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Algebraic, logical and stochastic reasoning for the automatic prediction of 3d building structures

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Kurzfassung

3D Gebäudemodelle sind heutzutage eine wichtige Voraussetzung für viele Anwendungen wie z.B. Rettungszenarien oder Navigationsaufgaben. Die meisten Ansätze für eine automatische Rekonstruktion von Gebäuden basieren jedoch auf hochaufgelösten Daten, welche durch Verdeckungen oder hohe Kosten der Datenerhebung nicht immer zur Verfügung stehen. Stattdessen müssen Reasoning-Methoden mit dünnen und möglicherweise unvollständigen Daten umgehen. In dieser Arbeit wird ein neuer Reasoning-Ansatz für die Prädiktion von Gebäuden und ihrer Bestandteile vorgestellt, welcher ohne dichte Messungen auskommt. Der entwickelte Reasoner profitiert von einem starken Vorwissen über funktionale Abhängigkeiten und Wahrscheinlichkeitsdichteverteilungen in einem modellgetriebenen Top-Down-Ansatz, der durch starke Regelmäßigkeiten und Symmetrien in von Menschenhand geschaffenen Objekten legitimiert ist. Dabei wird die Ansicht vertreten, dass es einfacher ist, prädizierte Hypothesen zu verifizieren oder zu falsifizieren, als Gebäude "bottom-up" aus Messungen zu rekonstruieren. Das Ergebnis ist eine kleine Anzahl qualifizierter Hypothesen, welche auf nur wenigen Beobachtungen basieren. Das mathematische Modell zeichnet sich jedoch a priori durch multimodale Wahrscheinlichkeitsdichtefunktionen sowie nichtlineare Relationen von diskreten als auch kontinuierlichen Parametern aus, was im Allgemeinen zu einer approximativen stochastischen anstelle einer exakten Inferenz führt. Eine wesentliche Entwurfsentscheidung, um etablierte exakte Algorithmen der Parameterschätzung zu verwenden, ist die Repräsentation von Verteilungen durch Gaußsche Mischverteilungen. Darauf aufbauend besteht die Kernidee dieser Arbeit darin, das Problem in einen kombinatorischen und einen stochastischen Teil zu unterteilen und logische Constraintprogrammierung mit Bayes'schen Netzwerken zu kombinieren. Die Constraintprogrammierung reduziert den Suchraum und bestimmt zunächst die diskreten Parameter. Anschließend wird dieses Zwischenergebnis durch stochastische Inferenz verfeinert, um die kontinuierlichen Parameter zu bestimmen und die wahrscheinlichsten Hypothesen aus einem a priori großen Hypothesenraum zu finden. Die Methode hat sich einerseits zur Prädiktion von Fassadenstrukturen bewährt. Da die nahtlose Außen-/Innenmodellierung immer mehr Beachtung findet, wurde der entwickelte Ansatz andererseits an die Prädiktion von Innenmodellen angepasst. Für das modellbasierte Reasoning werden darüber hinaus Werkzeuge benötigt, die die Entwicklung redundanzfreier und konsistenter Prototypmodelle erleichtern. Gleichzeitig ist es nützlich, implizite Constraints explizit zu machen, um die Interpretation von Messungen für die Gebäuderekonstruktion zu unterstützen. Da die Überprüfung von Redundanz und Konsistenz gleichbedeutend ist mit dem Nachweis, dass ein Fakt aus einer Reihe von Prämissen folgt, ergänzt diese Arbeit das Reasoning mit Methoden des automatischen Theorembeweisens. Um der zunehmenden Komplexität im 3D-Raum gerecht zu werden, wird ein neuartiger Ansatz vorgestellt, der algebraisches und logisches Reasoning unter Verwendung multivariater Polynome und Prädikatenlogik kombiniert. Das algebraische Reasoning basiert auf der Methode von Wu mittels Pseudodivision und "Characteristic Sets" und identifiziert Redundanz, Inkonsistenz und implizites Wissen. Das regelbasierte Schließen mit logischen Fakten und Regeln unterstützt das Reasoning durch bekannte Implikationen. Der Aspekt der Unsicherheit, der im Zusammenhang mit Geoinformationssystemen (GIS) unvermeidlich ist, wird in dieser Arbeit durch die Verwendung von Wahrscheinlichkeitsdichtefunktionen, graphischen Modellen und unsicherer projektiver Geometrie behandelt.

Abstract

3D building models are nowadays an important prerequisite for many applications such as rescue management or navigation tasks. However, most approaches for the automatic reconstruction of buildings rely on high-resolution data that cannot always be provided due to occlusions or high cost of the acquisition of data. Instead, reasoning methods have to cope with sparse and possibly incomplete data. This thesis presents a novel reasoning approach for the prediction of building substructures in the absence of dense measurements. The developed reasoner benefits from a strong profound prior knowledge of functional dependencies and probability density distributions in a model-driven top-down approach that is legitimated by strong regularities and symmetries in man-made objects. It thereby holds the view that it is easier to verify or falsify predicted hypotheses than to reconstruct buildings bottom-up from measurements and automatically generates a small number of qualified hypotheses based only on sparse observations such as footprints. However, the mathematical model for buildings is a priori characterized by multimodal probability density functions as well as non-linear relations with both discrete and continuous parameters that in general leads to approximate stochastic inference instead of exact inference. One substantial design decision in order to use well established exact algorithms of parameter estimation is the representation of distributions by Gaussian mixtures. For efficient reasoning in hybrid models, the key idea of this thesis is to divide the problem into a combinatorial (discrete) and stochastic (continuous) part and to combine constraint logic programming with Bayesian networks. Constraint programming reduces the search space by constraint propagation and intelligent search strategies and determines the discrete parameters first. Afterwards this intermediate result is refined by stochastic inference to evaluate and determine the continuous parameters and finding the most likely hypotheses out of an a priori large hypothesis space. The method has been demonstrated to predict façade structures on the one hand. As the seemless outdoor/indoor modeling gets more and more attention the developed approach was adapted to the prediction of indoor models on the other hand. As models are a prerequisite of model-based reasoning, tools are needed that facilitate the development of redundancy-free and consistent prototyped models providing prior knowledge during model prediction. At the same time, it is useful to make implicit constraints explicit for supporting the interpretation of measurements for building reconstruction. Recognizing that the task of checking redundancy and consistency is equivalent to proving that one constraint follows from a set of premises this thesis complements the reasoning with methods of automatic theorem proving. In order to handle the increasing complexity of symbolic reasoning in the 3D space a novel approach is presented that combines algebraic and logical reasoning based on an appropriate representation of the envisaged constraint-based model using multivariate polynomials and first-order predicate logic. Algebraic reasoning is based on Wu's method of pseudodivision and characteristic sets and identifies redundancy, inconsistency and implicit knowledge. Rule-based reasoning based on logical facts and rules supports the reasoning process using known implications. The aspect of uncertainty that is inevitable in the context of geoinformation systems (GIS) is handled in the developed reasoning methods by the incorporation of probability density functions, graphical models and uncertain projective geometry.

Contents

1 Introduction

Nowadays, 3D city models are used for a wide range of applications and the demand for a detailed building reconstruction constantly increases. While the pure visualization of buildings through textured façades is often easily available the semantic interpretation of laser scans or images such as the identification and modeling of façades, windows, doors or even rooms is more challenging. This however is an important prerequisite for many applications such as rescue management, navigation, facility management or the calculation of energy balances. An overview of applications of city models is provided by Biljecki et al. (2015).

Consequently, the fundamental aim is to identify and model the components of buildings and the relations between them. In this context, the representation of geometric, topological and semantic information of objects is enabled by the standardized markup language CityGML (Gröger et al. (2008); Gröger et al. (2012)). Different level of details (LOD) range from building footprints with extruded heights to models with detailed substructures including windows, doors, protrusions and roofs. Due to an increasing demand for indoor/outdoor seamless modeling the reconstruction of indoor environments is also an important aspect of building modeling. Applications such as rescue management or the navigation for the blinds are reliant on detailed information of the interior including the locations of rooms and their doors. Complementing CityGML, that uses a boundary representation for objects, building information models (BIM) are represented by the composition of solids especially developed for construction and facility management of a building complex. They thus go beyond the modeling of visible surfaces. Beside the rising interest for BIM the development of the new OGC standard IndoorGML used for indoor spatial information especially for navigation purposes (Lee et al., 2014) shows the need for models of indoor environments.

In order to avoid a manual modeling of the as-built state of buildings various approaches have been developed for the automatic reconstruction and modeling of buildings. They use more or less dense measurements for bottom up and prototyped models for top down identification of building substructures in laser scans or images. However, the collection of measured data often remains a tedious and expensive task. Data acquisition for façades can be made with mobile platforms such as passing cars or unmanned aerial vehicles (UAV). However, the interpretation algorithm has to cope with occlusions by e.g. vegetation. Furthermore, for indoor environments it is far costlier to have measurements since every room has to be entered, GPS signals may be weak and walls are often occluded by furniture that makes it difficult to identify the correct outlines of the detected objects in even high-resolution 3D point clouds or images.

Reasoning with sparse observations. While most approaches rely on dense data this thesis presents a reasoning method based on sparse observations. Figure 1.1 illustrates the prediction of façade and indoor models. For façades, windows can be predicted based on the

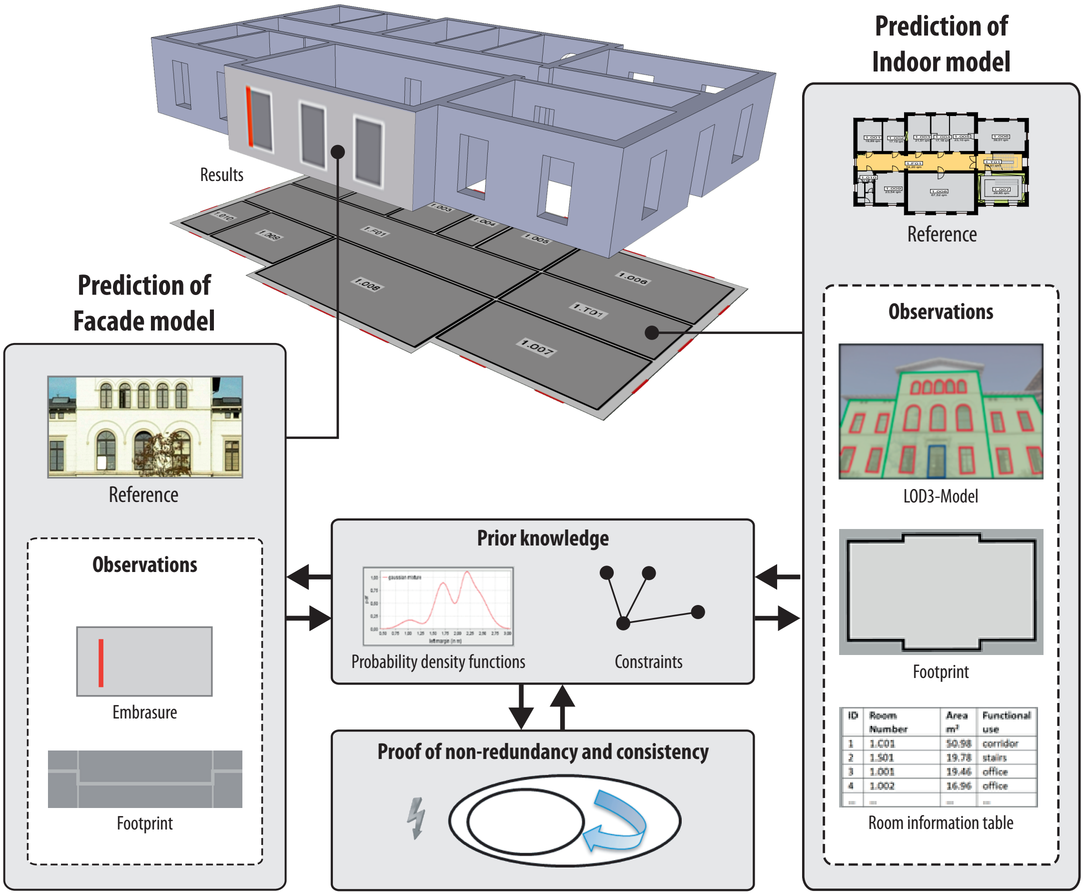


Figure 1.1: Overview of developed reasoning methods for the automatic prediction of 3D building structures in the absence of dense measurements.

footprint without the need of dense façade measurements. Similarly, 3D indoor models that include rooms and their doors are generated in the absence of costly interior observations. Instead, the predictions are based on the footprint and locations of windows from exterior reconstructions and an overview of rooms with functional uses and room areas.

The basic idea to waive on dense data is to use a model-driven approach that relies on the observation that model parameters follow certain aesthetic and architectural patterns being captured in probability density distributions or constraints between model parameters. In this way, this thesis shows that by the incorporation of profound prior knowledge the generation of hypotheses for the reconstruction of buildings of high quality becomes possible for façades as well as for indoor environments (cf. Figure 1.1). Legitimated by the regular character of buildings, the proposed method relies on a strong model specifying the constraints between the model parameters that have to be instantiated. The stochastic

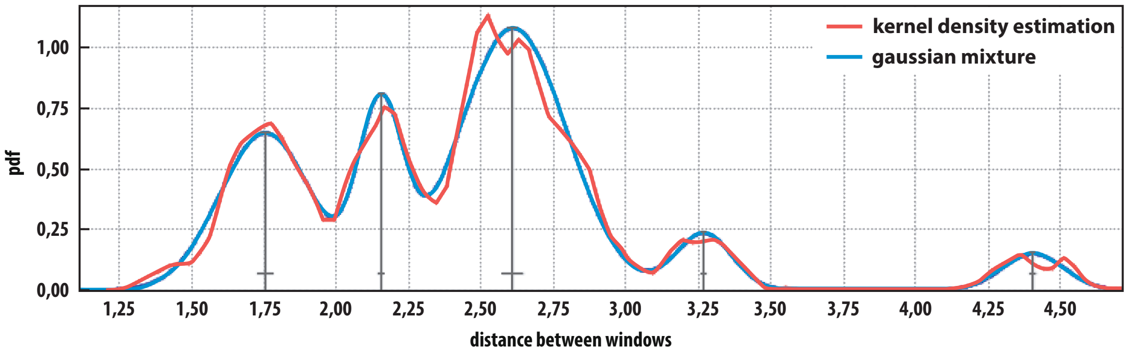


Figure 1.2: Kernel density estimation and approximated Gaussian mixture for the distance between windows. Gaussian mixtures for model parameters reveal clear peaks with small variances. Means of Gaussian mixture components are marked with vertical lines, the variances are represented with the length of horizontal lines.

model is based on probability density functions that are built up from a huge set of ground truth training data and structure the hypothesis space.

Probability density functions. Probability density functions are used to structure the hypotheses space in that they allow the derivation of plausible thresholds for parameter domains on the one hand and a ranking of hypotheses with respect to most probable parameter estimations on the other hand. Figure 1.2 shows a representative distribution for model parameters – the distance between windows – estimated by a kernel density estimation (Bowman and Azzalini (1997)). Obviously, the distribution is multimodal and hence not Gaussian. This however would be preferable in order to use well established reasoning methods for parameter estimation based on Gaussian distributions. Building the bridge to this, one substantial design decision in order to use classical algorithms of parameter estimation is the approximation of these distributions by Gaussian mixtures. It has been shown that each arbitrary distribution can be approximated by a Gaussian mixture (McLachlan and Peel (2000)). While most approaches for hybrid networks apply approximate inference that approximate intractable multimodal distributions by e.g. particles or unimodal Gaussians, this thesis presents a method that transforms the problem into a linear Gaussian problem without possibly unreliable approximations and enables estimations by prominent methods of exact stochastic inference.

By ruling out implausible predictions the presented reasoner yields a small number of qualified hypotheses. Probabilities that are derived by a well-defined stochastic model enable a ranking of the generated predictions. This even reveals that one hypothesis is often dominant and favored against other improbable ones. Herewith, the thesis takes the view that it is easier to verify or falsify a model than to reconstruct buildings bottom up from measurements. The decision between competing hypotheses can be enhanced by finding a discriminating feature in the different models that has to be examined. The two-staged top-down approach decreases the overhead for data acquisition and facilitates the interpretation that consequently saves costs and time.

Bi-linear Constraints. Regularities and symmetries between substructures that characterize man-made objects can be further exploited to define constraints on model parameters in order to restrict the search space for plausible model estimations. As an example for the prediction of façades, one constraint relates the distances between windows and their segments of the footprint, the façade width:

$$f_w = d_l + d_r + n_w w_w + (n_w - 1) w_d.$$

Here, the façade width (f_w) is the sum of left (d_l) and right (d_r) margins and the factorized window widths (w_w) and window distances (w_d) related to the (discrete) number of windows (n_w). This constraint illustrates the character of the used relations modeling a building. First, the constraints are non-linear and second, they contain discrete as well as continuous variables. Although these properties make reasoning in general more complex, for building façades a special structure of the equation can be recognized: The constraint is bi-linear and characterized by products of one discrete and one continuous variable. This reveals as an important property that is exploited in the presented reasoning process.

Exact inference in hybrid models. The addressed problem of building reconstruction can be modeled and solved by graphical models that are powerful graph-based tools to reason with uncertain data. Koller and Friedman (2009) give a detailed introduction to inference in graphical models. However, most approaches are restricted to discrete problems – solved by exact and approximate methods – while only few methods deal with continuous cases. One representative is the Kalman filter (Kalman, 1960). It has been shown to be equivalent to a Bayesian network with continuous variables and associated linear Gaussians and is an efficient method to determine the posterior distribution of the continuous model parameters according to a measurement update (Koller and Friedman (2009)). It is exact for linear systems with Gaussian distributions but nonetheless only covers continuous systems without any discrete variables.

In contrast, a hybrid model containing discrete as well as continuous parameters often leads to the use of approximate inference algorithms since the original multimodal distributions in general become intractable. The particle filter is one method that considers this topic but however is not exact. Lauritzen and Jensen (2001) tackle exact inference for special structured hybrid networks where distributions are correct for discrete variables and continuous variables are described in an exact way by the first and second moments of their distributions. While this method is restricted to linear relations the approach presented in this thesis is an extension of Lauritzen's algorithm to bilinear relations. The aim of this thesis is to show that exact inference in this case is feasible. The basic idea is based on the insight that can be drawn from the following formula for joint distributions in Bayesian networks (Kjærulff and Madsen (2008)):

$$p(X_\Delta = \tau) \cdot N_{|\mathcal{X}_\Gamma|}(\mu_\tau, \sigma_\tau^2) = \prod_{v \in V_\Delta} P(\tau_v | \tau_{pa(v)}) \prod_{w \in V_\Gamma} p(y_w | X_{pa(w)})$$

The formula over the discrete nodes in V_Δ and the continuous nodes in V_Γ defines the joint distribution with respect to a discrete instantiation τ and the corresponding normal distributions N with mean μ and variance σ^2. As one principle of Bayes networks it is described by distributions and probability density functions that depend on the associated parent nodes pa of the random variables, the preceding nodes in the network. The formula shows that the problem can be divided into a combinatorial (discrete) and a stochastic (continuous) product. Exploiting this fact the presented reasoning is a two-staged approach. In a first step the discrete problem is solved by using constraint (logic) programming. To this end, probability density functions are used to derive bounds for variables that together with constraints modeling the regularities of buildings restrict the search space. CLP instantiates the discrete variables so that the second step, the stochastic part, reduces to the estimation of solely continuous parameters according to plausible combinatorial results. Gaussians that are necessary for exact inference are extracted from the multimodal distributions by constraint propagation leading to a single Gaussian component that best fits the given observation. Constraint solvers are powerful tools to solve non-linear problems of discrete and continuous variables with more than one unknown. In this context, Frühwirth and Abdennadher (2003) state that constraint logic programming in general outperforms algebraic methods such as the Gröbner base method (Buchberger (1998)). In this thesis, the combinatorial part based on constraint logic programming is used to search for valid instantiations of discrete model parameters and consequently to linearize the problem and providing pure continuous relations. This enables the use of classical exact approaches of parameter estimation for continuous model parameters afterwards - e.g. by exploiting the simple structure of matrix multiplications of the Kalman filter.

From façades to indoor modeling. The presented approach is generic and is as well applied to floorplans in the absence of indoor measurements. The problem is to place n rectangular rooms of known areas within a polygonal building footprint that is assumed to follow the Manhattan World principle. The width and depth of each room are constrained by the basic constraint $area = width \cdot depth$ as well as by derived lower and upper bounds from probability density functions (PDF). As for the prediction of façades the reasoning within indoor models is divided into a combinatorial and a stochastic part. Gaussian distributions and the topology of the rooms – both represented by discrete parameters – are determined which results in preliminary topological models after constraint propagation. Afterwards, the posterior for continuous shape parameters such as the width, depth and locations of rooms are calculated by exact stochastic inference yielding final floor plan models that match the observations and known building information. Taking the correct hypothesis after model selection model accuracies for model parameters range between 10 and 20 cm – meeting the recommended LoD4 accuracy requirements of the OGC CityGML standard (Gröger et al. (2012)). The approach for verification or falsification of hypotheses is outlined, details however are beyond the scope of this thesis.

Reasoning on models. As model-driven approaches rely on well-defined prototyped building models, tools for their development are needed. Strong models play an important role for the automatic reconstruction of buildings but their consistent and redundancy-free development is a significant problem. In order to tackle this deficiency, another part of this thesis considers the reasoning on models during model development before reasoning with these models in a top-down approach. As illustrated in Figure 1.1, the prototyped constraint-based models serve as part of the fundamental prior knowledge used during prediction of building parts.

Constraint-based modeling is one common way of modeling where the functional model is built of building primitives such as planar faces that are connected by constraints e.g. orthogonality or parallelity. In the context of model development, there are three important questions. At first, it is often useful to have a redundancy-free model that contains a minimal number of constraints and thus is less complex and needs less memory. Moreover, it is a requirement that the set of constraints is consistent and constraints do not contradict each other. At last, the task of estimating building structures often needs an overdetermined equation system in order to compensate uncertain measurements. Instead of searching for a redundancy-free system the refinement of the model by deducing further implicit constraints may be helpful. All these questions are important but tools are not available. To this end, this thesis contributes to the reasoning with a novel method of logic, algebraic and stochastic reasoning.

The basic idea is that the mentioned questions lead to the problem whether one constraint follows from a set of constraints. This suggests the use of automatic theorem proving where implications are proven by the deduction of constraints. Since building models are mainly of geometric character the representation of constraints by multivariate polynomials and thus the use of algebraic methods is appropriate. Loch-Dehbi and Plümer (2011) show that man-made buildings are dominated by constraints of orthogonality and parallelity that can be easily defined by bilinear relations, i.e. the scalar product and the cross product, using homogeneous coordinates and thus simplify the representation and the subsequent reasoning. For instance two planes Π_1 and Π_2 with their normal form $a_i x + b_i y + c_i z + d_i = 0$ are orthogonal if the scalar product of their normal vectors equals zero:

$$\Pi_1 \perp \Pi_2 \Leftrightarrow a_1 a_2 + b_1 b_2 + c_1 c_2 = 0$$

While there exist efficient methods to solve these non-linear equations numerically the more complex decision whether a constraint set is satisfiable in general on a symbolic level is far more challenging. It can further be observed that there is a substantial increase in complexity with the transition from the 2D space to the 3D space. The novel aspect of this work is that it adresses the reasoning in the three-dimensional space. In contrast to Brenner and Sester (2005) who make use of algebraic theorem proving with Gröbner bases for the 2D space this thesis handles 3D problems with Wu's method (Wu, 1986), another variant of algebraic theorem proving. As Gröbner bases, it is based on multivariate polynomials but uses characteristic sets and the so called pseudodivision to proof the deducibility of constraints. The triangular form of characteristic sets are often more appropriate for theorems of constructive type as it is the case for building models. Further the method is able to generate degenerate cases, so called subsidiary conditions, under which the theorem remains true – assumptions on general positions of geometric objects as often mentioned in textbooks. A lightweight constraint-based reasoner was developed to show that reasoning is feasible as long as the problem is represented in an adequate manner. The algebraic method is supported by deductive rule based reasoning and the aspect of uncertainty is incorporated by the use of uncertain projective geometry.

Contribution. The main contribution of this thesis is a generic method for the prediction of building substructures based on sparse observations. The key idea is to break down the complex hybrid and non-linear problem into two subproblems of a combinatorial and a stochastic part. The developed approach is supported by a profound prior knowledge including Gaussian mixtures and bilinear constraints whose incorporation leads to a substantial restriction of the search space. Further, a novel approach of deductive and algebraic methods for automatic theorem proving is presented in order to identify inconsistency and redundancy in prototyped building models. In this context, the contributions of this thesis can be summarized as follows:

- *Prediction based on sparse observations.* The presented methods predict substructures in buildings based on sparse observations that are in general not sufficient for detailed reconstruction in common approaches. Together with a profound prior knowledge the reasoner yields a ranked set of high qualified hypotheses whose verification or falsification by single additional observations is easier than to build a model bottom up from measurements.
- *Exact inference in hybrid models with non-linear constraints.* The developed reasoner uses a novel approach that combines constraint programming and Bayesian networks and handles bilinear constraints of discrete as well as continuous variables. It herewith exploits the special structure of constraints to follow exact inference on hybrid models instead of approximating distributions in an unreliable way.
- *Gaussian mixtures.* As a substantial design decision, the stochastic model is described by Gaussian mixtures for continuous model parameters. Gaussian mixtures are used to structure the hypothesis space by providing plausible thresholds for parameter domains. Further, Conditional Linear Gaussian Models are extended to Multilinear Gaussian models in order to enable the use of Gaussian mixtures for efficient algorithms of exact inference.
- *Generic reasoner.* A generic implementation for the prediction of substructures in the absence of dense observations is provided. Herewith, the reasoning is not restricted to façade and indoor modeling but can be adapted to arbitrary problems of the same structure. A definition of problems that can be transferred is provided.
- *Algebraic reasoning in 3D.* The thesis presents a novel method for the reasoning on building models in the context of 3D model development. Increasing complexity in the three dimensional space is especially handled by an appropriate representation and the combination of logical and algebraic methods of automatic theorem proving. The problem is reduced to the proof of consistency and non-redundancy and Wu's method of characteristic sets and pseudodivision is used for algebraic reasoning on 3D building models. They are specially appropriate for theorems of constructive type but in contrast to Gröbner bases not yet applied in the context of 3D city models.
- *Assessing quality.* The quality of hypotheses is assessed by incorporating probability density functions and covariances together with statistical reasoning which enables a model selection under competing model hypotheses. Uncertain projective geometry is integrated in order to extend originally crisp methods to noisy data.

As illustrated by Figure 1.1 this thesis covers the prediction of façade and interior models on the one hand and the geometric reasoning on prototyped models on the other hand. In this context, the following publications are most relevant for this work and are appended to the thesis:

- Loch-Dehbi, S., Plümer, L., 2011. Automatic reasoning for geometric constraints in 3D city models with uncertain observations. ISPRS Journal of Photogrammetry and Remote Sensing 66, 177–187.
- Loch-Dehbi, S., Plümer, L., 2015. Predicting building façade structures with multilinear Gaussian graphical models based on few observations. Computers, Environment and Urban Systems 54, 68–81.
- Loch-Dehbi, S., Dehbi, Y., Plümer, L., 2017. Estimation of 3D indoor models with constraint propagation and stochastic reasoning in the absence of indoor measurements. ISPRS International Journal of Geo-Information 6.

Outline. The remainder of this thesis is structured as follows: Section 2 gives an overview of the related work while Section 3 introduces the methodological background. Constraint programming, Bayesian networks and methods of theorem proving are described for the automatic reasoning in the context of 3D building reconstruction. Section 4 addresses the developed approach for the prediction of 3D façade structures. Based on this model, Section 5 describes the transfer of this reasoning process to the prediction of 3D indoor models. Section 6 presents a method for the support of a consistent and redundancy-free model development. Finally, Section 7 concludes this work and gives an outlook.

2 Related Work

The reconstruction of as-built city models has nowadays a wide range of applications that go beyond visualization of the façades. This however comes along with different problems including the handling of occlusions, the minimization of distortions, the prevention of user interaction or the management of the complexity of the problem due to the envisaged level of detail or the diversity of models. Approaches differ in the degree of automation, the models and prior knowledge used for detecting building structures, the range of features and their level of detail that can be reconstructed or the methods used for interpretation.

One common approach for the automatic reconstruction of buildings is based on prototyped models. In contrast to data-driven approaches model-driven top-down approaches – as applied in this thesis – circumvent the deficiency of interpretability in noisy possibly incomplete data. They reconstruct buildings by selecting and instantiating an appropriate model that best matches the measurements. Haala and Kada (2010) give an overview of data-driven and model-driven approaches and emphasize the importance of detailed reconstruction of buildings.

Façade reconstruction. In the context of façade reconstruction various model-driven works have been published. Pu and Vosselman (2009) incorporated prior knowledge to extract building parts such as walls, roofs or windows from high density point clouds. Knowledge includes sizes, positions, orientations and topology. It supports the generation of outline polygons by least squares fitting, convex hull fitting or concave polygon fitting. Cohen et al. (2014) exploit architectural knowledge for a pixel-wise labeling task in images. A dynamic programming algorithm was developed for a labeling of pixels to semantic categories such as sky, roofs or windows. Windows are detected in a single image scenario by Recky and Leberl (2010) using k-means. Complex façades of historical buildings are interpreted despite perspective distortions.

Many model-driven approaches for building reconstruction rely on formal grammars. The possibility to define rules that can model a variety of aggregations and architectural styles makes grammar-based approaches an appropriate tool to cover a wide range of building models. One of the first works where grammars were applied to building models is presented by Müller et al. (2007). They use shape grammars to generate a wide range of 3D building scenes. However, they tackle procedural modeling for the construction of synthetic buildings rather than the reconstruction of existing ones. The modeling of the as-built state of façade objects is the opposite paradigm called inverse procedural modeling. In this context various works investigated the usefulness of formal grammars for reconstruction tasks. The reconstruction of façades by grammar rules was tackled in the work of Becker (2009) and Becker (2011) respectively. Images and point clouds are interpreted in a bottom-up and top-down approach where split grammars can be used to predict façade structures in case of missing data.

Formal grammars are a strong formalism for the interpretation of buildings but the need for grammar rules often leads to an expensive manual modeling. To this end, some approaches use machine learning algorithms to automatically construct grammars. Martinović et al. (2012) present a three-layered approach that uses recursive neural networks and Markov random fields for the semantic segmentation of building façades. Instead of relying on strong prior knowledge that restricts the variety of façade styles shape grammars are not provided as input but derived during façade parsing. They are thus not parametric but generated instance-based for each façade. For plausible and consistent results the reconstruction process is supported by weak architectural knowledge.

While this approach constructs grammars during the construction process – bound to a specific building, other works provide a generic model for subsequent façade parsing. Martinović and Van Gool (2013) developed an approach for the automatic learning of two-dimensional attributed stochastic context-free grammars from labeled images for buildings of grid-like design. Split grammars are learned with Bayesian model merging and used together with a reversible jump Markov chain Monte Carlo approach for façade parsing. Markov random fields (MRFs) together with split grammars were used by Kozinski et al. (2015) in order to segment an image of grid-structured patterns into architectural elements such as windows or doors. The algorithm also handles occlusions and incorporates a strong prior knowledge of semantic constraints that are defined by the user for a given dataset. In contrast to these procedural approaches Dehbi et al. (2017) developed a declarative method that has not to define the way rules have to be applied. They presented a method for statistical relational learning of grammar rules that serve as models for the reconstruction of 3D buildings. They use support vector machines to generate a weighted context free grammar and integrate Markov Logic Networks to enforce topological and geometric constraints and handle the aspect of uncertainty for probabilistic inference. Additionally, a parsing approach was presented that demonstrates the interpretation of 3D point clouds based on the learned grammar rules. The possibility to learn grammars seems to be a great step forward to avoid the manual modeling. However, the wide range of building types and styles requires a huge set of grammars.

In most of the approaches graphical models play an important role for reasoning in uncertain data. Graphical models are as well used by Fan and Wonka (2016). They present a learning algorithm for hierarchical graphical models with hidden variables in order to model the exterior of residential buildings based on aerial and street-view images. They further propose optimization methods for the reconstruction of partially interpreted buildings from images or the synthesis of new buildings. Wenzel and Förstner (2016) used marked point processes as an extension of Markov random fields for high-level façade image interpretation in a top-down manner. An energy model is presented that compared façade objects to the given image incorporating learned statistics.

Although these approaches yield good results they rely on dense measurements such as images or dense 3D point clouds or both. This requirement is often difficult to satisfy in an adequate way. Even if sensors are available for the acquisition of high-resolution data occlusions prevent a fully coverage of the façades so that an approach is needed that is able to cope with few observations, e.g. footprints. These are in general easily available using data sources such as OpenStreetMap [1] or building management services and are less costlier than the use of dense 3D point clouds. Biljecki et al. (2017) investigated the potential of generating buildings based on 2D footprints alone. Their approach extrudes footprints to a predicted height in order to reconstruct LoD1 city models. Façade substructures such as windows or doors are not the objective of this work. Dehbi et al. (2016a) address the problem of insufficient data and presented an approach for the identification of translational and axial symmetries as well as their hierarchical structures in building footprints. They use context-free grammars to model the symmetry and hereby are able to exploit this information to compensate occlusions or missing data. While these approaches based on footprints derive basic information about the building this thesis predicts semantic and geometric details of building structures such as windows. A survey by Musialski et al. (2013) summarizes the methods developed for urban reconstruction.

Indoor modeling. Beside reconstruction of the building exterior indoor modeling attracted more and more attention in the scientific environment in the last years. Zlatanova and Isikdag (2015) give an overview of applications for indoor models. A review of indoor modeling and mapping can be found in the work of Gunduz et al. (2016). In contrast to façade reconstruction, the reconstruction of indoor environments remains more difficult due to the need for entering each room and handling far more occlusions caused for example by furniture. Zlatanova et al. (2013) summarize the problems that involve the reconstruction of indoor environments.

Becker et al. (2015) present an iterative automatic learning process for the reconstruction of 3D indoor models from point clouds. A split grammar enables the prediction of indoor geometries based on LoD3 models with window structures and laser scans covering all rooms and hallways. Subsequently the model can be refined by additional indoor measurements while noisy or incomplete data can be compensated by the knowledge encoded in the grammar. Rosser et al. (2017) used a trained Bayesian network for the semi-automatic data-driven estimation of 2D floorplans in residential habits. Similar to the approach of this thesis they incorporated limited prior knowledge such as likely room dimensions and orientations as well as the building footprint or the existence of different room types. However, they assume that the user specifies the basic topology of rooms after an initial prediction of room shapes. Ochmann et al. (2016) emphasize the need for a volumetric, parametric building model beyond pure surface reconstructions and developed a high-level reconstruction method as a labeling problem with energy minimization. Xiong et al. (2013) described a method for the creation of semantically rich 3D building models from point clouds in the context of building information models (BIM). It detects main objects such as windows or walls. In contrast to this thesis, these approaches need dense measurements of the building interiors in order to build an indoor model.

Some approaches made a step forward towards avoiding expensive high-resolution measurements. Diakité and Zlatanova (2016) evaluate the reconstruction of building indoor environments by using 3D information that is captured by the low cost Android tablet from Google's Tango project. Although the data is not rich enough to produce detailed indoor models the approach benefits from the rapid interpretation of 3D meshes that is

[1] https://www.openstreetmap.org

necessary for indoor navigation in emergency cases. A single image per room captured by mobile devices is used by Pintore et al. (2016) together with a tracking of user movements to generate 2.5D indoor maps. The global optimization does not require a Manhattan world configuration but therefore is not reliant in case of many occlusions. Mobile phone sensors were used by Rosser et al. (2015) for semi-automatic reconstructions of residential building interiors. Together with hard and soft constraints they predict as-built building plans.

In contrast to the reasoner presented in this thesis with these approaches the problem remains that each room has to be entered and occlusions may distort the results. Further, with the low cost of the sensors the approaches have to cope with lower quality and distance ranges. An approach in the absence of indoor measurements was presented by Boeters et al. (2015). They automatically enhance LoD2 models of CityGML with their corresponding indoor geometry and predict the storeys of buildings in order to provide a LoD2+ model. The purpose of this work is to provide enough information for applications such as heat simulations or the estimation of inhabitants that do not need higher level of details. A floorplan with a prediction of shapes and locations of rooms that is covered in this thesis is not possible.

Related to the interpretation of as-built floorplans, some methods handle the generation of virtual indoor environments. Merrell et al. (2010) propose a method for the automatic generation of residential building layouts. They use a trained Bayesian network and stochastic optimization to construct visible plausible 3D floorplans for computer graphics applications. Mixed integer quadratic programming (MIQP) was applied by Wu et al. (2018) for the generation of building interiors. They used axis-aligned polygons and identified linear inequality constraints considering room size or room adjacencies. For optimization they present as well a rectangle-based layout.

Constraint Logic Programming for 3D city models. The reconstruction of a floorplan with its rooms can be seen as a combinatorial problem to place n rooms in a given footprint. To this end, Constraint Logic Programming (CLP) can be used to solve this kind of constraint satisfaction problem. While constraint programming is often used for combinatorial problems such as scheduling tasks its application in 3D city models is rare. Charman (1994) presented a knowledge-based system for floorplan design that considered geometric constraints on rooms such as non-overlap or adjacency. They used consistency techniques such as arc-consistency in order to generate all possible floorplans satisfying the given constraints. However, it does not deal with the reconstruction of existing floorplans and omits a ranking of possible floorplans as is proposed in this thesis. Although the strength of CLP is to solve non-linear problems with more than one unknown it is originally not made for uncertain relations that however is an important aspect in the context of GIS. Consequently, in the context of building reconstruction constraint programming is not widespread. Therefore, the work of Kolbe (2000) proposed the reconstruction of roofs by the use of constraint logic programming extended by uncertain constraints and an MAP classification. Observations of aerial images are compared to constraint-based models of roofs that are characterized by geometric constraints between primitives. A degree of knowledge is integrated in constraint programming by Saad et al. (2010) and extended by Saad (2015) in order to handle uncertainty. Intervals of domains are augmented by cumulative distribution functions (cdf-intervals). While these approaches integrate the stochastic knowledge into the algorithm for constraint logic programming, this thesis tries to profit from the strength of CLP in solving combinatorial problems on the one hand and the dominance of graphical models in stochastic reasoning on the other hand.

Automatic Theorem Proving. An important basis of successful reasoning with model-driven approaches are consistent and non-redundant prototyped building models. Hoffmann and Joan-Arinyo (2005) give an overview of methods for constraint solving and the closely related automatic theorem proving. Geometric and algebraic constraints are one way to represent building models. Due to the geometric character of buildings they can often be defined by multivariate polynomials that pave the way to use algebraic reasoning based on methods of automatic theorem proving such as Gröbner bases or Wu's method. Up to now these methods have hardly attracted attention in the context of GIS.

Brenner and Sester (2005) presented an approach for cartographic generalization in the two-dimensional space. During generalization a constraint may be added or removed after detecting consistency, redundancy or contradiction by the use of Gröbner bases or Jacobi matrices. The method is extended by the concept of weak primitives that were introduced by Brenner (2004) and reviewed by Brenner (2005) in the context of interactive modeling of building scenes. They enable to switch on and off model-defining constraints if geometric primitives such as points or lines are combined. In contrast to the method developed in this thesis the authors focus on the two dimensional space and question the feasibility for interactive systems – especially in the 3D space as handled in this thesis. A relaxation of constraints was handled in the context of this thesis by the concept of uncertain projective geometry and homogeneous coordinates that turned out to be an appropriate representation of constraints for geometric reasoning. This choice is confirmed by Meidow et al. (2009) who emphasize the benefits of using homogeneous coordinates for uncertain two-dimensional primitives in geometric reasoning processes. Meidow and Hammer (2016) described a workflow for the algebraic reasoning in the context of data-driven building reconstruction. Boundary representations are derived based on given point clouds and Gröbner bases are used to ensure independent and consistent constraints.

In contrast to these approaches that use Gröbner bases, this thesis uses Wu's method for algebraic reasoning that in general shows to be more successful – especially in the case of geometric construction as stated by Cox et al. (2007) and Quaresma (2010). Kapur and Mundy (1988) present the application of Wu's method to perspective viewing. In the context of building reconstruction it is not yet applied.

3 Methodological background

This chapter introduces the methodological background that is most relevant for this thesis. Constraint programming and Bayesian networks are introduced to reason with models for the prediction of building structures. While the strength of constraint logic programming is the combinatorial aspect with crisp constraints that do in general not consider the impreciseness of objects or measurements, Bayesian networks are powerful stochastic frameworks that incorporate uncertainty. Methods of automatic theorem proving are presented as a means for the consistent and redundancy-free development of prototyped models that are prerequisite for the developed model-driven approach.

3.1 Constraint programming

The reasoning method developed in this thesis has to solve combinatorial problems of non-linear equations with more than one unknown for which constraint programming is a powerful framework. In this context, it is used to reason with crisp constraints as a first step during model prediction. Uncertainty in form of probabilities are not considered so far.

Characteristic for constraint programming is that the information flow is not determined but the problem to solve is given in a declarative way by constraints on partially unknown discrete and continuous variables. The basic idea of solving a problem, i.e finding valid assignments for queried parameters, is following the principle of "constrain and generate" instead of "generate and test". The constraint solver does not generate instantiations of parameters and costly tests whether they are valid. Instead, as a first step constraints are used to restrict the search space before searching for one or all possible solutions afterwards.

Typically, there are two types of related problems solved by constraint programming: constraint satisfaction problems (CSP) and constraint optimization problems (COP). While CSPs yield one or more equally valuated solutions that satisfy defined constraints, COPs search for a (single) solution that respects the constraints and further minimizes a given objective function. The type of problems that are solved by constraint programming techniques in the context of this thesis are constraint satisfaction problems.

Formally, a constraint satisfaction problem (or constraint network) is defined by a set of constraints $\mathcal{C} = \{C_1, ..., C_q\}$ on a set of variables $\mathcal{X} = \{X_1, ..., X_n\}$ with corresponding domains $D_1, ..., D_n$ for each variable. The solution space is an n-dimensional search space that is initially defined by the Cartesian product $D_1 \times ... \times D_n$ of the domains. A *constraint* C_i is a relation on a subset of variables $\mathcal{X}' \subseteq \mathcal{X}$ and thus a subset of $D_1 \times ... \times D_n$. Figure 3.1 (top) illustrates a constraint network for a constraint satisfaction problem of three discrete variables $\{X_1, X_2, X_3\}$ constrained by two constraints $\{x_1 = x_2, x_1 + x_3 = 13\}$

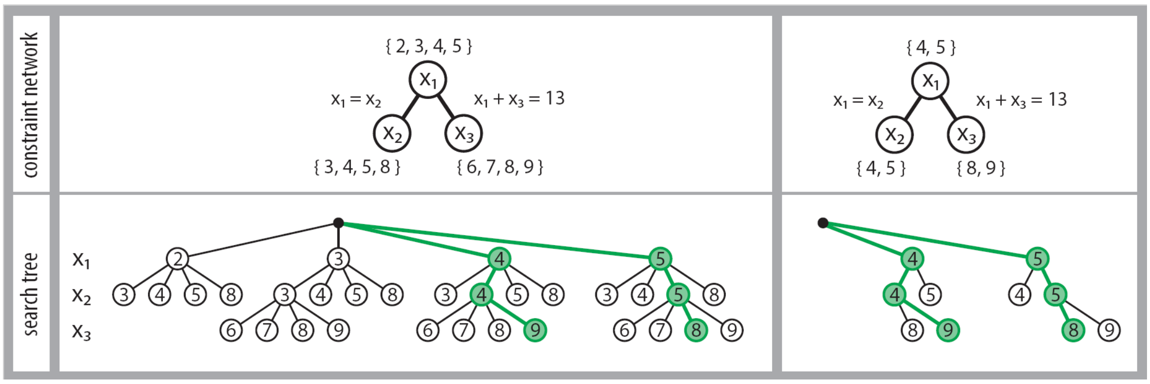


Figure 3.1: Constraint networks and search trees for CSP $(\mathcal{X},\mathcal{D},\mathcal{C})$ with variables $\mathcal{X} = \{X_1, X_2, X_3\}$, domains $\mathcal{D} = \{\{2,3,4,5\},\{3,4,5,8\},\{6,7,8,9\}\}$ and constraints $\mathcal{C} = \{X_1 = X_2, X_1+X_3 = 13\}$. Left) Naive depth-first search with backtracking. Right) Enhanced backtrack-free search after enforcing arc-consistency

and corresponding integer valued domains $D_1 = \{2,3,4,5\}$, $D_2 = \{3,4,5,8\}$ and $D_3 = \{6,7,8,9\}$. Finding a *solution* of a CSP means finding an instantiation of the variables that satisfies all constraints $C_i \in \mathcal{C}$, that is an assignment $\{(X_1,\alpha_1),...,(X_n,\alpha_n)\}$ of values for each variable X_i with $(\alpha_1,...,\alpha_n) \in D_1 \times ... \times D_n$.

The search space can be seen as a tree whose nodes are associated with a variable and one of its possible instantiations (cf. Figure 3.1 (bottom)). An arc represents an operator that expands the intermediate solution with an assignment of a value for an additional variable considering that together with prior instantiations the constraints are still satisfied. Solutions are found by performing a depth-first search in the search tree and are thus represented by solution paths to the leaf nodes.

Naive search would be characterized by backtracking, that is, it jumps back to preceding nodes in the tree (prior states) if a so called dead-end is reached meaning that the current solution is inconsistent with the constraints. Figure 3.1 (left) shows the original constraint network and, building on that, the search tree for finding all consistent solutions of the problem. White circles at the end of a path represent a dead-end where the algorithm has to go back to the previous node in order to try another value. Colored circles represent instantiations of a solution path. Since constraints are only used to test generated solutions for validity many unnecessary dead-ends are encountered. For an efficient search the basic aim of constraint programming is to achieve a backtracking-free search. For example, choosing $X_1 = 3$ and $X_2 = 3$ is consistent with the constraint $X_1 = X_2$ but no value can be found for X_3 satisfying $X_1 + X_3 = 13$ so that the algorithm has to backtrack. In order to avoid backtracking, the constraint solver uses constraint propagation and consistency-enforcing algorithms to derive new constraints from existing ones or to tighten these constraints. According to this principle of "constrain and generate", the search space is decreased before searching for valid instantiations of the unknown variables.

There exist different consistency-enforcing algorithms on a constraint network whose task is to extend a partial solution by another variable. Efficient computations exist for enforcing arc-consistency that consider constraints with two variables (binary constraints). A binary constraint is *arc-consistent* if for each value chosen for one variable of a constraint there is a consistent choice in the domain of the other variable. A constraint network is *arc-consistent* if all its binary constraints are arc-consistent. For constraints having i variables *i-consistency* can be enforced meaning that for every consistent instantiation of $(i-1)$ variables an instantiation of any ith variable can be found. While a naive algorithm of arc-consistency tests for all binary constraints of the constraint network every value of one variable against every value of the other variable until the domains do not change anymore there exist more intelligent algorithms that reduce the number of tests significantly. For example reduction of one domain implies only the repeated test of those constraints that are connected with the variable whose domain was currently changed. Algorithms for enforcing consistency are described in detail by Dechter (2003). The worst-case complexity of enforcing arc-consistency is $O(ek^2)$ for a constraint network of e binary constraints where k bounds the domain size. Figure 3.1 (right) shows the arc-consistent network of the exemplified CSP and illustrates the improved search as a result of constraint propagation. The domains of the values of the variables were checked for inconsistency and values were removed that cannot participate in a solution. As a consequence, search does not unnecessarily encounter a dead-end.

Although the search profits from a high level of consistency, there should be a trade-off between the time and space spent for consistency-enforcing and constraint propagation - depending on the level of consistency - and that spent for the search. In this context, since the complexity of i-consistency is bound by the domain size, bounds-consistency is often less costlier in case of integer variables with large domains or continuous intervals. A constraint is *bounds-consistent* if for each variable $X_i \in [A_i, B_i], i = 1...n$ of an n-ary constraint an instantiation for the other variables $X_j, j \neq i$ can be found so that the constraint with an assignment $X_i = A_i$ and $X_i = B_i$ respectively is still satisfied. In other terms, in contrast to arc-consistency values are only removed from their domain if the result is not split but still a single interval. The consistency condition of compatibility of values is thus satisfied for the lower and upper bound of the domain but not for all values and thus is a trade-off between efficiency of preprocessing and a high level of consistency.

For numeric constraints the search space can be reduced by using propagation rules and linear elimination that calculate new intervals for each domain of the variables. For a constraint $X = Y \times Z$ with $X \in [A, B], Y \in [C, D]$ and $Z \in [E, F]$ with $X, Y, Z > 0$ the domain of Z can for example be updated to $[A/D, B/C]$. Deducing additional constraints or tightening existing ones produces an equivalent constraint network that avoids to reach a dead-end involving backtracking.

Further improvements can be made in the subsequent search phase by different search strategies enhancing the two phases of backtracking, the forward phase and the backward phase. The forward phase of backtracking is enhanced by the *look-ahead* principle for variable and value ordering that chooses which variable is selected next and which value should be assigned. Going backward in case of a dead-end becomes more intelligent by the *look-back* principle. It tries to find out which node is the reason for a dead-end in order to jump back to this previous node (backjumping) and thus avoiding unnecessary backtrack points.

The same aim is reached by an analysis of encountering a dead-end is also used for constraint recording (learning) in that new constraints are added to avoid the same failure in subsequent search iterations.

To sum up, the search space and thus the performance of backtracking depends on the preprocessing that is the level of local consistency and the strategies used during variable instantiation.

Introduced in the 1980's, one prominent implementation of constraint solvers is constraint logic programming (CLP) that is based on logic programming. A logic program is defined by first-order predicate logic as a set of clauses of the form $H : -B_1, ..., B_n$. The *head* H as well as the B_is of the *body* are predicates $p(t_1, ..., t_m)$ with terms t_i being a variable, a constant or in turn the application of a function $f(t_1, ..., t_m)$. A clause is equivalent to an if-then-rule meaning if the conjunction of literals $B_1, ..., B_n$ is true then the head H is true, too. Since logic programming is characterized by depth-first search with backtracking and involves relations and predicates that control and constrain the set of solutions it is an appropriate language for implementing constraint solvers. Constraint programming extends logic programming by allowing the body of the rules to contain constraints. Further, constraint propagation is incorporated to enhance backtracking and thus to solve constraint satisfaction problems efficiently. Consequently, CLP provides powerful search strategies that enable the solvers to handle non-linear constraints with more than one unknown in a declarative way. For more details on constraint processing the reader is referred to Dechter (2003) and Marriott and Stuckey (1998).

3.2 Bayesian networks

The approach presented in this thesis addresses the uncertainty of both data and models explicitly. One of the most powerful frameworks for modeling and reasoning over complex domains with uncertain data are probabilistic graphical models. The prominent methods use graph-based representations in order to represent complex distributions in a compact way. An introduction to graphical models used in geodesy and photogrammetry is given by Förstner (2013). One prominent type of graphical models are Bayesian networks. They are directed acyclic graphs (DAGs) of a set of nodes V where each node $v \in V$ is associated with a random variable X_v and the edges define assumptions about conditional dependencies and independencies. Discrete random variables are associated with a conditional probability distribution (CPD) $P(X_v|X_{pa(v)})$ that denotes the probability for a discrete node $v \in V_\Delta$ given the state of its parent nodes $pa(v)$, that is, the immediate predecessors in the directed graph. The graph represents a full joint distribution over the set of random variables X_v that is – considering the independence assumptions – defined as

$$P(X_1, ..., X_n) = \prod_{v \in V_\Delta} P(X_v|X_{pa(v)})$$

The compact modeling of probabilistic dependencies and independencies enables the use of efficient inference algorithms for determining posterior distributions given an observation.

There are basically two common types of queries performed on graphical models both involving the joint distribution: conditional probability query and MAP queries. The first determines the posterior probability distribution $P(Y|E = e) = \frac{P(Y,e)}{P(e)}$ for a set Y of unknown variables given evidence on a subset of variables $E = e$. MAP queries – also known as most probable explanations (MPE) – seek the most probable joint assignment $\text{MAP}(W|e) = \arg\max_w P(w, e)$ to all unknown variables $W = X - E$ given evidence $E = e$.

Exact inference over the joint distribution in discrete Bayesian networks can be performed by variable elimination, that avoids repeated calculations of factors in the joint distribution by calculating common expressions once and reusing these new factors. Based on this idea clique trees (or junction trees) are used to specify the transformations and their partial order. New factors are calculated in corresponding cliques, subsets of variables, and sent as messages to the next clique towards the root of the tree that leads to the answer of the query.

Most problems found in the literature are modeled as discrete problems instead of defining continuous or even hybrid networks. The latter involves discrete variables $\mathcal{X}_\Delta$ as well as continuous variables $\mathcal{X}_\Gamma$ ($\mathcal{X} = \mathcal{X}_\Delta \cup \mathcal{X}_\Gamma$). Inference methods for hybrid models can be applied similar to the discrete case, nevertheless exact inference is in most cases far difficult (Koller and Friedman (2009)). One step towards exact inference can be made with additional assumptions such as that a continuous parameter $X \in \mathcal{X}_\Gamma$ is normally distributed with mean μ and variance σ^2:

$$p(x; \mu, \sigma^2) = N(\mu, \sigma^2) = \frac{1}{(2\pi\sigma^2)^{1/2}} exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)$$

Lauritzen and Jensen (2001) developed a method for exact inference in special Bayesian networks, so called conditional linear Gaussian (CLG) networks. Characteristic of these networks is that continuous variables are associated with conditional linear Gaussian CPDs and discrete nodes do not have any continuous parents. Similar to the discrete case the CLG CPD for a continuous variable X depends on its discrete parents $I \subseteq \mathcal{X}_\Delta$ and its continuous parents $Z \subseteq \mathcal{X}_\Gamma$. It is defined for each instantiated value $\tau \in \mathbb{Z}^{|I|}$ of the discrete parents as:

$$p(X|Z = z, I = \tau) = N(\mu_\tau + \beta_\tau^T z, \sigma_\tau),$$

with mean value μ_τ and variance σ_τ and a vector of regression coefficients β_τ. For the whole hybrid network the joint distribution can thus be defined as a $|\mathcal{X}_\Gamma|$-dimensional Gaussian distribution (Kjærulff and Madsen (2008)) for each instantiation $\tau \in \mathbb{Z}^{|\mathcal{X}_\Delta|}$:

$$p(X_\Delta = \tau) \cdot N_{|\mathcal{X}_\Gamma|}(\mu_\tau, \sigma_\tau^2) = \prod_{v \in V_\Delta} P(\tau_v|\tau_{pa(v)}) \prod_{w \in V_\Gamma} p(y_w|X_{pa(w)}). \tag{3.1}$$

Lauritzen's algorithm for CLG networks is similar to the clique tree algorithm for discrete networks. It is exact for discrete parameters and yields correct means and (co)variances for the continuous parameters. For non-linear relations and non-Gaussian distributions M-projection is a standard method for linearization that transforms a Gaussian mixture into

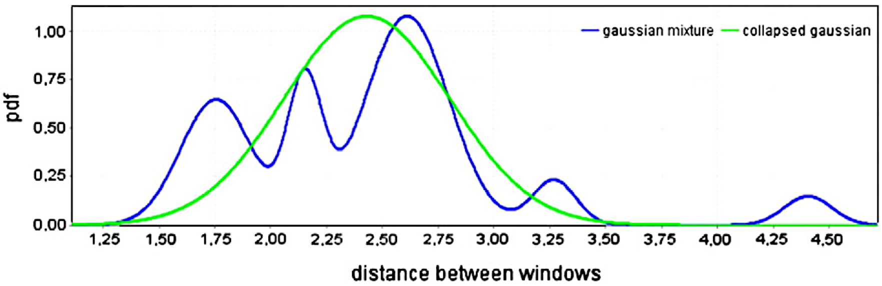


Figure 3.2: Collapsing a Gaussian mixture (blue) into a single Gaussian (green) does not yield a good approximation in this case.

a single Gaussian distribution (Koller and Friedman, 2009). This is especially a good approximation as long as the resulting multivariate Gaussian is close to the original Gaussian mixture. However, this thesis addresses continuous random variables whose Gaussian mixtures vary widely from the collapsed version using M-projection so that the proposed approximation reveals not to be practicable in this case (cf. Figure 3.2).

Another special structure of Bayesian networks are state-observation models where the state (of model parameters) evolves over time. The problem is described by a transition model that models the change of parameters μ from time step $t-1$ to t and an observation model that defines the mapping from model parameters μ to observation parameters o (Koller and Friedman (2009)).

In the special case of linear Gaussian systems with solely continuous parameters the posterior can be calculated by simple matrix operations using the Kalman filter (Kalman (1960)). This efficient implementation is exploited in the context of this thesis. The idea of the algorithm is to filter out noise and improve the prediction of model parameters with new observations. The first step is the estimation of the new state $\mu_i \in \mathbb{R}^n$ and the corresponding covariance matrix Σ_i with the transition matrix A that models the dependency on the prior value μ_{i-1}:

$$\begin{aligned} \mu_i &= A\mu_{i-1} \\ \Sigma_i &= A\Sigma_{i-1}A^T + R \end{aligned} \tag{3.2}$$

The noise of the model dynamics is incorporated by the covariance matrix R. The subsequent correction step or measurement update determines the a posteriori state and error covariance by incorporating the Kalman gain

$$K = \Sigma_i M^T (M\Sigma_i M^T + Q)^{-1} \tag{3.3}$$

and using a measurement matrix $M \in \mathbb{R}^{n\times m}$ that relates the observation vector $o \in \mathbb{R}^m$ to the model parameters μ by a mapping $o = M\mu$:

$$\begin{aligned} \mu &= \mu + K(o - M\mu) \\ \Sigma &= (Id - KM)\Sigma_i \end{aligned} \tag{3.4}$$

Here, Id is the identity matrix and $Q \in \mathbb{R}^{m\times m}$ defines the Gaussian noise of observations. The Kalman filter is equivalent to a dynamic Bayesian network (Koller and Friedman

(2009)), but saves time and space for the calculation of posterior beliefs due to the compact representation.

3.3 Theorem proving

Model-driven reasoning methods as presented in this thesis need prototyped building models in order to reconstruct buildings in a top-down manner. For an adequate development of these models the identification of redundancy and consistency is important. Further, making implicit latent constraints explicit is of particular interest. Checking the redundancy of constraints is equivalent to proving their deducibility from a (non-redundant) set of constraints. In the same way the method can be applied to check consistency or derive new constraints. For this aim, automatic theorem proving is a powerful means for showing the validity of a statement given its premises. Depending on the representation of the problem and applied methods it can be separated into two main categories: algebraic and deductive (rule-based) reasoning.

Deductive and algebraic reasoning are introduced in the next section and will be illustrated by the following small geometric theorem of three planes in the three-dimensional space:

Example 1. *If plane* Π_1 *is orthogonal to plane* Π_2 *and parallel to plane* Π_3*, then plane* Π_2 *is orthogonal to plane* Π_3 *(*$\Pi_1 \perp \Pi_2 \wedge \Pi_1 \| \Pi_3 \Rightarrow \Pi_2 \perp \Pi_3$*).*

3.3.1 Deductive reasoning

Deductive reasoning is based on previously known general implications (rules) and basic facts that often are represented using first-order predicate logic. A rule is equivalent to a logical implication $B_1 \wedge \ldots \wedge B_n \Rightarrow H$ and – as defined in Section 3.1 – expressed by so-called *Horn clauses*

$$H \leftarrow B_1, B_2, \ldots, B_n$$

where $H, B_1, \ldots, B_n$ are atoms and a logical $\wedge$ (AND) is expressed by a comma. Clauses without body, i.e. $n = 0$, are called facts. By applying rules new knowledge in form of new facts can be derived. In the context of automatic theorem proving the aim is then the derivation of the conclusion using the premises of the theorem together with the known knowledge of facts and rules given in the database.

The theorem of Example 1 can be expressed by the following rule

```
orthogonal(B,C) ← orthogonal(A,B), parallel(A,C).
```

where *A, B, C* are variables that can be substituted by constants and same variable names define the same object. As a consequence, if we have two facts

```
orthogonal(front,bottom).
parallel(front,back).
```

in our database, the reasoner can substitute the variable A with *front*, B with *bottom* and C with *back* and thus deduces *orthogonal(bottom, back)*. In this way, the knowledge can be extended by applying rules and propagating facts from the right side to the left side. An implementation of deductive reasoning based on first-order predicate logic can be found in deductive databases that are a combination of traditional relational databases and logic programming.

In order to draw all possible conclusions from a known set of rules and facts and herewith verify or falsify a theorem the fixpoint iteration is an efficient inference technique in deductive databases. A *fixpoint* is reached if the set of facts in the database does not change anymore although the given rules are iteratively applied to the data. If the knowledge base contains the conclusion the theorem is proven to be true. However, the concept is based on the closed world assumption that means that facts that are not present in the database and cannot be deduced are considered to be false (Russell and Norvig, 2009). Although deductive reasoning can deduce new facts very fast algebraic reasoning may prove implications where deductive inference techniques may fail since algebraic methods are not dependent on previously defined rules. Instead, they only require that theorems can be expressed by polynomial equations as presented in the next section.

3.3.2 Algebraic reasoning

Building models are dominated by geometric constraints that can be expressed by multivariate polynomials. This paves the way to deduce redundant or new constraints automatically by using algebraic approaches of automatic theorem proving. The introductory example of three planes will be used to illustrate the idea of algebraic theorem proving.

For algebraic reasoning, a plane Π_i can be represented by its normal form $a_i x + b_i y + c_i z + d_i = 0$, that is $\Pi_i = ((a_i, b_i, c_i), d_i) = (\mathbf{n}_i, d_i)$ where $\mathbf{n}_i$ denotes the normal vector of the plane. Using this representation, orthogonality and parallelity can easily be expressed by the dot and cross product:

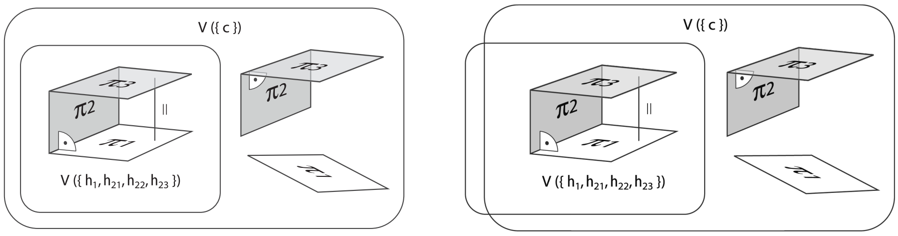
$$\Pi_i \perp \Pi_j \Leftrightarrow n_i^T n_j \Leftrightarrow a_i a_j + b_i b_j + c_i c_j = 0$$

$$\Pi_i \| \Pi_j \Leftrightarrow n_i \times n_j \Leftrightarrow \begin{pmatrix} b_i c_j - c_i b_j \\ c_i a_j - a_i c_j \\ b_i c_j - c_i b_j \end{pmatrix} = \mathbf{0}$$

Consequently the proof of deducing the conclusion c ($\Pi_2 \perp \Pi_3$) from the hypotheses H ($\Pi_1 \perp \Pi_2 \wedge \Pi_1 \| \Pi_3$) of Example 1 leads to the following algebraic representation[1]:

$$a_1 a_2 + b_1 b_2 + c_1 c_2 = 0 \wedge \begin{pmatrix} b_1 c_3 - c_1 b_3 \\ c_1 a_3 - a_1 c_3 \\ b_1 c_3 - c_1 b_3 \end{pmatrix} = \mathbf{0} \Rightarrow a_2 a_3 + b_2 b_3 + c_2 c_3 = 0$$

[1] The term *hypothesis* is equivalent to the premises under which condition the conclusion holds. It is a classical term in the context of automatic theorem proving and thus also used in this section - in contrast to the previously used *model hypothesis* in the context of predictions in model-based reasoning.



(a) $V(H) \subseteq V(c)$ (b) $V(H/d_1 \cdot \ldots \cdot d_t) \subseteq V(c)$ excluding degenerated cases with $d_i \neq 0$

Figure 3.3: relations between varieties for a theorem proven to be valid: the variety of the conclusion contains the variety of the premises.

The key idea of algebraic theorem provers is to reduce the problem to the analysis of common zeros of the multivariate polynomials that represent the premises and the conclusion. The common zeros of equations $h_1, \ldots, h_s$ in a polynomial ring $k[x_1, \ldots, x_n]$ are called its *variety*:

$$V(\{h_1, \ldots, h_s\}) := \{(a_1, \ldots, a_n) \in k^n : h_i(a_1, \ldots, a_n) = 0 \; \forall 1 \leq i \leq s\} \qquad (3.5)$$

In order to proof the validity of a theorem, it has to be shown that the zeros of a given set of premises $H = \{h_1, \ldots, h_s\}$ are a subset of the zeros of the conclusion c that is $V(\{h_1, \ldots, h_s\}) \subseteq V(c)$. If a set of polynomials has no common zeros it is inconsistent.

The relation between varieties of the conclusion and the premises is illustrated in Figure 3.3a. Here, Example 1 of three planes is represented by five constraint equations. The set of zeros $V(\{h_1; h_{21}, h_{22}, h_{23}\}) = V(\{n_1^T n_2, n_1 \times n_3^T\}) = V(\{a_1a_2 + b_1b_2 + c_1c_2; b_1c_3 - c_1b_3, c_1a_3 - a_1c_3, b_1c_3 - c_1b_3\})$ is not restricted further if the conclusion c – the second constraint of orthogonality – is added, since all zeros of H are also part of the zeros of c. As a consequence, the conclusion c is redundant. Note that actually non-degenerate cases have to be added.

There are two prominent approaches for automatic theorem proving that are based on comparing common zeros: the Gröbner base method and Wu's method. While the Gröbner base method (Buchberger (1998)) requires a Gröbner base that is tested with polynomial long division Wu's method (Wu (1986)) is based on a characteristic set where the theorem is proven by pseudodivision, a variant of polynomial long division. The advantage of Wu's method is that the characteristic set is in most cases easier to construct in the case geometric objects are introduced subsequently. Further, subsidiary conditions are produced automatically to accept the theorem under some non-degenerated cases. Cox et al. (2007) and Quaresma (2010) showed that Wu's method is in most cases superior to the Gröbner base method. In this thesis, Wu's method is used for geometric reasoning on 3D building models.

Wu's method

Wu's method proves whether a theorem is generically true that is true under some non-degenerated cases, so called subsidiary conditions. The advantage of this method is that degenerated cases d_i can be excluded by considering subsidiary conditions in order to declare an otherwise false theorem generically true. Consequently, the aim is to show that $V(h_1, \ldots, h_s/d_1 \cdot \ldots \cdot d_t) \in V(c)$ (cf. Figure 3.3b). The algorithm therefore uses characteristic sets and the so called pseudodivision of multivariate polynomial equations.

A characteristic set is a special triangular equation system on a given ordering of independent variables $x_1 \prec \ldots \prec x_s$,

$$h_1(u_1, \ldots, u_d, x_1) = 0 \in k[x_1]$$
$$\ddots$$
$$h_s(u_1, \ldots, u_d, x_1, \ldots, x_s) = 0 \in k[x_1, \ldots, x_s],$$

Being defined as a minimal ascending chain the characteristic set requires that for each hypothesis h_i the degree of its highest variable (according to the defined ordering of independent variables) is higher than in all subsequent hypotheses $h_j, j > i$. For geometric theorems this structure is often given with a sequential introduction of geometric objects and their corresponding relations. The same applies to the required ordering of independent variables that is crucial for the efficiency of triangulation. Geometrically seen they correspond to the parameters that can be chosen arbitrarily despite the involved constraints.

The calculation of the characteristic set and later on the proof of the theorem is performed by pseudodivision. The pseudodivision differs from the polynomial long division in allowing the multiplication of the dividend c with a factor $I(h_i)^{k_i}, k_i > 0$:

$$I(h_i)^{k_i} c = q_i h_i + r \tag{3.6}$$

where q_i is the quotient and r the pseudoremainder ($prem(c, h_i, x_i) = r$). $I(h_i)$ denotes the initial of h_i being in turn the coefficient of the highest variable of the polynomial in question. For more than one dividend the pseudodivision extends to

$$I(h_1)^{k_1} \ldots I(h_s)^{k_s} c = q_1 h_1 + \ldots + q_s h_s + r \tag{3.7}$$

A theorem is proven generically true if the result of the pseudodivision prem(c, H') equals zero where H' denotes the characteristic set of the original set of multivariate polynomials. Basically Wu's method consists of three steps:

1. *Theorem Formulation:* Define the theorem $\{h_1, \ldots, h_s\} \Rightarrow c$ with hypotheses $H = \{h_1, \ldots, h_s\}$ and the conclusion c in form of multivariate polynomial equations $h_i = 0, c = 0$.
2. *Triangulation to characteristic set:* Transform the hypothesis H into a characteristic set H' subject to the dependent variables of the geometric constraints.
3. *Proof:* Prove the implication H' $\Rightarrow$ c using the pseudodivision $prem(c, H')$ to show $V(h_1, \ldots, h_s/d_1 \cdot \ldots \cdot d_t) \in V(c)$. If the final pseudoremainder $prem(c, H')$ equals zero, the theorem is *generically* proven true with subsidiary conditions $d_i \neq 0$.

Referring back to the theorem of Example 1 the characteristic set $H' = \{h'_1, h'_2, h'_3\} = \{h_1, -h_{21}, -h_{23}\}$ with a chosen variable ordering $b_2 \prec c_3 \prec a_3$ is computed. The corresponding algorithm that uses as well pseudodivision can be found in Buchberger et al. (1988). Since the solution of the following pseudodivision

$$prem(c, H') = prem(prem(prem(c, h'_3, a_3), h'_2, c_3), h'_1, b_2) \tag{3.8}$$

yields zero, the theorem is generically proven true with the subsidiary condition that $b_1 \neq 0$, having $I(h'_1) = I(h'_2) = b_1$ and $I(h'_3) = -b_1$ as initials of the three polynomials. The specification of degenerated cases enables to rule out special configurations that otherwise would prevent to prove the corresponding theorem. An implementation of Wu's method and the pseudodivision can be found in Wang and Suter (2004).

4 Automatic reasoning for the prediction of 3D façade structures

This chapter presents a method for the prediction of 3D façade structures as published in Loch-Dehbi and Plümer (2015). As it is one of the main principles of this thesis the approach only relies on sparse observations. In the absence of dense measurements this thesis follows a model-driven approach that estimates shape and location parameters for building substructures such as windows and yields a ranked set of predicted façades. As a consequence, single additional measurements are sufficient to verify or falsify a model that is easier than to reconstruct buildings bottom-up from measurements.

Section 4.1 introduces the functional and stochastic model that is used as a basis for top-down façade reconstruction. The façade prediction in turn is presented in Section 4.2. A transfer of this approach for indoor models and the prediction of floorplans is explained in Chapter 5.

4.1 Constraint-based 3D modeling of buildings

The presented model-driven approach is backed by a prototyped building model of which parameters such as the width of windows have to be instantiated during the reconstruction process. Man-made objects such as buildings are often characterized by regularities and symmetries between their parts. In this context, constraints are an appropriate means to represent and enforce the relations between substructures and at the same time narrow the search space for plausible predictions of façades. The mathematical model for the identification and estimation of geometric objects from observations serves as prior knowledge and can be divided into a functional and a stochastic model. In this case, the functional model consists in constraints on model parameters and the stochastic model is defined by probability density functions.

The development of a mathematical model for buildings used in this thesis is based on an extensive analysis of existing city models in a spatial relational database of about 9 million buildings from North Rhine Westfalia, Germany. This data is enriched by training data that was collected by an annotation of ground truth data from about 1000 façades from images and laser scans. Figure 4.1 shows an extract of the designed database schema for annotated 3D buildings as used for this thesis in the analysis of characteristic properties. The hierarchical model includes buildings with their characteristics like footprint type, building type or architectural style. They in turn are composed of façades with windows. A wide spectrum of properties was gathered in order to develop a profound set of prior knowledge. This includes also derived properties such as distances between objects or correlations between

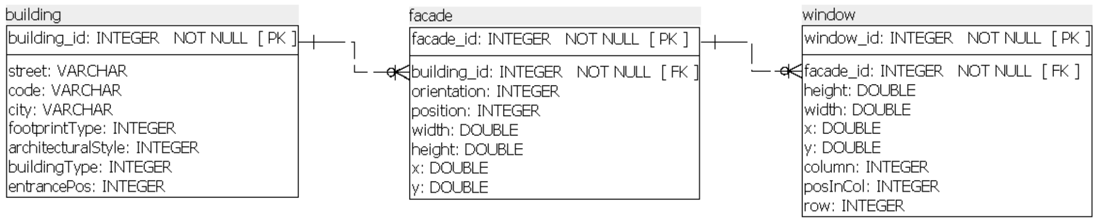


Figure 4.1: Extract of database schema for deriving universal prior knowledge used during reasoning in building façades

model parameters. It should be noted that this data is the basis of general valid and representative distributions and constraints but not used as direct input for the reconstruction of a specific building.

The functional model is represented by constraints that relate shape and location parameters of building substructures such as façades, windows or doors and restrict possible appearances of the façade. The presented method profits from strong regularities in man-made objects that consequently legitimate strong constraints. For example shape parameters for a row of windows cannot be chosen arbitrarily large since the width of the façade as an expected observation is the sum of the width of the windows and the distances between them and the margin of the façade. Façade widths are derived from the known footprint that is additionally analyzed with regard to symmetries. An approach for the identification of translational and axial symmetries in a building footprint can be found in Dehbi et al. (2016a). A symmetric footprint suggests the symmetry for the corresponding façade and avoids the need for measurements of the symmetric parts. Further, the width of a window is obviously related to the height of a window. This kind of dependency is a result of architectural, aesthetic or legal regulations that as well help to structure the search space.

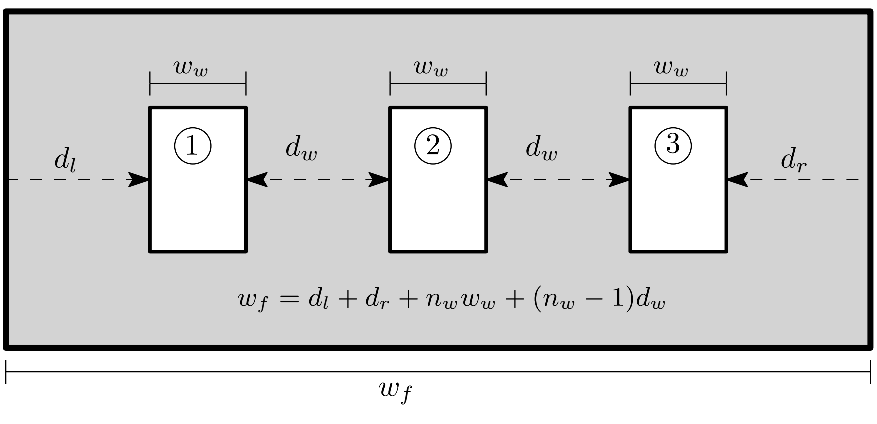


Figure 4.2: Functional model for façades illustrated by a row of windows

Figure 4.2 illustrates the representation of windows in a façade and shows the model parameters of the functional model exemplified for a row of windows. The vertical model of a façade including heights and vertical distances is represented likewise. The geometric character of the model leads to continuous shape and location parameters. Moreover, the model consists of discrete parameters such as the architectural style or the number of windows.

An overview of constraints and used parameters for predicting a row of windows can be found in Table 4.1. The model parameters can be categorized into discrete and continuous parameters on the one hand and model and observation parameters on the other hand. Central for this prediction of façade structures are the few observations o_i assumed to be known a priori and related to the model parameters. The façade width w_f as an observation is related to the continuous parameters d_l (distance to the left façade margin), d_r (distance to the right façade margin), w_w (window width) and d_w (distance between windows) in a bilinear constraint

$$o_1 = w_f = d_l + d_r + n_w w_w + (n_w - 1)d_w \tag{4.1}$$

with the discrete parameter n_w denoting the number of windows in a row.

For a possibly but not obligatory observed embrasure d_e, measured as the distance between left façade margin and the embrasure, the constraint equation is

$$o_2 = d_e = d_l + (c_w - 1)w_w + c_e w_w + (c_w - 1)d_w \tag{4.2}$$

where additionally c_w is the (unknown) index of the window containing the observed embrasure and a binary variable $c_e \in \{0, 1\}$ fixes whether the observation is related to a left or a right embrasure. A valid hypothesis is thus an instantiation of the unknown parameters on the right hand side of the equations. While most approaches handle pure discrete or continuous models, the presented method has to cope with a hybrid model of discrete and continuous parameters for which reasoning turns out to be more complex.

Special for this model is that for each observation o_i continuous model parameters are characterized by equations of the form

$$o_i = \sum_{j=1}^{|\mathcal{X}_\Gamma|} d_j c_j, \tag{4.3}$$

where the d_j's are discrete coefficients containing discrete model parameters, the c_j's are continuous model parameters and $|\mathcal{X}_\Gamma|$ is the cardinality of the set of continuous parameters. The key idea of the presented approach is that the non-linear problem can be reduced to a linear problem as soon as the discrete parameters are instantiated.

	continuous variable	**discrete variable**	
	in constraint		**for GM component**
(unknown) model parameter	w_w d_w d_l d_r	n_w c_w c_e r_1 r_2 r_3	gc_1 gc_2 gc_3 gc_4
observation parameter	w_f d_e p_1, p_2, p_3		

w_w	window width
d_w	distance between windows
d_l	distance between left façade margin and first window
d_r	distance between right façade margin and last window
n_w	number of windows
c_w	index of window corresponding to embrasure
c_e	index of left or right embrasure corresponding to observation
r_i	i-th ratio
gc_i	index of mixture component for i-th Gaussian mixture
w_f	façade width
d_e	distance between left façade margin and embrasure
p_i	i-th pseudo-observation ($= 0$)

constraints
$w_f = d_l + d_r + n_w w_w + (n_w - 1)d_w$
$d_e = d_l + (c_w - 1)w_w + c_e w_w + (c_w - 1)d_w$
$p_1 = r_1 w_w - d_w$
$p_2 = r_2 d_w - d_l$
$p_3 = r_3 d_l - d_r$

Table 4.1: Overview of constraints and used parameters for predicting a row of windows in façades

Beside these two basic constraints, the analysis of the ground truth data showed that there are high correlations between continuous model parameters that are exploited to constrain the search space further. A window, for example, that is very high is unlikely to be very narrow. More precise, it turns out that there are some dominant ratios r that can be represented categorically by some states of discrete nominators and denominators: $r = n/d$. By adding additional parameters as so called pseudo-observations p_i that are always observed as 0 the model incorporates the functional dependencies between two model parameters x_i and x_j: $0 = p_i = r_k x_i - x_j$. In this way, the constraints still fulfill the special form of equations 4.3 and further provide additional information for an overdetermined equation system used during the interpretation of measurements for the reconstruction of buildings. In order to consider the aspect of uncertainty the ratio is constrained up to a small ϵ: $n/d - \epsilon \leq r \leq n/d + \epsilon$.

The characteristics of the models are not only reflected in the constraints but also in distributions that can be learned from training data. As exemplified in Figure 1.2 none of these probability density functions are Gaussian but rather multimodal. However, it has been shown that each arbitrary distribution can be approximated by a Gaussian mixture of m components each weighted by its probability ω_i (McLachlan and Peel (2000)):

$$\sum_{i=1}^{m} \omega_i N(\mu_i, \sigma_i^2) \tag{4.4}$$

Gaussian mixtures are an appropriate way to model skew symmetric or multimodal distributions and thus allow for more efficient inference methods. In the case of façade modeling it has turned out that Gaussian mixtures are highly peaked with few components and small variances (cf. Figure 1.2) and thus help to structure and constrain the hypothesis space enormously. The knowledge of a building type or an architectural style yield even more precise distributions since the range of typical values for model parameters differentiate for different types and styles (cf. also Section 4.2).

With this basis of profound background knowledge the task of the reasoner is now to determine the most probable instantiations for the model parameters given the observations. For reasoning with uncertain data and modeling the dependencies and independencies between model parameters and its distributions efficiently Bayesian networks as a special variant of graphical models often turn out to be a powerful tool. Figure 4.3 shows the network for predicting a row of windows as is reflected in the constraints. Discrete nodes are expressed by simple ovals while a double line represents continuous nodes. Continuous variables related to the nodes of this network can be divided into observation parameters $O = (w_f, d_e)$ and (unknown) model parameters $\mathcal{X}_\Gamma = (d_l, d_r, w_w, d_w)$. As described by the constraints the functional dependencies of model parameters are modeled as converging connections (Kjærulff and Madsen (2008)). This inter-causal inference is a special property of graphical models: As soon as observations, e.g. the façade width, are known as evidence, model parameters become dependent and influence each other.

However, model parameters are a priori characterized by non-Gaussian probability density functions which makes inference for many well-known algorithms hard to cope with. In order to use efficient methods for linear Gaussian networks the developed reasoner incorporates

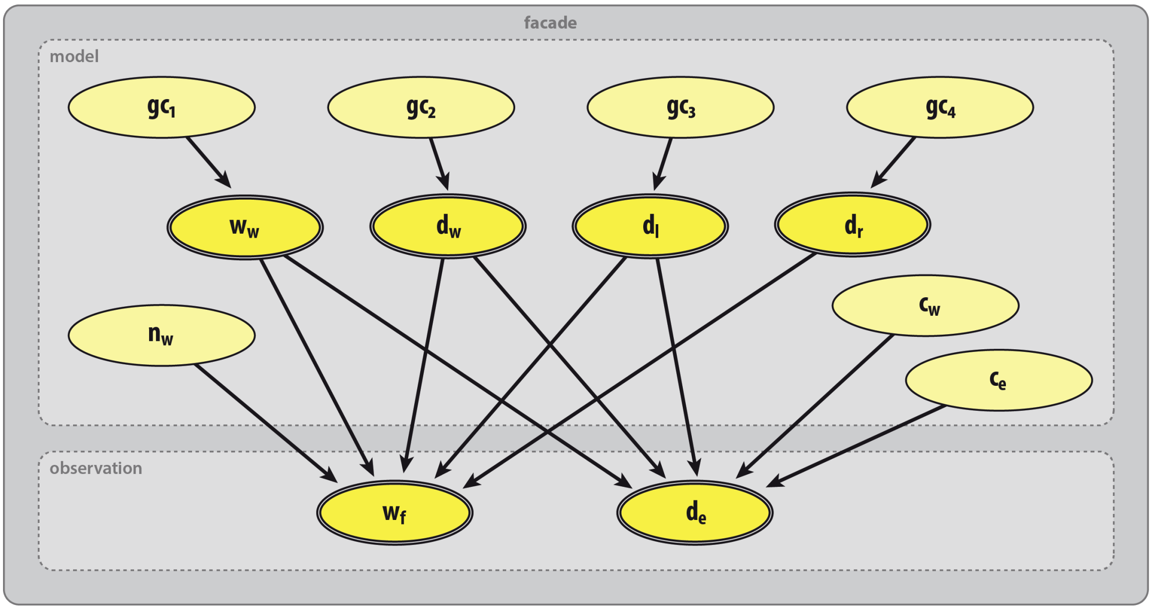


Figure 4.3: Extract of a Bayesian network for modeling a row of windows. Unknown model parameters get dependent by two observations: the façade width w_f and (optionally) the distance d_e between the left façade margin and an observed embrasure

Gaussian mixtures by adding an additional discrete parent node where each of the discrete states then represents the index of the Gaussian component with probability w_i of the corresponding mixture (Loch-Dehbi and Plümer, 2015). In this way, the Bayesian network is the basis for stochastically determining the posterior of the unknown model parameters that fit the observations and satisfy the related constraints. The following section describes how the introduced model is used for the prediction of building substructures based on sparse observations.

4.2 Predicting building façade structures based on sparse observations

The geometric and semantic interpretation of laser scans or images from buildings in general relies on measurements of high-density that are not always available or whose acquisition is expensive. In contrast, this section presents an approach that predicts substructures in building façades based on few observations such as footprints. The approach is illustrated by an example of predicting a row of windows as described in Section 4.1. Here, the building façade is characterized by a hybrid model of discrete and continuous parameters related by non-linear dependencies.

Referring to Section 4.1, the posteriors of model parameters can be determined exploiting converging connections in a Bayesian model as soon as an observation of the façade width and possibly an embrasure are given. While there are efficient algorithms for discrete systems

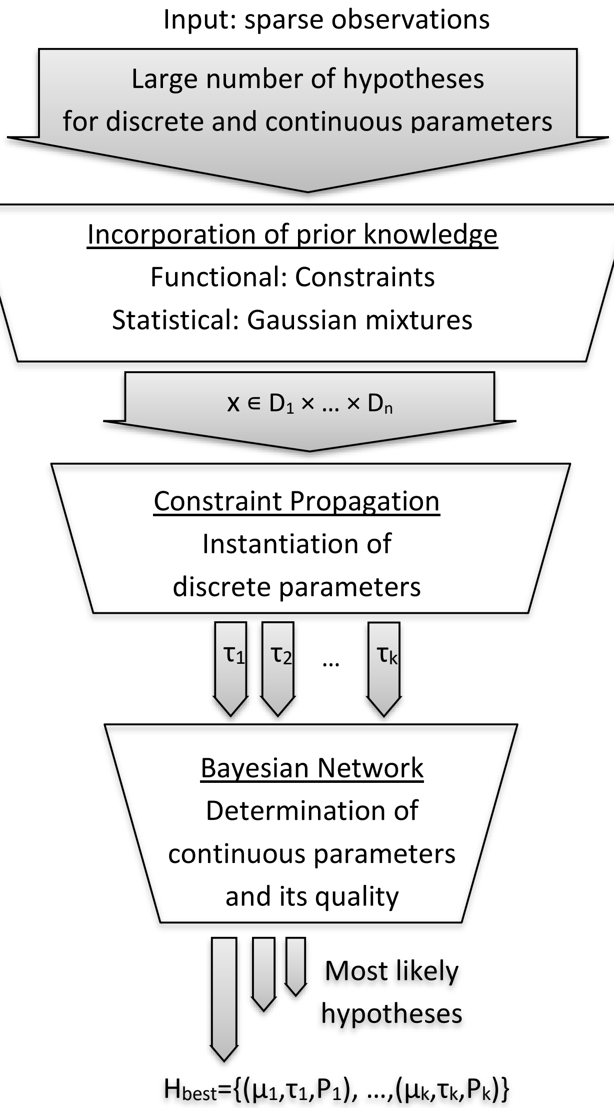


Figure 4.4: Overview of proposed cp-BN reasoner. A priori the reasoner has to cope with a large number of hypotheses. However, the incorporation of profound prior knowledge together with the combination of constraint propagation and Bayesian networks leads to a probabilistically high qualified and ranked set of hypotheses.

hybrid networks remain a challenging task. As stated by Koller and Friedman (2009) the resulting number of mixture components is in the worst case exponential in the number of unassigned discrete variables. However, if it becomes possible to determine the discrete parameters the problem reduces to a pure continuous and linear problem where distributions are no more multimodal but Gaussian and the Bayesian network represents a multivariate Gaussian distribution.

The joint distribution of equation 3.1 shows that the problem divides into a discrete and a continuous part which suggests the separation of the reasoning in a combinatorial (discrete) and a stochastical (continuous) part. To this end, this work presents a method that combines constraint logic programming with graphical models and profits from the strength of constraint programming in solving combinatorial problems and the power of Bayesian networks in reasoning with uncertain data. The result of the prediction is a set of ranked hypotheses for shape and location parameters that can be refined further in a top-down approach.

Figure 4.4 gives an overview of the developed cp-BN reasoner that basically consists of three reasoning steps:

1. Incorporation of prior knowledge
2. Constraint propagation
3. Bayesian network

The method only relies on few observations such as a building footprint or possibly available single embrasures of windows. Prior knowledge is incorporated to reduce the search space for valid values of model parameters. Therefore, probability density functions are derived from a ground truth data base using Expectation Maximization (McLachlan and Peel, 2000). They are used for deriving bounds of continuous parameters on the one hand and for providing distributions for the statistical component on the other hand. As described in Section 4.1 the model is further characterized by relations on continuous and discrete parameters in a Bayesian network where nodes of continuous parameters do not have any continuous parents.

In this way, the problem of this thesis is modeled similar to a conditional linear Gaussian network for which efficient inference methods exist (Lauritzen and Jensen, 2001). The difference lies in the fact that the states (especially their number) are not known a priori and that relations between model parameters are multilinear instead of linear. In order to avoid the unwanted case that there is no full assignment of discrete variables and the result of the reasoning process is a mixture of Gaussians instead of a multivariate Gaussian distribution, discrete parameters are determined in a first step using constraint propagation before reasoning with continuous parameters afterwards. The solution of the constraint satisfaction problem (CSP) is described by integers for discrete parameters τ_i including the indices of mixture components that represent relevant intervals of the continuous parameters.

Figure 4.5 illustrates this effect of determining the discrete parameters by the combinatorial component. With unknown discrete parameters the probability density functions of the model parameters are multimodal composed of several Gaussian components. The reduction after constraint propagation leads to single unimodal Gaussians whose means are appropriate for an initialization of the stochastic reasoning. By instantiating the discrete parameters

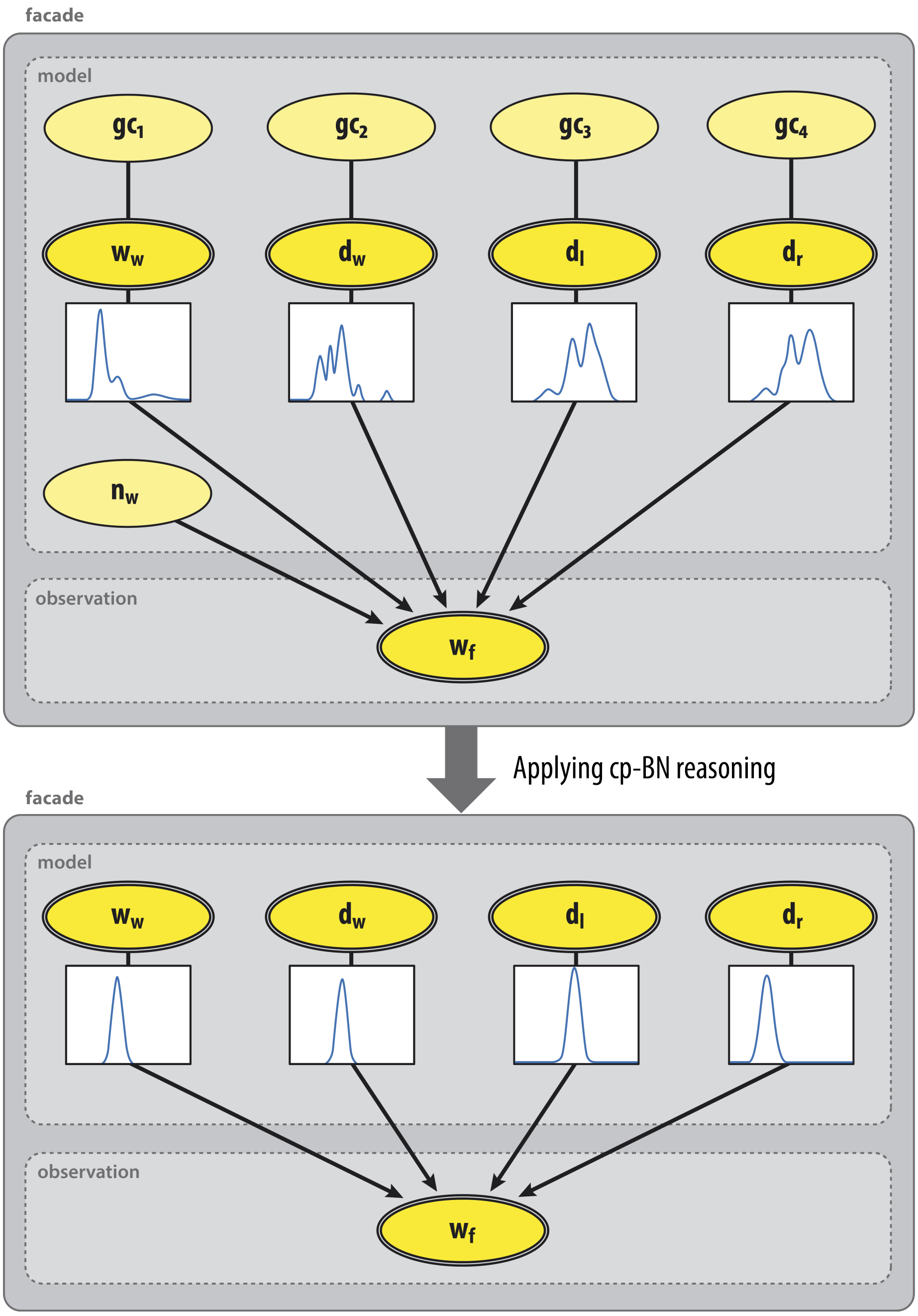


Figure 4.5: Effect of applying the proposed cp-BN reasoning on the illustrated example of predicting a row of windows. The Gaussian mixtures characterizing the model parameters are reduced to single Gaussian components after the instantiation of discrete parameters leading to a unimodal instead of a multimodal hypotheses space.

the stochastic reasoning has to cope with a linear Gaussian problem that enables the use of well-studied efficient inference algorithms for calculating the posterior belief.

The space of possible discrete solutions is restricted by constraint propagation using the constraints in Table 4.1. Furthermore, intervals can be derived from the different mixtures components to structure the hypothesis space. The j-th component is thus represented by 3λ intervals: $[\mu_j - \lambda\sigma_j, \mu_j + \lambda\sigma_j]$.

Together they define a CSP with constraints $\mathcal{C} = \{C_1, ..., C_q\}$ on variables $\mathcal{X} = \{X_1, ..., X_n\}$ with associated domains $\mathcal{D} = \{D_1, ..., D_n\}$. The constraint solver searches for a solution of the CSP by finding instantiations $\tau_i = (a_1, ..., a_n) \in D_1 \times ... \times D_n$ for each variable so that all constraints are satisfied. The incorporation of constraint logic programming yields a small number of qualified hypotheses.

The result of the combinatorial component are possible instantiations $\tau_1, ..., \tau_k$ for discrete parameters with $\tau_i \in \mathbb{N}^{|X_\Delta|}$, that is in the case of predicting a row of windows a set of solutions for

- the four indices of the selected Gaussian distributions from the mixtures $(gc_i, i = 1, ..., 4)$
- the number of windows (n_w)

and in case of an observed embrasure

- the index of the window that contains the observed embrasure (c_w)
- the assignment of the observation to the left or right embrasure (c_e).

After constraint propagation, the last step of reasoning consists in the determination of the continuous parameters and their quality. Stochastic reasoning is performed on a Bayesian network that is dynamically constructed for each solution of the constraint solver. For each discrete instantiation τ_l with $l \in 1, ..., k$ dependencies of variables are described with respect to the i-th observation by linear relations of the form

$$o_i = \sum_{j=1}^{|X_\Gamma|} \delta_{jl} c_j, \tag{4.5}$$

where the δ_{jl} are constants transformed by the instantiation of the discrete parameters. Only the continuous parameters c_j remain as random variables that finally have to be corrected by calculating the posterior belief. In order to exploit this special structure the intermediate model can be seen as a state-observation model, a dynamic Bayesian network where the state of variables evolves over time. In the case of linear Gaussian dependencies with solely continuous variables these networks are equivalent to linear dynamic systems and enable an efficient calculation of the posteriors by simple matrix multiplications using Kalman filters.

Since the building model does not evolve over time the transition from one state to another characteristic for dynamic systems is omitted. Instead, the measurement update, also called correction step, is applied to the vectors and matrices constructed from the output of the

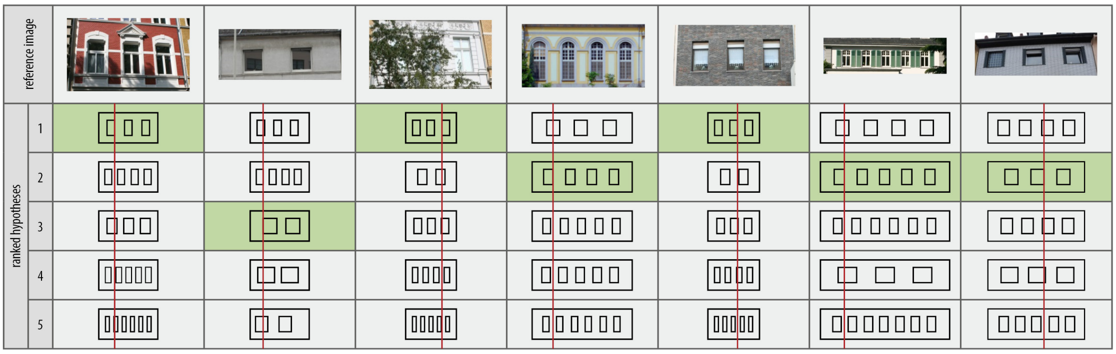


Figure 4.6: Result of predicting façades based on known façade widths. Green marked hypotheses match the correct solutions. Red vertical lines drawn across the ranked solutions mark the location suggested for a single measurement in order to decide between the different hypotheses.

combinatorial component. The reasoning determines a posterior distribution, i.e. $\mu \in \mathbb{R}^n$ and Σ, for the model parameters:

$$\begin{aligned} \mu &= \mu + K(o - M\mu) \\ \Sigma &= (Id - KM)\Sigma \end{aligned} \tag{4.6}$$

where $M \in \mathbb{R}^{n \times m}$ is the measurement matrix constructed according to the used constraints in Table 4.1 so that $o = M\mu$ and K is the Kalman gain defined by

$$K = \Sigma M^T (M \Sigma M^T + Q)^{-1} \tag{4.7}$$

Hereby, Id denotes the identity matrix and $Q \in \mathbb{R}^{m \times m}$ is the Gaussian noise of observations.

Applying this calculation to each intermediate result of the constraint solver, the reasoner outputs the k best hypotheses

$$H_{best} = (\mu_1, \tau_1, P_1), ..., (\mu_k, \tau_k, P_k)$$

with means $\mu_i \in \mathbb{R}^{|\mathcal{X}_\Gamma|}$ and discrete instantiations $\tau_i \in \mathbb{N}^{|\mathcal{X}_\Delta|}$ ranked by its (unnormalized) probabilities

$$P_i = \exp\left(\sum_{j=1}^{|\mathcal{X}_\Gamma|} \log(pdf_j^{01}(\mu_{ij}))\right) \tag{4.8}$$

where pdf_j^{01} is the on $[0,1]$ scaled density of the distribution corresponding to the j-th model parameter. By determining a set of ranked hypotheses instead of one single prediction it is avoided that the interpretation that best fits the real façade is rejected at an early stage due to a lower probability.

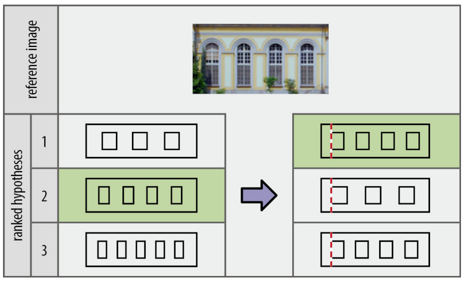


Figure 4.7: An observation of a single embrasure changes the ranking in favour of the correct hypothesis

Figure 4.6 shows the results of predicting a row of windows based on a single observation, namely the building footprint respectively the derived façade width. As depicted in the third column of Figure 4.6 occlusions can be compensated as well. Among 55 façades that were used for the evaluation of the approach more than half of the correct predictions were ranked as first hypothesis with an average error of 0.37 m. Observations of embrasures and the corresponding relation constraining the model parameters are not used in these examples. However, in order to select between different hypotheses, an additional measurement of for example an embrasure yields more accurate information about the building structures. Figure 4.7 demonstrates how the ranking is changed in favour of the correct hypothesis. The decision at which position a measurement best improves the reconstruction process and the final model selection is beyond the scope of this thesis.

The approach succeeds best for buildings that have a clearly defined appearance so that model parameters fit the characteristics covered by the probability density functions and the derived constraints. This is especially the case for office buildings or buildings of cultural heritage. In contrast, for example, some modern buildings may follow their own rules without the typical regular structure. This requires an extended modeling to capture the diversity of façades and possibly needs more measurements to verify or falsify a relaxed model.

A decrease of errors can be made, if the building type or the architectural style is incorporated. The knowledge of these characteristics results in more precise and stronger peaks of probability density functions and thus excludes implausible instantiations in an early stage. Figure 4.8 illustrates how distributions become more precise if the architectural style is known. Obviously, the Gaussian mixture components represent different styles and the knowledge of the architectural style can limit the search to a smaller range of probable values. Architectural style and building type can be automatically derived in a classification task by support vector machines as proposed by Römer and Plümer (2010) and Henn et al. (2012).

4.3 Cp-BN reasoner: A generic method

The presented approach is not restricted to building façades but is implemented independently from the domain of buildings and can thus be transferred to other applications. At

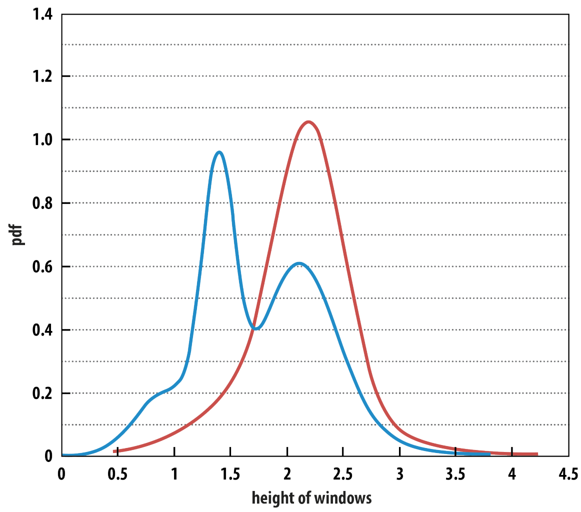


Figure 4.8: Impact of knowledge about architectural style. Comparison of the Gaussian mixture for the height of windows independent of the architectural style (blue) and for windows in buildings of Wilhelminian style (red).

all times the equations are build up dynamically dependent on the specific problem and the intermediate solutions. The object-oriented and modular manner covers a wide range of configurations.

The transfer to another problem requires that the problem can be defined as *cp-BN problem* $(\mathcal{X}, \mathcal{D}, \mathcal{C})$ on a cp-Bayesian network (cp-BN) as defined below:

A *cp-BN problem* $(\mathcal{X}, \mathcal{D}, \mathcal{C})$ is defined by a set of parameters $\mathcal{X}$ with corresponding domains $\mathcal{D}$ and a set of constraints $\mathcal{C}$ with the following properties:

- continuous model parameters $X \in \mathcal{X}$ are characterized by a Gaussian mixture
- observed continuous parameters $X \in \mathcal{X}$ are characterized by an observed value and a precision for this observation (σ)
- discrete model parameters $X \in \mathcal{X}$ are characterized by a domain with integral bounds
- (optional) ratios $r = \frac{X_i}{X_j}$ are characterized by a discrete domain of a list of numerators and a list of denominators
- constraints C are defined by a CLP expression on the set of variables X

Gaussian mixtures are not only used to handle probabilities for reasoning with uncertain data but provide a means to derive bounds for continuous model parameters. Together with integral bounds and constraints including relations on ratios constraint propagation algorithms are able to reduce the search space.

A *cp-Bayesian network* (cp-BN) is a Bayesian network where every discrete variable has only discrete parents and every continuous variable has a conditional probability distribution that can be characterized by a Gaussian mixture or whose dependency can be described by a bilinear relation with factors of discrete and continuous variables as defined in Equation 4.3.

Input: prior knowledge (prior distributions, integral domains, constraints), observations
Output: best k hypotheses $\mathcal{H}_{best} = (\mu_1, \tau_1, P_1), ..., (\mu_k, \tau_k, P_k)$
`// incorporation of prior knowledge`
derive information: observations, parameter bounds
initialize model: observations o with measurement noise Q, constraints and distributions
`// constraint-based reasoning`
construct constraint satisfaction problem CSP $(\mathcal{X}, \mathcal{D}, \mathcal{C})$ according to model initialization
propagate constraints and search for plausible intermediate hypotheses $\mathcal{H}' = (h_1, ...h_k)$ of the CSP (with instantiations for discrete parameters $X \in X_\Delta$)
`// statistical reasoning`
for $h_i \in H'$ **do**
construct matrices Σ, M, Q and vectors o and μ of Kalman filter
calculate posterior belief by updating measurement with
$K = \Sigma M^T (M \Sigma M^T + Q)^{-1}$
$\mu = \mu + K(o - M\mu)$
$\Sigma = (Id - KM)\Sigma$
end
determine k most likely assignments: $MAP^k(\mu|o)$
`// refinement (optional)`
refine results by updating beliefs and ranking as soon as additional information is available

Algorithm 1: Algorithm *cp-BN* combining constraint-based with statistical reasoning for a prediction of structures based on few observations

The latter can often be ensured by using pseudo-observations that are always observed as zero. The paradigm of restricting the search space by constraint propagation requires domains defined by one or more intervals, that is upper and lower bounds for model parameters that can be derived from available probability density functions.

The Algorithm 1 generalizes the presented approach and depicts the procedures that can be applied to arbitrary cp-BN problems. In all cases the reasoning passes three reasoning steps: the incorporation of problem specific prior knowledge, the combinatorial reasoning using constraint logic programming and the statistical reasoning using Bayesian networks. Constraints that are used in the combinatorial component are shipped over to the statistical component if they define a state-observation model and involve continuous parameters.

To sum up, the algorithm changes a problem of an infinite number of hypotheses to a small number of qualified hypotheses by the incorporation of prior knowledge and the combination of constraint logic programming and Bayesian networks. Hereby, it is able to perform exact inference on a non-linear problem with discrete as well as continuous parameters.

5 Automatic reasoning for the prediction of 3D indoor models

This section describes the prediction of 3D indoor models published by Loch-Dehbi et al. (2017). Similar to the approach for façade prediction sparse observations are sufficient for model generation. The proposed method avoids indoor measurements and thus circumvents expensive or difficult acquisition of data for which each room has to be entered and that is impeded by occlusions, e.g. from furniture. Instead it uses basic knowledge about the building footprint, the (exterior) locations of windows and available room characteristics such as the room area. Together with probability density functions and constraints the implemented reasoner predicts the shape and location of rooms and estimates floor heights and doors. As an adaption to the approach presented in chapter 4 it follows a model-driven top-down approach and combines constraint logic programming with Bayesian networks in order to make exact inference feasible despite few observations and a non-linear model with discrete as well as continuous parameters. A MAP based inference finally yields a ranked set of hypotheses.

The basic problem to solve is to place n rooms of known areas a_i in a given polygonal building footprint. The i-th room is represented by a reference point (x_i, y_i) and its width w_i and depth d_i that are in a first step constrained by a bilinear constraint $a_i = w_i * d_i$ (cf. Figure 5.1a). Herewith, it differs from the façade prediction in that the dependency is a product of two continuous parameters.

The problem of placing rooms in a rectangle is closely related to Perfect Rectangle Packing, which is known to be NP-hard (Garey and Johnson, 1979). Apparently, the floorplan

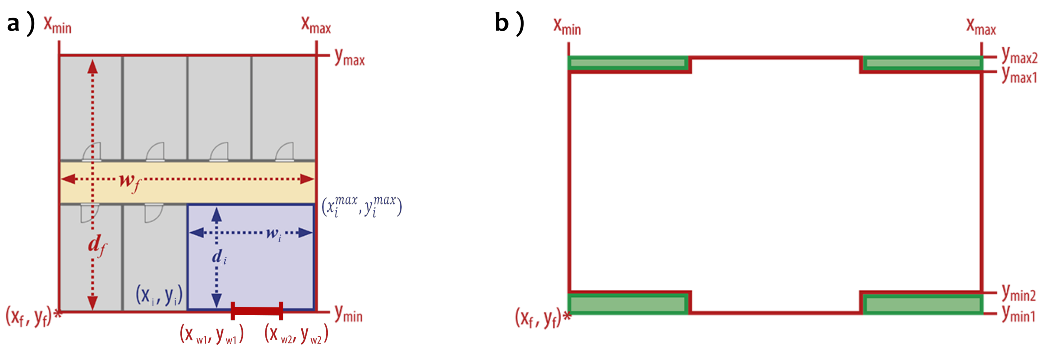


Figure 5.1: a) Overview of location and shape parameters for modeling floorplans b) Handling non-rectangular footprints by adding additional auxiliary rooms (green)

prediction is in the same way NP-hard as it is a generalization of the special case that the building footprints are rectangular and the lower bounds for shape parameters are equal to the upper bounds thus having a known width and depth. The NP-hardness suggests that it is difficult to achieve efficient algorithms in the worst case.

In the same way as for façade reconstruction the key idea is that regularities in the construction of buildings legitimate the use of strong building models during model prediction. Constraints on shape and location parameters and the incorporation of probability density functions narrow the search space of valid predicted floorplans. This also reduces the problem that measurements of furniture are misinterpreted as walls.

The dominance of orthogonality and parallelity in man-made objects leads to the simplifiying design decision that the model follows the Manhattan world assumption. The building footprint as well as the rooms are assumed to be rectangular. In order to handle non-rectangular footprints, this constraint can be relaxed by adding auxiliary virtual rooms as shown in Figure 5.1b.

Instead of an expensive acquisition of indoor measurements the reasoning is based on data that is usually available with house keepers or real estate managers. A priori known building footprints that can be extracted from open source data such as OpenStreetMap back the reasoner as well as categorical prior knowledge such as identifying room numbers, in particular for office rooms, or the functional use of rooms such as toilet or corridor. The latter for example advises information about typical dimensions of rooms such as long and narrow in case of a corridor. The developed reasoning method further exploits that rooms with consequent room numbers are likely to be adjacent. Measurements of the indoor environment are a priori not needed, they can though be used selectively to decide later on between competing hypotheses of floorplans.

Additionally, the reasoning profits from (two-dimensional) locations of window embrasures (x_{w1}, y_{w1}) and (x_{w2}, y_{w2}) that are assumed to be given from exterior measurements (cf. Figure 5.1a). The latter can be derived using existing approaches for the identification of building parts such as presented by Dehbi et al. (2016b), Dehbi et al. (2017) or Recky and Leberl (2010) and serve for constraining relations between room and window parameters as described later on.

In this context, for a floorplan prediction of high accuracy the location parameters of a room should be related to the parameters of its neighbour rooms and the parameters of the windows that they contain. However, a priori the correspondence of windows and the adjacency of rooms is unknown so that this information cannot be used to define constraining relations on location parameters. Instead, this becomes possible as soon as the topological model is given. To this end, the reasoning is based on a combinatorial and a stochastic part similar to the reasoning for façades presented in Section 4.2. The combinatorial component determines the discrete parameters such as the correspondence of windows to rooms and the bilateral relations between rooms, i.e. the horizontal and vertical neighborhood such as 'left to' or 'right to'. Finally, the stochastic component calculates the continuous parameters: the width and depth of rooms and their locations.

The uncertainty of the model is expressed in probability density functions for continuous model parameters that are derived from annotated floorplans with about 1160 rooms collected in a spatial relational ground truth database (cf. Figure 5.2). The hierarchical model

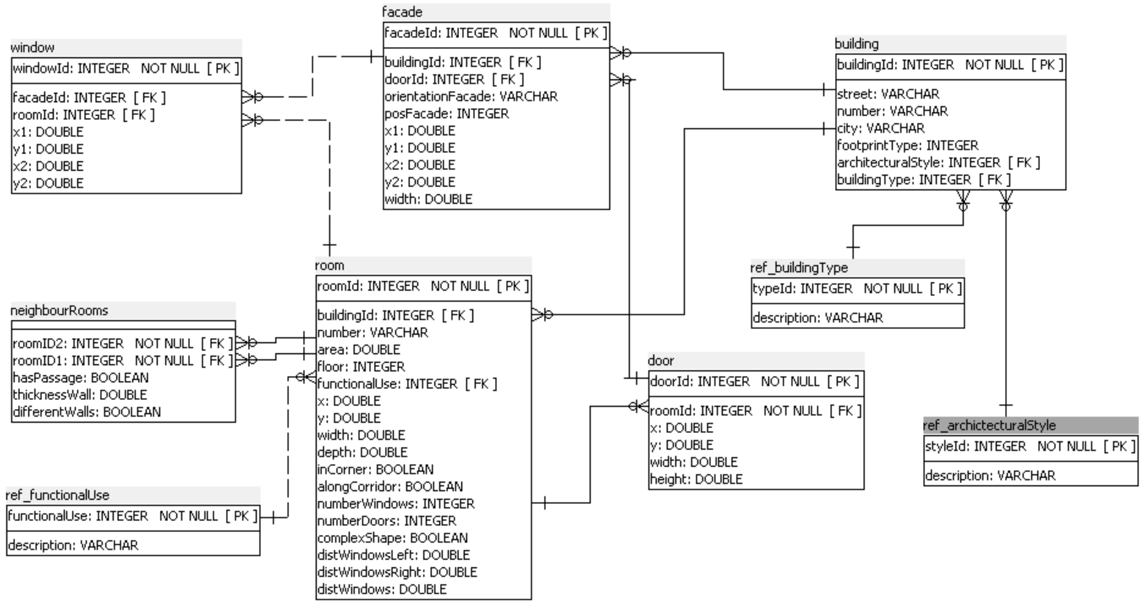


Figure 5.2: Database schema used for deriving probability density functions and constraints of the floorplan model

defines windows, doors, rooms, façades and buildings and records properties of and correlations between these objects. The data is used to develop the constraint-based mathematical floorplan model and to derive probability density functions. Alike for façades as described in Chapter 4 probability density functions are neither Gaussian nor unimodal, which a priori excludes the use of many well-studied efficient inference methods. In order to circumvent this deficiency the prepocessing of the combinatorial component selects only one appropriate Gaussian component of the Gaussian mixture.

The prediction of floorplans is implemented by three reasoning steps as depicted in Section 4.3. The reasoner incorporates prior knowledge and separates the problem into a discrete combinatorial and a continuous stochastic part.

Location and shape parameters of rooms are constrained by topological and statistical constraints that help to structure the hypothesis space. For an intermediate topological model the reasoning method incorporates hard and soft constraints on discrete as well as continuous parameters. Hard constraints have to be satisfied during model instantiation whereas soft constraints represent relations that should be valid but are allowed to be violated – bound by a maximum number of violations – to preserve plausible instantiations.

The following hard constraints are used by the combinatorial component to search for valid topological solutions:

- the product of room widths and depths is equal to the areas of rooms.
- The domain vertices for rooms that contain windows exclude ranges where windows are placed so that walls do not cross windows.
- at least one of the exterior walls of a room with a window contain the vertices of the corresponding window

- polygons of rooms do not overlap by forcing that vertices of one room cannot lie in a polygon of another room but on the same side outside of this polygon instead.
- polygons of rooms lie inside the footprint and the domains of room vertices are bounded according to the footprint dimensions.
- the number of rooms along a corridor is restricted by the width of the façade since in turn the room width is bound by a statistically derived minimum
- width and depth are bound by lower and upper bounds derived from probability density functions. They are in turn dependent on the building type or the functional use of the corresponding room.

The topological model is further constrained by the following two soft constraints:

- rooms with consecutive room numbers are mostly adjacent
- rooms are mostly adjacent to an existing corridor

As it can be seen these relations on location and shape parameters of rooms further constrain the domains of model parameters by upper and lower bounds or even more complex constraints. They thus exclude impossible instantiations and define a constraint satisfaction problem (CSP) that is solved by constraint logic programming. Table 5.1 gives an overview of model and observation parameters and used constraints similar to the prediction of façades (cf. Section 4.1).

It should be noted that the assignment of windows to rooms is not known a priori. Instead the combinatorial component solves the discrete labeling problem by fitting the corresponding constraints. The occurrence of windows in a room further depends on the functional use of the room so that not all rooms are forced to have a window. For example, it is required that a window is assigned to an office room but not necessary to a corridor.

The reasoning profits from a preliminary relaxation in that rooms in intermediate results do not have to fill the entire space and walls are modeled later on during statistical reasoning. Herewith, a spring model is used similar to the approach in Becker (2009) in order to provide buffers for fast intermediate results that are improved in a subsequent step. Without loss of generality, the footprint is rotated in such a way that its main axis is parallel to the x-axis of the coordinate system. Since the complexity of the algorithm is bound by the domain size, discrete values for x-coordinates are enumerated excluding those that fall in window ranges in order to avoid the a priori infinite continuous search space. This leads to an easier instantiation of other parameters since constraints can be transformed to functions. Since we are only interested in rough instantiations, more precisely ranges that represent a component of the Gaussian mixture, and further buffers are provided due to the omission of separating walls during constraint propagation this relaxation and discretization is sufficient for a correct topological model (see second row in Figure 5.3).

	continuous variable	discrete variable		
	in constraint		for GM component	
model parameter	w_i d_i x_i y_i $wall_{ext}$ $wall_{int}$	wc_w rel_{ij}	gc_{1_i} gc_{2_i} gc_{3_i} gc_{4_i} gc_{ext} gc_{int}	
observation parameter	$\mathbf{w_f}$ $\mathbf{d_f}$ $\mathbf{x_f}$ $\mathbf{y_f}$ $\mathbf{x_{w_1}}, \mathbf{y_{w_1}}$ $\mathbf{x_{w_2}}, \mathbf{y_{w_2}}$	$\mathbf{rno_i}$		

a_i	area of i-th room
w_i	width of i-th room
d_i	depth of i-th room
(x_i, y_i)	reference point of i-th room in left lower corner
$\mathbf{w_f}$	width of footprint
$\mathbf{d_f}$	depth of footprint
$(\mathbf{x_f}, \mathbf{y_f})$	reference point of footprint in left lower corner
$(\mathbf{x_{w_1}}, \mathbf{y_{w_1}})$	left reference point of w-th window
$(\mathbf{x_{w_2}}, \mathbf{y_{w_2}})$	right reference point of w-th window
$wall_{ext}$	widths of exterior walls
$wall_{int}$	width of interior walls
wc_w	window correspondence for w-th window
$\mathbf{rno_i}$	room number of i-th room
rel_{ij}	bilateral relation (neighborhood) between rooms
gc_{ext}, gc_{int}	index of Gaussian component for walls
gc_{k_i}	index of Gaussian component for k-th model parameter of i-th room

constraints (exemplarily)	
room area	$\mathbf{a_i} = w_i * d_i$
window correspondence	$wc_w = \mathbf{rno_r}$
room touches exterior wall (front)	$y_i = \mathbf{y_{w_1}} + wall_{ext}$
room walls between windows (front)	$((x_i \leq \mathbf{x_{w_1}}) \wedge (\mathbf{x_{w_2}} \leq (x_i + w_i)))$
rooms inside footprint	$((\mathbf{x_f} \leq x_i) \wedge (\mathbf{y_f} \leq y_i) \wedge ((x_i + w_i) \leq (\mathbf{x_f} + \mathbf{w_f})) \wedge ((y_i + d_i) \leq (\mathbf{y_f} + \mathbf{d_f})))$
non-overlapping of rooms	$(x_i + w_i \leq x_j) \vee (x_j + w_j \leq x_i) \vee (y_i + d_i \leq y_j) \vee (y_j + d_j \leq y_i)$
adjacency (room i left to room j)	$((x_i + w_i + wall_{int} = x_j) \wedge \neg((y_j + d_j \leq y_i) \vee (y_i + d_i \leq y_j)))$

Table 5.1: Model and observation parameters and constraints used for modeling floorplans. Bold marked parameters denote a priori known parameters. The text in brackets denotes the case for which the constraint is exemplarily written. Note that the adjacency of rooms and their assignment to an exterior wall and the containing window is not given but determined by the combinatorial component of the reasoner.

To sum up, the combinatorial component yields the initial states of continuous parameters for the statistical reasoning as means of selected Gaussian distributions and outputs a set of topological models with evidence for discrete parameters:

- indices of the selected Gaussian distributions from the mixtures
- indices of rooms mapped to corresponding windows
- the adjacencies between rooms.

Herewith, it provides the prerequisites for the definition of linear equations of a state-observation model for the exact stochastic reasoning. The stochastic reasoning applied afterwards produces a result that is topologically equivalent[1] but geometrically different in order to adapt the intermediate predictions to the as-is floorplans of the real world. The problem is set up dynamically for each solution in the output of the combinatorial component. In contrast to the prediction of façade substructures not all constraints are shipped over to the statistical component since they were mainly used to rule out values for discrete parameters. Further, walls were not modeled during constraint propagation in order to provide buffers for the topological model. Together with the continuous room parameters x_i, y_i, w_i and d_i two variables $wall_{int}$ and $wall_{ext}$ for the inner and outer walls are added for the stochastic reasoning.

There are basically two constraint types that are used to ensure geometrically correct solutions. In order to ensure the special structure of equation 4.3, where the observation is defined as a sum of products with one discrete and one continuous factor, zero-instantiated pseudo-observations are introduced.

The first constraint type expresses the adjacency of rooms. For an i-th room that lies left to the j-th room the relation can be expressed as follows:

$$o_{ij} = 0 = x_i + w_i + wall_{int} - x_j.$$

Likewise, the other vertical and horizontal neighborhoods are modeled.

Based on the a priori predicted assignments of rooms to windows the location of rooms is strongly limited by the corresponding known window coordinates. For a room related to the w-th window - in this case lying at the front side of the building - its coordinates have to be adjusted using the following constraint:

$$o_{iw} = 0 = y_w + wall_{ext} - y_i.$$

Obviously, the envisaged problem reduces to a special structured Bayesian network: a conditional linear Gaussian network. More precisely, the reasoner has to deal with a linear structure of constraints where initial beliefs for model parameters are represented by multivariate Gaussian distributions. In the same way as described in Section 4.2 the posterior belief is calculated using the correction step of the Kalman Filter and continuous shape and location parameters are adjusted. Bilateral relations of the topological model are coded in an adjacency graph that is used to built up the matrices dynamically in a recursive way.

[1] equivalent up to homeomorphic transformations (Worboys and Duckham, 2004)

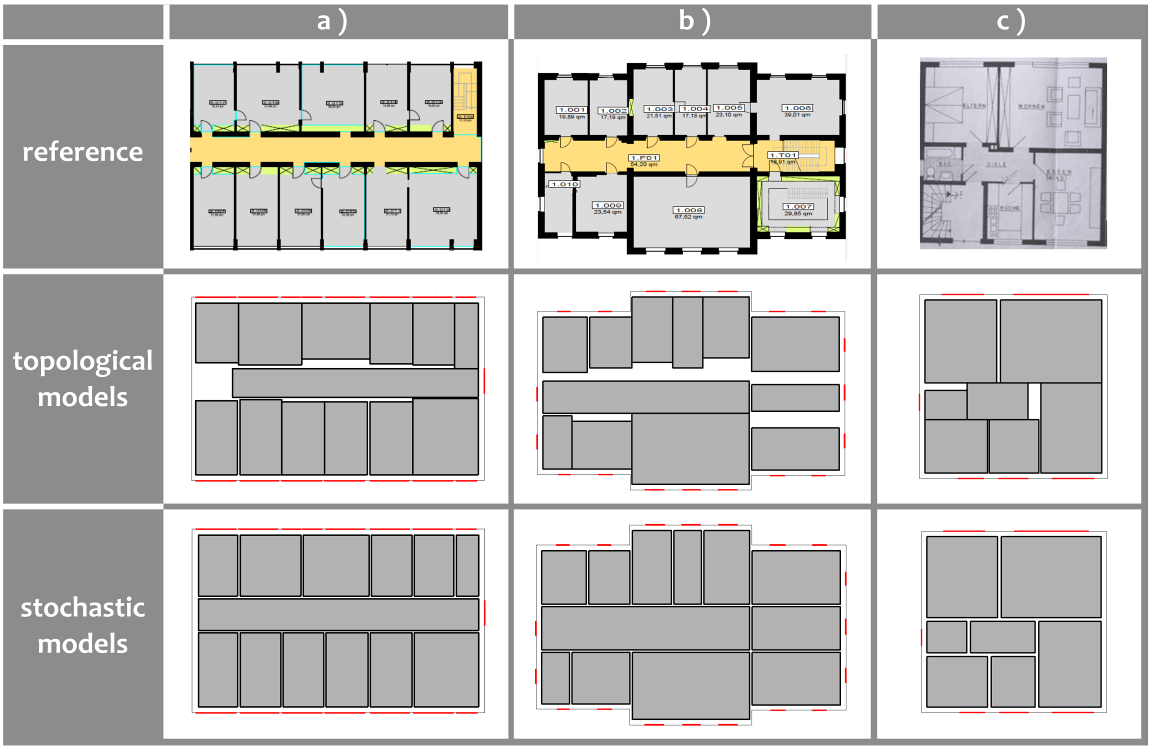


Figure 5.3: Result of predicting floorplans in the absence of indoor measurements. The topological model is the result of the combinatorial component by determining discrete parameters while the stochastic model is the final output after the subsequent statistical correction of the continuous parameters.

By incorporating the Kalman gain the reasoner outputs corrected means μ and covariances Σ for continuous model parameters of the final stochastic model.

Finally, the quality of the hypotheses is determined by calculating $\mathrm{MAP}^k(\mu|o)$, the k most probable explanations, and the resulting hypotheses are ranked by their (unnormalized) probabilities (cf. equation 4.8) .

Figure 5.3 shows the result of the reasoning process for three different types of buildings: a rectangular and a non-rectangular footprint of office buildings as well as a footprint stemming from a residential house. In contrast to the rectangular footprint auxiliary rooms have to be added for the non-rectangular footprint in order to ensure that the footprint satisfies the rectangle assumption (cf. Figure 5.1b). Nevertheless, the protrusions are beneficial in that they point to walls that are located at their end points and separate the rooms. As mentioned before, an important finding of the ground truth data is that rooms with consecutive room numbers are often adjacent and a large number of rooms can be placed with high topological accuracy if room numbers are known – meaning that the adjacency between the rooms is predicted in a correct way. The room numbers are only used to enhance the prediction but are not part of the final geometric result. Consequently, different assignments of room numbers may result in equivalent (geometric) solutions if room dimensions are the same. The result gets more precise if single rooms are known to belong to a specific window that thus constrain their order. For a limited number of rooms such as toilets or stairs this information can be derived based on the surface appearance or the location of windows. Additionally, the adjacency to a corridor automatically aligns the rooms along this special rectangle. The challenge of residential buildings is that rooms have no identifying room numbers that suggest the neighbourhood of rooms. Instead, the reasoner profits from a wider variety of functional uses and a smaller total area.

Figure 5.3 demonstrates the topological models as output of the combinatorial component. Temporarily walls are ignored and an alignment of rooms along the corridor is not forced in a first step to provide buffers for the prediction of the intermediate topological model that focuses on discrete parameters. Continuous parameters are initialized afterwards by the means of the chosen Gaussian mixture components and are updated by stochastic reasoning – more precisely by the measurement update of the Kalman filter – in order to fit optimal in the floorplan. As already addressed in Chapter 1 the stochastic model as finally selected meets the recommended OGC requirements of model accuracies between 10 and 20 cm. The reasoner predicts floorplans in the absence of indoor measurements and instead uses less expensive data such as available footprints or the room area. However, single additional observations from the interior can be used to select between competing hypotheses and increase the accuracies further.

While the presented constraints are limited to the description of 2D floorplans, a further processing of the result allows for the derivation of 3D indoor models as shown in Figure 5.4. The prediction of the 3D model becomes possible by extruding walls to predicted heights and estimating doors using statistics for the location of doors. Staircases or other details are beyond the scope of this thesis. False positives are marked with green dotted lines. The size of rooms as well as their functional use influences the location of doors. The histograms in Figure 5.5 show the dominance of specific locations dependent on the width of the corresponding rooms. Further, kernel density estimations based on 3D point clouds

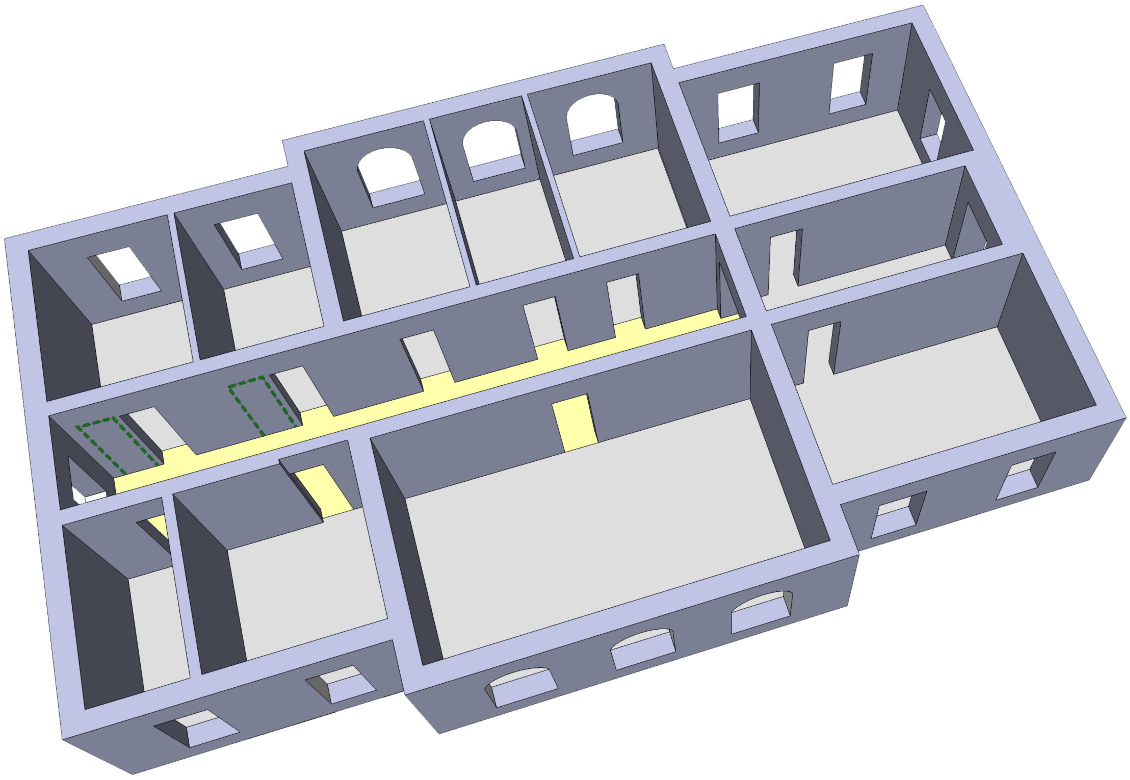


Figure 5.4: Derived 3D model as a postprocessing of the floorplan prediction. Dotted lines show false positives of the prediction of door locations.

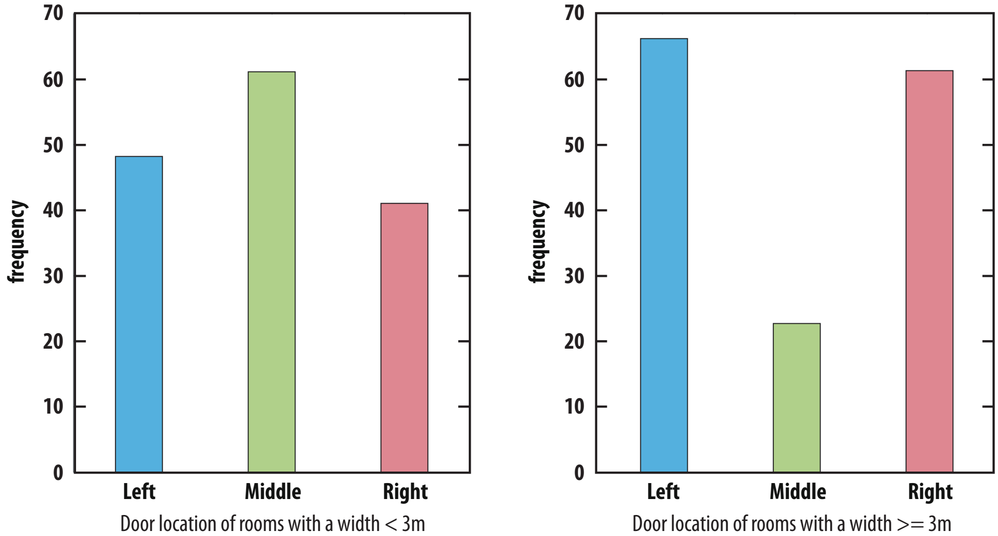


Figure 5.5: Histograms of door locations dependent on the width of rooms

of façades provide information about the height of each storey (Dehbi et al. (2016b)). Even more robust results could be achieved by the use of a classification for the location of doors.

The results show that the incorporation of profound prior knowledge enable plausible predictions of floorplans. This is especially the case for office buildings or other public buildings that usually provide room numbers as well as room types and exhibit a clear structure. The combinatorial complexity of locating rooms can thus be increased drastically. The approach is designed to be applicable to residential housings as well. The lower number of rooms compensates the absence of room numbers with regard to combinatorics. However, the floorplan layouts may be more diverse. Rooms are not necessarily aligned along a corridor and room shapes do not have to be a rectangle. In this case, the prediction of floorplan layouts demands the development of additional controlling mechanisms constraining models of residential houses.

6 Geometric reasoning on 3D building models

Model-driven approaches for the reconstruction of 3D buildings as presented in Chapter 4 and 5 depend on well-defined prototyped models. Therefore, as part of this thesis a prototyped reasoning tool was developed that ensures a consistent and redundancy-free representation, enables to reason within these models and is able to handle quality issues in uncertain data.

Constraint-based models are a common formalism for the representation of 3D buildings. Since man-made objects are often characterized by geometric constraints such as parallelity and orthogonality these constraints can be well represented by multivariate polynomials. While there are efficient methods to solve equations numerically for a concrete task of reconstruction this thesis presents a method that makes the analysis of constraint-based models feasible for general configurations on a symbolic level. The basic idea is that the proof of a consistent and redundancy-free model or the deduction of implicit constraints is equivalent to proving a theorem of one constraint being a conclusion of others. Following this concept, the developed approach uses algorithms of automatic theorem proving and integrates algebraic as well as deductive (rule-based) reasoning.

6.1 Geometric constraints for 3D building models

The geometric model can be specified by primitives of buildings such as planar faces for walls or roof halves. The presented approach is illustrated by a model of a gable roof house

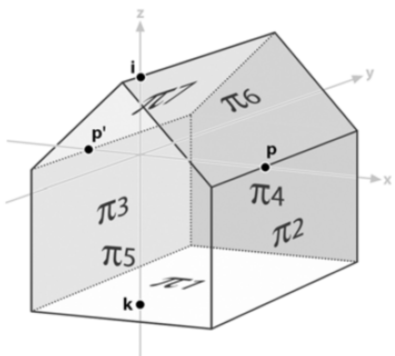


Figure 6.1: Prototyped model of gable roof

Figure 6.2: Orthogonality and Parallelity are dominant in man-made objects such as buildings

being an aggregation of a cuboid and a prism (see Figure 6.1). Since most man-made objects have a regular structure (Steadman (2006), Loch-Dehbi and Plümer (2011)), the primitives such as planes representing the walls are basically related by geometric constraints such as orthogonality and parallelity. Figure 6.2 shows the dominance of these constraints in 3D building models.

Object	Projective representation
plane	$A = (a, b, c, d) = (\mathbf{A}_h, A_0)$ $A = X \wedge L = \overline{\Pi}^T(X)L$
point	$X = (x, y, z, w) = (\mathbf{X}_0, X_h)$ $X = A \cap L = \Pi^T(A)L$
line	$L = (a, b, c, d, e, f) = (\mathbf{L}_h, \mathbf{L}_0)$ $L = A \cap B = \overline{\Pi}(A)B$ $L = X \wedge Y = \Pi(X)Y$

Table 6.1: Algebraic representation of geometric objects and possible constructions according to Heuel (2004) with $\Pi(X)^{(6x4)} := (\partial X \wedge Y/\partial Y)$, $\overline{\Pi}(X)^{(6x4)} := (\partial X \wedge L/\partial L)$ and $X \wedge Y = (X_h Y_0 - Y_h X_0, X_0 \times Y_0)$

In the context of algebraic modeling, a substantial increase of complexity can be observed from the 2D to the 3D space and many geometric constraints are characterized by multilinear or even quadratic equations of polynomials. As a consequence, an adequate choice of reasoning methods as well as an appropriate representation of the building model is crucial for a reliable method.

As a first step towards feasibility, the presented approach uses projective geometry and represents points, lines and planes as homogeneous coordinates so that constraints can be

Bi-relational constraint	Algebraic representation	degree of freedom (DoF)
plane A $\perp$ plane B	$A_h^T B_h = 0$	1
line L $\perp$ plane A	$S(L_h)A_h = L_h \times A_h = \mathbf{0}$	2
plane A $\parallel$ plane B	$S(A_h)B_h = A_h \times B_h = \mathbf{0}$	2
line L $\parallel$ plane A	$L_h^T A_h = 0$	1
point X $\in$ plane A	$X^T A = 0$	1

Table 6.2: Modeling orthogonality, parallelity and incidence with $S(x) = \partial x \wedge y/\partial y = \partial x \times y/\partial y$, the basic constraints that are needed to model a gable roof house.

represented by simple bilinear constraint equations. Table 6.1 and table 6.2 give an overview of the algebraic representation of geometric objects and constraints as proposed by Heuel (2004). Later on in this work, it can be seen how this representation also paves the way for the integration of uncertainty.

Beyond the use of projective geometry further simplifications of the representation lead to a reduction of running time and more interpretable results. With respect to the representation of the model a minimum of variables is at all times preferable. Without loss of the generality, one simplification is the rotation of the model so that one plane is parallel to the x-y-plane. To this end, the normal vector of the plane that is involved in most of the constraints can be set to $(0, 0, 1)$ so that many factors get eliminated. Further variables reduce to zero, if the origin of the coordinate system is placed where the most complex constraints are involved. For a gable roof house, the relation for constraining the orientation of the roof and its symmetry are more complex than the orthogonality and parallelity constraints of the walls so that the origin is placed in the roof of the building for a simplification of roof constraints.

Beside basic constraints of orthogonality and parallelity there are two high-level constraints that restrict the gable roof of the building. The first one constrains the orientation of the roof and the second one forces the roof to be symmetric. More details can be found in the work of Loch-Dehbi and Plümer (2011).

The implemented constraint-based reasoner uses two methods of automatic theorem proving: algebraic and logic reasoning. While algebraic reasoning uses multivariate polynomials to define constraints and theorems deductive reasoning uses logical facts and rules. While algebraic reasoning requires no knowledge about rules describing the dependencies between geometric properties the strength of deductive reasoning lies in the fast deduction of new constraints based on an a priori defined set of rules. The following section describes the combination of the logical and algebraic reasoner based on these representations. The assessment of quality under the aspect of uncertainty is an important aspect of GIS data and modeling. To this end, the numeric representation is as well integrated. Section 6.3 extends the representation by covariance matrices and describes the handling of uncertainty in order to apply the model in reconstruction tasks.

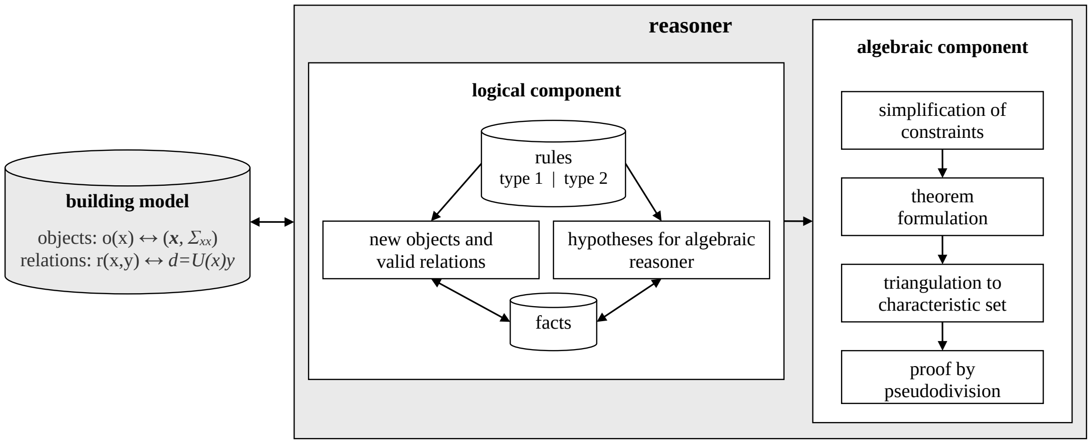


Figure 6.3: Overview of logical and algebraic methods for automatic theorem proving

6.2 Geometric reasoning using automatic theorem proving

The geometric reasoning is intended for two applications: Ensuring a redundancy-free and consistent representation of a building model and deducing new implicit objects and relations for the reconstruction from uncertain data. In both cases it has to be shown whether one constraint follows from another constraint that suggests the use of automatic theorem proving. While there exist efficient algorithms to solve the problems numerically, this thesis deals with proving theorems on a symbolic level that is to prove if a theorem is in general valid independent from real numeric values.

While most approaches of theorem proving handle two-dimensional problems, the presented method has to tackle 3D models. In this context, an increasing complexity can be observed in the transition from the two dimensional to the three dimensional space. The increasing number of variables and constraints that are needed to model a theorem in the 3D space requires a special representation and handling of the problem.

In order to tackle this challenge, logical reasoning is combined with algebraic reasoning and an appropriate representation of constraints is presented. To show the feasibility of this approach a constraint-based reasoner was developed. Figure 6.3 gives an overview of the implemented logical and algebraic component.

At first, the logical component uses fast deductive reasoning with first-order predicate logic to derive new facts based on known implications. It further extends the geometric scope of algebraic reasoning to constraints of topology and partonomy. The implemented deductive reasoner has two tasks and thus two types of rules:

1. derive new geometric objects and constraints based on known rules
2. generate hypotheses that have to be checked by the algebraic reasoner

Rules of type 1 derive new facts by applying construction rules and known axioms until a fixpoint is reached and no new facts are produced. This background knowledge is extended steadily with new proven theorems such as the example in Section 3.3 or proven theorems known from literature. As soon as they can be represented in a logical language the deductive reasoner profits from the possibility of a fast deduction of implicit relations. If for example objects $O1, O2$ and $O3$ are planes and $O1$ is orthogonal to $O2$ and parallel to $O3$ the fact that $O2$ is orthogonal to $O3$ can be deduced without any algebraic transformation:

```
orthogonal(O2,O3) ← plane(O1), plane(O2), plane(O3),
    O2 \= O3, orthogonal(O1,O2), parallel(O1,O3)
```

Rules of type 2 generate hypotheses that the algebraic reasoner has to verify. Due to the closed world assumption facts remain false as long as they are not in the database. As a consequence, it is worth to test whether this assumption really holds or the algebraic reasoner finds a proof of the contrary. As an example, having neither orthogonality nor parallelity of two planes, the algebraic component should check whether the negation as assumed in the closed world holds or on the contrary the following hypothesized fact can be deduced:

```
hyp_orthogonal(O1,O2) ← plane(O1), plane(O2),
    not(orthogonal(O1,O2)), not(parallel(O1,O2)), O1 \= O2
```

The result of the logical component is a prerequisite for the algebraic component which is a central component in the presented reasoning process. In contrast to the logical reasoner based on rules algebraic theorem proving needs neither rules nor prior knowledge. Instead the theorem alone is used to prove the conclusion from the premises. Based on multivariate polynomials the proof is more complex than simply applying rules but therefore is able to discover implications that could not be defined a priori in form of logical rules due to complexity or lack of knowledge.

Basically, the algebraic reasoning consists of two applications:

1. testing building models for consistency and redundancy for a minimum well-defined prototype
2. deducing new facts for the support of 3D building reconstruction from uncertain measured data.

Finding a redundancy-free and consistent representation leads to a more comprehensive and readable model as well as less disk space. Further, the combinatorial complexity is decreased and may thus be beneficial for further processing. In contrast, in the context of building reconstruction a redundant set of constraints may be of advantage in that additional deduced objects and implicit relations made explicit can be used to verify or adjust the interpretation of measurements.

In both cases the task is equivalent to prove whether one constraint follows from another constraint. For algebraic reasoning the second component of the developed reasoner uses constraints that are represented by multivariate polynomials and Wu's method is chosen as an algebraic method for automatic theorem proving (cf. Section 3.3). For each theorem a characteristic set is computed and it is checked by pseudodivision whether the conclusion can be deduced. If the method reveals that the polynomials have no common zeros the set of constraints is declared to be inconsistent. In general, Wu's method is preferred to other

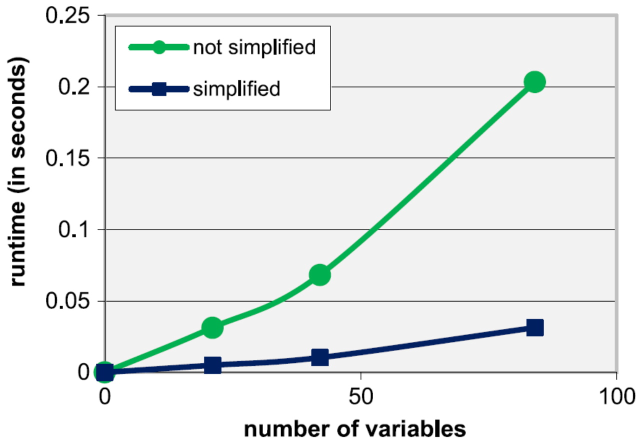


Figure 6.4: Runtime of proving a theorem with and without simplification of constraints

algebraic methods since they often turn out to be more efficient for geometric theorems (Cox et al. (2007)). Furthermore, the automatic generation of subsidiary conditions covers the assumption of geometric configurations being in a general position that is often mentioned in textbooks.

For the efficiency of the algorithm an appropriate representation as described in Section 6.1 turns out to be crucial. The use of projective geometry together with a reduction of variables reduces the complexity significantly. A limitation the reasoner has to cope with is that Wu's method is restricted to equations and does not allow inequalities. The constraints as described in Section 6.1 involve inequalities but however can be modeled by equivalent expressions with the introduction of an additional variable w (Kapur and Mundy (1988)):

$$\begin{aligned} x < 0 &\Leftrightarrow xw^2 + 1 = 0 \\ x > 0 &\Leftrightarrow xw^2 - 1 = 0 \end{aligned} \tag{6.1}$$

Beside an appropriate representation of constraints that avoids complex constraints with quadratic terms and many variables (cf. Section 6.1), the complexity of the problem depends on the choice and the ordering of dependent variables. They should be chosen in such a way that they support the construction of a characteristic set on the one hand and the proof by pseudodivision on the other hand. Geometrically seen, this often relates to the step-wise introduction of geometric objects and consequently their parameters. With a simplification of the problem by choosing the right representation and with the combination of deductive reasoning with algebraic reasoning the complexity and consequently the running time can be reduced significantly.

Feasibility of the proposed approach was proven on a set of constraints representing ten interrelated gable roof houses. It could be shown that 11 of 22 constraints that a gable roof exhibits are redundant and do not have to be modeled explicitly. Figure 6.4 shows the impact of the proposed simplification of constraints by measuring the runtime of proving a theorem with increasing number of constraints.

The following workflow exemplifies the two mentioned applications for the algebraic and deductive reasoning on 3D building models. Assuming we have a gable roof house that is modeled by 7 planes with 16 orthogonality and 3 parallelity constraints as suggested by Figure 6.1, the first task is to ensure a redundancy-free set of constraints. A set of k constraints can be tested with respect to redundancy by a recursive test of

$$c_1, ...c_{k-1} \leftarrow c_k$$

First, the reasoner tries to deduce the constraint c_k by fixpoint iteration in the logical component. It applies all existent rules to the known logical facts excluding c_k until the knowledge base does not change anymore. If c_k could be deduced, it is redundant and can be removed from the set of constraints. Implications that cannot be proven by the application of logical rules are checked with Wu's method. In this case, it can for example be shown that the theorem $front \perp bottom \wedge front \| back \Rightarrow bottom \perp back$ – not included in the set of logical rules – is true. Therefore, the theorem is repesented by multivariate polynomials

$$a_1a_2 + b_1b_2 + c_1c_2 = 0 \wedge \begin{pmatrix} b_1c_3 - c_1b_3 \\ c_1a_3 - a_1c_3 \\ b_1c_3 - c_1b_3 \end{pmatrix} = \mathbf{0} \Rightarrow a_2a_3 + b_2b_3 + c_2c_3 = 0$$

A characteristic set $H' = \{h'_1, h'_2, h'_3\} = \{h_1, h_{21}, h_{23}\}$ is constructed and finally the pseudoremainder is determined recursively:

$$\begin{aligned} r_3 &= prem(c, h'_3, a_3) = -b_1b_2b_3 - b_1c_2c_3 - b_3a_1a_2 \\ r_2 &= prem(r_2, h'_2, c_3) = -b_1^2b_2b_3 - b_1b_3a_1a_2 - b_1c_2b_3c_1 \\ r_1 &= prem(r_2, h'_1, b_2) = 0. \end{aligned}$$

Since the result equals zero the implication is true and the model can omit the deduced orthogonality constraint *orthogonal(bottom, back)*. As a consequence, the model needs less memory and reasoning methods have to cope with a less complex model.

The rule of general validity can be added as logical rule for the logical component:

```
orthogonal(B,C) ← orthogonal(A,B), parallel(A,C).
```

Expanding the knowledge of the deductive reasoner speeds up the proof of deduction in subsequent iterations. For example, by substituting A with $leftSide$, B with $bottom$ and C with $rightSide$ it can now be concluded by simply applying this new rule that the constraint $orthogonal(bottom, rightSide)$ is as well redundant.

The second application of the implemented reasoner concerns the reconstruction of buildings and therefore the deduction of implicit objects and constraints. In this case, redundancy helps to enhance the result in noisy data during model-driven reconstruction. Having a laserscan of 3D points representing a gable roof house, the task is to estimate planes that best match the observations and at the same time satisfy the constraints that define the model. Redundant information in an over-constrained system can be used to validate estimated objects. In a first step, new objects are created, for example a ridge line as the intersection of the left and right roof half of the gable roof (cf. Table 6.1):

$$ridge \equiv \begin{pmatrix} L_h \\ L_0 \end{pmatrix} = \begin{pmatrix} rR_h \times lR_h \\ rR_0 \cdot lR_h - lR_0 \cdot rR_h \end{pmatrix}$$

Consequently, it is the task to deduce new constraints concerning the ridge that so far were not part of the constraint set. The deductive component begins again with a fixpoint iteration. On this basis, it further generates hypotheses with rules of type 2 that should be proven by the algebraic component, for example:

```
hyp_orthogonal(O1,O2) ← line(O1), plane(O2),
    not(orthogonal(O1,O2)), not(parallel(O1,O2)), O1 \= O2.
```

The variables of this rule can be substituted by $O1 = ridge$ and $O2 = front$ which yields the new hypothesized fact

```
hyp_orthogonal(ridge,front).
```

It has to be proven by Wu's method whether this orthogonality constraint is an implication of the already existing constraint set. With this deduction the strength of Wu's method can be shown. Although the ridge does not occur explicitly in the premises, algebraic reasoning can conclude a new constraint concerning the ridge since the algebraic representation reveals the correlations between variables. Testing possible conclusions by logical and algebraic methods of automatic theorem proving, the reasoner finds two constraints of orthogonality and three constraints of parallelity concerning the ridge line. This shows that the ridge is a major object in the building model and its identification is beneficial for the building reconstruction. At the same time these new constraints can be used to verify estimated planes.

6.3 Checking validity for uncertain constraints

In contrast to the development of prototyped mathematical models for buildings the handling of uncertainty plays an important role in the context of GIS if these models are applied to the reconstruction of buildings. In this case, there is a need for evaluating uncertain conclusions since in the context of reconstructing buildings the reasoner has to consider that objects do not hold crisp constraints, that is, a constraint may only be valid up to a small ϵ as an effect of imprecise measurements. Furthermore, the quality of conclusions has to be quantified in a sound way. Deduced relations are not necessarily true in an uncertain configuration but have to be tested in the context of measured data and their precision. However, Wu's method is restricted to crisp constraints that do not consider imprecise measurements. To this end, the presented reasoning is extended to deal with noisy data and uncertain projective geometry is incorporated in order to address the statistical aspect of uncertainty. On the one hand, this ensures that constraints remain valid if the degree of

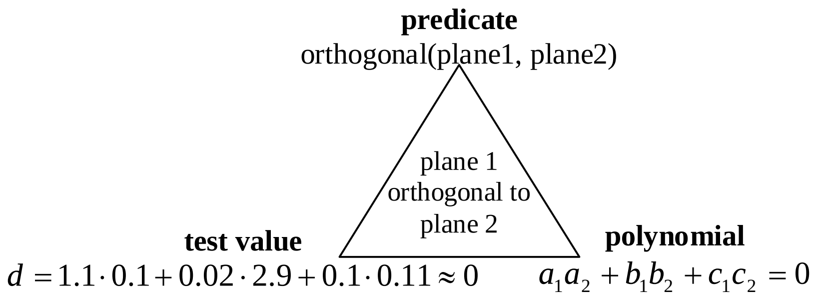


Figure 6.5: Three-fold representation of the orthogonality constraint with planes $\Pi_1 = ((1.1, 0.02, 0.1), 0)$ and $\Pi_2 = ((0.1, 2.9, 0.11), 0)$

uncertainty of the corresponding objects is negligible. On the other hand, the statistical component can check whether proven conclusions remain true under the perspective of uncertainty. For example, it might be that the constraints of the premises have a marginal degree of accepted uncertainty. As a consequence, the conclusion is not obligatory reliable due to error propagation.

In order to meet the requirements of uncertain data, geometric objects are extended by a covariance matrix to (x, Σ_{xx}). A chi-square distributed test statistics can be performed for a bilinear constraint whose matrix multiplications from Table 6.2 are represented of the form $c = U(a)b = V(b)a$. Here, $U(a)$ and $V(b)$ denote the Jacobians of the constraint with respect to a and b respectively. First order error propagation is applied to determine the covariance matrix of the constraint c:

$$\Sigma_{cc} = U(a)\Sigma_{bb}U^T(a) + V(b)\Sigma_{aa}V^T(b) \tag{6.2}$$

In this way, the proof by Wu's method is extended by a statistical test depending on a given significance level (1-α) and the degree of freedom (DoF) of the corresponding constraint (cf. Table 6.2) :

$$prem(c, H) = 0 \wedge I(h_i) \neq 0 \wedge c^T \Sigma_{cc}^{-1} c > \chi^2_{1-\alpha;DoF}.$$

Hereby, the conclusion is assessed by its degree of uncertainty. For deductive reasoning the proof is extended equivalently.

Figure 6.5 illustrates the three-fold representation of constraints used for logical, algebraic and statistical reasoning. In Section 6.2 the logical and algebraic reasoning was demonstrated with an example where a new object for the ridge was constructed and the fact $ridge \perp front$ was deduced automatically. However, it might be that for example the conclusion $ridge \perp front$ is not reliable for a validating test since the constraints of the premises were already very imprecise. A χ^2-distributed test statistics with test values $d_i = c_i \Sigma_{c_i c_i} c_i$

thus assesses the quality of the conclusion in a concrete (numeric) scenario:

$$
\begin{aligned}
leftRoofHalf \perp front :& d_1 = 2.52 < \chi^2_{0.95,1} = 3.748 \\
leftRoofHalf \perp front :& d_2 = 3.19 < \chi^2_{0.95,1} = 3.748 \\
&\not\Longrightarrow \\
ridge \perp front :& d_3 = 6.35 > \chi^2_{0.95,2} = 5.938
\end{aligned}
$$

In this case d_3 exceeds the value of the corresponding χ^2-distribution. The derived constraint differs too much from the strict orthogonality constraint, thus is not reliable for the use in further validations and cannot be considered true.

The feasibility of developing and analyzing prototyped models was illustrated in the context of 3D building modeling, but however can be transferred to arbitrary geometric objects. To this end, the approach of deductive, algebraic and statistical reasoning is implemented as a generic system. The reasoning accepts all constraint-based models that can be expressed by logical predicates or multivariate polynomials. Algebraic representations are restricted to equations but inequations can be transformed to equations after introducing an additional parameter. Crucial for the performance of the algebraic reasoning is an appropriate representation that minimizes the number of variables. The presented approach focuses on linear objects like points, lines or planes in the three-dimensional space and shows the feasibility of 3D theorem proving for interactive systems. Other primitives such as spheres that come along with a higher complexity need further investigations.

7 Conclusion and outlook

This thesis presented an approach for the prediction of 3D building structures based on sparse observations. In contrast to common approaches, dense measurements are not needed in order to generate plausible hypotheses of 3D building models. This becomes possible due to a highly structured hypothesis space together with an adequate combination of logical and stochastic reasoning methods. An a priori infinite search space is thus reduced to a few good hypotheses.

The presented reasoner follows a model-driven top down approach and holds the view that it is easier to verify or falsify a prediction top-down than to build a model bottom up from measurements. The overall task is to instantiate model parameters with most likely values that at the same time satisfy the given constraints. Bayesian networks have been used as a powerful framework for reasoning in stochastic models. However, the model is characterized by bilinear constraints with discrete and continuous parameters that in general suggests the use of approximate inference methods. To this end, this thesis proposed an approach that makes exact inference feasible. The idea of conditional linear Gaussian networks (CLG) was adapted to multilinear instead of linear relations and Gaussian mixtures instead of Gaussian distributions.

The key aspect is the separation of the problem into a combinatorial (discrete) and a stochastic (continuous) problem and the use of a strong prior knowledge that is legitimated by strong regularities in man-made objects and based on an extensive data analysis of a ground truth data base. The latter consists in constraints and probability density functions that in turn are approximated by Gaussian mixtures and thus pave the way to use well established reasoning methods. Clear peaks and small variances of the Gaussian mixtures often help to structure the hypotheses space.

The reasoner makes use of the strength of constraint logic programming for solving combinatorial problems and the advantages of Bayesian networks for reasoning with uncertain data. The combinatorial component transforms an a priori bilinear and multimodal problem in a linear unimodal one. It therefore determines the discrete parameters and initializes the continuous ones by means of Gaussian distributions. A Kalman filter as an efficient implementation of a special Bayesian network finally predicts the continuous parameters in accordance to known observations. Finally the reasoner outputs a ranked set of hypotheses according to a MAP-estimation that can be refined further by additional single measurements in order to decide between competing hypotheses. In this way, the proposed model-driven approach decreases time, space and costs for data acquisition and interpretation.

The thesis covers façade models and indoor models and thus bridges the gap between the exterior and interior reconstruction for a seamless indoor/outdoor modeling. For façade models the prediction is based only on measurements of the footprints, more precisely derived façade widths. Single observations of embrasures can enhance the result further. Despite the leak of fully observed objects occluded substructures are predicted as well and uncertain possibly erroneous data is compensated. The prediction of 3D indoor models gets along without the need for indoor measurements. Shape and location parameters of rooms are predicted based on the building footprint together with basic information of rooms such as the room area or the functional use that are usually available e.g by facility managers. If room numbers are available they are exploited to restrict the possible neighborhood of rooms and hence narrow the combinatorial possibilities of locating rooms. Exterior measurements of windows and their predicted correspondences to rooms help to localize the rooms within the footprint. This especially facilitates the correct prediction of occluded walls and reduces the misinterpretation of measurements. The 2D floorplan is extended to a 3D model by estimated heights and predicted doors.

In the context of model-driven approaches, consistent and redundancy-free prototyped mathematical models are an important prerequisite. To this end, an approach for geometric reasoning was presented that supports the development of models for the reconstruction of 3D buildings. The reasoner therefore uses methods of algebraic theorem proving, more precisely Wu's method of characteristic sets, and profits from the strength of algebraic reasoning based on multivariate polynomials and the advantages of deductive reasoning using logical facts. While deductive methods are based on pre-defined rules, perform fast deductions and further are able to handle a reasoning on topology and partonomy, algebraic methods prove implications without the need of any prior knowledge.

In order to demonstrate the feasibility of reasoning on 3D models a prototype was implemented that is able to check for consistency or redundancy and deduces new implicit objects or constraints automatically. The model is defined by constraints between primitives such as planes, lines or points where an appropriate representation of the model has been shown to be crucial for efficient reasoning. Algebraic relations are modeled using projective geometry in order to have multilinear rather than quadratic equations. Further, the explicit reference of points is avoided where possible and the invariance with respect to translation and rotation is exploited to reduce the model complexity.

As in the context of GIS the assessment of quality is important, Wu's method that originally does not consider any uncertainty is extended by a chi-square distributed test statistics. With objects and constraints augmented by covariances according to the concept of uncertain projective geometry, errors are propagated and the degree of uncertainty assesses the quality of deductions.

The reasoner is not restricted to building models. A generic framework was developed that can be applied to a wide range of applications. Further, the algorithm is implemented in a modular and object-oriented manner. Thus, substructures such as rooms, windows or façades can be combined in a manifold way to model a large variety of buildings.

While the developed approach focuses on the generation and the ranking of plausible hypotheses the efficient verification or falsification of models is outlined but beyond the scope of this work. In this context, it is desirable to suggest a minimal number of additional measurements needed in order to decide between competing hypotheses. This may be an observation that verifies an predicted object, e.g. an embrasure, or supports the interpretation of model properties such as the observation of the window appearance whose special structure for example may identify a bath room. An automatic determination of additionally needed measurements is thus a challenging and valuable task.

Due to the dominance of rectangular structures in man-made objects this thesis assumes that buildings and their substructures follow the Manhattan world assumption. This assumption is especially valid for cultural heritage buildings or other big buildings such as offices or hospitals. For indoor predictions, it was shown how non-rectangular shapes could be modeled by adding auxiliary rooms. The restriction of rooms being rectangular could be extended to other categories of rectilinear shapes such as L- or T-shapes by modeling a room as combination of two rectangles. The façade model may be extended to a grid of individual windows and doors instead of uniformed ones for modeling the variety that can be especially observed in modern residential houses. An extended analysis of characterizing relations between the substructures is needed in this case.

The prediction of floorplans is constrained by a non-linear relation that is not linearized during combinatorial reasoning. The Kalman filter used for determining the posterior of the shape and locations parameters can be extended for more precise results to non-linear systems such as the extended Kalman filter or the unscented Kalman filter.

Due to the fact that orthogonality, parallelity and symmetry are main organizing principles the presented algebraic reasoning covers only linear objects such as planes, lines and points. Primitives of higher order such as spheres have to be investigated.

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A List of own publications

A.1 List of publications appended to this thesis

The following list of publications is most relevant for this thesis and appended below.

- Loch-Dehbi, S., Plümer, L., 2011. Automatic reasoning for geometric constraints in 3D city models with uncertain observations. ISPRS Journal of Photogrammetry and Remote Sensing 66, 177–187.
- Loch-Dehbi, S., Plümer, L., 2015. Predicting building façade structures with multilinear Gaussian graphical models based on few observations. Computers, Environment and Urban Systems 54, 68–81.
- Loch-Dehbi, S., Dehbi, Y., Plümer, L., 2017. Estimation of 3D indoor models with constraint propagation and stochastic reasoning in the absence of indoor measurements. ISPRS International Journal of Geo-Information 6.

A.2 List of publications not appended to this thesis

Loch-Dehbi, S., Plümer, L., 2009. Geometric reasoning in 3D building models using multivariate polynomials and characteristic sets, in: Proceedings of ISPRS WG II/2+3+4 and Cost Workshop on Quality, Scale & Analysis Aspects of Urban City Models, Lund, Sweden.

Loch-Dehbi, S., Dehbi, Y., Plümer, L., 2013. Stochastic reasoning for UAV supported reconstruction of 3D building models. ISPRS Annals of Photogrammetry, Remote Sensing and Spatial Information Sciences XL-1/W2, 257–261.

Loch-Dehbi, S., Dehbi, Y., Gröger, G., Plümer, L., 2016. Prediction of building floorplans using logical and stochastic reasoning based on sparse observations. ISPRS Annals of Photogrammetry, Remote Sensing and Spatial Information Sciences IV-2/W1, 265–270.

Dehbi, Y., Loch-Dehbi, S., Plümer, L., 2017. Parameter estimation and model selection for indoor models based on sparse observations. ISPRS Annals of Photogrammetry, Remote Sensing and Spatial Information Sciences IV-2/W4, 303–310.

B Appended papers

B.1 Automatic reasoning for geometric constraints in 3D city models with uncertain observations

Loch-Dehbi, S., Plümer, L., 2011. Automatic reasoning for geometric constraints in 3D city models with uncertain observations. ISPRS Journal of Photogrammetry and Remote Sensing 66, 177–187.

Abstract

This paper presents a novel approach to automated geometric reasoning for 3D building models. Geometric constraints like orthogonality or parallelity play a prominent role in man-made objects such as buildings. Thus, constraint based modelling, that specifies buildings by their individual components and the constraints between them, is a common approach in 3D city models. Since prototyped building models allow one to incorporate a priori knowledge they support the 3D reconstruction of buildings from point clouds and allow the construction of virtual cities. However, high level building models have a high degree of complexity and consequently are not easily manageable. Interactive tools are needed which facilitate the development of consistent models that, for instance, do not entail internal logical contradictions. Furthermore, there is often an interest in a compact, redundancy-free representation. We propose an approach that uses algebraic methods to prove that a constraint is deducible from a set of premises. While automated reasoning in 2D models is practical, a substantial increase in complexity can be observed in the transition to the three-dimensional space. Apart from that, algebraic theorem provers are restricted to crisp constraints so far. Thus, they are unable to handle quality issues, which are, however, an important aspect of GIS data and models. In this article we present an approach to automatic 3D reasoning which explicitly addresses uncertainty. Hereby, our aim is to support the interactive modelling of 3D city models and the automatic reconstruction of buildings. Geometric constraints are represented by multivariate polynomials whereas algebraic reasoning is based on Wu's method of pseudodivision and characteristic sets. The reasoning process is further supported by logical inference rules. In order to cope with uncertainty and to address quality issues the reasoner integrates uncertain projective geometry and statistical hypothesis tests. Consequently, it allows one to derive uncertain conclusions from uncertain premises. The quality of such conclusions is quantified in a way which is sound both from a logical and a statistical perspective.

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Automatic reasoning for geometric constraints in 3D city models with uncertain observations

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ABSTRACT

This paper presents a novel approach to automated geometric reasoning for 3D building models. Geometric constraints like orthogonality or parallelity play a prominent role in man-made objects such as buildings. Thus, constraint based modelling, that specifies buildings by their individual components and the constraints between them, is a common approach in 3D city models. Since prototyped building models allow one to incorporate a priori knowledge they support the 3D reconstruction of buildings from point clouds and allow the construction of virtual cities. However, high level building models have a high degree of complexity and consequently are not easily manageable. Interactive tools are needed which facilitate the development of consistent models that, for instance, do not entail internal logical contradictions. Furthermore, there is often an interest in a compact, redundancy-free representation. We propose an approach that uses algebraic methods to prove that a constraint is deducible from a set of premises. While automated reasoning in 2D models is practical, a substantial increase in complexity can be observed in the transition to the three-dimensional space. Apart from that, algebraic theorem provers are restricted to crisp constraints so far. Thus, they are unable to handle quality issues, which are, however, an important aspect of GIS data and models. In this article we present an approach to automatic 3D reasoning which explicitly addresses uncertainty. Hereby, our aim is to support the interactive modelling of 3D city models and the automatic reconstruction of buildings. Geometric constraints are represented by multivariate polynomials whereas algebraic reasoning is based on Wu's method of pseudodivision and characteristic sets. The reasoning process is further supported by logical inference rules. In order to cope with uncertainty and to address quality issues the reasoner integrates uncertain projective geometry and statistical hypothesis tests. Consequently, it allows one to derive uncertain conclusions from uncertain premises. The quality of such conclusions is quantified in a way which is sound both from a logical and a statistical perspective.

1. Introduction

Due to an increasing demand of high resolution city models, automatic building construction and reconstruction in 3D has become an essential task. Since the use of city models for noise mapping, disaster management or the calculation of escape routes requires exact knowledge of the structure of buildings, the refinement of city models is a fundamental need. Though this is still a challenging and complex problem, its complexity can be reduced considerably by using 3D building models, such as a prototyped description of a gable roof house. These support the estimation of buildings or building parts in imagery or laser scanner data by the integration of strong a priori knowledge about the building structure.

The estimation of geometric objects from observations is generally solved by using a mathematical model that can be divided into a functional model and a stochastic model. The functional model describes the relations between unknown parameters whereas the stochastic model is characterized by an estimation of accuracy of the observations. In our context, the functional model is built of primitives of buildings, e.g. planar faces that represent walls or roof halves. Because most man-made objects have a regular structure, these primitives are specified by geometric constraints such as parallelity and orthogonality. An excerpt of constraints that characterise a gable roof house is depicted in Table 1. These constraints are expressed in terms of logical predicates which give both a formal and readable definition and furthermore support geometric and topological relations. Empirical investigations based on a database of LOD1 models show the dominance of orthogonality and parallelity in buildings. The diagram in Fig. 1 illustrates the

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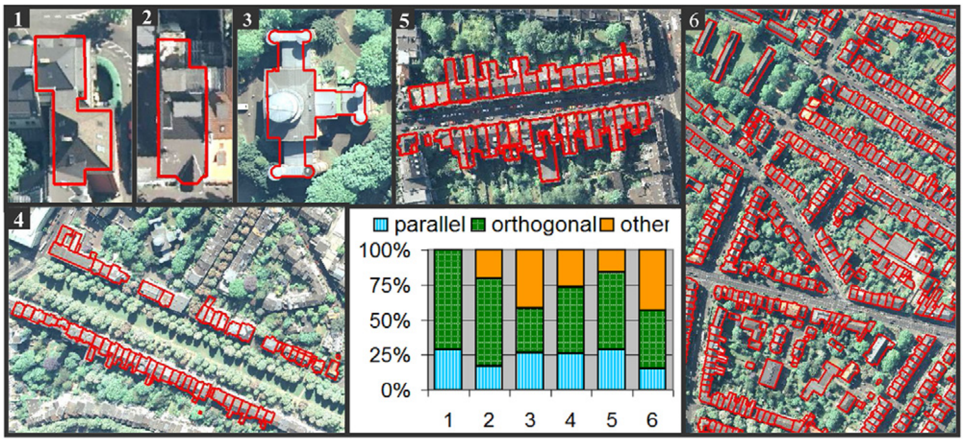

Fig. 1. Bi-relational constraint statistics show dominance of orthogonality and parallelity in 3D building models. Comparison is based on LOD1 models build upon the shown footprints.

Table 1
Excerpt of constraints describing a gable roof house.

Geometric constraints
parallel (leftWall, rightWall), parallel (front, back)
orthogonal (bottom, rightWall), orthogonal (bottom, leftWall), orthogonal (bottom, back), orthogonal (front, back), orthogonal (rightWall, front), orthogonal (rightWall, back), orthogonal (leftWall, front), orthogonal (leftWall, back), orthogonal (back, rightRoofHalf), orthogonal (back, leftRoofHalf), orthogonal (front, rightRoofHalf), orthogonal (front, leftRoofHalf) …
Topological constraints
meets (bottom, rightWall), meets (bottom, leftWall), meets (bottom, front), meets (bottom, back), meets (rightWall, front), meets (rightWall, back), meets (leftWall, front), meets (leftWall, back), meets (back, rightRoofHalf), meets (back, leftRoofHalf), meets (front, rightRoofHalf), …

percentage of bi-relational constraints between 3D planes in different extracted models of single buildings and of whole clusters of buildings respectively. We took imprecise measurements of geometric objects into account and performed pairwise comparisons between planes across all buildings. Obviously, it can be seen that in all cases more than 60% of coupled planar faces are either orthogonal or parallel. This is even valid for buildings with rounded shapes (no. 3) as well as for whole city districts where not all houses are aligned (no. 6).

In order to model complex 3D buildings, reasoning tools are needed which support the development of consistent building models. They are supposed to handle prototyped models as well as uncertain geometric objects from noisy observations which leads to several requirements. Apart from a consistent model it is often beneficial to provide a non-redundant specification of constraints. This allows not only for a compact and readable characterisation but also decreases the complexity for further processing of these models, e.g. the comparison of different models or the learning of structural details. In fact, our example in Table 1 contains redundant constraints so that a reduced description is composed of only less than half of these geometric constraints. The property, for example, that the right wall is orthogonal to the bottom has already been fulfilled by demanding that the right wall be parallel with the left wall, which is in turn orthogonal to the bottom.

In order to support the process of reconstruction, it is moreover conducive to enrich our knowledge of the model automatically. Thus it should be analyzed if further objects and constraints are deducible. Consequently, these predictions can be used to verify estimated hypotheses in order to support the 3D reconstruction of buildings from point clouds. Since we are dealing with uncertain observations the modeller has to deal with imprecise constraints and should be able to handle quality issues.

As we have several statements of the form A *and* $B \Rightarrow C$ we use approaches of automatic theorem proving to deduce the redundant or implicit constraints automatically. Since geometric constraints can be represented by algebraic equations one category of automatic proofs is expressed in terms of multivariate polynomials. As in many cases geometric constraints contain several parameters, the polynomials are often multilinear or even quadratic. While 2D models are manageable in many cases, a substantial increase in complexity can be observed in the transition to the three dimensional space, which has to be overcome by the modeller. There are efficient methods to solve non-linear equation systems numerically. However, we do not seek single solutions. Instead, we have to cope with the question whether or not a constraint set is valid in general. In other words, our interest does not lie in finding specific values but in proving theorems on a symbolic level.

Apart from algebraic approaches, there is another category of reasoning techniques which is based on logical facts and inference rules. They require the declaration of at least basic theorems but they are therefore very efficient in finding possible conclusions. Since logical reasoning is appropriate to represent structural and semantic information we integrate it in our reasoning tool.

As in the context of GIS uncertain relations are common it has to be ensured that proofs are still valid if the premises and the objects involved are inaccurate. Additionally, true conclusions have to be evaluated by their quality in order to enhance the reconstruction of 3D building models. Nevertheless, the approaches to algebraic theorem proving are designed for reasoning hard constraints. So far, the aspect of uncertainty has not been considered. In order to integrate automatic reasoning with uncertainty we make use of uncertain projective geometry which was proposed by Heuel (2004) and is able to model imprecise geometric constraints.

The main contribution of this article is a procedure for 3D reasoning in man-made objects. We use Wu's method, which is an algebraic approach of theorem proving based on multivariate polynomials and characteristic sets and which was introduced to prove geometric theorems, especially those of a constructive type. Additionally, we combine a logical rule-based reasoner with an algebraic reasoner. Hereby we benefit from the strength of deductive rules for deducing facts and the expressiveness of algebraic polynomial equations valuable for parameter-based geometric statements. We demonstrate that reasoning in 3D building models is feasible if constraints are represented in an suitable way. Uncertainty and quality issues are addressed by incorporating uncertain projective geometry and statistical hypothesis tests, thus allowing one to derive conclusions from uncertain premises and to quantify uncertainty of the former. We have implemented this procedure as a prototype and illustrate the feasibility of our approach by two applications thus demonstrating its suitability for interactive systems.

2. Related work

Constraints play an important role in the representation of man-made objects. Constraint graphs for geometric objects represent the geometric and topological relations between different primitives, such as parallelity between planes. Kolbe (2000) deals with these spatial relations between primitives. He describes roofs by geometric constraints and compares them to observations from aerial images to reconstruct buildings. Brenner (2005, 2004) discusses the modelling of complex objects by constraints and introduces weak primitives that allow for a relaxation of constraints between geometric primitives in observed objects.

Heuel (2004) uses a statistical approach for polyhedral object reconstruction which is based on relation tests in uncertain observations. For this aim, he developed statistical algorithms that are able to check geometric relations including uncertainty information. An extension of the mathematical estimation model by uncertain projective geometry yielded promising results for extracting objects from imagery data (Förstner, 2005; Meidow et al., 2009). Dehbi and Plümer (2011) proposed a framework to learn grammar rules of building parts from noisy observations. They make use of uncertain projective geometry to obtain characteristic geometric constraints between estimated objects that represent the concept to be learned.

Automatic theorem proving became popular in the late 1970s with the work of Wu (1986), who was able to prove numerous theorems automatically. Another widely used approach which is based on the construction of the Groebner Bases was proposed by Buchberger et al. (1998). These methods were successfully applied for perspective viewing (Kapur and Mundy, 1988) or formula derivation (Chou and Gao, 1989).

However, methods of automatic theorem proving have hardly been noticed in the context of GIS and building modelling. A notable extension is the work of Brenner and Sester (2005) who discuss the use of Groebner Bases for solving equation systems and identifying redundancy and consistency of constraints in the context of cartographic generalization. However, they emphasize the problem of complexity and state that the Groebner Base method may not be feasible for interactive systems. Alternatively, it has been shown that Wu's method, which is presented in the next subsection, can be more efficient in geometric theorems of construction type and is also able to solve more complex problems (Cox et al., 2007). Recently, Quaresma (2010) published a benchmark that compares algebraic methods of automatic theorem provers. This database confirms that most of the theorems can be proven by Wu's method even if Groebner Bases fail and if both methods succeed Wu's method is superior considering the runtime.

In the context of automatic theorem proving several interactive tools were implemented for the 2D space (e.g. Gao and Lin, 2002). Due to the lack of systems in the 3D space Roanes-Macías and Roanes-Lozano (2007) presented a software package for the mathematical tool Maple that allows for the investigation of problems in 3D geometry. They were able to prove geometric theorems by algebraic methods where conclusions are defined by point-on-object relations. In order to support polynomial elimination and the proving of algebraic theorems Wang (2004) implemented the Maple package Epsilon. It provides functions for calculating with arbitrary systems of multivariate polynomials and is able to interpret algebraic equations of theorems in the two-dimensional space. Chou et al. (2000) developed a deductive theorem prover for the 2D space that is specialized in geometric problems and able to prove and discover theorems.

3. Automatic theorem proving

Checking the redundancy of constraints is equivalent to proving their deducibility from other constraints. Automatic theorem proving is a powerful means to show the validity of a statement given its premises. Automatic theorem provers can be separated into two main categories: deductive (rule-based) and algebraic approaches. Deductive reasoners need predefined inference rules that are applied to initial facts in order to deduce new properties. Efficient methods allow for a fast deduction of new facts. However, the strength of deductive reasoning depends on the conclusiveness of its rules. It is based on the closest world assumption which means that facts that cannot be proven true are considered to be false. In contrast, algebraic reasoning is independent of any knowledge about deduction rules or geometric theorems but only analyses the constraints by computing with its algebraic representations. Matsuyama and Nitta (1995) state that algebraic theorem provers are particularly superior when theorems have an algebraic equivalent and include simple coordinate-based statements such as metric, parallelity or collinearity. Since these constraints are the main organizing principles in building models we will focus on this method in our approach. Nonetheless, we will benefit from features that are provided by logical reasoning techniques such as exploiting known geometric axioms and coping with topological and hierarchical relations.

In the following subsections deductive and algebraic reasoning techniques are presented and illustrated by a small 3D example of three planes, where the following theorem holds:

Theorem 1. *If plane A is orthogonal to plane B and parallel with plane C, then plane B is orthogonal to plane C* ($A \perp B \wedge A \parallel C \Rightarrow B \perp C$).

3.1. Deductive reasoning

Deductive reasoning is the process of making conclusions based on previously known general statements and facts. In general, deductive approaches work with first-order predicate logics. Consequently, the basis of deductive reasoning are relational facts and inference rules that have to be defined in advance. Table 1 contains a list of *atoms*, which express relations of the form `predicate(term1, term2, ..., termN)`. A *term* may be a constant or a variable. In the case of our example, they are called *facts* as they do not contain any variables. A rule is equivalent to a logical implication: $B_1 \wedge \ldots \wedge B_n \Rightarrow H$ and it is expressed by so-called *Horn clauses*

$$H \leftarrow B_1, B_2, \ldots, B_n$$

where $H, B_1, \ldots, B_n$ are atoms and a logical $\wedge$ (AND) is represented by a comma. The *head* H becomes true if the predicates in the *body* $B_1, \ldots, B_n$ are satisfied by the previously known facts. For example, the following facts represent geometric constraints mentioned in Theorem 1 and Table 1 respectively:

```
orthogonal(front,bottom).  parallel(front,back).
```

Additionally, an inference rule which expresses Theorem 1 is known to the reasoner, that is

```
orthogonal(B,C) ← orthogonal(A,B), parallel(A,C).
```

That means if there exist two facts, '*A is orthogonal to B*' and '*A is parallel with C*' it implies that the statement '*B is orthogonal to C*' is true as well. In this case, *A, B, C* are variables that can be replaced by constants, identical variable names define the same object. Applying this rule to the two facts above, the variables in the body of the rule will be substituted by the constants *front, back* and *bottom* and propagated to the head predicate. Consequently, the knowledge of the reasoner is extended by the fact `orthogonal(bottom,back)`. Thus, applying these rules to the data available, reasoning can be achieved. A useful feature of implemented inference techniques is the fixpoint iteration which allows to ask for all possible conclusions. A *fixpoint* is reached if the set of facts in the database does not change anymore and no new inferences can be made although the given rules are iteratively applied to the initial and deduced facts.

3.2. Algebraic reasoning

Since geometric constraints can be expressed by multivariate polynomials, redundant or new constraints can be deduced automatically by using symbolic approaches of automatic theorem proving. In the following, the idea of algebraic theorem provers is illustrated by the introductory example of three planes.

In order to express the theorem with multivariate polynomials, a plane Π_i can be represented by the algebraic equation $a_ix+b_iy+c_iz+d_i=0$, that is $\Pi_i=((a_i,b_i,c_i),d_i)=(\mathbf{n}_i,d_i)$ where n_i equals the normal vector of the plane. Hence, geometric constraints such as orthogonality or parallelity have their polynomial counterpart which leads to the following algebraic representation of Theorem 1 ($h_1:\Pi_1\perp\Pi_2\wedge h_2:\Pi_1\parallel\Pi_3\Rightarrow\hat{c}:\Pi_2\perp\Pi_3$):

$$h_1:a_1a_2+b_1b_2+c_1c_2=0\wedge h_2:\begin{pmatrix}b_1c_3-c_1b_3\\c_1a_3-a_1c_3\\a_1b_3-b_1a_3\end{pmatrix}=\mathbf{0} \quad (1)$$
$$\Rightarrow\hat{c}:a_2a_3+b_2b_3+c_2c_3=0.$$

With this algebraic representation of the theorem it is possible to answer the questions of deducibility, e.g. redundancy, by algebraic methods. It should be noted that the second geometric constraint, the parallelity, has to be expressed by three multivariate polynomials $h_2=\{h_{21},h_{22},h_{23}\}$ (see also Section 4.2 for algebraic representations of constraints). The key aspect is that the satisfaction of geometric relations $\{h_1,\dots,h_s\}$, i.e. possible geometric instances of the corresponding geometric objects, can be reduced to analysing common zeros of the polynomials. The common zeros of the equations $h_1,\dots,h_s$ in a polynomial ring $k[x_1,\dots,x_n]$ are called its *variety*:

$$V(\{h_1,\dots,h_s\}) := \{(\alpha_1,\dots,\alpha_n)\in k^n : h_i(\alpha_1,\dots,\alpha_n)=0\;\forall 1\le i\le s\}. \quad (2)$$

The relation between two varieties gives evidence of the deducibility of constraints: given a constraint set of hypotheses $H=\{h_1,\dots,h_s\}$ and a conclusion $\hat{c}$, the aim is to show that the zeros of $h_1,\dots,h_s$ are a subset of the zeros of $\hat{c}$: $V(\{h_1,\dots,h_s\})\subseteq V(\hat{c})$. Fig. 2 illustrates this fact by the given Theorem 1 of three planes and five constraint equations $h_1,h_{21},h_{22},h_{23},\hat{c}$. The set of zeros $V(\{h_1,h_{21},h_{22},h_{23}\})=V(\{n_1^Tn_2,(n_1\times n_3)^T\})$, that is the solution space of the exemplifying constraint equations, will not be restricted further if we add $\hat{c}$, the second constraint of orthogonality, because all zeros of H — excluding degenerated cases — are also part of the zeros of $\hat{c}$.

There are two well-known algebraic approaches of automatic theorem proving, namely the Groebner Base Method and Wu's Method. The Groebner Base Method uses polynomial long division of a Groebner Base whereas Wu's method uses pseudodivision of a characteristic set. In both cases a remainder of zero indicates that the conclusion follows from the premises. The main difference consists in the required transformation of the initial constraint set. The Groebner Base method requires that the polynomial set used

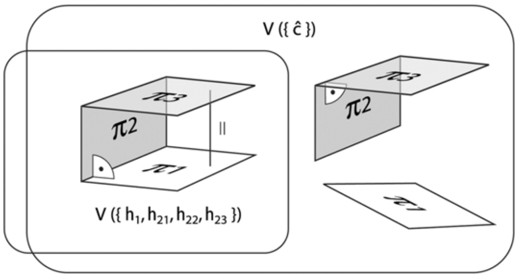

Fig. 2. Relations between varieties.

during the polynomial long division has to satisfy the property of a Groebner Base. This is necessary to ensure the uniqueness of the result since otherwise the result depends on the order of monomials and the divisibility of the leading terms. While the Groebner Base Method can solve several algebraic tasks, Wu's method was exclusively developed for proving theorems and is therefore mostly superior to the Groebner Base Method. This is mainly due to the fact that a characteristic set is computed which is a special triangular equation system on a given ordering of dependent variables $x_1\prec\dots\prec x_s$,

$$h_1(u_1,\dots,u_d,x_1)=0\in k[x_1]$$
$$\ddots$$
$$h_s(u_1,\dots,u_d,x_1,\dots,x_s)=0\in k[x_1,\dots,x_s].$$

Its structure is often almost satisfied with a step-by-step introduction of geometric objects and their constraints. Nonetheless, the feasibility still depends on how constraints are represented by polynomials, but together with a suitable representation Wu's method proves to be feasible for our domain of discourse (see Section 5).

Wu's method

Wu's Method verifies the relation between two constraint sets by using the so-called pseudodivision of multivariate polynomial equations. The output answers the question whether the theorem is generically true, that is, true under some non-degenerated conditions, the so-called subsidiary conditions. These often allow for a geometric interpretation or at least have got an algebraic meaning. Degenerated cases can therefore be excluded by considering subsidiary conditions in order to declare an otherwise false theorem generically true.

Thus, the main idea of Wu's method is to show that the zeros of the hypothesis $\{h_1,\dots,h_s\}$ which do not vanish on the degenerated cases are included in zeros of the conclusion $\hat{c}$. The used *pseudodivision* of two multivariate polynomials $\hat{c}$ and h_i can be considered as a division between univariate polynomials, e.g. in the highest variable x_i of the divisor h_i. It differs from the polynomial long division in allowing the multiplication of the dividend $\hat{c}$ with a factor $I(h_i)^{k_i}$, $k_i>0$:

$$I(h_i)^{k_i}\hat{c}=q_ih_i+r$$
$$I(h_1)^{k_1}\dots I(h_s)^{k_s}\hat{c}=q_1h_1+\dots+q_sh_s+r \quad (3)$$

where q_i denotes the quotient and r the pseudoremainder ($prem(\hat{c},h_i,x_i)=r$). $I(h_i)$ equals the initial of h_i, which is the coefficient of the highest variable of the polynomial in question.

A pseudoremainder of zero indicates that the polynomial c can be concluded from the hypotheses. Referring back to the introductory 3D example, recursive pseudodivision checks the validity of the Theorem 1:

$$prem(\hat{c},H')=prem(prem(prem(\hat{c},h_3',a_3),h_2',c_3),h_1',b_2).$$

Table 2
Outline of Wu's method.

1. *Theorem Formulation*: Define the theorem $H = \{h_1, \ldots, h_s\} \Rightarrow \hat{c}$ in form of multivariate polynomial equations $h_i = 0, \hat{c} = 0$.
2. *Triangulation to characteristic set*: Transform the hypothesis into a special triangulated equation system H' subject to the dependent variables of the geometric constraints.
3. *Proof*: Prove $H' \Rightarrow \hat{c}$, that is realized by pseudodivision to show that $V(h_1, \ldots, h_s/I_1 \cdot \cdots \cdot I_t) \subset V(\hat{c})$. If the final pseudoremainder $prem(\hat{c}, H')$ equals zero, the theorem is *generically* proven true under the subsidiary conditions $I_i \neq 0$.

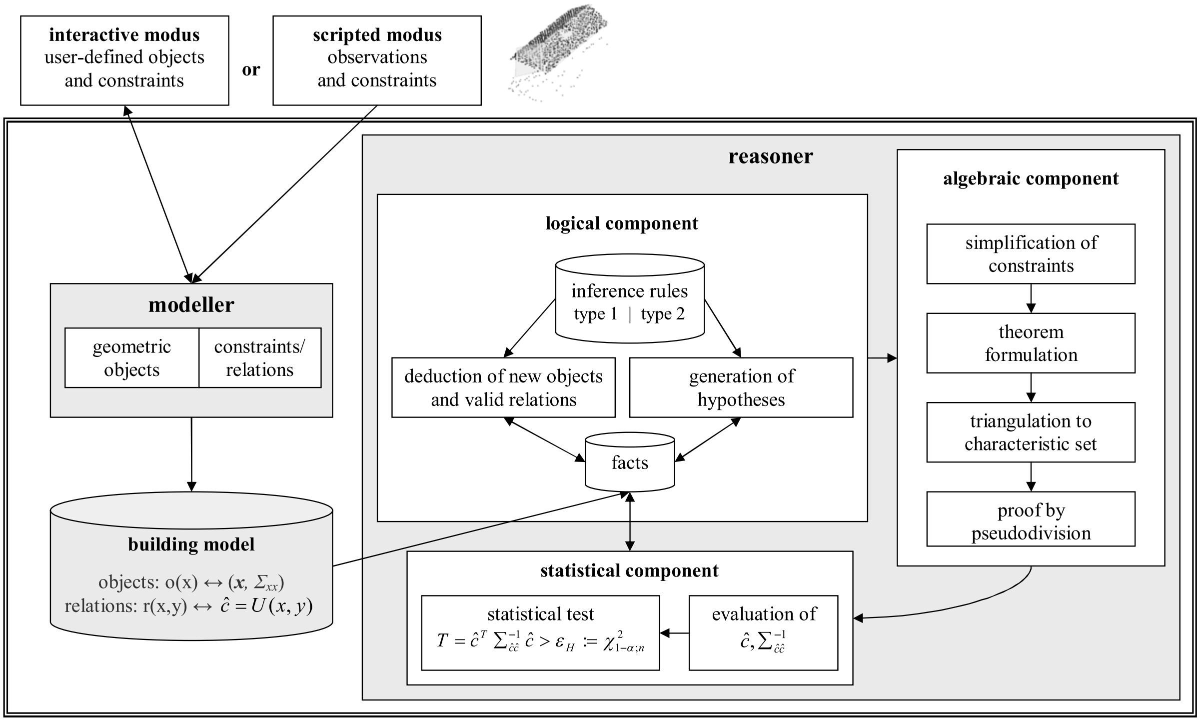

Fig. 3. System components of our prototype and their interactions.

This is done on basis of the computed characteristic set $\{h'_1, h'_2, h'_3\} = \{h_1, h_{21}, h_{23}\}$ with a chosen variable ordering $b_2 \prec c_3 \prec a_3$. It can be seen that the cardinality of the characteristic set does not equal the size of the initial constraint set but is oriented towards the dependent variables instead. The characteristic set is also computed by pseudodivision, an algorithm for this computation can be found in Buchberger et al. (1998).

The iterative computation of $prem(\hat{c}, H')$ in order to prove the theorem leads to the following results:

$$r_3 = prem(\hat{c}, h'_3, a_3) = -b_1b_2b_3 - b_1c_2c_3 - b_3a_1a_2$$
$$r_2 = prem(r_3, h'_2, c_3) = -b_1^2b_2b_3 - b_1b_3a_1a_2 - b_1c_2b_3c_1$$
$$r_1 = prem(r_2, h'_1, b_2) = 0.$$

Since the final pseudoremainder is zero ($prem(\hat{c}, H') = 0$), the theorem is true under the subsidiary condition that $b_1 \neq 0$, having $I(h'_1) = I(h'_2) = b_1$ and $I(h'_3) = -b_1$ as initials of the three polynomials. Further theoretical background and examples can be found in Chou (1988) and Loch-Dehbi and Plümer (2009).

Table 2 presents an outline of Wu's method. The basic algorithm of Wu's method and the pseudodivision have been implemented in Maple by Wang (2004).

4. Geometric reasoner

In order to demonstrate the feasibility of our approach and to provide a tool for reasoning in 3D building models we have implemented a constraint-based reasoning system. It draws upon the theories of automated theorem proving and uncertain projective geometry and is composed of two parts: a modeller and a reasoner. The data model becomes accessible by a database that connects both system parts. Fig. 3 shows the system components of our prototype and their interactions and thus illustrates our approach and its key issues.

The modeller provides the possibility for describing a constraint-based model. Our reasoning system handles prototyped building models as well as concrete geometric objects estimated from uncertain observations. The graphical user interface allows the user to define a complex geometric object by its components and the constraints between them. Alternatively, one can load a building model from a given external script that provides the basis for reasoning with estimated object and uncertain relations. Constraints as well as object parameters can thus be specified both by the manual and the automatic definition of a model but are at last stored in the same way in the underlying database. The defined model is then accessible for the reasoner which facilitates the deduction of constraints by defining and proving a geometric theorem.

The reasoner consists of a logical and an algebraic component in order to benefit from both approaches to deduction. In order to handle uncertain constraints from measured data we further integrate a statistical component which is able to complete the proof of exact constraints with regard to uncertainty. In correspondence to these components the building model has a threefold representation: a logical, a symbolic and a numerical one (see Fig. 4): our constraint based modelling system provides various predefined relations that are at first stored in the form of logical predicates. This has three fundamental advantages: first, it

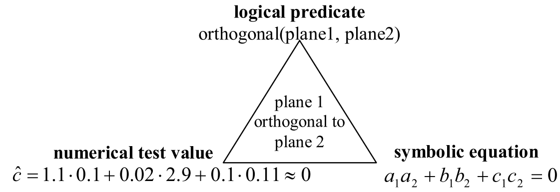

Fig. 4. Threefold representation of model: example for plane 1 ((1.1, 0.02, 0.1), 0) being orthogonal to plane 2 ((0.1, 2.9, 0.11), 0).

allows for a sound model description by providing the possibility of specifying topological and geometric constraints together with other arbitrary properties. Second, it is close to an ontology based description and even understandable for the user. Lastly, algebraic relations can be derived from these predicates uniquely. Algebraic relations are the second form of representation and are expressed by multivariate polynomials on a symbolic level. As a third form of representation we use a test value for statistical tests by substituting variables in the polynomial equations by real values of observations. All three representations are present at all stages of the reasoning process. In this way, the interaction of the three components is made continuously and the assignment of different representations remains unambiguous.

For our domain of discourse we have to deal with two major tasks. On the one hand a constraint-based modeller operating in the context of GIS has to deal with the uncertainty of measured data. Therefore, general validity has to consider uncertain observations which lead to conclusions with different degrees of quality. On the other hand, in the transition from 2D to 3D space, the reasoner has to cope with the increasing complexity of spatial theorems that at first results in a prolonged running time. The number of constraints that is necessary to deduce parallelity of planes from orthogonalities, for example, increases. Since three 3D planes can be orthogonal in pairs without having two of them parallel, in contrast to the 2D space we need five instead of two constraints to ensure that parallelity exists (Fig. 5). In order to tackle the problem of efficiency, an adequate combination of the logical and the algebraic reasoner as well as an appropriate algebraic representation is crucial. The following subsections present in detail the three different levels of reasoning realised by the interaction of the logical, algebraic and statistical component.

4.1. Applying inference rules

As a first step, the constraints are processed by the logical reasoner which is based on inference rules and has two domains of responsibility.

On one hand, it provides knowledge about structural topological details of geometric objects in order to construct new objects of existing ones. On the other hand, it tries to verify the basic theorem by deriving new facts. Therefore, construction rules and known axioms were rewritten in order to formulate them by logical formula. They are applied to the facts representing the building model until a fixpoint is reached. The verified theorem of Section 3 is an example for these rules that we denote as *type 1*:

```
orthogonal(O2,O3) ← plane(O1), plane(O2), plane(O3),
                    orthogonal(O1,O2), parallel(O1,O3)
```

For real observations and uncertain constraints statistical tests complete the reasoning process through a persistent communication (for details see Section 4.3).

Inference rules are not suitable for all kinds of geometric theorems. They perform well in high level and structured reasoning that is independent of real coordinates but they are dependent on their set of inference rules. As a consequence, the algebraic reasoner is used as a major tool for reasoning geometric constraints. Although the algebraic reasoner has no previous knowledge about valid conclusions it is not necessary to test obviously false theorems. For example, parallelity excludes orthogonality, and constraints that are already known do not have to be checked again. Occurrences of variables give hints for restricting the search space. Hence, the logical component supports the algebraic reasoner by generating hypotheses and thus ruling out impossible theorems by applying rules of *type 2* to both initial facts and already deduced relations, e.g.:

```
hyp_orthogonal(O1,O2) ← plane(O1), plane(O2),
    not(orthogonal(O1,O2)), O1 \=O2,
    not(parallel(O1,O2))
```

This exemplary rule generates hypotheses of orthogonality between two instantiated planes if at this stage neither orthogonality nor parallelity constraints between these planes exist. That is, e.g. `not(orthogonal(O1, O2))` expresses that the database

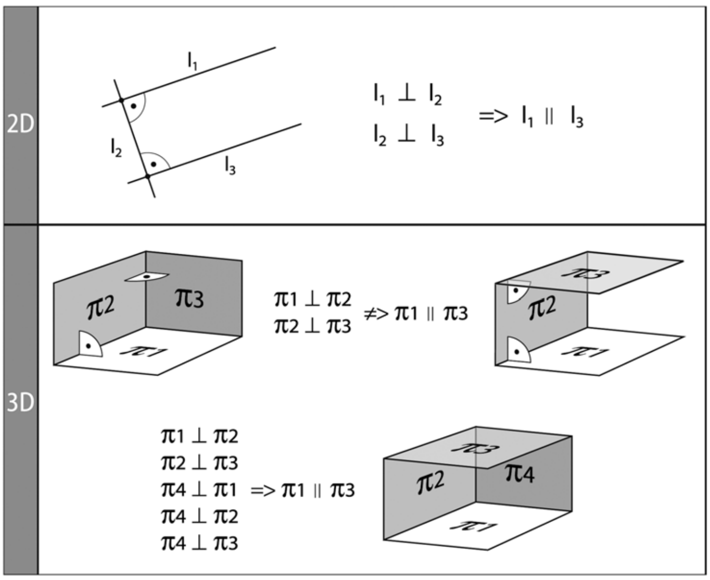

Fig. 5. Increasing complexity: deduction of parallelity in 2D and 3D.

Table 3
Orthogonality, parallelity and incidence with $S(x) = \partial x \wedge y/\partial y = \partial x \times y/\partial y$.

Bi-relational constraint	Algebraic representation	DOF (n)
plane $A \perp$ plane B	$A_h^T B_h = 0$	1
line $L \perp$ plane A	$S(L_h)A_h = L_h \times A_h = \mathbf{0}$	2
plane $A \parallel$ plane B	$S(A_h)B_h = A_h \times B_h = \mathbf{0}$	2
line $L \parallel$ plane A	$L_h^T A_h = 0$	1
point $X \in$ plane A	$X^T A = 0$	1

has not yet proven the fact and thus is worth testing for the algebraic reasoner. Finally, these hypotheses are passed to the algebraic reasoner.

4.2. Proving algebraic theorems

The algebraic component, which uses Wu's method, is a central element in our reasoner. It accesses the algebraic representation of the building model, more precisely a subset of objects and constraints that appear in the pre-selected hypotheses produced by the logical component.

As a first step it has to simplify the constraints in order to reduce complexity and formulates the theorem as it is required by Wu's method. Finally, it computes the characteristic set and checks the theorem by pseudodivision.

As depicted in Section 3.2 geometric entities and constraints can be expressed by multivariate polynomials. Nevertheless, geometric theorems in 3D space become more complex because the number of constraints and their variables as well as the subsidiary conditions increase. Since this influences the efficiency considerably, the geometric constraints of a building model have to be translated into an appropriate algebraic representation. This comprises the general choice of an algebraic equation for objects and constraints as well as further simplifications for specific constraint configuration. In the following we will show that building models can be expressed by multivariate polynomials. Furthermore, it will be demonstrated that simplifications of equations are possible in order to reduce complexity in the three dimensional space.

Euclidean geometry is the traditional way to represent geometric objects algebraically. For our purpose, projective geometry is a powerful formalism which allows the efficient representation of constraints and object constructions. Above all, it is able to integrate the aspect of uncertainty in a reliable way (see also Section 4.3).

Thus, we map all planes, lines and points into the projective space, that is, we analyse constraints between 3D points, lines and planes which are represented by homogeneous coordinates. While planes remain the same as in the Euclidean space, that is $A = (A_h; A_0) = (a, b, c; d)$, points are transformed to a 4-vector with $(x, y, z) \rightarrow (P_0; P_h) = (x, y, z; 1)$. A line is a 6-vector $L = (L_h; L_0) = (a, b, c; d, e, f)$ which contains its direction and the distance to the origin and can be obtained by a single matrix–vector-multiplication, e.g. as intersection of two planes or join of two points.

Using this algebraic representation the constraints can be expressed by simple constraint equations. Table 3 shows an excerpt of possible relations (see Heuel (2004) for comprehensive overviews of object constructions and relations). Although other representations are possible, the advantage of this algebraic representation (including the cross product and the scalar product) is that it does not contain any quadratic equations so far, but is bilinear instead. We further noticed that in the context of theorem proving even the cross product for expressing the parallelity is superior to an equation representing the linear dependency of normal vectors.

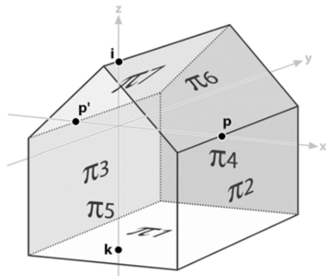

Fig. 6. Basic model of gable roof house.

Furthermore, we choose such algebraic representations that minimize the number of variables. Among others, this is reflected in the algebraic equation of a plane. Models are directly expressed by relations between planes so that the use of points is an exceptional case.

Apart from the basic relations mentioned we have two high-level constraints concerning the roof which cannot be expressed by a single equation. We assume the building to be represented by a cuboid and a prism (see Fig. 6). In order to ensure that the ridge is the top of the house and thus avoiding the roof being oriented downwards, we require that a point $i = (i_1, i_2, i_3)$ of the ridge line lies above the top of the cuboid. Although we cannot assume that planes are oriented, i.e. with normal vectors turned outwards, this property can be realized by incorporating oriented projective entities. Therefore the sign of the dot product between a plane A and a homogeneous point p is computed:

$$sign(\langle A, p \rangle) = \begin{cases} 1 \\ -1. \end{cases} \tag{4}$$

This provides information about the position of a point with regard to a plane. A value of 1 states that the point lies above the plane, otherwise it is located below. As a consequence another point $k = (k_1, k_2, k_3)$ is chosen in the bottom plane. The resulting constraints require that the two dot products $\langle T, k \rangle$ and $< T, i >$ with respect to the top plane of the cuboid T and the two points i and k must have different signs. Although Wu's method does not allow strict inequations, there are equivalent expressions which only use equations by introducing another variable w (Kapur and Mundy, 1988):

$$\begin{aligned} x < 0 &\Leftrightarrow xw^2 + 1 = 0 \\ x > 0 &\Leftrightarrow xw^2 - 1 = 0. \end{aligned} \tag{5}$$

The property of roof symmetry has to relate points p in the left roof plane to points p' in the right roof plane which are connected by a line that is orthogonal to the plane of symmetry (a_s, b_s, c_s, d_s):

$$\begin{aligned} (p'_1, p'_2, p'_3) = {} & (p_1, p_2, p_3)^T + 2((a_s, b_s, c_s)^T \\ & \times ((-1)(a_s, b_s, c_s)^T (p_1, p_2, p_3) - d_s)). \end{aligned}$$

Consequently, all geometric constraints can be expressed by multivariate polynomials.

The complexity of three-dimensional algebraic relations can further be reduced by a simplification of the polynomials. Obviously, our theorems are invariant to translation and rotation and are dominated by constraints of orthogonality and parallelity (cf. Fig. 1). Therefore, we make the plane that is related to most of the other objects without loss of generality parallel with the x-y-plane by setting its normal vector to (0, 0, 1). In turn, further values in the hypotheses are forced to be zero. In the case of the gable roof house, for example the normal vector of the bottom face is instantiated by real values and simplifies constraints by eliminating variables. Thus it enhances the calculation of the characteristic set and

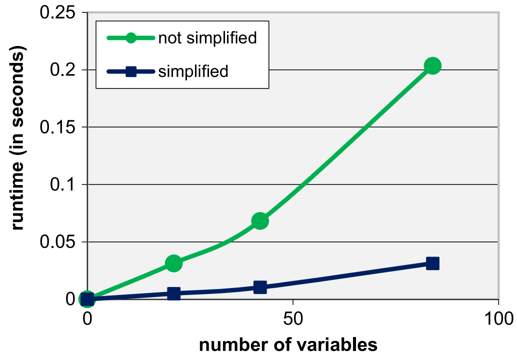

Fig. 7. Comparison of runtime for Wu's method: with and without simplification.

the formulation of subsidiary conditions. Fig. 7 shows the impact of exploiting structural properties of geometric objects. We evaluated our method on a changing number of building models and measured the running time of proving a theorem while extending the constraint set. In general, the complexity increases substantially with a growing number of variables.

Moreover, it is often possible to position the origin of the coordinate system in such a way that several variables are reduced to zero. This is especially possible if the process of building reconstruction involves one building at a time. In case of a gable roof house the origin lies in the top plane of the house block and contains the axis of symmetry for the roof. The resulting advantage is that it is possible to express the constraints of the roof by using four points with the special property that two of their coordinates equal zero. Consequently, the property of orientation reduces to $i_3w_1^2 - 1$ and $k_3w_2^2 + 1$ respectively. The symmetry is simply expressed by the incidence of three points $p = (p_1, 0, 0)$, $p' = (-p_1, 0, 0)$ and $i = (0, 0, i_3)$.

Besides, the efficiency of Wu's method depends on the choice and ordering of dependent variables. On one hand, they should support the construction of a triangular equation system. On the other, the polynomials in independent variables that appear in the denominators of the coefficients as well as the initials in the construction of the characteristic set do not have to equal zero. Consequently, they should be simplified or avoided.

We have observed that these simplifications do not only lead to a reduction of running time but also to interpretable subsidiary conditions. This is mainly due to the substantial reduction of the number of terms that occur in the constraints. In general, one single gable roof house contains at least seven planes with 28 variables. The corresponding constraint equations of the characteristic set are partly quadratic and have an average of four algebraic terms. Performance can be improved since simplifications reduce each equation of the theorems input to one to two terms without containing any computationally expensive quadratic equations.

4.3. *Evaluating uncertain conclusions*

So far, entities and their constraints were considered as certain. If we deal with the reconstruction of buildings our approach has to cope with data which do not yield crisp geometric relations. For example two estimated planes could share an angle of 89.9 instead of 90° which nonetheless should be considered to be orthogonal in a weak sense. However, the dot product for testing the orthogonality would not equal zero and thus the relation is rejected. The consideration of uncertain geometric objects also affects the reasoning process because the quality of conclusions has to be quantified in a sound way. Furthermore, proven conclusions are even in a weak sense not guaranteed to hold although the corresponding uncertain constraints in the hypotheses were satisfied. Hence, an additional test, that verifies to which extent the measured data supports the deduced conclusions, is indispensable.

In order to consider the aspect of uncertainty in geometric relations a common approach compares the result vector $\hat{c}$ of the relational operation to a chosen margin ϵ. Nevertheless, it is hard to choose an appropriate value for an error ribbon because it depends on the position of a specific geometric object and its confidence region. Alternatively, to cope with this type of uncertainty we use the concept of uncertain projective geometry which was proposed by Heuel (2004). It combines projective geometry with statistics and provides statistical tests to check geometric relations between uncertain geometric entities.

The idea behind this approach is to represent a geometric entity as a pair(x, Σ_{xx}) by adding a measure of accuracy to the algebraic representation, the covariance matrix Σ_{xx}. We assume that the uncertainty of observations is greater than the deviation between the mathematical model and its real implementation. Hence, we restrict attention to noisy observations. The error of measurement is propagated through all operations such as the transformation from Euclidean entities to projective entities or the construction of lines from two planes.

With a covariance matrix at hand, hypothesis tests of statistical testing theory can be integrated (Koch, 1999). According to Table 3 each concluded relation $R(x, y)$ can be expressed by a matrix–vector-multiplication. Thus, the null-hypothesis of a geometric relation corresponds to an algebraic constraint equation of the following form:

$$H_0 : \hat{c} = U(x)y = V(y)x = 0.$$

$U(x) = \partial\hat{c}/\partial y$ and $V(y) = \partial\hat{c}/\partial x$ are the Jacobians of the constraint equations with respect to y and x respectively and have to be determined for error propagation. In case of parallelity of two planes A and B, for example, where $x = A_h$ and $y = B_h$, $U(A_h)$ equals the Jacobian $S(A_h) = \partial(A_h \times B_h)/\partial B_h$, $V(B_h)$ the negated Jacobian $-S(B_h)$. Consequently, $\hat{c}$ is the result of the cross product with regard to planes A and B.

Finally, according to Heuel's statistical approach a chi-square distributed test statistics $\hat{c}^T\Sigma_{\hat{c}\hat{c}}^{-1}\hat{c}$ is computed where the covariance matrix of $\hat{c}$ can be obtained by first order error propagation $\Sigma_{\hat{c}\hat{c}} = U(x)\Sigma_{yy}U^T(x) + V(y)\Sigma_{xx}V^T(y)$. This statistical framework enables the automatic reasoning about uncertain constraints by quantifying the quality of the conclusion with its covariance matrix and a statistical value.

Given a significance level $(1-\alpha)$, the condition for Wu's method extends to

$$prem(\hat{c}, H) = 0 \wedge I(h_i) \neq 0 \wedge \hat{c}^T\Sigma_{\hat{c}\hat{c}}^{-1}\hat{c} < \epsilon_H := \chi^2_{1-\alpha;n}.$$

The logical deduced facts hold true if beside the geometric predicates an additional predicate is satisfied which confirms the relation in question on basis of the statistical hypothesis test.

5. Demonstrating feasibility: applications of automated deduction

This section presents possible applications of our reasoning system in the context of building modelling. We describe how a redundance-free and consistent representation can be obtained and how valid relations and predictions are found.

5.1. *Finding a redundance-free and consistent representation*

While describing a building model one major task is to obtain a redundance-free and consistent representation. On one hand, a compact representation leads to less disk space and a readable model as well as simplifying a model for further processing on the other hand. The latter will lead to less combinatorial complexity by ruling out redundant constraints as it is beneficial e.g. for learning structural properties as proposed by Dehbi and Plümer (2011). Thus, our aim is to reduce a constraint set automatically.

In Section 4, we presented a threefold internal model representation. Fig. 8 shows the graphical user interface for developing an underlying building model. It allows an incremental modelling of prototypes by creating 3D objects and constraints between these. If existing models such as estimations from point clouds are available, they can be loaded instead via the scripted modus. Iteratively added constraints may be discarded if they are redundant. A constraint is redundant if it is deducible from a constraint set so far declared as non-redundant. This leads to a theorem of hypothesis and conclusion: non-redundant constraint set $\Rightarrow$ new constraint.

First, the logical reasoner determines whether the conclusion is deducible by inference rules given the non-redundant part as initial facts. Being a query in terms of deductive reasoners it is an efficient procedure based on fixpoint iteration. As described in the last section the algebraic reasoner analyses hypotheses generated by inference rules. The algebraic reasoner becomes indispensable, if dealing with relations based on arithmetics. In case of the gable roof house, Wu's method is e.g. able to show that the roof symmetry implies the fact `orthogonal(rightRoofHalf, front)` if `orthogonal(leftRoofHalf, front)` holds. The proof is based on dependencies of parameters which is the strength of algebraic approaches.

Finally, referring back to Table 1 we obtain a non-redundant constraint set that still defines a gable roof house and is sufficient to ensure eight remaining geometric constraints. We preferred to retain constraints of parallelity and, as a consequence, the following final result is proven:

```
Redundant constraints:
orthogonal(rightRoofHalf, front),
orthogonal(leftRoofHalf, front),
orthogonal(bottom, leftWall),
orthogonal(rightWall, back),
orthogonal(leftWall, front),
orthogonal(rightWall, front),
orthogonal(bottom, front),
orthogonal(rightRoofHalf, front)
```

We further evaluated our reasoner by exploiting structural characteristics of whole groups of man-made objects. Therefore, a constraint set was reduced that represents several representative building models with roofs containing dormers: we processed 150 (uncertain) geometric constraints within an estimation of 32 planes found in a point cloud of 10 interrelated gable roof houses. Checking redundancy of a constraint took on average 0.29 s and at the longest 0.71 s (Intel®-Core™2-Duo @ 3.00 GHz). In general, not all houses will be correlated so that even a decomposition of the constraint set into smaller ones is possible.

In the case of finding a redundance-free representation we do not have to enable the statistical component although applying the process to observed data. As we only remove constraints from the initial constraint set it is ensured that these are already satisfied at the beginning.

5.2. Deducing new relations

Since the overall task is the reconstruction of buildings, we are not only interested in a redundancy-free representation. Instead, it is of great benefit to know which relations can be deduced from a given constraint set in order to enrich our knowledge of the model. The larger the building models become the more complex it is for humans to survey the implicit relations. Hence, the process of building reconstruction is supported by deduction as constructed objects and implicit relations can be used to verify estimated models.

Another need of finding relations relates to the development of a building description. If the user has to model a building it is much easier to use a predefined complex object that can be composed of a whole building or a building part. As an example, two parts of a gable house roof, a cuboid and a prism should build a complete house. By composing the two parts one surface vanishes in the inside of the house and thus has to be eliminated. Consequently, the constraints that were related to this object are also eliminated and it has to be guaranteed that the constraints that were deducible beforehand are added to the new constraint set. Apart from that, the user is able to control modelled prototypes since the tool helps to detect missing constraints or those that are not intended. Thus, deducing new relations facilitates the interactive modelling of complex buildings.

Our reasoner deduces relations from a given set of objects and constraints identified in real observations or from a predefined prototype of a building or a building part. In general, constraints are passed from the logical reasoner to the algebraic component and are finally processed by the statistical component. In order to find new relations the logical reasoner applies the rules to the input constraint set to reach a fixpoint. New objects can be constructed from the given model (e.g. a line as intersection of two planes), which in turn inherit new relations. Rules of type 2 are applied to initial and already deduced facts and forwarded to the algebraic reasoner. We use Wu's method to check whether further constraints of the predefined constraint types follow from a given constraint set. For each constraint a theorem is formulated and it is verified by pseudodivision whether the theorem holds. All results can be traced back to their geometric meaning since the threefold representation is never decoupled.

Since the strength of Wu's method lies in the capability of proving a theorem by computing polynomial consequences of the hypotheses, it is especially suitable for reasoning about new objects that are constructed from already known primitives. An example of conclusions our reasoner draws involves e.g. the ridge of a gable roof house whose representing line is the intersection of the planes $rR = (rR_h; rR_0)$ and $lR = (lR_h; lR_0)$ of the two roof halves:

$$ridge \equiv \begin{pmatrix} L_h \\ L_0 \end{pmatrix} = \begin{pmatrix} rR_h \times lR_h \\ rR_0 \cdot lR_h - lR_0 \cdot rR_h \end{pmatrix}.$$

The algebraic reasoner deduces that apart from the constraints between the planes of a gable roof house the following constraints hold:

```
orthogonal(ridge, front),
orthogonal(ridge, back),
parallel(ridge, leftWall),
parallel(ridge, rightWall),
parallel(ridge, bottom)
```

These correlations between the ridge and other primitives show that it is a major element in a building and contains considerable information about the corresponding geometric model. One can infer that the identification of a ridge in observations may substantially contribute to the reconstruction of buildings because its constraints determine various variables by polynomial elimination of the constraint equations (cf. Table 3). Vice versa, the conclusion can be used to verify the process of reconstruction. Deduced predictions including relational properties are beneficial to check geometric hypotheses from estimations in noisy observations. Hereby, they improve the correctness of 3D building models.

Likewise, reasoning in 3D building models becomes possible on a higher level of detail. For example, two dormers on a roof half are commonly aligned, that is they inhibit the property of translation symmetry along the direction of the ridge. For points p and p' in its

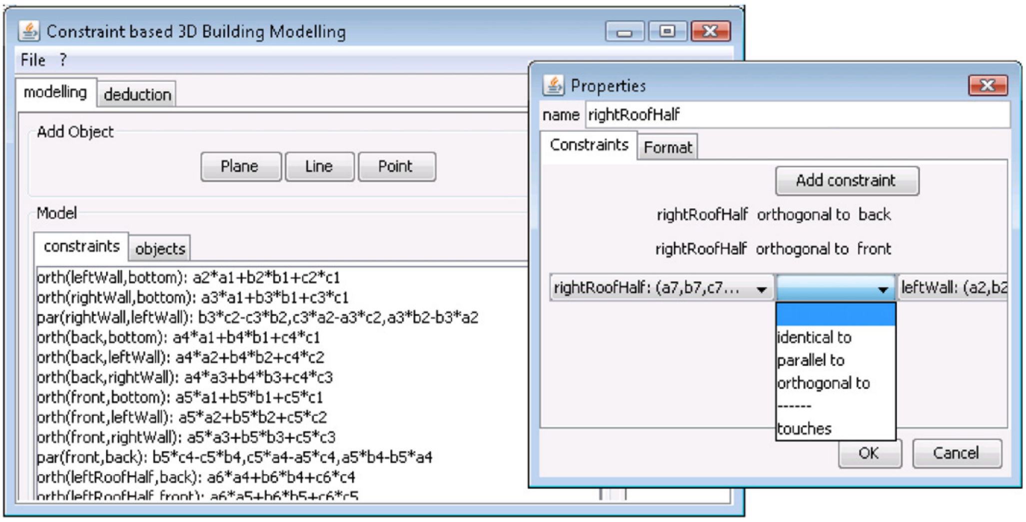

Fig. 8. Graphical user interface for developing a redundance-free and consistent building model.

components we obtain the relation: $p' = p + \lambda \cdot L_h$ which leads to statements about the mutual position of the dormers' ridge. Such prior knowledge enables the reasoner to show that they must have the same height and that their ridges are parallel with each other.

Finally, the rule-based reasoner can in turn benefit from this proof in that theorems can be used as inference rules for future fixpoint iterations.

If the input stems from noisy observations proven conclusions are continuously checked by statistical tests before they are added to the final result. The test value quantifies the data-dependent quality of the conclusion. The following concrete example with algebraic constraint equations $\hat{c}_i, i = 1, 2, 3$, depicts the consequences with regard to the χ^2-distributed test statistics (cf. Section 4.3):

$$\hat{c}_1 \equiv \mathit{leftRoofHalf} \perp \mathit{front}:$$
$$\hat{c}_1^T \Sigma_{\hat{c}_1\hat{c}_1}^{-1} \hat{c}_1 = 2.52 < \chi^2_{0.95,1} = 3.748$$
$$\hat{c}_2 \equiv \mathit{rightRoofHalf} \perp \mathit{front}:$$
$$\hat{c}_2^T \Sigma_{\hat{c}_2\hat{c}_2}^{-1} \hat{c}_2 = 3.19 < \chi^2_{0.95,1} = 3.748$$
$$\not\Longrightarrow$$
$$\hat{c}_3 \equiv \mathit{ridge} \perp \mathit{front}:$$
$$\hat{c}_3^T \Sigma_{\hat{c}_3\hat{c}_3}^{-1} \hat{c}_3 = 6.35 > \chi^2_{0.95,2} = 5.938.$$

This shows that with regard to this exemplified set of noisy data the validation of a model should not depend on this unreliable conclusion that was deduced correctly under exact conditions. In general, by integrating uncertain projective geometry we provide an indicator that scores the degree of uncertainty of our conclusion.

6. Conclusion

This paper presented a reasoning system that supports users in developing and analysing building models. It supports the development of redundance-free and consistent prototypes as well as the reasoning about estimated objects based on given observations. We have shown how methods of automatic theorem proving can successfully be applied to reduce constraint sets, predict related objects and find implicit relations in building models. Therefore, we combined the strength of algebraic methods with the advantages of deductive reasoners and demonstrated the feasibility for 3D building modelling. We have shown that geometric constraints of 3D buildings can be expressed by multivariate polynomials and that uncertainty can be integrated in a sound manner.

Our research was inspired by the work of Brenner and Sester (2005), who made use of Groebner Bases for 2D reasoning in the context of cartographic generalization and emphasize the complexity of the problem. Compared to 2D space, the complexity of theorems in 3D increases considerably. We developed a feasible approach which meets the requirements of interactive reasoning systems. We address building models and spatial reasoning by applying Wu's method in combination with deductive inference rules.

In order to cope with the increased complexity of the algebraic reasoner, the choice of an appropriate algebraic representation turned out to be crucial. While Roanes-Macías and Roanes-Lozano (2007) express theorems by point-on-object relations, we have chosen a different representation and avoided the explicit reference to point coordinates wherever possible. By using insights of projective geometry we were further able to construct polynomials that are multilinear rather than quadratic. We extended the reasoning with the construction of new geometric objects which in turn inherit new properties. Orthogonality, parallelity and symmetry are the main organizing principles in buildings. We focused on linear objects such as planes, lines and points. We made use of invariance with respect to rotation and translation and exploited structural properties to reduce the complexity. Our reasoner provides the construction of geometric primitives as well as predefined relations between them, especially orthogonality, parallelity, symmetry, incidence and touch. However, further investigations about primitives of higher order manifolds such as spheres which imply a higher complexity of theorems have still to be done.

In order to cope with real observations we integrated the theory of uncertain projective geometry and used a chi-square-distributed test statistics to obtain valid predictions in the context of noisy observations. Therefore, covariance matrices were added to the model representation and propagated throughout all operations. Deduced (strict) relations could thus be verified with regard to their objects' confidence regions so that we were able to quantify the quality of proven theorems. In contrast to the work of Heuel (2004) hypothesis tests are restricted to valid conclusions that were already deduced on a symbolic level and are finally valued considering the uncertainty of existing measurements.

We implicitly assumed that the uncertainty of observations is greater than the deviation between the geometrical model and its real counterpart. For this reason, we restricted attention to noisy observations and did not take the the mathematical model and its real implementation into account. However, independent from the impreciseness of measurement geometric objects such as walls not meeting the crisp constraints designed by architects. In order to represent both kinds of uncertainty, our approach can be extended by incorporating corresponding covariance matrices into the statistical tests. Therefore, further empirical evidence on the uncertainty of construction would be necessary.

We enhanced the process of algebraic reasoning by a rule-based reasoner which provides efficient techniques to deduce relational facts and reduces the complexity of the reasoner by ruling out impossible hypotheses beforehand. Thus, the reasoning is not only restricted to geometric constraints but allows also for drawing conclusions within other domains, such as topology and partonomy.

This article has a strong relation to the approach presented by Dehbi and Plümer (2011) whose goal is to learn semantic models of buildings and building parts from noisy observations. They use uncertain projective geometry to obtain basic relations of estimated objects while we use this framework to quantify the quality of drawn conclusions. They differentiate between high and low level learning. The latter yields a model of uncertain objects and geometric constraints that is the basis of further learning processes but has not to be redundance-free. In order to reduce the combinatorial complexity of correlations involving the concept to be learned, our geometric reasoner can be used to minimize this constraint set. Logic programming, the fundament of their approach, provides deductive mechanisms which are also applied within our approach. It should be investigated how these semantic rules which incorporate hierarchical, structural and geometric properties of building parts may enhance the deduction of predictions in our context.

The main contribution of this paper is to provide a method that handles automatic reasoning in three-dimensional constraint-based (building) models and includes the derivation of constraints from uncertain premises. We have implemented a prototype that checks the model with respect to redundancy and consistency and which is able to deduce new properties. The reasoning process works both on user-defined crisp constraints and uncertain constraints based on observations such as 3D point clouds. It is able to perform exact as well as uncertain reasoning. Although being specifically geared to the context of 3D building models, objects and constraints are not restricted to buildings but allow for a development of arbitrary geometric objects.

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B.2 Predicting building façade structures with multilinear Gaussian graphical models based on few observations

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This paper presents a new approach for the prediction of substructures in building façades based on sparse observations. We automatically generate a small number of most likely hypotheses and provide probabilities for each of them. Probability density functions of model parameters which in most cases are non Gaussian and multimodal are learned from training data and approximated by Gaussian mixtures. Relations between model parameters are represented by non-linear constraints. For stochastic reasoning we design and apply a special kind of Bayesian networks which involves both discrete as well as continuous variables, a scenario which often suggests the use of approximate inference which however is infeasible in the face of a huge number of competing model hypotheses. In order to be able to scan huge model spaces avoiding the pitfalls of approximate reasoning and to exploit the potential of both observations and models, we combined Bayesian networks with constraint logic programs. We designed a method which breaks down the problem into a feasible number of subproblems for which exact inference can be applied. We illustrate our approach with building façades and demonstrate that particularly for buildings with strong symmetries number and position of windows can be deduced on the basis of ground plans alone.

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Predicting building façade structures with multilinear Gaussian graphical models based on few observations

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ABSTRACT

This paper presents a new approach for the prediction of substructures in building façades based on sparse observations. We automatically generate a small number of most likely hypotheses and provide probabilities for each of them. Probability density functions of model parameters which in most cases are non Gaussian and multimodal are learned from training data and approximated by Gaussian mixtures. Relations between model parameters are represented by non-linear constraints. For stochastic reasoning we design and apply a special kind of Bayesian networks which involves both discrete as well as continuous variables, a scenario which often suggests the use of approximate inference which however is infeasible in the face of a huge number of competing model hypotheses. In order to be able to scan huge model spaces avoiding the pitfalls of approximate reasoning and to exploit the potential of both observations and models, we combined Bayesian networks with constraint logic programs. We designed a method which breaks down the problem into a feasible number of subproblems for which exact inference can be applied. We illustrate our approach with building façades and demonstrate that particularly for buildings with strong symmetries number and position of windows can be deduced on the basis of ground plans alone.

1. Motivation and context

Nowadays 3D building models are widely available but they mostly do not contain detailed structure and semantical information. Facades where details are only provided by texture mappings are often not sufficient for many applications. Semantics that can be encoded by the standardized data model CityGML (Gröger, Kolbe, Czerwinski, & Nagel, 2008) are important for many scenarios such as rescue scenarios or the calculation of energy balances. Detailed information about doors or windows in an apriori unknown building might be crucial for rescue teams to accelerate assistance.

Haala and Kada (2010) emphasize the importance of detailed building reconstruction and give an overview of approaches in the context of automatic city modeling. However, building models that represent building parts such as windows or doors explicitly are rare and up to now modeled manually or semi-automatically in most cases. Moreover, an automatic reconstruction in general relies on high-resolution measurements such as 3D point clouds from laser scans or features extracted from images. This requirement is often not able to be satisfied in an appropriate way so that we have to cope with an apriori small number of observations instead. While ground plans are already available by the use of data sources such as official data or Open Street Map the acquisition of 3D point clouds is far costlier.

As a consequence, our central motivation is to predict unknown substructures in buildings based only on few observations. While most approaches expect observations of adequate density, characteristic for our approach is that we are able to generate best hypotheses for a building model based on otherwise insufficient measurements, in particular ground plans. Additional data may lead to a verification or falsification of models which however is less expensive than reconstructing a building bottom-up from measurements. Fig. 1 illustrates our approach for predicting a row of windows in a complex façade of the Poppelsdorf Castle in Bonn, Germany (see Fig. 1a). The hypothesis about the windows in Fig. 1c is the result of the reasoning process that incorporated the ground plan of the building (red line) as well as measurements of single embrasures (dotted lines in Fig. 1b). Full observations of all windows are not necessary to generate a hypothesis of this quality. The width of façades is for example correlated to the number of windows, the width of windows and the distance between windows and together

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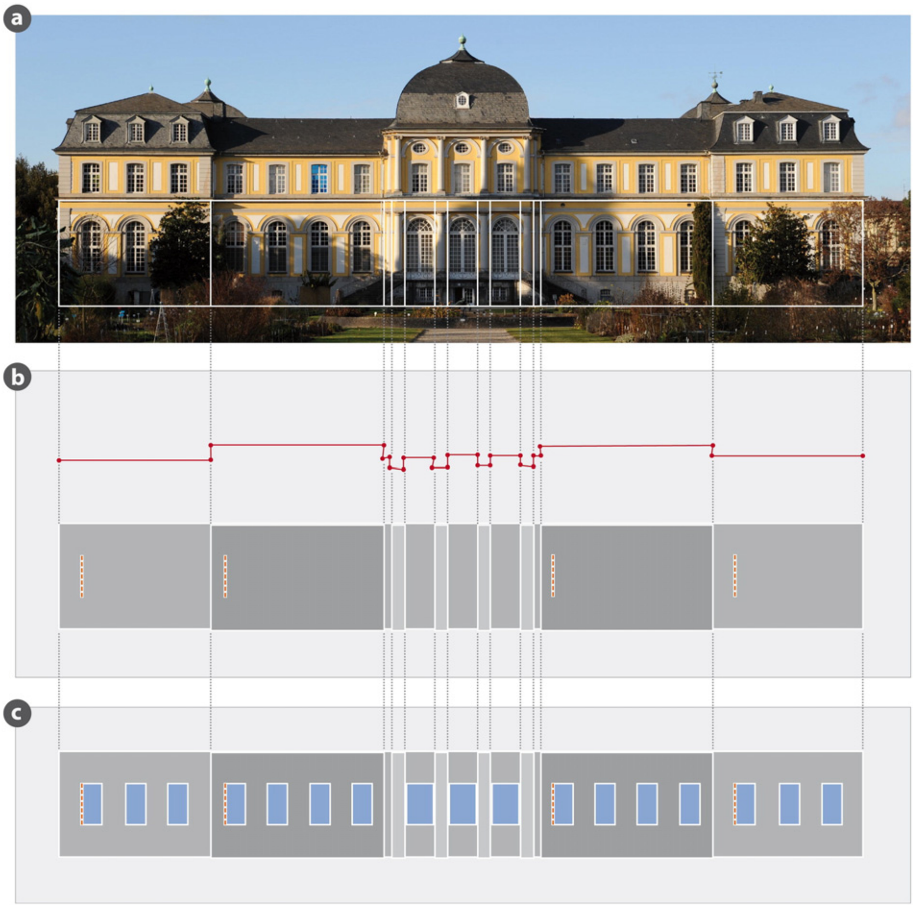


Fig. 1. Reasoning process for predicting a complex façade: (a) reference image: façade of Poppelsdorf Castle, (b) input: ground plan (solid line) and measurements of embrasures (dotted lines) as observations, and (c) output: resulting hypothesis.

with a model for these building parameters, especially probability distributions, the space of hypotheses becomes strongly constrained. The use of Gaussian mixture models, constraint solvers and stochastic models help to cope with the apriori infinite space of a hybrid building model.

Generating good hypotheses becomes possible due to the regularities that can be found in buildings. Poppelsdorf castle as illustrated in Fig. 1 is an example of a cultural heritage buildings which are characterized by symmetries and parallel or orthogonal structures. What is obvious and especially pronounced in façades of cultural heritage can be observed as well in modern buildings. Model parameters such as width of windows or height of floors follow certain architectural restrictions. In turn, the number of windows that can be placed within one floor is restricted by the width of the façade. A typical constraint characterizing this relation has the form $w_f = d_l + d_r + n_w * w_w + (n-1) * d_w$ where w_f denotes the width of the façade, d_l and d_r the distances to the left respectively right side of the façade, w_w the width of the windows and d_w the distance between windows and n_w the number of windows. Thus we get a bilinear formula with continuous parameters (w_f,d_l,d_r, w_w,d_w) and one discrete parameter (n_w) and products formed from a continuous and discrete factor. Beyond that, the values of the model parameters have characteristic distributions that can be learned from examples. Fig. 2 illustrates these distributions. Note that none of these distributions is Gaussian. Instead, both are multi-modal. We used a kernel density estimation (Bowman & Azzalini, 1997) that can, however, be approximated rather neatly by Gaussian mixtures with few components and small variances. It will turn out that this is an essential prerequisite for the possibility of using efficient inference algorithms. All in all, this prior knowledge together with a powerful reasoning algorithm allows to generate good hypotheses of buildings.

To generate the best hypotheses we make use of Bayesian networks as a special kind of probabilistic graphical models. Bayesian networks are the directed variant of graphical models and have been established to be powerful tools for reasoning with uncertain data. The domain model for buildings can be represented as a hybrid Bayesian model with discrete as well as continuous parameters. It is further characterized by multilinear equations with apriori unknown discrete variables and mixtures of Gaussians that make inference unfeasible. While there exist efficient inference algorithms for discrete networks, inference in hybrid networks remains to be a challenging task. Koller and Friedman (2009) pointed out that the resulting number of mixture components is exponential in the number of unassigned discrete variables in the worst case. Lauritzen and Jensen (2001) developed an efficient algorithm which is able to tackle the problem of exact inference in restricted hybrid networks. It provides a solution whose distributions are correct for discrete variables. For continuous variables first and second moments of the posterior distribution are correct while the true distribution might be a Gaussian mixture. It is sufficient for many applications since it is often the discrete variables that are queried or the resulting Gaussian distribution is close to the original Gaussian mixture. In contrast, the joint distributions of the continuous model parameters for façade prediction are multimodal and so are the marginal distributions. As a consequence, an approximate inference algorithm as proposed by Lauritzen and Jensen (2001) would prevent the computation

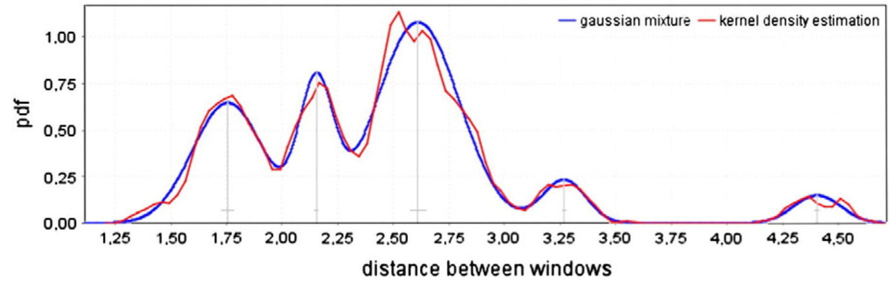


Fig. 2. Gaussian mixture (blue) and kernel density estimation (red) for distance between two neighboring windows, mixture model is good approximation with five means (gray vertical lines) and extremely small variances (gray horizontal lines).

of appropriate results since collapsing a Gaussian mixture means determining a single (probably worthless) Gaussian distribution using M-projection (Koller & Friedman, 2009).

Fig. 3 illustrates the problem in the context of buildings. It shows the Gaussian mixture for the distance between windows (blue). Applying the M-projection to this distribution yields a mean that is not valid in that it is unlikely having a building with this property. Instead, the most likely assignments to all model parameters, also known as MAP inference (Koller & Friedman, 2009), has to be found. While the common MAP task is to find a single solution that maximizes the probabilities of all query variables this work aims to provide a ranked set of most likely assignments.

To sum up, in the hybrid building model there are two main problems: the number of discrete states is unknown and the discrete states are never observed and remain uninstantiated. This leads to a hybrid model with discrete and continuous variables and – even worse – non-linear dependencies. In order to reason with uncertain data in an efficient and adequate way while profiting from the strength of Bayesian networks, we propose a new approach which handles the complex inference task in hybrid networks and still remains suitable for our application of building reconstruction. To this end, we combine Bayesian networks with constraint logic programming (CLP). The bridge between the two worlds CLP and Bayesian networks are on the one hand the discrete integer variables and on the other hand the individual components of the Gaussian mixture distributions. Gaussian distributions, the single components of the Gaussian mixture, are used to define 3 sigma or 4 sigma intervals on real numbers which serve as domains in the CLP algorithm. As will be demonstrated in Section 4 this allows to derive bounds for the hitherto unbound integral variables. While the constraint program linearizes the problem and instantiates the discrete parameters, the Bayesian network calculates the posterior given the observation in a compact way. For reasons of efficiency and convenience, we exploit the special structure of the resulting statistical problem and calculate posterior beliefs with matrix multiplications based on the concept of the Kalman Filter. The result is a multivariate Gaussian – not a mixture of Gaussians or its collapsed version. Our algorithm outputs the best instantiations for the model parameters given the observations and ranks the resulting hypotheses by their probabilities. We provide a whole set of (ranked) hypotheses instead of a single, most probable solution since the latter not always meets the real world configuration. Selecting the appropriate hypothesis in a second step opens the way to find nonetheless the correct interpretations.

Our main contributions are

- generation of few probable hypotheses for building façades based on sparse in general unsufficient observations
- description of the functional model by multilinear relations of mixed integral and continuous parameters
- description of the stochastic model by Gaussian mixtures and the extension of Conditional Linear Gaussian Models to Multilinear Gaussian Models
- development of an algorithm that efficiently restricts the solution space by solving the discrete problem using constraint logic programming and estimating the optimal continuous parameters using a Kalman filter.

The remainder of this paper is structured as follows: Section 3 illustrates the way buildings can be modeled by mixed graphical models. Our CLP approach for solving the discrete problem that occurs in this context is presented in Section 4. Section 5 describes the determination of continuous parameters. Section 6 presents the algorithm of our approach in detail and expounds the interaction of the combinatorial and the statistical part. It generalizes the presented approach and introduces a framework for predicting substructures for well-defined problems. Finally, Section 7 shows the results we achieve for generating a few best hypotheses.

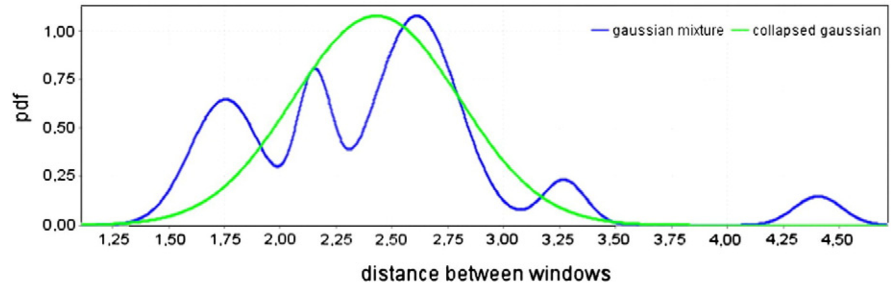


Fig. 3. Collapsing a Gaussian mixture (blue) yields a Gaussian distribution (green) with $\mu = 2.43$ and $\sigma^2 = 0.37$. Although it is an efficient approach for many application it is not appropriate for this case of multimodal distributions for predictions in buildings.

2. Related work

In the context of building reconstruction various approaches have been published that use a model-based top-down approach to detect symmetry or regular structures and to generate building models. Wenzel, Drauschke, and Förstner (2008) proposed an approach that detects repetitive structures in facade images in order to provide a mid-level feature for model-based learning. They therefore use heuristic search methods and the criterion of minimum description length and thus derive compact descriptions of facades. Alahmadi, Atkinson, and Martin (2013) exploit the regularity in facades and use functional dependencies for the building height to determine the number of floors in the context of population estimates. Ripperda and Brenner (2009) show the importance of the use of structural information for the reconstruction of building facades and present an approach that incorporates facade grammars in its reconstruction process. Schmittwilken and Plümer (2010) reconstruct and classify facade parts in 3d point clouds where the key feature of their approach is the usage of prior knowledge. It is incorporated in a step-wise classification that is composed of presampling, selection of the most likely sample and the estimation of boundaries. Becker (2011) presents an automatic approach for the geometric modeling of 3d buildings that uses grammar rules and combines a bottom-up and top-down knowledge propagation. Weissenberg, Riemenschneider, Prasad, and Gool (2013) present an automatic method that infers grammar rules from annotated facades. They show the benefit of these models for compression, facade comparison and synthesis of new virtual facade. The approach of Martinovic, Mathias, Weissenberg, and Gool (2012) uses three layers to semantically segment building facades. Beside the use of recursive neural networks and Markov random fields they introduce weak architectural knowledge to finally optimize the reconstruction. In Pu and Vosselman (2009) knowledge is incorporated to extract building parts such as walls, roofs or windows from high density point clouds or to make assumptions about occluded facade parts. Müller, Zeng, Wonka, and Van Gool (2007) use shape grammars for an image-based procedural modeling of facades. They try to match architectural elements with 3D elements in a library to generate high-quality geometric information. Drauschke and Förstner (2008) and Ali, Seifert, Jindal, Paletta, and Paar (2007) combine template matching with machine learning to find building parts such as windows in facades. Although all these model-based approaches yielded satisfying results they rely in contrast to this work on the existence of sufficient data and are unable to cope with very few observations.

Several approaches showed the benefit of employing machine learning to tasks related to architectural style. Hanna (2007) having the grouping of building block plans in different architectural styles in mind proposes an approach for classification based on axial graph spectra. Henn, Römer, Gröger, and Plümer (2012) predicts the type of a building by incorporating its spatial context using support vector machines.

Graphical models as used in our approach are well established in computer vision and 3D modeling. Yang and Förstner (2011) use conditional random fields (CRFs) as a special type of graphical models in combination with randomized decision forest classifiers to classify regions in images of building facades. Here, CRFs model the dependencies between neighboring regions and consider the differences in color to improve classification results. CRFs were introduced by Lafferty, McCallum, and Pereira (2001) for the segmentation and labeling of sequence data and adapted by Kumar and Hebert (2003) to natural image classification. Scholze, Moons, and Gool (2002) present a probabilistic approach for model-based reconstruction of building roofs. They extract 3D line segments from high resolution images and group these into planes by applying Bayesian networks. Batra, Yadollahpour, Guzmn-Rivera, and Shakhnarovich (2012) propose an approach for solving the Diverse M-best problem in Markov Random Fields. They constrain their work to discrete models but emphasize the need for diversity by using a dissimilarity function and generalizing the M-Best MAP problem.

Although originally not developed for domains with uncertain, inaccurate observations constraint programming was extended by several approaches to incorporate a stochastic component. Saad, Gervet, and Abdennadher (2010) propose a constraint programming algorithm using intervals with cumulative distribution functions (CDFs). They extended the formalism of interval bounds by CDFs that enable to represent a degree of knowledge for uncertain data. Flerova and Dechter (2010) solve the problem to find the m best solutions for optimization tasks in graphical models. To this end, combination and marginalization operators are adapted in order to generate a sorted list of solutions in tree decompositions. However, our approach relies on the classical CLP algorithm and thus exploits its strength in solving combinatorial problems with non-linear constraint equations, while Bayesian networks are used to reason with uncertain data.

3. Statistical reasoning in building models

As illustrated in Section 1 man-made objects such as buildings are characterized by a number of regularities. On the one hand geometric relations such as parallelity and orthogonality are dominant in buildings. Loch-Dehbi and Plümer (2011) studied the geometric rules that can be found in man-made objects and presented an approach for deducing geometric relations in 3D building models. On the other hand, buildings can be described by functional and statistical dependencies between model parameters. In this paper we focus on the latter properties of buildings. The knowledge of architectural design as well as available distributions about model parameters enable to generate good hypotheses in order to reconstruct buildings. The domain model of buildings includes probabilities for discrete parameters and probability density functions for continuous parameters. Due to the existence of relations between model parameters and the possibility to quantify them by conditional probabilities stochastic networks are an appropriate tool for reasoning within the uncertain world of building models.

Probabilistic graphical models are nowadays one of the most prominent and most powerful methods for reasoning in uncertain domains. One class of models are the Bayesian networks that are directed graphs whose nodes $v \in V$ represent random variables X_v and whose edges denote dependencies (Koller & Friedman, 2009). Each random variable is associated with a conditional probability distribution (CPD) $P(X_v|X_{pa(v)})$ that describes its status dependent on the values of its parent nodes $pa(v)$, that is its immediate predecessors as induced by the graph. The graph structure encodes complex distributions compactly and thus allows for efficient inference algorithms. Bayesian networks represent a joint distribution in a compact way and enable to answer questions about the posterior distribution given an observation.

Since building modeling is determined by discrete as well as continuous variables, hybrid networks are required that contain nodes associated with discrete as well as continuous variables $\mathcal{X} = \mathcal{X}_\Delta \cup \mathcal{X}_\Gamma$. In contrast to approximate reasoning exact inference within hybrid networks is in general not feasible. In order to reduce complexity for exact reasoning with continuous variables distributions of the building model are approximated by Gaussian distributions with mean μ and variance σ^2 so that a random variable X is characterized by

$$p(x;\mu,\sigma^2) = N(\mu,\sigma^2) = \frac{1}{(2\pi\sigma^2)^{1/2}} exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)$$

A special case of Bayesian network is the conditional linear Gaussian (CLG) Network (Lauritzen, 1992; Lauritzen & Jensen, 2001). The most important assumption here is that discrete nodes are not allowed to have continuous parents and that all continuous variables can be described by conditional linear Gaussian CPDs. A conditional linear Gaussian CPD with $I \subseteq \mathcal{X}_\Delta$ and $Z \subseteq \mathcal{X}_\Gamma$ is then defined as

$$p(X|Z = z, I = \tau) = N(\mu_\tau + \beta_\tau^T z, \sigma_\tau)$$

where μ_τ is a mean value for instantiation τ, β_τ a vector of regression coefficients and σ_τ the corresponding variance.

The joint distribution in a hybrid network can thus be defined as a $|X_\Gamma|$-dimensional Gaussian distribution

$$p(X_\Delta = \tau) \cdot N_{|\mathcal{X}_\Gamma|}(\mu_\tau, \sigma_\tau^2) = \prod_{v \in V_\Delta} P(\tau_v|\tau_{pa(v)}) \prod_{w \in V_\Gamma} p(y_w|X_{pa(w)})$$

for each instantiation τ of $\mathcal{X}_\Delta$ (Kjærulff & Madsen, 2008).

The most likely assignment (MAP assignment) for given evidence $E = e$ is found by maximizing the posterior probability for variables $W = \mathcal{X} - E$: $MAP(W|e) = \text{argmax}_\omega P(\omega, e)$. The presented work aims to find the k most probable explanations, denoted by $MAP^k(W|e)$.

The continuous parameters of a building model can be described by Gaussian mixtures of m components each weighted by its probability ω_i:

$$\sum_{i=1}^{m} \omega_i N(\mu_i, \sigma_i^2) \quad (1)$$

It has been shown that each arbitrary distribution can be approximated by a Gaussian mixture (McLachlan & Peel, 2000). Fig. 2 shows the probability density function of one of the building parameters – the distance between windows – estimated by the use of kernel densities – compared with its fitted Gaussian mixture. Gaussian mixtures are an appropriate way to model skew symmetric or multimodal distributions and make it possible to rely on a number of well-studied inference algorithms that are available in literature. Additionally, the estimated Gaussian mixtures for the building parameters reveal that distributions of continuous parameters are strongly peaked and mixture components only have small variances. This helps to structure the space of hypotheses.

Although Bayesian networks expect probability density functions to be Gaussian, Gaussian mixtures can be modeled by adding additional discrete nodes. Fig. 4 shows exemplarily the realization of a Gaussian mixture with two components. Each component of the Gaussian mixture is represented by a state of a discrete variable whose node is a parent of the continuous node. The probability density function of the continuous variables is determined by its means μ_i and variances σ_i^2 and the probability ω_i for each component.

Fig. 5 shows an excerpt of a hybrid domain model that serves as prototype for modeling a single building facade. Discrete nodes are represented by simple ovals while a double line expresses that it is a continuous one. Variables that are related to the nodes of the network can be divided into two groups: model parameters and observation parameters.

For each observation o_i there exist nonlinear functional dependencies for continuous model parameters

$$X = (d_l, d_r, w_w, d_w).$$

For the width of a facade (w_f) we assume for example that it is a result of a multilinear equation of the number of windows (n_w), the distance from the left facade margin to the first window (d_l), the distance from the right facade margin to the last window (d_r), the width of the window (w_w) and the distance between neighboring windows (d_w):

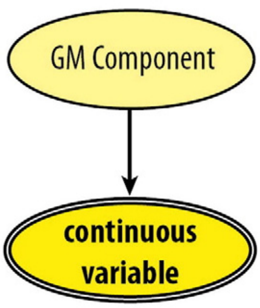
$$o_1 = w_f = d_l + d_r + n_w * w_w + (n_w - 1) * d_w$$

An observed part of an embrasure is additionally characterized by its correspondence to one of the windows c_w ($\leqslant n$) and the correspondence to the left or right embrasure c_e – marked as 0 for the left side and 1 for the right side of the window. The dependencies are determined by

$$o_2 = d_e = d_l + (c_w - 1) * w_w + c_e * w_w + (c_w - 1) * d_w$$

The functional dependencies of parameters are modeled by converging connections. This effect, in the literature often referred to as inter-causal inference or explaining away effect, is a special property of graphical models. As soon as observations are given, model parameters become dependent and influence each other. For instance, the width of a facade as an observation is dependent on the continuous model parameters and the number of windows. In turn, if the width of the facade is provided as evidence, the model parameters cannot be chosen arbitrarily anymore (cf. Fig. 5).

By observation, another structural property is that in many cases parameters are correlated. If the correlations are high, ratios of parameters turn out to be a good representation. It turns out that ratios between two continuous model parameters x_i and x_j can be modeled by discrete states since they can be summarized by some states of the form $ratio = n/d$ with $n, d \in \mathbb{N}$. The observation that the equation $x_j = ratio * x_i$ holds for some discrete state of the variable $ratio$ restricts the domains of the model parameters. To model this correlation in the Bayesian network a pseudo observation $p_i = 0$ is introduced: $0 =$



distribution for *GM component*

state	probability
first	w_1
second	w_2

distribution for *continuous variable*

state of parent variable	first	second
mean	μ_1	μ_2
variance	σ_1^2	σ_2^2

Fig. 4. Modeling Gaussian mixtures (GMs) $\sum_{i=1}^{m} \omega_i N(\mu_i, \sigma_i^2)$ in Bayesian networks by introducing an additional discrete node for selecting the component of the distribution, tables illustrate distributions for $m = 2$.

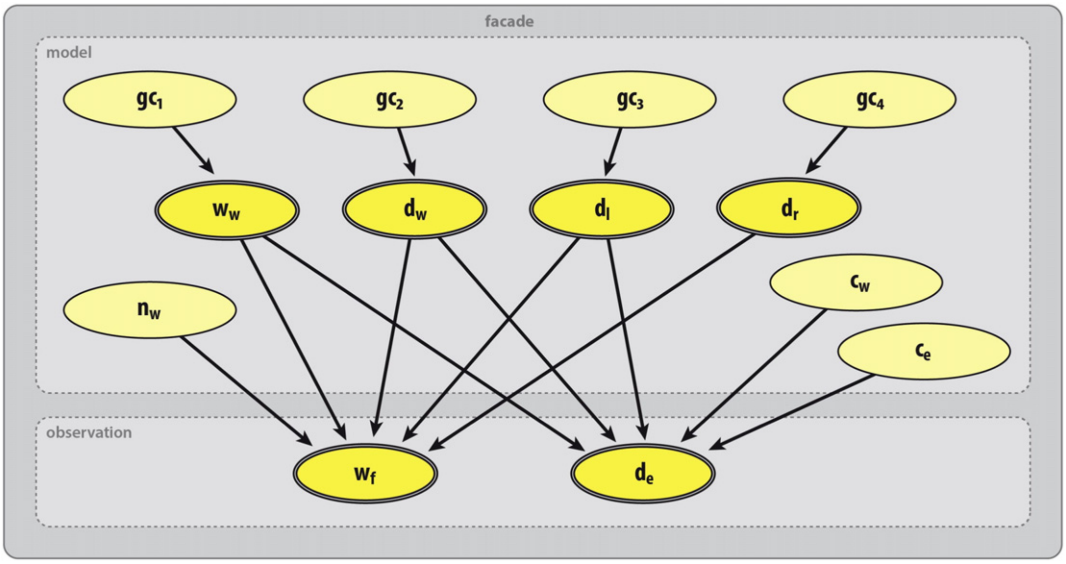


Fig. 5. Extract of generically constructed Bayesian network modeling a single (planar) building facade.

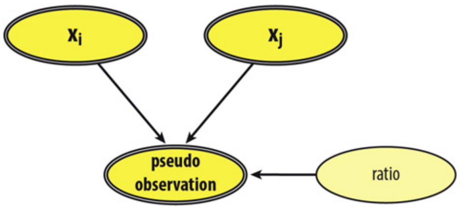


Fig. 6. Converging connection with introduced pseudo observation for modeling relations between two model parameters.

$p_i = ratio * x_i - x_j$ (Fig. 6). The dependency is defined by a "converging connection" (Kjærulff & Madsen, 2008). By setting evidence on the pseudo observation (always observed as 0) model parameters are forced to fulfil the relational properties. Thus, the special form that characterize the constraints of the statistical reasoning is preserved.

The search space can be tightened further by integrating further prior knowledge such as the building type or architectural style that influences the probability distributions. For instance, the width of window in a house of Wilhelminian style is in general smaller than in a typical modern single-family house. Further restriction due to architectural design are incorporated so that impossible configurations are ruled out at early stage.

A Bayesian network represents a multivariate Gaussian distribution if evidence for all discrete variables is available. As long as there is no full assignment of discrete variables the result of the reasoning process is a mixture of Gaussians. As mentioned in Section 1, Lauritzen and Jensen (2001) proposed an efficient algorithm for inference in conditional linear Gaussian networks that overcomes the problem of having Gaussian mixtures of exponential size but is not appropriate for the presented building model.

In our case, we have a special case of Bayesian networks that is similar to CLG networks. Discrete nodes do not have continuous parents but in contrast to CLG networks at first the functional dependencies are bilinear with products containing discrete integral and continuous variables since discrete variables are apriori unknown (especially the number of states). Table 1 summarizes the basic discrete and continuous variables and their nonlinear constraint equations that are used to predict a row of windows. They are divided into variables that occur in the constraints and those

Table 1
Basic variables and constraints for predicting a row of windows.

	continuous variable	discrete variable	
	in constraint		for GM component
model parameter	w_w d_w d_l d_r	n_w c_w c_e r_1 r_2 r_3	gc_1 gc_2 gc_3 gc_4
observation parameter	w_f d_e p_1, p_2, p_3		
constraints: $w_f = d_l + d_r + n_w * w_w + (n_w - 1) * d_w$ $d_e = d_l + (c_w - 1) * w_w + c_e * w_w + (c_w - 1) * d_w$ $p_1 = r_1 * w_w - d_w$ $p_2 = r_2 * d_w - d_l$ $p_3 = r_3 * d_l - d_r$			

w_w	window width
d_w	distance between windows
d_l	distance between left facade margin and first window
d_r	distance between right facade margin and last window
n_w	number of windows
c_w	window correspondence of embrasure
c_e	embrasure correspondence of observation
gc_i	index of mixture component for ith Gaussian mixture
r_i	ith ratio
w_f	facade width
d_e	distance between left facade margin and embrasure
p_i	ith pseudo observation (= 0)

that were only introduced to model Gaussian mixtures. The constraints elucidate that for each observed parameter o_i we have an equation that can be modeled by a special form of a bilinear constraint

$$o_i = \sum_i d_i c_i, \tag{2}$$

where the d_i's are discrete model parameters and the c_i's are continuous ones. As soon as the discrete parameters are determined the equation becomes linear.

To sum up, we define a cp-Bayesian network as follows:

A *cp-Bayesian network* (cp-BN) is a Bayesian network where every discrete variable has only discrete parents and every continuous variable has a CPD that can be characterized by a Gaussian mixture or whose dependency can be described by a relation as in formula (2).

As soon as there are valid instantiations for the discrete parameters, there is a parametric description of the continuous ones for each consistent solution that refers to a single hypothesis. The resulting small space of hypotheses together with the prior knowledge and given observations facilitates a dynamic construction of a graphical model afterwards. Having a full assignment of discrete parameters, inference in the Bayesian network only has to consider linear dependencies between continuous nodes and thus allows for efficient algorithms. The following section shows our approach to solve the discrete problem.

4. Solving the discrete problem

Although the instantiation of the discrete variables is apriori unknown the building model is characterized by various constraints that help to avoid combinatorial explosions. The space of parameters can be reduced drastically if only reasonable instantiations are considered.

Beside the functional dependencies described in Section 3 variables are further constrained by domains, i.e. bounded by intervals $[b_{lower}, b_{upper}]$. In the case of building modeling some discrete parameters such as the number of windows are apriori unrestricted, however the upper bounds can be derived dynamically during the reasoning process dependent on the width of the facade and the Gaussian mixtures which were used to describe the distributions of the model parameters – especially thresholds derived from their single components.

As described in Section 3 a continuous parameter x_i is determined by a Gaussian mixture with m components, means μ_j and small variances $\sigma_j^2, j = 1,\ldots,m$. Partitioning the space of a parameter into different components and introducing thresholds leads to a specification of possible domains as disjunction of m constraints such as

$$\begin{aligned} &\mu_1 - \lambda\sigma_1 \leqslant x_i \leqslant \mu_1 + \lambda\sigma_1 \\ &\vee \ldots \\ &\vee \mu_m - \lambda\sigma_m \leqslant x_i \leqslant \mu_m + \lambda\sigma_m \end{aligned} \tag{3}$$

The jth component is approximated by an interval of $[\mu_j - \lambda\sigma_j, \mu_j + \lambda\sigma_j]$. Setting $\lambda = 3$ implies that values are in this interval with a probability of 99.7%. The small error made by this discretization can be arbitrarily minimized by adapting λ. However, the chosen accuracy is sufficient enough for both reasoning and runtime issues.

Ratios are determined by discrete parameters, its numerator n and its denominator d ($r \approx n/d$), so that two model parameters x_i and x_j have to be related corresponding to the ratio r up to a small ϵ

$$x_i = r * x_j \wedge n/d - \epsilon \leqslant r \leqslant n/d + \epsilon.$$

Obviously, the problem is described by constraints on variables $X_1,\ldots,X_n$ with associated domains $D_1,\ldots,D_n$. Constraints described above restrict these domains so that the final solution leads to a small number of qualified hypotheses. The domains of the continuous variables are defined by intervals that in turn lead to finite domains of the discrete variables. Given the constraint equation for an observation, for example the width of the façade, the number of windows or the components of the Gaussian distributions cannot be combined arbitrarily. In conclusion, our problem can be seen as a constraint satisfaction problem (CSP). We are interested in possible instantiations of the discrete parameters before applying inference techniques with Bayesian networks.

Consequently the idea of our approach is to solve a constraint satisfaction problem with respect to valid values for the discrete parameters that can be used later on as evidence for statistical reasoning. For solving those combinatorial constraint satisfaction problems constraint programming is a powerful framework. In this context, the following concepts are of major value.

A *constraint satisfaction problem (CSP)* is defined by a set of variables $\mathcal{X} = \{X_1,\ldots,X_n\}$ with domains $\mathcal{D} = \{D_1,\ldots,D_n\}$ and a set of constraints $\mathcal{C} = \{C_1,\ldots,C_q\}$ on these domains. The search space for potential hypotheses is determined by the cartesian product of the domains, i.e. $D_1 \times \ldots \times D_n$. A *constraint* C_i is a relation on a subset of variables $\mathcal{X}' \subseteq \mathcal{X}$ and thus a subset of $D_1 \times \ldots \times D_n$. A *solution of a CSP* is an instantiation of the variables, i.e. an assignment of values for each variable $\{(X_1,\alpha_1),\ldots,(X_n,\alpha_n)\}$ with $(\alpha_1,\ldots,\alpha_n) \in D_1 \times \ldots \times D_n$ so that all constraints are satisfied.

There exist various implementations of constraint solvers of which one type is realized by the use of logic programming. Due to its declarative character and powerful search strategies logic programs are advantageous for defining constraint satisfaction problems (Dechter, 2003).

Logic programs are a collection of clauses described by first-order predicate logic. A *clause* is equivalent to a logical implication: $B_1 \wedge \ldots \wedge B_n \Rightarrow H$ and is expressed by

$$H \leftarrow B_1, B_2, \ldots, B_n$$

where $H, B_1,\ldots,B_n$ are literals and a logical AND ($\wedge$) is represented by a comma. A *literal* is a predicate of the form $p(t_1,\ldots,t_n)$ or its negation, where t_i is a term. A *term* can be a variable (in capital letters), a constant (lower case) or a function. The *head* H becomes true if the literals in the *body* $B_1,\ldots,B_n$ are satisfied. An example for a clause would be $p(X,Y,Z) \leftarrow q(X), r(X,Y,Z)$. This clause with variables X, Y and Z represents an if-then-rule, i.e. if both the literal $q(X)$ and $r(X,Y,Z)$ are true, then the literal $p(X,Y,Z)$ is true as well. Equal variable names have to be instantiated by equal values.

Constraint logic programs extend this formalism by adding constraints that restrict the search space following the definition on constraint satisfaction problems. As an example, the above example could be extended by a nonlinear multiplication constraint: $p(X,Y,Z) \leftarrow X = YZ, q(X), r(X,Y,Z)$.

Solving a constraint satisfaction problem means finding one solution or all solutions that is finding instantiations for the variables in the queried literal. A query $p(2,Y,Z)$ means finding an instantiation for Y and Z that satisfies $2 = YZ, q(2)$ and $r(2,Y,Z)$. Therefore the constraint solver follows the principle "constraint and generate", i.e. the algorithm begins with propagating the constraints and uses a depth-first search afterwards to find valid instantiations for the variables with respect to the constraints. Constraint propagation means deducing additional constraints or restricting existing ones such as narrowing the domains. An efficient and sufficiently powerful concept for such consistency-enforcing is to achieve bounds consistency. A constraint is *bounds-consistent* if all interval bounds participate in a solution of the constraint. Propagation rules are used for calculating new intervals for each variable. For the constraint $X = YZ$ (e.g. $x_i = ratio * x_j$) with $X \in [A,B], Y \in [C,D], Z \in [E,F]$ and $X,Y,Z > 0$ the interval bounds for Z can for example be updated to $[A/D, B/C]$. Domain splitting supports the search afterwards by splitting the intervals in order to check values for consistency and to determine the space of solutions for the variables.

Likewise, the constraint satisfaction problem is solved for the prediction of windows. For example a façade width of $W = 13.0$ m bounds the number of windows to a maximum of four due to given domains for the continuous model parameters that are based on available distributions. This in turn restricts the selection of mixture components. For more details on constraint processing the reader is referred to Dechter (2003) and Marriott and Stuckey (1998).

To conclude, (logic) constraint solvers are powerful tools to handle non-linear constraints and allow to solve problems which contain discrete as well as continuous variables. In contrast to Gröbner Bases (Buchberger, 1998) as a method for solving polynomial equations they can perform better in this case, i.e. they have polynomial time complexity (Frühwirth & Abdennadher, 2003). While the valuation of a discrete variable is an integer, continuous variables are determined by intervals dependent on a given precision. Bound propagation and interval reasoning enable to find a solution for non-linear continuous constraint satisfaction problems as being present in our approach and consequently transform the exponential problem into a linear one.

To sum up, the use of constraint programming in the presented prediction process leads to

1. Determination of discrete states
2. Gaussians instead of Gaussian mixtures
3. Linear constraints instead of multi-linear ones

Fig. 7 compares the size of resulting mixture components in the Bayesian network with and without applying the constraint satisfaction. It shows the effect of the presented *cp-BN* approach for the introduced example in the context of façade prediction (cf. Section 3). The resulting distribution for a continuous model parameter is not a mixture of Gaussians but a single Gaussian for each consistent solution that is appropriate for the prediction of reasonable façades. Since discrete parameters are instantiated by solving the constraint satisfaction problem they are treated as constants so that their nodes can be omitted in the subsequent statistical reasoning for determining the continuous parameters.

5. Finding the continuous values

Fig. 7 showed the Bayesian network that remains after eliminating the instantiated discrete nodes. Instead with non-linear dependencies on discrete and continuous variables, inference has now to consider linear dependencies on continuous variables. The stochastic component is mainly a special structured network: a state-observation model with a ρ-dimensional state vector x representing the model parameters and a s-dimensional observation vector o. One Gaussian distribution for each model parameter is selected by the CLP component as one part of the Gaussian mixtures so that distributions are no longer multimodal. With the instantiation of discrete parameters $I = \tau$ we have a linear problem for the lth hypothesis and ith observation of the form:

$$o_i = \sum_{j=1}^{\rho} \tau_{lj}^i x_{lj}^i \quad (4)$$

where the τ_{lj}^i's are the discrete (instantiated) coefficients of the continuous model parameters $\{x_{l1}, \ldots, x_{l\rho}\}$.

For such state estimations where model parameters are only observed indirectly the Kalman Filter is an efficient algorithm for calculating the posterior (Kalman, 1960). It assumes that state transition and measurement can be described linearly and initial beliefs are represented by multivariate Gaussian distributions. Being originally deployed for dynamic systems where states evolve over time, the Kalman filter is implemented in two steps, the prediction of time step t based on time step $t-1$ and the measurement update that corrects the prediction by incorporating the latest observations. Since the model of buildings is static and does not change over time the prediction step is omitted whereas the latter calculation, the measurement update, is implemented in the presented approach to determine the posterior distributions. Gaussian distributions represented by μ and Σ are carried over from the constraint solver of the reasoner. A linear observation model describes the mapping from the state vector $\mu \in \mathbb{R}^\rho$ to the observation vector $o \in \mathbb{R}^s$ by a multiplication with a measurement matrix $M \in \mathbb{R}^{\rho x s} : o = M\mu$. The Gaussian noise for observations is expressed by a s-by-s matrix Q so that the conditional probability is defined as

$$P(o|\mu) = N(M\mu, Q)$$

Table 2 gives an overview of the input parameters originating from prior input and combinatorial output and the assignment to the corresponding statistical variables in the measurement update of the Kalman Filter. Simple matrix multiplication for the state-observations model complete the reasoning process. By incorporating the Kalman gain

$$K = \Sigma M^T (M \Sigma M^T + Q)^{-1} \quad (5)$$

after the initialization of matrices the posterior distribution, i.e. μ and Σ, is calculated:

$$\begin{aligned} \mu &= \mu + K(o - M\mu) \\ \Sigma &= (Id - KM)\Sigma \end{aligned} \quad (6)$$

As a result, continuous parameters are determined for each instantiation of discrete parameters yielding a set of most likely hypotheses.

6. Prediction of building parts

In the following we delineate the implementation of the developed *cp-BN* reasoner for predicting unknown structures in building facades. Fig. 8 shows the two basic components the reasoner is composed of: a combinatorial component that uses constraint logic programming to restrict the number of possible hypotheses and to instantiate the discrete parameters and a statistical component based on bayesian networks that calculates the posterior distribution for the continuous parameters and allows for a final valuation of the prediction.

For reasons of generalisability the semantic formulation of the problem is encapsulated in a model so that the field of applications is not restricted to building facades but is flexible with respect to the input problem instead. Since the controlling units – the combinatorial and statistical component – are implemented independently from the model predictions can be generated for arbitrary problems as long as they represent a cp-BN problem. The following conventions have to hold for well-defined input data:

A *cp-BN problem* $(\mathcal{X}, \mathcal{C})$ is defined by a set of parameters $\mathcal{X}$ and a set of constraints $\mathcal{C}$ with the following properties:

- *continuous model parameters* $\in \mathcal{X}$ are characterized by a Gaussian mixture (with μ's, σ's, ω's)
- *observed continuous parameters* $\in \mathcal{X}$ are characterized by an observed value and a precision for this observation (σ)
- *discrete model parameters* $\in \mathcal{X}$ are (optionally) characterized by a domain with integral bounds
- *ratios* $\in \mathcal{X}$ are characterized by a list of numerators and a list of denominators
- *constraints* $\in \mathcal{C}$ are defined by an CLP expression whose variables $\in \mathcal{X}$

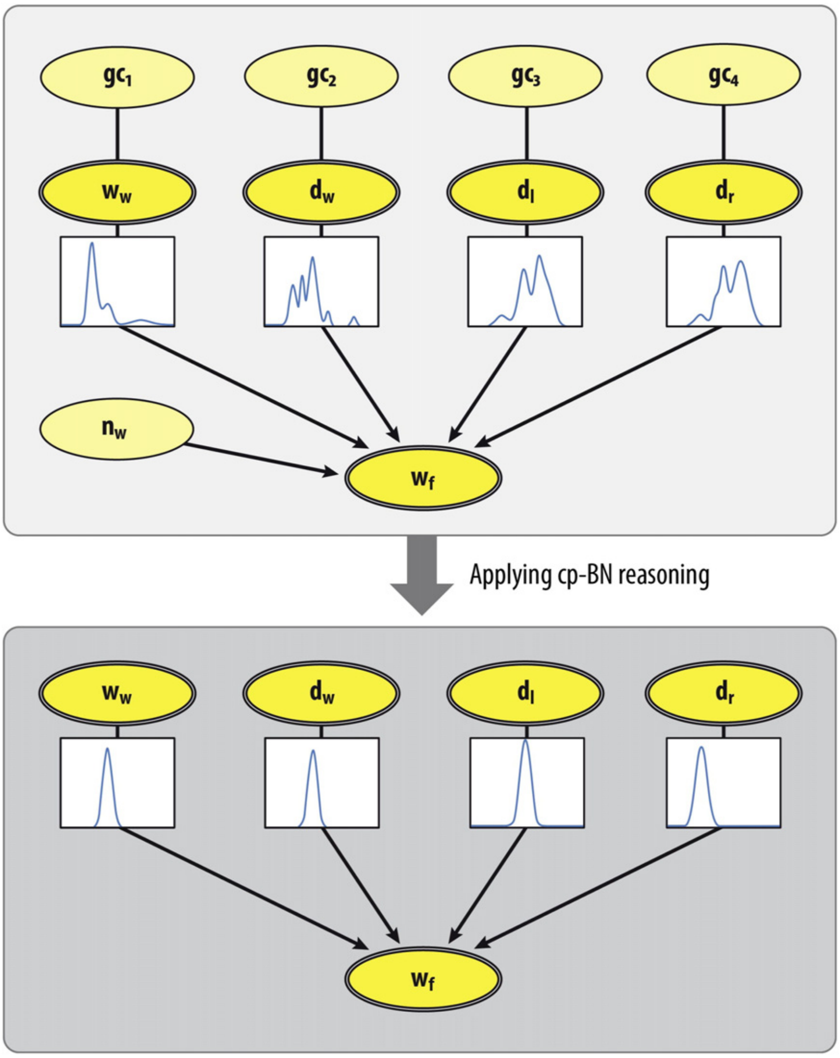


Fig. 7. Number of resulting mixture components for query variables of the building model in conventional BN and proposed cp-BN. Discrete states represent the number of windows on the one hand and the indices of mixture components on the other hand. The instantiation of discrete parameters by applying the cp-BN approach transforms the multimodal space of hypotheses into a unimodal one.

Table 2
Overview of input parameters for the measurement update and corresponding statistical variables.

origin of input	input parameters	corresponding statistical variables
prior input	s observations $(o_1, \dots, o_s)$	k observation vectors $o = \begin{pmatrix} o_1 \\ \dots \\ o_s \end{pmatrix} \in \mathbb{R}^s$
	uncertainty of s observations: $(\sigma_1, \dots, \sigma_s)$	k measurement noise matrices $Q = \begin{pmatrix} \sigma_1 & 0 & 0 \\ 0 & \dots & 0 \\ 0 & 0 & \sigma_s \end{pmatrix} \in \mathbb{R}^{sxs}$
combinatorial output for top k hypotheses	selected mixture components of ρ model parameters for each hypothesis l, $l = 1..k$: $((\mu_{l1}, \sigma_{l1}), \dots, (\mu_{l\rho}, \sigma_{l\rho}))$	k state vectors $\mu = \begin{pmatrix} \mu_{i1} \\ \dots \\ \mu_{i\rho} \end{pmatrix} \in \mathbb{R}^{\rho}$ and k matrices $\Sigma = \begin{pmatrix} \sigma_{l1} & 0 & 0 \\ 0 & \dots & 0 \\ 0 & 0 & \sigma_{l\rho} \end{pmatrix} \in \mathbb{R}^{\rho x \rho}$
	discrete coefficients $(\tau^i_{l1}, \dots, \tau^i_{l\rho})$ of ρ model parameters in constraints $o_i = \sum_{j=1}^{\rho} \tau^i_{lj} x^i_{lj}$ with $i = 1..s, l = 1..k$	k measurement matrices $M = \begin{pmatrix} \tau^1_{l1} & \dots & \tau^1_{l\rho} \\ & \dots & \\ \tau^s_{l1} & \dots & \tau^s_{l\rho} \end{pmatrix} \in \mathbb{R}^{sx\rho}$

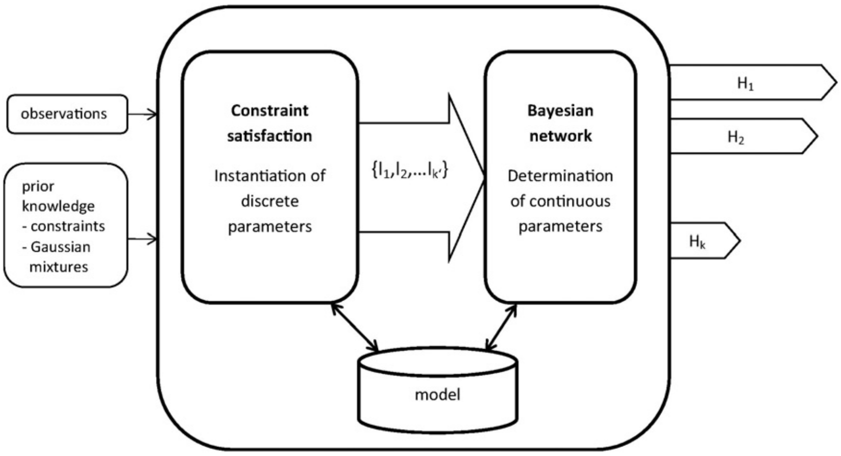


Fig. 8. Components of *cp-BN* reasoner.

Constraints are considered in the CLP component in order to restrict the search space. They are the basis of the statistical component if they define a state-observation-model and have the form as in formula (2). It should be noted that the latter requirement can often be fulfilled by adding pseudo observations (cf. Section 3).

Algorithm 1 depicts the *cp-BN* approach which is used to generate hypotheses. Combining constraint programming with probabilistic models the complexity is reduced and inference is adapted appropriately to the domain of buildings.

Algorithm 1. Algorithm *cp-BN* for prediction.

Input: prior knowledge (prior distributions, constraints), observations
Output: best k hypotheses
$\mathcal{H}_{best} = (\mu_1, \tau_1, P_1), \ldots, (\mu_k, \tau_k, P_k)$
`// model initialization`
1 `initialize model`: **observations** o **with measurement noise** Q **and derived information**
`// constraint-based reasoning`
2 construct query according to model initialization
3 solve constraint satisfaction problem CSP by querying discrete parameters I **yielding solutions** $\mathcal{S} = (S_1, \ldots S_{k'})$
4 for $S_j \in \mathcal{S}$ **do**
5 | **generate hypothesis** H_j **for model by incorporating instantiation for discrete parameters** I
6 | **eliminate similar hypotheses in set of hypotheses**
$\mathcal{H}' = [H_1, \ldots, H_{k'}]$ **yielding** $\mathcal{H} = [H_1, \ldots, H_k]$
`// statistical reasoning`
7 for $H_i \in H$ **do**
8 | **construct matrices** Σ, M, Q **and vectors** o **and** μ
9 | **calculate posterior belief by updating measurement with**
10 | $K = \Sigma M^T (M \Sigma M^T + Q)^{-1}$
11 | $\mu = \mu + K(o - M\mu)$
12 | $\Sigma = (Id - KM)\Sigma$
13 determine k most likely assignments: $MAP^k(\mu|o)$
`// refinement (optional)`
14 refine results as soon as additional information is available

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The reasoner is based on prior knowledge for the domain model that is given by a large set of training data. For buildings the training data consists of about 1000 annotated façades collected in a ground truth database of buildings. The database includes distances between building parts such as windows, their sizes and qualifying properties such as the building type. This allows to derive knowledge that can be used while reasoning about a queried building. Information includes Gaussian mixtures for model parameters, ratios between model parameters, type of building and characteristics of the ground plan such as symmetry.

Above, basic data about buildings are available in a database of about 9 million buildings of North-Rhine-Westfalia, Germany, with their ground plans and height. Footprints could as well be extracted from open source projects such as Open Street Map.

We illustrate our approach with an example of predicting a row of windows in a building façade where the only observation we might have is the width of the façade. As illustrated in Section 3, the model is described basically by four continuous parameters. These are characterized by Gaussian mixtures that are a good approximation of the distribution and at the same time allow for methods that require the distributions to be Gaussians. By using Expectation Maximization (McLachlan & Peel, 2000) a Gaussian mixture for each continuous parameter is provided for the statistical domain model (cf. Section 1).

The reasoner starts with defining a constraint satisfaction problem according to the prior knowledge and available observations. The domain model is represented by a logic program that incorporates constraints as described in Section 4. Known observations, e.g. for the width of a façade, are instantiated as input whereas the apriori unknown discrete model parameters remain as query variables and have to be found by the constraint solver. Therefore, the CLP component uses the ECLiPSe Constraint Programming System and its hybrid integer/real interval arithmetic constraint solver.

The constraint solver narrows the search space by propagating the given information and yields all possible solutions for the model specification. Thus, the CLP component instantiates the discrete variables such as selected component of the Gaussian mixtures and the number of windows (cf. Fig. 5) and linearizes the problem with the result that initial beliefs become normally distributed. Instead of a multimodal distribution for each model parameter we reduce the problem to one component of the Gaussian mixtures and thus are able to use well-studied efficient implementations. In order to minimize computation time similar hypotheses are identified in the CLP result as a post-processing and duplicates are eliminated by expecting a distance $\delta = max(x_{ki} - x_{lj}) \leq \epsilon, i, j = 1 \ldots |\mathcal{X}|$, between two hypotheses H_k and H_l for a given threshold ϵ.

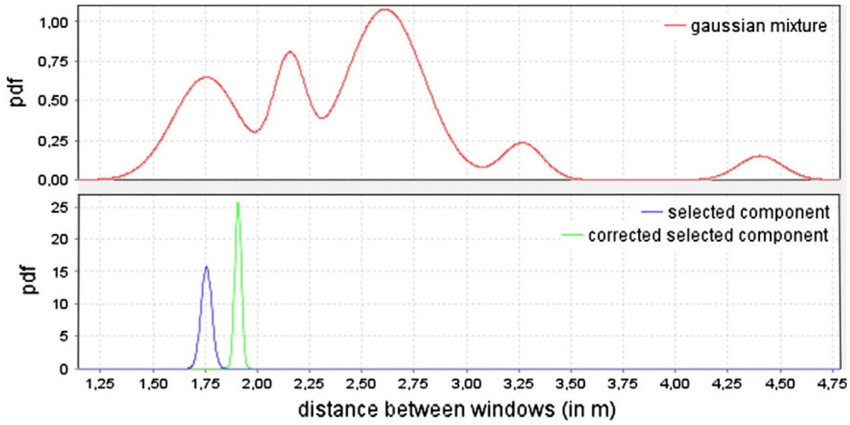

Fig. 9. Distributions for the distance between windows in different reasoning steps: top: prior as Gaussian mixture with 5 components, bottom: Gaussian distribution as selected component of the Gaussian mixture (blue) and resulting posterior (green).

Preferring a set of diverse hypotheses, this additionally leads to the fact that hypotheses differ from each other significantly.

In a final step, the posterior distribution for each hypothesis is calculated within the statistical component (lines 7–12). The probabilistic model is constructed generically dependent on the given problem and the determination of the discrete parameters. The resulting matrices are the input of the correction step of the Kalman filter (cf. Section 5) that yields the continuous parameters for each hypothesis.

Fig. 9 shows the results of the filter algorithm by comparing the prior and posterior beliefs. The beliefs are shown for three different steps. Apriori distributions (red) represented as Gaussian mixtures are the input of the algorithm. After solving the constraint satisfaction problem, the prior reduces to one component of the mixture (blue) as shown at the bottom of the figure. They are specific for one solution of the problem and are given to the statistical component that integrates observations and model assumption to calculate the posterior belief (green).

The reasoner provides means $\mu_i \in \mathbb{R}^{|\mathcal{X}_\Gamma|}$ for continuous model parameters and the related instantiations $\tau_i \in \mathbb{R}^{|\mathcal{X}_\Delta|}$ for discrete variables. Final hypotheses are ordered by their (unnormalized) probabilities P_i calculated on basis of the apriori known distributions:

$$P_i = exp\left(\sum_{j=1}^{|\mathcal{X}_\Gamma|} log(pd\,f_j^{01}(\mu_{ij}))\right)$$

where pdf_j^{01} is the on $[0\ldots1]$ scaled density of the distribution corresponding to the jth model parameter.

We finally get a set of hypotheses of the most probable façades given the observations:

$$H_{best} = (\mu_1,\tau_1,P_1),\ldots,(\mu_k,\tau_k,P_k)$$

The set of k best hypotheses does not only present the most probable hypothesis since this would probably prevent finding the best interpretation of the input data. Instead, a diverse but sorted set of predicted façades is the result.

7. Experimental results

The reasoner was evaluated with various constellations of input parameters. Buildings were chosen of different building types (cultural heritage, terraced building, …) and architectural style (Wilhelminian, Modern,…). The influence of prior knowledge was analysed. Ground truth values were used to assess the quality of the predictions. The prior knowledge and ground truth data is based on a relational database of about 1000 annotated façades. Fig. 10 shows the relevant extract of the database schema. It is a hierarchical model that characterizes buildings by their ground plan type, building type and architectural style and relates them to their measured parts, such as facades containing in turn windows. The data was acquired by the use of the annotation tool 'measureFacade' (Staat & Schmittwilken, 2010) that enables the measurement of location and form parameters in images and 3D laser scans.

Fig. 11 shows the output of the reasoner for predicting rows of windows in different façades. The only observation for the predictions was the ground plan of the building. That is, inference was based on the width of the façade and the prior knowledge about distributions and constraints given by the analysis of the ground truth data. Results were generated after 0.2 s on a Windows 64 Bit machine (3.4 GHz, 16 GB RAM) with a solution space of averagely 8 hypotheses. As shown in 11c, the prediction can also compensate occluded façades. Hypotheses can be distinguished by single discriminating measurements of embrasures as indicated by vertical lines. Additional laser scan measurements or the result of an edge detection in images yield more accurate information about the correct hypothesis.

Table 3 gives an overview of the quality of the results. It summarizes the results for hypotheses that have a rank lower than 1, 2, 4 and 8 respectively. The error is calculated as maximum norm of the difference between predicted and ground truth values. The quality depends on how much the facade meets the norm with respect to the distributions chosen according to the prior knowledge. For example, the prediction for the facade in 11a yields better results than in 11b since the form parameters such as the window distance of the latter facade are less common. It can be seen that for more than half of the tested façades the correct number of windows was predicted for its first hypotheses with an average error err_all of 0.374 m for the continuous model parameters. The width of windows is even predicted with an average error err_w_w of 0.089 m. For the whole test set the correct hypothesis was listed among the top eight hypotheses.

The generated hypotheses based on ground plans already include the accurate façade. However certainty can be increased by considering one additional measurement such as the position of an arbitrary embrasure (Fig. 12). The constraint satisfaction problem is extended to predict the correspondence of this observation to one of the windows and the decision between left or right embrasure (cf. Table 1). The functional dependency that comes with an equation for the distance between the left margin of the façade and the embrasure leads to clearer results since the location of an embrasure further restricts the space of valid hypotheses.

The quality of the results is enhanced if the type of building (e.g. cultural heritage, villa, terraced building) or the architectural style (e.g.

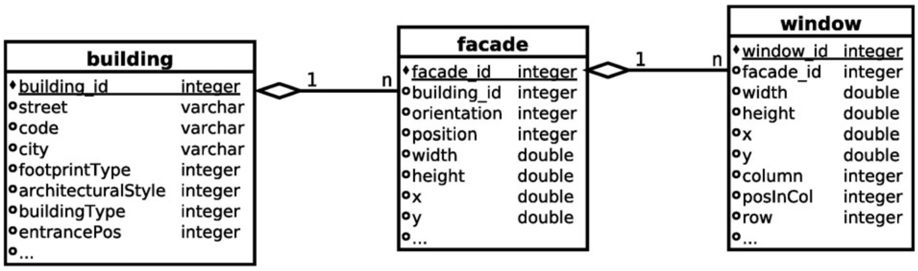

Fig. 10. Relevant extract of database schema. Measurements (location and form parameters) and categorical attributes are used for prior knowledge and evaluation.

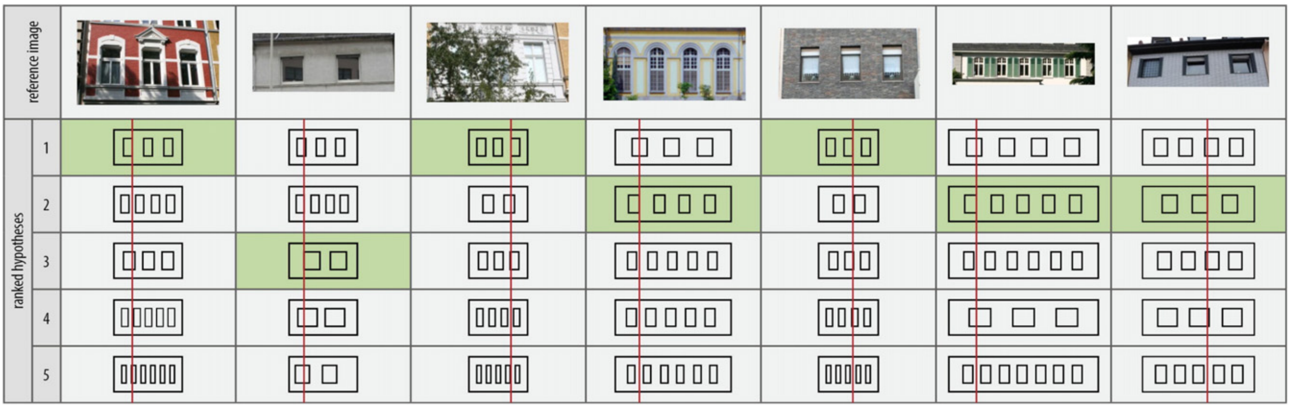

Fig. 11. Resulting ranked hypotheses for predicting a row of windows with given width of façade. Vertical red lines show how hypotheses can be discriminated by an additional test to select the hypothesis best matching the real-world scenario.

Table 3
Quality of results for predicting a row of window with given width of façade: number of hypotheses with correct number of windows and errors for continuous model parameters (in meters).

rank	#windows correct (correct/total)	avg(err)_w$_w$)	avg(err)_all)	min(err)_all)	max(err)_all)
1	28/55	0.089	0.374	0.078	0.817
≤ 2	38/55	0.094	0.435	0.052	1.857
≤ 4	45/55	0.128	0.509	0.052	1.857
≤ 8	55/55	0.198	0.573	0.052	1.857

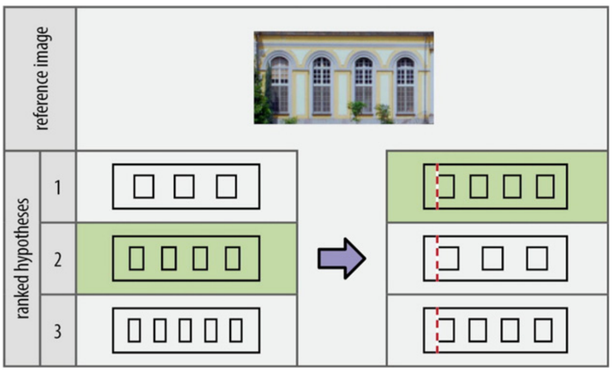

Fig. 12. Resulting ranked hypotheses with given width of façade (left) and additionally embrasure (right). Certainty is increased by considering a single observed embrasure.

Wilhelminian, modern) is known. Obviously, castles can have wider windows than terraced buildings. Buildings of Wilhelminian style are often characterized by high floors with high and narrow windows, whereas modern buildings have in general lower and wider windows. This fact is reflected in the Gaussian distributions that are used to restrict the domains of variables and can thus be exploited for the

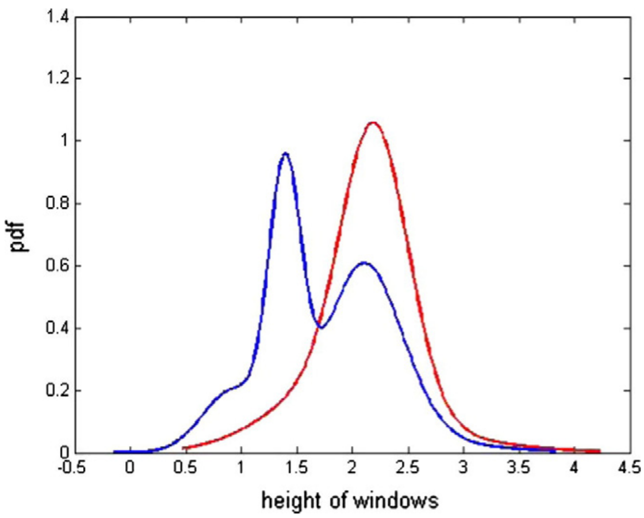

Fig. 13. Impact of different prior knowledge on Gaussian mixtures. blue: height of windows in façades of different architectural style, red: height of windows in façades of Wilhelminian style.

reasoning process. As the Gaussian mixtures in Fig. 13 suggest different components represent different architectural styles. A more precise reasoning is possible with known architectural style since parameters can be described with less mixture components and thus smaller domains. For the automatic derivation of building types from ground plans and LOD1 models with support vector machines see Römer and Plümer (2010) and Henn et al. (2012).

The horizontal model can be extended by a similar structured vertical model as soon as the height of a building is available (e.g. from a LOD1 model). As a result, the reasoner supports the process of reconstructing buildings in 3D. A verification of predictions that the reasoner provides supports a top-down approach where a data-driven identification of building parts in possibly sparse raw data can be omitted.

8. Conclusion

This paper presented an approach for predicting apriori unknown structures in buildings. In contrast to other approaches, the implemented reasoner only needs few observations such as the corresponding ground plan to generate appropriate hypotheses with high probability. To this end, statistical reasoning was combined with constraint logic programming. Prior knowledge was integrated that was obtained from a ground truth database of façades such as form parameters of windows.

For reasoning within uncertain data we followed a top-down approach and represented the domain model as a Bayesian network whose structure is similar to a conditional linear Gaussian (CLG) network. In contrast to CLG networks, some of the discrete parameters are apriori unknown and have an unrestricted number of values. As a consequence, we had to cope with bilinear functional dependencies between discrete and continuous variables. To tackle this problem the developed reasoner is extended by a component based on constraint logic programming that solved the combinatorial problem of discrete parameters. In this way, it was able to linearize the problem and to find an instantiation of the discrete parameters by solving constraint satisfaction problems. Being equivalent to a Bayesian network with continuous variables and associated linear Gaussians a Kalman Filter was applied to determine the posterior distribution of the continuous model parameters according to a measurement update.

Strong regularities in the appearance of man-made objects legitimate a top-down approach with a model characterized by strong constraints and distributions. Prior knowledge that supported the reasoning process was acquired by an extensive analysis of a ground truth database. Distributions of model parameters were described by non-parametric probability density functions and approximated by Gaussian mixture models. The distributions are characterized by a strongly peaked space of parameters that together with functional dependencies allowed for a generation of good hypotheses based only on few observations such as ground plans. A more precise reasoning becomes possible by exploiting prior knowledge, e.g. the architectural style. Hypotheses can be refined by incorporating further observations.

The field of application is not restricted to building façades but rather is extendable to other problems that are characterized by mixed integral and continuous values with multilinear constraints. To this end, the domain model is not represented by a static structure but instead build generically based on the given input.

By combining statistical and combinatorial reasoning the presented reasoner is able to perform exact inference. The reasoner provides means and variances for the continuous model parameters together with corresponding instantiations of the discrete parameters and outputs the best hypotheses given the observations.

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B.3 Estimation of 3D indoor models with constraint propagation and stochastic reasoning in the absence of indoor measurements

Loch-Dehbi, S., Dehbi, Y., Plümer, L., 2017. Estimation of 3D indoor models with constraint propagation and stochastic reasoning in the absence of indoor measurements. ISPRS International Journal of Geo-Information 6.

Abstract

This paper presents a novel method for the prediction of building floor plans based on sparse observations in the absence of measurements. We derive the most likely hypothesis using a maximum a posteriori probability approach. Background knowledge consisting of probability density functions of room shape and location parameters is learned from training data. Relations between rooms and room substructures are represented by linear and bilinear constraints. We perform reasoning on different levels providing a problem solution that is optimal with regard to the given information. In a first step, the problem is modeled as a constraint satisfaction problem. Constraint Logic Programming derives a solution which is topologically correct but suboptimal with regard to the geometric parameters. The search space is reduced using architectural constraints and browsed by intelligent search strategies which use domain knowledge. In a second step, graphical models are used for updating the initial hypothesis and refining its continuous parameters. We make use of Gaussian mixtures for model parameters in order to represent background knowledge and to get access to established methods for efficient and exact stochastic reasoning. We demonstrate our approach on different illustrative examples. Initially, we assume that floor plans are rectangular and that rooms are rectangles and discuss more general shapes afterwards. In a similar spirit, we predict door locations providing further important components of 3D indoor models.

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Estimation of 3D Indoor Models with Constraint Propagation and Stochastic Reasoning in the Absence of Indoor Measurements

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Abstract: This paper presents a novel method for the prediction of building floor plans based on sparse observations in the absence of measurements. We derive the most likely hypothesis using a maximum a posteriori probability approach. Background knowledge consisting of probability density functions of room shape and location parameters is learned from training data. Relations between rooms and room substructures are represented by linear and bilinear constraints. We perform reasoning on different levels providing a problem solution that is optimal with regard to the given information. In a first step, the problem is modeled as a constraint satisfaction problem. Constraint Logic Programming derives a solution which is topologically correct but suboptimal with regard to the geometric parameters. The search space is reduced using architectural constraints and browsed by intelligent search strategies which use domain knowledge. In a second step, graphical models are used for updating the initial hypothesis and refining its continuous parameters. We make use of Gaussian mixtures for model parameters in order to represent background knowledge and to get access to established methods for efficient and exact stochastic reasoning. We demonstrate our approach on different illustrative examples. Initially, we assume that floor plans are rectangular and that rooms are rectangles and discuss more general shapes afterwards. In a similar spirit, we predict door locations providing further important components of 3D indoor models.

Keywords: floor plan; 3D indoor models; automatic reasoning; graphical models; Constraint Logic Programming; Gaussian mixture

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1. Introduction

The automatic generation of 3D models of building exteriors such as facades or roofs in level of detail 3 (LoD3) according to CityGML [1] has been a subject of intensive research [2,3]. For indoor navigation, for example, interior models such as 3D models represented in LoD4 of CityGML [4] or models acquired from Building Information Modeling (BIM) are required [5,6]. In comparison to outdoor models, indoor models are not yet widely available. Indoor models, however, open up new fields of application with high relevance, including indoor navigation, evacuation planning and facility management. In addition, such models are an essential prerequisite for tasks like guide for the blind. While most approaches which derive indoor models rely on measured data such as images or 3D point clouds, we believe that this extensive data acquisition of additional indoor measurements is not necessary. To this aim, we propose a novel method for the derivation of 3D indoor models from sparse observations without the need of extra indoor measurements. If building information models are not available, previous approaches for indoor modeling require indoor measuring and modeling which is both expensive and difficult. Measurements are expensive because each single room has to be accessed.

The derivation of models from measurements is difficult due to the masking of walls by furniture. For the distinction between walls and for instance bookshelves or wardrobes, modeling affords prior knowledge and regularization. In this article, we demonstrate that prior knowledge together with outdoor models, especially footprints and information about the positions of windows and floors suffice in many cases if generally available data on room areas, functional use and room numbers are given. To structure and simplify the presentation, we start with the assumption that both floor plans and rooms are rectangular and discuss more general shapes in the end.

The problem we address is characterized by a set of N rectangular rooms that have to be placed within a polygonal footprint. In this context, a room is defined by a reference point and its width and depth. The width and depth of the rooms are bounded by upper and lower values and constrained by a bilinear constraint $area = width * depth$, where the area is known a priori and the two parameters $width$ and $depth$ are unknown. In the problem we solve, a building footprint as well as the area of each room are given. We assume that each room has a rectangular shape. Lower and upper bounds for the width and depth of each room are derived from probability density functions (PDF). The decision variant of our problem is to decide whether or not the building footprint can be partitioned into rooms that satisfy our specifications. In the special case that the building footprint is a rectangle and that, for each room, the lower bound is equal to the upper bound, this problem is related to Perfect Rectangle Packing. Since Perfect Rectangle Packing is known to be NP-hard [7], our more general problem seems NP-hard, too. For that reason, it is unlikely to find an efficient solution in the worst case. We understand, however, that an appropriate representation of background knowledge, the definition of domains and constraints on model parameters and an intelligent combination of constraint propagation and stochastic reasoning yields optimal solutions in a rather efficient way in most realistic scenarios. In order to meet these expectations, we propose a method that reduces the search space by a stepwise reasoning. Architectural constraints and regularities together with an initial relaxation of the problem lead to a fast intermediate result that is adapted to a qualified hypothesis in a second step. The relaxation is contributed to the fact that walls are initially not modeled and rooms do not have to fill the entire space. However, an important architectural constraint consists in the fact that interior walls do not intersect windows. In this step, this constraint is reduced to ensuring that interior room boundaries do not cross window ranges. Rooms are modeled in a topological correct way (non-overlapping, within footprint,...) but are not necessarily aligned along a corridor. In the sense of 2D-topological correctness, two rooms are in our context either disjoint or meet each other in common walls avoiding their overlapping. We used a spring model similar to the approach described in [8]. In this way, we do not consider wall elements in the first step, providing a buffer and enabling improvement of preliminary results in a subsequent step.

Based on the intermediate result, stochastic inference is used in order to deliver a qualified set of solutions that is topologically equivalent but is geometrically different. We use the notion of topological equivalence in the standard sense i.e., equivalent up to homeomorphic transformations (for details, see, for instance, [9] or [10]). In particular, topological equivalence preserves adjacency. The set of solutions is augmented by probabilities derived from a most probable estimation. The exact location of rooms—considering an alignment between rooms—, together with its width and depth and the width of walls are estimated in a subsequent step. The key point is the determination of hypotheses together with likelihood information which structures the hypotheses space. In our experiments, we stated that this space is dominated by a hypothesis with regard to others which describes an expected solution.

An extensive analysis of shape and location parameters such as width and depth of rooms leads to a prior knowledge represented by architectural constraints and probability density functions. Similar to the reasoning process performed in [11], estimation of floor plans is characterized by a bilinear model. The non-linearity is attributed to the fact that a room area is the product of its width and its depth. Besides not allowing that walls cross windows, the fact that windows are e.g., part of office or housing rooms is an important constraint which restricts the domains of shape and location parameters of rooms considerably. Obviously, the latter constraint is not necessary to be hold in the case of corridors or utility rooms. Furthermore, for office buildings with an usually large number of rooms, the available room number is an advantage not to be underestimated since rooms with consequent room numbers tend to be adjacent with high probability. This prior knowledge together with probability density functions make the problem of locating rooms within the footprint feasible. Assigning each window to a room and determining the bilateral relations between rooms turn out to be a combinatorial task that we solved by constraint propagation leading to preliminary topological models.

This paper presents a novel approach for the automatic prediction and generation of building floor plans. Based on sparse observations, we automatically generate a limited number of best hypothesis and provide likelihoods for each solution. The resulted hypotheses are ranked according to an MAP-estimation [12]. The probability density functions for each model parameter provide the possibility to assess the likelihood of each hypothesis and to order it w.r.t. competitive hypotheses accordingly. Dense observations like 3D point clouds are not required. We understand that it is easier to verify or falsify hypotheses than to reconstruct models from observations in a bottom-up way and follow a model-based top-down approach. While most approaches expect observations of adequate density, characteristic for our approach is that we are able to generate best hypotheses for a floor plan based on otherwise insufficient measurements. We start with geometric information on the building footprint as well as position and sizes of the windows. Furthermore, we exploit non-geometric data on rooms including areas and functional use, at least the distinction between office or housing rooms, corridors and toilet. This is important because, in contrast to office or housing rooms, corridors and toilets do not need windows. As illustrated in Figure 1, the input consists of a building footprint and available information about rooms (area of rooms, identifying number of each room and possibly the functional use of each room). Most of this information can be acquired from building management services. The location of the windows can be derived using existing methods for the identification of building parts from point clouds or images of facades such as, for instance, described in [13] or [14]. The algorithm does not require any indoor images or laser scans from walls to predict, nevertheless, floor plans such as those depicted in Figure 1. For the comparison, the associated reference floor plan is shown. Additional data may lead to a verification or falsification of models which, however, is less expensive than reconstructing a building interior bottom-up from measurements. The output of our method is an indoor model for the given floor including a layout for the rooms, location of the doors and the height of the rooms. Topology will be consistent and precision of geometry will be optimal w.r.t. the given information due to exact stochastic inference in the sense of [12]. In our test cases, we yielded accuracies for the model parameters between 10 and 20 cm.

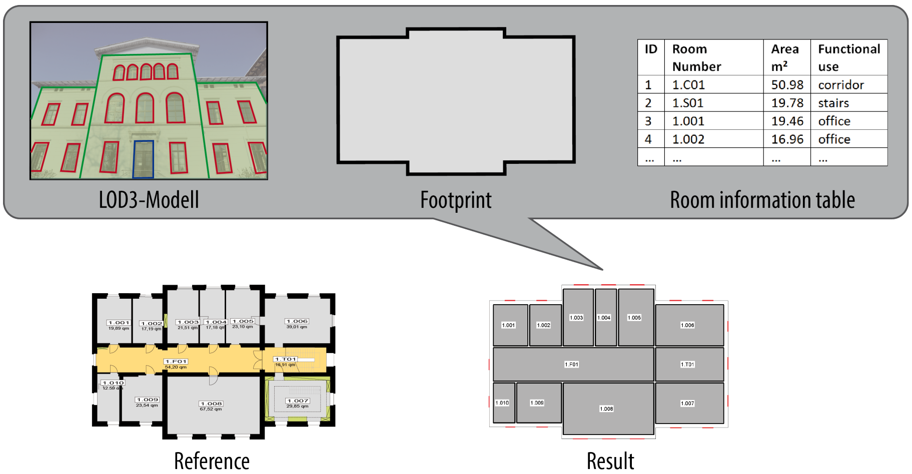

Figure 1. Our approach derives floor plans automatically (**bottom right**) from sparse observations like window locations from possibly LoD3 exterior models, footprint and room information such as room areas (**top**). No additional indoor measurements are needed. For the comparison, a reference floor plan is depicted (**bottom left**).

Despite the complex non-linear problem, the presented reasoner performs exact inferences. Therefore, Constraint Logic Programming (CLP) is combined with Graphical models. Figure 2a summarizes our general approach: we start with a problem characterized by discrete and continuous parameters and an infinite hypotheses space. This is restricted to a small number of feasible candidates by constraint propagation in a first step. In these hypotheses, model parameters are fixed from a topological point of view, but are intermediate. They are input to a stochastic reasoner in a second step. Parameters are adapted to the available observations and the background knowledge in the form of probability distributions of model parameters by using an MAP-based inference. Probability distributions of model parameters such as the width of office rooms can nicely be represented by kernel density estimations [15] or—for our purpose even better—Gaussian mixtures, as illustrated by Figure 2b. It can be seen that the Gaussian mixture is a good approximation to model skew symmetric or multimodal distributions and, at the same time, enables to using well established reasoning algorithms. As stated in [16], each arbitrary probability density function can be approximated by Gaussian mixture models. We use a special case of graphical models which is a Conditional Linear Gaussian model. This enables performing an exact stochastic inference.

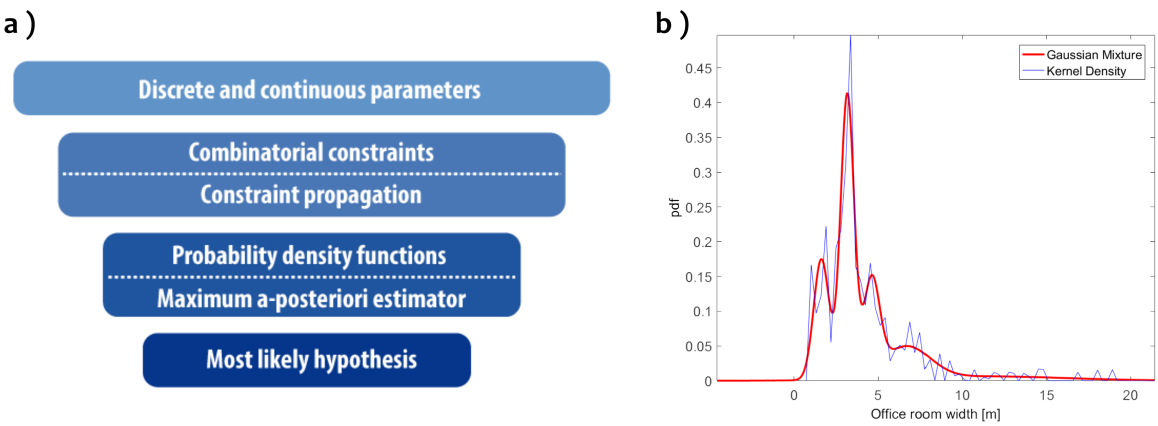

Figure 2. (**a**) reasoning process: the combination of constraint propagation and a maximum a posteriori probability inference reduces a huge search space with continuous and discrete parameters to a small set of solutions with the most likely hypothesis; (**b**) distribution of the width of office rooms: a Gaussian mixture is a good approximation to a skew symmetric or multimodal distribution.

Some of the results described in this paper have been presented at the indoor 3D workshop in the frame of the 11th 3D Geoinfo conference in Athens. This article extends [17] in several aspects:

- Whereas [17] describes the overall approach, this article discusses the relevant methods and algorithms in more depth.
- Whereas [17] represents position of rooms and windows in a continuous domain and performs reasoning with inequalities in these domains, in this paper, we represent geometry in discrete domains during the combinatorial part and apply a method of constraint propagation in finite domains that is considerably more efficient.
- This article deals with doors and estimates their sizes and positions and thus provides an access for indoor navigation.
- It generalizes from 2D floor plan layouts to 3D indoor models.

The main contribution of this paper is a novel approach which affords 3D indoor models and avoids extra measurements in the form of images or 3D point clouds. From a methodological point of view, we apply stochastic inference in the sense of graphical models and combine combinatorial reasoning/constraint propagation in a bilinear model with stochastic inference using Gaussian mixtures in a novel way. As extension of our previous publication [17], the approach is described and elaborated in the following sections.

2. Related Work

While 3D models of the exterior of buildings are widely available in different levels of detail, 3D building interior models (LoD4 in CityGML, [18]) are not yet widespread. Tasks such as rescue management, indoor navigation and guide for the blind have led to growing interest in the design and modeling of building interiors. In this context, the authors of [19] proposed an approach for the generation of building floor plans from laser range data based on a triangulation of a 2D sampling of wall positions. Becker et al. used shape grammars in [20] for the reconstruction of 3D indoor models from 3D point clouds. In [21], Ochmann et al. segmented a point cloud into rooms and outside area and reconstructed the scene by solving a labeling problem based on an energy minimization. For the derivation of indoor models, all mentioned approaches rely on dense observations such as 3D point clouds from laserscans or range cameras using mobile mapping systems. The necessary measurements are both cost and time extensive. Derivation of observations for indoor models is rather different from getting measurements for outdoor models using airborne or terrestrial platforms. Every single room must be entered and scanned. Furthermore, while one is interested in modeling walls, doors, windows and ceilings, they are concealed by all kinds of furniture. Strong model assumptions are needed in any case. In order to overcome the acquisition of dense observations as a time-consuming process, low cost sensors have been employed in several approaches. For instance, Diakité et al. investigated in [22] the usefulness of the low cost Android tablet from Google's Tango project for the acquisition of indoor building environments. The information extracted from the native models of this device are not rich enough in order to derive detailed indoor models. As a consequence, our central motivation is to predict unknown substructures in buildings such as floor plans based on strong model assumptions in the sense of background knowledge but only few observations like the area of rooms and footprints. For more information about the works in indoor modeling and mapping, we refer to a survey of recent research in this field [23].

A constraint-based approach for the generation of floor plans has already been designed in 1994. In [24], Charmann describes a knowledge-based system that generates all possible floor plans satisfying a set of geometric constraints on the rooms (non-overlap, adjacency, minimal/maximal area, minimal/maximal dimension, etc.). Therefore, he defines the semi-geometric arc-consistency in order to adapt consistency techniques to geometric problems. In comparison to our method, this approach does not address the reconstruction of floor plans for existing buildings and does not take probable configurations into consideration.

Constraint propagation is a powerful method to solve combinatorial problems. Approaches, however, that extend this framework by a stochastic component are rather rare. The authors of [25] adapt combination and marginalization operators to find the m-best solutions for optimization tasks in graphical models. Intervals with cumulative distribution functions are used in [26] to model a degree of knowledge for uncertain data. In order to address uncertainty, our approach combines the classical constraint propagation with Bayesian Networks and thus benefits from the strength of both paradigms.

3. Modeling Floor Plans with Constraints

Our approach to predict floor plans follows a model-based top-down approach. Therefore, the problem is modeled based on an extensive analysis of real floor plans as well as dimensional restrictions based on laws and architectural characteristics. We understand that man-made objects are characterized by a number of regularities. On the one hand, geometric relations such as parallelity and orthogonality are dominant in buildings. In [27], Loch-Dehbi et al. studied the geometric rules that can be found in man-made objects and presented an approach for deducing geometric relations in 3D building models. On the other hand, buildings can be described by functional and statistical dependencies between model parameters. In this paper, we focus on the latter properties of buildings.

The knowledge of architectural design as well as available distributions about model parameters enable generation of good hypotheses in order to reconstruct buildings.

The estimation of floor plans can be reduced to find the width w_i and depth d_i as well as the reference point (x_i, y_i) for each single *ith* room. Besides outdoor building models in LOD3, we only need data which is available with every housekeeper and every real estate manager without the need of any indoor measurements. We are given the area of the rooms in addition to the corresponding building footprint and possibly the functional use and the identifying numbers of rooms. Furthermore, we exploit the (two-dimensional) location parameters (x_{w1}, y_{w1}) and (x_{w2}, y_{w2}) of the windows stemming from exterior measurements of the facade. Figure 3a gives an overview of the domain model and its parameters. The figure further illustrates that, for rooms with windows, at least one of their walls has to take the value of the minimum or maximum values x_{min}, y_{min}, x_{max} and y_{max} within the corresponding footprint, respectively.

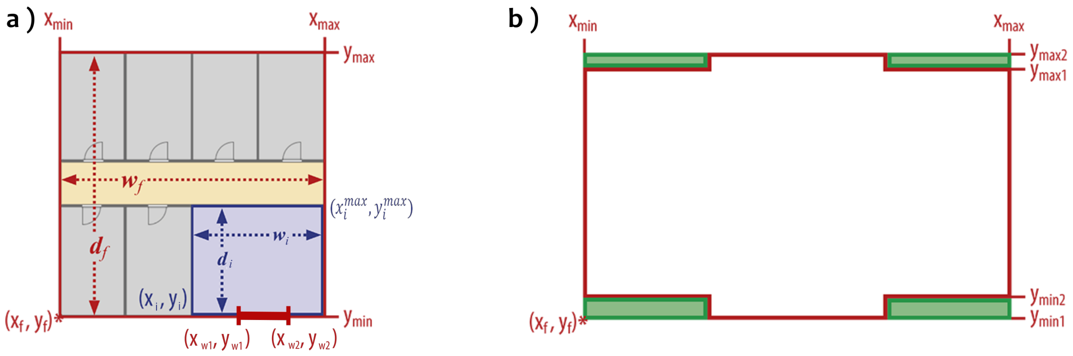

Figure 3. (**a**) illustration of location and shape parameters for a floor f and an *ith* room with a single window used during the reasoning process; (**b**) example for adding auxiliary fix rooms (green) in order to model floor plans with a non-rectangular footprint.

In a first step, we assume that rooms as well as the corresponding footprint have a rectangular shape. This assumption will later be relaxed in order to model buildings that are not rectangular but, however, follow the Manhattan world assumption. Therefore, auxiliary virtual rooms (green rectangles in Figure 3b can be added that fill the gaps in order to complete a rectangle. Without loss of generality, we further assume that the longest side of a footprint is parallel to the x-axis in order to have a consistent usage of width and depth for the rooms. For rectangular footprints, this is guaranteed based on an ortho-rectification after a main axis determination of the footprint.

3.1. Hard and Soft Constraints

The topological and architectural knowledge about buildings especially rooms is used to define constraints on the model parameters. In contrast to functions, the constraints usually have several unknown parameters that have to be determined. The first constraint relates the given parameter area of the *ith* room with the two unknowns width w_i and depth d_i which is a bilinear, that is non-linear, constraint:

$$area = w_i * d_i. \tag{1}$$

An obvious but important constraint is that all rooms have to be in the (rectangular) floor shape f derived from the building footprint. It simplifies to a test whether the *ith* room lies within a bounding box with reference point (x_f, y_f), width w_f and depth d_f. The index i corresponds to the room identifier in the column ID from the room information table from Figure 1:

$$((x_f \leq x_i) \wedge (y_f \leq y_i) \wedge ((x_i + w_i) \leq (x_f + w_f)) \wedge ((y_i + d_i) \leq (y_f + d_f))). \tag{2}$$

Another important constraint to maintain topological correctness is the non-overlapping of rooms. The fact that two distinct rooms i and j have to be disjunct can be modeled as follows:

$$(x_i + w_i \leq x_j) \vee (x_j + w_j \leq x_i) \vee (y_i + d_i \leq y_j) \vee (y_j + d_j \leq y_i). \tag{3}$$

Knowledge of the positions of windows can be used for two purposes: on the one hand, the coordinates of the rooms depend on the coordinates of the windows which are placed on a shared wall. Consequently, for instance a window w which lies in the front side of the floor, like the window in Figure 3, and corresponds to the *ith* room, constrains the possible values of the y-coordinate of this room in the following way:

$$y_i = y_{w1} + wall_{ext} = y_{min} + wall_{ext}, \tag{4}$$

where $wall_{ext}$ denotes the depth of the exterior wall. On the other hand, we use the fact that the walls separating rooms do not have to cross windows. Therefore, the x- and y-coordinates, respectively, cannot take values where windows are placed. For the same window lying on the front side of the floor, the constraint can be expressed as follows:

$$((x_i \leq x_{w1}) \wedge (x_{w2} \leq (x_i + w_i))). \tag{5}$$

For windows of the left, right and back side of a building equivalent constraints exist. Note that the correspondence of a window w to a room i expressed by $wc_w = rno_i$ is as well not known a priori and has to be determined during the reasoning process. This combinatorial task is formulated as a labeling problem. The existence of a window in a room depends on the functional use of this room. For office rooms, an assignment of a window is obligatory, while it is not the case for corridors or utility rooms. If a room is not assigned to a window, it is not constrained by relations (4) and (5) and thus can be considered as an interior room.

In addition to the described hard constraints that have to be fulfilled for all rooms obligatory, the floor plan model contains two soft constraints. They hold true in most of the cases but can be violated for exceptions. However, the number of violations is bounded avoiding implausible topological models. One soft constraint considers that the room number is highly correlated to the neighborhood of rooms. If two rooms have consequent room numbers, they should be adjacent where possible. In our context, the term adjacency refers to the neighborhood of rooms. This is expressed by the following constraint—exemplarily for an *ith* room i left to a *jth* room j:

$$((x_i + w_i + wall_{int} = x_j) \wedge \neg((y_j + d_j \leq y_i) \vee (y_i + d_i \leq y_j))), \tag{6}$$

where $wall_{int}$ represents the depth of the interior wall. The symbol "¬" stands for a logical negation. Right, top, and bottom adjacency are also possible and can be defined in an equivalent way. Since the relative relations between rooms are not known, an exclusive-or combines the four possibilities for the adjacency of rooms. This also turns out to be a combinatorial task that we solve.

Since the corridor is usually the entrance to a room, a further soft constraint describes the adjacency of a room to an existing corridor. In this case, the adjacency constraint used for the neighborhood above is conditional on the functional use—in this case "corridor"—of the rooms.

3.2. Probability Density Functions

Besides constraints, the reasoning process benefits from statistical prior knowledge that is derived from a groundtruth database of about 1600 rooms with different functional uses. Figure 4 shows a relevant excerpt of the underlying database model. Central to our analysis is the relation *room* with shape and location parameters of each room as well as their functional use and the location of *doors* and *windows*. Rooms reference their corresponding *buildings*, which enables access to available footprints. Finally, the *neighbourhood* of rooms is annotated in order to analyze the bilateral locations e.g., with respect to the functional use of rooms. Note that this data does not serve as direct input for the reasoning process but is a representative basis for the derivation of probability distributions and constraints for the floor plan model and its prediction.

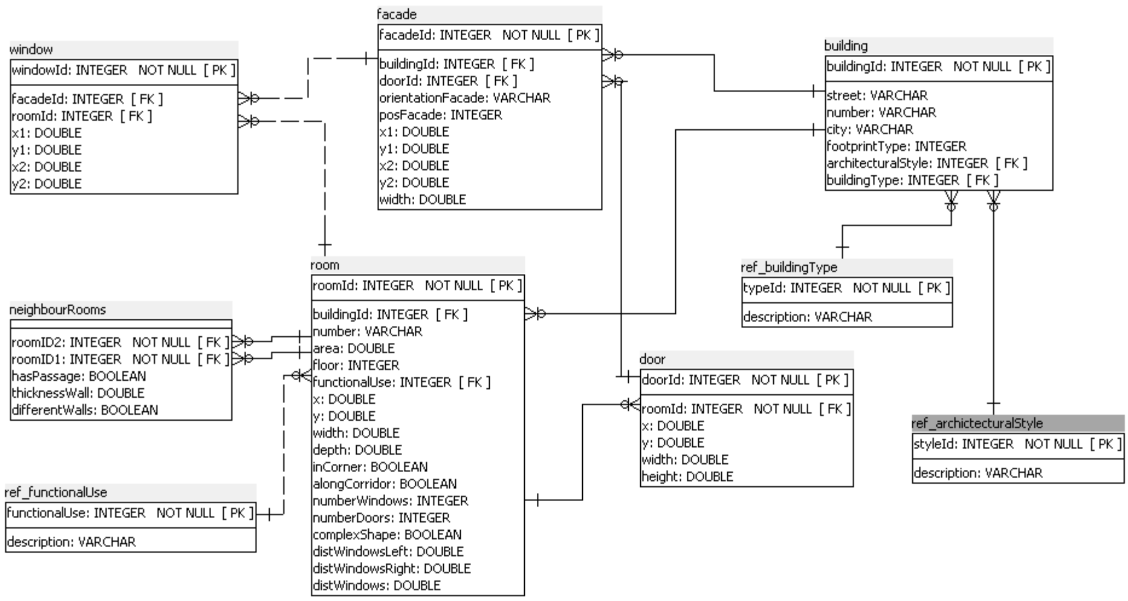

Figure 4. Excerpt from the relational database schema. Location and shape parameters of rooms are a.o. used for prior knowledge and evaluation.

As shown in the constraints before, the model is described basically by continuous parameters: x, y, width and depth of rooms as well as the depth of exterior and interior walls. Figure 2b shows the probability density function of the floor plan parameter width of office rooms estimated by the use of kernel densities and compared with its fitted Gaussian mixture. It has been shown that each arbitrary distribution can be approximated by a Gaussian mixture ([16]):

$$\sum_{i=1}^{m} \omega_i N(\mu_i, \sigma_i^2). \tag{7}$$

Gaussian mixtures are an appropriate way to model skew symmetric or multimodal distributions and make it possible to rely on a number of well-studied inference algorithms that are available in literature. By using Expectation Maximization ([16]), a Gaussian mixture of m components, each weighted by its probability ω_i for each continuous parameter, is estimated for the reasoning process. On the one hand, probability density functions are used to derive upper and lower bounds for the continuous model parameters during constraint propagation. On the other hand, they are an important knowledge for statistical inference.

4. Constraint Propagation for Topological Floor Plan Derivation

In a first step, the floor plan model is defined by constraints on several variables with associated domains. Constraints described above restrict these domains so that the final solution leads to a small number of qualified hypotheses. In conclusion, our problem can be seen as a *constraint satisfaction problem (CSP)*. Besides the values of the continuous model parameters defining the rooms, we are interested in further discrete parameters. In this context, the correspondence of windows to a room as well as the bilateral relations between rooms have to be determined. This turns out to be a complex combinatorial problem. The idea of our approach is to solve a constraint satisfaction problem with respect to valid values that can be used later on as initial instantiations for statistical reasoning. For solving those combinatorial constraint satisfaction problems, constraint programming is a powerful framework.

A constraint satisfaction problem is characterized by a set of constraints $\mathcal{C} = \{C_1, ..., C_q\}$ on domains $\mathcal{D} = \{D_1, ..., D_n\}$ of a set of variables $\mathcal{X} = \{X_1, ..., X_n\}$ that can be numeric-discrete as well as continuous or symbolic. The Cartesian product of the domains, i.e., $D_1 \times ... \times D_n$, defines the initial search space. A *constraint* C_i is defined as a relation on a subset of variables $\mathcal{X}' \subseteq \mathcal{X}$, i.e., a subset of $D_1 \times ... \times D_n$. They can be boolean or arithmetic—allowing linear as well as non-linear dependencies. A *solution of a CSP* is an instantiation of the variables, i.e., an assignment of values for each variable $\{(X_1, \alpha_1), ..., (X_n, \alpha_n)\}$ with $(\alpha_1, ..., \alpha_n) \in D_1 \times ... \times D_n$ so that all constraints are satisfied. Therefore, the constraint solver follows the "constrain and generate" principle in order to narrow the search space. Constraint inference is performed before finding valid instantiations. During inference, new constraints are derived from existing ones and existing constraints are tightened. This is done by using so called consistency-enforcing algorithms and constraint propagation [28].

The search benefits from the a priori known domains of the parameters and their constraints. In the context of floor plans, the model parameters of each room are restricted by the bounding box defined by the (rectangular) footprint of the building. The width and depth of the rooms depend on the functional use. For example, toilets usually cannot be as large as lecture halls. Their lower and upper bound can be derived from the Gaussian mixtures estimated by the use of the groundtruth database for rooms. The location of rooms is further bounded by the location of windows that are a priori known from exterior measurements. The constraints described in Section 3 are used to exclude impossible instantiations by constraint propagation. The latter means deducing additional constraints or restricting existing ones such as narrowing the domains. Herewith, the number of possible solutions is reduced. However, the set of possible solutions can contain those which are not consistent with regard to the constraints. The claim is to omit a subsequent search excluding inconsistencies as early as possible leading to single valued domains. In this context, the search space is affected by the level of consistency. Many domains of constraints can be updated as soon as a related domain changes by considering arc-consistency. A variable $X_i \in D_i, i = 1...n$ of a (binary) constraint is arc-consistent with respect to another variable $X_j, j \neq i$ if, for each value of X_i, there exists an instantiation for X_j not violating the constraint. However, in some cases, e.g., the equality constraint we use, a more efficient and but sufficiently powerful concept for such consistency-enforcing is to achieve bounds consistency. A constraint C with n variables $X_1, ..., X_n$ is *bounds-consistent* if for each variable $X_i \in [A, B], i = 1...n$, there exist instantiations for the other variables $X_j, j \neq i$ so that C is satisfiable with respect to the instantiations $X_i = A$ and $X_i = B$, respectively. A and B denote the lower and upper bound of the associated domain of the variable X_i represented as an interval. If a bound changes for one variable, new intervals for other variables are calculated by propagation rules in order to reduce the search space within the domains. For instance, for the constraint $X = Y * Z$ (e.g., $area = width * depth$) with $X \in [A, B], Y \in [C, D]$, $Z \in [E, F]$ and $X, Y, Z > 0$, the interval bounds for Z can be updated to $[A/D, B/C]$.

A search for solutions is performed by traversing a search graph and finding a solution path to the leaf nodes. A node represents a variable together with one possible instantiation. An arc represents an operator that augments the current solution with a value assignment to an additional variable that does not conflict the prior instantiations regarding the constraints. The search is in general characterized by backtracking. It uses a depth-first search and jumps back to prior states (nodes in the search graph) if the search leads to a dead-end. A dead-end is a leave that is inconsistent with the constraints which means that the domain of its corresponding variable became empty after the constraint propagation. In order to perform a search with as few failures as possible and to avoid backtracking, two principles are followed dynamically during search: Look-ahead and Look-back. The first determines the best choice for the next variable and its value, while the second deals with the level where the algorithm jumps back in case of a dead-end. More details on constraint processing can be found in [28,29].

There exist various implementations of constraint solvers [29]. In addition to algebraic, symbolic or graph-based algorithms, a prominent implementation is Constraint Logic Programming realized by the use of logic programming. CLP benefits from the declarative character of logic programs and the powerful search strategies of constraint programs in order to define and solve constraint satisfaction problems. The search strategy in logic programming is characterized by backtracking with depth-first search. The constraints correspond to relations and predicates in logical language. (Logic) constraint solvers as used in our implementation are powerful tools to handle non-linear constraints with more than one unknown.

The strength of CLP is to solve combinatorial problems in a declarative way. This enables in addition to predefined relations to define customized ones in a simple and flexible way. "Combinatorial problems can be tackled which usually have exponential complexity" [30]. In our context, we exploit the fact that separating walls of the rooms do not cross windows. We expressed this constraint as a logical relation. Before searching solutions, we enumerate possible (discrete) values for the x-coordinates of each room excluding those falling within window ranges. Exploiting architectural knowledge, we do not have to cope with the a priori infinite continuous search space. Instead, we transfer the problem into an enumeration of interesting values so that the search algorithm finds the arrangement of the rooms much faster than dealing with infinite continuous domains. The most important value is the one that lies in the middle between two windows. This is attributed to the fact that most of the walls are centered between windows. This high level of discretization is especially valid for rooms that obligatory have a window such as office rooms. Other rooms of different functional use e.g., corridors are excluded and their domains are represented by intervals with decimeter precision. This is necessary since the reference points of corridors can take values that are excluded by the windows for other rooms.

The discretization along the x-axis leads to an easier instantiation of the other parameters. If, for the ith room, x_i and x_i^{max} are determined, the width $w_i = x_i^{max} - x_i$ can be subsequently derived (cf. Figure 3). Likewise, the assignment of the depth follows from a previous assignment of the width according to the constraint $area = d_i * w_i$. Following this pattern, we are transforming constraints into functions with only one unknown parameter. The y-coordinate y_i is determined by the corresponding window, that is the minimum y_{min} or the maximum y_{max} of the footprint respectively, depending on the window location. If the window lies on the front side of the floor, $y_i - wall_{ext}$ equals the lower bound y_{min} of the floor. Otherwise, if the window is located on the opposite side, $y_i^{max} + wall_{ext}$ equals y_{max}.

We are aware that the transition from continuous to discrete values does not guarantee to match the expected values. We ignored temporary the walls in order to provide buffers for the deviations occurred after the value discretization. In this context, the variables $wall_{int}$ and $wall_{ext}$ have not to be determined in this step. As a consequence, irregular gaps exist between the rooms and the alignment of rooms along the corridor is not automatically given. Furthermore, uncertain measurements were not considered during the combinatorial reasoning. Figure 5 shows in the second row the intermediate result that is found by constraint propagation. Although the result is topologically correct and satisfies the given constraints defining the floor plan model, it has to be adapted geometrically in a subsequent step. This will be elaborated in the next section.

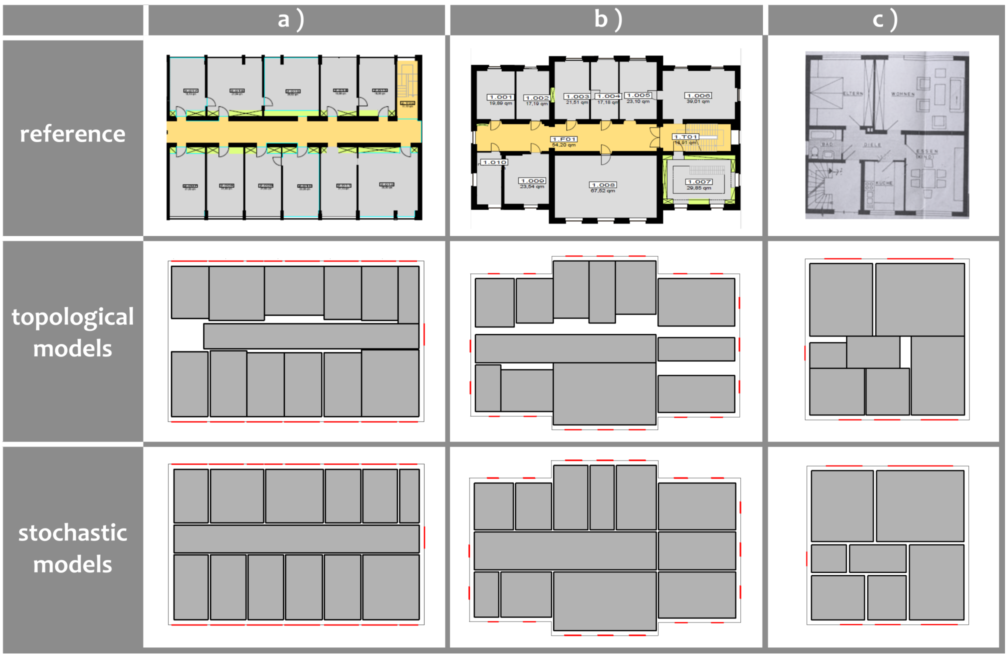

Figure 5. (**a–c**) Floor plan prediction demonstrated on three examples with regard to different requirements.

5. Conditional Linear Gaussian Models for Stochastic Floor Plan Prediction

As stated in the previous section, constraint propagation yields an intermediate result with a correct topology, but suboptimal geometrical parameters. The predicted preliminary rooms do not fill the entire footprint space. In particular, the alignment of rooms along a corridor is not guaranteed. The combinatorial reasoning provides initial values that are updated and refined. The result remains topologically equivalent but becomes geometrically different in order to match the provided measurements. After tackling the combinatorial task leading to a consistent solution from a topological point of view, we focus on closing the gaps between the rooms. In this stage, exact inference can be performed in order to satisfy this task.

The inference is realized in the frame of a stochastic reasoning which gives access to well known statistical algorithms. The functional model is defined basically by two types of constraints that ensure the geometrical consistency. The most important constraint describes the local relation between two adjacent rooms. We exploit the fact that the combinatorial component finds the bilateral relations between rooms. For example, if the *ith* room is left to the *jth* room, then their coordinates are related by:

$$x_i + w_i + wall_{int} - x_j = 0. \tag{8}$$

This constraint enforces not only the adjacency between rooms but also the alignment of rooms along corridors. This is a consequence of the rectangular shaped rooms according to the Manhattan world assumption. It should be noted that constraint (8) is a part of constraint (6) since topological aspects do not have to be considered anymore.

Similar to the constraint (4), the exact location of the reference point of each room is corrected exploiting the information about windows' correspondences to rooms from the combinatorial part:

$$y_{min} = y_i - wall_{ext}. \tag{9}$$

Again, this constraint exemplarily holds for windows (and their associated rooms) that lie on the front side of the floor.

In order to perform a stochastic reasoning, we make use of probability density functions addressing the uncertainty of the model parameters. It can be stated that the prior knowledge is in principle neither Gaussian nor unimodal. This statement can be confirmed in Figure 2b exemplified by the room width as an important part of our stochastic background knowledge. Non-Gaussian and multimodal distributions are known to not be suitable for efficient and exact reasoning. We mentioned already that we overcome this obstacle by using Gaussian mixtures. When we use the intermediate model of the combinatorial part as a starting point for the stochastic estimation of geometric parameters, each model parameter is related to one component of a Gaussian mixture. Using this approach, we can safely assume that parameters are normally distributed in the following and are no longer multimodal. We are now in a special, well-understood field of stochastic reasoning with probabilistic graphical models.

Probabilistic graphical models are nowadays one of the most prominent and most powerful methods for reasoning in uncertain domains. Bayesian Networks are one type of graphical model represented by directed graphs, where each node $v \in V$ is related to a random variable X_v and where the absence of edges indicates independencies between these model parameters ([12]). Random variables are characterized by conditional probability distributions (CPD) $P(X_v|X_{pa(v)})$ that give probabilities for each variable status dependent on the instantiation of its parent nodes $pa(v)$, which are its immediate predecessors as induced by the graph. The graph structure enables a compact representation of a joint distribution and paves the way for an efficient inference in order to determine the posterior distribution given an observation. The presented floor plan model is defined by constraints on discrete as well as continuous variables and thus has to be represented by hybrid networks that are characterized by nodes associated with discrete as well as continuous variables $\mathcal{X} = \mathcal{X}_\Delta \cup \mathcal{X}_\Gamma$. In contrast to approximate reasoning, exact inference within hybrid networks is in general not feasible. The feasibility is achieved with additional assumptions such as that a parameter X is normally distributed with mean μ and variance σ^2:

$$p(x;\mu,\sigma^2) = N(\mu,\sigma^2) = \frac{1}{(2\pi\sigma^2)^{1/2}} exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right).$$

A special case of Bayesian Networks is the conditional linear Gaussian (CLG) network ([31,32]) that assumes Gaussian distributions for the continuous parameters. All continuous variables have to be described by conditional linear Gaussian CPDs, and their corresponding continuous nodes are not allowed to be a parent of a discrete node. A conditional linear Gaussian CPD with $I \subseteq \mathcal{X}_\Delta$ and $Z \subseteq \mathcal{X}_\Gamma$ is defined as

$$p(X|Z = z, I = \tau) = N(\mu_\tau + \beta_\tau^T z, \sigma_\tau),$$

where μ_τ is a mean value for instantiation τ, β_τ a vector of regression coefficients and σ_τ the corresponding variance.

The joint distribution in a hybrid network can thus be defined as an $|X_\Gamma|$-dimensional Gaussian distribution

$$p(X_\Delta = \tau) \cdot N_{|\mathcal{X}_\Gamma|}(\mu_\tau, \sigma_\tau^2) = \prod_{v \in V_\Delta} P(\tau_v|\tau_{pa(v)}) \prod_{w \in V_\Gamma} p(y_w|X_{pa(w)})$$

for each instantiation τ of $\mathcal{X}_\Delta$ ([33]).

In the context of floor plan modeling, our problem fits well the assumptions on CLGs. Therefore, we use a CLG network in order to model uncertainty and improve the intermediate result from the CLP. The selection of the components of the Gaussian mixtures allows us to use a specially structured Bayesian Network: a state-observation model with an n-dimensional state vector $x \in \mathbb{R}^n$ representing the model parameters and an m-dimensional observation vector $o \in \mathbb{R}^m$ that can be described by the mapping $o = Mx$ with a measurement matrix $M \in \mathbb{R}^{n \times m}$. For such state estimations, the correction step of Kalman filter is an efficient algorithm for calculating the posterior. It assumes that state transition and measurement can be described linearly and initial beliefs are represented by multivariate Gaussian distributions.

In our context, Gaussian distributions with μ and σ are carried over from the constraint solver of the reasoner and are augmented by two dimensions in order to model the exterior and interior walls. The functional model represented by the measurement matrix M is defined according to Equations (8) and (9). It should be noted that in this case compared to Equation (6), the equality of parameters is converted into a substraction that equals zero. In this way, the state-observation model can be used with a pseudo-observation equal to zero. The posterior is computed by matrix multiplications similar to the correction step of the Kalman filter. The Kalman gain

$$K = \Sigma M^T (M \Sigma M^T + Q)^{-1} \tag{10}$$

is used to update, that is, to adjust the intermediate result of the combinatorial component and the Gaussian distributions are updated by

$$\begin{aligned} \mu &= \mu + K(o - M\mu), \\ \Sigma &= (Id - KM)\Sigma), \end{aligned} \tag{11}$$

where $Q \in \mathbb{R}^{m \times m}$ is the Gaussian noise of the observations and Id is the identity matrix.

Finally, the most likely assignment (MAP assignment) for given evidence $E = e$ is found by maximizing the posterior probability for variables $W = \mathcal{X} \setminus E$: $MAP(W|e) = \arg\max_\omega P(\omega, e)$. The presented work aims to find the k most probable explanations, denoted by $MAP^k(W|e)$, in order to assess the quality of the solutions.

The reasoner provides means $\mu_i \in \mathbb{R}^{|\mathcal{X}_\Gamma|}$ for continuous model parameters of the *ith* hypothesis and the related instantiations $\tau_i \in \mathbb{R}^{|\mathcal{X}_\Delta|}$ for discrete variables. Final hypotheses are ordered by their (unnormalized) probabilities P_i calculated on the basis of the a priori known distributions:

$$P_i = exp(\sum_{j=1}^{|\mathcal{X}_\Gamma|} log(pdf_j^{01}(\mu_{ij}))),$$

where pdf_j^{01} is the on [0...1] scaled density of the distribution corresponding to the *jth* model parameter. We finally get a set of hypotheses of the most probable floor plans given the observations:

$$H_{best} = (\mu_1, \tau_1, P_1), ..., (\mu_k, \tau_k, P_k).$$

Since, in this stage, we can assume Gaussian distributions, likelihood of hypotheses is calculated based on the given covariance matrices and the residuals in the ordinary way. The probability density functions for each model parameter provide the possibility to assess the likelihood of each hypothesis and order them accordingly.

6. Results and Discussion

As a result, the reasoner yields automatically the most probable floor plans in significantly less than a second in nearly all cases on a Windows 64 Bit machine (3.4 GHz, 16 GB RAM). We assumed standard deviations of 10 cm for the location of windows and yielded, in our test cases, accuracies for the model parameters between 10 and 20 cm. The results are summarized in Figure 5. Each predicted floor plan in the last row corresponds to the best ranked hypothesis found by our approach. The second row shows the intermediate results provided by the constraint propagation. For comparison, the reference floor plans are depicted in the first row. The quality of the results depends on how much the floor plans meet the norm with respect to the distributions derived from the prior knowledge.

It can be seen that the reasoning component in the combinatorial part estimates the correspondence of windows to rooms and determines the approximate location of each room. Not modeling the interior walls provides a buffer which supports preserving topological correctness. Geometric consistency is considered in the next step as well. The statistical component fills the gaps by adjusting the shape and location parameters and the depth of walls. The alignments of rooms along existing corridors are ensured as well. The listed results of our reasoning process show exemplarily floor plans predicted with regard to different requirements. The first column depicts a rectangular floor plan stemming from an office building. Most of the buildings are characterized by a rectangular shape. The second column shows a floor plan that does not satisfy the rectangularity assumption. In this case, it is adapted to a rectangular shape by adding four auxiliary rooms. Nevertheless, the non-rectangularity turns out to be beneficial and can be exploited to determine grid points for the discretization of the x-coordinate. While a simple unstructured rectangular footprint provides no information about walls, in this case, the left and right boundaries of the protrusions indicate an interior wall as a bound for a room at this position. The last row visualizes the results of the reasoning for a floor plan in a residential house. The particularity for this type of floor plan is that the rooms usually do not have an identifying number that could restrict their location.

How the knowledge about room numbers influences the prediction is shown in Figure 6. Due to the soft constraint of adjacent rooms in the case of consequent room numbers, the reasoner finds the correct ordering of the rooms and thus predicts the floor plan. The left result is avoided using an upper bound for the number of the violations of the soft constraint so that only rooms 1.006 and 1.007 are not adjacent. Furthermore, the knowledge of one single correspondence of a room to a window accelerates the reasoning process. This fact is especially beneficial for rooms whose associated windows are easily identifiable from outside, such as the case of stairs.

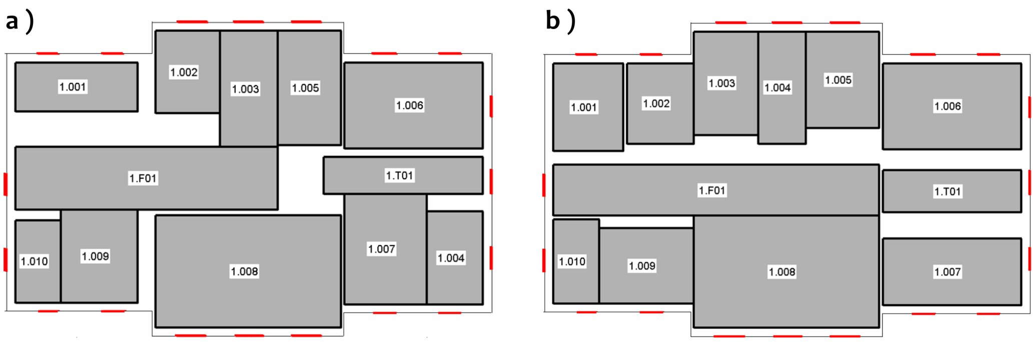

Figure 6. The incorporation of room numbers in (**b**) contributed to better results than without room number information in (**a**).

Derivation of a 3D Indoor Model from Predicted Floor Plans

Up to now, we have described how floor plans can be estimated without any indoor measurements. This 2D model is extended to a 3D model by extruding the walls of the rooms and inserting windows and doors. Windows are known a priori from outdoor measurements. Doors can be predicted due to available distributions of the location of doors from annotated data. Figure 7 shows these distributions, where possible locations of a door are divided into left, right or middle. Hypotheses are derived depending on the size of the rooms, which influences the distribution.

The height of the walls in turn is derived from the height of the facade and the location of windows. As described in [13], the height of each story can easily be derived by kernel density estimations based on 3D point clouds of facades. Since windows are usually located at hip height, the knowledge can be exploited to predict the location of the floor within the building.

Figure 8 shows the 3D indoor model corresponding to the floor plan derived in Figure 5b. In rooms with a width smaller than 3 m, doors are mostly located in the middle of the wall shared with the corridor. In larger rooms, the histogram shows that the left and right position of the door is more probable. The doors in our 3D model are predicted based on these distributions. Special rules are introduced in order to model architectural features. For example, for the rooms at the right side of the building, doors cannot be located close to the right wall, as indicated by the learned distribution, since the stairs would avoid this placing (cf. reference image in Figure 5b. Furthermore, in small toilets, such as room 1.010, the door does not lie in the middle since the space should be used to place a sink close to the door. A courtroom in cultural heritage buildings, such as the case here, usually has the door centered in the wall while lecture halls or auditoriums locate the door at the left or right side due to the rows of chairs and their accessing corridor. False positives of our door prediction are marked with green dotted lines. Future work will take this issue into account based on a supervised classification task in order to predict doors in a more robust way.

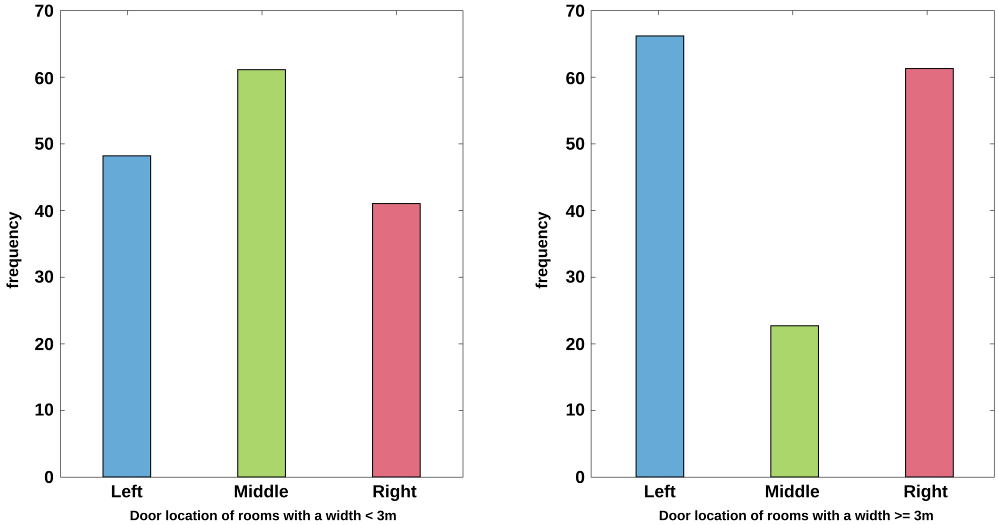

Figure 7. Histograms of the location of doors depending on the width of rooms. This information is used for the prediction of door locations in derived floor plans for 3D indoor modeling.

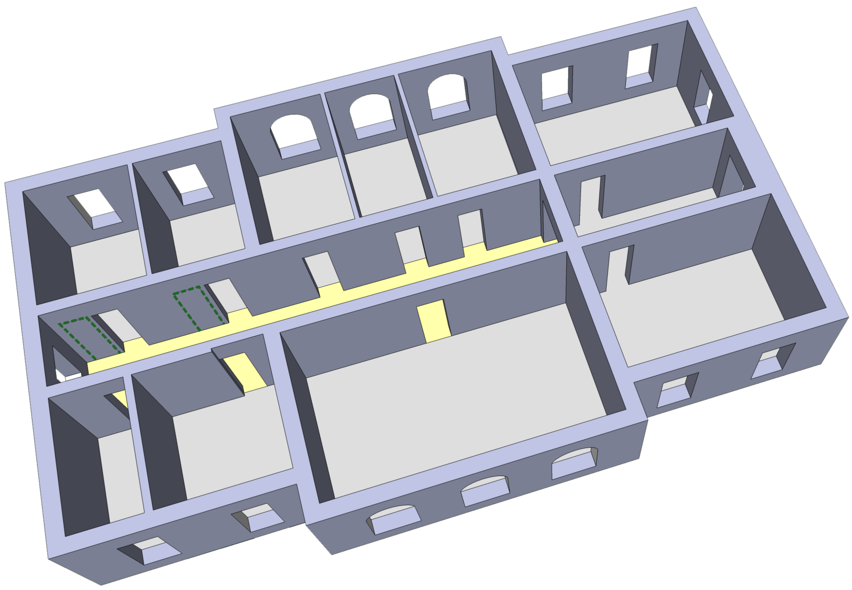

Figure 8. 3D indoor model of the resulted floor plan from Figure 5b. Door locations and story heights are derived and classified from background knowledge.

7. Conclusions

The paper presents a novel approach for the prediction of floor plans without the need of any indoor measurements. The algorithm represents prior knowledge with probability distributions of the model parameters on the one hand and linear as well as bilinear constraints on the other hand, using an extensive data analysis. This analysis is based on a large database of about 1600 buildings, which captures, among others, shape and location parameters of the rooms. Parameter distributions are represented by Gaussian mixtures that are flexible enough to model multi-modal distributions on the one hand, and can be reduced in our approach to single Gaussian distributions allowing exact stochastic inference on the other hand. Constraints together with probability density functions reduce the search space and enable reconstruction of floor plans based on otherwise insufficient data.

In order to be able to apply exact stochastic inference in complex models, we combine constraint processing with a conditional linear Gaussian graphical model. The combinatorial problem of assigning each window to a room and determining the bilateral relations between rooms is solved by constraint propagation, leading to preliminary topological models. This intermediate result is adjusted by a statistical component that aligns rooms along corridors where possible and estimates walls completing the floor plan model.

Initially, we had assumed that floor plans and rooms are rectangular. Figure 8, however, illustrates how an extension allows more general shapes of floor plans. We are aware that there are more general room layouts such as L, T or even U shaped rooms. Despite these shapes of rooms seeming to violate basic assumptions on general approaches, we believe that they can be modeled as a composition of two or more rectangular rooms. Whereas the stochastic modeling seems to be feasible, the modeling and design of constraints for a constraint solver will be a subject of future work.

Based on the predicted floor plans, a 3D indoor model is built with the use of architectural regularities and probability distributions exploited for the prediction of doors. The floor heights are extracted from the distances between windows from subsequent floors. The door locations are predicted using Gaussian mixtures, whereas the shape parameters of doors are derived from probability density functions depending on the building style.

In this paper, the examples have been chosen in such a way that they represent rectangular models that cover a significant part of office rooms as well as non-rectangular floor plans. We took not only post-war building styles into consideration, but also buildings stemming from the beginning of the last century. The paper dealt also with residential houses. More complex room layouts in this context will be the topic of a forthcoming paper.

Recently, the integration of BIM and outdoor models in the CityGML format have gained more attention. BIM is a hot topic in the context of construction industry and supports both the construction process and facility management. While, in the recent past, models in the geoinformation and CityGML context focused on the outer building surfaces, constructive aspects and building indoors are the main points of interest from a BIM point of view. We believe that, in cases where BIM models are not available, our approach that predicts indoor models without additional measurements provides a link between BIM and CityGML.

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Author Contributions: Sandra Loch-Dehbi and Lutz Plümer designed and implemented the reasoning method. Youness Dehbi contributed to the graphical model part. All authors took part in manuscript preparation.

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