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Contents

Introduction

This dissertation is composed of four chapters, two of them on the overarching theme of *Bidding in Common-Value Auctions with Multidimensional Uncertainty*, and the two others on *Decoupling, Vote Trading, and Corporate Governance*.

In the first two chapters, which are joint work with Stephan Lauermann, we analyze common-value auctions in which bidders are either uninformed about the number of their competitors or their competitors' additional private values for the good. Compared to canonical models of common-value auctions, the second dimension of uncertainty renders the value inference from the price non-monotone. This can significantly alter bidding behavior. In particular, bidders may fail to behave competitively and pool on common bids, affecting the allocational and informational efficiency of the auction: the good may not be allocated to the bidder with the highest valuation and the bid distribution is less informative about the common value of the good. Besides immediate consequences for the first- and second-price auctions studied, our results also shed light on the inner workings of centralized markets. Our analyses, thereby, help to understand the impact of multidimensional uncertainty on the price discovery and efficiency of centralized markets.

In Chapter 1, *Bidding in Common-Value Auctions with an Uncertain Number of Competitors*, we consider a standard common-value first-price auction in which bidders are uncertain about the number of their competitors. We show that this second dimension of uncertainty invalidates classic findings for common-value auctions with a known number of rival bidders (Milgrom and Weber, 1982). In particular, the inference from winning is no longer monotone, and intermediate bids suffer from the strongest "winner's curse." As a result, bidding strategies may not be strictly increasing, giving rise to atoms in the bid distribution. The location of the atoms is indeterminate, implying equilibrium multiplicity. Moreover, an equilibrium fails to exist when the expected number of competitors is large, and the bid space is continuous.

In Chapter 2, *Auctions with Multidimensional Signals*, we analyze auctions in which the bidders' valuation for the good depends on both common and private-value components with bidders receiving (conditionally) independent signals regarding each component. Signals regarding the common component are either fully revealing or pure noise. Due to the multidimensionality of signals, the value of the good and the bids are not affiliated, such that conventional arguments cannot be used to prove existence of an equilibrium. In fact, when the good is sold in a second-price auction and the distribution of the private values is discrete, the bid distribution needs to contain atoms, thwarting equilibrium existence (Jackson, 2009). Using an approach that does not rely on affiliation, we show that when the private-value distribution is continuous, no atoms can arise. Despite the non-monotone inference from winning, an equilibrium exists and every equilibrium is pure and strictly increasing in both dimensions. We also establish existence of an equilibrium in the first-price auction, independent of the private-value distribution.

Chapters 3 and 4, work that was jointly done with Paul Voß, deal with the effects of decoupling and vote trading on corporate governance. Especially since the Global Financial Crisis in 2008, regulators have strived to strengthen shareholder oversight and voice by simplifying the voting process and giving shareholders more explicit power, for instance through "say on pay" requirements. While regulatory authorities have been trying to foster shareholder democracy, the foundation of shareholder voting, the linking of each shareholder's voting power to his or her economic exposure, appears to be eroding. Financial innovation has created a myriad ways for activist investors to acquire voting rights far in excess of their stake in the company, breaking with the old and prudent rule that the number of voting rights should be aligned with a shareholder's "skin in the game." In two chapters, we investigate the effects of this decoupling on corporate governance.

In Chapter 3, *The Economics of Decoupling*, we set out to provide structure to the multitude of ways activist investors can use to acquire voting rights without assuming economic exposure. We do so by classifying them into Buy&Hedge, Hedge&Buy, and Vote Trading techniques. The possibility to swing the outcome of a vote without bearing the effect on share value is of particular interest to an activist who wants to push her private agenda instead of maximizing firm value. Thus, we analyze which classes of decoupling techniques can be exploited profitably by a hostile activist who seeks to prevent a value-increasing reform in order to obtain a private benefit. We find that Vote Trading techniques pose the largest threat to shareholder and overall welfare while being most profitable for the hostile activist. Buy&Hedge techniques are constrained efficient because the activist suffers from a commitment problem. Hedge&Buy techniques exhibit inefficient and constrained-efficient equilibria. The results match the empirical evidence on vote prices from options and equity lending markets.

In Chapter 4, *Shareholder Votes on Sale*, we build on the results from Chapter 3 and analyze Vote Trading techniques in greater detail, in a model with a finite number of shareholders. We show that Vote Trading techniques enable hostile activism because voting rights trade at inefficiently low prices, even when the activist's hostile motives are transparent. Our results explain the empirical findings of low vote prices (Christoffersen et al., 2007) and inefficient outcomes (Hu and Black, 2006). Though an activist with superior information can facilitate information transmission through Vote Trading techniques, traditional activist intervention techniques, such as proxy fights, provide the same information transmission without the downsides inherent in Vote Trading techniques. Our analysis of potential policy measures suggests that adopting simple majority rules and excluding bought votes offer the most promising intervention avenues.

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Chapter 1

Bidding in Common-Value Auctions with an Uncertain Number of Competitors

Joint with Stephan Lauermann

1.1 Introduction

In most auctions, bidders are uncertain about the number of competitors they face:

- At auction houses such as Christie's and Sotheby's, personal attendance is in decline as bidders prefer to phone in or place their bids online. Therefore, bidders "[...] know even less about who they're bidding against, which in some cases can leave them wondering how high they should go."[1]
- eBay reveals the number of bidders who place a bid but does not disclose how many prospective bidders follow the auction. In particular, the platform does not display how many bidders are online to "snipe," that is, to place their bid in the last seconds of the auction (Roth and Ockenfels, 2002).
- Considering auction-like trading mechanisms, the continuous order book at the New York Stock Exchange informs market participants about the stream of (un)filled buy and sell orders, but reveals neither the number nor the identity of (potential) buyers and sellers.

Although uncertainty about the number of competitors, or "numbers uncertainty," is ubiquitous, the subject has received little attention in the literature of auction theory. One reason may be its irrelevance in standard auction formats with pure and independent private values: by a revenue-equivalence argument, equilibrium bids are just a weighted average of the bids that are optimal when the number of rival bidders is known (Krishna, 2010, Chapter 3.2.2).

[1] The Wall Street Journal, "Why Auction Rooms Seem Empty These Days", June 15, 2014, https://www.wsj.com/articles/with-absentee-bidding-on-the-rise-auction-rooms-seem-empty-these-days-1402683887 cf. Akbarpour and Li (2020).

By contrast, in a common-value setting, numbers uncertainty significantly alters bidding behavior. Recall that when the number of rival bidders is known, the classic results going back to Milgrom and Weber (1982) establish that there exists a unique symmetric equilibrium in the first-price and second-price auctions, in which the bids are strictly increasing in the bidders' own value estimates. Uniqueness and strict monotonicity facilitate the revenue comparison of auction formats, welfare considerations (in general interdependent value settings), and empirical identification strategies. We show that these classic results no longer hold when the number of competitors is uncertain. Equilibria are generally not strictly increasing but contain atoms. The location of the atoms is often indeterminate, implying equilibrium multiplicity. Moreover, equilibrium payoffs are discontinuous at the atoms, invalidating standard methods for analyzing bidding behavior in these auctions. In particular, with a continuous bid space, equilibrium generally fails to exist.

To model an auction with numbers uncertainty, we start with a canonical common-value first-price auction. The value of the good is binary (high or low) and bidders receive conditionally independent and identically distributed signals, with higher signals indicating a higher value (affiliation). Each bidder simultaneously submits a bid, the highest bidder wins, and pays her bid. Ties are broken uniformly. The only difference from the textbook setting is that the number of (rival) bidders is not known, but instead a random variable which is assumed to be Poisson distributed. However, our results extend beyond this distributional assumption.

Numbers uncertainty affects bidding behavior with common values because it changes the value inference from winning. In a conventional common-value auction with a known number of bidders, the expected value conditional on winning is increasing in the relative position of the bid because a higher bid eases the "winner's curse." In fact, there is no winner's curse at the very top bid. This reduction reinforces price competition and implies the absence of pooling (atoms in the bid distribution). Note that at any bid below the top one, the winner's curse is more severe if there are more competitors.

With numbers uncertainty, winning is also informative about the number of rival bidders. In particular, winning with a low bid is more likely when there are fewer competitors which eases the winner's curse. Therefore, winning with a low bid is not necessarily bad news about the value of the good. In our model, the inference is U-shaped: intermediate bids are subject to the strongest winner's curse, while there is no winner's curse at the bottom or the top (Lemma 1.2 and 1.4).[2]

We show that every equilibrium is nondecreasing in the bidder's signal (Lemma 1.1), but the non-monotone inference implies that equilibria cannot be strictly in-

[2]The random number of competitors adds a second dimension of uncertainty. Thus, the value of the good is no longer affiliated with the first-order statistic of the signals.

creasing unless the expected number of competitors is sufficiently small (Propositions 1.1 and 1.2). Hence, the equilibrium bid distribution contains one or more atoms, as bidders with different signals pool on common bids. Numbers uncertainty incentivize bidders to pool because pooling shields them against the winner's curse: under a uniform tie-breaking rule, winning the auction with a bid that ties with positive probability is relatively more likely when there are fewer competitors, which reduces the negative inference from winning. An example in Appendix 1.B.1 demonstrates that atoms already occur in very small auctions, namely when the expected number of rival bidders is larger than one.

The presence of atoms in the bid distribution substantially alters the analysis of the auction. First, the location of atoms is often indeterminate, as illustrated by two examples in Appendices 1.B.2 and 1.B.3. Second, atoms create discontinuities in the bidders' payoffs. As a result of these discontinuities, no equilibrium exists when the expected number of bidders is sufficiently large (Proposition 1.3).

If the bid space is discrete rather than continuous, equilibria do exist by standard arguments (Lemma 1.9). To study the resulting bidding behavior on a fine grid, we utilize a "communication extension" of the auction, based on Jackson et al. (2002). In the communication extension, bidders not only submit a monetary bid from the continuous bid space but also a message that indicates their "eagerness" to win, which is used to break ties. The communication extension is useful because, in contrast to the standard auction, the limit of any converging sequence of equilibria on the ever-finer grid corresponds to an equilibrium of the communication extension. Since such an equilibrium inherits the properties of the equilibria on the fine grid, we can use the equilibrium characterization of the communication extension in Proposition 1.4 to derive the equilibria on a fine grid (Proposition 1.5).

Qualitatively, any equilibrium on a fine grid with increments $d > 0$ consists of three regions. Bidders with high signals essentially follow a strictly increasing strategy (as the grid permits), while bidders with intermediate signals pool on some bid b_p, and bidders with low signals bid one increment below it, $b_p - d$.[3]

The equilibria are shaped by a severe winner's curse at b_p, and a "winner's blessing" that arises at bids below b_p, so that, at these bids, the expected value conditional on winning is significantly higher than b_p. This induces bidders with low signals to compete for the largest bid strictly below b_p. On the grid, this competition leads them to pool on $b_p - d$; on the continuous bid space, the non-existence of a largest bid below b_p implies the non-existence of an equilibrium.

[3] In the limit of the ever-finer grid, the two bids b_p and $b_p - d$ "merge," such that low-signal bidders win with the same probability as intermediate-signal bidders. Hence, the limit strategy with $d = 0$ is generally not an equilibrium of the continuous bid space with the standard uniform tie-breaking rule. In contrast, the communication extension allows bidders with low and intermediate bids to send different messages, such that they can be differentiated.

We discuss the robustness of our results in Section 1.6. We argue that our findings do not depend on the Poisson distribution of the number of bidders, and that similar results hold in the second-price auction. Finally, we discuss the related literature on auctions with a non-constant number of bidders, especially recent contributions by Murto and Välimäki (2019) and Lauermann and Wolinsky (2018).

1.2 Model

A single, indivisible good is sold in a first-price, sealed-bid auction. The good's value is either high, v_h, or low, v_ℓ, with $v_h > v_\ell \geq 0$, depending on the unknown state of the world $\omega \in \{h, \ell\}$. The state is $\omega = h$ with probability ρ and $\omega = \ell$ with probability $1 - \rho$, where $\rho \in (0, 1)$. The number of bidders is a Poisson-distributed random variable with mean η, such that there are i bidders in the auction with probability $e^{-\eta}\frac{\eta^i}{i!}$. The realization of the variable is unknown to the bidders.

Every bidder receives a signal s from the compact set $[\underline{s}, \bar{s}]$. Conditional on the state, the signals are independent and identically distributed according to the cumulative distribution functions F_h and F_ℓ, respectively. Both distributions have continuous densities f_ω, and the likelihood ratio of these densities, $\frac{f_h(s)}{f_\ell(s)}$, satisfies the (weak) monotone likelihood ratio property: for all $s < s'$ it holds that $\frac{f_h(s)}{f_\ell(s)} \leq \frac{f_h(s')}{f_\ell(s')}$. Furthermore, $0 < \frac{f_h(\underline{s})}{f_\ell(\underline{s})} < \frac{f_h(\bar{s})}{f_\ell(\bar{s})} < \infty$, such that signals do contain information but never reveal the state perfectly. For convenience, let there be a unique neutral signal $\breve{s}$ at which $\frac{f_h(\breve{s})}{f_\ell(\breve{s})} = 1$.

Having received her signal, every bidder submits a bid b. Suppose that there is a reserve price at v_ℓ, and note that it is without loss to exclude bids above v_h, such that $b \in [v_\ell, v_h]$. The bidder with the highest bid wins the auction, receives the object, and pays her bid. Ties are broken uniformly. If there is no bidder, the good is not allocated. Bidders are risk neutral.

It is useful to recall two special properties of the Poisson distribution prior to beginning the analysis. A detailed derivation and discussion can be found in Myerson (1998). First, when participating in the auction, a bidder does not change her belief regarding the number of other bidders in the auction. Therefore, her belief about the number of her competitors is again a Poisson distribution with mean η. This property is analogous to a stationary Poisson process, in which an event does not allow for inferences about the number of other events.

Second, the Poisson distribution implies that attention can be restricted to symmetric equilibria.[4] Since the Poisson distribution has an unbounded support, it

[4] This fits our aim of analyzing how uncertainty about the number of competitors rather than their identity affects the equilibrium bidding behavior.

draws bidders from a hypothetical infinite urn. Any individual bidder and, thus, any individual strategy are thereby drawn with zero probability. One could imagine that certain fractions of the population in the urn follow different strategies, such that those are encountered with positive probability. However, this would be equivalent to drawing the bidders first and having them mix between strategies afterward.

Accordingly, we consider symmetric strategies, which are functions mapping from the signals into the set of probability distributions over bids $\beta : [\underline{s}, \bar{s}] \to \Delta[v_\ell, v_h]$. Let $\pi_\omega(b; \beta)$ denote the probability of winning the auction with a bid b in state ω, if the rival bidders follow strategy β. Using Bayes' rule, the interim expected utility for a bidder with signal s bidding b is

$$\begin{aligned} U(b|s;\beta) = & \frac{\rho f_h(s)}{\rho f_h(s) + (1-\rho) f_\ell(s)} \pi_h(b;\beta)(v_h - b) \\ & + \frac{(1-\rho) f_\ell(s)}{\rho f_h(s) + (1-\rho) f_\ell(s)} \pi_\ell(b;\beta)(v_\ell - b). \end{aligned} \tag{1.1}$$

A strategy β^* is a *best response* to a strategy β, if, for (almost) all s, a bid $b \in \operatorname{supp} \beta^*(s)$ implies that $b \in \arg\max_{\hat{b} \in [v_\ell, v_h]} U(\hat{b}|s;\beta)$. Henceforth, we distinguish between claims that hold everywhere and almost everywhere only when it is central to the argument. Unless specified otherwise, results hold for almost all s. Two strategies are *equivalent* if they correspond to the same distributional strategy after merging all signals that share the same likelihood ratio $\frac{f_h}{f_\ell}$. Thus, equivalent strategies imply the same distribution over bids and utilities.

Lemma 1.1 *Let β be some strategy and β^* a best response to it. If the likelihood ratio $\frac{f_h}{f_\ell}$ is strictly increasing, then β^* is essentially*[5] *pure and nondecreasing. If the likelihood ratio is only weakly increasing, then there exists an equivalent best response $\hat{\beta}^*$ that is pure and nondecreasing.*

The proof is in the appendix. Higher bids improve the prospects of winning, which is desirable in the high state in which the winner turns a profit ($b \leq v_h$), but disadvantageous in the low state in which the winner incurs a loss ($b \geq v_\ell$). Thus, more optimistic bidders are willing to bid more aggressively. If the likelihood ratio $\frac{f_h}{f_\ell}$ is constant along some interval, the bids can always be ordered to be nondecreasing along this interval.

We look for Bayes-Nash equilibria β^*, and, by Lemma 1.1, can restrict attention to pure and nondecreasing strategies. In the following, strategies are nondecreasing functions mapping signals into bids, $\beta : [\underline{s}, \bar{s}] \to [v_\ell, v_h]$.

[5] Up to a set of signals with measure zero.

1.3 Analysis of the standard auction

The analysis is structured into three parts. The first subsection focuses on the winning probability and inference from bids that never tie. We then use our findings to examine strictly increasing strategies, and show that there can be no strictly increasing equilibrium unless the expected number of bidders is sufficiently small. Hence, there have to be pooling bids—that is, atoms in the bid distribution. We investigate these atoms in the second subsection. Last, we argue that the atoms in the bid distribution necessarily prevent equilibrium existence.

1.3.1 Non-pooling bids

Fix some nondecreasing strategy β. A bid b is a *non-pooling bid* if it is selected with zero probability by any bidder. Given strategy β, this is the case if b is either not in the image of β, or if there is only a single signal s such that $\beta(s) = b$. In either situation, a bidder who chooses b wins whenever all of her competitors bid below b. Since β is nondecreasing, this implies that they all received lower signals than $\hat{s} = \sup\{s \colon \beta(s) \leq b\}$. Thus, the bidder wins in the event that $s_{(1)} \leq \hat{s}$, where $s_{(1)} = \sup\{s_{-i}\}$ is the highest of the competitors' signals. We employ the convention that $\sup\{\emptyset\} = -\infty$, which means that $s_{(1)} = -\infty$ in case there is no competitor. As a result, the generalized first-order statistic $s_{(1)}$ has a cumulative distribution function $F_{s_{(1)}}(s|\omega) = e^{-\eta(1-F_\omega(s))}$ for $s \in [\underline{s}, \bar{s}]$.[6] Since bid b wins whenever $s_{(1)} \leq \hat{s}$, it wins in state $\omega \in \{h, \ell\}$ with probability $\pi_\omega(b; \beta) = e^{-\eta(1-F_\omega(\hat{s}))}$.

A characteristic feature of common-value auctions is that winning is informative about the value of the good. When choosing a non-pooling bid, all that matters for this inference is the relative position of the bid, $\hat{s}$. Next, we analyze how this position $\hat{s}$ affects the conditional expected value, $\mathbb{E}[v|\text{win with } b; \beta] = \mathbb{E}[v|s_{(1)} \leq \hat{s}]$, with

$$\mathbb{E}[v|s_{(1)} \leq \hat{s}] = \frac{\rho e^{-\eta(1-F_h(\hat{s}))} v_h + (1-\rho) e^{-\eta(1-F_\ell(\hat{s}))} v_\ell}{\rho e^{-\eta(1-F_h(\hat{s}))} + (1-\rho) e^{-\eta(1-F_\ell(\hat{s}))}}. \tag{1.2}$$

Recall that $\breve{s}$ is the unique neutral signal, $\frac{f_h(\breve{s})}{f_\ell(\breve{s})} = 1$.

[6] Conditional on state ω, any competitor (independently) receives a signal larger than $\hat{s}$ with probability $1-F_\omega(\hat{s})$. By the decomposition and environmental equivalence properties of the Poisson distribution (Myerson, 1998), bidders believe that the number of rival bidders with signals larger than $\hat{s}$ is Poisson distributed with mean $\eta(1-F_\omega(\hat{s}))$. The probability that $s_{(1)} \leq \hat{s}$ is the probability that there is no competitor with a signal above $\hat{s}$.

Lemma 1.2 *The conditional expected value* $\mathbb{E}[v|s_{(1)} \leq \hat{s}]$ *is strictly decreasing in* $\hat{s}$ *when* $\hat{s} < \breve{s}$, *has its unique global minimum at* $\hat{s} = \breve{s}$, *and is strictly increasing when* $\hat{s} > \breve{s}$.

Proof. Note that $\frac{av_h+v_\ell}{a+1} > \frac{bv_h+v_\ell}{b+1}$ if and only if $a > b$. By (1.2), this means that $\mathbb{E}[v|s_{(1)} \leq \hat{s}]$ is strictly increasing if and only if $e^{\eta(F_h(\hat{s})-F_\ell(\hat{s}))}$ is strictly increasing. Its derivative is $e^{\eta(F_h(\hat{s})-F_\ell(\hat{s}))}\eta[f_h(\hat{s}) - f_\ell(\hat{s})]$ and so $e^{\eta(F_h(\hat{s})-F_\ell(\hat{s}))}$ is increasing if and only if $f_h(\hat{s}) > f_\ell(\hat{s})$. The uniqueness of the neutral signal $\breve{s}$ where $f_h(\breve{s}) = f_\ell(\breve{s})$ and the monotone likelihood ratio property imply that $f_h(\hat{s}) < f_\ell(\hat{s})$ for $\hat{s} < \breve{s}$, and $f_h(\hat{s}) > f_\ell(\hat{s})$ for $\hat{s} > \breve{s}$. □

Lemma 1.2 implies that $\mathbb{E}[v|s_{(1)} \leq \hat{s}]$ is U-shaped in $\hat{s}$ with its minimum at $\breve{s}$. The intuition behind the shape may be explained best with the help of Figure 1.1:

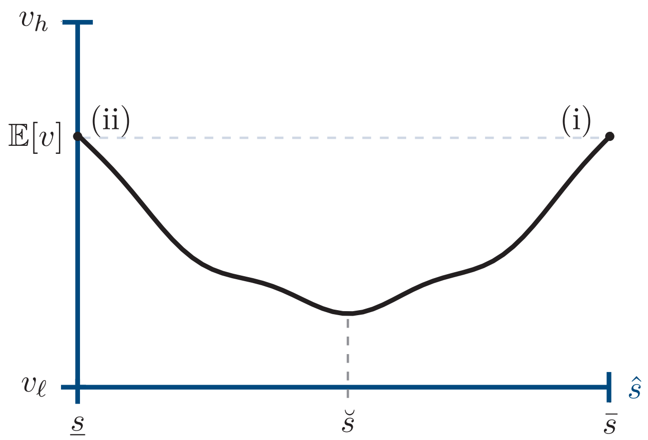

Figure 1.1 The conditional expected value $\mathbb{E}[v|s_{(1)} \leq \hat{s}]$.

First, consider point (i) on the top right, which marks $\mathbb{E}[v|s_{(1)} \leq \bar{s}]$. By definition, the highest signal, $s_{(1)}$, is always smaller than $\bar{s}$, independent of the state. Hence, the event that $s_{(1)} \leq \bar{s}$ is uninformative about the state and $\mathbb{E}[v|s_{(1)} \leq \bar{s}] = \mathbb{E}[v]$.

Second, consider point (ii) on the top left, denoting $\mathbb{E}[v|s_{(1)} \leq \underline{s}]$. The highest signal $s_{(1)}$ equals $\underline{s}$ with zero probability (the signal distribution has no atoms), while there are no competitor and $s_{(1)} = -\infty$ with positive probability. Consequently, $\mathbb{E}[v|s_{(1)} \leq \underline{s}] = \mathbb{E}[v|s_{(1)} = -\infty]$. Since the distribution of bidders is independent of the state, this event occurs with the same probability in both states. As a result, the event that $s_{(1)} \leq \underline{s}$ is also uninformative about the state and $\mathbb{E}[v|s_{(1)} \leq \underline{s}] = \mathbb{E}[v]$. Thus, there is no winner's curse at the bottom (ii) or at the top (i).

In the middle where $\hat{s} \in (\underline{s}, \bar{s})$, the winner's curse comes into play. With positive probability, there are competitor, all of which received signals below $\hat{s}$. These low signals are bad news about the value of the good. Consequently, for $\hat{s} \in (\underline{s}, \bar{s})$, the conditional expected value is smaller than the unconditional one, $\mathbb{E}[v|s_{(1)} \leq \hat{s}] < \mathbb{E}[v]$, with the global minimum at $\check{s}$, where $f_h(\check{s}) = f_\ell(\check{s})$.

Observe that as η increases, the winner expects to face more rival bidders, such that the winner's curse grows more severe. For $\hat{s} \in (\underline{s}, \bar{s})$, it follows that $\mathbb{E}[v|s_{(1)} \leq \hat{s}] \overset{\eta\to\infty}{\to} v_\ell$.[7] At the boundaries $\underline{s}$ and $\bar{s}$, on the other hand, the inference is independent of η; therefore, $\mathbb{E}[v|s_{(1)} \leq s]$ converges in η to a $\sqcup$-shape.

While the precise form of $\mathbb{E}[v|s_{(1)} \leq \hat{s}]$ follows from the Poisson distribution, the same effects are present under any distribution of bidders. Importantly, the non-monotonicity does not depend on the possibility that there is no rival bidder,[8] but is a consequence of the variation in the number of (rival) bidders. At any bid below the top, the winning bidder simultaneously updates her belief over two random variables: the number of competitors and their signal realization. Since these two can push the conditional expected value in opposite directions, the winning bidder's inference will generally not be monotone in $\hat{s}$. In other words, numbers uncertainty breaks the affiliation between the value of the good and the first-order statistic of (rivals') signals.

1.3.1.1 No strictly increasing equilibrium when η is large

The non-monotone inference from winning can substantially affect the equilibrium behavior of bidders. As a benchmark, consider the standard common-value auction with a fixed and known number of $n \geq 2$ bidders. In this setup, the inference is monotone, which implies that the unique symmetric equilibrium is strictly increasing.[9] When the numbers uncertainty causes a non-monotone inference, an equilibrium of this form generally does not exist.

Proposition 1.1 *When η is sufficiently large, no strictly increasing equilibrium exists.*

In Appendix 1.B.1 we provide an example which shows that strictly increasing equilibria can fail to exist for η as low as 1. Here, we first give an intuitive, verbal argument before sketching out the critical steps of the proof, which is also relegated to the appendix.

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[7] The monotone likelihood ratio property implies that $F_h(s) < F_\ell(s)$ for all $s \in (\underline{s}, \bar{s})$. Thus, $\eta(F_h(s) - F_\ell(s)) \to -\infty$ for all $s \in (\underline{s}, \bar{s})$ when $\eta \to \infty$. The convergence then follows by equation (1.2).

[8] For instance, if we consider a truncated Poisson distribution in which there are always at least $\underline{n} \geq 2$ bidders, $\mathbb{E}[v|s_{(1)} \leq \hat{s}]$ is still U-shaped when η is large. At the top, the inference from winning is unaffected by the truncation, and at the bottom, the winning bidder still updates her belief toward the lowest number of rival bidders possible, $\underline{n} - 1$. Thus, there is a bounded winner's curse at $\underline{s}$ which, however, does not depend on η. Since the winner's curse grows arbitrary large at any $\hat{s} \in (\underline{s}, \bar{s})$ when η increases, this results in the U-shape.

[9] There is an exception: If $\frac{f_h}{f_\ell}$ is constant along some interval at the bottom of the signal distribution, $[\underline{s}, s]$, then these signals choose the same bid (cf. Proposition 2 in Lauermann and Wolinsky (2017)).

Suppose to the contrary that there is a strictly increasing equilibrium β^* for an arbitrary large η arbitrary. In this case, a bidder with signal s, following the strategy β^* wins whenever $s_{(1)} \leq s$. Conditional on winning, and her own signal, she, therefore, expects the good to be of value $\mathbb{E}[v|\text{win with } \beta^*(s), s; \beta^*] = \mathbb{E}[v|s_{(1)} \leq s, s]$, with

$$\mathbb{E}[v|s_{(1)} \leq s, s] = \frac{\rho f_h(s) e^{-\eta(1-F_h(s))} v_h + (1-\rho) f_\ell(s) e^{-\eta(1-F_\ell(s))} v_\ell}{\rho f_h(s) e^{-\eta(1-F_h(s))} + (1-\rho) f_\ell(s) e^{-\eta(1-F_\ell(s))}}. \tag{1.3}$$

When η is large, the expected competition is fierce, which implies that equilibrium bids must be close to the expected value conditional on winning, $\beta^*(s) \approx \mathbb{E}[v|s_{(1)} \leq s, s]$. In addition to that, a large η makes the inference from winning more relevant for the expected value than the bidder's own signal. Consequently, when η is large, $\mathbb{E}[v|s_{(1)} \leq s, s]$ inherits the U-shape from $\mathbb{E}[v|s_{(1)} \leq s]$. Taken together, this means that $\beta^*(s)$ is decreasing below the neutral signal $\breve{s}$, which is a contradiction.[10]

To formalize this contradiction, fix three signals $s_- < s_\circ < s_+$ with $s_+ < \breve{s}$. The argument is structured into three steps. First, we derive an upper bound on the bid $\beta^*(s_+)$ from individual rationality (Step 1), and then a lower bound on $\beta^*(s_\circ)$ from the incentive constraints of s_- (Step 2). Step 3 shows that when η is large, the lower bound exceeds the upper bound.

Step 1 *An upper bound on $\beta^*(s_+)$ is given by*

$$\frac{\beta^*(s_+) - v_\ell}{v_h - \beta^*(s_+)} \leq \frac{\rho}{1-\rho} \frac{f_h(s_+)}{f_\ell(s_+)} \frac{e^{-\eta(1-F_h(s_+))}}{e^{-\eta(1-F_\ell(s_+))}}. \tag{1.4}$$

In equilibrium, it has to hold that $\beta^*(s) \leq \mathbb{E}[v|\text{win with } \beta^*(s), s; \beta^*]$ for (almost) any signal s. Otherwise, the utility

$$U(\beta^*(s)|s; \beta^*) = \mathbb{P}[\text{win with } \beta^*(s)|s; \beta^*]\Big(\mathbb{E}[v|\text{win with } b, s; \beta^*] - \beta^*(s)\Big)$$

is negative, which cannot be the case in equilibrium, because a bid of v_ℓ guarantees a non-negative payoff. Using (1.3), the condition $\beta^*(s) \leq \mathbb{E}[v|\text{win with } \beta^*(s), s; \beta^*]$ can be rearranged to

$$\frac{\beta^*(s) - v_\ell}{v_h - \beta^*(s)} \leq \frac{\rho}{1-\rho} \frac{f_h(s)}{f_\ell(s)} \frac{\pi_h(\beta^*(s); \beta^*)}{\pi_\ell(\beta^*(s); \beta^*)}. \tag{1.5}$$

Now, inequality (1.4) follows from (1.5) with s_+ and $\pi_\omega(\beta^*(s_+); \beta^*) = e^{-\eta(1-F_\omega(s_+))}$ because β^* is a strictly increasing strategy.

[10] The crucial step of the proof is to show that $\beta^*(s)$ converges to $\mathbb{E}[v|s_{(1)} \leq s, s]$ quick enough, such that the U-shape of $\mathbb{E}[v|s_{(1)} \leq s, s]$ can be exploited. Otherwise, the argument might fail because $\mathbb{E}[v|s_{(1)} \leq s, s]$ converges to v_ℓ for all $s \in (\underline{s}, \bar{s})$.

Step 2 *A lower bound on $\beta^*(s_\circ)$ is given by*

$$\frac{\beta^*(s_\circ) - v_\ell}{v_h - \beta^*(s_\circ)} \geq \frac{\rho}{1-\rho} \frac{f_h(s_-)}{f_\ell(s_-)} \frac{e^{-\eta(1-F_h(s_\circ))}}{e^{-\eta(1-F_\ell(s_\circ))}} A(\eta), \tag{1.6}$$

where $A(\eta)$ is a decreasing function with $\lim_{\eta\to\infty} A(\eta) = 1$.

In equilibrium, a bidder with a signal s_- cannot have an incentive to deviate from $\beta^*(s_-)$ to $\beta^*(s_\circ)$, which implies that $U(\beta^*(s_-)|s_-;\beta^*) \geq U(\beta^*(s_\circ)|s_-;\beta^*)$. In the appendix, we show that this condition can be used to derive (1.6). Observe that when $A(\eta) = 1$, the inequality rearranges to $\beta^*(s_\circ) \geq \mathbb{E}[v|s_{(1)} \leq s_\circ, s_-]$. Since the argument holds for any $s_- < s_\circ$, $A(\eta) \to 1$ captures the observation that when η is large, bids have to be close to the expected value conditional on winning.

Step 3 *When η is sufficiently large, the upper bound on $\beta^*(s_+)$ expressed by (1.4) is smaller than the lower bound on $\beta^*(s_\circ)$ given by inequality (1.6).*

Since $\beta^*(s_+) > \beta^*(s_\circ)$ and $\frac{b-v_\ell}{v_h-b}$ is increasing in b, a necessary condition for both inequalities to hold simultaneously is that

$$\frac{\rho}{1-\rho} \frac{f_h(s_+)}{f_\ell(s_+)} \frac{e^{-\eta(1-F_h(s_+))}}{e^{-\eta(1-F_\ell(s_+))}} > \frac{\rho}{1-\rho} \frac{f_h(s_-)}{f_\ell(s_-)} \frac{e^{-\eta(1-F_h(s_\circ))}}{e^{-\eta(1-F_\ell(s_\circ))}} A(\eta).$$

This can be rearranged to

$$\frac{f_h(s_+)}{f_\ell(s_+)} \left(\frac{f_h(s_-)}{f_\ell(s_-)}\right)^{-1} > \frac{e^{-\eta(1-F_h(s_\circ))}}{e^{-\eta(1-F_\ell(s_\circ))}} \left(\frac{e^{-\eta(1-F_h(s_+))}}{e^{-\eta(1-F_\ell(s_+))}}\right)^{-1} A(\eta). \tag{1.7}$$

The fractions $\frac{e^{-\eta(1-F_h(s_\circ))}}{e^{-\eta(1-F_\ell(s_\circ))}} \left(\frac{e^{-\eta(1-F_h(s_+))}}{e^{-\eta(1-F_\ell(s_+))}}\right)^{-1}$ capture the difference in the inference from winning when $s_{(1)} \leq s_\circ$ instead of $s_{(1)} \leq s_+$. Since $s_\circ < s_+ < \check{s}$, the signals are from the decreasing leg of $\mathbb{E}[v|s_{(1)} \leq s]$ such that the fraction is larger than one. In fact, the difference in inference grows without bound,[11]

$$\frac{e^{-\eta(1-F_h(s_\circ))}}{e^{-\eta(1-F_\ell(s_\circ))}} \left(\frac{e^{-\eta(1-F_h(s_+))}}{e^{-\eta(1-F_\ell(s_+))}}\right)^{-1} = e^{\eta([F_\ell(s_+)-F_\ell(s_\circ)]-[F_h(s_+)-F_h(s_\circ)])} \to \infty. \tag{1.8}$$

Since $A(\eta) \to 1$, this means that the right side of equation (1.7) becomes arbitrary large, while the left side stays constant. Hence, when η is large, the inference from winning (right side) dominates the inference from the signals (left side). This echoes the fact that $\mathbb{E}[v|s_{(1)} \leq s, s]$ becomes U-shaped as η grows. As a result, inequality (1.7) cannot hold, and β^* cannot be a strictly increasing equilibrium.

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[11] $[F_\ell(s_+) - F_\ell(s_\circ)] - [F_h(s_+) - F_h(s_\circ)] = \int_{s_\circ}^{s_+} [f_\ell(z) - f_h(z)]dz \geq \int_{s_\circ}^{s_+} f_\ell(z)(1 - \frac{f_h(s_+)}{f_\ell(s_+)})dz > 0$ since $s_+ < \check{s}$.

1.3.1.2 Unique strictly increasing equilibrium when η is small

When η is small, we can give necessary and sufficient conditions for the existence of a strictly increasing equilibrium. For $s, s' \in [\underline{s}, \bar{s}]$, let $F_{s_{(1)}}(s'|s)$ denote the expected cumulative distribution function of $s_{(1)}$ conditional on observing s,

$$F_{s_{(1)}}(s'|s) = \frac{\rho f_h(s)}{\rho f_h(s) + (1-\rho) f_\ell(s)} e^{-\eta(1-F_h(s'))} + \frac{(1-\rho) f_\ell(s)}{\rho f_h(s) + (1-\rho) f_\ell(s)} e^{-\eta(1-F_\ell(s'))},$$

and let $f_{s_{(1)}}(s'|s)$ be the associated density.

Proposition 1.2 *The ordinary differential equation*

$$\frac{\partial}{\partial s}\beta(s) = \Big(\mathbb{E}[v|s_{(1)} = s, s] - \beta(s)\Big)\frac{f_{s_{(1)}}(s|s)}{F_{s_{(1)}}(s|s)} \qquad \text{with } \beta(\underline{s}) = v_\ell \tag{1.9}$$

has a unique solution $\hat{\beta}$.

(i) If $\hat{\beta}$ *is strictly increasing, then it is the unique equilibrium in the class of strictly increasing equilibria.*

(ii) If $\hat{\beta}$ *is not strictly increasing, no strictly increasing equilibrium exists.*

The proof is provided in the appendix.[12] The argument that no strictly increasing equilibrium exists made use of two effects of a large η: that the winner's curse determines the shape of $\mathbb{E}[v|s_{(1)} \leq s, s]$ and that competition is fierce. Lemma 1.3 shows that both of these conditions are necessary; when the expected value conditional on winning is monotone, or competition is lax, a strictly increasing equilibrium exists.

Lemma 1.3 *A strictly increasing equilibrium exists if either*

(i) $\mathbb{E}[v|s_{(1)} = s, s]$ *is strictly increasing in* s*;*

(ii) or η *is sufficiently small.*

First, if $\mathbb{E}[v|s_{(1)} \leq s, s]$ is monotone, the existence problem described above does not arise. Even if bids are close to the conditional expected value, the bidding function can be strictly increasing. Indeed, there is a slightly tighter[13] bound and a strictly increasing equilibrium exists if $\mathbb{E}[v|s_{(1)} = s, s]$ is strictly increasing in s. This is the case if and only if

$$2\Big(\frac{\partial}{\partial s}\frac{f_h(s)}{f_\ell(s)}\Big)\frac{f_\ell(s)}{f_h(s)} + \eta f_h(s) - \eta f_\ell(s) > 0 \text{ for a.e. } s \in [\underline{s}, \bar{s}]. \tag{1.10}$$

[12] Apart from the slightly different definition of $s_{(1)}$, this is the standard ODE in the literature cf. (Krishna, 2010, Chapter 6.4).

[13] $\mathbb{E}[v|s_{(1)} \leq s, s]$ is strictly increasing if and only if $\Big(\frac{\partial}{\partial s}\frac{f_h(s)}{f_\ell(s)}\Big)\frac{f_\ell(s)}{f_h(s)} + \eta f_h(s) - \eta f_\ell(s) > 0$.

Observe that when $\frac{f_h}{f_\ell}$ is constant over some interval below the neutral signal $\check{s}$, condition (1.10) is never fulfilled. However, even in this case, a strictly increasing equilibrium exists when η is small. If competition is very weak, bids stay far below the expected value conditional on winning. Therefore, the problem described in Section 1.3.1.1 does not arise, and a strictly increasing equilibrium exists.

1.3.1.3 Strictly increasing equilibria and the second-price auction

In a second-price auction, standard arguments imply that the equilibrium bid in a symmetric and strictly increasing equilibrium is the expected value conditional on being tied at the top, $\mathbb{E}[v|s_{(1)} = s, s]$. Thus, condition (1.10) is necessary and sufficient for the existence of a strictly increasing equilibrium, and no such equilibrium exists when η is large. Similar problems also arise for other distributions of the number of bidders. For instance, Harstad et al. (2008) provide an example in which the distribution is binary and no strictly increasing equilibrium exists.

Wrapping up, Section 1.3.1 demonstrates that uncertainty over the number of competitors prevents the existence of a strictly increasing equilibrium unless η is sufficiently small. Combined with Lemma 1.1, this implies that if an equilibrium exists, it has to be piecewise flat. Next, we take a closer look at these flat parts to understand why bidders with different signals may have an incentive to pool on the same bid.

1.3.2 Pooling bids

Fix some nondecreasing strategy β, and suppose that $\beta(s) = b_p$ for all s from an interval, but $\beta(s) \neq b_p$ otherwise. We generally refer to the interval as a *pool*, to b_p as a *pooling bid* and, without loss, always consider the closure of interval which is denoted by $[s_-, s_+]$. We show by a simple computation (proof of Lemma 1.4) that the probability to win with b_p in state $\omega \in \{h, \ell\}$ is

$$\pi_\omega(b_p; \beta) = \frac{\mathbb{P}[s_{(1)} \in [s_-, s_+] \,|\omega]}{\mathbb{E}[\#s \in [s_-, s_+] \,|\omega]} = \frac{e^{-\eta(1-F_\omega(s_+))} - e^{-\eta(1-F_\omega(s_-))}}{\eta(F_\omega(s_+) - F_\omega(s_-))}, \tag{1.11}$$

where "$\mathbb{E}[\#s \in [s_-, s_+]$" denotes the expected number of signal realizations from the interval $[s_-, s_+]$.

Bidders have an incentive to pool because it insures them against the winner's curse, meaning that the expected value conditional on winning with the bid b_p is larger than the expected value conditional on winning with a bid marginally above b_p,

$$\mathbb{E}[v|\text{win with } b_p; \beta] > \lim_{\epsilon \to 0} \mathbb{E}[v|\text{win with } b_p + \epsilon; \beta] = \mathbb{E}[v|s_{(1)} \leq s_+].$$

If this wasn't he case, any bidder with a signal $s \in [s_-, s_+]$ would have an incentive to marginally overbid b_p, raising the expected profits conditional on winning.[14] In Appendices 1.B.2 and 1.B.3 we provide two examples of equilibria with atoms.

To gain intuition into how winning with b_p can ease the winner's curse compared to winning with a marginally higher bid, consider the following reasoning. With positive probability, multiple bidders tie on the pooling bid b_p, such that the winner is decided by the uniform tie-breaking rule. Consequently, the bid b_p is more likely to win when there are fewer competitors who also choose b_p, that is, when there are fewer competitors with signals from $[s_-, s_+]$. If those signals are low, meaning that they are more likely to realize in the low state, this implies that b_p wins less often in the low state than in the high state. Since the bid marginally above b_p never ties, it loses this blessing.

For this insurance to work, the number of competitor has to be uncertain. Otherwise, winning more often when there are fewer competitors with signals from $[s_-, s_+]$ means winning more often when there are more competitors with signals below s_-. This exacerbates the winner's curse. When the number of bidders is Poisson distributed, the number of bidders with signals below s_- is conditionally independent from the number of bidders with signals from $[s_-, s_+]$. Therefore, winning with b_p is more advantageous than winning with a marginally higher bid whenever the expected number of (rival) bidders with signals from $[s_-, s_+]$, that is, $\eta[F_\omega(s_+) - F_\omega(s_-)]$, is larger in the low state than in the high state.

Formalizing these observations (proof is in the appendix) gives us the following Lemma.

Lemma 1.4 *Assume that β is some strategy for which there exists an interval $[s_-, s_+]$ and a bid b_p such that $\beta(s) = b_p$ for all $s \in [s_-, s_+]$ and $\beta(s) < b_p < \beta(s')$ for all $s < s_- < s_+ < s'$.*

(i) If $\eta[F_h(s_+) - F_h(s_-)] < \eta[F_\ell(s_+) - F_\ell(s_-)]$, then

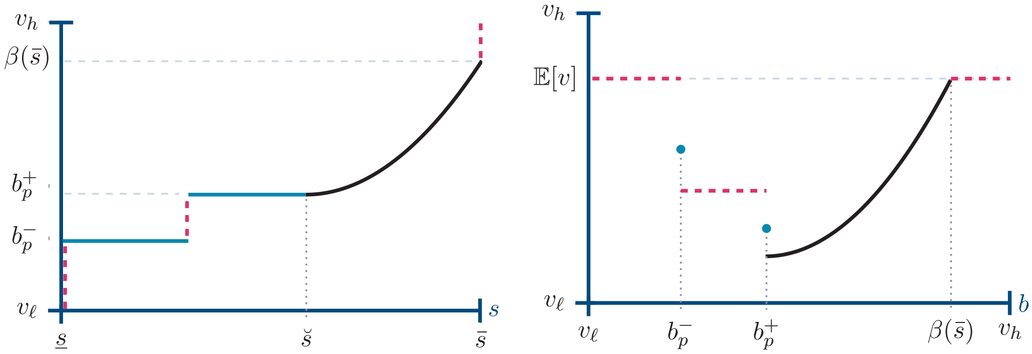
$$\mathbb{E}[v|s_{(1)} \leq s_-] > \mathbb{E}[v|\textit{win with } b_p; \beta] > \mathbb{E}[v|s_{(1)} \leq s_+]. \tag{1.12}$$

(ii) If $\eta[F_h(s_+) - F_h(s_-)] > \eta[F_\ell(s_+) - F_\ell(s_-)]$, then the inequalities in (1.12) reverse.

(iii) If β is an equilibrium strategy, then $\eta[F_h(s_+) - F_h(s_-)] < \eta[F_\ell(s_+) - F_\ell(s_-)]$.

[14] Reviewing the argument for inequality (1.5) highlights that the inequality is, in fact, strict, such that for all $s \in [s_-, s_+]$ it holds that $\beta(s) = b_p < \mathbb{E}[v|\text{win with } b_p, s; \beta]$, and winning more often raises the profit.

Combined, Lemmas 1.2 and 1.4 imply that the inference from winning is always U-shaped in the bid. Suppose, for instance, that all competitors follow strategy β depicted in the left panel of Figure 1.2 and consider the associated conditional expected value $\mathbb{E}[v|\text{win with } b; \beta]$ which is plotted in the right panel. Bids that are not in the image of β are colored pink (dashed), pooling bids are teal, and non-pooling bids are black.



A strategy $\beta(s)$ with two atoms. The expected value $\mathbb{E}[v|\text{win with b}; \beta]$.

Figure 1.2 The inference from winning.

Going through the bids from low to high, first, consider the inference from winning with a bid below the image of β, which is the first pink dashed interval. These bids can only win when there is no rival bidder, which is why there is no winner's curse and the conditional expected value is just $\mathbb{E}[v]$.

The first bid in the image of β is the pooling bid b_p^-, which can win when there are rival bidders. As a result, winning with b_p^- is bad news about the value of the good, such that $\mathbb{E}[v|\text{win with } b_p^-; \beta] < \mathbb{E}[v]$. Further, because b_p^- is exclusively chosen by bidders with signals below the neutral signal $\check{s}$, inequality (1.12) applies, and there is an even stronger winner's curse at bids between b_p^- and b_p^+ (second pink dashed interval) than at b_p^-.

The next bid in the image of β, that is b_p^+, is again a pooling bid exclusively chosen by signals below $\check{s}$. Thus, the winner's curse at b_p^+ is again stronger than at any lower bid, but a less severe one than winning with a marginally higher bid.

All bids above b_p^+ that are in the image of β are non-pooling bids and chosen by signals above $\check{s}$. Thus, Lemma 1.2 applies, and the conditional expected value is strictly increasing above $\beta(\check{s})$. At the top, there is no winner's curse since bids at or above $\beta(\bar{s})$ always win the auction.

When strategies contain atoms, the bidders' utilities are discontinuous in the bid. Winning with a pooling bid is discretely less likely than winning with a marginally higher bid, and since the probabilities change differently across states, the expected value conditional on winning with the pooling bid is discretely different, too. As a result, equilibria do not need to be unique. Instead of following a unique differential equation, they can consist of various mixtures of strictly increasing sections, pooling bids, and jumps. In Appendices 1.B.2 and 1.B.3, we provide two numerical examples of equilibrium multiplicity.

1.3.3 Non-existence of equilibria

In addition to the non-uniqueness, the discontinuities at atoms create an existence problem. In equilibrium, the U-shaped inference implies that there is an open set of bids below any pooling bid with a discretely higher expected value conditional on winning. When η is large, this induces bidders compete for the highest bid below the pooling bid, which prevents the existence of an equilibrium.

Proposition 1.3 *When η is sufficiently large, no equilibrium exists.*

Formally, this result is a corollary to Proposition 1.4 in the next section. However, the proof for Proposition 1.4 is fairly indirect. Thus, we sketch out the main idea here.

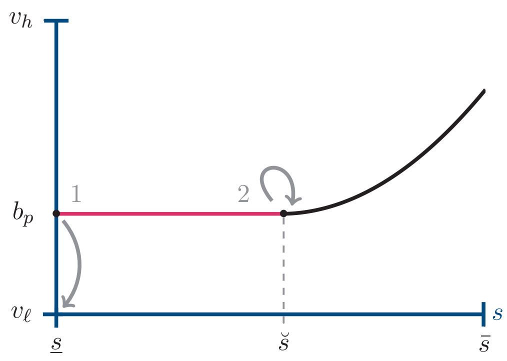
First, observe that Proposition 1.1 can be strengthened: when η is large, any strategy that is not locally constant below the neutral signal $\breve{s}$ can be excluded as an equilibrium. To be precise, for almost all $s < \breve{s}$, the winning probability must not converge to the probability of having the highest signal as η grows,

$$\lim_{\eta \to \infty} \frac{e^{-\eta(1-F_\omega(s))}}{\pi_\omega(\beta^*(s); \beta^*)} \neq 1 \quad \text{for } \omega \in \{h, \ell\},$$

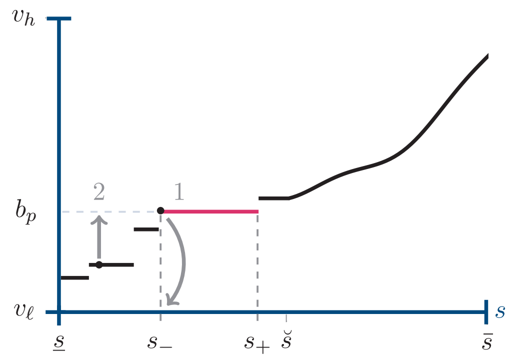
such that bidders with almost all signals $s < \breve{s}$ tie with positive probability.[15] As a result, any candidate equilibrium must essentially be a step function below $s < \breve{s}$.

In the following, we exclude two salient types of candidates: equilibria in which bidders with signals below $\breve{s}$ pool on the same bid, and equilibria in which only bidders with an interior subset of signals $[s_-, s_+] \subset (\underline{s}, \breve{s}]$ pool. Figure 1.3 sketches out both types. The two arrows in each panel depict two possible deviations. We show that one of them has to be profitable when η is large, such that equilibria cannot take either form. While this still leaves a large set of equilibrium candidates—in particular, equilibria in which the boundaries of the pools change as η increases—it turns out that similar arguments can be used to exclude those, too.

[15] If the fraction does converge to one for a set of signals $\breve{s}$ with positive mass, the proof of Proposition 1.1 yields exactly the same contradiction for any three signals $s_-, s_\circ, s_+$ from this set.



(a) Candidate β^* with a single, large pool.



(b) Candidate β^* with an interior pool.

Figure 1.3 Candidate equilibria.

(a) Single, large pool To begin, we show that when η is large, there can be no equilibrium shaped like the one in the left panel of Figure 1.3.

Lemma 1.5 *For a sufficiently large η, there is no equilibrium β^* in which $\beta^*(\underline{s}) = \beta^*(\breve{s})$ and $\beta^*(s) > \beta^*(\breve{s})$ for all $s > \breve{s}$.*

Suppose to the contrary such a β^* exists for an arbitrary large η. Denote the pooling bid by $b_p = \beta^*(\underline{s}) = \beta^*(\breve{s})$. The contradiction is derived in three steps. First, deviation 1 is used to derive an upper bound on b_p (Step 1), before deviation 2 is used to find a lower bound (Step 2). Last, Step 3 shows that when η is sufficiently large, the lower bound exceeds the upper bound, such that one deviation has to be profitable. As an abbreviation, we use $\pi_\omega^\circ = \pi_\omega(b_p; \beta^*)$ and $\pi_\omega^+ = \lim_{\epsilon \searrow 0} \pi_\omega(b_p+\epsilon; \beta^*)$ for $\omega \in \{h, \ell\}$.[16]

Step 1 *By (1.5), individual rationality (deviation 1) for signal $\underline{s}$ implies that*

$$\frac{b_p - v_\ell}{v_h - b_p} \leq \frac{\rho}{1-\rho} \frac{f_h(\underline{s})}{f_\ell(\underline{s})} \frac{\pi_h^\circ}{\pi_\ell^\circ}. \tag{1.13}$$

Step 2 *There exists a function $B(\eta) < 1$ with $B(\eta) \to 1$ such that*

$$\frac{b_p - v_\ell}{v_h - b_p} \geq \frac{\rho}{1-\rho} \frac{f_h(\breve{s})}{f_\ell(\breve{s})} \frac{\pi_h^+}{\pi_\ell^+} B(\eta). \tag{1.14}$$

Signal $\breve{s}$ has an incentive to deviate from b_p to a marginally higher bid (deviation 2), unless $U(b_p|\breve{s}; \beta^*) \geq \lim_{\epsilon \searrow 0} U(b_p + \epsilon|\breve{s}; \beta^*)$. Rearranging this inequality gives

$$\frac{b_p - v_\ell}{v_h - b_p} \geq \frac{\rho}{1-\rho} \frac{f_h(\breve{s})}{f_\ell(\breve{s})} \frac{\pi_h^+ - \pi_h^\circ}{\pi_\ell^+ - \pi_\ell^\circ}.$$

[16] Explicitly, $\pi_\omega^\circ = \frac{e^{-\eta(1-F_\omega(s_+))} - e^{-\eta(1-F_\omega(s_-))}}{\eta(F_\omega(s_+) - F_\omega(s_-))}$ and $\pi_\omega^+ = e^{-\eta(1-F_\omega(s_+))}$ for $\omega \in \{h, \ell\}$.

The bid marginally above b_p always wins when $s_{(1)} \leq \check{s}$, while b_p is also subject to a tie-break whenever there are competitors who also bid b_p. Since the expected number of competitors who choose b_p is $\eta F_\omega(\check{s})$, this means that $\frac{\pi_\omega^+}{\pi_\omega^\circ} \approx \eta F_\omega(\check{s})$. Because $\eta F_\omega(\check{s})$ grows in η without bound, this implies that $B(\eta) = \frac{\pi_h^+ - \pi_h^\circ}{\pi_\ell^+ - \pi_\ell^\circ}\left(\frac{\pi_h^+}{\pi_\ell^+}\right)^{-1} \to 1$, which gives the result. Observe that when $B(\eta) = 1$, equation (1.14) rearranges to $b_p \geq \mathbb{E}[v|s_{(1)} \leq \check{s}, \check{s}]$, meaning that b_p has to be at least the expected value conditional on winning with a marginally higher bid.

Step 3 *When η is sufficiently large, the lower bound (1.14) exceeds the upper bound (1.13). Thus, either deviation 1 or 2 is profitable.*

Combining inequalities (1.13) and (1.14) yields

$$\frac{\rho}{1-\rho}\frac{f_h(\underline{s})}{f_\ell(\underline{s})}\frac{\pi_h^\circ}{\pi_\ell^\circ} \geq \frac{\rho}{1-\rho}\frac{f_h(\check{s})}{f_\ell(\check{s})}\frac{\pi_h^+}{\pi_\ell^+}B(\eta).$$

By definition of the neutral signal $\frac{f_h(\check{s})}{f_\ell(\check{s})} = 1$, such that the inequality rearranges to

$$\frac{f_h(\underline{s})}{f_\ell(\underline{s})} \geq \frac{\pi_h^+}{\pi_\ell^+}\left(\frac{\pi_h^\circ}{\pi_\ell^\circ}\right)^{-1} B(\eta). \tag{1.15}$$

From $\frac{\pi_\omega^+}{\pi_\omega^\circ} \approx \eta F_\omega(\check{s})$, it follows that $\frac{\pi_h^+}{\pi_\ell^+}\left(\frac{\pi_h^\circ}{\pi_\ell^\circ}\right)^{-1} \to \frac{F_h(\check{s})}{F_\ell(\check{s})}$: the blessing from winning with b_p as opposed to a marginally higher bid is bounded and of order $\frac{F_h(\check{s})}{F_\ell(\check{s})}$. This blessing does not suffice to reconcile the lower bound (1.14) and the upper bound (1.13). Since $\frac{f_h(\underline{s})}{f_\ell(\underline{s})} < \frac{f_h(\bar{s})}{f_\ell(\bar{s})}$, the monotone likelihood ratio property implies that $\frac{f_h(\underline{s})}{f_\ell(\underline{s})} < \frac{F_h(\check{s})}{F_\ell(\check{s})}$.[17] Combined with the observation that $B(\eta) \to 1$, this means that condition (1.15) is violated when η is large. Thus, either deviation 1 or 2 is profitable, and there can be no equilibrium β^* in which all signals below $\check{s}$ pool on the same bid.

(b) Interior pool Suppose now that there is an equilibrium with an "interior pool", even when η is arbitrary large. This type of equilibrium is depicted qualitatively in the right panel of Figure 1.3.

Lemma 1.6 *Fix any s_-, s_+ with $\underline{s} < s_- < s_+ \leq \check{s}$. When η is sufficiently large, there is no equilibrium β^* in which $\beta^*(s_-) = \beta^*(s_+)$, $\beta^*(s) < \beta^*(s_-)$ for all $s < s_-$ and $\beta^*(s) > \beta^*(s_+)$ for all $s > s_+$.*

[17] Since $\frac{f_h(\underline{s})}{f_\ell(\underline{s})} < \frac{f_h(\bar{s})}{f_\ell(\bar{s})}$, the monotone likelihood ratio property implies that $\frac{f_h(\underline{s})}{f_\ell(\underline{s})} < \frac{f_h(\check{s})}{f_\ell(\check{s})} = 1$. Because $\frac{f_h(\underline{s})}{f_\ell(\underline{s})} < 1$ and the densities are continuous, $F_\ell(\check{s}) = \int_{\underline{s}}^{\check{s}} f_\ell(z)dz = \int_{\underline{s}}^{\check{s}} f_h(z)\frac{f_\ell(z)}{f_h(z)}dz < \int_{\underline{s}}^{\check{s}} f_h(z)\frac{f_\ell(\underline{s})}{f_h(\underline{s})}dz = \frac{f_\ell(\underline{s})}{f_h(\underline{s})}F_h(\check{s})$.

Suppose to the contrary that even when η is arbitrary large such a β^* exists. Denote the pooling bid by $b_p = \beta^*(s_-) = \beta^*(s_+)$. We proceed in the same way as before and use the deviation 1 to derive an upper bound on b_p (Step 1) and deviation 2 to derive a lower bound on b_p (Step 2). Step 3 shows that the lower bound exceeds the upper bound when η is large, such that one of the deviations has to be profitable. As abbreviations, we use $\pi_\omega^- = \lim_{\epsilon\searrow 0} \pi_\omega(b_p - \epsilon; \beta^*)$ and $\pi_\omega^\circ = \pi_\omega(b_p; \beta^*)$ for $\omega \in \{h, \ell\}$.[18]

Step 1 *By (1.5), individual rationality (deviation 1) for signal s_- implies that*

$$\frac{b_p - v_\ell}{v_h - b_p} \leq \frac{\rho}{1-\rho} \frac{f_h(s_-)}{f_\ell(s_-)} \frac{\pi_h^\circ}{\pi_\ell^\circ}. \tag{1.16}$$

Step 2 *There exists a function $E(\eta) > 1$ with $E(\eta) \to 1$ such that*

$$\frac{b_p - v_\ell}{v_h - b_p} \geq \frac{\rho}{1-\rho} \frac{f_h(\underline{s})}{f_\ell(\underline{s})} \frac{\pi_h^-}{\pi_\ell^-} E(\eta). \tag{1.17}$$

In equilibrium, no signal $s < s_-$ can have an incentive to deviate from $\beta^*(s)$ to any $b \in (\beta^*(s), b_p)$. In particular, there must not be an incentive to deviate a bid marginally below b_p (deviation 2), meaning that $U(\beta^*(s)|s;\beta^*) \geq \lim_{\epsilon\searrow 0} U(b_p - \epsilon|s;\beta^*)$. In the appendix, we use this condition for signal $\underline{s}$ to derive (1.17).

Step 3 *When η is sufficiently large, the lower bound (1.17) exceeds the upper bound (1.16). Thus, either deviation 1 or 2 is profitable.*

Combining inequalities (1.16) and (1.17) yields

$$\frac{\rho}{1-\rho} \frac{f_h(s_-)}{f_\ell(s_-)} \frac{\pi_h^\circ}{\pi_\ell^\circ} \geq \frac{\rho}{1-\rho} \frac{f_h(\underline{s})}{f_\ell(\underline{s})} \frac{\pi_h^-}{\pi_\ell^-} E(\eta).$$

This can be rearranged to

$$\frac{f_h(s_-)}{f_\ell(s_-)} \left(\frac{f_h(\underline{s})}{f_\ell(\underline{s})}\right)^{-1} \geq \frac{\pi_h^-}{\pi_\ell^-} \left(\frac{\pi_h^\circ}{\pi_\ell^\circ}\right)^{-1} E(\eta). \tag{1.18}$$

The product $\frac{\pi_h^-}{\pi_\ell^-}\left(\frac{\pi_h^\circ}{\pi_\ell^\circ}\right)^{-1}$ captures the difference in inference from winning with a bid marginally below b_p instead of b_p. From $s_+ < \check{s}$, it follows that $\eta[F_h(s_+) - F_h(s_-)] < \eta[F_\ell(s_+) - F_\ell(s_-)]$, such that equation (1.12) of Lemma 1.4 implies that the product is larger than one: winning with a marginally lower bid reduces the winner's curse.

[18] Explicitly, $\pi_\omega^- = e^{-\eta(1-F_\omega(s_-))}$ and $\pi_\omega^\circ = \frac{e^{-\eta(1-F_\omega(s_+))} - e^{-\eta(1-F_\omega(s_-))}}{\eta(F_\omega(s_+) - F_\omega(s_-))}$ for $\omega \in \{h, \ell\}$.

In fact, this effect becomes arbitrarily strong,

$$\frac{\pi_h^-}{\pi_\ell^-}\left(\frac{\pi_h^\circ}{\pi_\ell^\circ}\right)^{-1} \approx \underbrace{e^{\eta([F_\ell(s_+)-F_\ell(s_-)]-[F_h(s_+)-F_h(s_-)])}}_{\to\infty \text{ by } (1.8)} \frac{F_h(s_+)-F_h(s_-)}{F_\ell(s_+)-F_\ell(s_-)} \to \infty.$$

By $E(\eta) \to 1$, this means that when η is large, the inference from winning on the right side of inequality (1.18) dominates the inference from the signals on the left side of inequality (1.18). As a result, the inequality cannot hold, and either deviation 1 or 2 has to be profitable.

Since the argument against candidate (b) contains some of the key incentives which shape the bidding behavior, it is useful to repeat it verbally. First, equilibrium bids can at most be the expected value conditional on winning, such that $\mathbb{E}[v|\text{win with } b_p, s_-; \beta^*]$ puts an upper bound on b_p (1.16). When η is large, there is a "winner's blessing" on bids below b_p, such that this upper bound is dwarfed by the expected value conditional on winning with any lower bid $b < b_p$. In particular, b_p has to be a lot smaller than the expected value conditional on winning with a marginally lower bid, which wins whenever $s_{(1)} \leq s_-$. Hence, the expected profits at this marginally lower bid are strictly positive. When η is large, a Bertrand competition emerges among bidders with signals below s_-: the rivals compete for the highest bid below b_p which maximizes their chances to win the auction but is subject to a strictly smaller winner's curse than b_p. On the continuous bid space, a largest bid below b_p does not exist, such that no equilibrium exists.

As noted above, the arguments presented do not constitute a comprehensive proof. We restricted attention to pools which do not change in size as η increases and only considered equilibria in which the pools end at $\check{s}$ and $s_+ < \check{s}$, respectively. As it turns out, however, none of these simplifications are significant, and existence always fails due to an interior atom and the "open set problem" it creates. Naturally, this open set is a feature of the continuous bid space; when considering auctions on a grid, there is a maximal bid below any pooling bid, and an equilibrium exists. At the same time, however, a discrete bid space makes the equilibrium characterization more challenging. Therefore, we take an indirect approach and first analyze an extended auction on the continuous bid space, which will help us to characterize equilibria on a fine grid afterward.

1.4 Communication extension

In this section, we augment the auction mechanism by a communication dimension similar to Jackson et al. (2002).[19] We denote this *communication extension* by $\Gamma^{\mathfrak{c}}$, whereas we label the standard auction mechanism by Γ. As we show in the next section, sequences of equilibria on an ever-finer grid converge to an equilibrium of the communication extension. Therefore, the communication extension always has an equilibrium, which we can use in Section 1.5 to characterize the equilibria on a fine grid.

In the communication extension, every bidder simultaneously selects three actions: a message space $M \subseteq [0,1]$, a message $m \in [0,1]$, and a bid $b \in [v_\ell, v_h]$. We consider strategies of the form $\sigma : [\underline{s}, \bar{s}] \to \mathcal{P}[0,1] \times \Delta\Big([0,1] \times [v_\ell, v_h]\Big)$ mapping the signals into a message space and a distribution over messages and bids.[20]

The auction mechanism selects the winner according to the following rule. First, it checks whether all bidders report the same message space M; if not, the good is not allocated. Afterwards, it discards all bidders who report messages $m \notin M$. Among the remaining bidders, the good is allocated to the one with the highest bid. If multiple bidders tie on the highest bid, the tie is broken uniformly among those who report the highest message $m \in M$. The winner receives the object and pays her bid.

Denote the probability to win in state $\omega \in \{h, \ell\}$ with action-tuple (M, m, b) when all rival bidders follow strategy σ by $\pi^{\mathfrak{c}}_\omega(M, m, b; \sigma)$. Then, the interim expected utility for a bidder with signal s who selects (M, m, b) is

$$\begin{aligned} U^{\mathfrak{c}}(M, m, b|s; \sigma) = {} & \frac{\rho f_h(s)}{\rho f_h(s) + (1-\rho) f_\ell(s)} \pi^{\mathfrak{c}}_h(M, m, b; \sigma)(v_h - b) \\ & + \frac{(1-\rho) f_\ell(s)}{\rho f_h(s) + (1-\rho) f_\ell(s)} \pi^{\mathfrak{c}}_\ell(M, m, b; \sigma)(v_\ell - b). \end{aligned} \tag{1.19}$$

A strategy σ^* is a *best response* to a strategy σ if for (almost) every s an action-tuple $(M, m, b) \in \operatorname{supp} \sigma^*(s)$ implies that $(M, m, b) \in \arg\max_{\hat{M}, \hat{m}, \hat{b}} U(\hat{M}, \hat{m}, \hat{b}|s; \sigma)$. As in the case of the standard auction, unless specified otherwise, all following results hold for almost every s. Again, we analyze symmetric Bayes-Nash equilibria, but restrict attention to *concordant* equilibria in which all bidders report the same message space M.[21]

[19] We discuss the relation to Jackson et al. (2002) in footnote 21.

[20] We immediately restrict attention to equilibria in which all bidders choose the same M so that the restriction to pure strategies with respect to the message space is without further loss.

[21] The outcomes of concordant equilibria are a subset of the outcomes of solutions to the communication extension in Jackson et al. (2002). In their communication extension, the tie-breaking is part of the solution which can be interpreted as introducing the auctioneer as a player who selects

Note that, conditional on a bid b, different messages m, m' may induce the same winning probability, such that they are equivalent. Two strategies are *m-equivalent*, if after merging all signals s that share the same likelihood ratio, $\frac{f_h}{f_\ell}$, they correspond to the same distributional strategy, up to equivalent messages.

Lemma 1.7 *Let σ^* be a concordant equilibrium of the communication extension. Then, there exists an m-equivalent, concordant equilibrium $\hat{\sigma}^*$ that is pure and where*

(i) bids b are nondecreasing in s;

(ii) for any given bid b, the report $m \in M$ is nondecreasing in s.

This implies that in both states $\omega \in \{h, \ell\}$

(a) $\pi^{\mathfrak{c}}_{\omega}(\hat{\sigma}^*(s); \hat{\sigma}^*)$ is nondecreasing in s;

(b) $\hat{\sigma}^*(s) = \hat{\sigma}^*(s')$ if and only if $\pi^{\mathfrak{c}}_{\omega}(\hat{\sigma}^*(s); \hat{\sigma}^*) = \pi^{\mathfrak{c}}_{\omega}(\hat{\sigma}^*(s'); \hat{\sigma}^*)$ for $s, s' \in [\underline{s}, \bar{s}]$.

In essence, Lemma 1.7 is analogous to Lemma 1.1. Bidders with higher signals are more optimistic, select weakly higher bids/messages, and win weakly more often. If multiple signals induce the same belief, the actions can be reordered such that they are monotone, without altering the implied distribution of bids and (payoff-relevant) messages. Implication *(b)* establishes that the problem of equivalent messages can be ignored. If two distinct action-tuples are in the image of the strategy, then they win with different probabilities. This simplifies later statements.

Henceforth, we restrict attention to concordant strategies that are pure, in which bidders with higher signals win weakly more often and where *(b)* holds. We denote these by $\sigma : [\underline{s}, \bar{s}] \to \mathcal{P}[0,1] \times [0,1] \times [v_\ell, v_h]$.

We can now explicitly state the winning probabilities. To do so, fix some concordant strategy σ and functions μ and β such that $\sigma(s) = (M, \mu(s), \beta(s))$ for all s. Suppose a bidder chooses the action-tuple (M, m, b). If $m \in M$ and (M, m, b) is selected with zero probability by a competitor, then she wins whenever $s_{(1)} \leq \hat{s}$, where $\hat{s} = \sup(\{s\colon \beta(s) < b\} \cup \{s\colon \beta(s) = b \text{ and } \mu(s) < m\})$ is the highest signal that chooses a lower bid, or the same bid but lower message. This happens in state $\omega \in \{h, \ell\}$ with probability $\pi^{\mathfrak{c}}_{\omega}(M, m, b; \sigma) = e^{-\eta(1-F_\omega(\hat{s}))}$.

If $\sigma(s) = (M, m, b)$ for all $s \in [s_-, s_+]$ and $\sigma(s) \neq (M, m, b)$ for all other signals, then the action-tuple wins in state $\omega \in \{h, \ell\}$ with probability

$$\pi^{\mathfrak{c}}_{\omega}(M, m, b; \sigma) = \frac{e^{-\eta(1-F_\omega(s_+))} - e^{-\eta(1-F_\omega(s_-))}}{\eta(F_\omega(s_+) - F_\omega(s_-))}.$$

the tie-breaking rule. By contrast, we fully specify the mechanism without introducing such an additional player. Roughly speaking, in our mechanism, the bidders report the tie-breaking rule. This is possible in our setting because a misreport can be punished by a uniformly worst outcome (no allocation) while such outcome may not exist in the more general payoff setting in Jackson et al. (2002).

These probabilities are analogous to those in the standard auction, and are derived in the same manner.

If a bidder chooses an action-tuple (M', m', b) with $M' \neq M$, but $m' \in M'$ then she only wins when the deviation to M' is not detected. This only happens when she is alone, which occurs in state $\omega \in \{h, \ell\}$ with probability $\pi^{\mathfrak{c}}_{\omega}(M', m', b; \sigma) = e^{-\eta}$.

If $M' = M$ but $m' \notin M$, the probability to win is zero.

To fix ideas, note that every equilibrium of the standard auction, Γ, is also an equilibrium of the communication extension, $\Gamma^{\mathfrak{c}}$. If all bidders report $M = \{0\}$ and $m = 0$, the messages do not affect the outcome of the auction, and deviations in M or m are (weakly) dominated by bidding (M, m, v_ℓ); the lowest bid also only wins when the bidder is alone, but at the lowest possible cost. Thus, the winner is solely determined by the bids, and we only need to consider deviations in the bid. Obviously, this makes following the equilibrium strategy of Γ an equilibrium of $\Gamma^{\mathfrak{c}}$.

Since every equilibrium of the standard auction is an equilibrium of the communication extension, the set of equilibria of $\Gamma^{\mathfrak{c}}$ is a superset of the equilibria of Γ. Indeed, it can be a proper superset, because the communication extension always has an equilibrium.

Lemma 1.8 *The communication extension $\Gamma^{\mathfrak{c}}$ always has a concordant equilibrium.*

The result follows as a corollary to Lemmas 1.9 and 1.10 found in the next section. For now, we just take existence as given. Even though equilibria are not unique, it is possible to characterize their form up to some ϵ environment around $\underline{s}$ and $\breve{s}$.

Proposition 1.4 *Fix any $\epsilon \in (0, \frac{\breve{s}-\underline{s}}{2})$. When η is sufficiently large (given ϵ), any concordant equilibrium σ^* of $\Gamma^{\mathfrak{c}}$ takes the following form:*
there are two disjoint, adjacent intervals of signals I, J such that

(i) $[\underline{s} + \epsilon, \breve{s} - \epsilon] \subset I \cup J$;

(ii) $\sigma^*(s_I) = (M, m_I, b_p)$ *for all* $s_I \in I$ *and* $\sigma^*(s_J) = (M, m_J, b_p)$ *for all* $s_J \in J$, *with* $m_I < m_J$;

(iii) there is no $m \in M$ *s.t.* $\pi^{\mathfrak{c}}_{\omega}(\sigma^*(s_I); \sigma^*) < \pi^{\mathfrak{c}}_{\omega}(M, m, b_p; \sigma^*) < \pi^{\mathfrak{c}}_{\omega}(\sigma^*(s_J); \sigma^*)$ *for* $\omega \in \{h, \ell\}$;

(iv) $\int_I \eta f_\omega(z) dz > \frac{1}{\epsilon}$ *and* $\int_J \eta f_\omega(z) dz > \frac{1}{\epsilon}$ *for* $\omega \in \{h, \ell\}$;

(v) on $s \in (\breve{s} + \epsilon, \bar{s}]$ *the bids are strictly increasing, such that the message* m *is irrelevant.*

The proof is in the appendix. The following figure summarizes the results:

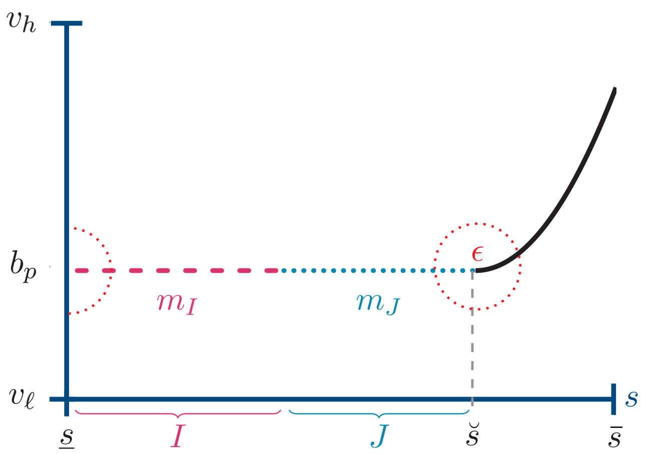


Figure 1.4 Equilibria σ^* of the communication extension.

By part *(i)*, there are two adjacent intervals I and J (pink/dashed, teal/dotted) that span the signals between $\underline{s} + \epsilon$ and $\breve{s} - \epsilon$. Bidders with signals from either interval bid b_p but separate by sending messages m_I and m_J, *(ii)*. Importantly, m_I and m_J are adjacent, meaning that there is no $m \in M$ with $m_I < m < m_J$. Thus, *(iii)* holds and there is no action-tuple that wins more often than (M, m_I, b_p) but less often than (M, m_J, b_p). The intervals I and J can vary in length as η increases, but the expected number of bidders in both intervals grows without bound, as asserted by *(iv)*. Above $\breve{s} + \epsilon$, bids are strictly increasing and follow the ordinary differential equation (1.9) with the proper initial value, *(v)*. The message m is irrelevant in this region. Observe that Figure 1.4 is only a qualitative sketch: J may be contained in the ϵ-environment around $\breve{s}$, and the equilibrium may assume a different form within the ϵ-environments.

The form of the equilibrium is a direct consequence of the results in Section 1.3.3. There, we reasoned that in any equilibrium of the standard auction, bids cannot be strictly increasing below the neutral signal $\breve{s}$ and $\underline{s}$ and $\breve{s}$ cannot pool. The logic behind these two results remains valid in the communication extension. Hence, there has to be an interior atom, b_p, on which bidders with intermediate signals, J, pool to insure against the winner's curse, as depicted in candidate equilibrium (b). Since the inference from winning is U-shaped (cf. Lemma 1.4 and Figure 1.2), compared to b_p, winning with any bid below b_p is a blessing for the conditional expected value. When η is large, this incentivizes bidders with low signals, I, to compete for the highest bid below b_p. In the standard auction, Γ, no such bid exists, such that no equilibrium exists. With an endogenous tie-breaking rule, the problem can be solved. By sending messages m_I and m_J, bidders with signals from I and J can differentiate themselves, while leaving no room for bidders with signals from I to marginally deviate upwards, as stated in part *(iii)*.

One immediate implication of Proposition 1.4 is that there can be no equilibrium in the standard auction (Proposition 1.3). By our earlier observation, all equilibria of Γ are also equilibria of a communication extension, $\Gamma^{\mathfrak{c}}$, in which the message space is a singleton. Since Proposition 1.4 describes every equilibrium of $\Gamma^{\mathfrak{c}}$, and the intervals I and J cannot be separated without two distinct messages, Γ cannot have an equilibrium.

1.5 Standard auction on the grid

Consider a variation of the standard auction in which the bids are constrained to a set of $k \geq 2$ equidistant[22] values

$$B_k = \{v_\ell, v_\ell + d, ..., v_\ell + (k-2)d, v_h\},$$

where $d = \frac{v_h - v_\ell}{k-1}$. We denote such an auction by $\Gamma(k)$.

Lemma 1.9 *Any auction on the grid, $\Gamma(k)$, has an equilibrium in pure and nondecreasing strategies.*

The proof builds on Myerson (2000) and is in the appendix.[23] The monotonicity directly follows from Lemma 1.1, which does not rely on the form of the bid space.

While the discretization solves the existence problem, the discontinuous bid space makes the equilibrium characterization more challenging. Here, the communication extension, $\Gamma^{\mathfrak{c}}$, proves helpful. We are going to show that the equilibria on a fine grid must have the same structure as the equilibria of $\Gamma^{\mathfrak{c}}$. A first step shows that the limit of a converging sequence of equilibria on the ever-finer grid can be represented as a concordant equilibrium of the communication extension. For a deterministic population, this corresponds to a special case of Theorem 2 in Jackson et al. (2002).

Lemma 1.10 *Consider any sequence of auctions on the ever-finer grid, $(\Gamma(k))_{k\in\mathbb{N}}$, and any corresponding sequence of equilibria, $(\beta_k^*)_{k\in\mathbb{N}}$. There exists a subsequence of auctions $(\Gamma(n))_{n\in\mathbb{N}}$ with equilibria $(\beta_n^*)_{n\in\mathbb{N}}$ and a concordant equilibrium σ^* of $\Gamma^{\mathfrak{c}}$ such that, for all $s \in [\underline{s}, \bar{s}]$,*

(i) $\sigma^*(s) = (M, \mu(s), \lim_{n\to\infty} \beta_n^*(s))$ *for some M and function $\mu : [\underline{s}, \bar{s}] \to M$;*

(ii) $\lim_{n\to\infty} \pi_\omega(\beta_n^*(s); \beta_n^*) = \pi_\omega^{\mathfrak{c}}(\sigma^*(s); \sigma^*)$ *for $\omega \in \{h, \ell\}$;*

(iii) $\lim_{n\to\infty} U(\beta_n^*(s)|s; \beta_n^*) = U^{\mathfrak{c}}(\sigma^*(s)|s; \sigma^*)$.

[22] The assumption of equidistance is for expositional purposes only. The following results hold for any discretization, as long as the grid becomes dense on $[v_\ell, v_h]$ as $k \to \infty$.

[23] Since best responses are monotone, the existence proof can be simplified, and applies even if the likelihood ratio $\frac{f_h(s)}{f\ell(s)}$ contains jumps.

The proof is in the appendix. Combined with Lemma 1.9, Lemma 1.10 establishes the existence of equilibria of $\Gamma^{\mathfrak{c}}$ (Lemma 1.8). Next, we compare the structure of equilibria on the ever-finer grid with the corresponding limit equilibrium of the communication extension.

Lemma 1.11 *Consider a sequence of auctions on the ever-finer grid, $(\Gamma(n))_{n\in\mathbb{N}}$, with corresponding equilibria, $(\beta_n^*)_{n\in\mathbb{N}}$, that converge to an equilibrium of $\Gamma^{\mathfrak{c}}$, denoted σ^*, in the sense of Lemma 1.10. Then it holds for (almost) any two signals $s_- < s_+$ that*

(i) $\sigma^*(s_-) = \sigma^*(s_+)$, *if and only if* $\beta_n^*(s_-) = \beta_n^*(s_+)$ *for any n sufficiently large;*

(ii) $\sigma^*(s_-) \neq \sigma^*(s_-)$, *if and only if* $\beta_n^*(s_-) < \beta_n^*(s_+)$ *for any n sufficiently large.*

Due to this close relationship, equilibria on a fine grid have to be similar to those of the communication extension. Thus, the characterization from Proposition 1.4 can be used to derive properties of equilibria on a fine grid.

Proposition 1.5 *Fix any $\epsilon \in (0, \frac{\breve{s}-\underline{s}}{2})$. When η is sufficiently large (given ϵ) and k is sufficiently large (given ϵ and η), any equilibrium β^* of $\Gamma(k)$ takes the following form: there are two disjoint, adjacent intervals of signals I, J such that*

(i) $[\underline{s}+\epsilon, \breve{s}-\epsilon] \subset I \cup J$;

(ii) $\beta^*(s_I) = b$ *for all* $s_I \in I$ *and* $\beta^*(s_J) = b + d$ *for all* $s_J \in J$;

(iii) $\int_I \eta f_\omega(z)dz > \frac{1}{\epsilon}$ *and* $\int_J \eta f_\omega(z)dz > \frac{1}{\epsilon}$ *for* $\omega \in \{h, \ell\}$;

(iv) on $s \in (\breve{s}+\epsilon, \bar{s}]$ *the bids tie with a probability smaller than* ϵ.[24]

Again, the result is summarized best with the help of a figure:

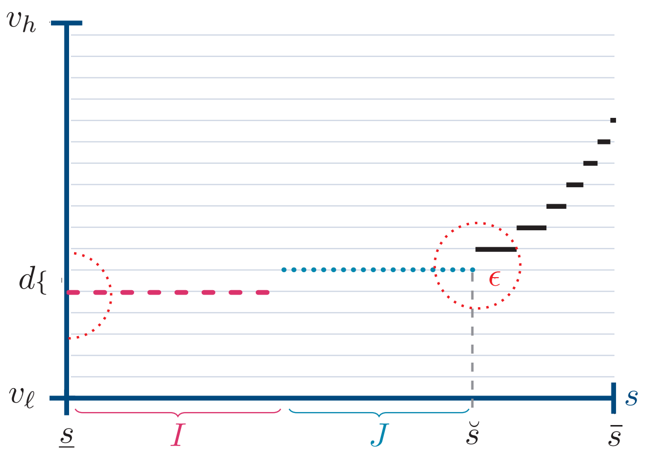


Figure 1.5 Equilibria β^* of the auction on the grid.

[24]Take any $\hat{s} > \breve{s}+\epsilon$ and let $\hat{b} = \beta^*(\hat{s})$. If there exists an interval $[s_-, s_+]$ such that $\beta^*(s) = \hat{b}$ for all $s \in [s_-, s_+]$, then $\eta \int_{s_-}^{s_+} f_\omega(z)dz < \epsilon$ for $\omega \in \{h, \ell\}$.

There are two adjacent intervals I and J (pink/dashed and teal/dotted). By *(i)*, any signal between $\underline{s}+\epsilon$ and $\breve{s}-\epsilon$ is part of one of the two intervals. By *(ii)*, bidders with signals from interval I pool on a bid b_p, while bidders on the interval J select the next bid on the grid, $b_p + d$. The intervals can vary in length as η increases, but the expected number of bidders in both intervals grows without bound, *(iii)*. Assertion *(iv)* states that there are no significant atoms above $\breve{s}+\epsilon$; in fact, the bidding function becomes smooth and strictly increasing as grid $d \to 0$.

The characterization highlights why the standard auction, Γ, is not the limit of the auctions on an arbitrarily fine grid. As $d \to 0$, the difference between the two pooling bids b_p and $b_p + d$ vanishes. In the limit, I and J can no longer be separated such that they win with the same probability, and the utility changes discontinuously. Therefore, the limit of a sequence of equilibria on the ever-finer grid is generally not an equilibrium of the limit auction Γ.[25] However, the limit outcome can be represented as an equilibrium of $\Gamma^{\mathfrak{c}}$, because the tie-breaking rule can be chosen to preserve the different winning probabilities in I and J. Thereby, equilibria of $\Gamma^{\mathfrak{c}}$ inherit the characteristics of equilibria on a fine grid, which is why the communication extension can be used to characterize the equilibria on a fine grid.

We now turn to the proof of Proposition 1.5:

Proof. Suppose that for every k at least one of the properties *(i)*-*(iv)* is violated. Then, there exists a sequence of auctions on the ever-finer grid, $(\Gamma(k))_{k\in\mathbb{N}}$, with equilibria, $(\beta_k^*)_{k\in\mathbb{N}}$, along which one property never holds. By Lemma 1.10, this sequence has a subsequence $(\beta_n^*)_{n\in\mathbb{N}}$ converging to an equilibrium of the communication extension, σ^*. When η is large, strategy σ^* takes the form detailed in Proposition 1.4. We use this form of σ^* and the convergence of $(\beta_n^*)_{n\in\mathbb{N}}$ to find contradictions for the violations of properties *(i)*-*(iv)* for infinitely many n.

First, consider property *(iv)*. If the bids in σ^* are strictly increasing over some interval, so is $\beta^* = \lim_{n\to\infty} \beta_n^*$. Thus, when n is sufficiently large (d sufficiently small), the bids tie with a probability smaller than ϵ on $s \in (\breve{s}+\epsilon, \bar{s}]$, and property *(iv)* cannot be violated.

Next, consider the intervals I and J of σ^*, and fix some $s_I \in int(I)$ and $s_J \in int(J)$. Further, define $I^n = \{s\colon \beta_n^*(s) = \beta_n^*(s_I)\}$ as well as $J^n = \{s\colon \beta_n^*(s) = \beta_n^*(s_J)\}$. By Lemma 1.11, $I^n \to I$ and $J^n \to J$. Thus, property *(iii)* cannot be violated when n is large.

What remains to be shown is that $\beta_n^*(s_I)+d = \beta_n^*(s_J)$ when n is sufficiently large *(ii)*. If this is the case, then $(\underline{s}+\epsilon, \breve{s}-\epsilon) \subset I^n \cup J^n$, such that property *(i)* follows,

[25] In the limit, the strategy becomes roughly the one we ruled out in candidate equilibrium (a) of Section 1.3.3 in which all signals below the neutral signal $\breve{s}$ pool on a single bid.

completing the proof. Suppose to the contrary that $\beta_n^*(s_I) + d < \beta_n^*(s_J)$ for every n. Then, it follows from $I^n \to I$ and $J^n \to J$ that $\lim_{n\to\infty} \pi_\omega(\beta_n^*(s_I) + d; \beta_n^*) = e^{-\eta(1-F_\omega(\hat{s}))}$. Since strategy β_n^* is an equilibrium, $U(\beta_n^*(s_n)|s_n; \beta_n^*) \geq U(\beta_n^*(s_I) + d|s_n; \beta_n^*)$ for all $s_n \in I^n \cup J^n$. Hence, Lemma 1.10 implies that

$$\lim_{n\to\infty} U(\beta_n^*(s)|s; \beta_n^*) = U^{\mathfrak{c}}(\sigma^*(s)|s; \sigma^*) \geq \lim_{n\to\infty} U(\beta_n^*(s_I) + d|s; \beta_n^*) \quad \forall s \in I \cup J.$$

This means that in equilibrium σ^*, bidders prefer $\sigma^*(s_I)$ or $\sigma^*(s_J)$ over some hypothetical action-tuple that wins whenever $s_{(1)} \leq \hat{s}$. Thus, there could be an $m \in M$ with $m_I < m < m_J$ because bidders would not deviate to such a message. This is a contradiction to property *(iii)* of Proposition 1.4, which completes the proof. □

The proof illustrates how the communication extension can be employed to characterize equilibria on a fine grid. This contrasts with standard auctions on the continuous bidding space that cannot fully handle non-vanishing atoms in the equilibrium bid distribution, thereby acting as an equilibrium refinement. The communication extension is, hence, the "correct" mechanism to analyze auctions on the fine grid.

1.6 Discussion

1.6.1 State-dependent competition

One natural modification of the model is the introduction of state-dependent participation, expressed by a state-dependent mean η_ω. This extension combines numbers uncertainty with the deterministic but state-dependent participation in Lauermann and Wolinsky (2017). When the number of bidders depends on the state, being solicited to participate in the auction contains information about the state. Conditional on participation, a bidder updates her belief to

$$\mathbb{P}[\omega = h|\text{participation}] = \frac{\rho\eta_h}{\rho\eta_h + (1-\rho)\eta_\ell}.$$

Further, knowledge of the number of competitor now has two additional effects. Apart from determining the intensity of the winner's curse, it is also directly informative about the state. This changes the inference from winning, and, thus, the form of $\mathbb{E}[v|s_{(1)} \leq \hat{s}]$. Specifically, consider the effect state-dependent participation has on the inference at the bottom, $\mathbb{E}[v|s_{(1)} \leq \underline{s}]$. As we argued in Section 1.3.1, if $s_{(1)} \leq \underline{s}$, then there is no competitor. When participation is state dependent, this is either good news about the value of the good ($\eta_h < \eta_\ell$) or bad news ($\eta_h > \eta_\ell$). Thus, there is either a winner's blessing, or winner's curse at the bottom. As long as $\frac{\eta_h}{\eta_\ell} \in (\frac{f_\ell(\bar{s})}{f_h(\bar{s})}, \frac{f_\ell(\underline{s})}{f_h(\underline{s})})$, however, this effect does not change the general shape of the conditional expected value: $\mathbb{E}[v|s_{(1)} \leq \hat{s}]$ is decreasing in $\hat{s}$ when $\eta_h f_h(\hat{s}) < \eta_\ell f_\ell(\hat{s})$, has its minimum where $\frac{\eta_h}{\eta_\ell}\frac{f_h(\hat{s})}{f_\ell(\hat{s})} = 1$, and is increasing when $\eta_h f_h(\hat{s}) > \eta_\ell f_\ell(\hat{s})$. As a result, state-dependent participation leaves our results mostly unaltered. One merely needs to replace $f_\omega(s)$ with $\eta_\omega f_\omega(s)$ in every expression and redefine the neutral signal $\breve{s}$ such that $\frac{\eta_h}{\eta_\ell}\frac{f_h(\breve{s})}{f_\ell(\breve{s})} = 1$. In the working paper version (Lauermann and Speit, 2019), we prove every result for this more general case. Only when $\frac{\eta_h}{\eta_\ell} \notin (\frac{f_\ell(\bar{s})}{f_h(\bar{s})}, \frac{f_\ell(\underline{s})}{f_h(\underline{s})})$ such that no neutral signal $\breve{s}$ exists are Propositions 1.3, 1.4, and 1.5 vacuous. If $\frac{\eta_h f_h(\underline{s})}{\eta_\ell f_\ell(\underline{s})} \geq 1$, claim *(iii)* of Proposition 1.2 ensures the existence of a strictly increasing strategy; by Lemma 1.4, this is the only symmetric equilibrium.[26] If, on the other hand, $\frac{\eta_h f_h(\bar{s})}{\eta_\ell f_\ell(\bar{s})} < \frac{f_h(\underline{s})}{f_h(\underline{s})}$ and η_h, η_ℓ are sufficiently large, then there exists an equilibrium in which every bidder selects the same bid.

1.6.2 Distribution of the number of bidders

Generally, numbers uncertainty breaks the affiliation between the first-order statistic of bidders' signals and the value of the good. Without affiliation, however, one cannot expect the equilibrium strategy to be strictly increasing. At the same time, the lack of affiliation creates room for atoms in the bid distribution. Thus, the results do not hinge on the distributional assumptions. The Poisson distribution only serves as a transparent example to illustrate the effects because it allows for closed-form solutions and is characterized by a single parameter. It is not clear, however, whether there is a class other than Poisson for which the equilibrium existence necessarily fails. At the very least, the Poisson distribution is not a "knife-edge" case, in the sense that we can truncate the distribution to always have at least $\underline{n} \geq 2$ bidders (cf. footnote 8), or marginally change the probabilities without changing the results.

1.6.3 Signal structure

The assumption of a unique neutral signal $\breve{s}$ is for convenience only. If there is an interval of signals along which $f_h(s) = f_\ell(s)$, the propositions just become lengthier.[27] Also, unboundedly informative signals leave our results unchanged but complicate some proofs.

While all results are given for continuous densities, we can allow for a finite number of jumps in f_h and f_ℓ. This nests problems with a finite number of discrete signals as these can be modeled as intervals of signals sharing the same likelihood ratio. When the densities are discontinuous, all results except of Propositions 1.3, 1.4, and 1.5 still apply. The equilibrium characterizations and non-existence result, however, rely on the existence of an interval of signals S such that $\frac{f_h(s)}{f_\ell(s)} \leq 1$, but $\frac{f_h(\underline{s})}{f_\ell(\underline{s})}\frac{F_\ell(s)}{F_h(s)} < \frac{f_h(s)}{f_\ell(s)}$ for all $s \in S$. If there is no such interval, and η is sufficiently large,

[26] Compare Lauermann et al. (2018).

[27] For example, in Proposition 1.4 the bids are constant between $\underline{s}+\epsilon$ and $\inf\{s\colon f_h(s) = f_\ell(s)\}-\epsilon$, and strictly increasing at or above $\sup\{s\colon f_h(s) = f_\ell(s)\} + \epsilon$.

an equilibrium of the form of candidate equilibrium (a) exists: all signals below $\breve{s}$ pool on the same bid, and all higher signals follow a strictly increasing strategy. Note that this is always true when signals are binary, which makes this signal structure a special case.[28]

1.6.4 Reserve price

The assumption of a reserve price at v_ℓ is used in the proof of Lemma 1.1, which shows that, without loss, any equilibrium strategy is nondecreasing. If η is sufficiently large, the assumption can be dropped. As η increases, the probability of being alone in the auction vanishes, such that, by Bertrand logic, bidders with signals above some $\underline{s} + \epsilon$ choose a bid at or above v_ℓ and follow a nondecreasing strategy.[29] We prove the result formally in the working paper version (Lauermann and Speit, 2019). Alternatively, if one assumes that the good is only allocated when there are at least two bidders, or if one truncates the Poisson distribution at $\underline{n} \geq 2$ (cf. 1.6.2), the Bertrand logic applies, and all equilibrium bids have to be above v_ℓ. The condition of a minimal amount of competition leaves our results qualitatively unaltered.

1.6.5 Second-price auction

As noted in Section 1.3.1.3, whenever η is sufficiently large, there is no strictly increasing equilibrium in the second-price auction because condition (1.10) does not hold. Thus, any equilibrium bid distribution necessarily contains atoms, which are problematic for the standard auctions. In fact, one can check that when η is sufficiently large, no nondecreasing equilibrium exists in the second-price auction, either. However, it is possible to construct an analogous communication extension for the second-price auction that captures the bidding behavior on a fine grid.

1.6.6 Literature

There is a small literature on numbers uncertainty in private-value auctions, notably Matthews (1987), McAfee and McMillan (1987), and Harstad et al. (1990), studying, e.g., the interaction of numbers uncertainty and risk aversion.

Moreover, there is a recent strand of literature on common-value auctions and non-constant numbers of bidders. Murto and Välimäki (2019) consider a common-

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[28] Lauermann and Wolinsky (2017) make use of this fact.

[29] If, to the contrary, the reserve price is $0 < v_\ell$, participation is state dependent with $\eta_h << \eta_\ell$ and if η_h, η_ℓ are small, then equilibrium strategies can be strictly decreasing. In this case, bidders with high signals expect less competition and are, therefore, bid less. Bidders with signal $\bar{s}$ bid 0, betting to be alone in the auction.

value auction with costly entry.[30] After observing a binary signal, potential bidders have to decide whether to pay a fee to bid in the auction. When the pool of potential bidders is arbitrary large, the number of participating bidders is Poisson distributed with a signal-dependent mean. The signal dependent entry decision precludes atoms in the bid distribution, which enables revenue comparisons.

In Lauermann and Wolinsky (2017, 2018) the participation is deterministic, but state dependent due to a solicitation decision by an informed auctioneer. The interest is in how the outcome of a large first-price auction is affected by the ratio of bidders in the high to the low state. If this the ratio is sufficiently high, the outcome resembles the usual outcome in large auctions, whereas, when the ratio is small, there are necessarily atoms at the top. Atoms are the result of a "participation curse" that arises when there are fewer bidders in the high than in the low state. The atom at the top prevents information aggregation.

In a setting with many goods, Harstad et al. (2008) and Atakan and Ekmekci (2019) consider the effect of numbers uncertainty on the information aggregation properties of a k-th price auction (Pesendorfer and Swinkels, 1997). In Harstad et al. (2008), the distribution of bidders is exogenously given. They find that even if the equilibrium strategy is strictly increasing (which aids aggregation), information aggregation fails unless the numbers uncertainty is negligible. They also provide an example in which equilibrium is not strictly increasing, but they do not study this question further. Atakan and Ekmekci (2019) assume that bidders have a type-dependent outside option such that the numbers uncertainty arises endogenously and is correlated with the state, showing that this also upsets information aggregation.

1.7 Conclusion

We have studied a canonical common-value auction in which the bidders are uncertain about the number of their competitors. This numbers uncertainty invalidates classic findings for common-value auctions (Milgrom and Weber, 1982). In particular, it breaks the affiliation between the first-order statistic of the signals and the value of the good. As a consequence, bidding strategies are generally not strictly increasing but contain atoms. The location of the atoms is indeterminate, implying equilibrium multiplicity. Moreover, no equilibrium exists in the standard auction on the continuous bid space when the expected number of bidders is sufficiently large.

Many of the known failures of equilibrium existence in auctions require careful crafting of the setup, and rely on a discrete type space to generate atoms in the bid distribution (cf. Maskin and Riley (2000), Jackson (2009)). By contrast, we identify

[30] Auctions with endogenous entry are also examined by, among others, Levin and Smith (1994) and Harstad (1990).

a failure of equilibrium existence in an otherwise standard auction setting in which the type space is continuous and atoms in the bid distribution arise endogenously.

We solve the existence problem by analyzing auctions on the grid, which we then characterize with the help of a communication extension based on Jackson et al. (2002). While previous applications of the communication extension used it largely to provide abstract existence proofs, we show how it can be utilized as a solution method.

The communication extension captures all limit outcomes of equilibria on the ever-finer grid. Hence, equilibria on the fine grid have to share the characteristic properties of the equilibria of the communication extension. In particular, we show the emergence of an interior atom and a "winner's blessing" at bids below it. This incentivizes bidders with low signals to compete for the highest bid below the atom. Since such a bid does not exist on the continuous bidding space, none of the equilibria of the communication extension are compatible with the uniform tie-breaking of the standard auction.

Pooling and the equilibrium multiplicity that arise from numbers uncertainty have interesting implications. For example, even though the model is purely competitive, bidders with low signals engage in cooperative behavior to reduce the winner's curse. Contrary to a common-value auction with affiliation, they have an incentive to coordinate on certain bids. Consequently, equilibria resemble collusive behavior, even though they are the outcome of independent, utility-maximizing behavior of the bidders.[31] Moreover, the presence of atoms in the bid distribution invalidates empirical identification strategies that rely on the bidder's first-order condition (cf. Athey and Haile (2007)) and, hence, on a strictly increasing bidding strategy.

Future research may examine the consequences of pooling and equilibrium multiplicity for classic questions such as revenue comparisons across auction formats. Since atoms arise at the bottom of the bid distribution, they are particularly relevant for the determination of the optimal reserve price. Finally, with atoms, the auction sometimes fails to sell to the bidder with the highest signal, suggesting negative welfare consequences in general interdependent value settings with a small private-value component.

[31] Compare also Lauermann and Wolinsky (2017).

Appendices

1.A Proofs

1.A.1 Proof of Lemma 1.1

Step 1 *If $b' > b \geq v_\ell$ and $U(b'|s;\beta) \geq U(b|s;\beta)$, then $U(b'|s';\beta) \geq U(b|s';\beta)$ for $s' > s$. The second inequality is strict if $\frac{f_h(s')}{f_\ell(s')} > \frac{f_h(s)}{f_\ell(s)}$.*

Since $b' > b \geq v_\ell$ it follows that $(v_\ell - b') < (v_\ell - b) \leq 0$. Because the winning probability π_ω is weakly increasing in the bid and never zero (the bidder is alone with positive probability), $\pi_\omega(b';\beta) \geq \pi_\omega(b;\beta) \geq \pi_\omega(v_\ell;\beta) > 0$. Together, this yields $\pi_\ell(b';\beta)(v_\ell - b') < \pi_\ell(b;\beta)(v_\ell - b) \leq 0$. Hence, $U(b'|s;\beta) \geq U(b|s;\beta)$ requires that $\pi_h(b';\beta)(v_h - b') > \pi_h(b;\beta)(v_h - b)$. Rearranging $U(b'|s;\beta) \geq U(b|s;\beta)$ gives

$$\frac{\rho f_h(s)}{(1-\rho) f_\ell(s)}[\pi_h(b';\beta)(v_h - b') - \pi_h(b;\beta)(v_h - b)] \geq \pi_\ell(b;\beta)(v_\ell - b) - \pi_\ell(b';\beta)(v_\ell - b').$$

If $s' > s$ is such that $\frac{f_h(s')}{f_\ell(s')} > \frac{f_h(s)}{f_\ell(s)}$, the left side is strictly larger for s' and $U(b'|s',\beta) > U(b|s',\beta)$.

Step 2 *The set of interim beliefs that imply indifference between two bids, $L = \{\frac{f_h(s)}{f_\ell(s)} : \exists b, b' \text{ with } b \neq b' \text{ and } U(b|s;\beta) = U(b'|s;\beta)\}$, is countable.*

By construction, $\forall l \in L$ there exist two bids $b^l_- < b^l_+$ and a bidder with signal s^l such that $\frac{f_h(s^l)}{f_\ell(s^l)} = l$ who is indifferent between these two bids, $U(b^l_-|s^l;\beta) = U(b^l_+|s^l;\beta)$. Furthermore, there exists a $q^l \in \mathbb{Q}$ s.t. $b^l_- < q^l < b^l_+$. By Step 1, $b^l_+ \leq b^{l'}_-$ for all $l < l'$, which implies that $q^l < q^{l'}$. Because $\mathbb{Q}$ is countable, so is L.

Step 3 *Fix any strategy β. If the likelihood ratio $\frac{f_h}{f_\ell}$ is constant on some interval I, then there is an equivalent strategy $\hat{\beta}$ that is pure and nondecreasing over I and equal to β at every other signal.*

Compare Pesendorfer and Swinkels (1997) footnote 8.

Now combine the steps to prove the lemma. First, suppose that the MLRP holds strictly. Then, for every element $l \in L$, there is only a single signal s_l such that $\frac{f_h(s_l)}{f_\ell(s_l)} = l$ which is indifferent between two bids and may mix. Since L is countable, the set of signals which potentially mix has zero measure and we can assign them the lowest bid in the support of their strategies. The resulting strategy is pure and, by Step 1, nondecreasing. Since the strategy is only changed on a set of measure zero, the resulting distribution of bids is unchanged.

Next, suppose that signal structure is such that it contains intervals I along which the likelihood ratio is constant. In this case, apply Step 3 sequentially to any such I to receive a strategy which is pure and nondecreasing. Furthermore, the reordering leaves the distribution of bids and thereby outcomes and utilities unaltered.

1.A.2 Proof of Proposition 1.1

We follow the steps as in the body of the text and only derive Step 2.

Step 2 *A lower bound on* $\beta^*(s_\circ)$ *is given by*

$$\frac{\beta^*(s_\circ)-v_\ell}{v_h-\beta^*(s_\circ)} \geq \frac{\rho}{1-\rho}\frac{f_h(s_-)}{f_\ell(s_-)}\frac{e^{-\eta_h(1-F_h(s_\circ))}}{e^{-\eta_\ell(1-F_\ell(s_\circ))}}A(\eta), \tag{1.20}$$

where $A(\eta)$ *is a decreasing function with* $\lim_{\eta\to\infty} A(\eta)=1$.

In equilibrium, there is no profitable deviation, such that $U(\beta^*(s_-)|s_-;\beta^*) \geq U(\beta^*(s_\circ)|s_-;\beta^*)$, that is

$$\begin{aligned}&\frac{\rho f_h(s_-)\pi_h(\beta^*(s_-);\beta^*)(v_h-\beta^*(s_-))+(1-\rho)f_\ell(s_-)\pi_\ell(\beta^*(s_-);\beta^*)(v_\ell-\beta^*(s_-))}{\rho f_h(s_-)+(1-\rho)f_\ell(s_-)}\\ \geq\ &\frac{\rho f_h(s_-)\pi_h(\beta^*(s_\circ);\beta^*)(v_h-\beta^*(s_\circ))+(1-\rho)f_\ell(s_-)\pi_\ell(\beta^*(s_\circ);\beta^*)(v_\ell-\beta^*(s_\circ))}{\rho f_h(s_-)+(1-\rho)f_\ell(s_-)}.\end{aligned}$$

Since $\beta^*(s_-)\geq v_\ell$, a necessary condition for the inequality is that

$$\begin{aligned}&\rho f_h(s_-)\pi_h(\beta^*(s_\circ);\beta^*)(v_h-v_\ell)\\ \geq&\rho f_h(s_-)\pi_h(\beta^*(s_\circ);\beta^*)(v_h-\beta^*(s_\circ))+(1-\rho)f_\ell(s_-)\pi_\ell(\beta^*(s_\circ);\beta^*)(v_\ell-\beta^*(s_\circ)).\end{aligned}$$

Rearranging the inequality gives a lower bound on $\beta^*(s_\circ)$

$$\frac{\beta^*(s_\circ)-v_\ell}{v_h-\beta^*(s_\circ)} \geq \frac{\rho}{1-\rho}\frac{f_h(s_-)}{f_\ell(s_-)}\frac{\pi_h(\beta^*(s_\circ);\beta^*)}{\pi_\ell(\beta^*(s_\circ);\beta^*)}\Big(1-\frac{\pi_h(\beta^*(s_-);\beta^*)}{\pi_h(\beta^*(s_\circ);\beta^*)}\frac{v_h-v_\ell}{v_h-\beta^*(s_\circ)}\Big). \tag{1.21}$$

Because $s_\circ > s_-$ and $\eta\to\infty$, it follows that $\frac{\pi_h(\beta^*(s_-);\beta^*)}{\pi_h(\beta^*(s_\circ);\beta^*)} = e^{-\eta(F_h(s_\circ)-F_h(s_-))} \to 0$. Thus, $A(\eta) = 1-\frac{\pi_h(\beta^*(s_-);\beta^*)}{\pi_h(\beta^*(s_\circ);\beta^*)}\frac{v_h-v_\ell}{v_h-\beta^*(s_\circ)} \to 1$ unless $\beta^*(s_\circ)\to v_h$. However, by individual rationality, $\beta^*(s_\circ) < \mathbb{E}[v|\text{win with } \beta^*(s_\circ), s_\circ;\beta^*] \leq \mathbb{E}[v|s_\circ] \leq \mathbb{E}[v]$, yielding a contradiction. Therefore, $A(\eta)\to 1$.

1.A.3 Proof of Proposition 1.2

For $s, \hat{s} \in [\underline{s}, \bar{s}]$, let $F_{s_{(1)}}(\hat{s}|s)$ denote the cdf of $s_{(1)}$ conditional on observing s, and let $f_{s_{(1)}}$ be the associated density

$$F_{s_{(1)}}(\hat{s}|s) = \frac{\rho f_h(s)e^{-\eta(1-F_h(\hat{s}))} + (1-\rho)f_\ell(s)e^{-\eta(1-F_\ell(\hat{s}))}}{\rho f_h(s) + (1-\rho)f_\ell(s)}, \tag{1.22}$$

$$f_{s_{(1)}}(\hat{s}|s) = \frac{\rho\eta f_h(s)f_h(\hat{s})e^{-\eta(1-F_h(\hat{s}))} + (1-\rho)\eta f_\ell(s)f_\ell(\hat{s})e^{-\eta(1-F_\ell(\hat{s}))}}{\rho f_h(s) + (1-\rho)f_\ell(s)}. \tag{1.23}$$

Note that because signal distribution is atomless, the probability that there is no (other) bidder, $s_{(1)} = -\infty$, conditional on observing signal s is $\mathbb{P}[s_{(1)} = -\infty|s] = F_{s_{(1)}}(\underline{s}|s)$. As a result, for $\hat{s} \in [\underline{s}, \bar{s}]$ it holds that $F_{s_{(1)}}(\hat{s}|s) = \int_{\underline{s}}^{\hat{s}} f_{s_{(1)}}(z|s)dz + F_{s_{(1)}}(\underline{s}|s)$.

As a further abbreviation, define $v(\hat{s}|s) = \mathbb{E}[v|s_{(1)} = \hat{s}, s]$, that is

$$v(\hat{s}|s) = \begin{cases} \frac{\rho f_h(s)f_h(\hat{s})e^{-\eta(1-F_h(\hat{s}))}v_h + (1-\rho)f_\ell(s)f_\ell(\hat{s})e^{-\eta(1-F_\ell(\hat{s}))}v_\ell}{\rho f_h(s)f_h(\hat{s})e^{-\eta(1-F_h(\hat{s}))} + (1-\rho)f_\ell(s)f_\ell(\hat{s})e^{-\eta(1-F_\ell(\hat{s}))}} & \text{if } \hat{s} \in [\underline{s}, \bar{s}] \\ \frac{\rho f_h(s)e^{-\eta}v_h + (1-\rho)f_\ell(s)e^{-\eta}v_\ell}{\rho f_h(s)e^{-\eta} + (1-\rho)f_\ell(s)e^{-\eta}} & \text{if } \hat{s} = -\infty. \end{cases} \tag{1.24}$$

If β is strictly increasing and continuous, $\pi_\omega(b;\beta) = \mathbb{P}[s_{(1)} \le \beta^{-1}(b)|\omega;\beta]$ for all b in β's image. As a result, for all b in the image, the utility (1.1) can be rewritten as

$$U(b|s;\beta) = \int_{\underline{s}}^{\beta^{-1}(b)=s} \Big[v(z|s) - b\Big] f_{s_{(1)}}(z|s)dz + \Big[v(-\infty|s) - b\Big] F_{s_{(1)}}(\underline{s}|s). \tag{1.25}$$

Step 1 *If β is a strictly increasing equilibrium, then β is differentiable and solves the ODE* $\frac{\partial\beta(s)}{\partial s} = \Big(\mathbb{E}[v|s_{(1)} = s, s] - \beta(s)\Big)\frac{f_{s_{(1)}}(s|s)}{F_{s_{(1)}}(s|s)}$ *with* $\beta(\underline{s}) = v_\ell$.

If β is a strictly increasing equilibrium (economizing on the $*$), then it is continuous. If β would jump upwards, any bid just above a jump would be dominated by a bid just below the jump, which wins with the same probability but at a lower price. By the same reason, $\beta(\underline{s}) = v_\ell$.

Next, take any point $s \in (\underline{s}, \bar{s})$ and show that β is differentiable at this point. Let $(s_n)_{n\in\mathbb{N}}$ be a sequence converging to s from below. Then, the sequence with elements $b_n = \beta(s_n)$ converges to $b = \beta(s)$ from below, too. Because $b_n < b$ is a best response for $s_n < s$, it follows that $U(b_n|s_n;\beta) \ge U(b|s_n;\beta)$. Using (1.25), gives

$$\begin{aligned}&\int_{\underline{s}}^{\beta^{-1}(b_n)=s_n} \Big[v(z|s_n) - b_n\Big] f_{s_{(1)}}(z|s_n)dz + \Big[v(-\infty|s_n) - b_n\Big] F_{s_{(1)}}(\underline{s}|s_n) \\ &\qquad \ge \int_{\underline{s}}^{\beta^{-1}(b)=s} \Big[v(z|s_n) - b\Big] f_{s_{(1)}}(z|s_n)dz + \Big[v(-\infty|s_n) - b\Big] F_{s_{(1)}}(\underline{s}|s_n),\end{aligned}$$

which can be rearranged to

$$\int_{\underline{s}}^{s_n} \Big[b - b_n\Big] f_{s_{(1)}}(z|s_n)dz + \Big[b - b_n\Big] F_{s_{(1)}}(\underline{s}|s_n) \geq \int_{s_n}^{s} \Big[v(z|s_n) - b\Big] f_{s_{(1)}}(z|s_n)dz.$$

Dividing by $s - s_n > 0$ as well as $F_{s_{(1)}}(s|s_n) = \int_{\underline{s}}^{s} f_{s_{(1)}}(z|s_n)dz + F_{s_{(1)}}(\underline{s}|s_n) > 0$ and taking the $\liminf$ yields

$$\liminf_{n\to\infty} \frac{b - b_n}{s - s_n} \geq \liminf_{n\to\infty} \frac{1}{s - s_n} \int_{s_n}^{s} \Big[v(z|s_n) - b\Big] \frac{f_{s_{(1)}}(z|s_n)}{F_{s_{(1)}}(s|s_n)} dz.$$

By inspection of equations (1.23) and (1.24), the continuity of f_h and f_ℓ ensures that $v(z|s_n)$, $f_{s_{(1)}}(z|s_n)$ and $F_{s_{(1)}}(s|s_n)$ are continuous in both arguments such that

$$\liminf_{n\to\infty} \frac{b - b_n}{s - s_n} \geq [v(s|s) - b] \frac{f_{s_{(1)}}(s|s)}{F_{s_{(1)}}(s|s)}. \tag{1.26}$$

At the same time, bid b is a best response for signal s, implying that $U(b_n|s;\beta) \leq U(b|s;\beta)$, which rearranges to

$$\begin{aligned}\int_{\underline{s}}^{\beta^{-1}(b_n)=s_n} &\Big[v(z|s) - b_n\Big] f_{s_{(1)}}(z|s)dz + \Big[v(-\infty|s) - b_n\Big] F_{s_{(1)}}(\underline{s}|s) \\ &\leq \int_{\underline{s}}^{\beta^{-1}(b)=s} \Big[v(z|s) - b\Big] f_{s_{(1)}}(z|s)dz + \Big[v(-\infty|s) - b\Big] F_{s_{(1)}}(\underline{s}|s).\end{aligned}$$

Repeating the steps as before, but taking the $\limsup$ instead, yields

$$\limsup_{n\to\infty} \frac{b - b_n}{s - s_n} \leq [v(s|s) - b] \frac{f_{s_{(1)}}(s|s)}{F_{s_{(1)}}(s|s)}, \tag{1.27}$$

and because $\liminf \leq \limsup$, it follows from equations (1.26) and (1.27) that

$$\lim_{n\to\infty} \frac{b - b_n}{s - s_n} = \lim_{n\to\infty} \frac{\beta(s) - \beta(s_n)}{s - s_n} = [v(s|s) - \beta(s)] \frac{f_{s_{(1)}}(s|s)}{F_{s_{(1)}}(s|s)}.$$

We can repeat the construction for any sequence of signals and bids which converges from above instead of below and obtain the same result. Therefore, β is differentiable and can be written as (replacing v)

$$\frac{\partial \beta(s)}{\partial s} = \Big(\mathbb{E}[v|s_{(1)} = s, s] - \beta(s)\Big) \frac{f_{s_{(1)}}(s|s)}{F_{s_{(1)}}(s|s)},$$

or, fully spelled out for future reference,

$$= \frac{\rho\eta f_h(s)^2 e^{-\eta(1-F_h(s))}(v_h - \beta(s)) + (1-\rho)\eta f_\ell(s)^2 e^{-\eta(1-F_\ell(s))}(v_\ell - \beta(s))}{\rho f_h(s) e^{-\eta(1-F_h(s))} + (1-\rho) f_\ell(s) e^{-\eta(1-F_\ell(s))}}. \tag{1.28}$$

Step 2 *If β is strictly increasing and solves the ODE* $\frac{\partial\beta(s)}{\partial s} = \Big(\mathbb{E}[v|s_{(1)} = s, s] - \beta(s)\Big)\frac{f_{s_{(1)}}(s|s)}{F_{s_{(1)}}(s|s)}$ *with initial value* $\beta(\underline{s}) = v_\ell$*, then* β *is an equilibrium.*

Suppose that β is strictly increasing and solves the ODE. We want to show that $U(\beta(s)|s;\beta) \geq U(\beta(s')|s;\beta)$ for all $s' \in [\underline{s}, \bar{s}]$. This suffices because $\beta(\underline{s}) = v_\ell$ denotes the lower bound of bids and any bid $b > \beta(\bar{s})$ is dominated by bidding $\beta(\bar{s})$. We show that $U(\beta(s)|s;\beta) \geq U(\beta(s')|s;\beta)$ by proving that $\frac{\partial U(\beta(s')|s;\beta)}{\partial s'} \geq 0$ for all $s' < s$ and $\frac{\partial U(\beta(s')|s;\beta)}{\partial s'} \leq 0$ for all $s' > s$ such that the utility is hump-shaped with a global maximum for signal s at $\beta(s)$.

Substituting b by $\beta(s')$ in the utility function (1.25) and taking the derivative wrt. s' yields (note that β is differentiable by the assumption of the step)

$$\frac{\partial}{\partial s'}U(\beta(s')|s;\beta) = \Big([v(s'|s) - \beta(s')]\frac{f_{s_{(1)}}(s'|s)}{F_{s_{(1)}}(s'|s)} - \beta'(s')\Big)F_{s_{(1)}}(s'|s),$$

which is positive if and only if

$$[v(s'|s) - \beta(s')]\frac{f_{s_{(1)}}(s'|s)}{F_{s_{(1)}}(s'|s)} > \beta'(s').$$

Because β solves the ODE $\beta'(s') = [v(s'|s') - \beta(s')]\frac{f_{s_{(1)}}(s'|s')}{F_{s_{(1)}}(s'|s')}$, this means that $\frac{\partial}{\partial s'}U(\beta(s')|s,\beta)$ is positive if and only if

$$[v(s'|s) - \beta(s')]\frac{f_{s_{(1)}}(s'|s)}{F_{s_{(1)}}(s'|s)} > [v(s'|s') - \beta(s')]\frac{f_{s_{(1)}}(s'|s')}{F_{s_{(1)}}(s'|s')}.$$

Fully expanded, the left side of the equation becomes (cf. equations (1.22)-(1.24))

$$\frac{\rho f_h(s)e^{-\eta(1-F_h(s'))}}{\rho f_h(s)e^{-\eta(1-F_h(s'))} + (1-\rho)f_\ell(s)e^{-\eta(1-F_\ell(s'))}}\underbrace{f_h(s')(v_h - \beta(s'))}_{>0}$$
$$+ \frac{(1-\rho)f_\ell(s)e^{-\eta(1-F_\ell(s'))}}{\rho f_h(s)e^{-\eta(1-F_h(s'))} + (1-\rho)f_\ell(s)e^{-\eta(1-F_\ell(s'))}}\underbrace{f_\ell(s')(v_\ell - \beta(s'))}_{<0}.$$

As a result, the expression is nondecreasing in s, and strictly increasing in s if $\frac{f_h(s)}{f_\ell(s)}$ is increasing. This means that

$$[v(s'|s) - \beta(s')]\frac{f_{s_{(1)}}(s'|s)}{F_{s_{(1)}}(s'|s)} > [v(s'|s') - \beta(s')]\frac{f_{s_{(1)}}(s'|s')}{F_{s_{(1)}}(s'|s')}$$

if and only if $\frac{f_h(s')}{f_\ell(s')} < \frac{f_h(s)}{f_\ell(s)}$.

It follows that

- $\frac{\partial}{\partial s'} U(\beta(s')|s,\beta) > 0$ for all $s' < s : \frac{f_h(s')}{f_\ell(s')} < \frac{f_h(s)}{f_\ell(s)}$,
- $\frac{\partial}{\partial s'} U(\beta(s')|s,\beta) < 0$ for all $s' > s : \frac{f_h(s')}{f_\ell(s')} > \frac{f_h(s)}{f_\ell(s)}$,
- $\frac{\partial}{\partial s'} U(\beta(s')|s,\beta) = 0$ for all $s' : \frac{f_h(s')}{f_\ell(s')} = \frac{f_h(s)}{f_\ell(s)}$,

such that $\beta(s)$ is a global maximizer for s.

Step 3 *$\hat{\beta}$ is a strictly increasing equilibrium if an only if it is strictly increasing and solves the ODE $\frac{\partial \hat{\beta}(s)}{\partial s} = \Big(\mathbb{E}[v|s_{(1)} = s, s] - \beta(s)\Big) \frac{f_{s_{(1)}}(s|s)}{F_{s_{(1)}}(s|s)}$ with initial value $\hat{\beta}(\underline{s}) = v_\ell$. If $\hat{\beta}$ is an equilibrium, it is unique in the class of strictly increasing equilibria. Thus, if $\hat{\beta}$ is not strictly increasing, no strictly increasing equilibrium exists.*

Because the signal densities are continuous, the likelihood ratio $\frac{f_h}{f_\ell}$, bids, as well as values v_ω are bounded and $F_{s_{(1)}}(s|s) > 0$ as well as the ODE $\frac{\partial \hat{\beta}(s)}{\partial s} = [\mathbb{E}[v|s_{(1)} = s, s] - \hat{\beta}(s)] \frac{f_{s_{(1)}}(s|s)}{F_{s_{(1)}}(s|s)}$ is Lipschitz continuous (cf. (1.23) and (1.24)). Thus, there exists a unique solution to the initial value problem $\beta(\underline{s}) = v_\ell$. Combining this with Step 1 (necessary condition) and 2 (sufficient condition), the result follows.

1.A.4 Proof of Lemma 1.3

Proposition 1.2 shows that a strictly increasing equilibrium exists if and only if the unique solution $\hat{\beta}$ to the ODE (1.28) is strictly increasing. Thus, we have to show that $\hat{\beta}$ is strictly increasing.

Step 1 *If $\mathbb{E}[v|s_{(1)} = s, s]$ is strictly increasing in s, then $\hat{\beta}$ is strictly increasing. This is the case if and only if $2\Big(\frac{\partial}{\partial s} \frac{f_h(s)}{f_\ell(s)}\Big) \frac{f_\ell(s)}{f_h(s)} + \eta f_h(s) - \eta f_\ell(s) > 0$ for a.e. s.*

Since $\frac{f_h(\underline{s})}{f_\ell(\underline{s})} > 0$, it follows that $\mathbb{E}[v|s_{(1)} = \underline{s}, \underline{s}] = v(\underline{s}, |\underline{s}) > v_\ell$. In combination with the initial value $\hat{\beta}(\underline{s}) = v_\ell$, this means that $\hat{\beta}'(\underline{s}) > 0$ (cf. (1.28)). Because the densities f_h and f_ℓ are continuous, so is $\hat{\beta}$ and $\hat{\beta}'$. Thus, $\hat{\beta}'$ can only become negative if it intersects the 0 from above. In that case, there exists a $\hat{s}$ such that $\hat{\beta}'(\hat{s}) = 0$, meaning that $v(\hat{s}|\hat{s}) - \hat{\beta}(\hat{s}) = 0$. Since $\hat{\beta}'(\hat{s}) = 0$, marginally increasing $\hat{s}$ will not change $\hat{\beta}$. Hence, the marginal change of $v(\hat{s}|\hat{s})$ decides whether $\hat{\beta}'$ is just tangent, or intersects the 0 at $\hat{s}$. Thus, it suffices that $\mathbb{E}[v|s_{(1)} = \hat{s}, \hat{s}] = v(\hat{s}|\hat{s})$ is strictly increasing in $\hat{s} \in (\underline{s}, \bar{s})$.

The expected value $v(s|s)$ is increasing at (almost) every s if and only if $\frac{f_h(s)^2 e^{-\eta(1-F_h(s))}}{f_\ell(s)^2 e^{-\eta(1-F_\ell(s))}}$, is increasing in s (cf. (1.24)). Differentiating the function with respect to s yields

$$2\Big(\frac{\partial}{\partial s} \frac{f_h(s)}{f_\ell(s)}\Big) \frac{f_h(s)}{f_\ell(s)} \frac{e^{-\eta(1-F_h(s))}}{e^{-\eta(1-F_\ell(s))}} + \frac{f_h(s)^2}{f_\ell(s)^2} \frac{e^{-\eta(1-F_h(s))} e^{-\eta(1-F_\ell(s))}}{(e^{-\eta(1-F_\ell(s))})^2} (\eta f_h(s) - \eta f_\ell(s)) > 0.$$

Dividing by $\frac{e^{-\eta(1-F_h(s))}}{e^{-\eta(1-F_\ell(s))}} > 0$ and $\frac{f_h(s)^2}{f_\ell(s)^2} > 0$ yields the result. Since $\frac{f_h}{f_\ell}$ is monotone, it is differentiable almost everywhere.

Step 2 *When η is sufficiently small, $\hat{\beta}$ is strictly increasing.*

$$
\begin{aligned}
v(s|s) &= \frac{\rho f_h(s)^2 e^{-\eta(1-F_h(s))} v_h + (1-\rho) f_\ell(s)^2 e^{-\eta(1-F_\ell(s))} v_\ell}{\rho f_h(s)^2 e^{-\eta(1-F_h(s))} + (1-\rho) f_\ell(s)^2 e^{-\eta(1-F_\ell(s))}} \\
&\overset{\eta\to 0}{\to} \frac{\rho f_h(s)^2 v_h + (1-\rho) f_\ell(s)^2 v_\ell}{\rho f_h(s)^2 + (1-\rho) f_\ell(s)^2} = \phi(s) \geq \phi(\underline{s}) > v_\ell.
\end{aligned}
$$

Using that $\hat{\beta}(s) \geq v_\ell$ and equation (1.28), $\hat{\beta}'(s)$ can be bounded above by $\eta f_h(s)(v_h - v_\ell)$. Therefore, $\hat{\beta}(s) = \int_{\underline{s}}^{s} \hat{\beta}'(z)dz + v_\ell < \phi(\underline{s})$ when η is small. Thus, if η is small, $\hat{\beta}'(s) = [v(s|s) - \hat{\beta}(s)]\frac{f_{s_{(1)}}(s|s)}{F_{s_{(1)}}(s|s)} \geq [\phi(\underline{s}) - \hat{\beta}(s)]\frac{f_{s_{(1)}}(s|s)}{F_{s_{(1)}}(s|s)} > 0$ for all s.

1.A.5 Proof of Lemma 1.4

Step 1 $\pi_\omega(b_p;\beta) = \frac{\mathbb{P}[s_{(1)} \in [s_-, s_+]\,|\omega]}{\mathbb{E}[\# s \in [s_-, s_+]\,|\omega]} = \frac{e^{-\eta(1-F_\omega(s_+))} - e^{-\eta(1-F_\omega(s_-))}}{\eta(F_\omega(s_+) - F_\omega(s_-))}$ *for $\omega \in \{h, \ell\}$.*

$$
\begin{aligned}
\pi_\omega(b_p;\beta) &= \mathbb{P}[\text{no bid} > b_p|\omega] \sum_{n=0}^{\infty} \frac{1}{n+1} \mathbb{P}[\text{n competitors bid } b_p|\omega] \\
&= e^{-\eta(1-F_\omega(s_+))} \Big(\sum_{n=0}^{\infty} \frac{1}{n+1} e^{-\eta(F_\omega(s_+) - F_\omega(s_-))} \frac{[\eta(F_\omega(s_+) - F_\omega(s_-))]^n}{n!} \Big) \\
&= e^{-\eta(1-F_\omega(s_+))} \Big(\sum_{n=0}^{\infty} e^{-\eta(F_\omega(s_+) - F_\omega(s_-))} \frac{[\eta(F_\omega(s_+) - F_\omega(s_-))]^n}{(n+1)!} \Big) \\
&= \frac{e^{-\eta(1-F_\omega(s_+))}}{\eta(F_\omega(s_+) - F_\omega(s_-))} \Big(\sum_{n=1}^{\infty} e^{-\eta(F_\omega(s_+) - F_\omega(s_-))} \frac{[\eta(F_\omega(s_+) - F_\omega(s_-))]^n}{n!} \Big) \\
&= \frac{e^{-\eta(1-F_\omega(s_+))}}{\eta(F_\omega(s_+) - F_\omega(s_-))} \Big(1 - e^{-\eta(F_\omega(s_+) - F_\omega(s_-))} \Big) \\
&= \frac{e^{-\eta(1-F_\omega(s_+))} - e^{-\eta(1-F_\omega(s_-))}}{\eta(F_\omega(s_+) - F_\omega(s_-))}.
\end{aligned}
$$

The numerator is $\mathbb{P}[s_{(1)} \in [s_-, s_+]|\omega]$ and the denominator is the expected number of signals from $[s_-, s_+]$ in state ω i.e. $\mathbb{E}[\# s \in [s_-, s_+]\,|\omega]$.

Step 2 *If $\eta[F_h(s_+) - F_h(s_-)] < \eta[F_\ell(s_+) - F_\ell(s_-)]$, then*

$$
\mathbb{E}[v|s_{(1)} \leq s] > \mathbb{E}[v|\textit{win with } b_p;\beta] > \mathbb{E}[v|s_{(1)} \leq s_+].
$$

If $\eta[F_h(s_+) - F_h(s_-)] > \eta[F_\ell(s_+) - F_\ell(s_-)]$, the inequalities reverse.

For any two events ϕ and ϕ', $\mathbb{E}[v|\phi] > \mathbb{E}[v|\phi']$ if and only if $\frac{\mathbb{P}[\phi|h]}{\mathbb{P}[\phi|\ell]} > \frac{\mathbb{P}[\phi'|h]}{\mathbb{P}[\phi'|\ell]}$. Therefore, we have to show that when $\eta[F_h(s_+) - F_h(s_-)] < \eta[F_\ell(s_+) - F_\ell(s_-)]$ it holds that

$$\frac{e^{-\eta(1-F_h(s_-))}}{e^{-\eta(1-F_\ell(s_-))}} > \frac{\frac{e^{-\eta(1-F_h(s_+))}-e^{-\eta(1-F_h(s_-))}}{\eta[F_h(s_+)-F_h(s_-)]}}{\frac{e^{-\eta(1-F_\ell(s_+))}-e^{-\eta(1-F_\ell(s_-))}}{\eta[F_\ell(s_+)-F_\ell(s_-)]}} > \frac{e^{-\eta(1-F_h(s_+))}}{e^{-\eta(1-F_\ell(s_+))}}. \tag{1.29}$$

As an abbreviation, define $x_\omega = \eta[F_\omega(s_+) - F_\omega(s_-)]$ for $\omega \in \{h, \ell\}$. Dividing the left inequality of (1.29) by $\frac{e^{-\eta(1-F_h(s_-))}}{e^{-\eta(1-F_\ell(s_-))}}$, it becomes $1 > \frac{\frac{e^{x_h}-1}{x_h}}{\frac{e^{x_\ell}-1}{x_\ell}}$, which holds because $\frac{e^z-1}{z}$ is strictly increasing in z.

If, on the other hand, the right inequality of (1.29) is divided by $\frac{e^{-\eta(1-F_h(s_+))}}{e^{-\eta(1-F_\ell(s_+))}}$, it becomes $\frac{\frac{1-e^{x_h}}{x_h}}{\frac{1-e^{x_\ell}}{x_\ell}} > 1$, which is true because $\frac{1-e^z}{z}$ is strictly decreasing in z.

Step 3 *β can only be an equilibrium if* $\eta[F_h(s_+) - F_h(s_-)] < \eta[F_\ell(s_+) - F_\ell(s_-)]$.

Suppose to the contrary that β is an equilibrium, but $\eta[F_h(s_+) - F_h(s_-)] \geq \eta[F_\ell(s_+) - F_\ell(s_-)]$.[32] Consider a deviation to $b + \epsilon$ by any $s \in [s_-, s_+]$. There are two possibilities:

First, $b_p + \epsilon$ can be a pooling bid meaning that there exists an interval of signals $[s'_-, s'_+]$ such that $\forall s \in [s'_-, s'_+]$ it holds that $\beta(s) = b_p + \epsilon$ and $\neq b_p + \epsilon$ otherwise. Since $\breve{s} < s_+ \leq s'_-$, this implies that $\eta[F_h(s'_+) - F_h(s'_-)] \geq \eta[F_\ell(s'_+) - F_\ell(s'_-)]$ and

$$\begin{aligned}\mathbb{E}[v|\text{win with } b_p + \epsilon; \beta] &\overset{\text{Step 2}}{\geq} \mathbb{E}[v|s_{(1)} \leq s'_-] \\ &\overset{\text{Lemma 1.2}}{\geq} \mathbb{E}[v|s_{(1)} \leq s_+] \overset{\text{Step 2}}{\geq} \mathbb{E}[v|\text{win with } b_p; \beta].\end{aligned}$$

If $b_p + \epsilon$ is not played with positive probability, then it wins whenever $s_{(1)} \leq y$ for some $y \geq s^+$, which means that $\mathbb{E}[v|\text{win with } b_p + \epsilon, s; \beta] = \mathbb{E}[v|s_{(1)} \leq y, s]$. This implies that

$$\begin{aligned}\mathbb{E}[v|\text{win with } b_p + \epsilon; \beta] &= \mathbb{E}[v|s_{(1)} \leq y] \\ &\overset{\text{Lemma 1.2}}{\geq} \mathbb{E}[v|s_{(1)} \leq s_+] \overset{\text{Step 2}}{\geq} \mathbb{E}[v|\text{win with } b_p; \beta].\end{aligned}$$

In either case, it follows that $\mathbb{E}[v|\text{win with } b_p + \epsilon, s; \beta] \geq \mathbb{E}[v|\text{win with } b_p, s; \beta] \geq b_p$, where the latter inequality follows by individual rationality (cf. (1.5)). Since a deviation to $b_p + \epsilon$ discretely increases the winning probability by avoiding the tie-break, it is always profitable for ϵ sufficiently small. Thus, β cannot be an equilibrium when $\eta[F_h(s_+) - F_h(s_-)] \geq \eta[F_\ell(s_+) - F_\ell(s_-)]$ which proves the last assertion.

[32] Note that because $\breve{s} : \frac{\eta f_h(\breve{s})}{\eta f_\ell(\breve{s})} = 1$, it follows from the MLRP that $s_+ > \breve{s}$.

1.A.6 Proof of Lemma 1.5

We follow the steps as in the body of the text and only derive Step 2.

Step 2 *There exists a function $B(\eta) < 1$ with $B(\eta) \to 1$ such that*

$$\frac{b_p - v_\ell}{v_h - b_p} \geq \frac{\rho}{1-\rho} \frac{f_h(\breve{s})}{f_\ell(\breve{s})} \frac{\pi_h^+}{\pi_\ell^+} B(\eta). \tag{1.30}$$

Signal $\breve{s}$ has an incentive to deviate from b_p to a marginally higher bid (deviation 2), unless $U(b_p|\breve{s};\beta^*) \geq \lim_{\epsilon \searrow 0} U(b_p + \epsilon|\breve{s};\beta^*)$, that is

$$\begin{aligned} &\frac{\rho f_h(\breve{s})\pi_h^\circ(v_h - b_p) + (1-\rho) f_\ell(\breve{s})\pi_\ell^\circ(v_\ell - b_p)}{\rho f_h(\breve{s}) + (1-\rho) f_\ell(\breve{s})} \\ &\qquad \geq \frac{\rho f_h(\breve{s})\pi_h^+(v_h - b_p) + (1-\rho) f_\ell(\breve{s})\pi_\ell^+(v_\ell - b_p)}{\rho f_h(\breve{s}) + (1-\rho) f_\ell(\breve{s})} \end{aligned}$$

Rearranging this inequality gives

$$\frac{b_p - v_\ell}{v_h - b_p} \geq \frac{\rho}{1-\rho} \frac{f_h(\breve{s})}{f_\ell(\breve{s})} \frac{\pi_h^+ - \pi_h^\circ}{\pi_\ell^+ - \pi_\ell^\circ}.$$

The last fraction can be replaced by

$$\frac{\pi_h^+ - \pi_h^\circ}{\pi_\ell^+ - \pi_\ell^\circ} = \frac{\pi_h^+ - \pi_h^\circ}{\pi_\ell^+ - \pi_\ell^\circ} \left(\frac{\pi_h^+}{\pi_\ell^+}\right)^{-1} \left(\frac{\pi_h^+}{\pi_\ell^+}\right) = \frac{1 - \frac{1-e^{-\eta F_h(\breve{s})}}{\eta F_h(\breve{s})}}{1 - \frac{1-e^{-\eta F_\ell(\breve{s})}}{\eta F_\ell(\breve{s})}} \left(\frac{\pi_h^+}{\pi_\ell^+}\right) = B(\eta) \frac{\pi_h^+}{\pi_\ell^+},$$

where $B(\eta) = \frac{1 - \frac{1-e^{-\eta F_h(\breve{s})}}{\eta F_h(\breve{s})}}{1 - \frac{1-e^{-\eta F_\ell(\breve{s})}}{\eta F_\ell(\breve{s})}} \to 1$ because $\eta F_\omega(\breve{s}) \to \infty$ for $\omega \in \{h, \ell\}$.

1.A.7 Proof of Lemma 1.6

We follow the steps as in the body of the text and only derive Step 2.

Step 2 *There exists a function $E(\eta) > 1$ with $E(\eta) \to 1$ such that*

$$\frac{b_p - v_\ell}{v_h - b_p} \geq \frac{\rho}{1-\rho} \frac{\eta f_h(\underline{s})}{\eta f_\ell(\underline{s})} \frac{\pi_h^-}{\pi_\ell^-} E(\eta). \tag{1.31}$$

In equilibrium, no signal $s < s_-$ has an incentive to deviate from $\beta^*(s)$ to any $b \in (\beta^*(s), b_p)$. In particular, there is no incentive to deviate to a bid marginally below b_p (deviation 2), that is, $U(\beta^*(s)|s;\beta^*) \geq \lim_{\epsilon \searrow 0} U(b_p - \epsilon|s;\beta^*)$. Using equation (1.21), this rearranges to

$$\frac{b_p - v_\ell}{v_h - b_p} \geq \frac{\rho}{1-\rho} \frac{\eta f_h(s)}{\eta f_\ell(s)} \frac{\pi_h^-}{\pi_\ell^-} \Big(1 - \frac{\pi_h(\beta^*(s);\beta^*)}{\pi_h^-} \frac{v_h - v_\ell}{v_h - b_p}\Big). \tag{1.32}$$

Observe that the right side of equation (1.32) is decreasing in $\pi_h(\beta^*(s);\beta^*)$. Thus, we can find the most conservative lower bound on b_p by bounding $\pi_h(\beta^*(s);\beta^*)$ from above.

Consider now $\underline{s}$ and note that monotonicity implies that

$$\pi_h(\beta^*(\underline{s});\beta^*) \leq \frac{e^{-\eta(1-F_h(s_-))} - e^{-\eta(1-F_h(\underline{s}))}}{\eta[F_h(s_-) - F_h(\underline{s})]} = \frac{e^{-\eta(1-F_h(s_-))} - e^{-\eta}}{\eta F_h(s_-)} = \bar{\pi}_h.$$

Thus, if we plug $\underline{s}$ into equation (1.32), the lower bound becomes

$$\frac{b_p - v_\ell}{v_h - b_p} \geq \frac{\rho}{1-\rho}\frac{f_h(\underline{s})}{f_\ell(\underline{s})}\frac{\pi_h^-}{\pi_\ell^-}\Big(1 - \frac{\bar{\pi}_h}{\pi_h^-}\frac{v_h - v_\ell}{v_h - b_p}\Big).$$

By inspection, $\frac{\bar{\pi}_h}{\pi_h^-} \to 0$ such that $E(\eta) = 1 - \frac{\pi_h}{\pi_h^-}\frac{v_h - v_\ell}{v_h - b_p} \to 1$, unless $b_p \to v_h$. However, by individual rationality, $b_p \leq \mathbb{E}[v|\text{win with } b_p, s_+; \beta^*] < \mathbb{E}[v|s_+] \leq \mathbb{E}[v]$, such that this cannot be the case and $E(\eta) \to 1$.

1.A.8 Proof of Lemma 1.7

Since we only deal with concordant strategies, all signals report the same message space M. Further, reporting a different space is weakly dominated by reporting M, any $m \in M$ and bidding v_ℓ. To keep notation cleaner, we, hence, drop the explicit reference to M from all expressions.

Step 1 *Consider any concordant strategy σ and two actions (m, b) and (m', b') s.t. $\pi_h^{\mathfrak{c}}(m', b'; \sigma) > \pi_h^{\mathfrak{c}}(m, b; \sigma)$. If $U^{\mathfrak{c}}(m', b'|s; \sigma) \geq U^{\mathfrak{c}}(m, b|s; \sigma)$, then $U^{\mathfrak{c}}(m', b'|s'; \sigma) \geq U^{\mathfrak{c}}(m, b|s'; \sigma)$ for $s' > s$. The second inequality is strict if and only if $\frac{f_h(s')}{f_\ell(s')} > \frac{f_h(s)}{f_\ell(s)}$.*

Note that $\pi_h^{\mathfrak{c}}(m', b'; \sigma) > \pi_h^{\mathfrak{c}}(m, b; \sigma)$ implies that $\pi_\ell^{\mathfrak{c}}(m', b'; \sigma) > \pi_\ell^{\mathfrak{c}}(m, b; \sigma)$ since the winning probabilities are isomorphic across states.

From $\pi_h^{\mathfrak{c}}(m', b'; \sigma) > \pi_h^{\mathfrak{c}}(m, b; \sigma)$, it follows that $b' \geq b \geq v_\ell$ which implies that $(v_\ell - b') \leq (v_\ell - b) \leq 0$. If $b' = b$, then $\pi_h^{\mathfrak{c}}(m', b'; \sigma)(v_h - b') > \pi_h^{\mathfrak{c}}(m', b; \sigma)(v_h - b)$ directly. If $b' > b$, on the other hand, $\pi_\ell^{\mathfrak{c}}(m', b'; \sigma) > \pi_\ell^{\mathfrak{c}}(m, b; \sigma)$ that $\pi_\ell^{\mathfrak{c}}(m', b'; \sigma)(v_\ell - b') < \pi_\ell^{\mathfrak{c}}(m, b; \sigma)(v_\ell - b)$. In this case, it follows from $U^{\mathfrak{c}}(m', b'|s; \sigma) \geq U^{\mathfrak{c}}(m, b|s; \sigma)$ also requires that $\pi_h^{\mathfrak{c}}(m', b'; \sigma)(v_h - b') > \pi_h^{\mathfrak{c}}(m, b; \sigma)(v_h - b)$.

Rearranging $U^{\mathfrak{c}}(m', b'|s; \sigma) \geq U^{\mathfrak{c}}(m, b|s; \sigma)$ yields

$$\begin{aligned} &\frac{\rho\eta f_h(s)}{(1-\rho)\eta f_\ell(s)}[\pi_h^{\mathfrak{c}}(m', b'; \sigma)(v_h - b') - \pi_h^{\mathfrak{c}}(m, b; \sigma)(v_h - b)] \\ &\geq \pi_\ell^{\mathfrak{c}}(m, b; \sigma)(v_\ell - b) - \pi_\ell^{\mathfrak{c}}(m', b'; \sigma)(v_\ell - b'). \end{aligned}$$

If $s' > s$ is such that $\frac{f_h(s')}{f_\ell(s')} > \frac{f_h(s)}{f_\ell(s)}$, the left side is strictly larger for s' and thus $U^{\mathfrak{c}}(m', b'|s; \sigma) > U^{\mathfrak{c}}(m, b|s; \sigma)$.

Step 2 *The set of interim beliefs that imply indifference between two actions which win with different probabilities,* $L = \{\frac{f_h(s)}{f_\ell(s)} \colon \exists (m,b),(m',b')$ *with* $\pi_h^{\mathfrak{c}}(m,b;\hat{\sigma}^*) \neq \pi_h(m',b';\hat{\sigma}^*)$ *and* $U^{\mathfrak{c}}(m,b|s;\sigma^*) = U^{\mathfrak{c}}(m',b'|s;\sigma^*)\}$, *is countable.*

By construction, $\forall l \in L$ there exist two tuples (m_-^l, b_-^l), (m_+^l, b_+^l) with $\pi_h^{\mathfrak{c}}(m_-^l, b_-^l;\hat{\sigma}^*) < \pi_h^{\mathfrak{c}}(m_+^l, b_+^l;\hat{\sigma}^*)$ such that signals $s^l \colon \frac{f_h(s^l)}{f_\ell(s^l)} = l$ are indifferent between these two bids, $U^{\mathfrak{c}}(m_-^l, b_-^l|s^l;\sigma^*) = U^{\mathfrak{c}}(m_+^l, b_+^l|s^l;\sigma^*)$. Furthermore, there exists a $q^l \in \mathbb{Q}$ s.t. $\pi_h^{\mathfrak{c}}(m_-^l, b_-^l;\hat{\sigma}^*) < q^l < \pi_h^{\mathfrak{c}}(m_+^l, b_+^l;\hat{\sigma}^*)$. By Step 1, $\pi_h^{\mathfrak{c}}(m_+^l, b_+^l;\hat{\sigma}^*) \leq \pi_h^{\mathfrak{c}}(m_-^{l'}, b_-^{l'};\hat{\sigma}^*)$ for all $l < l'$, which implies that $q^l < q^{l'}$. Because $\mathbb{Q}$ is countable, so is L.

Step 3 *Let* σ^* *be a concordant equilibrium. Then, there exists a m-equivalent, concordant equilibrium* $\hat{\sigma}^*$ *with the following property: If* (m,b) *and* (m',b') *are in the support of* $\hat{\sigma}^*$ *and* $\pi_h^{\mathfrak{c}}(m,b;\hat{\sigma}^*) = \pi_h^{\mathfrak{c}}(m',b';\hat{\sigma}^*)$, *then* $(m,b) = (m',b')$.

If (m,b) and (m',b') are in the support of σ^* and $\pi_h^{\mathfrak{c}}(m,b;\sigma^*) = \pi_h^{\mathfrak{c}}(m',b';\sigma^*)$ (and thereby $\pi_\ell^{\mathfrak{c}}(m,b;\sigma) = \pi_\ell^{\mathfrak{c}}(m',b';\sigma)$), then $b = b'$. Otherwise, the action-tuple with the higher bid is dominated and, hence, cannot be part of a best response.

If (m,b) and (m',b) are in the support of σ^*, and $\pi_h^{\mathfrak{c}}(m,b;\sigma^*) = \pi_h^{\mathfrak{c}}(m',b;\sigma^*)$, then the report m is, conditional on b, irrelevant. Thus, any (m',b) can be replaced by (m,b) without altering winning probabilities or payoffs, receiving a m-equivalent equilibrium $\hat{\sigma}^*$ which has the asserted properties.

Using Step 3, in equilibrium, any winning probability can be identified with a unique message/bid combination, (m,b). By Step 1, bidders with higher beliefs choose actions tuples which win with higher probabilities and by Step 2 there are at most countably many beliefs which are indifferent between multiple action-tuples. We, thus, can proceed as in Lemma 1.1 and reorder the actions in such a way, that the strategies are pure and the probability to win is nondecreasing in s. In the resulting strategy $\hat{\sigma}^*$ bids are nondecreasing in the signal, and, given a bid, the reports are nondecreasing in the signal.

1.A.9 Proof of Proposition 1.4

Consider a sequence of communication extensions $(\Gamma_n^{\mathfrak{c}})_{n \in \mathbb{N}}$, along which $\eta^n \to \infty$. By Lemma 1.8, there exists an equilibrium for each n denoted (economizing on the $*$) σ_n. For any σ_n, we adopt the notation that $\sigma_n(s) = (M_n, \mu_n(s), \beta_n(s))$ for some M_n and functions $\mu_n : [\underline{s}, \bar{s}] \to M_n$ and $\beta_n : [\underline{s}, \bar{s}] \to [v_\ell, v_h]$. Since we look at concordant equilibria, we drop the explicit reference M_n, unless its central to the argument.

Step 1 *Fix two signals $s_- < s_+$ with $s_+ > \breve{s}$. If $\beta_n(s_-) = \beta_n(s_+)$ on exactly $[s_-, s_+]$, then, $s_- < \breve{s}$. Further, there is a signal $s_\circ$ with $s_- \leq s_\circ < \breve{s}$ such that $\sigma_n(s_\circ) = \sigma_n(s_+)$.*

First, suppose that $\mu_n(s)$ is strictly increasing on some sub-interval $[s'_-, s'_+] \subseteq [s_-, s_+]$. Then, any $\hat{s} \in [\hat{s}'_-, \hat{s}'_+]$ wins whenever $s_{(1)} \leq \hat{s}$, that is with probability $e^{-\eta^n_\omega(1-F_\omega(\hat{s}))}$. Since the conditional expected value $\mathbb{E}[v|s_{(1)} \leq s, s]$ is strictly increasing above $\breve{s}$, it has to be that $s'_+ \leq \breve{s}$. Otherwise, any signal $s' \in [\breve{s}, s'_+)$ would have a strict incentive to mimic s'_+, winning more often, paying the same, and receiving a higher expected value.[33]

Since $\mu_n(s)$ cannot be strictly increasing above $\breve{s}$, there has to be a signal $s_\circ < s_+$ such that $\mu_n(s) = \mu_n(s_+)$ for all $s \in [s_\circ, s_+]$, meaning that $\sigma_n(s) = \sigma_n(s_+)$ for all $s \in [s_\circ, s_+]$.

Now suppose that $s_\circ \geq \breve{s}$, such that $[F_h(s_+) - F_h(s_\circ)] < [F_\ell(s_+) - F_\ell(s_\circ)]$. By inspection, the argument in the proof of Step 2 of Lemma 1.4 is also valid in the communication extension. Thus, there is a profitable deviation for all $s \in [s_\circ, s_+]$. By choosing bid marginally above $\beta_n(s) = \beta_n(s_+)$ and an arbitrary report $m \in M_n$, signal s wins more often and receives a good of a higher expected value.

Step 2 *Fix any $\epsilon > 0$. If n is sufficiently large, $\beta_n(s)$ is strictly increasing on $(\breve{s}+\epsilon, \bar{s}]$ and $\mu_n(s)$ is irrelevant on that interval.*

Suppose to the contrary that there is a sequence of equilibria $(\sigma_n)_{n\in\mathbb{N}}$ for which the claim is violated: for any n, there is an interval of signals $[s^n_-, s^n_+]$ with $s^n_+ > \breve{s}+\epsilon$ along which the bid is constant and equal to b_n. By Step 0, there is a signal $s^n_\circ < \breve{s}$ such that $\sigma_n(s^n_\circ) = \sigma_n(s^n_+) = (m_n, b_n)$. Without loss, let $s^n_\circ = s^n_-$. Note that by construction, $\beta_n(s) > b_n$ for all $s > s^+_n$.

The proof now follows by contradiction, which is structured into three parts. First, Substep 1 derives an upper bound on b_n, and Substep 2 a lower bound on b_n. Then, Substep 3 shows that when n is sufficiently large, the lower bound exceeds the upper bound which completes the proof.

Note that because $\beta_n(s) > b_n$ for all $s > s^+_n$, a bid marginally above b_n wins the auction whenever $s_{(1)} \leq s^n_+$, independent of the signal $m \in M_n$. To simplify notation, we abbreviate the implied winning probabilities from bidding (m_n, b_n) and bidding marginally more in state $\omega \in \{h, \ell\}$ by

$$\pi^n_\omega = \pi^{\mathfrak{c}}_\omega(m_n, b_n; \sigma_n) = \frac{e^{-\eta^n_\omega(1-F_\omega(s^n_+))} - e^{-\eta^n_\omega(1-F_\omega(s^n_-))}}{\eta^n_\omega(F_\omega(s^n_+) - F_\omega(s^n_-))},$$
$$\pi^{+,n}_\omega = \lim_{\epsilon\to 0} \pi^{\mathfrak{c}}_\omega(m_n, b_n + \epsilon; \sigma_n) = e^{-\eta^n_\omega(1-F_\omega(s^n_+))}.$$

[33] Individual rationality would imply that $\beta_n(s_-) = \beta_n(s') \leq \mathbb{E}[s_{(1)} \leq s', s'] < \mathbb{E}[v|s_{(1)} \leq s'_+, s']$.

Substep 1 *An upper bound on b_n is given by*

$$\frac{b_n - v_\ell}{v_h - b_n} \leq \frac{\rho f_h(s_-^n)}{(1-\rho) f_\ell(s_-^n)} \frac{\pi_h^n}{\pi_\ell^n}. \tag{1.33}$$

The individual rationality argument (1.5) remains unaltered in the communication extension. Since $\beta_n(s_-^n) = b_n$, this means that $b_n \leq \mathbb{E}[v|\text{win with } (m_n, b_n), s_-^n; \sigma_n]$, which rearranges to equation (1.33).

Substep 2 *A lower bound on b_n is given by*

$$\frac{b_n - v_\ell}{v_h - b_n} \geq \frac{\rho}{1-\rho} \frac{f_h(s_+^n)}{f_\ell(s_+^n)} \frac{\pi_h^{+,n} - \pi_h^n}{\pi_\ell^{+,n} - \pi_\ell^n}. \tag{1.34}$$

Since σ_n is an equilibrium, there can not be a profitable deviation. In particular, $U^\mathfrak{c}(m_n, b_n | s_+^n; \sigma_n) \geq \lim_{\epsilon \to 0} U^\mathfrak{c}(m_n, b_n + \epsilon | s_+^n; \sigma_n)$, that is

$$\begin{aligned} &\frac{\rho f_h(s_+^n)\pi_h^n(v_h - b_n) + (1-\rho) f_\ell(s_+^n)\pi_\ell^n(v_\ell - b_n)}{\rho f_h(s_+^n) + (1-\rho) f_\ell(s_+^n)} \\ \geq\ &\frac{\rho f_h(s_+^n)\pi_h^{+,n}(v_h - b_n) + (1-\rho) f_\ell(s_+^n)\pi_\ell^{+,n}(v_\ell - b_n)}{\rho f_h(s_+^n) + (1-\rho) f_\ell(s_+^n)}, \end{aligned}$$

which rearranges to (1.34).

Substep 3 *When n is sufficiently large, the upper bound on b_n expressed by (1.33) is smaller than the lower bound on b_n given by inequality (1.34).*

Combining inequalities (1.33) and (1.34) yields

$$\frac{\rho f_h(s_-^n)}{(1-\rho) f_\ell(s_-^n)} \frac{\pi_h^n}{\pi_\ell^n} \geq \frac{\rho}{1-\rho} \frac{f_h(s_+^n)}{f_\ell(s_+^n)} \frac{\pi_h^{+,n} - \pi_h^n}{\pi_\ell^{+,n} - \pi_\ell^n} \iff \frac{f_h(s_-^n)}{f_\ell(s_-^n)} \geq \frac{f_h(s_+^n)}{f_\ell(s_+^n)} \frac{\frac{\pi_h^{+,n}}{\pi_h^n} - 1}{\frac{\pi_\ell^{+,n}}{\pi_\ell^n} - 1}.$$

Note that because $s_+^n > \breve{s}$, it follows that $\frac{f_h(s_+^n)}{f_\ell(s_+^n)} > 1$, such that is has to hold that

$$\frac{f_h(s_-^n)}{f_\ell(s_-^n)} > \frac{\frac{\pi_h^{+,n}}{\pi_h^n} - 1}{\frac{\pi_\ell^{+,n}}{\pi_\ell^n} - 1}. \tag{1.35}$$

Since $s_-^n < \breve{s} < \breve{s} + \epsilon \leq s_+^n$, it has to be true that $\eta[F_\omega(s_+^n) - F_\omega(s_-^n)] \to \infty$ for $\omega \in \{h, \ell\}$. Thus, it follows from

$$\frac{\pi_\omega^{+,n}}{\pi_\omega^n} = \frac{\eta^n[F_\omega(s_+^n) - F_\omega(s_-^n)]}{1 - e^{-\eta^n[F_\omega(s_+^n) - F_\omega(s_-^n)]}} \quad \text{that} \quad \lim_{n\to\infty} \frac{\frac{\pi_h^{+,n}}{\pi_h^n} - 1}{\frac{\pi_\ell^{+,n}}{\pi_\ell^n} - 1} = \lim_{n\to\infty} \frac{[F_h(s_+^n) - F_h(s_-^n)]}{[F_\ell(s_+^n) - F_\ell(s_-^n)]}.$$

Further, the MLRP implies that

$$[F_h(s_+^n) - F_h(s_-^n)] = \int_{s_-^n}^{s_+^n} f_h(z)dz = \int_{s_-^n}^{s_+^n} f_\ell(z)\frac{f_h(z)}{f_\ell(z)}dz$$
$$\geq \int_{s_-^n}^{s_+^n} f_\ell(z)\frac{f_h(s_-^n)}{f_\ell(s_-^n)}dz = \frac{f_h(s_-^n)}{f_\ell(s_-^n)}[F_\ell(s_+^n) - F_\ell(s_-^n)],$$

which rearranges to $\frac{F_h(s_+^n)-F_h(s_-^n)}{F_\ell(s_+^n)-F_\ell(s_-^n)} \geq \frac{f_h(s_-^n)}{f_h(s_-^n)}$. Thus, equation (1.35) is necessarily violated when n is large.

Step 3 *Fix any $\epsilon \in (0, \frac{\breve{s}-\underline{s}}{2})$. When n is sufficiently large, $\nexists(M, m, b)$ s.t. $\pi_\omega^{\mathfrak{c}}(\sigma_n(\underline{s}+\epsilon); \sigma_n) < \pi_\omega^{\mathfrak{c}}(M, m, b; \sigma_n) < \pi_\omega^{\mathfrak{c}}(\sigma_n(\breve{s}-\epsilon); \sigma_n)$ for $\omega \in \{h, \ell\}$. As a result, $\beta_n(s)$ is constant on $[\underline{s}+\epsilon, \breve{s}-\epsilon]$.*

Since the winning probabilities are isomorphic across states, if the claim is violated in one state, it is also violated in the other. Suppose to the contrary that there exists an $\epsilon \in (0, \frac{\breve{s}-\underline{s}}{2})$ and a subsequence of equilibria $(\sigma_n)_{n\in\mathbb{N}}$ for which such a deviation denoted by (M_n', m_n', b_n') exists. It follows immediately that $M_n' = M_n$. Otherwise, the deviation only wins when the bidder is alone. Henceforth, let $M_n = M_n'$ and drop the explicit reference from the expressions.

The rest of the for this step is structured into three substeps which yield a contradiction. First, Substep 1 derives an upper bound on b_n' and Substep 2 a lower bound. Then, Substep 3 shows that when n is sufficiently large, the lower exceeds the upper bound, which yields the contradiction. To simplify notation, define signals $s_-^n = \sup\{s\colon \sigma_n(s) = \sigma_n(\underline{s}+\epsilon)\}$ and $s_+^n = \inf\{s\colon \sigma_n(s) = \sigma_n(\breve{s}-\epsilon)\}$ and fix some signal $s_{++} > \breve{s}$.

Substep 1 *An upper bound on b_n' is given by*

$$\frac{b_n' - v_\ell}{v_h - b_n'} \leq \frac{\rho f_h(s_{++})}{(1-\rho)f_\ell(s_{++})}\frac{e^{-\eta^n(1-F_h(s_{++}))}}{e^{-\eta^n(1-F_\ell(s_{++})}}. \tag{1.36}$$

By Step 2, $s_{++} > \breve{s}$ does not pool when n is sufficiently large, and, hence, wins whenever $s_{(1)} \leq s_{++}$, such that $\pi_\omega^{\mathfrak{c}}(\sigma_n(s_{++}); \sigma_n) = e^{-\eta^n(1-F_\omega(s_{++}))}$ for $\omega \in \{h, \ell\}$. Since the individual rationality argument for equation (1.5) remains unaltered in the communication extension, an upper bound on $\beta_n(s_{++})$ is given by

$$\frac{\beta_n(s_{++}) - v_\ell}{v_h - \beta_n(s_{++})} \leq \frac{\rho f_h(s_{++})}{(1-\rho)f_\ell(s_{++})}\frac{e^{-\eta^n(1-F_h(s_{++}))}}{e^{-\eta^n(1-F_\ell(s_{++})}}.$$

Because the left side of the inequality is increasing in b_n and $\beta_n(s_{++}) \geq b_n'$, the upper bound (1.36) follows.

Substep 2 *A lower bound on b'_n is given by*

$$\frac{b'_n - v_\ell}{v_h - b'_n} \geq \frac{\rho}{1-\rho} \frac{f_h(\underline{s})}{f_\ell(\underline{s})} \frac{\pi^{\mathfrak{c}}_h(m'_n, b'_n; \sigma_n)}{\pi^{\mathfrak{c}}_\ell(m'_n, b'_n; \sigma_n)} Q(n), \tag{1.37}$$

where $Q(n)$ is an increasing function with $\lim_{n\to\infty} Q(n) = 1$.

First, we want to find a lower bound on $\pi^{\mathfrak{c}}_\omega(\sigma_n(\underline{s}); \sigma_n)$. By monotonicity, this probability is maximal if $\sigma_n(\underline{s}) = \sigma_n(\underline{s} + \epsilon)$. Further, if $\pi^{\mathfrak{c}}_\omega(\sigma_n(\underline{s} + \epsilon); \sigma_n) = \pi^{\mathfrak{c}}_\omega(\sigma_n(\underline{s}); \sigma_n)$, then it attains the highest value in either state $\omega \in \{h, \ell\}$ if all signals up to s^n_- pool on the same action-tuple, that is if $\sigma_n(\underline{s} + \epsilon) = \sigma_n(s)$ for $s < s^n_-$. As a result,

$$\pi^{\mathfrak{c}}_\omega(\sigma_n(\underline{s}); \sigma_n) \leq \frac{e^{-\eta^n_\omega(1-F_\omega(s^n_-))} - e^{-\eta}}{\eta^n_\omega F_\omega(s^n_-)} = \pi^{-,n}_\omega.$$

Given this lower bound and because $\beta(\underline{s}) \geq v_\ell$, we can bound

$$U^{\mathfrak{c}}(\sigma_n(\underline{s})|\underline{s}; \sigma_n) \leq \frac{\rho f_h(\underline{s})\pi^{-,n}_h(v_h - v_\ell) + (1-\rho) f_\ell(\underline{s})\pi^{-,n}_\ell(v_\ell - v_\ell)}{\rho f_h(\underline{s}) + (1-\rho) f_\ell(\underline{s})}. \tag{1.38}$$

In equilibrium, it has to hold that $U^{\mathfrak{c}}(\sigma_n(\underline{s})|\underline{s}; \sigma_n) \geq U^{\mathfrak{c}}(m_n, b'_n|\underline{s}; \sigma_n)$. Using the lower bound on (1.38), this rearranges to

$$\frac{b'_n - v_\ell}{v_h - b'_n} \geq \frac{\rho}{1-\rho} \frac{f_h(\underline{s})}{f_\ell(\underline{s})} \frac{\pi^{\mathfrak{c}}_h(m'_n, b'_n; \sigma_n)}{\pi^{\mathfrak{c}}_\ell(m'_n, b'_n; \sigma_n)} \Big(1 - \frac{\pi^{-,n}_h}{\pi^{\mathfrak{c}}_h(m'_n, b'_n; \sigma_n)} \frac{v_h - v_\ell}{v_h - b'_n}\Big).$$

Note that (m'_n, b'_n) wins at least whenever $s_{(1)} \leq s^n_-$, such that $\pi^{\mathfrak{c}}_h(m'_n, b'_n; \sigma_n) \geq e^{-\eta(1-F_h(s^n_-))}$. Thus $\pi^{-,n}_h/\pi^{o,n}_h \to 0$ and $Q(n) = 1 - \frac{\pi^{-,n}_h}{\pi^{\mathfrak{c}}_h(m'_n, b'_n; \sigma_n)} \frac{v_h - v_\ell}{v_h - b'_n} \to 1$, unless $b'_n \to v_h$. However, by individual rationality, $b'_n \leq \mathbb{E}[v|\text{win with } m_n, b'_n, s^n_-; \sigma_n] < \mathbb{E}[v|s^n_-] < \mathbb{E}[v]$, such that $Q(n) \to 1$.

Substep 3 *When n is sufficiently large, the upper bound on b'_n expressed by inequality (1.36) is smaller than the lower bound on b'_n given by inequality (1.37).*

Combining inequalities (1.36) and (1.37) yields

$$\frac{\rho f_h(s_{++})}{(1-\rho) f_\ell(s_{++})} \frac{e^{-\eta^n(1-F_h(s_{++}))}}{e^{-\eta^n(1-F_\ell(s_{++})}} \geq \frac{\rho}{1-\rho} \frac{f_h(\underline{s})}{f_\ell(\underline{s})} \frac{\pi^{\mathfrak{c}}_h(m'_n, b'_n; \sigma_n)}{\pi^{\mathfrak{c}}_\ell(m'_n, b'_n; \sigma_n)} Q(n),$$

which rearranges to

$$\frac{f_h(s_{++})}{f_\ell(s_{++})} \frac{f_\ell(\underline{s})}{f_h(\underline{s})} \geq \Big(\frac{e^{-\eta^n(1-F_h(s_{++}))}}{e^{-\eta^n(1-F_\ell(s_{++})}}\Big)^{-1} \frac{\pi^{\mathfrak{c}}_h(m_n, b_n; \sigma_n)}{\pi^{\mathfrak{c}}_\ell(m_n, b_n; \sigma_n)} Q(n). \tag{1.39}$$

Observe that because $\pi^{\mathfrak{c}}_\omega(m_n, b_n; \sigma_n) < e^{-\eta^n(1-F_\omega(s^n_+))}$ for $\omega \in \{h, \ell\}$, inequality

(1.29) and $s_+^n < \breve{s}$ imply that

$$\frac{\pi_h^{\mathfrak{c}}(m_n, b_n; \sigma_n)}{\pi_\ell^{\mathfrak{c}}(m_n, b_n; \sigma_n)} \geq \frac{e^{-\eta^n(1-F_h(s_+^n))}}{e^{-\eta^n(1-F_\ell(s_+^n))}} \geq \frac{e^{-\eta^n(1-F_h(\breve{s}-\epsilon))}}{e^{-\eta^n(1-F_\ell(\breve{s}-\epsilon))}}.$$

Thus, a necessary condition for inequality (1.39) to hold is

$$\frac{f_h(s_{++})}{f_\ell(s_{++})}\frac{f_\ell(\underline{s})}{f_h(\underline{s})} \geq \Big(\frac{e^{-\eta^n(1-F_h(s_{++}))}}{e^{-\eta^n(1-F_\ell(s_{++})}}\Big)^{-1}\frac{e^{-\eta^n(1-F_h(\breve{s}-\epsilon))}}{e^{-\eta^n(1-F_\ell(\breve{s}-\epsilon))}}Q(n).$$

However, since $F_\ell(s_{++})-F_\ell(\breve{s}-\epsilon) > F_h(s_{++})-F_h(\breve{s}-\epsilon)$ it follows that $\eta^n[F_\ell(s_{++})-F_\ell(\breve{s}-\epsilon)-F_h(s_{++})+F_h(\breve{s}-\epsilon)] \to \infty$, such that the right side grows without bound while the left stays constant. Thus, inequality (1.39) is violated when n is large, which proves the claim.

Step 4 *Fix any $\epsilon \in (0, \frac{\breve{s}-\underline{s}}{2})$. When n is sufficiently large, there are two disjoint, adjacent intervals I_n and J_n with $[\underline{s}+\epsilon, \breve{s}-\epsilon] \subset I_n \cup J_n$. Signals $s_I \in I_n$ choose $\sigma_n(s_I) = (M_n, m_I^n, b_n)$ and signals $s_J \in J_n$ choose $\sigma_n(s_J) = (M_n, m_J^n, b_n)$. There is no $m_n \in M_n$ s.t. $m_I^n < m_n < m_J^n$, which implies that $\nexists(M, m, b)$ s.t. $\pi_\omega^{\mathfrak{c}}(\sigma_n(s_I); \sigma_n) < \pi_\omega^{\mathfrak{c}}(M, m, b; \sigma_n) < \pi_\omega^{\mathfrak{c}}(\sigma_n(s_J); \sigma_n)$ for $\omega \in \{h, \ell\}$. Last, the expected number of bidders in both intervals is larger than $\frac{1}{\epsilon}$.*

Fix any $\epsilon > 0$ sufficiently small such that $\frac{f_h(\underline{s}+\epsilon)}{f_\ell(\underline{s}+\epsilon)}\frac{F_\ell(\breve{s}-\epsilon)}{F_h(\breve{s}-\epsilon)} < \frac{f_h(\breve{s}-\epsilon)}{f_\ell(\breve{s}-\epsilon)}$. Notice that such an ϵ exists, because $\frac{f_h(\underline{s})}{f_\ell(\underline{s})}\frac{F_\ell(\breve{s})}{F_h(\breve{s})} < 1 = \frac{f_h(\breve{s})}{f_\ell(\breve{s})}$ and the expressions are continuous in its arguments.

Define $(m_n^I, b_n) = \sigma_n(\underline{s}+\epsilon)$ and let $I_n = \{s\colon (\mu_n(s), \beta_n(s)) = (m_n^I, b_n)\}$ be the interval of signals which choose the same action-tuple as $\underline{s}+\epsilon$. Further, let $m_n^J = \inf\{m \in M_n : m > m_n^I\}$ be the next higher report from M_n and $J_n = \{s\colon (\mu_n(s), \beta_n(s)) = (m_n^J, b_n)\}$ be the interval of signals which choose the same bid as $\underline{s}+\epsilon$, but this higher report.[34]

By Step 3 $[\underline{s}+\epsilon, \breve{s}-\epsilon] \subset I_n \cup J_n$ and, apart from the last, all the other properties follow by construction. Thus, we only need to check that $\int_{I_n}\eta^n f_\omega(s)ds, \int_{J_n}\eta^n f_\omega(s)ds \to \infty$ for $\omega \in \{h, \ell\}$. Observe that it suffices to show the convergence in state h.

First, consider interval I_n with bounds denoted $s_-^{I,n} = \inf I_n$ and $s_+^{I,n} = \sup I_n$ and suppose to the contrary that $\eta^n(F_h(s_+^{I,n}) - F_h(s_+^{I,n})) \not\to \infty$. Then, there exists a subsequence along which $s_+^{I,n} - s_-^{I,n} \to 0$, which, by construction, means that $s_+^{I,n}, s_-^{I,n} \to \underline{s}+\epsilon$. This implies, however, that when n is sufficiently large

[34] It might happen that there is no $m \in M_n : m > m_n^I$. In this case, all $s > \sup I_n$ bid $\beta_n(s) > b_n$, such that choosing a $m > m_n^I$ is equivalent to choosing a marginally higher bid. Thus, it is without loss to assume that the report exists and, if necessary, approximate the action-tuple by choosing a marginally higher bid.

$\pi_h^{\mathfrak{c}}(\sigma_n(\underline{s}+\frac{\epsilon}{2});\sigma_n) < \pi_h^{\mathfrak{c}}(\sigma_n(\underline{s}+\epsilon);\sigma_n) < \pi_h^{\mathfrak{c}}(\sigma_n(\breve{s}-\frac{\epsilon}{2});\sigma_n)$, which is a contradiction to Step 3. Thus, $\int_{I_n} \eta^n f_h(s)ds \to \infty$.

Second, consider interval J_n with bounds denoted $s_-^{J,n} = \inf J_n = \sup I_n$ as well as $s_+^{J,n} = \sup J_n$.[35] Suppose to the contrary that $\eta^n(F_h(s_+^{J,n}) - F_h(s_+^{J,n})) \not\to \infty$. In this case, there is a subsequence along which $s_-^{J,n}, s_+^{J,n}$ converge to some common limit s^J. Without loss, let the original sequence be this subsequence. Notice that it cannot be that $s^J < \breve{s}-\epsilon$. Otherwise, $\pi_h^{\mathfrak{c}}(\sigma_n(\underline{s}+\epsilon);\sigma_n) < \pi_h^{\mathfrak{c}}(m_n^J, b_n;\sigma_n) < \pi_h^{\mathfrak{c}}(\sigma_n(\breve{s}-\epsilon);\sigma_n)$, which is a contradiction to Step 3. Since the same is true for any $\epsilon' < \epsilon$ and $s^J < \breve{s}-\epsilon'$, it follows that $s^J \geq \breve{s}$. In the following, we only concentrate on this remaining case.

We, hence, suppose that $s_-^{J,n}, s_+^{J,n}$ converge to some $s^J \geq \breve{s}$, such that $\eta^n(F_h(s_+^{J,n}) - F_h(s_+^{J,n})) \not\to \infty$. The contradiction is created in four steps. First, Substep 0 shows that the inference from winning with (m_n^J, b_n) is approximately the same as from winning whenever $s_{(1)} \leq s_+^{I,n}$. Then, Substep 1 derives an upper bound on b_n and Substep 2 a lower bound. Substep 3 shows that the lower exceeds the upper bound when n is sufficiently large, which yields a contradiction. To simplify notation, abbreviate the probabilities to win with action-tuple $(m_n^{I,n}, b_n)$ by $\pi_\omega^{I,n} = \pi_\omega^{\mathfrak{c}}(m_n^{I,n}, b_n);\sigma_n)$ and with action-tuple $(m_n^{J,n}, b_n)$ by $\pi_\omega^{J,n} = \pi_\omega^{\mathfrak{c}}(m_n^{J,n}, b_n);\sigma_n)$ for $\omega \in \{h,\ell\}$.

Substep 0

$$\frac{\pi_h^{J,n}}{\pi_\ell^{J,n}} = \frac{e^{-\eta^n(1-F_h(s_-^{J,n}))}}{e^{-\eta^n(1-F_h(s_-^{J,n}))}} D(n), \tag{1.40}$$

where $D(n)$ is a function with $\lim_{n\to\infty} D(n) = 1$.

If $\eta^n[F_h(s_+^{J,n}) - F_h(s_-^{J,n})] \to 0$, then $\eta^n[F_\ell(s_+^{J,n}) - F_\ell(s_-^{J,n})] \to 0$ and

$$\begin{aligned}
D(n) &= \frac{\pi_h^{J,n}}{\pi_\ell^{J,n}} \Big(\frac{e^{-\eta^n(1-F_h(s_-^{J,n}))}}{e^{-\eta^n(1-F_\ell(s_-^{J,n}))}} \Big)^{-1} \\
&= \frac{\frac{e^{-\eta^n(1-F_h(s_+^{J,n}))} - e^{-\eta^n(1-F_h(s_-^{J,n}))}}{\eta^n(F_h(s_+^{J,n}) - F_h(s_-^{J,n}))}}{\frac{e^{-\eta^n(1-F_\ell(s_+^{J,n}))} - e^{-\eta^n(1-F_\ell(s_-^{J,n}))}}{\eta^n(F_\ell(s_+^{J,n}) - F_\ell(s_-^{J,n}))}} \Big(\frac{e^{-\eta^n(1-F_h(s_-^{J,n}))}}{e^{-\eta^n(1-F_\ell(s_-^{J,n}))}} \Big)^{-1} \\
&= \frac{\frac{e^{\eta^n(F_h(s_+^{J,n}) - F_h(s_-^{J,n}))} - 1}{\eta^n(F_h(s_+^{J,n}) - F_h(s_-^{J,n}))}}{\frac{e^{\eta^n(F_\ell(s_+^{J,n}) - F_\ell(s_-^{J,n}))} - 1}{\eta^n(F_\ell(s_+^{J,n}) - F_\ell(s_-^{J,n}))}} \to 1. \quad \text{by l'Hospital.}
\end{aligned}$$

If $\lim \eta^n[F_h(s_+^{J,n}) - F_h(s_-^{J,n})] > 0$, then $s^J = \breve{s}$. Otherwise, there is a signal

[35] If J_n is empty, define $s_-^{J,n} = s_+^{J,n} = \sup I_n$.

$s^J > \check{s}$ which ties with positive probability, which is at odds with Step 3 when n is sufficiently large. Since $s^J = \check{s}$, this means that $\frac{[F_h(s_+^{J,n})-F_h(s_-^{J,n})]}{[F_\ell(s_+^{J,n})-F_\ell(s_-^{J,n})]} \to 1$ and $\eta^n[F_h(s_+^{J,n}) - F_h(s_-^{J,n}) - F_\ell(s_+^{J,n}) + F_\ell(s_-^{J,n})] \to 0$, such that

$$\begin{aligned}
D(n) &= \frac{\pi_h^{J,n}}{\pi_\ell^{J,n}} \left(\frac{e^{-\eta^n(1-F_h(s_-^{J,n}))}}{e^{-\eta^n(1-F_\ell(s_-^{J,n}))}}\right)^{-1} \\
&= \frac{\frac{e^{-\eta^n(1-F_h(s_+^{J,n}))} - e^{-\eta^n(1-F_h(s_-^{J,n}))}}{\eta^n(F_h(s_+^{J,n})-F_h(s_-^{J,n}))}}{\frac{e^{-\eta^n(1-F_\ell(s_+^{J,n}))} - e^{-\eta^n(1-F_\ell(s_-^{J,n}))}}{\eta^n(F_\ell(s_+^{J,n})-F_\ell(s_-^{J,n}))}} \left(\frac{e^{-\eta^n(1-F_h(s_-^{J,n}))}}{e^{-\eta^n(1-F_\ell(s_-^{J,n}))}}\right)^{-1} \\
&= \frac{F_\ell(s_+^{J,n}) - F_\ell(s_-^{J,n})}{F_h(s_+^{J,n}) - F_h(s_-^{J,n})} \\
&\qquad \cdot \frac{e^{\eta^n[F_h(s_+^{J,n})-F_h(s_-^{J,n})-F_\ell(s_+^{J,n})+F_\ell(s_-^{J,n})]} - e^{-\eta^n[F_\ell(s_+^{J,n})-F_\ell(s_-^{J,n})]}}{1-e^{-\eta^n[F_\ell(s_+^{J,n})-F_\ell(s_-^{J,n})]}} \to 1.
\end{aligned}$$

Substep 1 *An upper bound on b_n is given by*

$$\frac{b_n - v_\ell}{v_h - b_n} \leq \frac{\rho f_h(\underline{s}+\epsilon)}{(1-\rho)f_\ell(\underline{s}+\epsilon)} \frac{\pi_h^{I,n}}{\pi_l^{I,n}}. \tag{1.41}$$

The individual rationality argument for equation (1.5) remains unaltered in the communication extension. Applied to signal $\underline{s}+\epsilon$, which chooses (b_n, m_n^I), it provides the inequality.

Substep 2 *A lower bound on b_n is given by*

$$\frac{b_n - v_\ell}{v_h - b_n} \geq \frac{\rho f_h(s_+^{I,n})}{(1-\rho)f_\ell(s_+^{I,n})} \frac{\pi_h^{J,n} - \pi_h^{I,n}}{\pi_\ell^{J,n} - \pi_l^{I,n}}. \tag{1.42}$$

Consider signal $s_+^{I,n} = s_-^{J,n}$, which is indifferent (if J_n is non-empty) or prefers (if J_n is empty) action (m_n^I, b_n) over (m_n^J, b_n). Then, $U^{\mathfrak{c}}(m_n^I, b_n | s_+^{I,n}; \sigma_n) \geq U^{\mathfrak{c}}(m_n^J, b_n | s_+^{I,n}; \sigma_n)$ implies that

$$\begin{aligned}
&\frac{\rho f_h(s_+^{I,n})\pi_h^{I,n}(v_h - b_n) + (1-\rho)f_\ell(s_+^{I,n})\pi_\ell^{I,n}(v_\ell - b_n)}{\rho f_h(s_+^{I,n}) + (1-\rho)f_\ell(s_+^{I,n})} \\
&\qquad \geq \frac{\rho f_h(s_+^{I,n})\pi_h^{J,n}(v_h - b_n) + (1-\rho)f_\ell(s_+^{I,n})\pi_\ell^{J,n}(v_\ell - b_n)}{\rho f_h(s_+^{I,n}) + (1-\rho)f_\ell(s_+^{I,n})},
\end{aligned}$$

which rearranges to inequality (1.42).

Substep 3 *When n is sufficiently large, the lower bound (1.42) exceeds the upper bound (1.41).*

Combining equations (1.41) and (1.42) yields

$$\frac{\rho f_h(\underline{s}+\epsilon)}{(1-\rho)f_\ell(\underline{s}+\epsilon)}\frac{\pi_h^{I,n}}{\pi_l^{I,n}} \geq \frac{\rho}{1-\rho}\frac{f_h(s_+^{I,n})}{f_\ell(s_+^{I,n})}\frac{\pi_h^{J,n}-\pi_h^{I,n}}{\pi_\ell^{J,n}-\pi_\ell^{I,n}},$$

which rearranges to

$$\frac{f_h(\underline{s}+\epsilon)}{f_\ell(\underline{s}+\epsilon)}\Big(\frac{f_h(s_+^{I,n})}{f_\ell(s_+^{I,n})}\Big)^{-1} \geq \frac{\frac{\pi_h^{J,n}}{\pi_h^{I,n}}-1}{\frac{\pi_\ell^{J,n}}{\pi_\ell^{I,n}}-1}. \tag{1.43}$$

Since $s_+^{I,n} = s_-^{J,n} \to \check{s}$, it has to hold in either state $\omega \in \{h, \ell\}$ that $\eta^n(F_\omega(s_+^{n,I}) - F_\omega(\underline{s}+\epsilon)) \to \infty$. Combined with the observations that

$$\pi_\omega^{I,n} \leq \frac{e^{-\eta^n(1-F_\omega(s_+^{I,n}))}}{\eta^n(F_\omega(s_+^{n,I})-F_\omega(\underline{s}+\epsilon))} \quad \text{and} \quad \pi_\omega^{J,n} \geq e^{-\eta^n(1-F_\omega(s_+^{I,n}))},$$

this implies that $\frac{\pi_\omega^{I,n}}{\pi_\omega^{J,n}} \to 0$. Hence, the right side of inequality (1.43) converges to

$$\begin{aligned}
\lim_{n\to\infty}\frac{\frac{\pi_h^{J,n}}{\pi_h^{I,n}}-1}{\frac{\pi_\ell^{J,n}}{\pi_\ell^{I,n}}-1} &= \lim_{n\to\infty}\frac{\pi_h^{J,n}}{\pi_l^{J,n}}\Big(\frac{\pi_h^{I,n}}{\pi_l^{I,n}}\Big)^{-1} \overset{\text{Step 1}}{=} \lim_{n\to\infty} D(n)\frac{e^{-\eta^n(1-F_h(s_-^{J,n}))}}{e^{-\eta^n(1-F_h(s_-^{J,n}))}}\Big(\frac{\pi_h^{I,n}}{\pi_l^{I,n}}\Big)^{-1}\\
&= \lim_{n\to\infty}\frac{e^{-\eta^n(1-F_h(s_-^{J,n}))}}{e^{-\eta^n(1-F_h(s_-^{J,n}))}}\frac{F_\ell(s_+^{I,n})-F_\ell(s_-^{I,n})}{F_h(s_+^{I,n})-F_h(s_-^{I,n})}\frac{e^{-\eta^n(1-F_\ell(s_+^{I,n}))}-e^{-\eta^n(1-F_\ell(s_-^{I,n}))}}{e^{-\eta^n(1-F_h(s_+^{I,n}))}-e^{-\eta^n(1-F_h(s_-^{I,n}))}}\\
&= \lim_{n\to\infty}\frac{F_\ell(s_+^{I,n})-F_\ell(s_-^{I,n})}{F_h(s_+^{I,n})-F_h(s_-^{I,n})}\frac{1-e^{-\eta^n(F_\ell(s_+^{I,n})-F_\ell(s_-^{I,n}))}}{1-e^{-\eta^n(F_h(s_+^{I,n})-F_h(s_-^{I,n}))}} = \lim_{n\to\infty}\frac{F_\ell(s_+^{I,n})-F_\ell(s_-^{I,n})}{F_h(s_+^{I,n})-F_h(s_-^{I,n})}.
\end{aligned}$$

Since $s_+^{I,n} = s_-^{J,n} \to s^J \geq \check{s}$, when n is large, the MLRP implies that $\frac{F_\ell(s_+^{I,n})-F_\ell(s_-^{I,n})}{F_h(s_+^{I,n})-F_h(s_-^{I,n})} \leq \frac{F_\ell(s_-^{I,n})}{F_h(s_-^{I,n})} < \frac{F_\ell(\check{s}-\epsilon)}{F_h(\check{s}-\epsilon)}$. Furthermore, we chose $\epsilon > 0$ s.t. $\frac{f_h(\underline{s}+\epsilon)}{f_\ell(\underline{s}+\epsilon)}\frac{F_\ell(\check{s}-\epsilon)}{F_h(\check{s}-\epsilon)} < \frac{f_h(\check{s}-\epsilon)}{f_\ell(\check{s}-\epsilon)} < \frac{f_h(s_+^{I,n})}{f_\ell(s_+^{I,n})}$ such that $\frac{f_h(\underline{s}+\epsilon)f_\ell(s_+^{I,n})}{f_\ell(\underline{s}+\epsilon)f_h(s_+^{I,n})} < \frac{F_\ell(s_+^{I,n})-F_\ell(s_-^{I,n})}{F_h(s_+^{I,n})-F_h(s_-^{I,n})}$ when n is large. Thus, inequality (1.43) is violated when n is large, which means that the lower bound on b_n (1.42) exceeds the upper bound (1.41), such that b_n cannot exist. Since $(\sigma_n)_{n\in\mathbb{N}}$ is a sequence of equilibria, it, therefore, cannot be that $\eta^n(F_h(s_+^{J,n}) - F_h(s_+^{J,n})) \not\to \infty$.

1.A.10 Proof of Lemma 1.8

Denote the bid space with $k \geq 2$ equidistant bids by B_k. Existence is shown by a fixed point argument on the distribution of bids. Since those are Poisson distributed and thereby fully described by the mean, we look at the compact set of vectors

$$\Lambda = \Big\{ \Big(\lambda(b_1|h) \quad \ldots \quad \lambda(b_k|h) \quad \lambda(b_1|\ell) \quad \ldots \quad \lambda(b_k|\ell) \Big) : \sum_{b \in B_k} \lambda(b|\omega) = \eta \Big\} \subset R^{2k},$$

where $\lambda(b|\omega)$ denotes the expected number of bids b in state ω.

Let $F : \Lambda \rightrightarrows \mathcal{P}(\Lambda)$ be the correspondence which maps any λ into the set of vectors $\{\tilde{\lambda}\}$ that are induced by a pure and nondecreasing best response $\beta : [\underline{s}, \bar{s}] \to B_k$ meaning that $\tilde{\lambda}(b|\omega) = \int_{\beta^{-1}(b)} \eta f_\omega(s) ds$ for all $b \in B_k$, and $\beta(s) \in \arg\max_b U(b|s, \lambda)$ for almost all s. Here, $U(b|s, \lambda)$ is the interim expected utility from bidding b, given the bidders signal s and distribution of (other) bids described by the Poisson parameter λ, which fully determines the probability to win with the bid b.

Because Λ is compact, to apply Kakutani's Fixed-Point Theorem, we need to show that $F(\lambda)$ is non-empty, convex valued, and that F has a closed graph.

$F(\lambda)$ is non-empty because on the finite set there exists a best response for any signal s. By Lemma 1.1, these best responses can be reordered such that the resulting β is pure and nondecreasing.

To show that $F(\lambda)$ is convex valued, consider $\tilde{\lambda}$ and $\tilde{\lambda}'$ from its image. We have to show that $\forall \alpha \in [0, 1]$, $\alpha\tilde{\lambda} + (1 - \alpha)\tilde{\lambda}' = \tilde{\lambda}^* \in F(\lambda)$. $\tilde{\lambda}$ and $\tilde{\lambda}'$ are induced by two best responses $\tilde{\beta}$ and $\tilde{\beta}'$. Consider a mixed strategy that follows $\tilde{\beta}$ with probability α and $\tilde{\beta}'$ with probability $1-\alpha$. Such a strategy is optimal for the bidders and result in a distribution of bids $\tilde{\lambda}^*$. By Lemma 1.1, there is a pure, nondecreasing strategy inducing the same distribution and utilities. Thus $\tilde{\lambda}^* \in F(\lambda)$.

What remains to be shown is that F has a closed graph. Take any two sequences $\lambda_n \to \lambda$ and $\tilde{\lambda}_n \to \tilde{\lambda}$ where $\tilde{\lambda}_n \in F(\lambda_n)$. We have to show that $\tilde{\lambda} \in F(\lambda)$. For every λ_n there is a nondecreasing best response β_n inducing $\tilde{\lambda}_n$. By Helly's Selection Theorem, there is a point-wise converging subsequence of those β_n with a nondecreasing limit β. Obviously, β induces $\tilde{\lambda}$. Furthermore, because $U(b|s, \lambda_n)$ is continuous in both λ_n and b, β is a best response to λ. Thus, F has a closed graph.

Kakutani's Fixed-Point Theorem guarantees an equilibrium vector $\lambda \in \Lambda$, and, by construction, there exists a pure, nondecreasing strategy β which is a best response and induces this λ. Thus, β is a pure, nondecreasing and symmetric equilibrium.

1.A.11 Proof of Lemma 1.10

Take the sequence of auctions on the ever-finer grid $(\Gamma(k))_{k\in\mathbb{N}}$, denote the sequence of respective bid spaces by $(B_k)_{k\in\mathbb{N}}$ and equilibria by $(\beta_k)_{k\in\mathbb{N}}$ (economizing on the $*$). For every k, denote the on-path winning probability in the high state by $\pi_h^k(s) = \pi_h(\beta_k(s); \beta_k)$ and define an auxiliary function $\delta_k(b) = \max\{b' \in B_k \colon b' \leq b\}$.

Since $(\beta_k)_{k\in\mathbb{N}}$, $(\pi_h^k)_{k\in\mathbb{N}}$ and $(\delta_k)_{k\in\mathbb{N}}$ are sequences of nondecreasing functions, by Helly's Selection Theorem, there is a subsequence along which these functions converge pointwise to some nondecreasing limit β, π_h and δ, respectively. We denote this subsequence by n and, henceforth, consider it exclusively.

Construct M by including $m \in M$ if and only if there exists a signal $s \in [\underline{s}, \bar{s}]$ such that $\pi_h(s) = m$. Further, define function $\sigma^*(s) = (M, \pi_h(s), \beta(s))$ for every s.

By construction, properties *(i)* and *(ii)* of Lemma 1.10 are fulfilled. Steps 1 and 2 proceed by showing properties *(iii)* and *(iv)*, before Steps 3 and 4 show that σ^* is an equilibrium of communication extension.

Step 1 $\pi^{\mathfrak{c}}_\omega(\sigma^*(s); \sigma^*) = \lim_{n\to\infty} \pi_\omega(\beta_n(s); \beta_n)$ *for every* s *and* $\omega \in \{h, \ell\}$.

We focus on state h, the result follows for ℓ because the winning probabilities are isomorphic across states. Fix any $\hat{s} \in [\underline{s}, \bar{s}]$ and define the sets $W_n = \{s\colon \pi_h^n(s) < \pi_h^n(\hat{s})\}$, $T_n = \{s\colon \pi_h^n(s) = \pi_h^n(\hat{s})\}$, and $L_n = \{s\colon \pi_h^n(s) > \pi_h^n(\hat{s})\}$. Furthermore, define $W = \{s\colon \pi_h(s) < \pi_h(\hat{s})\}$, and T as well as L analogously. Because π_h^n is nondecreasing and converges pointwise, $W_n \to W, T_n \to T$ and $L_n \to L$.

Given strategy β_n, signal $\hat{s}$ wins against signals from the set W_n, loses against signals L_n and ties with signals from T_n. We want to show that under strategy σ^*, signal $\hat{s}$ wins against signals from the set W, loses against signals L, and ties with signals from T. If this is true, the convergence of the sets and atomless signal distribution ensures that the winning probabilities converge.

Fix any $s_L \in L$. When n is sufficiently large, $s_L \in L_n$. Further, it follows from $\pi_h^n(\hat{s}) < \pi_h^n(s_L)$ that $\beta_n(\hat{s}) < \beta_n(s_L)$. This and the convergence of β_n implies that $\beta(\hat{s}) \leq \beta(s)$. Further, by definition of L and M, $\mu(s_L) = \pi_h(s_L) > c(\hat{s}) = \pi_h(\hat{s})$. Thus, either s_L chooses a higher bid and/or a higher report than $\hat{s}$. Thus, $\hat{s}$ never wins the auction when s_L is present.

The symmetric argument can be made for all signals $s_W \in W$, such that signal $\hat{s}$ following σ^* wins against all signals from W.

Last, fix any $s_T \in T$. Again, when n is sufficiently large, $s_T \in T_n$ meaning that $\pi_h^n(s_T) = \pi_h^n(\hat{s})$. This implies that $\beta_n(s_T) = \beta_n(\hat{s})$ for all n large, which means that in the limit $\beta(s_T) = \beta(\hat{s})$. Further, by definition of T and M, $\mu(s_T) = \pi_h(s_T) = c(\hat{s}) = \pi_h(\hat{s})$. Thus, s_T and $\hat{s}$ choose the same bid and same report, $\sigma^*(s_T) = \sigma^*(\hat{s})$, meaning that they tie.

Step 2 *For every* s*, it holds that*

$$\lim_{n\to\infty} U(\beta_n(s)|s; \beta_n) = U^{\mathfrak{c}}(\sigma^*(s)|s; \sigma^*) = U^{\mathfrak{c}}(M, \pi_h(\sigma^*(s); \sigma^*), \beta(s)|s; \sigma^*).$$

Since $\lim_{n\to\infty} \pi_\omega(\beta_k(s); \beta_k) = \pi^{\mathfrak{c}}_\omega(\sigma^*(s); \sigma^*)$ in both states $\omega \in \{h, \ell\}$ and for every s, and because $\beta_n(s)$ converges to $\beta(s)$ for every s, the convergence is immediate from (1.19).

Step 3 *For every* (M, m, b) *with* $m \in M$*, there is a sequence of bids* $(b_n)_{n \in \mathbb{N}}$ *with* $b_n \in B_n$ *such that* $b_n \to b$ *and* $\pi_\omega(b_n; \beta_n) \to \pi^{\mathfrak{c}}_\omega(m, b; \sigma^*)$ *in either state* $\omega \in \{h, \ell\}$*.*

Since M is kept fixed throughout the proof, it is dropped from the expressions for ease of notation. Further, we focus on state h. For state ℓ the result follows because the winning probabilities are isomorphic across states. By construction of M, there exists a signal s_m such that $\pi^{\mathfrak{c}}_h(\sigma^*(s_m); \sigma^*) = m$. The proof is structured into three cases:

Case 1 $\pi^{\mathfrak{c}}_h(m, b; \sigma^*) = m$ If $\sigma^*(s_m) = (m, b)$, then it follows from $\pi_h(\beta_n(s_m); \beta_n) = \pi^n_h(s_m) \to m$ and $\beta_n(s_m) \to b$, that $(\beta_n(s_m))_{n \in \mathbb{N}}$ is the desired sequence.

If $\sigma^*(s_m) \neq (m, b)$, it follows from $b > \beta(s_m)$ that $\pi^{\mathfrak{c}}_h(m, \beta(s_m); \sigma^*) = \mathbb{P}[s_1 \leq s_m | h]$. Otherwise, there is a non-trivial interval of signals $I = \{s \colon \sigma^*(s) = (m, \beta(s_m))\}$ that is outbid by (m, b), such that (m, b) wins strictly more often, which violates the definition of m. The same is true, if there is a non-trivial interval $I = \{s \colon b > \beta(s) > \beta(s_m)\}$. Thus, $\beta(s') > b$ for all $s' > s_m$. Take any $\lambda \in (0, 1)$ and consider $b_n = \delta_n(\lambda b + (1 - \lambda)\beta(s'))$. Whenever n is sufficiently large, $\beta_n(s_m) < b_n < \beta_n(s')$, such that in the limit

$$\pi^{\mathfrak{c}}_h(m, b; \sigma^*) = \mathbb{P}[s_1 \leq s_m | h] \leq \lim_{n \to \infty} \pi^n_h(b_n; \beta_n) \leq \mathbb{P}[s_1 \leq s' | h].$$

Since this is true for any $s' > s_m$ and λ arbitrary close to 1, the desired sequence exists.

If $\sigma^*(s_m) \neq (m, b)$ because $b < \beta(s_m)$ the construction can be repeated symmetrically.

Case 2: $\pi^{\mathfrak{c}}_h(m, b; \sigma^*) < m$ In this case, m breaks any potential tie in on b in the bidders favor, such that (m, b) wins whenever there is no bid above b. Let $s_+ = \inf\{s \colon \beta(s) > b\}$. Then, $\pi^{\mathfrak{c}}_h(m, b; \sigma^*) = \mathbb{P}[s_{(1)} \leq s_+ | h]$ and for all $s' > s_+$ it holds that $\beta(s') > b$. Take any $\lambda \in (0, 1)$ and consider $b_n = \delta_n(\lambda b + (1 - \lambda)\beta(s'))$. When n is large, $\beta_n(s_+) < b_n < \beta_n(s')$, meaning that in the limit

$$\mathbb{P}[s_{(1)} \leq s_+ | h] \leq \lim_{n \to \infty} \pi^n_h(b_n; \beta_n) \leq \mathbb{P}[s_1 \leq s' | h].$$

Since this is true for any $s' > s_+$ and λ arbitrary close to 1, the desired sequence exists.[36]

Case 3: $\pi^{\mathfrak{c}}_h(m, b; \sigma^*) > m$ The proof is symmetric to Case 2, with an approximation from below.

Step 4 σ^* *is a concordant equilibrium of the communication extension,* $\Gamma^{\mathfrak{c}}$*.*

[36] Note that if $s_+ = \bar{s}$, then $\lim_{n \to \infty} \pi^n_h(b_n; \beta_n) \to 1$ with any $b_n = \delta_n(\lambda b + (1 - \lambda)v_h)$.

First, deviations to a different M' and/or a $m' \notin M$ are dominated by reporting M, an arbitrary $m \in M$, and bidding v_ℓ. This action-tuple wins at least whenever the bidder is alone and generates a strictly positive profit in the high state. Thus, we restrict attention to deviations (m', b') where $m' \in M$ and do not explicitly reference M in the expressions.

Suppose now there is a signal $\hat{s}$ and a profitable deviation (m', b') such that $U^{\mathfrak{c}}(m', b'|\hat{s}; \sigma^*) > U^{\mathfrak{c}}(\sigma^*(\hat{s})|\hat{s}; \sigma^*)$. By Step 3, there exists a sequence of bids $(b_n)_{n\in\mathbb{N}}$ with $b_n \to b'$ such that $\pi_\omega(b_n; \beta_n) \to \pi^{\mathfrak{c}}_\omega(m', b'; \sigma^*)$ in either state $\omega \in \{h, \ell\}$. This means that $U(b_n|\hat{s}; \beta_n) \to U^{\mathfrak{c}}(m', b'|\hat{s}; \sigma^*)$. But then $U(\beta_n(\hat{s})|\hat{s}; \beta_n) \to U^{\mathfrak{c}}(\sigma^*(\hat{s})|\hat{s}; \sigma^*)$ (Step 2) implies that when n is sufficiently large, a deviation from $\beta_n(\hat{s})$ to b_n must have been profitable for $\hat{s}$.[37] This is a contradiction.

1.A.12 Proof of Lemma 1.11

The "if" part of the statement follows directly by (ii) and (iii) of Lemma 1.10. Thus, we only show that "only if" part.

Step 1 *If $\sigma^*(s_-) = \sigma^*(s_+)$, then $\beta^*_n(s_-) = \beta^*_n(s_+)$ whenever n is sufficiently large.*

Suppose to the contrary that there is a sequence along which $\beta^*_n(s_-) \neq \beta^*_n(s_+)$. Since any β^*_n is nondecreasing, it has to hold that $\{s\colon \beta^*_n(s) \in [\beta^*_n(s_-), \beta^*_n(s_+)]\} \not\to \emptyset$. We show that combined, these two conditions imply that $|\pi_h(\beta^*_n(s_+); \beta^*_n) - \pi_h(\beta^*_n(s_-); \beta^*_n)| \not\to 0$.

If $\{s\colon \beta^*_n(s) \in (\beta^*_n(s_-), \beta^*_n(s_+))\} \not\to \emptyset$, this follows immediately. Otherwise, either $\{s\colon \beta^*_n(s) = \beta^*_n(s_-)\} \not\to \emptyset$, in which case $\pi_h(\beta^*_n(s_+); \beta^*_n)$ stays bounded above $\pi_h(\beta^*_n(s_-); \beta^*_n)$ because it wins the uniform tie-break on $\beta^*_n(s_-)$ with certainty; and/or $\{s\colon \beta^*_n(s) = \beta^*_n(s_+)\} \not\to \emptyset$, in which case $\pi_h(\beta^*_n(s_-); \beta^*_n)$ stays bounded below $\pi_h(\beta^*_n(s_+); \beta^*_n)$ because $\beta^*_n(s_-)$ only wins when no bid at or above $\beta^*_n(s_+)$ is made.

If $|\pi_h(\beta^*_n(s_+); \beta^*_n) - \pi_h(\beta^*_n(s_-); \beta^*_n)| \not\to 0$, it follows that $|\pi_h(\beta^*_n(s_+); \beta^*_n) - \pi^{\mathfrak{c}}_h(\sigma^*(s_+); \sigma^*)| + |\pi_h(\beta^*_n(s_-); \beta^*_n) - \pi^{\mathfrak{c}}_h(\sigma^*(s_-); \sigma^*)| \geq |\pi_h(\beta^*_n(s_+); \beta^*_n) - \pi_h(\beta^*_n(s_-); \beta^*_n)| \not\to 0$. This is a contradiction to property *(iii)* of Lemma 1.10, however, which implies that if $\sigma^*(s_-) = \sigma^*(s_+)$, then $\pi_h(\beta^*_n(s_-); \beta^*_n)$ and $\pi_h(\beta^*_n(s_+); \beta^*_n)$ converge to some common limit $\pi^{\mathfrak{c}}_h(\sigma^*(s_-); \sigma^*)$.

Step 2 *If $\sigma^*(s_-) \neq \sigma^*(s_+)$, then $\beta^*_n(s_-) < \beta^*_n(s_+)$ whenever n is sufficiently large.*

Suppose to the contrary that $\beta^*_n(s_-) = \beta^*_n(s_+)$ for infinitely many n, and hence $\beta^*(s_-) = \beta^*(s_+)$. This implies that $\lim_{n\to\infty} \pi_h(\beta^*_n(s_-); \beta^*_n) = \lim_{n\to\infty} \pi_h(\beta^*_n(s_+); \beta^*_n)$. Since $\lim_{n\to\infty} \pi_h(\beta^*_n(s); \beta^*_n) = \pi^{\mathfrak{c}}_h(\sigma^*(s); \sigma^*)$ for all s, this means that $\pi^{\mathfrak{c}}_h(\sigma^*(s_-); \sigma^*) = \pi^{\mathfrak{c}}_h(\sigma^*(s_+); \sigma^*)$.

[37] Observe that $\beta_n(\hat{s}) \neq b_n$ for infinitely many n. Otherwise, $\lim_{n\to\infty} \pi_h(b_n; \beta_n) = \lim_{n\to\infty} \pi_h(\beta_n(\hat{s}); \beta_n) = \pi_h(\hat{s})$ and, by construction, $(m', b') = \sigma^*(\hat{s})$.

Denote the report which strategy σ^* assigns to signal s by $\mu(s)$. Because $\beta^*(s_-) = \beta^*(s_+)$, Lemma 1.7 implies that $\mu(s_-) \leq \mu(s_+)$. We now show that it cannot be that $\mu(s_-) < \mu(s_+)$ because then $\pi_h^{\mathfrak{c}}(\sigma^*(s_-); \sigma^*) < \pi_h^{\mathfrak{c}}(\sigma^*(s_+); \sigma^*)$. If $\{s \in [s_-, s_+] \colon \mu(s) \in (\mu(s_-), \mu(s_+))\}$ has positive mass, then $\pi_h^{\mathfrak{c}}(\sigma^*(s_-); \sigma^*) < \pi_h^{\mathfrak{c}}(\sigma^*(s_+); \sigma^*)$ follows immediately. Otherwise, either $\{s \in [s_-, s_+] \colon \mu(s) = \mu(s_-)\}$ has positive mass, in which case $\pi_h^{\mathfrak{c}}(\mu(s_+), \beta^*(s_-); \sigma^*) > \pi_h^{\mathfrak{c}}(\mu(s_-), \beta^*(s_-); \sigma^*)$ because $\mu(s_+)$ wins the uniform tie-break on $(\mu(s_-), \beta^*(s_-))$ with certainty; and/or $\{s \in [s_-, s_+] \colon \mu(s) = \mu(s_-)\}$ has positive mass, in which case $\pi_h^{\mathfrak{c}}(\mu(s_-), \beta^*(s_-); \sigma^*) < \pi_h^{\mathfrak{c}}(\mu(s_+), \beta^*(s_-); \sigma^*)$ because $\mu(s_-)$ never wins when an action-tuple $(\mu(s_+), \beta^*(s_-))$ is played.

Thus, $\mu(s_-) = \mu(s_+)$, such that $\sigma^*(s_-) = \sigma^*(s_+)$, which is a contradiction.

1.B Numerical examples

1.B.1 No strictly increasing when $\eta > 1$

Lemma 1.12 *Suppose that $v_\ell = 0$, $v_h = 1$ and both states are equally likely. For any $\eta > 1$, there are signal distributions such that no strictly increasing equilibrium exists.*

Proof. Without loss, let the signal space be $[0,1]$. In a strictly increasing equilibrium, the lowest bid equals the reserve price $v_\ell = 0$. Otherwise, the lowest signal, $s = 0$, can lower her bid and win in the same situations (when she is alone) paying less. Suppose that $\frac{f_h(s)}{f_\ell(s)}$ is constant on $s \in [0, \frac{1}{2}]$, meaning that bidders with these signals are essentially identical and have to be indifferent about each other's bids, i.e.

$$\begin{aligned}
&U(0|0;\beta) = U(\beta(s)|s;\beta) \quad \forall s \in [0, \frac{1}{2}] \\
\iff &\frac{\rho f_h(0)\pi_h(0;\beta)}{\rho f_h(0) + (1-\rho) f_\ell(0)} \\
&= \frac{\rho f_h(s)\pi_h(\beta(s);\beta)(1-\beta(s)) + (1-\rho) f_\ell(s)\pi_\ell(\beta(s);\beta)(-\beta(s))}{\rho f_h(s) + (1-\rho) f_\ell(s)}.
\end{aligned}$$

Note that $f_\omega(s) = f_\omega(0)$ for all $s \in [0, \frac{1}{2}]$, $\omega \in \{h, \ell\}$ and $\rho = \frac{1}{2}$, such that we can rearrange the fraction to

$$\begin{aligned}
\iff \beta(s) &= \frac{f_h(s)}{f_\ell(s)} \frac{\pi_h(\beta(s);\beta) - \pi_h(\beta(0);\beta)}{\pi_\ell(\beta(s);\beta) + \frac{f_h(s)}{f_\ell(s)}\pi_h(\beta(s);\beta)} \\
&= \frac{f_h(s)}{f_\ell(s)} \frac{1 - e^{-\eta F_h(s)}}{e^{\eta(F_\ell(s) - F_h(s))} + \frac{f_h(s)}{f_\ell(s)}}.
\end{aligned}$$

The derivative of β is positive on $s \in [0, \frac{1}{2}]$ if

$$\frac{f_h(s)}{f_\ell(s)} \geq e^{\eta F_\ell(s)}(\frac{f_\ell(s)}{f_h(s)} - 1 - \frac{f_\ell(s)}{f_h(s)} e^{-\eta F_h(s)}). \tag{1.44}$$

Suppose now that $f_h(s), f_\ell(s)$ are constant for $s \leq \frac{1}{2}$, and consider the point $s = \frac{1}{2}$. At this point, the inequality (1.44) becomes

$$\frac{f_h(0.5)}{f_\ell(0.5)} \geq e^{\eta 0.5 f_\ell(0.5)}(f_\ell(0.5)\frac{1 - e^{-\eta 0.5 f_h(0.5)}}{f_h(0.5)} - 1). \tag{1.45}$$

If, for $s \leq \frac{1}{2}$, density $f_\ell(s) \in (\frac{1}{\eta 0.5}, 2)$ and $f_h(s)$ becomes arbitrary small, the left side of (1.45) converges to zero, while

$$\lim_{f_h(0.5) \to 0} \frac{1 - e^{-\eta 0.5 f_h(0.5)}}{f_h(0.5)} = 0.5\eta,$$

such that right sides of (1.45) remains bounded above 0. Thus, inequality (1.45) is violated if $f_h(0.5)$ is sufficiently small. The densities above $\frac{1}{2}$ can be chosen freely as long $F_\omega(1) = 1$ and the MLRP holds. □

1.B.2 Equilibria with atoms—binary signals

In this subsection, we construct an example of equilibrium multiplicity in a binary signal structure. While this violates our standing assumption that signal densities are continuous, the example is more transparent. We give an example with continuous densities in the next subsection.

Suppose that $v_\ell = 0$, $v_h = 1$ and both states are equally likely. Let the signal space be $[0, 1]$ and suppose that

$$f_h(s) = \begin{cases} \frac{1}{2} & s \in [0, \frac{1}{2}) \\ \frac{3}{2} & s \in [\frac{1}{2}, 1] \end{cases} \qquad f_\ell(s) = \begin{cases} \frac{3}{2} & s \in [0, \frac{1}{2}) \\ \frac{1}{2} & s \in [\frac{1}{2}, 1], \end{cases}$$

such that the likelihood ratio is constant and equal to $\frac{1}{3}$ on $s \leq \frac{1}{2}$ and $\frac{3}{1}$ on $s > \frac{1}{2}$.

By inspection of inequality (1.44), for $\eta = 3$, no strictly increasing equilibrium exists. However, there is an equilibrium in which all signals $s < \frac{1}{2}$ pool, while all higher signals follow a strictly increasing strategy. In particular, set $\beta^*(s) = 0.036$ for all $s < \frac{1}{2}$ and suppose that for $s \geq \frac{1}{2}$, strategy β^* follows the ODE (1.28) with initial value $\beta^*(0.5) = 0.036$. To ensure that this is an equilibrium, low signal bidders $s < \frac{1}{2}$ must have no incentive to deviate to 0 or a bid marginally above 0.0036. Further, $\beta^*(s)$ has to be strictly increasing above 0.5 and that high signals $s \geq \frac{1}{2}$ have to prefer the high bids over the pooling bid 0.036.

By simple computation, for $s < \frac{1}{2}$

$$U(0|s;\beta^*) = \frac{\frac{1}{2}e^{-3}v_h + \frac{3}{2}e^{-3}v_\ell}{\frac{1}{2}+\frac{3}{2}} = \frac{e^{-3}}{4} < 0.0125,$$

whereas

$$U(0.036|s;\beta^*) = \frac{\frac{1}{2}\frac{e^{-3(1-0.5\cdot 0.5)}-e^{-3}}{3\cdot 0.5\cdot 0.5}[0.964] + \frac{3}{2}\frac{e^{-3(1-0.5\cdot 1.5)}-e^{-3}}{3\cdot 0.5\cdot 1.5}[-0.036]}{\frac{1}{2}+\frac{3}{2}} > 0.0127,$$

such that a deviation to any bid $b \in [0, 0.036)$ is not profitable for low signal bidders. At the same time, the utility from bidding marginally above 0.036 is

$$\lim_{\epsilon \searrow 0} U(0.036+\epsilon|s;\beta^*) = \frac{\frac{1}{2}e^{-3(1-0.5\cdot 0.5)}[0.964] + \frac{3}{2}e^{-3(1-0.5\cdot 1.5)}[-0.036]}{\frac{1}{2}+\frac{3}{2}} < 0.0127,$$

such that this is no profitable deviation, either.

Since $f_h(s) > f_\ell(s)$ for all $s \geq \frac{1}{2}$, the ODE (1.28) is strictly increasing above $\frac{1}{2}$ if $\beta^*(0.5) = 0.036 \leq \mathbb{E}[v|s_{(1)} = 0.5, 0.5]$. This is fulfilled because

$$\mathbb{E}[v|s_{(1)} = 0.5, 0.5] = \frac{(\frac{3}{2})^2 e^{-3(1-0.5\cdot 0.5)}}{(\frac{3}{2})^2 e^{-3(1-0.5\cdot 0.5)} + (\frac{1}{2})^2 e^{-3(1-0.5\cdot 1.5)}} > 0.6.$$

Last, for $s \geq \frac{1}{2}$ it holds that

$$U(0.036|s;\beta^*) = \frac{\frac{3}{2}\frac{e^{-3(1-0.5\cdot 0.5)}-e^{-3}}{3\cdot 0.5\cdot 0.5}[0.964] + \frac{1}{2}\frac{e^{-3(1-0.5\cdot 1.5)}-e^{-3}}{3\cdot 0.5\cdot 1.5}[-0.036]}{\frac{3}{2}+\frac{1}{2}} < 0.06,$$

and

$$\lim_{\epsilon \searrow 0} U(0.036+\epsilon|s;\beta^*) = \frac{\frac{3}{2}\frac{e^{-3(1-0.5\cdot 0.5)}-e^{-3}}{3\cdot 0.5\cdot 0.5}[0.964] + \frac{1}{2}\frac{e^{-3(1-0.5\cdot 1.5)}-e^{-3}}{3\cdot 0.5\cdot 1.5}[-0.036]}{\frac{3}{2}+\frac{1}{2}} > 0.07,$$

such that all high signals prefer to follow the ODE (1.28).

Hence, β^* is an equilibrium. Further, all inequalities are strict, and utilities are continuous in the payment, such that there is a continuum of equilibria with different pooling bids around 0.036.

1.B.3 Equilibria with atoms—continuous signals

In this subsection we construct another example of equilibrium multiplicity. Suppose that $v_\ell = 0$, $v_h = 1$ and both states are equally likely. Let the signal space be $[0, 1]$

and suppose that for $s \in [0.36, 0.37]$

$$f_h(s) = \frac{2s}{100} \qquad\qquad f_\ell(s) = \frac{3-4s}{100}$$
$$F_h(s) = \frac{199+16s^2}{1600} \qquad\qquad F_\ell(s) = \frac{395+24s-16s^2}{800},$$

as well as $\frac{f_h(0)}{f_\ell(0)} = \frac{1}{4}$. The details of the rest of the distribution is arbitrary, as long as the MLRP is fulfilled. Let $\eta = 7$.

We want to show that there is a continuum of equilibria with an atom at the bottom and is strictly increasing above. An equilibrium of this form, β^*, is characterized by a cutoff $\hat{s}$ and a bid b_p, such that $\beta^*(0) = \beta^*(\hat{s}) = b_p$ whereas $\beta^*(s)$ follows ODE (1.28) for $s > \hat{s}$ with initial value $\beta^*(\hat{s}) = b_p$. We restrict attention to equilibria where $\hat{s} \in [0.361, 0.365]$. A combination $(\hat{s}, b_p)$ describes an equilibrium if bidders with signal $\hat{s}$ are indifferent between bidding b_p and marginally higher bid. Further, bidders with the lowest signal $s = 0$ can have no incentive to deviate to 0, and $\beta^*(s)$ has to be strictly increasing above $\hat{s}$.

For $\hat{s} \in [0.361, 0.365]$, a bidder with signal $\hat{s}$ is indifferent between b_p and a marginally higher bid if $U(b_p|\hat{s};\beta^*) = \lim_{\epsilon \searrow 0} U(b_p + \epsilon|\hat{s};\beta^*)$, which rearranges to

$$b_p = \frac{f_h(\hat{s})(e^{-7(1-F_h(\hat{s}))} - \frac{e^{-7(1-F_h(\hat{s}))}-e^{-7}}{7F_h(\hat{s})})}{f_h(\hat{s})(e^{-7(1-F_h(\hat{s}))} - \frac{e^{-7(1-F_h(\hat{s}))}-e^{-7}}{7F_h(\hat{s})}) + f_\ell(\hat{s})(e^{-7(1-F_\ell(\hat{s}))} - \frac{e^{-7(1-F_\ell(\hat{s}))}-e^{-7}}{7F_\ell(\hat{s})})}.$$

Plugging in $F_h(\hat{s}), F_\ell(\hat{s})$, the indifference gives rise to an increasing function $b_p(\hat{s})$ with $b_p(0.361) = 0.0151769$, and $b_p(0.365) = 0.0154979$.

Bidders with signal $s = 0$ prefer $b_p(\hat{s})$ over 0 if $U(0|0;\beta^*) \le U(b_p(\hat{s})|0;\beta^*)$. This rearranges to

$$b_p(\hat{s}) \le \frac{\frac{1}{4}(\frac{e^{-7(1-F_h(\hat{s}))}-e^{-7}}{7F_h(\hat{s})} - e^{-7})}{\frac{1}{4}\frac{e^{-7(1-F_h(\hat{s}))}-e^{-7}}{7F_h(\hat{s})} + \frac{e^{-7(1-F_\ell(\hat{s}))}-e^{-7}}{7F_\ell(\hat{s})}},$$

where the right side is larger than 0.0155 for $\hat{s} \in (0.361, 0.365)$. Thus, the lowest signal, $s = 0$, never wants to deviate to 0.

Next, one can check that $\mathbb{E}[v|s_{(1)} = s, s]$ is increasing on $s \in [\hat{s}, 1]$ because inequality (1.10) holds. Hence, the ODE (1.28) is strictly increasing on $[\hat{s}, 1]$ if $b_p(\hat{s}) \le \mathbb{E}[v|s_{(1)} = \hat{s}, \hat{s}]$. This is the case if

$$b_p(\hat{s}) \le \frac{f_h(\hat{s})^2 e^{-7(1-F_h(\hat{s}))}}{f_h(\hat{s})^2 e^{-7(1-F_h(\hat{s}))} + f_\ell(\hat{s})^2 e^{-7(1-F_\ell(\hat{s}))}}.$$

The right side is increasing in $\hat{s}$, and equal to 0.0152206 for $\hat{s} = 0.361$ and equal to 0.0158707 for $\hat{s} = 0.365$. Indeed, one can check that the upper bound is never violated for $\hat{s} \in (0.361, 0.365)$.

Hence, we constructed a continuum of equilibria: for any $\hat{s} \in (0.361, 0.365)$, there is an equilibrium in which all signals $s \leq \hat{s}$ pool on $b_p(\hat{s})$, and all higher signals follow a strictly increasing strategy.

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Chapter 2

Auctions with Multidimensional Signals

Joint with Stephan Lauermann

2.1 Introduction

In almost all applications of auctions, bidders' valuations for the good are interdependent, combining common and private-value components. Moreover, bidders have information on both their own value and the common value, thereby implying that their private information is multidimensional. Without the natural ordering of types of the one-dimensional setting, auctions with multidimensional signals are much more complicated to analyze, because solution methods building on a one-to-one relationship between signals and bids are not applicable. When the signals are two-dimensional, a high bid can either be a sign of a bidder with positive information regarding the common value component or an indicator of a large private value. Hence, high bids are not necessarily good news regarding the common value of the good: in auctions with multidimensional signals, bids and the common value of the good are generally not affiliated. In the absence of affiliation, equilibrium strategies do not need to be monotone[1] and the equilibrium bid distribution can contain atoms, that is, bids that bidders with different types pool on.[2]

While it does complicate the analysis, the lack of affiliation makes auctions with multidimensional signals an interesting object of study, because atoms and non-monotone strategies are an important source of allocational and informational inefficiencies: bidders with the highest valuation may not receive the object and the distribution of bids is less informative about the common value of the good.[3] Further, atoms can significantly complicate the equilibrium analysis. If the equilibria of an auction on an arbitrary fine grid contain nonvanishing atoms, these equilibria

[1]Reny and Perry (2006) provide an example of a non-monotone best response in a double auction.

[2]Consider, for example, Chapter 1.

[3]Among others, compare Atakan and Ekmekci (2014) and Lauermann and Wolinsky (2017).

may have no counterpart on the continuous bid space. Therefore, atoms can prevent equilibrium existence.[4]

In this chapter, we analyze a second-price auction in which the value of the good depends on both common and a private-value components, and bidders are either informed or uninformed about the common component. Jackson (2009) shows that in this setup, a discrete private-value distribution may thwart equilibrium existence: in a second-price auction, it is dominant for informed bidders to bid their valuation. If the private-value distribution is discrete, this valuation takes finitely many values, such that any equilibrium bid distribution must contain atoms. At these atoms, the bidders' payoffs are discontinuous, which can prevent equilibrium existence. This non-existence result has been taken to indicate a general existence problem in multidimensional auctions—for example in Pesendorfer and Swinkels (2000, p. 501), Tsetlin and Pekeč (2006, p. 64), Tan and Xing (2011, p. 99), and Heumann (2019, p. 4).

We show that despite the lack of affiliation, no such problem arises when the private-value distribution is continuous. In this case, the bidding behavior by informed bidders does not create atoms in the bid distribution, and uninformed bidders have an incentive to bid away from any bid that ties with positive probability because winning a random tie-break would intensify the winner's curse. Establishing this curse from the random tie-break is the crucial step in our proof, and we provide a more detailed intuition for this in the body of the text. Consequently, when the private-value distribution is continuous, there can be no atoms in the bid distribution, such that the second-price auction has an equilibrium, and any equilibrium strategy is pure and strictly increasing in both dimensions. Further, we prove existence in the first-price auction, independent of the private-value distribution.[5]

The results are remarkable, in that an equilibrium exists and all equilibria are "well-behaved," even though the equilibrium bids and the common-value component are not affiliated and basic single-crossing properties fail. Consequently, bidders in a second-price auction may incur an expected loss when winning at a low price and enjoy a profit only when winning at a higher price.[6] This goes to show that while the lack of affiliation can give rise to non-monotone strategies or atoms in the bid distribution, it does not necessarily have to. Hence, the non-existence that Jackson (2009) uncovers appears to be an artifact of the specific assumptions on the auction format and private-value distribution, rather than an indicator of a

[4] In particular, existence proofs that rely on a discretization approach like the one by Athey (2001) are not applicable. Therefore, an equilibrium may not exist, as in Chapter 1.

[5] In a first-price auction, informed bidders have a strict incentive to outbid any atom because marginally raising their bid discretely increases their probability to win at essentially the same cost.

[6] We demonstrate this by an example in Section 2.7. As one consequence, the corresponding social choice function is not posterior-implementable (Jehiel et al., 2007).

fundamental problem with the existence of well-behaved equilibria in auctions with multidimensional signals. Further, the lack of affiliation and non-standard solution method sets our results apart from other existence proofs in the literature on auctions with multidimensional signals, such as Jehiel et al. (2007) and Heumann (2019), which usually recover some sort of affiliation.

As a technical contribution, we demonstrate how the endogenous tie-breaking rule of Jackson et al. (2002) can be made amenable and used to prove existence and form of equilibria in the auction. In particular, we restrict attention to a simple class of endogenous tie-breaking rules and use insights from the equilibrium strategies on a bid grid to show that Jackson et al. (2002) guarantee the existence of an equilibrium. In a second step, we analyze our simplified "communication extension" and prove that equilibrium strategies must be strictly increasing, such that the tie-breaking rule is irrelevant and the equilibrium strategy of the communication extension is also an equilibrium of the standard auction. This solution method lends itself to related problems.

The remainder of the chapter is structured in the following manner. In Section 2.2, we set up the model, and in Section 2.3, we revisit the non-existence result by Jackson (2009). In Section 2.4, we prove equilibrium existence when the private-value distribution is continuous. In Section 2.5, we turn to the first-price auction and sketch the existence proof. In Section 2.6, we discuss the effect of an uncertain number of competitors, and in Section 2.7, we present the conclusion.

2.2 Model

A single, indivisible good is sold in a second-price sealed-bid auction with $n \geq 2$ risk-neutral bidders. The value of the good depends on both common and a private-value components: the value is $u(v_\omega, \theta)$, where v_ω is the unknown, state-dependent common-value component, θ is the bidder's private-value component, and $u(v_\omega, \theta) \geq 0$ is the value function. Let u be bounded and strictly increasing in both arguments.

The private value $\theta \in \Theta \subseteq \mathbb{R}$ is independently and identically distributed across bidders, according to some distribution with a cumulative distribution function F. The common value v_ω is either high, v_h, or low, v_ℓ, with $v_h > v_\ell$, depending on an unknown state of the world $\omega \in \{h, \ell\}$. The state is high, h, with probability $\rho > 0$ and low, ℓ, with probability $1 - \rho > 0$.

Every bidder knows her private value and receives a conditionally independent and identically distributed signal $s \in \{h, \ell, \emptyset\}$ about the state. With probability $q \in (0, 1)$ the signal s is uninformative, $s = \emptyset$, with probability $1 - q$ it reveals the state, $s = \omega$.

Having observed her private value and signal, every bidder simultaneously submits a bid $b \geq 0$. The bidder with the highest bid wins the auction, receives the object, and pays the second-highest bid. Ties are broken uniformly.

We analyze symmetric strategies represented as three functions $\{\beta_h, \beta_\ell, \beta_\emptyset\}$ that describe the bidding behavior conditional on observing signal $s \in \{h, \ell, \emptyset\}$. Every function $\beta_s(\theta) : \Theta \to \Delta\mathbb{R}_+$ maps the private values into a distribution over bids. To simplify the notation, we collect the functions in $\beta = \{\beta_h, \beta_\ell, \beta_\emptyset\}$.

Given a signal $s \in \{h, \ell, \emptyset\}$ and the strategy β, we denote by $G(p|s;\beta)$ the (expected) cumulative distribution function of the second highest bid—that is the price, p. If a bidder with signal s and private value θ bids b, this implies that her interim expected utility is

$$\begin{aligned} U(b|s,\theta;\beta) = &\int_{[0,b)} \mathbb{E}[u(v_\omega,\theta)|p = z, s;\beta] - z \, dG(z|s;\beta) \\ &+ \mathbb{P}[p = b, \text{win tb}|s;\beta]\Big(\mathbb{E}[u(v_\omega,\theta)|p = b, \text{win tb}, s;\beta] - b\Big), \end{aligned} \tag{2.1}$$

where "$p = b, \text{win tb}$" denotes the event that the bidder ties on b and wins the random tie-break.

We study symmetric Bayes-Nash equilibria: a strategy profile β is an equilibrium if $b \in \text{supp } \beta_s(\theta)$ implies that $b \in \arg\max_{\hat{b}} U(\hat{b}|s,\theta;\beta)$ for $s \in \{h, \ell, \emptyset\}$ and almost all θ.

2.3 Discrete private-value distribution

Before proving the existence of an equilibrium when the private-value distribution is continuous, it is helpful to revisit the non-existence result by Jackson (2009) when the distribution of the private-value component is discrete.

Definition 2.1 *In the discrete model, denoted Γ_d^{SPA}, the distribution of private values is discrete.*

Jackson (2009) shows that when there are $n = 2$ bidders, the support of discrete private-value distribution is sufficiently dense and the probability of receiving an informative signal, q, is low, no symmetric equilibrium may exist.

Proposition 2.1 (Jackson (2009)) *When there are $n = 2$ bidders, the discrete model, Γ_d^{SPA}, may not have an equilibrium.*

Proof. Suppose that the space of private values is $\Theta = \{0, \epsilon, 2\epsilon, ..., \bar{\theta}\}$ and that every realization has equal probability. Let $u(v_\omega, \theta) = v_\omega + \theta$ and $v_\ell \geq 0$. We show that there is an $\bar{\epsilon} > 0$ and a function $\bar{q}(\epsilon) > 0$, such that for any $\epsilon < \bar{\epsilon}$ and any $q < \bar{q}(\epsilon)$, no equilibrium exists.

In a second-price auction, every symmetric equilibrium is an equilibrium in undominated strategies (cf. Jackson (2009)). Thus, equilibrium bids in a symmetric equilibrium must lie between the lowest and the highest feasible valuation (given signal s). Bidders who receive a perfectly informative signal $s \in \{h, \ell\}$ face no uncertainty regarding the value of the good, such that they bid their valuation, $v_\omega + \theta$. Uninformed bidders know that the common value is v_ℓ at worst and v_h at best, such that their bids can be restricted to the interval $[v_\ell + \theta, v_h + \theta]$.

For the remainder of the proof, we focus on uninformed bidders with private value $\theta = 0$. First, note that $(\emptyset, 0)$ bidders select bids $b < v_h$ with probability 1. If they were to bid v_h with positive probability, they would tie with and win against another $(\emptyset, 0)$ bidder with positive probability, decreasing the expected value conditional on tying on $p = v_h$ from v_h toward the prior expected value. Thus, tying on v_h would result in an expected loss, such that any $(\emptyset, 0)$ bidder would be strictly better off marginally reducing her bid.

When q is small, a $(\emptyset, 0)$ bidder believes that her competitor is informed with probability close to 1. Because informed bidders only bid below v_h in state ℓ, the expected value conditional on winning with a bid $b < v_h$ is close to v_ℓ. This implies that when q is sufficiently small (given ϵ), bids by $(\emptyset, 0)$ bidders are bounded below $v_\ell + \epsilon$. Otherwise, they would occasionally win and pay a price above $v_\ell + \epsilon$ while receiving a good with an expected value close to v_ℓ. In fact, $(\emptyset, 0)$ bidders will only select bids strictly below $v_\ell + \epsilon$ because a small q implies that the bid $v_\ell + \epsilon$ is almost certainly made by a (ℓ, ϵ) competitor. If a $(\emptyset, 0)$ bidder would tie on $p = v_\ell + \epsilon$, her expected valuation for the good would only be $\approx v_\ell$ and she would incur a strict loss. Hence, she would be strictly better off marginally reducing her bid.

In the last step, we argue that this confinement of bids to $[v_\ell, v_\ell + \epsilon)$ creates a contradiction. If a $(\emptyset, 0)$ bidder bids in this interval, she can only win against two types of competitors: a $(\ell, 0)$ bidder or another $(\emptyset, 0)$ bidder. All other types select higher bids. When the competitor is of type $(\ell, 0)$, the price is $p = v_\ell$. This case is insignificant for bidder $(\emptyset, 0)$'s incentives because her payoff is exactly 0 in this situation. Therefore, we only need to consider the case in which her competitor is also of type $(\emptyset, 0)$. When both bidders in the auction are uninformed, there can be no inference regarding the common-value component of the good, such that the expected value is $\rho v_h + (1-\rho) v_\ell$. When ϵ is sufficiently small, $v_\ell + \epsilon < \rho v_h + (1-\rho) v_\ell$, such that the $(\emptyset, 0)$ bidder strictly prefers to win against another $(\emptyset, 0)$ type at any price $p < v_\ell + \epsilon$. Now, a Bertrand competition among the $(\emptyset, 0)$ bidders emerges. They have an incentive to outbid any bid $b < v_\ell + \epsilon$ and would settle only on the highest bid from this set—which, however, does not exist. $\square$

The critical step in the argument is the last one. Conditional on the pivotal event that the competitor is also of the $(\emptyset, 0)$ type, the two $(\emptyset, 0)$ bidders are engaged in a Bertrand competition. Due to the lack of affiliation between the bids and the common value of the good, the conditional expected value drops discretely at $v_\ell + \epsilon$. Therefore, bidders compete on an open set, such that no equilibrium exists. The source of the discrete downward jump at $v_\ell + \epsilon$ is the atom in the bid distribution, which follows directly from the discrete value distribution and the auction format: the perfectly informed bidders are basically mechanical and bid their valuation. Since they are of type (ℓ, ϵ) with strictly positive probability, there has to be an atom at $v_\ell + \epsilon$.[7]

However, we must indicate that this argument only works for $n = 2$ bidders. When there are $n \geq 3$ bidders, and the winning bid is below $v_\ell + \epsilon$, multiple competitors may either be of the $(\ell, 0)$ or of the $(\emptyset, 0)$ type, such that the contradiction does not arise. Unfortunately, the general proof of Proposition 1 in Jackson (2009) misses a step, which we were unable to patch (cf. Appendix 2.B).

2.4 Continuous private-value distribution

Having established why the existence of an equilibrium in a model with a discrete private-value distribution may fail, we now turn to the model with a continuous private-value distribution.

Definition 2.2 *In the continuous model, the distribution of private values is is continuous. For simplicity, assume that the distribution has full support on $\Theta = [\underline{\theta}, \bar{\theta}]$. We denote this model by Γ_c^{SPA}.*

When the private-value distribution is continuous, it is obvious that informed bidders bidding their valuation no longer cause atoms in the bid distribution. By utilizing an endogenous tie-breaking rule as in Jackson et al. (2002), we show that, in fact, the bid distribution is continuous, such that an equilibrium exists.

Proposition 2.2 *In the continuous model, Γ_c^{SPA}, an equilibrium β^* exists, and every equilibrium is pure and strictly increasing in both dimensions.*[8]

In the next section, we prove the existence and form of equilibrium strategies in multiple steps. The idea and main challenge of the proof is to establish that uninformed bidders have a strict incentive to bid away from any bid that ties with positive probability.

[7] In an auction with an endogenous tie-breaking rule (Jackson et al., 2002), this problem can be solved such that an equilibrium exists. In equilibrium, types $(\emptyset, 0)$ and (ℓ, ϵ) bid $v_\ell + \epsilon$, but $(\emptyset, 0)$ bidders only win when there are no (ℓ, ϵ) bidders. In our specific version of the communication extension (Definition 2.3), this would correspond to a rule τ^* in which $\tau^*(\ell, \epsilon) > \tau^*(\emptyset, 0)$.

[8] In other words, $\beta_s^*(\theta)$ is strictly increasing in θ for all $s \in \{h, \emptyset, \ell\}$ and $\beta_\ell^*(\theta) < \beta_\emptyset^*(\theta) < \beta_h^*(\theta)$ for all $\theta \in \Theta$.

2.4.1 Proof of Proposition 2.2

We begin the proof by introducing a simple endogenous tie-breaking rule to the auction, one that is easy to work with. In a second step, we then argue that the results by Jackson et al. (2002) guarantee the existence of an equilibrium in this particular communication extension. Thereafter, we show that the equilibrium bid distribution of our communication extension cannot contain atoms, such that the tie-breaking rule is irrelevant, and the equilibrium strategy of the communication extensions is also an equilibrium strategy of the standard auction, Γ_c^{SPA}. Last, we show that any equilibrium of Γ_c^{SPA} must be pure and strictly increasing in both dimensions.

Definition 2.3 *In the communication extension of the continuous model, Γ_c^{CE-SPA}, bidders report their signal, $\hat{s} \in \{h, \ell, \emptyset\}$, as well as their private value, $\hat{\theta} \in \Theta$, alongside their bid, b. The tie-breaking rule is characterized by a function $\tau : S \times \Theta \to [0,1]$. If multiple bidders tie on the same bid, the rule awards the good to the bidder with the highest $\tau(\hat{s}, \hat{\theta})$. If multiple bidders tie on the same bid and report the same $\tau(\hat{s}, \hat{\theta})$, the winner is selected randomly among them.*

A pair (β^*, τ^*) constitutes an equilibrium of Γ_c^{CE-SPA}, if, given τ^*, bidding $\beta_s^*(\theta)$ and reporting truthfully are a best response for almost every type (s, θ). As we show, Γ_c^{CE-SPA} has an equilibrium with a simple structure.

Proposition 2.3 *There is an equilibrium (β^*, τ^*) of the communication extension, Γ_c^{CE-SPA}, in which*

(i) $\beta_\omega^*(\theta) = u(v_\omega, \theta)$ *for* $\omega \in \{h, \ell\}$;

(ii) $\beta_\emptyset^*$ *is pure and nondecreasing;*

(iii) $\tau^*(\emptyset, \hat{\theta})$ *is nondecreasing in* $\hat{\theta}$.

To show that our simple communication extension, Γ_c^{CE-SPA}, has an equilibrium of this particular form, we make use of Theorem 2 of Jackson et al. (2002), which guarantees that the limit of a sequence of equilibria on the ever-finer grid (where an equilibrium exists) converges to an equilibrium of a game with some endogenous tie-breaking rule. Therefore, we can use insights regarding the bidding behavior in the standard auction on the grid to ensure that the simple endogenous tie-breaking rule of Γ_c^{CE-SPA} suffices. First, in the symmetric equilibrium of the standard auction, informed bidders bid their valuation (as the grid permits), so that they also have to do so in Γ_c^{CE-SPA} *(i)*. Further, we show that on the grid, the strategy of the uninformed bidders has to be pure and nondecreasing. Hence, their strategy also needs to be pure and nondecreasing in Γ_c^{CE-SPA} *(ii)* and higher private values need to win more often, such that the tie-breaking rule can be represented by a nondecreasing function *(iii)*. The formal proof can be found in the appendix.

Observe that because β_h^* and β_ℓ^* are strictly increasing in θ, informed bidders almost never tie and the tie-breaking rule is irrelevant to them. In the next step, we want to show that $\beta_\emptyset^*$ is also strictly increasing in θ, meaning that the (endogenous) tie-breaking rule plays no role and β^* is an equilibrium of Γ_c^{SPA}, as well.

Suppose, on the contrary, that there exists a bid $b_a \in (u(v_\ell, \underline{\theta}), u(v_h, \bar{\theta}))$ and a nontrivial interval I such that $\beta_\emptyset^*(\theta) = b_a$ for all $\theta \in I$.[9] This is qualitatively illustrated in Figure 2.1.

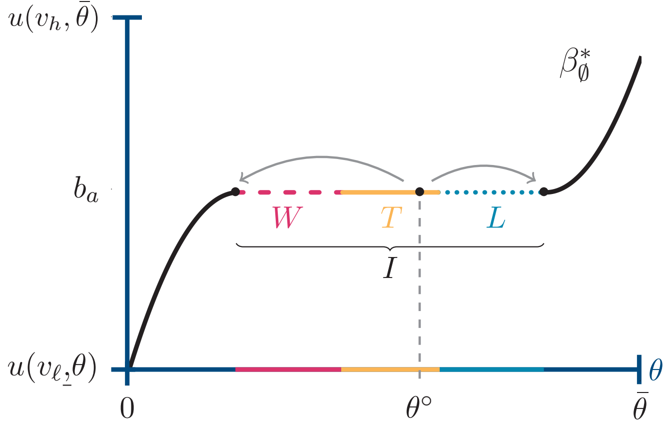

Figure 2.1 Candidate equilibrium strategy $\beta_\emptyset^*$ of Γ_c^{CE-SPA} with an atom at b_a.

Pick any θ° from the interior of I. We want to show that an uninformed bidder with private value θ° either has an incentive to marginally overbid or underbid b_a, such that she either always wins or loses when $p = b_a$. When $p \neq b_a$, her payoffs are unchanged.

Denote the highest private value of an uninformed competitor by $\theta_\emptyset^{(1)}$ and of an informed competitor by $\theta_\omega^{(1)}$. If there is no (un-)informed competitor, we associate the respective first-order statistic with $-\infty$. Since informed bidders follow a strictly increasing strategy, they bid b_a with zero probability. This implies that when $p = b_a$, with probability 1, there is at least one uninformed competitor who also bids b_a, meaning that $\theta_\emptyset^{(1)} \in I$, and all informed competitors select a bid below b_a, which is the case when $\theta_\omega^{(1)} < \beta_\omega^{*-1}(b_a) = \theta_\omega$. Note that because $b_a \in (u(v_\ell, \underline{\theta}), u(v_h, \bar{\theta}))$, it follows from $\beta_\omega^*(\theta) = u(v_\omega, \theta)$ that $F(\theta_h) < F(\theta_\ell)$.

[9] When there is an atom at $b_a = u(v_\ell, \underline{\theta})$, uninformed bidders with private values $\theta > \underline{\theta}$ turn a strict profit when the price is $p = b_a$. Since marginally higher bid wins discretely more often (no random tie-break), it is a profitable deviation. When the atom is at $b_a = u(v_h, \bar{\theta})$, tying on $p = b_a$ and winning the random tie-break is uninformative about the state since only uninformed bidders tie and b_a always wins against informed bidders. Thus, the expected common value conditional on tying on $b_a = u(v_h, \bar{\theta})$ and winning the random tie-break is just the prior, such that the winning uninformed bidder incurs a loss and would be better off marginally lowering her bid.

When multiple uninformed bidders tie on b_a, the tie-breaking rule has to decide on the winner among the uninformed bidders: either $\tau^*(\emptyset, \theta_\emptyset^{(1)}) > \tau^*(\emptyset, \theta^\circ)$, in which case θ° loses the auction, or $\tau^*(\emptyset, \theta_\emptyset^{(1)}) < \tau^*(\emptyset, \theta^\circ)$, in which case θ° wins. If $\tau^*(\emptyset, \theta_\emptyset^{(1)}) = \tau^*(\emptyset, \theta^\circ)$, the uninformed bidder with private value θ° wins in case the random tie-break decides in her favor. Therefore, τ^* partitions the interval of uninformed bidders that tie on b_a, that is I, into three (potentially empty) intervals: those values that θ° wins against, W, those values that θ° ties with, T, and those she loses against, L; these are depicted in Figure 2.1. Taken together, an uninformed bidder with private value θ° who reports her type truthfully derives a payoff of

$$\begin{aligned}&\mathbb{P}[\theta_\emptyset^{(1)} \in W, \theta_\omega^{(1)} \leq \theta_\omega | \emptyset]\Big(\mathbb{E}[u(v_\omega, \theta^\circ) | \theta_\emptyset^{(1)} \in W, \theta_\omega^{(1)} \leq \theta_\omega, \emptyset] - b_a\Big)\\&+ \mathbb{P}[\theta_\emptyset^{(1)} \in T, \text{win tb}, \theta_\omega^{(1)} \leq \theta_\omega | \emptyset]\Big(\mathbb{E}[u(v_\omega, \theta^\circ) | \theta_\emptyset^{(1)} \in T, \text{win tb}, \theta_\omega^{(1)} \leq \theta_\omega, \emptyset] - b_a\Big)\end{aligned}$$

from the pivotal event that $p = b_a$.

To show that θ° has an incentive to either overbid or underbid b_a, we first consider auctions with $n = 2$ bidders, and then turn to $n \geq 3$ bidders. Given that informed bidders never tie, the result is relatively straightforward when there are only $n = 2$ bidders because then, tying on b_a reveals that the competitor is also uninformed. The problem becomes more difficult when we analyze auctions with $n \geq 3$ bidders. Although only uninformed bidders tie on b_a, when there is more than one competitor, tying on b_a and the random tie-break is informative regarding the (relative) number of informed and uninformed bidders in the auction. This affects the inference regarding the common component of the good.

Case $n = 2$ When there are two bidders and $p = b_a$, with probability 1, the competitor is also uninformed. Consequently, the expected value conditional on $p = b_a$ is just the prior. This implies that an uninformed bidder with private value θ° has an incentive to marginally deviate,[10] unless

$$b_a = \rho u(v_h, \theta^\circ) + (1 - \rho) u(v_\ell, \theta^\circ). \tag{2.2}$$

If b_a is smaller, θ° can strictly raise her profits by bidding marginally more, thereby ensuring a win whenever $\theta_\emptyset^{(1)} \in L$ or when $\theta_\emptyset^{(1)} \in T$ and she would have lost the random tie-break. If b_a is larger, θ° wants to bid marginally less, thereby circumventing an expected loss when $\theta_\emptyset^{(1)} \in W$ or when $\theta_\emptyset^{(1)} \in T$ and she would have won the ran-

[10] A marginal up- or downward-deviation is available because there are at most countably many atoms in the bid distribution. Thereby, there is a bid arbitrary close to b_a that only wins whenever $p < b_a$ (underbid) or guarantees a victory whenever $p \leq b_a$ (overbid). Because the deviation bid never ties, the tie-breaking rule and, thus, the report are irrelevant.

dom tie-break.[11] However, if (2.2) holds for θ°, it will not hold for any $\theta \in I \setminus \{\theta^\circ\}$. Hence, no nontrivial interval I can exist and $\beta^*_\emptyset$ has to be strictly increasing θ.

Case $n \geq 3$ For the case with more than two bidders, we derive two preliminary lemmas. Both are surprising because they show that when there are $n \geq 3$ bidders, the order statistics of actions by other uninformed bidders are informative with regard to the value of the good. Here, we provide an intuition for the results and move the formal proofs to the appendix.

Lemma 2.1 *Suppose that $n \geq 3$ and T is nontrivial. Then,*

$$\mathbb{E}[u(v_\omega, \theta)|\theta^{(1)}_\emptyset \in T, win\ tb, \theta^{(1)}_\omega \leq \theta_\omega, \emptyset] < \mathbb{E}[u(v_\omega, \theta)|\theta^{(1)}_\emptyset \in T, \theta^{(1)}_\omega \leq \theta_\omega, \emptyset],$$

for all $\theta \in \Theta$.

First, we find that winning the random tie-break intensifies the winner's curse, relative to merely being tied. The common value shifts the bids by informed bidders to a higher level in the high state. Therefore, winning at any price $b_a \in (u(\underline{\theta}, v_\ell), u(\bar{\theta}, v_h))$ is more likely in the low than in the high state, $F(\theta_h) < F(\theta_\ell)$. This is bad news regarding the value of the good: the winner's curse. Note that the winner's curse is stronger if there are more informed competitors. If $\theta^{(1)}_\emptyset \in T$, which implies that the winner is selected by the random tie-breaking rule, winning the auction is more likely when there are only a few competitors who also bid b_a and report $\tau^*(\emptyset, \hat{\theta})$. Since only uninformed bidders tie on b_a, this implies that winning the random tie-break is more likely when there are fewer uninformed competitors and, thus, more informed competitors. This intensifies the winner's curse, such that winning the random tie-break is bad news regarding the value of the good.

Lemma 2.2 *Let J and J' be two nontrivial intervals with $J < J'$.*[12] *Then,*

$$\mathbb{E}[u(v_\omega, \theta)|\theta^{(1)}_\emptyset \in J, \theta^{(1)}_\omega \leq \theta_\omega, \emptyset] < \mathbb{E}[u(v_\omega, \theta)|\theta^{(1)}_\emptyset \in J', \theta^{(1)}_\omega \leq \theta_\omega, \emptyset],$$

for all $\theta \in \Theta$.

Lemma 2.2 shows that, holding everything else constant, outbidding more uninformed bidders reduces the winner's curse. Higher realizations of the first-order statistic of uninformed types, $\theta^{(1)}_\emptyset$, are more likely when the number of uninformed bidders is large. Hence, the conditional expected number of uninformed bidders is

[11] Note that marginally overbidding or underbidding strictly changes the winning probability of the uninformed bidder with private value θ° because θ° is from the interior of I.

[12] $J < J'$ if $\inf J \leq \inf J'$ and $\sup J \leq \sup J'$, where at least one inequality is strict.

higher conditional on J' than conditional on J. If there are more uninformed competitors, there are fewer informed ones, which eases the winner's curse.

With the two lemmas in place, we can now turn to the main argument. Let "Δunderbid" denote the additional events the uninformed bidder with private value θ° wins in if she bids b_a and reports her type truthfully instead of selecting a marginally lower bid[13]—that is, when $\theta_\emptyset^{(1)} \in W$ or when $\theta_\emptyset^{(1)} \in T$ and she wins the random tie-break. Symmetrically, let "Δoverbid" be the additional events the bidder wins in when selecting a bid marginally above b_a—when $\theta_\emptyset^{(1)} \in T$ and she loses the random tie-break or when $\theta_\emptyset^{(1)} \in L$. Both deviations are depicted in Figure 2.1. Since θ° is from the interior of I, both $\mathbb{P}[\Delta\text{underbid}, \theta_\omega^{(1)} \leq \theta_\omega | \emptyset] > 0$ and $\mathbb{P}[\Delta\text{overbid}, \theta_\omega^{(1)} \leq \theta_\omega | \emptyset] > 0$. As we show in the appendix, Lemmas 2.1 and 2.2 give rise to the following corollary.

Corollary 2.1

$$\mathbb{E}[u(v_\omega, \theta) | \Delta \textit{underbid}, \theta_\omega^{(1)} \leq \theta_\omega, \emptyset] < \mathbb{E}[u(v_\omega, \theta) | \Delta \textit{overbid}, \theta_\omega^{(1)} \leq \theta_\omega, \emptyset],$$

for all $\theta \in \Theta$.

Since $\mathbb{P}[\Delta\text{overbid}, \theta_\omega^{(1)} \leq \theta_\omega | \emptyset] > 0$, an uninformed bidder with private value θ° benefits strictly from marginally overbidding b_a, unless

$$b_a \geq \mathbb{E}[u(v_\omega, \theta^\circ) | \Delta\text{overbid}, \theta_\omega^{(1)} \leq \theta_\omega, \emptyset]. \tag{2.3}$$

However, if equation (2.3) holds, it follows from Corollary 2.1 that

$$b_a > \mathbb{E}[u(v_\omega, \theta^\circ) | \Delta\text{underbid}, \theta_\omega^{(1)} \leq \theta_\omega, \emptyset].$$

Because $\mathbb{P}[\Delta\text{underbid}, \theta_\omega^{(1)} \leq \theta_\omega | \emptyset] > 0$, this implies that the uninformed bidder with private value θ° would strictly benefit from marginally lowering her bid. Thus, either overbidding or underbidding b_a has to be strictly profitable for the uninformed bidder with private value θ° (and every other $\theta^{\circ\prime}$ from the interior of I), creating a contradiction. Thereby, no interval I can exist, which proves that $\beta_\emptyset^*$ is indeed strictly increasing in θ. Consequently, β^* is pure and strictly increasing in θ, such that the tie-breaking rule is irrelevant and β^* is an equilibrium of Γ_c^{SPA}.

[13] Again, such a deviation is available because there are, at most, countably many atoms in the bid distribution. Thus, there has is a bid arbitrarily close to b_a that only wins whenever $p < b_a$ (underbid) or guarantees a victory whenever $p \leq b_a$ (overbid). Because the deviation never ties, the tie-breaking rule and, thus, the report are irrelevant.

We now show that all symmetric equilibria of the continuous model, Γ_c^{SPA}, are pure and strictly increasing in both dimensions.

First, note that any symmetric equilibrium of Γ_c^{SPA} is also an equilibrium of Γ_c^{CE-SPA} in the form specified in Proposition 2.3. By setting τ^* to be constant, *(iii)* is satisfied with the endogenous tie-breaking "disabled." Further, in Γ_c^{SPA}, symmetric equilibria are equilibria in undominated strategies (cf. Jackson (2009)), such that informed bidders bid their valuation *(i)*. Last, the argument that uninformed bidders follow a nondecreasing strategy in the proof to Proposition 2.3 did not rely on the continuous bid space, such that it also holds for Γ_c^{SPA}, *(ii)*. Having shown that any equilibrium of Proposition 2.3 must be pure and strictly increasing in θ, this completes the proof in the first dimension.

What remains to be shown is that any β^* is also strictly increasing with respect to the signal, that is, $\beta_\ell^*(\theta) < \beta_\emptyset^*(\theta) < \beta_h^*(\theta)$ for almost all θ. Because bids for uninformed bidders are confined to $[u(v_\ell,\theta), u(v_h,\theta)]$, the equilibrium strategy β^* is nondecreasing in s. The proof for strict monotonicity is slightly tedious, which is why we move it to a separate lemma in the appendix; Lemma 2.4. Roughly speaking, in equilibrium, uninformed bidders have to bid their expected valuation conditional on being tied. If β^* is constant in s, uninformed bidders select the same bids as informed bidders, such that the inference from being tied cannot be perfectly revealing of the state, resulting in a contradiction.

2.5 First-price auction

In this section, we change the auction format to a first-price auction and analyze both the discrete and the continuous private-value distribution.

Definition 2.4 *Holding everything else fixed, let Γ_d^{FPA} and Γ_c^{FPA} be the analogs of Γ_d^{SPA} and Γ_c^{SPA} when the good is sold in a first-price auction.*

As evident from the last section, when the good is sold in a second-price auction, a continuous private-value distribution implies that there can be no atom in the bid distribution, which ensures that an equilibrium exists. Compared to a discrete private-value distribution, the crucial difference originates from the fact that under the continuous private-value distribution, informed bidders bidding their valuation no longer tie with positive probability.

When the good is sold in a first-price auction, the auction format incentivizes informed bidders to bid away from any bid that ties with positive probability; independent of the private-value distribution. Employing a similar communication extension as earlier, we can show that, thereby, uninformed bidders must follow a strictly increasing strategy as well, such that the first-price auction always has an equilibrium and the equilibrium bid distribution contains no atoms.

Proposition 2.4 *The discrete, Γ_d^{FPA}, and continuous, Γ_c^{FPA}, model of the first-price auction have an equilibrium.*

- *In all equilibria of Γ_d^{FPA}, bidders with signal $s = \ell$ and the lowest private value, $\underline{\theta} = \inf \Theta$, bid $u(v_\ell, \underline{\theta})$. All other types mix continuously, and bidders with higher private values mix over sets of strictly higher bids. Therefore, the only atom in the equilibrium bid distribution is at $u(v_\ell, \underline{\theta})$.*
- *All equilibria of Γ_c^{FPA} are pure and strictly increasing in θ.*

Having established that informed bidders do not tie with positive probability in a first-price auction, the proof follows along the same lines and with the same intuition as the proof for the second-price auction. To save on redundancies and keep the chapter short, we only outline the central steps of the proof. We focus on the discrete case, Γ_d^{FPA}. The continuous case follows analogously.

Proof sketch.

1. The communication extension of the first-price auction, Γ_d^{CE-FPA}, has an equilibrium.
2. In any equilibrium of Γ_d^{CE-FPA}, informed bidders except $(\ell, \underline{\theta})$ types earn strictly positive profits.
   - In state ℓ, $(\ell, \underline{\theta})$ types bid at most $u(v_\ell, \underline{\theta})$. Thus, all other informed bidders can secure a positive payoff by bidding marginally above $u(v_\ell, \underline{\theta})$, winning whenever all competitors are of the $(\ell, \underline{\theta})$ type. In state h, informed bidders earn information rents relative to uninformed bidders. Since the uninformed bidders make at least zero profits, informed bidders in state h make positive profits.
3. In any equilibrium of Γ_d^{CE-FPA}, informed bidders except $(\ell, \underline{\theta})$ types play a fully mixed strategy.
   - Suppose there is an informed bidder, other than the $(\ell, \underline{\theta})$ type, who selects a bid b_a with strictly positive probability. Then, she ties on this bid with strictly positive probability. Consequently, bidding marginally more than b_a discretely raises her probability to win, and since she makes strictly positive profits when winning, this deviation is strictly profitable.
4. This implies that if there is an atom above $u(v_\ell, \underline{\theta},)$, it has to be the result of uninformed bidders pooling.

5. Using the same argument as earlier, one can show that pooling on the atom intensifies the winner's curse for uninformed bidders. Thus, uninformed bidders have a strict incentive to bid away from it and follow a "strictly increasing" strategy (bidders with higher private values mix over sets of strictly higher bids).

6. Except for the atom at $u(v_\ell, \underline{\theta})$, which only $(\ell, \underline{\theta})$ types tie on, the equilibrium bid distribution is atom free. Hence, the tie-breaking rule is irrelevant and the equilibrium strategy of Γ_d^{CE-FPA} is also an equilibrium of Γ_d^{FPA}.

□

2.6 Uncertain number of competitors

In case the number of bidders is Poisson-distributed, the environmental equivalence property of the Poisson distribution (Myerson, 1998) implies that knowledge of the number of uninformed bidders does not allow for an inference regarding the number of informed bidders. Ergo, tying and winning the random tie-break does not affect the conditional expected valuation, such that the inequalities of Lemmas 2.1 and 2.2 become equalities. In this case, the proof follows similarly as for $n = 2$ bidders and Proposition 2.2 still holds.

2.7 Conclusion

In this chapter, we studied auctions in which the value of the good depends on both private and common-value components and bidders are either informed or uninformed regarding the common component. We showed that when the good is sold in a second-price auction and the private-value distribution has no mass points, or when the good is sold in a first-price auction, any equilibrium is strictly increasing in the private value. The monotonicity and the resulting absence of atoms in the equilibrium bid distribution have multiple effects: apart from implications for the allocational and informational efficiency of the auction, it means that the bidders' utility is continuous in the bid and the tie-breaking rule is irrelevant. This is important for empirical estimation methods that usually rely on continuity and, if the payoff function is differentiable, first-order conditions to identify the bidders' types (compare, for example, Athey and Haile (2007)). Further, the absence of atoms in the equilibrium bid distribution entails that the limit of any sequence of equilibria on an ever-finer bid grid can be represented as an equilibrium on the continuous bid space with the standard tie-breaking rule. Therefore, equilibria on the fine grid can be analyzed by solving the game on the continuous bid space, avoiding complications arising from a coarse bid set.

Nevertheless, although all equilibria are well-behaved, the bids and the common value of the good are not affiliated. This implies that in the second-price auction, with positive probability, the payment of the winning bidder may exceed her conditional expected valuation for the good, such that she incurs a loss. Contrary to auctions with affiliation in which a single-crossing property ensures that the winning bidder earns an expected profit when winning at any price below her bid, the bidder may prefer not to win at low prices.[14] The following example demonstrates this effect.

Example To simplify the argument, we provide an example in which the private-value distribution is discrete. It is easy to see that a similar continuous distribution will imply the same qualitative bidding behavior. Suppose that there are $n = 2$ bidders, the good is sold in a second-price auction and the common value is either $v_h = 1$ or $v_\ell = 0$ with equal probability. Let the value function be $u(v_\omega, \theta) = v_\omega + \theta$. There are four private values, $\Theta = \{0, \frac{1}{2}, \frac{3}{4}, 1\frac{1}{4}\}$ and $q \approx 0$, such that bidders are almost certainly informed. Since informed bidders bid their valuation and bidders are almost always informed, there are essentially eight bids in the equilibrium bid distribution:

$$\begin{array}{l|cccccccc} \text{bid } b & 0 & \frac{1}{2} & \frac{3}{4} & 1 & 1\frac{1}{4} & 1\frac{1}{2} & 1\frac{3}{4} & 2\frac{1}{4} \\ \mathbb{E}[v_\omega | p = b; \beta^*] \approx & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \end{array}.$$

Obviously, the price and the conditional expected value are not affiliated. Now consider the bidding incentives for an uninformed bidder with private value $\frac{3}{4}$. This type will bid strictly above 1 because at all lower bids, she earns a profit. If she bids above $1\frac{1}{2}$, she trades off a profit of $\frac{1}{4}$ when the price is $1\frac{1}{2}$ against a loss of $\frac{1}{2}$ when the price is $1\frac{1}{4}$. However, if the private value $\theta = \frac{1}{2}$ is sufficiently likely and private value $\theta = 1\frac{1}{4}$ is sufficiently unlikely, the expected gain outweighs the loss, such that the $(\emptyset, \frac{3}{4})$ bidder has a strict incentive to bid more than $1\frac{1}{2}$. Consequently, $(\emptyset, \frac{3}{4})$ bidders will bid at least $1\frac{1}{2}$ and incur a strict loss if the competitor is a $(\ell, 1\frac{1}{4})$ type and the price is $1\frac{1}{4}$.

The lack of affiliation results in non-concave payoff functions for the bidders, implying that local optimality is not a sufficient condition to determine a bidder's equilibrium bid. Further, it implies that the corresponding social choice functions are not posterior-implementable: conditional on winning at a price that results in an expected loss, bidders experience regret and would like to revise their bid (cf. Jehiel et al. (2007)).

[14] In an ascending auction, bidders would have an incentive to be inactive at intermediate bids.

Last, our findings indicate that the non-existence identified in Jackson (2009) is an artifact of the specific assumptions—namely, the discrete private value distribution and the second-price auction format. Thereby, the result is unlikely to hint at a more fundamental reason why auctions with multidimensional signals should not have equilibria or why equilibria should not be well-behaved. This is good news for the efficiency result obtained by Pesendorfer and Swinkels (2000), which relies on the existence of a strictly increasing equilibrium and suggests that the positive result by Heumann (2019) does not hinge on the assumption of Gaussian signals.[15]

[15] Heumann (2019) shows that in a setting with two-dimensional Gaussian signals, the signals can be translated into a single-dimensional sufficient statistic that is affiliated with the value of the good, such that the results by Milgrom and Weber (1982) hold and an equilibrium exists. Further, Heumann (2019) shows that there can be multiple of these sufficient statistics, resulting in equilibrium multiplicity.

Appendices

2.A Proofs

2.A.1 Proof of Proposition 2.3

Consider a version of Γ_c^{SPA} on a bid grid, in which bidders are restricted to choose a bid from a finite set B^k. Suppose that B^k becomes dense on $[0, u(v_h, \bar{\theta})]$ as $k \to \infty$. Any such discretized auction has an equilibrium which we denote by $\beta^k = \{\beta_h^k, \beta_\ell^k, \beta_\emptyset^k\}$.

Theorem 2 of Jackson et al. (2002) guarantees that here is a subsequence of discretized auctions, as well as an endogenous tie-breaking rule, such that (a) $\beta_s^k \to \beta_s^*$ for all $s \in \{h, \emptyset, \ell\}$, (b) $\beta^* = \{\beta_h^*, \beta_\ell^*, \beta_\emptyset^*\}$ and the endogenous tie-breaking rule form an equilibrium of a communication extension, and (c) outcomes and payoffs converge. We want to show that the specific class of endogenous tie-breaking rules of our communication extension (Definition 2.3) captures one of these equilibria and that it takes the form as described in Proposition 2.3.

Since symmetric equilibria of the SPA are equilibria in undominated strategies, on any grid B^k, informed bidder select a bid $\beta_\omega^k(\theta) \in \big\{\max\{b \in B^k : b \leq u(v_\omega, \theta)\}, \min\{b \in B^k : b \geq u(v_\omega, \theta)\}\big\}$. Consequently, the limit strategy is $\beta_\omega^*(\theta) = u(v_\omega, \theta)$, which pure and strictly increasing. This means that at most a countable number of informed types tie with positive probability. Since those do not affect the outcome of the auction and informed bidders are indifferent about tying on their valuation, the tie-breaking rule is irrelevant for them.

We now turn to uninformed bidders. We first show that, without loss, $\beta_\emptyset^k$ is pure and nondecreasing. Fix any strategy β^k and consider any two $b' > b$ such that $G(b'|\emptyset; \beta^k) - G(b|\emptyset; \beta^k) > 0$. Note that the change in utility from moving from bid b to b' is

$$
\begin{aligned}
&U(b|\theta, \emptyset; \beta^k) - U(b'|\theta, \emptyset; \beta^k) \\
&= \mathbb{P}[\text{win with b' but not b}|\emptyset; \beta^k] \\
&\cdot \Big(\mathbb{E}[u(v_\omega, \theta)|\text{win with b' but not b}, \emptyset; \beta^k] - \mathbb{E}[p|\text{win with b' but not b}, \emptyset; \beta^k]\Big).
\end{aligned}
$$

Because $G(b'|\emptyset; \beta^k) - G(b|\emptyset; \beta^k) > 0$, it follows that $\mathbb{P}[\text{win with b' but not b}|\emptyset; \beta^k] > 0$, such that $U(b|\emptyset, \theta; \beta^k) - U(b'|\emptyset, \theta; \beta^k)$ is strictly increasing in θ. Thus, if θ prefers b' over b, so does any $\theta' > \theta$.

Let $\hat{\Theta} = \{\theta : \exists b \neq b' \ s.t.\ U(b|\emptyset, \theta; \beta^k) = U(b'|\emptyset, \theta; \beta^k)\}$ be the set of uninformed bidders that is indifferent between two bids. We want to show that this set is countable. By construction, $\forall \theta \in \hat{\Theta}$ there are two bids $b_-^\theta < b_+^\theta$ such that a bidder θ is indifferent between these two bids, $U(b_-^\theta|\emptyset, \theta; \beta^k) = U(b_+^\theta|\emptyset, \theta; \beta^k)$. Furthermore, there exists a $q^\theta \in \mathbb{Q}$ s.th. $b^\theta_- < q^\theta < b^\theta_+$. By the observation above, $b^\theta_+ \leq b^{\theta'}_-$ for all $\theta < \theta'$, which implies that $q^\theta < q^{\theta'}$. Because $\mathbb{Q}$ is countable, so is $\hat{\Theta}$. Since $\hat{\Theta}$ is countable, it has zero measure. As a result, any best response and, thereby, any equilibrium is pure and nondecreasing almost everywhere. Thereby, it is without loss to restrict attention to pure and nondecreasing strategies.

Since $\beta^k_\emptyset$ are pure and nondecreasing, the limit, $\beta^*_\emptyset$, is pure and nondecreasing. This means that in the limit, if there is an atom in the bid distribution, it is the result of uninformed bidders tying. Thus, the tie-breaking rule has to decide which tying, uninformed bidder wins, which can result in three cases: a certain win, a certain loss, or a random tie-break. Because bidders with higher private values win (weakly) more often along the sequence of auctions on the ever-finer grid, the tie-breaking rule has to be such that bidders with higher private values win (weakly) more often. Thus, the limit outcome of any sequence of auctions on the ever-finer grid can be captured by an endogenous tie-breaking rule $\tau^*(\emptyset, \hat{\theta})$ which is nondecreasing in $\hat{\theta}$. Since τ^* captures the limit of equilibrium outcomes, deviations in the report $\hat{\theta}$ are either equivalent to mimicking some type, or to choosing a marginally larger bid or larger bid which does not tie, such that the tie-breaking rule is irrelevant. Since $\beta^*_\emptyset$ is the limit of equilibria, none of these deviations in the report can be strictly profitable.

What remains to be shown is that τ^* can actually deter bidders from lying about their signal s. First, note that because informed bidders almost never tie, on-path, they are indifferent about the tie-breaking rule. Further, any deviation to a bid $b_a \neq \beta^*_\omega(\theta)$ where the report may matter can only reduce the informed bidder's payoff. Next, turn to uninformed bidders. Because informed bidders are indifferent about $\tau^*(\omega, \hat{\theta})$, we can deter any misreport by an uninformed bidder by setting $\tau^*(\omega, \hat{\theta}) < \tau^*(\emptyset, \hat{\theta})$ for all $\hat{\theta}$. Thereby, for an uninformed bidder, misreporting her signal becomes identical to a marginal downward deviation in her bid. Since such a marginal deviation is not strictly profitable in equilibrium, neither is misreporting under τ^*.

2.A.2 Probabilities and expected values of the uninformed bidders

We derive the probabilities and expected values of the uninformed bidders for later use. Let $J = [\theta_-, \theta_+] \subseteq [0, \bar{\theta}]$ be any nontrivial interval.

Probabilities $\mathbb{P}[\theta_{\emptyset}^{(1)} \in J, \theta_{\omega}^{(1)} \leq \theta_{\omega}|\emptyset]$
$\mathbb{P}[\theta_{\emptyset}^{(1)} \in J, \text{win tb}, \theta_{\omega}^{(1)} \leq \theta_{\omega}|\emptyset]$

$$\begin{aligned}\mathbb{P}[\theta_{\emptyset}^{(1)} \in J, \theta_{\omega}^{(1)} \leq \theta_{\omega}|\omega] &= \sum_{i=0}^{n-1} \mathbb{P}[\text{i uninf}] \cdot \mathbb{P}[\theta_{\emptyset}^{(1)} \in J, \theta_{\omega}^{(1)} \leq \theta_{\omega}|\text{i uninf}; \omega] \\ &= \sum_{i=0}^{n-1} \binom{n-1}{i} q^i (1-q)^{n-1-i} \cdot F(\theta_{\omega})^{n-1-i}[F(\theta_+)^i - F(\theta_-)^i].\end{aligned} \tag{2.4}$$

$[F(\theta_+)^i - F(\theta_-)^i]$ is the probability that at least one of i uninformed bidders has a private value $\theta \in J$. Expanding this probability gives

$$[F(\theta_+)^i - F(\theta_-)^i] = \sum_{j=1}^{i} \binom{i}{j} \underbrace{(F(\theta_+) - F(\theta_-))^j}_{j \text{ uninformed } \in J} F(\theta_-)^{i-j}.$$

As a result, the probability to tie on J and win the random tie-break is

$$\begin{aligned}\mathbb{P}[\theta_{\emptyset}^{(1)} \in J, \text{win tb}, \theta_{\omega}^{(1)} \leq \theta_{\omega}|\omega] &= \sum_{i=1}^{n-1} \binom{n-1}{i} q^i (1-q)^{n-1-i} F(\theta_{\omega})^{n-1-i} \\ &\cdot [\sum_{j=1}^{i} \binom{i}{j} (F(\theta_+) - F(\theta_-))^j F(\theta_-)^{i-j} \frac{1}{j+1}].\end{aligned} \tag{2.5}$$

$\mathbb{P}[\theta_{\emptyset}^{(1)} \in J, \theta_{\omega}^{(1)} \leq \theta_{\omega}|\emptyset]$ and $\mathbb{P}[\theta_{\emptyset}^{(1)} \in J, \text{win tb}, \theta_{\omega}^{(1)} \leq \theta_{\omega}|\emptyset]$ are the averages of (2.4) and (2.5), weighted by the prior.

Expected values $\mathbb{E}[u(v_{\omega}, \theta)|\theta_{\emptyset}^{(1)} \in J, \theta_{\omega}^{(1)} \leq \theta_{\omega}, \emptyset]$
$\mathbb{E}[u(v_{\omega}, \theta)|\theta_{\emptyset}^{(1)} \in J, \text{win tb}, \theta_{\omega}^{(1)} \leq \theta_{\omega}, \emptyset]$

Given any event A and the uninformative signal $\emptyset$, the conditional expected value for private value θ is

$$\mathbb{E}[u(v_{\omega}, \theta)|A, \emptyset] = \frac{\rho \frac{\mathbb{P}[A|h]}{\mathbb{P}[A|\ell]} u(v_h, \theta) + (1-\rho) u(v_{\ell}, \theta)}{\rho \frac{\mathbb{P}[A|h]}{\mathbb{P}[A|\ell]} + (1-\rho)}.$$

Plugging in (2.4) and (2.5), respectively, yields $\mathbb{E}[u(v_{\omega}, \theta)|\theta_{\emptyset}^{(1)} \in J, \theta_{\omega}^{(1)} \leq \theta_{\omega}, \emptyset]$ as well as $\mathbb{E}[u(v_{\omega}, \theta)|\theta_{\emptyset}^{(1)} \in J, \text{win tb}, \theta_{\omega}^{(1)} \leq \theta_{\omega}, \emptyset]$.

Expected values $\mathbb{E}[u(\theta, v_{\omega})|\text{i uninf}, \theta_{\emptyset}^{(1)} \in J, \theta_{\omega}^{(1)} \leq \theta_{\omega}, \emptyset]$
$\mathbb{E}[u(\theta, v_{\omega})|\text{i uninf}, \theta_{\emptyset}^{(1)} \in J, \text{win tb}, \theta_{\omega}^{(1)} \leq \theta_{\omega}, \emptyset]$

We can expand $\mathbb{E}[u(v_\omega,\theta)|A,\emptyset] = \sum_{i=0}^{n-1} \mathbb{P}[\text{i uninf}|A,\emptyset] \cdot \mathbb{E}[u(v_\omega,\theta)|\text{i uninf}, A, \emptyset]$ to

$$\mathbb{E}[u(\theta, v_\omega)|\text{i uninf}, A, \emptyset] = \frac{\rho\frac{\mathbb{P}[A,\text{i uninf}|h]}{\mathbb{P}[A,\text{i uninf}|\ell]}u(v_h,\theta) + (1-\rho)u(v_\ell,\theta)}{\rho\frac{\mathbb{P}[A,\text{i uninf}|h]}{\mathbb{P}[A,\text{i uninf}|\ell]} + (1-\rho)}, \tag{2.6}$$

which is strictly increasing in $\frac{\mathbb{P}[A,\text{i uninf}|h]}{\mathbb{P}[A,\text{i uninf}|\ell]}$. Using the summands of (2.4) and (2.5), we note that for any nontrivial J

$$\begin{aligned}\frac{\mathbb{P}[\theta_\emptyset^{(1)} \in J, \theta_\omega^{(1)} \leq \theta_\omega, \text{i uninf}|h]}{\mathbb{P}[\theta_\emptyset^{(1)} \in J, \theta_\omega^{(1)} \leq \theta_\omega, \text{i uninf}|\ell]} &= \frac{\mathbb{P}[\theta_\emptyset^{(1)} \in J, \text{win tb}, \theta_\omega^{(1)} \leq \theta_\omega, \text{i uninf}|h]}{\mathbb{P}[\theta_\emptyset^{(1)} \in J, \text{win tb}, \theta_\omega^{(1)} \leq \theta_\omega, \text{i uninf}|\ell]} \\ &= \Big(\frac{F(\theta_h)}{F(\theta_\ell)}\Big)^{n-1-i}.\end{aligned} \tag{2.7}$$

Thus, for any nontrivial J, it follows by (2.6) that

$$\begin{aligned}&\mathbb{E}[u(v_\omega,\theta)|\text{i uninf}, \theta_\emptyset^{(1)} \in J, \theta_\omega^{(1)} \leq \theta_\omega, \emptyset] \\ &\qquad = \mathbb{E}[u(v_\omega,\theta)|\text{i uninf}, \theta_\emptyset^{(1)} \in J, \text{win tb}, \theta_\omega^{(1)} \leq \theta_\omega, \emptyset].\end{aligned} \tag{2.8}$$

Further, we observe that because (2.8) is strictly increasing in $\left(\frac{F(\theta_h)}{F(\theta_\ell)}\right)^{n-1-i}$, when $\frac{F(\theta_h)}{F(\theta_\ell)} < 1$, this means that (2.8) is nondecreasing[16] in i with

$$\begin{aligned}&\mathbb{E}[u(v_\omega,\theta)|1 \text{ uninf}, \theta_\emptyset^{(1)} \in J, \theta_\omega^{(1)} \leq \theta_\omega, \emptyset] \\ &\qquad < \mathbb{E}[u(v_\omega,\theta)|n-1 \text{ uninf}, \theta_\emptyset^{(1)} \in J, \theta_\omega^{(1)} \leq \theta_\omega, \emptyset].\end{aligned} \tag{2.9}$$

2.A.3 Lemma 2.3—Ranking of conditional expected values

Lemma 2.3 *Suppose that for $j \in \{1, ..., n-2\}$ and any $k > j$ it holds that*

$$\frac{\mathbb{P}[j \textit{ uninf}|A,\emptyset]}{\mathbb{P}[j \textit{ uninf}|B,\emptyset]} < \frac{\mathbb{P}[k \textit{ uninf}|A,\emptyset]}{\mathbb{P}[k \textit{ uninf}|B,\emptyset]}, \tag{2.10}$$

and

$$\mathbb{E}[u(v_\omega,\theta)|i \textit{ uninf}, A, \emptyset] = \mathbb{E}[u(v_\omega,\theta)|i \textit{ uninf}, B, \emptyset]$$

is nondecreasing i with $\mathbb{E}[u(v_\omega,\theta)|1 \textit{ uninf}, A, \emptyset] < \mathbb{E}[u(v_\omega,\theta)|n-1 \textit{ uninf}, A, \emptyset]$.

Then,

$$\mathbb{E}[u(v_\omega,\theta)|A,\emptyset] > \mathbb{E}[u(v_\omega,\theta)|B,\emptyset].$$

Proof. We can expand $\mathbb{E}[u(v_\omega,\theta)|A,\emptyset] = \sum_{i=1}^{n-1} \mathbb{P}[\text{i uninf}|A,\emptyset] \cdot$

[16] If $0 < \frac{F(\theta_h)}{F(\theta_\ell)} < 1$ this is obvious. If $\frac{F(\theta_h)}{F(\theta_\ell)} = 0$, then $\left(\frac{F(\theta_h)}{F(\theta_\ell)}\right)^{n-1-i} = 0$ for all $i < n-1$ and $\left(\frac{F(\theta_h)}{F(\theta_\ell)}\right)^{n-1-i} = 1$ for $i = n-1$.

$\mathbb{E}[u(v_\omega,\theta)|\text{i uninf}, A, \emptyset]$ and $\mathbb{E}[u(v_\omega,\theta)|B,\emptyset]$ respectively. Then,

$$\begin{aligned}&\mathbb{E}[u(v_\omega,\theta)|A,\emptyset] - \mathbb{E}[u(v_\omega,\theta)|B,\emptyset]\\ =&\sum_{i=1}^{n-1}(\mathbb{P}[\text{i uninf}|A,\emptyset] - \mathbb{P}[\text{i uninf}|B,\emptyset])\cdot \mathbb{E}[u(v_\omega,\theta)|\text{i uninf}, A,\emptyset].\end{aligned}$$

Since $\sum_{i=1}^{n-1}\mathbb{P}[\text{i uninf}|A,\emptyset] = \sum_{i=1}^{n-1}\mathbb{P}[\text{i uninf}|B,\emptyset] = 1$, the strict monotone likelihood ratio property (2.10) implies that $\mathbb{P}[\text{i uninf}|A,\emptyset]-\mathbb{P}[\text{i uninf}|B,\emptyset]$ is strictly increasing in i.

If $\mathbb{E}[u(v_\omega,\theta)|\text{i uninf}, A,\emptyset]$ is constant in i, $\mathbb{E}[u(v_\omega,\theta)|A,\emptyset] - \mathbb{E}[u(v_\omega,\theta)|B,\emptyset] = 0$. However, if $\mathbb{E}[u(v_\omega,\theta)|\text{i uninf}, A,\emptyset]$ is nondecreasing in i and strictly increases at (at least) one i, it follows that $\mathbb{E}[u(v_\omega,\theta)|A,\emptyset] - \mathbb{E}[u(v_\omega,\theta)|B,\emptyset] > 0$, completing the proof. $\square$

2.A.4 Proof of Lemma 2.1

From $b_a \in (u(v_\ell,\underline{\theta}), u(v_h,\bar{\theta}))$ it follows that $\frac{F(\theta_h)}{F(\theta_\ell)} < 1$, such that (2.8) is nondecreasing in i and (2.9) holds. By Lemma 2.3, we only need to show that

$$\begin{aligned}&\frac{\mathbb{P}[\text{i uninf}|\theta_\emptyset^{(1)}\in T, \theta_\omega^{(1)}\le\theta_\omega,\emptyset]}{\mathbb{P}[\text{i uninf}|\theta_\emptyset^{(1)}\in T, \text{win tb}, \theta_\omega^{(1)}\le\theta_\omega,\emptyset]} \qquad (2.11)\\ =&\underbrace{\frac{\mathbb{P}[\theta_\emptyset^{(1)}\in T, \text{win tb}, \theta_\omega^{(1)}\le\theta_\omega]}{\mathbb{P}[\theta_\emptyset^{(1)}\in T, \theta_\omega^{(1)}\le\theta_\omega]}}_{\text{independent of i}}\underbrace{\frac{\mathbb{P}[\theta_\emptyset^{(1)}\in T, \theta_\omega^{(1)}\le\theta_\omega|\text{i uninf}]}{\mathbb{P}[\theta_\emptyset^{(1)}\in T, \text{win tb}, \theta_\omega^{(1)}\le\theta_\omega|\text{i uninf}]}}_{\phi(i)}\frac{\mathbb{P}[\text{i uninf}]}{\mathbb{P}[\text{i uninf}]}\end{aligned}$$

is strictly increasing in i.

Using the summands of (2.4) and (2.5), we can write $\phi(i)$ as

$$\begin{aligned}\phi(i) &= \frac{(F(\theta_+)-F(\theta_-))^i}{\sum_{j=1}^i\binom{i}{j}(F(\theta_+)-F(\theta_-))^j\frac{1}{j+1}}\frac{\rho F(\theta_h)^{n-1-i}+(1-\rho)F(\theta_\ell)^{n-1-i}}{\rho F(\theta_h)^{n-1-i}+(1-\rho)F(\theta_\ell)^{n-1-i}}\\ &= (i+1)\frac{\left(\frac{F(\theta_+)}{F(\theta_-)}\right)^i - 1}{\sum_{j=2}^{i+1}\binom{i+1}{j}(\frac{F(\theta_+)}{F(\theta_-)}-1)^{j-1}}\\ &= (i+1)\frac{\left(\frac{F(\theta_+)}{F(\theta_-)}\right)^i - 1}{(\frac{F(\theta_+)}{F(\theta_-)}-1)^{-1}\sum_{j=2}^{i+1}\binom{i+1}{j}(\frac{F(\theta_+)}{F(\theta_-)}-1)^{j}}\\ &= (i+1)(\frac{F(\theta_+)}{F(\theta_-)}-1)\cdot\frac{\left(\frac{F(\theta_+)}{F(\theta_-)}\right)^i - 1}{\left(\frac{F(\theta_+)}{F(\theta_-)}\right)^{i+1}-1-(i+1)(\frac{F(\theta_+)}{F(\theta_-)}-1)}.\end{aligned}$$

For ease of notation, we replace $\frac{F(\theta_+)}{F(\theta_-)}$ by d. ϕ is strictly increasing in i if and only if $\frac{\phi(i)}{\phi(i-1)} > 1$ for all $i \geq 2$ which can be expanded to

$$\begin{aligned}\frac{\phi(i)}{\phi(i-1)} = \frac{(i+1)(d^i-1)}{d^{i+1}-1-(i+1)(d-1)} \frac{d^i-1-i(d-1)}{i(d^{i-1}-1)} > 1 \\ \iff d^i(d^i-(d-1)^2 i^2 - 2d) > d^{2i} - ((d-1)^2 i^2 + 2d)d^{i-1} \\ \iff d + d^{1+2i} - d^i(2d + (-1+d)^2 i^2) > 0 \\ \iff \frac{d^{\frac{i}{2}} - d^{-\frac{i}{2}}}{i} > \frac{d-1}{\sqrt{d}}. \end{aligned} \tag{2.12}$$

For $i = 1$ both sides of (2.12) equal. Hence, the inequality holds for $i \geq 2$ if the left side is strictly increasing in i. We can rewrite the left side as[17]

$$\frac{d^{\frac{i}{2}} - d^{-\frac{i}{2}}}{i} = \ln(d) + \sum_{j=1}^{\infty} \frac{\frac{i}{2}^{2j}}{(2j+1)!} \ln(d)^{2j+1}.$$

Since $d = \frac{F(\theta_+)}{F(\theta_-)} > 1$, it follows that $\ln(d) > 0$, such that the left side of inequality (2.12) is indeed strictly increasing in i. Thus, (2.11) is strictly increasing in i which proves the result.

2.A.5 Proof of Lemma 2.2

Since (2.7) is independent of J, it follows that

$$\begin{aligned}&\mathbb{E}[u(v_\omega, \theta) | \text{i uninf}, \theta_\emptyset^{(1)} \in J, \theta_\omega^{(1)} \leq \theta_\omega, \emptyset] \\ &\quad = \mathbb{E}[u(v_\omega, \theta) | \text{i uninf}, \theta_\emptyset^{(1)} \in J', \theta_\omega^{(1)} \leq \theta_\omega, \emptyset].\end{aligned} \tag{2.13}$$

From $b_a \in (u(v_\ell, \underline{\theta}), u(v_h, \bar{\theta}))$ it follows that $\frac{F(\theta_h)}{F(\theta_\ell)} < 1$, such that $\mathbb{E}[u(v_\omega, \theta) | \text{i uninf}, \theta_\emptyset^{(1)} \in J, \theta_\omega^{(1)} \leq \theta_\omega, \emptyset]$ is nondecreasing in i and (2.9) holds. By Lemma 2.3, we only have to show that for any $j \in \{1, ..., n-2\}$ and any $k > j$

$$\frac{\mathbb{P}[\text{k uninf} | \theta_\emptyset^{(1)} \in J, \theta_\omega^{(1)} \leq \theta_\omega, \emptyset]}{\mathbb{P}[\text{j uninf} | \theta_\emptyset^{(1)} \in J, \theta_\omega^{(1)} \leq \theta_\omega, \emptyset]} < \frac{\mathbb{P}[\text{k uninf} | \theta_\emptyset^{(1)} \in J', \theta_\omega^{(1)} \leq \theta_\omega, \emptyset]}{\mathbb{P}[\text{j uninf} | \theta_\emptyset^{(1)} \in J', \theta_\omega^{(1)} \leq \theta_\omega, \emptyset]}. \tag{2.14}$$

[17]Since $\frac{d^{\frac{i}{2}}}{i} = \frac{e^{\ln(d)\frac{i}{2}}}{i}$ and $\frac{d^{\frac{-i}{2}}}{i} = \frac{e^{\ln(d)\frac{-i}{2}}}{i}$, we can use $e^x = \sum_j \frac{x^j}{j!}$ to write $\frac{d^{\frac{i}{2}} - d^{-\frac{i}{2}}}{i}$ as $\sum_{j=0} \frac{\ln(d)^j (\frac{i}{2})^j}{j!i} - \sum_{j=0} \frac{\ln(d)^j (\frac{-i}{2})^j}{j!i} = \sum_{j=0} \frac{\ln(d)^j \frac{i^j - (-i)^j}{2}}{j!i} = \sum_{j=1,3,...} \frac{\ln(d)^j i^j}{j!i} = \sum_{j=1,3,...} \frac{\ln(d)^j i^{j-1}}{j}$.

Note that for $J = [\theta_-, \theta_+]$ the probability $\mathbb{P}[\text{i uninf}|\theta_\emptyset^{(1)} \in J, \theta_\omega^{(1)} \leq \theta_\omega, \emptyset]$ is

$$\frac{\binom{n-1}{i} q^i (1-q)^{n-1-i} (\rho F(\theta_h)^{n-1-i} + (1-\rho) F(\theta_\ell)^{n-1-i})(F(\theta_+)^i - F(\theta_-)^i)}{\sum_{j=1}^{n-1} \binom{n-1}{j} q^j (1-q)^{n-1-j} (\rho F(\theta_h)^{n-1-j} + (1-\rho) F(\theta_\ell)^{n-1-j})(F(\theta_+)^j - F(\theta_-)^j)}.$$

Using this, we prove (2.14) by showing that for any $j \in \{1, ..., n-2\}$ and any $k > j$

$$\begin{aligned} &\frac{\mathbb{P}[\text{k uninf.}|\theta_\emptyset^{(1)} \in J, \theta_\omega^{(1)} \leq \theta_\omega, \emptyset]}{\mathbb{P}[\text{j uninf.}|\theta_\emptyset^{(1)} \in J, \theta_\omega^{(1)} \leq \theta_\omega, \emptyset]} \\ &= \frac{q^k (1-q)^{n-1-k} (\rho F(\theta_h)^{n-1-k} + (1-\rho) F(\theta_h)^{n-1-k})}{q^j (1-q)^{n-1-j} (\rho F(\theta_h)^{n-1-j} + (1-\rho) F(\theta_\ell)^{n-1-j})} \cdot \frac{F(\theta_+)^k - F(\theta_-)^k}{F(\theta_+)^j - F(\theta_-)^j} \end{aligned} \tag{2.15}$$

is strictly increasing in θ_+. To do so, we only need to consider the rightmost fraction of (2.15) and show that

$$\begin{aligned} &\frac{\partial}{\partial \theta_+} \frac{F(\theta_+)^k - F(\theta_-)^k}{F(\theta_+)^j - F(\theta_-)^j} > 0 \\ &\iff F(\theta_+)[jF(\theta_+)^j F(\theta_-)^k + F(\theta_+)^k((k-j)F(\theta_+)^j - kF(\theta_-)^j)] > 0 \\ &\iff [1 - \Big(\frac{F(\theta_-)}{F(\theta_+)}\Big)^k]\frac{1}{k} < [1 - \Big(\frac{F(\theta_-)}{F(\theta_+)}\Big)^j]\frac{1}{j}. \end{aligned}$$

Because $\frac{1-a^x}{x}$ is decreasing in x when $a = \frac{F(\theta_-)}{F(\theta_+)} < 1$, this is always satisfied.

In the same manner, we show that (2.15) is strictly increasing in θ_-. Taking the derivative of the last fraction of (2.15) with respect to θ_- yields

$$\begin{aligned} &\frac{\partial}{\partial \theta_-} \frac{F(\theta_+)^k - F(\theta_-)^k}{F(\theta_+)^j - F(\theta_-)^j} > 0 \\ &\iff F(\theta_-)[jF(\theta_-)^j F(\theta_+)^k + F(\theta_+)^k((k-j)F(\theta_-)^j - kF(\theta_+)^j)] > 0 \\ &\iff [\Big(\frac{F(\theta_+)}{F(\theta_-)}\Big)^k - 1]\frac{1}{k} > [\Big(\frac{F(\theta_+)}{F(\theta_-)}\Big)^j - 1]\frac{1}{j}. \end{aligned}$$

Because $\frac{d^x - 1}{x}$ is increasing in x when $d = \frac{F(\theta_+)}{F(\theta_-)} > 1$, this is always satisfied.

Since $J < J'$, i.e. $\inf J \leq \inf J'$ and $\sup J \leq \sup J'$ with at least one strict inequality, this implies that (2.14) is satisfied, which completes the proof.

2.A.6 Proof of Corollary 2.1

For ease of notation, we suppress the explicit reference to $\theta_\omega^{(1)} \leq \theta_\omega$ and the uninformative signal $\emptyset$.

The expected value $\mathbb{E}[u(v_\omega,\theta)|\Delta overbid]$ expands to

$$\frac{1}{\mathbb{P}[\theta_\emptyset^{(1)} \in T] - \mathbb{P}[\theta_\emptyset^{(1)} \in T, \text{win tb}] + \mathbb{P}[\theta_\emptyset^{(1)} \in L]} \Big(\mathbb{E}[u(v_\omega,\theta)|\theta_\emptyset^{(1)} \in T]\mathbb{P}[\theta_\emptyset^{(1)} \in T] \\ - \mathbb{E}[u(v_\omega,\theta)|\theta_\emptyset^{(1)} \in T, \text{win tb}]\mathbb{P}[\theta_\emptyset^{(1)} \in T, \text{win tb}] + \mathbb{E}[u(v_\omega,\theta)|\theta_\emptyset^{(1)} \in L]\mathbb{P}[\theta_\emptyset^{(1)} \in L]\Big),$$

and $\mathbb{E}[u(v_\omega,\theta)|\Delta underbid]$ expands to

$$\frac{1}{\mathbb{P}[\theta_\emptyset^{(1)} \in T, \text{win tb}] + \mathbb{P}[\theta_\emptyset^{(1)} \in W]} \Big(\mathbb{E}[u(v_\omega,\theta)|\theta_\emptyset^{(1)} \in T, \text{win tb}]\mathbb{P}[\theta_\emptyset^{(1)} \in T, \text{win tb}] \\ + \mathbb{E}[u(v_\omega,\theta)|\theta_\emptyset^{(1)} \in W]\mathbb{P}[\theta_\emptyset^{(1)} \in W]\Big).$$

If $\mathbb{P}[\theta_\emptyset^{(1)} \in T] = 0$, then $\mathbb{P}[\theta_\emptyset^{(1)} \in T, \text{win tb}] = 0$, but because θ° is from the interior of I, $\mathbb{P}[\theta_\emptyset^{(1)} \in T] = 0$ implies that both L and W are non trivial. As a result,

$$\mathbb{E}[u(v_\omega,\theta)|\Delta overbid] = \mathbb{E}[u(v_\omega,\theta)|\theta_\emptyset^{(1)} \in L] \\ \overset{L2.2}{>} \mathbb{E}[u(v_\omega,\theta)|\theta_\emptyset^{(1)} \in W] = \mathbb{E}[u(v_\omega,\theta)|\Delta underbid].$$

If $\mathbb{P}[\theta_\emptyset^{(1)} \in T] > 0$ and, thereby, $\mathbb{P}[\theta_\emptyset^{(1)} \in T, \text{win tb}] > 0$, then

$$\frac{1}{\mathbb{P}[\theta_\emptyset^{(1)} \in T] - \mathbb{P}[\theta_\emptyset^{(1)} \in T, \text{win tb}] + \mathbb{P}[\theta_\emptyset^{(1)} \in L]} \Big(\mathbb{E}[u(v_\omega,\theta)|\theta_\emptyset^{(1)} \in T]\mathbb{P}[\theta_\emptyset^{(1)} \in T] \\ - \mathbb{E}[u(v_\omega,\theta)|\theta_\emptyset^{(1)} \in T, \text{win tb}]\mathbb{P}[\theta_\emptyset^{(1)} \in T, \text{win tb}] + \mathbb{E}[u(v_\omega,\theta)|\theta_\emptyset^{(1)} \in L]\mathbb{P}[\theta_\emptyset^{(1)} \in L]\Big) \\ \overset{L2.1,L2.2}{>} \frac{\mathbb{E}[u(v_\omega,\theta)|\theta_\emptyset^{(1)} \in T]\Big(\mathbb{P}[\theta_\emptyset^{(1)} \in T] - \mathbb{P}[\theta_\emptyset^{(1)} \in T, \text{win tb}] + \mathbb{P}[\theta_\emptyset^{(1)} \in L]\Big)}{\mathbb{P}[\theta_\emptyset^{(1)} \in T] - \mathbb{P}[\theta_\emptyset^{(1)} \in T, \text{win tb}] + \mathbb{P}[\theta_\emptyset^{(1)} \in L]} \\ = \mathbb{E}[u(v_\omega,\theta)|\theta_\emptyset^{(1)} \in T] \\ = \frac{\mathbb{E}[u(v_\omega,\theta)|\theta_\emptyset^{(1)} \in T]\mathbb{P}[\theta_\emptyset^{(1)} \in T, \text{win tb}] + \mathbb{E}[u(v_\omega,\theta)|\theta_\emptyset^{(1)} \in T]\mathbb{P}[\theta_\emptyset^{(1)} \in W]}{\mathbb{P}[\theta_\emptyset^{(1)} \in T, \text{win tb}] + \mathbb{P}[\theta_\emptyset^{(1)} \in W]} \\ \overset{L2.1,L2.2}{>} \frac{1}{\mathbb{P}[\theta_\emptyset^{(1)} \in T, \text{win tb}] + \mathbb{P}[\theta_\emptyset^{(1)} \in W]} \\ \cdot\Big(\mathbb{E}[u(v_\omega,\theta)|\theta_\emptyset^{(1)} \in T, \text{win tb}]\mathbb{P}[\theta_\emptyset^{(1)} \in T, \text{win tb}] + \mathbb{E}[u(v_\omega,\theta)|\theta_\emptyset^{(1)} \in W]\mathbb{P}[\theta_\emptyset^{(1)} \in W]\Big),$$

so that $\mathbb{E}[u(v_\omega,\theta)|\Delta overbid] > \mathbb{E}[u(v_\omega,\theta)|\Delta underbid]$.

2.A.7 Lemma 2.4—Strict monotonicity in s

Lemma 2.4 *Any equilibrium of* Γ_c^{SPA} *is strictly increasing in* s*, meaning that* $\beta_\ell^*(\theta) < \beta_\emptyset^*(\theta) < \beta_h^*(\theta)$ *for almost all* θ.

Proof. Let μ be the measure induced by F and suppose to the contrary that there is a set $K \subseteq \Theta$ with $\mu(K) > 0$ such that $\beta_\emptyset^*(\theta) = \beta_\omega^*(\theta) = u(v_\omega, \theta)$ for all $\theta \in K$ and some $\omega \in \{h, \ell\}$. Without loss, let K be such that $\inf K > \underline{\theta}$ and $\sup K < \bar{\theta}$.

We want to show that if $\beta_\emptyset^*(\theta) = \beta_h^*(\theta)$ for all $\theta \in K$, uninformed bidders overpay with positive probability, meaning that they have an incentive to lower their bid, and that if $\beta_\emptyset^*(\theta) = \beta_\ell^*(\theta)$ for all $\theta \in K$, uninformed bidders miss out on profitable events, meaning that they have an incentive to raise their bid. In either case, this results in a contradiction since β^* is an equilibrium.

We prove the lemma for state $\omega = h$, i.e. when $\beta_\emptyset^*(\theta) = u(v_h, \theta)$ for all $\theta \in K$. The argument for state ℓ follows symmetrically.

First, because $\beta_\emptyset^*(\theta) = \beta_h^*(\theta)$ for all $\theta \in K$, no perfect updating is possible. In particular, there is a $\phi > 0$ such that for every subset $L \subseteq K$ with $\mu(L) > 0$, the probability $\mathbb{P}[h|p \in \beta_\emptyset^*(L)|\emptyset; \beta^*] < 1 - \phi$.[18]

Further, since u is strictly increasing in v_ω, there is a $\psi > 0$ and a subset $K' \subset K$ with $\mu(K') > 0$ such that $u(v_h, \theta) - u(v_\ell, \theta) > \psi$ for all $\theta \in K'$.

Last, because u is bounded, there are at most finitely many jump points $m \in M$ at which $u(v_h, m + \epsilon) - u(v_h, m - \epsilon) \geq \eta = \psi\phi^2$ for all $\epsilon > 0$.

Now, for any $\rho \in (0, 1)$, there is an interval I_ρ with $\mu(I_\rho) > 0$ such that $\mu(K' \cap I_\rho) > \rho\mu(I_\rho)$. It is without loss to assume that for $\rho' > \rho$, $I_{\rho'} \subset I_\rho$ and that $\sup I_\rho \in K'$.[19] Further, let $\mu(I_\rho) \to 0$ as $\rho \to 1$ (if necessary, we chop up the interval). Since M is finite, when is ρ sufficiently large, we can choose I_ρ such that $I_\rho \cap M = \emptyset$.[20]

Denote $\theta_- = \inf I_\rho$ and $\theta_+ = \sup I_\rho$. When ρ is sufficiently large, i.e. the interval I_ρ is sufficiently short, $u(v_h, \theta_+) - u(v_h, \theta_-) < \eta = \phi^2\psi$. Further, because $\frac{\mu(K' \cap I_\rho)}{\mu(I_\rho)} \to 1$, when ρ is sufficiently large, $\mathbb{P}[h|p \in [\beta_\emptyset^*(\theta_-), \beta_\emptyset^*(\theta_+)]|\emptyset; \beta^*] < 1 - \phi^2$.

We want to show that a downward deviation of type θ_+ from $\beta_\emptyset^*(\theta_+) = u(v_h, \theta_+)$ to $\beta_\emptyset^*(\theta_-) = u(v_h, \theta_-)$ is profitable. Since $\mu(I_\rho) > 0$, it follows that $\mathbb{P}[p \in [\beta_\emptyset^*(\theta_-), \beta_\emptyset^*(\theta_+)]|\emptyset; \beta^*] > 0$. Therefore, the deviation is strictly profitable if

$$\mathbb{E}[u(v_\omega, \theta_+)|p \in [\beta_\emptyset^*(\theta_-), \beta_\emptyset^*(\theta_+)], \emptyset; \beta^*] < \mathbb{E}[p|p \in [\beta_\emptyset^*(\theta_-), \beta_\emptyset^*(\theta_+)], \emptyset; \beta^*].$$

---

[18] Since $q > 0$ and $\beta_\emptyset^*(\inf K) > \beta_\emptyset^*(\underline{\theta})$, conditional on any price $p \in \beta_\emptyset^*(L)$, the probability that all bidders are uninformed is bounded away from zero, which pushes the probability toward the prior.

[19] Otherwise, we shrink the interval from the top, removing points $\notin K'$ and raising $\frac{\mu(K' \cap I_\rho)}{\mu(I_\rho)}$.

[20] When ρ is sufficiently large, there is at most one $m \in I_\rho$, which splits I_ρ into two. We take the left half of I_ρ^l if $\mu(K' \cap I_\rho^l) \geq \rho\mu(I_\rho^l)$, and the right half, otherwise.

Because $\theta_+ \in K'$, it has to hold that $u(v_h, \theta_+) - u(v_\ell, \theta_+) > \psi$. Since $\mathbb{P}[h|p \in [\beta^*_\emptyset(\theta_-), \beta^*_\emptyset(\theta_+)]|\emptyset; \beta^*] < 1 - \phi^2$, it follows that

$$\mathbb{E}[u(v_\omega, \theta_+)|p \in [\beta^*_\emptyset(\theta_-), \beta^*_\emptyset(\theta_+)], \emptyset; \beta^*] < u(v_h, \theta_+) - \phi^2\psi.$$

Further, because $u(v_h, \theta_+) - u(v_h, \theta_-) < \eta$ and $\beta^*_\emptyset(\theta) \geq u(v_h, \theta_-)$ for all $\theta \geq \theta_-$, it has to hold that

$$\mathbb{E}[p|p \in [\beta^*_\emptyset(\theta_-), \beta^*_\emptyset(\theta_+)], \emptyset; \beta^*] \geq u(v_h, \theta_-) \geq u(v_h, \theta_+) - \eta.$$

By construction, the deviation is profitable, completing the contradiction. □

2.B Proposition 1 in Jackson (2009)

In this section, we quickly point out the contradictory step in the proof of Proposition 1 ($n \geq 3$) in Jackson (2009) which we were unable to resolve.

- Claim 3 requires that $\epsilon < \frac{1-a}{2a} \iff 2a\epsilon < 1 - a$.
- Hence, there is type $T = 2a\epsilon, S = 0$ who bids $2a\epsilon \in (a\epsilon, 1 - a)$.
- Since $T = 2a\epsilon, S = 0$ is an informed bidder, even as $m \to 0$, the probability that $W_i \in (a\epsilon, 2a\epsilon)$ does not vanish.
- As a result, a bidder of type $T_i = \epsilon$, $S_i = \frac{1}{2}$ who deviates to a bid slightly above $(1-a)$ incurs a loss with strictly positive probability. Thus, the deviation does not need to be profitable.

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Chapter 3

The Economics of Decoupling

Joint with Paul Voß

3.1 Introduction

Even if a company formally adheres to the "one-share one-vote" principle, this does not imply that the number of votes a shareholder can cast is actually aligned with his or her stake in the company. Financial innovation has created a vast set of "decoupling techniques" for activist investors to acquire votes without taking a long position, decoupling their voting power from their economic exposure. As the cases collected by Hu and Black (2008a),[1] the aggregate evidence found by Christoffersen et al. (2007) as well as Kalay et al. (2014), and the recent fight for control over Premier Foods (2018) show,[2] these decoupling techniques are very popular with activist investors. Thereby, it comes as no surprise that the practice caught the eye of the press and regulatory authorities alike.[3]

What stands out about the public cases of decoupling is the variety of techniques employed, ranging from the usage of repo contracts to the acquisition of shares and hedges. While all these techniques ultimately resulted in a misalignment of voting power and economic exposure, they differed substantially in the transactions, timing, and parties involved. This begs the question if from the activists' perspective, different decoupling techniques are mere substitutes or whether there are meaningful economic differences in the cost and incentives they impose on activist investors.

The second, complementary question is what motivates activist investors to employ these decoupling techniques. While decoupling has been used to improve corporate governance, the prospect of voting without bearing the effect on share value

[1] Henceforth, we quote Hu and Black (2008a) as the most recent overview of their extensive documentation of decoupling, Hu and Black (2006, 2008a,b, 2007).

[2] Financial Times, July 15, 2018, "Market reverberates with accusations of empty voting," https://www.ft.com/content/0e28929e-85dd-11e8-a29d-73e3d454535d.

[3] See, for instance, the ESMA's "Call for evidence on empty voting" (September 2011), https://www.esma.europa.eu/press-news/consultations/call-evidence-empty-voting, or the "SEC Staff Roundtable on the Proxy Process" (July 2018), https://www.sec.gov/news/public-statement/statement-announcing-sec-staff-roundtable-proxy-process.

is undoubtedly of particular interest to activists who want to push their private agenda, instead of maximizing firm value. *"[Therefore,] [it] is a source of some concern that [...] important corporate actions [...] might be decided by persons who could have the incentive to [...] block actions that are in the interests of the shareholders as a whole"* (SEC, Concept Release on the U.S. Proxy System, p. 139).[4]

In this chapter, we give structure to the vast amount of decoupling techniques by deriving three classes of economically equivalent decoupling techniques: *Buy&Hedge* techniques, *Hedge&Buy* techniques and *Vote Trading* techniques.[5]

Afterward, we analyze which of these three classes can be exploited profitably by a hostile activist who opposes a firm value increasing reform, and we uncover a clear ranking in welfare implications. We find that Vote Trading techniques allow the activist to push her private agenda and expropriate shareholders at zero costs, whereas Buy&Hedge techniques are constrained efficient because the activist suffers from a commitment problem. Hedge&Buy techniques fall in between, exhibiting inefficient and constrained-efficient equilibria.

By categorizing the decoupling techniques, we develop a framework to assess existing and novel financial transactions in their potential to promote hostile activism.[6] Thereby, we provide guidance on which financial transactions need the closest monitoring and, potentially, regulation. Further, our results match and help to better understand differences in empirical findings of decoupling via equity lending markets Christoffersen et al. (2007) and options markets Kalay et al. (2014).

3.1.1 Shareholder voting processes and decoupling techniques

Before we can classify the decoupling techniques and preview our results, we need to provide a short overview of the shareholder voting process and highlight how it is vulnerable to decoupling.

Shareholders can exercise their voting rights in ordinary, annual meetings, and special meetings. Proceedings conducted at a record date, held roughly 30 days prior to the meeting, determine which shareholders are eligible to vote how many shares:[7] doing so, the shareholder structure is locked-in, such that later changes are not taken into account. At the meeting day, decisions are made either with a simple majority or a supermajority.

[4]See https://www.sec.gov/rules/concept/2010/34-62495.pdf.

[5]In Chapter 4, we analyze the pros and cons of Vote Trading techniques as means of activist intervention compared to traditional forms of shareholder activism. In this chapter, we consider Vote Trading techniques as a benchmark.

[6]It is worth pointing out that our classification does not square with the one suggested in the 2010 SEC Concept Release on the U.S. Proxy System, https://www.sec.gov/rules/concept/2010/34-62495.pdf.

[7]The details of the process can vary across countries. However, it is easy to check that the lead time is irrelevant for the outcomes and incentives of the decoupling techniques analyzed.

There are different features of this process that allow an activist investor to decouple her voting power from her economic exposure. First, the allocation of voting rights is agnostic toward coupled assets in the activist's portfolio. For example, the allocation does not take any hedges into account, allowing an activist to shed her economic exposure to retain only the voting right. Further, the shareholder structure is fixed after the record date, such that trades between the record date and meeting do not affect the number of votes a shareholder can cast. By acquiring shares before the record date (cum voting rights) and offloading them right after (ex voting rights), the activist can acquire voting rights without the economic exposure. Even more significant, the number of votes is determined by the temporary possession of the shares. Hence, the activist is eligible to vote borrowed shares, or shares that she has already sold for later delivery at the time of the record date.

Combined, these three features open the possibility for a multitude of decoupling techniques, which can substantially diverge in their economic implications depending on the timing, order of transactions, and counterparties involved. In any of these decoupling techniques, however, the activist has to achieve two goals. First, she has to obtain possession of the shares for the record date, either by buying or borrowing them. In case she purchased the share, she then has to shed the associated economic exposure. This can be done by either selling the shares after the record date or by hedging them. In fact, a hedge can be bought before or after acquiring the shares. Taken together, this gives rise to three classes of decoupling techniques.

Buy&Hedge In the first class of decoupling techniques, the activist buys the shares she wants to vote (prior to the record date) before hedging them. This hedging can be done, for instance, by acquiring options or simply selling the shares after the record date, retaining only the voting rights. In this class of decoupling techniques, the activist assumes positive economic exposure before reducing it again.

Hedge&Buy The second class of decoupling techniques simply flips the order of transactions of Buy&Hedge techniques. By hedging her economic exposure first, the activist is essentially short before acquiring the shares, such that she never takes a long position in the company.

Vote Trading The third class of decoupling techniques is composed of those which are equivalent to the outright trade of voting rights. Essentially, in these techniques, the shares and hedge are both provided by the same shareholder. Thereby, the economic exposure remains with the shareholder at all times, and only the voting rights ever change hand. Most importantly, Vote Trading techniques include the common practice of borrowing shares over the record date (Christoffersen et al., 2007), but also the usage of repos or synthetic assets. For instance, in a repo contract, the shares posted as collateral are already set to be repurchased, such that only the voting rights are reallocated.

3.1.2 Preview of results

To analyze which classes of decoupling techniques can be exploited profitably by a hostile activist to push her private agenda, we consider a simple model in which dispersed shareholders vote on the implementation of a reform. Shareholders know the reform to be value increasing and, thus, support it. The hostile activist, on the other hand, derives a private benefit from the status quo and wants to prevent the reform. The activist's motives are common knowledge.

We find that because the activist's hostile motives are known, she does not benefit from hedging her economic exposure after acquiring the shares (Buy&Hedge technique): any rational and competitive market providing her with a hedge will charge her the fair value, taking into account the activist's motives. Consequently, the hedging market is irrelevant to the activist's incentives. The shares commit her to implement the reform unless her private benefit from the status quo exceeds the loss in share value on the blocking minority of shares. Thus, the outcome of decoupling via a Buy&Hedge technique is constrained efficient.

Still, a hedge may be beneficial to the activist when the order of transitions is flipped, that is when the activist uses a Hedge&Buy technique. By acquiring the hedge first, the activist builds a short position, which commits her to block the reform whenever she gets the chance. If shareholders anticipate that the activist will be successful in acquiring a blocking minority, they are willing to sell their shares at the depressed "no reform"-price. Thereby, shareholders suffer a loss in share value, and the activist can prevent the reform while earning a profit. On the other hand, if shareholders do expect the reform to pass, they demand the high "successful reform"-price, which the activist may not be willing to pay when her private benefit from the status quo is small. Thus, when the activist's private benefit is small, there are two types of self-fulfilling equilibria: ones in which the reform is blocked and ones in which the reform passes.

Last, Vote Trading techniques have a unique equilibrium in which the activist acquires the necessary voting rights at zero prices and always blocks the reform. When employing a Vote Trading technique, the activist essentially bundles the buy and hedge transaction and only trades with the shareholders. Thereby, shareholders always retain the economic exposure and only sell their voting right. When evaluating the offer by the hostile activist, shareholders value their voting right according to their expectation of whether it will change the outcome of the vote. When there are many shareholders, no individual shareholder is pivotal with positive probability, such that the voting right holds no value to him. As a result, there is no monetary transfer from the activist to the shareholders.

In conclusion, we can rank the three classes of decoupling techniques in order of their implications on (shareholder) welfare as

$$\text{Buy\&Hedge} \succ \text{Hedge\&Buy} \succ \text{Vote Trading.}$$

While Buy&Hedge techniques are constrained efficient, Hedge&Buy techniques have two types of equilibria: ones which are constrained efficient and inefficient ones, which allow the hostile activist to block the reform and earn a profit, even when her private benefit from the status quo is small. Vote Trading techniques only have inefficient equilibria and result in the lowest (zero) transfer from the activist to the shareholders.

We also analyze the interaction of decoupling techniques and dual-class structures. In dual-class structures, the activist only has to acquire voting-shares, reducing the economic exposure she has to assume to block the reform. Thereby, dual-class structures foster hostile activism through Buy&Hedge and Hedge&Buy techniques by reducing the private benefit required to make a hostile intervention profitable. In contrast, we find that dual-class structures have no impact on the inefficiency of Vote Trading techniques.

The rest of the chapter is structured into eight sections. After discussing the related literature in Section 3.2, we set up the model in Section 3.3. In Section 3.4 we analyze Buy&Hedge techniques, and in Section 3.5 Hedge&Buy techniques. In Section 3.6 we analyze Vote Trading techniques. We discuss the effect of dual-class structures in Section 3.7, relate our results to previous empirical findings in Section 3.8, and conclude in Section 3.9.

3.2 Literature

The early papers on the optimal design of voting rights in the corporation are primarily concerned with dual-class structures. Grossman and Hart (1988) as well as Harris and Raviv (1988) provide conditions under which a single share class is optimal in corporate control contests. The subsequent literature has also shown that dual-class structures can be useful in the context of corporate takeovers to overcome the free-rider problem (Grossman and Hart, 1980). In particular, non-voting shares can be used to increase private benefits of control (Burkart et al., 1998), or solve problems of asymmetric information (At et al., 2011), thereby enabling value-increasing takeovers. In a model with finitely many shareholders, Gromb (1992) shows that reducing the number of voting shares increases the pivotality probability and, thus, mitigates shareholders' free-riding behavior. For a detailed overview of the literature on dual-class structures, see Burkart and Lee (2008). Recently, Burkart and Lee (2015) demonstrate how synthetic assets can be used to overcome adverse selection problems and free-riding in takeovers.

As far as decoupling techniques go, Vote Trading techniques have received by far the most attention. In the context of corporate governance, Brav and Mathews (2011) and Eso et al. (2015) show that Vote Trading techniques may be beneficial for corporate governance when information about the optimal decision is dispersed. On the other hand, Casella et al. (2012) shows that there is generally no competitive equilibrium in the market for voting rights when market participants have different preferences about the outcome of the vote. Neeman (1999), Bó (2007), and Chapter 4 show in different models that Vote Trading techniques generally lead to inefficiently low vote prices, which can be exploited by a hostile activist. Further, in Chapter 4, we demonstrate that shareholders can learn from activist's willingness to employ a Vote Trading technique but that traditional forms of activist interventions are superior in communicating information. Blair et al. (1989) as well as Dekel and Wolinsky (2012) consider the effect of Vote Trading techniques on control contests. Blair et al. (1989) analyze the effect of taxation on the choice of vehicle by the contestants. Dekel and Wolinsky (2012) prove that Vote Trading techniques can be socially harmful by fostering welfare decreasing takeovers.

Levit et al. (2019) consider a model with heterogeneous shareholder preferences in which shareholders can trade shares before the voting stage. Trading opportunities render the shareholder base endogenous, introducing a feedback loop and self-fulfilling equilibria. In Kalay and Pant (2009), shareholders use the options market as a commitment device to improve their bargaining position in a subsequent control contest. This effect is similar to the one the activist exploits in our model when employing a Hedge&Buy technique.

3.3 Model

Investors Consider a public company owned by a continuum of shareholders with mass 1. Every shareholder owns one share, consisting of a cash-flow claim and a voting right. Further, there is an activist investor who owns no shares. All investors are risk neutral.

Shareholder meeting The company has an upcoming shareholder meeting with a single, exogenously given reform-proposal on the agenda. The vote is binding, and the reform is implemented if at least $\lambda \in (0, 1)$ votes are cast in favor of it. Otherwise, the status quo prevails.

Payoffs If the company sticks with the status quo, the company's total value remains unchanged at $v > 0$; if the reform is implemented, the company's value increases by $\Delta > 0$ to $v + \Delta$. In spite of its positive effect on the firm value, the activist opposes the reform as she gains private benefits $b > 0$ if the status quo remains. These private benefits may, for instance, stem from other assets of her portfolio: debt in the same company reducing the risk appetite or cross ownership leading to different supplier preferences. Alternatively, the status quo may allow the activist to (continue to) extract b at a cost to the firm of $\Delta \geq b$. In any case, we take b to be exogenously given. If $b < \Delta$, the reform increases overall welfare, whereas the status quo is efficient whenever $b > \Delta$.

3.3.1 Voting stage

We ignore the peculiar equilibria in which voters play weakly dominated strategies. Thus, investors always vote in favor of their preferred alternative, and the outcome of the vote is uniquely determined by who owns how many voting rights at the time of the meeting. In the following, we do not explicitly model the voting stage, but only use that the activist can block the reform if she controls at least $(1 - \lambda)$ of the voting rights.

3.4 Buy&Hedge techniques

We first consider the class of decoupling techniques we call "Buy&Hedge" techniques. In this simplest form of decoupling, the hostile activist buys shares from the shareholders and hedges her position afterward, for instance, by procuring put options or reselling the shares after the record date has passed.

3.4.1 Order of transactions

Suppose that the activist can make a public take-it-or-leave-it offer $p \in \mathbb{R}_+$ per share. She can restrict her offer to be valid for m shares she is willing to buy. If more shareholders decide to sell, they are rationed. It is without loss to assume that the activist makes offers for up to $m = 1 - \lambda$ shares.

Shareholders observe the offer p and decide whether they want to sell their share. To capture the predominant anonymity among shareholders, we consider symmetric strategies, denoted by their mixing probability $q : \mathbb{R}_+ \to [0, 1]$.

Having acquired $\min\{q(p), m\}$ shares for p, the activist then has the option to hedge her entire position, guaranteeing her the "successful reform"-value $v + \Delta$. For instance, this can be done by buying put options with a strike price of $v + \Delta$.[8] We

[8] If the activist could choose the strike price and size of the hedge, insuring all of her shares at

assume that the hedging market is rational and competitive, such that the activist needs to pay the fair value.

An explicit overview of the payoffs can be found in Appendix 3.A.1. Here, and henceforth in this chapter, we analyze subgame perfect equilibria.

3.4.2 Hedging stage

Solving the model from the back, suppose that the activist acquired $q^*(p) < 1 - \lambda$ shares in the buying stage. In this case, she cannot swing the decision and the share value is $v + \Delta$. As a result, the hedge is free, and the activist is indifferent between acquiring or not.

Alternatively, suppose that the activist bought the necessary $1 - \lambda$ shares and also the hedge. Then, the value of her portfolio is fixed at $(1 - \lambda)(v + \Delta)$, meaning that it is strictly optimal for her to block the reform. In this case, the hedge has to pay out $(1 - \lambda)\Delta$. The rational and fully informed market providing the hedge expects this and charges $(1 - \lambda)\Delta$ for the hedge. As a result, the activist is, again, indifferent about hedging her shares, and her decision whether to block the reform is unaffected. Consequently, she will only block the reform if $b \geq (1 - \lambda)\Delta$.

Wrapping up, since hedging markets ask for the fair price, the ability to hedge does not affect the activist's payoffs or her decision: the activist will only block the reform in case she acquired $1 - \lambda$ shares (the blocking minority) and $b \geq (1 - \lambda)\Delta$ (blocking is profitable).

3.4.3 Buying stage

When $b < (1 - \lambda)\Delta$, shareholders anticipate that the activist will never block the reform and are not willing to sell their share unless the activist pays them the "successful reform"-price of $v + \Delta$ per share. Therefore, the activist is indifferent between buying the shares and not. In any equilibrium, the reform passes, the firm value rises to $v + \Delta$, and the payoffs of the shareholders and the activist are unchanged.

When $b > (1 - \lambda)\Delta$, shareholders correctly anticipate that the reform is blocked if the activist can acquire sufficiently many shares, $q^*(p) \geq 1 - \lambda$. Depending on how shareholders coordinate, this gives rise to a continuum of equilibria where $p^* \in [v, v + \Delta]$ and reform is always blocked. Details can be found in the proof in the appendix.

$v + \Delta$ would constitute a best response. Note that in contrast to the share market, the activist cannot exploit any potential coordination failure in the market for hedges (e.g. by splitting and randomizing her purchase of options) since non-shareholders make at least zero profits by standard participation constraints.

Proposition 3.1 *Suppose that the activist employs a Buy&Hedge technique,*

- *if $b < (1 - \lambda)\Delta$, the reform passes and the firm value increases to $v + \Delta$ in any equilibrium. The shareholders' and the activist's payoffs are unchanged;*
- *if $b > (1 - \lambda)\Delta$, the reform is blocked and the firm value remains at v in any equilibrium. Shares trade at prices between v and $v + \Delta$, such that the total loss incurred by shareholders is between $\Delta - (1 - \lambda)\Delta$ and Δ. The activist's profit is between b and $b - (1 - \lambda)\Delta$.*

If $b < (1-\lambda)\Delta$, shareholders are fully protected against hostile activism through Buy&Hedge techniques. Absent of asymmetric information, the activist cannot fool the hedging market and is, thereby, stuck with the economic exposure of the shares she seeks to vote. When the private benefit from the status quo is small, these shares commit her to implement the reform.

If $b > (1 - \lambda)\Delta$, the economic exposure of the blocking minority of shares does not commit the activist to implement the reform, such that the reform is blocked. Depending on the coordination among shareholders, their aggregate loss is between $\lambda\Delta = \Delta - (1 - \lambda)\Delta$ and Δ.

Note that the inefficient outcome in case $b > (1-\lambda)\Delta$ and $b < \Delta$ stems from the externality of voting. If a fraction $(1-\lambda)$ of voters were to equally share the benefit $b > (1-\lambda)\Delta$, they would block the reform without any regard to their externality on the other λ voters. In that sense, Buy&Hedge techniques result in efficient outcomes, constrained only by the inefficiency from the voting process itself.

For coherent exposition, we phrase the transaction in which the activist sheds her economic exposure in terms of a hedge, e.g., put options. As we mention in the introduction to this section, the same outcome can be achieved via share sales after the record date. In this case, a competitive and rational outside market will pay the activist the fair value for her share position, anticipating her actions.[9] In particular, when the activist sells all of her shares or none (cf. footnote 8), the outside market will pay her v per share. Therefore, the activist does not benefit from selling her shares, and she only blocks the reform if $b \geq (1 - \lambda)\Delta$.

[9] Alternatively, the activist could sell her shares to existing shareholders. In our model with a continuum of shareholders, existing shareholders have the same willingness to pay for the shares as an outside market. If the number of shareholders was finite, such that their decision whether to buy shares could affect the outcome of the vote, they would pay less: the incumbent shareholders would internalize that, with positive probability, their acquisition encourages the activist to block the reform, reducing the value of their existing share portfolio.

3.5 Hedge&Buy techniques

In this section, we consider "Hedge&Buy" techniques. In this class of decoupling techniques, the hostile activist switches the order of transactions of the Buy&Hedge techniques, such that she uses the hedge to build a short position before acquiring the shares.

3.5.1 Order of transactions

Suppose that the activist can buy a hedge from the outside market that guarantees her a firm value of $v+\Delta$; for instance, in the form of put options with a strike price at $v+\Delta$. It is without loss to assume that she either buys no hedge or insures $(1-\lambda)$ shares (cf. footnote 8). The hedging market is rational and competitive, so that the activist can acquire the hedge for its fair value

After deciding whether to buy a hedge, the activist can make a public take-it-or-leave-it offer $p \in \mathbb{R}_+$ for which she is willing to acquire shares. She can further set an upper bound on the number of shares she is willing to acquire, m. If more shareholders decide to sell their shares, they are rationed. Assume that the activist makes offers for up to $m = 1-\lambda$ shares. The activist conditions her offer on whether she acquired a hedge, such that her strategy becomes $p : \{0, 1-\lambda\} \to \mathbb{R}_+$.

Shareholders observe whether the activist hedged her position as well as the offer p and decide whether they want to sell their share. We denote shareholders' symmetric strategy by $q : \{0, 1-\lambda\} \times \mathbb{R}_+ \to [0, 1]$.

An explicit overview of the payoffs is in Appendix 3.A.1.

3.5.2 Buying stage

In the body of text, we solve the game when the activist's private benefit is small, $b < (1-\lambda)\Delta$. The solution to the game with a large private benefit, $b > (1-\lambda)\Delta$, can be found in the proof to Proposition 3.2 in the appendix. Again we solve the game from the back.

The activist can only block the reform in case she offers a price p such that shareholders sell with probability $q^*(\cdot, p) \geq (1-\lambda)$. Further, she only wants to do so if she hedged her position beforehand. Otherwise, the economic exposure of the shares commits her to implement the value-increasing reform (cf. Section 3.4.2). If the activist does not own a hedge, shareholders know that the activist will implement the reform and demand the "successful reform"-price of $v+\Delta$. Thus, when the activist owns no hedge, the reform passes, the activist is indifferent between acquiring the shares or not, and her payoff is 0.

Now, suppose that the activist hedged her shares which commits her to block the reform. Shareholders anticipate this and base their decision whether to sell on the other shareholders' equilibrium decision. Since no shareholder is pivotal with positive probability, it is optimal for any shareholder to sell his share if $p \geq v$ and $q^*(1-\lambda, p) \geq 1-\lambda$, so that the reform is blocked, or whenever $p \geq v + \Delta$.[10] Not selling is optimal for the shareholder whenever $p \leq v + \Delta$ and $q^*(1-\lambda, p) < 1-\lambda$, meaning that the reform passes. The activist, on the other hand, has an incentive to pay any price $p \leq \frac{b}{1-\lambda} + v + \Delta$ as long as $q^*(1-\lambda, p) \geq (1-\lambda)$ because this provides her with a payoff of

$$V_{hedge}(p) = b + (1-\lambda)v + \underbrace{(1-\lambda)\Delta}_{\text{payout hedge}} - (1-\lambda)p > 0,$$

whereas any price p such that $q^*(1-\lambda, p) < (1-\lambda)$ results in a payoff of at most zero. Since a price $p > v + \Delta$ guarantees her $q^*(1-\lambda, p) \geq 1-\lambda$, the activist will always choose a price p^* such that $q^*(1-\lambda, p^*) \geq 1-\lambda$. This gives rise to a continuum of equilibria in the buying subgame in which the activist owns a hedge. For any $p^* \in [v, v+\Delta]$ there is an equilibrium in which $q^*(1-\lambda, p^*) \geq 1-\lambda$ and $q^*(1-\lambda, p) < (1-\lambda)$ for all $p < p^*$. Consequently, the value from owning a hedge is $V_{hedge}(p^*) \in [b, b + (1-\lambda)\Delta]$.[11]

Combined, there are two outcomes of the buying subgame: when the activist did not acquire a hedge, she does not block the reform and her payoff is 0. In case she did buy a hedge, she always blocks the reform and her payoff is $V_{hedge}(p^*) \in [b, b + (1-\lambda)\Delta]$.

3.5.3 Hedging stage

If the activist decides to buy a hedge, she will always block the reform, such that the sellers of the hedge incur a loss of $(1-\lambda)\Delta$. The rational outside market anticipates this and demands the fair value for the hedge, $(1-\lambda)\Delta$.

As a result, it only pays for the activist to buy a hedge and block the reform in case the value from owning a hedge is $V_{hedge}(p^*) \geq (1-\lambda)\Delta$. Since $b < (1-\lambda)\Delta$, this means that there are two types of equilibria, depending on the equilibrium of the subsequent subgame: when $V_{hedge}(p^*) > (1-\lambda)\Delta$, the activist acquires the hedge and blocks the reform, whereas if $V_{hedge}(p^*) < (1-\lambda)\Delta$, she does not buy the hedge and the reform is implemented.

[10] If $q^*(1-\lambda, p) \leq 1-\lambda$ and $p \geq v+\Delta$, selling shareholders are not rationed and any shareholder is better off selling. If $q^*(1-\lambda, p) \geq 1-\lambda$, the reform is blocked which is compatible with any price $p \geq v$.

[11] Note that $p^* \leq v + \Delta$ because at any $p > v + \Delta$, $q^*(1-\lambda, p) = 1$, meaning that the activist is strictly better off lowering her offer to $p' = \frac{p+v+\Delta}{2}$.

Proposition 3.2 *Suppose that the activist employs a Hedge&Buy technique,*

- *if $b < (1-\lambda)\Delta$, there are two types of equilibria:*

  1. *either the activist buys the hedge for $(1-\lambda)\Delta$, acquires $(1-\lambda)$ shares, and blocks the reform. In this case, the firm value remains at v. Shares trade at prices between v and $v + \frac{b}{1-\lambda}$, such that the total loss incurred by shareholders is between $\Delta - b$ and Δ. The activist's profit is between b and 0.*
  2. *or the activist does not buy a hedge, so that the reform passes and the firm value increases to $v+\Delta$. The shareholders' and the activist's payoffs are unchanged.*

- *if $b > (1-\lambda)\Delta$, the reform is blocked and the firm value remains at v in any equilibrium. Shares trade at prices between v and $v + \Delta$, such that the total loss incurred by shareholders is between $\Delta - (1-\lambda)\Delta$ and Δ. The activist's profit is between b and $b - (1-\lambda)\Delta$.*

Since the hedging market anticipates the activist's actions, it charges the correct fair value for the hedge. Thus, the activist does not benefit directly from hedging her shares (cf. equation (3.1)). Nevertheless, acquiring a hedge before the shares can be beneficial for her because it ensures that the activist never holds a long position. Whereas in a Buy&Hedge technique, the interim ownership of the shares commits the activist with a low private value, $b < (1-\lambda)\Delta$, to pass the reform, buying the hedge first lifts this commitment. This gives rise to two types of self-fulfilling equilibria when $b < (1-\lambda)\Delta$.

In both types of equilibria, conditional on owning a hedge, the activist offers a price p^* such that she acquires the blocking minority of shares, $q^*(1-\lambda, p^*) \geq 1-\lambda$. Thus, if the activist buys the hedge and prevents the reform, her ex ante payoff is

$$(1-\lambda)v + b - (1-\lambda)p^* + \underbrace{(1-\lambda)\Delta}_{\text{payoff hedge}} - \underbrace{(1-\lambda)\Delta}_{\text{price hedge}}. \tag{3.1}$$

However, only when $p^* < v + \frac{b}{1-\lambda}$, it pays for the activist to buy the hedge and the blocking minority of shares. This is the first type of equilibrium. In the other type of equilibrium, $p^* > v + \frac{b}{1-\lambda}$, meaning that the activist's profits from acquiring the shares and blocking the reform do not suffice to cover the cost of the hedge, preventing her from doing so.

When $b > (1-\lambda)\Delta$, the case we mostly ignored in this section, the result is unchanged relative to the result of the Buy&Hedge technique. Since the activist has an incentive to prevent the reform independent of a hedge, the reform is blocked in any equilibrium and the price the activist pays is $p^* \in [v, v+\Delta]$, as in Section 3.4.

3.6 Vote Trading techniques

Last, we turn to the class of decoupling techniques that are equivalent to the outright trade of voting rights, such as borrowing shares over the record date via the equity lending market. A more thorough analysis with a finite number of shareholders can be found in Chapter 4. Here, we keep the analysis Vote Trading techniques short and treat it primarily as a benchmark.

Suppose that before the record date, the activist can make a public take-it-or-leave-it offer $p \in \mathbb{R}_+$ per voting right.[12] Shareholders observe the offer and sell their voting right with probability $q : \mathbb{R}_+ \to [0, 1]$.

Appendix 3.A.2 provides an explicit overview of the payoffs.

Proposition 3.3 *In any equilibrium, the activist offers $p^* = 0$, shareholders sell with probability $q^*(0) \geq 1 - \lambda$, and the activist always blocks the reform.*

When the activist employs a Vote Trading technique, the economic exposure never leaves the original shareholders. Hence, the activist only needs to compensate shareholders for their voting rights. Since there are many shareholders, they correctly anticipate that their individual sale is not going to change the outcome of the vote, such that shareholders do not value their voting rights—the curse of pivotality. Thus, they are willing to sell their voting rights at any positive price. The activist, on the other hand, never assumes economic exposure herself, making it optimal for her to block the reform, independent of her private value, $b > 0$. As a result, the activist can always acquire the voting rights for free and prevent the reform.

3.7 Dual-class structures

Up to now, we assumed that all shares are identical voting shares. To also cover dual-class structures, suppose there are $\phi \in (0, 1]$ voting and $1 - \phi$ non-voting shares. Every shareholder holds either one or the other. Given the dual-class structure, the activist can block the reform if she controls $(1 - \lambda)\phi$ shares.

Corollary 3.1 *All previous results remain valid for dual-class structures when replacing $(1 - \lambda)$ by $(1 - \lambda)\phi$.*

The proofs hold verbatim, replacing $(1 - \lambda)$ by $(1 - \lambda)\phi$. In dual-class structures, holders of non-voting shares get no say in the outcome of the vote, meaning that the inefficiency of voting increases: if $(1 - \lambda)\phi$ shareholders prefer a particular course of action, they ignore the effect on the $(1 - \phi) + \lambda\phi$ minority. As a result, Buy&Hedge techniques, as well as the first type of equilibria in Hedge&Buy techniques, remain

[12] The activist might restrict her offer to $(1 - \lambda)$ voting rights as in the other decoupling techniques analyzed, but this does not affect the results.

constrained efficient given the inefficiency of voting in dual-class structures. Still, the private benefit required for a hostile activist to profit from blocking the reform decreases from $(1-\lambda)\Delta$ to $\phi(1-\lambda)\Delta$. Further, the total compensation to shareholders decreases. Vote Trading techniques, on the other hand, are unaffected by dual-class structures.[13]

Note that our analysis of Buy&Hedge techniques concluded that hedging after the acquisition of shares is never strictly profitable, such that the Buy&Hedge techniques are, essentially, "Buy" techniques. Thereby, the results for the Buy&Hedge techniques also cover the simple form of hostile activism in which the activist blocks the reform through the acquisition of (few) voting shares.

3.8 Empirical implications

Our model predicts that the (implicit) prices for voting rights vary substantially, depending on the decoupling technique employed. When voting rights are acquired via a Vote Trading technique, prices are zero. This is in line with the empirical evidence from the equity lending market, which finds a significant trade volume and close to zero prices (Christoffersen et al., 2007).[14] Turning to Buy&Hedge and Hedge&Buy techniques, when $b < (1-\lambda)\Delta$, Buy&Hedge techniques are not profitable for the activist. Depending on the equilibrium selection, however, the activist may be able to block the reform using a Hedge&Buy technique. In this case, the implicit price of a voting right, i.e. the difference between the price offered by the activist and the value of the cash flow entitlement, is between 0 and $\frac{b}{1-\lambda}$. When $b > (1-\lambda)\Delta$, Buy&Hedge techniques as well as Hedge&Buy techniques, allow the activist to block the reform. Here, the implicit price of a voting right is between 0 and Δ, depending on the equilibrium selected. The positive prices are consistent with the findings by Kalay et al. (2014) who detect a spike in the options trading around the record date and find that the implicit prices for voting rights derived from options are strictly positive.

Moreover, our results show that hostile activism via Buy&Hedge techniques and Hedge&Buy techniques are particularly likely when λ is large, that is when the reform requires a supermajority. This is in line with most of the cases collected by Hu and Black (2008a), which predominantly involved supermajority decisions.

[13] In the context of corporate takeovers, Hart (1995) points out that dual-class structures are irrelevant if voting rights and cash flow claims can be unbundled.

[14] Christoffersen et al. (2007) attribute their findings to the supposedly common interests of shareholders. However, this explanation seems to be at odds with the evidence by Hu and Black (2008a). As we argue more extensively in Chapter 4, low prices are the result of a market failure in the market for voting rights and not necessarily a sign of aligned interests.

3.9 Conclusion

Our analysis focuses on hostile activism in an environment without hidden motives. Thereby, we seek to bound the threat of hostile activism through decoupling techniques and abstract from any inefficiencies stemming from asymmetric information.[15] We find that the three classes of decoupling techniques can be ranked in terms of their implications on shareholder and overall welfare as

$$\text{Buy\&Hedge} \succ \text{Hedge\&Buy} \succ \text{Vote Trading.}$$

When $b < (1-\lambda)\Delta$, the activist cannot use a Buy&Hedge technique to block the reform, such that overall welfare is maximized. Hedge&Buy techniques, on the other hand, have two types of equilibria: equilibria in which the reform passes, reducing shareholder and overall welfare, and equilibria in which the reform is blocked. Thus, the result is ambiguous and relies on equilibrium selection. Last, Vote Trading techniques always result in a blocked reform and zero transfer to the shareholders. Therefore, this class of decoupling techniques is the worst in terms of shareholder and overall welfare.

When $b > (1-\lambda)\Delta$, all three classes of decoupling techniques allow the activist to block the reform. However, Vote Trading techniques guarantee that there is zero transfer from the activist to the shareholders, whereas Buy&Hedge as well as Hedge&Buy techniques can result in strictly positive transfers.

By ranking the three classes of decoupling techniques, we provide insights into which current and future transactions need the most rigorous monitoring and, potentially, regulation.[16] Further, we find that dual-class structures increase the threat of hostile activism via Buy&Hedge and Hedge&Buy techniques, whereas Vote Trading techniques, already least efficient, remain unaffected. Last, we note that simple majority rules are most robust to hostile activism via Buy&Hedge and Hedge&Buy techniques by maximizing the constrained-efficient parameter regions and minimize the loss to shareholders, independent of the labeling of the options.

[15] Whereas activists with an aligned agenda have ample opportunity to communicate and verify their best interests to implement value-increasing reforms, hostile activists must rely on methods that allow them to gain control of the company without bearing the full economic costs. Thus, while decoupling may also aid friendly activists, hostile activists set the benchmark for the efficiency loss from decoupling, cf. Chapter 4.

[16] For instance, our results show that share-blocking systems which prevent one type of Buy&Hedge technique have no benefit when there is no asymmetric information.

Appendices

3.A Payoffs

3.A.1 Buy&Hedge and Hedge&Buy techniques

Shareholders When the activist offers p per share, a shareholder who sells his share and is not rationed receives a payoff of p. If the shareholder is rationed or rejects the offer, his payoff is equal to the firm value: if the reform is implemented it is $v + \Delta$, if the status quo remains it is v.

Activist If the activist does not buy a hedge, offers p per share, and receives $q(p)$ of the shares, her payoff is

$$\min\{q(p), 1 - \lambda\}(v - p) + b$$

in case she blocks the reform (which requires $q(p) \geq 1 - \lambda$), and

$$\min\{q(p), 1 - \lambda\}(v + \Delta - p)$$

when she does not block the reform.

If the activist buys a hedge for p_h, offers p per share, and receives $q(p)$ of the shares, her payoff is

$$\min\{q(p), 1 - \lambda\}(v - p) + b + (1 - \lambda)\Delta - p_h$$

when she blocks the reform (which requires $q(p) \geq 1 - \lambda$), and

$$\min\{q(p), 1 - \lambda\}(v + \Delta - p) - p_h$$

in case she does not.

Note that depending on the timing, in the second stage of the game, either the cost of the hedge, p_h, or the cost of the shares, $p \min\{q(p), 1 - \lambda\}$, are sunk.

3.A.2 Vote Trading techniques

Shareholders When the activist offers p per voting right, a shareholder who sells his voting right and is not rationed receives a payoff of p plus the firm value: if the reform is implemented, his total payoff is $p + v + \Delta$, if the status quo remains, it its $p + v$. If the shareholder is rationed or rejects the offer, his payoff is equal to the firm value v or $v + \Delta$, respectively.

Activist If the activist offers p per voting right and receives $q(p)$ of the voting rights, her payoff is

$$b - q(p)p$$

when she blocks the reform (which requires that $q(p) \geq 1 - \lambda$), and

$$-q(p)p$$

in case she does not.

3.B Proofs

3.B.1 Proof of Proposition 3.1

The case in which $b < (1-\lambda)\Delta$ is covered in the body of the text.

Since the outside market charges the fair value for the hedge, the activist is indifferent between hedging her position and not, and because $b > (1-\lambda)\Delta$, she always blocks the reform. Shareholders anticipate this. Since no shareholder is pivotal, they are willing to sell their shares for v if they anticipate that the activist will block the reform, $q^*(p) \geq 1-\lambda$, or require $v+\Delta$ if they anticipate that the activist will not block the reform, $q^*(p) < 1-\lambda$. This means that when $q^*(p) < 1-\lambda$ but $p \leq v+\Delta$, they are (weakly) better off not selling, such that $q^*(p) \leq 1-\lambda$ is a best response. If $q^*(p) \geq 1-\lambda$ and $p \geq v$, they are (weakly) better off selling, such that $q^*(p) \geq 1-\lambda$ is a best response.

Since $b > (1-\lambda)\Delta$, the activist makes a strict profit by offering p marginally above $v+\Delta$, where $q^*(p) = 1$. Therefore, it cannot be that the equilibrium price p^* is such that $q^*(p^*) < 1-\lambda$ and the activist makes (weakly) negative profits. Further, it has to hold that $p^* \leq v+\Delta$. Otherwise, $p' = \frac{p^*+v+\Delta}{2}$ would always be a profitable deviation. Thus, the equilibrium price has to be $p^* \leq v+\Delta$ and $q^*(p^*) \geq 1-\lambda$, which implies that $p^* \geq v$.

The continuum of equilibria can be constructed by fixing any $p^* \in [v, v+\Delta]$. If $q^*(p^*) = 1$ and $p^* \geq v$, then selling is a best response for shareholders. For all $p < p^*$ and $q^*(p) = 0$, not selling is a best response. Since the activist chooses the lowest p such that $q^*(p) \geq 1-\lambda$, the result follows.

3.B.2 Proof of Proposition 3.2

The case in which $b < (1-\lambda)\Delta$ is covered in the body of the text.

If $b > (1-\lambda)\Delta$ and the activist acquired $(1-\lambda)$ shares, the activist always blocks the reform, independent of any hedge. Let her payoff from the buying stage be $W_{hedge}(p_h^*)$ in case she owns a hedge and $W_{nohedge}(p_{nh}^*)$ in case she does not own a hedge.

If p^* is such that $q^*(\cdot, p^*) \geq 1-\lambda$, then the activist's payoff from paying p^* is $W_{hedge}(p^*) = V_{hedge}(p^*)$ and $W_{nohedge}(p^*) = V_{hedge}(p^*) - (1-\lambda)\Delta$. Note that for p marginally above $v+\Delta$, it must be true that $q^*(\cdot, p) = 1$ such that $W_{hedge}(p) >$

$(1-\lambda)\Delta$, and $W_{nohedge}(p) > 0$. This means that in equilibrium, it has to hold for $p^* \in \{p_h^*, p_{nh}^*\}$ that $q^*(\cdot, p^*) \geq 1-\lambda$. Otherwise, the activist would make a (weakly) negative profit and could profitably deviate to a p marginally above $v+\Delta$. Further, because $p > v+\Delta$ guarantees $q^*(\cdot, p) = 1$, it follows that $p^* \leq v+\Delta$. Otherwise, the activist could always lower her offer to $p' = \frac{p^*+v+\Delta}{2}$ and achieve the same outcome at lower cost. Thereby, the equilibrium price has to be $p^* \leq v+\Delta$ and $q^*(\cdot, p^*) \geq 1-\lambda$, which implies that $p^* \geq v$.

For any $p^* \in [v, v+\Delta]$, there is an equilibrium in which $q^*(\cdot, p^*) \geq 1-\lambda$ and $q^*(\cdot, p) < 1-\lambda$ for all $p < p^* \leq v+\Delta$. Given that $p \leq v+\Delta$, if $q^*(\cdot, p) < 1-\lambda$, shareholders anticipate that the reform will pass and are (weakly) better off not selling. If $p \geq v$ and $q^*(\cdot, p) \geq 1-\lambda$, shareholders anticipate that the reform will pass and are (weakly) better off selling. As a result, there is a continuum of continuation payoffs: $W_{hedge}(p_h^*) \in [b, b+(1-\lambda)\Delta]$ and $W_{nohedge}(p_{nh}^*) \in [b-(1-\lambda)\Delta, b]$.

The outside market correctly anticipates that a hedged activist blocks the reform and charges the fair value $(1-\lambda)\Delta$ for the hedge. The activist buys it, depending on the value of the continuation game. Thus, the hedge has no direct effect on the activist's payoff, but may affect it through equilibrium selection in the continuation game. Taken together, the payoff of the activist is between $b-(1-\lambda)\Delta$ and b.

3.B.3 Proof of Proposition 3.3

Since no shareholder is pivotal with positive probability, the vote's outcome is independent of any individual shareholder's sale. As a result, no shareholder values his voting right, such that $q^*(p) = 1$ for any $p > 0$. It follows that $p^* = 0$. Otherwise, $p' = \frac{p^*}{2} > 0$ would be a profitable deviation for the activist because p' would also guarantee her the voting right, $q^*(p') = 1$, but at a lower cost. Further, $q^*(0) \geq 1-\lambda$. If it was the case that $q^*(0) < (1-\lambda)$, the activist would make zero profits. Hence, she could profitably deviate to a price p marginally above 0 at which $q^*(p) = 1$, securing her all the voting rights at essentially zero cost, thereby guaranteeing her a profit.

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Chapter 4

Shareholder Votes on Sale

Joint with Paul Voß

4.1 Introduction

Shareholder voting is one of the cornerstones of corporate governance. It equips shareholders with the power to enforce their demands, laying the foundation for shareholder activism. Typically, a shareholder's voting rights are determined by her shares on a pro-rata basis—one share, one vote—thereby linking a shareholder's influence to his economic interest. However, activist investors can subvert this principle by acquiring voting rights far in excess of their cash flow claims. While the outright trade of voting rights is illegal in most jurisdictions, financial innovation has created new techniques to decouple voting power and economic exposure—for instance, via the equity lending market. Activist investors were happy to add these new techniques to their toolbox,[1] whereas the decoupling raised eyebrows among policymakers[2] and the press.[3]

In this chapter, we analyze how decoupling techniques relate to traditional forms of shareholder activism, and examine the consequences for corporate governance. We focus on the class of decoupling techniques that are economically equivalent to the outright trade of voting rights, that is, the class of Vote Trading techniques (cf. Chapter 3). In the remainder of the chapter, we simply refer to (the usage of) these techniques as *vote trading*. Importantly, this class includes the most common practice of acquiring voting rights by borrowing shares over the record date (Christoffersen et al., 2007). Our analysis reveals that vote trading unilaterally

[1]Hu and Black (2006, 2008a,b, 2007) document anecdotal evidence of decoupling. Henceforth, we reference Hu and Black (2008a) as the most recent overview.

[2]Consider, for example, the "SEC Concept Release on the U.S. Proxy System" (July 2010), https://www.sec.gov/rules/concept/2010/34-62495.pdf, the "SEC Staff Roundtable on the Proxy Process" (July 2018), https://www.sec.gov/news/public-statement/statement-announcing-sec-staff-roundtable-proxy-process, or the ESMA's "Call for evidence on empty voting" (September 2011), https://www.esma.europa.eu/press-news/consultations/call-evidence-empty-voting.

[3]The New York Times, April 26, 2012, "The Curious Case of the Telus Proxy Battle", https://dealbook.nytimes.com/2012/04/26/the-curious-case-of-the-telus-proxy-battle/.

benefits hostile activists and is not needed for friendly activists to guide corporate decision making as they can rely on traditional intervention techniques such as proxy campaigns.[4]

In a first analysis, we build a simple model in which a finite number of shareholders vote on the implementation of a reform. Shareholders know the reform to be value increasing and, thus, support it. In this setting, there is no need for value-increasing activism. Therefore, we concentrate on the case in which a hostile activist who derives private benefits from the company sticking with the status quo. Shareholders are fully aware of the activist's motives.

We show that despite the activist's transparent motives, the activist can acquire voting rights at prices close to zero and prevent the value increasing reform. This is the result of a market failure in the market for voting rights. The value of a voting right depends on the trading and voting decisions of the other market participants: it only bears value if it is decisive (pivotal) in the outcome of the vote, which is unlikely for any individual voting right. Thus, rational shareholders are willing to sell their voting rights at a price significantly below their individual loss from the blocked reform. This allows the hostile activist to block the value-increasing reform without compensating shareholders.

Competition in the market for voting rights does not fix the market failure and, hence, does not prevent hostile activism. Even if a blockholder is willing to act as a white knight and make a competing offer, he may be at a disadvantage depending on the majority rule. In particular, if the reform requires a supermajority to pass, it may be too expensive for the blockholder to acquire the necessary fraction of voting rights. Therefore, competition reduces the threat of hostile activism, but inefficient outcomes remain.

Our results give a new interpretation of the empirical and anecdotal observations on vote trading. Christoffersen et al. (2007) find that voting rights trade at near-zero prices, which they attribute to common interests of investors. On the other hand, Hu and Black (2008a) present anecdotal evidence of vote trading which yields—prima facie—inefficient outcomes. We reconcile these two seemingly contradictory findings in that we show that low prices need not be a sign of common interests, and inefficient outcomes do not require hidden motives. Instead, our analysis suggests that low prices are caused by a more fundamental market failure. Further, the competitive advantage of a hostile activist in supermajority decisions delivers an explanation for the disproportionate occurrence of vote trading in these decisions, as documented by Hu and Black (2008a).

[4]Our results imply that activist chooses her intervention method as a function of her motives. Hence, the model explains why studies investigating "traditional" shareholder activism (such as Brav et al. (2008)) find positive effects of activism on shareholder value, whereas the evidence on vote trading suggests adverse effects on shareholder value (Hu and Black, 2008a).

In a second step, we consider the more complex setup in which the activist possesses superior information about the effect of the reform. We ask the question of whether vote trading may be advantageous for corporate governance by fostering information transmission from the activist to other shareholders,[5] and we compare vote trading to other traditional forms of activist interventions. To this end, we extend the model by an uncertain state that determines whether the reform proposal increases or decreases shareholder value. The activist privately knows the state.

If the activist and shareholders have aligned interests, that is, if the activist's private benefit from the status-quo is negligible, vote trading is not necessary for information transmission: the activist can also communicate her superior information via cheap talk, such as public endorsements.

We, thus, focus on the case in which the activist's private benefit from the status quo leads her to oppose the reform in either state, preventing cheap talk. Interestingly, despite the misaligned interests of shareholders and activist, vote trading can facilitate information, and improve firm value in this situation. Shareholders can learn from the activist's vote acquisition: when the activist is endowed with some shares, her willingness to pay for the voting rights is correlated with the state. This gives rise to a separating equilibrium in which the activist prevents the reform more often when it is in the shareholders' interest, increasing firm value. However, the ability to improve shareholder value depends on significant prices for voting rights since those are needed as a costly signal. When shareholdings are dispersed, the emerging low prices prevent an informational benefit. Absent of vote trading, the activist might use other costly signals to achieve the same, or even superior outcomes. Activist investors' traditional methods—the acquisition of a minority stake in the company, or costly proxy fights, for example—can achieve first-best communication, independent of the shareholder structure.

We conclude that vote trading benefits only hostile activists because they cannot rely on traditional forms of activist interventions. As a result, vote trading threatens corporate governance and shareholder value. This is true even in the (unlikely) best-case scenario in which shareholders are fully informed about the activist's motives. Thus, we advocate the regulation of vote trading.

Because inefficient outcomes from the market for voting rights occur even when motives are transparent, policy measures aimed at increasing transparency are not sufficient to restore efficiency and prevent hostile activism. At the same time, vote trading often emerges as a byproduct, such that banning transactions that may be used for vote trading is costly. Consequently, we recommend policy measures that regulate the eligibility to vote. In particular, we propose regulating entities instead

[5] The informational advantage of vote trading is stressed by Brav and Mathews (2011) as well as Eso et al. (2015).

of securities. That is, we argue that any entity who acquires voting rights through vote trading (through a Vote Trading technique, cf. Chapter 3) should not be eligible to vote. Further, our analysis reveals that decisions taken by supermajority rule are especially likely to be blocked by hostile activists. Consequently, our model suggests that simple majority voting helps to prevent hostile activism.

4.1.1 Trading votes for shareholder meetings

In this chapter, we analyze the empirically most relevant decoupling techniques,[6] which are the ones that are economically identical to the outright trade of voting rights (Vote Trading techniques, cf. Chapter 3). For simplicity, we refer to (the usage of) these techniques as *vote trading*. When engaging in vote trading, the activist trades directly with the shareholders, and the economic exposure remains with the shareholders at all times. Only the voting right changes hands for a flat transfer.

In practice, the bulk of vote trading occurs via the equity lending market. Since the possession of a share at the record date suffices to obtain the voting right, an activist investor seeking to acquire voting rights only needs to borrow the shares she wants to vote over the record date. When the lending fee is independent of the share value, as is usually the case, the shareholder (lender) retains the economic exposure and only sells the voting right. The lending fee captures the cost of acquiring the voting right. Alternatively, the same outcome can be achieved through a repo contract in which the cash-providing side (the activist) obtains the shares for a limited amount of time, before selling it back to the collateral providing side (shareholder) at pre-negotiated terms. Again, the initial shareholder fully retains the economic exposure, whereas the activist only secures her right to vote, at a flat price. Last, it is easy to design synthetic assets that are economically equivalent to vote trading.[7]

4.1.2 Empirical insights from the equity lending market

Christoffersen et al. (2007) provide the first evidence of vote trading via the equity lending market. They find that a significant spike in the volume of share lending over the record date. Kalay et al. (2014) validate this result with a different estimation

[6] Financial innovation has created a multitude of decoupling techniques that diverge in their economic implications depending on the timing, order of transactions, and counterparties. For a more detailed account of the shareholder voting process and an overview over (other) decoupling techniques, see Chapter 3.

[7] For instance, the activist could engage in voting trading by buying synthetic calls, i.e. bundles of shares and a put option, from the shareholder. If the put option is at the money, the activist can exercise it right after the record date, such that she only retains the voting right. In case the activist is hostile and seeks to reduce share value, she will always exercise the option and the economic exposure remains with the shareholder.

approach.[8] Hu and Black (2008a), collect anecdotal evidence of decoupling between 1988 and 2008. They register over 40 decoupling cases, many of which rely on share lending. In those cases, the additional voting rights were used to influence decisions over diverse issues, ranging from management entrenchment to takeover approvals. The practice continues to be popular with activists, as the recent fight for control of Premier Foods (2018) highlights.[9] Arguably, one of the reasons for this popularity of the equity lending market as a platform for vote trading is its size and liquidity. Within the U.S. stock market, for instance, an average of 20% of a company's shares is available for borrowing (Campello et al., 2019).[10][11]

Besides providing empirical evidence of an active and sizable market for voting rights, Christoffersen et al. (2007) and Kalay et al. (2014) also estimate the market price of voting rights. Christoffersen et al. (2007) find no significant prices for voting rights, whereas Kalay et al. (2014) estimate significant but small prices. Christoffersen et al. (2007) interpret their findings as a sign of common values. Because all investors supposedly share the same interests, there is no need to charge positive prices for voting rights. Instead, investors are willing to delegate their voting rights to more informed parties. This argument, however, seems to be at odds the findings of Hu and Black (2008a); most of the their cases resulted in—prima facie—–inefficient outcomes and reduced shareholder value. While different in detail, the cases share a common feature in that voting rights acquired by a single hostile activist were used to block supermajority decisions.

Our theory reconciles the empirical findings of positive trading volume, low prices, and inefficient outcomes. We show that a market failure in the market for voting rights leads to low prices, enabling hostile activism. Those inefficient outcomes do not require hidden motives by the activist.[12]

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[8]Kalay et al. (2014) focus on decoupling techniques that work via the options market and are not equivalent to the outright trade of voting rights (i.e. no Vote Trading techniques, cf. Chapter 3), i.e. the class of decoupling techniques analyzed in this chapter. However, they also use their methodology to analyze data from the equity lending market.

[9]Financial Times, July 15, 2018, "Market reverberates with accusations of empty voting", https://www.ft.com/content/0e28929e-85dd-11e8-a29d-73e3d454535d.

[10]With the continuing growth in popularity of ETFs, which use share lending as an integral part of their business model, the size of this market is likely to expand—see, for example, Deutsche Bundesbank Monthly Report, October 2018, https://www.bundesbank.de/resource/blob/766600/2fd3ae4f0593fb2ce465c092ce40888b/mL/2018-10-exchange-traded-funds-data.pdf.

[11]Campello et al. (2019) show that companies try to limit the number of lendable shares with share buybacks, and argue that they do so to limit short-selling opportunities. Our results give another rationale for the buyback—namely that placing a limit on the number of lendable shares limits the number of votes that can be bought via the equity lending market.

[12]Hu and Black (2007) point out that there may be other issues, such as lack of transparency in the market for voting rights and pivotality considerations. We pick up on this issue of pivotality and formalize it.

The chapter is structured into five sections. Section 4.2 reviews the literature. In Section 4.3 we show that vote trading in a symmetric information setting uniquely benefits a hostile activist who can exploit a market failure in the market for voting rights. In Section 4.4 we investigate the effect of vote trading when the activist has superior information about the correct course of action. We compare vote trading with traditional forms of activist interventions. We draw conclusion from our findings in Sections 4.5, before developing policy recommendations in Section 4.6. All proofs are relegated to the appendix.

4.2 Literature

Decoupled economic interest and voting power has been studied in the context of dual-class share structures and takeovers. Grossman and Hart (1988) as well as Harris and Raviv (1988) provide conditions under which a single share class is optimal. Gromb (1992) proves that reducing the number of voting shares increases shareholders' likelihood of being pivotal, thereby reducing shareholders' free-riding incentives. Burkart et al. (1998) shows that if private benefits are an endogenous choice by the winning bidder after the takeover, reducing the number of voting shares necessary for control can increase welfare. When bidders have private information about the post-takeover value of the firm, At et al. (2011) show that dual-class shares can facilitate value-increasing corporate takeovers. For a detailed overview of the theoretical literature on dual-class shares and takeovers, compare Burkart and Lee (2008). Adams and Ferreira (2008) summarize the empirical literature on dual-class shares, stock pyramids, and cross-ownership. They find that the value of voting rights differs substantially across countries, time frames, and analysis, but can be quite significant. However, trading dual-class shares to decouple voting rights and economic interests is not equivalent to the outright trade of voting rights (i.e. no Vote Trading technique, cf. Chapter 3), such that it has different economic implications.

Burkart and Lee (2015) show how the free-rider problem and asymmetric information can be overcome by the usage of option contracts. In the context of contests for corporate control, Dekel and Wolinsky (2012) find that allowing for vote trading in addition to share trading may increase the probability that an inefficient bidder takes over the company. Neeman and Orosel (2006) consider a repeated control contest among an incumbent manager and a challenger in which vote trading can be used as a signaling device. Blair et al. (1989) analyze the effect of taxation in a takeover contest where shares and votes can be traded separately. In political economy, Dekel et al. (2008) consider a contest between two political party's which can either buy votes or bribe voters. The authors find that overall payments are substantially higher when parties can pay bribes. Dekel et al. (2009) introduce budget constraints to this setting. Their model is related to our competition game, as we discuss in Section 4.3.3. Casella et al. (2012) demonstrate that the market for voting rights does not have a competitive equilibrium; thus, they introduce an "ex-ante vote trading equilibrium." They identify conditions under which vote trading fails to aggregate preferences and generates welfare losses relative to simple majority voting.

Neeman (1999) and Bó (2007) argue that a single buyer can acquire voting rights at zero prices. Neeman (1999) shows that when the number of voters grows large, a zero-price equilibrium is the only pure strategy equilibrium robust to noise voters. Bó (2007) shows that when an activist can write arbitrary, outcome-dependent contracts, she can bribe voters to vote for her at zero cost.

Brav and Mathews (2011) examine the effects of vote trading in the presence of an informed activist who can either buy shares or sell them short. Shareholders are no strategic players, but are noise voters. By assumption, the activist can acquire a certain fraction of their voting rights for free. The activist is more likely to be pivotal when she has aligned interests because additional shares also provide her with additional voting rights. As a consequence, vote trading increases the expected welfare. In Eso et al. (2015), only shareholders with (conditionally) aligned interests participate in the market for voting rights. They use the market as a way to delegate their voting rights to the most informed parties, aiding information aggregation and ensuring that partisans are out-voted.

This chapter is also related to the literature on shareholder voting. Yermack (2010) summarizes the empirical literature on shareholder voting in United States based companies, whereas Iliev et al. (2015) present evidence for the importance of shareholder voting in non-U.S. firms. Bar-Isaac and Shapiro (2020) show that a blockholder may optimally abstain from voting all of his shares to not crowd out information of other shareholders. This requires alignment of interests among the blockholder and other shareholders. Levit and Malenko (2011) show that non-binding shareholder voting may fail to aggregate information when interests between management and shareholders only partially align. Malenko and Malenko (2019) study the effect of proxy advisors on information acquisition and voting behavior of shareholders. Levit et al. (2019) analyze the effect of share trading opportunities on shareholder voting, the shareholder base, and the optimal board design.

4.3 Symmetric information

4.3.1 Model

We revisit the model of Chapter 3 but with a finite number of shareholders and an activist who may own shares in the company.

Investors Consider a public company with $n \in \mathbb{N}$ shares outstanding. Each share consists of a cash-flow claim and a voting right. The company is owned by two types of investors: an activist investor who owns $\alpha n \in \mathbb{N}_0$ shares and $(1-\alpha)n = n_S \geq 3$ ordinary shareholders who hold a single share each. Henceforth, we will refer to the activist shareholder as *activist*, A, and to the ordinary shareholders as *shareholders*, S, although the activist can be a shareholder herself. All investors are risk neutral.

Shareholder meeting The company has an upcoming shareholder meeting with a single, exogenously given reform proposal on the agenda. The vote is binding,[13] and the reform is implemented if at least $\lambda n \in \mathbb{N}$ votes are cast in favor of it. Otherwise, the status quo prevails. We assume that $1 - \lambda > \alpha$, such that the activist cannot block the reform unilaterally and that $1 < \lambda n < n_S$, meaning that an individual shareholder can neither block, nor implement the reform.

Payoffs If the company sticks with the status quo, the company's total share value remains unchanged at $v > 0$; if the reform is implemented, the company's value increases by $\Delta > 0$ to $v + \Delta$.

Despite the positive effect of the proposed reform on firm value, the activist may oppose it as she gains private benefits $b > 0$ from the status quo. These private benefits can, for instance, stem from other assets in her portfolio.[14] Debt in the same company may reduce the risk appetite, common ownership leading to anti-competitive preferences[15] or different supplier choices. Alternatively, the status-quo may allow the activist to (continue to) extract b at a cost to the firm of Δ. In any case, we take b to be exogenously given and fixed. In summary, the payoffs are

	activist	shareholder
status quo	$\alpha v + b$	$\frac{v}{n}$
reform	$\alpha(v + \Delta)$	$\frac{v+\Delta}{n}$.

When $b < \alpha\Delta$, the activist and the shareholders have aligned interests and both prefer to implement the reform; the activist is *friendly*. If $b > \alpha\Delta$, the activist prefers the company to stick with the status quo, in which case she is *hostile*. Since

[13] In the US binding shareholder voting occurs in the context of by-law amendments, acquisitions, and equity restructuring. In other countries, such as countries of the EU, shareholder decisions are usually binding.

[14] In 2004, during the acquisition of MONY by AXA, bond holdings introduced a wedge in the interest of MONY shareholders, compare https://www.nytimes.com/2004/05/19/business/holders-of-mony-approve-1.5-billion-sale-to-axa.html.

[15] Compare Azar et al. (2018) for empirical evidence on the effects of common ownership.

the friendly activist has no effect on the outcome of the decision under symmetric information, in this section we focus on this case of a hostile activist. Further, we think of the private benefit b as relatively small compared to the overall change in firm value Δ. In particular, we assume that $b < \Delta$, such that the reform increases welfare.[16]

4.3.1.1 Voting stage

As usual, the voting stage has degenerate equilibria in which all investors either vote for the status quo or the reform. When no voter can swing the outcome of the vote unilaterally, voting independent of the own preferences is a best response. However, these strategies are weakly dominated and yield peculiar equilibria, such that we rule them out. We assume that if an investor's voting decision does not affect the outcome of the vote, she votes for her preferred alternative. Hence, the activist casts all of her votes in favor of the status quo and the shareholders in favor of the reform. The outcome of the vote is, thereby, uniquely determined by who owns how many voting rights at the time of the meeting.

In the following, we do not model the voting stage explicitly but only use that the activist can block the reform if she controls at least $(1-\lambda)n+1$ voting rights. Given that $\alpha < (1-\lambda)$, this means that she needs $m = (1-\lambda-\alpha)n+1$ additional voting rights to prevent the reform. Otherwise, the efficient reform is implemented.

4.3.2 Vote trading

We now allow the activist to acquire voting rights, for instance by borrowing shares over the record date.

Suppose the activist can make a public take-it-or-leave-it offer $p \in \mathbb{R}_+$ per voting right. The offer is restricted, meaning that the activist can set an upper bound on the number of voting rights she is willing to acquire. If more shareholders sell to her, they are rationed. It is without loss to assume that the activist sets an upper bound at $m = (1-\lambda-\alpha)n+1$ voting rights. Having observed the offer p, shareholders simultaneously decide whether to sell. To capture the anonymity among shareholders, we consider symmetric strategies represented by a response function $q : \mathbb{R}_+ \to [0,1]$ mapping any offer p into an acceptance probability $q(p)$. As a result, the number of shareholders who accept is a binomial random variable $M(n_S, q(p)) \sim Bin(n_S, q(p))$. Since shareholders are rationed when $M(n_S, q(p)) > m$, the activist acquires $\bar{M}(n_S, q(p)) = \min\{M(n_S, q(p)), m\}$ voting rights.

Suppose that the activist offers price p and the shareholders respond by mixing with probability $q(p)$. If the activist buys fewer than m votes, the company's value

[16] If $b \geq \Delta$, the activist could simply take over the company and block the reform, maximizing welfare.

rises to $v + \Delta$. As a result, her payoff is $\alpha(v + \Delta) - pM(n_S, q(p))$. To the contrary, if $M(n_S, q(p)) \geq m$, the firm value remains at v and the activist receives the private benefit b, such that her payoff is $\alpha v + b - pm$. Together, this yields an expected payoff of

$$\Pi_A(p; q) = \alpha(v + \Delta) + \mathbb{P}[M(n_S, q(p)) \geq m](b - \alpha\Delta) - p\mathbb{E}[\bar{M}(n_S, q(p))]. \tag{4.1}$$

A shareholder's payoff depends on her selling decision as well as the behavior of the other $n_S - 1$ shareholders. Fix one shareholder, suppose that the activist offers price p and that the other shareholders respond by mixing with probability $q(p)$. If the shareholder decides to sell his voting right but fewer than $m - 1$ other shareholders also sell, the reform passes and the shareholder's payoff is $p + \frac{v+\Delta}{n}$. Conversely, if at least $m - 1$ of the other shareholders also sell their voting rights, the reform is blocked and the share value remains at $\frac{v}{n}$. Further, if more than $m - 1$ other shareholders sell, i.e. $M(n_S - 1, q(p)) > m - 1$, the shareholder is rationed. In this case, his payoff is

$$p\frac{m}{M(n_S - 1, q(p)) + 1} + \frac{v}{n}.$$

If the shareholder does not sell his voting right, but at least m other shareholders do, the reform is blocked and his payoff is $\frac{v}{n}$. Otherwise, it rises to $\frac{v+\Delta}{n}$. In expectation, this means that a shareholder's payoff is

$$\Pi_S(\text{sell}; p, q) = \frac{v + \Delta}{n} - \mathbb{P}[M(n_S - 1, q(p)) \geq m - 1]\frac{\Delta}{n} + p\frac{\mathbb{E}[\bar{M}(n_S, q(p))]}{n_S q(p)}$$

if he sells his voting right and

$$\Pi_S(\text{keep}; p, q) = \frac{v + \Delta}{n} - \mathbb{P}[M(n_S - 1, q(p)) \geq m]\frac{\Delta}{n}$$

if he keeps his voting right. The fraction $\frac{\mathbb{E}[\bar{M}(n_S, q(p))]}{n_S q(p)}$ is the probability not to be rationed.[17]

We consider subgame perfect equilibria.

Proposition 4.1 *For any n, an equilibrium (p^*, q^*) exists. If $q^*(p^*) > 0$ and, thereby, $\mathbb{P}[M(n_S, q^*(p^*)) \geq m] > 0$, then*

$$\underbrace{p^*\mathbb{E}[\bar{M}(n_S, q^*(p^*))]}_{\mathbb{E}[total\ transfer]} < m \underbrace{\frac{\Delta}{n} \cdot \mathbb{P}[M(n_S, q^*(p^*)) \geq m]}_{\mathbb{E}[loss\ per\ shareholder]}. \tag{4.2}$$

Further,

[17] Compare (4.7) in the appendix for an explicit derivation of the expression.

- *there always is an equilibrium in which* $p^* = 0$ *and* $q^*(0) = 1$;
- *as* n *grows large, along any sequence of equilibria,*

$$\lim_{n\to\infty} \mathbb{P}[M(n_S, q^*(p^*)) \geq m] = 1 \quad \textit{and} \quad \lim_{n\to\infty} p^*\mathbb{E}[\bar{M}(n_S, q^*(p^*))] = 0.$$

Proposition 4.1 establishes that the activist can obtain the blocking minority without the need to (fully) compensate the shareholders (4.2). Whenever there is trade,[18] shareholders suffer a strict loss. This is possible because the activist can exploit two inefficiencies, which create a market failure in the market for voting rights.

First, there is the externality of voting. The λ-majority-rule implies that only $(1-\lambda)n+1$ votes have to be cast against the reform to block it. This blocking minority does not internalize the effect of their behavior on the rest of the shareholders. As a result, it would suffice if the activist compensated m shareholders for their individual loss of $\frac{\Delta}{n}$.

However, the activist can do even better and pays less than m times the expected loss of a shareholder (4.2). A shareholder's valuation for her voting right depends on the selling decisions of the other shareholders. The voting right is only valuable if it is decisive or pivotal in the vote—that is, if exactly $m-1$ other shareholders sell their voting rights. Therefore, any shareholder compares the expected payment the activist offers with the expected loss in case the reform is blocked, but weighs the expected loss with the probability to be pivotal. In particular, if the activist offers p and the other shareholders sell with probability $q(p)$, a shareholder prefers to sell if $\Pi_S(\text{sell}; p, q) \geq \Pi_S(\text{keep}; p, q)$, which rearranges to

$$\underbrace{p\frac{\mathbb{E}[\bar{M}(n_S, q(p))]}{n_S q(p)}}_{\mathbb{E}[payment]} \geq \underbrace{\mathbb{P}[M(n_S - 1, q(p)) = m-1]}_{\mathbb{P}[pivotal]} \underbrace{\frac{\Delta}{n}}_{loss} . \tag{4.3}$$

As $m = (1-\lambda-\alpha)n+1 \in \{2, ..., n_S - 1\}$, the probability of being pivotal is always strictly smaller than 1.[19] Hence, there is a dilution of control and the activist can acquire the voting rights at a discount.

The proof of Proposition 4.1 further shows that as the population of shareholders grows, the probability that any single shareholder is pivotal quickly converges to zero. Therefore, any equilibrium outcome approaches the most extreme one in which every shareholder sells his voting rights to the activist for free, and the activist always

[18] Whenever n and b are sufficiently small, there may also be an equilibrium in which $p^* = 0$ and $q^*(p^*) = 0$.

[19] If $q \in \{0, 1\}$, such that every other or no other shareholder sells, $\mathbb{P}[pivotal] = 0$ and the shareholder sells at any positive price. For all $q \in (0, 1)$, every or no shareholder sells with strictly positive probability, such that $\mathbb{P}[pivotal] < 1$.

blocks the reform.

When the number of shareholders is sufficiently large, the market failure that creates inefficient outcomes occurs across all symmetric equilibria, such that our result does not rely on an equilibrium selection. Further, Neeman (1999) shows that the zero-price equilibrium is the only asymmetric equilibrium robust to noise voters; this highlights the robustness of our results.

4.3.2.1 Conditional or unrestricted offers

Restricted offers are natural since an activist only needs to acquire a fraction of the voting rights. Further, shareholders correctly anticipate the possibility to be rationed (left side of (4.3)), and demand a higher price to compensate for the possibility. Thereby, the restriction has no effect on the transfers, and Proposition 4.1 is completely driven by the shareholders' pivotality considerations. If we were to consider unrestricted offers, the results would remain unchanged for large n. For small n and large b, the activist may choose a price that gives her, in expectation, more than m voting rights, to guarantee that she can block the reform. As a result, when there are few shareholders, the total transfer can exceed $m\frac{\Delta}{n}$. In the alternative case in which the activist can restrict the offer and condition it on the event that at least m shareholders agree to sell their voting right, the result of Proposition 4.1 is strengthened: for any n, only the zero-price equilibrium survives. We prove the results in Lemmas 4.7 and 4.8 in the appendix.

4.3.3 Competing offers

We now investigate how the market failure and the resulting threat of hostile activism reacts to competition by a friendly blockholder. To that end, suppose that there is such a blockholder B who owns $\beta n \in \mathbb{N}$ shares but $\beta < \lambda$ such that he cannot implement the reform unilaterally. The number of ordinary shareholders is $n_S = (1-\alpha-\beta)n \in \mathbb{N}$. As before, activist A first makes an offer p_A for $m_A = (1-\lambda-\alpha)n+1$ voting rights. After observing A's offer, blockholder B, acting as a white knight who wants to implement the reform, jumps in and makes a counteroffer p_B for up to $m_B = (\lambda - \beta)n = n_S - m_A + 1$ voting rights. Thus, B's strategy is a function $p_B : \mathbb{R}_+ \to \mathbb{R}_+$ which maps any offer p_A into a counteroffer $p_B(p_A)$.

Note that for the shareholders, selling the voting rights to the blockholder dominates holding onto them. Thus, every shareholder (tries to) sell his voting right to either the activist or the blockholder. The symmetric best response function of shareholders is given by $q : \mathbb{R}_+ \times \mathbb{R}_+ \to [0, 1]$, where q is the probability that shareholders sell to A and $1 - q$ the probability that they sell to B. Further, define $\bar{M}_A = \max\{M(n_S, q), m_A\}$ and $\bar{M}_B = \max\{n_S - M(n_S, q), m_B\}$ as the random number of shares A and B actually acquire. Again, we consider subgame perfect equilibria.

Proposition 4.2 *For any n, an equilibrium (p_A^*, p_B^*, q^*) exists.*

1. *If $\frac{b-\alpha\Delta}{1-\lambda-\alpha} > \frac{\beta\Delta}{\lambda-\beta}$ and n is sufficiently large, the reform is always blocked, $q^*(p_A^*, p_B^*(p_A^*)) = 1$. Further,*
$$p_A^*\mathbb{E}[\bar{M}_A(n_S, q^*(p_A^*, p_B^*(p_A^*)))] = p_A^* m_A < \frac{1-\lambda-\alpha}{\lambda-\beta}\beta\Delta$$
but $\lim_{n\to\infty} p_A^*\mathbb{E}[\bar{M}_A(n_S, q^*(p_A^*, p_B^*(p_A^*)))] = \frac{1-\lambda-\alpha}{\lambda-\beta}\beta\Delta$.
2. *If $\frac{b-\alpha\Delta}{1-\lambda-\alpha} < \frac{\beta\Delta}{\lambda-\beta}$, as n grows large, along any sequence of equilibria, the reform becomes certain, $\lim_{n\to\infty} \mathbb{P}[M_A(n_S, q^*(p_A^*, p_B^*(p_A^*)) \geq m_A] = 0$, and transfers converge to zero, $\lim_{n\to\infty} p_A^* n_S = \lim_{n\to\infty} p_B^*(p_A^*) n_S = 0$.*

When the shareholdings are dispersed, i.e. n and m_A, m_B are large, no individual shareholder is pivotal with substantial probability. Thus, he simply sells to the investor who offers the higher expected payment, anticipating the different probabilities to be rationed. How much A and B are willing to offer depends on their willingness to pay, $b - \alpha\Delta$ and $\beta\Delta$, as well as the number of shares they have to acquire, $(1 - \lambda - \alpha)n + 1$ and $(\lambda - \beta)n$. In particular, the activist has a comparative advantage when she has to acquire fewer shares than the blockholder, $\frac{(1-\lambda-\alpha)n+1}{(\lambda-\beta)n} \approx \frac{1-\lambda-\alpha}{\lambda-\beta} < 1$. Note that this is true whenever λ is large, such that competition is unlikely to deter hostile activism in supermajority decisions. Further, the compensation shareholders receive when the activist blocks the reform is decreasing in λ. Surprisingly, when λ is large, the total transfer from the activist to the shareholders can be substantially below the expected loss of the blockholder. When the hostile activist succeeds and blocks the reform, welfare is reduced, although small shareholders may be (partially) compensated.

If the blockholder deters the activist from making an offer, vote prices in our model are close to zero. On the other hand, if the blockholder cannot deter the activist, the activist has to pay a strictly positive transfer. The analysis by Dekel et al. (2009) suggest that strictly positive prices may be the result of the offer structure. Dekel et al. (2009) show that the unique trading price is zero if the activist and the blockholder can sequentially adjust their offer upwards, and if there is a continuum of shareholders.[20] Therefore, (close to) zero prices and a positive trade volumes do not signal an absence of competition or aligned interests.

[20] Dekel et al. (2009) analyze a game with a continuum of voters in which the two contestants make alternating, increasing offers until one stops. By an unraveling argument, the loser does not compete because she would acquire a strictly positive fraction of the voting rights at a positive price without changing the outcome of the vote.

4.3.4 Discussion

In markets for standard assets without externalities, voluntary trade produces Pareto improvements. We show that this intuition cannot be transferred to the market for voting rights. Not only does voting create an externality of the majority on the minority, but there is a market failure in the voting right market that goes beyond the externality of voting. The activist does not even compensate m shareholders for their loss; she pays close to zero compensation. This market failure is the result of the relative value of a voting right, which depends entirely on the other investors' trading and voting decisions and is close to zero when the shareholdings are dispersed. Importantly, it does not depend on hidden motives by the activist or the details of the modeling approach.[21] As long as shareholders do not believe that they are pivotal with probability one, the voting rights trade at inefficiently low prices.

As we show further, competition in the market for voting rights does not eliminate the market failure and, by extension, cannot solve the problem of hostile activism. The threat of competition by a blockholder may deter hostile activists without raising voting right prices, but relies on the blockholder's willingness to pay as well as the number of voting rights he and the activist must acquire.

As pointed out previously, we do not consider a friendly activist in this section since she would not change the outcome of the vote. When the optimal decision is common knowledge, an activist only plays a role if she has misaligned interests, i.e. is hostile. Hence, in a symmetric information setting, vote trading uniquely aids hostile activists. In Section 4.4 we investigate the situation with asymmetric information.

Empirical predictions Our model jointly explains low prices for voting rights (Christoffersen et al. (2007), Kalay et al. (2014)) and inefficient outcomes caused by hostile activists which engage in vote trading (Hu and Black, 2008a).

Moreover, we show that active blockholders may deter hostile activists from acquiring voting rights, such that it is less likely to occur in companies with large, active blockholders. Interestingly, the competition does not need to increase prices in order to deter vote trading. Hence, the observed low prices in the market for voting rights do not necessarily indicate a lack of competition.

Last, our results imply that supermajority decisions are particularly likely to be targeted by hostile activists. In addition to market frictions, decisions that require a supermajority for approval give her a distinct advantage over any potential competitor. This fits the anecdotal evidence of Hu and Black (2008a) showing that most incidents of vote trading occurred when a hostile activist blocked a reform that required a supermajority.

[21] Casella et al. (2012) show that a competitive equilibrium does not exist. Instead, they consider a novel equilibrium concept, and show that vote trading can reduce (expected) welfare.

4.4 Asymmetric information

In the previous section, we established that in a symmetric information setting, vote trading promotes hostile activism, threatening corporate governance and shareholder value. Certainly, activism can also be put to good use.[22] Shareholders often rely on activist investors for their professional insights and analysis to identify value-increasing reforms. However, this mutually beneficial relationship is hindered by ulterior motives of the activist which can make it hard for her to communicate with the shareholders. To solve this problem, activists engage in proxy fights and disclose their share position to convince shareholders of their best intentions.

In this section, we investigate the possibilities of vote trading to improve corporate governance under asymmetric information.[23] To this end, we consider a version of the model in which the activist possesses private information about the optimal reform decision. We compare vote trading with traditional forms of intervention, which we identify by their potential to (credibly) communicate the information. The analysis is split into two cases: when the activist and the shareholders have common interests (friendly activist), and when the activist always wants to block the reform (hostile activist).

4.4.1 Model

States and payoffs We extend the model by introducing an uncertain state $\omega \in \{Q, R\}$ with prior probability $\rho \in (0, \frac{1}{2})$ that the state is Q. The activist investor, A, knows the state, the shareholders, S, do not. Throughout Section 4.4, the activist has a strictly positive share endowment, $\alpha > 0$.

Again, the activist obtains private benefits whenever the status quo remains. The reform, however, is not uniformly beneficial for shareholders. In state Q, the reform reduces firm value by Δ, such that shareholders also prefer the status quo over the reform; in state R the reform raises firm value by Δ. As a result, the payoffs are

Q	activist	shareholder	R	activist	shareholder
status quo	$\alpha v + b$	$\frac{v}{n}$	status quo	$\alpha v + b$	$\frac{v}{n}$
reform	$\alpha(v - \Delta)$	$\frac{v-\Delta}{n}$	reform	$\alpha(v + \Delta)$	$\frac{v+\Delta}{n}$.

[22] Compare Brav et al. (2015, 2008) for an empirical analysis of the effects of hedge fund activism.
[23] Brav and Mathews (2011) and Eso et al. (2015) stress the positive effect of vote trading on information transmission and aggregation.

4.4.1.1 Voting stage

Shareholders try to maximize their (expected) share value by matching the state. Let ξ be the shareholders' belief that the state is Q at the time of the vote. As before, we ignore degenerate equilibria where voters play weakly dominated strategies. This means that if $\xi < \frac{1}{2}$, shareholders vote for the reform, and if $\xi > \frac{1}{2}$, they vote for the status quo. Absent of any additional information $\xi = \rho < \frac{1}{2}$, meaning that shareholders vote for the reform. The activist knows the state and matches it whenever $b < \alpha\Delta$, but she always votes in favor of the status quo whenever $b > \alpha\Delta$. As noted before, we refer to these two cases as a *friendly activist* and *hostile activist*, respectively.

4.4.2 Friendly activist, $b < \alpha\Delta$

When the activist has superior information valuable to shareholders, she can potentially improve corporate decision making. Therefore, we also need to analyze the friendly activist, who did not change the outcome of the decision in the symmetric information case.

4.4.2.1 Vote trading

Suppose the activist can make a public take-it-or-leave-it offer $p \geq 0$ for up to m voting rights. Alternatively, the activist may make no offer, which we denote by $\emptyset$.[24] Since the activist's offer depends on the state, her strategy becomes $p : \{Q, R\} \to \mathbb{R}_+ \cup \emptyset$. Having observed the offer, any individual shareholder updates her belief to $\xi(p)$ and sells with probability $q(p) \in [0, 1]$.

Because the activist votes for the firm-value maximizing decision, shareholders benefit from selling their voting right to her. The activist, on the other hand, tries to acquire the voting rights or steer the decision at the lowest possible cost. The payoffs are stated explicitly in the proof of Lemma 4.1.

We solve the game for perfect Bayesian equilibria.

Lemma 4.1 *An equilibrium $(p^*, q^*; \xi^*)$ exists. In any equilibrium,*

- *the activist offers $p^*(\omega) = 0$ in at least one state $\omega \in \{h, \ell\}$;*
- *the reform is implemented in state R and the status quo remains in state Q.*

By Lemma 4.1, vote trading increases the probability that the state is matched from $1 - \rho$ to 1. Since this is in the best interest of both shareholders and the activist, welfare rises from $v + (1 - 2\rho)\Delta$ to $v + (1 - \rho)\Delta + \rho b$. This improvement

[24] Such an action would be (weakly) dominated by offering zero in the symmetric information game.

is achieved through one of two types of equilibria. In the “delegation equilibrium,” the activist acquires all voting rights for $p^*(Q) = p^*(R) = 0$. Shareholders know that the activist has aligned interests and that she implements the correct decision, such that they cede their voting rights to her.[25] In a “signaling equilibrium,” the friendly activist only offers to purchase the voting rights in one state. Therefore, the presence (or lack) of an offer reveals the state to the shareholders and they vote in favor of the correct decision.

4.4.2.2 Costless communication

Whenever the activist is friendly, there are other forms of activist interventions by which she can ensure that the correct decision is implemented. She just has to communicate the optimal decision to the shareholders.

Formally, suppose that the activist cannot acquire voting rights but communicates with the shareholders before the meeting by sending a message from $\{0, 1\}$. Thus, a strategy for the activist is a mapping from the state into the binary message space $\mu : \{Q, R\} \to \{0, 1\}$. Having observed $\mu(\omega)$, shareholders form posterior $\xi(\mu(\omega))$ and vote for the status quo if $\xi(\mu(\omega)) > \frac{1}{2}$, and vote for the reform if $\xi(\mu(\omega)) < \frac{1}{2}$. We consider perfect Bayesian equilibria.

Lemma 4.2 *There is an equilibrium $(\mu^*; \xi^*)$ in which the activist sends $\mu^*(Q) \neq \mu^*(R)$, such that shareholders learn the state. Thereby, the reform is implemented in state R and the status quo remains in state Q.*

Since shareholders and the friendly activist have aligned interests, they follow her recommendation, such that the correct decision is taken and welfare is maximized. Thus, vote trading does not have a unique upside when the activist is friendly.

In practice, means of (cheap talk) communication are readily available and there is a long-standing tradition of activist investors endorsing company policies or publicly venting their discontent with management, be it through public statements, interviews, or 13D attachments. Further, the internet significantly simplifies the communication among shareholders, and regulatory authorities have deliberately removed legal obstacles to foster communication. For example, proxy rule amendments made in 2007 by the U.S. Securities and Exchange Commission (SEC) encourage electronic shareholder forums with this in mind. Christopher Cox, who served as SEC chairman at that time, summarized the reform, saying,[26]

[25] Observe that while such an equilibrium also exists in the game with a hostile activist, the rationale here is different. Shareholders benefit from delegating their voting rights, such that they strictly prefer to do so, independent of pivotality considerations.

[26] SEC press release, November 28, 2007, http://www.sec.gov/news/press/2007/2007-247.htm.

"Today's action is intended to tap the potential of technology to help shareholders communicate with one another and express their concerns to companies in ways that could be more effective and less expensive. The rule amendments are intended to remove legal concerns, such as the risk that discussion in an online forum might be viewed as a proxy solicitation, that might deter shareholders and companies from using this new technology."

Ultimately, there is another channel by which the correct decision can be implemented by the friendly activist: delegation. Uniformed shareholders have an incentive to give a proxy to the informed, friendly activist free of charge. This allows the friendly activist to implement the correct decision in their place, resulting in the same Pareto improvement that vote trading offers.

4.4.3 Hostile activist, $b > \alpha\Delta$

As we have seen in the last section, vote trading as well as other forms of costless communication or delegation can improve corporate governance and shareholder value when the activist is friendly. When the activist is hostile, however, she always wants to block the reform, such that she cannot transmit information to shareholders via cheap talk, and shareholders are unwilling to delegate their voting rights. However, we show in this section that vote trading might still improve corporate governance and the expected firm value. We then investigate whether traditional forms of intervention can have similar benefits.

4.4.3.1 Vote trading

Again, the activist can make a public take-it-or-leave-it offer p for up to m voting rights. Shareholders update their belief to $\xi(p)$ and decide with which probability to sell, $q(p)$. Thus, strategies are $p : \{Q, R\} \to \mathbb{R}_+$ and $q : \mathbb{R}_+ \to [0, 1]$.

The shareholders' posterior belief about the state, $\xi(p)$, affects their expected loss when the activist blocks the reform. When $\xi(p) > \frac{1}{2}$, shareholders actually prefer the status quo, fixing the firm value at v. In this case, shareholders' incentives are aligned with those of the activist and selling to the activist does not change the outcome of the vote, such that there is no expected loss in firm value when the activist blocks the reform. On the other hand, when $\xi(p) < \frac{1}{2}$, shareholders prefer the reform since it increases the expected firm value to $v + (1 - 2\xi(p))\Delta$. Thus, when the activist blocks the reform, shareholders incur a loss of $(1 - 2\xi(p))\frac{\Delta}{n}$.[27]

[27] Fully spelled out, this means that

$$\Pi_S(\text{sell}; p, q, \xi) = \frac{v}{n} + (1 - \mathbb{P}[M(n_S - 1, q(p)) \geq m - 1]) \max\{0, 1 - 2\xi(p)\}\frac{\Delta}{n} + p\frac{\mathbb{E}[\bar{M}(n_S, q(p))]}{n_S q(p)},$$
$$\Pi_S(\text{keep}; p, q, \xi) = \frac{v}{n} + (1 - \mathbb{P}[M(n_S - 1, q(p)) \geq m]) \max\{0, 1 - 2\xi(p)\}\frac{\Delta}{n}.$$

The activist's payoff is also influenced by the shareholders' belief, $\xi(p)$, because it determines their voting behavior. Suppose that $\xi(p) < \frac{1}{2}$, such that shareholders who do not sell their voting right vote for the reform. In state R, the activist's payoff is given by equation (4.1), whereas in state Q, it is

$$\Pi_A(p;q,\xi,Q) = \alpha(v-\Delta) + \mathbb{P}[M(n_S,q(p)) \geq m](b+\alpha\Delta) - p\mathbb{E}[\bar{M}(n_S,p(q))].$$

If $\xi(p) \geq \frac{1}{2}$ and shareholders who do not sell their voting right vote against the reform, the activist's payoff is $\alpha v + b - p\mathbb{E}[\bar{M}(n_S,p(q))]$, independent of the state.

Since $\alpha > 0$, the activist's willingness to pay for the voting rights is higher in state Q than in state R. As a result, there are separating perfect Bayesian equilibria in which vote trading can be welfare increasing. The following exemplary equilibrium illustrates this effect.

Example Suppose there are $n = 4$ shares and that the activist and three other shareholders each own one share. The reform changes firm value by $\Delta = 1$, whereas the status quo provides the activist with a private benefit of $b = \frac{1}{2}$. The prior probability of state Q is $\rho = \frac{1}{4}$, such that, in expectation, the shareholders benefit from the reform. The activist, on the other hand, wants to block the reform in either state. The reform requires a simple majority; in case of a tie, it is implemented as well. Thus, the activist needs to acquire $m = 2$ voting rights to prevent the reform.

There is an equilibrium in which $p^*(Q) = \frac{1}{8}, p^*(R) = 0, q^*(p^*(Q)) = 1$ and $q^*(p^*(R)) = 0$. In this separating equilibrium, the reform is implemented only in state R and welfare is maximized. Figure 4.1 illustrates the equilibrium strategies (p^*, q^*).

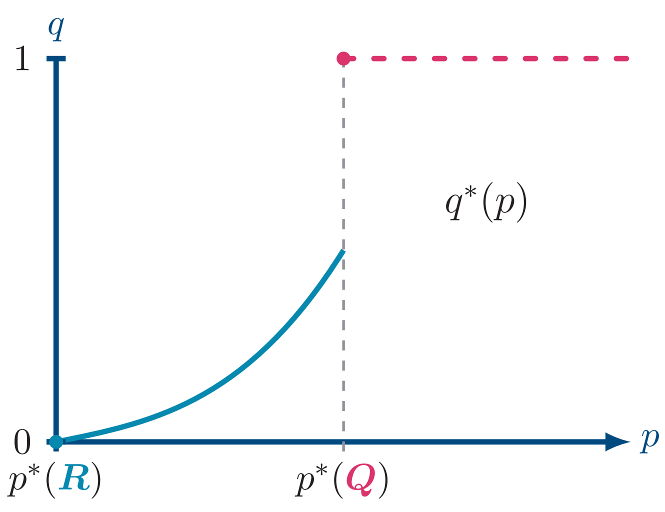

Figure 4.1 Example of a fully separating equilibrium.

To construct this equilibrium, suppose that $\xi^*(p) = 0$ for all $p \in [0, p^*(Q))$. Given this belief, let $q^*(p)$ be the smallest solution to the condition that shareholders are indifferent between selling and retaining their voting right

$$\underbrace{2(1-q)q}_{\mathbb{P}[M(n_S-1,q)=m-1]} \frac{\Delta}{n} = p \underbrace{[(1-q)^2 + 2q(1-q) + q^2\frac{2}{3}]}_{\frac{\mathbb{E}[\bar{M}(n_S,q)]}{n_S q}}.$$

For all $p \geq p^*(Q)$, let $\xi^*(p) = 1$, such that it is strictly optimal for shareholders to sell, $q^*(p) = 1$. Naturally, the resulting q^* is a best response given their belief ξ^*.

When shareholders respond with q^*, in state R, the activist is indifferent between $p^*(R) = 0$ and $p^*(Q) = \frac{1}{8}$, $\Pi_A(p^*(R); q^*, \xi^*, R) = \Pi_A(p^*(Q); q^*, \xi^*, R) = \frac{1}{4}$. Further, we show in Appendix 4.B.8 that all prices except 0 and $\frac{1}{8}$ are dominated. Thus, $p^*(R)$ is a best response. In state Q, the activist's payoff from blocking the reform is higher than in state R, such that $p^*(Q) = \frac{1}{8}$ is the unique best response.

By construction, all investors play best responses and the beliefs are consistent, such that the proposed strategies and beliefs form a perfect Bayesian equilibrium.

As the next proposition shows, a separating perfect Bayesian equilibrium always exists but fails to improve expected firm value when n is large.

Proposition 4.3 *There always exists a separating equilibrium $(p^*, q^*; \xi^*)$, i.e. an equilibrium in which $p^*(Q) \neq p^*(R)$, such that shareholders learn the state. Further,*

1. *in any separating equilibrium $p^*(R) < p^*(Q)$ and $q^*(p^*(R)) < q^*(p^*(Q)) = 1$;*
2. *as n grows large, along any sequence of equilibria and for $\omega \in \{Q, R\}$,*

$$\lim_{n\to\infty} \mathbb{P}[M(n_S, q^*(p^*(\omega))) \geq m] = 1 \quad \textit{and} \quad \lim_{n\to\infty} p^*(\omega)\mathbb{E}[\bar{M}(n_S, q^*(p^*(\omega)))] = 0.$$

When the number of shareholders is small, the separating equilibrium can, as Example 4.1 demonstrates, raise the probability that the correct decision is implemented beyond the ex-ante probability of $1 - 2\rho$. Thus, vote trading can increase welfare, even when the activist is hostile, and even if the private benefit does not suffice to make up for the expected loss in firm value when the reform is blocked, $b < (1-\rho)\Delta - \rho\Delta = (1-2\rho)\Delta$.

This effect, however, utilizes vote trading as a costly signal, which can only work in case the voting rights are sufficiently expensive. As established by Proposition 4.1, vote prices quickly converge to zero when the firm is owned by more shareholders; if shareholdings are dispersed, the activist can acquire a blocking minority of voting rights at negligible cost and block the reform in either state. As a result, the expected firm value converges to $v < \rho(v - \Delta) + (1-\rho)(v + \Delta) = v + (1-2\rho)\Delta$, while the expected transfer converges to zero. When $b < (1-2\rho)\Delta$, overall welfare is reduced compared to the situation without vote trading.

4.4.3.2 Costly communication

Vote trading may improve communication by acting as a costly signal, but so does any traditional form of costly intervention, yielding (weakly) superior outcomes.

To formalize the idea, suppose that, instead of buying voting rights, the activist can spend amount $\kappa \in \mathbb{R}_+$, for example, on running a costly but non-informative public proxy campaign. Thus, her strategy is $\kappa : \{Q, R\} \to \mathbb{R}_+$. Shareholders observe κ, form posterior $\xi(\kappa)$, and vote for the status quo if $\xi(\kappa) > \frac{1}{2}$; they vote for the reform if $\xi(\kappa) < \frac{1}{2}$. Again, we consider perfect Bayesian equilibria.

Proposition 4.4

1. *There is an equilibrium $(\kappa^*; \xi^*)$ in which the activist spends $\kappa^*(Q) = b - \alpha\Delta$ and $\kappa^*(R) = 0$. Shareholders learn the state, block the reform in state Q, and implement the reform in state R.*

2. *In every (other) equilibrium, the state is matched with probability of at least $1 - \rho$.*

Proposition 4.4 shows that a costly signal can also be used to credibly communicate that the state is Q. In any separating equilibrium, the activist needs to spend at least $\kappa^*(Q) = b - \alpha\Delta$ to signal that the state is Q, preventing the reform. At $\kappa^*(Q) = b - \alpha\Delta$ the activist in state R is exactly indifferent between spending $\kappa^*(Q)$ and remaining passive, $\kappa^*(R) = 0$: both yield her a payoff of v. In state Q, the activist strictly benefits from spending $\kappa^*(Q)$ because $\alpha v + b - \kappa^*(Q) > \alpha(v - \Delta)$.

Different from vote trading, in any separating equilibrium of the costly communication game, the first-best firm value is attained. In case the costly signal is not wasteful, this implies that welfare is maximized. Further, costly signaling can never reduce shareholder value relative to the pure voting benchmark. It, therefore, circumvents the risks of hostile activism inherent to vote trading.

Traditional forms of costly intervention include public proxy campaigns or the public acquisition of shares. Our results generate two new insights regarding the usage of these tools. First, even if the activist cannot provide evidence of her claims during the proxy fight, the fact that she is willing to engage in a costly proxy fight can suffice as a credible signal. Proxy fights are valuable not because they directly transmit information but because the associated costs give credence to the activist. Further, the public acquisition of shares not only aligns the activist and the shareholders' interests by raising α, but can be a credible signal that the activist wants to maximize shareholder value. Hence, the public disclosure of these acquisitions—through regulatory filings, for instance—serves an important function in the communication between investors.

4.5 Conclusion

Financial innovation has created manifold new ways to exchange voting rights; most notably using the equity lending market. Vote trading became a new force in shareholder activism, raising the question whether regulators should embrace or worry about vote trading. Our results show that regulators have reason to be concerned.

Vote trading does not yield Pareto improvements, but renders shareholders vulnerable to hostile activism—even in a best-case environment with transparent motives by the activist. It is true that when the activist has private information about the optimal decision, vote trading can be beneficial despite the activist's ulterior motives. Nevertheless, compared with traditional forms of intervention such as public endorsements, proxy campaigns, or share acquisitions, vote trading creates inferior outcomes. Note that we even consider a lower bound on the efficiency of these traditional forms of interventions by reducing them to their capacity to act as a costless or costly signal. For instance, we analyze models of non-verifiable information only. In practice, activist investors not only suggest certain courses of action but also (try to) provide evidence for their claims, which can be scrutinized by shareholders and outside analysts alike.

In conclusion, claims of more efficient corporate governance via vote trading seem unconvincing when compared with the traditional forms of intervention by activist investors. Instead, vote trading threatens shareholder value by enabling hostile activism. This goes to show that the long-standing tradition of outlawing the outright trade of voting rights in most countries is well founded. To prevent the new, indirect ways of vote trading, regulation has to be updated. We discuss some salient policy proposals in the final section.

4.6 Policy implications

4.6.1 Transparency measures

The market failure in the market for voting rights does not depend on hidden motives of the activist. As a result, policies aimed at increasing transparency, such as extended disclosure requirements[28] or rules of informed consent, do not suffice to prevent inefficient market outcomes and hostile activism. Nevertheless, additional transparency rules might be helpful, to prevent problems of asymmetric information and to monitor the extent of vote trading.

[28] Compare Hu and Black (2006) for a discussion of disclosure requirements with the SEC.

4.6.2 Self-regulation by shareholders

Because shareholders *collectively* bear the cost of vote trading, they have an incentive to self-regulate. In this spirit, large asset managers such as BlackRock claim to recall shares in case of an "economically relevant vote."[29] Further, non-binding regulations such as stewardship codes have extended asset managers' "best practice" recommendations in the same direction. However, without some form of commitment, none of these self-imposed rules or "shareholder-cartels" are stable. Since it is *individually* optimal for shareholders to sell their voting rights if others do not, there can be no collective abstention from vote trading.

4.6.3 Forced recalls

Regulatory authorities could require shareholders to recall their shares for the record date, forcing them to change the collateral their repo and cancel their lending agreements. While such measures would prevent the most relevant forms of vote trading, they would also come at a substantial cost. For instance, such regulation would imply a temporary shutdown of the equity lending market, thereby preventing (non-naked) short sales over the record date.

4.6.4 Excluding bought votes

One way to substantially reduce the ease of vote trading would be to suspend the voting rights of shares that were acquired in a way that can be exploited for vote trading. Shares borrowed or posted as collateral would, thus, lose their voting right until they were returned or resold to a third party.[30] This would leave the equity lending and repo markets unaffected in terms of their capacity to enable short selling or financing. However, this exclusion would not be a comprehensive solution since a hostile activist with a positive share endowment could still obtain control. When owning $\alpha > 0$ shares, the activist could borrow a fraction $\sigma > \frac{1-\alpha-\lambda}{1-\lambda}$ of the shares, implicitly voting σ as abstentions, thereby blocking the reform.

4.6.5 Excluding vote buyers

A more reliable solution than excluding bought votes would be to exclude the vote buyer from voting any of her shares. This solution not only has the same upsides as excluding borrowed votes but also prevents the acquisition of voting rights to void them.

[29] See https://www.ft.com/content/0e28929e-85dd-11e8-a29d-73e3d454535d.

[30] If a borrowed share would not regain its voting right, share lending would endogenously create non-voting shares, leading to additional problems.

4.6.6 Share blocking, lead time of the record date

Prior to 2007, it was common in many EU countries that shares, when voted on, were blocked from trading before the meeting.[31] This was done in an effort to prevent investors from voting shares they no longer owned, aligning the economic interest and voting power. However, the class of decoupling techniques discussed in this chapter (Vote Trading techniques, cf. Chapter 3) is unaffected by such measures. In the case of vote trading via the equity lending market, for example, share blocking would only require the activist to borrow the shares for the whole lead time of the record date. The economic exposure would still remain with the initial shareholders whereas the activist would only receive the voting right.

Similarly, the lead time of the record date has no effect on the economic forces of vote trading and, thereby, the possibility to use vote trading for hostile activism. Consider, for instance, the most extreme case, in which the voting and the record date coincide. Such an arrangement would not prevent the activist from borrowing shares before the record/voting date and returning them afterwards, yielding the same outcome as the current practice.

4.6.7 Majority rules

The anecdotal evidence of Hu and Black (2008a) suggests that decisions that require a supermajority are particularly vulnerable to hostile activism via vote trading. In Section 4.3.3 we give one reason for this effect: if the reform requires a supermajority, a blockholder is not able to deter a hostile activist because he is at a disadvantage relative to the activist, and the transfers from the activist to shareholders is particularly low. In addition to that, though the depth of the equity lending market may be sizeable, it is still limited. For both reasons, reducing the required majority towards a simple majority will help to deter hostile activism.

[31] See European Commission Staff Working Document SEC(2006) 181, https://ec.europa.eu/transparency/regdoc/rep/2/2006/EN/2-2006-181-EN-1-0.pdf.

Appendices

4.A Identities

Lemma 4.3 $\mathbb{P}[M(n_S-1,q)=m-1]<1$ *and* $\lim_{n\to\infty}\mathbb{P}[M(n_S-1,q)=m-1]=0$.

Proof. The first assertion follows because $0<m<n_S-1$, such that $1=\sum_{i=0}^{n_S-1}\mathbb{P}[M(n_S-1,q)=i]>\mathbb{P}[M(n_S-1,q)=m-1]$.

For the second, note that $\mathbb{P}[M(n_S-1,q)=m-1]=\binom{n_S-1}{m-1}q^{m-1}(1-q)^{n_S-m}$ is maximized if

$$\begin{aligned}0&=\binom{n_S-1}{m-1}q^{m-2}(m-n_Sq+q-1)(1-q)^{-m+n_S-1}\\ \iff q&=\frac{m-1}{n_S-1}.\end{aligned}$$

Thus,

$$\mathbb{P}[M(n_S-1,q)=m-1]\le\binom{n_S-1}{m-1}\left(\frac{m-1}{n_S-1}\right)^{m-1}\left(\frac{n_S-m}{n_S-1}\right)^{n_S-m}. \tag{4.4}$$

Using Stirling's formula, $\binom{a}{b}=(1+o(1))\sqrt{\frac{a}{2\pi(a-b)b}}\frac{a^a}{(a-b)^{a-b}b^b}$, the right side of (4.4) becomes

$$=(1+o(1))\sqrt{\frac{n_S-1}{2\pi(n_S-m)(m-1)}}=(1+o(1))\sqrt{\frac{1}{2\pi(1-\eta)(n_S-1)\eta}}, \tag{4.5}$$

with $\eta=\frac{m-1}{n_S-1}$ (implying that $\eta\approx\frac{1-\lambda-\alpha}{1-\alpha}$). When n, n_S, and $m\to\infty$, the second assertion follows. □

Lemma 4.4

$$\sum_{i=m-1}^{n_S-1}\mathbb{P}[M(n_S-1,q)=i]\frac{m}{i+1}=\mathbb{P}[M(n_S,q)\ge m]\frac{m}{n_Sq}. \tag{4.6}$$

$$\sum_{i=0}^{m-2}\mathbb{P}[M(n_S-1,q)=i]+\sum_{i=m-1}^{n_S-1}\mathbb{P}[M(n_S-1,q)=i]\frac{m}{i+1}=\frac{\mathbb{E}[\bar{M}(n_S,q)]}{n_Sq}. \tag{4.7}$$

$$\mathbb{P}[M(n_S-1,q)=m-1]=\frac{m}{n_Sq}\mathbb{P}[M(n_S,q)=m]. \tag{4.8}$$

Proof.

$$\begin{aligned}\sum_{i=m-1}^{n_S-1} \mathbb{P}[M(n_S-1,q)=i]\frac{m}{i+1} &= \sum_{i=m-1}^{n_S-1} \binom{n_S-1}{i} q^i(1-q)^{n_S-1-i}\frac{m}{i+1}\\ &= \sum_{i=m-1}^{n_S-1} \frac{1}{n_S q}\binom{n_S}{i+1} q^{i+1}(1-q)^{n_S-(i+1)}m\\ &= \sum_{k=m}^{n_S} \frac{1}{nq}\binom{n_S}{k} q^k(1-q)^{n_S-k}m\\ &= \frac{m}{n_S q}\cdot \mathbb{P}[M(n_S,q)\geq m].\end{aligned}$$

$$\begin{aligned}\mathbb{E}[\bar{M}(n_S,q)] &= \mathbb{P}[M(n_S,q)\geq m]m + \sum_{i=0}^{m-1}\mathbb{P}[M(n_S,q)=i]i\\ &= \mathbb{P}[M(n_S,q)\geq m]m + \sum_{i=1}^{m-1}\binom{n_S}{i}q^i(1-q)^{n_S-i}i\\ &= \mathbb{P}[M(n_S,q)\geq m]m + \sum_{i=1}^{m-1}\binom{n_S-1}{i-1}n_S\cdot q\cdot q^{i-1}(1-q)^{n_S-i}\\ &= \mathbb{P}[M(n_S,q)\geq m]m + \sum_{k=0}^{m-2}\binom{n_S-1}{k}n_S\cdot q\cdot q^{k}(1-q)^{n_S-1-k}\\ &= n_S q\Big(\mathbb{P}[M(n_S,q)\geq m]\frac{m}{n_S q} - \mathbb{P}[M(n_S-1,q)\leq m-2]\Big),\end{aligned}$$

and plugging (4.6) into the equation, (4.7) follows.

$$\begin{aligned}\mathbb{P}[M(n_S-1,q)=m-1] &= \binom{n_S-1}{m-1}q^{m-1}(1-q)^{n_S-m}\\ &= \frac{(n_S-1)!}{(n_S-m)!(m-1)!}q^{m-1}(1-q)^{n_S-m}\\ &= \frac{(n_S)!}{(n_S-m)!(m)!}\frac{m}{n_S q}q^{m}(1-q)^{n_S-m}\\ &= \frac{m}{n_S q}\mathbb{P}[M(n_S,q)=m].\end{aligned}$$

□

Lemma 4.5

$$\phi(q) = \frac{\mathbb{P}[M(n_S-1,q)=m-1]n_S q}{\mathbb{E}[\bar{M}(n_S,q)]}$$

is continuous, strictly concave, with a unique maximum $\bar{\phi} < 1$, *and* $\phi(0) = \phi(1) = 0$. *Also,* $\lim_{n\to\infty}\phi(q) = 0$ *for all* q. *Further, there are two continuous functions*

$q_-(\phi), q_+(\phi)$ with domain $[0,\bar{\phi}]$ of which q_- is strictly increasing and q_+ is strictly decreasing. For all $\phi \in [0,\bar{\phi})$ it holds that $q_-(\phi) < q_+(\phi)$ but $\bar{\phi} = \phi(q_-) = \phi(q_+)$. In particular, $q_-(0) = 0$ and $q_+(0) = 1$.

Proof.

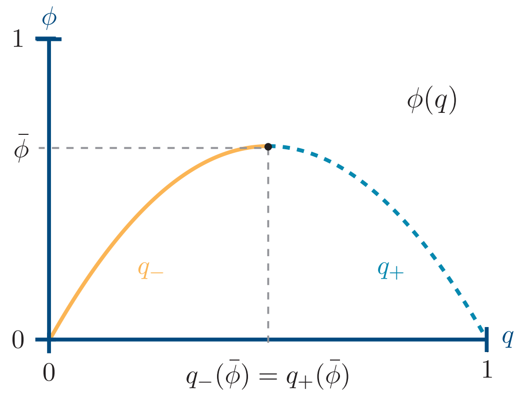

Figure 4.A.1 Form of $\phi(q)$ and definition of q_- and q_+.

Using (4.8), $\frac{1}{\phi(q)}$ can be rewritten as

$$\Longleftrightarrow \quad \frac{1}{\phi(q)} = \frac{\sum_{i=0}^{m} \mathbb{P}[M(n_S,q)=i]i + \mathbb{P}[M(n_S,q)>m]m}{m\mathbb{P}[M(n_S,q)=m]}$$

$$\Longleftrightarrow \quad \frac{1}{\phi(q)} = \frac{\sum_{i=1}^{m} \binom{n_S}{i} q^i(1-q)^{n_S-i}i + \sum_{i=m+1}^{n_S} \binom{n_S}{i} q^i(1-q)^{n_S-i}m}{m\binom{n_S}{m}q^m(1-q)^{n_S-m}}$$

$$\Longleftrightarrow \quad \frac{1}{\phi(q)} = \frac{1}{m\binom{n_S}{m}}[\sum_{i=1}^{m} \binom{n_S}{i} q^{i-m}(1-q)^{m-i}i + \sum_{i=m+1}^{n_S} \binom{n_S}{i} q^{i-m}(1-q)^{m-i}m]$$

$$\Longleftrightarrow \quad \frac{1}{\phi(q)} = \frac{1}{m\binom{n_S}{m}}[\sum_{i=1}^{m} \binom{n_S}{i} (\frac{q}{1-q})^{i-m}i + \sum_{i=m+1}^{n_S} \binom{n_S}{i} (\frac{q}{1-q})^{i-m}m].$$

Both summands are strictly convex in q such that $\frac{1}{\phi(q)}$ is strictly convex in q. Further, $\lim_{q\to 0} \frac{1}{\phi(q)} = \lim_{q\to 1} \frac{1}{\phi(q)} = \infty$, such that $\frac{1}{\phi(q)}$ is U-shaped. Since $\frac{1}{\phi(q)} \geq 0$, it follows that ϕ is hump-shaped with $\phi(0) = \phi(1) = 0$ and a unique maximum $\bar{\phi}$. Further, because

$$\phi(q) = \frac{\mathbb{P}[M(n_S-1,q)=m-1]n_S q}{\mathbb{E}[\min\{m, M(n_S,q)\}]} < \frac{\mathbb{P}[M(n_S-1,q)=m-1]n_S q}{n_S q},$$

Lemma 4.3 implies that $\bar{\phi} < 1$ and $\lim_{n\to\infty} \phi(q) = 0$.

Last, since ϕ is hump-shaped, with $\phi(0) = \phi(1) = 0$ and a unique maximum $\bar{\phi}$, for all $p < \bar{\phi}$, there are exactly two functions $q_-(p) < q_+(p)$, such that $p = \phi(q_-(p)) = \phi(q_+(p))$. Since ϕ is continuous, so are q_- and q_+. □

4.B Proofs

4.B.1 Proof of Proposition 4.1

Note that $\Pi_S(\text{sell}; p, q) = \Pi_S(\text{keep}; p, q)$ rearranges to

$$\mathbb{P}[M(n_S - 1, q(p)) = m - 1]\frac{\Delta}{n} = p\frac{\mathbb{E}[\bar{M}(n_S, q(p))]}{n_S q(p)}. \tag{4.9}$$

Step 1 *There is always an equilibrium in which $p^* = 0$ and $q^*(0) = 1$.*

Since $1 < m < n_S$ and $n_S \geq 3$, if $q^*(0) = 1$, no shareholder is pivotal and selling the voting right is a best response. Since this is the lowest possible price, the activist has no profitable deviation.

Step 2 *If $q^*(p^*) > 0$ and, thereby, $\mathbb{P}[M(n_S, q^*(p^*)) \geq m] > 0$, then it has to hold that $p^*\mathbb{E}[\bar{M}(n_S, q^*(p^*))] < m\frac{\Delta}{n}\mathbb{P}[M(n_S, q^*(p^*)) \geq m]$.*

If $q^*(p^*) \in (0, 1)$, then (4.9) holds with equality. Further, by (4.8), equation (4.9) can restated as

$$p^*\mathbb{E}[\bar{M}(n_S, q^*(p^*))] = \mathbb{P}[M(n_S, q^*(p^*)) = m]m\frac{\Delta}{n} < \mathbb{P}[M(n_S, q^*(p^*)) \geq m]m\frac{\Delta}{n}.$$

Now suppose that $q^*(p^*) = 1$. Using Lemma 4.5, let $\bar{p} = \max_q \phi(q)\frac{\Delta}{n} < \frac{\Delta}{n}$. At any $p > \bar{p}$, equation (4.9) cannot hold with equality, such that $q^*(p) = 1$. It follows that if $q^*(p^*) = 1$, then $p^* \leq \bar{p}$, otherwise a deviation to a price $\frac{\bar{p}+p^*}{2}$ would be strictly profitable. Thereby, $p^*\mathbb{E}[\bar{M}(n_S, q^*(p^*))] < \frac{\Delta}{n}m = \frac{\Delta}{n}\mathbb{P}[M(n_S, q^*(p^*)) \geq m]$.

Step 3 $\lim_{n\to\infty} \mathbb{P}[M(n_S, q^*(p^*)) \geq m] = 1$ *and* $\lim_{n\to\infty} p^*\mathbb{E}[\bar{M}(n_S, q^*(p^*))] = 0$.

Suppose to the contrary that one of the statements was violated. In this case

$$\alpha(v + \Delta) + \mathbb{P}[M(n_S, q^*(p^*)) \geq m](b - \alpha\Delta) - p^*\mathbb{E}[\bar{M}(n_S, q^*(p^*))] < \alpha v + b$$

for n arbitrary large. Using Lemma 4.5, let $\bar{p} = \max_q \phi(q)\frac{\Delta}{n}$, and consider a deviation to $p' = \bar{p} + \frac{\epsilon}{m}$. Since $n \cdot \bar{p} \to 0$ and $q^*(p') = 1$, it follows that

$$\lim_{n\to\infty}\alpha(v + \Delta) + \mathbb{P}[M(n_S, q^*(p')) \geq m](b - \alpha\Delta) - p'\mathbb{E}[\bar{M}(n_S, q^*(p'))] = \alpha v + b - \epsilon,$$

such that the deviation is profitable when ϵ is small and n is large.

Step 4 *When b and n are small, there are equilibria in which there is no trade.*

Using Lemma 4.5, there is a best response $q^*(p) = q_-(p)$, which is continuous and strictly increasing on $[0, \bar{p}]$ with $\bar{p} = \max_q \phi(q)\frac{\Delta}{n}$. Further, $q^*(0) = 0$ and $q^*(p) = 1$ for all $p > \bar{p}$.

Suppose that the activist offers a price $p^* \in (0, \bar{p})$ such that $q^*(p^*) \in (0,1)$ and equality (4.9) holds. Since p^* is a best response, $\Pi_A(p^*; q^*) \geq \Pi_A(0; q^*) = \alpha(v+\Delta)$. Plugging (4.9) into $\Pi_A(p; q^*)$ and using (4.8), this can be rearranged to

$$\alpha(v+\Delta) - \mathbb{P}[M(n_S, q) = m]\Delta\frac{m}{n} + \mathbb{P}[M(n,q) \geq m](b - \alpha\Delta) \geq \alpha(v+\Delta)$$
$$\iff -\Delta\frac{m}{n} + \frac{\mathbb{P}[M(n_S, q) \geq m]}{\mathbb{P}[M(n_S, q) = m]}(b - \alpha\Delta) > 0.$$

The likelihood ratio

$$\frac{\mathbb{P}[M(n_S, q) \geq m]}{\mathbb{P}[M(n_S, q) = m]} = \frac{\sum_{i=m}^{n_S} \binom{n_S}{i} q^i (1-q)^{n-i}}{\binom{n_S}{m} q^m (1-q)^{n_S - m}} = \frac{1}{\binom{n_S}{m}} \sum_{i=m}^{n_S} \binom{n_S}{i} (\frac{q}{1-q})^{i-m} \tag{4.10}$$
$$= \frac{1}{\binom{n_S}{m}} \binom{n_S}{m} (\frac{q}{1-q})^0 + \sum_{i=m+1}^{n_S} \binom{n_S}{i} (\frac{q}{1-q})^{i-m} \overset{q \to 0}{\to} 1.$$

Thus, for p (and, hence, $q^*(p)$) sufficiently low, $\Pi_A(p; q^*) < \Pi_A(0; q^*)$ when $-\Delta\frac{m}{n} + \frac{\mathbb{P}[M(n_S,q) \geq m]}{\mathbb{P}[M(n_S,q)=m]}(b - \alpha\Delta) \approx b - (1-\lambda)\Delta - \frac{1}{n}\Delta < 0$. Further, any price above $\frac{b}{m}$ is dominated by offering $p = 0$ and not trading. If b is sufficiently small, this means that we found a contradiction and $p = 0$ is the unique best response.

4.B.2 Proof of Proposition 4.2

To enhance clarity, we prove equilibrium existence separately in Lemma 4.6 and characterize the equilibrium first.

Suppose the activist offers p_A, the blockholder p_B, and shareholders mix with probability $q(p_A, p_B)$. Then, an individual shareholder (weakly) prefers to sell to A if and only if

$$\mathbb{P}[M(n_S - 1, q(p_A, p_B)) < m_A - 1]\frac{\Delta}{n} + p_A \frac{\mathbb{E}[\bar{M}_A(n_S, q(p_A, p_B))]}{n_S q(p_A, p_B)}$$
$$\geq \mathbb{P}[M(n_S - 1; q(p_A, p_B)) < m_A]\frac{\Delta}{n} + p_B \frac{\mathbb{E}[\bar{M}_B(n_S, q(p_A, p_B))]}{n_S(1 - q(p_A, p_B))}$$
$$\iff p_A \frac{\mathbb{E}[\bar{M}_A(n_S, q)]}{n_S q} - \mathbb{P}[M(n_S - 1; q) = m_A - 1]\frac{\Delta}{n} \geq p_B \frac{\mathbb{E}[\bar{M}_B(n_S, q)]}{n_S(1-q)}. \tag{4.11}$$

The expected payoffs for the activist and blockholder are

$$\Pi_A(p_A; p_B, q)$$
$$= \alpha(v+\Delta) + \mathbb{P}[M(n_S, q(p_A, p_B)) \geq m_A](b - \alpha\Delta) - p_A \mathbb{E}[\bar{M}_A(n_S, q(p_A, p_B))],$$
$$\Pi_B(p_B; p_A, q)$$
$$= \beta(v+\Delta) + \mathbb{P}[M(n_S, q(p_A, p_B)) \geq m_A](-\beta\Delta) - p_B \mathbb{E}[\bar{M}_B(n_S, q(p_A, p_B))].$$

In an effort to keep notation cleaner, we henceforth drop the explicit reference to the shareholders' strategy q.

For any n, let $p_{A;n}$ and $p_{B;n}$ be any two prices and let q_n^* be a best responses. Given q_n^*, let $p_{B;n}^*$ be a best response, and, given q_n^* and $p_{B;n}^*$, let $p_{A;n}^*$ be an equilibrium price. We take converging (sub)sequences of prices and probabilities as needed.

Step 0 *Suppose that* $\lim p_{A;n}n > 0$ *and/or* $\lim p_{B;n}n > 0$.

1. *If* $\lim \frac{p_{A;n}}{p_{B;n}} > \frac{1-\alpha-\beta}{1-\lambda-\alpha}$, *then* $q_n^*(p_{A;n}, p_{B;n}) = 1$ *when n is sufficiently large;*
2. *If* $\lim \frac{p_{A;n}}{p_{B;n}} > 1$ *but* $\lim \frac{p_{A;n}}{p_{B;n}} \leq \frac{1-\alpha-\beta}{1-\lambda-\alpha}$, *then* $\lim q_n^*(p_{A;n}, p_{B;n}) = \lim \frac{p_{A;n}}{p_{B;n}} \frac{1-\lambda-\alpha}{1-\alpha-\beta}$ *and* $\lim \mathbb{P}[M(n_S, q_n^*(p_{A;n}, p_{B;n})) \geq m_A] = 1$;
3. *If* $\lim \frac{p_{A;n}}{p_{B;n}} = 1$, *then* $\lim q_n^*(p_{A;n}, p_{B;n}) = \frac{1-\lambda-\alpha}{1-\alpha-\beta}$ *as well as* $\lim \mathbb{P}[M(n_S, q_n^*(p_{A;n}, p_{B;n})) \geq m_A] = \frac{1}{2}$;
4. *If* $\lim \frac{p_{A;n}}{p_{B;n}} < 1$ *but* $\lim \frac{p_{A;n}}{p_{B;n}} \geq \frac{\lambda-\beta}{1-\alpha-\beta}$, *then* $\lim q_n^*(p_{A;n}, p_{B;n}) = 1 - \lim \frac{p_{B;n}}{p_{A;n}} \frac{\lambda-\beta}{1-\alpha-\beta}$ *and* $\lim \mathbb{P}[M(n_S, q_n^*(p_{A;n}, p_{B;n})) \geq m_A] = 0$;
5. *If* $\lim \frac{p_{A;n}}{p_{B;n}} < \frac{\lambda-\beta}{1-\alpha-\beta}$, *then* $q_n^*(p_{A;n}, p_{B;n}) = 0$ *when n is sufficiently large.*

For ease of notation, let $q_n^* = q^*(p_{A,n}, p_{B,n})$.

By Lemma 4.3, for any q, $\lim \mathbb{P}[M(n_S - 1, q_n^*) = m_A - 1] = 0$. Further, by the LLN, if $\lim q_n^* > \frac{1-\lambda-\alpha}{1-\alpha-\beta}$, then $\lim \mathbb{P}[M(n_S, q_n^*) \geq m_A] = 1$, $\lim \frac{\mathbb{E}[\bar{M}_A(n_S, q_n^*)]}{n_S q_n^*} = \lim \frac{1-\lambda-\alpha}{q_n^*(1-\alpha-\beta)}$, and $\lim \frac{\mathbb{E}[\bar{M}_B(n_S, q_n^*)]}{n_S(1-q_n^*)} = 1$. If, on the other hand, $q_n^* < \frac{1-\lambda-\alpha}{1-\alpha-\beta}$, then $\lim \mathbb{P}[M(n_S, q_n^*) < m_A] = 1$, $\lim \frac{\mathbb{E}[\bar{M}_B(n_S, q_n^*)]}{n_S(1-q_n^*)} = \lim \frac{\lambda-\beta}{(1-q_n^*)(1-\alpha-\beta)}$, and $\lim \frac{\mathbb{E}[\bar{M}_A(n_S, q_n^*)]}{n_S q_n^*} = 1$. Last, if $\lim q_n^* = \frac{1-\lambda-\alpha}{1-\alpha-\beta}$, then $\lim \frac{\mathbb{E}[\bar{M}_A(n_S, q_n^*)]}{n_S q_n^*} = \lim \frac{\mathbb{E}[\bar{M}_B(n_S, q_n^*)]}{n_S(1-q_n^*)} = 1$ and $\lim \mathbb{P}[M(n_S, q_n^*(p_{A;n}, p_{B;n})) \geq m_A] = \frac{1}{2}$.

If $q_n^* = 1$ and n is arbitrary large, then inequality (4.11), $\lim \mathbb{P}[M(n_S - 1, q_n^*) = m_A - 1] = 0$, and $\lim p_{A;n}n > 0$ or $\lim p_{B;n}n > 0$ imply that $\lim \frac{p_{A;n}}{p_{B;n}} \geq \frac{\lambda-\beta}{1-\alpha-\beta}$. If $q_n^* = 0$ for n arbitrary large, the inequality of (4.11) reverses. Since $\lim \mathbb{P}[M(n_S - 1, q_n^*) = m_A - 1] = 0$, and $\lim p_{A;n}n > 0$ or $\lim p_{B;n}n > 0$, it follow that $\lim \frac{p_{A;n}}{p_{B;n}} \leq \frac{\lambda-\beta}{1-\alpha-\beta}$.

Suppose that $\lim \frac{p_{A;n}}{p_{B;n}} = \gamma > 1$. When $q_n^* < 1$ s.th. (4.11) holds with equality, $\lim \mathbb{P}[M(n_S - 1, q_n^*) = m_A - 1] = 0$, and $\lim p_{A;n}n > 0$ or $\lim p_{B;n}n > 0$, it follows that $\lim \frac{\mathbb{E}[\bar{M}_A(n_S, q_n^*)]}{\mathbb{E}[\bar{M}_B(n_S, q_n^*)]} \frac{1-q_n^*}{q_n^*} = \frac{1}{\gamma}$. By our earlier observation, this means that $\lim q_n^* > \frac{1-\lambda-\alpha}{1-\alpha-\beta}$, such that equality (4.11) implies that $\lim q_n^* = \gamma \frac{1-\lambda-\alpha}{1-\alpha-\beta} = \lim \frac{p_{A;n}}{p_{B;n}} \frac{1-\lambda-\alpha}{1-\alpha-\beta}$. If $\gamma > \frac{1-\alpha-\beta}{1-\lambda-\alpha}$, equality (4.11) cannot hold when n is large, such that $q_n^* = 1$. In either case $\lim \mathbb{P}[M(n_S, q_n^*) \geq m_A] = 1$. This proves properties *1* and *2*.

Next, consider the case in which $\lim \frac{p_{A;n}}{p_{B;n}} = \gamma < 1$. When $q_n^* > 0$ s.th. (4.11) holds with equality, $\lim \mathbb{P}[M(n_S - 1, q_n^*) = m_A - 1] = 0$, and $\lim p_{A;n}n > 0$ or $\lim p_{B;n}n > 0$, it follows that $\lim \frac{\mathbb{E}[\bar{M}_A(n_S, q_n^*)]}{\mathbb{E}[\bar{M}_B(n_S, q_n^*)]} \frac{1-q_n^*}{q_n^*} = \frac{1}{\gamma}$. By our earlier observation, this means that $\lim q_n^* < \frac{1-\lambda-\alpha}{1-\alpha-\beta}$, such that equality (4.11) implies that $\lim 1 - q_n^* = \lim \frac{p_{B;n}}{p_{A;n}} \frac{\lambda-\beta}{1-\alpha-\beta}$.

If $\gamma < \frac{\lambda-\beta}{1-\alpha-\beta}$, equality (4.11) cannot hold when n is large, such that $q_n^* = 0$. In either case $\lim \mathbb{P}[M(n_S, q_n^*) \geq m_A] = 0$. This proves properties *4* and *5*.

Last, if $\lim \frac{p_{A;n}}{p_{B;n}} = 1$, then equality (4.11), $\lim \mathbb{P}[M(n_S-1, q_n^*) = m_A - 1] = 0$, and $\lim p_{A;n} n > 0$ or $\lim p_{B;n} n > 0$ imply that $\lim \frac{\mathbb{E}[\bar{M}_A(n_S,q_n^*)]}{\mathbb{E}[\bar{M}_B(n_S,q_n^*)]}\frac{1-q_n^*}{q_n^*} = 1$. By our observation, this is the case if and only if $\lim q_n^* = \frac{1-\lambda-\alpha}{1-\alpha-\beta}$. This proves property *3*.

Step 1 *If* $\frac{b-\alpha\Delta}{1-\lambda-\alpha} > \frac{\beta\Delta}{\lambda-\beta}$ *and* n *is sufficiently large, then* $q_n^*(p_{A;n}^*, p_{B;n}^*(p_{A;n}^*)) = 1$. *Further,* $p_{A;n}^* \mathbb{E}[\bar{M}_A(n_S, q_n^*(p_{A;n}^*, p_{B;n}^*(p_{A;n}^*)))] = p_{A;n}^* m_A < \frac{1-\lambda-\alpha}{\lambda-\beta}\beta\Delta$, *but* $\lim_{n\to\infty} \mathbb{E}[\bar{M}_A(n_S, q_n^*(p_{A;n}^*, p_{B;n}^*(p_{A;n}^*)))]p_{A;n}^* = \frac{1-\lambda-\alpha}{\lambda-\beta}\beta\Delta$.

Suppose to the contrary that $q_n^*(p_{A,n}^*, p_{B,n}^*(p_{A,n}^*)) < 1$ even when n is arbitrary large. When there is no room for confusion, we employ the convention that $q_n^* = q_n^*(p_{A,n}^*, p_{B,n}^*(p_{A,n}^*))$ and $p_{B,n}^* = p_{B,n}^*(p_{A,n}^*)$.

First, we consider the case in which $\lim q_n^* > \frac{1-\lambda-\alpha}{1-\alpha-\beta}$. Observe that

$$\lim \frac{1-q_n^*}{\sum_{i=0}^{m_A-1} \mathbb{P}[M(n_S, q_n^*) = i]} = \infty.$$

For $\lim q_n^* < 1$, this follows directly, when $\lim q_n^* = 1$, we apply L'Hopital[32] to receive

$$\lim \frac{1-q_n^*}{\sum_{i=0}^{m_A-1} \mathbb{P}[M(n_S, q_n^*) = i]} = \lim \frac{1}{\mathbb{P}[M(n_S-1, q_n^*) = m_A - 1]} = \infty.$$

Since $\lim q_n^* > \frac{1-\lambda-\alpha}{1-\alpha-\beta}$ and $\lim \sum_{i=m_A}^{n_S-1} \mathbb{P}[M(n_S-1, q_n^*) = i] = 1$, this means that

$$\begin{aligned}
&\frac{\mathbb{E}[\bar{M}_B(n_S, q_n^*)]}{n(1-\mathbb{P}[M(n_S, q_n^*) \geq m_A])} \\
=&\frac{\sum_{i=0}^{m_A-1} \mathbb{P}[M(n_S, q_n^*) = i]m_B + \sum_{i=m_A}^{n_S} \mathbb{P}[M(n_S, q_n^*) = i](n_S - i)}{n\sum_{i=0}^{m_A-1} \mathbb{P}[M(n_S, q_n^*) = i]} \\
=&\frac{\sum_{i=0}^{m_A-1} \mathbb{P}[M(n_S, q_n^*) = i]m_B + \sum_{i=m_A}^{n_S-1} \binom{n_S-1}{i}(q_n^*)^i(1-q_n^*)^{n_S-1-i}(1-q_n^*)n_S}{n\sum_{i=0}^{m_A-1} \mathbb{P}[M(n_S, q_n^*) = i]} \\
=&\frac{m_B}{n} + (1-\alpha-\beta)\sum_{i=m_A}^{n_S-1} \mathbb{P}[M(n_S-1, q_n^*) = i]\frac{1-q_n^*}{\sum_{i=0}^{m_A-1} \mathbb{P}[M(n_S, q_n^*) = i]}
\end{aligned}$$

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$$\begin{aligned}
&\frac{\partial}{\partial q}\sum_{i=0}^{m_A-1} \mathbb{P}[M(n_S, q_n^*) = i] \\
=&\sum_{i=1}^{m_A-1}\binom{n_S}{i}[i(q_n^*)^{i-1}(1-q_n^*)^{n_S-i}] - \sum_{i=0}^{m_A-1}\binom{n_S}{i}[(q_n^*)^i(1-q_n^*)^{n_S-i-1}(n_S-i)] \\
=&\sum_{i=0}^{m_A-2}\mathbb{P}[M(n_S-1, q_n^*) = i]n_S - \sum_{i=0}^{m_A-1}\mathbb{P}[M(n_S-1, q_n^*) = i]n_S = -\mathbb{P}[M(n_S-1, q_n^*) = m_A - 1].
\end{aligned}$$

grows without bound. This growth implies that $\lim p^*_{B;n} n = 0$, because when $\lim p^*_{B;n} n > 0$ and n is large

$$\beta(v+\Delta) + \mathbb{P}[M(n_S, q^*_n) \geq m_A](-\beta\Delta) - p_{B;n}\mathbb{E}[\bar{M}_B(n_S, q^*_n)] < \beta v$$
$$\iff \beta\Delta < \frac{\mathbb{E}[\bar{M}_B(n_S, q^*_n)]}{n(1 - \mathbb{P}[M(n_S, q^*_n) \geq m_A])} p_{B;n} n,$$

such that a deviation by B to $p_B = 0$ is strictly profitable. If $\lim p^*_{B;n} n = 0$, then $\lim q^*_n \geq \frac{1-\lambda-\alpha}{1-\alpha-\beta}$ and Step 0 imply that $\lim p^*_{A;n} n = 0$. This means that when n is sufficiently large, B has an incentive to deviate to $p'_{B;n} = p^*_{A;n} + \frac{\epsilon}{n}$. By Step 0, when n is sufficiently large, $q^*_n(p^*_{A;n}, p'_{B;n}) = 0$, implying that

$$\Pi^n_B(p'_{B;n}; p^*_{A;n}) = \beta(v+\Delta) - n(\lambda - \beta)(p^*_{A;n} + \frac{\epsilon}{n}),$$

which is obviously larger than $\Pi^n_B(p^*_{B;n}; p^*_{A;n})$ when ϵ is sufficiently small and n is large. Consequently, it cannot be that $q^*_n < 1$ for n arbitrary large, but $\lim q^*_n > \frac{1-\lambda-\alpha}{1-\alpha-\beta}$.

In a second step, suppose that $\lim q^*_n = \frac{1-\lambda-\alpha}{1-\alpha-\beta}$. If $\lim p^*_{A;n} n > 0$ or $\lim p^*_{B;n} n > 0$, then Step 0 implies that $\lim p^*_{A;n} n = \lim p^*_{B;n} n$ and $\lim \mathbb{P}[M(n_S, q^*_n) \geq m_A] = \frac{1}{2}$, such that

$$\lim \Pi^n_B(p^*_{B;n}; p^*_{A;n}) = \beta v + \frac{1}{2}\beta\Delta - \lim p_{B;n} n(\lambda - \beta).$$

Now consider a deviation by B to $p'_{B;n} = p^*_{B;n} + \frac{\epsilon}{n}$ which, by Step 0, guarantees that $\lim \mathbb{P}[M(n_S, q^*(p_{A;n}, p'_{B;n})) \geq m_A] = 0$ and, hence, yields

$$\lim \Pi^n_B(p'_{B;n}; p^*_{A;n}) = \beta v + \beta\Delta - \lim p^*_{B;n} n(\lambda - \beta) - \epsilon(\lambda - \beta).$$

When n is sufficiently large and ϵ sufficiently small, such a deviation is always profitable. When $\lim p^*_{A;n} n = \lim p^*_{B;n} n = 0$, the same deviation is profitable.

Last, suppose that $\lim q^*_n < \frac{1-\lambda-\alpha}{1-\alpha-\beta}$. Then $\lim \mathbb{P}[M(n_S, q^*_n) \geq m_A] = 0$, such that $\lim \Pi^n_A(p^*_{A;n}; p^*_{B;n}) \leq \alpha(v+\Delta)$. Now consider a deviation by A to $p'_{A;n} = \frac{\beta\Delta}{n(\lambda-\beta)}$ and B's possible responses. If B offers $p^*_{B;n}(p'_A)$ such that $\lim \frac{p'_{A;n}}{p^*_{B;n}(p'_{A;n})} < 1$, then, by Step 0, $\lim \mathbb{P}[M(n_S, q^*_n(p'_{A;n}, p^*_{B;n}(p'_{A;n}))) \geq m_A] = 0$, and because $\lim p^*_{B;n}(p'_{A;n}) n > \frac{\beta\Delta}{(\lambda-\beta)}$, it follows that

$$\lim \Pi^n_B(p^*_{B;n}(p'_{A;n}); p'_{A;n}) < \beta(v+\Delta) - (\lambda - \beta)\frac{\beta\Delta}{(\lambda - \beta)} = \beta v,$$

which is dominated by $p_B = 0$ when n is sufficiently large. If B offers $p^*_{B;n}(p'_{A;n})$ such that $\lim \frac{p'_{A;n}}{p^*_{B;n}(p'_{A;n})} = 1$, then, by our observation above, B has a strict incentive to deviate upwards. Thus, B has to respond by offering $p^*_{B;n}(p'_{A;n})$ such that

$\lim \frac{p'_{A;n}}{p^*_{B;n}(p'_{A;n})} > 1$. As a result, $\lim \mathbb{P}[M(n_S, q^*(p'_{A;n}, p^*_{B;n}(p'_{A;n}))) \geq m_A] = 1$ and, in the limit, the deviation yields A the payoff

$$\lim \Pi^n_A(p'_{A;n}; p^*_{B;n}(p'_{A;n})) = \alpha v + b - (1-\lambda-\alpha)\frac{\beta\Delta}{(\lambda-\beta)},$$

which is larger than $\alpha(v+\Delta)$ by assumption. Hence, the deviation is profitable for A when n is sufficiently large. This proves that $q^*_n = 1$ when n is sufficiently large.

When $q^*_n = 1$ and $p^*_{A;n} m_A = p^*_{A;n}\mathbb{E}[\bar{M}_A(n_S, q^*(p^*_{A;n}, p^*_{B;n}))] \geq \frac{1-\lambda-\alpha}{\lambda-\beta}\beta\Delta$, then $p^*_{A;n} \geq \frac{\beta\Delta}{n(\lambda-\beta)} - \frac{\beta\Delta}{m_A n(\lambda-\beta)}$. Suppose A chooses or deviates to $p'_{A;n} = \frac{\beta\Delta}{n(\lambda-\beta)} - \frac{\beta\Delta}{m_A n(\lambda-\beta)}$. If B offers $p^*_{B;n}(p'_{A;n})$ such that $\lim \frac{p'_{A;n}}{p^*_{B;n}(p'_{A;n})} < 1$, then $\lim p^*_{B;n}(p'_{A;n})n > \lim \frac{\beta\Delta}{(\lambda-\beta)}$ and by Step 0, it follows that $\lim \mathbb{P}[M(n_S, q^*_n(p'_{A;n}, p^*_{B;n}(p'_{A;n}))) \geq m_A] = 0$. However, in this case,

$$\lim \Pi^n_B(p^*_{B;n}(p'_{A;n}); p'_{A;n}) < \beta(v+\Delta) - (\lambda-\beta)\frac{\beta\Delta}{(\lambda-\beta)} = \beta v,$$

such that $p^*_{B;n}(p'_{A;n})$ is dominated by $p_B = 0$ when n is sufficiently large. If B offers $p^*_{B;n}(p'_{A;n})$ such that $\lim \frac{p'_{A;n}}{p^*_{B;n}(p'_{A;n})} = 1$, then, by our observation above, B would have a strict incentive to deviate upwards. This means that B has to choose a $p^*_{B;n}(p'_{A;n})$ such that $\lim \frac{p'_{A;n}}{p^*_{B;n}(p'_{A;n})} > 1$, which implies, by our previous argument, that $q^*_n(p'_{A;n}, p^*_{B;n}(p'_{A;n})) = 1$ when n is large. Thereby, the deviation to $p'_{A;n}$ is profitable for A when n is sufficiently large. Further, because all expressions are continuous and inequalities strict, the same can be achieved with a $p'_{A;n}$ marginally below $\frac{\beta\Delta}{n(\lambda-\beta)} - \frac{\beta\Delta}{m_A n(\lambda-\beta)}$, meaning that $p'_{A;n} m_A = p'_{A;n}\mathbb{E}[\bar{M}_A(n_S, q^*(p'_{A;n}, p^*_{B;n}(p'_{A;n})))] < \frac{1-\lambda-\alpha}{\lambda-\beta}\beta\Delta$.

Last, if $\lim p^*_A \mathbb{E}[\bar{M}_A(n_S, q^*(p^*_A, p^*_B))] < \frac{1-\lambda-\alpha}{\lambda-\beta}\beta\Delta$, this means that $p^*_{A;n} < \frac{\beta\Delta}{n(\lambda-\beta)} - \frac{\epsilon}{n}$ for some $\epsilon > 0$ and any n sufficiently large. In this case, however, B could deviate to $p'_{B;n} = \frac{\beta\Delta}{n(\lambda-\beta)} - \frac{\epsilon}{2n}$. As a result, $\lim \mathbb{P}[M(n_S, q^*_n(p^*_{A;n}, p'_{B;n})) \geq m_A] = 0$ and

$$\lim \Pi^n_B(p'_{B;n}; p^*_{A;n}) = \beta(v+\Delta) - \beta\Delta + \epsilon\frac{\beta\Delta}{2(\lambda-\beta)} > \beta v,$$

such that the deviation is profitable when n is sufficiently large.

Step 2 *If* $\frac{b-\alpha\Delta}{1-\lambda-\alpha} < \frac{\beta\Delta}{\lambda-\beta}$, *as n grows large, along any sequence of equilibria,* $\lim_{n\to\infty} \mathbb{P}[M_A(n_S, q^*_n(p^*_{A;n}, p^*_{B;n}(p^*_{A;n}))) \geq m_A] = 0$ *and* $\lim_{n\to\infty} p^*_{A;n} n_S = \lim_{n\to\infty} p^*_{B;n}(p^*_{A;n}) n_S = 0$.

For ease of notation, let $q^*_n = q^*_n(p^*_{A,n}, p^*_{B,n}(p^*_{A,n}))$. When there is no room for confusion, we employ the convention that $p^*_{B,n} = p^*_{B,n}(p^*_{A,n})$.

First, suppose to the contrary that $\lim \mathbb{P}[M(n_S, q_n^*) \geq m_A] > 0$. Since $\Pi_A^n(p_{A;n}^*; p_{B;n}^*) \geq \Pi_A^n(0; p_{B;n}^*(0)) \geq \alpha(v+\Delta)$, it follow that

$$\begin{aligned}\Pi_A^n(p_{A;n}^*; p_{B;n}^*) &= \alpha(v+\Delta) + (b-\alpha\Delta)\mathbb{P}[M(n_S, q_n^*) \geq m_A] - p_{A;n}^*\mathbb{E}[\bar{M}_A(n_S, q_n^*)] \\ &\geq \alpha(v+\Delta).\end{aligned}$$

Since $\frac{\mathbb{E}[\bar{M}_A(n_S,q_n^*)]}{\mathbb{P}[M(n_S,q_n^*)\geq m_A]} \geq m_A$ and $m_A = n(1-\lambda-\alpha)+1$, it follows that in the limit

$$\lim p_{A;n}^* n \leq \frac{b-\alpha\Delta}{1-\lambda-\alpha}.$$

Now consider a deviation by B from $p_{B;n}^*$ to $p'_{B;n} = p_{A;n}^* + \frac{\epsilon}{n}$. Because $\lim q_n^*(p_{A;n}^*, p'_{B;n}) > \frac{1-\lambda-\beta}{1-\alpha-\beta}$, it follows that $\lim \mathbb{P}[M(n_S, q_n^*(p_{A;n}^*, p'_{B;n})) \geq m_A] = 0$. Such deviation is profitable when $\epsilon > 0$ is small and n is large because

$$\begin{aligned}&\lim \Pi_B^n(p'_{B;n}; p_{A;n}^*) - \Pi_B^n(p_{B;n}^*; p_{A;n}^*) \\ &\geq \lim(1-\mathbb{P}[M(n_S, q_n^* \geq m_A])[\beta\Delta - (\lambda-\beta)np_{A;n}^*] - \epsilon,\end{aligned}$$

where

$$\beta\Delta - (\lambda-\beta)np_{A;n}^* \geq \beta\Delta - (\lambda-\beta)\frac{b-\alpha\Delta}{1-\lambda-\alpha} > 0.$$

This establishes that $\lim \mathbb{P}[M(n_S, q_n^*) \geq m_A] = 0$.

We now show that $\lim p_{A;n}^* n = \lim p_{B;n}^* n = 0$. First, suppose to the contrary that $\lim p_{A;n}^* n > 0$. In this case, it has to hold that $\lim q_n^* > 0$. Assume this was not true either, that is $\lim p_{A;n}^* n > 0$ and $\lim q_n^* = 0$. Then, there is a small $\epsilon > 0$ such that $\lim \frac{p_{A;n}^*}{p_{B;n}^* - \frac{\epsilon}{m_B}} \in (\frac{\lambda-\beta}{1-\alpha-\beta}, 1)$, which still implies that $\lim q_n^*(p_{A;n}^*, p_{B;n}^* - \frac{\epsilon}{m_B}) < \frac{1-\lambda-\alpha}{1-\alpha-\beta}$, and, thereby,

$$\lim \Pi_B^n(p_{B;n}^* - \frac{\epsilon}{m_B}; p_{A;n}^*) - \lim \Pi_B(p_{B;n}^*; p_{A;n}^*) = (\lambda-\beta)\epsilon,$$

making it a profitable deviation when n is large. Now, if $\lim q_n^* > 0$ and $\lim p_{A;n}^* n > 0$ but $\lim \mathbb{P}[M(n_S, q_n^*) \geq m_A] = 0$, then

$$\lim \Pi_A^n(p_{A;n}^*; p_{B;n}^*) = \alpha(v+\Delta) - \lim p_{A;n}^* q_n^* n < \alpha(v+\Delta) \leq \lim \Pi_A^n(0; p_{B;n}^*),$$

such that A would have profitable deviation to 0. Last, if $\lim p_{A;n}^* n = 0$, then $\lim p_{B;n}^* n = 0$. Otherwise, a deviation to $\frac{p_{B;n}^*}{2}$ would always be profitable for B when n is sufficiently large.

Lemma 4.6 *The competition game always has an equilibrium (p_A^*, p_B^*, q^*).*

Proof. We are going to show existence by construction. Fix some p_A. Then, shareholders are indifferent between selling to A and B if $p_B = \psi(q; p_A)$ where

$$\psi(q;p_A) = (p_A \frac{\mathbb{E}[\bar{M}_A(n_S,q)]}{n_S q} - \mathbb{P}[M(n_S-1;q) = m_A - 1]\frac{\Delta}{n})\frac{n_S(1-q)}{\mathbb{E}[\bar{M}_B(n_S,q)]}$$

is a polynomial of q and strictly increasing and continuous in p_A. For later use, we further note that the slope of $\psi(q; p_A)$ with respect to p_A is decreasing in q ($-\psi$ is supermodular): for any $p_A < p_A'$ and $q < q'$, it holds that

$$\psi(q;p_A') - \psi(q;p_A) > \psi(q';p_A') - \psi(q';p_A).$$

We can use $\psi(q; p_A)$ to define a best response for shareholders as

$$q^*(p_A,p_B) = \begin{cases} 1 \text{ for } p_B < \psi(1;p_A) \\ \min\{q\colon \psi(q;p_A) = p_B\} \text{ for } \psi(1;p_A) \le p_B < \psi(0;p_A) \\ 0 \text{ for } p_B \ge \psi(0;p_A). \end{cases}$$

By construction, q^* is (weakly) decreasing and right-continuous in p_B. Note that $q^*(p_A, \psi(q; p_A)) \le q$.

Step 1 *Given any offer p_A, B has at least one best response $p_B^*(p_A)$.*

Since q^* is (weakly) decreasing and right-continuous in p_B and all expressions are bounded, B's problem has at least one solution. We denote an arbitrary one by $p_B^*(p_A)$.

Step 2 *B's problem can be restated as*

$$\begin{aligned} &\operatorname*{arg\,max}_{q \in \text{supp } q^*(p_A,\cdot)} \hat{\Pi}_B(q;p_A) \\ = &\operatorname*{arg\,max}_{q \in \text{supp } q^*(p_A,\cdot)} \beta(v+\Delta) - \mathbb{P}[M(n_S,q) \ge m]\beta\Delta - \psi(q;p_A)\mathbb{E}[\bar{M}_B(n_S,q)]. \end{aligned}$$

If $\hat{\Pi}_B(q; p_A) \ge \beta v$, and $q' < q$ s.th. $\psi(q; p_A) = \psi(q'; p_A)$, then $\hat{\Pi}_B(q'; p_A) > \hat{\Pi}_B(q; p_A)$.

The first restatement follows directly from the definition of ψ and q^*. For the second, note that $\hat{\Pi}_B(q; p_A) \ge \beta v$ can be rearranged to

$$\begin{aligned} &(1 - \mathbb{P}[M(n_S,q) \ge m_A])\beta\Delta - \psi(q;p_A)\mathbb{E}[\bar{M}_B(n_S,q)] \ge 0 \\ \iff &\mathbb{P}[M(n_S,q) < m_A](\beta\Delta - \psi(q;p_A)\frac{\mathbb{E}[\bar{M}_B(n_S,q)]}{\mathbb{P}[M(n_S,q) < m_A]}) \ge 0. \end{aligned} \tag{4.12}$$

We want to show that the left side of (4.12) is strictly decreasing in q. Since $\mathbb{P}[M(n_S, q) < m_A]$ is strictly decreasing in q and (4.12) is positive, it suffices to show that $\frac{\mathbb{E}[\bar{M}_B(n_S,q)]}{\mathbb{P}[M(n_S,q)<m_A]}$ is strictly increasing in q. Note that

$$
\begin{aligned}
\frac{\mathbb{E}[\bar{M}_B(n_S,q)]}{\mathbb{P}[M(n_S,q)<m_A]} &= \frac{\mathbb{P}[M(n_S,q)<m_A]m_B + \sum_{i=0}^{m_B-1}\mathbb{P}[M(n_S,1-q)=i]i}{\mathbb{P}[M(n_S,q)<m_A]} \\
&= m_B + \frac{\sum_{i=0}^{m_B-1}\mathbb{P}[M(n_S,1-q)=i]i}{\sum_{i=m_B}^{n_S}\mathbb{P}[M(n_S,1-q)=i]} \\
&= m_B + \frac{\sum_{i=0}^{m_B-1}\binom{n_S}{i}i(1-q)^i q^{n_S-i}i}{\sum_{i=m_B}^{n_S}\binom{n_S}{i}i(1-q)^i q^{n_S-i}} \\
&= m_B + \frac{\sum_{i=0}^{m_B-1}\binom{n_S}{i}i(\frac{1-q}{q})^{i-(m_B-1)}i}{\sum_{i=m_B}^{n_S}\binom{n_S}{i}(\frac{1-q}{q})^{i-(m_B-1)}},
\end{aligned}
$$

where the numerator is increasing in q for all $i \in (0, ..., m_B-1)$, and the denominator is strictly decreasing in q for all $i \in (m_B, ..., n_S)$. Thereby, the assertion follows.

Step 3 *Any best response $p_B^*(p_A)$ is such that $q(p_A, p_B^*(p_A))$ is nondecreasing in p_A.*

Suppose to the contrary that $p'_A > p_A$, but $q' = q^*(p'_A, p_B^*(p'_A)) < q = q^*(p_A, p_B^*(p_A))$.

If $q \in \text{supp } q^*(p'_A, \cdot)$ and $q' \in \text{supp } q^*(p_A, \cdot)$, then, by revealed preferences,

$$\hat{\Pi}_B(q'; p'_A) \geq \hat{\Pi}_B(q; p'_A) \quad \text{and} \quad \hat{\Pi}_B(q; p_A) \geq \hat{\Pi}_B(q'; p_A). \tag{4.13}$$

Suppose that $q \notin \text{supp } q^*(p'_A, \cdot)$ but $\hat{\Pi}_B(q; p'_A) \geq \beta v$. Then, $q^*(p'_A, \psi(q; p'_A)) < q$ and revealed preferences imply that $\hat{\Pi}_B(q'; p'_A) \geq \hat{\Pi}_B(q^*(p'_A, \psi(q; p'_A)); p'_A) > \hat{\Pi}_B(q; p'_A)$. If $\hat{\Pi}_B(q; p'_A) < \beta v$, then $\hat{\Pi}_B(q'; p'_A) \geq \hat{\Pi}_B(q^*(0; p_A), p'_A) \geq \beta v$ implies that $\hat{\Pi}_B(q'; p'_A) \geq \hat{\Pi}_B(q; p'_A)$. The argument for q' follows symmetrically, such that (4.13) holds.

Rearranging equation (4.13) using Step 2 gives

$$
\begin{aligned}
&(\mathbb{P}[M(n_S,q) \geq m_A] - \mathbb{P}[M(n_S,q') \geq m_A])\beta\Delta \\
&\qquad \geq \mathbb{E}[\bar{M}_B(n_S,q')]\psi(q'; p'_A) - \mathbb{E}[\bar{M}_B(n_S,q)]\psi(q; p'_A), \\
&(\mathbb{P}[M(n_S,q) \geq m_A] - \mathbb{P}[M(n_S,q') \geq m_A])\beta\Delta \\
&\qquad \leq \mathbb{E}[\bar{M}_B(n_S,q')]\psi(q'; p_A) - \mathbb{E}[\bar{M}_B(n_S,q)]\psi(q; p_A).
\end{aligned}
$$

Combined, these yield

$$
\begin{aligned}
&\mathbb{E}[\bar{M}_B(n_S,q')]\psi(q'; p_A) - \mathbb{E}[\bar{M}_B(n_S,q)]\psi(q; p_A) \\
&\qquad \geq \mathbb{E}[\bar{M}_B(n_S,q')]\psi(q'; p'_A) - \mathbb{E}[\bar{M}_B(n_S,q)]\psi(q; p'_A),
\end{aligned}
$$

which rearranges to

$$\mathbb{E}[\bar{M}_B(n_S, q')](\psi(q'; p'_A) - \psi(q'; p_A)) \le \mathbb{E}[\bar{M}_B(n_S, q)](\psi(q; p'_A) - \psi(q; p_A)).$$

Now, because $q' < q$, it follows that $\mathbb{E}[\bar{M}_B(n_S, q')] > \mathbb{E}[\bar{M}_B(n_S, q)]$ and since $\psi(q; p_A)$ is more increasing for lower q, $\psi(q'; p'_A) - \psi(q'; p_A) \ge \psi(q; p'_A) - \psi(q; p_A)$, such that (4.13) is violated. This completes the contradiction.

Step 4 *Without loss, $q^*(p_A, p^*_B(p_A))$ is right-continuous in p_A. Since $q^*(p_A, p^*_B(p_A))$ is nondecreasing in p_A (Step 3), A's maximization problem has at least one solution and an equilibrium exists.*

Suppose to the contrary that there exists a decreasing sequence $(p_{A;n})_{n\mathbb{N}}$ with $\lim p_{A;n} = p_A$, and that $\lim q^*(p_{A;n}, p^*_B(p_{A;n})) = q^+$, but $q^+ > q^*(p_A, p^*_B(p_A)) = q^-$.

We argue that it has to hold that

$$\hat{\Pi}_B(q^-; p_A) \ge \hat{\Pi}_B(q^*(p_A, \psi(q^+; p_A)), p_A) \ge \hat{\Pi}_B(q^+; p_A)$$
$$\hat{\Pi}_B(q^*(p_{A;n}; p^*_B(p_{A;n})); p_{A;n}) \ge \hat{\Pi}_B(q^*(p_{A;n}, \psi(q^-; p_{A;n})), p_{A;n}) \ge \hat{\Pi}_B(q^-; p_{A;n}).$$

By construction of q^*, for any q it is true that $q^*(\psi(q, p_A), p_A)$ is in the support of $q^*(p_A, \cdot)$ and $q^*(\psi(q, p_A), p_A) \le q$. Thereby, the first inequality of either line is a result of p^*_B being a best response of B and the second inequality follows by Step 2.

Since ψ and, thereby, $\hat{\Pi}_B$ are continuous in p_A and q, and because $q^*(p_{A;n}, p^*_B(p_{A;n}))$ as well as $p_{A;n}$ converge, it follows that $\hat{\Pi}_B(q^-; p_A) = \hat{\Pi}_B(q^+; p_A)$. Therefore, it's without loss to change B's response function at p_A to $p^*_B(p_A) = \psi(q^+; p_A)$ and $q^*(p_A, p^*_B(p_A)) = q^+$.

Since $q^*(p_A, p^*_B(p_A))$ is nondecreasing and right-continuous in p_A and all expressions are bounded, $\Pi_A(p^*_A; p^*_B, q)$ has at least one maximizer, such that an equilibrium exists. □

4.B.3 Proof of Lemma 4.1

When the activist makes no offer, $\emptyset$, no shareholder can sell, $q(\emptyset) = 0$.

In state Q, the activist's payoff is

$$\Pi_A(p; q, \xi, Q) = \alpha v + b - p\mathbb{E}[\bar{M}(n_S, q(p))]$$

if $\xi(p) \le \frac{1}{2}$ and shareholders vote against the reform, and

$$\Pi_A(p; q, \xi, Q) = \alpha(v - \Delta) + \mathbb{P}[M(n_S, q(p)) \ge m](b + \alpha\Delta) - p\mathbb{E}[\bar{M}(n_S, q(p))]$$

when $\xi(p) \ge \frac{1}{2}$ and shareholders vote in favor of the reform.

In state R, the activist's payoff is

$$\Pi_A(p;q,\xi,R) = \alpha v + b + \mathbb{P}[M(n_S,q(p)) \geq m](\alpha\Delta - b) - p\mathbb{E}[\bar{M}(n_S,q(p))]$$

in case $\xi(p) \leq \frac{1}{2}$ and shareholders vote against the reform, and

$$\Pi_A(p;q,\xi,R) = \alpha(v+\Delta) + b - p\mathbb{E}[\bar{M}(n_S,q(p))]$$

if $\xi(p) \geq \frac{1}{2}$ and shareholders vote in favor of the reform.

When $\xi(p) \geq \frac{1}{2}$ and shareholders block the reform, firm value is v; if the activist dictates the outcome of the vote, it rises in expectation by $(1-\xi(p))\Delta$. If $\xi(p) \leq \frac{1}{2}$ and shareholders implement the reform, expected firm value is $v+(1-2\xi(p))\Delta$, and rises in expectation by $\xi(p)\Delta$ when the activist dictates the outcome of the vote. Therefore, the shareholders' payoffs can be written as

$$\begin{aligned}\Pi_S(\text{sell};p,q,\xi) &= \frac{v}{n} + \max\{0,1-2\xi(p)\}\frac{\Delta}{n} \\ &\quad + \mathbb{P}[M(n_S-1,q(p)) \geq m-1]\min\{\xi(p),1-\xi(p)\}\frac{\Delta}{n} + p\frac{\mathbb{E}[\bar{M}(n_S,q(p))]}{n_S q(p)}, \\ \Pi_S(\text{keep};p,q,\xi) &= \frac{v}{n} + \max\{0,1-2\xi(p)\}\frac{\Delta}{n} \\ &\quad + \mathbb{P}[M(n_S-1,q(p)) \geq m]\min\{\xi(p),1-\xi(p)\}\frac{\Delta}{n}.\end{aligned}$$

Step 1 *There cannot be an equilibrium with $p^*(\omega) > 0$ in either state $\omega \in \{Q,R\}$.*

If A offers any price $p > 0$, all shareholders sell because they know that the friendly activist matches the state. Thus, if $p^*(\omega) > 0$, the activist has a profitable deviation to any $p' \in (0,p^*)$ because it reduces her transfer.

Step 2 *There cannot be an equilibrium where $p^*(\omega) \neq 0$ in both states $\omega \in \{Q,R\}$.*

Suppose A never offers $p^* = 0$. By Step 1, $p^*(Q) = p^*(R) = \emptyset$. Thus, shareholders do not learn from the activists action and implement the reform. In state Q, this means that the activist's payoff is $\alpha(v-\Delta)$. Consider a deviation to $\frac{\epsilon}{m} > 0$ in state Q. Being offered this positive price, all shareholders sell because they know that the friendly activist matches the state. Thus, the activist's payoff is $b+\alpha v-\epsilon$, such that the deviation is profitable when ϵ i sufficiently small. By Step 1 and Step 2, it follows that the activist offers $p^*(\omega) = 0$ in at least one state $\omega \in \{h,\ell\}$.

Step 3 *In any equilibrium, the reform is implemented in state R, but status quo remains in state Q.*

Given Step 1 and 2, there are two possibilities. If $p^*(Q) = p^*(R) = 0$, shareholders do not learn from the offer, $\xi^*(p^*(Q)) = \xi^*(p^*(R)) = \rho$. If they do not sell and implement the reform, they choose the wrong action with probability $1-\rho$. Since the friendly activist always matches the state, if $q^*(0) > 0$ and shareholders are pivotal with positive probability, it is strictly optimal for them to sell. In case $q^*(0) = 0$, the activist has a profitable deviation in state Q by offering a small positive price $\frac{\epsilon}{m}$, securing all voting rights, and blocking the reform (compare Step 2).

When $p^*(Q) = 0$ and $p^*(R) = \emptyset$, or $p^*(R) = 0$ and $p^*(Q) = \emptyset$, shareholders learn the state from the offer, and vote for the reform in state R and for the status quo in state Q. The activist also matches the state. Thus, when $p^*(\omega) = 0$, shareholders are indifferent between voting themselves or delegating their voting right to activist. Since the firm value is maximized and the activist has no cost, there are no profitable deviations.

4.B.4 Proof of Lemma 4.2

Suppose that $\mu^*(Q) = 1$ and $\mu^*(R) = 0$. Conditional on observing the message, shareholders learn the state, $\xi^*(0) = 0$ and $\xi^*(1) = 1$, implement the reform in state R, and block it in state Q. Since this maximizes firm value and the activist has aligned incentives, no investor has an incentive to deviate.

4.B.5 Proof of Proposition 4.3

Step 1 *There always exists a separating equilibrium.*

We construct an equilibrium of the following form:

- The activist offers $p^*(Q) > p^*(R) \geq 0$.
- Shareholders sell with probability $q^*(p) \begin{cases} = 1 & \text{if } p \geq p^*(Q) \\ < 1 & \text{if } p < p^*(Q). \end{cases}$
- On path beliefs are correct, $\xi^*(p^*(Q)) = 1$ and $\xi^*(p^*(R)) = 0$. Off-path beliefs are $\xi^*(p) = 0$ for all $p < p^*(Q)$ (shareholders believe that the state is R), and $\xi^*(p) = 1$ for all $p > p^*(Q)$.

Let $q^*(p) = q_-(p)$ as defined by (4.9) and Lemma 4.5 for all $p < \bar{p} = \max_q \phi(q)\frac{\Delta}{n}$ (where $\xi^*(p) = 0$), and $q^*(p) = 1$ for all $p \geq \bar{p}$.

If $\bar{p} \notin \arg\max_p \Pi_A(p; q^*, R)$, reduce $\bar{p}$ and modify q^* till it is. This has to be possible, because $\Pi_A(\bar{p}; q^*, R) = \alpha v + b - m\bar{p}$ is continuous and strictly decreasing in $\bar{p}$, whereas for any $p < \bar{p}$ it holds that $\Pi_A(p; q^*, R) \leq \alpha(v + \Delta) + \mathbb{P}[M(n_S, q^*(p)) \geq m](b - \alpha\Delta)$ and $\mathbb{P}[M(n_S, q^*(p)) \geq m]$ is bounded away from one.

When $\bar{p} \in \arg\max \Pi_A(p; q^*, R)$, select a $p' < \bar{p}$ and $q^*(p') = q_+(p')$ as defined in Lemma 4.5 such that $\Pi_A(\bar{p}; q^*, R) = \Pi_A(p'; q^*, R)$. Such a p' has to exist, because

$q_+(p')$ is continuous and strictly decreasing in p' with $q_+(0) = 1$, and $\Pi_A(p; q, R)$ is continuous in both, p and q. Notice that $p' < \bar{p}$ and $q^*(p') < 1 = q^*(\bar{p})$.

Let $p^*(R) = p'$, which, by construction, is a best response. Further, let $p^*(Q) = \bar{p}$ and notice that

$$\begin{aligned}\Pi_A(p; q^*, Q) &= \alpha(v - \Delta) + \mathbb{P}[M(n_S, q^*(p)) \geq m](b + \alpha\Delta) - p\mathbb{E}[\bar{M}(n_S, q^*(p))]\\ &= \Pi_A(p; q^*, R) - 2(1 - \mathbb{P}[M(n_S, q^*(p)) \geq m])\alpha\Delta\\ &< \Pi_A(\bar{p}; q^*, R) = \alpha v + b - \bar{p}m = \Pi_A(\bar{p}; q^*, Q)\end{aligned}$$

for all $p \neq \bar{p}$. All prices above $\bar{p}$ are dominated by $\bar{p}$. Thus, the activist has no profitable deviation in either state.

Last, shareholders do not want to deviate. If the price is $p > \bar{p}$, then $q^*(p) = 1$, such that no shareholder is pivotal and selling is a best response. At any price below $\bar{p}$, shareholders play a best response given their belief that the state is R. When the price is $p^*(R)$, this belief is correct.

Step 2 *In any separating equilibrium,* $p^*(R) < p^*(Q)$ *and* $q^*(p^*(R)) < q^*(p^*(Q)) = 1$.

Suppose to the contrary that $p^*(R) \neq p^*(Q)$ but $q^*(p^*(R)) \geq q^*(p^*(Q))$. In any separating equilibrium, after observing $p^*(Q)$, shareholders know that the activist has aligned interests.

If $p^*(Q) > 0$, shareholders sell with probability $q^*(p^*(Q)) = 1$. Thus, the claim can only be violated if $q^*(p^*(R)) = 1$. However, this contradicts the separation, $p^*(R) \neq p^*(Q)$, because the lower price dominates the higher price, such that the activist would want to deviate in one state.

If $p^*(Q) = 0$, shareholders either sell or vote to block the reform. In either case, the reform does not pass, meaning that any $p^*(R) > 0$ is dominated by $p^*(Q) = 0$, which contradicts the separation. Thereby, $q^*(p^*(R)) < q^*(p^*(Q))$.

If $p^*(R) \geq p^*(Q)$, then $p^*(Q)$ dominates $p^*(R)$ because $q^*(p^*(R)) < q^*(p^*(Q))$. Thereby, $p^*(R) < p^*(Q)$, completing the proof.

Step 3 *As n grows large, along any sequence of equilibria and for* $\omega \in \{Q, R\}$,

$$\mathbb{P}[M(n_S, q^*(p^*(\omega))) \geq m] \to 1 \quad \textit{and} \quad p^*(\omega)\mathbb{E}[\bar{M}(n_S, q^*(p^*(\omega)))] \to 0.$$

In the proof of Proposition 4.1, we derived that there is a price $\bar{p}$ such that $q^*(p) = 1$ for all $p > \bar{p}$, even when shareholders believe the state is R, $\xi^*(\bar{p}) = 0$, such that their expected loss is maximal. Further, $n\bar{p} \to 0$. Without loss, suppose that $q^*(\bar{p}) = 1$ as well. Then, $\lim \Pi_A(\bar{p}; q^*, \xi^*, \omega) = \alpha v + b$ in both state $\omega \in \{h, \ell\}$.

Suppose the assertion was violated and consider a sequence of separating equilibria. By Step 2, it suffices to show that $p^*(Q)\mathbb{E}[\bar{M}(n_S, q^*(p^*(Q)))] \to 0$ and

$\mathbb{P}[M(n_S, q^*(p^*(R))) \geq m] \to 1$. In a separating equilibrium, $\xi^*(p^*(Q)) = 1$, such that shareholders vote for the status quo and

$$\Pi_A(p^*(Q); q^*, \xi^*, Q) = \alpha v + b - p^*(Q)\mathbb{E}[\bar{M}(n_S, q^*(p^*(Q)))].$$

If $p^*(Q)\mathbb{E}[\bar{M}(n_S, q^*(p^*(Q)))] \not\to 0$, a deviation to $\bar{p}$ is profitable when n is sufficiently large. In state R, the belief is $\xi^*(p^*(R)) = 0$, meaning that shareholders vote for the reform and

$$\begin{aligned}\Pi_A(p^*(R); q^*, \xi^*, R) = {} & \alpha(v + \Delta) + \mathbb{P}[M(n_S, q^*(p^*(R))) \geq m](b - \alpha\Delta) \\ & - p^*(R)\mathbb{E}[\bar{M}(n_S, q^*(p^*(R)))].\end{aligned}$$

If $\mathbb{P}[M(n_S, q^*(p^*(R))) \geq m] \not\to 0$, a deviation to $\bar{p}$ is profitable when n is sufficiently large.

Next, consider a sequence of pooling equilibria, where $p^*(Q) = p^*(R) = p^*$, meaning that $\xi^*(p^*) = \rho$ and shareholders vote for the reform. Then,

$$\Pi_A(p^*; q^*, \xi^*, Q) = \alpha(v - \Delta) + \mathbb{P}[M(n_S, q^*(p^*)) \geq m](b + \alpha\Delta) - p^*\mathbb{E}[\bar{M}(n_S, q^*(p^*))]$$
$$\Pi_A(p^*; q^*, \xi^*, R) = \alpha(v + \Delta) + \mathbb{P}[M(n_S, q^*(p^*)) \geq m](b - \alpha\Delta) - p^*\mathbb{E}[\bar{M}(n_S, q^*(p^*))].$$

When either assertion is violated, then $\Pi_A(p^*; q^*, \xi^*, \omega) < \alpha v + b$ for n arbitrary large, such that a deviation to $\bar{p}$ is profitable.

4.B.6 Proof of Proposition 4.4

The equilibrium is supported by off-path beliefs $\xi^*(\kappa) < \frac{1}{2}$ for any $\kappa \in (0, b - \alpha\Delta)$ and the correct on-path belief $\xi^*(0) = 0$. Thus, after any $\kappa < b - \alpha\Delta$, the reform is implemented, such that $\kappa = 0$ dominates all $\kappa < b - \alpha\Delta$. After observing $\kappa = b - \alpha\Delta$, the shareholders believe that the state is Q, $\xi^*(b - \alpha\Delta) = 1$, and the reform is blocked. Above $\kappa = b - \alpha\Delta$, the off-path beliefs are arbitrary. Thus, any $\kappa > b - \alpha\Delta$ is also dominated by either $\kappa = 0$ or $\kappa = b - \alpha\Delta$.

In state Q, the activist has an incentive to spend $\kappa = b - \alpha\Delta$, yielding a payoff of $b + \alpha v - \kappa = b + \alpha v - (b - \alpha\Delta) = \alpha(v + \Delta)$ instead of spending $\kappa = 0$, which yields her a profit of $\alpha(v - \Delta)$. In state R, the activist spends $\kappa = 0$ and receives $\alpha(v + \Delta)$ which yields the same payoff as spending $\kappa = b - \alpha\Delta$. Hence, $\kappa^*(R) = 0$ and $\kappa^*(Q) = b - \alpha\Delta$ is optimal for the activist and the on-path beliefs are consistent.

There cannot be an equilibrium in which the state is matched with probability strictly smaller than $(1 - \rho)$. In any separating equilibrium, shareholders learn the state and, therefore, the probability of matching the state is one. In any pooling equilibrium, shareholders vote according to their prior and implement the reform, such that the probability of matching the state is $(1 - \rho)$.

4.B.7 Unrestricted and conditional offers

Lemma 4.7 *When the activist cannot set a restriction, there are equilibria in which* $\mathbb{P}[M(n_S, q^*(p^*)) \geq m] > 0$ *but*

$$p^*\mathbb{E}[M(n_S, q^*(p^*))] > m\frac{\Delta}{n}\mathbb{P}[M(n_S, q^*(p^*)) \geq m].$$

Proof. Suppose that there is no restriction, such that the activist has to buy from all shareholder who sell to her. Given offer p and response $q(p)$, shareholders are willing to sell if

$$p \geq \mathbb{P}[M(n_S - 1, q(p)) = m - 1]\frac{\Delta}{n}. \tag{4.14}$$

We prove the result by an example.

Suppose that $\alpha = 0$, $n = 11$, and $m = 2$. Further, $\Delta = 1$ and $b = \frac{3}{4}$. In this case

$$\mathbb{P}[M(n_S - 1, q(p)) = m - 1] \leq \mathbb{P}[M(10, 0.1) = 1] = 0.38742.$$

Solving for $q(p)$ when (4.14) holds with equality, there is a continuous, strictly increasing best response q^* with $q^*(p) < 0.1$ for all $p < 0.38742\frac{\Delta}{n}$ and $q^*(p) = 1$ for all $p \geq 0.38742\frac{\Delta}{n}$.

It now follows that $p^* = 0.38742\frac{\Delta}{n}$ because for all $p < p^*$

$$\begin{aligned}\Pi_A^{\text{nr}}(p; q^*) &< b * \mathbb{P}[M(n_S, 0.1) \geq 2] \\ &= b * 0.302643 < b - n * 0.38742\frac{\Delta}{n} = b - 0.38742 = \Pi_A^{\text{nr}}(p^*; q^*).\end{aligned}$$

Any $p > p^*$ is dominated by p^*. Further, $\mathbb{E}[M(n_S, q^*(p^*))]p^* = 0.38742 > \frac{2}{11}\Delta$, completing the proof. □

Lemma 4.8 *When the activist can condition her restricted offer on success, in the unique equilibrium* $p^* = 0$ *and* $q^*(p^*) = 1$.

Proof. As in the case without the condition, $p^* = 0$ and $q^*(0) = 1$ constitute an equilibrium. We show that there is no other equilibrium.

Given any q and the conditional restricted offer p, a shareholder is indifferent between selling the and retaining the share if

$$p\sum_{i=m-1}^{n_S-1}\mathbb{P}[M(n_S - 1, q) = i]\frac{m}{i+1} = \frac{\Delta}{n}\mathbb{P}[M(n_S - 1, q) = m - 1]. \tag{4.15}$$

With (4.6) and (4.8) this rearranges to

$$\begin{aligned} p\mathbb{P}[M(n_S,q)\geq m]\frac{m}{qn_S} &= \frac{\Delta}{n}\mathbb{P}[M(n_S-1,q)=m-1] \\ \iff p\mathbb{P}[M(n_S,q)\geq m] &= \frac{\Delta}{n}\mathbb{P}[M(n_S,q)=m] \\ \iff p &= \frac{\Delta}{n}\frac{\mathbb{P}[M(n_S,q)=m]}{\mathbb{P}[M(n_S,q)\geq m]}. \end{aligned}$$

We now note that by (4.10), $\frac{\mathbb{P}[M(n_S,q)=m]}{\mathbb{P}[M(n_S,q)\geq m]}$ is monotonically decreasing in q with

$$\lim_{q\searrow 0}\frac{\mathbb{P}[M(n_S,q)=m]}{\mathbb{P}[M(n_S,q)\geq m]}=1 \qquad \lim_{q\nearrow 1}\frac{\mathbb{P}[M(n_S,q)=m]}{\mathbb{P}[M(n_S,q)\geq m]}=0.$$

By offering $p>0$, either $q^*(p)=1$ or q^* is determined by (4.15). In either case, for any $p>0$ and any $\epsilon>0$, there is a price $p_\epsilon<\epsilon$ such that $\frac{q^*(p_\epsilon)}{q^*(p)}\geq 1-\epsilon$. Hence, a profitable deviation always exists. This means that in equilibrium, it has to hold that $p^*=0$ and $q^*(0)=1$. □

4.B.8 Proof of the example

Most of the proof can be found in the body of the text. What remains to be shown is that in state R, the activist does not want to deviate from 0 to any $p\in(0,\bar{p})$.

At any $p\in(0,\bar{p})$, the shareholders' belief is $\xi^*(p)=0$, and because $q^*(p)\in(0,1)$, q^* is determined by the shareholders' indifference condition (4.9). In state R, the activist's payoff function is given by (4.1). Plugging in (4.9), and using (4.8) gives

$$\begin{aligned} &\Pi_A(p;q^*,\xi^*,R) \\ &=\alpha(v+\Delta)+\mathbb{P}[M(n_S,q^*(p))\geq m](b-\alpha\Delta)-m\mathbb{P}[M(n_S,q^*(p))=m]\frac{\Delta}{n} \\ &=\alpha(v+\Delta)+\mathbb{P}[M(n_S,q^*(p))\geq m]((b-\alpha\Delta)-m\frac{\mathbb{P}[M(n_S,q^*(p))=m]}{\mathbb{P}[M(n_S,q^*(p))\geq m]}\frac{\Delta}{n}) \end{aligned}$$

for all $p\in(0,\bar{p})$.

Since $\mathbb{P}[M(n_S,q)\geq m]$ is increasing in q, $\frac{\mathbb{P}[M(n_S,q)=m]}{\mathbb{P}[M(n_S,q)\geq m]}$ is decreasing in q (cf. equation (4.10)), and q^* is strictly increasing in p, there can be no interior optimum $p^*\in(0,\bar{p})$. Since every $p>\bar{p}$ is also dominated by $\bar{p}$, it follows that 0 and $\bar{p}$ are the only two non-dominated actions. Since the activist is indifferent between 0 and $\bar{p}$ when the state is R, there can be no profitable deviations.

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