Embeddings of group rings and L^2-invariants

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The most exciting phrase to hear in science, the one that heralds new discoveries, is not "Eureka!" but "That's funny...".

— Isaac Asimov

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Introduction

Given a ring R and a group G, a fundamental construction in algebra is that of the group ring RG, that is, of the free R-module with basis the elements of G and multiplication extended R-linearly from that on G. Even though the group ring provides a very simple way to obtain a ring from a group, it is a major object of study and plays a role in numerous fields of mathematics, including but not limited to algebraic topology, dynamical systems, representation theory, and theoretical computer science.

For a finite group G and a field K, studying the group ring KG amounts to understanding the linear representations of G, for which ordinary (if K is of characteristic 0) and modular (if K is of prime characteristic) character theory provide plenty of tools. In the particularly well-behaved case of a coefficient field of characteristic prime to the order of G, the group ring KG decomposes into matrix rings over division K-algebras, i.e., it is semisimple, and the individual factors can be understood in terms of characters.

Much less is known in the case where G is an infinite group. Without the help of character theory, the methods used to study the group ring KG naturally become more ring-theoretic in nature. Whereas the decomposition into matrix rings over division rings provides a full classification of the non-trivial idempotents, zero divisors, and units in the case of a finite group, the analogous questions for torsion-free groups are open in general:

Conjecture (Kaplansky conjectures)**.** *Let G be a torsion-free group and K a field. Then*

(a) *all idempotents in KG are trivial, i.e., if $x^2 = x$ for $x \in KG$, then $x = 0$ or $x = 1$;*

(b) *all zero divisors in KG are trivial, i.e., if $xy = 0$ for $x, y \in KG$, then $x = 0$ or $y = 0$;*

(c) *all units in KG are trivial, i.e., if $xy = yx = 1$ for $x, y \in KG$, then $x = kg$ with $k \in K$ and $g \in G$.*

The Kaplansky conjecture on units implies that on zero divisors, which in turn implies that on idempotents. The Kaplansky idempotent conjecture has been approached quite successfully with methods from C^*-algebras and algebraic K-theory, relying on the fact that it is implied by both the Baum–Connes conjecture (see [Val02]) and the Farrell–Jones conjecture (see [BLR08]). The unit conjecture has yet to be embedded into a conceptual algebraic framework and progress has so far mostly gone through the strictly stronger unique product property, which is known to be false for general torsion-free groups [Pro88]. In the following, we will thus focus on the zero divisor conjecture, which represents an interesting middle ground between the two other conjectures.

For a commutative ring R without non-trivial zero divisors, there is always an associated algebraic object that certifies the absence of such elements: By adjoining inverses of all non-zero elements to R, we obtain a field, the so-called field of fractions, into which R embeds. Clearly, a ring that is contained in a field cannot contain non-trivial zero divisors and the same holds true more generally for a subring of a division ring, i.e., a not necessarily commutative ring in which every non-zero element is invertible. Thus, a natural strengthening of the Kaplansky zero divisor conjecture is the subject of the following question:

Open Problem ([Kourovka, 1.5]). *Let G be a torsion-free group and K a field. Does KG embed into a division ring, i.e., a ring in which every non-zero element is invertible?*

Starting well before its first inclusion in the Kourovka notebook in 1965, many positive and no negative answers to this question have been obtained. In particular, it has been answered in the positive for the free metabelian group on two generators [Mou37], free and more generally biorderable groups [Mal48; Neu49], and torsion-free one-relator groups [LL78]. All of these results have in common that they give explicit constructions of the embedding division rings.

More recently, initiated by an influential paper of Linnell [Lin93] from 1993, the question whether the group ring of a torsion-free group embeds into a division ring has seen tremendous progress via methods stemming from an interplay between algebraic topology and functional analysis. In order to motivate this connection, we will first consider a potential application of the embedding question to a common situation in algebraic topology. Given a topological space X with an action by a group G, the singular chain complex $C_*(X)$ admits the structure of a $\mathbb{Z}G$-chain complex. If we assume that G is torsion-free and embeds into a division ring D, then it could be expected that the possibly infinite natural numbers

$$b_n^D(X) \coloneqq \dim_D H_n(C_*(X) \otimes_{\mathbb{Z}G} D),$$

where D is viewed as a $\mathbb{Z}G$-module via the embedding, bear topological significance similar to that of ordinary Betti numbers. Even though this is opposite to how the theory of L^2-invariants evolved historically, we will review in Chapter 2 that the most prominent conjecture in the field, the *strong Atiyah conjecture*, ensures that the so-called *L^2-Betti numbers* can be expressed in this way and in particular that $\mathbb{Z}G$ embeds into a division ring.

The key difference between classical embedding results and those obtained as a consequence of the strong Atiyah conjecture is that the latter start with a naturally defined von Neumann regular overring of the group ring, the *$*$-regular closure* $\mathcal{R}_{KG}$, that exists for all groups. This overring is then shown to be a division ring, but it already has some convenient properties to start with. This additional structure enables proofs both for larger classes of groups and of inheritance properties, such as permanence under certain types of extensions and (co-)limits. Furthermore, $\mathcal{R}_{KG}$ is also defined for groups G with torsion and the strong Atiyah conjecture implies the characteristic 0 case of the following more general embedding conjecture:

Open Problem. *Let G be a group with a finite bound on the order of its finite subgroups and K a field. Does KG embed into a semisimple ring?*

The aim of this thesis is to study the structure of the ring $\mathcal{R}_{KG}$ both in general and in restricted settings, such as assuming certain variants of the strong Atiyah conjecture or considering only particular classes of groups.

Structure of the thesis

Chapter 1 sets up the methods and notions from ring theory that will be used throughout the thesis. Both crossed products, which are generalizations of group rings, as well as non-commutative localizations of rings are introduced here and play a fundamental role in all following chapters. The class of rings that will be most important for us is that of von Neumann regular rings. These rings are easily defined as those in which for every element x there exists an element y satisfying $xyx = x$, that is, where y acts as a two-sided inverse of x after being multiplied by x. This innocuous property turns out to have a profound impact on the homological algebra of these rings, making them almost as convenient as division rings in many aspects.

The key construction reviewed in this chapter is the $*$-regular closure of a ring R inside a von Neumann regular ring S with a compatible involution. As opposed to other classical notions of ring closures, such as division and rational closures, the $*$-regular closure always enjoys good ring-theoretic properties: it is again von Neumann regular and its finitely presented modules are formal difference of modules induced from finitely presented R-modules. Intuitively, this type of closure enjoys the convenient properties of S while still being very close to R in all matters related to finitely presented modules.

In Chapter 2, we introduce the theory of L^2-invariants, first and foremost L^2-Betti numbers, which can be assigned to topological spaces with an action by a group G. Their construction involves the *group von Neumann algebra* $\mathcal{N}(G)$, which can be viewed as a completion of the complex group ring $\mathbb{C}G$. Given that our general focus lies on algebraic aspects of the theory, we quickly pass from $\mathcal{N}(G)$ to a suitable localization, the *algebra of affiliated operators* $\mathcal{U}(G)$, which enjoys even better ring-theoretic properties by being von Neumann regular. Both of these rings are introduced in Section 2.1. The particular usefulness of $\mathcal{N}(G)$ and $\mathcal{U}(G)$ stems from the existence of a real-valued additive dimension function for modules over these rings, known as the *von Neumann dimension.* We will review its general properties in Section 2.2.

Even though our introduction to L^2-invariants may, apart from a few black boxes from functional analysis, seem quite algebraic, the origins of L^2-Betti numbers lie in the spectral analysis of heat kernels on Riemannian manifolds. It may therefore come as a surprise that these numbers, defined as von Neumann dimensions of homology groups with coefficients in the algebra of affiliated operators, often turn out to be integers. This was already remarked by Atiyah when he first introduced L^2-Betti numbers in [Ati76] and led to the famous conjecture about the rationality of L^2-Betti numbers that is now firmly attached to his name. We refer to it as the *weak Atiyah conjecture* given that we will discuss quite a few variations of the "Atiyah question" in this thesis. The conjecture together with counterexamples will be discussed in Section 2.3.

The *strong Atiyah conjecture*, which we have already alluded to in the introduction, is formally introduced in Section 2.4. It goes beyond the weak Atiyah conjecture in that it prescribes the possible denominators of L^2-Betti numbers for a particular group G, but only applies to groups with a uniform bound on the orders of their finite subgroups. Further consequences for the values of von Neumann dimensions are discussed in Section 2.4.1. Even though most of the fundamental results on the Atiyah conjecture that hold for all groups are well-known, we aim to give self-contained proofs that highlight the role of the $*$-regular closure $\mathcal{R}_{KG}$ as an algebraically well-behaved intermediary between the group ring KG and the algebra of affiliated operators $\mathcal{U}(G)$. As an advantage of this approach, we in some cases obtain results that are slightly stronger than those recorded in the literature. Among these results are bounds on all structure constants of the Artin–Wedderburn decomposition of $\mathcal{R}_{KG}$ assuming only the strong Atiyah conjecture (see Proposition 2.4.6) as well as an equivalent formulation of the conjecture in terms of von Neumann dimensions of arbitrary $\mathcal{R}_{KG}$-modules (see Proposition 2.4.10). Following an introduction to commonly used classes and properties of groups, a detailed overview of the current status of the strong Atiyah conjecture over arbitrary subfields of $\mathbb{C}$ is given in Section 2.4.3. The first two major results on the conjecture include a proof by Linnell [Lin93] for a class of groups $\mathcal{C}$ that contains all free-by-{elementary amenable} groups as well as a proof by Schick [Sch01] for a class of groups $\mathcal{D}$ that contains all residually {torsion-free elementary amenable} groups. More recently, the conjecture has been resolved for locally indicable groups by Jaikin-Zapirain and López-Álvarez [JL20]. Our presentation of the current status puts particular emphasis on those inheritance properties that are not just enjoyed by a particular constrained subclass of groups, but rather hold for the classes of all torsion-free, sofic, or arbitrary groups satisfying the strong Atiyah conjecture.

We conclude with a brief introduction to the *center-valued Atiyah conjecture* in Section 2.5, which is even stronger than the strong Atiyah conjecture and allows for a full description of the semisimple structure of $\mathcal{R}_{KG}$ in terms of the finite subgroups of G. The implications between this and other variants of the Atiyah conjecture treated in this thesis are summarized in a diagram in Section 2.6.

Having set the stage around the strong Atiyah conjecture and the $*$-regular closure $\mathcal{R}_{KG}$ in the first two chapters, the remaining chapters branch out into two natural and mutually orthogonal directions of research: Groups with torsion are the focus of Chapter 3, whereas torsion-free groups are treated in Chapters 4 and 5.

Chapter 3 opens to an investigation of the so-called *algebraic Atiyah conjecture*, which was introduced by Jaikin-Zapirain in [Jai19a]. The unique characteristic of this conjecture is its purely K-theoretic formulation in terms of the elements of $K_0(\mathcal{R}_{KG})$ that are induced from finite subgroups. Our first result on the algebraic Atiyah conjecture answers one of the questions Jaikin-Zapirain raised in his survey:

Theorem (Theorem 3.1.4)**.** *The algebraic Atiyah conjecture is equivalent to the center-valued Atiyah conjecture.*

Even though this means that the algebraic Atiyah conjecture does not constitute a new variant of the Atiyah conjecture from a logical point of view, its algebraically convenient formulation nonetheless makes it very helpful for proving inheritance properties. As an example of such an application, we combine the techniques underlying the base change result for the strong Atiyah conjecture for sofic groups obtained by Jaikin-Zapiran [Jai19c] with a careful analysis of the K-theoretic effects of a change of coefficients for semisimple algebras to obtain the following result:

Theorem (Special case of Theorem 3.1.4)**.** *Let G be a sofic group with* $\mathrm{lcm}(G) < \infty$*. If G satisfies the center-valued Atiyah conjecture over $\overline{\mathbb{Q}}$, then it satisfies the center-valued Atiyah conjecture over $\mathbb{C}$.*

We conclude the chapter with results on $\mathcal{R}_{KG}$ that are not conditional on the strong Atiyah conjecture. A slight generalization of an argument of Lück based on the Hattori–Stallings rank provides an unconditional lower bound, matching the upper bound implied by the center-valued Atiyah conjecture, on the rank of $K_0(\mathcal{R}_{KG})$ in terms of the elements of G of finite order. Focusing on the conjectured torsion-freeness of $K_0(\mathcal{R}_{KG})$ instead of its rank, we again make use of Jaikin-Zapirain's base change techniques to give a partial answer to a question of Ara and Goodearl raised in [AG17]:

Theorem (Theorem 3.4.6)**.** *Let $K \leqslant \mathbb{C}$ be of infinite transcendence degree over $\mathbb{Q}$ and closed under conjugation and let G be a sofic group. Then $\mathcal{R}_{KG}$ is unit-regular.*

In Chapter 4, the focus lies exclusively on torsion-free groups. Given that the strong Atiyah conjecture over $\mathbb{Q}$ for a group G implies that the group ring $\mathbb{Z}G$ embeds into a division ring, it is a natural question to ask whether a theory analogous to that of L^2-invariants can be developed based on any such embedding and without additional input from functional analysis. Starting with nothing but a fixed map from a group ring $\mathbb{Z}G$ to a division ring D, we define analogues of L^2-Betti numbers, universal L^2-torsion, twisted L^2-Euler characteristics, and the L^2-polytope, where the latter have been introduced by Friedl and Lück in a series of papers [FL19; FL17]. In the case where the chosen map is the embedding of KG into the division ring $\mathcal{R}_{\mathbb{Q}G}$ provided by the strong Atiyah conjecture, our so-called *agrarian* invariants recover the classical L^2-invariants.

Apart from providing evidence for the point of view that most of the structural properties of L^2-Betti numbers are indeed rooted in the group ring itself rather than special properties of the rings $\mathcal{U}(G)$ and $\mathcal{R}_{\mathbb{Q}G}$, we also offer an application of agrarian invariants to one-relator groups. In [FT20], Friedl and Tillmann introduced an invariant of two-generator one-relator groups that is defined in terms of a group presentation and takes two-dimensional integral polytopes as values. They also relate the thickness of the polytope to the minimal complexity of HNN splittings of the group. The question whether this invariant is independent of the choice of the presentation was picked up by Friedl and Lück in [FL17], where a positive answer is given assuming that the group is torsion-free and satisfies the strong Atiyah conjecture. At the time, this conjecture was not yet known to hold for all one-relator groups even though it had already been established in [LL78] that the group rings of such groups embed into division rings. Using agrarian invariants, specifically the agrarian polytope and twisted agrarian Euler characteristics, we can remove the assumption on the strong Atiyah conjecture:

Theorem (Precise formulation in Theorems 4.6.16 and 4.6.21)**.** *The Friedl–Tillmann polytope invariant for two-generator one-relator groups admits a construction that is intrinsic to the group and in particular does not depend on a choice of a group presentation. The thickness of the polytope in a given direction corresponds to the minimal complexity of an HNN splitting of the group with that direction as its character.*

The strong Atiyah conjecture for one-relator groups has since been resolved by Jaikin-Zapirain and López-Álvarez [JL20], which provides an alternative proof of this result.

In Chapter 5, we study group rings of free-by-{infinite cyclic} groups, which are always torsion-free and satisfy the strong Atiyah conjecture as members of Linnell's class $\mathcal{C}$. As this already provides us with an embedding of the group rings into division rings, we can further analyze the way in which the division ring is constructed out of the group ring, with the aim of identifiying this process as a particular kind of non-commutative localization. We exploit the fact that these group rings can be expressed as skew Laurent polynomial rings over group rings of free groups, where the homological algebra of the latter is particularly constrained as in such rings every ideal turns out to be free of unique rank. This very strong property ensures that the group rings of free groups embed into division rings over which all matrices that possibly could become invertible, i.e., that they are *Sylvester domains*. We modify a homological criterion for this property due to Jaikin-Zapirain [Jai19b] and combine it with recent results on the Farrell-Jones conjecture for normally poly-free groups by Brück, Kielak, and Wu [BKW19] to prove that the group rings of free-by-{infinite cyclic} groups satisfy this property stably:

Theorem (Special case of Theorem 5.B)**.** *Let K be a field of arbitrary characteristic and G a group arising as an extension*

$$1 \to F \to G \to \mathbb{Z} \to 1$$

where F is a free group. Then the group ring KG is a pseudo-Sylvester domain unconditionally and a Sylvester domain if and only if every stably free KG-module is free.

Using the theorem, we provide new examples of group rings that are pseudo-Sylvester domains but not Sylvester domains.

Relation to published work

Chapters 4 and 5, exclusively, are based on published joint work as indicated at their respective beginnings.

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Notation and conventions

We write $A \leqslant B$ for two objects A and B of the same category that is clear from the context to indicate that A is a subobject of B. For example, we will use this notation for subgroups, subrings, subfields and submodules.

Groups are understood to be discrete. The neutral element of a group is denoted by e. If G is a group, then we write $N \trianglelefteqslant G$ to indicate that N is a normal subgroup of G.

Rings are associative and unital, but not necessarily commutative. Morphisms of rings are understood to be unital. If R is a ring, then $R^\times$ denotes the group of units, i.e., of two-sided invertible elements, of R. The center of R, i.e., the subring consisting of elements x such that $xy = yx$ for all $y \in R$, is denoted by $\mathcal{Z}(R)$.

Modules are understood to be left modules if not specified otherwise. If R is a ring, then we use $M_{m\times n}(R)$ to denote the R-module of $m \times n$-matrices with entries in R and if $A \in M_{m\times n}(R)$, then A_{ij} denotes the entry of A in the i-th row and j-th column. Most of the time, we will consider $M_n(R) = M_{n\times n}(R)$, the ring of square matrices with entries in R.

A *zero divisor* in a ring R is an element $z \in R$ for which there exists a non-zero element $z' \in R$ such that $zz' = 0$ or $z'z = 0$. Since 0 is a zero divisor in every non-trivial ring, we usually speak of *non-trivial zero divisors*, which exclude 0. A ring (commutative or non-commutative) without non-trivial zero divisors is called a *domain*.

A *division ring* is a ring in which every non-zero element is a unit. A *field* is a commutative division ring. If K is a field, then a *division K-algebra* is a division ring that is also a K-algebra.

Chapter 1

Ring theory

This chapter reviews several notions and constructions from non-commutative ring theory that will be used throughout the thesis. The most important notions are that of a $*$-regular ring, treated in Section 1.5, and the associated $*$-regular closure, which is defined in Section 1.7. While none of the proofs in this chapter are original, we extract a structural result on finitely presented modules over $*$-regular closures from the proof of a result by Jaikin-Zapirain.

1.1 Crossed products

Let K be a field and consider a short exact sequence of groups:

$$1 \to N \to G \xrightarrow{\mathrm{pr}} Q \to 1.$$

Then the group ring KN is naturally a subring of KG. Viewed as a KN-module, we can then express KG as the internal direct sum $\bigoplus_{q \in Q} \tilde{q}KN$, where for each $q \in Q$ we fix a choice $\tilde{q}$ of an element in the preimage $\mathrm{pr}^{-1}(q)$. Written in this way, the ring KG starts to resemble the group ring $(KN)Q$. However, there are two notable differences:

- For $n \in N$ and $q \in Q$, the elements n and $\tilde{q}$ of KG do not necessarily commute. In general, we only have the tautological identity $\tilde{q}n = (\tilde{q}n\tilde{q}^{-1})\tilde{q}$, with the bracketed term representing an element of KN given that N is a normal subgroup of G. Forgetting the existence of the ambient group G for a moment, we observe that $\tilde{q}$ and n commute up to an *action* of Q (viewed as a set) on KN, i.e., a map of sets $Q \to \mathrm{Aut}(KN)$.

- For $q_1, q_1 \in Q$, we do not necessarily have that $\widetilde{q_1 q_2} = \tilde{q}_1\tilde{q}_2$. This is because we chose the preimages of elements of q independently and did not demand any coherence properties such as the assignment $q \mapsto \tilde{q}$ being a group homomorphism. Of course, if the short exact sequence above is not split, we cannot do any better and have to introduce correction terms: Certainly

  $$\mathrm{pr}(\tilde{q}_1\tilde{q}_2) = \mathrm{pr}(\tilde{q}_1)\,\mathrm{pr}(\tilde{q}_2) = q_1 q_2 = \mathrm{pr}(\widetilde{q_1 q_2})$$

  and thus $\tilde{q}_1\tilde{q}_2\widetilde{q_1 q_2}^{-1} \in N = KN^{\times}$. In this way, the multiplication of representatives $\tilde{q}$ is *twisted* by a map of sets $Q \times Q \to KN^{\times}$.

Abstracting away the concrete situation of a group extension, we can turn the observed structure of KG as a "product" of KN and Q into a general definition:

Definition 1.1.1. Let R be a ring and G a group. A *crossed product* $R * G$ is a ring that as a left R-module is free on a copy of G usually denoted by $\tilde{G} = \{\tilde{g} \mid g \in G\}$ and such that the ring multiplication is determined by the following two properties:

(1) There is a map of sets $\sigma\colon G \to \operatorname{Aut}(R)$, called the *action*, such that $\tilde{g} \cdot r = \sigma(g)(r) \cdot \tilde{g}$ for every $r \in R$ and $g \in G$.

(2) There is a map of sets $\alpha\colon G \times G \to R^\times$, called the *twisting*, such that $\tilde{g} \cdot \tilde{h} = \alpha(g, h) \cdot \widetilde{gh}$ for every $g, h \in G$.

Note that even though verifying that a given ring is a crossed product requires the choice of an R-basis as well as producing the auxiliary action and twisting maps, we do not consider this additional data to be part of what constitutes a crossed product. We will use the term *crossed product structure* to refer to a ring that is a crossed product together with particular choices of a basis and action and twisting maps.

Whenever we use a crossed product, we will additionally assume that $\sigma(e) = \mathrm{id}_R$ and $\alpha(g, e) = \alpha(e, g) = 1$, for every $g \in G$ and $e \in G$ the neutral element, which makes $\tilde{e}$ the unit of the crossed product. The map $r \mapsto r \cdot \tilde{e}$ is then an embedding of R into $R * G$. For any given crossed product together with a choice of basis and structure maps, this can always be arranged by a diagonal change of basis and modifications to the twisting and action, but without changing the ring.

As we will also want to construct rings as crossed products out of a ring R and a group G, we need sufficient conditions for given action and twisting maps as in Definition 1.1.1 to assemble to a crossed product structure. This is achieved by the following classical result on crossed products:

Proposition 1.1.2 ([Pas89, Lemma 1.1])**.** *The associativity of the ring multiplication of a crossed product $R * G$ is equivalent to the following conditions on the action and twisting maps for all $g, h, k \in G$:*

(1) $\alpha(g, h)\alpha(gh, k) = \sigma_g(\alpha(h, k))\alpha(g, hk)$;

(2) $\sigma_g \circ \sigma_h = c_{\alpha(g,h)}\sigma_{gh}$, *where c_u for $u \in R^\times$ denotes the conjugation map $c_u(r) = uru^{-1}$.*

For every crossed product $R * G$ and every subgroup $H \leqslant G$ we obtain an induced crossed product $R * H$ by restricting the basis as well as the action and twisting. Our initial example of a crossed product can then be generalized as follows:

Lemma 1.1.3 ([Pas89, Lemma 1.3])**.** *Let $1 \to N \to G \to Q \to 1$ be a short exact sequence of groups and let $R * G$ be a crossed product for an arbitrary ring R. Then*

$$R * G = (R * N) * Q.$$

Proof. For every section $s\colon Q \to G$ of the homomorphism $G \to Q$ viewed as a map of sets, we obtain a crossed product structure for $(R * N) * Q$, with $R * N$-basis $\{s(q) \mid q \in Q\}$, action $\sigma(q)(r) = s(q)rs(q)^{-1}$ and twisting $\alpha(q_1, q_2) = s(q_1)s(q_2)s(q_1q_2)^{-1}$. □

Example 1.1.4. Let R be a ring and τ an automorphism of R. The skew Laurent polynomial rings $R[t^{\pm 1}; \tau]$, in which $tr = \tau(r)t$ for all $r \in R$, are particular instances of crossed products with $\sigma(t^n) = \tau^n$ and trivial α. In fact, every crossed product $R * \mathbb{Z}$ is isomorphic to such a skew Laurent polynomial ring for some choice of τ (see [Sán08, Remark 4.6] and [Haz16, 1.1.4]).

We will conclude our introduction to crossed products with an equivalent definition that does not use auxiliary maps. Whereas the precise restrictions on action and twisting map vary in the literature, this point of view on crossed products is helpful in verifying that they are all equivalent.

Definition 1.1.5. Let Γ be a group. A ring R is called a Γ-*graded ring* if its underlying additive group can be expressed as a direct sum $\bigoplus_{g \in \Gamma} R_g$ where each R_g is an additive subgroup of R and $R_g R_h \subseteq R_{gh}$ for all $g, h \in \Gamma$. The subring R_e of R is called the *base ring* of R.

If R is a Γ-graded ring, then every R_g for $g \in \Gamma$ is an R_e-R_e-bimodule and $1_R \in R_e$.

Definition 1.1.6. A Γ-graded ring R is called a *crossed product* (of R_0 and Γ) if $R_g \cap R^\times \neq \emptyset$ for every $g \in \Gamma$, i.e., if every R_g contains a unit of R.

A ring is a crossed product in the sense of Definition 1.1.1 if and only if it is one in the sense of Definition 1.1.6, see [Haz16, 1.1.4].

1.2 Non-commutative localization

A natural approach to constructing a division ring into which a given ring R embeds is to study rings obtained from R obtained by adjoining inverses to a prescribed subset of elements of R. This procedure can be formalized as follows:

Definition 1.2.1. Let R be a ring and $S \subseteq R$ a multiplicatively closed subset. The *localization of R at S* is the universal ring homomorphism $\varphi \colon R \to S^{-1}R$ such that $\varphi(s)$ is invertible for every $s \in S$.

The localization of R at S always exists and is well-defined up to unique isomorphism, but may be the zero ring.

If R is a commutative ring and $S \subset R$ is a subset without zero divisors, then the map $R \to S^{-1}R$ is injective and the elements of $S^{-1}R$ can all be taken to be of the form $\frac{r}{s}$ with $r \in R, s \in S$. In particular, if $S = R \setminus \{0\}$, then $S^{-1}R$ is a field, the *field of fractions* of R.

The situation is much more complicated for a general ring R. We will start with a condition on R and the subset S which ensures that the localization $S^{-1}R$ behaves analogously to the commutative setting.

Definition 1.2.2. Let R be a ring and $S \subset R$ a multiplicatively closed subset that contains no zero divisors. Then R is said to satisfy the *left Ore condition with respect to S* if for every $a \in R$ and every $s \in S$ there exist $b \in R$ and $t \in S$ such that

$$ta = bs.$$

The motivation behind the Ore condition is that if R' were any ring containing R in which elements of S are invertible, then the condition would allow us to rewrite the left fraction as^{-1} as the right fraction $t^{-1}b$. If it is satisfied, then the elements of the localization $S^{-1}R$ can indeed all be represented as left fractions:

Theorem 1.2.3 ([Row88, Theorem 3.1.4])**.** *Let R be a ring and $S \subset R$ a multiplicatively closed subset that contains no zero divisors. If R satisfies the left Ore condition with respect to S, then the localization $R \to S^{-1}R$ is injective and all elements of $S^{-1}R$ are of the form $s^{-1}r$ for $s \in S, r \in R$. In particular, if R does not contain non-trivial zero divisors, then $S^{-1}R$ is a division ring.*

If the Ore condition is satisfied, then localization is an exact functor, just as in the commutative case:

Lemma 1.2.4 ([GW04, Corollary 10.13])**.** *If R satisfies the left Ore condition with respect to S, then $S^{-1}R$ is a flat right R-module, i.e., the functor $S^{-1}R \otimes_R ?$ is exact.*

While there are even more general versions of the Ore condition that allow the subset S to contain zero divisors, we will usually contend ourselves with the following special case:

Definition 1.2.5. A ring R is said to satisfy the *left Ore condition* if it satisfies the left Ore condition with respect to the subset S_R of elements that are not zero divisors. If this is the case, then we also denote the ring $S_R^{-1}R$ by $\mathrm{Ore}(R)$ and call it the *Ore ring of fractions* of R.

Assuming that R does not contain non-trivial zero divisors, the ring $\mathrm{Ore}(R) = S^{-1}R$ is a division ring if R satisfies the Ore condition. If this is the case, we will call R an *Ore domain* and $\mathrm{Ore}(R)$ its *Ore division ring of fractions.*

Completely analogously, one can consider the right Ore condition and arrive at a representation of the localization $S^{-1}R$ in which all elements are represented by right fractions. While there are rings that satisfy the Ore condition only one one side, this will not be the case for the rings of interest to us in this thesis. For this reason, we will usually omit the side in the following.

The Ore condition for a ring R implies that for a matrix ring over R:

Proposition 1.2.6 ([Rei98, Proposition 13.7])**.** *Suppose that R satisfies the Ore condition with respect to the set S. Then $M_n(R)$ satisfies the Ore condition with respect to the set $S \cdot I_n$ and the canonical embedding $M_n(R) \hookrightarrow M_n(S^{-1}R)$ induces an isomorphism*

$$(S \cdot I_n)^{-1} M_n(R) \xrightarrow{\cong} M_n(S^{-1}R).$$

The following remarkable result is a very useful way to verify the Ore condition for an abstract ring:

Theorem 1.2.7 (Goldie's theorem [Lam99, (11.13)])**.** *If R is a left Noetherian ring that is also left* semiprime*, i.e., that has no non-zero nilpotent left ideals, then R satisfies the left Ore condition and* $\mathrm{Ore}(R)$ *is semisimple.*

There is also a direct analogue where "left" is replaced with "right".

Example 1.2.8. Let R be a left and right Noetherian ring and τ an automorphism of R. Then the skew Laurent polynomial ring of the form $R[t^{\pm 1}; \tau]$ is again two-sided Noetherian by [GW04, Corollary 1.15] as well as a domain and hence semiprime. By Goldie's theorem, the polynomial ring is a left and right Ore domain and admits an Ore division ring of fractions.

For a certain class of group rings, the Ore condition is satisfied automatically if the ring does not contain non-trivial zero divisors:

Theorem 1.2.9. *Let $D * G$ be a crossed product of a division ring D and a group G. Assume that $D * G$ does not contain non-trivial zero divisors. Then $D * G$ satisfies the Ore condition with respect to its non-zero elements if G is amenable. Furthermore, the reverse implication holds if D is a field and $D * G$ is an ordinary group ring.*

Proof. For the first statement see [Kie20, Theorem 2.14]. The second statement is proved in [Bar19, Appendix A]. □

While Theorem 1.2.9 is very useful for the study of group rings of amenable groups, it also serves to show that embeddings of group rings of torsion-free non-amenable groups, e.g., free groups, into division rings cannot be constructed simply by adjoining inverses of ring elements. The definition of the non-commutative analogue of a field of fractions that also applies to such rings will use the following generalization of a surjective ring homomorphism:

Definition 1.2.10. A ring homomorphism $f\colon R \to S$ is called *epic* if $\alpha \circ f = \beta \circ f$ for a pair of ring homomorphisms $\alpha, \beta\colon S \to T$ implies that $\alpha = \beta$.

The two main classes of examples of epic ring homomorphisms are given by surjective ring homomorphisms and by maps $R \to \mathrm{Ore}(R)$ for rings R that satisfy the Ore condition. A ring homomorphism $R \to D$ to a division ring is an if and only if its image generates D as a division ring (see [Coh06, Corollary 7.2.2]). If we were only interested in maps to division rings, this would have allowed use to introduce epicity in a radically simpler fashion. However, the notion of epicity will not only be useful for maps to division rings, so the more elaborate definition is warranted. For later use, we record the following important equivalent characterization:

Proposition 1.2.11 ([Ste75, Proposition XI.1.2])**.** *A ring homomorphism $f\colon R \to S$ is epic if and only if the multiplication map $S \otimes_R S \to S, s \otimes t \mapsto st$ is an isomorphism, where S is viewed as an R-R-bimodule via f.*

We can now define the non-commutative analogue of a field of fractions:

Definition 1.2.12. Let R be a ring. A ring S together with an epic ring homomorphism $R \to S$ is called an *epic R-ring*. If $R \to S$ is additionally injective and S is a division ring, then it is called a *division R-ring of fractions*.

An example of a division ring of fractions that does not arise as an Ore division ring of fractions is given by the inclusion of the group ring $\mathbb{Q}F_2$ of the free group on two generators into its *universal field of fractions*, which will be introduced and studied in more detail in Chapter 5.

1.3 K_0, G_0 and the Farrell–Jones conjecture

Definition 1.3.1. For a ring R, denote by $K_0(R)$ the abelian group on generators $[A]$ for every finitely generated projective R-module A and with a relation $[A] = [B] + [C]$ for every short exact sequence $0 \to B \to A \to C \to 0$.

If $f\colon R \to S$ is a ring homomorphism and A is a finitely generated projective R-module, then $S \otimes_R A$ is a finitely generated projective S-module. Since short exact sequences of projective modules split, this makes $K_0(?)$ a functor from the category of rings to the category of abelian groups.

The *Farrell–Jones conjecture* makes far-reaching claims about the K-theory (and L-theory) of group rings or, more generally, additive categories with group actions. It is known for many classes of groups and satisfies a number of useful inheritance properties. For a full statement of the Farrell–Jones conjecture and an overview of the groups for which it is known, we refer the reader to the surveys [BLR08] and [RV18], and also to [Lüc10; Lüc19]. We will only record the following basic consequence of the Farrell–Jones conjecture:

Theorem 1.3.2. *If the group G satisfies the Farrell–Jones conjecture, then the map*

$$\operatorname*{colim}_{\substack{\longrightarrow \\ F \leqslant G \\ |F| < \infty}} K_0(KF) \xrightarrow{\cong} K_0(KG)$$

is an isomorphism for every field K.

By considering all finitely generated modules instead of just the projective ones, we obtain another invariant of rings that takes values in abelian groups:

Definition 1.3.3. For a ring R, denote by $G_0(R)$ the abelian group on generators $[A]$ for every finitely generated R-module A and with a relation $[A] = [B] + [C]$ for every short exact sequence $0 \to B \to A \to C \to 0$.

If R is a semisimple ring, e.g., a group ring KF for a field K and a finite group F, then every R-module is projective and $K_0(R) = G_0(R)$. In general, there is only the forgetful map $K_0(R) \to G_0(R)$, which will usually not be an isomorphism.

As opposed to $K_0(?)$, the assignment $R \mapsto G_0(R)$ is not functorial in general as ring homomorphisms need not preserve short exact sequences of finitely generated modules. However, if $f\colon R \to S$ is flat, then it induces a well-defined map $G_0(f)\colon G_0(R) \to G_0(S)$.

That $G_0(?)$ falls short of being a functor just serves as an example of the many ways in which $G_0(R)$ behaves more intricately than $K_0(R)$. Whereas non-trivial computations of $K_0(KG)$ for infinite groups G have been carried out via the Farrell–Jones conjecture, it remains consistent with current knowledge that $G_0(\mathbb{C}G) = 0$ for every non-amenable group G (see [Lüc02, Remark 9.69]).

1.4 Von Neumann regular rings

Definition 1.4.1. A ring R is called *von Neumann regular* if for every $x \in R$ there is an element $y \in R$ such that $xyx = x$.

The notion of a von Neumann regular ring as defined above should not be confused with that of a regular ring from commutative algebra. The latter notion will however only play a minor role in this thesis, appearing only in Chapter 5, which should limit the potential for misunderstandings.

If $x \in R$ is invertible, its inverse x^{-1} could be taken as the element y in Definition 1.4.1. For a general ring element x, an element y such that $xyx = x$ can intuitively be viewed as an inverse of x "away from its kernel". For example, if R is the endomorphism ring of a finite-dimensional Hilbert space and $x \in R$ is an endomorphism, then y could be taken to be the inverse of $x|_{\ker(x)^\perp}$ on $\operatorname{im}(x)$ and 0 on $\operatorname{im}(x)^\perp$.

Example 1.4.2. The following rings are von Neumann regular:

- division rings;
- rings of square matrices over von Neumann regular rings;
- endomorphism rings of not necessarily finite-dimensional vector spaces.

While Definition 1.4.1 is certainly the most elementary way to define von Neumann regular rings, there are a number of equivalent definitions that make this class of rings very useful for the purposes of homological algebra:

Proposition 1.4.3. *The following statements are equivalent for a ring R:*

(a) *R is von Neumann regular.*

(b) *Every finitely generated left (resp. right) ideal of R is generated by an idempotent.*

(c) *Every finitely generated submodule of a finitely generated left (resp. right) R-module is a direct summand.*

(d) *Every finitely presented left (resp. right) R-module is projective.*

(e) *Every left (resp. right) R-module is flat.*

The following structure theorem for projective modules over von Neumann regular rings originally appeared as [Kap58, Theorem 4]. Our formulation is obtained by combining the result with Proposition 1.4.3 (b).

Theorem 1.4.4. *Every projective left (resp. right) module over a von Neumann regular ring R is a direct sum of modules isomorphic to left (resp. right) ideals of R, each of which is generated by a single idempotent.*

As a consequence of these properties, we can see that von Neumann regular rings are as rich in zero divisors as possible:

Proposition 1.4.5. *An element of a von Neumann regular ring is either a unit or a zero divisor.*

Proof. Let R be a von Neumann regular ring and consider the ideal xR generated by an element $x \in R$. By Proposition 1.4.3 (b), there is an idempotent $e \in R$ such that $xR = eR$. In particular, there is $z \in R$ such that $x = ez$. If x is not a unit, the ideal xR does not contain 1, thus $e \neq 1$. But then $(1-e)x = (1-e)ez = 0$ where $1 - e \neq 0$, so x is a zero divisor. □

We also record the following elementary consequence of our definition of a von Neumann regular ring:

Lemma 1.4.6. *Every left (resp. right) ideal I in a von Neumann regular ring is idempotent, i.e., $I^2 = I$. In particular, if J is a nilpotent left (resp. right) ideal, i.e., $J^n = (0)$ for some $n \in \mathbb{N}$, then $J = (0)$.*

Proof. Let I be a left ideal in a von Neumann regular ring R and consider an element $x \in I$. Since R is von Neumann regular, there is $y \in R$ such that $xyx = x$. Thus $x = x(yx) \in I^2$. □

1.5 $*$-regular rings

While von Neumann regular rings are already quite convenient to work with, they lack a technical property that will be crucial for our purposes: If $\{R_i\}_{i\in I}$ is a family of von Neumann regular subrings of an ambient von Neumann regular ring R, then there is no reason why the intersection $\bigcap_{i\in I} R_i$ should again be a von Neumann regular ring. We will fix this deficiency by extending Definition 1.4.1 such that, given an element x of the ring, there is a preferred choice of y such that $xyx = x$. This will require the following additional structure:

Definition 1.5.1. A *$*$-ring* is a ring R together with an involution $?^* \colon R \to R$, i.e., a map that has the following properties:

(1) $1^* = 1$;

(2) $(x + y)^* = x^* + y^*$;

(3) $(xy)^* = y^* x^*$;

(4) $(x^*)^* = x$.

A subring S of a $*$-ring R is called a *$*$-subring* if $*$ restricts to an involution of S. A $*$-ring R is called *proper* if $x^*x = 0$ implies $x = 0$ for every $x \in R$.

A $*$-subring of a proper $*$-ring is again proper. If R is a $*$-ring, then the matrix ring $M_n(R)$ becomes a $*$-ring by setting $(A^*)_{ij} := A^*_{ji}$.

A $*$-ring admits a refined notion of an idempotent:

Definition 1.5.2. An element $x \in R$ in a $*$-ring R is called a *projection* if it is an idempotent and $x^* = x$.

The relation between idempotents and projections in a $*$-ring is similar to that of (arbitrary) projections and orthogonal projections in a Hilbert space: While there are usually many different projection onto a given subspace, there is only one orthogonal projection. Following this intuitive picture, we are able to resolve our technical difficulties by adding the structure of a proper $*$-ring to a von Neumann regular ring:

Definition 1.5.3. A *$*$-regular ring* is a proper $*$-ring that is also von Neumann regular. A *$*$-regular subring* of a $*$-regular ring is a von Neumann regular $*$-subring, which is automatically $*$-regular.

Example 1.5.4. The following $*$-rings are $*$-regular:

- subfields of $\mathbb{C}$ that are closed under complex conjugation, with the involution given by complex conjugation;
- rings of square matrices over $*$-regular rings, with the involution given by transposition followed by element-wise application of the involution.

The most important property of $*$-regular rings is that for every $x \in R$ there is a canonical choice of $y \in R$ such that $xyx = x$:

Lemma 1.5.5. *Let R be a $*$-regular ring and let $x \in R$.*

(a) *There exist unique projections $\mathrm{LP}(x), \mathrm{RP}(x) \in R$ such that $\mathrm{LP}(x)R = xR$ and $R\,\mathrm{RP}(x) = Rx$, respectively.*

(b) *There exists a unique element $x^{[-1]} \in \mathrm{RP}(x)R\,\mathrm{LP}(x)$ such that $xx^{[-1]} = \mathrm{LP}(x)$ and $x^{[-1]}x = \mathrm{RP}(x)$.*

(c) $xx^{[-1]}x = x$.

(d) $x^{[-1]} = (x^*x)^{[-1]}x^*$.

Proof. For the proofs of (a), (b), and (d) see [Jai19c, Proposition 3.2 (3), (4) & (6)].

Let $z \in R$ such that $\mathrm{LP}(x)z = x$. Then

$$xx^{[-1]}x = \mathrm{LP}(x)x = \mathrm{LP}(x)\,\mathrm{LP}(x)z = \mathrm{LP}(x)z = x,$$

which proves (c). □

Definition 1.5.6. Let R be a $*$-regular ring. For every $x \in R$, we call the unique element $x^{[-1]}$ of Lemma 1.5.5 (b) the *relative inverse* of x.

If one considers a matrix ring $M_n(\mathbb{C})$ equipped with the $*$-structure coming from complex conjugation on $\mathbb{C}$, then the relative inverse of a matrix agrees with the so-called Moore–Penrose inverse of the matrix.

Lemma 1.5.7. *Let R be a $*$-regular ring and $\{R_i\}_{i\in I}$ a family of $*$-subrings of R, i.e., of subrings of R that are preserved by $*$. Then $\bigcap_{i\in I} R_i$ is a $*$-regular ring.*

Proof. Let $x \in \bigcap_{i\in I} R_i$ be any element in the intersection. Since R is $*$-regular, we obtain from Lemma 1.5.5 (b) that $x^{[-1]} \in R_i$ for every $i \in I$. Thus $x^{[-1]} \in \bigcap_{i\in I} R_i$ and $\bigcap_{i\in I} R_i$ is von Neumann regular by Lemma 1.5.5 (c). The intersection of proper $*$-subrings is clearly again a proper $*$-subring, hence $\bigcap_{i\in I} R_i$ is $*$-regular. □

1.6 Semisimple Artinian rings

We will now look at a particularly well-behaved class of von Neumann regular rings that intuitively are not too far from being division rings. Recall that a ring R is a division ring if and only if every R-module is free.

Definition 1.6.1. A ring R is called *semisimple* if every R-module is projective.

Every semisimple ring is von Neumann regular by Proposition 1.4.3 (d).

The following fundamental result on semisimple rings provides a complete classification and implies that it does not matter whether one considers left or right R-modules in the definition of semisimplicity:

Theorem 1.6.2 (Artin–Wedderburn theorem [Gri07, IX, Theorem 3.3 & Corollary 3.11])**.** *Let R be a semisimple ring. Then there is $s \in \mathbb{N}$ as well as $n_i \in \mathbb{N}$ and division rings D_i for $i = 1, \dots, s$ such that:*

$$R \cong M_{n_1}(D_1) \times \cdots \times M_{n_s}(D_s).$$

Furthermore, the numbers s and n_i and the division rings D_i are uniquely determined by R up to permutations of the indices and every such choice gives rise to a semisimple ring.

If K is a field and R is a K-algebra, then D_i is also a K-algebra for every $i = 1, \dots, s$.

Definition 1.6.3. Let R be a ring. A non-trivial R-module S is called *simple* if it has no submodules other than 0 and S.

Proposition 1.6.4. *Let R be a semisimple ring and let s, n_i and D_i be as in Theorem 1.6.2.*

(a) *Every simple R-module is isomorphic to a minimal left ideal of some $M_{n_i}(D_i)$ and all minimal left ideals of $M_{n_i}(D_i)$ are isomorphic as R-modules.*

(b) *Every simple R-module is finitely presented.*

(c) *Every R-module is isomorphic to $S_1^{m_1} \oplus \cdots \oplus S_s^{m_s}$ for unique cardinal numbers $m_1, \dots, m_s$, where S_i is some fixed choice of a minimal left ideal of $M_{n_i}(D_i)$ for every $i = 1, \dots, s$.*

(d) *$M_{n_i}(D_i)$ is a direct sum of n_i minimal left ideals.*

Proof. (a) and (c) follow from [Gri07, IX, Proposition 1.8, 3.6 & 3.7] and (d) is the statement of [Gri07, IX, Proposition 1.7].

For (b), note that every minimal left ideal is necessarily principal and in particular finitely generated. Since every R-module is projective, the simple R-modules are therefore finitely generated projective and thus finitely presented. □

As a consequence of Propositions 1.6.4 (c) and 1.6.4 (d), we obtain the following computation of K_0 of a semisimple ring:

Corollary 1.6.5. *Let R be a semisimple ring and let s, n_i and D_i be as in Theorem 1.6.2, so that $R \xrightarrow{\cong} M_{n_1}(D_1) \times \cdots \times M_{n_s}(D_s)$ via an isomorphism Φ. For every $i = 1, \dots, s$, denote by e_i the central idempotent corresponding to the projection onto the i-th factor of the right-hand side, and by s_i any choice of a generator of a minimal left ideal of $M_{n_i}(D_i)$. Then $K_0(R)$ is freely generated as an abelian group by the elements*

$$[\Phi^{-1}(s_i)] = \frac{1}{n_i}[\Phi^{-1}(e_i)], \ i = 1, \dots, s.$$

In the following, we will frequently identify a semisimple ring R with a fixed choice of an Artin–Wedderburn decomposition as a product of matrix rings over division rings. In this situation, by virtue of Corollary 1.6.5, there is a canonical identification of $K_0(R)$ with

$$\frac{1}{n_1}\mathbb{Z} \oplus \cdots \oplus \frac{1}{n_s}\mathbb{Z}.$$

1.7 Division and $*$-regular closure

Having introduced several particularly convenient classes of rings, such as $*$-regular and semisimple rings, we will consider closures of arbitrary subrings of such rings. More specifically, for a subring R of a $*$-regular ring S, we want to construct an intermediate ring R' with $R \subseteq R' \subseteq S$ such that R' inherits structural properties such as regularity from S, but is otherwise "close" to R. We begin with a classical construction that does however falls short of our goal in general:

Definition 1.7.1. Let S be a ring and $R \leqslant S$ a subring. Then R is *division closed* in S if the inverse of every element of R which is invertible in S already lies in R.

Since inverses are unique if they exist, arbitrary intersections of division closed subrings are again division closed. This enables the following construction:

Definition 1.7.2. Let S be a ring and $R \leqslant S$ a subring. The *division closure* of R in S, denoted by $\mathcal{D}(R, S)$, is the smallest division closed subring of S containing R.

The division closure of a subring in a division ring is the division ring generated by the subring. In the case of a more general ambient ring, the division closure is not certain to inherit desirable properties, as the following example shows:

Example 1.7.3. Let $S = M_n(\mathbb{C})$ for some $n \in \mathbb{N}, n \geqslant 2$, which is semisimple. Consider the subring $R \leqslant S$ given by upper triangular matrices. Since every upper triangular matrix can be brought into diagonal form via elementary row operations that are themselves upper triangular matrices, the subring R is division closed in S. However, R is not even von Neumann regular for the following reason: The matrices in R with a single non-zero entry in the upper right corner form a non-zero ideal I such that $I^2 = (0)$. By Lemma 1.4.6, this is impossible in a von Neumann regular ring.

For this reason, we will now introduced a larger closure that will always be a $*$-regular ring if the ambient ring is. It has first been considered in [LS12].

Definition 1.7.4. Let S be a $*$-regular ring and $R \leqslant S$ a $*$-subring. The *$*$-regular closure* of R in S, denoted by $\mathcal{R}(R, S)$, is the smallest $*$-regular subring of S containing R.

The $*$-regular closure always contains the division closure:

Lemma 1.7.5. *A von Neumann regular ring is division closed in every overring.*

Proof. If an element of the von Neumann regular ring is not a unit, then it is a zero divisor by Proposition 1.4.5 and hence cannot become a unit in an overring. □

In the situation of Example 1.7.3, the $*$-regular closure behaves much better than the division closure:

Example 1.7.6. Let R and S be as in Example 1.7.3. Then $\mathcal{R}(R, S) = S$, which can be seen as follows: Let $E_{i,j}$ for $1 \leqslant i, j \leqslant n$ denote the matrix with all zero entries except for a one at the position (i, j). Viewed as an element of the $*$-regular ring S, we obtain in the notation of Lemma 1.5.5 that $\mathrm{LP}(E_{i,j}) = E_{i,i}$, $\mathrm{RP}(E_{i,j}) = E_{j,j}$ and $E_{i,j}^{[-1]} = E_{j,i}$. Since R contains E_{ij} for $i \leqslant j$, the $*$-regular closure of R in S consequently also contains $E_{i,j}$ for all $j \leqslant i$ and thus coincides with S.

Remark 1.7.7. The abstractly defined division and $*$-regular closure can also be constructed explicitly as follows. Starting with the subring $R_0 := R$ of S, inductively define R_{i+1} for $i \geqslant 0$ to be the subring of S generated by R_i and $\{x^{-1} \mid x \in R, x \in S^\times\}$ (resp. $\{x^{[-1]} \mid x \in R\}$). Then the division closure (resp. $*$-regular closure) of R in S is given by $\bigcup_{i\geqslant 0} R_i$, see [AG17, Proposition 6.2].

The following result is implicit in the proof of [Jai19c, Proposition 5.11].

Proposition 1.7.8. *Let S be a $*$-regular ring and R a $*$-subring such that $S = \mathcal{R}(R, S)$. For every $t \in S$, there exist a finitely presented R-module L and an element $1 \otimes t' \in S \otimes_R L$ such that $St \cong S(1 \otimes t')$ as left S-modules, where the isomorphism is compatible with the canonical maps $S \to St$ and $S \to S(1 \otimes t')$.*

Proof. In this proof, all tensor products are taken over R.

Every R-module can be expressed as a directed colimit over finitely presented modules. In particular, we can write $S = \varinjlim_{j\in J} L_j$ for a directed set $(J, \leqslant)$, a family of finitely presented R-modules $\{L_j \mid j \in J\}$ and a family of homomorphisms $\{\varphi_{ij} \colon L_i \to L_j \mid i \leqslant j\}$ satisfying $\varphi_{jj} = \mathrm{id}_{L_j}$ and $\varphi_{ik} = \varphi_{jk} \circ \varphi_{ij}$ for all $i \leqslant j \leqslant k$. The directed colimit comes with canonical maps $\varphi_j \colon L_j \to S$ for $j \in J$. Since tensor products preserve colimits, we obtain an induced isomorphism $S \otimes S \cong \varinjlim_{j\in J} S \otimes L_j$, where the structure homomorphisms are given by the family $\{\mathrm{id}_S \otimes \varphi_{ij} \mid i \leqslant j\}$.

Consider the surjective homomorphism $p \colon S \to St$ given by $s \mapsto st$. Since S is von Neumann regular and St is a finitely generated left S-submodule of S, we obtain from Proposition 1.4.3 (c) that p splits. Hence, the kernel of p, which we denote by C, is of the form Sd for an element $d \in S$.

We now choose $k \in J$ large enough such that $t = \varphi_k(t_k)$ for some $t_k \in L_k$. Since the embedding of R into S is epic by [Jai19c, Proposition 6.1], we obtain from Proposition 1.2.11 that

$$(\mathrm{id}_S \otimes \varphi_k)(d \otimes t_k) = d \otimes \varphi_k(t_k) = d \otimes t = dt \otimes 1 = 0.$$

As $d \otimes t_k$ maps to 0 in the colimit $\varinjlim_{j\in J} S \otimes L_j$, there is $l \in J, l \geqslant k$ such that $(\mathrm{id}_S \otimes \varphi_{kl})(d \otimes t_k) = 0$ in $S \otimes L_k$. We abbreviate $t' := 1 \otimes \varphi_{kl}(t_k) \in S \otimes L_k$ and obtain the following characterization of C:

Claim.

$$C = \{s \mid s \in S,\ st' = 0\}$$

We first show that C is contained in the right-hand side. For this, let $c \in C = Sd$ and choose an element $s_c \in S$ such that $c = s_c d$. We obtain

$$ct' = c \cdot (\mathrm{id}_S \otimes \varphi_{kl})(1 \otimes t_k) = s_c \cdot (\mathrm{id}_S \otimes \varphi_{kl})(d \otimes t_k) = 0.$$

To prove the other containment, let $s \in S$ such that $st' = 0$ and observe that

$$0 = (\mathrm{id}_S \otimes \varphi_l)(s \cdot (1 \otimes \varphi_{kl}(t_k))) = s \otimes \varphi_k(t_k) = s \otimes t \in S \otimes S.$$

Using the epicity of $R \hookrightarrow S$ a second time, we conclude that $s \otimes t = 0$ implies $st = 0$. It follows that $s \in C$, which establishes the claim.

The claim allows us to conclude that the image St' of the map $S \to S \otimes L_l$ given by $s \mapsto st'$ is isomorphic as an S-module to $S/C = S/\{s \mid s \in S,\ st' = 0\}$, which by definition of C is then seen to be isomorphic to St. $\square$

As a corollary, we obtain that finitely presented modules over a $*$-regular closure can be expressed as formal differences of modules induced from the base ring. This property of the closure will prove particularly useful when we later study additive dimension functions for modules over these rings.

Corollary 1.7.9. *Let S be a $*$-regular ring and R a $*$-subring such that $S = \mathcal{R}(R,S)$. Then every finitely presented S-module is virtually induced from finitely presented R-modules, i.e., for every finitely presented S-module M there exist finitely presented R-modules N_1 and N_2 such that*

$$M \oplus S \otimes_R N_1 \cong S \otimes_R N_2.$$

Proof. Since S is von Neumann regular, the finitely presented S-module M is also projective by Proposition 1.4.3 (d). Using Theorem 1.4.4, we obtain that $M \cong \sum_{i=1}^n Se_i$ for certain $e_i \in S, i = 1, \ldots, n$. We apply Proposition 1.7.8 to this embedding and each of the e_i to obtain finitely presented R-modules L_i and elements $t_i \in S \otimes_R L_i$ such that $Se_i \cong S(1 \otimes_R t_i)$ for $i = 1, \ldots, n$. For each i, we have the following exact sequence:

$$0 \longrightarrow S(1 \otimes_R t_i) \longrightarrow S \otimes_R L_i \longrightarrow S \otimes_R L_i/(1 \otimes_R t_i) \longrightarrow 0$$

Since the third module is a quotient of a finitely presented S-module, it is projective by Proposition 1.4.3 (d) and the sequence splits. Considering the direct sum of all these split sequences, we obtain that

$$M \oplus (S \otimes_R \bigoplus_{i=1}^n L_i/t_i) \cong \bigoplus_{i=1}^n S(1 \otimes_R t_i) \oplus (S \otimes_R L_i/(1 \otimes_R t_i)) \cong S \otimes_R \bigoplus_{i=1}^n L_i,$$

which concludes the proof. $\square$

Chapter 2

L^2-Betti numbers and the Atiyah conjecture

This chapter begins with a streamlined introduction to the theory of L^2-Betti numbers that is meant to provide the shorted route to a convenient formulation of the Atiyah conjecture on the possible values of these numbers. Whereas L^2-Betti numbers were originally defined via analytic methods, we will isolate the input from functional analysis in a few black boxes discussed in Sections 2.1 and 2.2.

Our study of the Atiyah conjecture is picked up in Section 2.3, where we also mention the known examples of groups that admit non-rational L^2-Betti numbers. The commonly used strong Atiyah conjecture, to which no counterexample is known as of today, is introduced in Section 2.4 together with a detailed discussion of its status. The even stronger center-valued Atiyah conjecture is formulated in Section 2.5 and will play an important role in our study of groups with torsion in Chapter 3. We end in Section 2.6 with a diagrammatic overview of the implications between the various variants of the Atiyah conjecture introduced in this and the following chapters.

2.1 The group von Neumann algebra and the algebra of affiliated operators

Definition 2.1.1. Let G be a group. Denote by $\ell^2(G)$ the complex Hilbert space of square-summable formal $\mathbb{C}$-linear combinations of elements of G, i.e.,

$$\left\{ \sum_{g\in G} \lambda_g g \mid \lambda_g \in \mathbb{C}, g \in G, \sum_{g\in G} \|\lambda_g\|^2 < \infty \right\},$$

together with the scalar product

$$\left\langle \sum_{g\in G} \lambda_g g, \sum_{g\in G} \mu_g g \right\rangle := \sum_{g\in G} \lambda_g \overline{\mu_g},$$

where $\bar{?}$ denotes complex conjugation.

Formal multiplication by elements of G equips the Hilbert space $\ell^2(G)$ with both a left and a right action of G by $\mathbb{C}$-linear isometries.

Definition 2.1.2. Let $\mathcal{H}$ be a Hilbert space. Denote by $\mathcal{B}(\mathcal{H})$ the Banach algebra of norm-continuous linear operators from $\mathcal{H}$ to iself, where multiplication is given by composition.

Definition 2.1.3. Let G be a group. The *group von Neumann algebra* $\mathcal{N}(G)$ of G is the subalgebra of $\mathcal{B}(\ell^2(G))$ of operators that are equivariant with respect to the left action of G on $\ell^2(G)$.

For every $x \in \mathbb{C}G$, right multiplication by x defines a G-equivariant norm-continuous linear operator on $\ell^2(G)$, and in this way $\mathbb{C}G$ can be viewed as a subalgebra of $\mathcal{N}(G)$.

The rings $\mathbb{C}G$ and $\mathcal{N}(G)$ can be equipped with compatible $*$-ring structures:

Definition 2.1.4. Let G be a group and $K \leqslant \mathbb{C}$ a subfield closed under complex conjugation. Define the map $?^*\colon KG \to KG$ by $\sum_{g\in G} \lambda_g g \mapsto \sum_{g\in G} \overline{\lambda_g} g^{-1}$. Similarly, the map $?^*\colon \mathcal{N}(G) \to \mathcal{N}(G)$ is defined by mapping an operator to its adjoint.

Both maps are well-defined involutions and agree on $\mathbb{C}G$. Furthermore, the $*$-ring $\mathcal{N}(G)$ is proper since $f^*f = 0$ implies that

$$0 = \langle (f^* \circ f)(x), x\rangle_{\ell^2(G)} = \langle f(x), f(x)\rangle_{\ell^2(G)} = \|f(x)\|^2_{\ell^2(G)}$$

for each $x \in \ell^2(G)$, from which it follows that $f = 0$.

The group von Neumann algebra $\mathcal{N}(G)$, viewed as a ring, is *semihereditary* by [Lüc02, Theorem 6.5 & 6.7], which means that every finitely generated submodule of a projective $\mathcal{N}(G)$-module is again projective. As we have seen in Proposition 1.4.3 (c), von Neumann regular rings satisfy a slightly stronger property. While the group von Neumann algebra is in general not von Neumann regular, it turns out not to be too far away:

Theorem 2.1.5 ([Lüc02, Theorem 8.22 (1) & (3)])**.** *Let G be a group. The group von Neumann algebra $\mathcal{N}(G)$ satisfies the Ore condition and* $\mathrm{Ore}(\mathcal{N}(G))$ *is a von Neumann regular ring.*

The resulting localization of $\mathcal{N}(G)$ can alternatively be constructed as an algebra of densely defined unbounded operators affiliated to the von Neumann algebra $\mathcal{N}(G)$, which explains its name:

Definition 2.1.6. Let G be a group. The ring $\mathrm{Ore}(\mathcal{N}(G))$ is called the *algebra of affiliated operators* of G and is denoted by $\mathcal{U}(G)$.

Since $*\colon \mathcal{N}(G) \to \mathcal{N}(G)$ maps units to units, the universal property of localizations ensures that the $*$-ring structure can be extended to $\mathcal{U}(G)$. Furthermore, the $*$-ring $\mathcal{U}(G)$ is proper and thus $*$-regular, which can be seen as follows: Since $\mathcal{U}(G)$ satisfies the Ore condition, every element of $\mathcal{U}(G)$ is of the form fg^{-1} for $f, g \in \mathcal{N}(G)$. If $0 = (fg^{-1})^* fg^{-1} = (g^*)^{-1} \cdot (f^*f) \cdot g$, then necessarily $f^*f = 0$ as g and g^* are units in $\mathcal{U}(G)$. But then $f = 0$ since $\mathcal{N}(G)$ is a proper $*$-ring.

The $*$-regularity of $\mathcal{U}(G)$ enables the following construction of rings that lie between the group ring and the algebra of affiliated operators and play the most important role in this thesis:

Definition 2.1.7. Let G be a group and $K \leqslant \mathbb{C}$ a field closed under complex conjugation. We denote the $*$-regular closure (resp. the division closure) of KG in $\mathcal{U}(G)$ by $\mathcal{R}_{KG}$ (resp. $\mathcal{D}_{KG}$).

Starting with the group ring KG for a field $K \leqslant \mathbb{C}$ closed under complex conjugation, we have constructed the following commutative square of ring inclusions:

$$\begin{tikzcd} KG \arrow[r, hook] \arrow[d, hook] & \mathcal{N}(G) \arrow[d, hook] \\ \mathcal{R}_{KG} \arrow[r, hook] & \mathcal{U}(G) \end{tikzcd}$$

Example 2.1.8 (Fundamental example). If $G = \mathbb{Z}$, then the Fourier transform provides a G-equivariant isomorphism between $\ell^2(\mathbb{Z})$ and the space $L^2(S^1)$ of square-integrable functions on the circle, where $n \in \mathbb{Z}$ acts on $L^2(S^1)$ via multiplication by the function

$z \mapsto z^n$. Under this identification, the commutative square above takes the following form:

$$\begin{tikzcd} K[z^{\pm 1}] \arrow[r, hook] \arrow[d, hook] & L^\infty(S^1) \arrow[d, hook] \\ K(z) \arrow[r, hook] & L(S^1) \end{tikzcd}$$

Here, $L^\infty(S^1)$ and $L(S^1)$ denote the commutative $\mathbb{C}$-algebras of essentially bounded functions and all measurable functions on S^1, respectively, with ring multiplication given by pointwise multiplication. The $*$-operation on $L(S^1)$ is given by $f \mapsto \overline{f}$ and restricts to $\sum_{n\in\mathbb{Z}} a_n z^n \mapsto \sum_{n\in\mathbb{Z}} \overline{a_n} z^{-n}$ on KG since $\bar{z} = z^{-1}$ for $z \in S^1$.

This example provides useful general intuition for the objects $\mathcal{R}_{KG}$,$\mathcal{N}(G)$ and $\mathcal{U}(G)$: $L^\infty(S^1)$ contains many zero divisors (all functions whose support is not all of S^1) and every element that is not a zero divisors becomes invertible in $L(S^1)$. Moreover, all projections in $L(S^1)$, which are given by the functions that take the values ± 1 only, are already contained in $L^\infty(S^1)$. The $*$-regular closure $K(z)$ is a field. We will later see that the so-called strong Atiyah conjecture for a torsion-free group implies that $\mathcal{R}_{KG}$ is a division ring.

2.2 The von Neumann dimension

The algebra of affiliated operators $\mathcal{U}(G)$ is not only useful for its von Neumann regularity, but also because it admits a highly non-trivial additive dimension function, which we will now introduce.

Definition 2.2.1. Let G be a group. The *von Neumann trace* is the $\mathbb{C}$-linear map

$$\begin{aligned} \operatorname{tr}_{\mathcal{N}(G)} \colon \mathcal{N}(G) &\to \mathbb{C} \\ f &\mapsto \langle f(e), e \rangle_{\ell^2(G)}, \end{aligned}$$

where $e \in G$ is the neutral element. For every $n \in \mathbb{N}$, we also use the same notation for the following extension:

$$\begin{aligned} \operatorname{tr}_{\mathcal{N}(G)} \colon M_n(\mathcal{N}(G)) &\to \mathbb{C} \\ A &\mapsto \sum_{i=1}^{n} \operatorname{tr}_{\mathcal{N}(G)}(A_{ii}). \end{aligned}$$

Example 2.2.2. The map $\operatorname{tr}_{\mathcal{N}(\mathbb{Z})} \colon L^\infty(S^1) \to \mathbb{C}$ is given by

$$f \mapsto \int_{S^1} f \, d\mu,$$

where μ denotes the Lebesgue measure on S^1.

Using the trace, we can define a dimension for finitely generated projective $\mathcal{N}(G)$-modules:

Definition 2.2.3. Let G be a group and P a finitely generated projective $\mathcal{N}(G)$-module. The *von Neumann dimension* of P is given by

$$\dim_{\mathcal{N}(G)}(P) := \operatorname{tr}_{\mathcal{N}(G)}(A) \in [0, \infty)$$

for any matrix $A \in M_n(\mathcal{N}(G))$ such that $A^2 = A^* = A$, and the image of the map $r_A \colon \mathcal{N}(G)^n \to \mathcal{N}(G)^n$ given by right multiplication by A is $\mathcal{N}(G)$-isomorphic to P.

The existence of a matrix A as in Definition 2.2.3 is the subject of [Lüc02, Lemma 6.23] and the independence of the choice of A is verified in [Lüc02, (6.4)]. If $f \in \mathcal{N}(G)$ is such that $f^2 = f^* = f$, then

$$\mathrm{tr}_{\mathcal{N}(G)}(f) = \langle f(e), e\rangle = \langle (f^* \circ f)(e)), e\rangle = \langle f(e), f(e)\rangle \geqslant 0,$$

so $\mathrm{tr}_{\mathcal{N}(G)}(f)$ is real and non-negative. This property then easily extends to matrices A satisfying $A^2 = A^* = A$.

Example 2.2.4. The map $\mathrm{tr}_{\mathcal{N}(\mathbb{Z})}\colon L^\infty(S^1) \to \mathbb{C}$ is given by

$$f \mapsto \int_{S^1} f \, d\mu,$$

where μ denotes the Lebesgue measure on S^1.

Based on an axiomatic framework for the extension of dimension functions to arbitrary modules introduced in [Lüc98] and a careful analysis of the lattice of projections in $\mathcal{U}(G)$, the dimension function $\dim_{\mathcal{U}(G)}$ has been extended from finitely generated projective to arbitrary $\mathcal{U}(G)$-modules in [Rei01]. We summarize its main properties in the following theorem:

Theorem 2.2.5. *For every group G, there exists a unique function* $\dim_{\mathcal{U}(G)}$ *that assigns to every $\mathcal{U}(G)$-module a non-negative real number or infinity, called the* von Neumann dimension, *such that the following conditions are satisfied:*

Invariance under isomorphisms. $\dim_{\mathcal{U}(G)}(M)$ *depends only on the isomorphism class of M.*

Extension property. *If $M = \mathcal{U}(G) \otimes_{\mathcal{N}(G)} P$ for a finitely generated projective $\mathcal{N}(G)$-module P, then*

$$\dim_{\mathcal{U}(G)}(\mathcal{U}(G) \otimes_{\mathcal{N}(G)} P) = \dim_{\mathcal{N}(G)}(P).$$

Additivity. *If $\mathcal{U}(G)$-modules M_0, M_1, M_2 fit into an exact sequence*

$$0 \to M_0 \to M_1 \to M_2 \to 0,$$

their von Neumann dimensions satisfy

$$\dim_{\mathcal{U}(G)}(M_1) = \dim_{\mathcal{U}(G)}(M_0) + \dim_{\mathcal{U}(G)}(M_2).$$

Cofinality. *If a $\mathcal{U}(G)$-module $M = \bigcup_{i\in I} M_i$ is a directed union of $\mathcal{U}(G)$-submodules M_i, then*

$$\dim_{\mathcal{U}(G)}(M) = \sup\{\dim_{\mathcal{U}(G)}(M_i) \mid i \in I\}.$$

Furthermore, the dimension function satisfies the following additional properties:

Monotonicity. *If $N \leqslant M$ is a submodule of a $\mathcal{U}(G)$-module M, then* $\dim_{\mathcal{U}(G)}(N) \leqslant \dim_{\mathcal{U}(G)}(M)$.

Faithfulness. *If P is a projective $\mathcal{U}(G)$-module, then $\dim_{\mathcal{U}(G)}(P) = 0$ if and only if $P = 0$.*

Invariance under induction. *If $H \leqslant G$ is a subgroup and M is a $\mathcal{U}(H)$-module, then*

$$\dim_{\mathcal{U}(H)}(M) = \dim_{\mathcal{U}(G)}(\mathcal{U}(G) \otimes_{\mathcal{U}(H)} M).$$

Proof. All properties in the statement of the theorem except for Faithfulness and Induction are direct consequences of [Rei01, Theorem 3.11]. The a priori different Extension property stated there is equivalent to the the one we use by [Lüc02, Theorem 8.22 (7)], which says that every finitely generated projective $\mathcal{U}(G)$-module is up to isomorphism induced from a finitely generated projective $\mathcal{N}(G)$-module.

The induction property is proved in [Lüc02, Theorem 6.29 (2)] for $\dim_{\mathcal{N}(?)}$, but also follows for $\dim_{\mathcal{U}(?)}$ by the Extension property.

For the proof of Faithfulness, see [Lüc02, Theorem 8.29]. □

For a subring S of $\mathcal{U}(G)$, we set $\dim_S(M) \coloneqq \dim_{\mathcal{U}(G)}(\mathcal{U}(G) \otimes_S M)$ for every S-module M and call this value the *von Neumann dimension* of M whenever the particular subring S is clear from the context. For example, understood in this way, we obtain a dimension function $\dim_{\mathcal{N}(G)}$ for arbitrary $\mathcal{N}(G)$-modules that agrees with the one defined in [Lüc02, Chapter 6].

2.3 L^2-Betti numbers and the Atiyah question

We will now come to the main application of the construction of a dimension function for $\mathcal{U}(G)$-modules. For a G-space X, the singular chain complex $C_*^{\mathrm{sing}}(X)$ inherits a linear action by G that also descends to its homology. However, since $\mathbb{Z}G$ is in general a very complicated ring, it is usually impractical to extract any useful information, such as Betti numbers, out of these homology groups. For example, the group ring $\mathbb{Z}F_2$ of the free group on two generators admits an injection $(\mathbb{Z}F_2)^2 \hookrightarrow \mathbb{Z}F_2$ and thus cannot have a non-trivial additive dimension function. Given its convenient algebraic properties, most notably its $*$-regularity, and the availability of a dimension function, at this point the overring $\mathcal{U}(G)$ of $\mathbb{Z}G$ would seem well-suited to act as a replacement for $\mathbb{Z}G$. Following through on this idea, we arrive at the following rather algebraic definition of L^2-Betti numbers.

Definition 2.3.1. For a G-space X and $n \in \mathbb{N}$, the *n-th L^2-Betti number* of X, denoted by $b_n^{(2)}(X)$, is given by

$$b_n^{(2)}(X) = \dim_{\mathcal{U}(G)} \left(H_n(\mathcal{U}(G) \otimes_{\mathbb{Z}G} C_*^{\mathrm{sing}}(X))\right).$$

If X admits a G-CW-structure, the L^2-Betti numbers can alternatively be computed in terms of the cellular chain complex of X (see [Lüc98, Lemma 4.2]), which is more amenable to explicit computations.

For an overview of the computational properties of L^2-Betti numbers, most notably their homotopy invariance and multiplicativity under finite coverings, we refer the reader to [Lüc02, Chapter 6].

The historically first construction of L^2-Betti numbers was carried out by Atiyah in [Ati76]. He defined L^2-Betti numbers for free and cocompact actions of a discrete group G on a non-compact Riemannian manifold X using L^2-index theory. That the almost fully algebraic definition of L^2-Betti numbers of Definition 2.3.1 agrees with the classical one whenever the latter makes sense is the content of the L^2-Hodge–de Rham theorem, which was proved by Dodziuk in [Dod77]. A textbook presentation of the proof can be found in [Lüc02, Section 1.4].

Rephrased in our terminology, Atiyah posed the following seminal problem about the possible values of the L^2-Betti numbers of X:

> *A priori the numbers $b_n^{(2)}(\tilde{X})$ are real. Give examples where they are not integral and even perhaps irrational.*

Even though Atiyah left open whether he expected such examples to exist or not, the conjecture that L^2-Betti numbers of coverings of Riemannian manifolds or, equivalently, of finite free G-CW-complexes are rational is invariably named after him: the *Atiyah conjecture*. Due to the algebraic focus of this work, our precise formulation of (one version of) this conjecture involves von Neumann dimensions of finitely presented KG-modules for a subfield $K \leqslant \mathbb{C}$ rather than L^2-Betti numbers of manifolds.

Definition 2.3.2. Let G be a group and $K \leqslant \mathbb{C}$ a field. We say that the *weak Atiyah conjecture for G holds over K* if every finitely presented KG-module M satisfies

$$\dim_{\mathcal{U}(G)}(\mathcal{U}(G) \otimes_{KG} M) \in \mathbb{Q}.$$

While the present formulation of the (weak) Atiyah conjecture restricts the von Neumann dimensions of finitely presented KG-modules, the usual formulation given in e.g. [Lüc02, Conjecture 10.3] restricts the von Neumann dimensions of kernels of linear operators on $\mathcal{U}(G)^n$ given by multiplication by matrices over KG. Since finitely presented KG-modules are precisely the *co*kernels of maps given by right multiplication by matrices over KG, it should not be surprising that the two formulations turn out to be equivalent:

Lemma 2.3.3. *Let G be a group and $K \leqslant \mathbb{C}$ a field. Then the two additive subgroups of $\mathbb{R}$ generated by*

$$\{\dim_{\mathcal{U}(G)}(\mathcal{U}(G) \otimes_{KG} M) \mid M \textit{ finitely presented } KG\textit{-module}\}$$

and

$$\{\dim_{\mathcal{U}(G)}(\ker(\mathcal{U}(G)^m \xrightarrow{\cdot A} \mathcal{U}(G)^n)) \mid A \in M_{m\times n}(KG)\}$$

coincide.

Proof. Taking M to be a free module and A to be a zero matrix, both additive subgroups are seen to contain $\mathbb{Z}$. In a similar way as in the proof of [Lüc02, Lemma 10.7], we will prove that both subgroups differ only by an integral shift, which combined with the first observation implies that they coincide.

For a matrix $A \in M_{m\times n}(KG)$, we consider the homomorphisms $r_A \colon KG^m \to KG^n$ and $r_A^{\mathcal{U}(G)} \colon \mathcal{U}(G)^m \to \mathcal{U}(G)^n$ given by right multiplication by A. We consider the following exact sequence associated to the map $r_A^{\mathcal{U}(G)}$:

$$0 \to \ker(r_A^{\mathcal{U}(G)}) \to \mathcal{U}(G)^m \xrightarrow{r_A^{\mathcal{U}(G)}} \mathcal{U}(G)^n \to \operatorname{coker}(r_A^{\mathcal{U}(G)}) \to 0.$$

Since $\mathcal{U}(G) \otimes_{KG} ?$ is right exact, we conclude that the $\mathcal{U}(G)$-modules $\operatorname{coker}(r_A^{\mathcal{U}(G)})$ and $\mathcal{U}(G) \otimes_{KG} \operatorname{coker}(r_A)$ are isomorphic. By splitting the exact sequence into two short exact sequences and using the additivity of the von Neumann dimension for each, we thus obtain:

$$\dim_{\mathcal{U}(G)}(\ker(r_A^{\mathcal{U}(G)})) = m - n + \dim_{\mathcal{U}(G)}(\mathcal{U}(G) \otimes_{KG} \operatorname{coker}(r_A)).$$

The proof is concluded by observing that every finitely presented KG-module M is of the form $\operatorname{coker}(KG^m \xrightarrow{\cdot A} KG^n)$ for some $A \in M_{m\times n}(KG)$ and for every such A, the resulting KG-module is finitely presented. ☐

The equivalence of the weak Atiyah conjecture for G over $\mathbb{Q}$ and Atiyah's original question about the rationality of L^2-Betti numbers of coverings of Riemannian manifolds is now implied by Lemma 2.3.3, [Lüc02, Lemma 10.5] and the L^2-Hodge–de Rham-Theorem.

Since the question of rationality of L^2-Betti numbers had first become an area of research, many positive results on the weak Atiyah conjecture have been obtained. A survey of such results is provided in Section 2.4.3. The first negative result, disproving the weak Atiyah conjecture for arbitrary groups, has been obtained by Austin:

Theorem 2.3.4 ([Aus13]). *There exist an uncountable index set I and families of finitely generated groups G_i and elements $q_i \in \mathbb{Q}G_i$ in their rational group rings such that the values*

$$v_i \coloneqq \dim_{\mathcal{U}(G)}(\mathcal{U}(G_i) \otimes_{\mathbb{Q}G} \mathbb{Q}G_i/q_i) \in \mathbb{R}$$

for $i \in I$ are pairwise distinct. In particular, since there are only countably many algebraic numbers, there are uncountably many indices $i \in I$ for which v_i is transcendental.

Grabowski later studied connections between Turing machines and values of L^2-Betti numbers, thereby obtaining the following characterization of possible values:

Theorem 2.3.5 ([Gra14]). *The set of von Neumann dimensions arising from finitely generated groups is equal to the set of non-negative real numbers. The set of von Neumann dimensions arising from finitely presented groups contains all numbers with computable binary expressions.*

In their work, both Austin and Grabowski relied on the following construction of finitely generated groups with arbitrarily large finite subgroups:

Definition 2.3.6. Let G and H be groups. The *wreath product* of G and H, denoted by $G \wr H$, is the group

$$\left(\bigoplus_{h \in H} G\right) \rtimes H,$$

where H acts on the direct sum by left translation on the index set.

The easiest class of non-trivial examples of wreath products is given by the so-called lamplighter groups. Their name stems from an intuitive way of describing their elements, which can be imagined as tracking the state of a doubly infinite sequence of lamps that can be in a finite number of states as well as the position of a lamplighter standing at one of the lamps.

Definition 2.3.7. For every natural number $p \in \mathbb{N}, p > 1$, the *p-state lamplighter group* is the finitely generated group $L_p \coloneqq \mathbb{Z}/p \wr \mathbb{Z}$.

Grabowski gave the first explicit examples of elements of the rational group rings of unmodified p-state lamplighter groups that realize transcendental von Neumann dimensions:

Theorem 2.3.8 ([Gra16, Theorem 2]). *For every natural number $p \in \mathbb{N}, p > 1$, there exists an explicit matrix $T_p \in M_n(\mathbb{Q}L_p)$ with entries in $\mathbb{Q}L_p$ such that*

$$\dim_{\mathcal{U}(G)}(\mathcal{U}(G) \otimes_{\mathbb{Q}G} \mathbb{Q}G^n/\mathbb{Q}G^n T_p)$$

is transcendental.

All counterexamples to the weak Atiyah conjecture known to date involve groups that contain arbitrarily large finite subgroups.

2.4 The strong Atiyah conjecture

As we have seen in the previous section, the weak Atiyah conjecture cannot be expected to hold in the presence of arbitrarily large finite subgroups. We will thus restrict our attention to groups that admit a finite bound on the orders of their finite subgroups. Under this additional assumption, we can study an even stronger version of the weak Atiyah conjecture that further restricts the possible denominators of L^2-Betti numbers.

Definition 2.4.1. For a group G, denote by $\operatorname{lcm}(G)$ the least common multiple of the set $\{|F| \mid F \leqslant G, |F| < \infty\}$ if that set is finite and ∞ otherwise.

Definition 2.4.2. Let G be a group with $\operatorname{lcm}(G) < \infty$ and $K \leqslant \mathbb{C}$ a field. We say that the *strong Atiyah conjecture for G holds over K* if every finitely presented KG-module M satisfies

$$\dim_{\mathcal{U}(G)}(\mathcal{U}(G) \otimes_{KG} M) \in \frac{1}{\operatorname{lcm}(G)}\mathbb{Z}.$$

Remark 2.4.3. The statement of the strong Atiyah conjecture is optimal in the following sense: The additive subgroup of $\mathbb{R}$ generated by the values of $\dim_{\mathcal{U}(G)}(\mathcal{U}(G)\otimes_{KG} M)$ where M runs through all finitely presented KG-modules always contains $\frac{1}{\operatorname{lcm}(G)}\mathbb{Z}$. Namely, if $F \leqslant G$ is any finite subgroup, then the trivial KF-module K is finitely presented and $\dim_{\mathcal{U}(F)}(\mathcal{U}(F) \otimes_{KF} K) = \frac{1}{|F|}$. Consider the induced KG-module $M_F := KG \otimes_{KF} K$, which is again finitely presented. Using [Lüc02, Theorem 6.29 (2)] in the penultimate step, we obtain

$$\begin{aligned}
\dim_{\mathcal{U}(G)}(\mathcal{U}(G) \otimes_{KG} M_F) &= \dim_{\mathcal{U}(G)}(\mathcal{U}(G) \otimes_{\mathbb{C}G} \mathbb{C}G \otimes_{KG} KG \otimes_{KF} K) \\
&= \dim_{\mathcal{U}(G)}(\mathcal{U}(G) \otimes_{\mathbb{C}G} \mathbb{C}G \otimes_{\mathbb{C}F} \mathbb{C}) \\
&= \dim_{\mathcal{U}(G)}(\mathcal{U}(G) \otimes_{\mathcal{U}(F)} \mathbb{C}) \\
&= \dim_{\mathcal{U}(F)}(\mathbb{C}) = \frac{1}{|F|}.
\end{aligned}$$

The additive subgroup of $\mathbb{R}$ generated by the fractions $\frac{1}{|F|}$ is $\frac{1}{\operatorname{lcm}(G)}\mathbb{Z}$.

2.4.1 Consequences for $\dim_{\mathcal{U}(G)}$ and $\mathcal{R}_{KG}$

Even though we formulated the strong Atiyah conjecture as a condition on the possible values of von Neumann dimensions of finitely presented KG-modules, we will see now that it has strong implications on the structure of the $*$-regular closure as well as topological consequences. It will also become evident that the restriction to finitely presented modules is redundant.

Recall from Definition 2.1.7 that $\mathcal{R}_{KG}$ denotes the $*$-regular closure of KG in $\mathcal{U}(G)$. The statement of the strong Atiyah conjecture can be extended to cover finitely presented $\mathcal{R}_{KG}$-modules:

Proposition 2.4.4. *Let G be a group with $\operatorname{lcm}(G) < \infty$ and $K \leqslant \mathbb{C}$ a field closed under complex conjugation. Then the strong Atiyah conjecture for G holds over K if and only if every finitely presented $\mathcal{R}_{KG}$-module N satisfies*

$$\dim_{\mathcal{U}(G)}(\mathcal{U}(G) \otimes_{\mathcal{R}_{KG}} N) \in \frac{1}{\operatorname{lcm}(G)}\mathbb{Z}.$$

Proof. If M is a finitely presented KG-module, then $N := \mathcal{R}_{KG} \otimes_{KG} M$ is a finitely presented $\mathcal{R}_{KG}$-module with

$$\dim_{\mathcal{U}(G)}(\mathcal{U}(G) \otimes_{KG} M) = \dim_{\mathcal{U}(G)}(\mathcal{U}(G) \otimes_{\mathcal{R}_K} N),$$

which proves one implication.

We can apply Corollary 1.7.9 to $KG \leqslant \mathcal{R}_{KG}$ and any finitely presented $\mathcal{R}_{KG}$-module N to obtain two finitely presented KG-modules N^+ and N^- such that $N \oplus \mathcal{R}_{KG} \otimes_{KG} N^- \cong \mathcal{R}_{KG} \otimes_{KG} N^+$. Using the additivity of $\dim_{\mathcal{U}(G)}$, we compute that

$$\begin{aligned}
&\dim_{\mathcal{U}(G)}(\mathcal{U}(G) \otimes_{\mathcal{R}_{KG}} N) \\
&= \dim_{\mathcal{U}(G)}(\mathcal{U}(G) \otimes_{KG} N^+) - \dim_{\mathcal{U}(G)}(\mathcal{U}(G) \otimes_{KG} N^-)
\end{aligned}$$

which is contained in $\frac{1}{\operatorname{lcm}(G)}\mathbb{Z}$ since the strong Atiyah conjecture for G is assumed to hold over K. $\square$

The following lemma shows that the von Neumann dimension is faithful for projective $\mathcal{R}_{KG}$-modules:

Lemma 2.4.5. *Let P be a projective $\mathcal{R}_{KG}$-module. If $\dim_{\mathcal{U}(G)}(\mathcal{U}(G) \otimes_{\mathcal{R}_{KG}} P) = 0$, then $P = 0$.*

Proof. Since $\mathcal{U}(G) \otimes_{\mathcal{R}_{KG}} P$ is a projective $\mathcal{U}(G)$-module, we conclude from the assumption and faithfulness of the von Neumann dimension that $\mathcal{U}(G) \otimes_{\mathcal{R}_{KG}} P = 0$. We now use that P is in particular flat as an $\mathcal{R}_{KG}$-module, which implies that the $\mathcal{R}_{KG}$-linear map $\mathcal{R}_{KG} \otimes_{\mathcal{R}_{KG}} P \to \mathcal{U}_{KG} \otimes_{\mathcal{R}_{KG}} P$ induced from the injective map $\mathcal{R}_{KG} \hookrightarrow \mathcal{U}(G)$ is again injective. The codomain of the map is trivial and the domain is isomorphic to P, thus $P = 0$. $\square$

The following proposition and its corollary connect the strong Atiyah conjecture to the ring-theoretic structure of $\mathcal{R}_{KG}$. The statement about semisimplicity is not known to hold for the division closure $\mathcal{D}_{KG}$ assuming just the strong Atiyah conjecture.

Proposition 2.4.6. *Let G be a group with $\operatorname{lcm}(G) < \infty$ and $K \leqslant \mathbb{C}$ a field closed under complex conjugation. If the strong Atiyah conjecture for G holds over K, the ring $\mathcal{R}_{KG}$ is semisimple. Furthermore, the parameters s and n_i of its Artin–Wedderburn decomposition satisfy*

$$\sum_{i=1}^{s} n_i \leqslant \operatorname{lcm}(G).$$

Proof. We first assume that $\mathcal{R}_{KG}$ is semisimple and prove the second statement. Let $M_{n_1}(D_1) \times \cdots \times M_{n_s}(D_s)$ be the Artin–Wedderburn decomposition of $\mathcal{R}_{KG}$. We denote by S_i some choice of a minimal left ideal of $M_{n_i}(D_i)$ for every $i = 1, \ldots, s$ as in Proposition 1.6.4 and obtain from this proposition that as $\mathcal{R}_{KG}$-modules

$$\mathcal{R}_{KG} \cong S_1^{n_1} \oplus \cdots \oplus S_s^{n_s}.$$

For every $i = 1, \ldots, s$, the $\mathcal{R}_{KG}$-module S_i is finitely presented by Proposition 1.6.4 (b), and hence $\dim_{\mathcal{U}(G)}(\mathcal{U}(G) \otimes_{\mathcal{R}_{KG}} S_i) \in \frac{1}{\operatorname{lcm}(G)}\mathbb{Z}$ as a consequence of Proposition 2.4.4. Furthermore, since S_i is projective and non-trivial, we conclude from Lemma 2.4.5 that $\dim_{\mathcal{U}(G)}(\mathcal{U}(G) \otimes_{\mathcal{R}_{KG}} S_i) > 0$. All in all, we obtain that

$$1 = \dim_{\mathcal{U}(G)}(\mathcal{U}(G) \otimes_{\mathcal{R}_{KG}} \mathcal{R}_{KG}) = \sum_{i=1}^{s} n_i \dim_{\mathcal{U}(G)}(\mathcal{U}(G) \otimes_{\mathcal{R}_{KG}} S_i) \geqslant \frac{\sum_{i=1}^{s} n_i}{\operatorname{lcm}(G)},$$

and thus $\sum_{i=1}^{s} n_i \leqslant \operatorname{lcm}(G)$.

We now return to the proof of the first statement. Since $\mathcal{R}_{KG}$ is von Neumann regular, every finitely presented $\mathcal{R}_{KG}$-module is projective by Proposition 1.4.3 (d). Thus, the last paragraph in fact proves the more general statement that whenever $\mathcal{R}_{KG}$ contains a direct sum of non-trivial finitely presented $\mathcal{R}_{KG}$-submodules, then the number of summands is at most $\operatorname{lcm}(G)$ and in particular finite.

Now assume for the sake of contradiction that $\mathcal{R}_{KG}$ is not semisimple. Then some ideal J of $\mathcal{R}_{KG}$ is not a direct summand, see [Gri07, Proposition 3.1]. This is impossible for finitely generated ideals by Proposition 1.4.3 (c), so we find a non-finitely generated ideal J of $\mathcal{R}_{KG}$. By repeatedly adjoining an element not contained in J_i, we obtain a chain of finitely generated ideals J_i of $\mathcal{R}_{KG}$, where $i \in \mathbb{N}$, with strict inclusions:

$$\{0\} = J_0 \subsetneq J_1 \subsetneq J_2 \subsetneq \cdots \subsetneq J.$$

We use Proposition 1.4.3 (c) again to conclude that every J_i is a direct summand in J_{i+1}, with the non-trivial finitely generated projective, and hence finitely presented, complement denoted by K_i. But then $\oplus_{i\in\mathbb{N}} K_i$ is an infinite direct sum of non-trivial finitely presented $\mathcal{R}_{KG}$-submodules of $\mathcal{R}_{KG}$ and we have reached a contradiction. □

The following corollary shows that the strong Atiyah conjecture for a torsion-free group implies the Kaplansky zero divisor conjecture. For the class of torsion-free amenable groups, the two conjectures are equivalent by [Lüc02, Lemma 10.16].

Corollary 2.4.7. *Let G be a torsion-free group and $K \leqslant \mathbb{C}$ a field closed under complex conjugation. Then the strong Atiyah conjecture for G holds over K if and only if the ring $\mathcal{R}_{KG}$ is a division ring. If this is the case, then $\mathcal{D}_{KG} = \mathcal{R}_{KG}$.*

Proof. We first assume the strong Atiyah conjecture for G over K and prove that $\mathcal{D}_{KG} = \mathcal{R}_{KG}$ is a division ring. Since $\operatorname{lcm}(G) = 1$, Proposition 2.4.6 implies that $s = 1$ and $n_1 = 1$ in the Artin–Wedderburn decomposition of $\mathcal{R}_{KG}$. Hence, $\mathcal{R}_{KG}$ is a division ring, which is in particular division closed in every overring. As it is also a subring of $\mathcal{U}(G)$ and contains KG, we obtain

$$\mathcal{D}_{KG} = \mathcal{D}(KG,\mathcal{U}(G)) = \mathcal{D}(KG,\mathcal{R}_{KG}) = \mathcal{R}(KG,\mathcal{R}_{KG}) = \mathcal{R}_{KG}.$$

If $\mathcal{R}_{KG}$ is a division ring, every $\mathcal{R}_{KG}$-module is free and thus has integral von Neumann dimension by addivity. We conclude from the easy direction of Proposition 2.4.4 that this implies the strong Atiyah conjecture for G over K. □

The restriction on the field K can in fact be dropped if one replaces the $*$-regular closure in Corollary 2.4.7 by the division closure. We refer the reader to [Lüc02, Lemma 10.39] for the slightly more technical proof.

Theorem 2.4.8. *Let G be a torsion-free group and $K \leqslant \mathbb{C}$ a field. Then the strong Atiyah conjecture for G holds over K if and only if the ring $\mathcal{D}_{KG}$ is a division ring.*

Since the subgroup $\frac{1}{\operatorname{lcm}(G)}\mathbb{Z}$ of $\mathbb{R}$ is discrete, the finiteness assumption on the KG-module M in Definition 2.4.2 can in fact be dropped. The proof of this fact requires a special case of the following lemma:

Lemma 2.4.9. *Subquotients of finitely generated $\mathcal{U}(G)$-modules have finite von Neumann dimension.*

Proof. We first show that finitely generated $\mathcal{U}(G)$-modules have finite von Neumann dimension. Let M be a finitely generated $\mathcal{U}(G)$-module and choose a surjection $p\colon \mathcal{U}(G)^n \to M$ for some $n \in \mathbb{N}$. Since $\dim_{\mathcal{U}(G)}$ is additive on exact sequences, applying it to the short exact sequence

$$0 \to \ker(p) \to \mathcal{U}(G)^n \to M \to 0$$

shows that

$$\begin{aligned}\infty > n &= \dim_{\mathcal{U}(G)}(\mathcal{U}(G)^n)\\ &= \dim_{\mathcal{U}(G)}(\ker(p)) + \dim_{\mathcal{U}(G)}(M)\\ &\geqslant \dim_{\mathcal{U}(G)}(M).\end{aligned}$$

Now let $N \leqslant M$ be a submodule of the finitely generated $\mathcal{U}(G)$-module M. Using additivity again, we obtain that

$$\infty > \dim_{\mathcal{U}(G)}(M) = \dim_{\mathcal{U}(G)}(N) + \dim_{\mathcal{U}(G)}(M/N) \geqslant \dim_{\mathcal{U}(G)}(N).$$

The same argument then shows that quotients of N have finite von Neumann dimension. □

Proposition 2.4.10. *Let G be a group with $\operatorname{lcm}(G) < \infty$ and $K \leqslant \mathbb{C}$ a field. Then the strong Atiyah conjecture for G holds over K if and only if every (arbitrary) $\mathcal{R}_{KG}$-module N satisfies*

$$\dim_{\mathcal{U}(G)}(\mathcal{U}(G) \otimes_{\mathcal{R}_{KG}} N) \in \frac{1}{\operatorname{lcm}(G)}\mathbb{Z} \cup \{\infty\}.$$

Proof. If M is an arbitrary finitely presented KG-module, then its von Neumann dimension $\dim_{\mathcal{U}(G)}(\mathcal{U}(G) \otimes_{KG} M)$ is finite by Lemma 2.4.9. Hence, from the assumption on arbitrary $\mathcal{R}_{KG}$-modules, we obtain that

$$\dim_{\mathcal{U}(G)}(\mathcal{U}(G) \otimes_{\mathcal{R}_{KG}} \mathcal{R}_{KG} \otimes_{KG} M) \in \frac{1}{\operatorname{lcm}(G)}\mathbb{Z},$$

which confirms the strong Atiyah conjecture for G over K.

We now assume that the strong Atiyah conjecture for G holds over K and consider an arbitrary $\mathcal{R}_{KG}$-module N. By Proposition 2.4.6, our assumption implies that the ring $\mathcal{R}_{KG}$ is semisimple. Consequently, Proposition 1.6.4 implies that $N \cong S_1^{m_1} \oplus \cdots \oplus S_s^{m_s}$ for some fixed choice $\{S_1, \ldots, S_s\}$ of a set of representatives for the isomorphism classes of simple $\mathcal{R}_{KG}$-modules and suitable cardinal numbers $m_1, \ldots, m_s$.

By Proposition 1.6.4 (b), simple $\mathcal{R}_{KG}$-modules are finitely presented. We can therefore apply Proposition 2.4.4 to each S_i and get that

$$b_i \coloneqq \dim_{\mathcal{U}(G)}(\mathcal{U}(G) \otimes_{\mathcal{R}_{KG}} S_i) \in \frac{1}{\operatorname{lcm}(G)}\mathbb{Z}.$$

Using the additivity and cofinality of $\dim_{\mathcal{U}(G)}$, we obtain that

$$\dim_{\mathcal{U}(G)}(\mathcal{U}(G) \otimes_{\mathcal{R}_{KG}} S_i^{m_i}) = \dim_{\mathcal{U}(G)}((\mathcal{U}(G) \otimes_{\mathcal{R}_{KG}} S_i)^{\oplus m_i}) = m_i \cdot b_i,$$

where $m_i \cdot b_i$ is understood to be 0 if $b_i = 0$, ∞ if $b_i \neq 0$ and m_i is an infinite cardinal, and the result of ordinary multiplication otherwise. With the additional convention that the sum of two cardinal numbers is infinite if any of the two summands is, we finally conclude that

$$\dim_{\mathcal{U}(G)}(\mathcal{U}(G) \otimes_{\mathcal{R}_{KG}} N) = m_1 \cdot b_1 + \cdots + m_s \cdot b_s \in \frac{1}{\operatorname{lcm}(G)}\mathbb{Z} \cup \{\infty\}. \qquad \square$$

Although our formulation of the strong Atiyah conjecture is quite algebraic, its version with coefficients in $\mathbb{Q}$ can in fact be formulated equivalently in terms of topological invariants:

Theorem 2.4.11. *The following statements are equivalent for a group G with $\operatorname{lcm}(G) < \infty$:*

(a) *The strong Atiyah conjecture for G holds over $\mathbb{Q}$.*

(b) *For every finite free G-CW-complex X and every $n \in \mathbb{N}$, the n-th L^2-Betti number $b_n^{(2)}(X)$ is contained in $\frac{1}{\operatorname{lcm}(G)}\mathbb{Z}$.*

(c) *For every G-space X and every $n \in \mathbb{N}$, the n-th L^2-Betti number $b_n^{(2)}(X)$ is either infinite or contained in $\frac{1}{\operatorname{lcm}(G)}\mathbb{Z}$.*

Proof. (a)$\Rightarrow$(c): Given that $\mathcal{R}_{\mathbb{Q}G}$ is a von Neumann regular ring, $\mathcal{U}(G)$ is flat as an $\mathcal{R}_{\mathbb{Q}G}$-module by Proposition 1.4.3 (e). We thus obtain that

$$\begin{aligned}
&H_n(\mathcal{U}(G) \otimes_{\mathbb{Z}G} C_*^{\mathrm{sing}}(X)) \\
\cong\, &H_n(\mathcal{U}(G) \otimes_{\mathcal{R}_{\mathbb{Q}G}} \mathcal{R}_{\mathbb{Q}G} \otimes_{\mathbb{Z}G} C_*^{\mathrm{sing}}(X)) \\
\cong\, &\mathcal{U}(G) \otimes_{\mathcal{R}_{\mathbb{Q}G}} H_n(\mathcal{R}_{\mathbb{Q}G} \otimes_{\mathbb{Z}G} C_*^{\mathrm{sing}}(X)),
\end{aligned}$$

which in particular implies that

$$\begin{aligned} b_n^{(2)}(X) &= \dim_{\mathcal{U}(G)} \big(H_n(\mathcal{U}(G) \otimes_{\mathbb{Z}G} C_*^{\text{sing}}(X))\big) \\ &= \dim_{\mathcal{U}(G)} \big(\mathcal{U}(G) \otimes_{\mathcal{R}_{\mathbb{Q}G}} H_n(\mathcal{R}_{\mathbb{Q}G} \otimes_{\mathbb{Z}G} C_*^{\text{sing}}(X))\big). \end{aligned}$$

Given our assumption that the strong Atiyah conjecture holds for G over $\mathbb{Q}$, Proposition 2.4.10 allows us to conclude that the last term is either infinite or contained in $\frac{1}{\operatorname{lcm}(G)}\mathbb{Z}$.

(c)⇒(b): If X is a G-CW-complex, we have

$$H_n(\mathcal{U}(G) \otimes_{\mathbb{Q}G} C_*(X)) \cong H_n(\mathcal{U}(G) \otimes_{\mathbb{Q}G} C_*^{\text{cell}}(X))$$

by [Lüc98, Lemma 4.2]. The latter $\mathcal{U}(G)$-module is obtained as a subquotient of the finitely generated $\mathcal{U}(G)$-module $\mathcal{U}(G)^{\beta_n(X)}$, where $\beta_n(X)$ denotes the number of equivariant n-cells of X, and thus has finite von Neumann dimension by Lemma 2.4.9.

(b)⇒(a): This is implied by [Lüc02, Lemma 10.5] since every free G-CW-complex is automatically proper and a G-CW-complex is cocompact if and only if it is finite. ☐

Corollary 2.4.12. *Let G be a group with $\operatorname{lcm}(G) < \infty$. If the strong Atiyah conjecture for G holds over $\mathbb{Q}$, then Atiyah's question has a positive answer, that is, the L^2-Betti numbers $b_n^{(2)}(X)$ are rational (or infinite) for every G-space X.*

2.4.2 Classes of groups

Before we can formulate the current state of knowledge on the strong Atiyah conjecture, we need to introduce a few commonly used classes of groups.

Definition 2.4.13. Let P_N and P_Q be two properties of groups, e.g., being finite. A group G is called a *P_N-by-P_Q* group if there exist a group N satisfying P_N and a group Q satisfying P_Q such that G fits into a short exact sequence

$$1 \to N \to G \to Q \to 1.$$

We will sometimes put curly brackets around properties that are expressed with more than one word, e.g., when we consider free-by-{infinite cyclic} groups.

Definition 2.4.14. Let P be a property of groups. A group G is called *locally P* if every finitely generated subgroup of G satisfies P.

Definition 2.4.15. A group G is called *amenable* if it admits a finitely additive left-G-invariant probability measure.

The class of amenable groups contains all abelian and all finite groups and is closed under taking subgroups, quotients, extensions, and directed unions (see [CC10, Section 4.5]). The groups that arise from these base groups via this list of inheritance properties are called *elementary amenable.* Every free group on at least two generators is not amenable (see [CC10, Corollary 4.5.2]) and there are amenable groups that are not elementary amenable (see [Gri84]).

Definition 2.4.16. Let P be a property of groups. A group G is called *residually P* if for every $g \in G, g \neq 1$ there exists an epimorphism $p\colon G \to Q$ to a group Q satisfying P such that $p(g) \neq 1$. G is called *fully residually P* if for every finite subset $F \subset G$ there exists an epimorphism $p\colon G \to Q$ to a group Q satisfying P such that p is injective when restricted to F.

A group that is fully residually P is clearly residually P. The converse holds if the property P is preserved under taking subgroups and finite products, e.g., this holds for the properties of being finite or amenable. For example, free groups are residually finite (see [CC10, Theorem 2.3.1]) and therefore fully residually finite.

We will now discuss a model for considering limits of finitely generated groups. Recall that F_k denotes the free group on k generators.

Definition 2.4.17. For every natural number $k \in \mathbb{N}$, a *k-marked group* is an equivalence class of epimorphisms $p\colon F_k \twoheadrightarrow G$, where $p_1\colon F_k \to G_1$ and $p_2\colon F_k \to G_2$ are equivalent if there exists a group isomorphism $u\colon G_1 \to G_2$ such that $p_2 = u \circ p_1$.

Note that k-marked group are necessarily finitely generated.

The k-marked groups are in natural bijective correspondence to normal subgroups of F_k. Via this correspondence, we view the set of n-marked groups as a subset of the set $\mathcal{P}(F_k) \cong \{0,1\}^{F_k}$ of all subsets of F_k, which allows for the following definition:

Definition 2.4.18. For every natural number $k \in \mathbb{N}$, the *space of k-marked groups* is the subspace $\mathcal{M}_k$ of $\{0,1\}^{F_k}$ corresponding to the k-marked groups.

The space $\mathcal{M}_k$ is a totally disconnected compact Hausdorff space (see [CC10, Proposition 3.4.1]) that is metrizable (see [CC10, Remark 3.4.2]). Intuitively, two k-marked groups are close to each other in $\mathcal{M}_k$ if the intersections of the corresponding normal subgroups with a large finite subset of F_k coincide.

Definition 2.4.19. Let $k \in \mathbb{N}$ be a fixed natural number and let $(F_k \twoheadrightarrow G_n)_{n\in\mathbb{N}}$ be a convergent sequence of k-marked groups. The codomain of the limit of the sequence in $\mathcal{M}_k$ is called the *marked limit* of $(F_k \twoheadrightarrow G_n)_{n\in\mathbb{N}}$.

For example, a residually finite group is the marked limit of its finite quotients.

Marked limits generalize the concept of a fully residually P group for finitely generated groups and coincide with it when considering finitely presented groups:

Proposition 2.4.20. *Let P be a property of groups. A finitely generated group G is fully residually P if and only if it is a marked limit of quotients of itself satisfying P. If G is finitely presented, these conditions are further equivalent to G being an arbitrary marked limit of marked groups satisfying P.*

Proof. The first statement is proved simply by unraveling the definition of convergence of quotients in a space of marked groups. The second statement is proved analogously to [CC10, Corollary 7.1.21]. □

If S is a set, we denote by $\mathrm{Sym}(S)$ the group of permutations on S. For two permutations $\sigma_1, \sigma_2 \in \mathrm{Sym}(F)$, where F is a finite set, we set

$$d_F(\sigma_1, \sigma_2) := \frac{\{x \in F \mid \sigma_1(x) \neq \sigma_2(x)\}}{|F|}.$$

Definition 2.4.21. A group G is called *sofic* if for every finite subset $K \subset G$ and every $\epsilon > 0$ there exist a non-empty finite set F and a map of sets $\varphi\colon G \to \mathrm{Sym}(F)$ such that:

(1) $d_F(\varphi(xy), \varphi(x)\varphi(y)) \leqslant \epsilon$ for all $x, y \in K$;

(2) $d_F(\varphi(x), \varphi(y)) \geqslant 1 - \epsilon$ for all $x, y \in K, x \neq y$.

The class of sofic groups contains all amenable groups and all residually sofic groups and is closed under taking subgroups, extensions by amenable groups, products, coproducts, directed unions, colimits of directed systems, limits of directed inverse systems, and marked limits (see [CC10, Section 7.5] and [ES06]). There exists a finitely presented sofic group that is not a marked limit of amenable groups or, equivalently, not residually amenable (see [Cor11, Corollary 3]). An important open question is whether there exist non-sofic groups.

We will now introduce a class of groups that is not modeled on finite groups, but rather on infinite cyclic groups:

Definition 2.4.22. A group G is called *indicable* if it admits a homomorphism onto $\mathbb{Z}$ or is the trivial group.

The class of locally indicable groups includes all one-relator groups (see [Bro84]) and (left) orderable amenable groups (see [Mor06]), is closed under subgroups, extensions, and directed unions. It is is possible that there exists a non-sofic locally indicable (even one-relator) group.

2.4.3 Current status

In order to simplify the statements, we will in this section only refer to the following classes of groups:

Definition 2.4.23. For a field $K \leqslant \mathbb{C}$, we denote by

- $\mathcal{SAC}_K$ the class of groups satisfying the strong Atiyah conjecture over K;
- $\mathcal{SAC}_K^{tf}$ the subclass of torsion-free groups in $\mathcal{SAC}_K$;
- $\mathcal{SAC}_K^{s}$ the subclass of sofic groups in $\mathcal{SAC}_K^{s}$.

Especially within the class of sofic groups many results on the strong Atiyah conjecture use some version of Lück approximation, which often allows to compute L^2-Betti numbers over limits or colimits of directed (inverse) systems of groups in terms of L^2-Betti numbers over the members of the systems. In order to treat such approximation results in a uniform way, we introduce some notation:

Definition 2.4.24. Let G be a group and $(G_i)_{i\in I}$ a family of groups for a directed, possibly inverse system I. The pair $(G, (G_i)_{i\in I})$ is said to satisfy the *Lück approximation condition* if for every finitely presented KG-module M there exist finitely presented KG_i-modules M_i for every $i \in I$ such that

$$\dim_{\mathcal{U}(G)}(\mathcal{U}(G) \otimes_{KG} M) = \lim_{i\in I} \dim_{\mathcal{U}(G)}(\mathcal{U}(G) \otimes_{KG_i} M_i).$$

Definition 2.4.25. Let $(G_i)_{i\in I}$ be a family of groups for an index set I. We set

$$\operatorname{lcm}((G_i)_{i\in I}) := \operatorname{lcm}\{\operatorname{lcm}(G_i) \mid i \in I\},$$

which is a finite natural number unless the set on the right-hand side contains ∞ or there is no upper bound on the prime divisors or exponents of the elements of the set.

We now have the following abstract criterion that allows to deduce the strong Atiyah conjecture from an approximation statement:

Lemma 2.4.26. *Let $K \leqslant \mathbb{C}$ be a field and consider a pair $(G, (G_i)_{i\in I})$ of a group and a directed, possibly inverse system that satisfies the Lück approximation condition. If additionally $\operatorname{lcm}(G) < \infty$, $\operatorname{lcm}((G_i)_{i\in I}) \mid \operatorname{lcm}(G)$, and every G_i for $i \in I$ satisfies the strong Atiyah conjecture over K, then so does G.*

Proof. Since every G_i satisfies the strong Atiyah conjecture over K, the von Neumann dimension of a finitely presented KG_i-module lies in $\frac{1}{\operatorname{lcm}(G_i)}\mathbb{Z}$, which by assumption is a subgroup of $\frac{1}{\operatorname{lcm}(G)}\mathbb{Z}$. The latter is a discrete subgroup of $\mathbb{R}$ and thus the Lück approximation condition implies that the von Neumann dimension of an arbitrary finitely presented KG-module lies in $\frac{1}{\operatorname{lcm}(G)}$, i.e., the strong Atiyah conjecture over K holds for G. $\square$

The following theorem summarizes most of the known results on the strong Atiyah conjecture. For classes of groups that we have not introduced, we refer the reader to the references listed in the proof.

Theorem 2.4.27. *The following is known about the classes of groups $\mathcal{SAC}_K$, $\mathcal{SAC}_K^{tf}$, and $\mathcal{SAC}_K^s$:*

(a) ***Base change.***

  (a1) *If $K \leqslant L \leqslant \mathbb{C}$ are fields, then $\mathcal{SAC}_L \subset \mathcal{SAC}_K$.*

  (a2) $\mathcal{SAC}_{\overline{\mathbb{Q}}}^s = \mathcal{SAC}_{\mathbb{C}}^s$.

(b) ***Specific groups.*** *Groups G with $\operatorname{lcm}(G) < \infty$ that are contained in any of the following classes of groups are in $\mathcal{SAC}_K$ for every subfield $K \leqslant \mathbb{C}$:*

  (b1) *free-by-{elementary amenable} groups;*

  (b2) *residually {torsion-free elementary amenable} groups;*

  (b3) *braid groups;*

  (b4) *right-angled Artin and Coxeter groups;*

  (b5) *virtually special groups;*

  (b6) *fundamental groups of connected orientable compact irreducible 3-manifolds with empty or toroidal boundary that is not a closed graph manifold;*

  (b7) *primitive link groups;*

  (b8) *virtual congruence subgroups;*

  (b9) *torsion-free pro-p groups of finite rank;*

  (b10) *locally indicable groups;*

  (b11) *one-relator groups.*

(c) ***Inheritance properties.*** *For every field $K \leqslant \mathbb{C}$ that is closed under complex conjugation, a group G with $\operatorname{lcm}(G) < \infty$ is contained in $\mathcal{SAC}_K$ if it is*

  (c1) *a subgroup of a group $G' \in \mathcal{SAC}_K$ with $\operatorname{lcm}(G) = \operatorname{lcm}(G')$;*

  (c2) *a directed union of groups in $\mathcal{SAC}_K$ with a uniform bound on the order of finite subgroups;*

  (c3) *an extension of a group by an elementary amenable group A such that the preimage of each finite subgroup of A in G lies in $\mathcal{SAC}_K$;*

  (c4) *an extension of a group in $\mathcal{SAC}_K$ by a torsion-free elementary amenable group;*

  (c5) *an extension of a group $H \in \mathcal{SAC}_K$ by an elementary amenable group, where H has a finite classifying space, enough torsion-free quotients, and is cohomologically complete (e.g., if H is a pure braid, primitive link, or right-angled Artin group);*

  (c6) *an extension of a group $H \in \mathcal{SAC}_K$ by an elementary amenable group Q whose finite subgroups are all p-groups, where H has a finite classifying space, enough torsion-free quotients, and is cohomologically p-complete (e.g., if H is the commutator subgroup of a right-angled Coxeter group);*

(c7) *an extension of a group $H \in \mathcal{SAC}_K$ by an elementary amenable group, where H has a finite classifying space, is a good group and has the factorization property (e.g., if H is cocompact special).*

(d) ***Inheritance properties for sofic groups.*** *The class $\mathcal{SAC}^s_{\mathbb{C}} = \mathcal{SAC}^s_{\overline{\mathbb{Q}}}$ contains a group G with $\operatorname{lcm}(G) < \infty$ if it*

(d1) *is the colimit (also called "direct limit") of a directed system $(G_i)_{i\in I}$, assuming that $\operatorname{lcm}((G_i)) | \operatorname{lcm}(G)$ and $G_i \in \mathcal{SAC}^s_{\overline{\mathbb{Q}}}$ for all $i \in I$;*

(d2) *is the limit (also called "inverse limit") of a directed inverse system $(G_i)_{i\in I}$, assuming that $\operatorname{lcm}((G_i)) | \operatorname{lcm}(G)$ and $G_i \in \mathcal{SAC}^s_{\overline{\mathbb{Q}}}$ for all $i \in I$;*

(d3) *is the marked limit of a sequence of marked groups $(F_k \twoheadrightarrow G_n)_{n\in\mathbb{N}}$, assuming that $\operatorname{lcm}((G_n)) | \operatorname{lcm}(G)$ and $G_n \in \mathcal{SAC}^s_{\overline{\mathbb{Q}}}$ for all $n \in \mathbb{N}$.*

(d4) *admits a chain of subgroups $G = N_0 \geqslant N_1 \geqslant N_2 \geqslant \dots$, each of which is normal in G, such that $\bigcap_{n\in\mathbb{N}} N_n = \{1\}$, $\operatorname{lcm}((G/N_n)) | \operatorname{lcm}(G)$ and $G/N_n \in \mathcal{SAC}^s_{\overline{\mathbb{Q}}}$ for each $n \in \mathbb{N}$.*

(e) ***Inheritance properties for torsion-free groups.*** *For every field $K \leqslant \mathbb{C}$, the class $\mathcal{SAC}^{tf}_K$ is closed under taking*

(e1) *subgroups;*

(e2) *directed unions;*

(e3) *extensions by locally indicable groups.*

The arguably most desirable inheritance property missing from Theorem 2.4.27 is that of passing to products or coproducts of groups. At least for torsion-free groups, this, however, is not much of an issue in practice, as has first been remarked by Schick in [Sch01]:

Theorem 2.4.28. *Let $K \leqslant \mathbb{C}$ be a field that is closed under complex conjugation. The subclass of $\mathcal{SAC}^{tf}_K \cap \mathcal{SAC}^s_K$ obtained from the groups listed in Theorem 2.4.27 (b1)–(b7) via the inheritance properties (c), (d1), (d2), (d4), (e1), and (e2) is closed under arbitrary products and coproducts.*

Proof. We show that the subclass coincides with the class $\mathcal{D}$ introduced in [Sch01, Definition 1.10], which has the desired property by [Sch01, Proposition 1.13]. Properties (c1) and (e1) as well as (c2), (d1), (d2), and (e2) are built into the definition of the class $\mathcal{D}$ considered in [Sch01] as properties (2) and (3). In the presence of (e1), property (d4) follows from (d2) as the group G is a subgroup of the inverse limit of the quotients by the normal subgroups. The proof of [Sch01, Proposition 1.13] also covers properties (b1), (b3)–(b7), and (c4)–(c7) as they all go through (b2) and (c3), which are consequences of [Sch01, Corollary 2.7] and property (1) of the class $\mathcal{D}$ considered in [Sch01], respectively. □

Note that it is not clear whether Theorem 2.4.28 remains valid if the subclass is also assumed to be closed under (e3). This is because locally indicable groups are not known to be sofic and while the Lück approximation condition is known for both sofic groups and locally indicable groups individually, the latter by [JL20, Theorem 1.5], it is not known for products of groups from both classes.

Proof of Theorem 2.4.27: (a1) If M is a finitely presented KG-module, then $LG \otimes_{KG} M$ is a finitely presented LG-module with the same von Neumann dimension.

(a2) [Jai19c, Theorem 1.1]

(b1) [Lin93, Theorem 1.5] (see also [Lüc02, Chapter 10]) (b2) By [Dod+03, Theorem 1.4], residually {torsion-free elementary amenable} groups satisfy the strong Atiyah conjecture over $\overline{\mathbb{Q}}$. Since such groups are sofic, they also satisfy the strong Atiyah conjecture over $\mathbb{C}$ by (a2) and thus over all subfields of $\mathbb{C}$ by (a1).

(b3) Pure braid groups are residually {torsion-free nilpotent} by [LS07, Theorem 5.40] and thus satisfy the strong Atiyah conjecture over $\mathbb{C}$ by (b2). Furthermore, by the same result, they satisfy the conditions of (c5).

(b4) By [LOS12, Proposition 9], both right-angled Artin groups and the commutator subgroups of right-angled Coxeter groups are residually {torsion-free nilpotent} and thus satisfy the strong Atiyah conjecture over $\mathbb{C}$ by (b2). They also satisfy the conditions of (c5) and (c6), respectively, as is shown in the proof of [LOS12, Theorem 2]. As is observed there, right-angled Coxeter groups are extensions of their commutator subgroups by finite 2-groups.

(b5) By a remarkable result of Haglund and Wise in [HW08], cocompact special groups are subgroups of right-angled Artin groups and thus satisfy the strong Atiyah conjecture over $\mathbb{C}$ by (b4) and (e1). They satisfy the conditions of (c7) by [Sch14, Corollary 4.3], which then also covers virtually special groups.

(b6) These groups are virtually special by [FL19, Theorem 3.2 (3)], thus (b5) applies.

(b7) Since primitive links are in particular not splittable, their complements in S^3 are connected, orientable, compact, and irreducible 3-manifolds with non-empty toroidal boundary. It thus follows from (b6) that primitive link groups satisfy the strong Atiyah conjecture over $\mathbb{C}$. They also satisfy the properties of (c5) by [LS07, Proposition 5.34] and [BLS08, Theorem 1.4].

(b8) [FL06, Theorem 1.1]

(b9) [FL06, Theorem 7.4]

(b10) [JL20, Theorem 1.1]

(b11) One-relator groups are either locally indicable (if they are torsion-free) or virtually special (if they contain torsion), so this follows from (b7) and (b10) (see also [JL20, Corollary 1.3]).

(c1) If M is a finitely presented KG-module, then $KG' \otimes_{KG} M$ is a finitely presented KG'-module with the same von Neumann dimension because the dimension is invariant under induction.

(c2) Let G be a directed union of subgroups G_i for some index set $i \in I$ and assume that there is a uniform bound on the order of finite subgroups of the groups G_i. Since every finite subgroup of G arises as a finite subgroup of some G_i and thus has bounded order, the group G satisfies $\operatorname{lcm}(G) < \infty$. We now consider an arbitrary finitely presented KG-module M and a fixed choice of a presentation matrix $A \in M_{m\times n}(KG)$. Since A has only finitely many entries and each entry has finite support, there exists $i \in I$ such that $A \in M_{m\times n}(KG_i)$. We denote by $\overline{M}$ the finitely presented KG_i-module with A as its presentation matrix. Using the invariance under induction of the von Neumann dimension, we conclude that

$$\dim_{\mathcal{U}(G)}(\mathcal{U}(G) \otimes_{KG} M) = \dim_{\mathcal{U}(G)}(\mathcal{U}(G) \otimes_{KG_i} \overline{M}) = \dim_{\mathcal{U}(G_i)}(\mathcal{U}(G_i) \otimes_{KG_i} \overline{M}).$$

The right-hand side is contained in $\frac{1}{\operatorname{lcm}(G_i)}\mathbb{Z}$ by the assumption that G_i satisfies the strong Atiyah conjecture over K. Since every finite subgroup of G_i is also a finite subgroup of G, it is thus contained in $\frac{1}{\operatorname{lcm}(G)}\mathbb{Z}$.

(c3) [LS07, Corollary 2.7]

(c4) This follows from (c3) since the only finite subgroup of a torsion-free group is the trivial group, the preimage of which is assumed to satisfy the strong Atiyah conjecture over K.

(c5) [LS07, Corollary 4.62] (see also the proofs of (b3), (b4), and (b7))

(c6) [LOS12, Proposition 10] (see also the proof of (b4)) (c7) [Sch14, Theorem 1.1] (see also the proof of (b5))

(d1) & (d2) These appear as situations (1) and (2) of [Dod+03, Situation 3.5]. By Lemma 2.3.3 and [Dod+03, Theorem 3.26], the Lück approximation condition is satisfied in these cases if the groups G_i satisfy the determinant bound conjecture over $\overline{\mathbb{Q}}$, which is formulated in [Dod+03, Definition 3.1]. This conjecture is proved for all sofic groups in [Jai19a, Theorem 10.10]. The Lück approximation condition and the assumptions imply the strong Atiyah conjecture for G over $\overline{\mathbb{Q}}$ by Lemma 2.4.26.

(d3) The case of a marked limit is also a consequence of Lemma 2.4.26 since the Lück approximation condition holds by [Jai19c, Corollary 1.4].

(d4) is a special case of (d3).

(e1) & (e2) These properties are direct consequences of (c1) and (c2) since $\operatorname{lcm}(G) = 1$ for every torsion-free group G.

(e3) [JL20, Proposition 6.5] □

2.5 The center-valued Atiyah conjecture

As we have observed in Proposition 2.4.6, the strong Atiyah conjecture for G over K implies certain restrictions on the structure of the ring $\mathcal{R}_{KG}$. In particular, it is always semisimple and all numeric parameters in its Artin–Wedderburn decomposition are bounded above by $\operatorname{lcm}(G)$. However, if G is not torsion-free, these conditions are not sufficient to fully determine the parameters.

In this section, we will discuss a slightly stronger version of the strong Atiyah conjecture that, among its many equivalent statements, admits a complete description of the semisimple ring $\mathcal{R}_{KG}$, assuming its division ring components are treated as black boxes.

At the same time, this version will answer another question one might reasonably ask about the Atiyah conjecture: To what extent do L^2-Betti numbers remain restricted if one replaces the standard trace $\operatorname{tr}_{\mathcal{N}(G)}$ underlying the construction of the dimension function $\dim_{\mathcal{U}(G)}$ by a different trace function? For example, if $g \in G$ has finite conjugacy class (g), then

$$\begin{aligned} \operatorname{tr}^{(g)}_{\mathcal{N}(G)} \colon \mathcal{N}(G) &\to \mathbb{C} \\ f &\mapsto \sum_{h \in (g)} \langle f(e), h \rangle_{\ell^2} \end{aligned}$$

satisfies all of the required properties and could be used to define an alternative dimension function for $\mathcal{U}(G)$-modules. Fortunately, there is a universal choice of a trace-like function, the *center-valued trace* $\operatorname{tr}^u_{\mathcal{N}(G)}$ on $\mathcal{N}(G)$. It takes values in the center $\mathcal{Z}(\mathcal{N}(G))$ and all possible traces on $\mathcal{N}(G)$ factor through it. The various properties of $\operatorname{tr}^u_{\mathcal{N}(G)}$ are listed in [Lüc02, Theorem 9.5].

Just as $\dim_{\mathcal{N}(G)}$ is constructed from $\operatorname{tr}_{\mathcal{N}(G)}$, one can define a dimension function $\dim^u_{\mathcal{N}(G)}$, the *center-valued von Neumann dimension*, for finitely generated projective $\mathcal{N}(G)$-modules using $\operatorname{tr}^u_{\mathcal{N}(G)}$. This dimension function takes values in the subspace of $\mathcal{Z}(\mathcal{N}(G))$ of elements satisfying $a = a^*$ and provides a full classification of finitely generated projective $\mathcal{N}(G)$-modules up to isomorphism: two such modules are isomorphic if and only if their center-valued dimensions agree (see [Lüc02, Theorem 9.13]).

Definition 2.5.1. Let G be a group with $\operatorname{lcm}(G) < \infty$ and $K \leqslant \mathbb{C}$ a field. We say that the *center-valued Atiyah conjecture for G holds over K* if every finitely presented KG-module M satisfies

$$\dim^u_{\mathcal{N}(G)}(\mathcal{N}(G) \otimes_{KG} M) \in L_K(G) \subset \mathcal{Z}(\mathcal{N}(G)),$$

where $L_K(G)$ is the additive subgroup of $\mathcal{Z}(\mathcal{N}(G))$ generated by

$$\{\operatorname{tr}^u_{\mathcal{N}(G)}(p) \mid p \in KF, p = p^2 = p^*, F \leqslant G, |F| < \infty\}.$$

The subgroup $L_K(G) \leqslant \mathcal{Z}(\mathcal{N}(G))$ is discrete per [KLS17, Corollary 3.7], which furthers the similarity between the center-valued and the strong Atiyah conjecture and implies that the values of any dimension function on $\mathcal{N}(G)$ constructed out of a trace function will form a discrete subset of $\mathbb{R}$.

Since finite groups satisfy the strong Atiyah conjecture, the image of $L_K(G)$ in $\mathbb{C}$ under $\operatorname{tr}_{\mathcal{N}(G)}$ is given by $\frac{1}{\operatorname{lcm}(G)}\mathbb{Z}$. We thus conclude:

Proposition 2.5.2. *The center-valued Atiyah conjecture for G over K implies the strong Atiyah conjecture for G over K.*

While Definition 2.5.1 is most similar to the statement of the strong Atiyah conjecture, other equivalent formulations given in [KLS17] provide more insight into the strength of the center-valued Atiyah conjecture:

Theorem 2.5.3 ([KLS17, Theorem 3.7])**.** *Let G be a group with $\operatorname{lcm}(G) < \infty$ and $K \leqslant \mathbb{C}$ a subfield closed under complex conjugation. The following statements are equivalent:*

(a) *The center-valued Atiyah conjecture holds for G over K.*

(b) *$\mathcal{D}_{KG}$ is Atiyah-expected Artinian.*

(c) *$\mathcal{D}_{KG}$ is semisimple and the map*

$$\bigoplus_{\substack{F \leqslant G \\ |F| < \infty}} K_0(KF) \to K_0(\mathcal{D}_{KG})$$

is surjective.

(d) *The map*

$$\bigoplus_{\substack{F \leqslant G \\ |F| < \infty}} G_0(KF) \to G_0(\mathcal{D}_{KG})$$

is surjective.

Due to its quite technical nature, we will not give the full definition of what it means for $\mathcal{D}_{KG}$ to be *Atiyah-expected Artinian* and instead refer the reader to [KLS17, Definition 3.6] for all details. If $\mathcal{D}_{KG}$ is Atiyah-expected Artinian, then it is semisimple and the number of factors in its Artin–Wedderburn decomposition is bounded above by the number of finite conjugacy classes of elements of finite order in G. More precisely, the number of factors is given by the number of finite K-conjugacy classes of elements of finite order in G, a notion that will be introduced and studied in detail in Section 3.3. Furthermore, the dimension of every matrix ring factor can be expressed entirely in terms of the Artin–Wedderburn decompositions of the group rings KF for finite subgroups F of G.

Theorem 2.5.3 (c) implies that for a torsion-free group G, the strong Atiyah conjecture implies the center-valued Atiyah conjecture. This implication remains valid in the more general case of a group G that does not contain elements that are simultaneously of finite order and have only a finite number of conjugates (see [LS12, Theorem 1.3]).

In Section 3.1, we will add an equivalent statement to Theorem 2.5.3 that involves only the K-theory of $\mathcal{R}_{KG}$.

The following theorem summarizes most of what is known about the center-valued Atiyah conjecture and uses an extension of Theorem 2.4.27 (a2) that will be proved in Corollary 3.2.14:

Theorem 2.5.4. *Let $K \leqslant \mathbb{C}$ be a field closed under complex conjugation and assume that either $K \leqslant \overline{\mathbb{Q}}$ or $\overline{\mathbb{Q}} \leqslant \mathbb{C}$. The center-valued Atiyah conjecture over K holds for the groups listed in Theorem 2.4.27 (b), except possibly for virtual congruence subgroups with torsion, and has the inheritance properties corresponding to (c2)–(c7) as well as (d4), where the condition on* $\operatorname{lcm}(G)$ *in the latter is replaced by the stronger condition that for every finite subgroup $F \leqslant G/N_i$ there is a finite subgroup of G which is mapped to F isomorphically by the projection $G \twoheadrightarrow G/N_i$.*

Proof. We first consider the case $K \leqslant \overline{\mathbb{Q}}$. That the groups listed in Theorem 2.4.27 (b1) and (b3)–(b7) satisfy the center-valued Atiyah conjecture over K is the content of [KLS17, Corollary 4.6]. The groups in (b2), (b9), (b10), and (b11) as well as torsion-free virtual congruence subgroups satisfy the strong Atiyah conjecture over K by Theorem 2.4.27 and thus the center-valued Atiyah conjecture over K as they are torsion-free. Properties (c2) and (c3) are proved in [KLS17, Lemma 4.2 & 4.3]. The proofs of properties (c4)–(c7) all go through (b2) and (c3). Finally, property (d4) with the extra condition on finite subgroups appears as [KLS17, Proposition 4.4].

In the case that $K \leqslant \overline{\mathbb{Q}}$ we apply Corollary 3.2.14 to obtain the corresponding statements for sofic groups. Of the groups considered in this theorem, only the locally indicable groups are potentially not sofic so that the corollary does not apply. However, these groups are always torsion-free and satisfy the strong Atiyah conjecture over $\mathbb{C}$. $\square$

2.6 An overview of variants of the Atiyah conjecture

The following diagram visualizes the implications and equivalences between the various variants of the Atiyah conjecture. Variants and implications marked with an asterisk next to their reference are introduced or proved, respectively, in this thesis.

$$
\begin{tikzcd}
\boxed{\begin{array}{c}\textbf{Center-valued Atiyah conjecture (2.5.1)}\\ \dim^u_{\mathcal{U}(G)}(\mathcal{U}(G)\otimes_{KG} M)\in L_K(G)\end{array}} \arrow[r, Leftrightarrow, "(3.1.4^*)"] \arrow[d, Rightarrow, "(2.5.2)"] & \boxed{\begin{array}{c}\textbf{Algebraic Atiyah conjecture (3.1.1)}\\ \bigoplus K_0(KF)\to K_0(\mathcal{R}_{KG})\end{array}} \arrow[d, Rightarrow, "(3.3.8^*)"] \\
\boxed{\begin{array}{c}\textbf{Strong Atiyah conjecture (2.4.2)}\\ \dim_{\mathcal{U}(G)}(\mathcal{U}(G)\otimes_{KG} M)\in \frac{1}{\operatorname{lcm}(G)}\mathbb{Z}\end{array}} \arrow[d, Rightarrow] & \boxed{\begin{array}{c}\textbf{Rank of } K_0(\mathcal{R}_{KG}) \textbf{ (3.3.8*)}\\ \operatorname{rk}_{\mathbb{Z}}(K_0(\mathcal{R}_{KG})) = |\operatorname{con}_K(G)_{f,cf}|\end{array}} \arrow[d, Rightarrow, "(3.3.8^*)"] \\
\boxed{\begin{array}{c}\textbf{Weak Atiyah conjecture (2.3.2)}\\ \dim_{\mathcal{U}(G)}(\mathcal{U}(G)\otimes_{KG} M)\in \mathbb{Q}\end{array}} & \boxed{\begin{array}{c}\textbf{Rationalized algebraic}\\ \textbf{Atiyah conjecture (3.1.5*)}\\ \bigoplus K_0(KF)\otimes_{\mathbb{Z}}\mathbb{Q}\to K_0(\mathcal{R}_{KG})\otimes_{\mathbb{Z}}\mathbb{Q}\end{array}} \arrow[l, Rightarrow, "(3.1.6^*)"']
\end{tikzcd}
$$

Chapter 3

The structure of the ring $\mathcal{R}_{KG}$

We have seen in the previous chapter that the structure of the $*$-regular closure $\mathcal{R}_{KG}$ is determined in large parts by suitable variants of the Atiyah conjectures. If G is torsion-free, than the story ends here, with $\mathcal{R}_{KG}$ being a division ring. However, if G contains non-trivial torsion, there is more to say about this ring.

The aim of the current chapter is to further study $\mathcal{R}_{KG}$ and its zeroth K-group in the presence of torsion. In Section 3.1, we discuss another variant of the Atiyah conjecture, the *algebraic Atiyah conjecture*, that has been introduced by Jaikin-Zapirain and is formulated entirely in terms of the K-theory of $\mathcal{R}_{KG}$ and the group rings of finite subgroups of G. Relying on results of Knebusch, Linnell, and Schick, we prove that this conjecture is equivalent to the center-valued Atiyah conjecture. Even though it is not new from this point of view, its K-theoretic formulation turns out to be rather useful when studying inheritance properties. In Section 3.2, we thus benefit from using the algebraic Atiyah conjecture in an analysis of center-valued Atiyah conjecture's behavior under a change of the coefficient field. Based on a base change result of Jaikin-Zapirain for the strong Atiyah conjecture for sofic groups, we prove that the center-valued Atiyah conjecture for such groups over $\overline{\mathbb{Q}}$ implies that over any field with sufficiently many roots of unity or transcendental extensions thereof.

In the second part of the chapter, we turn to structural results on $\mathcal{R}_{KG}$ that do not depend on the strong Atiyah conjecture. Building on work of Lück, we show in Section 3.3 that the rank of $K_0(\mathcal{R}_{KG})$ always admits a lower bound in terms of generalized conjugacy classes of elements of finite order in G. This lower bound matches the rank predicted by the center-valued Atiyah conjecture. Finally, in Section 3.4, we discuss an open question of Handelman about the unit-regularity of $*$-regular rings for the specific case of $\mathcal{R}_{KG}$. We prove that for a sofic group G, the ring $\mathcal{R}_{KG}$ is always unit-regular if K has infinite transcendence degree over $\mathbb{Q}$, thereby providing a partial answer to a question by Ara and Goodearl.

3.1 The algebraic Atiyah conjecture

In his survey article [Jai19b], Jaikin-Zapirain introduces the following variant of the Atiyah conjecture:

Definition 3.1.1 ([Jai19a, Conjecture 6.2]). Let G be a group and $K \leqslant \mathbb{C}$ a subfield closed under complex conjugation. We say that the *algebraic Atiyah conjecture for G holds over K* if the map

$$\bigoplus_{\substack{F \leqslant G \\ |F| < \infty}} K_0(KF) \to K_0(\mathcal{R}_{KG})$$

is surjective. We call this map the *algebraic Atiyah map.*

By the universal property of the colimit, the algebraic Atiyah map factors as

$$\bigoplus_{\substack{F\leqslant G\\ |F|<\infty}} K_0(KF) \twoheadrightarrow \operatorname*{colim}_{\substack{F\leqslant G\\ |F|<\infty}} K_0(KF) \to K_0(KG) \to K_0(\mathcal{R}_{KG}),$$

where the first map is always surjective. It follows that the algebraic Atiyah conjecture for G holds over K if and only if the composition of the second and third map is surjective.

Remark 3.1.2. If the group G satisfies the K-theoretic Farrell–Jones conjecture, the second map above is an isomorphism. In this case, the algebraic Atiyah conjecture for G over K is thus equivalent to the surjectivity of the map $K_0(KG) \to K_0(\mathcal{R}_{KG})$. The latter condition notably no longer directly involves the finite subgroups of G. In this sense, it can be viewed as a generalization of Corollary 2.4.7 to groups with torsion.

Lemma 3.1.3. *Let G be a group and $K \leqslant \mathbb{C}$ a subfield closed under complex conjugation. Then $\mathcal{D}_{KG}$ is a $*$-subring of $\mathcal{U}(G)$.*

Proof. Consider the subrings $R_i \leqslant \mathcal{U}(G)$ defined in Remark 1.7.7 in the situation $R \coloneqq KG$ and $S \coloneqq \mathcal{U}(G)$. Since K is closed under complex conjugation, the ring $R_0 = KG$ is a $*$-subring of $\mathcal{U}(G)$. Now assume that R_i is a $*$-subring of $\mathcal{U}(G)$ for some $i \geqslant 0$. The set $U_i \coloneqq \{x^{-1} \mid x \in R_i, x \in \mathcal{U}(G)^\times\}$ is closed under $*$ since the anti-automorphism maps units to units. Thus, also the ring R_{i+1}, which is generated by R_i and U_i, is a $*$-subring of $\mathcal{U}(G)$. By induction, all subrings $R_i \leqslant \mathcal{U}(G)$ are $*$-subrings, and therefore also their directed union $\mathcal{D}_{KG}$. □

The following theorem answers a question of Jaikin-Zapirain raised in [Jai19a, 6.1].

Theorem 3.1.4. *Let G be a group with $\operatorname{lcm}(G) < \infty$ and $K \leqslant \mathbb{C}$ a subfield closed under complex conjugation. The algebraic Atiyah conjecture for G holds over K if and only if the center-valued Atiyah conjecture for G holds over K.*

Proof. Assume that G satisfies the algebraic Atiyah conjecture. We adapt the proof of [KLS17, Theorem 3.7] to use $\mathcal{R}_{KG}$ instead of $\mathcal{D}_{KG}$ and K-theory instead of G-theory. Let M be a finitely presented KG-module. The $\mathcal{R}_{KG}$-module $\overline{M} \coloneqq \mathcal{R}_{KG} \otimes_{KG} M$ is again finitely presented and, since $\mathcal{R}_{KG}$ is von Neumann regular, also projective by Proposition 1.4.3 (d). It thus represents a class in $K_0(\mathcal{R}_{KG})$, which by assumption is an integer linear combination of classes in $K_0(KF_i)$ for finitely many finite subgroups $F_i \leqslant G, i = 1, \dots, k$. Recall that KF is semisimple for a finite group F, which implies that $K_0(KF)$ is generated by classes that are represented by idempotents in KF. Using the assumption, we thus deduce that there are idempotents $x_i^+, x_i^- \in KF_i$, $i = 1, \dots, k$, which may be 0 or 1 and where the same finite subgroup can appear multiple times, such that

$$\overline{M} \oplus \bigoplus_{i=1}^k \mathcal{R}_{KG} x_i^- \cong \bigoplus_{i=1}^k \mathcal{R}_{KG} x_i^+.$$

By inducing up further to $\mathcal{U}(G)$ and applying the dimension function $\dim_G^u$ for finitely presented $\mathcal{U}(G)$-modules, we obtain

$$\begin{aligned}
\dim^u_{\mathcal{U}(G)}(\mathcal{U}(G) \otimes_{KG} M) &= \dim^u_{\mathcal{U}(G)} \left(\mathcal{U}(G) \otimes_{\mathcal{R}_{KG}} \overline{M}\right) \\
&= \dim^u_{\mathcal{U}(G)} \left(\bigoplus_{i=1}^k \mathcal{U}(G) x_i^+\right) - \dim^u_{\mathcal{U}(G)} \left(\bigoplus_{i=1}^k \mathcal{U}(G) x_i^-\right) \\
&= \sum_{i=1}^k \dim^u_{\mathcal{U}(G)}(\mathcal{U}(G) x_i^+) - \sum_{i=1}^k \dim^u_{\mathcal{U}(G)}(\mathcal{U}(G) x_i^-) \in L_K(G).
\end{aligned}$$

Now assume that G satisfies the center-valued Atiyah conjecture over K. By Theorem 2.5.3 (c), $\mathcal{D}_{KG}$ is semisimple and in particular von Neumann regular. Since $K \leqslant \mathbb{C}$ is closed under complex conjugation, Lemma 3.1.3 implies that it is even $*$-regular. But $KG \leqslant \mathcal{D}_{KG} \leqslant \mathcal{R}_{KG} \leqslant \mathcal{U}(G)$ by construction and $\mathcal{R}_{KG}$ is the smallest $*$-regular subring of $\mathcal{U}(G)$ containing KG, hence $\mathcal{D}_{KG} = \mathcal{R}_{KG}$ and the algebraic Atiyah conjecture for G is implied by Theorem 2.5.3 (c). □

Given its purely K-theoretic formulation, the algebraic Atiyah conjecture lends itself to being considered rationally:

Definition 3.1.5. Let G be a group and $K \leqslant \mathbb{C}$ a subfield closed under complex conjugation. We say that the *rationalized algebraic Atiyah conjecture for G holds over K* if the map

$$\bigoplus_{\substack{F\leqslant G\\ |F|<\infty}} K_0(KF) \otimes_{\mathbb{Z}} \mathbb{Q} \to K_0(\mathcal{R}_{KG}) \otimes_{\mathbb{Z}} \mathbb{Q}$$

is surjective. We call this map the *rationalized algebraic Atiyah map.*

Theorem 3.1.6. *Let G be a group and $K \leqslant \mathbb{C}$ a subfield closed under complex conjugation. Then the rationalized algebraic Atiyah conjecture for G over K implies the weak Atiyah conjecture for G over K.*

Proof. Let M be a finitely presented KG-module. As in the proof of Theorem 3.1.4, the $\mathcal{R}_{KG}$-module $\overline{M} \coloneqq \mathcal{R}_{KG} \otimes_{KG} M$ represents a class in $K_0(\mathcal{R}_{KG})$. Since we assumed the algebraic Atiyah map to be rationally surjective, we can find $n \in \mathbb{Z}$ such that $n[\overline{M}] \in K_0(\mathcal{R}_{KG})$ lies in its image. As in the proof of Theorem 3.1.4, we thus find finitely many finite subgroups $F_i \leqslant G$ and idempotents $x_i^-, x_i^+ \in KF_i$, where $i = 1, \ldots, k$, such that

$$\overline{M}^n \oplus \bigoplus_{i=1}^{k} \mathcal{R}_{KG} x_i^- \cong \bigoplus_{i=1}^{k} \mathcal{R}_{KG} x_i^+.$$

We now apply the classical von Neumann dimension and obtain

$$\begin{aligned}
&n \cdot \dim_{\mathcal{U}(G)}(\mathcal{U}(G) \otimes_{KG} M)\\
&= \dim_{\mathcal{U}(G)}(\mathcal{U}(G) \otimes_{KG} M^n)\\
&= \dim_{\mathcal{U}(G)}(\mathcal{U}(G) \otimes_{\mathcal{R}_{KG}} \overline{M}^n)\\
&= \dim_{\mathcal{U}(G)}\left(\bigoplus_{i=1}^{k} \mathcal{U}(G) x_i^+\right) - \dim_{\mathcal{U}(G)}\left(\bigoplus_{i=1}^{k} \mathcal{U}(G) x_i^-\right)\\
&= \sum_{i=1}^{k} \dim_{\mathcal{U}(G)}(\mathcal{U}(G) x_i^+) - \sum_{i=1}^{k} \dim_{\mathcal{U}(G)}(\mathcal{U}(G) x_i^-)\\
&= \sum_{i=1}^{k} \dim_{\mathcal{U}(G)}(\mathcal{U}(G) \otimes_{KF_i} KF_i x_i^+) - \sum_{i=1}^{k} \dim_{\mathcal{U}(G)}(\mathcal{U}(G) \otimes_{KF_i} KF_i x_i^-).
\end{aligned}$$

Since finite groups satisfy the strong Atiyah conjecture over K, we conclude that

$$\dim_{\mathcal{U}(G)}(\mathcal{U}(G) \otimes_{KG} M) \in \frac{1}{n \cdot \operatorname{lcm}\{|F_i| \mid i = 1, \ldots, k\}} \mathbb{Z}.$$

In particular, it is a rational number. □

Corollary 3.1.7. *For the lamplighter group L_2 introduced in Definition 2.3.7 and every subfield $K \leqslant \mathbb{C}$, the map*

$$K_0(KL_2) \otimes_{\mathbb{Z}} \mathbb{Q} \to K_0(\mathcal{R}_{KL_2}) \otimes_{\mathbb{Z}} \mathbb{Q}$$

is not surjective.

Proof. For every $K \leqslant \mathbb{C}$, the lamplighter group L_2 does not satisfy the weak Atiyah conjecture over K by Theorem 2.3.8. We obtain from Theorem 3.1.6 that the rational algebraic Atiyah map is not surjective. Since L_2 satisfies the K-theoretic Farrell–Jones conjecture, see [Lüc02, Remark 10.25], this completes the proof by Remark 3.1.2. □

3.2 The center-valued Atiyah conjecture over subfields of $\mathbb{C}$

Having seen that the algebraic Atiyah conjecture is equivalent to the center-valued Atiyah conjecture, we will now use the former as a tool to study how the latter behaves under a change of the coefficient field. Such a base change result was obtained by Jaikin-Zapirain in [Jai19c] for the strong Atiyah conjecture for sofic groups. We crucially rely on his methods, but face some additional technical complications.

Most importantly, as opposed to the strong Atiyah conjecture, the center-valued Atiyah conjecture does not pass to subfields, which means that the case of the conjecture over $\mathbb{C}$ no longer implies the general case. As a consequence, in order to obtain a result that applies to as many coefficient fields as possible, we are not be able to start over $\overline{\mathbb{Q}}$ as Jaikin-Zapirain did, but instead consider all base fields with sufficiently many roots of unity.

As was shown in the proof of the base change result for the strong Atiyah conjecture, for a sofic group G, a change of fields from K to an extension field L changes the $*$-regular closure $\mathcal{R}_{KG}$ in a well-behaved way: The tensor product $\mathcal{R}_{KG} \otimes_K L$ satisfies the Ore condition and the Ore ring of fractions is $\mathcal{R}_{LG}$. Given that assuming the strong Atiyah conjecture over K both the ring $\mathcal{R}_{KG}$ and the group rings KF of finite subgroups F of G are semisimple, we start in Section 3.2.1 with a unified analysis of the K-theoretic effects of scalar base changes for semisimple rings.

The proof of our base change results for the center-valued Atiyah conjecture, Corollary 3.2.14 and Proposition 3.2.15 are then obtained from a mostly formal argument, using both the abstract base change results for semisimple rings as well as additional results on central division ring extensions.

3.2.1 Base change for semisimple rings

Definition 3.2.1. A ring is *simple* if it has no two-sided ideals other than the zero ideal and the ring itself.

Note that simple rings are not necessarily semisimple as they might not have a minimal left ideal. A semisimple ring is simple if and only if its Artin–Wedderburn decomposition consists of a single direct factor, i.e., if it is of the form $M_n(D)$ for $n \in \mathbb{N}$ and a division ring D.

We need the following simplified version of [Jai99, Lemma 2.15]:

Lemma 3.2.2. *Let R_1 and R_2 be simple rings and C a ring that is isomorphic to subrings of $\mathcal{Z}(R_1)$ and $\mathcal{Z}(R_2)$. Then $R_1 \otimes_C R_2$ is* almost simple*, i.e., every non-trivial ideal has non-trivial intersection with its center.*

The proof of the following lemma is mostly contained in that of [Jai99, Lemma 10.7 (2)], but we present it in detail in order to clarify the required assumptions.

Lemma 3.2.3. *Let K be a field of characteristic 0 and R a semisimple K-algebra. For every field extension L/K, the Ore condition is satisfied for the L-algebra $R \otimes_K L$. If L/K is finitely generated, then* $\operatorname{Ore}(R \otimes_K L)$ *is semisimple, and in general* $\operatorname{Ore}(R \otimes_K L)$ *is a directed union of semisimple rings.*

Proof. By Theorem 1.6.2, the semisimple K-algebra R is isomorphic to a finite product of matrix rings over division K-algebras. Since localization commutes with direct sums and a finite direct sum of semisimple rings is again semisimple, we can restrict to the case that $R = M_n(D)$ for a division ring D, i.e., that R is simple Artinian.

Then, since any field is simple, we conclude from Lemma 3.2.2 that $R \otimes_K L$ is almost simple. This means that every non-trivial ideal of $R \otimes_K L$ has a non-trivial intersection with $\mathcal{Z}(R \otimes_K L) = \mathcal{Z}(D) \otimes_K L$. Since $K \subset R$ has characteristic 0, it is perfect, and thus $\mathcal{Z}(D) \otimes_K L$ is reduced by [Bou90, V.§15, Theorem 3 d)]. Thus, if $R \otimes_K L$ were to contain a non-trivial nilpotent ideal I, then $I \cap \mathcal{Z}(R \otimes_K L)$ would be a non-trivial nilpotent ideal in a reduced ring, which is not possible. It follows that $R \otimes_K L$ is semiprime.

We assume for the moment that L/K is finitely generated as a field extension. Then $R \otimes_K L$ is Noetherian by the Hilbert basis theorem (see [Row88, Proposition 3.5.2]), where we use additionally that Noetherianity passes to localizations at central elements (see [Row88, Proposition 3.1.13]). As it is also semiprime, Theorem 1.2.7 implies that $R \otimes_K L$ satisfies the left Ore condition and $\operatorname{Ore}(R \otimes_K L)$ is a semisimple ring.

We now return to the case of a general field extension L/K, which can always be expressed as the directed union of its finitely generated subextensions. Since a directed union of rings R_i satisfying the Ore condition again satisfies the Ore condition and $\operatorname{Ore}(\bigcup R_i) = \bigcup \operatorname{Ore}(R_i)$, this concludes the proof. ☐

We will now study the effect of a base change on the zeroth K-group. In the situation of Lemma 3.2.3, we can consider the map $\Phi_K^L \colon K_0(R) \to K_0(\operatorname{Ore}(R \otimes_K L))$ induced by the embedding $R \hookrightarrow R \otimes_K L$. Recall from Corollary 1.6.5 that $K_0(R)$ of a semisimple ring R has a particularly simple structure: If $R \cong M_{n_1}(D_1) \times \cdots \times M_{n_s}(D_s)$, then $K_0(R) \cong \frac{1}{n_1}\mathbb{Z} \oplus \cdots \oplus \frac{1}{n_s}\mathbb{Z}$, where the generator of each direct summand corresponds to a minimal left ideal in $M_{n_i}(D_i)$.

Before we continue with general structural results on the possible map Φ_K^L, it will be instructive to consider the following two prototypical examples:

Example 3.2.4. If $L/K = \mathbb{Q}(i)/\mathbb{Q}$ and $R = \mathbb{Q}(i)$, then $R_L \coloneqq R \otimes_K L$ can be identified with $\mathbb{Q}(i) \times \mathbb{Q}(i)$, with the map $R \hookrightarrow R_L$ given by the diagonal embedding. Since R_L is semisimple, it agrees with $\operatorname{Ore}(R_L)$. Denote by e_1 and e_2 the central idempotents $(1,0)$ and $(0,1)$ in $\mathbb{Q}(i) \times \mathbb{Q}(i)$. Then $K_0(R) = \{k \cdot [R] \mid k \in \mathbb{Z}\} = \mathbb{Z}$, $K_0(R \otimes_K L) = \{k \cdot [R_L e_1] + l \cdot [R_L e_2] \mid k, l \in \mathbb{Z}\} = \mathbb{Z}^2$, and the map Φ_K^L is the diagonal embedding $\mathbb{Z} \hookrightarrow \mathbb{Z}^2, k \mapsto (k,k)$.

Example 3.2.5. If $K/L = \mathbb{Q}(i)/\mathbb{Q}$ and $R = \mathbb{H} = \{a + bi + cj + dk \mid a,b,c,d \in \mathbb{R}\}$, the quaternions, then $R_L \coloneqq R \otimes_K L$ can be identified with $M_2(\mathbb{Q}(i))$, with the map $R \hookrightarrow R_L$ given by

$$a + bi + cj + dk \mapsto \begin{pmatrix} a + bi & c + di \\ -c + di & a - bi \end{pmatrix}.$$

Since R_L is semisimple, it agrees with $\operatorname{Ore}(R_L)$. Denote by $e \in R_L$ the idempotent matrix with ones in the first and zeros in the second column. Then $K_0(R) = \{k \cdot [R] \mid k \in \mathbb{Z}\} = \mathbb{Z}$, $K_0(R_L) = \{k \cdot \frac{[R_L]}{2} = k \cdot [R_L e] \mid k \in \mathbb{Z}\} = \frac{1}{2}\mathbb{Z}$, and the map Φ_K^L is the index 2 embedding $\mathbb{Z} \hookrightarrow \frac{1}{2}\mathbb{Z}, k \mapsto k$.

In both examples, the map Φ_K^L is injective, but fails to be surjective due to a splitting phenomenon: A division ring splits either into a product of division rings, which increases the rank of K_0, or into a matrix ring over a new division ring, which makes Φ_K^L an embedding of non-trivial finite index. In slightly more abstract terms, in these examples, the increased complexity of $K_0(R_L)$ compared to $K_0(R)$ stems from the appearance of (additional) zero divisors in $R \otimes_K L$. We will now see that this holds more generally: The map Φ_K^L is always injective and it is surjective if and only if the division rings in the Artin–Wedderburn decomposition remain non-commutative domains after base change. This result will be key to our study of the base change in the algebraic and center-valued Atiyah conjecture in the next section.

Proposition 3.2.6. *Let L/K be an extension of fields of characteristic 0 and R a semisimple K-algebra. Then $\Phi_K^L\colon K_0(R) \to K_0(\operatorname{Ore}(R \otimes_K L))$*

(a) *is injective;*

(b) *is surjective if and only if for every division ring D in the Artin–Wedderburn decomposition of R the ring $D \otimes_K L$ is a domain.*

Proof. For the sake of comprehensibility, we abbreviate $\operatorname{Ore}(R \otimes_K L)$ to R_L and proceed in steps of increasing generality, where we list in each step all of the restrictions on R and L that are assumed in addition to those in the statement of the proposition.

Step 1: (a) for $R = D$ division ring, L/K finitely generated. We have $K_0(D) = \{k \cdot [D] \mid k \in \mathbb{Z}\}$, so that Φ_K^L is injective if $k \cdot [D_L] \neq 0$ in $K_0(D_L)$ for every $k \neq 0$. Since L/K is finitely generated, the ring D_L is semisimple by Lemma 3.2.3. Thus, using the calculation of $K_0(D_L)$ of Corollary 1.6.5, we conclude that $[D_L]$ always generates an infinite cyclic subgroup of $K_0(D_L)$.

Step 2: (a) for $R = D$ division ring. We can write L/K as a directed union over its finitely generated subextensions L_i/K for $i \in I$ for some index set I. This gives rise to a directed union of rings D_{L_i}, together with compatible embeddings of D inducing the maps $\Phi_K^{L_i}$ on K_0. Since $K_0(?)$ commutes with directed colimits and all the $\Phi_K^{L_i}$ are injective by Step 1, the map Φ_K^L is injective as well.

Step 3: (a) for $R = M_n(D)$, D division ring. By [Ros94, Theorem 1.2.4], also known as Mortia invariance, there is a natural isomorphism $K_0(D) \xrightarrow{\cong} K_0(M_n(D))$ for every $n \in \mathbb{N}$. Since both $? \otimes_K L$ and $\operatorname{Ore}(?)$ commute with $M_n(?)$, the latter by Proposition 1.2.6, this step reduces to the previous one.

Step 4: (a). We identify R with $M_{n_1}(D_1) \times \cdots \times M_{n_s}(D_s)$ for $n_i \in \mathbb{N}$ and D_i a division K-algebra for every $i = 1, \ldots, s$. Denote by $e_i \in R$ the central idempotent corresponding to the projection onto the i-th factor, so that $K_0(R)$ is free abelian on $\frac{1}{n_i}[Re_i]$ for $i = 1, \ldots, s$.

Since L is commutative, the idempotents e_i are also central in $R \otimes_K L$. They remain so in $R_L \coloneqq \operatorname{Ore}(R \otimes_K L)$ since in every ring $as = sa$ implies $as^{-1} = s^{-1}sas^{-1} = s^{-1}ass^{-1} = s^{-1}a$. We conclude that R_L can be expressed as the product $R_L e_1 \times \cdots \times R_L e_s$, where every factor $R_L e_i$ is a non-trivial ring with unit e_i.

In K_0, these decompositions as products of rings give rise to direct sum decompositions $K_0(R) \cong K_0(Re_1) \oplus \cdots \oplus K_0(Re_s)$ and $K_0(R_L) \cong K_0(R_L e_1) \oplus \cdots \oplus K_0(R_L e_s)$ which are natural in the sense that the map Φ_K^L is given by the direct sum of the maps $K_0(Re_i) \to K_0(R_L e_i)$. These maps are of the form Φ_K^L for rings as in Step 3, which concludes the proof of (a) since the direct sum of injective maps is again injective.

Step 5: (b) for $R = D$ division ring, L/K finitely generated.

We observe that by construction, $D_L = \operatorname{Ore}(D \otimes_K L)$ admits non-trivial zero divisors if and only if $D \otimes_K L$ does. Since both $R_1 \times R_2$ and $M_n(R_1)$ contain non-trivial zero divisors for non-zero rings R_1 and R_2 and $n \geqslant 2$, the Artin–Wedderburn theorem implies that the semisimple ring D_L has no non-trivial zero divisors if and only if it is a division ring. We are thus left to show that Φ_K^L is surjective if and only if D_L is a division ring.

If D_L is a division ring, then Φ_K^L is clearly an isomorphism of infinite cyclic groups and in particular surjective. If D_L is not a division ring, then by Corollary 1.6.5 there exists an element $x \in K_0(D_L)$ that cannot be expressed as $k \cdot [D_L]$ and is thus not contained in the image of Φ_K^L.

Step 6: (b) for $R = D$ division ring. We again write L/K as a directed union over its finitely generated subextensions L_i/K for $i \in I$ for some index set I, as in Step 2.

If Φ_K^L is not surjective, then there is some $i \in I$ and $x \in K_0(D_{L_i})$ such that x is not in the image of $\Phi_K^{L_i}$. By Step 5, there is a non-trivial zero divisor in $D \otimes_K L_i$. Since the maps $D \otimes_K L_i \to D \otimes_K L$ are all injective, this element remains a non-trivial zero divisor in $D \otimes_K L$.

At last, we consider a non-trivial zero divisor $z \in D \otimes_K L$, witnessed by an element $z' \in D \otimes_K L, z' \neq 0$ such that $zz' = 0$. Then there exists $i \in I$ such that $z, z' \in D \otimes_K L_i$, which means that $D \otimes_K L_i$ has non-trivial zero divisors. By Step 5, there is $x \in K_0(D_{L_i})$ that is not contained in the image of $\Phi_K^{L_i}$. We additionally observe that, since the canonical map

$$\mathrm{Ore}(\mathrm{Ore}(D \otimes_K L_i) \otimes_{L_i} L_j) \to \mathrm{Ore}(D \otimes_K L_j)$$

is an isomorphism, the maps on K_0 induced by the connecting maps in the directed union of rings D_{L_i} are themselves of the form $\Phi_{L_i}^{L_j}$ for $i, j \in I, i \leqslant j$. If Φ_K^L were surjective, there would be $y \in K_0(D)$ such that $\Phi_K^L(y) = \Phi_{L_i}^L(x)$. But then $\Phi_{L_i}^L(\Phi_K^{L_i}(y)) = \Phi_K^L(y) = \Phi_{L_i}^L(x)$, which implies that $\Phi_K^{L_i}(y) = x$ since $\Phi_{L_i}^L$ is injective by (a) applied to the semisimple L_i-algebra D_{L_i} and the field extension L/L_i. This contradicts the fact that x is not contained in the image of $\Phi_K^{L_i}$ and we have established that Φ_K^L is not surjective.

Step 7: (b). This reduces to the situation of Step 5 as in Step 3 and 4 by noting that a direct sum of surjective maps between abelian groups is surjective if and only if every individual map is. □

3.2.2 Base change in the algebraic and center-valued Atiyah conjecture

Now that we have studied the K-theoretic effects of a base change on semisimple rings, we will apply our results to the two classes of semisimple rings relevant to the center-valued Atiyah conjecture for a group G over a field K: the group rings KF for finite subgroups F of G as well as the $*$-regular closure $\mathcal{R}_{KG}$. The former rings are inherently linked to the linear representation theory of finite groups, which will play a role in the following form:

Definition 3.2.7. Let G be a group. We say that a field $K \leqslant \mathbb{C}$ *realizes* G if every linear $\mathbb{C}$-representation of a finite subgroup of G is isomorphic to $\mathbb{C} \otimes_K V$ for a linear K-representation V.

The usefulness of this notion in the present context lies in the following K-theoretic reformulation:

Lemma 3.2.8. *If a field $K \leqslant \mathbb{C}$ realizes a group G, then the map*

$$K_0(KF) \to K_0(LF)$$

is an isomorphism for every $K \leqslant L \leqslant \mathbb{C}$ and every finite subgroup F of G.

Proof. Since the group ring of a finite group is semisimple by Maschke's theorem, we can apply the results of the previous section. Taking $R = KF$, the map in the statement turns out to be of the form Φ_K^L and is thus always injective by Proposition 3.2.6.

As K realizes G, the map $\Phi_K^{\mathbb{C}}$ is also surjective since by Proposition 1.6.4 (c) and Corollary 1.6.5 two $\mathbb{C}$-representations are isomorphic if and only if they represent the same element in K_0. The same argument applies to $\Phi_L^{\mathbb{C}}$. As both $\Phi_K^{\mathbb{C}}$ and $\Phi_L^{\mathbb{C}}$ are isomorphisms and $\Phi_K^{\mathbb{C}} = \Phi_L^{\mathbb{C}} \circ \Phi_K^L$, the map Φ_K^L is also an isomorphism □

For a group G, we denote the order of an element $g \in G$ by $\mathrm{ord}(g)$. Recall that $\exp(G) \coloneqq \mathrm{lcm}\{\mathrm{ord}(g) \mid g \in G, \mathrm{ord}(g) < \infty\}$ is the *exponent* of G. The exponent of G may be infinite, but we always have $\exp(G) \leqslant \mathrm{lcm}(G)$.

Proposition 3.2.9. *Let G be a group with $\exp(G) < \infty$ and $\omega \in \mathbb{C}$ a primitive root of unity ω of order $\exp(G)$. Then $\mathbb{Q}(\omega)$ realizes G.*

Proof. For any finite subgroup F of G we have that $\exp(F) | \exp(G)$ by definition. Thus, $\mathbb{Q}(\omega)$ contains a primitive $\exp(F)$-th root of unity for any finite subgroup F, which suffices to realize every linear representation of F by the Brauer induction theorem, see [Ser77, 12.3]. □

We will also need the following generalization of a well-known notion from field theory that will allow us to invoke Proposition 3.2.6:

Definition 3.2.10. Let D be a division ring of characteristic 0 and $K \leqslant \mathcal{Z}(D)$ a subfield of its center. We call K *totally algebraically closed in D* if for every field extension L/K the ring $D \otimes_K L$ is a domain.

A reader well-versed in field theory will notice that being relatively algebraically closed usually means something else and that what we call a totally algebraically closed extension is called a *regular* extension in the literature. We make this deliberate choice since the term "regular" is already attached to two other concepts of relevance to this thesis. There should be no potential for confusion as we are exclusively working in characteristic 0 in this chapter, where an extension D/K is totally algebraically closed in our sense if and only if it is totally algebraically closed in the usual sense.

Checking whether a given central subfield K of a division ring D is totally algebraically closed reduces to understanding the base change to the algebraic closure of K:

Theorem 3.2.11 ([CD80, Corollary 6])**.** *Let D be a division ring of characteristic 0 and $K \leqslant \mathcal{Z}(D)$ a subfield of its center. Then K is totally algebraically closed in D if and only if $D \otimes_K \overline{K}$ is a division ring.*

In [Jai19c], Jaikin-Zapirain has studied the base change from $\overline{\mathbb{Q}}$ to $\mathbb{C}$ in the Lück approximation and the strong Atiyah conjecture for sofic groups, showing that the latter holds over $\mathbb{C}$ if it holds over $\overline{\mathbb{Q}}$. He achieves this by showing that for a sofic group G the $*$-regular closure $\mathcal{R}_{\mathbb{C}G}$ coincides with $\mathrm{Ore}(\mathcal{R}_{\overline{\mathbb{Q}}} \otimes_{\overline{\mathbb{Q}}} \mathbb{C})$ and is thus obtained via the type of central base change of a semisimple algebra studied in the previous section. In the final step, he uses the assumption that the strong Atiyah conjecture holds over $\overline{\mathbb{Q}}$ to apply the special case of Theorem 3.2.11 where $K = \overline{K} = \overline{\mathbb{Q}}$. The first part of this proof applies more generally:

Theorem 3.2.12. *Let G be a sofic group with $\mathrm{lcm}(G) < \infty$. Let $K \leqslant L \leqslant \mathbb{C}$ be subfields closed under complex conjugation. Then the inclusion $KG \hookrightarrow LG$ extends to an isomorphism*

$$\mathrm{Ore}(\mathcal{R}_{KG} \otimes_K L) \xrightarrow{\cong} \mathcal{R}_{LG}.$$

If the field extension L/K is algebraic, the same statement holds also without applying $\mathrm{Ore}(?)$.

Proof. The inductive strategy used to show that $\mathcal{R}_{\mathbb{C}G} \cong \mathrm{Ore}(\mathcal{R}_{\overline{\mathbb{Q}}G} \otimes_{\overline{\mathbb{Q}}} \mathbb{C})$ in the proof of [Jai19c, Theorem 10.7 (2)] can be applied verbatim to the more general situation in which $\overline{\mathbb{Q}}$ is replaced by K and $\mathbb{C}$ is replaced by L. Specific properties of $\overline{\mathbb{Q}}$ are only used to conclude that extending the coefficients of a division ring from $\overline{\mathbb{Q}}$ to some extension field does not introduce zero divisors, which is not needed for our purposes (and also far from true in our more general setting).

The second statement appears in the proof of [Jai19c, Theorem 10.7 (2)] in the inductive step for the algebraic closure and again does not use any specific properties of the base field. $\square$

We can now put the pieces together and obtain a base change result for the algebraic Atiyah conjecture in the presence of sufficiently many roots of unity:

Theorem 3.2.13. *Let G be a sofic group with $\mathrm{lcm}(G) < \infty$ and $K \leqslant \overline{\mathbb{Q}}$ a subfield closed under complex conjugation. Assume that K realizes G and that G satisfies the algebraic Atiyah conjecture over $\overline{\mathbb{Q}}$. Then G satisfies the algebraic Atiyah conjecture over every $L \leqslant \mathbb{C}$ that contains K and is closed under complex conjugation.*

Proof. We consider the following commutative diagram, where all maps are induced from the corresponding embedding of rings:

$$\begin{tikzcd}
\bigoplus\limits_{\substack{F \leqslant G \\ |F| < \infty}} K_0(KF) \arrow[r, "f_1"] \arrow[d, "f_2"] & K_0(\mathcal{R}_{KG}) \arrow[d] \arrow[dr, "f_5"] & \\
\bigoplus\limits_{\substack{F \leqslant G \\ |F| < \infty}} K_0(\overline{\mathbb{Q}}F) \arrow[r, "f_3"] & K_0(\mathcal{R}_{\overline{\mathbb{Q}}G}) & K_0(\mathcal{R}_{KG} \otimes_K \overline{\mathbb{Q}}) \arrow[l, "f_4"']
\end{tikzcd}$$

By Lemma 3.2.8 combined with the assumption that K realizes G, the map f_2 is an isomorphism and in particular surjective. The map f_3 is surjective since G satisfies the algebraic Atiyah conjecture over $\overline{\mathbb{Q}}$. The map f_4 is an isomorphism as it is induced by the map $\mathcal{R}_{KG} \otimes_K \overline{\mathbb{Q}} \to \mathcal{R}_{\overline{\mathbb{Q}}G}$, which is an isomorphism by Theorem 3.2.12 applied to the algebraic extension $\overline{\mathbb{Q}}/K$. Taken together, these facts imply that the concatenation $f_4^{-1} \circ f_3 \circ f_2$ of the maps along the left and lower edge of the diagram is surjective. By commutativity, we conclude that also the concatenation $f_5 \circ f_1$ of the maps along the upper and right edge is surjective. In particular, the map f_5 is surjective.

Since G satisfies the algebraic Atiyah conjecture over $\overline{\mathbb{Q}}$, it also satisfies the strong Atiyah conjecture over $\overline{\mathbb{Q}}$ by Theorem 3.1.4 and Proposition 2.5.2. As opposed to the algebraic Atiyah conjecture, the strong Atiyah conjecture clearly descends to subfields, so that it holds over every subfield of $\overline{\mathbb{Q}}$. We conclude from Proposition 2.4.6 that $\mathcal{R}_{KG}$ and $\mathcal{R}_{KG} \otimes_K \overline{\mathbb{Q}} \cong \mathcal{R}_{\overline{\mathbb{Q}}}$ are both semisimple. Note that a semisimple ring is in particular von Neumann regular and hence, by Proposition 1.4.5, coincides with its localization at the non-trivial zero divisors. We have thus identified f_5 as the map $\Phi_K^{\overline{\mathbb{Q}}}$ appearing in Proposition 3.2.6 for the semisimple ring $R = \mathcal{R}_{KG}$ and $L = \overline{\mathbb{Q}}$. In particular, f_5 is also injective, which means that the surjectivity of $f_5 \circ f_1$ implies that f_1 is surjective. We have thus verified that the algebraic Atiyah conjecture for G holds over K and are left to show that this extends to all fields L between K and $\mathbb{C}$.

If $M_{n_1}(D_1) \oplus \cdots \oplus M_{n_s}(D_s)$ is the Artin–Wedderburn decomposition of $\mathcal{R}_{KG}$, then, using Proposition 3.2.6, we obtain as a consequence of the surjectivity of $\Phi_K^{\overline{\mathbb{Q}}}$ that $D_i \otimes_K \overline{\mathbb{Q}}$ is a domain for every $i = 1, \ldots, s$. By Theorem 3.2.11, the field K is totally algebraically closed in each of the D_i. We are now in a position to apply the other direction of Proposition 3.2.6 to any extension L/K as in the statement of the theorem. In that case, since K is totally algebraically closed in each of the D_i, we conclude that the map $\Phi_K^L\colon K_0(\mathcal{R}_{KG}) \to K_0(\mathrm{Ore}(\mathcal{R}_{KG} \otimes_K L))$ is surjective. Another application of Theorem 3.2.12, this time to the extension L/K, yields that $\mathrm{Ore}(\mathcal{R}_{KG} \otimes_K L) \to \mathcal{R}_{LG}$ is an isomorphism of rings. Combined with the surjectivity of Φ_K^L, we conclude that $K_0(\mathcal{R}_{KG}) \to K_0(\mathcal{R}_{LG})$ is surjective.

The situation we have arrived at can be summarized in the following commutative diagram, where the upper horizontal map is surjective since we have already verified above that the algebraic Atiyah conjecture for G holds over K:

$$\begin{tikzcd} \bigoplus\limits_{\substack{F \leqslant G \\ |F|<\infty}} K_0(KF) \arrow[r, two heads] \arrow[d] & K_0(\mathcal{R}_{KG}) \arrow[d, two heads] \\ \bigoplus\limits_{\substack{F \leqslant G \\ |F|<\infty}} K_0(LF) \arrow[r] & K_0(\mathcal{R}_{LG}) \end{tikzcd}$$

Since the diagram is commutative, we read off that the lower horizontal map is surjective, which is precisely the statement of the algebraic Atiyah conjecture for G over L. □

Corollary 3.2.14. *Let G be a sofic group such that $\mathrm{lcm}(G) < \infty$. If G satisfies the center-valued Atiyah conjecture over $\overline{\mathbb{Q}}$, then G satisfies the center-valued Atiyah conjecture over any $K \geqslant \mathbb{Q}(\omega)$ that is closed under complex conjugation, where ω is a primitive $\exp(G)$-th root of unity. In particular, it satisfies the center-valued Atiyah conjecture over $\mathbb{C}$.*

Proof. The field $\mathbb{Q}(\omega)$ realizes G by Proposition 3.2.9, hence the corresponding statement for the algebraic Atiyah conjecture follows from Theorem 3.2.13. Now use the equivalence to the center-valued Atiyah conjecture proved in Theorem 3.1.4. □

While Corollary 3.2.14 allows us to deduce the center-valued Atiyah conjecture over most subfields of $\mathbb{C}$ once we know it for $\overline{\mathbb{Q}}$, in certain situations the existence of sufficiently many roots of unity may not be guaranteed. For this reason, we also want to mention the following result, which allows passing to purely transcendental extensions without any assumption on the base field.

Proposition 3.2.15. *Let G be a sofic group such that $\mathrm{lcm}(G) < \infty$. Let $K \leqslant \mathbb{C}$ be a subfield and assume that G satisfies the algebraic Atiyah conjecture over K. Then G satisfies the algebraic Atiyah conjecture over every purely transcendental extension L of K. The corresponding statements hold for the center-valued Atiyah conjecture.*

Proof. As in the proof of Theorem 3.2.13, we conclude from the assumption that $\mathcal{R}_{KG}$ is semisimple. Take its Artin–Wedderburn decomposition to be $M_{n_1}(D_1) \oplus \cdots \oplus M_{n_s}(D_s)$ and choose a transcendence basis X of L over K. Then Theorem 3.2.12 implies that

$$\mathcal{R}_{LG} \cong \mathrm{Ore}(\mathcal{R}_{KG} \otimes_K L) \cong \oplus_{i=1}^{s} M_{n_i}(\mathrm{Ore}(D_i \otimes_K K(X)).$$

Since $D_i \otimes_K K(X)$ embeds into the domain $D_i(X)$, i.e., the ring of rational functions in central indeterminants X with coefficients in D_i, it is itself a domain for $i = 1, \ldots, s$. We now deduce from Proposition 3.2.6 that the canonical map $K_0(\mathcal{R}_{KG}) \to K_0(\mathcal{R}_{LG})$ is surjective, which finishes the proof exactly as in the proof of Theorem 3.2.13. □

Corollary 3.2.16. *Let K be a subfield of $\mathbb{C}$ that*

(1) is a purely transcendental extension of a subfield of $\overline{\mathbb{Q}}$, or

(2) contains all roots of unity.

Then the center-valued Atiyah conjecture over K is true for all elementary amenable extensions of pure braid groups, of right-angled Artin groups, of primitive link groups, of cocompact special groups, or of products of such, assuming that the lcm *of the orders of their finite subgroups is finite. In particular, the center-valued Atiyah conjecture over $\mathbb{C}$ holds for these groups.*

Proof. First note that all groups in the classes under consideration are residually elementary amenable, as is explained in the proof of [KLS17, Corollary 4.6], and in particular sofic. If $K \leqslant \overline{\mathbb{Q}}$, then the center-valued Atiyah conjecture over K holds by [KLS17, Corollary 4.6]. It extends to purely transcendental extensions by Proposition 3.2.15 and Theorem 3.1.4 and holds for subfields of $\mathbb{C}$ that contain all roots of unity by Corollary 3.2.14. □

3.3 The algebraic Atiyah map and $\mathrm{rk}_{\mathbb{Z}}(K_0(\mathcal{R}_{KG}))$

Given that the algebraic Atiyah conjecture highlights the important role played by the map $\operatorname{colim} K_0(KF) \to K_0(\mathcal{R}_{KG})$, we will now take a closer look at this map and derive unconditional lower bounds on the rank of its image and codomain. These bounds are obtained via an extension of the methods used in [Lüc02, p. 9.5]to detect non-trivial elements in $G_0(\mathbb{C}G)$ for amenable groups. They match the formula for the rank of $K_0(\mathcal{R}_{KG})$ that is implied by the algebraic Atiyah conjecture and, in the opposite direction, if these bounds are attained the weak Atiyah conjecture holds.

The lower bound on $\mathrm{rk}_{\mathbb{Z}}(K_0(\mathcal{R}_{KG}))$ will be expressed in terms of the following generalization of conjugacy that is defined in the presence of finite subgroups:

Definition 3.3.1. Let $K \leqslant \mathbb{C}$ be a field. Fix an integer $m \geqslant 1$ and a primitive m-th root of unity $\zeta_m \in \mathbb{C}$. Let $K(\zeta_m) \geqslant K$ be the Galois extension obtained by adjoining ζ_m to K and denote the Galois group of this extension by $\Gamma(m, K)$. Identify $\Gamma(m, K)$ with a subgroup of $\mathbb{Z}/m^{\times}$ by mapping $\sigma \in \Gamma(m, K)$ to the unique element $u(\sigma) \in \mathbb{Z}/m^{\times}$ such that $\sigma(\zeta_m) = \zeta_m^{u(\sigma)}$. Two elements $g_1, g_2 \in G$ of finite order are called *K-conjugate* if for some, and hence all, positive integers m with $g_1^m = g_2^m = 1$ there exists an element $\sigma \in \Gamma(m, K)$ such that $g_1^{u(\sigma)}$ and g_2 are conjugate in G.

For example, if $K = \mathbb{C}$, then the extension $K(\zeta_m) \geqslant K$ is trivial and so is $\Gamma(m, K)$, which means that two elements of G are $\mathbb{C}$-conjugate if and only if they are conjugate. Two elements of G are $\mathbb{Q}$-conjugate if and only if the cyclic subgroups they generate are conjugate.

Definition 3.3.2. Let G be a group and $K \leqslant \mathbb{C}$ a field. Denote by $\mathrm{con}(G)$ the set of conjugacy classes (g) of elements $g \in G$. Furthermore, denote by $\mathrm{con}_K(G)_f$ the set of K-conjugacy classes $(g)_K$ of finite order elements $g \in G$ and by $\mathrm{con}_K(G)_{f,fc}$ the subset of $\mathrm{con}_K(G)_f$ of those K-conjugacy classes that are finite.

For a group homomorphism $f\colon G \to G'$ we get induced maps on $\mathrm{con}_K(G)$, $\mathrm{con}_K(G)_f$ and $\mathrm{con}_K(G)_{f,fc}$ by applying f to any representative of a conjugacy class.

Informally speaking, the larger the field K, the smaller the Galois group $\Gamma(m, K)$ and thus the more K-conjugacy classes exist. More precisely, if K is a subfield of L, then we can consider the following diagram of fields:

$$\begin{tikzcd}[column sep=small]
 & L(\zeta_m) \arrow[dl, dash] \arrow[dr, dash] & \\
K(\zeta_m) \arrow[dr, dash] & & L \arrow[dl, dash] \\
 & K(\zeta_m) \cap L \arrow[d, dash] & \\
 & K &
\end{tikzcd}$$

By standard Galois theory, see [Lan02, Theorem VI.1.12], there is an isomorphism

$$\mathrm{Gal}(L(\zeta_m)/L) \to \mathrm{Gal}(K(\zeta_m)/L \cap K(\zeta_m)).$$

As a consequence, the group $\mathrm{Gal}(L(\zeta_m)/L)$ is a subgroup of $\mathrm{Gal}(K(\zeta_m)/K)$ and this containment is compatible with the identification of the latter group with $\mathbb{Z}/m^\times$. In particular, every K-conjugacy class of G is a disjoint union of L-conjugacy classes.

We will now consider linear combinations of conjugacy classes, which will allow us to study the way K-conjugacy classes split into L-conjugacy classes in more detail:

Definition 3.3.3. Let G be a group and $K \leqslant \mathbb{C}$ a field. Denote by $\mathrm{class}_K(G)$, $\mathrm{class}_K(G)_f$, and $\mathrm{class}_K(G)_{f,fc}$ the K-vector spaces with bases $\mathrm{con}(G)$, $\mathrm{con}_K(G)_f$, and $\mathrm{con}_K(G)_{f,fc}$, respectively.

Since it is a likely source of confusion, the reader should pause at this point and take note of the fact that while the basis of $\mathrm{class}_K(G)$ consists of ordinary conjugacy classes in G, the bases of $\mathrm{class}_K(G)_f$ and $\mathrm{class}_K(G)_{f,cf}$ consist of K-conjugacy classes.

If $K \leqslant L \leqslant \mathbb{C}$ are fields, we can consider the K-linear map

$$\begin{aligned} c_K^L \colon \mathrm{class}_K(G)_f &\to \mathrm{class}_L(G)_f \\ (g)_K &\mapsto \sum_{(h)_L \in \mathrm{con}_L(G)_f} (h)_L. \end{aligned}$$

The map c_K^L is well-defined since $(g)_K$ can split up into the at most finitely many L-conjugacy classes $(g)_L, (g^2)_L, \ldots, (g^n)_L$, where n is the order of g. It is also injective since every element of G of finite order lies in exactly one K-conjugacy class and every element in its L-conjugacy class shares the same K-conjugacy class. We also define analogous maps on $\mathrm{class}_K(G)_{f,cf}$ and observe that these are compatible with the projections $\mathrm{class}_K(G)_f \twoheadrightarrow \mathrm{class}_K(G)_{f,cf}$.

We further consider the chain of inclusions

$$\mathrm{class}_K(G)_f \xhookrightarrow{c_K^{\mathbb{C}}} \mathrm{class}_{\mathbb{C}}(G)_f \hookrightarrow \mathrm{class}_{\mathbb{C}}(G) = \mathrm{class}_K(G) \otimes_K \mathbb{C}.$$

It is clear from the definition of $c_K^{\mathbb{C}}$ that its image consists purely of elements of $\mathrm{class}_{\mathbb{C}}(G)$ that are K-linear combinations of the basis elements $\mathrm{con}(F)$. We will view $\mathrm{class}_K(G)_f$ as a K-subspace of $\mathrm{class}_K(G)$ in this way.

After this setup, we can now study the following entirely algebraic trace-like function on KG:

Definition 3.3.4. Let G be a group and K a field. The *universal trace on KG* is the map $\mathrm{tr}^u_{KG} \colon KG \to \mathrm{class}_K(G)$ given by

$$\mathrm{tr}^u_{KG}\left(\sum_{g \in G} \lambda_g g\right) := \sum_{g \in G} \lambda_g (g).$$

The universal trace is clearly natural in G and can be extended to square matrices $A \in M_n(KG)$ by setting

$$\mathrm{tr}^u_{KG}(A) := \sum_{i=1}^{n} \mathrm{tr}^u_{KG}(A_{ii}).$$

Definition 3.3.5. Let G be a group and K a field. The *Hattori–Stallings rank* HS_{KG} of a finitely generated projective KG-module P is defined to be $\mathrm{tr}^u_{KG}(A)$, where A is any idempotent in $M_n(KG)$ such that the image of the map $r_A \colon KG^n \to KG^n$ given by right multiplication by A is KG-isomorphic to P.

The Hattori-Stallings rank is well-defined (see [Lüc02, (6.4)]). Since a representative of a direct sum of modules can be taken to be a block diagonal matrix, the Hattori–Stallings rank induces the following *Hattori–Stallings homomorphism*:

$$\begin{aligned} K_0(KG) \otimes_{\mathbb{Z}} K &\xrightarrow{\mathrm{HS}_{KG}} \mathrm{class}_K(G) \\ [P] \otimes \lambda &\longmapsto \lambda \cdot \mathrm{HS}_{KG}(P). \end{aligned}$$

The Hattori–Stallings homomorphism is natural in G and K. For a finite group F, it is an isomorphism onto its image:

Lemma 3.3.6. *Let F be a finite group and K a field of characteristic 0. Then*

$$K_0(KF) \otimes_{\mathbb{Z}} K \xrightarrow{\mathrm{HS}_{KF}} \mathrm{class}_K(F)$$

is an isomorphism onto its image $\mathrm{class}_K(F)_f$.

Proof. While the proof is that of [BLR08, Lemma 5.1], it is important to note that the statement of that lemma is not correct: In our notation, it claims that $\mathrm{class}_K(F) = \mathrm{class}_K(F)_f$, which is not true in general since the left-hand side has as dimension the number of conjugacy classes, whereas the right-hand side has as dimension the number of K-conjugacy classes. A concrete counterexample is given by the alternating group A_4, which has 4 ($\mathbb{C}$-)conjugacy classes, but only 3 different $\mathbb{Q}$-conjugacy classes. □

If one takes the colimit over all finite subgroups of a potentially infinite group G, the Hattori–Stallings homomorphisms for the individual subgroups assemble to an isomorphism onto $\mathrm{class}_K(G)_f$:

Lemma 3.3.7 ([BLR08, Lemma 5.2])**.** *Let G be a group and K a field of characteristic* 0. *Then the composite*

$$h_K \colon \operatorname*{colim}_{\substack{F \leqslant G \\ |F| < \infty}} K_0(KF) \otimes_{\mathbb{Z}} K \xrightarrow{a_K \otimes \mathrm{id}_K} K_0(KG) \otimes_{\mathbb{Z}} K \xrightarrow{\mathrm{HS}_{KG}} \mathrm{class}_K(G)$$

is an isomorphism onto its image $\mathrm{class}_K(G)_f$.

We will now show that the following diagram is commutative by proving that the subdiagrams (1) through (7) are commutative:

$$\begin{tikzcd}
 & \operatorname{colim} K_0(KF) \arrow[r, "h_K"] \arrow[d] \arrow[ddl, "a_K"'] \arrow[dr, phantom, "(1)"] & \operatorname{class}_K(G)_f \arrow[r, two heads] \arrow[d, hook, "c_K^{\mathbb{C}}"] \arrow[dr, phantom, "(2)"] & \operatorname{class}_K(G)_{f,cf} \arrow[d, hook] \\
 & \operatorname{colim} K_0(\mathbb{C}F) \arrow[r, "h_{\mathbb{C}}"] \arrow[d, "a_{\mathbb{C}}"] \arrow[dl, phantom, "(3)"] \arrow[dr, phantom, "(4)"] & \operatorname{class}_{\mathbb{C}}(G)_f \arrow[r, two heads] \arrow[d, hook] \arrow[dr, phantom, "(5)"] & \operatorname{class}_{\mathbb{C}}(G)_{f,cf} \arrow[d, hook] \\
K_0(KG) \arrow[r] \arrow[d] \arrow[dr, phantom, "(6)"] & K_0(\mathbb{C}G) \arrow[r, "\mathrm{HS}_{\mathbb{C}G}"] \arrow[d] & \operatorname{class}_{\mathbb{C}}(G) \arrow[r, two heads] \arrow[d, phantom, "(7)"] & \operatorname{class}_{\mathbb{C}}(G)_{cf} \arrow[d, hook, "k"] \\
K_0(\mathcal{R}_{KG}) \arrow[dr] & K_0(\mathcal{N}(G)) \arrow[r, hook, "\dim^u_{\mathcal{N}(G)}"] \arrow[d, "\cong"] & \mathcal{Z}(\mathcal{N}(G))^{\mathbb{Z}/2} \arrow[r, hook, "j"] & \mathcal{Z}(\mathcal{N}(G)) \\
 & K_0(\mathcal{U}(G)) & &
\end{tikzcd}$$

We will introduce the maps that make up the diagram during our verification of its commutativity.

The maps in square (1) are obtained by restricting the map h_K (resp. $h_{\mathbb{C}}$) introduced in Lemma 3.3.7 to $\operatorname{colim} K_0(KF)$ and corestricting it to its image. It commutes since HS_{KG} and a_K are natural in K.

The maps that make up the squares (2) and (5) have already been introduced above, where they have also been seen to commute.

The subdiagrams (3) and (6) clearly commute since all maps are induced from the respective inclusions of rings. That $K_0(\mathcal{N}(G)) \to K_0(\mathcal{U}(G))$ is an isomorphism is the content of [Rei01, Theorem 3.7].

The square (4) is introduced and proved to be commutative in [Lüc02, Theorem 9.49], where the maps $a_{\mathbb{C}}$ and $h_{\mathbb{C}}$ are called a and h, respectively, and additionally the functor $? \otimes_{\mathbb{Z}} \mathbb{C}$ is applied to the left vertical map.

The entire subdiagram (7) is introduced and proved to be commutative in [Lüc02, Lemma 9.56]. Furthermore, it is shown there that j, k, and $\dim^u_{\mathcal{N}(G)}$ are injective.

Having checked the commutativity of the diagram, we can now prove the main result of this section:

Theorem 3.3.8. *Let G be a group and $K \leqslant \mathbb{C}$ a subfield closed under complex conjugation. Set $r_{KG} \coloneqq \operatorname{rk}_{\mathbb{Z}}(K_0(\mathcal{R}_{KG}))$. Then:*

(a) $r_{KG} \geqslant |\operatorname{con}_K(G)_{f,cf}|$.

(b) *If G satisfies the center-valued Atiyah conjecture over K, then $r_{KG} = |\operatorname{con}_K(G)_{f,cf}|$.*

(c) *If $r_{KG} = |\operatorname{con}_K(G)_{f,cf}|$, then G satisfies the weak Atiyah conjecture over K.*

Proof. (a) According to the commutative diagram, the map $u\colon \operatorname{colim} K_0(KF) \to \mathcal{Z}(\mathcal{N}(G))$ factors through $K_0(\mathcal{R}_{KG})$. We will thus obtain a lower bound for r_{KG} from any lower bound on the rank of the image $\operatorname{im}(u)$.

We conclude from Lemma 3.3.7 that the image of the $\mathbb{Z}$-linear map

$$h_k\colon \operatorname{colim} K_0(KF) \to \operatorname{class}_K(G)_f$$

generates the codomain as a K-vector space. Following along the upper and right-hand edge of the diagram, we thus read off that $\operatorname{im}(u)$ generates a K-subspace of $\mathcal{Z}(\mathcal{N}(G))$ of dimension $|\operatorname{con}_K(G)_{f,cf}|$. But this is only possible if

$$\operatorname{rk}_{\mathbb{Z}}(\operatorname{im}(u)) \geqslant |\operatorname{con}_K(G)_{f,cf}|.$$

(b) This can be extracted from [KLS17, Lemma 3.2 & 3.3] by unraveling the use of Galois descent and rephrasing it in terms of K-conjugacy.

Alternatively, it is possible to argue in a different way, which will additionally allow us to identify $K_0(\mathcal{R}_{KG})$ with $\mathrm{im}(u)$. If the center-valued Atiyah conjecture holds, then $\mathcal{R}_{KG}$ is semisimple and we can use [Lüc02, Lemma 10.87] to conclude that the map $K_0(\mathcal{R}_{KG}) \to K_0(\mathcal{U}(G))$ is injective. While this result is formulated only for $\mathcal{D}_{\mathbb{C}G}$, its proof only uses that $\mathcal{D}_{\mathbb{C}G}$ is a semisimple ring between KG and $\mathcal{U}(G)$ and that every $e \in \mathcal{U}(G)$ that commutes with every element of $\mathcal{D}_{\mathbb{C}G}$ already commutes with every element of $\mathbb{C}G$. While the latter is tautological for $\mathcal{D}_{\mathbb{C}G}$, it remains true for $\mathcal{R}_{KG}$ since $\mathbb{C}$ is central in $\mathcal{U}(G)$.

Since $\dim^u_{\mathcal{N}(G)}$ and j are also injective and G satisfies the algebraic Atiyah conjecture over K by Theorem 3.1.4, we arrive at the epi-mono factorization

$$\operatorname{colim} K_0(KF) \twoheadrightarrow K_0(\mathcal{R}_{KG}) \hookrightarrow \mathcal{Z}(\mathcal{N}(G)).$$

The uniqueness of this factorization implies that $K_0(\mathcal{R}_{KG}) = \mathrm{im}(u)$ when viewed as subgroups of $\mathcal{Z}(\mathcal{N}(G))$.

(c) Our proof of (a) obtained a lower bound of $|\operatorname{con}_K(G)_{f,cf}|$ for $\mathrm{rk}_{\mathbb{Z}}(\mathrm{im}(u))$. Since u factors through the algebraic Atiyah map $\operatorname{colim} K_0(KF) \to K_0(\mathcal{R}_{KG})$, the rank of the image of this map also satisfies the lower bound. By the assumption, the image has full rank, which means that the algebraic Atiyah map is rationally surjective. This implies the weak Atiyah conjecture by Theorem 3.1.6. □

3.4 Unit-regularity of $\mathcal{R}_{KG}$

Continuing our unconditional study of the ring $\mathcal{R}_{KG}$, we now focus on the following stronger notion of von Neumann regularity:

Definition 3.4.1. A ring R is called *unit-regular* if for every $x \in R$ there is a unit $u \in R^\times$ such that $xux = x$.

Example 3.4.2. The following rings are unit-regular:

- semisimple rings;
- $\mathcal{U}(G)$, see [Rei01, Proposition 2.1(v)].

In particular, for a group G and a subfield $K \leqslant \mathbb{C}$ closed under complex conjugation, the $*$-regular closure $\mathcal{R}_{KG}$ is unit-regular if the strong Atiyah conjecture holds for G over K as a consequence of Proposition 2.4.6. This should not come as a surprise given the following long-standing open problem:

Open Problem (Handelman, [Goo91, Open Problem 48])**.** *Is every $*$-regular ring unit-regular?*

Since $\mathcal{R}_{KG}$ is $*$-regular, it is a natural candidate to validate this open question on:

Open Problem ([AG17, Question 6.4])**.** *Let $K \leqslant \mathbb{C}$ be countable and closed under complex conjugation, and let G be a group. Is $\mathcal{R}_{KG}$ unit-regular?*

The following theorem, which is mentioned in [AG17], is a direct consequence of the general unit-regularity result [GM88, Corollary 5.3], which applies to von Neumann regular K-algebras with a faithful rank function for every uncountable field K:

Theorem 3.4.3. *Let $K \leqslant \mathbb{C}$ be uncountable and closed under complex conjugation, and let G be a group. Then $\mathcal{R}_{KG}$ is unit-regular.*

A von Neumann regular K-algebra with a faithful rank function for a countable field K will not necessarily be unit-regular, see [CL90] for the construction of counterexamples for any such field. In light of the general unit-regularity result for uncountable fields and the genericity of the counterexamples for countable fields, an extension of Theorem 3.4.3 to at least some countable fields would provide some evidence that it should hold in general.

In the special case of $\mathcal{R}_{KG}$ for a sofic group G, we can slightly improve upon Theorem 3.4.3 and provide examples of countable fields K for which $\mathcal{R}_{KG}$ is unit-regular. Since we will make use of the part of the unit-regularity criterion underlying the proof of [GM88, Corollary 5.3] that is not a cardinality argument, we repeat its short proof for the sake of completeness:

Lemma 3.4.4 ([GM88, Lemma 5.1])**.** *Let R be a von Neumann regular ring. Suppose that for every $x, y \in R$ there exists a unit $u \in R^\times$ such that $x - u$ and $y - u^{-1}$ are both units. Then R is unit-regular.*

Proof. Let $x \in R$ be given and choose $y \in R$ such that $xyx = x$. By assumption, there is a unit $u \in R^\times$ such that $x - u$ and $y - u^{-1}$ are both units. Then

$$x(u^{-1} - y)u = x - xyu = xy(x - u),$$

and hence we get that the unit $v \coloneqq (u^{-1} - y)u(x - u)^{-1}$ satisfies

$$xvx = x(u^{-1} - y)u(x - u)^{-1}x = xy(x - u)(x - u)^{-1}x = xyx = x.$$

Since x was arbitrary, we conclude that R is unit-regular. □

The following lemma is a consequence of Jaikin-Zapirain's solution to the algebraic eigenvalue conjecture in [Jai19c, Corollary 1.5]:

Lemma 3.4.5. *Let $L \leqslant \mathbb{C}$ be closed under complex conjugation and G a sofic group. Then a number $\lambda \in \mathbb{C}$ that is transcendental over L cannot be an eigenvalue of $x \in \mathcal{R}_{LG}$, i.e., the element $x - \lambda$ is invertible in $\mathcal{R}_{L(\lambda)G}$.*

Proof. Considering x as a 1×1 matrix, we obtain from [Jai19c, Corollary 1.5] that $x - \lambda$ is invertible in $\mathcal{U}(G)$. Since $\mathcal{R}_{L(\lambda)G}$ contains $\mathcal{R}_{LG}$ and λ, it also contains $x - \lambda$. Furthermore, as a von Neumann regular ring, it is divison closed in $\mathcal{U}(G)$ by 1.7.5, and thus $(x - \lambda)^{-1} \in \mathcal{R}_{L(\lambda)G}$. □

Theorem 3.4.6. *Let $K \leqslant \mathbb{C}$ be of infinite transcendence degree over $\mathbb{Q}$ and closed under complex conjugation and let G be a sofic group. Then $\mathcal{R}_{KG}$ is unit-regular.*

Proof. We will use Lemma 3.4.4 and thus consider two arbitrary elements $x, y \in \mathcal{R}_{KG}$.

By the explicit construction of the $*$-regular closure described in Remark 1.7.7, every fixed element of $\mathcal{R}_{KG}$ can be obtained from finitely many elements of KG by applying ring operations and taking relative inverses finitely many times. This allows us to find a finitely generated field extension $L/\mathbb{Q}, L \leqslant K$ such that $x, y \in \mathcal{R}_{LG} \leqslant \mathcal{R}_{KG}$, which we can assume to be closed under complex conjugation.

As a finitely generated extension, L has finite transcendence degree over $\mathbb{Q}$. Thus, by our assumption on K, there exists $\lambda \in K$ that is transcendental over L. We can now apply Lemma 3.4.5 to x and λ and obtain that $x - \lambda$ is a unit in $\mathcal{R}_{L(\lambda)G} \leqslant \mathcal{R}_{KG}$. In the same way, we obtain that $y - \lambda^{-1}$ is also a unit in $\mathcal{R}_{KG}$, which concludes the proof. □

Example 3.4.7. Since π is transcendental, its powers $1, \pi, \pi^2, \ldots$ are linearly independent over $\mathbb{Q}$. By the Lindemann–Weierstrass theorem (see [Bak75, Theorem 1.4]), the numbers $e, e^\pi, e^{\pi^2}, \ldots$ are algebraically independent over $\mathbb{Q}$. Thus, for the countable field $K = \mathbb{Q}(e, e^\pi, e^{\pi^2}, \ldots)$ and every countable sofic group G, the ring $\mathcal{R}_{KG}$ is unit-regular by Theorem 3.4.6.

Whereas most von Neumann regular rings appearing in practice have torsion-free K_0, arbitrary countable abelian torsion groups can arise as subgroups of $K_0(R)$ for a unit-regular ring R, see [Goo95]. If G satisfies the strong Atiyah conjecture over K, then $\mathcal{R}_{KG}$ is semisimple by Proposition 2.4.6, and hence $K_0(\mathcal{R}_{KG})$ is a torsion-free abelian group. We are thus led to the following question, which is a priori weaker than the strong Atiyah conjecture:

Open Problem. *Let G be a group and $K \leqslant \mathbb{C}$ a subfield closed under complex conjugation. Is $K_0(\mathcal{R}_{KG})$ torsion-free?*

For sofic groups, we can at least show that in order to answer this question for all fields $K \leqslant \mathbb{C}$ it suffices to prove the strong Atiyah conjecture over $\mathbb{Q}$:

Proposition 3.4.8. *Let G be a sofic group and assume that G satisfies the strong Atiyah conjecture over $\mathbb{Q}$. Then $\mathcal{R}_{KG}$ is unit-regular and $K_0(\mathcal{R}_{KG})$ is torsion-free for every $K \leqslant \mathbb{C}$ that is closed under complex conjugation conjugation.*

Proof. The assumption on G implies by Proposition 2.4.6 that $\mathcal{R}_{\mathbb{Q}[G]}$ is semisimple. Assuming for the moment that $K/\mathbb{Q}$ is a finitely generated extension, we obtain from Theorem 3.2.12 an isomorphism $\mathrm{Ore}(\mathcal{R}_{\mathbb{Q}[G]} \otimes_{\mathbb{Q}} K) \xrightarrow{\cong} \mathcal{R}_{KG}$ of semisimple rings, which are always unit-regular and have torsion-free K_0.

If $K/\mathbb{Q}$ is now taken to be any extension, it can be expressed as the directed union of its finitely generated subextensions, and thus $\mathcal{R}_{KG}$ is a directed union of semisimple rings. Since K_0 commutes with and unit-regularity is preserved under directed unions, the result follows. $\square$

Chapter 4

Agrarian invariants and two-generator one-relator groups

This chapter is based on the paper "The agrarian polytope of two-generator one-relator groups" [HK20]*, the corresponding preprint [HK19b], and the preprint* "Agrarian and L^2-invariants" [HK19a]*, all of which report on joint work with Dawid Kielak.*

The story of L^2-invariants does not end with the L^2-Betti numbers introduced in Section 2.3, but rather continues with other examples such as L^2-torsion and Novikov–Shubin invariants. In [FL19; FL17], Friedl and Lück added twisted L^2-Euler characteristics, universal L^2-torsion, and the L^2-polytope to the list of L^2-invariants. While their constructions have a very algebraic flavor throughout and, assuming the strong Atiyah conjecture, mostly play out within the Linnell division ring $\mathcal{D}_{\mathbb{Q}G}$, certain crucial steps rely on input from functional analysis.

In this chapter, we will propose fully algebraic analogues of their invariants starting with nothing more than a ring homomorphism from a group ring to *any* division ring D. Recall that a group G is *agrarian* if its integral group ring $\mathbb{Z}G$ embeds in a division ring. This terminology was introduced in [Kie20], but the idea dates back to Malcev [Mal48], and is a central theme of the work of Cohn [Coh95]. Taking a specific *agrarian embedding* $\mathbb{Z}G \hookrightarrow D$ for some division ring D, or more generally an *agrarian map* $\mathbb{Z}G \to D$, allows us to define the notion of *(D-)agrarian Betti numbers*: when G acts cellularly on a CW-complex X, we simply compute the D-dimension of the homology of $D \otimes_{\mathbb{Z}G} C_*$, where C_* is the cellular chain complex of X. When G is torsion-free and satisfies the strong Atiyah conjecture over $\mathbb{Q}$, D can be taken to be $\mathcal{D}_{KG}$ and the D-agrarian Betti numbers are precisely the L^2-Betti numbers. We show in Proposition 4.2.8 that for two non-equivalent agrarian embeddings, there is always a CW complex whose agrarian Betti numbers with respect to the two embeddings differ.

When the agrarian Betti numbers vanish and G acts on X cocompactly, we define the *agrarian torsion*, in essentially the same way as Whitehead or Reidemeister torsion is defined. Again, when division ring D is taken to be Linnell's division ring $\mathcal{D}_{KG}$, we obtain an invariant very closely related to the universal L^2-torsion. In fact, in this case agrarian and universal L^2-torsion often contain the same amount of information by a theorem of Linnell–Lück [LL18].

The vanishing of L^2-Betti numbers is guaranteed when X fibres over the circle due to a celebrated theorem of Lück; the agrarian Betti numbers also vanish in this setting, as we show in Theorem 4.2.12, provided that the agrarian map used satisfies the additional technical condition of being *rational* (see Definition 4.1.5). Let us remark here that every agrarian map can be turned into a rational one, whose target we will usually denote by D_r.

The final invariant, the *agrarian polytope*, is a little more involved. In the context of

L^2-invariants, one can write the universal L^2-torsion as a fraction of two elements of a (twisted) group ring of the free part of the abelianization of G. Both the numerator and the denominator can be converted into polytopes, using the Newton polytope construction, and the L^2-torsion polytope is defined as the formal difference of these Newton polytopes. The L^2-torsion polytope naturally lives in the polytope group of G, defined in [FL17] and investigated further by Funke [Fun19]. In the agrarian setting it is precisely the notion of rationality which allows us to express the agrarian torsion as a fraction of two elements of a (twisted) group ring of the free part of the abelianization of G, in complete analogy to the L^2 case. The agrarian polytope is then constructed in the same way as the L^2-torsion polytope.

An advantage of agrarian invariants over L^2-invariants lies in the fact that they are defined for a group G as long as $\mathbb{Z}G$ maps to *any* division ring – not necessarily the one known to exist if G were to satisfy the Atiyah conjecture. Even when we require the agrarian map to be injective, the class of agrarian groups is *a priori* larger than the class of torsion-free groups satisfying the Atiyah conjecture.

Furthermore, even if one is not interested in this additional generality, the perspective offered by agrarian invariants can provide more formal answers to questions about the origins of the many convenient properties enjoyed by the L^2-invariants: Are they rooted in the group ring $\mathbb{Z}G$, potentially applying on a more fundamental level, or are they specific to the particular analytic constructions involved in the definitions of L^2-invariants? This question is picked up in Section 4.7.1.

An inconvenience that comes with our more general approach is that for a torsion-free group G not known to satisfy the Atiyah conjecture, there is no longer a canonical choice of an agrarian embedding of G. In general, different agrarian embeddings will lead to differing values for the associated agrarian invariants, which makes it important to keep track of the embedding used to define them.

After the more theoretical groundwork has been laid, we present an application of agrarian invariants to two-generator one-relator groups in Section 4.6. In [FT20], Friedl–Tillmann assigned a marked polytope to a fixed presentation of a torsion-free two-generator one-relator group. By recognizing this polytope as an agrarian polytope, we are able to prove that their construction does in fact not depend on the chosen presentation. We can also relate the thickness of the polytope in a given direction to another agrarian invariant, which in the case of two-generator one-relator groups will turn out to compute a measure of complexity for possible HNN splittings of the group.

After the work on the main results of this chapter had been concluded, the strong Atiyah conjecture for torsion-free one-relator groups was proved by Jaikin-Zapirain and López-Álvarez in [JL20]. We refer the reader to Section 4.7.2 for a discussion of alternative proofs of the main results that have become possible as a result of this achievement.

4.1 Agrarian maps and groups

Let G be a group and denote by $\mathbb{Z}G$ the integral group ring of G.

Definition 4.1.1. Let G be a group. A ring homomorphism $\alpha\colon \mathbb{Z}G \to D$ to a division ring D is called an *agrarian map* for G. A morphism between two agrarian maps is an inclusion of division rings that together with the maps from $\mathbb{Z}G$ forms a commutative triangle.

While many formal properties of the agrarian Betti numbers we will introduce below hold in the situation of an arbitrary agrarian map, concrete calculations and definitions of higher invariants usually require the map to be injective:

Definition 4.1.2. Let G be a group. An *agrarian embedding* for G is an injective agrarian map. If G admits an agrarian embedding (into a division ring D), it is called a *(D-)agrarian group.*

An agrarian group G is necessarily torsion-free; also, it satisfies the Kaplansky zero divisor conjecture, that is, $\mathbb{Z}G$ has no non-trivial zero divisors.

At present, there are no known torsion-free examples of groups which are not agrarian. There is however a plethora of positive examples of agrarian groups:

- Torsion-free groups satisfying the strong Atiyah conjecture over $\mathbb{Q}$ are agrarian as they embed into the Linnell division ring $\mathcal{D}(G) := \mathcal{D}_{\mathbb{Q}G}$ by Corollary 2.4.7.
- Extensions of the groups from 1 by a torsion-free amenable group A are agrarian, assuming that the crossed products $D * A$ for an arbitrary division ring D do not contain non-trivial zero divisors.
- Countable fully residually agrarian groups are again agrarian by an ultraproduct construction.

A more comprehensive list of examples and inheritance properties, including proofs, is given in [Kie20, Section 4]. It should be noted however that some of the results mentioned there have meanwhile been subsumed by the recent advancements of Jaikin-Zapirain and López-Álvarez on the strong Atiyah conjecture in [JL20]. More specifically, as stated in Theorem 2.4.27 (e3), the class of torsion-free groups satisfying the strong Atiyah conjecture is now known to be closed under extensions by locally indicable groups, improving tremendously upon previous considerations about extensions of agrarian groups by biorderable groups. Furthermore, the new result recovers the classical result of [LL78] that torsion-free one-relator groups are agrarian.

4.1.1 The rationalization of an agrarian map

The construction of a crossed product out of a short exact sequence of groups as in Section 1.1 can be extended to agrarian maps. This technique is formulated in the following lemma, which will be our main source of crossed products.

Lemma 4.1.3. *Let $\alpha\colon \mathbb{Z}G \to D$ be an agrarian map for a group G. Let $N \leqslant G$ be a normal subgroup and set $Q := G/N$. Then α restricts to an agrarian map $\mathbb{Z}N \to D$ for N that is equivariant with respect to the conjugation action of G. Moreover, for any set-theoretic section $s\colon Q \to G$ of the quotient map, this restriction of α extends to a ring homomorphism*

$$\mathbb{Z}N *_s Q \to D *_s Q,$$

where $\mathbb{Z}N *_s Q$ is the crossed product structure constructed out of s in Lemma 1.1.3 and $D *_s Q$ is a crossed product structure with the same basis elements and action and twisting maps extended from those of $\mathbb{Z}N * Q$. The ring $D *_s Q$ is independent of the choice of the section s up to ring isomorphism and the basis of the crossed product structure is independent up to a diagonal change of basis, i.e., the ring isomorphism can be chosen to map $\sum_{q\in Q} u_q q$ to $\sum_{q\in Q} v_q q$ such that for every $q \in Q$, the elements u_q and v_q differ only by an element of $D^\times$.

Proof. By definition, α restricts to an agrarian map for N. Note that an element $g \in G$ acts on D by conjugation with $\alpha(g)$, which is always invertible in D since g is invertible in $\mathbb{Z}G$. Since N is normal in G, the conjugation action of G on $\mathbb{Z}G$ preserves $\mathbb{Z}N$ and hence induces an action on $\mathbb{Z}N$. The restricted agrarian map $\alpha\colon \mathbb{Z}N \to D$ is equivariant with respect to these actions by construction.

Let $s\colon Q \to G$ be a set-theoretic section of the group epimorphism $\mathrm{pr}\colon G \to Q$ and denote by $\mathbb{Z}N *_s Q$ the crossed product with the basis and structure maps associated to the section s as in Lemma 1.1.3. Since the automorphism $\mathbb{Z}N$ given by conjugation by $s(q)$ extends to an inner automorphism of $D \supset \mathbb{Z}N$, we can apply the crossed product construction of Proposition 1.1.2 also to D and Q in such a way that the map $(\mathbb{Z}N) *_s Q \to D *_s Q$ extends $\mathbb{Z}N \to D$.

Let s_1 and s_2 be two set-theoretic sections of $\mathrm{pr}\colon G \to Q$. Denote by $D *_{s_1} Q$ and $D *_{s_2} Q$ the associated crossed product structures. We claim that the map $\Phi\colon D *_{s_1} Q \to D *_{s_2} Q$ given by

$$\sum_{q\in Q} u_q \cdot q \mapsto \sum_{q \in Q} (u_q s_1(q) s_2(q)^{-1}) \cdot q$$

is a ring isomorphism. Since $s_1(q)s_2(q)^{-1} \in G \subset D^\times$ for all $q \in Q$, it is clear that Φ is an isomorphism between the underlying free D-modules and changes the coefficients by a unit in D only. We omit the straightforward verification that Φ respects the multiplications (see [HK19a, Lemma 2.5]). □

We now consider the case of an agrarian map $\alpha\colon \mathbb{Z}G \to D$ and a normal subgroup $K \leqslant G$ such that G/K is a finitely generated free abelian group H. Lemma 4.1.3 then provides us with a crossed product $D * H$ and a ring homomorphism $\mathbb{Z}G \cong (\mathbb{Z}K) * H \to D * H$. Since H is free abelian, it is in particular biorderable and hence $D * H$ contains no non-trivial zero divisors (this is a standard fact following from the existence of an embedding of $D * H$ into its Malcev–Neumann completion; for details see [Kie20, Theorem 2.6]). It then follows from Theorem 1.2.9 that $D * H$ satisfies the Ore condition and thus has an Ore division ring of fractions.

Our construction is summarized in

Definition 4.1.4. Let $\alpha\colon \mathbb{Z}G \to D$ be an agrarian map for a group G. Let K be a normal subgroup of G such that $H := G/K$ is finitely generated free abelian. The K-*rationalization* of α is the composite agrarian map

$$\alpha_K\colon \mathbb{Z}G \cong (\mathbb{Z}K) * H \to D * H \hookrightarrow \mathrm{Ore}(D * H),$$

where $\mathrm{Ore}(D * H)$ is the Ore division ring of fractions of the crossed product $D * H$ of Lemma 4.1.3.

The construction of the K-rationalization of an agrarian map $\mathbb{Z}G \to D$ of course depends on a choice of a set-theoretic section of the projection $G \to G/K$, which we will assume to be fixed once and for all for any group G being considered. By Lemma 4.1.3, at least the target division ring of the K-rationalization is independent of this choice up to isomorphism.

Also note that the K-rationalization of an agrarian embedding is again an embedding.

The typical situation in which we consider the K-rationalization of a given agrarian map is that where K is the kernel of the projection of G onto the free part of its abelianization.

Definition 4.1.5. Let G be a finitely generated group and let $\alpha\colon \mathbb{Z}G \to D$ be an agrarian map for G. Denote the free part of the abelianization of G by H and the kernel of the canonical projection of G onto H by K. For this particular choice of K, we simply call the K-rationalization of α the *rationalization* and denote it by α_r. The agrarian map α is called *rational* if there exists a division subring $D' \subseteq D$ such that α is of the form

$$\mathbb{Z}G \cong \mathbb{Z}K * H \to \mathrm{Ore}(D' * H),$$

where the crossed product structure on $D' * H$ is obtained from that of the rationalization by restriction.

The rationalization of an agrarian map is rational with $D' = D$.

The term *rational* is chosen to indicate that the target of a rational agrarian map should be viewed as a division ring of rational functions in finitely many variables with coefficients in a division ring. While the special structure of rational functions is crucial for the development of the theory of agrarian invariants, the specific choice of the division ring of coefficients is mostly immaterial.

Remark 4.1.6. Let G be a finitely generated group and $\alpha\colon \mathbb{Z}G \to D$ a rational agrarian map for G. If we restrict the codomain of α to the division subring generated by the image of α, then the resulting agrarian map will again be rational. In fact, if we denote the free part of the abelianization of G by H and the kernel of the projection of G onto H by K, then the division subring of D generated by $\alpha(\mathbb{Z}G)$ is $\operatorname{Ore}(D' * H)$, where D' is the division subring of D generated by $\alpha(\mathbb{Z}K)$.

For later use, we record a result allowing us to pass to the "full" K-rationalization by performing two "partial" rationalizations whenever we are given a chain $K \trianglelefteq K' \trianglelefteq G$ of normal subgroups:

Lemma 4.1.7. *Let G be a finitely generated group with agrarian map $\alpha\colon \mathbb{Z}G \to D$. Denote by $\mathrm{pr}\colon G \to H$ the projection onto the free part H of the abelianization of G. Let $\varphi\colon G \to H'$ be an epimorphism onto a finitely generated free abelian group, inducing the following commutative diagram of epimorphisms:*

$$\begin{tikzcd} G \arrow[r, two heads, "\mathrm{pr}"] \arrow[dr, two heads, "\varphi"'] & H \arrow[d, two heads, "\overline{\varphi}"] \\ & H' \end{tikzcd}$$

Denote the kernels of pr, φ and $\overline{\varphi}$ by K, K_φ and $K_{\overline{\varphi}}$, respectively. Further let s and t be sections of the epimorphisms pr and $\overline{\varphi}$, respectively. Then

$$\begin{aligned} \beta\colon (D *_t K_{\overline{\varphi}}) *_s H' &\to D *_{s\circ t} H \\ \sum_{h' \in H'} \Big(\sum_{k \in K_{\overline{\varphi}}} u_{k,h'} \cdot k \Big) \cdot h' &\mapsto \sum_{\substack{h' \in H' \\ k \in K_{\overline{\varphi}}}} u_{k,h'} \cdot kt(h') \end{aligned}$$

is a ring isomorphism presersing the crossed product structures defined in terms of s and t. It extends to an isomorphism

$$\beta\colon \operatorname{Ore}(\operatorname{Ore}(D * K_{\overline{\varphi}}) * H') \xrightarrow{\cong} \operatorname{Ore}(D * H)$$

of division rings.

Proof. The left D-bases of $(D *_t K_{\overline{\varphi}}) *_s H'$ and $D *_{s\circ t} H$ are given by $k * h'$ and $kt(h')$ respectively for $k \in K_{\overline{\varphi}}$ and $h' \in H'$. These bases are identified bijectively by β with inverse $h \mapsto ht(\overline{\varphi}(h)^{-1}) \cdot \overline{\varphi}(h)$. It follows that β is an isomorphism of left D-modules. Checking that β is a ring homomorphism is a tedious but direct computation that we will omit.

Since $D * K_{\overline{\varphi}}$ is a subring of $D * H$, and since the rings have no non-trivial zero divisors, β extends to an injection $\operatorname{Ore}(D * K_{\overline{\varphi}}) * H' \hookrightarrow \operatorname{Ore}(D * H)$ that contains $D * H$ in its image. Passing to the Ore division ring of fractions, this implies that β extends to an isomorphism $\operatorname{Ore}(\operatorname{Ore}(D * K_{\overline{\varphi}}) * H') \to \operatorname{Ore}(D * H)$. □

4.2 Agrarian Betti numbers

From now on, G will denote a group with a fixed agrarian map $\alpha\colon \mathbb{Z}G \to D$. Unless indicated explicitly, all tensor products will be taken over $\mathbb{Z}G$.

4.2.1 Definition of agrarian Betti numbers

Let C_* be a $\mathbb{Z}G$-chain complex and let $n \in \mathbb{Z}$. Viewing D as a D-$\mathbb{Z}G$-bimodule via the agrarian map α, the chain complex $D \otimes C_*$ becomes a D-chain complex. Since D is a division ring, the D-module $H_p(D \otimes C_*)$ is free and we can consider its dimension $\dim_D H_p(D \otimes C_*) \in \mathbb{N} \sqcup \{\infty\}$. This leads to the following definition of agrarian Betti numbers:

Definition 4.2.1. Let G be a group with an agrarian map $\alpha\colon \mathbb{Z}G \to D$ and C_* a $\mathbb{Z}G$-chain complex. For $n \in \mathbb{Z}$, the *n-th D-Betti number* of C_* with respect to α is defined as

$$b_n^D(C_*) \coloneqq \dim_D H_n(D \otimes C_*) \in \mathbb{N} \sqcup \{\infty\},$$

A $\mathbb{Z}G$-chain complex is called *D-acyclic* if all of its D-Betti numbers are equal to 0.

If the agrarian map α is chosen to be the augmentation homomorphism

$$\mathbb{Z}G \to \mathbb{Z} \hookrightarrow \mathbb{Q},$$

the associated agrarian Betti numbers of a $\mathbb{Z}G$-chain complex C_* reduce to the ordinary Betti numbers of the quotient complex C_*/G. In the special case where G satisfies the strong Atiyah conjecture over $\mathbb{Q}$ and the agrarian map is chosen to be the agrarian embedding $\mathbb{Z}G \hookrightarrow \mathcal{D}(G) = \mathcal{D}_{\mathbb{Q}G}$, the D-Betti numbers of any $\mathbb{Z}G$-chain complex C_* agree with its L^2-Betti numbers $b_*^{(2)}(C_*; G)$ by [FL19, Theorem 3.6 (2)]. Note that the assumption that C_* is projective is not used in the proof, the theorem thus holds for arbitrary $\mathbb{Z}G$-chain complexes.

We will mainly be concerned with the agrarian Betti numbers assigned to *G-CW-complexes*, which are equivariant analogues of CW-complexes and very convenient models for G-spaces. A typical example of a G-CW-complex is the universal covering of a connected CW-complex X with $G = \pi_1(X)$.

Definition 4.2.2. Let G be a (discrete) group. A *G-CW-complex* is a CW-complex X together with an implicit (left) G-action mapping p-cells to p-cells and such that any cell mapped into itself is fixed pointwise by the action. An action satisfying these properties is called *cellular*. The *p-skeleton* of a G-CW-complex X, denoted by X_p, is the p-skeleton of the underlying CW-complex together with the restriction of the G-action. If $X = X_p$ for some p and p is minimal with this property, then X is said to be *of dimension* p. Any G-orbit of a p-dimensional cell of the underlying CW-complex constitutes a *p-dimensional G-cell* of X. A G-CW-complex is *connected* if the underlying CW-complex is connected. The *cellular ($\mathbb{Z}G$-)chain complex* $C_*(X)$ is obtained from the cellular chain complex of the underlying CW-complex by considering the induced left action by G. All differentials are $\mathbb{Z}G$-linear.

Definition 4.2.3. A G-CW-complex X is called *free* if its G-action is free. It is called *of finite type* if for every $p \geqslant 0$ there are only finitely many p-dimensional G-cells in X. If the total number of G-cells of any dimension in X is finite, the G-CW-complex X is called *finite*.

Definition 4.2.4. A $\mathbb{Z}G$-chain complex C_* is called *free* (*of finite type*) if C_n is a free (finitely generated) $\mathbb{Z}G$-module for every $n \in \mathbb{Z}$. It is called *bounded* if there is $N \in \mathbb{N}$ such that $C_i = 0$ for $i > N$ and $i < -N$.

If a G-CW-complex is free (of finite type), then its associated cellular chain complex $C_*(X)$ is free (of finite type). If it is finite, then its cellular chain complex is bounded and of finite type.

We now define agrarian Betti numbers for G-CW-complexes:

Definition 4.2.5. Let X be an G-CW-complex. For $p \geqslant 0$, the *p-th D-Betti number* of X with respect to α is defined as

$$b_p^D(X) := b_p^D(C_*(X)) \in \mathbb{N} \sqcup \{\infty\}.$$

A G-CW-complex X is called *D-acyclic* if all of its D-Betti numbers are equal to 0.

If X is an G-CW-complex of finite type, then the D-Betti numbers of X will all be non-negative integers.

4.2.2 Dependence on the agrarian map

Note that the agrarian Betti numbers may depend not only on the division ring D but also on the particular choice of agrarian map. We will first consider a typical situation in which two different agrarian maps define the same agrarian Betti numbers.

Remark 4.2.6. Recall that a ring homomorphism $f\colon R \to S$ is called *epic* if $g_1 \circ f = g_2 \circ f$ implies $g_1 = g_2$ for any ring homomorphisms $g_1, g_2 \colon S \to T$. Any agrarian map $\alpha\colon \mathbb{Z}G \to D$ factors in a canonical way through an epic agrarian map, namely the map to the division subring D' of D generated by $\alpha(\mathbb{Z}G)$. Now any D'-module, and in particular the overfield D, is flat as a right module over the division ring D'. Hence, we get

$$\begin{aligned} \dim_D H_n(D \otimes C_*) &= \dim_D H_n(D \otimes_{D'} D' \otimes C_*) \\ &= \dim_D D \otimes_{D'} H_n(D' \otimes C_*) \\ &= \dim_{D'} H_n(D' \otimes C_*) \end{aligned}$$

for any $\mathbb{Z}G$-chain complex C_* and $n \in \mathbb{Z}$, i.e., D and D' yield the same agrarian Betti numbers. We can thus restrict our attention to epic agrarian maps when computing agrarian Betti numbers.

Remark 4.2.7. We conclude from Remark 4.1.6 that the epic agrarian map obtained from any rational agrarian map $\alpha\colon \mathbb{Z}G \to D$ is again rational. In fact, the division subring D' of D considered in Definition 4.1.5 will in this case be generated by $\alpha(\mathbb{Z}K)$, which is the minimal choice.

The epic agrarian map $\mathbb{Z}G \to \mathbb{Z} \to \mathbb{Q}$ given by the augmentation homomorphism always produces positive zeroth agrarian Betti numbers, whereas the zeroth agrarian Betti numbers with respect to any other epic agrarian map is always zero, as we will see in Theorem 4.2.9 (d). For agrarian embeddings, the situation is as follows:

Proposition 4.2.8. *Let G be a finitely generated agrarian group. The agrarian Betti numbers for any connected finite free G-CW-complex are independent of the choice of agrarian embedding if and only if there exists an epic agrarian embedding for G that is unique up to isomorphism.*

Proof. If there is a unique isomorphism type of epic agrarian embeddings for G, then by the preceding discussion every choice of an agrarian embedding factors through an epic agrarian embedding of this type and hence gives the same agrarian Betti numbers even for all $\mathbb{Z}G$-complexes.

Now let $\mathbb{Z}G \hookrightarrow D_1$ and $\mathbb{Z}G \hookrightarrow D_2$ be non-isomorphic epic agrarian embeddings. By [Coh95, Theorem 4.3.5] there exists an $m \times n$-matrix A over $\mathbb{Z}G$ which becomes invertible when viewed as a matrix over D_1, but becomes singular over D_2 (without loss of generality). We realise A topologically by constructing the skeleta of a connected finite free G-CW-complex X step by step as follows: Choose a generating set $S = \{x_1, \ldots, x_k\}$ for G and consider the Cayley graph $C(G, S)$ as a 1-dimensional G-CW-complex X_1. We now attach m 2-dimensional spheres to each vertex, and extend the action of G in the obvious way; this way we arrive at the 2-skeleton X_2 of X. For every $1 \leqslant j \leqslant n$, we then attach a 3-dimensional G-cell to X_2 in such a way that the resulting boundary map $C_3(X) \to C_2(X)$ in the cellular chain complex of the resulting space coincides with the matrix A. This concludes the construction of the connected finite free G-CW-complex X.

The cellular chain complex $C_*(X)$ is concentrated in degrees 0, 1, 2 and 3 and looks as follows:

$$\mathbb{Z}G^n \xrightarrow{A} \mathbb{Z}G^m \xrightarrow{0} \mathbb{Z}G^k \xrightarrow{(x_1-1,\ldots,x_k-1)} \mathbb{Z}G$$

As A becomes invertible over D_1, but singular over D_2, we have

$$H_2(X; D_1) = 0 \neq H_2(X; D_2),$$

and hence the D_1- and D_2-Betti numbers of X differ. □

Translated into our setting, in [Lew74, Section V], Lewin constructs two non-isomorphic epic agrarian embeddings of F_6, the free group on six generators. Using the previous lemma, we conclude that the notion of *the* agrarian Betti numbers of an F_6-CW-complex is not well-defined. Nonetheless, we will later give examples of complexes for which the D-Betti numbers can be shown to not depend on D.

4.2.3 Computational properties

In order to formulate and prove agrarian analogues of the properties of L^2-Betti numbers, as collected by Lück in [Lüc02, Theorem 1.35], we have to introduce a few classical constructions on G-CW-complexes and chain complexes.

Recall that for a free G-CW-complex X and a subgroup $H \leqslant G$ of finite index, the H-space $\operatorname{res}_G^H X$ is obtained from X by restricting the action to H. A free (finite, finite type) H-CW-structure for this space can be obtained from a free (finite, finite type) G-CW-structure of X by replacing a G-cell with $|G : H|$ many H-cells.

If $H \leqslant G$ is any subgroup and Y is a free H-CW-complex, then $G \times_H Y$ is the H-space $G \times Y/(g, y) \sim (gh^{-1}, hy)$. A free (finite, finite type) H-CW-structure of Y determines a free (finite, finite type) G-CW-structure of $G \times_H Y$ by replacing an H-cell with a G-cell.

We now consider a chain complex C_* with differentials c_*. Its *suspension* ΣC_* is the chain complex with C_{n-1} as the module in degree n and n-th differential equal to $-c_{n-1}$. If $f_*\colon C_* \to D_*$ is a chain map between chain complexes with differentials c_* and d_*, the *mapping cone* $\operatorname{cone}_*(f_*)$ is the chain complex with $\operatorname{cone}_n(f_*) = C_{n-1} \oplus D_n$ and n-th differential given by

$$C_{n-1} \oplus D_n \xrightarrow{\begin{pmatrix} -c_{n-1} & 0 \\ f_{n-1} & d_n \end{pmatrix}} C_{n-2} \oplus D_{n-1}.$$

The mapping cone of f_* fits into the following short exact sequence:

$$0 \to D_* \to \operatorname{cone}_*(f_*) \to \Sigma C_* \to 0.$$

The following theorem covers all the properties of agrarian Betti number we will use in computations:

Theorem 4.2.9. *The following properties of D-Betti numbers hold, where we fix an agrarian map $\alpha\colon \mathbb{Z}G \to D$ for a group G:*

(a) **(Homotopy invariance).** *Let $f\colon X \to Y$ be a G-map of free G-CW-complexes of finite type. If the map $H_p(f;\mathbb{Z})\colon H_p(X;\mathbb{Z}) \to H_p(Y;\mathbb{Z})$ induced on cellular homology with integral coefficients is bijective for $p \leqslant d-1$ and surjective for $p = d$, then*

$$\begin{aligned} b_p^D(X) &= b_p^D(Y) \quad \text{for } p \leqslant d-1; \\ b_d^D(X) &\geqslant b_d^D(Y). \end{aligned}$$

In particular, if f is a weak homotopy equivalence, we get for all $p \geqslant 0$:

$$b_p^D(X) = b_p^D(Y).$$

(b) **(Euler-Poincaré formula).** *Let X be a finite free G-CW-complex. Let $\chi(X/G)$ be the Euler characteristic of the finite CW-complex X/G, i.e.,*

$$\chi(X/G) := \sum_{p \geqslant 0} (-1)^p \cdot \beta_p(X/G),$$

where $\beta_p(X/G)$ denotes the number of p-cells of X/G. Then

$$\chi^D(X) := \sum_{p \geqslant 0} (-1)^p \cdot b_p^D(X) = \chi(X/G).$$

(c) **(Upper bound).** *Let X be a free G-CW-complex. With $\beta_p(X/G)$ as above, for all $p \geqslant 0$ we have*

$$b_p^D(X) \leqslant \beta_p(X/G).$$

(d) **(Zeroth agrarian Betti number).** *Let X be a connected free G-CW-complex of finite type and assume that the agrarian map $\alpha\colon \mathbb{Z}G \to D$ does not factor through the augmentation homomorphism $\mathbb{Z}G \to \mathbb{Z} \hookrightarrow \mathbb{Q}$. Then*

$$b_0^D(X) = 0.$$

(e) **(Induction).** *Let $H \leqslant G$ be a subgroup of G. If X is a free H-CW-complex, then for $p \geqslant 0$*

$$b_p^D(G \times_H X) = b_p^D(X),$$

where the agrarian map for H is chosen as the restriction of α to $\mathbb{Z}H$.

(f) **(Amenable groups).** *Let X be a free G-CW-complex of finite type and assume that the agrarian map $\alpha\colon \mathbb{Z}G \to D$ is actually an agrarian embedding. Further assume that G is amenable. Then*

$$b_p^D(X) = \dim_D(D \otimes H_p(X;\mathbb{Z}G)).$$

Proof. (a) We replace f by a homotopic cellular map. Consider the $\mathbb{Z}G$-chain map

$$f_*\colon C_*(X) \to C_*(Y)$$

induced by f on the cellular chain complexes and its mapping cone $\operatorname{cone}_*(f_*)$, which fits into a short exact sequence

$$0 \to C_*(Y) \to \operatorname{cone}_*(f_*) \to \Sigma C_*(X) \to 0$$

of $\mathbb{Z}G$-chain complexes. Applying the assumptions on the map $H_p(f;\mathbb{Z})$ to the long exact sequence in homology associated to this short exact sequence, we obtain that $H_p(\operatorname{cone}_*(f_*)) = 0$ for $p \leqslant d$.

Claim. *The homology of $D \otimes \operatorname{cone}_*(f_*)$ vanishes in degrees $p \leqslant d$.*

Assume for the moment that this indeed holds. Since $D \otimes \operatorname{cone}_*(f_*) = \operatorname{cone}_*(\mathrm{id}_D \otimes f_*)$ and $D \otimes \Sigma C_*(X) = \Sigma(D \otimes C_*(X))$, the short sequence

$$0 \to D \otimes C_*(Y) \to D \otimes \operatorname{cone}_*(f_*) \to D \otimes \Sigma C_*(X) \to 0$$

is also exact. We now consider the associated long exact sequence in homology, in which the terms $H_p(D \otimes \operatorname{cone}_*(f_*))$ for $p \leqslant d$ vanish by the claim. The exactness of the sequence then implies that the differentials $H_p(D \otimes \Sigma C_*(Y)) \xrightarrow{\cong} H_{p-1}(D \otimes C_*(X))$ are isomorphisms for $p \leqslant d$ and the differential $H_{d+1}(D \otimes \Sigma C_*(Y)) \twoheadrightarrow H_d(D \otimes C_*(X))$ is an epimorphism. Applying $\dim_D$ and using the definition of the suspension then yields the desired statement.

We are left with proving the claim. Since $\operatorname{cone}_*(f_*)$ is bounded below and consists of free modules, we can inductively construct a $\mathbb{Z}G$-chain homotopy equivalent $\mathbb{Z}G$-chain complex Z_* which vanishes in degrees $p \leqslant d$. Tensoring with D then yields a D-chain homotopy equivalence between $D \otimes \operatorname{cone}_*(f_*)$ and $D \otimes Z_*$. As $Z_p = 0$ for $p \leqslant d$, the same holds true for $D \otimes Z_*$ and hence $H_p(D \otimes \operatorname{cone}_*(f_*)) = H_p(D \otimes Z_*) = 0$ for $p \leqslant d$.

(b) This is a consequence of two immediate facts: first, the Euler characteristic of a chain complex over a division ring does not change when passing to homology; second, we have the identity $\beta_p(X/G) = \dim_D D \otimes C_p(X)$.

(c) This holds since $H_p(X; D)$ is a subquotient of $D \otimes C_p(X)$ and the latter has dimension $\beta_p(X/G)$ over D (as remarked above).

(d) If X is empty, then the claim is trivially true. Otherwise, we will first argue that, without loss of generality, we may assume X/G to have exactly one 0-cell. Let T be a maximal tree in the 1-skeleton of the CW-complex X/G and denote by $q\colon X/G \to (X/G)/T$ the associated cellular quotient map, which is a homotopy equivalence. Note that $(X/G)/T$ has a single 0-cell. Let $p\colon (X/G)/T \to X/G$ be a cellular homotopy inverse of q. We denote by X' the total space in the following pullback of the G-covering $X \to X/G$ along p:

$$\begin{tikzcd} X \arrow[d] & X' \arrow[l, dashed] \arrow[d, dashed] \\ X/G \arrow[r, "q"] & (X/G)/T \arrow[l, bend left, "p"] \end{tikzcd}$$

Alternatively, we can view X' as being obtained from X by collapsing each lift of T individually to a point. Since $(X/G)/T$ is a connected free G-CW-complex of finite type, $X \to X/G$ is a G-covering and X is connected, the G-CW-complex X' is also connected, free and of finite type. Furthermore, X' is G-homotopy equivalent to X via any G-equivariant lift of the homotopy equivalence p. By Theorem 4.2.9 (a), the D-Betti numbers of X and X' agree, so we may assume without loss of generality that X has a single equivariant 0-cell.

Since X is a free G-CW-complex of finite type which has a single 0-cell, the differential $c_1\colon C_1(X) \to C_0(X)$ in its cellular chain complex is of the form

$$\mathbb{Z}G^n \xrightarrow{\oplus_{i=1}^n (1-g_i)} \mathbb{Z}G$$

for $g_i \in G, i = 1, \dots, n, n \in \mathbb{N}$ for any choice of a $\mathbb{Z}G$-basis of $C_*(X)$ consisting of cells. The image of the differential is contained in the augmentation ideal $I = \langle g-1 \mid g \in G\rangle$ of $\mathbb{Z}G$, and, as X is assumed to be connected, has to coincide with it for $H_0(X;\mathbb{Z})$ to be isomorphic to $\mathbb{Z}$. By our additional assumption on the agrarian map, there is thus an element in the image of the differential that does not lie in the kernel of the agrarian map. But the image of this element is invertible in D, and hence $H_0(X;D) = 0$ as claimed.

(e) On cellular chain complexes, $G \times_H ?$ translates into applying the functor $\mathbb{Z}G \otimes_{\mathbb{Z}H} ?$. The claim thus follows from the canonical identification

$$D \otimes_{\mathbb{Z}G} \mathbb{Z}G \otimes_{\mathbb{Z}H} C_*(X) \cong D \otimes_{\mathbb{Z}H} C_*(X).$$

(f) As G is agrarian, its group ring $\mathbb{Q}G$ does not admit zero divisors (this is immediate, since $\mathbb{Q}G$ embeds into the same division ring D as $\mathbb{Z}G$ does). Since G is amenable, we conclude from Theorem 1.2.9 that $\mathbb{Q}G$ and hence $\mathbb{Z}G$ admits an Ore division rings of fractions F. In particular, F is flat over $\mathbb{Z}G$ and every embedding of $\mathbb{Z}G$ into a division ring, such as the agrarian embedding $\alpha\colon \mathbb{Z}G \hookrightarrow D$, factors through the natural inclusion $\mathbb{Z}G \hookrightarrow F$. We thus obtain the following for any $p \geqslant 0$, using first that $F \hookrightarrow D$ is flat and then that $\mathbb{Z}G \hookrightarrow F$ is flat:

$$\begin{aligned} b_p^D(X) &= \dim_D H_p(X;D) = \dim_D D \otimes_F H_p(X;F) \\ &= \dim_D D \otimes_F F \otimes H_p(X;\mathbb{Z}G) = \dim_D D \otimes H_p(X;\mathbb{Z}G). \end{aligned}$$ □

The behavior of L^2-Betti numbers under restriction to finite-index subgroups carries over to agrarian invariants under an additional assumption on the agrarian map:

Proposition 4.2.10. *Let $H \leqslant G$ be a subgroup of G of finite index $|G:H| < \infty$. Let $\alpha\colon \mathbb{Z}G \to D$ be an epic agrarian map for G and denote the division subring of D generated by $\alpha(\mathbb{Z}H)$ by D'. Assume that the map*

$$\Psi\colon D' \otimes_{\mathbb{Z}H} \mathbb{Z}G \to D, x \otimes g \mapsto x \cdot \alpha(g^{-1})$$

of D'-$\mathbb{Z}G$-bimodules is an isomorphism. If X is a free G-CW-complex of finite type, then for $p \geqslant 0$

$$b_p^D(\operatorname{res}_G^H X) = |G:H| \cdot b_p^D(X).$$

Proof. Since $D' \otimes_{\mathbb{Z}H} \mathbb{Z}G \otimes_{\mathbb{Z}G} C_*(X) \cong D' \otimes_{\mathbb{Z}H} C_*(\operatorname{res}_G^H X)$, the map Ψ induces an isomorphism

$$D' \otimes_{\mathbb{Z}H} C_*(\operatorname{res}_G^H X) \xrightarrow{\cong} \operatorname{res}_D^{D'} D \otimes_{\mathbb{Z}G} C_*(X)$$

of D'-chain complexes. Passing to agrarian Betti numbers on both sides, we obtain

$$b_p^D(\operatorname{res}_G^H X) = \dim_{D'} \operatorname{res}_D^{D'} H_p(D \otimes_{\mathbb{Z}G} C_*(X)). \tag{4.1}$$

Since $\mathbb{Z}G$ is a free left $\mathbb{Z}H$-module of rank $|G:H|$, the isomorphism Ψ exhibits D as a left D'-vector space of dimension $|G:H|$. As a consequence,

$$\dim_{D'} \operatorname{res}_D^{D'} V = |G:H| \cdot \dim_D V$$

holds for any left D-vector space V. We arrive at the claimed formula by applying this identity to the right-hand side of (4.1). □

Mapping tori

In subsequent sections, we will study invariants of CW complexes with vanishing agrarian Betti numbers. In the context of L^2-invariants, an extremely useful way of showing the vanishing of L^2-Betti numbers comes from a theorem of Lück [Lüc94, Theorem 2.1] concerning mapping tori. Below, we offer a straightforward adaption of Lück's result to the setting of agrarian Betti numbers. If G satisfies the strong Atiyah conjecture over $\mathbb{Q}$, then our version reduces to the classical L^2-formulation if one considers the agrarian embedding $\mathbb{Z}G \hookrightarrow \mathcal{D}(G)$ into the Linnell division ring.

Definition 4.2.11. Let $f\colon X \to X$ be a selfmap of a path-connected space. The *mapping torus* T_f of f is obtained from the cylinder $X \times [0,1]$ by identifying $(x,1)$ with $(f(x),0)$ for every $x \in X$. The *canonical projection* is the map $T_f \to S^1$ sending (x,t) to $\exp(2\pi i t)$. It induces an epimorphism $\pi_1(T_f) \to \pi_1(S^1) = \mathbb{Z}$.

If X has the structure of a CW-complex with $\beta_p(X)$ cells of dimension p and f is cellular, then T_f can be endowed with a CW-structure with $\beta_p(T_f) = \beta_p(X) + \beta_{p-1}(X)$ cells of dimension p for each $p \geqslant 0$.

Theorem 4.2.12. *Let $f\colon X \to X$ be a cellular selfmap of a connected CW-complex X and $\pi_1(T_f) \xrightarrow{\varphi} G \xrightarrow{\psi} \mathbb{Z}$ any factorisation into epimorphisms of the epimorphism induced by the canonical projection. Let $\overline{T_f}$ be the covering of the mapping torus T_f associated to φ, endowed with the structure of a connected free G-CW-complex. Let $\alpha\colon \mathbb{Z}G \to D$ be a rational agrarian map for G. If the d-skeleton of X (and thus of $\overline{T_f}$) is finite for some $d \geqslant 0$, then for all $p \leqslant d$*

$$b_p^D\left(\overline{T_f}\right) = 0.$$

Proof. The topological part of the proof is the analogue of the proof for L^2-Betti numbers, see [Lüc02, Theorem 1.39].

By Remarks 4.2.6 and 4.2.7, we may assume that α is epic. Fix $p \geqslant 0$. For any $n \geqslant 1$, define $G_n \leqslant G$ to be the preimage of the subgroup $n \cdot \mathbb{Z} \leqslant \mathbb{Z}$ under $\psi\colon G \to \mathbb{Z}$, for which we consider the induced agrarian map $\mathbb{Z}G_n \hookrightarrow \mathbb{Z}G \xrightarrow{\alpha} D$. Further denote the kernel of ψ by K and the division subring of D generated by $\mathbb{Z}K$ by D'. Since the agrarian map α is epic and rational, the division subring D_n of D generated by $\alpha(\mathbb{Z}G_n)$ is given by $\operatorname{Ore}(D'(G_n/K))$.

Claim. *For our choice of $\alpha\colon \mathbb{Z}G \to D$ and $H \coloneqq G_n$, the map Ψ of Proposition 4.2.10 is an isomorphism.*

We first conclude the proof assuming the claim. Since G_n has index n in G, we deduce from the claim and Proposition 4.2.10 that

$$b_p^D\left(\overline{T_f}\right) = \frac{1}{n} \cdot b_p^D\left(\operatorname{res}_G^{G_n} \overline{T_f}\right). \tag{4.2}$$

Reparametrizing yields a homotopy equivalence $h\colon T_{f^n} \xrightarrow{\simeq} \overline{T_f}/G_n$ of CW-complexes, where f^n denotes the n-fold composition of f. Let $\overline{T_{f^n}}$ be the G_n-space obtained as the following pullback, or equivalently, as the covering of T_{f^n} corresponding to the kernel of $\pi_1(T_{f^n}) \cong \pi_1(\overline{T_f}/G_n) \to G_n$:

$$\begin{tikzcd}
\overline{T_{f^n}} \arrow[r, "\bar{h}"] \arrow[d] & \operatorname{res}_G^{G_n} \overline{T_f} \arrow[d] \\
T_{f^n} \arrow[r, "h"] & \overline{T_f}/G_n
\end{tikzcd}$$

Since h is a homotopy equivalence between base spaces of G_n-coverings, $\bar{h}$ is a G_n-homotopy equivalence. By Theorem 4.2.9 (a), we obtain

$$b_p^D\left(\overline{T_{f^n}}\right) = b_p^D\left(\operatorname{res}_G^{G_n} \overline{T_f}\right) \tag{4.3}$$

for $p \geqslant 0$. Since T_{f^n} has a CW-structure with $\beta_p(X) + \beta_{p-1}(X)$ cells of dimension p and this number is finite by assumption, using Theorem 4.2.9 (c) we conclude:

$$b_p^D\left(\overline{T_f}\right) \overset{(4.2)}{=} \frac{1}{n} \cdot b_p^D\left(\operatorname{res}_G^{G_n} \overline{T_f}\right) \overset{(4.3)}{=} \frac{1}{n} \cdot b_p^D\left(\overline{T_{f^n}}\right) \leqslant \frac{\beta_p(X) + \beta_{p-1}(X)}{n}.$$

Letting $n \to \infty$ finishes the proof of the theorem assuming the claim.

Proof of the claim. For this proof, it is instructive to reinterpret the objects we are dealing with. Recall that $D = \operatorname{Ore}(D'(G/K))$, and hence its elements are twisted rational functions in one variable, say t, with coefficients in D'. Similarly, D_n consists of such rational functions in a single variable t^n, and the embedding $D_n \to D$ is obtained by identifying the variable t^n in the former ring of rational functions with the n^{th} power of t in the latter (as the notation suggests).

Now it becomes clear that $D_n \otimes_{\mathbb{Z}G_n} \mathbb{Z}G$ is generated by elements of the form $pq^{-1} \otimes t^m$ where $m \in \{0, \dots, n-1\}$ and where p, q are twisted polynomials in t^n with $q \neq 0$. Therefore we may view $D_n \otimes_{\mathbb{Z}G_n} \mathbb{Z}G$ as consisting of elements of the form pq^{-1} where q is a non-zero polynomial in t^n, and p is a polynomial in t. Viewed in this way, the map $\Psi\colon D_n \otimes_{\mathbb{Z}G_n} \mathbb{Z}G \to D$ maps identically into D.

We are left to see that Ψ is surjective, which we will achieve by equipping its domain with a ring structure. If we denote the cyclic group G/G_n of order n by $\mathbb{Z}_n$, then $D_n \otimes_{\mathbb{Z}G_n} \mathbb{Z}G$ is identified with the crossed product $D_n\mathbb{Z}_n$ via the map $pq^{-1} \otimes t^m \mapsto pq^{-1} * m$, where $m \in \{0, \dots, n-1\}$ and p, q are twisted polynomials in t^n with $q \neq 0$. We can thus replace the domain of Ψ with $D_n\mathbb{Z}_n$ and note that the resulting map, which we again denote by Ψ, is in fact an injective ring homomorphism. Since $\mathbb{Z}_n$ is a finite group and D_n is a division ring of characteristic 0, the crossed product $D_n\mathbb{Z}_n$ is semisimple by [Lüc02, Lemma 10.55] – note that this is a version of Maschke's theorem for crossed products. Since a semisimple subring of a division ring is a division ring and D is assumed to be generated by $\mathbb{Z}G \subset D_n\mathbb{Z}_n$, we conclude that Ψ is also surjective and hence an isomorphism. □

4.3 Agrarian torsion

Having introduced agrarian Betti numbers together with computational tools allowing us to prove their vanishing for certain spaces, we will now present a secondary invariant for such spaces. This invariant will be called *agrarian torsion* and arises as Reidemeister torsion with values in the abelianized units of the division ring D. It is motivated by the construction of *universal L^2-torsion* by Friedl and Lück [FL17]. We will reference the rather general treatment of torsion by Cohen [Coh73] throughout this section.

As usual, in this section G will always be a group with a fixed agrarian map $\alpha\colon \mathbb{Z}G \to D$.

4.3.1 Non-commutative Reidemeister torsion

In order to define agrarian torsion, we require a contractible D-chain complex. In our case, contractibility is governed by the agrarian Betti numbers because of

Proposition 4.3.1 ([Ros94, Proposition 1.7.4])**.** *Let R be a ring and C_* an R-chain complex. If C_* is acyclic, vanishes in sufficiently small degree and consists of projective R-modules, then C_* is contractible.*

Lemma 4.3.2. *A finite $\mathbb{Z}G$-chain complex C_* is D-acyclic if and only if $D \otimes C_*$ is contractible.*

Proof. Since C_* is finite, the D-chain complex $D \otimes C_*$ is in particular bounded below. All its modules are free because D is a division ring, and hence the statement follows from Proposition 4.3.1. □

Agrarian torsion, being constructed as non-commutative Reidemeister torsion, naturally takes values in the first K-group of D:

Definition 4.3.3. Let R be a ring. Denote by $\mathrm{GL}(R)$ the direct limit of the groups $\mathrm{GL}_n(R)$ of invertible $n \times n$ matrices over R with the embeddings given by adding an identity block in the bottom-right corner. The *K_1-group* $K_1(R)$ is defined as the abelianization of $\mathrm{GL}(R)$. The *reduced K_1-group* $\widetilde{\mathrm{K}}_1(R)$ is defined as the quotient of $K_1(R)$ by the subgroup $\{(\pm 1)\}$.

We now consider a D-acyclic finite free $\mathbb{Z}G$-chain complex (C_*, c_*). Such a complex will be called *based* if it comes with a choice of preferred bases for all chain modules. By the previous lemma, we can find a chain contraction γ_* of $D \otimes C_*$. Set $C_{\mathrm{odd}} := \bigoplus_{i\,\mathrm{odd}} C_i$ and $C_{\mathrm{even}} := \bigoplus_{i\,\mathrm{even}} C_i$. Note that D-acyclicity guarantees that $\dim_D D \otimes C_{\mathrm{odd}} = \dim_D D \otimes C_{\mathrm{even}}$.

Lemma 4.3.4. *In the situation above, the map $c_* + \gamma_* \colon D \otimes C_{\mathrm{odd}} \to D \otimes C_{\mathrm{even}}$ is an isomorphism of finitely generated based free D-modules and the class in $\widetilde{\mathrm{K}}_1(D)$ defined by the matrix representing it in the preferred basis does not depend on the choice of γ.*

Proof. That the map is an isomorphism is the content of [Coh73, (15.1)], the independence is covered by [Coh73, (15.3)]. □

4.3.2 The Dieudonné determinant

The K_1 groups of division rings can be determined using a generalization of the classical determinant of a matrix over a field to matrices over division rings, which is known as the *Dieudonné determinant*. As opposed to the situation for fields, there is no longer a polynomial expression in terms of the entries of the matrix; rather, the Dieudonné determinant is defined by an inductive procedure:

Definition 4.3.5. Let $A = (a_{ij})$ be an $n \times n$ matrix over a division ring D. The *canonical representative of the Dieudonné determinant* $\det^c A \in D$ is defined inductively as follows:

(a) If $n = 1$, then $\det^c A := a_{11}$.

(b) If the last row of A consists of zeros only, then $\det^c A := 0$.

(c) If $a_{nn} \neq 0$, then we form the $(n-1) \times (n-1)$ matrix $A' = (a'_{ij})$ by setting $a'_{ij} := a_{ij} - a_{in} a_{nn}^{-1} a_{nj}$, and declare $\det^c A := \det^c A' \cdot a_{nn}$.

(d) Otherwise, let $j < n$ be maximal such that $a_{nj} \neq 0$. Let A' be obtained from A by interchanging rows j and n. Then set $\det^c A := -\det^c A'$.

The *Dieudonné determinant* $\det A$ of A is defined to be the image of $\det^c A$ in $D^\times/[D^\times, D^\times]$, i.e., in the abelianized unit group of D, if $\det^c A \neq 0$, and is understood to be 0 otherwise. We also write $D^\times_{\mathrm{ab}}$ for the abelianized unit group of D.

As a convention, we will write the group operation of the abelian group $D^\times_{\mathrm{ab}}$ (and its quotients) additively.

If D is a commutative field, then the Dieudonné determinant coincides with the usual determinant as the matrix A is brought into upper-diagonal form during the inductive procedure defining $\det^c A$.

The Dieudonné determinant is multiplicative on all matrices and takes non-zero values on invertible matrices [Die43].

Proposition 4.3.6 ([Ros94, Corollary 2.2.6]). *Let D be a division ring. Then the Dieudonné determinant* $\det\colon \mathrm{GL}(D) \to D^\times_{ab}$ *induces group isomorphisms*

$$\det\colon K_1(D) \xrightarrow{\cong} D^\times_{ab} \text{ and}$$
$$\det\colon \widetilde{\mathrm{K}}_1(D) \xrightarrow{\cong} D^\times_{ab}/\{\pm 1\}.$$

4.3.3 Definition and properties of agrarian torsion

Relying on the explicit description of $\widetilde{\mathrm{K}}_1(D)$ obtained above, we can motivate

Definition 4.3.7. The *D-agrarian torsion* of a D-acyclic finite based free $\mathbb{Z}G$-chain complex (C_*, c_*) is defined as

$$\rho_D(C_*) \coloneqq \det([c_* + \gamma_*]) \in D^\times_{\mathrm{ab}}/\{\pm 1\},$$

where $[c_* + \gamma_*] \in \widetilde{\mathrm{K}}_1(D)$ is the class determined by the (representing matrix of the) isomorphism constructed in Lemma 4.3.4.

The usual additivity property for torsion invariants directly carries over to the agrarian setting in the following form:

Lemma 4.3.8 ([Coh73, (17.2)]). *Let $0 \to C'_* \to C_* \to C''_* \to 0$ be a short exact sequence of finite based free $\mathbb{Z}G$-chain complexes such that the preferred basis of C_* is composed of the preferred basis of C'_* and preimages of the preferred basis elements of C''_*. Assume that any two of the complexes are D-acyclic. Then so is the third and*

$$\rho_D(C_*) = \rho_D(C'_*) + \rho_D(C''_*).$$

The difference in agrarian torsion between $\mathbb{Z}G$-chain homotopy equivalent chain complexes is measured by the Whitehead torsion of the chain homotopy equivalence, analogously to the statement of [FL17, Lemma 2.10] for universal L^2-torsion:

Lemma 4.3.9. *Let $f\colon C_* \to E_*$ be a $\mathbb{Z}G$-chain homotopy equivalence of finite based free $\mathbb{Z}G$-chain complexes. Denote by $\rho(\mathrm{cone}(f_*)) \in \widetilde{\mathrm{K}}_1(\mathbb{Z}G)$ the Whitehead torsion of the contractible finite based free $\mathbb{Z}G$-chain complex* $\mathrm{cone}(f_*)$. *If one of C_* and E_* is D-acyclic, then so is the other and we get*

$$\rho_D(E_*) - \rho_D(C_*) = \det\nolimits_D \Big(\alpha_*\big(\rho(\mathrm{cone}(f_*))\big)\Big),$$

where $\alpha_*\colon \widetilde{\mathrm{K}}_1(\mathbb{Z}G) \to \widetilde{\mathrm{K}}_1(D)$ is induced by $\alpha\colon \mathbb{Z}G \to D$.

Proof. Since f_* is a $\mathbb{Z}G$-chain homotopy equivalence, the finite free $\mathbb{Z}G$-chain complex $\mathrm{cone}(f_*)$ is contractible and hence its Whitehead torsion $\rho(\mathrm{cone}(f_*))$ is defined. The finite free D-chain complex $D \otimes \mathrm{cone}(f_*)$ is again contractible and since the matrix defining its agrarian torsion are already invertible over $\mathbb{Z}G$, we get that $\rho_D(\mathrm{cone}(f_*)) = \det_D(\alpha_*(\rho(\mathrm{cone}(f_*))))$.

We now apply Lemma 4.3.8 to the short exact sequence

$$0 \to E_* \to \mathrm{cone}_*(f_*) \to \Sigma C_* \to 0$$

with $\mathrm{cone}_*(f_*)$ and one of ΣC_* and E_* being D-acyclic. Since $\rho_D(\Sigma C_*) = -\rho_D(C_*)$, as is readily observed from the definition of ρ_D, we obtain that $\rho_D(E_*) - \rho_D(C_*) = \rho_D(\mathrm{cone}(f_*)) = \det_D(\alpha_*(\rho(\mathrm{cone}(f_*))))$. □

Our goal is to apply the concept of D-agrarian torsion to G-CW-complexes. Since the free cellular chain complexes associated to such complexes do not admit a canonical $\mathbb{Z}G$-basis, but only a canonical $\mathbb{Z}$-basis (up to orientation), we have to account for this additional indeterminacy by passing to a further quotient of $D^\times_{\text{ab}}$:

Definition 4.3.10. Let X be a D-acyclic finite free G-CW-complex. The *D-agrarian torsion* of X is defined as

$$\rho_D(X) \coloneqq \rho_D(C_*(X)) \in D^\times_{\text{ab}}/\{\pm g \mid g \in G\},$$

where $C_*(X)$ is endowed with any $\mathbb{Z}G$-basis that projects to a $\mathbb{Z}$-basis of $C_*(X/G)$ consisting of unequivariant cells.

That $\rho_D(X)$ is indeed well-defined can be seen from [Coh73, (15.2)].

4.3.4 Comparison with universal L^2-torsion

A rich source of agrarian groups is the class of torsion-free groups that satisfy the strong Atiyah conjecture over $\mathbb{Q}$. For these groups, there is a canonical division ring $\mathcal{D}(G)$ in which the group ring $\mathbb{Z}G$ embeds. In the case of $D = \mathcal{D}(G)$, agrarian torsion coincides with the determinant of the universal L^2-torsion introduced by Friedl and Lück in [FL17], as we will see now.

Universal L^2-torsion naturally lives in a weak version of the K_1-group of the group ring, which is defined as follows:

Definition 4.3.11 ([FL17, Definition 2.1])**.** Let G be a group. Denote by $\mathrm{K}_1^\omega(\mathbb{Z}G)$ the *weak K_1-group*, which is defined to be an abelian groups with the following generators and relations:

Generators $[A]$ for square matrices A over $\mathbb{Z}G$ that become invertible after the change of rings $\mathbb{Z}G \hookrightarrow \mathcal{D}(G)$

Relations
- $[AB] = [A] + [B]$ for matrices A and B of compatible sizes and such that A and B become invertible over $\mathcal{D}(G)$.
- $[D] = [A] + [C]$ for a block matrix

$$D = \begin{pmatrix} A & B \\ 0 & C \end{pmatrix}$$

with A and C square and invertible over $\mathcal{D}(G)$.

Define the *weak Whitehead group* $\mathrm{Wh}^\omega(G)$ as the quotient of $\mathrm{K}_1^\omega(\mathbb{Z}G)$ by the subgroup generated by the 1×1-matrices $(\pm g)$ for all $g \in G$.

Note that there are canonical maps $K_1(\mathbb{Z}G) \to \mathrm{K}_1^\omega(\mathbb{Z}G)$ and $\mathrm{K}_1^\omega(\mathbb{Z}G) \to K_1(\mathcal{D}(G))$ given by $[A] \mapsto [A]$ and $[A] \mapsto [1 \otimes A]$ on generators, respectively.

The following result by Linnell and Lück indicates that the abelian groups in which agrarian torsion and universal L^2-torsion take values coincide up to isomorphism for a large class of groups:

Theorem 4.3.12 ([LL18])**.** *Let $\mathcal{C}$ be the smallest class of groups which contains all free groups and is closed under directed union and extensions by elementary amenable groups. Let G be a torsion-free group which belongs to $\mathcal{C}$. Then $\mathcal{D}(G)$ is a division ring and the composite map*

$$\mathrm{K}_1^\omega(\mathbb{Z}G) \to K_1(\mathcal{D}(G)) \xrightarrow{\det} \mathcal{D}(G)^\times_{ab}$$

is an isomorphism.

Let X be a finite free G-CW-complex that is L^2*-acyclic*, i.e., whose L^2-Betti numbers vanish. Friedl and Lück [FL17, Definition 3.1] associate to such a G-CW-complex an element $\rho_u^{(2)}(X) \in \mathrm{Wh}^\omega(G)$ called the *universal* L^2*-torsion* of X. We can obtain from this an element

$$\det(\rho_u^{(2)}(X)) \in \mathcal{D}(G)^\times_{\mathrm{ab}}/\{\pm g \mid g \in G\},$$

which by Theorem 4.3.12 carries an equivalent amount of information as $\rho_u^{(2)}$ for many groups G.

The statement of the following theorem is implicit in [FLT19, Section 2.3] by Friedl, Lück and Tillmann.

Theorem 4.3.13. *Let G be a torsion-free group that satisfies the strong Atiyah conjecture over $\mathbb{Q}$. Then G is $\mathcal{D}(G)$-agrarian. Furthermore, if X is any finite free G-CW-complex, then X is $\mathcal{D}(G)$-acyclic if and only if it is L^2-acyclic. If this is the case, we have*

$$\rho_{\mathcal{D}(G)}(X) = \det(\rho_u^{(2)}(X)) \in \mathcal{D}(G)^\times_{ab}/\{\pm g \mid g \in G\}.$$

Proof. During the proof, we will use the notion of universal L^2-torsion for L^2-acyclic finite based free $\mathbb{Z}G$-chain complexes as defined in [FL17, Definition 2.7]. The universal L^2-torsion of a finite free G-CW-complex is then obtained as the universal L^2-torsion of the associated cellular chain complex together with any basis consisting of G-cells. We will also abuse notation in that we consider classes in $\widetilde{\mathrm{K}}_1^\omega(\mathbb{Z}G)$ to be represented by both square matrices over $\mathbb{Z}G$ (our convention) and $\mathbb{Z}G$-endomorphisms of some $\mathbb{Z}G^n$ (the convention in [FL17]).

The first statement is proved analogously to one direction of [Lüc02, Lemma 10.39], the second statement then follows from Lemma 4.3.2 and [FL17, Lemma 2.21].

In order to prove the last statement, we want to make use of the universal property of universal L^2-torsion (see [FL17, Remark 2.16]). To this end, we first consider $\mathbb{Z}G$-chain complexes of the following simple form: Let $[A] \in \widetilde{\mathrm{K}}_1^\omega(\mathbb{Z}G)$ be represented by an $n \times n$ matrix A over $\mathbb{Z}G$, and let C_*^A be the $\mathbb{Z}G$-chain complex concentrated in degrees 0 and 1 with the only non-trivial differential given by $r_A\colon \mathbb{Z}G^n \to \mathbb{Z}G^n, x \mapsto x \cdot A$. Since A becomes an isomorphism over $\mathcal{D}(G)$, such a complex is always $\mathcal{D}(G)$- and thus L^2-acyclic.

The universal L^2-torsion of C_*^A is computed from a weak chain contraction (δ_*, v_*) of C_*^A as defined in [FL17, Definition 2.4]. In this particular case, we can take $\delta_0 = \mathrm{id}_{\mathbb{Z}G^n}, \delta_p = 0$ for $p \neq 0$ and $v_0 = v_1 = r_A, v_p = 0$ for $p \notin \{0, 1\}$. According to [FL17, Definition 2.7], the universal L^2-torsion of C_*^A is thus given by

$$\rho_u^{(2)}(C_*^A) = [v_1 \circ r_A + 0] - [v_1] = [r_A^2] - [r_A] = [A] \in \widetilde{\mathrm{K}}_1^\omega(\mathbb{Z}G)$$

and hence $\det(\rho_u^{(2)}(C_*^A)) = \det A \in \mathcal{D}(G)^\times_{\mathrm{ab}}/\{\pm 1\}$.

The $\mathcal{D}(G)$-agrarian torsion of C_*^A is computed from a (classical) chain contraction of $\mathcal{D}(G) \otimes C_*^A$; let γ_* be such a contraction with $\gamma_0 = (\mathrm{id}_{\mathcal{D}(G)} \otimes r_A)^{-1}$ and $\gamma_p = 0$ for $p \neq 0$. Since γ vanishes in odd degrees, the construction of $\mathcal{D}(G)$-agrarian torsion yields

$$\rho_{\mathcal{D}(G)}(C_*^A) = \det([\mathrm{id}_{\mathcal{D}(G)} \otimes r_A + 0]) = \det A \in \mathcal{D}(G)^\times_{\mathrm{ab}}/\{\pm 1\},$$

and hence $\det(\rho_u^{(2)}(C_*^A)) = \rho_{\mathcal{D}(G)}(C_*^A)$.

The pair $(\mathcal{D}(G)^\times_{\mathrm{ab}}/\{\pm 1\}, \rho_{\mathcal{D}(G)})$ consists of an abelian group and an assignment that associates to a $\mathcal{D}(G)$-acyclic (i.e., L^2-acyclic) finite based free $\mathbb{Z}G$-chain complex an element $\rho_{\mathcal{D}(G)} \in \mathcal{D}(G)^\times_{\mathrm{ab}}/\{\pm 1\}$. The assignment is additive by Lemma 4.3.8 and maps complexes of the shape $\mathbb{Z}G \xrightarrow{\pm \mathrm{id}_{\mathbb{Z}G}} \mathbb{Z}G$ to $1 \in \mathcal{D}(G)^\times_{\mathrm{ab}}/\{\pm 1\}$ by construction. It hence constitutes an example of an *additive* L^2*-torsion invariant* in the sense of [FL17, Remark 2.16]. Since by [FL17, Theorem 2.12] the pair $(\widetilde{\mathrm{K}}_1^\omega(\mathbb{Z}G), \rho_u^{(2)})$ is the universal such invariant, there is a unique group homomorphism $f\colon \widetilde{\mathrm{K}}_1^\omega(\mathbb{Z}G) \to \mathcal{D}(G)^\times_{\mathrm{ab}}/\{\pm 1\}$ satisfying $f \circ \rho_u^{(2)} = \rho_{\mathcal{D}(G)}$.

It is left to check that f and det agree as maps $\widetilde{\mathrm{K}}_1^\omega(\mathbb{Z}G) \to \mathcal{D}(G)^\times_{\mathrm{ab}}/\{\pm 1\}$. We have seen already that $\det(\rho_u^{(2)}(C_*^A)) = \rho_{\mathcal{D}(G)}(C_*^A)$. But $\rho_u^{(2)}(C_*^A) = [A]$, and hence $\{\rho_u^{(2)}(C_*^A) \mid [A] \in \widetilde{\mathrm{K}}_1^\omega(\mathbb{Z}G)\}$ generates $\widetilde{\mathrm{K}}_1^\omega(\mathbb{Z}G)$ as a group. Since f agrees with det on this generating set, we conclude that $f = \det$. □

4.3.5 Agrarian torsion via matrix chains

While the construction of agrarian torsion described so far is well-suited for the comparison to L^2-torsion, a more computational approach based on matrix chains will be more suitable for applications.

We will use concepts and notation from [Tur01, p. I.2.1]. Assume that we are given a D-acyclic finite free $\mathbb{Z}G$-chain complex C_* concentrated in degrees 0 through m, which is equipped with a choice of a preferred basis. By fixing an ordering of the preferred basis, we identify subsets of $\{1, \dots, \operatorname{rk} C_p\}$ with subsets of the preferred basis elements of C_p. We then denote by A_p, for $p = 0, \dots, m-1$, the matrix representing the differential $c_{p+1}\colon C_{p+1} \to C_p$ in the preferred bases. Note the shift in grading between A_p and c_{p+1}, which is needed in order to bring our notation in line with that of Turaev. The matrix A_p consists of the entries $a^p_{jk} \in \mathbb{Z}G$, where $j = 1, \dots, \operatorname{rk} C_{p+1}$ and $k = 1, \dots, \operatorname{rk} C_p$.

Definition 4.3.14. A *matrix chain* for C_* is a collection of sets $\gamma = (\gamma_0, \dots, \gamma_m)$, where $\gamma_p \subseteq \{1, \dots, \operatorname{rk} C_p\}$ and $\gamma_0 = \emptyset$. Write $S_p = S_p(\gamma)$ for the submatrix of A_p formed by the entries a^p_{jk} with $j \in \gamma_{p+1}$ and $k \notin \gamma_p$. A matrix chain γ is called a *τ-chain* if S_p is a square matrix for $p = 0, \dots, m-1$. A τ-chain γ is called *non-degenerate* if $\det_D(S_p) \neq 0$ for all $p = 0, \dots, m-1$.

We want to point out that the reference [Tur01, p. I.2.1] only considers chain complexes over a commutative field $\mathbb{F}$. Nonetheless, all statements and proofs directly carry over to our setting of chain complexes over a division ring D if we throughout replace the commutative determinant $\det_{\mathbb{F}}\colon \mathrm{GL}(\mathbb{F}) \to \mathbb{F}^\times$ with the Dieudonné determinant $\det_D$. In particular, there is still a well-behaved notion of the *rank* of a matrix A over a division ring D, which can be defined in any of the following equivalent ways:

- the largest number r such that A contains an invertible $r \times r$-submatrix;
- the D-dimension of the image of the linear map of left D-vector spaces given by right multiplication by A;
- the D-dimension of the right D-vector space spanned by the columns of A (the *column rank*);
- the D-dimension of the left D-vector space spanned by the rows of A (the *row rank*).

With this convention, the proofs in [Tur01, p. I.2.1] carry over verbatim.

Taken together, [Tur01, Theorem I.2.2 & Remark I.2.7] imply that any non-degenerate τ-chain can be used to compute the agrarian torsion of C_* and such a τ-chain always exists if the complex is D-acyclic. Note though that, compared to our definition of torsion in Definition 4.3.7, Turaev's conventions differ in that he writes torsion multiplicatively instead of additively and uses the inverse of the torsion element in $\widetilde{\mathrm{K}}_1(D)$ we construct, see [Tur01, Theorem I.2.6]. Correcting for these differences by inserting a sign, we obtain

Theorem 4.3.15. *For any non-degenerate τ-chain γ of a D-acyclic finite free $\mathbb{Z}G$-chain complex C_* with a choice of a preferred basis, we have*

$$\rho_D(C_*) = \sum_{p=0}^{m-1} (-1)^p \det_D(S_p(\gamma)) \in D_{ab}^\times/\{\pm 1\}.$$

Furthermore, any D-acyclic finite free $\mathbb{Z}G$-chain complex with a choice of a preferred basis admits a non-degenerate τ-chain.

4.4 Agrarian polytope

Building on the notions of agrarian Betti numbers and agrarian torsion, we are now able to associate to a D-acyclic finite G-CW-complex X a polytope. This polytope, called the *agrarian polytope* of X, arises as the convex hull of the support of the associated agrarian torsion, viewed as a quotient of suitable twisted polynomials. The idea to study the Newton polytope of a torsion invariant goes back to [FL17].

4.4.1 The polytope group

We begin with polytope-specific terminology:

Definition 4.4.1. Let V be a finite-dimensional real vector space. A *polytope* in V is the convex hull of finitely many points in V. For a polytope $P \subset V$ and a linear map $\varphi\colon V \to \mathbb{R}$ we define

$$F_\varphi(P) \coloneqq \{p \in P \mid \varphi(p) = \min_{q \in P} \varphi(q)\}$$

and call this polytope the *φ-face* of P. The elements of the collection

$$\{F_\varphi(P) \mid \varphi\colon V \to \mathbb{R}\}$$

are the *faces* of P. A face is called a *vertex* if it consists of a single point.

In the following, the ambient vector space V will always be $\mathbb{R} \otimes_{\mathbb{Z}} H$ for some finitely generated free abelian group H. For such V, we will consider a special type of polytope:

Definition 4.4.2. A polytope P in V is called *integral* if its vertices lie on the lattice $H \subset V$.

Given two integral polytopes P and Q in V, their pointwise or *Minkowski sum* $P + Q = \{p + q \mid p \in P, q \in Q\}$ is again an integral polytope. Any vertex of the resulting polytope is a pointwise sums of a vertex of P and a vertex of Q. Equipped with the Minkowski sum the set of all integral polytopes in V becomes a cancellative abelian monoid with neutral element $\{0\}$, see [Råd52, Lemma 2]. Hence, the monoid embeds into its Grothendieck group, which was first considered in [FT20, p. 6.3]:

Definition 4.4.3. Let H be a finitely generated free abelian group. Denote by $\mathcal{P}(H)$ the *polytope group* of H, that is the Grothendieck group of the cancellative abelian monoid given by all integral polytopes in $\mathbb{R} \otimes_{\mathbb{Z}} H$ under Minkowski sum. In other words, let $\mathcal{P}(H)$ be the abelian group with generators the formal differences $P - Q$ of integral polytopes and relations $(P - Q) + (P' - Q') = (P + P') - (Q - Q')$ as well as $P - Q = P' - Q'$ if $P + Q' = P' + Q'$. The neutral element is given by the one-point polytope $\{0\}$, which we will drop from the notation. We view H as a subgroup of $\mathcal{P}(H)$ via the map $h \mapsto \{h\}$.

An element of the polytope group that is of the form $P - 0$, for which we also just write P, is called a *single polytope* and is uniquely represented in this form. Any other element is called a *virtual polytope*.

In order to later get well-defined invariants with values in the polytope group, we will mostly be dealing with the following quotient of the full polytope group:

Definition 4.4.4. The *translation-invariant polytope group* of H, denoted by $\mathcal{P}_T(H)$, is defined to be the quotient group $\mathcal{P}(H)/H$.

4.4.2 The polytope homomorphism

As is the case for the L^2-torsion polytope, the following simple construction underpins the definition of the agrarian polytope:

Definition 4.4.5. Let D be a division ring and let H be a finitely generated free abelian group. Let $D * H$ denote some crossed product structure formed out of D and H, fixing in particular its left D-basis that is identified with H. The *Newton polytope* $P(p)$ of an element $p = \sum_{h \in H} u_h \cdot h \in D * H$ is then defined as the convex hull of the *support* $\operatorname{supp}(p) = \{h \in H \mid u_h \neq 0\}$ in $H_1(H; \mathbb{R})$.

Since H is finitely generated free abelian, as in Definition 4.1.4 we can consider the Ore division ring of fractions $\operatorname{Ore}(D * H)$ of the crossed product $D * H$. The previous definition can then be extended to elements of $\operatorname{Ore}(D * H)$ as follows:

Definition 4.4.6. The group homomorphism

$$\begin{aligned} P\colon \operatorname{Ore}(D * H)^{\times}_{\mathrm{ab}} &\to \mathcal{P}(H) \\ pq^{-1} &\mapsto P(p) - P(q) \end{aligned}$$

is called the *polytope homomorphism* of $\operatorname{Ore}(D*H)$ (with respect to a fixed crossed product structure on $D * H$). It induces a homomorphism

$$P\colon \operatorname{Ore}(D * H)^{\times}_{\mathrm{ab}}/\{\pm h \mid h \in H\} \to \mathcal{P}_T(H).$$

The well-definedness of P is immediate from the construction of $\mathcal{P}(H)$. The fact that P is a group homomorphisms is not hard, and has been shown in [Kie20, Lemma 3.12] (see also the discussion following the lemma).

4.4.3 Definition of the agrarian polytope for agrarian maps

With the polytope homomorphism at our disposal, we will now define the agrarian polytope of an appropriate chain complex in two steps: We first restrict our attention to the special case of a rational agrarian map and then show that we obtain a well-defined polytope for all agrarian maps by passing to any rationalization.

The agrarian polytope for rational agrarian maps

We now consider a finitely generated group G and take the free abelian group H to be the free part of the abelianization of G. Furthermore, we denote the canonical projection onto H by pr and its kernel by K.

In [FL17], the Newton polytope is constructed for the Linnell division ring $\mathcal{D}(G)$, which can be expressed as the Ore division ring of fractions of a crossed product $\mathcal{D}(K) * H$. While the target of an arbitrary agrarian map for G is of course not always an Ore division ring of fractions of a suitable crossed product, this will be the case for the *rational* agrarian maps introduced in Definition 4.1.5.

Definition 4.4.7. Let $\alpha\colon \mathbb{Z}G \to D$ be a rational agrarian map for G and consider a D-acyclic finite based free $\mathbb{Z}G$-chain complex C_*. The *D-agrarian polytope* of C_* is defined as

$$P^D(C_*) \coloneqq P(-\rho_D(C_*)) \in \mathcal{P}(H).$$

The purpose of the sign in the definition of the D-agrarian polytope is to get a single polytope in many cases of interest.

We will mostly be interested in the agrarian polytope associated to the cellular chain complex of a G-CW-complex, where we have to account for the indeterminacy caused by the need to choose a basis of cells by considering the resulting polytope only up to translation:

Definition 4.4.8. Let $\alpha\colon \mathbb{Z}G \to D$ be a rational agrarian map for G, i.e., one of the form $\mathbb{Z}G \to \mathbb{Z}K * H \to \operatorname{Ore}(D' * H) = D$ for some division subring D' of D. Consider a D-acylic finite free G-CW-complex X. The *D-agrarian polytope* of X is defined as

$$P^D(X) \coloneqq P(-\rho_D(X)) \in \mathcal{P}_T(H).$$

Proposition 4.4.9. *The D-agrarian polytope $P^D(X)$ is a G-homotopy invariant of X.*

Proof. Let X and X' be D-acyclic finite free G-CW-complexes G-homotopy equivalent via $f\colon X \to X'$. We denote the induced homotopy equivalence between X/G and X'/G by $\overline{f}$. By Lemma 4.3.9, the agrarian torsions of X and X' are related via

$$\rho_D(X') - \rho_D(X) = \det_D(\rho(\overline{f})).$$

After applying the polytope homomorphism, we obtain

$$P^D(X') - P^D(X) = P(\det_D(\rho(\overline{f}))).$$

The latter polytope is a singleton by [Kie20, Corollary 5.16] and hence $P^D(X') = P^D(X) \in \mathcal{P}_T(H)$. □

Because of the previous proposition, the agrarian polytope of the universal covering of the classifying space of a group, which is only well-defined up to G-homotopy equivalence, does not depend on the choice of a particular G-CW-model. We are thus led to

Definition 4.4.10. Assume that G is *of type* $\mathtt{F}$, i.e., let there be an unequivariantly contractible finite free G-CW-complex EG. Let $\alpha\colon \mathbb{Z}G \to D$ be a rational agrarian map for G. We say that G is *D-acyclic* if any such G-CW complex is D-acyclic. If this is the case, we define the *D-agrarian polytope of G* to be

$$P^D(G) \coloneqq P^D(EG).$$

For future reference, we record the following direct consequence of Lemma 4.3.8:

Lemma 4.4.11. *Let $0 \to C'_* \to C_* \to C''_* \to 0$ be a short exact sequence of finite based free $\mathbb{Z}G$-chain complexes such that the preferred basis of C_* is composed of the preferred basis of C'_* and preimages of the preferred basis elements of C''_*. Assume that any two of the complexes are D-acyclic. Then so is the third and*

$$P^D(C_*) = P^D(C'_*) + P^D(C''_*).$$

The agrarian polytope for arbitrary agrarian maps

Let $\alpha\colon \mathbb{Z}G \to D$ be an agrarian map for a finitely generated group G, and denote the free part of the abelianization of G by H. In this situation, we can pass to the rationalization $\mathbb{Z}G \to \mathrm{Ore}(D * H)$ of α as introduced in Definition 4.1.5, which is always rational. Via this replacement, we can extend Definitions 4.4.7, 4.4.8 and 4.4.10 to arbitrary agrarian maps. Three remarks are in order.

First, passing to the rationalization involves a choice of a section of the projection $G \twoheadrightarrow H$. A priori, the polytope may depend on this choice. However, by Lemma 4.1.3, the crossed product structures obtained from any two choices differ by an isomorphism preserving supports and thus give rise to the same agrarian polytope.

Second, note that it is not clear that a chain complex C_* that is D-acyclic is also $\mathrm{Ore}(D * H)$-acyclic (nor vice versa). Hence, in order to compute an agrarian polytope with respect to an arbitrary agrarian map, it is always necessary to check acyclicity with respect to its rationalization. In our application, we will obtain a computation of the agrarian Betti numbers that is uniform across all possible agrarian embeddings, so that this point will not be an issue for us.

Third, we have now introduced two potentially different definitions of the agrarian polytope for an agrarian map $\mathbb{Z}G \to D$ that is already rational: We could calculate the polytope directly with respect to this map or first replace it by its rationalization. As it turns out, these two a priori different approaches lead to the same agrarian polytope. By verifying that our two definitions are compatible, we will as a byproduct establish a comparison with the L^2-torsion polytope.

We will first show that passing to the rationalization of an agrarian map $\alpha\colon \mathbb{Z}G \to D$ that is already rational only changes the agrarian torsion by pushing forward along an inclusion of division rings:

Lemma 4.4.12. *Let G be a finitely generated group and $\alpha\colon \mathbb{Z}G \to D$ a rational agrarian map. Denote the rationalization of α by α_r and its target division ring by D_r. Then α_r factors through α, and hence any finite free G-CW-complex is D-acylic if and only if it is D_r-acyclic. If this is the case, then*

$$\rho_{D_r}(X) = j_*(\rho_D(X)) \in {D_r}^{\times}_{ab}/\{\pm g \mid g \in G\},$$

where $j_*\colon D^{\times}_{ab}/\{\pm g \mid g \in G\} \to {D_r}^{\times}_{ab}/\{\pm g \mid g \in G\}$ is induced by an injective map $j\colon D \hookrightarrow D_r$ between the respective agrarian maps.

Proof. We again write H for the free part of the abelianization of G and K for the kernel of the projection of G onto H. Recall that since α is assumed to be rational, it is of the form

$$\mathbb{Z}G \cong \mathbb{Z}K * H \to D' * H \hookrightarrow \mathrm{Ore}(D' * H) = D$$

for some division subring $D' \leqslant D$. Analogously, the construction of α_r exhibits it as the composition

$$\mathbb{Z}G \cong \mathbb{Z}K * H \to D * H \hookrightarrow \mathrm{Ore}(D * H) = D_r.$$

Here, the structure functions of the twisted group ring $D' * H$ are determined by the images of $h \in H$ under $\alpha' \circ s\colon H \to D'$, where s is the section of the projection $G \to H$ chosen to define $D' * H$. We consider D' as a division subring of D via $d' \mapsto \frac{d' * 1}{1}$, which is a ring homomorphism since $s(1) = 1$. The same choice of a section results in the same basis and structure functions (up to enlargening their codomains to D) being used for the construction of $D * H$, and hence we get an induced inclusion of rings $D' * H \to D * H$ that together with the maps $\mathbb{Z}G \to D' * H$ and $\mathbb{Z}G \to D * H$ forms a commutative triangle. Passing to Ore fields of fractions, we obtain the desired injective map $j\colon D \hookrightarrow D_r$ between the agrarian maps α and α_r.

As discussed in Remark 4.2.6, the agrarian maps α and α_r yield the same agrarian Betti numbers, which proves the acyclicity statement.

We now turn to the statement on agrarian torsion. By [Lüc02, Lemma 10.34 (1)], a division ring (and more generally, a von Neumann regular ring) is *rationally closed* in any overring, i.e., every matrix over the division ring that becomes invertible over the overring is already invertible in the division ring. Applied to our situation, we obtain that the invertible matrices appearing in the construction of ρ_{D_r} following Definition 4.3.7 are already invertible over the division subring D. Since the Dieudonné determinant is by construction natural with respect to inclusions of division rings, the second statement holds.

In our case, one can also replace the use of the lemma by the more direct observation that we may put the matrices appearing in the construction of ρ_{D_r} into an upper-triangular form using elementary matrices over the division ring D since the entries of the matrices lie in $\mathbb{Z}G$, and hence in D. A (square) matrix in upper-triangular form over a division ring is invertible if and only if its diagonal elements are non-zero, in particular invertibility over D_r implies invertibility over D. $\square$

Theorem 4.4.13. *Let G be a finitely generated group and $\alpha\colon \mathbb{Z}G \to D$ a rational agrarian map. Denote the rationalization of α by α_r and its target division ring by D_r. Let X be a D- or D_r-acyclic finite free G-CW complex. Then X is both D- and D_r-acyclic and*

$$P^{D_r}(X) = P^D(X) \in \mathcal{P}_T(H).$$

Proof. By Lemma 4.4.12, the agrarian torsions of X with respect to D and D_r are related via

$$\rho_{D_r}(X) = j_*(\rho_D(X)) \in {D_r}^\times_{\mathrm{ab}}/\{\pm g \mid g \in G\}, \tag{4.4}$$

where j_* is induced by the inclusion $j\colon D \hookrightarrow D_r$. Recall that we defined the agrarian polytope with respect to D_r as $P^{D_r}(X) = P(-\rho_{D_r}(X))$, where we use that D_r is the Ore division ring of fractions of the twisted group ring $D * H$. Analogously, the agrarian polytope with respect to D is defined as $P^D(X) = P(-\rho_D(X))$, where we use that D is the Ore division ring of fractions of a twisted group ring $D' * H$ for some division ring D'. In light of 4.4, we are thus left to check that taking the support over $D' * H$ gives the same result as pushing forward to $D * H$ using j and taking supports there. But j was constructed in Lemma 4.4.12 specifically to preserve the crossed product structures and in particular the support. $\square$

4.4.4 Comparison with the L^2-torsion polytope

We will now apply the results of the previous section to the Linnell division ring $\mathcal{D}(G)$ of a finitely generated torsion-free group G that satisfies the strong Atiyah conjecture. As before, we denote the free part of the abelianization of G by H and the kernel of the projection of G onto H by K. Our aim is to compare the L^2-torsion polytope introduced in [FL17] to the agrarian polytope associated to the agrarian embedding $\mathbb{Z}G \hookrightarrow \mathcal{D}(G)$.

The most important feature of the Linnell division ring $\mathcal{D}(G)$ is that it is the Ore division ring of fractions of a twisted group ring in which $\mathbb{Z}K$ embeds, namely $\mathcal{D}(K) * H$, see [Lüc02, Lemma 10.69]. Translated into the agrarian language, this means that the agrarian embedding $\mathbb{Z}G \hookrightarrow \mathcal{D}(G)$ is rational. We have already observed in Theorem 4.3.13 that universal L^2-torsion and agrarian torsion define the same element in the abelianized units of the Linnell division ring, whenever both are defined. The construction of the L^2-torsion polytope in [FL17, Definition 4.21] is thus equivalent to our definition of the agrarian polytope with respect to the rational agrarian embedding $\mathbb{Z}G \hookrightarrow \mathcal{D}(G)$. By Theorem 4.4.13, the resulting polytope agrees with the agrarian polytope constructed using the rationalization of $\mathbb{Z}G \hookrightarrow \mathcal{D}(G)$, i.e., without using the special structure of the Linnell division ring.

Summarizing our discussion, we have established

Theorem 4.4.14. *Let G be a finitely generated torsion-free group that satisfies the strong Atiyah conjecture over $\mathbb{Q}$ and consider the agrarian embedding $\mathbb{Z}G \hookrightarrow \mathcal{D}(G)$ into the Linnell division ring. Denote by $\mathcal{D}(G)_r$ the target of the rationalization of this embedding. Let X be an L^2-acyclic finite free G-CW complex. Then X is both $\mathcal{D}(G)$- and $\mathcal{D}(G)_r$-acyclic and*

$$P_{L^2}(X) = P^{\mathcal{D}(G)}(X) = P^{\mathcal{D}(G)_r}(X) \in \mathcal{P}_T(H).$$

4.4.5 Thickness of Newton polytopes

The agrarian polytope is usually rather difficult to compute for a concrete group. Its thickness along a given line is often more accessible. With an approach similar to [FL19], we will see in Section 4.5 that it can be computed in terms of agrarian Betti numbers of a suitably restricted chain complex.

Definition 4.4.15. Assume that G is finitely generated and denote the free part of its abelianization by H. Let $\varphi\colon G \to \mathbb{Z}$ be a homomorphism factoring through H as $\overline{\varphi}\colon H \to \mathbb{Z}$. Let $P \in \mathcal{P}(H)$ be a single polytope. The *thickness* of P along φ is given by

$$\mathrm{th}_\varphi(P) := \max\{\overline{\varphi}(x) - \overline{\varphi}(y) \mid x, y \in P\} \in \mathbb{Z}_{\geqslant 0}.$$

Since it respects the Minkowski sum and vanishes on polytopes consisting of a single point, the assignment $P \mapsto \mathrm{th}_\varphi(P)$ extends to a group homomorphism $\mathrm{th}_\varphi\colon \mathcal{P}_T(H) \to \mathbb{Z}$.

An equivalent way of thinking of a twisted group ring $D * H$ constructed from an agrarian map $\mathbb{Z}G \to D$ in the case $H = \mathbb{Z}$ is as a twisted Laurent polynomial ring $D[t^{\pm 1}]$. In order to see the correspondence, note that since $\mathbb{Z}$ is free with one generator, we can choose a section s of the epimorphism $\varphi\colon G \to \mathbb{Z}$ which is itself a homomorphism. By Lemma 4.1.3, the resulting twisted group ring will be independent of the choice of the (group-theoretic or not) section. If we stipulate that $tdt^{-1} = s(1)ds(1)^{-1}$ for $d \in D$, then the ring $D[t^{\pm 1}]_\varphi$, with φ added as an index to indicate the origin of the twisting, will be canonically isomorphic to $D * \mathbb{Z}$.

For elements of the Laurent polynomial ring, the Newton polytope will be a line of length equal to the degree of the polynomial. Here, the *degree* $\deg(x)$ of a non-trivial Laurent polynomial x is the difference of the highest and lowest degree among its monomials. In particular, the degree of a single monomial is always 0 and the degree of a polynomial with non-vanishing constant term coincides with its degree as a Laurent polynomial.

Let now G be a finitely generated agrarian group with agrarian map $\mathbb{Z}G \to D$ and denote by K the kernel of the projection of G onto the free part of its abelianization, which we denote H. Further let $\varphi\colon G \to \mathbb{Z}$ be an epimorphism with kernel K_φ, and denote the induced map $H \to \mathbb{Z}$ by $\overline{\varphi}$ with kernel $K_{\overline{\varphi}}$. Recall that by Lemma 4.1.7, the iterated Ore field $\mathrm{Ore}(\mathrm{Ore}(D * K_{\overline{\varphi}}) * \mathbb{Z})$ can be identified with the Ore field $\mathrm{Ore}(D * H)$ via the isomorphism β. We write $\mathrm{Ore}(D * K_{\overline{\varphi}}) * \mathbb{Z}$ as a twisted Laurent polynomial ring $\mathrm{Ore}(D * K_{\overline{\varphi}})[t^{\pm 1}]_\varphi$. The idea behind the following lemma is now based on the fact that the Newton polytope of a multi-variable Laurent polynomial x determines all the Newton 'lines' of x when viewed as a single-variable Laurent polynomial with more complicated coefficients.

Lemma 4.4.16. *In the situation above, for any $x \in \mathrm{Ore}(D * K_{\overline{\varphi}})[t^{\pm 1}]_\varphi$ with $x \neq 0$, we have*

$$\mathrm{th}_\varphi(P(\beta(x))) = \deg(x).$$

Proof. Since multiplying by a common denominator of all $\mathrm{Ore}(D * K_{\overline{\varphi}})$-coefficients of x does neither change its degree nor the support of its image under β, we can restrict to the case $x \in D * K_{\overline{\varphi}}[t^{\pm 1}]_\varphi$. Thus x will be of the form $x = \sum_{n\in\mathbb{Z}}(\sum_{k\in K_{\overline{\varphi}}} u_{k,n} \cdot k)t^n$ with $u_{k,n} \in D$. Denoting the group-theoretic section of $\overline{\varphi}$ used to construct the twisted Laurent polynomial ring by s, we obtain:

$$\beta(x) = \sum_{\substack{n\in\mathbb{Z}\\ k\in K_{\overline{\varphi}}}} u_{k,n} \cdot ks(n).$$

The elements $ks(n)$ form a basis of the free D-module $D * H$, and thus no cancellation can occur between the individual $u_{k,n}$. By the analogous argument for the twisted group ring $D * K_{\overline{\varphi}}$, cancellation can also be ruled out for the sum $\sum_{k\in K_{\overline{\varphi}}} u_{k,n} \cdot k$ for each $n \in \mathbb{Z}$. We conclude:

$$\begin{aligned}
\mathrm{th}_\varphi(P(\beta(x))) &= \max\{\overline{\varphi}(k_1 s(n_1)) - \overline{\varphi}(k_2 s(n_2)) \mid k_1, k_2 \in K_{\overline{\varphi}}, n_1, n_2 \in \mathbb{Z}, u_{k_i,n_i} \neq 0\}\\
&= \max\{n_1 - n_2 \mid k_1, k_2 \in K_{\overline{\varphi}}, n_1, n_2 \in \mathbb{Z}, u_{k_i,n_i} \neq 0\}\\
&= \max\{n_1 - n_2 \mid \exists k_i \in K_{\overline{\varphi}} \colon u_{k_i,n_i} \neq 0 \text{ for } i = 1, 2\}\\
&= \max\{n_1 - n_2 \mid \sum_{k_i\in K_{\overline{\varphi}}} u_{k_i,n_i} \cdot k_i \neq 0 \text{ for } i = 1, 2\}\\
&= \deg(x).
\end{aligned}$$

□

4.5 Twisted agrarian Euler characteristic

While the shape of the agrarian polytope introduced in the previous section is often hard to determine, there is a convenient equivalent description of its thickness along a given line. To this end, we will introduce the agrarian analogue of the twisted L^2-Euler characteristic introduced by Friedl and Lück in [FL19]. We assume that G is a finitely generated D-agrarian group with a fixed agrarian map $\alpha\colon \mathbb{Z}G \to D$. We use H to denote the free part of the abelianization of G, and let K be the kernel of the canonical projection of G onto H.

4.5.1 Definition of the twisted agrarian Euler characteristic

We now introduce twisted agrarian Euler characteristics, which arise as ordinary agrarian Euler characteristics of cellular $\mathbb{Z}G$-chain complexes twisted by an epimorphism from G to the integers:

Definition 4.5.1. Let X be a finite free G-CW-complex and let $\varphi\colon G \to \mathbb{Z}$ be a homomorphism. We denote by $\varphi^*\mathbb{Z}[t^{\pm 1}]$ the $\mathbb{Z}G$-module obtained from the $\mathbb{Z}$-module $\mathbb{Z}[t^{\pm 1}]$ by letting G act as $g \cdot t^n = t^{n+\varphi(g)}$. Consider the $\mathbb{Z}G$-chain complex $C_*(X) \otimes_\mathbb{Z} \varphi^*\mathbb{Z}[t^{\pm 1}]$ equipped with the diagonal G-action and set

$$\begin{aligned}
b_p^D(X;\varphi) &:= b_p^D(C_*(X) \otimes_\mathbb{Z} \varphi^*\mathbb{Z}[t^{\pm 1}]) \in \mathbb{N} \cup \{\infty\},\\
h^D(X;\varphi) &:= \sum_{p\geqslant 0} b_p^D(X;\varphi) \in \mathbb{N} \cup \{\infty\},\\
\chi^D(X;\varphi) &:= \sum_{p\geqslant 0} (-1)^p b_p^D(X;\varphi) \in \mathbb{Z}, \text{ if } h^D(X;\varphi) < \infty.
\end{aligned}$$

We say that X is *φ-D-finite* if $h^D(X;\varphi) < \infty$, and in this case $\chi^D(X;\varphi)$ is called the *φ-twisted D-agrarian Euler characteristic* of X. More generally, we will also consider the φ-twisted agrarian Euler characteristic $\chi^D(C_*;\varphi)$ for any finite free $\mathbb{Z}G$-chain complex C_*, with C_* taking the role of the cellular chain complex $C_*(X)$.

The aim of this section is to prove that the thickness of the agrarian polytope in a prescribed direction can be computed as a twisted agrarian Euler characteristic. Recall that G is a finitely generated group with a fixed agrarian map $\alpha\colon \mathbb{Z}G \to D$ and that we denote by $\alpha_r\colon \mathbb{Z}G \to D_r$ the rationalization of α as introduced in Definition 4.1.5.

Theorem 4.5.2. *Let X be a D_r-acyclic finite free G-CW-complex and $\varphi\colon G \to \mathbb{Z}$ a homomorphism. Then*

$$\operatorname{th}_\varphi(P^{D_r}(X)) = -\chi^{D_r}(X;\varphi).$$

For universal L^2-torsion, the analogous statement has been proved by Friedl and Lück in [FL17, Remark 4.30]. Their proof is based on the fact that universal L^2-torsion is the universal abelian invariant of L^2-acyclic finite based free $\mathbb{Z}G$-chain complexes C_* that is additive on short exact sequences and satisfies a certain normalisation condition. While large parts of the verification of this universal property are purely formal, in the proof of [FL17, Lemma 1.5] it is used that the combinatorial Laplace operator on C_* induces the L^2-Laplace operator on $\mathcal{N}(G) \otimes C_*$, which has no analogue over a general division ring D. We instead establish Theorem 4.5.2 using the matrix chain approach to the computation of Reidemeister torsion explained in [Tur01, I.2.1].

4.5.2 Reduction to ordinary Euler characteristics

Before we get to the proof, we will transfer some of the helpful lemmata in [FL19, Sections 2.2 & 3.3] to the agrarian setting.

The following lemma allows us to restrict our attention to surjective twists $\varphi\colon G \to \mathbb{Z}$ in the proof of Theorem 4.5.2:

Lemma 4.5.3. *Let X be a finite free G-CW-complex and let $\varphi\colon G \to \mathbb{Z}$ be a group homomorphism.*

(a) *For any integer $k \geqslant 1$ we have that X is $(k\cdot\varphi)$-D-finite if and only if X is φ-D-finite, and if this is the case we get*

$$\chi^D(X; k\cdot\varphi) = k \cdot \chi^D(X;\varphi).$$

(b) *Denote the trivial homomorphism $G \to \mathbb{Z}$ by c_0. The complex X is c_0-D-finite if and only if X is D-acylic, and if this is the case we get*

$$\chi^D(X; c_0) = 0.$$

Proof. (a) This follows from the direct sum decomposition $(k\cdot\varphi)^*\mathbb{Z}[t^{\pm1}] \cong \bigoplus_{i=1}^k \varphi^*\mathbb{Z}[t^{\pm1}]$ and additivity of Betti numbers.

(b) This is a direct consequence of $C_*(X) \otimes_{\mathbb{Z}} c_0^*\mathbb{Z}[t^{\pm1}] \cong \bigoplus_{\mathbb{Z}} C_*(X)$ and additivity of Betti numbers. ☐

We will now see that twisted D-agrarian Euler characteristics over G can equivalently be viewed as ordinary D-agrarian Euler characteristics over the kernel of the twist homomorphism.

Lemma 4.5.4. *Let X be a finite free G-CW-complex and let $\varphi\colon G \to \mathbb{Z}$ be an epimorphism. Denote the kernel of φ by K_φ. Then X is φ-D-finite if and only if $\sum_{p\geqslant 0} b_p^D(\operatorname{res}_G^{K_\varphi} X) < \infty$, and in this case we have*

$$\chi^D(X;\varphi) = \chi^D(\operatorname{res}_G^{K_\varphi} X).$$

Proof. The proof is based on the following isomorphism of $\mathbb{Z}G$-chain complexes:

$$\begin{aligned}\mathbb{Z}G \otimes_{\mathbb{Z}K_\varphi} \operatorname{res}_G^{K_\varphi} C_*(X) &\xrightarrow{\cong} C_*(X) \otimes_{\mathbb{Z}} \varphi^*\mathbb{Z}[t^{\pm 1}] \\ g \otimes x &\longrightarrow gx \otimes t^{\varphi(g)},\end{aligned}$$

the inverse of which is given by $y \otimes t^q \mapsto g \otimes g^{-1}y$ for any choice of $g \in \varphi^{-1}(q)$. Using the isomorphism, we obtain for every $p \geqslant 0$:

$$\begin{aligned}H_p(D \otimes C_*(X) \otimes_{\mathbb{Z}} \varphi^*\mathbb{Z}[t^{\pm 1}]) &\cong H_p(D \otimes \mathbb{Z}G \otimes_{\mathbb{Z}K_\varphi} \operatorname{res}_G^{K_\varphi} C_*(X)) \\ &= H_p(D \otimes_{\mathbb{Z}K_\varphi} \operatorname{res}_G^{K_\varphi} C_*(X)).\end{aligned}$$

We conclude that $b_p^D(X;\varphi) = b_p^D(\operatorname{res}_G^{K_\varphi} X)$ by applying $\dim_D$, which yields the claim after taking the alternating sum over $p \geqslant 0$. □

Remark 4.5.5. Let G be a group of type $\mathtt{F}$ with an agrarian map $\alpha\colon \mathbb{Z}G \to D$. Let $\varphi\colon G \to \mathbb{Z}$ be an epimorphism with kernel K_φ. If K_φ is also of type $\mathtt{F}$, then by Lemma 4.5.4 and Theorem 4.2.9 (b)

$$\chi^D(EG;\varphi) = \chi^D(\operatorname{res}_G^{K_\varphi} EG) = \chi^D(EK_\varphi) = \chi(K_\varphi).$$

In particular, in this case the value of $\chi^D(EG;\varphi)$ does not depend on the choice of agrarian map.

Lemma 4.5.6. *Let C_* be a D-acyclic $\mathbb{Z}G$-chain complex of finite type. Let $\varphi\colon G \to \mathbb{Z}$ be an epimorphism with kernel K_φ. Consider $D[t^{\pm 1}]_\varphi$ as a $\mathbb{Z}G$-module via the map $\mathbb{Z}G \cong (\mathbb{Z}K_\varphi)\mathbb{Z} \to D\mathbb{Z} = D[t^{\pm 1}]_\varphi$ constructed in Lemma 4.1.3 for $K := K_\varphi$, where we use that $G/K \cong \mathbb{Z}$ via φ. Then*

$$b_n^D(\operatorname{res}_G^{K_\varphi} C_*) = \dim_D H_n(D[t^{\pm 1}]_\varphi \otimes C_*) < \infty.$$

In particular, the $D[t^{\pm 1}]_\varphi$-modules $H_n(D[t^{\pm 1}]_\varphi \otimes C_*)$ are torsion.

Proof. The proof is analogous to that of [FL19, Theorem 3.8 (4)] with D taking the role of $\mathcal{D}(K)$. The assumption that C_* be projective is in fact not used in the proof of the theorem and hence is not part of the statement of Lemma 4.5.6. □

Corollary 4.5.7. *Let X be a D-acyclic finite free G-CW-complex. Let $\varphi\colon G \to \mathbb{Z}$ be an epimorphism with kernel K_φ. Then X is φ-D-finite and*

$$\chi^D(X;\varphi) = \sum_{p\geqslant 0} (-1)^p \dim_D H_p(D[t^{\pm 1}]_\varphi \otimes C_*(X)).$$

Proof. Apply Lemmata 4.5.4 and 4.5.6. □

4.5.3 Thickness of the agrarian polytope

We are now able to give the proof of Theorem 4.5.2:

Proof of Theorem 4.5.2. We will actually prove the more general statement that for every D_r-acyclic finite based free $\mathbb{Z}G$-chain complex C_* concentrated in degrees 0 through m

$$\operatorname{th}_\varphi\left(P(-\rho_{D_r}(C_*))\right) = -\chi^{D_r}(C_*;\varphi). \tag{4.5}$$

Since $\operatorname{th}_\varphi$ and P are homomorphisms, we can drop the signs from both sides. Using Lemma 4.5.3, we can further assume that φ is an epimorphism.

By Theorem 4.3.15, we find a non-degenerate τ-chain γ such that

$$\operatorname{th}_\varphi\Big(P\big(\rho_{D_r}(C_*)\big)\Big) = \operatorname{th}_\varphi\Big(P\Big(\sum_{p=0}^{m}(-1)^p \det_{D_r}\big(S_p(\gamma)\big)\Big)\Big).$$

Crucially,

$$\operatorname{Ore}(\operatorname{Ore}(D * K_{\overline{\varphi}})[t^{\pm 1}]_\varphi) \cong \operatorname{Ore}(D * H) = D_r$$

via the isomorphism β constructed in Lemma 4.1.7, where $K_{\overline{\varphi}}$ is the kernel of the epimorphism $\overline{\varphi}\colon H \to \mathbb{Z}$ induced by φ. The subring

$$\operatorname{Ore}(D * K_{\overline{\varphi}})[t^{\pm 1}]_\varphi$$

of the left-hand side, which contains $\beta^{-1}(\mathbb{Z}G)$ and thus all entries of $S_p = S_p(\gamma)$, is a (non-commutative) Euclidean domain. This means that we can diagonalize the matrices S_p by multiplying them from the left and right with permutation matrices and elementary matrices over this twisted Laurent polynomial ring. This diagonalization procedure occurs as part of an algorithm that brings a matrix into Jacobson normal form, which is a non-commutative analogue of the better-known Smith normal form for matrices over commutative PIDs. For details, we refer to the proof of [Jac43, Theorem 3.10]. Recall that a *permutation matrix* is a matrix obtained from an identity matrix by permuting rows and columns. An *elementary matrix* over a ring R is a matrix differing from the identity matrix in a single off-diagonal entry. The determinant of either type of matrix is 1 or -1, and thus the thickness in direction of φ of their polytopes vanish. Hence, $\operatorname{th}_\varphi(P(\det(S_p))) = \operatorname{th}_\varphi(P(\det(T_p)))$ for the diagonal matrix T_p obtained from S_p in this way. We denote the diagonal entries of T_p by $\lambda_{p,i} \in \operatorname{Ore}(D * K_{\overline{\varphi}})[t^{\pm 1}]_\varphi$ for $i = 1, \ldots, |\gamma_p|$ and note that all the entries $\lambda_{p,i}$ are non-zero since all matrices S_p become invertible over D_r. Using that both $\operatorname{th}_\varphi$ and P are homomorphisms, and applying Lemma 4.4.16 once more, we compute:

$$\begin{aligned}
\operatorname{th}_\varphi\big(P(\rho_{D_r}(C_*))\big) &= \operatorname{th}_\varphi\Big(P\Big(\sum_{p=1}^{m}(-1)^p \det_{D_r}\big(S_p(\gamma)\big)\Big)\Big) \\
&= \sum_{p=0}^{m-1}(-1)^p \sum_{i=1}^{|\gamma_p|} \operatorname{th}_\varphi(P(\beta(\lambda_{p,i}))) \\
&= \sum_{p=0}^{m-1}(-1)^p \sum_{i=1}^{|\gamma_p|} \deg(\lambda_{p,i}).
\end{aligned}$$

We will now consider the right-hand side of (4.5). For this, we use that the agrarian map $\mathbb{Z}K_\varphi \to D_r = \operatorname{Ore}(\operatorname{Ore}(D * K_{\overline{\varphi}}) * \mathbb{Z})$ factors through the agrarian map $\mathbb{Z}K_\varphi \to \operatorname{Ore}(D * K_{\overline{\varphi}})$, and thus the map $\mathbb{Z}G \cong (\mathbb{Z}K_\varphi)\mathbb{Z} \to D_r[t^{\pm 1}]_\varphi$ introduced in Lemma 4.5.6 factors through $\mathbb{Z}G \cong (\mathbb{Z}K_\varphi)\mathbb{Z} \to \operatorname{Ore}(D * K_{\overline{\varphi}})[t^{\pm 1}]_\varphi$. Since D_r is flat over the division ring $\operatorname{Ore}(D * K_{\overline{\varphi}})$, we conclude from Corollary 4.5.7 that

$$\begin{aligned}
\chi^{D_r}(C_*; \varphi) &= \sum_{p=0}^{m}(-1)^p \dim_{D_r} H_p(D_r[t^{\pm 1}]_\varphi \otimes C_*) \\
&= \sum_{p=0}^{m}(-1)^p \dim_{\operatorname{Ore}(D * K_{\overline{\varphi}})} H_p(\operatorname{Ore}(D * K_{\overline{\varphi}})[t^{\pm 1}]_\varphi \otimes C_*).
\end{aligned}$$

Since C_* is D_r-acyclic, we have $H_m(D_r \otimes C_*) = 0$. But C_{m+1} is trivial, which means that the differential c_m must be injective. In particular, the summand corresponding to $p = m$ vanishes.

In order to establish (4.5), we are now left to prove that

$$\sum_{i=1}^{|\gamma_p|} \deg(\lambda_{p,i}) = \dim_{\mathrm{Ore}(D*K_{\overline{\varphi}})} H_p(\mathrm{Ore}(D * K_{\overline{\varphi}})[t^{\pm 1}]_\varphi \otimes C_*) \tag{4.6}$$

holds for $p = 0, \dots, m-1$. In order to not overload notation, we abbreviate $\mathrm{Ore}(D * K_{\overline{\varphi}})[t^{\pm 1}]_\varphi$ as R. Recall that the homology modules $H_p(R \otimes C_*)$ consist solely of R-torsion elements by Lemma 4.5.6. Furthermore, since $R \otimes C_{p-1}$ is a free R-module, any R-torsion maps into it trivially. We are thus able to express the homology modules as torsion submodules of a cokernel in the following way:

$$\begin{aligned} H_p(R \otimes C_*) &= \ker(\mathrm{id}_R \otimes c_p)/\operatorname{im}(\mathrm{id}_R \otimes c_{p+1}) \\ &\cong \ker\big(\mathrm{id}_R \otimes c_p \colon (R \otimes C_p)/\operatorname{im}(\mathrm{id}_R \otimes c_{p+1}) \to R \otimes C_{p-1}\big) \\ &= \mathrm{tors}_R((R \otimes C_p)/\operatorname{im}(\mathrm{id}_R \otimes c_{p+1})) \\ &= \mathrm{tors}_R(\operatorname{coker}(\mathrm{id}_R \otimes c_{p+1})). \end{aligned}$$

Instead of performing elementary operations on the matrix S_p to obtain the diagonal matrix T_p, we can instead apply them to the entire matrix A_p representing $\mathrm{id}_R \otimes c_{p+1}$. This procedure will not change the isomorphism type of the cokernel of the map given by right multiplication with this matrix. Applying further elementary operations over R, we can achieve that all the entries not contained in S_p consist only of zeros with the submatrix S_p now being of the form T_p. This is possible since S_p has the same rank as A_p over the division ring of fractions of R by the same rank counting argument used to prove [Tur01, p. I.2.2]. Hence

$$H_p(R \otimes C_*) \cong \mathrm{tors}_R(\operatorname{coker}(\mathrm{id}_R \otimes c_{p+1})) \cong \oplus_{i=1}^{|\gamma_p|} R/(\lambda_{p,i}),$$

which yields (4.6) after applying $\dim_{\mathrm{Ore}(D*K_{\overline{\varphi}})}$. □

4.6 Application to two-generator one-relator groups

Now that we have successfully set up the framework of agrarian invariants, we will compute the invariants for two-generator one-relator groups. This will allow us to give intrinsic and presentation-independent definitions of invariants first considered by Friedl and Tillmann in [FT20] as well as to remove amenability assumptions from the results obtained in their work.

4.6.1 The Bieri–Neumann–Strebel invariants and HNN extensions

With the agrarian polytope and the twisted agrarian Euler characteristic, we have developed useful tools to study (kernels of) homomorphisms of groups to free abelian groups of finite rank. Before we apply these tools to two-generator one-relator groups, we need to introduce the group-theoretic invariants and constructions based on such homomorphisms.

Definition 4.6.1. Let G be a group generated by a finite subset S, and let X denote the Cayley graph of G with respect to S. Recall that the vertex set of X coincides with G. We define the *Bieri–Neumann–Strebel* (or *BNS*) *invariants* $\Sigma^1(G)$ to be the subset of $H^1(G;\mathbb{R}) \smallsetminus \{0\}$ consisting of the non-trivial homomorphisms (the *characters*) $\varphi \colon G \to \mathbb{R}$ for which the full subgraph of X spanned by $\varphi^{-1}([0,\infty)) \subseteq G$ is connected.

The BNS invariants were introduced by Bieri, Neumann and Strebel in [BNS87] via a different, but equivalent definition. It is an easy exercise to see that $\Sigma^1(G)$ is independent of the choice of the finite generating set S.

We now aim to give an interpretation of lying in the BNS invariant for integral characters $\varphi \colon G \to \mathbb{Z}$. To do so, we need to introduce the notion of HNN extensions.

Definition 4.6.2. Let A be a group and let $\alpha\colon B \xrightarrow{\cong} C$ be an isomorphism between two subgroups of A. Choose a presentation $\langle S \mid R\rangle$ of A and let t be a new symbol not in S. Then the group $A*_\alpha$ defined by the presentation

$$\langle S, t \mid R, tbt^{-1} = \alpha(b)\ \forall b \in B\rangle$$

is called the *HNN extension of A relative to $\alpha\colon B \xrightarrow{\cong} C$*. We call A the *base group* and B the *associated group* of the HNN extension.

The HNN extension is called *ascending* if $B = A$.

The homomorphism $\varphi\colon A*_\alpha \to \mathbb{Z}$ given by $\varphi(t) = 1$ and $\varphi(s) = 0$ for every $s \in S$ is the *induced character*.

Proposition 4.6.3 ([BNS87, Proposition 4.3]). *Let G be a finitely generated group, and let $\varphi\colon G \to \mathbb{Z}$ be a non-trivial character. We have $\varphi \in \Sigma^1(G)$ if and only if G is isomorphic to an ascending HNN extension with finitely generated base group and induced character φ.*

Definition 4.6.4. Suppose that G is finitely generated. Let P be a single polytope in the $\mathbb{R}$-vector space $H_1(G;\mathbb{R})$, and let F be a face of P. A *dual* of F is a connected component of the subspace

$$\{\varphi \in H^1(G;\mathbb{R}) \setminus \{0\} \mid F_\varphi(P) = F\}.$$

A *marked polytope* is a pair (P, m), where P is a single polytope in $H_1(G;\mathbb{R})$, and m is a *marking*, that is a function $m\colon H^1(G;\mathbb{R}) \to \{0,1\}$, which is constant on duals of faces of F, and such that $m^{-1}(1)$ is open.

The pair (P, m) is a *polytope with marked vertices* if $m^{-1}(1)$ is a union of some duals of vertices of P.

The marking m will usually be implicit, and the characters φ with $m(\varphi) = 1$ will be called *marked*.

In [FT20], Friedl–Tillmann use a different notion of a marking of a polytope, which corresponds to a polytope with marked vertices in our terminology where the marking m is additionally required to be constant on all duals of a given vertex. Thus, our notion is more general, and the two notions differ when the polytope in question is a singleton in a 1-dimensional ambient space: with our definition of marking, such a polytope admits four distinct markings (just as every compact interval of non-zero length does), whereas with the Friedl–Tillmann definition such a polytope admits only two markings in which either every character is marked or none is.

4.6.2 The agrarian invariants of two-generator one-relator groups

Definition 4.6.5. A $(2,1)$-*presentation* is a group presentation of the form $\langle x, y \mid r\rangle$, i.e., with two generators and a single relator. A group that admits a $(2,1)$-presentation is called a *two-generator one-relator group.*

The story of the usefulness of agrarian invariants for two-generator one-relator groups begins with the following result:

Theorem 4.6.6 ([LL78, Theorem 1]). *Torsion-free one-relator groups are agrarian.*

In the following, for a group presentation π, we will denote the groups it presents by G_π.

In order to describe the cellular chain complex of the universal coverings of classifying spaces for two-generator one-relator groups, we will use Fox derivatives, which were originally defined in [Fox53]. Let F be a free group on generators $x_i, i \in I$. The *Fox derivative* *with respect to* x_i is then defined to be the unique $\mathbb{Z}$-linear map $\frac{\partial}{\partial x_i}\colon \mathbb{Z}F \to \mathbb{Z}F$ satisfying the conditions

$$\frac{\partial 1}{\partial x_i} = 0, \frac{\partial x_i}{\partial x_j} = \delta_{ij} \text{ and } \frac{\partial uv}{\partial x_i} = \frac{\partial u}{\partial x_i} + u\frac{\partial v}{\partial x_i}$$

for all $u, w \in F$, where δ_{ij} denotes the Kronecker delta. The fundamental formula for Fox derivatives [Fox53, (2.3)] states that for every $u \in \mathbb{Z}F$ we have

$$u - 1 = \sum_{i \in I} \frac{\partial u}{\partial x_i} \cdot (x_i - 1).$$

In the particular case of a two-generator one-relator group $G = \langle x, y \mid r \rangle$, the fundamental formula applied to r implies that the following identity holds in $\mathbb{Z}G$, since there $r - 1 = 0$:

$$\frac{\partial r}{\partial x} \cdot (x - 1) = -\frac{\partial r}{\partial y} \cdot (y - 1). \tag{4.7}$$

We will need the following non-triviality result for Fox derivatives in two-generator one-relator groups:

Lemma 4.6.7. *Let* $\pi = \langle x, y \mid r \rangle$ *be a* $(2,1)$*-presentation with cyclically reduced relator* r*, and take* z *to denote either* x *or* y*. Denote the number of times* z *or* z^{-1} *appears in the word* r *by* s*. Then the Fox derivative* $\partial r/\partial z \in \mathbb{Z}F_2$ *is a sum of the form* $\sum_{j=1}^{s} \pm w_j$ *for words* w_j *representing mutually distinct elements* $g_j \in G_\pi$*. In particular,* $\partial r/\partial z \neq 0$ *in* $\mathbb{Z}G_\pi$ *if* $s > 0$*.*

Proof. This follows from [FT20, Corollary 3.4]. While the statement of the corollary only asserts the distinctness of the group elements g_j together with their scalar factors of ± 1, the proof actually shows that the elements themselves are distinct. Also note that, in the proof of the corollary, n_s is actually always strictly smaller than l, which is crucial for the correctness of the penultimate sentence. □

We are now able to show that the agrarian torsion of torsion-free two-generator one-relator groups is defined and can be calculated explicitly:

Lemma 4.6.8. *Let* $\pi = \langle x, y \mid r \rangle$ *be a* $(2,1)$*-presentation with* r *cyclically reduced. Denote the universal covering of the presentation* 2*-complex of* G_π *associated to this presentation by* EG_π*. Then* EG_π *is contractible and* D*-acyclic with respect to any agrarian embedding* $\mathbb{Z}G_\pi \hookrightarrow D$*. If* x *or* x^{-1} *appears as a letter in* r*, then*

$$\rho_D(EG_\pi) = -\left[\frac{\partial r}{\partial x}\right] + [y - 1] \in D^\times_{ab},$$

where $[-]\colon D^\times \to D^\times_{ab}$ *is the canonical quotient map. If* y *or* y^{-1} *appears in* r*, then the analogous statement holds with the roles of* x *and* y *interchanged.*

Proof. That EG_π is contractible follows from [LS01, Chapter III, Proposition 11.1]. The cellular $\mathbb{Z}G_\pi$-chain complex of EG_π takes the following form in terms of the Fox derivatives $\frac{\partial r}{\partial x}$ and $\frac{\partial r}{\partial y}$, see [Fox53]:

$$\mathbb{Z}G_\pi \xrightarrow{\begin{pmatrix} \frac{\partial r}{\partial x} & \frac{\partial r}{\partial y} \end{pmatrix}} \mathbb{Z}G_\pi^2 \xrightarrow{\begin{pmatrix} x - 1 \\ y - 1 \end{pmatrix}} \mathbb{Z}G_\pi.$$

We will now construct a non-degenerate τ-chain for the associated D-chain complex and simultaneously obtain that the complex is acyclic. Note that acyclicity is also a general consequence of the existence of a non-degenerate τ-chain by [Tur01, Lemma I.2.5].

Since r is assumed to be cyclically reduced, the only case in which any of the generators is trivial in $\mathbb{Z}G_\pi$ is when r consists of a single letter. Let us suppose for now that this is the case, and without loss of generality let us take $r = x$. In this case, the chain complex under investigation becomes

$$\mathbb{Z}G_\pi \xrightarrow{\begin{pmatrix}1 & 0\end{pmatrix}} \mathbb{Z}G_\pi^2 \xrightarrow{\begin{pmatrix}0 \\ y-1\end{pmatrix}} \mathbb{Z}G_\pi.$$

Since $y \neq 1$ as $G = \langle y \rangle$, we immediately see that the complex is D-acyclic and comes with an obvious choice of a non-degenerate τ-chain.

We will now assume that both generators represent non-trivial elements of G_π. By Lemma 4.6.7, the Fox derivative $\frac{\partial r}{\partial x}$ resp. $\frac{\partial r}{\partial y}$ represents the trivial element of $\mathbb{Z}G_\pi$ and hence of D only if x resp. y does not appear in the word r, possibly inverted. Since G_π is not the free group on two generators, at least one of the letters x and y appears in this way, and hence at least one of the Fox derivatives represents an invertible element in D.

In conclusion, both differentials in $D \otimes C_*(EG_\pi)$ have maximal rank, namely 1, and so the complex is acyclic, since it is a complex of modules over a division ring.

We obtain a non-degenerate τ-chain by choosing the submatrices S_1 and S_0 to correspond to a non-trivial Fox derivative and the generator which is not the one with respect to which that Fox derivative was taken, respectively. With this choice, the formula for the agrarian torsion is obtained from Theorem 4.3.15. □

By the work of Waldhausen [Wal78, Theorem 17.5 & Theorem 19.4], two presentation complexes associated to two $(2,1)$-presentations of isomorphic torsion-free two-generator one-relator groups are always simple homotopy equivalent. Since agrarian Betti numbers are homotopy invariant and agrarian torsion is a simple homotopy invariant by Lemma 4.3.9, Lemma 4.6.8 actually implies that EG is D-acyclic for every torsion-free two-generator one-relator group G and its agrarian torsion can be calculated from any $(2,1)$-presentation $\langle x, y \mid r\rangle$ with r cyclically reduced.

Since the agrarian polytope is homotopy invariant by Proposition 4.4.9, we obtain the following result even without appealing to the work of Waldhausen:

Proposition 4.6.9. *Let G be a torsion-free two-generator one-relator group that is not isomorphic to the free group on two generators, and let $\mathbb{Z}G \hookrightarrow D$ be an agrarian embedding. Denote the free part of the abelianization of G by H. If $\pi = \langle x, y \mid r\rangle$ is any $(2,1)$-presentation of G such that r is cyclically reduced and x or x^{-1} appears as a letter in r, we have*

$$P^{D_r}(G) = P^{D_r}(EG_\pi) = P([\partial r/\partial x]) - P([y-1]) \in \mathcal{P}_T(H).$$

If y or y^{-1} appears in r, then the analogous statement holds with the roles of x and y interchanged.

Since the space EG is unique up to G-homotopy equivalent, the polytope $P^{D_r}(G)$ is an invariant of the group G and does not depend on the choice of a $(2,1)$-presentation.

In [FT20], Friedl and Tillmann associate a polytope to *nice* $(2,1)$-presentations, which are defined as follows:

Definition 4.6.10. A $(2,1)$-presentation $\pi = \langle x, y \mid r\rangle$ giving rise to a group G_π is called *nice* if

(a) r is a non-empty word,

(b) r is cyclically reduced and

(c) $b_1(G_\pi) = 2$.

Their construction of the polytope is equivalent to the following definition by [FT20, Proposition 3.5]:

Definition 4.6.11. Let $\pi = \langle x, y \mid r\rangle$ be a nice $(2,1)$-presentation giving rise to a group G_π. Denote by H the free part of the abelianization of G and write $\overline{w}$ for the image of an element $w \in \mathbb{Z}G$ under the projection to $\mathbb{Z}H$. Then we set

$$\mathcal{P}_\pi := P\left(\overline{\frac{\partial r}{\partial x}}\right) - P\big(\overline{y-1}\big) = P\left(\overline{\frac{\partial r}{\partial y}}\right) - P\big(\overline{x-1}\big) \in \mathcal{P}_T(H).$$

It is shown in [FT20, Proposition 3.5] that the element $\mathcal{P}_\pi \in \mathcal{P}_T(H)$ defined in this way is indeed a single polytope.

For a nice $(2,1)$-presentation π, Friedl and Tillmann also endow $\mathcal{P}_\pi$ with a marking of vertices, turning it into a marked polytope $\mathcal{M}_\pi$. A vertex of $\mathcal{P}_\pi$ is declared marked if any of its duals contains a character lying in $\Sigma^1(G)$. Friedl–Tillmann prove in [FT20, Theorem 1.1] that every character lying in any dual of a marked vertex lies in $\Sigma^1(G)$, and hence the markings of $\mathcal{P}_\pi$ and $\Sigma^1(G)$ determine one another.

If $\pi = \langle x, y \mid r\rangle$ and $\pi' = \langle x, y \mid r'\rangle$ are two $(2,1)$-presentations such that there exists an automorphism $f\colon \langle x, y\rangle \to \langle x, y\rangle$ of the free group on two generators satisfying $f(r) = r'$, then the two presentations clearly define isomorphic groups. The automorphism f induces an isomorphism $\overline{f}\colon H_\pi \to H_{\pi'}$ between the free parts of the abelianizations of G_π and $G_{\pi'}$.

Proposition 4.6.12. *Let $\pi = \langle x, y \mid r\rangle$ and $\pi' = \langle x, y \mid r'\rangle$ be two nice $(2,1)$-presentations. Assume that there exists an automorphism $f\colon \langle x, y\rangle \to \langle x, y\rangle$ with $f(r) = r'$. Then*

$$\mathcal{P}_{\pi'} = \mathcal{P}_T(\overline{f})(\mathcal{P}_\pi) \in \mathcal{P}_T(H_{\pi'}).$$

Proof. The automorphism group of a finitely generated free group is generated by the elementary Nielsen transformations, which in the case of two generators x and y consist of the following operations:

- Interchange x and y: $f_1(x) = y, f_1(y) = x$.
- Replace x with x^{-1}: $f_2(x) = x^{-1}, f_2(y) = y$.
- Replace x with xy: $f_3(x) = xy, f_3(y) = y$.

Since the statement of the proposition is functorial in f, we are thus left to show that $\mathcal{P}_{\pi'} = \mathcal{P}_T(\overline{f})(\mathcal{P}_\pi)$ holds whenever f is one of f_1, f_2 and f_3.

The chain rule for Fox derivatives [Fox53, (2.6)] applied to f takes the following form:

$$\frac{\partial}{\partial x} f(r) = f\Big(\frac{\partial}{\partial x} r\Big) \cdot \frac{\partial}{\partial x} f(x) + f\Big(\frac{\partial}{\partial y} r\Big) \cdot \frac{\partial}{\partial x} f(y).$$

For the three elementary Nielsen transformations, we obtain

$$\begin{aligned}
\frac{\partial}{\partial x} f_1(r) &= f_1\Big(\frac{\partial}{\partial x} r\Big) \cdot 0 &&+ f_1\Big(\frac{\partial}{\partial y} r\Big) \cdot 1 = f_1\Big(\frac{\partial}{\partial y} r\Big)\\
\frac{\partial}{\partial x} f_2(r) &= f_2\Big(\frac{\partial}{\partial x} r\Big) \cdot (-x^{-1}) &&+ f_2\Big(\frac{\partial}{\partial y} r\Big) \cdot 0 = f_2\Big(\frac{\partial}{\partial x} r\Big) \cdot (-x^{-1})\\
\frac{\partial}{\partial x} f_3(r) &= f_3\Big(\frac{\partial}{\partial x} r\Big) \cdot 1 &&+ f_3\Big(\frac{\partial}{\partial y} r\Big) \cdot 0 = f_3\Big(\frac{\partial}{\partial x} r\Big).
\end{aligned}$$

When $f = f_2$ or $f = f_3$, we read off that $\partial r'/\partial x$ and $f(\partial r/\partial x)$ differ only by a factor of the form $\pm g$ for some $g \in G_{\pi'}$. It follows that $P(\partial r'/\partial x)$ and $\mathcal{P}(\overline{f})(P(\partial r/\partial x))$ agree up to translation and hence define the same class in $\mathcal{P}_T(H_{\pi'})$. Since $f(y) = y$ in these cases, the same holds true for the polytopes $\mathcal{P}_\pi$ and $\mathcal{P}_{\pi'}$.

For f_1, we obtain using (4.7) that

$$\begin{aligned}
\mathcal{P}_{\pi'} &= P\Big(\frac{\partial}{\partial x} f_1(r)\Big) - P(y-1) = \mathcal{P}_T(\overline{f_1})\Big(P\Big(\frac{\partial}{\partial y} r\Big)\Big) - \mathcal{P}_T(\overline{f_1})\Big(P(x-1)\Big) \\
&= \mathcal{P}_T(\overline{f_1})\Big(P\Big((\frac{\partial}{\partial y} r)(y-1)\Big) - P\Big(y-1\Big) - P\Big(x-1\Big)\Big) \\
&= \mathcal{P}_T(\overline{f_1})\Big(P\Big((\frac{\partial}{\partial x} r)(x-1)\Big) - P\Big(x-1\Big) - P\Big(y-1\Big)\Big) \\
&= \mathcal{P}_T(\overline{f_1})\Big(P\Big(\frac{\partial}{\partial x} r\Big) - P\Big(y-1\Big)\Big) = \mathcal{P}_T(\overline{f_1})(\mathcal{P}_\pi),
\end{aligned}$$

which concludes the proof also in this case. □

There are $(2,1)$-presentations $\pi = \langle x, y \mid r \rangle$ and $\pi' = \langle x, y \mid r' \rangle$ giving rise to isomorphic groups, such that no isomorphism lifts to an automorphism of $\langle x, y \rangle$ mapping r to r'. The first examples of such pairs of presentations appeared in [MP73], one of which is $\langle x, y \mid x^2y^{-2}x^2y^{-3} \rangle \cong \langle x, y \mid x^2y^{-5} \rangle$. This raises the question whether the (marked) polytopes associated to π and π' are still related. A possible answer to this question has been formulated as a conjecture by Friedl and Tillmann:

Conjecture 4.6.13 ([FT20, Conjecture 1.2]). *If G is a group admitting a nice $(2,1)$-presentation π, then $\mathcal{M}_\pi \subset H_1(G;\mathbb{R})$ is an invariant of G (up to translation).*

In more formal terms, the conjecture asserts that if $f\colon G_\pi \to G_{\pi'}$ is an isomorphism of two groups associated to $(2,1)$-presentations π and π', then $\mathcal{P}_{\pi'} = \mathcal{P}_T(\overline{f})(\mathcal{P}_\pi) \in \mathcal{P}_T(H_\pi)$, where $\overline{f}\colon H_\pi \to H_{\pi'}$ is the isomorphism of the free parts of the abelianizations of G_π and G'_π induced by f.

As evidence for their conjecture, Friedl and Tillmann prove:

Theorem 4.6.14 ([FT20, Theorem 1.3]). *If G is a torsion-free group admitting a nice $(2,1)$-presentation π and G is residually {torsion-free elementary amenable}, then $\mathcal{M}_\pi \subset H_1(G;\mathbb{R})$ is an invariant of G (up to translation).*

They further remark that the polytope does not change (up to translation) when the relator is permuted cyclically.

Making use of their construction of universal L^2-torsion, Friedl and Lück resolved this conjecture and provided a construction of $\mathcal{M}_\pi$ intrinsic to the group G under the additional assumption that G is torsion-free and satisfies the Atiyah conjecture:

Theorem 4.6.15 ([FL17, Remark 5.5]). *If G is a torsion-free group admitting a nice $(2,1)$-presentation π and G satisfies the strong Atiyah conjecture over $\mathbb{Q}$, then $\mathcal{M}_\pi \subset H_1(G;\mathbb{R})$ is an invariant of G (up to translation). Moreover, $\mathcal{P}_\pi = P_{L^2}(G)$.*

By using agrarian torsion instead of universal L^2-torsion, we are able to remove the additional assumptions on G, thereby resolving Conjecture 4.6.13:

Theorem 4.6.16. *If G is a group admitting a nice $(2,1)$-presentation π, then $\mathcal{M}_\pi \subset H_1(G;\mathbb{R})$ is an invariant of G (up to translation). Moreover, if G is torsion-free then $\mathcal{P}_\pi = P^{D_r}(G) \in \mathcal{P}_T(\mathbb{Z}^2)$ for any choice of an agrarian embedding $\mathbb{Z}G \hookrightarrow D$.*

Proof. We start by looking at the case of G containing torsion. The solution to this case was found and communicated by Alan Logan.

First note that in this case, the BNS invariant $\Sigma^1(G)$ is empty – this follows immediately from Brown's algorithm [Bro87], or equivalently, from the construction of the marking of $\mathcal{M}_\pi$. An alternative way to see this is to observe that the first L^2-Betti number of G is not zero, see [DL07].

Since $\Sigma^1(G) = \emptyset$, we need only worry about $\mathcal{P}_\pi$. If one alters the presentation π by applying an automorphism f of the free group $F_2 = \langle x, y\rangle$ to the relator r, the polytope remains invariant in the sense of Conjecture 4.6.13 by Proposition 4.6.12. But it was shown by Pride [Pri77] that when G contains torsion, every two two-generator one-relator presentations of G are related by an automorphism of F_2, up to possibly replacing the relator r in one of the presentations by r^{-1}. This last operation does not alter the class of the polytope since, as a consequence of the product rule for Fox derivatives, we get $\partial r^{-1}/\partial x = -r^{-1}\partial r/\partial x$, and thus the polytopes associated to $\partial r^{-1}/\partial x$ and $\partial r/\partial x$ agree up to translation.

Now suppose that G is torsion-free. Then the equality $\mathcal{P}_\pi = P^{D_r}(G)$ follows directly from the definitions of $\mathcal{P}_\pi$ and $P^{D_r}(G)$ by the computation done in Proposition 4.6.9, and the agrarian polytope is an invariant of the group by construction. We conclude from [FT20, Theorem 1.1] that once $\mathcal{P}_\pi$ is known to be an invariant of G, the same is true for the marked version $\mathcal{M}_\pi$ since marked vertices are determined by the BNS invariant $\Sigma^1(G)$ of the group G. □

As a consequence of the equality $\mathcal{P}_\pi = P^{D_r}(G_\pi)$ for a $(2,1)$-presentation π giving rise to a torsion-free group we conclude that $P^{D_r}(G_\pi)$ is actually independent of the choice of agrarian embedding.

Simple $(2,1)$-presentations

Friedl and Tillmann claim in [FT20, Proposition 8.1] and the subsequent two paragraphs that they can associate a single polytope $\mathcal{P}_\pi$ to any $(2,1)$-presentation $\pi = \langle x, y \mid r\rangle$ where r is non-trivial and cyclically reduced, even without assuming the presentation to be nice. If $b_1(G_\pi) = 1$, x represents a generator of the free part of the abelianization of G_π and y represents the trivial element therein, they call such a presentation *simple*. For a simple presentation π, the polytope $\mathcal{P}_\pi$ is computed by the formula involving the Fox derivative of r with respect to x from Definition 4.6.11, and therefore agrees with $P^{D_r}(G_\pi)$ if G_π is torsion-free.

The statement and proof of [FT20, Proposition 8.1] are not fully correct, as the following example shows:

Example 4.6.17. Consider the simple $(2,1)$-presentation $\pi = \langle x, y \mid y^2\rangle$. Then the associated polytope $\mathcal{P}_\pi$ is only a virtual polytope, more specifically the additive inverse of the class of a unit interval in $\mathcal{P}_T(\mathbb{Z}) = \mathcal{P}_T(\langle \bar{x}\rangle)$.

In the proof of [FT20, Proposition 8.1], the assumption that the relator r is either of the form $x^{m_1}y^{n_1}\cdots x^{m_k}y^{m_k}$ or $y^{n_1}x^{m_1}\cdots y^{m_k}x^{m_k}$ for *non-zero* integers $m_1, n_1, \ldots, m_k, n_k$ is incorrect; in our example $k = 1$, $m_1 = 0$ and $n_1 = 2$.

In order to fix the statement and the proof of the proposition, it is necessary to consider the case of group presentations $\langle x, y \mid y^n\rangle$, $n \in \mathbb{Z}, n \neq 0$ separately. These presentations are the only simple ones for which any of the m_i is zero. In this case, the polytope $P(\overline{\partial r/\partial x})$ is an interval of length $D = 0$, which means that $\mathcal{P}_\pi$ is the additive inverse of a unit interval in $\mathcal{P}_T(\mathbb{Z})$.

With this additional case considered, we now observe that the correct result of [FT20, Proposition 8.1] should be that $\mathcal{P}_\pi$ is a single polytope for a simple $(2,1)$-presentation π if and only if G_π is not isomorphic to $\mathbb{Z} * \mathbb{Z}/n\mathbb{Z}$ for any $n \in \mathbb{Z}$. The polytope $\mathcal{P}_\pi$ can be turned into a marked polytope $\mathcal{M}_\pi$ in the Friedl–Tillmann sense if and only if G_π is neither isomorphic to $\mathbb{Z} * \mathbb{Z}/n\mathbb{Z}$ nor to $B(\pm 1, n) := \langle x, y \mid xy^{\pm 1}x^{-1}y^{-n}\rangle$ for $n \in \mathbb{Z}$.

The problem with the Baumslag–Solitar groups $B(\pm 1, n)$ is that the resulting polytope is a singleton lying in a 1-dimensional $\mathbb{R}$-vector space. Since $\Sigma^1(B(\pm 1, n))$ is non-trivial and proper in $H^1(B(\pm 1, n); \mathbb{R})$, there is no marking of $\mathcal{P}_\pi$ in the Friedl–Tillmann sense which would correctly control the BNS invariant. Our notion of marking of vertices of a polytope circumvents this problem, and allows for a definition of $\mathcal{M}_\pi$ also for these groups by marking one of the duals of the only face and not marking the other.

The groups $\mathbb{Z} * \mathbb{Z}/n\mathbb{Z}$ arising from the presentations $\langle x, y \mid y^n \rangle$ all admit a virtual polytope which is the additive inverse of the unit interval in $\mathcal{P}_T(\mathbb{Z}\langle \bar{x} \rangle)$. The notion of a marked polytope readily extends to additive inverses of single polytopes by describing a marking for the single polytope. Since $\mathbb{Z} * \mathbb{Z}/n\mathbb{Z}$ is an ascending HNN extension along any of the two possible epimorphisms to $\mathbb{Z}$ if $n = \pm 1$ and contains torsion otherwise, the polytope will have all duals of its only face marked if $n = \pm 1$ and not marked if $n \neq \pm 1$.

4.6.3 Polytope thickness and splitting complexity

We continue with the notation of the previous section. Our aim now is to show that the thickness of $\mathcal{P}_\pi$ controls the minimal complexity of certain expressions of G as an HNN extension over a finitely generated group. Before we state the precise connection, we need to introduce the following concept:

Definition 4.6.18 ([FLT19, Section 5.1])**.** Let Γ be a finitely presented group and let $\varphi \colon \Gamma \to \mathbb{Z}$ be an epimorphism. A *splitting of* (Γ, φ) is a presentation of Γ as an HNN extension with induced character φ and finitely generated base and associated groups.

It is proved in [BS78, Theorem A] that any pair (Γ, φ) admits a splitting. Hence we can define the *splitting complexity of* (Γ, φ) as

$$c(\Gamma, \varphi) \coloneqq \min\{\mathrm{rk}(B) \mid (\Gamma, \varphi) \text{ splits with associated group } B\},$$

where $\mathrm{rk}(B)$ denotes the minimal number of generators of B. We also define the *free splitting complexity of* (Γ, φ) as

$$c_f(\Gamma, \varphi) \coloneqq \min\{\mathrm{rk}(F) \mid (\Gamma, \varphi) \text{ splits with associated free group } F\},$$

which may be infinite. We always have $c(\Gamma, \varphi) \leqslant c_f(\Gamma, \varphi)$.

Friedl and Tillmann observed the following connection between the thickness of $\mathcal{P}_\pi$ and the (free) splitting complexity of G:

Theorem 4.6.19 ([FT20, Theorem 7.3])**.** *Let G be a residually {torsion-free elementary amenable} group admitting a nice $(2,1)$-presentation π. Then for any epimorphism $\varphi \colon G \to \mathbb{Z}$ we have*

$$c(G, \varphi) = c_f(G, \varphi) = \mathrm{th}_\varphi(\mathcal{P}_\pi) + 1.$$

Note that every residually {torsion-free elementary amenable} group must itself be torsion-free. Friedl, Lück, and Tillmann then noted in [FLT19, Theorem 5.2] that the original proof could be adapted to the setting of [FL19], thereby giving the same formula for groups satisfying the Atiyah conjecture.

We will now extend these results to general torsion-free two-generator one-relator groups. For this, we require the following strengthened form of a proposition of Harvey, which is evident from the last sentence of its original proof:

Proposition 4.6.20 ([Har05, Proposition 9.1])**.** *Let D be a division ring and $D[t^{\pm 1}]$ a twisted Laurent polynomial ring with coefficients in D. Let $M = A + tB$ where A and B are two $l \times m$ matrices over D. Then the map $r_M \colon D[t^{\pm 1}]^l \to D[t^{\pm 1}]^m$ given by right multiplication by M satisfies*

$$\dim_D \mathrm{tors}(\mathrm{coker}(r_M)) \leqslant \mathrm{rk}_D B.$$

We are now in a position to improve upon both [FT20, Theorem 7.3] and [FLT19, Theorem 5.2] by recasting the proof of [FT20, Theorem 7.3] in the agrarian world. In the statement of the following theorem, the agrarian polytope $P^{D_r}(G)$ can be replaced by $\mathcal{P}_\pi$ for any nice or simple $(2,1)$-presentation π of G in the sense of [FT20, Section 8.1].

Theorem 4.6.21. *Let G be a torsion-free two-generator one-relator group other than the free group on two generators. Then for every epimorphism $\varphi\colon G \to \mathbb{Z}$ we have*

$$c(G,\varphi) = c_f(G,\varphi) = \mathrm{th}_\varphi(P^{D_r}(G)) + 1.$$

Proof. The inequality $c_f(G,\varphi) \leqslant \mathrm{th}_\varphi(\mathcal{P}_\pi) + 1$ is proved in [FT20, Proposition 7.4] for all nice $(2,1)$-presentations. The proof of [FT20, Lemma 7.7] also applies to any simple $(2,1)$-presentation $\langle x, y \mid r\rangle$ for which r is not a word in just one of the generators and its inverse, since then the numbers m_1 and n_1 appearing in the proof are non-zero. Any other simple $(2,1)$-presentation π is, up to renaming the generators, of the form $\langle x, y \mid x^n\rangle$ for $n \in \mathbb{N}, n \neq 0$, and there are only two different epimorphisms $G_\pi \to \mathbb{Z}$. It is then easy to see right from the definitions that the splitting complexity and thickness of $\mathcal{P}_\pi$ with respect to any of the two epimorphisms are given by 0 and -1, respectively.

Since every torsion-free two-generator one-relator group G that is not the free group on two generators admits either a nice or a simple presentation π and $P^{D_r}(G) = \mathcal{P}_\pi$ by Proposition 4.6.9, we are left to show that $c(G,\varphi) \geqslant \mathrm{th}_\varphi(P^{D_r}(G)) + 1$. By Theorem 4.5.2, this is further reduced to the following statement about the φ-twisted D_r-agrarian Euler characteristic of G:

$$c(G,\varphi) - 1 \geqslant -\chi^{D_r}(G;\varphi).$$

Recall from the proof of Lemma 4.6.8 that the Cayley 2-complex X associated to a $(2,1)$-presentation of G serves as a model of EG and that the application of Theorem 4.5.2 is justified since we constructed a non-degenerate τ-chain. By Lemmata 4.5.4 and 4.5.6, we can thus compute $\chi^{D_r}(G;\varphi)$ from the Betti numbers of the complex $D_r[t^{\pm1}]_\varphi \otimes C_*(X)$:

$$D_r[t^{\pm1}]_\varphi \xrightarrow{\begin{pmatrix}\frac{\partial r}{\partial x} & \frac{\partial r}{\partial y}\end{pmatrix}} D_r[t^{\pm1}]_\varphi^2 \xrightarrow{\begin{pmatrix} x-1 \\ y-1\end{pmatrix}} D_r[t^{\pm1}]_\varphi.$$

Since $D_r[t^{\pm1}]_\varphi$ is a (non-commutative) principal ideal domain, the kernel of the differential originating from degree 2 is free. It is also seen to be torsion by Lemma 4.5.6 and hence $\dim_{D_r} H_p(D_r[t^{\pm1}]_\varphi \otimes C_*(X)) = 0$ for $p \geqslant 2$.

We let $c = c(G,\varphi)$ and choose a splitting

$$G = \langle A, t \mid \mu(B) = tBt^{-1}\rangle$$

of (G,φ) with associated group B generated by $x_1, \ldots, x_c$; in particular $A \subseteq \ker(\varphi)$ is finitely generated. We pick a presentation $A = \langle g_1, \ldots, g_k \mid r_1, r_2, \ldots\rangle$, which is possible since G and thus A are countable. Denote the number of relations in this presentation by $l \in \mathbb{Z}_{\geqslant 0} \cup \{\infty\}$. The splitting of (G,φ) then gives the following alternative presentation of G:

$$G = \langle g_1, \ldots, g_k, t \mid r_1, r_2, \ldots, \mu(x_1)^{-1}tx_1t^{-1}, \ldots, \mu(x_c)^{-1}tx_ct^{-1}\rangle.$$

Note that the words r_i, x_j and $\mu(x_j)$ are words in the generators g_i of A. Denote by Y the Cayley 2-complex associated to this presentation. By construction, $\pi_1(Y/G) = \pi_1(X/G)$, and thus Y can be turned into a model for EG by attaching G-cells in dimension 3 and higher only. Hence, its homology with arbitrary coefficients agrees with that of X up to dimension 1, which in particular implies that $\dim_{D_r} H_p(D_r[t^{\pm1}]_\varphi \otimes C_*(X)) = \dim_{D_r} H_p(D_r[t^{\pm1}]_\varphi \otimes C_*(Y))$ for $p = 0, 1$.

In conclusion, we will know $\chi^{D_r}(G;\varphi)$ if we compute the first two $D_r[t^{\pm 1}]_\varphi$-Betti numbers of the G-CW-complex Y. For this, we need to consider its shape in more detail. The complex Y is a two-dimensional free G-CW-complex with one zero-cell, $k+1$ one-cells and $l+c$ two-cells, and its cellular chain complex takes the form

$$\cdots \to 0 \to \mathbb{Z}G^{l+c} \xrightarrow{\begin{pmatrix} M_0 & M_1 \end{pmatrix}} \mathbb{Z}G \oplus \mathbb{Z}G^k \xrightarrow{\begin{pmatrix} v_0 \\ v_1 \end{pmatrix}} \mathbb{Z}G,$$

where the (potentially infinite) block matrix $M = \begin{pmatrix} M_0 & M_1 \end{pmatrix}$ representing the second differential consists of the Fox derivatives of the relations with respect to t and the g_i, respectively, and $v_0 = t-1, v_1 = (g_1 - 1, \ldots, g_k - 1)^t$. Since the relations $r_1, r_2, \ldots$ are words in $\mathbb{Z}A$, their Fox derivatives with respect to t are trivial and their derivatives with respect to each g_i again lie in $\mathbb{Z}A$. For the other relations, we obtain

$$\begin{aligned}
&\frac{\partial}{\partial t}(\mu(x_j)^{-1} t x_j t^{-1}) = \mu(x_j)^{-1} - \mu(x_j)^{-1} t x_j t^{-1} \in \mathbb{Z}A \text{ and}\\
&\frac{\partial}{\partial g_i}(\mu(x_j)^{-1} t x_j t^{-1}) = \frac{\partial}{\partial g_i}(\mu(x_j)^{-1}) + \mu(x_j)^{-1} t \frac{\partial}{\partial g_i} x_j \in \mathbb{Z}A + t \cdot \mathbb{Z}A.
\end{aligned}$$

Hence, the matrix M is of the shape

$$\begin{array}{r} l\left\{ \begin{array}{c} \\ \\ \\ \end{array} \right. \\ c\left\{ \begin{array}{c} \\ \\ \\ \end{array} \right. \end{array}
\begin{pmatrix}
0 & \overbrace{\phantom{\in \mathbb{Z}A + t \cdot \mathbb{Z}A}}^{k} \\
\vdots & \in \mathbb{Z}A \\
0 & \\
\in \mathbb{Z}A & \\
\vdots & \in \mathbb{Z}A + t \cdot \mathbb{Z}A \\
\in \mathbb{Z}A &
\end{pmatrix}$$

with the block M_0 consisting of the first column of M. Now consider the following chain map of $D_r[t^{\pm 1}]_\varphi$-chain complexes, where the vertical maps are given by projections and both complexes continue trivially to the left and right:

$$\begin{tikzcd}
D_r[t^{\pm 1}]_\varphi^{l+c} \arrow[r, "\begin{pmatrix} M_0 & M_1 \end{pmatrix}"] \arrow[d] & D_r[t^{\pm 1}]_\varphi \oplus D_r[t^{\pm 1}]_\varphi^k \arrow[r, "\begin{pmatrix} v_0 \\ v_1 \end{pmatrix}"] \arrow[d] & D_r[t^{\pm 1}]_\varphi \arrow[d] \\
0 \arrow[r] & D_r[t^{\pm 1}]_\varphi^k / (D_r[t^{\pm 1}]_\varphi^{l+c} M_1) \arrow[r, "\begin{pmatrix} v_1 \end{pmatrix}"] & D_r[t^{\pm 1}]_\varphi / (t-1)
\end{tikzcd}$$

Since multiplication from the right with $t-1$ is injective on $D_r[t^{\pm 1}]_\varphi$, the chain map induces an isomorphism on homology in degrees 0 and 1. Since all the homology modules $H_i(D_r[t^{\pm 1}]_\varphi \otimes C_*(Y))$ are torsion by Lemma 4.5.6, the same holds true for the homology of the lower chain complex. Using Proposition 4.6.20, we thus get the bound

$$\dim_{D_r} D_r[t^{\pm 1}]_\varphi^k / (D_r[t^{\pm 1}]_\varphi^{l+c} M_1) = \dim_{D_r} \operatorname{tors}(\operatorname{coker}(r_{M_1})) \leqslant c.$$

As $\deg(t-1) = 1$, we also get

$$\dim_{D_r} D_r[t^{\pm 1}]_\varphi / (t-1) = 1.$$

In particular, the lower chain complex consists of finite D_r-vector spaces. Applying the rank-nullity theorem to its only non-trivial differential, we obtain

$$\begin{aligned}
\operatorname{th}_\varphi(\mathcal{P}_\pi) &= -\chi^{D_r}(X;\varphi) \\
&= \dim_{D_r} H_1(D_r[t^{\pm1}]_\varphi \otimes C_*(Y)) - \dim_{D_r} H_0(D_r[t^{\pm1}]_\varphi \otimes C_*(Y)) \\
&= \dim_{D_r} D_r[t^{\pm1}]^k_\varphi/(D_r[t^{\pm1}]^{l+c}_\varphi M_1) - \dim_{D_r} D_r[t^{\pm1}]_\varphi/(t-1) \\
&\leqslant c-1. \qquad \square
\end{aligned}$$

We end with an example of a two-generator one-relator group that is not residually solvable and thus not covered by the results of [FT20]. Its $(2,1)$-presentation is a nice version of the non-residually finite two-generator one-relator group constructed by Baumslag in [Bau69].

Example 4.6.22. For words $x, y \in \langle a, b\rangle$, we define $x^y := y^{-1}xy$ and $[x,y] := x^{-1}y^{-1}xy$. Consider the two-generator one-relator group G defined by

$$\left\langle a, b \;\middle|\; [a,b] = \Big[[a,b],[a,b]^b\Big] \right\rangle,$$

which can be presented in cyclically reduced form as

$$\pi := \left\langle a, b \;\middle|\; a^{-1}bab^{-1}a^{-1}bab^{-2}a^{-1}baba^{-1}b^{-2}ab \right\rangle.$$

We see directly from the first presentation of G that the relator becomes trivial in the abelianization, hence $b_1(G) = 2$ and π is a nice $(2,1)$-presentation. By [LS01, Proposition II.5.18], the group G is also torsion-free since the single relator is not a proper power.

We claim that G is not residually solvable, i.e., not every element maps non-trivially into a solvable quotient of G. Since the element $[a,b]$ can be written as an arbitrarily deeply nested iterated commutator (using the relation of the first presentation above), it is contained in all derived subgroups of G and hence of every quotient. But if a quotient is solvable, some derived subgroup and hence the image of $[a,b]$ will be trivial. It is thus left to show that $[a,b]$ is non-trivial in G. Assume that $[a,b] = 1$ in G. Then G is abelian and hence also $[b,a] = b^{-1}a^{-1}ba = 1$ in G. But $[b,a]$ appears as a proper subword of the relator in π and thus represents a non-trivial element by [LS01, Proposition II.5.29].

We conclude that a method such as the one employed in [FT20, Lemma 6.1] cannot be used to deduce that G is residually {torsion-free elementary amenable} and hence satisfies the assumptions of Theorem 4.6.14. We deem it plausible that G is even not residually {torsion-free elementary amenable} and is thus not covered by Theorem 4.6.14, but to the best of the authors' knowledge no two-generator one-relator group has been shown to have this property.

If we denote the single relator of π by r, an easy but tedious computation shows that

$$\begin{aligned}
\frac{\partial r}{\partial a} = &-\overbrace{b^{-1}a^{-1}}^{(-1,-1)} + \overbrace{b^{-1}a^{-1}b}^{(-1,0)} - \overbrace{b^{-1}a^{-1}bab^{-1}a^{-1}}^{(-1,-1)} + \overbrace{b^{-1}a^{-1}bab^{-1}a^{-1}b}^{(-1,0)} \\
&- \overbrace{(b^{-1}a^{-1}ba)^2b^{-2}a^{-1}}^{(-1,-2)} + \overbrace{(b^{-1}a^{-1}ba)^2b^{-2}a^{-1}b}^{(-1,-1)} - \overbrace{(b^{-1}a^{-1}ba)^2b^{-2}a^{-1}baba^{-1}}^{(-1,0)} \\
&+ \overbrace{(b^{-1}a^{-1}ba)^2b^{-2}a^{-1}baba^{-1}b^{-2}}^{(-1,-2)},
\end{aligned}$$

with the image in the abelianization of each summand noted in brackets. The convex hull of these points in $\mathbb{R}^2$ corresponds to an interval of length 2 in the b-direction, hence

$\mathcal{P}_\pi = P^{D_r}(G)$ is an interval of length 1 in the b-direction. The marked polytope $\mathcal{M}_\pi$ has no markings since all abelianized monomials appear multiple times.

Let $\varphi_b \colon G \to \mathbb{Z}$ be the homomorphism sending a to 0 and b to 1. Since $\mathrm{th}_{\varphi_b}(P^{D_r}(G)) = 1$, we conclude from Theorem 4.6.21 that $c_f(G, \varphi_b) = c(G, \varphi_b) = 2$. A (free) splitting of G along φ_b of minimal rank is thus given by

$$\begin{aligned} G &= \left\langle a, b, x, y \;\middle|\; x = [x, y], y = x^b, x = [a, b] \right\rangle \\ &= \left\langle a, x, y, b \;\middle|\; x = [x, y], y = x^b, ax = a^b \right\rangle. \end{aligned}$$

4.7 Concluding remarks

Looking back and contemplating on the results obtained in the main part of this chapter, we will now try to understand what makes L^2-invariants special from the agrarian perspective. We will further remark on how the recent proof of the strong Atiyah conjecture for one-relator groups provides an alternative route to our main application.

4.7.1 What makes $\mathbb{Z}G \hookrightarrow \mathcal{D}(G)$ special?

Let G be a torsion-free group that satisfies the strong Atiyah conjecture over $\mathbb{Q}$. As a consequence of Corollary 2.4.7, there is the agrarian embedding $\mathbb{Z}G \hookrightarrow \mathcal{D}(G) = \mathcal{D}_{\mathbb{Q}G} = \mathcal{R}_{\mathbb{Q}G}$ into the Linnell division ring. Applied to this particular embedding, agrarian invariants recover most of the information contained in the L^2-invariants they are modeled on. From this abstract perspective, the Linnell division ring is as good as any other, but it still enjoys a number of special properties that make it particularly useful for both computations and theoretic considerations. At certain points in the proofs given in this chapter, we have already encountered these properties because we had to work around their absence in order to construct agrarian invariants for general agrarian maps. We will now revisit these points hoping that their discussion will provide an improved understanding of why L^2-invariants have been so successful and computationally tractable.

$\mathbb{Z}G \hookrightarrow \mathcal{D}(G)$ is an embedding

As a careful review of the preceding sections shows, there is no need to consider only those agrarian maps that are embeddings anywhere in the construction of the agrarian invariants or the proofs of their general properties. However, as soon as we needed to prove that a certain chain complex was indeed D-acyclic in Lemma 4.6.8, we crucially used that no elements of the group ring map to 0 in the division ring to conclude that the differentials attain the ranks required for acyclicity. Given that the agrarian invariants other than Betti numbers can only be defined for D-acyclic chain complexes, agrarian maps thus do not seem to offer significantly increased generality over agrarian embeddings in the context of higher invariants.

$\mathbb{Z}G \hookrightarrow \mathcal{D}(G)$ is strongly Hughes-free

By [Lüc02, Lemma 10.58], the $\mathbb{Z}G$-ring $\mathcal{D}(G)$ is *strongly Hughes-free*, i.e., whenever $N \trianglelefteq H$ is a normal subgroup of a subgroup $H \leqslant G$, elements of $H \subset \mathcal{D}(G)$ that are in mutually distinct cosets of H with respect to N are $\mathcal{D}(N)$-linearly independent. This property can be traced back to the construction of $\mathcal{D}(G)$ out of the free linear action of $\mathbb{Z}G$ on $\ell^2(G)$. It comes in useful in two ways related to agrarian invariants.

First, the agrarian embedding $\mathbb{Z}G \hookrightarrow \mathcal{D}(G)$ is rational to start with and thus does not need to be replaced by its rationalization before the agrarian polytope can be constructed. However, this does not seem to confer much of a benefit in practice: Even though there is no direct comparison in terms of specializations, a given agrarian embedding and its rationalization appear to be equally tractable (or intractable), so that computations for one should be expected to also apply for the other. In our application, the particular structure of the embedding division ring was completely immaterial for the computations.

Second, when applied to finite-index normal subgroups, the fact that $\mathcal{D}(G)$ is strongly Hughes-free implies that the condition of Proposition 4.2.10 is satisfied and thus that the $\mathcal{D}(G)$-agrarian Betti numbers satisfy the restriction identity enjoyed by L^2-Betti numbers. Even though we were able to work around this issue in our application as the passage to a subgroup played out entirely in the rational part of the agrarian embedding, it is likely the most severe limitation faced when considering general agrarian embeddings.

L^2-acyclicity is witnessed by the combinatorial Laplacian

As explained in detail in [Lüc02, Lemma 2.5], the L^2-acyclicity of a $\mathbb{Z}G$-chain complex C_* is witnessed by the combinatorial Laplacian inducing a level-wise isomorphism on $\mathcal{D}(G) \otimes_{\mathbb{Z}G} C_*$. Equivalently, the chain complex C_* admits a weak type of chain contraction that consists of maps defined entirely over $\mathbb{Z}G$. The origin of this convenient property of $\mathcal{D}(G)$ can be traced back all the way to the Hodge decomposition of the Laplacian on $\ell^2(G)$ and thus to the existence of a scalar product.

This extra structure has no analogue for a general agrarian map and, consequently, a chain contraction for a $\mathbb{Z}G$-chain complex that is D-acyclic may need to involve elements from D that are not in $\mathbb{Z}G$. Furthermore, the lack of this convenient type of chain contraction in the general setting is why the agrarian torsion we construct does not appear to enjoy a universal property in the way universal L^2-torsion does. In [FL19], this property is used to prove that the thickness of the L^2-polytope can be expressed as a twisted L^2-Euler characteristic. Luckily, it turns out that it is not required for the correctness of this statement, allowing us to give a different, albeit more laborious proof based on matrix chains.

Apart from the lack of a universal property, the missing connection to the combinatorial Laplacian also means that the combinatorial approach to calculating L^2-torsion described in [Lüc02, 3.7] is not applicable to agrarian invariants.

Analytic expressions of L^2-invariants

Apart from computing L^2-invariants such as L^2-Betti numbers and L^2-torsion from their basic computational properties, one can also calculate them by analytic means, using the L^2-Hodge–de Rham theorem (see [Lüc02, 1.3.2]) and the deep result that analytic and topological L^2-torsion agree ([Bur+96], see also [BFK98]). Computations obtained via these bridges into analysis have no counterpart in the agrarian world.

In summary, assuming the strong Atiyah conjecture, the convenient properties of L^2-invariants can be, for the most part, traced back to the strongly Hughes-free agrarian embedding $\mathbb{Z}G \hookrightarrow \mathcal{D}(G)$. Both the combinatorial as well as the analytic approach to L^2-Betti numbers and L^2-torsion serve as notable exceptions to this statement and provide important ways of computing these L^2-invariants that are not at all applicable to agrarian invariants.

4.7.2 The proof of the strong Atiyah conjecture for one-relator groups

After the work on Theorem 4.6.16 and Theorem 4.6.21 had been concluded, Jaikin-Zapirain and López-Álvarez publicized a proof of the conjecture for locally indicable groups, which has meanwhile appeared as [JL20]. Since torsion-free one-relator groups are locally indicable by a result of Brodskii [Bro84], the torsion-free part of Theorem 4.6.16 thus follows directly from Theorem 4.6.15. Furthermore, a proof of Theorem 4.6.21 assuming the strong Atiyah conjecture is indicated in [FLT19, Theorem 5.2].

The proofs provided in this chapter as well as the proof of the agrarianicity of torsion-free one-relator groups of [LL78] are more laborious, but serve to show that the additional structure enjoyed by the Linnell division ring is in fact not required to obtain the desired results.

Chapter 5

Pseudo-Sylvester domains and skew Laurent polynomials over firs

This chapter is based on the preprint "Pseudo-Sylvester domains and skew Laurent polynomials over firs" [HL20]*, which reports on joint work with Diego López-Álvarez.*

Our discussion of non-commutative localizations in Section 1.2 ended with the observation that we cannot construct division rings out of group rings of non-amenable groups simply by inverting elements. It was P. M. Cohn who realized that, in the same way that we can obtain a field from a commutative ring by localizing at a prime ideal (and then taking the residue field), we can obtain a division ring $\mathcal{D}$ from any ring R by means of universal localization at prime matrix ideals (see [Coh06]). Similarly to the commutative case, the division ring obtained in this way is generated as a division ring by the image of R under the corresponding map $R \to \mathcal{D}$. The pair given by $\mathcal{D}$ and the map $R \to \mathcal{D}$, or sometimes just $\mathcal{D}$ if the map is clear from the context, is usually referred to as *epic division R-ring*.

Adopting the previous terminology, recall that a homomorphism from a commutative ring R to an epic field K is completely characterized by its kernel, which is a prime ideal of R, in the sense that K can be recovered as mentioned above, i.e., by localizing at the kernel and taking the residue field. This is equivalent to saying that such a homomorphism is determined by the set of elements that become invertible in K, the ones outside the kernel. In the very same spirit, P. M. Cohn showed that a prescribed epic division R-ring is completely characterized by its singular kernel, which is a prime matrix ideal, or equivalently by the set Σ of matrices becoming invertible under the homomorphism. The latter point of view is particularly useful since the map will be injective if and only if Σ contains every non-zero element of R.

Assume that we are given an embedding $R \hookrightarrow \mathcal{D}$ of the domain R into the division ring $\mathcal{D}$. Then, a natural necessary condition for an $n \times n$ matrix A over R to become invertible over $\mathcal{D}$ is that it cannot be expressed as a product $A = BC$ for some matrices B, C of sizes $n \times m$ and $m \times n$, respectively, where $m < n$. Otherwise, the usual rank $\mathrm{rk}_{\mathcal{D}}(A)$ of A over $\mathcal{D}$ would be less or equal than m, and hence A would not be invertible. A matrix satisfying this necessary condition is called *full*. Therefore, one may wonder whether, among the division rings in which R can be embedded, there exists one in which we can invert every full matrix. The rings for which this is possible, originally studied by W. Dicks and E. Sontag ([DS78]) as those satisfying the law of nullity with respect to the inner rank function, comprise the family of *Sylvester domains*. The first examples of Sylvester domains were the *free ideal rings (firs)* (see [Coh06, Section 5.5]).

In addition, observe that if the matrix A is to become invertible in a division ring, then the same holds true for $A \oplus I_m$, the block diagonal matrix with blocks A and I_m, where I_m denotes the $m \times m$ identity matrix. Thus, $A \oplus I_m$ must in fact be full for every non-negative integer m. A matrix with this property is called *stably full* and, of course, in a Sylvester domain it is the case that every full matrix is stably full. Nevertheless, in general there may be full matrices that are not stably full, and hence, the question of whether there exists a division ring $\mathcal{D}$ in which R embeds and in which we can invert every stably full matrix over R is interesting in its own right. The rings with this property are the *pseudo-Sylvester domains*, which were introduced in [CS82] as the family of stably finite rings satisfying the law of nullity with respect to the stable rank function. Notice that if such a division ring $\mathcal{D}$ exists, then it is necessarily *universal* in the sense of P. M. Cohn (see Section 5.1.1), meaning that if a matrix A over R becomes invertible over some division ring, then it is also invertible over $\mathcal{D}$.

Recently, in [Jai19b], Jaikin-Zapirain introduced a new homological criterion for a ring to be a Sylvester domain. In this chapter, we provide a similar recognition principle for pseudo-Sylvester domains and use it to prove the following result:

Theorem 5.A. *Let $\mathfrak{F}$ be a fir with universal division $\mathfrak{F}$-ring of fractions $\mathcal{D}_{\mathfrak{F}}$, and consider a crossed product ring $\mathcal{S} = \mathfrak{F} * \mathbb{Z}$. Then, the following holds:*

(a) *$\mathcal{S}$ is a pseudo-Sylvester domain if and only if every finitely generated projective $\mathcal{S}$-module is stably free.*

(b) *$\mathcal{S}$ is a Sylvester domain if and only if it is projective-free.*

In any of the previous situations, $\mathcal{D}_{\mathcal{S}} = \operatorname{Ore}(\mathcal{D}_{\mathfrak{F}} * \mathbb{Z})$ is the universal division $\mathcal{S}$-ring of fractions and it is isomorphic to the universal localization of $\mathcal{S}$ with respect to the set of all stably full (resp. full) matrices.

As a particular application of Theorem 5.A, we obtain the next result through the recent advances on the Farrell–Jones conjecture by Bestvina–Fujiwara–Wigglesworth and Brück–Kielak–Wu:

Theorem 5.B. *Let E be a division ring and G a group arising as an extension*

$$1 \to F \to G \to \mathbb{Z} \to 1$$

where F is a free group. Then every crossed product $E * G$ is a pseudo-Sylvester domain. In particular, $\mathcal{D}_{E*G} = \operatorname{Ore}(\mathcal{D}_{E*F} * \mathbb{Z})$ is the universal division $E * G$-ring of fractions and is isomorphic to the universal localization of $E * G$ with respect to the set of all stably full matrices. Moreover, $E * G$ is a Sylvester domain if and only if it has stably free cancellation.

Some examples of groups as in Theorem 5.B with and without stably free cancellation are discussed in Section 5.3.2.

Note that Jaikin-Zapirain already showed in [Jai19d, Theorem 1.1 & Theorem 3.7] that $E * G$ has a universal division ring of fractions. With Theorem 5.B, we provide an independent proof of this fact as well as a description of the matrices that become invertible over $\mathcal{D}_{E*G}$. Furthermore, in [LL18, Theorem 2.17], it has already been shown that KG, where K is a subfield of $\mathbb{C}$, admits a universal localization that is a division ring.

This chapter is organized as follows. In Section 5.1 we recall the major notions that are going to play a role in the proof of our main result. We recall in Section 5.1.1 and Section 5.1.3 the basics on localization, stably freeness and stably finiteness, three notions needed to introduce properly (pseudo-)Sylvester domains in Section 5.1.4. In Section 5.1.2 we introduce the main homological tools that we are going to work with.

Section 5.2 is devoted to prove Theorem 5.A. We first state the criteria for Sylvester and pseudo-Sylvester domains in Section 5.2.1 and obtain additional input from homological algebra in Section 5.2.2.

In Section 5.3, we prove Theorem 5.B as an application of Theorem 5.A and the recent proof of the Farrell–Jones conjecture for the family of groups considered. In Section 5.3.2 we end with some examples of group rings to which our results apply.

5.1 Definitions and background

In this section, we will review the required notions and basic results related to non-commutative localizations and homological algebra that are relevant for the formulation and proofs of our main results.

5.1.1 Universal localization

We have seen in Section 1.2, specifically in Theorem 1.2.9, that division rings of fractions of general non-commutative domains cannot be obtained simply by passing to fractions of elements. We will now see that one can get much further by adjoining inverses of matrices instead of just ring elements. This generalized notion of non-commutative localization, known as *universal localization* goes back to P. M. Cohn and is part of this theory of epic division R-rings (see [Coh06, Chapter 7]). It builds on the notion of *prime matrix ideals*, certain subsets of the set of all square matrices over a ring that behave similarly to a prime ideal in a commutative ring. We refer the reader to [Coh06, Section 7.3] for the details of their definition.

Definition 5.1.1. Given a set Σ of (square) matrices over R, and a homomorphism of rings $\varphi\colon R \to S$, we say that the map φ is Σ*-inverting* if every element of Σ becomes invertible over S. We say that φ is *universal* Σ*-inverting* if any other Σ-inverting homomorphism factors uniquely through φ. In this latter case, we denote $S = R_\Sigma$ and we call R_Σ the *universal localization* of R with respect to Σ.

If we allow R_Σ to be the zero ring, the existence of the universal localization can always be proved by taking a presentation of R as a ring and formally adding the necessary generators and relations. Moreover, the universal Σ-inverting homomorphism will be injective if and only if there exists a Σ-inverting embedding to some ring ([Coh06, Theorem 7.2.4]).

Cohn's main result is that epic division R-rings, as defined in Definition 1.2.12, are completely characterized (up to R-isomorphism) by the set Σ of matrices over R that become invertible in the division ring, and that they always arise as residue fields of a universal localization R_Σ ([Coh06, Theorem 7.2.5 & Theorem 7.2.7]). In addition, such sets Σ are precisely the complements in the set of square matrices over R of prime matrix ideals $\mathcal{P}$ ([Coh06, Theorem 7.4.3]). Thus, we would obtain a division R-ring of fractions if we could construct such a set Σ including all non-zero elements in R.

Finally, if among all the possible division R-rings of fractions, there exists one in which we can invert "the most" (relative to R) matrices possible, we call it the universal division R-ring of fractions. More precisely:

Definition 5.1.2. The division R-ring of fractions $R \hookrightarrow \mathcal{D}$ is called the *universal division R-ring of fractions* if, for any other epic division R-ring $\mathcal{D}'$, the set Σ' of matrices that become invertible over $\mathcal{D}'$ is contained in the set Σ of matrices that become invertible over $\mathcal{D}$.

In Section 5.1.4 we will introduce two families of rings, namely Sylvester and pseudo-Sylvester domains, for which there exists a universal division ring of fractions and for which the set Σ of matrices becoming invertible under the embedding can be characterized in a natural way only depending on R. Our main result will be build on a homological criterion for a ring to belong to one of these families, which is why we need to introduce parts of the dimension theory of (non-commutative) rings in the following.

5.1.2 Weak and global dimensions

Recall that a module N over a ring R has *projective dimension* at most n (abbreviated $\mathrm{pd}(N) \leqslant n$) if N admits a resolution

$$0 \to P_n \to \ldots \to P_0 \to N \to 0$$

of projective R-modules. In particular, observe that N is projective if and only if $\mathrm{pd}(N) = 0$. The supremum among the projective dimensions of all left (resp. right) R-modules is called the *left* (resp. *right*) *global dimension* of R, and it is not left-right symmetric in general. This concept is deeply related to Ext functors.

Lemma 5.1.3 ([Rot09, Proposition 8.6]). *Let N be a left R-module. Then* $\mathrm{pd}(N) \leqslant n$ *if and only if* $\mathrm{Ext}_R^{n+1}(N, N') = 0$ *for all left R-modules N'.*

Analogously, we say that the *flat dimension* of N is at most n, and we write $\mathrm{fd}(N) \leqslant n$, if it admits a resolution of flat R-modules

$$0 \to Q_n \to \ldots \to Q_0 \to N \to 0,$$

and define the *left* (resp. *right*) *weak dimension* of R as the supremum of the flat dimensions of all left (resp. right) R-modules. It turns out that this notion is always left-right symmetric ([Rot09, Theorem 8.19]) and hence we can just talk about the *weak dimension* of R. As it happens with $\mathrm{pd}(N)$ and $\mathrm{Ext}_R^*(N, ?)$, the flat dimension of N (resp. of a right R-module M) can be characterized in terms of $\mathrm{Tor}_*^R(?, N)$ (resp. $\mathrm{Tor}_*^R(M, ?)$). Observe though that, unlike the previous case, here we need to change the argument while considering left or right modules.

Lemma 5.1.4 ([Rot09, Proposition 8.17]). *Let N be a left R-module. Then* $\mathrm{fd}(N) \leqslant n$ *if and only if* $\mathrm{Tor}_{n+1}^R(M, N) = 0$ *for all right R-modules M.*

We finish this section with the following result regarding Tor, sometimes referred to as Shapiro's Lemma (see [Rot09, Corollary 10.61] for a generalization).

Lemma 5.1.5. *Let R be a subring of S such that S is flat as a left R-module. Then, for any right R-module M, for any left S-module N and for any $n \geqslant 0$, we have*

$$\mathrm{Tor}_n^R(M, {}_RN) \cong \mathrm{Tor}_n^S(M \otimes_R S, N)$$

where ${}_RN$ denotes N considered as a left R-module.

5.1.3 Stably freeness and stably finite rings

The criteria we are going to introduce in Section 5.2 rely on proving that certain submodules are finitely generated free or stably free, respectively. Therefore, we need to deal with the latter concept and its relation with the notion of stably finite rings.

Definition 5.1.6. A module M over a ring R is called *stably free* if there exists $n \geqslant 0$ such that $M \oplus R^n$ is a free R-module.

By a result of Gabel, a proof of which is given in [Lam78, Proposition 4.2], any stably free module that is not finitely generated is already free. For this reason, we will restrict our attention to finitely generated stably free modules in the following.

If M is a finitely generated stably free R-module and $M \oplus R^n$ is free, then this free module is necessarily finitely generated and hence isomorphic to some R^m. In general, the difference $m - n$ needs neither be positive nor uniquely determined by M. It is here where the stably finite property enters the scene. Recall that a ring R is said to be *stably finite* (or *weakly finite*) if whenever A and B are two $n \times n$-matrices over R such that $AB = I_n$, then also $BA = I_n$. This can be reformulated in terms of modules by saying that if $R^n \oplus K \cong R^n$, then $K = 0$. For example, every division ring is stably finite. Also, if K is a field of characteristic 0 and G is any group, or if K has positive characteristic and G is sofic, the group ring KG is stably finite (see [Jai19a, Corollary 13.7]). Furthermore, any subring of a stably finite ring is clearly again stably finite.

If M is a non-trivial module over a stably finite ring R and $M \oplus R^n \cong R^m$, then the difference $m - n$ is positive and constant among all such representations. We call this positive number the *stably free rank* of M and denote it by $\mathrm{rk}_{sf}(M)$.

To finish this subsection, let P be a finitely generated projective module over R. We will recall in the next subsection that if R is a Sylvester domain then P is necessarily free, while if R is just a pseudo-Sylvester domain, we can only deduce that P is stably free. Thus, a first (and in fact, the only) obstruction for a pseudo-Sylvester domain to be a Sylvester domain is the following property:

Definition 5.1.7. A stably finite ring R is said to have *stably free cancellation (SFC)* if every finitely generated stably free R-module M is free of rank $\mathrm{rk}_{sf}(M)$.

Examples of group rings with and without stably free cancellation will be given in Section 5.3.2.

Remark 5.1.8. Let R be a ring. If M is a left (right) R-module, then $M^* := \mathrm{Hom}_R(M, R)$, called the *dual of* M, is naturally a right (left) R-module. For every ring R, the functor $P \mapsto P^*$ defines an equivalence between the category of finitely generated projective left R-modules and the opposite of the category of finitely generated projective right R-modules, with the inverse functor given in the same way. To see that $P \cong P^{**}$, note that taking the dual commutes with finite direct sums and the claim thus needs to be checked only for R itself viewed as an R-module, where it is clear. The equivalence defined in this way restricts to equivalences of the respective subcategories of finitely generated stably free and finitely generated free modules.

As a consequence, every property of rings that can be expressed in terms of these categories in a way that is invariant under passing to an equivalent or opposite category will hold for left modules if and only if it holds for right modules. In particular, whether or not any of the classes of finitely generated projective, stably free or free modules coincide for a ring does not depend on whether left or right modules are considered.

5.1.4 (Pseudo-)Sylvester domains

In this section we introduce the main families of rings we are going to deal with throughout the chapter, namely, Sylvester domains and pseudo-Sylvester domains, which requires us to first introduce the notions of inner and stable rank.

Let R be a ring, and A an $m \times n$ matrix over R. Recall that the *inner rank* $\rho(A)$ is defined as the least k such that A admits a decomposition $A = B_{m\times k} C_{k\times n}$. We say that a square matrix A of size $n \times n$ is *full* if $\rho(A) = n$. Recall also that the *stable rank* $\rho^*(A)$ is given by

$$\rho^*(A) = \lim_{s\to\infty} \left[\rho(A \oplus I_s) - s\right],$$

whenever the limit exists, where $A \oplus I_s$ denotes the block diagonal matrix with blocks A and I_s. We analogously say that a square matrix is *stably full* if it has maximum stable rank. When R is stably finite, $\rho^*(A)$ is well-defined and non-negative, and it is positive if A is a non-zero matrix ([Coh06, Proposition 0.1.3]). For this reason, in the following we restrict our attention to stably finite rings.

Observe that from the definition of the inner rank it follows that the sequence in the limit is always non-increasing and bounded above by $\rho(A)$. In particular, for an $n \times n$ matrix A we obtain that $\rho^*(A) \leqslant \rho(A) \leqslant n$ and that $\rho^*(A) = n$ if and only if the sequence is constantly n. Thus, A is stably full if and only if $\rho(A \oplus I_s) = n + s$ for every $s \geqslant 0$.

We summarize useful properties of the stable rank over stably finite rings.

Lemma 5.1.9. *Let R be a stably finite ring. Then the following holds for every matrix A over R:*

(a) *For every $k \geqslant 0$, $\rho^*(A \oplus I_k) = \rho^*(A) + k$.*

(b) *There exists $N \geqslant 0$ such that for every $l \geqslant N$, $\rho^*(A \oplus I_l) = \rho(A \oplus I_l)$.*

(c) $0 \leqslant \rho^*(A) \leqslant \rho(A)$.

Proof. Since R is stably finite, we know that $\rho^*(A) = r \geqslant 0$. This means that there exists $N \geqslant 0$ such that for any $l \geqslant N$ we have $\rho(A \oplus I_l) - l = r$. Thus, for $k \geqslant 0$,

$$\rho^*(A \oplus I_k) = \lim_{s \to \infty} \big[\rho(A \oplus I_k \oplus I_s) - (s+k) + k\big] = r + k = \rho^*(A) + k.$$

From here, we also deduce that for $l \geqslant N$ one has

$$\rho(A \oplus I_l) = l + r = l + \rho^*(A) = \rho^*(A \oplus I_l).$$

The last statement has already been observed above. □

We can now introduce the main notions of the subsection. Let us define first the notion of Sylvester domain, together with the main examples and properties.

Definition 5.1.10. A non-zero ring R is a *Sylvester domain* if R is stably finite and satisfies the law of nullity with respect to the inner rank, i.e., if $A \in \mathrm{Mat}_{m\times n}(R)$ and $B \in \mathrm{Mat}_{n \times k}(R)$ are such that $AB = 0$, then

$$\rho(A) + \rho(B) \leqslant n$$

In fact, it can be shown that the condition that R is stably finite is redundant here, but we keep it as a requirement to show the symmetry with the upcoming definition of pseudo-Sylvester domain. The following rings serve as the most prominent examples of Sylvester domains ([Coh06, Proposition 5.5.1]):

Definition 5.1.11. A *free ideal ring (fir)* is a ring in which every left and every right ideal is free of unique rank (as a module).

As a consequence, in a fir every submodule of a free module is again free (see [Coh06, Corollary 2.1.2] and note that every submodule of a free R-module of rank κ is $\max(|R|, \kappa)$-generated). For instance, a division ring $\mathcal{D}$ is a fir, and the inner rank over $\mathcal{D}$ is just its usual rank, which will be denoted by $\mathrm{rk}_{\mathcal{D}}$. An important example is the group ring KF, where K is a field and F is a free group. This result was originally proved by P. M. Cohn, and we refer the reader to [Lew69, Theorem 1] for a concise treatment. More generally, for any division ring E and free group F, the crossed product $E * F$ is a fir. This is a consequence of Bergman's coproduct theorem (see [Sán08, Theorem 4.22 (i)]).

The following property of a ring, which by Remark 5.1.8 is left-right symmetric, is intimately related to Sylvester domains.

Definition 5.1.12. A ring R is called *projective-free* if every finitely generated projective R-module is free of unique rank.

Note, for instance, that if K is a field, then the polynomial ring $K[t_1, \dots, t_n]$ in n indeterminates is projective-free, a result known as the Quillen–Suslin theorem.

Every Sylvester domain is projective-free and has weak dimension at most 2 (see [DS78, Theorem 6] and the subsequent discussion). In Theorem 5.A, we will provide a class of rings of weak dimension at most 2 which are Sylvester domains if and only if they are projective-free.

In the same way that Sylvester domains are defined in terms of inner rank, pseudo-Sylvester domains are defined in terms of stable rank.

Definition 5.1.13. A non-zero ring R is a *pseudo-Sylvester domain* if R is stably finite and satisfies the law of nullity with respect to the stable rank, i.e., if $A \in \mathrm{Mat}_{m\times n}(R)$ and $B \in \mathrm{Mat}_{n\times k}(R)$ are such that $AB = 0$, then

$$\rho^*(A) + \rho^*(B) \leqslant n.$$

Example 5.1.14. The following rings are pseudo-Sylvester domains, but not Sylvester domains:

- The polynomial ring $D[x, y]$ in two variables over a division ring D is a pseudo-Sylvester domain by [CS82, Proposition 6.5] and [Bas68, Theorem XII.3.1]. It is not projective-free by [OS71, Proposition 1] if D is non-commutative.

- The Weyl algebra $A_1(K)$ for a field K, which is the quotient of the free algebra on two generators x and y by the ideal generated by $xy - yx - 1$, is a pseudo-Sylvester domain by [CS82, Proposition 6.5] and [Sta77b, Theorem 2.2]. An example of a projective non-free ideal is provided in [Sta77a, Section 6].

In analogy to the case of Sylvester domains, any finitely generated projective module over a pseudo-Sylvester domain is stably free [Coh06, Proposition 5.6.2]. Moreover, a pseudo-Sylvester domain is a Sylvester domain if and only if the ring has stably free cancellation by [CS82, Proposition 6.1].

Several characterizations of Sylvester and pseudo-Sylvester domains can be found in [Coh06, Theorem 7.5.13] and [Coh06, Theorem 7.5.18], respectively. In particular, they can be defined in terms of universal localizations and universal division rings of fractions. In this flavour, observe that for an $n \times n$ matrix A to become invertible over a division ring $\mathcal{D}$, we need A to be stably full, since otherwise there would exists $s \geqslant 0$ such that $\rho(A \oplus I_s) < n + s$ and hence $A \oplus I_s$ would not be invertible over $\mathcal{D}$. Thus, one can wonder whether there exists a division ring in which R embeds and in which every stably full matrix can be inverted. The family of rings for which this is possible is precisely the family of pseudo-Sylvester domains.

For a Sylvester domain, the inner rank is additive, in the sense that $\rho(A \oplus B) = \rho(A) + \rho(B)$ holds for any matrices A and B (see [Coh06, Lemma 5.5.3]), and thus the inner and stable rank coincide. Indeed, if $\rho^*(A) = r$, then by Lemma 5.1.9 (b) there exists $s \geqslant 0$ such that $\rho(A \oplus I_s) = \rho^*(A \oplus I_s)$, from where Lemma 5.1.9 (a) and additivity tell us that $\rho^*(A) = \rho(A)$. As a consequence, every full matrix is actually stably full, and hence Sylvester domains will form the family of rings embeddable into a division ring in which we can invert all full matrices.

We record this in the following proposition, whose statement is implicit in [Coh06, Theorem 7.5.13 & Theorem 7.5.18].

Proposition 5.1.15. *For a non-zero ring R, the following are equivalent:*

(a) *R is a Sylvester (resp. pseudo-Sylvester) domain.*

(b) *There exists a division R-ring of fractions $R \hookrightarrow \mathcal{D}$ such that every full (resp. stably full) matrix over R becomes invertible over $\mathcal{D}$.*

Moreover, if R satisfies one, and hence each of the previous properties, $\mathcal{D}$ is the universal division R-ring of fractions, and it is isomorphic to the universal localization of R with respect to the set of all full (resp. stably full) matrices over R.

5.2 Proof of Theorem 5.A

This section is devoted to prove Theorem 5.A by verifying the conditions of Theorems 5.2.3 and 5.2.4, both of which will be stated in Section 5.2.1. The former is a particular case of a homological criterion introduced by Jaikin-Zapirain in [Jai19b] to determine when a ring with a prescribed embedding into a division ring is a Sylvester domain. The latter is the analogous recognition principle adapted to pseudo-Sylvester domains.

Throughout this section, $\mathfrak{F}$ will always denote a fir with universal division $\mathfrak{F}$-ring of fractions $\mathcal{D}_{\mathfrak{F}}$, and we will consider any crossed product ring $\mathcal{S} = \mathfrak{F} * \mathbb{Z}$.

The following lemma tells us in particular that the crossed product structure $\mathcal{S} = \mathfrak{F} * \mathbb{Z}$ can always be extended to a crossed product structure $\mathcal{D}_{\mathfrak{F}} * \mathbb{Z}$, and that this ring is an Ore domain.

Lemma 5.2.1. *Let R be a (pseudo-)Sylvester domain with universal division R-ring of fractions $\mathcal{D}_R$ and let $R * \mathbb{Z}$ be a crossed product. Then we can form a crossed product $\mathcal{D}_R * \mathbb{Z}$ together with an embedding $R * \mathbb{Z} \hookrightarrow \mathcal{D}_R * \mathbb{Z}$ such that*

(1) The composition $R * \mathbb{Z} \hookrightarrow \mathcal{D}_R * \mathbb{Z} \hookrightarrow \operatorname{Ore}(\mathcal{D}_R * \mathbb{Z})$ is a division $R * \mathbb{Z}$-ring of fractions.

(2) The left $R * \mathbb{Z}$-modules $\mathcal{D}_R * \mathbb{Z}$ and $(R * \mathbb{Z}) \otimes_R \mathcal{D}_R$ (resp. the right $R * \mathbb{Z}$-modules $\mathcal{D}_R * \mathbb{Z}$ and $\mathcal{D}_R \otimes_R (R * \mathbb{Z})$) are isomorphic.

Proof. First, we are going to see that every automorphism φ of R extends uniquely to an automorphism of $\mathcal{D}_R$. Indeed, let Σ denote the set of (stably) full matrices over R and notice that φ preserves Σ (i.e., $\varphi(\Sigma) = \Sigma$). Thus, the composition $R \xrightarrow{\varphi} R \hookrightarrow \mathcal{D}_R$ is a Σ-inverting embedding, and hence the universal property of universal localization gives us a unique injective map $\varphi\colon R_\Sigma = \mathcal{D}_R \to \mathcal{D}_R$ such that the diagram

$$\begin{tikzcd} R \arrow[r, hook] \arrow[d, "\varphi", "\cong"'] & \mathcal{D}_R \arrow[d, dotted, "\tilde{\varphi}"] \\ R \arrow[r, hook] & \mathcal{D}_R. \end{tikzcd}$$

commutes. Since $\mathcal{D}_R$ is generated by R as a division ring, $\tilde{\varphi}$ is also surjective, and hence an automorphism of $\mathcal{D}_R$.

As mentioned in Example 1.1.4, we have a ring isomorphism $R * \mathbb{Z} \cong R[t^{\pm 1}; \tau]$ for some automorphism τ of R, and taking the automorphism $\tilde{\tau}$ of $\mathcal{D}_R$ that extends τ, we can form the ring $\mathcal{D}_R[t^{\pm 1}; \tilde{\tau}]$, so that we have a commutative diagram

$$\begin{tikzcd} R \arrow[r, hook] \arrow[d, hook] & \mathcal{D}_R \arrow[d, hook] \\ R[t^{\pm 1}; \tau] \arrow[r, hook] & \mathcal{D}_R[t^{\pm 1}; \tilde{\tau}]. \end{tikzcd}$$

To see that the bottom map is epic, let S be any ring and $f, g\colon \mathcal{D}_R[t^{\pm 1}; \tilde{\tau}] \to S$ ring homomorphisms that agree on $R[t^{\pm 1}; \tau]$. They induce ring homomorphisms $\mathcal{D}_R \to S$ that coincide on R, and hence, since the embedding $R \hookrightarrow \mathcal{D}_R$ is epic, f and g agree on $\mathcal{D}_R$. Since they also agree in the indeterminate, we deduce that $f = g$.

Thus, we have an epic embedding $R * \mathbb{Z} \cong R[t^{\pm 1}; \tau] \hookrightarrow \mathcal{D}_R[t^{\pm 1}; \tilde{\tau}] = \mathcal{D}_R * \mathbb{Z}$, the latter ring is an Ore domain by Theorem 1.2.9 and since the map $\mathcal{D}_R * \mathbb{Z} \hookrightarrow \mathrm{Ore}(\mathcal{D}_R * \mathbb{Z})$ is also epic and $\mathrm{Ore}(\mathcal{D}_R * \mathbb{Z})$ is a division ring, the composition $R * \mathbb{Z} \hookrightarrow \mathrm{Ore}(\mathcal{D}_R * \mathbb{Z})$ is a division $R * \mathbb{Z}$-ring of fractions. This finishes the proof of (1).

Finally, note that since $R * \mathbb{Z}$ is isomorphic to $R[t^{\pm 1}; \tau]$ as a ring, (2) follows from the fact that the left $R[t^{\pm 1}; \tau]$-linear map

$$\begin{aligned} R[t^{\pm 1}; \tau] \otimes_R \mathcal{D}_R &\to \mathcal{D}_R[t^{\pm 1}; \tilde{\tau}] \\ t^n \otimes \lambda &\mapsto \tilde{\tau}^n(\lambda) t^n \end{aligned}$$

is an isomorphism since it is also right $\mathcal{D}_R$-linear and maps the basis $\{t^n \otimes 1 \mid n \in \mathbb{Z}\}$ to the basis $\{t^n \mid n \in \mathbb{Z}\}$. The statement for right modules is proved analogously. ☐

We are interested in the homological properties of $\mathcal{D}_{\mathcal{S}} = \mathrm{Ore}(\mathcal{D}_{\mathfrak{F}} * \mathbb{Z})$, to which we will dedicate Section 5.2.2. In the previous lemma we explored the $\mathcal{S}$-module structure of $\mathcal{D}_{\mathfrak{F}} * \mathbb{Z}$. The next one, applied to the case $R \coloneqq \mathcal{D}_{\mathfrak{F}} * \mathbb{Z}$, $\mathcal{O} \coloneqq \mathcal{D}_{\mathcal{S}}$ and $S \coloneqq \mathcal{S}$, will allow us later to restrict our attention to $\mathcal{S}$-submodules of $\mathcal{D}_{\mathfrak{F}} * \mathbb{Z}$.

Lemma 5.2.2. *Let R be a right Ore domain with Ore division ring of fractions $\mathcal{O}$ and S a subring of R. Then every finitely generated S-submodule M of the left S-module $\mathcal{O}$ is isomorphic to a finitely generated S-submodule of R.*

Proof. Let M be generated as a left S-module by $x_1, \dots, x_m \in \mathcal{O}$. We find $p_i, q_i \in R$ such that $x_i = p_i q_i^{-1}$ for $i = 1, \dots, m$. If $m \geqslant 2$ we can use the Ore condition to find non-zero $a, b \in R$ such that $q_1 a = q_2 b$, and hence $x_1 = (p_1 a)(q_1 a)^{-1}$ and $x_2 = (p_2 b)(q_2 b)^{-1}$ can be expressed as fractions with common denominators. By repeatedly applying this procedure we produce $p_i', q \in R, q \neq 0$ such that $x_i = p_i' q^{-1}$ for all i.

We now consider the left S-submodule M' of R generated by $x_1 q, \dots, x_m q$. The map $f \colon M \to M'$ given by $y \mapsto yq$ is S-linear since $\mathcal{O}$ is associative and surjective since its image contains the generators. Finally, it is injective, since $\mathcal{O}$ is a division ring and hence $zq \neq 0$ for every $z \neq 0$. We conclude that f is an S-linear isomorphism. ☐

5.2.1 Recognition principles for (pseudo-)Sylvester domains

As mentioned above, we are going to use the next two theorems to prove Theorem 5.A. The first one is a direct consequence of a criterion for a ring to be a Sylvester domain which was recently formulated by Jaikin-Zapirain in [Jai19b, Theorem 2.3].

Theorem 5.2.3. *Let $R \hookrightarrow \mathcal{D}$ be a division R-ring of fractions. Assume that*

(1) $\mathrm{Tor}_1^R(\mathcal{D}, \mathcal{D}) = 0$ and

(2) for any finitely generated left or right R-submodule M of $\mathcal{D}$ and any exact sequence $0 \to J \to R^n \to M \to 0$, the R-module J is free of finite rank.

Then R is a Sylvester domain and $\mathcal{D}$ is the universal division R-ring of fractions.

The second theorem is an analogue for pseudo-Sylvester domains, involving stably free modules instead of free modules. The proof proceeds similarly, but we include it here for the sake of completeness. Given an embedding $R \hookrightarrow \mathcal{D}$ of R into a division ring and a matrix A over R, we will denote by $\mathrm{rk}_{\mathcal{D}}(A)$ the usual $\mathcal{D}$-rank of A considered as a matrix over $\mathcal{D}$. Similarly, if M is a left R-module, we take $\dim_{\mathcal{D}}(M)$ to denote the $\mathcal{D}$-dimension $\dim_{\mathcal{D}}(\mathcal{D} \otimes_R M)$ of the left $\mathcal{D}$-module $\mathcal{D} \otimes_R M$.

Theorem 5.2.4. *Let $R \hookrightarrow \mathcal{D}$ be a division R-ring of fractions. Assume that*

(1) $\operatorname{Tor}_1^R(\mathcal{D}, \mathcal{D}) = 0$ *and*

(2) for any finitely generated left or right R-submodule M of $\mathcal{D}$ and any exact sequence $0 \to J \to R^n \to M \to 0$, the R-module J is finitely generated stably free.

Then R is a pseudo-Sylvester domain and $\mathcal{D}$ is the universal division R-ring of fractions.

Proof. Notice that by Proposition 5.1.15 it suffices to show that every stably full matrix over R becomes invertible over $\mathcal{D}$. Thus, let A be an $n \times n$ matrix over R with $\rho^*(A) = n$, and assume that A is not invertible over $\mathcal{D}$, i.e., $\operatorname{rk}_{\mathcal{D}}(A) < n$. Since R is a subring of a division ring, it is necessarily stably finite.

Let N be the left R-module $N = R^n / R^n A$. Then A is also the presentation matrix of $\mathcal{D} \otimes_R N$, and therefore $\dim_{\mathcal{D}}(N) = n - \operatorname{rk}_{\mathcal{D}}(A)$, which is finite and positive. This implies that $\mathcal{D} \otimes_R N \cong \mathcal{D}^k$ as $\mathcal{D}$-modules for some $k \geqslant 1$ and, thus, composing the R-homomorphism $N \to \mathcal{D} \otimes_R N$ given by $x \to 1 \otimes x$ with an appropriate projection, we obtain a non-trivial R-homomorphism $N \to \mathcal{D}$. Therefore, if M is the image of this map, the surjection $N \to M$ gives us a commutative diagram with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & R^n A \arrow[r] \arrow[d, dotted] & R^n \arrow[r] \arrow[d, equal] & N \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & J \arrow[r] & R^n \arrow[r] & M \arrow[r] & 0.
\end{tikzcd}$$

Here, J is the kernel of the map $R^n \to M$ and the dotted arrow is such that the left square commutes (see [Rot09, Proposition 2.71]) and therefore injective. Moreover, notice that $\mathcal{D} \otimes_R M$ is non-trivial since the multiplication map to $\mathcal{D}$ is non-trivial. We conclude that $\dim_{\mathcal{D}}(M) > 0$.

Now we have by (2) that J is stably free, i.e., there exists $s \geqslant 0$ such that $J \oplus R^s$ is free. Moreover, since J is finitely generated and R, as a subring of a division ring, is stably finite, we conclude that $J \oplus R^s \cong R^{\operatorname{rk}_{sf}(J)+s}$, where rk_{sf} denotes the stably free rank. In fact, we obtain that $\operatorname{rk}_{sf}(J) = \dim_{\mathcal{D}}(J)$ by applying $\mathcal{D} \otimes_R ?$. Notice also that the previous diagram remains exact and commutative if we add $0 \to R^s \to R^s \to 0 \to 0$ to both rows. Thus, setting $t \coloneqq \dim_{\mathcal{D}}(J)$, the situation can be summarized in the following commutative diagram:

$$\begin{tikzcd}
R^{n+s} \arrow[d, two heads, "r_{A\oplus I_s}"'] \arrow[dr, "r_{A\oplus I_s}"] & \\
R^n A \oplus R^s \arrow[r, hook] \arrow[d, hook] & R^{n+s} \arrow[d, equal] \\
R^{t+s} \cong J \oplus R^s \arrow[r, hook] & R^{n+s}.
\end{tikzcd}$$

Here, $r_{A\oplus I_s}$ denotes the homomorphism given by right multiplication by $A \oplus I_s$, so that all maps except the isomorphism behave identically on the R^s summand. In terms of matrices, this factorization of $r_{A\oplus I_s}$ allows us to express $A \oplus I_s$ as a product of two matrices of dimensions $(n+s) \times (t+s)$ and $(t+s) \times (n+s)$, respectively. Thus, $\rho(A \oplus I_s) \leqslant t + s$ right by definition. Since A is stably full, we have $\rho(A \oplus I_s) = n + s$ for every s, so we conclude that $n \leqslant t$.

We are going to show on the other hand that $t < n$, a contradiction. Observe first that the condition (2) tells us in particular that the flat (in fact, projective) dimension of any finitely generated right R-submodule of $\mathcal{D}$ is at most 1. Hence, using Lemma 5.1.4 and the fact that Tor commutes with directed colimits (see [Rot09, Proposition 7.8]), we obtain that for any left R-module Q,

$$\operatorname{Tor}_2^R(\mathcal{D}, Q) = \operatorname{Tor}_2^R\left(\varinjlim L_i, Q\right) \cong \varinjlim \operatorname{Tor}_2^R(L_i, Q) = 0,$$

where L_i runs through all finitely generated R-submodules of the right R-module $\mathcal{D}$. Again by Lemma 5.1.4, this means that $\mathcal{D}$ itself has flat dimension at most 1 as a right R-module.

Now, since M is an R-submodule of $\mathcal{D}$, we have an exact sequence of left R-modules $0 \to M \to \mathcal{D} \to Q \to 0$ for some left R-module Q, and hence, applying $\mathcal{D}\otimes_R ?$ we can construct a long exact sequence containing the following exact part:

$$\cdots \to \operatorname{Tor}_2^R(\mathcal{D}, Q) \to \operatorname{Tor}_1^R(\mathcal{D}, M) \to \operatorname{Tor}_1^R(\mathcal{D}, \mathcal{D}) \to \cdots .$$

The first term is trivial by the previous argument, while the third term is trivial because of (1). Thus, we deduce that $\operatorname{Tor}_1^R(\mathcal{D}, M) = 0$. From here, it follows that applying $\mathcal{D}\otimes_R ?$ to the exact sequence $0 \to J \to R^n \to M \to 0$ returns an exact sequence of left $\mathcal{D}$-modules

$$0 \to \mathcal{D} \otimes_R J \to \mathcal{D}^n \to \mathcal{D} \otimes_R M \to 0,$$

from which we obtain

$$t = \dim_{\mathcal{D}}(J) = n - \dim_{\mathcal{D}}(M) < n.$$

This is the desired contradiction, which shows that necessarily $\operatorname{rk}_{\mathcal{D}}(A) = n$. □

In the case of $\mathfrak{F} * \mathbb{Z}$, the role of $\mathcal{D}$ will be played by the Ore division ring of fractions $\mathcal{D}_{\mathcal{S}} = \operatorname{Ore}(\mathcal{D}_{\mathfrak{F}} * \mathbb{Z})$.

5.2.2 The homological properties of $\mathcal{D}_{\mathcal{S}}$

We will now study the homological properties of the $\mathcal{S}$-module $\mathcal{D}_{\mathcal{S}}$ and its submodules. In particular, we will derive vanishing results for Tor and Ext, which will allow us to verify condition (1) and a weak version of condition (2) of Theorems 5.2.3 and 5.2.4. From this, we will finally derive Theorem 5.A.

The following theorem, which combines Theorem 4.7 and 4.8 of [Sch85], will be very useful in verifying condition (1):

Theorem 5.2.5. *Let $R \to S$ be an epic ring homomorphism. Then the following are equivalent:*

(a) $\operatorname{Tor}_1^R(S, S) = 0$.

(b) $\operatorname{Tor}_1^R(M, N) = \operatorname{Tor}_1^S(M, N)$ *for every right S-module M and every left S-module N.*

(c) $\operatorname{Ext}_R^1(M, M') = \operatorname{Ext}_S^1(M, M')$ *for all right S-modules M and M'.*

(d) $\operatorname{Ext}_R^1(N, N') = \operatorname{Ext}_S^1(N, N')$ *for all left S-modules N and N'.*

If $S = R_\Sigma$ is a universal localization of R, then all of these properties are satisfied.

The importance of this theorem lies in the fact that, since firs are Sylvester domains, the universal division $\mathfrak{F}$-ring of fractions $\mathcal{D}_{\mathfrak{F}}$ is precisely the universal localization of $\mathfrak{F}$ with respect to the set of all full matrices. Therefore, each of the statements in Theorem 5.2.5 holds for the epic embedding $\mathfrak{F} \hookrightarrow \mathcal{D}_{\mathfrak{F}}$, which will serve as the starting point for the proof of the main result. The other crucial property in our setting is the following:

Lemma 5.2.6. *Let R be a ring of right (resp. left) global dimension at most 1. Then any crossed product $R * \mathbb{Z}$ has right (resp. left) global dimension at most 2. In particular, if $\mathfrak{F}$ is a fir, then $\mathfrak{F} * \mathbb{Z}$ has right and left global dimension at most 2.*

Proof. This can be found in [MR01, Corollary (ii) on page 265] noting the symmetry in the definition of a crossed product. The last statement follows because firs have right and left global dimension at most 1. □

We are now ready to study the homological properties of $\mathcal{D}_{\mathcal{S}}$ and its submodules.

Lemma 5.2.7.

(a) $\operatorname{Ext}^3_{\mathcal{S}}(M, M') = 0$ *for all left (resp. right) $\mathcal{S}$-modules M and M'.*

(b) $\mathcal{D}_{\mathfrak{F}} * \mathbb{Z}$ *has projective dimension at most 1 as a left and right $\mathcal{S}$-module.*

(c) *Every left or right $\mathcal{S}$-submodule of $\mathcal{D}_{\mathfrak{F}} * \mathbb{Z}$ has projective dimension at most 1.*

(d) *Every finitely generated left or right $\mathcal{S}$-submodule of $\mathcal{D}_{\mathcal{S}}$ has projective dimension at most 1.*

Proof. (a) Since $\mathcal{S}$ has global dimension at most 2 by Lemma 5.2.6, this is a consequence of Lemma 5.1.3.

(b) Since $\mathfrak{F}$ has global dimension at most 1, the left $\mathfrak{F}$-module $\mathcal{D}_{\mathfrak{F}}$ admits a resolution $0 \to P_1 \to P_0 \to \mathcal{D}_{\mathfrak{F}} \to 0$ with P_1 and P_0 projective left $\mathfrak{F}$-modules. We now apply the functor $\mathcal{S} \otimes_{\mathfrak{F}} ?$ to this short exact sequence, where we view $\mathcal{S}$ as an $\mathcal{S}$-$\mathfrak{F}$-bimodule. Since $\mathcal{S}$ is a free right $\mathfrak{F}$-module, the resulting sequence is a projective resolution of the left $\mathcal{S}$-module $\mathcal{S} \otimes_{\mathfrak{F}} \mathcal{D}_{\mathfrak{F}}$, and thus the projective dimension of this module is at most 1. This finishes the proof, since the left $\mathcal{S}$-modules $\mathcal{S} \otimes_{\mathfrak{F}} \mathcal{D}_{\mathfrak{F}}$ and $\mathcal{D}_{\mathfrak{F}} * \mathbb{Z}$ are isomorphic by Lemma 5.2.1. The corresponding statement for the right $\mathcal{S}$-module $\mathcal{D}_{\mathfrak{F}} * \mathbb{Z}$ follows analogously.

(c) For every left (resp. right) $\mathcal{S}$-module M', the Ext long exact sequence obtained by applying the functor $\operatorname{Hom}_{\mathcal{S}}(?, M')$ to the short exact sequence $0 \to M \to \mathcal{D}_{\mathfrak{F}} * \mathbb{Z} \to Q \to 0$ for an appropriate $\mathcal{S}$-module Q contains the following exact part:

$$\ldots \to \operatorname{Ext}^2_{\mathcal{S}}(\mathcal{D}_{\mathfrak{F}} * \mathbb{Z}, M') \to \operatorname{Ext}^2_{\mathcal{S}}(M, M') \to \operatorname{Ext}^3_{\mathcal{S}}(Q, M') \to \ldots$$

Here, the first term vanishes by (b) and Lemma 5.1.3, and the third term vanishes by property (a). By exactness, we conclude that the term in the middle also vanishes. Thus, the claim follows from Lemma 5.1.3.

(d) This follows directly from (c) and Lemma 5.2.2. □

Lemma 5.2.8.

(a) $\operatorname{Tor}^{\mathfrak{F}}_1(\mathcal{D}_{\mathfrak{F}}, \mathcal{D}_{\mathfrak{F}}) = 0$.

(b) $\operatorname{Tor}^{\mathcal{S}}_2(\mathcal{D}_{\mathfrak{F}} * \mathbb{Z}, N) = 0$ *for every left $\mathcal{S}$-module N.*

(c) $\operatorname{Tor}^{\mathcal{S}}_1(\mathcal{D}_{\mathfrak{F}} * \mathbb{Z}, N) = 0$ *for every left $\mathcal{D}_{\mathfrak{F}} * \mathbb{Z}$-module N.*

(d) $\operatorname{Tor}^{\mathcal{S}}_1(\mathcal{D}_{\mathfrak{F}} * \mathbb{Z}, N) = 0$ *for every left $\mathcal{S}$-submodule $N \leqslant \mathcal{D}_{\mathcal{S}}$.*

(e) $\operatorname{Tor}^{\mathcal{S}}_1(\mathcal{D}_{\mathcal{S}}, N) = 0$ *for every left $\mathcal{S}$-submodule $N \leqslant \mathcal{D}_{\mathcal{S}}$.*

(f) $\operatorname{Tor}^{\mathcal{S}}_1(N, \mathcal{D}_{\mathcal{S}}) = 0$ *for every right $\mathcal{S}$-submodule $N \leqslant \mathcal{D}_{\mathcal{S}}$.*

(g) $\operatorname{Tor}^{\mathcal{S}}_1(\mathcal{D}_{\mathcal{S}}, \mathcal{D}_{\mathcal{S}}) = 0$.

Proof. (a) Since $\mathfrak{F}$ is a fir, we know that $\mathcal{D}_{\mathfrak{F}}$ is the universal localization of $\mathfrak{F}$ with respect to the set of all full matrices, so this follows from Theorem 5.2.5.

(b) The flat dimension of a module is at most its projective dimension, so this follows from Lemma 5.2.7 (b) and Lemma 5.1.4.

(c) Observe that $\mathcal{D}_{\mathfrak{F}} * \mathbb{Z}$ is isomorphic to $\mathcal{D}_{\mathfrak{F}} \otimes_{\mathfrak{F}} \mathcal{S}$ as a right $\mathcal{S}$-module by Lemma 5.2.1 and that $\mathcal{S}$ is a free left $\mathfrak{F}$-module (in particular flat). Thus, Lemma 5.1.5, together with (a) and Theorem 5.2.5 (b), tells us that

$$\operatorname{Tor}_1^{\mathcal{S}}(\mathcal{D}_{\mathfrak{F}} * \mathbb{Z}, N) \cong \operatorname{Tor}_1^{\mathfrak{F}}(\mathcal{D}_{\mathfrak{F}}, N) \cong \operatorname{Tor}_1^{\mathcal{D}_{\mathfrak{F}}}(\mathcal{D}_{\mathfrak{F}}, N) = 0.$$

(d) We have a short exact sequence $0 \to N \to \mathcal{D}_{\mathcal{S}} \to Q \to 0$ for some left $\mathcal{S}$-module Q. Applying $\mathcal{D}_{\mathfrak{F}} * \mathbb{Z} \otimes_{\mathcal{S}} ?$ to this sequence, we obtain a long exact sequence that contains the following subsequence:

$$\ldots \to \operatorname{Tor}_2^{\mathcal{S}}(\mathcal{D}_{\mathfrak{F}} * \mathbb{Z}, Q) \to \operatorname{Tor}_1^{\mathcal{S}}(\mathcal{D}_{\mathfrak{F}} * \mathbb{Z}, N) \to \operatorname{Tor}_1^{\mathcal{S}}(\mathcal{D}_{\mathfrak{F}} * \mathbb{Z}, \mathcal{D}_{\mathcal{S}}) \to \ldots$$

Since the first and third term vanish by (b) and (c), respectively, we obtain the result.

(e) Let

$$\ldots \to P_k \to \ldots \to P_0 \to N \to 0$$

be a projective resolution of N. We can compute $\operatorname{Tor}_1^{\mathcal{S}}(\mathcal{D}_{\mathcal{S}}, N)$ as the first homology group of the $\mathcal{S}$-chain complex

$$\ldots \to \mathcal{D}_{\mathcal{S}} \otimes_{\mathcal{S}} P_k \to \ldots \to \mathcal{D}_{\mathcal{S}} \otimes_{\mathcal{S}} P_0 \to 0.$$

Since $\mathcal{D}_{\mathcal{S}} \otimes_{\mathcal{S}} ? \cong \mathcal{D}_{\mathcal{S}} \otimes_{\mathcal{D}_{\mathfrak{F}} * \mathbb{Z}} \mathcal{D}_{\mathfrak{F}} * \mathbb{Z} \otimes_{\mathcal{S}} ?$, this complex is $\mathcal{S}$-isomorphic to:

$$C_*\colon \quad \ldots \to \mathcal{D}_{\mathcal{S}} \otimes_{\mathcal{D}_{\mathfrak{F}} * \mathbb{Z}} \mathcal{D}_{\mathfrak{F}} * \mathbb{Z} \otimes_{\mathcal{S}} P_k \to \ldots \to \mathcal{D}_{\mathcal{S}} \otimes_{\mathcal{D}_{\mathfrak{F}} * \mathbb{Z}} \mathcal{D}_{\mathfrak{F}} * \mathbb{Z} \otimes_{\mathcal{S}} P_0 \to 0.$$

Using that $\mathcal{D}_{\mathcal{S}}$ is the Ore division ring of fractions of $\mathcal{D}_{\mathfrak{F}} * \mathbb{Z}$, which implies that the functor $\mathcal{D}_{\mathcal{S}} \otimes_{\mathcal{D}_{\mathfrak{F}} * \mathbb{Z}} ?$ is exact, we obtain that $H_*(C_*) \cong \mathcal{D}_{\mathcal{S}} \otimes_{\mathcal{D}_{\mathfrak{F}} * \mathbb{Z}} H_*(D_*)$, where

$$D_*\colon \quad \ldots \to \mathcal{D}_{\mathfrak{F}} * \mathbb{Z} \otimes_{\mathcal{S}} P_k \to \ldots \to \mathcal{D}_{\mathfrak{F}} * \mathbb{Z} \otimes_{\mathcal{S}} P_0 \to 0.$$

But the homology of this complex computes $\operatorname{Tor}_k^{\mathcal{S}}(\mathcal{D}_{\mathfrak{F}} * \mathbb{Z}, N)$, and thus

$$\operatorname{Tor}_1^{\mathcal{S}}(\mathcal{D}_{\mathcal{S}}, N) \cong H_1(C_*) \cong \mathcal{D}_{\mathcal{S}} \otimes_{\mathcal{D}_{\mathfrak{F}} * \mathbb{Z}} H_1(D_*) \cong \mathcal{D}_{\mathcal{S}} \otimes_{\mathcal{D}_{\mathfrak{F}} * \mathbb{Z}} \operatorname{Tor}_1^{\mathcal{S}}(\mathcal{D}_{\mathfrak{F}} * \mathbb{Z}, N) \stackrel{\text{(d)}}{=} 0.$$

(f) Every step in the proof of (e) can be adapted for right modules since $\mathcal{S}$ is also a free right $\mathfrak{F}$-module, and we can apply Lemma 5.2.7, Lemma 5.2.1 and the corresponding version of Lemma 5.1.5 for right modules.

(g) This is a special case of (e). □

We obtain from the previous results a weaker version of conditions (2) of Theorem 5.2.3 and Theorem 5.2.4:

Proposition 5.2.9. *For every finitely generated left or right $\mathcal{S}$-submodule M of $\mathcal{D}_{\mathcal{S}}$ and every exact sequence $0 \to J \to \mathcal{S}^n \to M \to 0$, the $\mathcal{S}$-module J is finitely generated projective.*

Proof. Since M has projective dimension at most 1 by Lemma 5.2.7 (d) and $\mathcal{S}^n$ is projective, it follows from Schanuel's lemma that J is projective.

If M is a left $\mathcal{S}$-module and we apply the functor $\mathcal{D}_{\mathcal{S}} \otimes_{\mathcal{S}} ?$ to the short exact sequence defining J, the sequence remains exact by Lemma 5.2.8 (e). In particular, $\mathcal{D}_{\mathcal{S}} \otimes_{\mathcal{S}} J$ is isomorphic to a $\mathcal{D}_{\mathcal{S}}$-submodule of the finitely generated $\mathcal{D}_{\mathcal{S}}$-module $(\mathcal{D}_{\mathcal{S}})^n$. But $\mathcal{D}_{\mathcal{S}}$ is a division ring, thus $\mathcal{D}_{\mathcal{S}} \otimes_{\mathcal{S}} J$ is itself finitely generated. Since J is projective, [LLS03, Lemma 4] applies and we obtain that J is finitely generated. □

We finally have all the necessary ingredients for the proof of Theorem 5.A.

Proof of Theorem 5.A. By Lemma 5.2.8 (g), the conditions (1) of Theorem 5.2.4 and Theorem 5.2.3 are satisfied for $\mathcal{S} \hookrightarrow \mathcal{D}_{\mathcal{S}}$, while we obtain from Proposition 5.2.9 that the module I appearing in the conditions (2) is finitely generated and projective. Therefore, if every finitely generated projective $\mathcal{S}$-module is stably free (resp. free), we deduce that $\mathcal{S}$ is a pseudo-Sylvester domain (resp. Sylvester domain). Conversely, over a pseudo-Sylvester domain every finitely generated projective module is stably free (see [Coh06, Proposition 5.6.2]), while Sylvester domains are always projective-free (see [Coh06, Proposition 5.5.7]).

In any of the previous cases, we conclude from the criteria that $\mathcal{D}_{\mathcal{S}} = \mathrm{Ore}(\mathcal{D}_{\mathfrak{F}} * \mathbb{Z})$ is the universal division $\mathfrak{F} * \mathbb{Z}$-ring of fractions, and hence isomorphic to the universal localization of $\mathfrak{F} * \mathbb{Z}$ with respect to the set of all stably full (resp. full) matrices. $\square$

As we mentioned in Section 5.1.4, one could also use the results of Cohn and Schofield in [CS82] to deduce Theorem 5.A b) from a).

5.3 Application to free-by-{infinite cylic} groups

The aim of this section is to prove Theorem 5.B. Thus, throughout this section the main object of study will be a crossed product $E * G$, where E is a division ring and G denotes a group that fits into a short exact sequence

$$1 \to F \to G \to \mathbb{Z} \to 1$$

with F a non-necessarily finitely generated free group. Since $\mathbb{Z}$ is a free group, any such extension splits and G arises as a semi-direct product $F \rtimes \mathbb{Z}$.

The crossed product $E * G$ can be expressed as an iterated crossed product $(E * F) * \mathbb{Z}$ by [Pas89, Lemma 1.3], using that the free subgroup F is normal in G. Since $E * F$ is a fir, we are in the situation of Theorem 5.A with $\mathfrak{F} = E * F$ and $\mathcal{S} = E * G$.

In this section, we use $\mathcal{D}_{E*F}$ to denote the universal division $E*F$-ring of fractions and set $\mathcal{D}_{E*G} = \mathrm{Ore}(\mathcal{D}_{E*F} * \mathbb{Z})$. The apparent notational collision with the division closure of a group ring in the algebra of affiliated operators introduced in Definition 2.1.7 is not an oversight. To see why, observe that if $K \leqslant \mathbb{C}$ is a field, then the division closure of the group ring KF of the free group F in the algebra of affiliated operators $\mathcal{U}(G)$ is isomorphic as an epic KF-ring to the universal division ring of fractions $\mathcal{D}_{KF}$ considered in this chapter ([Lin93], see also [Lüc02, Lemma 10.81]). Then, since both $\mathcal{D}_{KG}$ and the division closure of KG in $\mathcal{U}(G)$ are obtained from these isomorphic division rings by extending the crossed product structure of $KF * \mathbb{Z}$ and passing to Ore division rings of fractions, also these division rings are isomorphic (as KG-rings).

In Section 5.3.1 we use the Farrell–Jones conjecture in algebraic K-theory to show that $E * G$ is always a pseudo-Sylvester domain. Whether this ring is even a Sylvester domain is a much more delicate question and not much can be said in general. In Section 5.3.2 we give examples of group rings for which this question has a known answer.

5.3.1 The Farrell–Jones conjecture and the proof of Theorem 5.B

In this subsection we use recent results on the Farrell–Jones conjecture to prove that the finitely generated projective $E*G$-module J that appears in condition (2) of Theorem 5.2.4 is actually stably free, which will conclude the first part of the proof of Theorem 5.B. The following piece of the algebraic K-theory of a ring is needed to phrase the results:

Definition 5.3.1. Let R be a ring. Then we denote by $K_0(R)$ the abelian group generated by the isomorphism classes $[P]$ of finitely generated projective R-modules together with the relations

$$[P \oplus Q] - [P] - [Q] = 0$$

for all finitely generated projective R-modules P and Q.

Every element of $K_0(R)$ is of the form $[P] - [P']$ for finitely generated projective R-modules P and P'. The identity $[P] = [P'] \in K_0(R)$ holds for two finitely generated projective R-modules P and P' if and only if there is a finitely generated projective R-module Q such that $P \oplus Q \cong P' \oplus Q$, where Q can even be taken to be free.

If $f\colon R \to S$ is a ring homomorphism and P is a finitely generated projective R-module, then $S \otimes_R P$ is a finitely generated projective S-module. In this way, $K_0(?)$ becomes a functor from rings to abelian groups.

The conditions of Remark 5.1.8 are satisfied for $K_0(?)$ and thus it does not depend on whether we use left or right modules in its definition.

We will need the following consequence of the Farrell–Jones conjecture which is certainly well-known, but has not been made explicit in the literature. For further references on the conjecture, see Section 1.3.

Proposition 5.3.2. *Let E be a division ring, Γ a torsion-free group and $E * \Gamma$ a crossed product. If the K-theoretic Farrell–Jones conjecture with coefficients in an additive category holds for Γ, then the embedding $E \hookrightarrow E * \Gamma$ induces an isomorphism*

$$K_0(E) \xrightarrow{\cong} K_0(E * \Gamma).$$

In particular, since $K_0(E) = \{n[E] \mid n \in \mathbb{Z}\}$, every finitely generated projective $E * \Gamma$-module is stably free.

Proof. For a given crossed product $E * \Gamma$, we will denote the additive category defined in [BR07, Corollary 6.17] by $\mathcal{A}_{E*\Gamma}$. We will freely use the terminology and notation of that paper. Furthermore, we will denote the family of virtually cyclic subgroups of a given group by VCyc and the family consisting just of the trivial subgroup by Triv. The K-theoretic Farrell–Jones conjecture for the group Γ with coefficients in the additive category $\mathcal{A}_{E*\Gamma}$ arises as an instance of the more general meta-isomorphism conjecture [Lüc19, Conjecture 13.2] for the Γ-homology theory $\mathcal{H}_*^\Gamma(?; \mathbf{K}_{\mathcal{A}_{E*\Gamma}})$ introduced in [BR07] and the family $\mathcal{F} = \mathrm{VCyc}$. It states that the assembly map

$$\mathcal{H}_*^\Gamma(\mathcal{E}_{\mathrm{VCyc}}(\Gamma); \mathbf{K}_{\mathcal{A}_{E*\Gamma}}) \to \mathcal{H}_*^\Gamma(\mathrm{pt}; \mathbf{K}_{\mathcal{A}_{E*\Gamma}})$$

is an isomorphism, where the right-hand side is isomorphic to $K_*(E * \Gamma)$ by [BR07, Corollary 6.17].

In order to arrive at the desired conclusion, we need to reduce the family from VCyc to Triv. Since Γ is assumed to be torsion-free and hence all its virtually cyclic subgroups are infinite cyclic, we can arrange for this via the transitivity principle of [Lüc19, Theorem 13.13 (i)] if the meta-isomorphism conjecture holds for the $\mathbb{Z}$-homology theory $\mathcal{H}_*^{\mathbb{Z}}(?; \mathbf{K}_{\mathcal{A}_{E*\mathbb{Z}}})$ and the family $\mathcal{F} = \mathrm{Triv}$. A model for the classifying space $\mathcal{E}_{\mathrm{Triv}}(\mathbb{Z})$ is given by $\mathbb{R}$ and we may again assume that the crossed product $E * \mathbb{Z}$ is a skew Laurent polynomial ring $E[t^{\pm 1}; \tau]$. In this situation, since E is regular (not to be confused with von Neumann regular), the assembly map coincides with the map provided by the analogue of the Fundamental Theorem of algebraic K-theory for skew Laurent polynomial rings, which is an isomorphism (see [BL20, Theorems 6.8 & 9.1] or [Gra88] for a more classical treatment).

Since the K-theoretic Farrell–Jones conjecture with coefficients in an additive category is assumed to hold for Γ, we now obtain from the transitivity principle that the assembly map

$$\mathcal{H}_*^\Gamma(\mathcal{E}_{\mathrm{Triv}}(\Gamma); \mathbf{K}_{\mathcal{A}_{E*\Gamma}}) \to \mathcal{H}_*^\Gamma(\mathrm{pt}; \mathbf{K}_{\mathcal{A}_{E*\Gamma}}) \cong K_*(E*\Gamma)$$

is an isomorphism. The space $\mathcal{E}_{\mathrm{Triv}}$ is a free Γ-space and the value at the coset $\Gamma/\{1\}$ of the $\mathrm{Or}(\Gamma)$-spectrum $\mathbf{K}_{\mathcal{A}_{E*\Gamma}}$ is $\mathbb{K}^{-\infty}(\mathcal{A}_{E*\Gamma} * \Gamma/\{1\})$. We can thus simplify the left-hand side of the assembly map as follows:

$$\mathcal{H}_*^\Gamma(\mathcal{E}_{\mathrm{Triv}}(\Gamma); \mathbf{K}_{\mathcal{A}_{E*\Gamma}}) \cong H_*(B\Gamma; \mathbb{K}^{-\infty}(\mathcal{A}_{E*\Gamma} * \Gamma/\{1\})).$$

Here, $B\Gamma$ denotes the standard classifying space of the group Γ and homology is taken with local coefficients. Using [BR07, Corollary 6.17] once more, we observe that $\mathbb{K}^{-\infty}(\mathcal{A}_{E*\Gamma} * \Gamma/\{1\})$ is weakly equivalent to $\mathbb{K}^{-\infty}(E)$, which is connective by [Lüc19, Theorem 3.6] since E is a regular ring. In particular, the Atiyah–Hirzebruch spectral sequence provides the following natural isomorphism:

$$H_0(B\Gamma; \mathbb{K}^{-\infty}(\mathcal{A}_{E*\Gamma} * \Gamma/\{1\})) \cong H_0(B\Gamma; \pi_0(\mathbb{K}^{-\infty}(\mathcal{A}_{E*\Gamma} * \Gamma/\{1\}))),$$

where homology is again taken with local coefficients. Since $\pi_0(\mathbb{K}^{-\infty}(\mathcal{A}_{E*\Gamma} * \Gamma/\{1\})) \cong K_0(\mathcal{A}_{E*\Gamma} * \Gamma/\{1\})$ and the Γ-action on $\mathcal{A}_{E*\Gamma} * \Gamma/\{1\}$, which is induced from that on the Γ-space $\Gamma/\{1\}$, preserves isomorphism types, the local coefficients are in fact constant. We conclude that

$$H_0(B\Gamma; \mathbb{K}^{-\infty}(\mathcal{A}_{E*\Gamma} * \Gamma/\{1\})) \cong H_0(B\Gamma; K_0(E)),$$

and thus the assembly map in degree 0 simplifies to

$$K_0(E) \cong H_0(B\Gamma; K_0(E)) \xrightarrow{\cong} K_0(E*\Gamma).$$

This proves the first statement.

The second statement is a direct consequence since every finitely generated projective $E*\Gamma$-module P represents an element $n[E*\Gamma]$ in $K_0(E*\Gamma)$ for some $n \geqslant 0$ and thus there exists a finitely generated free $E*\Gamma$-module Q such that $P \oplus Q \cong (E*\Gamma)^n \oplus Q$, which is free. □

The following is the K-theoretic part of [BFW19, Theorem 1.1] in the case of a finitely generated free group F and [BKW19, Theorem A] in the general case:

Theorem 5.3.3. *The K-theoretic Farrell–Jones conjecture with coefficients in an additive category holds for every group that arises as an extension*

$$1 \to F \to G \to \mathbb{Z} \to 1$$

with F a (not necessarily finitely generated) free group.

The previous result provides the final step in the proof of Theorem 5.B.

Proof of Theorem 5.B. Since G satisfies the K-theoretic Farrell–Jones conjecture with coefficients in additive categories by Theorem 5.3.3, we obtain from Proposition 5.3.2 that every finitely generated projective $E*G$-module is stably free. Therefore, the statement follows from Theorem 5.A. □

5.3.2 Examples and non-examples

The main examples of groups of the form $1 \to F \to G \to \mathbb{Z} \to 1$ are the free-by-{infinite cyclic} groups (terminology usually reserved in the literature for the case where F is finitely generated) and fundamental groups of connected closed surfaces with genus $g \geqslant 1$ other than the projective plane (which has to be excluded since its fundamental group has torsion). In the latter family, we have to distinguish the fundamental groups S_g of orientable closed surfaces of genus $g \geqslant 1$, which admit the presentations

$$S_g = \langle a_1, b_1, \ldots, a_g, b_g \mid [a_1, b_1] \cdot \ldots \cdot [a_g, b_g] \rangle,$$

and the fundamental groups of non-orientable closed surfaces of genus $g \geqslant 2$, which admit the presentations

$$\mathfrak{S}_g = \langle a_1, \ldots, a_g \mid a_1^2 \cdot \ldots \cdot a_g^2 \rangle.$$

That these groups contain a normal free subgroup F such that G/F is infinite cyclic is a consequence of the fact that their infinite index subgroups are free (see [HKS72]) and that their abelianizations contain an infinite cyclic summand.

Within these families, there are some cases of group rings for which it is known whether they admit stably free cancellation. In the following examples, K is any field of characteristic 0.

- Examples of group rings of free-by-{infinite cyclic} groups with stably free cancellation and thus of Sylvester domains are $K[\mathbb{Z}^2] = K[S_1]$ (see [Swa78]) and $K[F_2 \times \mathbb{Z}]$ (see [Bas68, IV.6.4], using that $K[\mathbb{Z}]$ is a PID and thus a projective-free Dedekind domain).
- Examples of group rings which do admit non-free stably free modules are given by $K[\mathbb{Z} \rtimes \mathbb{Z}] = K[\mathfrak{S}_2]$ (see [Sta85, Theorem 2.12]) and $\mathbb{Q}[\langle x, y \mid x^3 = y^2 \rangle] = \mathbb{Q}[F_2 \rtimes \mathbb{Z}]$ (see [Lew82] and note that the non-free projective ideal in the main theorem is actually stably free). Here, the latter example is the rational group ring of the fundamental group of the complement of the trefoil knot, which fibers over the circle and hence admits a free-by-{infinite cyclic} fundamental group (see [BZH14, Corollary 4.12]). Both group rings serve as examples of pseudo-Sylvester domains that are not Sylvester domains.

There do not seem to be any similar results for surface groups of higher genus:

Open Problem. *Do the rings $\mathbb{C}[S_g]$ for $g \geqslant 2$ and $\mathbb{C}[\mathfrak{S}_g]$ for $g \geqslant 3$ have stably free cancellation?*

As we have seen, this is equivalent to asking whether the group rings are Sylvester domains.

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