdot)\|_{L^\infty(\mathbb{R}^3)} |X_\lambda(s) - X_*(s)| ds + \frac{C}{\lambda} \\ & \leq C \int_0^t d_{\lambda, \delta}(t) + \delta + |X_\lambda(s) - X_*(s)| ds + \frac{C}{\lambda} \\ & \leq C \int_0^t \eta(t) + \frac{1}{\delta\lambda} + \delta ds + \frac{C}{\lambda}. \end{aligned}$$

Taking the supremum over $(x, v) \in \text{supp } f_0$ yields for $\delta \leq 1$

$$\eta(t) \leq C \int_0^t \eta(s) ds + C \left(\frac{1}{\delta\lambda} + \delta \right).$$

Grönwall's lemma implies

$$\eta(t) \leq C \left(\frac{1}{\delta\lambda} + \delta \right) e^{Ct}.$$

Lemma 7.4.2 yields

$$d_{\lambda,\delta}(t) \leq C \left(\frac{1}{\delta\lambda} + \delta \right) e^{Ct}.$$

Taking the limits $\lambda \rightarrow \infty$ followed by $\delta \rightarrow 0$ finishes the proof. \square

Now, we have all the estimates needed to prove the statement of Theorem 7.1.1 up to times $T < T_*$.

Proposition 7.4.5. *Let $T < T_*$. Then, for all $\alpha < 1$,*

$$\rho_\lambda \rightarrow \rho_* \quad \text{in } C^{0,\alpha}((0, T) \times \mathbb{R}^3).$$

Moreover, for all $0 < t < T$,

$$u_\lambda \rightarrow u_* \quad \text{in } L^\infty((t, T); W^{1,\infty}(\mathbb{R}^3)) \text{ and in } L^1((0, T); W^{1,\infty}(\mathbb{R}^3)). \quad (7.4.24)$$

Proof. By Lemma 7.3.6, the sequence ρ_λ is uniformly bounded in $W^{1,\infty}((0, T) \times \mathbb{R}^3)$ for large enough λ . Therefore, for any $\alpha < 1$, ρ_λ has a subsequence that converges in $C^{0,\alpha}((0, T) \times \mathbb{R}^3)$ to some function σ . We need to show $\sigma = \rho_*$. We claim that for all cubes $Q \subset \mathbb{R}^3$ and all $t < T$,

$$\int_Q \rho_\lambda(t, x) dx \rightarrow \int_Q \rho_*(t, x) dx. \quad (7.4.25)$$

Clearly, (7.4.25) implies $\sigma = \rho_*$. In order to prove (7.4.25), let $\varepsilon > 0$. Then, by Proposition 7.4.4, there exists $\delta_0 > 0$ such that for all $\delta < \delta_0$ and all $x \in \mathbb{R}^3$

$$\lim_{\lambda \rightarrow \infty} d_{\lambda,\delta} = \left| \int_{Q_{\delta,x}} \rho_\lambda(t, x) - \rho_*(t, x) dx \right| < \varepsilon.$$

Up to a nullset, we can write Q as the disjoint union of cubes $Q_i \in \cup_{\delta < \delta_0} \mathcal{Q}_\delta$. Thus, since ε is arbitrary, (7.4.25) follows.

In order to prove (7.4.24), we notice that by Lemma 7.4.1 it suffices to prove

$$\tilde{u}_\lambda \rightarrow u_* \quad \text{in } L^\infty((0, T); W^{1,\infty}(\mathbb{R}^3)).$$

However, by regularity theory of the Stokes equations

$$\|\tilde{u}_\lambda - u_*\|_{L^\infty((0,T);W^{1,\infty}(\mathbb{R}^3))} \leq C \|\rho_\lambda - \rho_*\|_{L^\infty((0,T) \times \mathbb{R}^3)},$$

where we used that by Lemma 7.3.6 we have uniform control of the support of ρ_λ . \square

7.4.3 Convergence for arbitrary times

In view of Proposition 7.4.5, it only remains to prove $T_* = \infty$ to finish the proof of Theorem 7.1.1. The idea of the proof is the following. Due to Lemma 7.3.4, it is sufficient to control the quantity $M_\lambda(t)$ defined in (7.3.13) uniformly in λ . Indeed, arguing similarly to Lemma 7.3.5, $\limsup_{\lambda \rightarrow \infty} M_\lambda(t)$ has to blow up at time T_* . However, Proposition 7.4.5 shows, that for large enough values of λ , $M_\lambda(t)$ is controlled by the corresponding quantity of the limit equation. This gives a contradiction.

Proof of Theorem 7.1.1. By Proposition 7.4.5, it suffices to prove $T_* = \infty$. By Lemma 7.3.5, we have $T_* > 0$. Assume $T_* < \infty$ and let $T < T_*$. By definition of T_* and Lemma 7.3.2, the assumption $\lambda \geq 4\|\nabla u_\lambda\|_{L^\infty((0,T)\times\mathbb{R}^3)}$ is satisfied for all $\lambda \geq \lambda_0(T)$. Recall the definition of $M(T)$ from Lemma 7.3.4, which we will now denote by $M_\lambda(T)$ to emphasize the dependence on λ . Moreover, we denote by M_* the corresponding quantity for the solution of the limit problem, i.e.,

$$M_*(t) := \exp\left(\int_0^t 2\|\nabla u(s, \cdot)\|_{L^\infty(\mathbb{R}^3)} ds\right).$$

By Proposition 7.4.5, we have

$$M_\lambda(T) \rightarrow M_*(T) \leq M_*(T_*).$$

In particular, for all $\lambda \geq \lambda_0(T)$ (possibly enlarging $\lambda_0(T)$),

$$M_\lambda(T) \leq 2M_*(T_*).$$

Therefore, Lemma 7.3.4 implies for all $\lambda \geq \lambda_0(T)$

$$\sup_{s \leq t} \|\rho_\lambda(s, \cdot)\|_{L^\infty(\mathbb{R}^3)}^2 \leq 2M_*(T_*).$$

The rest of the proof is very similar to the proof of Lemma 7.3.5. We define

$$T_\lambda := \sup\{t \geq 0 : \sup_{s \leq t} \|\rho_\lambda(s, \cdot)\|_{L^\infty(\mathbb{R}^3)} \leq 2C_0(2M_*(T_*))^3\}.$$

Then, $T_\lambda > T$ as ρ_λ is continuous. Analogously as we have shown (7.3.21) in Lemma 7.3.5, we find that for all $t > 0$ and $\lambda \geq C \sup_{s \leq T+t} \|\rho_\lambda(s, \cdot)\|_{L^\infty(\mathbb{R}^3)}^2$

$$\sup_{s \leq T+t} \|\rho_\lambda(s, \cdot)\|_{L^\infty(\mathbb{R}^3)} \leq C_0(2M_*(T_*))^3 \exp\left(Ct \sup_{s \leq T+t} \|\rho_\lambda(s, \cdot)\|_{L^\infty(\mathbb{R}^3)}^2\right).$$

This implies for all $\lambda \geq \max\{\lambda_0(T), CC_0^2(M_*(T_*))^6\}$ and all $T + t < T_\lambda$

$$\sup_{s \leq T+t} \|\rho_\lambda(s, \cdot)\|_{L^\infty(\mathbb{R}^3)} \leq C_0(2M_*(T_*))^3 \exp(CC_0^2(M_*(T_*))^6 t).$$

As ρ_λ is continuous, this yields for all $\lambda \geq \max\{\lambda_0, CC_0^2(M_*(T_*))^6\}$

$$T_\lambda \geq T + \frac{\log(2)}{CC_0^2(M_*(T_*))^6}.$$

In particular, if we choose $T < T_*$ large enough, we deduce

$$T_\lambda > T_* \quad \text{for all } \lambda \geq \max\{\lambda_0, CC_0^2(M_*(T_*))^6\},$$

which gives a contradiction to the definition of T_* . □

Chapter 8

Well-posedness results for rod models

In this chapter, we study well-posedness of the rod model (1.1.3). As argued in Chapter 1.7, the existence proof of solutions globally in time seems much harder than the one for the transport-Stokes equations (1.1.1) or the Vlasov-Stokes equations (1.1.2), where we have shown global existence in Chapters 4 and 7, respectively. For the rod model (1.1.3), we are only able to establish local well-posedness. Furthermore, we consider cylindrically symmetric solutions to (1.1.3) and prove global existence in this case.

8.1 Introduction

We consider the rod model (1.1.3) which we repeat here:

$$\begin{aligned} \partial_t f + (u + (\text{Id} + \xi \otimes \xi)g) \cdot \nabla_x f + \text{div}_\xi (P_{\xi^\perp}(\xi \cdot \nabla u)f) &= 0, \\ -\Delta u + \nabla p &= \int_{S^2} f d\xi g, \quad \text{div } u = 0. \end{aligned} \tag{8.1.1}$$

Here, $f(t, x, \xi) \geq 0$ denotes the density of particles at time t and position $x \in \mathbb{R}^3$ which have orientation $\xi \in S^2$. Moreover, u is the fluid velocity and g the gravitational acceleration. The operator P_{ξ^\perp} denotes the orthogonal projection to the orthogonal complement of ξ and may be expressed as $P_{\xi^\perp} = \text{Id} - \xi \otimes \xi$, where Id is the identity matrix.

We denote the spatial mass-density by

$$\rho(t, x) := \int_{S^2} f(t, x, \xi) d\xi.$$

Then, the fluid equation reads

$$-\Delta u + \nabla p = \rho g, \quad \text{div } u = 0. \tag{8.1.2}$$

The characteristic equations associated with the system (8.1.1) are

$$\begin{aligned} \dot{X} &= u + (\text{Id} + \Xi \otimes \Xi)g, \\ \dot{\Xi} &= P_{\Xi^\perp}(\Xi \cdot \nabla u), \\ \dot{Z} &= -\text{div}_\Xi (P_{\Xi^\perp}(\Xi \cdot \nabla u))Z = -5\Xi \otimes \Xi : \nabla u Z, \end{aligned} \tag{8.1.3}$$

where Z denotes the values of f along the (projected) characteristic curves.

Our local well-posedness result, that we prove in Section 8.2, is based on the analysis of the characteristic curves and a fixed point argument. Roughly speaking, we control the characteristics

well if we have good estimates on u and ∇u (and $\nabla^2 u$). We can use that u solves (8.1.2). By standard estimates (see Remark 4.4.4), we have

$$\|u(s, \cdot)\|_{W^{1,\infty}(\mathbb{R}^3)} \lesssim \|\rho(s, \cdot)\|_{L^1(\mathbb{R}^3)}^{\frac{1}{3}} \|\rho(s, \cdot)\|_{L^\infty(\mathbb{R}^3)}^{\frac{2}{3}}.$$

Since u is divergence free, the L^1 -norm of ρ is conserved. Moreover, since the measure of S^2 is finite, the L^∞ -norm may be estimated by

$$\|\rho(s, \cdot)\|_{L^\infty(\mathbb{R}^3)} \lesssim \|f(s, \cdot)\|_{L^\infty(\mathbb{R}^3 \times S^2)}.$$

Using the characteristic equation for Z , we thus find

$$\|f(s, \cdot)\|_{L^\infty(\mathbb{R}^3 \times S^2)} \leq \|f(0, \cdot)\|_{L^\infty(\mathbb{R}^3 \times S^2)} + C \int_0^t \|f(s, \cdot)\|_{L^\infty(\mathbb{R}^3 \times S^2)}^{\frac{5}{3}} ds,$$

where the constant C depends on the L^1 -norm of $f(0, \cdot)$. This estimate allows to control the L^∞ -norm of f for short times, but it is not sufficient to rule out a blowup in finite time.

We have encountered a related problem in the case of the Vlasov-Stokes equations (1.1.2) in Chapter 7. We have argued in Chapter 1.7 that, due to the lack of an appropriate energy and the different structures of the equations, it does not seem possible for the rod model (8.1.1) to apply the strategies used for the well-posedness proof of the Vlasov-Stokes equations in Chapter 7 or for the Vlasov-Poisson equations used in [Pfa92; Sch91; Gla96]. We are therefore not able to rule out a blowup of the L^∞ -norm of f in finite time.

In comparison with the Vlasov-Stokes equations, the Vlasov equation in the rod-model (8.1.1) also involves the gradient of the fluid velocity u . This means that in the analysis of the characteristics, also the second gradient of u appears. More precisely, $\nabla^2 u$ appears when one tries to estimate the differences of characteristics starting at different points (or which are driven by different fluid velocities). Unfortunately, it is not possible to control the L^∞ -norm of $\nabla^2 u$ in terms of the L^∞ -norm of ρ . Therefore, in Lemma 8.2.2, we give an estimate for $\nabla^2 u$ in terms of $\nabla \rho$. Using this lemma, the proof of our local well-posedness result implies that, at the maximal time of existence T , the L^∞ -norm of f or of its gradient blows up if $T < \infty$. However, it seems possible to refine the argument in order to prove that it has to be the L^∞ -norm of f that blows up. Indeed, an analogous result has been given for the Vlasov-Poisson equations (see [Gla96]). It is based on a more careful estimate of the L^∞ -norm of $\nabla^2 u$ which only involves $\nabla \rho$ logarithmically.

It seems to be a rather delicate question, whether blowup in finite time of solutions to the rod model (8.1.1) may occur. In fact, there seems to be a mechanism that leads to concentration of the particle density, which might produce blowup. This mechanism is the same that has been argued in [KS89] to be responsible for the experimentally observed instability of homogeneous suspensions of rods which we briefly discussed in Chapter 2.5. Roughly speaking, the fluid velocity u is larger in regions of larger concentration of particles and u is almost parallel to the gravity in these regions. This produces a gradient of u that causes particles near a region of large concentration to rotate in such a way that they tend to move towards this region. It is not clear, though, if this mechanism is strong enough to produce blowup in finite time.

We will therefore study this phenomenon and existence of global solutions for a simplified system. In Section 8.3, we consider cylindrically symmetric solutions f to (8.1.1). More precisely, we consider particle densities f that satisfy

$$f(t, Ox + \lambda g, O\xi) = f(t, x, \xi)$$

for all $t \in (0, T)$, $\lambda \in \mathbb{R}$ and all rotations $O \in SO(3)$ such that $O(g) = g$. In some sense, this is the maximal symmetry which is preserved by the evolution of the system and still leads to a system with

the same potential blowup scenario as for the full problem. In particular, in this system, the velocity of the particles still depends on their orientation, and the orientation undergoes changes due to the gradient of the fluid velocity.

Nevertheless, we prove global well-posedness for such cylindrically symmetric solutions. Again, the proof is based on the method of characteristics. As argued above, the crucial step for global existence is to establish an a priori estimate on the L^∞ -norm of f . This is achieved using the additional structure provided by the symmetry assumptions. In fact, the symmetry assumptions imply that the characteristic curves of the solution are straight lines in space.

8.2 Short-time existence for the full problem

Similar to the transport-Stokes equations (1.1.1), where we proved well-posedness in Chapter 4, we work in weighted L^∞ -spaces. More precisely, for $\beta > 0$, we define

$$\begin{aligned} \|h\|_{X_\beta(\mathbb{R}^3)} &:= \|(1 + |x|^\beta)|h(x)|\|_{L^\infty(\mathbb{R}^3)}, \\ X_\beta(\mathbb{R}^3) &:= \{h \in W^{1,\infty}(\mathbb{R}^3) : \|h\|_{X_\beta(\mathbb{R}^3)} < \infty\}. \end{aligned} \quad (8.2.1)$$

Analogously, we define the space $X_\beta(\mathbb{R}^3 \times S^2)$ by introducing

$$\|h\|_{X_\beta(\mathbb{R}^3 \times S^2)} := \|(1 + |x|^\beta)|h(x, \xi)|\|_{L^\infty(\mathbb{R}^3 \times S^2)}.$$

We state the local well-posedness result in the following theorem.

Theorem 8.2.1. *Let $\beta > 2$ and $f_0 \in X_\beta(\mathbb{R}^3 \times S^2)$ with $\nabla f_0 \in X_\beta(\mathbb{R}^3 \times S^2)$. Then, there exists $T > 0$ such that there exists a unique solution $f \in W^{1,\infty}([0, T] \times \mathbb{R}^3 \times S^2) \cap L^\infty(0, T; X_\beta(\mathbb{R}^3 \times S^2))$ to (8.1.1) with $f(0, \cdot) = f_0$ and $u \in L^\infty((0, T); W^{2,\infty} \cap \dot{H}^1(\mathbb{R}^3))$.*

The following lemma provides estimates for the fluid velocity in terms of the particle density. The lemma directly follows from Lemma 4.4.3

Lemma 8.2.2. *Let $\beta > 2$ and $\rho \in X_\beta(\mathbb{R}^3)$ with $\nabla \rho \in X_\beta$. Then there exists a unique solution $u \in W^{2,\infty}(\mathbb{R}^3) \cap \dot{H}^1(\mathbb{R}^3)$ to (8.1.2) which satisfies*

$$\begin{aligned} \|u(t)\|_{W^{1,\infty}(\mathbb{R}^3)} &\lesssim \|\rho(t)\|_{X_\beta(\mathbb{R}^3)}, \\ \|\nabla u(t)\|_{W^{1,\infty}(\mathbb{R}^3)} &\lesssim \|\nabla \rho(t)\|_{X_\beta(\mathbb{R}^3)}. \end{aligned} \quad (8.2.2)$$

Proof of Theorem 8.2.1. Step 1: Setup.

We will apply the Banach fixed point theorem. We define the metric space, where we prove contractiveness. Let $T, L > 0$. We write $\Omega = \mathbb{R}^3 \times S^2$ and $U_T = (0, T) \times \mathbb{R}^3$, and define

$$Y_{T,L} := \left\{ h \in L^\infty(0, T; X_\beta(\Omega)) : \|h\|_{L^\infty(0,T;X_\beta(\Omega))} + \|\nabla_x h\|_{L^\infty(0,T;X_\beta(\Omega))} \leq L \right\}.$$

Then, $Y_{T,L}$ is a complete metric space with norm $\|\cdot\|_{L^\infty(0,T;X_\beta(\Omega))}$.

We now define an operator $S: Y_{T,L} \rightarrow L^\infty(0, T; X_\beta(\Omega))$. Let $h \in Y_{T,L}$, and define $u \in L^\infty(0, T; W^{2,\infty}(\mathbb{R}^3))$ to be the solution to (8.1.2) with $\rho = \int_{S^2} h d\xi$. Then, we define $f := Sh$ as the solution to the first equation in (8.1.1) with this given fluid velocity u .

In order to apply the Banach fixed point theorem, we have to show that S is contractive and $S(Y_{T,L}) \subset Y_{T,L}$.

Step 2: $S(Y_{T,L}) \subset Y_{T,L}$. In view of (8.1.3), we define the characteristics $(X_i, \Xi_i)(s, t, x, \xi)$ by

$$\begin{aligned}\partial_s(X, \Xi)(s, t, x, \xi) &= \left(u_i(s, X) + (I + \Xi \otimes \Xi)g, P_{\Xi_i^\perp}(\Xi \cdot \nabla u(s, X)) \right), \\ (X, \Xi)(t, t, x, \xi) &= (x, \xi).\end{aligned}$$

Then, $f = Sh$ is given by

$$f(t, x, \xi) = \exp \left(-5 \int_0^t \Xi(s, t, x, \xi) \otimes \Xi(s, t, x, \xi) : \nabla u(s, X(s, t, x, \xi)) ds \right) f_0((X, \Xi)(0, t, x, \xi)). \quad (8.2.3)$$

We observe that Lemma 8.2.2 implies for $t < T$ (assuming without loss of generality $|g| = 1$)

$$\begin{aligned}|(X)(t, 0, x, \xi) - x| &\leq \int_0^t |u(s, X(t, s, x, \xi))| + 2 ds \\ &\leq CT \left(1 + \|h\|_{L^\infty(0, T; X_\beta(\Omega))} + \|\nabla h\|_{L^\infty(0, T; X_\beta(\Omega))} \right) \\ &\leq CT(1 + L).\end{aligned}$$

Thus,

$$1 + |x|^\beta \leq C_1(1 + T^\beta(1 + L)^\beta)(1 + |X(t, 0, x, \xi)|^\beta), \quad (8.2.4)$$

where C_1 is a constant which only depends on β . Equation (8.2.3) implies

$$|f(t, x, \xi)| \leq e^{CTL} |f_0((X_i, \Xi_i)(0, t, x, \xi))|. \quad (8.2.5)$$

Combining (8.2.4) and (8.2.5) yields

$$\|f\|_{L^\infty(0, T; X_\beta(\Omega))} \leq C_1 e^{CTL} (1 + T^\beta(1 + L)^\beta) \|f_0\|_{X_\beta(\Omega)}.$$

In order to estimate the gradient of f , we define $Y_i(s) := (X, \Xi)(t, s, x_i, \xi_i)$, $i = 1, 2$, to be the characteristics starting from (x_i, ξ_i) at time t . Then,

$$\begin{aligned}|Y_1(s) - Y_2(s)| &\leq |(x_1, \xi_1) - (x_2, \xi_2)| + \int_0^t C(1 + \|u(\tau, \cdot)\|_{W^{2, \infty}(\mathbb{R}^3)}) |Y_1(\tau) - Y_2(\tau)| d\tau \\ &\leq |(x_1, \xi_1) - (x_2, \xi_2)| + C(1 + L) \int_s^t |Y_1(\tau) - Y_2(\tau)| d\tau.\end{aligned}$$

Thus, Grönwall's estimate implies

$$|Y_1(0) - Y_2(0)| \leq |(x_1, \xi_1) - (x_2, \xi_2)| e^{CT(1+L)}.$$

Combining this bound with (8.2.4) yields

$$\|\nabla f\|_{L^\infty(0, T; X_\beta(\Omega))} \leq C_1 e^{CT(L+1)} (1 + T^\beta(1 + L)^\beta) \|\nabla f_0\|_{X_\beta(\Omega)}. \quad (8.2.6)$$

Hence, for all $L > C_1(\|f_0\|_{X_\beta(\Omega)} + \|\nabla f_0\|_{X_\beta(\Omega)})$ there exists $T > 0$ such that $S(Y_{T,L}) \subset Y_{T,L}$.

Step 3: Contraction.

Let $h_1, h_2 \in Y_{T,L}$. For $i = 1, 2$, we define u_i to be the solution to

$$-\Delta u_i + \nabla p = \int_{S^2} h_i d\xi g, \quad \operatorname{div} u_i = 0.$$

We denote by $Y_i(s, t, x, \xi) = (X_i, \Xi_i)(s, t, x, \xi)$ the characteristics associated to $f_i = Sh_i$ and abbreviate $X_i(s) = X_i(s, t, x, \xi)$ and analogously for Ξ_i . Moreover, we write $C(T, L) := C_1(1 + T^\beta(1 + L)^\beta)$ where C_1 is the constant in (8.2.4). Then, we estimate, repeatedly applying the triangle,

$$\begin{aligned}
& (1 + |x|^\beta) |f_1(t, x, \xi) - f_2(t, x, \xi)| \\
& \stackrel{(8.2.3)}{=} (1 + |x|^\beta) \left| \exp \left(-5 \int_0^t \Xi_1(s) \otimes \Xi_1(s) : \nabla u_1(s, X_1(s)) ds \right) f_0(Y_1(0)) \right. \\
& \quad \left. - \exp \left(-5 \int_0^t \Xi_2(s) \otimes \Xi_2(s) : \nabla u_2(s, X_2(s)) ds \right) f_0(Y_2(0)) \right| \\
& \stackrel{(8.2.4)}{\leq} C(T, L) \exp(5T \|\nabla u_1\|_{L^\infty(U_T)}) \|\nabla f_0\|_{L^\infty(X_\beta(\Omega))} |(Y_1)(0) - (Y_2)(0)| \\
& \quad + C(T, L) CT \exp(CT(\|\nabla u_1\|_{L^\infty(U_T)} + \|\nabla u_2\|_{L^\infty(U_T)})) \|f_0\|_{X_\beta(\Omega)} \\
& \quad \left(\|\nabla^2 u_1\|_{L^\infty(U_T)} \sup_{s \leq t} |X_1(s) - X_2(s)| + \|\nabla(u_1 - u_2)\|_{L^\infty(U_T)} + \|\nabla u_1\|_{L^\infty(U_T)} \sup_{s \leq t} |\Xi_1(s) - \Xi_2(s)| \right) \\
& \stackrel{(8.2.2)}{\leq} C(T, L) \exp(CTL) \left(\|f_0\|_{X_\beta(\Omega)} + \|\nabla f_0\|_{X_\beta(\Omega)} \right) \\
& \quad \left((1 + CTL) |(Y_1)(0) - (Y_2)(0)| + CT \|h_1 - h_2\|_{L^\infty(0, T; X_\beta(\Omega))} \right).
\end{aligned} \tag{8.2.7}$$

Moreover, for all $0 \leq s \leq t \leq T$

$$\begin{aligned}
& |(X_1)(s) - (X_2)(s)| \\
& \leq C \int_s^t |(\Xi_1(\tau) - \Xi_2(\tau)) + |u_1(\tau, X_1(\tau)) - u_2(\tau, X_2(\tau))| d\tau \\
& \leq C \int_s^t |\Xi_1(\tau) - \Xi_2(\tau)| + \|\nabla u_1\|_{L^\infty(U_T)} |X_1(\tau) - X_2(\tau)| + \|u_1 - u_2\|_{L^\infty(U_T)} d\tau \\
& \leq C(1 + L) \int_s^t |Y_1(\tau) - Y_2(\tau)| d\tau + CT \|h_1 - h_2\|_{L^\infty(0, T; X_\beta(\Omega))},
\end{aligned}$$

and

$$\begin{aligned}
& |(\Xi_1)(s) - (\Xi_2)(s)| \\
& \leq C \int_s^t |X_1(\tau) - X_2(\tau)| \|\nabla^2 u_1\|_{L^\infty(U_T)} + \|\nabla(u_1 - u_2)\|_{L^\infty(U_T)} + \|\nabla u_1\|_{L^\infty(U_T)} |\Xi_1(\tau) - \Xi_2(\tau)| d\tau \\
& \leq CL \int_s^t |Y_1(\tau) - Y_2(\tau)| d\tau + CT \|h_1 - h_2\|_{L^\infty(0, T; X_\beta(\Omega))}.
\end{aligned}$$

Thus,

$$|Y_1(s) - Y_2(s)| \leq C(1 + L) \int_s^t |Y_1(\tau) - Y_2(\tau)| d\tau + CT \|h_1 - h_2\|_{L^\infty(0, T; X_\beta(\Omega))}.$$

Grönwall's lemma implies

$$|Y_1(0) - Y_2(0)| \leq CT \|h_1 - h_2\|_{L^\infty(0, T; X_\beta(\Omega))} e^{C(1+L)T}.$$

Inserting this in (8.2.7) yields

$$\begin{aligned}
& \|f_1 - f_2\|_{L^\infty(0, T; X_\beta(\Omega))} \\
& \leq C(T, L)(1 + CTL) T e^{C(L+1)T} \left(\|f_0\|_{X_\beta(\Omega)} + \|\nabla f_0\|_{X_\beta(\Omega)} \right) \|h_1 - h_2\|_{L^\infty(0, T; X_\beta(\Omega))} \\
& \leq C_1(1 + (T(1 + L))^{\beta+1}) T e^{C(L+1)T} \left(\|f_0\|_{X_\beta(\Omega)} + \|\nabla f_0\|_{X_\beta(\Omega)} \right) \|h_1 - h_2\|_{L^\infty(0, T; X_\beta(\Omega))},
\end{aligned}$$

where we used the definition of $C(T, L) = C_1(1 + T^\beta(1 + L)^\beta)$. Thus, for all L , there exists $T > 0$ such that the mapping $h \mapsto f$ is contractive with respect to $L^\infty(0, T; X_\beta(\Omega))$.

Step 4: Conclusion. By the Banach fixed point theorem, there exists a unique fixed point of $Y_{T,L}$ of S if we choose first L sufficiently large and then T sufficiently small. It is easily verified that $f \in W^{1,\infty}([0, T] \times \mathbb{R}^3 \times S^2)$. Indeed, it only remains to prove Lipschitz continuity in time. Let $0 < t_1 < t_2 < T$ and consider $(X, \Xi)(t_i, s, x, \xi)$, the characteristics starting from (x, ξ) at times t_i . Then, we know

$$(X, \Xi)(t_2, s, x, v) = (X, \Xi)(t_1, s, X(t_2, t_1, x, \xi), \Xi(t_2, t_1, x, \xi)).$$

Using (8.2.2), we have

$$|X(t_2, t_1, x, \xi), \Xi(t_2, t_1, x, \xi) - (x, v)| \lesssim L(t_2 - t_1).$$

Combining this estimate with (8.2.3) yields, similarly to (8.2.6),

$$\sup_{x, \xi} \|f(\cdot, x, \xi)\|_{W^{1,\infty}(0, T)} \leq e^{CT(L+1)} \left(CTL + \|f_0\|_{W^{1,\infty}(\mathbb{R}^3 \times S^2)} \right).$$

The regularity for u follows immediately from Lemma 8.2.2. □

8.3 Global existence for cylinder symmetric solutions

8.3.1 Setting and main result

We consider the symmetry group of all rotations around the axis parallel to g ,

$$\mathcal{O} := \{O \in SO(3) : Og = g\},$$

and consider initial data $f_0 : \mathbb{R}^3 \times S^2 \rightarrow \mathbb{R}$ such that

$$f_0(x, \xi) = f(Ox + \lambda g, O\xi) \quad \text{for all } \lambda \in \mathbb{R}, O \in \mathcal{O}.$$

Since we require f_0 to be invariant under translations parallel to the gravity, we cannot have $f_0 \in L^1(\mathbb{R}^3 \times S^2)$ unless $f_0 = 0$.

Instead, we impose that $f_0 \geq 0$ has finite mass on planes perpendicular to the gravity, which is given by

$$H := \{x \in \mathbb{R}^3 : x \cdot g = 0\}. \tag{8.3.1}$$

By translation invariance, it is sufficient to consider the function $h_0 : H \times S^2 \rightarrow \mathbb{R}$ defined by

$$h_0(y, \xi) := f_0(y, \xi) \quad \text{for all } y \in H.$$

We require

$$h_0 \in X_\beta(H \times S^2).$$

For translation invariant ρ , i.e. $\rho(x + \lambda g) = \rho(x)$ we have $\operatorname{div}(\rho g) = 0$. Thus, the fluid equation simplifies to

$$-\Delta u = \rho g \quad \text{in } \mathbb{R}^3,$$

and we can reduce this to the two dimensional problem in H by

$$-\Delta v = \rho \quad \text{in } H, \quad u(x) = v(P_H x)g, \quad (8.3.2)$$

where P_H is the orthogonal projection to the plane H defined in (8.3.1). In general there is no solution in $v \in L^p(H)$ to the Poisson equation for any $1 \leq p \leq \infty$, even if the right-hand side satisfies $\rho \in C_c^\infty(H)$. However, defining $X_\beta(H)$ analogously to (8.2.1), the analog of Lemma 8.2.2 holds.

Lemma 8.3.1. *Let $\beta > 1$ and $\rho \in X_\beta(H)$ with $\nabla \rho \in X_\beta(H)$. Then, there exists a solution $v \in L_{\text{loc}}^\infty(H)$ to (8.3.2) such that $\nabla v \in L^p(H)$ for all $2 < p < \infty$. This solution is unique up to the addition of constants. Moreover, $\nabla v \in W^{1,\infty}(H)$ with*

$$\begin{aligned} \|\nabla v\|_{L^\infty(H)} &\lesssim \|\rho\|_{X_\beta(H)}, \\ \|\nabla^2 v\|_{L^\infty(H)} &\lesssim \|\nabla \rho\|_{X_\beta(H)}. \end{aligned}$$

Since the fluid velocity u is translation invariant for translation invariant particle densities f , the translation invariance is preserved by the evolution under (8.1.1). Therefore, since u is parallel to g , the fluid velocity u in the term $u + (\text{Id} + \xi \otimes \xi)g \cdot \nabla f$ has no influence on the dynamics.

Making use of the translational invariance, we can rewrite the dynamics (8.1.1) in $(0, T) \times H \times S^2$ as

$$\begin{aligned} \partial_t h + (\xi \cdot g)P_H \xi \cdot \nabla_x h + \text{div}_\xi((\xi \cdot \nabla v)P_{\xi^\perp} g h) &= 0 \quad \text{in } (0, T) \times H \times S^2, \\ -\Delta v &= \int_{S^2} \int h \, d\xi \quad \text{in } (0, T) \times H. \end{aligned} \quad (8.3.3)$$

It should be emphasized that the gradient of the fluid velocity is still relevant for changing the orientation of the particles, and that this is the most dangerous term regarding possible blowup. However, we prove global existence of cylindrically symmetric solution, as stated in the following theorem.

Theorem 8.3.2. *Let $\beta > 2$ and assume $h_0 \in X_\beta(H)$ with $\nabla h_0 \in X_\beta(H)$ satisfies*

$$h_0(x, \xi) = h_0(Ox, O\xi) \quad \text{for all } \lambda \in \mathbb{R}, \, O \in \mathcal{O}. \quad (8.3.4)$$

Then, for all $T > 0$, there exists a unique solution $h \in W^{1,\infty}([0, T] \times H \times S^2) \cap L^\infty(0, T; X_\beta(H \times S^2))$ to (8.3.3) such that $h(0, \cdot) = h_0$ and $h(t, \cdot)$ satisfies (8.3.4) for all $0 < t < T$ and $v \in L^\infty(0, T; L_{\text{loc}}^\infty(H))$ with $\nabla v \in W^{1,\infty}(H) \cap L^p(H)$ for all $2 < p < \infty$.

8.3.2 The characteristic equations of the system

For $h(t, \cdot)$ that satisfies (8.3.4), we know that the spatial particle density ρ is invariant under rotations, i.e. there exists a function $\sigma: (0, T) \times (0, \infty) \rightarrow \mathbb{R}$ such that

$$\rho(t, x) := \int_{S^2} h(t, x, \xi) \, d\xi = \sigma(t, |x|).$$

This form of the spatial particle density considerably reduces the complexity of the fluid equation (8.3.2) to an explicitly solvable ODE. Indeed, v is given as the solution¹ to

$$v(t, x) = \psi(t, |x|), \quad -\partial_r^2 \psi - \frac{1}{r} \partial_r \psi = \sigma, \quad (8.3.5)$$

¹Since only the gradient of v , which is unique in $W^{1,\infty}(H) \cap L^p(H)$, is relevant for the dynamics, we will refer to v as the solution to (8.3.2) even though v itself is not unique.

which is solved by

$$\partial_r \psi(t, r) = -\frac{1}{r} \int_0^r r' \sigma(t, r') dr' =: -\frac{1}{r} M(t, r), \quad (8.3.6)$$

where the function $M(t, r)$ is (up to a factor 2π) the total particle mass in $B_r(0) \subset H$ at time t . In particular, we find

$$\nabla v(t, x) = -\frac{1}{|x|} M(t, |x|) \hat{x}, \quad \hat{x} := \frac{x}{|x|}.$$

We write the characteristic equations for the system (8.3.3):

$$\begin{aligned} \dot{X} &= (\Xi \cdot g) P_H \Xi, \\ \dot{\Xi} &= -\frac{M(t, |X|)}{|X|} (\Xi \cdot \hat{X}) P_{\Xi^\perp} g, \\ \dot{Z} &= 5 \frac{M(t, |X|)}{|X|} (\Xi \cdot g) (\Xi \cdot \hat{X}) Z. \end{aligned} \quad (8.3.7)$$

For the analysis of the characteristics, it is useful to write them in cylinder coordinates in space and spherical coordinates in the orientation. More precisely, we assume without loss of generality $g = e_3$ and write for $X \in H$, $\Xi \in S^2$

$$X = \begin{pmatrix} R \cos \Phi_x \\ R \sin \Phi_x \\ 0 \end{pmatrix}, \quad \Xi = \begin{pmatrix} \sin \Theta \cos \Phi_\xi \\ \sin \Theta \sin \Phi_\xi \\ \cos \Theta \end{pmatrix}.$$

Then, with

$$e_r = \begin{pmatrix} \cos \Phi_x \\ \sin \Phi_x \\ 0 \end{pmatrix}, \quad e_{\phi_x} = \begin{pmatrix} -\sin \Phi_x \\ \cos \Phi_x \\ 0 \end{pmatrix}, \quad e_\theta = \begin{pmatrix} \cos \Theta \cos \Phi_\xi \\ \cos \Theta \sin \Phi_\xi \\ -\sin \Theta \end{pmatrix}, \quad e_{\phi_\xi} = \begin{pmatrix} -\sin \Phi_\xi \\ \cos \Phi_\xi \\ 0 \end{pmatrix},$$

we have

$$\dot{X} = \dot{R} e_r + R \dot{\Phi}_x e_{\phi_x}, \quad \dot{\Xi} = \dot{\Theta} e_\theta + \sin \Theta \dot{\Phi}_\xi e_{\phi_\xi}.$$

Therefore, the characteristic equations take the form

$$\begin{aligned} \dot{R} &= \cos \Theta \sin \Theta \cos(\Phi_\xi - \Phi_x), \\ \dot{\Phi}_x &= \frac{\cos \Theta \sin \Theta \sin(\Phi_\xi - \Phi_x)}{R}, \\ \dot{\Theta} &= M(t, R) \frac{\sin^2 \Theta \cos(\Phi_\xi - \Phi_x)}{R}, \\ \dot{\Phi}_\xi &= 0, \\ \dot{Z} &= 5M(t, R) \frac{\cos \Theta \sin \Theta \cos(\Phi_\xi - \Phi_x)}{R}. \end{aligned} \quad (8.3.8)$$

The identity $\dot{\Phi}_\xi = 0$ expresses the fact that the particle orientations only change in the polar angle. The reason why the azimuthal angle Φ_ξ is constant is that the fluid velocity u is parallel to g everywhere. Thus, the orientation cannot change in the direction perpendicular to $g = e_3$. However, the polar angle Θ of the orientation is only relevant for the sign and the absolute value of the particle

velocity, and not for its direction. Therefore, $\Phi_\xi = \text{const}$ implies that the projected characteristics are straight lines in space. More precisely, along the characteristics we have

$$R \sin(\Phi_\xi - \Phi_x) = \text{const}. \quad (8.3.9)$$

as can be checked by a direct computation of the derivative.

We thus introduce $\Phi := \Phi_\xi - \Phi_x$, which contains the full information of both angles due to the symmetry under rotations. Then, we rewrite the characteristic equations as

$$\begin{aligned} \dot{R} &= \cos \Theta \sin \Theta \cos \Phi, \\ \dot{\Theta} &= \frac{\sin^2 \Theta \cos \Phi}{R} M(R, \cdot), \\ \dot{\Phi} &= -\frac{\cos \Theta \sin \Theta \sin \Phi}{R}, \\ \dot{Z} &= 5 \frac{\sin \Theta \cos \Theta \cos \Phi}{R} M(R, \cdot) Z. \end{aligned} \quad (8.3.10)$$

Let us consider a characteristic curve with $\Phi = 0$ initially. Then, the equation for Φ implies $\Phi = \text{const}$. Hence, $\dot{\Theta} \geq 0$ and $\dot{\Theta} > 0$ unless $\Theta \in \{0, \pi\}$. Consequently $\Theta \rightarrow 0$ (unless $\Theta = \pi$). Then, as soon as $\Theta < \pi/2$, we have $\dot{R} < 0$. Thus, the characteristic curve eventually approaches the origin. This resembles the mechanism explained in the introduction that could produce blowup of the particle density: due to the symmetry, the fluid flow is largest at the origin. Therefore, the particles turn in such a way that they approach the origin.

The particles behave similarly if $\Phi(0) \neq 0$. However, due to (8.3.9), we have

$$R(t) \geq R(0) |\sin(\Phi(t))| = R(0) |\sin(\Phi(0))| > 0 \quad (8.3.11)$$

if $\Phi(0) \notin \{0, \pi\}$ assuming $R(0) > 0$. This means that those characteristic curves cannot approach the origin farther than to a certain threshold. Therefore, the concentration mechanism is not strong enough in this case in order to produce blowup in finite time.

8.3.3 Proof of global existence of solutions

We have argued in the introduction that blowup of solutions is equivalent to blowup of the L^∞ -norm of h . On the level of the characteristics, this is expressed by blowup of Z . From the characteristic equations (8.3.10), we deduce that we control Z if we control $\sup_{r \in (0, \infty)} r^{-1} M(r, t)$. By definition (see (8.3.6)), $M(r, t)$ is proportional to the particle mass in $B_r(0) \times S^2$ at time t . In particular, since the total mass $\|h(t, \cdot)\|_{L^1(H \times S^2)}$ is conserved, we only need to estimate $r^{-1} M(r, t)$ for small r .

For the proof of this a priori estimate of $r^{-1} M(r, t)$, we use the fact that the system (8.3.3) preserves mass transported by the characteristics. This is a general property of continuity equations such as the Vlasov equation (8.3.3), which is also true for the full system (8.1.1) because u is divergence free. We summarize this property in the following lemma.

Lemma 8.3.3. *Let $h \in W^{1,\infty}([0, T] \times H \times S^2) \cap L^\infty(0, T; L^1(H \times S^2))$ be a solution to the first equation (8.3.3) with some given $v \in L^\infty(0, T; L^\infty_{\text{loc}}(H) \cap W^{1,\infty}(H))$. Let $\Gamma_t = (X(t, x, v), \Xi(t, x, v))$ be the diffeomorphism induced by the characteristics starting from (x, ξ) at time zero. Then, for any measurable set $\Omega \subset H \times S^2$ and for all $0 < t < T$,*

$$\int_{\Omega} h(t, x, \xi) = \int_{\Gamma_t^{-1}(\Omega)} h(0, x, \xi).$$

In particular, for all $0 < t < T$

$$\|h(t, \cdot)\|_{L^1(H \times S^2)} = \|h(0, \cdot)\|_{L^1(H \times S^2)}.$$

Proposition 8.3.4. *Let $T > 0$ and $\beta > 2$ and $h \in W^{1,\infty}([0, T] \times H \times S^2) \cap L^\infty(0, T; X_\beta(H \times S^2))$ be a non-negative solution to (8.3.3) such that $h(t, \cdot)$ satisfies (8.3.4) for all $t \in [0, T]$. Then, for all $0 < t < T$,*

$$\|h(t, \cdot)\|_{X_\beta(H \times S^2)} \leq e^{C(1+t)\|h(0, \cdot)\|_{X_\beta(H \times S^2)}} \|h(0, \cdot)\|_{X_\beta(H \times S^2)}.$$

Proof. By the definition of $M(t, r)$ in (8.3.6), we have

$$2\pi M(t, r) = \int_{B_r(0)} \int_{S^2} h(t, x, \xi) d\xi dx.$$

In particular, by Lemma 8.3.3 above, we have

$$2\pi M(t, r) = \int_{H \times S^2} h(0, x, \xi) \mathbf{1}_{|X(t)| \leq r} \leq \|h(0, \cdot)\|_{L^\infty(H \times S^2)} \int_{H \times S^2} \mathbf{1}_{|X(t)| \leq r}.$$

By the smoothness of the coordinate change from cylindrical and spherical coordinates to Cartesian coordinates away from the origin, the formulation of the characteristic equations (8.3.8) is equivalent to (8.3.7) for all characteristics that do not pass through the origin. We already know from (8.3.9) that the only characteristics passing through the origin are the ones with $R(0) \sin(\Phi_\xi(0) - \Phi_x(0)) = 0$. Since the set of those initial data are a nullset, the characteristic equations (8.3.8) contain the same information as (8.3.7).

We can thus rewrite

$$\begin{aligned} \int_{H \times S^2} \mathbf{1}_{|X(t)| \leq r} &= \int_0^\infty \int_0^{2\pi} \int_0^\pi \int_0^{2\pi} s \sin \theta \mathbf{1}_{R(t) \leq r} d\phi_\xi d\theta d\phi_x ds \\ &= 2\pi \int_0^\infty \int_0^\pi \int_0^{2\pi} s \sin \theta \mathbf{1}_{R(t) \leq r} d\phi d\theta ds. \end{aligned}$$

In the second line, we substituted ϕ_ξ by $\phi = \phi_\xi - \phi_x$ and integrated out one of the angles using the symmetry.

We use $|\dot{R}| \leq 1$ and (8.3.11) to deduce

$$\left\{ (s, \theta, \phi) : R(t) \leq r \right\} \subset \left\{ (s, \theta, \phi) : s \leq r + t, |\sin \phi| \leq \frac{s}{r} \right\}.$$

Moreover, we observe

$$\left| \left\{ \phi \in (0, 2\pi) : |\sin \phi| \leq \frac{s}{r} \right\} \right| \lesssim \frac{s}{r}.$$

Combining the above equations yields

$$M(t, r) \lesssim \|h(0, \cdot)\|_{L^\infty(H \times S^2)} \int_0^{r+t} r ds = 2r(r+t) \|h(0, \cdot)\|_{L^\infty(H \times S^2)}.$$

Using also that by Lemma 8.3.3

$$M(t, r) \leq 2\pi \|h(t, \cdot)\|_{L^1(H \times S^2)} = 2\pi \|h(0, \cdot)\|_{L^1(H \times S^2)} \lesssim \|h(0, \cdot)\|_{X_\beta(H \times S^2)},$$

we find

$$\begin{aligned} \max_{r \in \mathbb{R}_+} \frac{M(t, r)}{r} &\lesssim \max_{r \in \mathbb{R}_+} \min\{(r+t), r^{-1}\} \|h(0, \cdot)\|_{X_\beta(H \times S^2)} \\ &\lesssim (1+t) \|h(0, \cdot)\|_{X_\beta(H \times S^2)}. \end{aligned}$$

From the characteristic equations, we deduce

$$|h(t, x, \xi)| \leq e^{C(1+t)\|h(0, \cdot)\|_{X_\beta(H \times S^2)}} |h_0(X(0), \Xi(0)).$$

Using that $|\dot{X}| \leq 1$ we deduce (dealing with the weight of the L^∞ -norm as in Step 2 of the proof of Theorem 8.2.1)

$$\|h(t, \cdot)\|_{X_\beta(H \times S^2)} \lesssim e^{C(1+t)\|h(0, \cdot)\|_{X_\beta(H \times S^2)}} \|h(0, \cdot)\|_{X_\beta(H \times S^2)}.$$

This finishes the proof. \square

Proof of Theorem 8.3.2. With the help of the a priori estimate given in Proposition 8.3.4, the proof of global well-posedness is not difficult. First, local well-posedness is proved in the same manner as in the proof of Theorem 8.2.1. Then, one has to show that the quantities that determine the time of existence do not blow up in finite time. The first part is almost completely analogous to the proof of Theorem 8.2.1. We therefore do not repeat the technical details here.

One observes that the existence time only depends on

$$\|h(0, \cdot)\|_{X_\beta(H \times S^2)} + \|\nabla h(0, \cdot)\|_{X_\beta(H \times S^2)}.$$

We thus need to make sure that this quantity does not blow up in finite time. We already know from Proposition 8.3.4 that the X_β -norm of h does not blow up in finite time. It remains to argue that the same also holds true for the gradient.

This is true, because the estimates from Lemma 8.2.2 can be improved due to the symmetry. We recall from (8.3.5) that $v(t, x) = \psi(t, |x|)$, and from (8.3.6) that

$$|\partial_r^2 \psi(t, r)| = \left| -\frac{d}{dr} \frac{M(t, r)}{r} \right| = \left| \frac{M(t, r)}{r^2} - \sigma(r, t) \right| \lesssim \|h(t, \cdot)\|_{L^\infty(H \times S^2)}.$$

With this estimate, we are able to deduce a bound for $\|\nabla h(0, \cdot)\|_{X_\beta(H \times S^2)}$. Indeed, using the above estimate in the characteristic equations (8.3.7), we find

$$|(X, \Xi)(0, t, x_1, \xi_1) - (X, \Xi)(0, t, x_2, \xi_2)| \leq e^{Ct} |(x_1, \xi_1) - (x_2, \xi_2)|,$$

where C only depends on the L^∞ -norm of $h(s, \cdot)$, which we already control. Using the characteristic equation for Z in (8.3.7) and again the estimates for $M(r, t)r^{-1}$ and its derivative in terms of the L^1 -norm and the L^∞ -norm of h , we deduce

$$\|\nabla h(t, \cdot)\|_{X_\beta(H \times S^2)} \leq C(t),$$

where $C(t)$ only depends on t and $\|h(0, \cdot)\|_{X_\beta(H \times S^2)}$. This concludes the proof. \square

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