Natural maps in higher Teichmüller theory

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Summary

In this thesis we consider harmonic maps and barycentric maps in the context of higher Teichmüller theory. We are particularly interested in how these maps can be used to study Hitchin representations. The main results of this work are as follows.

Our first result states that equivariant harmonic maps into non-compact symmetric spaces that satisfy suitable non-degeneracy conditions depend in a real analytic fashion on the metric of the domain manifold and the representations they are associated to.

For our second result we consider the energy functional on Teichmüller space that is associated to a Hitchin representation. We prove that this functional is strictly plurisubharmonic for Hitchin representations into either $\mathrm{PSL}(n, \mathbb{R})$, $\mathrm{PSp}(2n, \mathbb{R})$, $\mathrm{PSO}(n, n+1)$ or G_2.

In the third part of this thesis we examine the energy functional on Teichmüller space that is associated to a metric on a surface. We prove that the simple length spectrum of a non-positively curved metric is determined by its energy functional. We use this to prove that hyperbolic metrics and singular flat metrics induced by quadratic differentials are determined, up to isotopy, by their energy functional.

Our next result concerns the harmonic heat flow for maps from a compact Riemannian manifold into a Riemannian manifold of non-positive curvature. We prove that if the harmonic heat flow converges to a harmonic map that is a non-degenerate critical point of the Dirichlet energy, then it converges exponentially fast.

In the final part of this thesis we study the barycenter construction of Besson–Courtois–Gallot. We prove that for any Fuchsian representation and Hitchin representation into $\mathrm{SL}(n, \mathbb{R})$ there exists a natural map $\mathbb{H}^2 \to \mathrm{SL}(n, \mathbb{R}) / \mathrm{SO}(n)$ that intertwines the actions of the two representations. We put these maps forward as a new way to parametrise and study Hitchin components.

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Contents

Introduction

Higher Teichmüller theory is, in a broad sense, the study of representations of surface groups into Lie groups of higher rank. A particular emphasis is placed on the study of connected components of representation varieties that consist of representations that exhibit nice geometric and dynamical behaviour. Suppose that S is a closed, connected and orientable surface of genus at least two and that G is a simple Lie group that has rank two or higher. Then the representation variety

$$\mathrm{Rep}(\pi_1(S), G) = \mathrm{Hom}(\pi_1(S), G)/G$$

is the set of reductive[1] representations of $\pi_1(S)$ into G, considered up to conjugation by elements in G. It can be equipped with a topology that is induced from the topology of the Lie group G. Of particular interest are the connected components of these spaces that consist entirely of representations that are both discrete and faithful. A space that consist of a union of such components is called a higher Teichmüller space (here we follow [Wie18]).

These higher Teichmüller spaces can be seen as a direct generalisation of the classical Teichmüller space. Namely, the classical Teichmüller space of the surface S can be realised as a connected component of the representation variety $\mathrm{Rep}(\pi_1(S), \mathrm{PSL}(2, \mathbb{R}))$ that consists entirely of discrete and faithful representations. The word 'higher' in higher Teichmüller theory refers to the fact that the rank one Lie group $\mathrm{PSL}(2, \mathbb{R})$ is replaced by a Lie group of higher rank. In recent years many interesting parallels have been found between the higher Teichmüller spaces and classical Teichmüller space. In addition, a number of phenomena have been observed that occur only in higher rank. An excellent introduction to higher Teichmüller theory and an account of recent developments can be found in [Wie18].

The first instances of higher Teichmüller spaces were discovered by Hitchin in [Hit92]. He identified certain connected components of the representation varieties $\mathrm{Rep}(\pi_1(S), G)$ (when G is a split real simple Lie group) that contain a copy of the Teichmüller space of S. These connected components are now called Hitchin components. Representations whose conjugacy class lies in such a component are called Hitchin representations. The tools Hitchin used to describe these components were predominantly (complex) analytic in nature. The fact that Hitchin representations are discrete and faithful (and hence, that the Hitchin components are higher Teichmüller spaces in the sense of [Wie18]) was proved later by Labourie in [Lab06] and, independently, by Fock and Goncharov in [FG06]. The methods used in these works are, in contrast to those employed by Hitchin, of a geometrical and dynamical nature. The fact that the Hitchin

[1] A representation is called reductive if the Zariski closure of its image in G is a reductive subgroup.

components can be approached in so many different ways (analytic, geometric and dynamical) makes them a particularly fruitful area of research.

In this thesis our focus lies mostly on the analytical aspects of Hitchin representations. We will study in particular the equivariant harmonic maps that appear in the theory. A harmonic map is a map between Riemannian manifolds that is a critical point of the Dirichlet energy functional. A foundational paper in the theory of harmonic maps is [ES64]. In order to discuss equivariant harmonic maps we first introduce some notation. Let $X = (S, J)$ denote the Riemann surface[2] obtained by equipping the surface S with a complex structure J. Denote by $\widetilde{X}$ its universal cover. We take G to be a semisimple Lie group without compact factors, $K \subset G$ a maximal compact subgroup and we denote by G/K the associated symmetric space. Now, if $\rho\colon \pi_1(S) \to G$ is a representation of $\pi_1(S)$ in G, then a ρ-equivariant harmonic map is a map

$$f\colon \widetilde{X} \to G/K$$

that is harmonic and satisfies $f(\gamma x) = \rho(\gamma) f(x)$ for all $x \in \widetilde{X}$ and $\gamma \in \pi_1(S)$. Before we discuss the results of this thesis let us first give a brief account of how these maps fit into the analytical theory of Hitchin representations.

The analytic tools that were used by Hitchin in [Hit92] to study representation varieties were provided by the Non-Abelian Hodge correspondence. This correspondence provides an identification between representation varieties of complex algebraic groups and moduli spaces of Higgs bundles on Riemann surfaces. It was developed by Donaldson, Corlette, Hitchin and Simpson ([Don87, Cor88, Hit87, Sim88]). In this introduction we restrict ourselves to the discussion of the Non-Abelian Hodge correspondence for the algebraic group $G = \mathrm{SL}(n, \mathbb{C})$. An $\mathrm{SL}(n, \mathbb{C})$-Higgs bundle over a Riemann surface X is a pair (E, ϕ) consisting of a holomorphic vector bundle E over X and a Higgs field ϕ which is a holomorphic section of $K_X \otimes \mathrm{End}_0(E)$. Here K_X denotes the canonical bundle of X and $\mathrm{End}_0(E)$ denotes the vector bundle of trace free endomorphisms of E. The Non-Abelian Hodge correspondence gives an identification between $\mathrm{Rep}(\pi_1(S), G)$ and $\mathcal{M}_{\mathrm{Higgs}}(G)$, the moduli space of gauge equivalence classes of polystable (see Definition 2.2.1) $\mathrm{SL}(n, \mathbb{C})$-Higgs bundles.

Equivariant harmonic maps provide an important intermediate step in the procedure that assigns to each representation a corresponding Higgs bundle. Let us briefly describe this procedure (a more thorough account is given in Section 2.2). Given a representation $\rho\colon \pi_1(S) \to G$ we consider the flat vector bundle $E = (\widetilde{X} \times \mathbb{C}^n)/\pi_1(S)$, where the action of $\pi_1(S)$ on the $\mathbb{C}^n$ factor is determined by ρ. Metrics on this bundle are in one-to-one correspondence with ρ-equivariant maps $\widetilde{X} \to G/K$ (with $K = \mathrm{SU}(n)$). Any such metric determines a splitting of the flat connection D on E into two parts. Namely, $D = \nabla + \Psi$ where ∇ is a connection compatible with the metric and $\Psi \in \Omega^1(\mathrm{End}_0(E))$ is a Hermitian endomorphism. It turns out that the ρ-equivariant map associated to

[2]When the domain manifold is a surface the Dirichlet energy depends only on the conformal class of the metric on the domain. Hence, we choose a complex structure on S rather than a Riemannian metric.

the metric is harmonic if and only if $\nabla^{0,1}\Psi^{1,0} = 0$. The existence of a harmonic equivariant map is provided by a result of Corlette ([Cor88]) when ρ is a reductive representation. So, using the existence of a harmonic equivariant map we find a splitting $D = \nabla + \Psi$ with $\nabla^{0,1}\Psi^{1,0} = 0$. It follows that if we equip E with the complex structure determined by $\nabla^{0,1}$, then $\phi = \Psi^{1,0}$ is a holomorphic section of the bundle $K_X \otimes \mathrm{End}_0(E)$. Hence, $(E, \nabla^{0,1}, \phi)$ is an $\mathrm{SL}(n,\mathbb{C})$-Higgs bundle. This assignment of the Higgs bundle $(E, \nabla^{0,1}, \phi)$ to a representation ρ is one direction of the Non-Abelian Hodge correspondence.

Hitchin identified the Hitchin components by giving an explicit parametrisation of a component of the representation variety in terms of Higgs bundles. Let us again restrict to the case $G = \mathrm{SL}(n,\mathbb{C})$. By applying the Chern-Weil construction to Higgs fields, Hitchin defined a projection map $p_X \colon \mathcal{M}_{\mathrm{Higgs}}(G) \to \bigoplus_{i=2}^n H^0(X; K_X^i)$. For some choice of holomorphic vector bundle $K_X^{1/2}$ over X that satisfies $(K_X^{1/2})^{\otimes 2} = K_X$ he then considered the bundle

$$E = K_X^{\frac{n-1}{2}} \oplus K_X^{\frac{n-3}{2}} \oplus \ldots \oplus K_X^{\frac{3-n}{2}} \oplus K_X^{\frac{1-n}{2}}.$$

Then $K_X \otimes \mathrm{End}_0(E) \subset \bigoplus_{i,j}^n K_X^{i-j+1}$. A section $s_X \colon \bigoplus_{i=2}^n H^0(X; K_X^i) \to \mathcal{M}_{\mathrm{Higgs}}(G)$ of the projection p_X can be constructed by setting

$$s_X(q_2, \ldots, q_n) = \left(E,\ \phi = \begin{pmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ 0 & 0 & 0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 1 \\ q_n & \ldots & q_3 & q_2 & 0 \end{pmatrix} \right).$$

Hitchin proved in [Hit92] that the representations corresponding to the image points of this section via the Non-Abelian Hodge correspondence take values in $\mathrm{SL}(n,\mathbb{R})$. Furthermore, he proved that these representations constitute precisely a connected component of $\mathrm{Rep}(\pi_1(S), \mathrm{SL}(n,\mathbb{R}))$. This component is the Hitchin component.

The connection (via the Non-Abelian Hodge correspondence) between Hitchin representations and these very explicitly defined Higgs bundles allows for investigating Hitchin representations by analytic methods. Moreover, because the Higgs fields are closely linked to the equivariant harmonic maps, it is possible to derive properties of these maps by studying the Higgs fields. We use this fact, for example, in Chapter 2. A more detailed description of these connections can be found in the survey [Li19].

The parametrisation of the Hitchin component by the space $\bigoplus_{i=2}^n H^0(X; K_X^i)$ that we described here does, however, carry a disadvantage. Namely, it does not respect the natural action of the mapping class group on the Hitchin component. The mapping class group of S can be realised as the group of outer automorphisms of the group $\pi_1(S)$. By composing representations of $\pi_1(S)$ with such an automorphism we obtain a natural action of the mapping class group on the Hitchin component. This symmetry is broken in Hitchin's parametrisation because of the need to pick a fixed complex structure on the surface S (which amounts to picking a basepoint in the Teichmüller space of S).

A method to modify the parametrisation, such that it becomes equivariant for the action of the mapping class group, was proposed by Labourie in [Lab08]. Let us denote by $\mathcal{T}(S)$ the Teichmüller space of S, which we realise as the space of complex structures on S up to isotopy (see Section 3.2.1). Labourie considered, given a Hitchin representation $\rho\colon \pi_1(S) \to G$, the energy functional $E\colon \mathcal{T}(S) \to \mathbb{R}$ that assigns to each point $[J] \in \mathcal{T}(S)$ the energy of the associated equivariant harmonic map $(\widetilde{S}, J) \to G/K$. He proved that this is a proper function on Teichmüller space and hence it has a global minimum. He then made the conjecture that this minimum is always unique. If this conjecture holds true, then we obtain a mapping class group equivariant projection from the Hitchin component to Teichmüller space. Namely, if we denote by $\mathrm{Hit} \subset \mathrm{Rep}(\pi_1(S), G)$ the Hitchin component, then we can define $\pi\colon \mathrm{Hit} \to \mathcal{T}(S)$ by sending each representation to the unique point in Teichmüller space that minimises the energy functional.

For a complex structure J on S let us denote by $p_J\colon \mathrm{Hit} \to \bigoplus_{i=2}^n H^0(X; K_X^i)$ the Hitchin parametrisation with basepoint $X = (S, J)$. It is a classical observation that if a point $[J] \in \mathcal{T}(S)$ is a minimiser of the energy function of some representation $\rho\colon \pi_1(S) \to G$, then the ρ-equivariant harmonic map $(\widetilde{S}, J) \to G/K$ is a conformal mapping. This turns out to be equivalent to the condition that $q_2 = 0$ when we write $p_J(\rho) = (q_2, \ldots, q_n)$. We apply this observation as follows. Let us define the vector bundle $\mathcal{Q} \to \mathcal{T}(S)$ with fibres $\mathcal{Q}_{[J]} = \bigoplus_{i=3}^n H^0(X; K_X^i)$ (here $X = (S, J)$). Then, if Labourie's conjecture holds, we obtain a mapping class group equivariant parametrisation

$$\mathrm{Hit} \to \mathcal{Q}\colon \rho \mapsto p_{\pi(\rho)}(\rho).$$

Consequently, we can, in cases where Labourie's conjecture is true, realise the quotient of the Hitchin component by the mapping class group as a vector bundle over the moduli space of S.

The Labourie conjecture is known to be true for split real Lie groups of rank two. Namely, for $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$ it follows from results of Schoen in [Sch93]. For $\mathrm{SL}(3,\mathbb{R})$ it was proved, independently, by Loftin in [Lof01] and Labourie in [Lab07]. The remaining cases where proved in [Lab17]. An analogue of Labourie's conjecture has been proven to hold for maximal representations into rank two Hermitian Lie groups (see [CTT19] and the references therein). A case where Labourie's conjecture does not hold has been found by Marković in a recent preprint ([Mar21]). He proved the conjecture is false[3] for the group $\Pi_{i=1}^3 \mathrm{PSL}(2,\mathbb{R})$. However, for simple split real Lie groups of rank three and higher the conjecture remains open.

The main subjects of study in this thesis are the equivariant harmonic maps and the energy functionals on Teichmüller space associated to Hitchin

[3]Actually, Marković disproved a stronger formulation of the Labourie conjecture than we have stated here. Namely, he showed that critical points (rather than minima) of the energy functional are not unique.

representations. As we have outlined here, these objects play an important role in the study of the Hitchin components. We now give a brief summary of the contents of this thesis.

Contents of the thesis

This thesis is cumulative in nature and most chapters correspond to papers that have appeared previously as preprints. Other chapters will appear as papers at a later time. Consequently, all chapters (except for Chapter 4) can be read independently. Each chapter is accompanied by an introduction that gives a synopsis of the results of that chapter. So we will not give a detailed overview of the results of the thesis here. Rather, we briefly describe the results of each chapter and discuss how they fit into the broader story that we have sketched in this introduction.

Chapter 1

The first chapter of this thesis corresponds to the paper [Sle20a].

In this chapter we prove that, under certain non-degeneracy conditions, equivariant harmonic maps depend in a real analytic fashion on the representation they are associated to. Our main motivation for proving this result is to apply it to harmonic maps that are equivariant for Hitchin representations. Our proof, however, is valid in a more general context. Namely, we consider representations of fundamental groups of manifolds of any dimension.

Let M be a compact manifold and $\widetilde{M}$ its universal cover. Let G be a semisimple Lie group without compact factors, $K \subset G$ a maximal compact subgroup and let G/K be the associated symmetric space. We consider a Riemannian metric g_0 on M and a representation $\rho_0 \colon \pi_1(S) \to G$. We assume that there exists a ρ_0-equivariant harmonic map $(\widetilde{M}, g_0) \to G/K$ and we impose the non-degeneracy condition that the centraliser of the image of ρ_0 in G contains no semi-simple elements. Our main result (Theorem 1.2.7) states that for small deformations g_t of the metric g_0 and ρ_t of the representation ρ_0 there exist ρ_t-equivariant harmonic maps $(\widetilde{M}, g_t) \to G/K$ that depend real analytically on these deformations.

This result can be applied to the equivariant harmonic maps we considered in the first part of this introduction. For this we let $M = S$ and use that any Hitchin representation $\rho_0 \colon \pi_1(S) \to G$ satisfies the non-degeneracy condition we imposed in our main theorem. It follows that harmonic maps equivariant for Hitchin representations depend in a real analytic fashion on the representation. As a result we find that any quantity that can be expressed in terms of these harmonic maps also depends real analytically on the representation. For example, it follows that the energy functional that was introduced before is a real analytic function on Teichmüller space and depends in a real analytic way on the Hitchin representation.

Chapter 2

The second chapter of the thesis corresponds to the paper [Sle20b].

In this chapter we study the energy functional $E\colon \mathcal{T}(S) \to \mathbb{R}$ associated to a Hitchin representation (as introduced in the first part of this introduction). Our primary result (Theorem 2.1.1) is that if $\rho\colon \pi_1(S) \to G$ is a Hitchin representation into $G = \mathrm{PSL}(n, \mathbb{R})$, $\mathrm{PSp}(2n, \mathbb{R})$, $\mathrm{PSO}(n, n+1)$ or the exceptional group G_2, then the associated energy functional is strictly plurisubharmonic.

For the Hitchin representations we consider in this chapter the Labourie conjecture is still open (when the rank of G is larger than two). Our result provides some information regarding the critical points, and hence in particular the minima, of the energy functional. Namely, a corollary of our result is that the Hessian of the energy functional at any point in Teichmüller space is positive-definite on a subspace that has half the dimension of Teichmüller space (cf. Corollary 2.4.1). This puts, in particular, a limit on how degenerate a critical point of the energy functional can be, i.e. its nullity is bounded by half the dimension of Teichmüller space.

Chapter 3

The third chapter of the thesis corresponds to the paper [Sle21a].

The leading question in this chapter is how much information about a representation is encoded in its energy functional[4]. More specifically, we consider whether a representation is uniquely determined by its energy functional. We develop an approach to this question by first looking at a simpler situation. Namely, we let ρ be a metric of non-positive curvature on the surface S. We consider the function that assigns to each complex structure J on S the infimum of the energies of all Lipschitz maps $(S, J) \to (S, \rho)$ that are homotopic to the identity. This function descends to a function $\mathscr{E}\colon \mathcal{T}(S) \to \mathbb{R}$ that we will call the energy spectrum of ρ.

The main results of this chapter relate the energy spectrum of the metric ρ to its simple length spectrum (for a definition see Section 3.2.2). First, we prove that the energy spectrum determines the simple length spectrum (Theorem 3.3.3). Secondly, we show that the converse does not hold by exhibiting two metrics with equal simple length spectrum but different energy spectrum (Proposition 3.4.1). By combining our first theorem with results from the literature, we show that the set of hyperbolic metrics and the set of singular flat metrics induced by quadratic differentials satisfy energy spectrum rigidity, i.e. a metric in these sets is determined, up to isotopy, by its energy spectrum.

In the second part of the chapter we prove a similar result for Kleinian representations and find that, also in this case, the simple length spectrum of such a representation is determined by its energy spectrum (Theorem 3.6.1). We combine this with a result of Bridgeman and Canary to conclude that a Kleinian surface group is uniquely determined, up to conjugation, by its energy spectrum.

[4] In this chapter we use different terminology and refer to this function as the energy spectrum rather than the energy functional.

Considering the important role that the energy spectrum plays in the study of the Hitchin components, it would be interesting to know whether a similar statement holds true for Hitchin representations. We put forward our approach as a possible strategy to prove this. Our current methods do, however, not suffice. We end the chapter with a discussion on what is needed to complete such a proof.

Chapter 4

In the fourth chapter we examine the energy spectrum of metrics on a surface that are obtained by a grafting procedure. Our goal is to obtain insights into the general properties of the energy spectrum by studying it in this particular setting.

We let σ be a hyperbolic metric on S and $\gamma \subset S$ a simple closed geodesic loop. These choices determine a family of grafted surfaces $\{\mathrm{Gr}_{t\cdot\gamma}(\sigma)\}_{t\geq 0}$ (see Section 4.2.3 for a definition). For each $t \geq 0$ we consider the energy spectrum $\mathscr{E}(\cdot, t)\colon \mathcal{T}(S) \to \mathbb{R}$ of the surface $\mathrm{Gr}_{t\cdot\gamma}(\sigma)$. We will study points in Teichmüller space which almost minimise these energy spectra. It is an easy observation (Lemma 4.3.3) that, as a function of t, the minimum of $\mathscr{E}(\cdot, t)$ behaves asymptotically as $t \mapsto t \cdot \ell_\sigma(\gamma)$. Here $\ell_\sigma(\gamma)$ is the length of γ with respect to σ. So, given a constant $A > \ell_\sigma(\gamma)$, we consider $X \in \mathcal{T}(S)$ with $\mathscr{E}(X, t) \leq A \cdot t$. The main result (Proposition 4.3.4) of this chapter concerns the Fenchel–Nielson coordinates associated to γ of such points. Let us denote by $X \mapsto \ell_X(\gamma)$ the length parameter and by $X \mapsto s_X(\gamma)$ the twisting parameter (see Section 4.2.2). We prove that there exists constants $t_0 > 0$ and $c > 0$ depending only on A and $\ell_\sigma(\gamma)$ such that for all $t \geq t_0$, if a point $X \in \mathcal{T}(S)$ satisfies $\mathscr{E}(X, t) \leq A \cdot t$, then $1/(c \cdot t) \leq \ell_X(\gamma) \leq c/t$ and $|s_\sigma(\gamma) - s_X(\gamma)| \leq c \cdot t$.

The result shows (Remark 4.3.5) that, in an appropriate sense, points that almost minimise the energy spectrum stay uniformly close to the true minimising point. We hope that similar methods can, perhaps, be used to obtain information about the minimisers of the energy spectra associated to Hitchin representations.

Chapter 5

The fifth chapter of the thesis corresponds to the paper [Sle21b].

In this chapter we look at the harmonic heat flow. We identify circumstances under which it converges exponentially fast.

Let M be a compact Riemannian manifold and let N be a Riemannian manifold that is complete and non-positively curved. Let $(f_t\colon M \to N)_{t\in[0,\infty)}$ be a family of smooth maps that satisfy the harmonic heat flow equation (see Section 5.1). We assume that the maps f_t converge to a limiting harmonic map f_∞ and assume that f_∞ is a non-degenerate critical point of the Dirichlet energy functional. The main result of this chapter (Theorem 5.1.1) states that, under these assumptions, the rate at which the maps f_t converge to f_∞ is exponential (in the L^2 norm).

It is an interesting observation that the non-degeneracy condition we put on the limiting harmonic map is the same condition that is considered in Chapter 1. This is consistent with the hope of the author that the results of this chapter can, perhaps, be used as an ingredient in an alternative proof that harmonic maps depend smoothly on the Riemannian metrics used to define them.

The results of this chapter apply to harmonic maps in general and are not directly tied to higher Teichmüller theory. Let us mention, however, that a connection between the main theorem of this chapter and equivariant harmonic maps does exist. Namely, Labourie used the harmonic heat flow in [Lab91] to give a proof (different from the proof of Corlette) of the existence of equivariant harmonic maps. If a representation $\rho\colon \pi_1(S) \to G$ induces an action on the symmetric space G/K that is free and proper, then the harmonic heat flow considered by Labourie coincides with the one considered in this chapter (with $M = S$ and $N = \rho(\pi_1(S)) \setminus G/K$). Labourie proved that, in this setting, the heat flow converges to a limiting harmonic map if and only if the representation is reductive. Hence, it follows from the results of this chapter, that if the representation is reductive and satisfies the conditions of Lemma 1.2.1, then the convergence rate of the heat flow considered in Labourie's proof is exponential.

Chapter 6

In the final chapter we move away from the study of harmonic maps and instead consider the Hitchin components from a novel perspective. Namely, we consider barycentric maps that are equivariant for Hitchin representations.

The barycentric construction is a method to extend maps from the boundaries of symmetric spaces to maps between the symmetric spaces themselves. The method was introduced in [DE86] and [BCG95]. Barycentric maps have been studied in several different settings (references are given in Section 6.1). However, they have not yet been examined in the context of higher Teichmüller theory. In this chapter we begin this investigation by proving an existence result for barycentric maps in this context.

It follows from the work of Labourie in [Lab06] that Hitchin representations induce natural equivariant boundary maps. We apply the barycentric method to these boundary maps to produce equivariant barycentric maps. The main result of this chapter (Theorem 6.5.2) is as follows. If $\theta\colon \pi_1(S) \to \mathrm{SL}(2,\mathbb{R})$ is a Fuchsian representation and $\rho\colon \pi_1(S) \to \mathrm{SL}(n,\mathbb{R})$ is a Hitchin representation, then there exists a barycentric map $f_{\theta,\rho}\colon \mathbb{H}^2 \to \mathrm{SL}(n,\mathbb{R})/\,\mathrm{SO}(n)$ that intertwines the actions of θ and ρ. Furthermore, we prove that these maps depend smoothly on the representations θ and ρ (Theorem 6.8.1). Finally, we construct a novel parametrisation of the Hitchin component by assigning to each representation the corresponding barycentric map (Theorem 6.9.1).

The barycentric maps should be compared with the equivariant harmonic maps. The latter are most amenable to investigation via methods that are analytical in nature. It has proven to be much harder to study these maps from a geometrical point of view. The barycentric maps, on the other hand, are constructed using relatively explicit geometrical methods. For this reason, we believe that they could offer additional ways to study geometric aspects of Hitchin representations.

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Chapter 1

Equivariant harmonic maps depend real analytically on the representation

Abstract

We prove that when assuming suitable non-degeneracy conditions equivariant harmonic maps into symmetric spaces of non-compact type depend in a real analytic fashion on the representation they are associated to. The main tool in the proof is the construction of a family of deformation maps which are used to transform equivariant harmonic maps into maps mapping into a fixed target space so that a real analytic version of the results in [EL81] can be applied.

1.1 Introduction

In this article we prove that equivariant harmonic maps into symmetric spaces of non-compact type depend in a real analytic fashion on the representation they are associated to. Throughout this article we let M be a closed real analytic Riemannian manifold, $\widetilde{M}$ its universal cover and $\Gamma = \pi_1(M)$ its fundamental group. Also we let G be a real semisimple Lie group without compact factors and $X = G/K$ its associated symmetric space. If $\rho\colon \Gamma \to G$ is a reductive representation of Γ in G, then by work of Corlette ([Cor88]) there exists a ρ-equivariant harmonic map $f\colon \widetilde{M} \to X$. A map f is called ρ-equivariant if

$$f(\gamma m) = \rho(\gamma) f(m) \text{ for all } m \in \widetilde{M} \text{ and } \gamma \in \Gamma.$$

These maps were used by Corlette to prove a version of super rigidity. They were also used by Hitchin and Simpson in the development of the Non-Abelian Hodge correspondence which gives an identification between representation varieties and moduli spaces of Higgs bundles over Kähler manifolds.

In [EL81] Eells and Lemaire proved that, under suitable non-degeneracy conditions, harmonic maps between closed Riemannian manifolds depend smoothly on the metrics on both the domain and the target (see also [Sam78]). Similarly, one expects that equivariant harmonic maps depend smoothly (or even real analytically) on the representation when a similar non-degeneracy condition is imposed. The purpose of the current article is to prove that this is indeed true.

The main result (Theorem 1.2.7) of this article is as follows. If $(\rho_t)_t$ is a real analytic family of representations of Γ in G such that ρ_0 is reductive and the centraliser of its image contains no semi-simple elements, then for all t in a neighbourhood of zero there exist ρ_t-equivariant harmonic maps depending real analytically on t. Similarly, we also prove real analytic dependence on the metric on the domain M. Furthermore, we prove a real analytic version of the results in [EL81] which will serve as the central analytic ingredient in the proof of the main theorem (see Proposition 1.3.3).

In Section 1.2.4 we apply these results to families of Hitchin representations. Such families satisfy the assumptions of the main theorem (see Proposition 1.2.10). As a result we can characterise certain sets as real analytic subsets of Teichmüller space and the set of Hitchin representations. Namely in Corollary 1.2.11 we prove that the set of points at which the equivariant harmonic maps are not immersions is a real analytic subvariety of the universal Teichmüller curve crossed with the set of Hitchin representations. Similarly, in Corollary 1.2.12 we prove that the set of points Y in Teichmüller space and representations ρ such that Y can be realised as a minimal surface in $X/\rho(\Gamma)$ is a real analytic subvariety of Teichmüller space crossed with the set of Hitchin representations.

1.2 Statement of the results

We first collect some preliminary definitions and results needed to give a statement of the main theorem.

1.2.1 Harmonic maps

If (M, g) and (N, h) are Riemannian manifolds with M compact, then a C^1 map $f\colon (M, g) \to (N, h)$ is called harmonic if it is a critical point of the Dirichlet energy functional

$$E(f) = \frac{1}{2}\int_M \|df\|^2 \operatorname{vol}_g .$$

Here we view df as a section of the bundle $T^*M \otimes f^*TN$. A metric on this vector bundle is induced by the metrics g and h. A harmonic map satisfies the Euler–Lagrange equation $\tau(f) = 0$ where $\tau(f) = \operatorname{tr}_g \nabla df$ is the tension field of f. Here ∇ denotes the connection on $T^*M \otimes f^*TN$ induced by the Levi-Civita connections of g and h. If the domain M is not compact, then a map is called harmonic if its tension field vanishes identically.

At a critical point the Hessian of the energy functional is given by

$$\nabla^2 E(f)(X, Y) = \int_M [\langle \nabla X, \nabla Y\rangle - \operatorname{tr}_g \langle R^N(X, df\cdot)df\cdot, Y\rangle] \operatorname{vol}_g$$

for $X, Y \in \Gamma^1(f^*TN)$. Here ∇ denotes the connection on f^*TN induced by the Levi–Civita connection on TN and R^N denotes the Riemannian curvature tensor of (N, h). The non-degeneracy condition imposed on harmonic maps in [EL81] is that a harmonic map f is a non-degenerate critical point of the energy, i.e. $\nabla^2 E(f)$ is a non-degenerate bilinear form. In [Sun79] Sunada proved that if the target is a locally symmetric space of non-positive curvature this condition is satisfied if and only if the harmonic map is unique. We collect these non-degeneracy conditions in the following lemma.

Lemma 1.2.1 (Sunada)**.** *Suppose $N = X/\Lambda$ (with $\Lambda \subset G$) is a locally symmetric space of non-positive curvature and $f\colon M \to N$ a harmonic map. Then the following are equivalent:*

i. f is a non-degenerate critical point of E.

ii. f is the unique harmonic map in its homotopy class.

iii. The centraliser of Λ in G contains no semi-simple elements.

Proof. We can write $\nabla^2 E(f) = -\int_M \langle J_f \cdot, \cdot \rangle \operatorname{vol}_g$ where

$$J_f(X) = \operatorname{tr}_g \nabla^2 X + \operatorname{tr}_g R^N(X, df \cdot) df \cdot$$

is the Jacobi operator at f. As discussed in [EL81, p.35] the Hessian $\nabla^2 E(f)$ is non-degenerate precisely when $\ker J_f = 0$. It follows from [Sun79, Proposition 3.2] that the set

$$\operatorname{Harm}(M, N, f) = \{h\colon M \to N \mid h \text{ is harmonic and homotopic to } f\}$$

is a submanifold of $W^k(M, N)$ (the space of maps from M to N equipped with the $W^{k,2}$ Sobolev topology) with tangent space at h equal to $\ker J_h$. Because N has non-positive curvature the space $\operatorname{Harm}(M, N, f)$ is connected ([Jos11, Theorem 8.7.2]). We see that f is a non-degenerate critical point of the energy if and only if $\operatorname{Harm}(M, N, f)$ contains only f. If $g \in G$ is a semi-simple isometry centralising Λ, then it is clear that $h = g \cdot f$ is a distinct harmonic map homotopic to f. Conversely, if h is a harmonic map homotopic to f, then there exists a semi-simple $g \in G$ contained in the centraliser of Λ such that $h = g \cdot f$ by [Sun79, Lemma 3.4]. We conclude that f is the unique harmonic map in its homotopy class if and only if the centraliser of Λ in G contains no semi-simple elements. □

The main existence result in the theory of equivariant harmonic maps is the following theorem of Corlette.

Proposition 1.2.2 ([Cor88])**.** *A representation $\rho\colon \Gamma \to G$ is reductive if and only if there exists a ρ-equivariant harmonic map $f\colon \widetilde{M} \to X$.*

A representation $\rho\colon \Gamma \to G$ is called reductive if the Zariski closure of its image in G is a reductive subgroup.

1.2.2 Families of representations and metrics

We will index families of representations or metrics by open balls in $\mathbb{R}^n$. For $\epsilon > 0$ we denote by D_ϵ the open ϵ ball in $\mathbb{R}^n$ centred at 0. Let $(\rho_t)_{t \in D_\epsilon}$ be a family of representations $\rho_t\colon \Gamma \to G$. Such a family induces a natural action of Γ on $X \times D_\epsilon$ given by $\gamma \cdot (x, t) = (\rho_t(\gamma)x, t)$. We make the following properness and freeness assumption on the families of representations we will consider.

Definition 1.2.3. *We call a family of representations uniformly free and proper if the induced action on $X \times D_\epsilon$ is free and proper.*

In particular, each representation in such a family acts freely and properly on X. On families of representations we will make the following regularity assumption.

Definition 1.2.4. *A family of representations $(\rho_t)_{t\in D_\epsilon}$ of Γ in G is called real analytic if for every $\gamma \in \Gamma$ the map $D_\epsilon \to G\colon t \mapsto \rho_t(\gamma)$ is real analytic.*

Remark 1.2.5. A family of representations can be seen as a map from D_ϵ into $\mathrm{Hom}(\Gamma, G)$, the set of representations of Γ into G. If G is an algebraic subgroup of $\mathrm{GL}(n, \mathbb{R})$ and if S is a generating set of Γ with relations R, then $\mathrm{Hom}(\Gamma, G)$ can be realised as the closed subset of $\mathrm{GL}(n, \mathbb{R})^S$ consisting of tuples $(g_1, ..., g_n)$ satisfying the relations $r(g_1, ..., g_n) = 1$ for $r \in R$. In this way we realise $\mathrm{Hom}(\Gamma, G)$ as a real algebraic variety. We note that in this case a family of representations is real analytic if and only if the map $D_\epsilon \to \mathrm{Hom}(\Gamma, G)$ is real analytic.

Finally, for families of metrics we make the following regularity assumption.

Definition 1.2.6. *We call a family $(g_t)_{t\in D_\epsilon}$ of Riemannian metrics on M a real analytic family of metrics if $(x, t) \mapsto g_t(x)$ induces a real analytic map $M \times D_\epsilon \to S^2T^*M$.*

1.2.3 Mapping spaces

If M and N are real analytic manifolds we denote by $C^{k,\alpha}(M, N)$ ($k \in \mathbb{N}, 0 < \alpha < 1$) the space of k-times differentiable maps from M to N such that the k-th derivatives are α-Hölder continuous. We equip these spaces with the topology of uniform $C^{k,\alpha}$ convergence on compact sets. If the domain manifold M is compact, then these spaces can be equipped with a natural real analytic Banach manifold structure.

There is no such Banach manifold structure when M is not compact. It is possible to instead give a Fréchet manifold structure where a chart around a point $f\colon M \to N$ is modelled on spaces of sections of f^*TN with compact support. Such structures are not useful when considering convergence of equivariant maps $\widetilde{M} \to X$ because variations will necessarily not be compactly supported. We will instead make use of the fact that equivariant maps are determined by their values on a fundamental domain which allows us to state our results using Banach manifolds after all.

When M is a closed manifold we let $\Omega \subset \widetilde{M}$ be a bounded domain containing a fundamental domain for the action of Γ on $\widetilde{M}$. We note that a map $\widetilde{M} \to X$ that is equivariant with respect to any representation is completely determined by its restriction to Ω. Furthermore, ρ_n-equivariant maps f_n converge to a ρ-equivariant map f uniformly on compacts if and only if the restrictions $f_n|_\Omega$ converge uniformly to $f|_\Omega$.

We will consider the space of bounded functions from Ω to X. For this we equip M with a background metric and for simplicity we identify X with $\mathbb{R}^n$ via the exponential map $\exp_o\colon T_oX \to X$ based at some basepoint $o \in X$. The metric on M induces a $C^{k,\alpha}$ norm on the space of functions $\Omega \to \mathbb{R}^n \cong X$. We denote by $C_b^{k,\alpha}(\Omega, X)$ the space of functions for which this norm is bounded. This space can be equipped with the structure of a real analytic Banach manifold. For this we observe that equipped with the $C^{k,\alpha}$ norm the space $C_b^{k,\alpha}(\Omega, X)$ becomes a Banach space (note that the linear structure comes from the identification $X \cong \mathbb{R}^n$ and carries no direct geometric meaning). The Banach manifold structure is obtained by declaring this to be a global chart. One can check that the Banach manifold structure is independent of the choice of background metric on M and basepoint in X (see Lemma 1.3.4). We would like to note that, although the use of the identification $X \cong \mathbb{R}^n$ is somewhat ad hoc, if we replace the domain Ω by a closed manifold M, then the above construction yields the usual Banach manifold structure on the space $C^{k,\alpha}(M, X)$.

1.2.4 Main result

The main result of this article can be stated as follows.

Theorem 1.2.7. *Let $(g_t)_{t\in D_\epsilon}$ be a real analytic family of metrics on M and let $(\rho_t)_{t\in D_\epsilon}$ be a real analytic family of representations of Γ in G. We assume that the family $(\rho_t)_{t\in D_\epsilon}$ is uniformly free and proper. Suppose ρ_0 is reductive and the centraliser $Z_G(\operatorname{im}\rho_0)$ contains no semi-simple elements. Then for every $k \in \mathbb{N}, 0 < \alpha < 1$ there exists a $\delta > 0$ smaller then ϵ and a unique continuous map $F\colon D_\delta \to C^{k,\alpha}(\widetilde{M}, X)$ such that each $F(t)$ is a ρ_t-equivariant harmonic map $(\widetilde{M}, g_t) \to X$ and the restricted map $F(\cdot)|_\Omega\colon D_\delta \to C_b^{k,\alpha}(\Omega, X)$ is real analytic.*

Remark 1.2.8. The above result is also true in the smooth category. More precisely we can define, analogous to Definitions 1.2.4 and 1.2.6, the notion of smooth families of metrics and representations. Then Theorem 1.2.7 also holds when we replace 'real analytic' by 'smooth'. For brevity we will not prove the smooth case here but the reader can easily check that the proof goes through also in this case.

If each ρ_t is reductive and has trivial centraliser, then by applying the above theorem at each $t \in D_\epsilon$ we obtain immediately the following corollary.

Corollary 1.2.9. *Let $(g_t)_{t\in D_\epsilon}$, $(\rho_t)_{t\in D_\epsilon}$ be as in Theorem 1.2.7. Suppose that for every $t \in D_\epsilon$ the representation ρ_t is reductive and $Z_G(\operatorname{im}\rho_t) = 0$. Then for all $k \in \mathbb{N}, 0 < \alpha < 1$ there exists a unique continuous map $F\colon D_\epsilon \to C^{k,\alpha}(\widetilde{M}, X)$ such that each $F(t)\colon (\widetilde{M}, g_t) \to X$ is a ρ_t-equivariant harmonic map and the restricted map $F(\cdot)|_\Omega\colon D_\epsilon \to C_b^{k,\alpha}(\Omega, X)$ is real analytic.*

Hitchin representations

We briefly mention how the above results can be applied when we consider Hitchin representations. In this section we let $M = S$ be a closed surface of genus $g \geq 2$ and as before $\Gamma = \pi_1(S)$. In this case the harmonicity of a map $f\colon S \to N$ depends only on the conformal class of the metric on S. Also, in this section we let G be a split real Lie group.

In [Hit92] Hitchin proved that each representation variety $\mathrm{Rep}(\Gamma, G) = \mathrm{Hom}_{\mathrm{red}}(\Gamma, G)/G$ contains a connected component, now called the Hitchin component, which contains $\mathcal{T}(S)$, the Teichmüller space of S, in a natural way. We denote by $\mathrm{Hom}_{\mathrm{Hit}}(\Gamma, G)$ the component of $\mathrm{Hom}(\Gamma, G)$ consisting of Hitchin representations, i.e. representations of Γ in G whose conjugacy class lies in the Hitchin component.

Hitchin representations enjoy many special properties. Relevant to our discussion is that they are reductive and the centraliser of their image is trivial. Also, in [Lab06] Labourie showed that Hitchin representations act freely and are Anosov representations. It follows from [KLP14, Theorem 7.33] that continuous families of Anosov representations satisfy the uniformly free and proper assumption as in Definition 1.2.3.

It follows that there exists a map $F\colon \mathcal{T}(S) \times \mathrm{Hom}_{\mathrm{Hit}}(\Gamma, G) \to C^{k,\alpha}(\widetilde{S}, X)$ assigning to each (J, ρ) the unique ρ-equivariant harmonic map $(\widetilde{S}, J) \to X$. A chart of $\mathrm{Hom}_{\mathrm{Hit}}(\Gamma, G)$ modelled on D_ϵ can be seen as real analytic family of representations that is uniformly free and proper. Furthermore, it follows from [Wol91] that it is possible to choose metrics g_J on S representing points in Teichmüller space $J \in \mathcal{T}(S)$ depending on J in a real analytic fashion. By applying Corollary 1.2.9 to charts around points $(J, \rho) \in \mathcal{T}(S) \times \mathrm{Hom}_{\mathrm{Hit}}(\Gamma, G)$ we obtain the following proposition.

Proposition 1.2.10. *For all $k \in \mathbb{N}, 0 < \alpha < 1$ the map*

$$F\colon \mathcal{T}(S) \times \mathrm{Hom}_{\mathrm{Hit}}(\Gamma, G) \to C^{k,\alpha}(\widetilde{S}, X)$$

assigning to each (J, ρ) the unique ρ-equivariant harmonic map $(\widetilde{S}, J) \to X$ is continuous and the restricted map $F(\cdot, \cdot)|_\Omega\colon \mathcal{T}(S) \times \mathrm{Hom}_{\mathrm{Hit}}(\Gamma, G) \to C^{k,\alpha}_b(\Omega, X)$ is real analytic.

We discuss three corollaries to this result.

First we observe that the family of harmonic maps given by F can also be interpreted as a single map with the universal Teichmüller curve as domain. Namely, let $\Xi(S)$ be the universal Teichmüller curve of S. It is a trivial fibre bundle over $\mathcal{T}(S)$ with fibres homeomorphic to S. It is equipped with a complex structure such that the fibre $\Xi(S)_J$ over $J \in \mathcal{T}(S)$ together with the marking provided by the trivialization $\Xi(S)_J \cong \mathcal{T}(S) \times S$ determines the point J in Teichmüller space (see [Hub06, section 6.8]). The universal cover $\widetilde{\Xi}(S)$ is a trivial fibre bundle over $\mathcal{T}(S)$ with fibres homeomorphic to $\widetilde{S}$. Let $F'\colon \widetilde{\Xi}(S) \times \mathrm{Hom}_{\mathrm{Hit}}(\Gamma, G) \to X$ be the map which on each fibre $\widetilde{\Xi}(S)_J \times \{\rho\} \cong \widetilde{S}$ is given by the ρ-equivariant harmonic map $(\widetilde{\Xi}(S)_J, J) \to X$. It follows from Proposition 1.2.10 that this map is real analytic.

Corollary 1.2.11. *The set*

$$\begin{aligned} I = \{((J,x),\rho) \in \widetilde{\Xi}(S) \times \mathrm{Hom}_{\mathrm{Hit}}(\Gamma, G) \mid \\ F'(J, \cdot, \rho)\colon \widetilde{\Xi}(S)_J \to X \textit{ is not an immersion at } x\} \end{aligned}$$

is a real analytic subvariety of $\widetilde{\Xi}(S) \times \mathrm{Hom}_{\mathrm{Hit}}(\Gamma, G)$.

It is conjectured (see for example [Li19, Conjecture 9.3]) that equivariant harmonic maps associated to Hitchin representations are immersions which would correspond to the set I being empty.

Proof. We equip $\Xi(S)$ with a choice of real analytic metric. Given a point $((J,x),\rho) \in \widetilde{\Xi}(S) \times \mathrm{Hom}_{\mathrm{Hit}}(\Gamma, G)$ we consider the derivative of F' in the fibre direction

$$dF(J,\cdot,\rho) \colon T_x(\widetilde{\Xi}(S)_J) \to \operatorname{im} dF(J,\cdot,\rho).$$

Because the spaces $T_x(\widetilde{\Xi}(S)_J)$ and $\operatorname{im} dF(J,\cdot,\rho) \subset T_{F'(J,x,\rho)}X$ are equipped with inner products we can consider the determinant of this map. We let $d \colon \widetilde{\Xi}(S) \times \mathrm{Hom}_{\mathrm{Hit}}(\Gamma, G) \to \mathbb{R}$ be the map which at a point $((J,x),\rho)$ is given by the determinant of the above map. Because F' is real analytic the map d is real analytic as well. We observe that $I = d^{-1}(0)$ from which the result follows. $\square$

In a similar vein we also have the following corollary.

Corollary 1.2.12. *The set*

$$T = \{(J,\rho) \in \mathcal{T}(S) \times \mathrm{Hom}_{\mathrm{Hit}}(\Gamma, G) \mid \\ (S,J) \text{ can be realised in } X/\rho(\Gamma) \text{ as a branched minimal surface}\}$$

is a real analytic subvariety of $\mathcal{T}(S) \times \mathrm{Hom}_{\mathrm{Hit}}(\Gamma, G)$.

Proof. For $J \in \mathcal{T}(S)$ and $\rho \in \mathrm{Hom}_{\mathrm{Hit}}(\Gamma, G)$ we consider the Hopf differential of the harmonic map $F(J,\rho) \colon (\widetilde{S}, J) \to X$ which is given by $\phi_{J,\rho} = (F(J,\rho)^* m)^{2,0}$. Here m is the Riemannian metric of the symmetric space X. The Hopf differential is a holomorphic quadratic differential on $(\widetilde{S}, J)$ which vanishes if and only if the harmonic map $F(J,\rho)$ is a (branched) minimal surface. The Hopf differential $\phi_{J,\rho}$ is Γ-invariant and descends to S since $F(J,\rho)$ is ρ-equivariant. Consider the function $V \colon \mathcal{T}(S) \times \mathrm{Hom}_{\mathrm{Hit}}(\Gamma, G) \to \mathbb{R}$ given by the L^2-norm of $\phi_{J,\rho}$, namely

$$V(J,\rho) = \int_S \frac{|\phi_{J,\rho}|^2}{\sqrt{\det g_J}} |dz|^2.$$

Here g_J is a metric in the conformal class of J depending real analytically on J. It follows from Proposition 1.2.10 that this function is real analytic (it is for this reason that we choose the L^2-norm rather than the L^1-norm). The statement of the corollary follows from $T = V^{-1}(0)$. $\square$

The space of Fuchsian representations $\Gamma \to \mathrm{SL}(2,\mathbb{R})$ can be included in $\mathrm{Hom}_{\mathrm{Hit}}(\Gamma, G)$ in the following way. There exists an irreducible representation $\iota \colon \mathrm{SL}(2,\mathbb{R}) \to G$ that is unique up to conjugation. Then to a Fuchsian representation $\rho_0 \colon \Gamma \to \mathrm{SL}(2,\mathbb{R})$ we associate a so-called quasi-Fuchsian representation $\rho = \iota \circ \rho_0$ which lies in $\mathrm{Hom}_{\mathrm{Hit}}(\Gamma, G)$. This inclusion descends to the natural inclusion of Teichmüller space into the Hitchin component. A quasi-Fuchsian representation stabilises the totally geodesically embedded copy of $\mathbb{H}^2$ in G/K given by the inclusion $\iota' \colon \mathrm{SL}(2,\mathbb{R})/\mathrm{SO}(2) \to G/K = X$ that is induced by ι.

By uniqueness of equivariant harmonic maps we have that the harmonic map $(\widetilde{S}, J) \to X$ equivariant for $\rho = \iota \circ \rho_0$ is given by the composition $\iota' \circ f_0$ where $f_0 \colon (\widetilde{S}, J) \to \mathbb{H}^2$ is the unique ρ_0-equivariant map. It follows from [Sam78, Theorem 11] that f_0 is a diffeomorphism. Hence, equivariant harmonic maps associated to quasi-Fuchsian representations are immersions. Because being an immersion is an open condition we immediately obtain the following corollary to Proposition 1.2.10

Corollary 1.2.13. *There exists an open neighbourhood of*

$$\mathcal{T}(S) \times \{\textit{quasi-Fuchsian representations}\} \subset \mathcal{T}(S) \times \mathrm{Hom}_{\mathrm{Hit}}(\Gamma, G)$$

such that for any pair (J, ρ) in this neighbourhood the ρ-equivariant harmonic map $(\widetilde{S}, J) \to X$ is an immersion.

1.3 Proof of the main result

As in the proof of Eells and Lemaire in [EL81], our main analytical tool will be the implicit function theorem for maps between Banach manifolds. The main difficulty to overcome is that a priori the equivariant harmonic maps are not elements of the same space of mappings. Namely, if $(\rho_t)_t$ is a family of representations, then a ρ_t-equivariant map is an element of the space $C^{k,\alpha}(M, X/\rho_t(\Gamma))$. Since the target manifold is different for each t these spaces are not equal (although they are likely to be diffeomorphic). Our aim is to modify these maps so that they can be seen as elements of a single mapping space. This will be achieved by means of a family of deformation maps which intertwine the representations ρ_0 and ρ_t. By composing with these deformation maps we can view each ρ_t-equivariant map as element of (a subset of) $C^{k,\alpha}(M, X/\rho_0(\gamma))$.

We first fix some notation. We let $(\rho_t)_{t \in D_\epsilon}$ be a real analytic family of representations that is uniformly free and proper. We denote $X_\epsilon = X \times D_\epsilon$ and by $\alpha \colon \Gamma \times X_\epsilon \to X_\epsilon, \alpha(\gamma)(x,t) = (\rho_t(\gamma)x, t)$ the natural action induced by $(\rho_t)_t$. We fix a base point $o \in X$ of the symmetric space and denote by $U_R = \cup_{\gamma \in \Gamma} B(\rho_0(\gamma)o, R)$ the R-neighbourhood of the $\rho_0(\Gamma)$-orbit of o.

Our deformation maps will be provided by the following proposition.

Proposition 1.3.1. *For every $R > 0$ there exists a $\delta = \delta(R) > 0$ smaller than ϵ and family of maps $(\Phi_t \colon U_R \to X)_{t \in D_\delta}$ satisfying the following properties:*

i. *The induced map $U_R \times D_\delta \to X \colon (x,t) \mapsto \Phi_t(x)$ is real analytic.*

ii. *For each $t \in D_\delta$ the set $\Phi_t(U_R)$ is open and $\Phi_t \colon U_R \to \Phi_t(U_R)$ is a real analytic diffeomorphism.*

iii. $\Phi_0 = \mathrm{id} \colon U_R \to U_R$.

iv. *For each $t \in D_\delta$ the map Φ_t intertwines the actions of ρ_0 and ρ_t, i.e. $\rho_t(\gamma) \circ \Phi_t = \Phi_t \circ \rho_0(\gamma)$ for $\gamma \in \Gamma$.*

The content of this proposition is closely related to Ehresmann's fibration theorem. In fact, when the actions of the representations ρ_t on X are cocompact Proposition 1.3.1 can be obtained from it. Consequently the proof of Proposition 1.3.1 is along the same lines as the proof of the fibration theorem.

We denote by $\mathrm{pr}_X : X_\epsilon \to X$ and $\pi : X_\epsilon \to D_\epsilon$ the projections onto the first and second factor of $X_\epsilon = X \times D_\epsilon$ respectively. By $(t_1, ..., t_n)$ we denote the coordinates on D_ϵ and also the coordinates on the D_ϵ factor in X_ϵ. So in this notation we have $d\pi(\frac{\partial}{\partial t_i}(x,t)) = \frac{\partial}{\partial t_i}(t)$.

We first prove the following lemma.

Lemma 1.3.2. *Let $R > 0$. On an $\alpha(\Gamma)$-invariant neighbourhood of $U_R \times \{0\}$ in X_ϵ there exist $\alpha(\Gamma)$-invariant real analytic vector fields ξ_i (for $i = 1, ..., n$) that satisfy $d\pi(\xi_i(x,t)) = \frac{\partial}{\partial t_i}(t)$.*

Proof. It is possible to give a more or less explicit construction for such vector fields. However, proving they are indeed real analytic is rather cumbersome. Instead we opt to explicitly construct smooth vector fields which we then approximate by real analytic ones.

We let $\varphi\colon [0,\infty) \to [0,1]$ be a smooth function satisfying $\varphi|_{[0,R]} \equiv 1$ and $\varphi|_{[R+1,\infty)} \equiv 0$. For $i = 1, ..., n$ we define smooth vector fields η_i on X_ϵ by

$$\eta_i(x,t) = \varphi(d(o,x)) \cdot \frac{\partial}{\partial t_i}.$$

Now let

$$\xi_i' = \sum_{\gamma \in \Gamma} (\alpha(\gamma))_* \eta_i.$$

The sum on the right hand side is locally finite by the uniform properness assumption on $(\rho_t)_t$. Hence, each ξ_i' is $\alpha(\Gamma)$-invariant smooth vector field on X_ϵ. We observe that $d\pi(\eta_i(x,t)) = s(x,t)\frac{\partial}{\partial t_i}$ with

$$s(x,t) = \sum_{\gamma \in \Gamma} \varphi(d(o, \rho_t(\gamma)^{-1}x)).$$

On $B(o,R) \times \{0\}$ we have that $s(x,t) \geq \varphi(d(o,x)) = 1$ and by $\alpha(\Gamma)$-invariance we have that $s \geq 1$ on $U_R \times \{0\}$.

We now approximate the smooth vector fields ξ_i' by real analytic ones. By the uniformly free and proper assumption on $(\rho_t)_t$ we have that $X_\epsilon/\alpha(\Gamma)$ is a real analytic manifold. The vector fields ξ_i' descend to smooth vector fields. On compact subsets these vector fields can be approximated arbitrarily closely in C^0 norm by real analytic vector fields (see [Whi34] and [Roy60]). The set $U_R \times \{0\}$ maps to a precompact subset of $X_\epsilon/\alpha(\Gamma)$. Hence, by pulling back approximating vector fields we see that on a neighbourhood of $U_R \times \{0\}$ we can approximate ξ_i' arbitrarily closely by $\alpha(\Gamma)$-invariant real analytic vector fields. Let ξ_i'' be such approximating vector fields. For some real analytic functions s_i' we have $d\pi(\xi_i''(x,t)) = s_i'(x,t)\frac{\partial}{\partial t_i}$. By choosing the approximating vector fields ξ_i'' close enough to ξ_i' we can arrange that each s_i' is close to s and hence satisfies

$s_i' > 0$ on a neighbourhood of $U_R \times \{0\}$. For $i = 1, ..., n$ we can now define $\xi_i = \xi_i''/s'$. $\square$

Proof of Proposition 1.3.1. Let ξ_i for $i = 1, ..., n$ be the vector fields constructed in Lemma 1.3.2. We denote by ψ_i^s their flows which are defined on a neighbourhood of $U_R \times \{0\}$. We consider the maximal flow domain for a combination of these flows starting at points in $X \times \{0\}$, i.e. the set

$$\Omega = \{(x, s) \in X \times \mathbb{R}^n \mid \psi_1^{s_1} \circ \cdots \circ \psi_n^{s_n}((x, 0)) \text{ is defined}\}.$$

This is an open set containing $U_R \times \{0\}$. On Ω we set

$$\Psi(x, (s_1, ..., s_n)) = \psi_1^{s_1} \circ \cdots \circ \psi_n^{s_n}((x, 0)).$$

Because $d\pi(\xi_i) = \frac{\partial}{\partial t_i}$ (when defined) we see that

$$t \mapsto \pi \circ \Psi(x, (s_1, ..., s_{i-1}, t, s_{i+1}, ..., s_n))$$

is an integral line for the vector field $\frac{\partial}{\partial t_i}$. Since these integral lines are unique and $\pi \circ \Psi(x, 0) = 0$ we find $\pi \circ \Psi(x, s) = s$ when $(x, s) \in \Omega$, i.e. $\pi \circ \Psi = \pi$. Because the vector fields ξ_i are $\alpha(\Gamma)$-invariant we observe for $\gamma \in \Gamma$ that

$$\begin{aligned}\Psi(\rho_0(\gamma)x, s) &= \psi_1^{s_1} \circ \cdots \circ \psi_n^{s_n}(\alpha(\gamma)(x, 0)) \\ &= \alpha(\gamma)[\psi_1^{s_1} \circ \cdots \circ \psi_n^{s_n}(x, 0)] = \alpha(\gamma)\Psi(x, s)\end{aligned}$$

whenever both sides are defined. We define $\beta\colon \Gamma \times \Omega \to \Omega$ as an action of Γ on Ω by $\beta(\gamma)(x, s) = (\rho_0(\gamma)x, s)$ which is the action of $\rho_0(\Gamma)$ on X times the trivial action. By the above the we see that the set Ω is $\beta(\Gamma)$-invariant and Ψ intertwines the β and α actions, i.e. $\alpha(\gamma) \circ \Psi = \Psi \circ \beta(\gamma)$ for all $\gamma \in \Gamma$.

On $X \times \{0\} \cap \Omega$ the map Ψ is simply the inclusion into X_ϵ. Combined with the fact that $\pi \circ \Psi = \pi$ we see for each $(x, 0) \in X \times \{0\} \cap \Omega$ the tangent map $d\Psi|_{(x,0)}\colon T_xX \times T_0\mathbb{R}^n \to T_xX \times T_0D_\epsilon$ is the identity map. Hence, we can shrink Ω to a smaller open neighbourhood of $U_R \times \{0\}$ such that Ψ is a local diffeomorphism on Ω. By shrinking Ω further we can also assume Ψ to be injective. For if not, then there exist two distinct sequences $(x_n, s_n), (x_n', s_n') \in \Omega$ satisfying $\Psi(x_n, s_n) = \Psi(x_n', s_n')$ with $s_n, s_n' \to 0$ and x_n, x_n' converging to points x and x' in U_R. By $\pi \circ \Psi = \pi$ we see $s_n = s_n'$. By continuity $\Psi(x, 0) = \Psi(x', 0)$ and because when restricted to $X \times \{0\} \cap \Omega$ the map Ψ is an injection we must have $x = x'$. However, this contradicts the fact that Ψ is a local diffeomorphism. So we can arrange that Ψ is a diffeomorphism onto its image. Since Ψ intertwines β and α this can be done in such a way that Ω is still $\beta(\Gamma)$-invariant.

Since Ω is a neighbourhood of $U_R \times \{0\}$ we can, by compactness, find a $\delta > 0$ such that $\overline{B(o, R)} \times D_\delta \subset \Omega$. By $\beta(\Gamma)$-invariance we then have $U_R \times D_\delta \subset \Omega$. We now define the family of deformation maps $\Phi_t\colon U_R \to X$ as $\Phi_t(x) = \mathrm{pr}_X \circ \Psi(x, t)$ for $t \in D_\delta$. We check that indeed $(\Phi_t)_{t \in D_\delta}$ satisfies Properties (i)-(iv). Property (i) follows since flows of real analytic vector fields are real analytic. Property (ii) follows since $\Psi\colon \Omega \to \Psi(\Omega)$ is a diffeomorphism and satisfies $\pi \circ \Psi = \pi$ hence induces diffeomorphisms between the fibres $\pi^{-1}(t) \cap \Omega$ and $\pi^{-1}(t) \cap \Psi(\Omega)$.

Property (iii) follows from the fact that Ψ restricted to $X \times \{0\} \cap \Omega$ is the inclusion map and Property (iv) follows from the fact that Ψ intertwines the actions of β and α. $\square$

Using the deformation maps the problem of dependence on representations can be reduced to the problem of dependence on metrics on a fixed target manifold. In this case the results of [EL81] can be used. In their paper Eells and Lemaire only prove smooth dependence so for completeness we prove a version of their result in the real analytic category.

Proposition 1.3.3. *Let M, N be real analytic manifolds with M compact. Let $(g_t)_{t\in D_\epsilon}$ and $(h_t)_{t\in D_\epsilon}$ be real analytic families of metrics on M and N, respectively. If $f_0\colon (M, g_0) \to (N, h_0)$ is a harmonic map such that $\nabla^2 E(f_0)$ is non-degenerate, then for every $k \in \mathbb{N}, 0 < \alpha < 1$ there exists a $\delta > 0$ and a unique real analytic map $F\colon D_\delta \to C^{k,\alpha}(M, N)$ such that $F(0) = f_0$ and each $F(t)$ is a harmonic map $(M, g_t) \to (N, h_t)$.*

Proof. For each $t \in D_\epsilon$ a C^2 map $\phi\colon (M, g_t) \to (N, h_t)$ is harmonic if and only if $\tau_t(\phi) = \operatorname{tr}_{g_t} \nabla d\phi = 0$ where ∇ is the connection on $T^*M \otimes \phi^*TN$ induced by g_t and h_t. In local coordinates $(x^i)_i$ on M and $(u^\alpha)_\alpha$ on N, $\tau_t(\phi)$ is given by

$$\tau_t(\phi)^\gamma = (g_t)_{ij} \left\{ \frac{\partial^2 \phi^\gamma}{\partial x^i \partial x^j} - (\Gamma_{g_t})^k_{ij} \frac{\partial \phi^\gamma}{\partial x^k} + (\Gamma_{h_t})^\gamma_{\alpha\beta}(\phi) \frac{\partial \phi^\alpha}{\partial x_i} \frac{\partial \phi^\beta}{\partial x_j} \right\}$$

here Γ_g denote the Christoffel symbols of a metric g. We combine the tension fields for different $t \in D_\epsilon$ in a map

$$\tau\colon C^{k+2+\alpha}(M, N) \times D_\epsilon \to TC^{k+\alpha}(M, N).$$

We claim this map is real analytic. To see this we write τ as a composition of two real analytic map. First we consider the second prolongation map

$$J\colon C^{k+2+\alpha}(M, N) \to C^{k+\alpha}(M, J^2(M, N))$$

mapping a map $\phi\colon M \to N$ to its 2-jet $j^2\phi$. A diffeomorphism between a neighbourhood of the zero section in ϕ^*TN and a neighbourhood of the image of graph(ϕ) in $M \times N$ induces charts of the two mapping spaces modelled on $\Gamma^{k+2+\alpha}(\phi^*TN)$ and $\Gamma^{k+\alpha}(J^2(M, \phi^*TN))$ respectively. In these charts the second prolongation map is a bounded linear map so in particular it is real analytic. Secondly, we consider the map $T\colon J^2(M, N) \times D_\epsilon \to TN$ given in local coordinates (induced by $(x^i)_i$ on M and $(u^\alpha)_\alpha$ on N) by

$$(x^i, u^\alpha, u^\alpha_i, u^\alpha_{ij}, t) \mapsto (g_t)_{ij} \left\{ u^\gamma_{ij} - (\Gamma_{g_t})^k_{ij} u^\gamma_k + (\Gamma_{h_t})^\gamma_{\alpha\beta}(u) u^\alpha_i u^\beta_j \right\}.$$

By assumption the coefficients $(g_t)_{ij}, (\Gamma_{g_t})^k_{ij}$ and $(\Gamma_{h_t})^\gamma_{\alpha\beta}$ are real analytic functions so the map T is real analytic. It now follows from the Ω-lemma (see Lemma 1.3.4 below) that T induces a real analytic map

$$\begin{aligned} T_*\colon C^{k+\alpha}(M, J^2(M, N)) \times D_\epsilon &\to C^{k+\alpha}(M, TN) \\ (\psi, t) &\mapsto T(\psi(\cdot), t) \end{aligned}$$

The map τ is a composition of T_* and J and is therefore real analytic.

As discussed in [EL81, p. 35] the partial derivative of τ with respect to the first factor $(D_1\tau)_{(f_0,0)}\colon TC^{k+2+\alpha}(M,N) \to T_0TC^{k+\alpha}(T,N)$ is an isomorphism of Banach spaces precisely when $\nabla^2 E(f_0)$ is non-degenerate. Hence, we can apply a real analytic version of the implicit function theorem for Banach spaces (e.g. [Whi65]) to obtain, for $\delta > 0$ small enough, a unique real analytic map $F\colon D_\delta \to C^{k+2+\alpha}(M,N)$ such that $F(0) = f_0$ and $\tau_t(F(t)) = 0$ for all $t \in D_\delta$. $\square$

Lemma 1.3.4 (The Ω-Lemma)**.** *Let M, N and P be real analytic manifolds with M compact. Suppose $F\colon N \to P$ is real analytic. Then F induces a real analytic map*

$$\Omega_F\colon C^{k,\alpha}(M,N) \to C^{k,\alpha}(M,P)\colon \phi \mapsto F \circ \phi$$

for all $k \in \mathbb{N}, 0 < \alpha < 1$.

Compare with [Abr63, Theorem 11.3]. Unfortunately, a proof of the real analytic case as stated here does not seem to be available in the literature. We give a sketch of the proof.

Sketch of proof of Lemma 1.3.4. By following the same steps as in [Abr63] the statement can be reduced to a local version (cf. [Abr63, Theorem 3.7]), i.e. it is enough to prove that if $K \subset \mathbb{R}^n$ is compact, $V \subset \mathbb{R}^p$ open and $F\colon K \times V \to \mathbb{R}^q$ real analytic, then $\Omega_F\colon C^{k,\alpha}(K,V) \to C^{k,\alpha}(K,\mathbb{R}^q)$ given by $[\Omega_F(\phi)](x) = F(x,\phi(x))$ is a real analytic map between Banach spaces. To this end we observe that since F is real analytic it can be extended to a complex analytic map $\widetilde{F}\colon U \to \mathbb{C}^q$ on an open set $U \subset \mathbb{C}^n \times \mathbb{C}^p$ containing $K \times V$. Let $\widetilde{V} \subset \mathbb{C}^p$ be an open such that $K \times V \subset K \times \widetilde{V} \subset U$. Then $\widetilde{F}$ induces a map $\Omega_{\widetilde{F}}\colon C^{k,\alpha}(K,\widetilde{V}) \to C^{k,\alpha}(K,\mathbb{C}^q)$ between complex Banach spaces which extends Ω_F. Applying the smooth version of the Ω-Lemma yields that $\Omega_{\widetilde{F}}$ is a C^1 map with derivative given by $D\Omega_{\widetilde{F}} = \Omega_{D_2\widetilde{F}}$. Since $\widetilde{F}$ is holomorphic we see that this derivative is complex linear. It now follows from [Hub06, Theorem A5.3] that $\Omega_{\widetilde{F}}$ is a complex analytic map. We conclude that Ω_F, which is a restriction of $\Omega_{\widetilde{F}}$ to $C^{k,\alpha}(K,V)$, is real analytic. $\square$

We can now prove the statement of our main theorem.

Proof of Theorem 1.2.7. We set $N = X/\rho_0(\Gamma)$ (recall that by assumption the action of ρ_0 on X is free and proper so N is a manifold). Since ρ_0 is reductive and $Z_G(\operatorname{im}\rho_0)$ contains no semi-simple elements there exists a unique ρ_0-equivariant harmonic map $f\colon \widetilde{M} \to X$. This map descends to a harmonic map $f\colon M \to N$. We denote by o' the point in N covered by the base point o in X. The set U_R descends to the set $V = B(o', R)$ in N. We choose $R > 0$ large enough such that the image of f is contained in $B(o', R)$.

Let $(\Phi_t)_{t \in D_\delta}$ be the family of deformation maps as in Proposition 1.3.1. We denote by m the Riemannian metric on the symmetric space X. Define a family of metrics $(h_t)_{t\in D_\delta}$ on U_R by $h_t = \Phi_t^* m$. By Property (1.3.1.i) this is a real analytic family of metrics. We observe for $\gamma \in \Gamma$ that

$$\rho_0(\gamma)^* h_t = \rho_0^* \Phi_t^* m = \Phi_t^* \rho_t(\gamma)^* m = \Phi_t^* m = h_t.$$

Here we used Property (1.3.1.iv) and the fact that each ρ_t acts on X by isometries. We conclude that each h_t is $\rho_0(\Gamma)$-invariant hence the family of metrics descends to a family of metrics, also denoted $(h_t)_{t\in D_\delta}$, on V. By Lemma 1.2.1 the Hessian $\nabla^2 E(f)$ is non-degenerate so Proposition 1.3.3 yields, after shrinking δ, a unique real analytic map $G\colon D_\delta \to C^{k,\alpha}(M, V)$ such that $G(t)$ is a harmonic map from (M, g_t) to (V, h_t) for each $t \in D_\delta$. By choosing for each t a ρ_0-equivariant lift we can view G as a continuous map $G\colon D_\delta \to C^{k,\alpha}(\widetilde{M}, U_R)$. We define F by composing with the deformation maps, $F(t) = \Phi_t \circ G(t)$. By Property (1.3.1.iv) every map $F(t)$ is ρ_t-equivariant. By construction, each Φ_t is an open isometric embedding of (V, h_t) into (X, m) hence each $F(t)$ is also harmonic. Finally, by Property (1.3.1.i) we see that the map $F\colon D_\delta \to C^{k,\alpha}(\widetilde{M}, X)$ is continuous and real analytic as a map $F(\cdot)|_\Omega\colon D_\delta \to C^{k,\alpha}(\Omega, X)$. $\square$

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Chapter 2

Strict plurisubharmonicity of the energy on Teichmüller space associated to Hitchin representations[1]

Abstract

Let Σ be a closed surface of genus at least two and $\rho\colon \pi_1(\Sigma) \to G$ a Hitchin representation into $G = \mathrm{PSL}(n,\mathbb{R})$, $\mathrm{PSp}(2n,\mathbb{R})$, $\mathrm{PSO}(n,n+1)$ or G_2. We consider the energy functional E on the Teichmüller space of Σ which assigns to each point in $\mathcal{T}(\Sigma)$ the energy of the associated ρ-equivariant harmonic map. The main result of this paper is that E is strictly plurisubharmonic. As a corollary we obtain an upper bound of $3 \cdot \mathrm{genus}(\Sigma) - 3$ on the index of any critical point of the energy functional.

2.1 Introduction

Let Σ be a closed surface of genus at least two and let $\rho\colon \pi_1(\Sigma) \to G$ be a Hitchin representation. In this paper we take G to be either $\mathrm{PSL}(n,\mathbb{R}), \mathrm{PSp}(2n,\mathbb{R})$, $\mathrm{PSO}(n,n+1)$ or the exceptional group G_2. Let K be a maximal compact subgroup of G. For every complex structure J on Σ there exists a (unique) ρ-equivariant harmonic map $f_J\colon (\widetilde{\Sigma}, J) \to G/K$. Recall that a map $f\colon \widetilde{\Sigma} \to G/K$ is called ρ-equivariant if $f(\gamma x) = \rho(\gamma) f(x)$ for all $\gamma \in \pi_1(\Sigma)$. The energy density of each f_J is $\pi_1(\Sigma)$-invariant. Hence, it descends to Σ and can be integrated to obtain the Dirichlet energy of f_J. Assigning to a complex structure J the energy of the harmonic map f_J gives us an energy functional on the Teichmüller space of Σ. We will denote this functional by $E\colon \mathcal{T}(\Sigma) \to \mathbb{R}$. The main result of this paper is the following theorem.

Theorem 2.1.1. *Let G be one of the following Lie groups:* $\mathrm{PSL}(n,\mathbb{R})$, $\mathrm{PSp}(2n,\mathbb{R})$, $\mathrm{PSO}(n,n+1)$ *or the exceptional group* G_2. *If $\rho\colon \pi_1(\Sigma) \to G$ is a Hitchin representation, then the energy functional $E\colon \mathcal{T}(\Sigma) \to \mathbb{R}$ is strictly plurisubharmonic.*

This theorem extends the results of Tromba ([Tro92, Theorem 6.2.6]) to a wider class of energy functionals. Tromba considers a fixed hyperbolic metric g on Σ and studies the energy functional that assigns to each complex structure J the energy of the harmonic map $(\Sigma, J) \to (\Sigma, g)$ that is homotopic to the identity. He proves that this functional is strictly plurisubharmonic. This corresponds to our result if we take $\rho\colon \pi_1(\Sigma) \to \mathrm{PSL}(2,\mathbb{R})$ to be a Fuchsian representation.

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Hitchin representations form a larger class of representations (that contains the Fuchsian representations) so Theorem 2.1.1 can be seen as a natural extension of Tromba's results.

The energy functional E is studied by Labourie in [Lab08]. He proved E is a proper function on Teichmüller space and hence has a global minimum. Labourie conjectured that this critical point of the energy functional is unique. This conjecture has been proved in the case that the Lie group G has rank 2 (see [Lof01] and [Lab17]) but remains open in higher rank. Our result puts a limit on how degenerate a critical point of E can be. More precisely, it implies that the Hessian of E at any critical point is positive definite on a subspace of dimension at least $3 \cdot \mathrm{genus}(\Sigma) - 3$ (cf. Corollary 2.4.1).

Various examples of plurisubharmonic functions on Teichmüller space have been constructed. Notably, in [Yeu03] it is proved that Teichmüller space admits a bounded and strictly plurisubharmonic exhaustion function. In contrast, the energy functionals we consider in this paper provide interesting examples of strictly plurisubharmonic functions that are proper.

Our proof of Theorem 2.1.1 is based on the work of Toledo in [Tol12]. Our main innovation is the use of Higgs bundles techniques to sharpen the results of that paper in the particular case we consider. Toledo considers a Riemannian manifold N of non-positive Hermitian curvature (see Section 2.3) and makes the assumption that for every complex structure J there exists a unique harmonic map $(\Sigma, J) \to N$ in a given homotopy class. He then proves that the functional that assigns to each J the energy of this harmonic map is a plurisubharmonic function on Teichmüller space. The setting we consider amounts to taking $N = \rho(\pi_1(\Sigma)) \setminus G/K$. Our proof of Theorem 2.1.1 combines the result of Toledo with the Higgs bundle description of Hitchin representations to obtain the *strict* plurisubharmonicity of E.

We obtain two corollaries to Theorem 2.1.1. The first, Corollary 2.4.1, gives an upper bound on the index of critical points of E. Namely, if g is the genus of Σ, then the index of a critical point of E is at most $\dim_{\mathbb{C}} \mathcal{T}(\Sigma) = 3g - 3$. The second corollary, Corollary 2.4.2, states that the set of points where E attains its minimal value is locally contained in a totally real submanifold of $\mathcal{T}(\Sigma)$.

The proof of Theorem 2.1.1 and its corollaries will be given in Section 2.4. In Section 2.2 we recall the aspects of the Non-Abelian Hodge correspondence and the construction of the Hitchin component that we need for our proof. In Section 2.3 we describe the results of [Tol12] on which our proof will be based.

2.2 Non-Abelian Hodge correspondence

We briefly recall the Non-Abelian Hodge correspondence and the construction of the Hitchin component for the case $G = \mathrm{SL}(n, \mathbb{C})$. We follow parts of the expositions found in [Mau15] and [Li19]. In this section we will denote $G = \mathrm{SL}(n, \mathbb{C})$ and $K = \mathrm{SU}(n)$. The Lie algebras of these groups we denote by $\mathfrak{g} = \mathfrak{sl}(n, \mathbb{C})$ and $\mathfrak{k} = \mathfrak{su}(n)$ and we let $\mathfrak{p} \subset \mathfrak{g}$ be the subspace of Hermitian matrices. We have $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$. Furthermore, let X be a Riemann surface of genus at least two, let $\widetilde{X}$ be its universal cover and denote by K_X the canonical bundle of X. Finally, if $E \to X$ is a vector bundle we denote by $\mathrm{End}_0(E)$ the vector bundle of trace free endomorphisms of E.

Definition 2.2.1. *A G-Higgs bundle over X is a pair (E, ϕ) where E is a rank n holomorphic vector bundle over X with trivial determinant bundle and ϕ is a holomorphic section of $K_X \otimes \mathrm{End}_0(E)$. We call (E, ϕ) stable if any proper sub-G-Higgs bundle has negative degree and we call (E, ϕ) polystable if it is a direct sum of stable G-Higgs bundles.*

We denote by $\mathcal{M}_{\mathrm{Higgs}}(G)$ the moduli space of gauge equivalence classes of polystable G-Higgs bundles over X. The representation variety $\mathrm{Rep}(\pi_1(X), G)$ is the set of conjugacy classes of reductive representations of $\pi_1(X)$ into G. The Non-Abelian Hodge correspondence describes an identification between $\mathrm{Rep}(\pi_1(X), G)$ and $\mathcal{M}_{\mathrm{Higgs}}(G)$.

We first describe how to construct a G-Higgs bundle from a representation. Let $\rho \colon \pi_1(X) \to G$ be a reductive representation and consider the G-bundle $P_G = (\widetilde{X} \times G)/\pi_1(X) \to X$ where $\pi_1(X)$ acts on the second factor via the representation ρ. Let $\omega \in \Omega^1(G, \mathfrak{g})$ be the left Cartan form on G. Then the form $\pi^*\omega$ on $\widetilde{X} \times G$ is the connection form of the flat connection of $\widetilde{X} \times G$ (where $\pi \colon \widetilde{X} \times G \to G$ is the projection to the second factor). This form descends to P_G inducing a flat connection on P_G which we will denote by D.

Since ρ is reductive it follows from [Cor88] that there exists a ρ-equivariant harmonic map $f \colon \widetilde{X} \to G/K$ (unique up to composition with an element in the centraliser of $\operatorname{im} \rho$). We consider the reduction of the structure group of P_G to K determined by f. The projection $G \to G/K$ is a K-bundle which we pull back via f to obtain the K-subbundle $f^*G \subset \widetilde{X} \times G$. By ρ-equivariance of f this bundle descends to a K-bundle $P_K \subset P_G$ over X. We denote by $\omega^{\mathfrak{k}}$ and $\omega^{\mathfrak{p}}$ the composition of the Cartan form on G with the projections $\mathfrak{g} \to \mathfrak{k}$ and $\mathfrak{g} \to \mathfrak{p}$ respectively. Note that on f^*G we have $\pi^*\omega = f^*\omega$ hence $\pi^*\omega = f^*\omega^{\mathfrak{k}} + f^*\omega^{\mathfrak{p}}$. The form $f^*\omega^{\mathfrak{k}}$ descends to P_K and is a connection form. We will denote the connection it determines on P_K by ∇. The form $f^*\omega^{\mathfrak{p}}$ descends to a $\mathfrak{p}$-valued one-form on P_K. This form is basic and hence determines a section of $T^*X \otimes (P_K \times_{\mathrm{Ad}\, K} \mathfrak{p})$ which we will call Φ. From the above observations follows that $D = \nabla + \Phi$.

From the data of (P_K, ∇) and Φ a G-Higgs bundle can be constructed. We consider $E = P_K \times_K \mathbb{C}^n$ where $K = \mathrm{SU}(n)$ acts on $\mathbb{C}^n$ via the canonical action. The connection ∇ on P_K induces a connection on E, that we will also denote by ∇. The $(0,1)$ part of ∇ determines a holomorphic structure on E. We note that $\mathfrak{p}^{\mathbb{C}} = \mathfrak{sl}(n, \mathbb{C}) = \mathrm{End}_0(\mathbb{C}^n)$ hence $P_K \times_K \mathfrak{p}^{\mathbb{C}} = \mathrm{End}_0(E)$. It follows that the $(1,0)$ part of Φ, which we will denote by $\phi = \Phi^{1,0}$, is a section of $K_X \otimes (P_K \times_{\mathrm{Ad}\, K} \mathfrak{p}^{\mathbb{C}}) = K_X \otimes \mathrm{End}_0(E)$. Finally, we use that the harmonicity condition on the map f translates to $\nabla^{0,1}\phi = \nabla^{0,1}\Phi^{1,0} = 0$. So ϕ is a holomorphic section of $K_X \otimes \mathrm{End}_0(E)$. We conclude that the pair (E, ϕ) is a G-Higgs bundle.

Conversely, if (E, ϕ) is a polystable G-Higgs bundle, then it follows from a theorem of Hitchin [Hit87] and Simpson [Sim88] that there exists a Hermitian metric H on E such that

$$F^{\nabla^H} + [\phi, \phi^{*^H}] = 0.$$

Here ∇^H denotes the Chern connection of H, F^{∇^H} is its curvature and ϕ^{*^H} is the adjoint of ϕ with respect to H. The above condition implies that if we define a connection by setting $D = \nabla^H + \phi + \phi^{*^H}$, then D is flat. We now obtain a representation $\rho\colon \pi_1(X) \to \mathrm{SL}(n, \mathbb{C})$ by taking a holonomy representation of the flat bundle E around any point $x \in X$.

The Non-Abelian Hodge correspondence states that the two constructions described above are inverses of each other and describe a homeomorphism between $\mathcal{M}_{Higgs}(G)$ and $\mathrm{Rep}(\pi_1(X), G)$.

In the following lemmas we collect two observations about the above construction that we will use in later arguments. By ρ-equivariance the bundle $f^*T(G/K)$ defined over $\widetilde{X}$ descends to a bundle over X. We will denote this bundle also by $f^*T(G/K)$. We denote by ∇^{lc} the Levi-Civita connection on $T(G/K)$.

Lemma 2.2.2. *The bundles* $(\mathrm{End}_0(E), \nabla)$ *and* $(f^*T_{\mathbb{C}}(G/K), f^*\nabla^{lc})$ *are affine isomorphic. That is there is a vector bundle isomorphism* $\beta\colon f^*T_{\mathbb{C}}(G/K) \to \mathrm{End}_0(E)$ *with* $\beta^*\nabla = f^*\nabla^{lc}$.

Proof. We first observe that $T(G/K) = G \times_{\mathrm{Ad}\,K} \mathfrak{p}$ and hence on $\widetilde{X}$ we have

$$f^*T(G/K) = f^*(G \times_{\mathrm{Ad}\,K} \mathfrak{p}) = (f^*G) \times_{\mathrm{Ad}\,K} \mathfrak{p}.$$

Both these bundles descend to X so on X we have

$$f^*T(G/K) = P_K \times_{\mathrm{Ad}\,K} \mathfrak{p}.$$

In the above discussion we saw $P_K \times_{\mathrm{Ad}\,K} \mathfrak{p}^{\mathbb{C}} = \mathrm{End}_0(E)$ so we find that $f^*T_{\mathbb{C}}(G/K) = \mathrm{End}_0(E)$. Finally, we observe that $\omega^{\mathfrak{k}}$ on G is the connection form that induces the Levi-Civita connection on $G \times_{\mathrm{Ad}\,K} \mathfrak{p}$. So $f^*\omega^{\mathfrak{k}}$ induces the connection $f^*\nabla^{lc}$ on $f^*T_{\mathbb{C}}(G/K)$ and also, by construction, induces the connection ∇ on $\mathrm{End}_0(E)$. We conclude that the two bundles are indeed affine isomorphic. □

Lemma 2.2.3. *Consider the derivative of the map f as a section $df \in T^*X \otimes f^*T(G/K)$. Then under the above described correspondence of vector bundles we have the following equality of $P_K \times_{\mathrm{Ad}\,K} \mathfrak{p}$ valued one-forms*

$$\beta(df) = \Phi$$

As a consequence we obtain, if we denote $d'f = (df)^{1,0}$, that

$$\beta(d'f) = \phi.$$

Proof. We consider the vector bundle valued one-form $\Psi \in T^*X \otimes (P_K \times_{\mathrm{Ad}\,K} \mathfrak{p})$ defined by $\Psi = \beta(df)$. We lift Ψ first to P_K and then to f^*G to obtain a $\mathfrak{p}$-valued one-form $\widetilde{\Psi}$ on f^*G. Let $p\colon G \to G/K$ be the quotient map. By unrolling the definition of β we can describe $\widetilde{\Psi}$ as follows. Let $(x,g) \in f^*G$ (i.e. $f(x) = p(g)$) and $(X,A) \in T_{(x,g)}f^*G$. Then $\widetilde{\Psi}((X,A)) = \xi$ where $\xi \in \mathfrak{p}$ is the unique element such that $g_*dp(\xi) = df(X)$. We now consider the form $f^*\omega^{\mathfrak{p}}$. Here $f\colon f^*G \to G$ is the map induced by the pull back construction and is given by $f(x,g) = g$. We have $(f^*\omega^{\mathfrak{p}})((X,A)) = \omega^{\mathfrak{p}}(A)$. The condition $(X,A) \in T_{(x,g)}f^*G$ implies $df(X) = dp(A)$ hence we observe that

$$g_*dp(f^*\omega^{\mathfrak{p}}((X,A))) = g_*dp(\omega^{\mathfrak{p}}(A)) = dp(A) = df(X).$$

We find that $f^*\omega^{\mathfrak{p}}((X,A)) = \xi$ and hence $f^*\omega^{\mathfrak{p}} = \widetilde{\Psi}$ on f^*G. Since $f^*\omega^{\mathfrak{p}}$ descends to Φ and $\widetilde{\Psi}$ descends to $\Psi = \beta(df)$ we conclude that indeed $\beta(df) = \Phi$. $\square$

2.2.1 Hitchin component

If $(p_2, \ldots, p_n)$ is a basis for the space of conjugation invariant polynomials on $\mathfrak{sl}(n,\mathbb{C})$ we can construct a map

$$p\colon \mathcal{M}_{\mathrm{Higgs}}(G) \to \oplus_{i=2}^{n} H^0(X; K_X^i)\colon (E,\phi) \mapsto (p_2(\phi), \ldots, p_n(\phi))$$

via the Chern-Weil construction. This map is called the Hitchin fibration. A section of this map can be constructed as follows. Let $K_X^{1/2}$ be a choice of holomorphic line bundle over X that squares to K_X. We set

$$E = K_X^{\frac{n-1}{2}} \oplus K_X^{\frac{n-3}{2}} \oplus \cdots \oplus K_X^{\frac{3-n}{2}} \oplus K_X^{\frac{1-n}{2}}.$$

Then $K_X \otimes \mathrm{End}_0(E) \subset \oplus_{i,j=1}^{n} K_X^{i-j+1}$. For $(q_2, \ldots, q_n) \in \oplus_{i=2}^{n} H^0(X; K_K^i)$ we define

$$s(q_2, \ldots, q_n) = \left(E,\ \phi = \begin{pmatrix} 0 & q_2 & q_3 & \cdots & q_n \\ r_1 & 0 & q_2 & \cdots & q_{n-1} \\ 0 & r_2 & 0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & q_2 \\ 0 & \cdots & 0 & r_{n-1} & 0 \end{pmatrix} \right) \tag{2.1}$$

where $r_i = \frac{i(n-i)}{2}$. For a suitable choice of $(p_2, \ldots, p_n)$ we have that s is indeed a section of p. Hitchin proved in [Hit92] that representations determined (via the Non-Abelian Hodge correspondence) by Higgs bundles in the image of this section take values in $\mathrm{SL}(n,\mathbb{R})$. Furthermore, these representations constitute precisely a connected component of the space $\mathrm{Rep}(\pi_1(X), \mathrm{SL}(n,\mathbb{R}))$. We call this connected component the *Hitchin component* and representations contained in it *Hitchin representations*. We note that the exact form of the section ϕ in Equation (2.1) depends on a choice of irreducible embedding of $\mathrm{SL}(2,\mathbb{R})$ into

$\mathrm{SL}(n,\mathbb{R})$. The resulting sections for different choices can be related by a gauge transformation. We follow the choice made in [Li19] and hence ϕ differs slightly from the section that appears in [Hit92].

By composing with the projection $\mathrm{SL}(n,\mathbb{R}) \to \mathrm{PSL}(n,\mathbb{R})$ a Hitchin representation induces a representation into $\mathrm{PSL}(n,\mathbb{R})$. Hitchin proved ([Hit92, Section 10]) that $\mathrm{Rep}(\pi_1(\Sigma), \mathrm{PSL}(n,\mathbb{R}))$ contains a connected component consisting entirely of representations that are obtained in this way (i.e. each can be lifted to a Hitchin representations into $\mathrm{SL}(n,\mathbb{R})$). We call representations of $\pi_1(\Sigma)$ into $\mathrm{PSL}(n,\mathbb{R})$ that lie in this component also *Hitchin representations.*

If G^r is an adjoint group of the split real form of a complex simple Lie group it is also possible to identify a Hitchin component in $\mathrm{Rep}(\pi_1(\Sigma), G^r)$ using a Higgs bundle argument ([Hit92]). However, for convenience we give an alternative definition. Namely, for such G^r there exists an irreducible representation $\iota_{G^r}\colon \mathrm{PSL}(2,\mathbb{R}) \to G^r$ that is unique up to conjugation. Composing a Fuchsian representation $\rho_0\colon \pi_1(\Sigma) \to \mathrm{PSL}(2,\mathbb{R})$ that corresponds to a point in Teichmüller space with ι_{G^r} yields a representation into G^r. The Hitchin component of $\mathrm{Rep}(\pi_1(\Sigma), G^r)$ can be defined as the connected component containing $\iota_{G^r} \circ \rho_0$.

The cases $G^r = \mathrm{PSp}(2n,\mathbb{R})$, $\mathrm{PSO}(n, n+1)$ or G_2 have the special feature that if we consider G^r as a subset of $\mathrm{PSL}(m,\mathbb{R})$ (for $m = 2n, 2n+1$ or 7 respectively), then $\iota_{G^r} = \iota_{\mathrm{PSL}(m,\mathbb{R})}$. Hence, the Hitchin component for G^r can be realised as a subset of the Hitchin component for $\mathrm{PSL}(m,\mathbb{R})$.

2.3 Plurisubharmonicity

In this section we explain some of the results of [Tol12] and introduce some notation used in that paper that we will also use. Let

$$\mathcal{C} = \{J \in C^\infty(\Sigma, \mathrm{End}(T\Sigma)) \mid J^2 = -\,\mathrm{id}\}$$

be the set of almost complex structures on Σ. We let N be a Riemannian manifold of non-positive Hermitian sectional curvature. This condition means that $R(X, Y, \overline{X}, \overline{Y}) \leq 0$ for all $X, Y \in TN \otimes \mathbb{C}$ where R is the complex multilinear extension of the Riemannian curvature tensor of N.

If A is an endomorphism of $T\Sigma$, then for any one-form $\alpha \in \Omega^1(\Sigma)$ we denote $A\alpha = -\alpha \circ A$. In particular, if $J \in \mathcal{C}$, then $Jdf = -df \circ J = df \circ J^{-1}$. For any $J \in \mathcal{C}$ the Dirichlet energy of a map $f\colon \Sigma \to N$ is given by

$$\mathcal{E}(J, f) = \frac{1}{2} \int_\Sigma \langle df \wedge Jdf \rangle \,.$$

A map is harmonic if it is a critical point of this functional. We fix a homotopy class of maps $\Sigma \to N$. We make the assumption that for each $J \in \mathcal{C}$ there exists a unique harmonic map $f_J\colon (\Sigma, J) \to N$ in this homotopy class. We assume further that the maps f_J depend smoothly on J. These assumptions will be satisfied in the situation we will consider. Define $E\colon \mathcal{C} \to \mathbb{R}$ by $E(J) = \mathcal{E}(J, f_J)$. This map descends to Teichmüller space because if $\phi \in \mathrm{Diff}_0(\Sigma)$, then $f_{\phi^* J} = \phi^* f_J$ hence $E(\phi^* J) = \mathcal{E}(\phi^* J, \phi^* f_J) = \mathcal{E}(J, f_J) = E(J)$. The main result of [Tol12] is that E is a plurisubharmonic function on Teichmüller space.

To state this result formally we consider a small disk $D \subset \mathbb{C}$ centred around 0 and a holomorphic family of complex structures $J\colon D \to \mathcal{C}$. Denote by $u = s + it$ the complex coordinates on D. Set $E(s,t) = E(J(s,t))$ and $f(s,t) = f_{J(s,t)}$. We define

$$W = \frac{\partial f}{\partial s} + i\frac{\partial f}{\partial t} \in \Gamma^\infty(f^* T_{\mathbb{C}} N).$$

We equip Σ with the complex structure $J_0 = J(0,0)$ and denote by $T_{\mathbb{C}}\Sigma = T_{1,0}\Sigma \oplus T_{0,1}\Sigma$ and $T^*_{\mathbb{C}}\Sigma = T^{1,0}\Sigma \oplus T^{0,1}\Sigma$ the induced splittings of the tangent and cotangent space into $+i$ and $-i$ eigenspaces of J_0. The complexification of the derivative df splits into a $(1,0)$ and $(0,1)$ part denoted by $d'f$ and $d''f$ respectively. Similarly, if s is a section of a vector bundle equipped with a connection ∇, then we denote by $d'_\nabla s$ and $d''_\nabla s$ respectively the $(1,0)$ and $(0,1)$ part of ∇s. Finally, we consider

$$H = \frac{\partial J}{\partial s}(0,0) \in T_{J_0}\mathcal{C}.$$

The endomorphism H of TM anti-commutes with J_0 hence its complexification can be written as $H = \mu + \overline{\mu}$ with μ a smooth section of $T^{0,1}\Sigma \otimes T_{1,0}\Sigma$.

Theorem 2 of [Tol12] now states:

Theorem 2.3.1. *We have*

$$\Delta E(0,0) \geq 0$$

and in case of equality we have

$$d''_\nabla W = \pm \mu d' f. \tag{2.2}$$

The last statement in this theorem is not explicitly stated in [Tol12] but follows from the arguments used to prove the first statement. We briefly clarify how Equation (2.2) is obtained when $\Delta E(0,0) = 0$ (see also the proof of [Tol12, Theorem 3]). In this section any reference to a numbered equation will to refer to an equation in [Tol12].

Toledo first calculates (Equation 16) that $\Delta E(0,0) = -a + b$ where

$$a = -\int_\Sigma \left\langle d_\nabla \frac{\partial f}{\partial s} \wedge H df \right\rangle + \left\langle d_\nabla \frac{\partial f}{\partial t} \wedge J_0 H df \right\rangle \text{ and } b = \int_\Sigma \left\langle df \wedge J_0 H^2 df \right\rangle.$$

We denote also

$$\alpha = \int_\Sigma \langle d'_\nabla \overline{W} \wedge J_0 d''_\nabla W \rangle \text{ and } \rho = \int_\Sigma R\left(\frac{\partial f}{\partial z}, W, \overline{\frac{\partial f}{\partial z}}, \overline{W}\right) dx \wedge dy.$$

Inequality (26) yields that $a \leq \alpha + \frac{b}{2}$ and Equation (29) gives that $\alpha = \frac{a}{2} + \rho$. Putting these together gives $\alpha \leq \frac{1}{2}(a+b) + 2\rho$ or equivalently $a \leq b + 4\rho$ (which is Inequality (30)). The non-positive Hermitian curvature condition implies $\rho \le 0$ hence $a \le b$ from which follows that $\Delta E(0,0) \ge 0$.

If the family J is such that $\Delta E(0,0) = 0$, then we see that equality holds in inequalities (26) and (30). The remarks made by Toledo after Inequality (26) tell us that Inequality (26) is an equality if and only if $d''_\nabla W = \pm \mu d' f$.

We note that in this case we also have $\rho = 0$ which means $R\left(\frac{\partial f}{\partial z}, W, \overline{\frac{\partial f}{\partial z}}, \overline{W}\right) = 0$ everywhere. However, we do not use this in our proof.

Remark 2.3.2. We note that in the statements of Theorems 1 and 2 in [Tol12] the manifold N is assumed to be compact. This is something that will not be true in the application we have in mind. The compactness assumption is used to guarantee the existence of a harmonic map $(\Sigma, J) \to N$ in a given homotopy class for every J. This is not necessarily true when N is not compact. However, in the situation we consider the existence of such harmonic maps follows from the results of Corlette ([Cor88]). An inspection of the proof in [Tol12] shows that the compactness of N plays no further role. This means we are free to apply Theorem 2.3.1 even if N is not compact, as long as the existence of a (unique) harmonic map for each $J \in \mathcal{C}$ is guaranteed.

2.4 Proof

We turn now to the proof of Theorem 2.1.1. We observe first that it is enough to give a proof for $G = \mathrm{PSL}(n, \mathbb{R})$. Namely, if G equals $\mathrm{PSp}(2n, \mathbb{R})$, $\mathrm{PSO}(n, n+1)$ or G_2, then the inclusion $G \subset \mathrm{PSL}(m, \mathbb{R})$ (for $m = 2n, 2n+1$ or 7 respectively) induces an inclusion of the Hitchin component for G into the Hitchin component for $\mathrm{PSL}(m, \mathbb{R})$. Moreover, via the totally geodesic embedding $G/K \subset \mathrm{PSL}(m, \mathbb{R}) / \mathrm{PSO}(m)$ a harmonic map $\widetilde{X} \to G/K$ equivariant for a representation $\rho\colon \pi_1(X) \to G$ can be seen as a harmonic map into $\mathrm{PSL}(m, \mathbb{R}) / \mathrm{PSO}(n)$ equivariant for ρ as a representation into $\mathrm{PSL}(m, \mathbb{R})$. In particular, the energy functional E is unchanged if we view ρ as a representation into $\mathrm{PSL}(m, \mathbb{R})$ rather then into G.

We consider now the energy function associated to a $\mathrm{PSL}(n, \mathbb{R})$-Hitchin representation. We lift this representation to a representation into $\mathrm{SL}(n, \mathbb{R})$ which we denote by $\rho\colon \pi_1(\Sigma) \to \mathrm{SL}(n, \mathbb{R})$. From now on we denote $G = \mathrm{SL}(n, \mathbb{R})$ and $K = \mathrm{SO}(n)$. Hitchin representations act freely and properly on G/K ([Lab06]) so we can consider the locally symmetric space $N = \rho(\pi_1(\Sigma)) \setminus G/K$. The representation ρ determines a homotopy class of maps $\Sigma \to N$ that lift to ρ-equivariant maps $\widetilde{\Sigma} \to G/K$. Equivariant harmonic maps for Hitchin representations are unique and depend smoothly on J (see [EL81] or [Sle20]). Hence, we can consider the energy functional $E\colon \mathcal{T}(\Sigma) \to \mathbb{R}$ as defined in Section 2.3. We note that this coincides with the energy functional as described in Section 2.1. In [Sam86] Sampson proved that locally symmetric spaces of non-compact type have non-positive Hermitian sectional curvature. So Theorem 2.3.1 applies to E.

We now give a proof of Theorem 2.1.1. The strategy is similar to the proof of [Tol12, Theorem 3] in which strict plurisubharmonicity is proved when the target is assumed to have strictly negative Hermitian sectional curvature. It is interesting that this strictly negative curvature condition can be replaced by the explicit information about the form of the harmonic map that is provided by the Higgs bundle picture.

Proof of Theorem 2.1.1. We use the notation introduced in Section 2.3. Suppose that $J\colon D \to \mathcal{C}$ is a holomorphic family of complex structures such that $\Delta E(0,0) = 0$. It then follows from Theorem 2.3.1 that Equation (2.2) holds.

We note that W is a smooth section of $f^*T_{\mathbb{C}}N$. Using Lemma 2.2.2 we can view it as a section of $\mathrm{End}_0(E)$ by considering $\nu = \beta(W)$. Since β is an affine isomorphism we have $d''_\nabla \nu = \beta(d''_\nabla W)$. Taking into account Lemma 2.2.3 we see that Equation (2.2) is equivalent to

$$d''_\nabla \nu = \pm \mu \phi. \tag{2.3}$$

We write $\nu = (\nu_{i,j})_{i,j}$ with each $\nu_{i.j}$ a smooth section of K^{j-i}. Keeping in mind the expression for ϕ as given in Equation (2.1) we consider now the (2,1) component of the matrices on both sides of Equation (2.3). This gives

$$\overline{\partial}\nu_{2,1} = \pm\mu(r_1 \cdot 1) = \pm\frac{1}{2}\mu.$$

Here $\nu_{2,1}$ is a section of $K^{-1} = T_{1,0}\Sigma$. The above equality implies that $[\mu] = 0$ in $H^1(X, T_{1,0}\Sigma)$ which means precisely that the tangent vector $H \in T_{J_0}\mathcal{C}$ projects to zero in $T_{[J_0]}\mathcal{T}(\Sigma)$.

We conclude that for any family J of complex structures inducing a non-zero tangent vector in Teichmüller space we have $\Delta E(0,0) > 0$. This concludes the proof. $\square$

As a first corollary of Theorem 2.1.1 we obtain a bound on the index of the critical points of E. We recall that if $g = \mathrm{genus}(\Sigma)$, then $\dim_{\mathbb{R}} \mathcal{T}(\Sigma) = 6g - 6$.

Corollary 2.4.1. *Under the assumptions of Theorem 2.1.1 the index of a critical point of E is at most* $\dim_{\mathbb{C}} \mathcal{T} = 3g - 3$.

Proof. Assume $[J] \in \mathcal{T}(\Sigma)$ is a critical point of E. Let H be the Hessian of E at this point and denote by $\widetilde{H}$ its sesquilinear extension of the complexified tangent space of $\mathcal{T}(\Sigma)$. The forms H and $\widetilde{H}$ have the same index. If $(z^1, \ldots, z^{3g-3})$ are complex coordinates around $[J]$, then the strict plurisubharmonicity property of E implies that

$$\widetilde{H}(u,v) = \frac{\partial^2 E}{\partial z^\alpha \partial \overline{z}^\beta} u^\alpha \overline{v^\beta}$$

is positive definite. This means that $\widetilde{H}$ is positive definite on the subspace of dimension $3g-3$ that is spanned by the vectors $\frac{\partial}{\partial z^\alpha}$ and as a result has index at most $3g-3$. $\square$

Finally, we obtain the following corollary by applying the results of [HW73] to the function $f = E - \min_{[J]\in\mathcal{T}(\Sigma)} E([J])$. We call a submanifold P of $\mathcal{T}(\Sigma)$ *totally real* if T_pP contains no non-zero complex subspaces of $T_p\mathcal{T}(\Sigma)$ for all $p \in P$.

Corollary 2.4.2. *The set*

$$M = \{[J] \in \mathcal{T}(\Sigma) \mid E \text{ attains its global minimum at } [J]\}.$$

is locally contained in totally real submanifolds of $\mathcal{T}(\Sigma)$. More precisely for every $[J] \in M$ there exists an open neighbourhood $U \subset \mathcal{T}(\Sigma)$ of $[J]$ and a totally real submanifold $P \subset U$ such that $M \cap U \subset P$. In particular, at smooth points of M its tangent space is totally real. It follows that the Hausdorff dimension of M is at most $3g - 3$.

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Chapter 3

The energy spectrum of metrics on surfaces

Abstract

Let (N,ρ) be a Riemannian manifold, S a surface of genus at least two and let $f\colon S \to N$ be a continuous map. We consider the energy spectrum of (N,ρ) (and f) which assigns to each point $[J] \in \mathcal{T}(S)$ in the Teichmüller space of S the infimum of the Dirichlet energies of all maps $(S,J) \to (N,\rho)$ homotopic to f. We study the relation between the energy spectrum and the simple length spectrum. Our main result is that if $N = S$, $f = \mathrm{id}$ and ρ is a metric of non-positive curvature, then the energy spectrum determines the simple length spectrum. Furthermore, we prove that the converse does not hold by exhibiting two metrics on S with equal simple length spectrum but different energy spectrum. As corollaries to our results we obtain that the set of hyperbolic metrics and the set of singular flat metrics induced by quadratic differentials satisfy energy spectrum rigidity, i.e. a metric in these sets is determined, up to isotopy, by its energy spectrum. We prove that analogous statements also hold true for Kleinian surface groups.

3.1 Introduction

In this paper we study, what we will call, the energy spectrum of a Riemannian manifold (see Section 3.3). Let S be a closed surface of genus at least two, let $\mathcal{T}(S)$ be its Teichmüller space, let (N,ρ) be a Riemannian manifold and let $[f]$ be a homotopy class of maps $S \to N$. In brief, the energy spectrum of (N,ρ) and $[f]$ is the function on Teichmüller space that assigns to each $[J] \in \mathcal{T}(S)$ the infimum of the energies of all Lipschitz maps $(S,J) \to (N,\rho)$ that lie in $[f]$. It gives a measure of how compatible (N,ρ) and a point in Teichmüller space are.

The energy spectrum has been considered (under a different name[1]) by several authors. Toledo proved in [Tol12] that the energy spectrum (for any $[f]$) is a plurisubharmonic function on Teichmüller space if (N,ρ) is a compact manifold of non-positive Hermitian curvature. He used this result to give an alternative formulation of the rigidity theory of Siu and Sampson. In [Lab08] Labourie used the energy spectrum to study Hitchin components in representation varieties. Given a Hitchin representation $\rho\colon \pi_1(S) \to \mathrm{PSL}(n,\mathbb{R})$ he considered the energy spectrum of $N = \rho(\pi_1(S)) \setminus \mathrm{PSL}(n,\mathbb{R}) / \mathrm{PSO}(n)$ and the homotopy class of maps that lift to ρ-equivariant maps $\widetilde{S} \to \mathrm{PSL}(n,\mathbb{R}) / \mathrm{PSO}(n)$. He proved that it is a proper function on Teichmüller space. Furthermore, he made the conjecture

[1] In [Lab08] and [Tol12] it is called the energy function or energy functional.

that it has a unique minimum. The author showed in [Sle20] that in this same setting the energy spectrum is strictly plurisubharmonic.

In this paper we examine to what extend a Riemannian manifold is determined by its energy spectrum. We begin by restricting ourselves to the case $N = S$ and $[f] = [\mathrm{id}]$. We will define, by analogy with simple length spectrum rigidity, the notion of energy spectrum rigidity. We will say a set $\mathcal{M}$ of metrics on S, determined up to isotopy, satisfies energy spectrum rigidity if the map $\mathcal{M} \to C^0(\mathcal{T}(S))$, assigning to each metric its energy spectrum, is an injection. We will study the question which sets of metrics satisfy this type of rigidity.

The main results of this paper offer a comparison between the energy spectrum and the simple length spectrum. Our first result states that the energy spectrum determines the simple length spectrum.

Theorem (Theorem 3.3.3)**.** *Let ρ, ρ' be non-positively curved Riemannian metrics on a surface S of genus at least two. If the energy spectra of (S, ρ) and (S, ρ') (with $[f] = [\mathrm{id}]$) coincide, then the simple length spectra of ρ and ρ' coincide.*

Our second second results shows that the converse is not true. Namely, the energy spectrum carries strictly more information and hence is not determined by the simple length spectrum.

Proposition (Proposition 3.4.1)**.** *For every hyperbolic metric on a surface there exists a negatively curved Riemannian metric on that surface with equal simple length spectrum but different energy spectrum.*

In summary, the energy spectrum is a strictly more sensitive way to tell metrics on a surface apart. This raises the following interesting question: how does the energy spectrum compare to the (full) marked length spectrum? It is, at the moment, unknown to the author whether the energy spectrum carries the same information as the marked length spectrum or whether it carries strictly less information. We discuss this question in more depth in Section 3.4.

As a corollary to our results we obtain that the set of hyperbolic metrics satisfies energy spectrum rigidity.

Corollary (Corollary 3.5.1)**.** *The set of hyperbolic metrics on S, defined up to isotopy, satisfies energy spectrum rigidity.*

A quadratic differential on S induces a singular flat metric (see Section 3.2.4). It is proved in [DLR10] that the set of these metrics satisfies simple length spectrum rigidity. It then follows from our results that this set also satisfies energy spectrum rigidity.

Corollary (Corollary 3.5.3)**.** *The set of singular flat metrics that are induced by quadratic differentials, defined up to isotopy, satisfies energy spectrum rigidity.*

Our interest in these questions surrounding the energy spectrum stems from the work of Labourie in [Lab08] (as described above). He asked whether it is possible to assign to each Hitchin representation an associated point in Teichmüller space, in a mapping class group invariant way. In cases where the aforementioned Labourie conjecture is true such a projection can be constructed by mapping a Hitchin representation to the unique minimiser of its energy spectrum. The Labourie conjecture has been proved for real split simple Lie groups of rank two ([Lab17]). Marković showed in a recent preprint ([Mar21]) that for the semisimple Lie group $G = \Pi_{i=1}^{3} \mathrm{PSL}(2,\mathbb{R})$ the analogue of Labourie's conjecture does not hold. The conjecture, however, remains open for simple Lie groups of rank at least three.

Considering this situation from a slightly different angle we ask ourselves how much information about a Hitchin representation is actually encoded in its energy spectrum. More concretely, we ask whether a Hitchin representation is determined, up to conjugacy, by its energy spectrum. We hope that the results of this paper are a step towards answering this question in the affirmative. We illustrate this by applying our results to the simpler setting of Kleinian surface groups. We prove the following result.

Theorem (Theorem 3.6.1)**.** *Let $\rho, \rho' \colon \Gamma \to \mathrm{PSL}(2,\mathbb{C})$ be two Kleinian surface groups. If the energy spectra of ρ and ρ' coincide, then their simple simple length spectra coincide.*

Combined with the results of Bridgeman and Canary in [BC17] we obtain the following corollary.

Corollary (Corollary 3.6.2)**.** *If $\rho, \rho' \colon \Gamma \to \mathrm{PSL}(2,\mathbb{C})$ are Kleinian surface groups with equal energy spectrum, then ρ' is conjugate to either ρ or $\overline{\rho}$.*

Unfortunately, the results obtained in this paper are not enough to conclude the same for Hitchin representations. In Section 3.7 we discuss briefly the further steps that would be required to do so.

3.2 Prerequisites

We let S be a closed and oriented surface. We will denote its genus by g.

3.2.1 Teichmüller space

We recall the definition of the Teichmüller space of a surface. A general reference for the concepts discussed in this section is [Hub06].

A *marked complex structure* on S is a pair (X, ϕ) where X is an Riemann surface and $\phi \colon S \to X$ is an orientation preserving diffeomorphism. Two marked complex structures (X, ϕ) and (X', ϕ') are equivalent if there exists a biholomorphism $\psi \colon X' \to X$ such that $\phi^{-1} \circ \psi \circ \phi' \colon S \to S$ is isotopic to the identity map.

Definition 3.2.1. *The Teichmüller space of S, denoted $\mathcal{T}(S)$, is the set of equivalence classes of marked complex structures on S.*

Teichmüller space can be equipped with a smooth structure (or even a complex structure) and if S is a surface of genus $g \geq 2$, then $\mathcal{T}(S)$ is diffeomorphic to $\mathbb{R}^{6g-6}$.

We will describe here some alternative ways to describe $\mathcal{T}(S)$ which will be more practical to work with in the applications we have in mind. The complex structure on a Riemann surface X is uniquely determined by an automorphism $J_X\colon TX \to TX$ that satisfies $J_X^2 = -\mathrm{id}$. We note that in general such an automorphism is only an almost complex structure, however on surfaces every almost complex structure is integrable and hence determines a complex structure. We see that each marking (X, ϕ) determines a complex structure $J = \phi^* J_X$ on S. It follows that we can alternatively take

$$\mathcal{T}(S) = \{J \mid J\colon TS \to TS \text{ is complex structure on } S\}/\sim$$

as definition of Teichmüller space. Here we define that $J \sim J'$ if and only if a diffeomorphism $\psi\colon S \to S$ isotopic to the identity exists such that $J' = \psi^* J$. Furthermore, on a surface a complex structure is uniquely determined by a conformal class of metrics and vice versa. So we could also describe $\mathcal{T}(S)$ as the set of conformal structures up to isotopy. Finally, if S is a surface of genus at least two, then in each conformal class of metrics on S there exists a unique hyperbolic metric. So we can also take

$$\mathcal{T}(S) = \{\rho \mid \rho \text{ is a hyperbolic metric on } S\}/\sim$$

where $\rho \sim \rho'$ if $\rho' = \psi^* \rho$ for some diffeomorphism ψ of S that is isotopic to the identity.

The different views on Teichmüller space will be useful at different points in our discussion. If we consider a point $X \in \mathcal{T}(S)$ we will think of this as the surface S equipped with either a complex structure or a hyperbolic metric, each determined up to isotopy.

3.2.2 Length of curves

Let ρ be a Riemannian metric on S. If $\gamma \subset S$ is a path in S, then we denote by $l_\rho(\gamma)$ its length measured with respect to ρ. If $[\gamma]$ is a free homotopy class of closed loops on S, then we denote

$$\ell_\rho([\gamma]) = \inf_{\gamma' \in [\gamma]} l_\rho(\gamma').$$

Often we will not distinguish between a closed loop on S and the free homotopy class it determines and simply write $\ell_\rho(\gamma)$ for $\ell_\rho([\gamma])$.

We will denote by $\mathcal{C}$ the set of homotopy classes of closed curves on S and by $\mathcal{S} \subset \mathcal{C}$ the set of homotopy classes of simple closed curves. The *marked length spectrum* of a metric ρ is the vector

$$(\ell_\rho(\gamma))_{\gamma \in \mathcal{C}} \in (\mathbb{R}_{>0})^{\mathcal{C}}.$$

Similarly, the *(marked) simple length spectrum* of a metric ρ is

$$(\ell_\rho(\gamma))_{\gamma\in\mathcal{S}} \in (\mathbb{R}_{>0})^{\mathcal{S}}.$$

If $\mathcal{M}$ is a set of metrics on S, defined up to isometry, then we can ask whether the marked length spectrum or even the simple length spectrum distinguishes metrics in that set. If $\rho \mapsto (\ell_\rho(\gamma))_{\gamma\in\mathcal{C}}$ is an injection of $\mathcal{M}$ into $(\mathbb{R}_{>0})^{\mathcal{C}}$, then we say $\mathcal{M}$ satisfies *length spectrum rigidity.* If the map $\rho \mapsto (\ell_\rho(\gamma))_{\gamma\in\mathcal{S}}$ injects $\mathcal{M}$ into $(\mathbb{R}_{>0})^{\mathcal{S}}$, then we say $\mathcal{M}$ satisfies *simple length spectrum rigidity.*

If $[\gamma], [\eta]$ are conjugacy classes of simple closed curves on S, then we define their *intersection number* as

$$i([\gamma],[\eta]) = \min\{|\gamma' \cap \eta'| \mid \gamma' \in [\gamma], \eta' \in [\eta]\}.$$

If γ and η are simple closed curves, then, for convenience, we will write $i(\gamma,\eta)$ rather than $i([\gamma],[\eta])$. When γ and η are simple closed geodesics for a non-positively curved metric on S, then $|\gamma\cap\eta|$ realises $i(\gamma,\eta)$.

3.2.3 Dehn twists

Assume S has genus at least one and let $\gamma \subset S$ a simple closed curve. Let $N \subset S$ be a closed collar neighbourhood of γ which we will identify, in an orientation preserving way, with $[0,1]\times\mathbb{R}/\mathbb{Z}$. The *Dehn twist* around γ is the orientation preserving homeomorphism T_γ of S that is equal to the identity map outside of N and is given by

$$(t,[\theta]) \mapsto (t,[\theta+t])$$

on $N \cong [0,1]\times\mathbb{R}/\mathbb{Z}$. Since these definitions coincide on the boundary of N, we see that T_γ is indeed continuous. Note that the isotopy class of T_γ is independent of the choice of representative of $[\gamma]$ and the choice of collar neighbourhood N. In general we will refer to any homeomorphism in the isotopy class determined by T_γ as a Dehn twist around γ. By a slight modification to the above construction it is possible to find a smooth representative of the isotopy class.

A Dehn twist defines a mapping on Teichmüller space. Namely, if $[(X,\phi)] \in \mathcal{T}(S)$, then $T_\gamma\cdot[(X,\phi)] = [(X,\phi\circ T_\gamma^{-1})]$. To put this in a slightly broader context we note that the Dehn twist is an element of the mapping class group of the surface S. The mapping class group has a natural action on Teichmüller space which is given by precisely the mapping defined here for the Dehn twist.

If $\eta \subset S$ is a closed loop (resp. a homotopy class of closed loops), then we define $T_\gamma\eta$ to be the loop $T_\gamma\circ\eta$ (resp. the homotopy class containing this loop).

In our proof of Theorem 3.3.3 we will need a lower bound on the length of a loop that has been Dehn twisted often. The following lemma provides such an estimate.

Lemma 3.2.2. *Let (S,ρ) be an oriented surface of genus at least two equipped with a metric of non-positive curvature. For every pair $\gamma,\eta\subset S$ of simple closed curves there exists a constant $C = C(\gamma,\eta) > 0$ such that*

$$\ell_\rho(T_\gamma^n\eta) \geq n\cdot i(\gamma,\eta)\cdot\ell_\rho(\gamma) - C$$

for all $n \geq 1$.

Let $M = \widetilde{S}$ be the universal cover of S equipped with the pullback metric. In our proof of Lemma 3.2.2 we will use that M is non-positively curved, both in a local sense and in a global sense. We will use [BH99] as our reference for the facts on metric spaces of non-positive curvature that we will need. Because ρ is a metric of non-positive curvature, it follows that M is a CAT(0) space ([BH99, Section II.1]). Moreover, it is also a Gromov δ-hyperbolic space ([BH99, Section III.H.1]) for some $\delta > 0$ because, by the Švarc-Milnor lemma, it is quasi-isometric to the Cayley graph of $\pi_1(S)$.

We first prove two auxiliary lemmas. For any two points $x, y \in M$ let us denote by $[x, y]$ the (directed) geodesic segment connecting x to y. Furthermore, for $x, y, z \in M$ we denote by $\angle_z(x, y)$ the angle the geodesic segments $[x, z]$ and $[z, y]$ make at z.

Lemma 3.2.3. *For all $x, y, z \in M$ with $\angle_z(x, y) \geq \pi/2$ we have*

$$d(x, y) \geq d(x, z) + d(y, z) - 4\delta.$$

Proof. Because M is Gromov δ-hyperbolic, it follows that the triangle with vertices x, y, z is δ-thin (see [BH99, Definition III.1.16]) and hence there exist points $w_{x,y} \in [x, y], w_{x,z} \in [x, z], w_{y,z} \in [y, z]$ such that $\operatorname{diam}(\{w_{x,y}, w_{x,z}, w_{y,z}\}) \leq \delta$. We compare the triangle with vertices $w_{x,z}, w_{y,z}, z$ to a triangle in the Euclidean plane with vertices a, b, c that satisfy $d(a, c) = d(w_{x,z}, z), d(b, c) = d(w_{y,z}, z)$ and $\angle_c(a, b) = \angle_z(w_{x,z}, w_{y,z}) = \angle_z(x, y) \geq \pi/2$. From the CAT(0) condition follows (see [BH99, Proposition II.1.7(5)]) that

$$\delta \geq d(w_{x,z}, w_{y,z}) \geq d(a, b) \geq \sqrt{d^2(w_{x,z}, z) + d^2(w_{y,z}, z)}.$$

From this we conclude that that $d(z, w_{x,z}) \leq \delta$. The triangle inequality then yields that

$$d(w_{x,y}, z) \leq d(w_{x,y}, w_{x,z}) + d(w_{x,z}, z) \leq 2\delta.$$

Using again the triangle inequality now gives

$$\begin{aligned} d(x, y) = d(x, w_{x,y}) + d(w_{x,y}, y) &\geq d(x, z) - d(w_{x,y}, z) + d(y, z) - d(w_{x,y}, z) \\ &\geq d(x, z) + d(y, z) - 4\delta. \end{aligned}$$

□

Consider three points $x, y, z \in M$ and let $\gamma_{x,y} \colon [0, 1] \to M$ be a parametrisation of $[x, y]$ with $\gamma_{x,y}(0) = x$ and $\gamma_{x,y}(1) = y$. Similarly, let $\gamma_{y,z}$ be a parametrisation of $[y, z]$. We say the angle that $[x, y]$ and $[y, z]$ make at y is positively oriented if $(\dot{\gamma}_{x,y}(1), \dot{\gamma}_{y,z}(0))$ is a positively oriented frame of T_yM (recall that S is oriented and hence also M). We say it is negatively oriented otherwise.

Consider a continuous path consisting of a concatenation of geodesic segments $[x_0, x_1], [x_1, x_2], \ldots, [x_{n-1}, x_n]$ with pairwise distinct points $x_i \in M$. We call such a path a *stairstep path* if all successive segments meet each other orthogonally and the orientation of the angle between segments at points x_i is alternately positive and negative. So either each angle at even numbered points is positively oriented and negatively oriented at odd numbered points or it is the other way around.

Lemma 3.2.4. *If the segments* $[x_0, x_1], [x_1, x_2], \ldots, [x_{n-1}, x_n]$ *form a stairstep path, then*

$$d(x_0, x_n) \geq \sum_{i=0}^{n} d(x_i, x_{i+1}) - 4(n-1)\delta.$$

Proof. For $i = 0, \ldots, n-1$ let L_i be the geodesic in M that contains the segment $[x_i, x_{i+1}]$. A pair of geodesics L_i, L_{i+2} is connected by a segment $[x_{i+1}, x_{i+2}]$ that meets both geodesics orthogonally. It follows from convexity of the distance function that this is the unique geodesic segment that realises the shortest path between L_i and L_{i+2}. Because we assumed that the points x_i are pairwise distinct it follows that L_i and L_{i+2} are a positive distance apart. In particular, they do not intersect.

Each L_i divides the manifold M into two halves. For $i = 0, \ldots, n-2$ let H_i be the component of $M - L_i$ that contains x_n. From the assumption that successive angles have opposite orientation it follows that x_n and x_{n-3} lie on opposite sides of L_{n-2} and hence $x_{n-3} \notin H_{n-2}$. Because the segment $[x_{n-4}, x_{n-3}]$ is contained in L_{n-4} which is disjoint from L_{n-2}, we also have $x_{n-4} \notin H_{n-2}$. We claim the same holds for x_{n-5}. Since L_{n-4} and L_{n-2} do not intersect, it follows that $L_{n-2} \cup H_{n-2} \subset H_{n-4}$. Note that $x_{n-2} \in L_{n-2} \subset H_{n-4}$. Using again the assumption that successive angles have opposite orientation we find that x_{n-2} and x_{n-5} lie on opposite sides of L_{n-4}, hence we must have $x_{n-5} \notin H_{n-4}$. Because $H_{n-2} \subset H_{n-4}$ we conclude that in particular $x_{n-5} \notin H_{n-2}$. Continuing this argument inductively we find that $x_0 \notin H_{n-2}$ or, in other words, x_0 and x_n lie on opposite sides of L_{n-2}.

We now prove the lemma by induction on n, the number of segments. For $n = 1$ the statement is trivial and for $n = 2$ it follows directly from Lemma 3.2.3. Assume the lemma holds for some $n \geq 2$. Consider a stairstep path $[x_0, x_1], \ldots, [x_n, x_{n+1}]$ consisting of $n+1$ segments. Let L_{n-1} as defined above. Then the segments $[x_0, x_n]$ and $[x_n, x_{n+1}]$ lie on opposite sides of L_{n-1} and meet at $x_n \in L_{n-1}$. Because the segment $[x_n, x_{n-1}]$ is orthogonal to L_{n-1}, it follows that $\angle_{x_n}(x_0, x_{n+1}) \geq \pi/2$. We apply Lemma 3.2.3 to find

$$\begin{aligned} d(x_0, x_{n+1}) &\geq d(x_0, x_n) + d(x_n, x_{n+1}) - 4\delta \\ &\geq \sum_{i=0}^{n} d(x_i, x_{i+1}) + d(x_n, x_{n+1}) - 4(n-1)\delta - 4\delta \\ &= \sum_{i=0}^{n+1} d(x_i, x_{i+1}) - 4n\delta. \end{aligned}$$

Here the second inequality follows from the induction assumption. We see that the lemma also holds for paths consisting of $n+1$ segments. This concludes the argument. $\square$

Proof of Lemma 3.2.2. The statement is trivial if $i(\gamma, \eta) = 0$. Hence, from now on we assume that $i(\gamma, \eta) > 0$. Take γ and η to be geodesic representatives in (S, ρ) of their free homotopy class. These loops realise the minimal number of intersections so $k := i(\gamma, \eta) = |\gamma \cap \eta|$. We label the intersection points $\gamma \cap \eta = \{p_1, \dots, p_k\}$ in order of appearance along some parametrisation of η. Cut η into k pieces $\eta^1, \dots, \eta^k$, where each η^i is the subarc connecting p_i to p_{i+1} (and η^k connects p_k to p_1).

For each $i = 1, \dots, k$ let A_i be the geodesic arc of minimal length in the homotopy class of η^i with endpoints sliding freely over γ. Each arc A_i meets γ orthogonally because it is length minimising. The loop η is homotopic to a unique loop ω_0 consisting of a concatenation of geodesic arcs

$$A_1, B_{1,0}, A_2, B_{2,0}, \dots, A_k, B_{k,0}$$

where each $B_{i,0}$ is an arc that lies along the geodesic γ. Similarly, the Dehn twisted loops $T_\gamma^n \eta$ are homotopic to a unique loop ω_n consisting of segments $A_1, B_{1,n}, \dots, A_k, B_{k,n}$. Each $B_{i,n}$ differs from $B_{i,0}$ by n turns around γ.

After untwisting any turns that η made around γ in the opposite direction of the Dehn twist we find that for n high enough the angle between each A_i and $B_{i,n}$ is positively oriented and the angle between each $B_{i,n}$ and A_{i+1} is negatively oriented. It follows that if we lift ω_n to M it is a stairstep path. We also see there exists a constant $c > 0$ such that $l_\rho(B_{i,n}) \geq n \cdot \ell_\rho(\gamma) - c$ for all $i = 1, \dots, k$ and $n \geq 1$.

Consider the geodesic representatives η_n of the homotopy classes $T_\gamma^n \eta$. Because for n high enough the arc $B_{1,n}$ winds around γ at least once, it follows that η_n and ω_n intersect at least once. Parametrise $\eta_n \colon [0,1] \to S$ to start at such an intersection point and consider a lift $\widetilde{\eta}_n$ to M. The endpoints of $\widetilde{\eta}_n$ are connected by the stairstep path that is a lift of ω_n. We use Lemma 3.2.4 to conclude that

$$\begin{aligned}
\ell_\rho(\eta) = d(\widetilde{\eta}_n(0), \widetilde{\eta}_n(1)) &\geq \sum_{i=0}^{k} (l_\rho(A_i) + l_\rho(B_i)) - 4k\delta \\
&\geq n \cdot k \cdot \ell_\rho(\gamma) - (4\delta + c) \cdot k \\
&= n \cdot i(\gamma, \eta) \cdot \ell_\rho(\gamma) - C
\end{aligned}$$

where we take $C = (4\delta + c) \cdot k$. $\square$

3.2.4 Conformal geometry of surfaces

In this section we will consider some of the conformal aspects of the geometry of a closed surface. We let X be a closed Riemann surface.

Definition 3.2.5. *Let $\gamma \subset X$ be a closed curve. We define the extremal length of γ in X to be*

$$E_X(\gamma) = \sup_\sigma \frac{\ell_\sigma^2(\gamma)}{\mathrm{Area}(\sigma)}. \tag{3.1}$$

Here the supremum runs over all metrics in the conformal class determined by X.

In case γ is a simple closed curve a second equivalent definition for its extremal length exists. We will denote the modulus of an annulus $A \subset X$ by $M(A)$.

Definition 3.2.6. *If $\gamma \subset X$ is a simple closed curve, then*

$$E_X(\gamma) = \inf_A \frac{1}{M(A)} \tag{3.2}$$

where the infimum runs over all annuli in X whose core curve is homotopic to γ.

If γ is a simple closed curve, then the metric realising the supremum in Equation (3.1) and the annulus realising the infimum in Equation (3.2) can be explicitly described. In order to do this we need to consider Strebel differentials on X which we will describe here. We refer to [Str84] as a reference on Strebel differentials and quadratic differentials in general.

A *quadratic differential* ϕ on X is a differential that in any local coordinates can be written as $\phi = \phi(z)dz^2$ with $\phi(z)$ a holomorphic function. A quadratic differential determines two singular foliations of X. Namely, away from the zeroes of ϕ, lines that have tangent directions $v \in TS$ with $\phi(v, v) > 0$ form a foliation called the *horizontal foliation* of ϕ and lines with $\phi(v, v) < 0$ form its *vertical foliation*. The leaves of these foliations are called *singular* if they terminate in a zero of ϕ and are called *non-singular* otherwise. Furthermore, a quadratic differential also determines a flat singular metric on S which can be expressed as $|\phi(z)||dz|^2$ in local coordinates. Around any point on S that is not a zero of ϕ there exist complex coordinates in which $\phi = dz^2$. In these coordinates the singular flat metric is simply the Euclidean metric $|dz|^2$, the horizontal foliation consists of the lines with constant $\mathrm{Im}\, z$ and the vertical foliation consists of the lines with constant $\mathrm{Re}\, z$.

For every simple closed curve $\gamma \subset X$ there exists a unique quadratic differential, called the *Strebel differential*, such that every non-singular leaf of the horizontal foliation of the differential is closed and homotopic to γ. The annulus obtained by taking the union of these non-singular leaves realises the infimum in Equation (3.2). The singular flat metric that is determined by the Strebel differential realises the supremum in Equation (3.1).

We will prove here some results on the extremal length of intersecting curves that we will need in our proofs below.

Lemma 3.2.7. *Let $\gamma, \eta \subset X$ be simple closed curves. Then*

$$E_X(\gamma)E_X(\eta) \geq i(\gamma, \eta)^2. \tag{3.3}$$

Proof. Consider the Strebel differential of γ on X. Let $A \subset X$ be the annulus consisting of the union of all non-singular leaves of its horizontal foliation. Then we have $M := M(A) = 1/E_X(\gamma)$. Consider on X the singular flat metric σ determined by the Strebel differential. Normalise such that the annulus A has circumference 1 and height M. Any curve homotopic to η crosses the annulus at least $i(\gamma,\eta)$ times and hence $\ell_\sigma(\eta) \geq i(\gamma,\eta) \cdot M$. Then from Equation (3.1) we see that

$$E_X(\eta) \geq \frac{\ell_\sigma(\eta)^2}{\mathrm{Area}(\sigma)} \geq \frac{i(\gamma,\eta)^2 M^2}{M} = \frac{1}{E_X(\gamma)} i(\gamma,\eta)^2.$$

This proves the result. □

Lemma 3.2.8. *Let S be a surface of genus at least two and let $\gamma \subset S$ be a simple closed curve. Then there exists a simple closed curve $\eta \subset S$, satisfying $i(\gamma,\eta) \in \{1,2\}$, such that for every $\epsilon > 0$ there exists a complex structure X on S with*

$$E_X(\gamma)E_X(\eta) \leq i(\gamma,\eta)^2 + \epsilon$$

and

$$1 - \epsilon \leq E_X(\gamma) \leq 1 + \epsilon.$$

Proof. We construct the complex structure on S by cutting and pasting together several pieces. The main idea is to start with a smaller Riemann surface and curves γ, η for which Equation (3.3) is an equality. Then we add pieces to this surface to make it of the same topological type as S in a way that does not disturb the quantity $E_X(\gamma)E_X(\eta)$ to much.

For our construction we need to distinguish between two cases, namely whether γ is a separating curve or not. We will start with the case that γ is separating which is the more complicated case. The curve γ separates S into two surfaces S', S'' with border. Denote by $g', g'' \geq 1$ their respective genus. Then the genus of S equals $g = g' + g''$.

We start by considering a square with side lengths 1 in $\mathbb{C}$. We glue the boundary according to the gluing pattern given in Figure 3.1 to obtain the 2-sphere. We denote by X_0 the 2-sphere equipped with the complex structure determined by this gluing. We consider two simple closed loops γ' and η' on the sphere as specified in Figure 3.1. Fix a small constant $\delta > 0$. In each of the four components of the complement of $\gamma' \cup \eta'$ we cut a slit of length δ at the locations as indicated in Figure 3.1 (the slits are marked by (I) through (IV)). We let X' be an arbitrary closed Riemann surface of genus $g' - 1$. At arbitrary points in X' we cut two slits. We glue one of these slits to the slit marked (I) in X_0. The other slit we glue to the slit marked (II). Similarly, we take X'' an arbitrary Riemann surface of genus $g'' - 1$, again cut two slits and glue X'' to X_0 by gluing one of these slits to the slit marked (III) and the other to the slit marked (IV).

We denote by $X = X_0 \sqcup X' \sqcup X''/\sim$ the Riemann surface that is obtained from these gluings. Let us first make the observation that in X the curves γ' and η' are no longer null homotopic (as they were on the sphere) and they satisfy $i(\gamma',\eta') = 2$. Secondly, we note that the genus of X equals g. Namely,

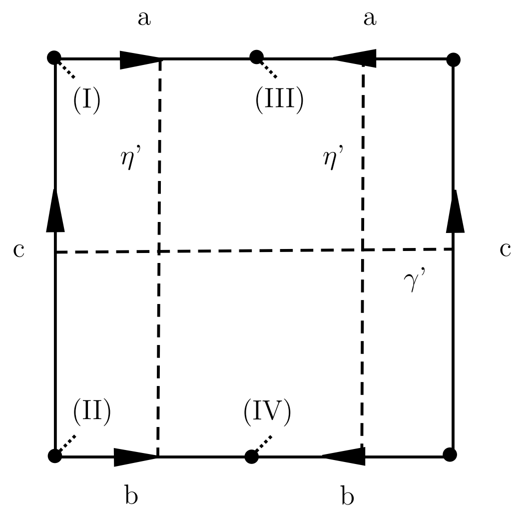

Figure 3.1: A gluing pattern on the boundary of a square. Edges labelled with the same letter are glued together according to the orientation indicated by the arrows. We cut slits of length δ at the places indicated by (I) through (IV).

the combined genus of X' and X'' contributes $g' + g'' - 2$ to the genus of X and the fact that we glued each surface along two slits contributes 2 more (see Figure 3.2).

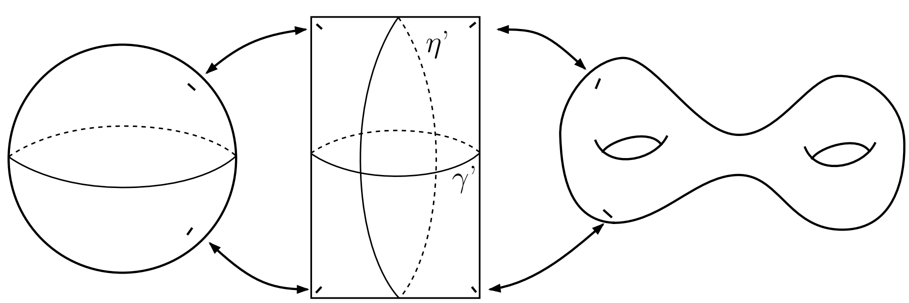

Figure 3.2: Example of a gluing as described above with $g' - 1 = 0$ and $g'' - 1 = 2$.

Consider the square in $\mathbb{C}$ from which we glue X_0. We note that the $1/2 - \delta$ neighbourhood of the curve γ' in the square intersects no slits. This neighbourhood descends to an annulus in X around γ' that has modulus $1 - 2\delta$. From Equation (3.2) it follows that $E_X(\gamma') \leq 1/(1 - 2\delta)$. Similarly, the $1/4 - \delta$ neighbourhood of η' in the square intersects no slits and descends to an annulus in X around η'. Its modulus equals $1/4 - \delta$ and hence $E_X(\eta') \leq 1/(1/4 - \delta)$. We now see that for any $\epsilon > 0$ there is a δ small enough such that

$$E_X(\gamma')E_X(\eta') \leq \frac{1}{1 - 2\delta} \cdot \frac{1}{1/4 - \delta} \leq 4 + \epsilon = i(\gamma, \eta)^2 + \epsilon$$

and $E_X(\gamma') \leq 1+\epsilon$. For the lower bound on $E_X(\gamma')$ we combine Equation (3.3) with $E_X(\eta') \leq 1/(1/4-\delta)$ to find that also $E_X(\gamma') \geq 1-\epsilon$ for δ small enough.

Finally, we note that γ' separates X into two surfaces with border that have genus g' and g'' respectively. It follows from the classification of surfaces that these two subsurfaces are diffeomorphic to the two corresponding subsurfaces of S. By gluing these diffeomorphisms together we find that there exists a diffeomorphism between X and S that sends the homotopy class of γ' to that of γ. We let η be the simple closed curve in S that corresponds to η' under this diffeomorphism. We note that the homotopy class of η only depends on the placement of the slits in X_0 along which we glued and not on the constant δ. Hence, we can take η the same for all choices of ϵ. Using this diffeomorphism we equip S with a complex structure that satisfies the bounds on the extremal lengths of γ and η.

The case where γ is non-separating is easier. In this case we take X_0 to be a torus and γ' and η' a pair of simple closed curves with $i(\gamma', \eta') = 1$. By picking a suitable complex structure on the torus we can realise equality in Equation (3.3) and $E_X(\gamma') = 1$. We glue an arbitrary Riemann surface of genus $g-1$ to the torus along a single small slit to obtain a Riemann surface X of genus g. Again by the classification of surfaces we can find a diffeomorphism between X and S that takes γ' to γ. The estimate on the extremal lengths γ and η in this case is similar to the previous case. □

3.2.5 Harmonic maps

Let (M, σ) and (N, ρ) be Riemannian manifolds. Consider a Lipschitz continuous map $f\colon N \to N$. We define its *energy density* $e(f)\colon M \to \mathbb{R}$ to be

$$e(f) = \frac{1}{2}\|df\|^2$$

where the norm $\|\cdot\|$ is the Hilbert-Schmidt norm on the vector bundle $T^*M \otimes f^*TN$ induced by the metrics σ and ρ. The energy density is a pointwise measure of the amount of stretching that a map does. We note that as f is Lipschitz continuous it is differentiable almost everywhere and hence $e(f)$ is defined almost everywhere. The *Dirichlet energy* of f is defined as

$$\mathcal{E}(f) = \int_M e(f)\,\mathrm{vol}_\sigma\,.$$

A critical point of this energy functional is called a *harmonic map*. If σ and ρ are smooth Riemannian metrics, then a harmonic map is also smooth.

A straightforward calculation shows that if M is a surface, then the Dirichlet energy of a map is independent of conformal scalings of the metric σ. It follows that in this case the harmonicity of a map and its energy depend only on the conformal structure on the surface. If we want to stress the dependence of the energy on a complex structure J on M and the metric ρ on N we will write $e(f; J, \rho)$ for the energy density and $\mathcal{E}(f; J, \rho)$ for the Dirichlet energy of a map f.

We will make use of the following lemma by Minsky.

Lemma 3.2.9 ([Min92, Proposition 3.1])**.** *Let X be a Riemann surface and (N, ρ) be a Riemannian manifold. For any map $f\colon X \to (N, \rho)$ and any simple closed curve $\gamma \subset X$ we have*

$$\mathcal{E}(f) \geq \frac{1}{2} \frac{\ell_\rho^2(f \circ \gamma)}{E_X(\gamma)}.$$

3.3 The energy spectrum

In this section we introduce the energy spectrum of a Riemannian manifold and study its relation to the simple length spectrum.

Let S be a surface of genus at least two and let (N, ρ) be a Riemannian manifold. We fix a homotopy class $[f] \in [S, N]$ of maps from S to N. For every complex structure J on S we consider the quantity

$$\mathscr{E}(J) = \inf_{h \in [f]} \mathcal{E}(h; J, \rho).$$

Here the infimum is taken over all Lipschitz continuous maps in the homotopy class $[f]$. If $\phi\colon S \to S$ is a diffeomorphism, then $\phi\colon (S, \phi^* J) \to (S, J)$ is a biholomorphism. In particular, we have $\mathcal{E}(h \circ \phi; \phi^* J, \rho) = \mathcal{E}(h; J, \rho)$. It follows that if ϕ is isotopic to the identity, then $\mathscr{E}(J) = \mathscr{E}(J \circ \phi)$ and we see that the function $\mathscr{E}$ descends to a well-defined function on Teichmüller space.

Definition 3.3.1. *The energy spectrum of (N, ρ) and $[f]$ is the function*

$$\mathscr{E}\colon \mathcal{T}(S) \to \mathbb{R}, \ \mathscr{E}([J]) = \inf_{h \in [f]} \mathcal{E}(h; J, \rho)$$

where the infimum is taken over all Lipschitz continuous maps in $[f]$.

We will often suppress the dependence on a choice of the homotopy class $[f]$ and simply refer to the energy spectrum of (N, ρ).

The energy spectrum gives a rough measure of the compatibility between (N, ρ) and points in Teichmüller space. Namely, the quantity $\mathscr{E}([J])$ measures how much the complex surface (S, J) must be stretched for it to be mapped into (N, ρ).

Proposition 3.3.2. *The energy spectrum $\mathscr{E}\colon \mathcal{T}(S) \to \mathbb{R}$ is a continuous function on Teichmüller space.*

Proof. If σ is a Riemannian metric on S and $h\colon S \to N$ a Lipschitz continuous map, then the energy density of h with respect to σ is given, at a point $x \in S$ where h is differentiable, by

$$e(f; \sigma, \rho) = \frac{1}{2} \sum_{i=1}^{2} \|dh(e_i)\|_\rho^2 \tag{3.4}$$

where (e_1, e_2) is an orthonormal basis of T_xS with respect to σ. If σ' is a second Riemannian metric, then by compactness of S there exists a Lipschitz constant $C(\sigma, \sigma') \geq 1$ such that

$$\frac{\sigma(v,v)}{C(\sigma,\sigma')} \leq \sigma'(v,v) \leq C(\sigma,\sigma') \cdot \sigma(v,v) \text{ for all } v \in TS.$$

For any $x \in S$ we can simultaneously diagonalise the metrics at x to find a basis (e_1, e_2) of T_xS that is orthonormal for σ and orthogonal for σ'. If we denote $\lambda_i = \sigma'(e_i, e_i)$, then $1/C(\sigma,\sigma') \leq \lambda_i \leq C(\sigma, \sigma')$. The basis $(e_1/\sqrt{\lambda_1}, e_2/\sqrt{\lambda_2})$ is orthonormal for σ' and from the expression of the energy density given in Equation (3.4) now follows that

$$\frac{e(h;\sigma',\rho)}{C(\sigma,\sigma')} \leq e(h;\sigma,\rho) \leq C(\sigma,\sigma') \cdot e(h;\sigma',\rho).$$

By integrating we see that similar inequalities hold true for $\mathcal{E}(h;\sigma,\rho)$ and $\mathcal{E}(h;\sigma',\rho)$. Then taking the infimum over all $h\colon S \to N$ Lipschitz continuous in the homotopy class $[f]$ gives

$$\frac{\mathscr{E}([\sigma'])}{C(\sigma,\sigma')} \leq \mathscr{E}([\sigma]) \leq C(\sigma,\sigma') \cdot \mathscr{E}([\sigma']). \tag{3.5}$$

Now suppose X_n is a sequence in Teichmüller space converging to a point $X \in \mathcal{T}(S)$. The points X_n and X can be represented by hyperbolic metrics σ_n and σ such that $\sigma_n \to \sigma$ uniformly on S as $n \to \infty$. It follows that the Lipschitz constants can be taken such that $C(\sigma_n, \sigma) \to 1$. Then Equation (3.5) gives that $\mathscr{E}(X_n) \to \mathscr{E}(X)$ for $n \to \infty$ and thus $\mathscr{E}\colon \mathcal{T}(S) \to \mathbb{R}$ is indeed a continuous function. □

If we assume that for every complex structure there exists an energy minimising harmonic map $f_J\colon (S, J) \to (N, \rho)$ in the homotopy class $[f]$, then $\mathscr{E}([J]) = \mathcal{E}(f_J; J, \rho)$. By the classical results of [ES64] this is for example the case if (N, ρ) is compact and has non-positive curvature. If the harmonic maps f_J are unique and satisfy certain non-degeneracy conditions, then they depend smoothly on the complex structure (see [EL81]). This happens for example if (N, ρ) is negatively curved and the map f can not be homotoped into the image of a closed geodesic. In this case the energy spectrum $\mathscr{E}$ is a smooth map on Teichmüller space.

To state our main result we will restrict to the situation where $N = S$ is a surface of genus at least two, $[f] = [\mathrm{id}]$ and ρ is a non-positively curved Riemannian metric on S.

Theorem 3.3.3. *Let ρ, ρ' be non-positively curved Riemannian metrics on a surface S of genus at least two. If the energy spectra of (S, ρ) and (S, ρ') (with $[f] = [\mathrm{id}]$) coincide, then the simple length spectra of ρ and ρ' coincide.*

Simply put, the energy spectrum of a metric determines its simple length spectrum. In fact, we will detail a procedure that recovers the length of a simple closed curve from the information given by the energy spectrum. Our principal observation is that when repeatedly Dehn twisting around a simple closed curve the quadratic growth rate of the energy is proportional to the square of the length of that curve in (S, ρ).

We now start our proof of Theorem 3.3.3. For this we fix a non-positively curved Riemannian metric ρ on S. We let $\mathscr{E}\colon \mathcal{T}(S) \to \mathbb{R}$ be its energy spectrum.

Definition 3.3.4. *For $\gamma \subset S$ a simple closed curve, $X \in \mathcal{T}(S)$ and $n \in \mathbb{N}$ we define*

$$\tau(X, \gamma, n) = \frac{\mathscr{E}(T_\gamma^n X)}{n^2}$$

and

$$\tau^-(X, \gamma) = \liminf_{n\to\infty} \tau(X, \gamma, n) \text{ and } \tau^+(X, \gamma) = \limsup_{n\to\infty} \tau(X, \gamma, n).$$

Remark 3.3.5. The value of the energy spectrum at the point $T_\gamma^n X$ can alternatively be characterised as

$$\mathscr{E}(T_\gamma^n X) = \inf_{h' \in [T_\gamma^n]} \mathcal{E}(h'; J, \rho)$$

where the infimum runs over all Lipschitz continuous maps $h'\colon S \to S$ homotopic to T_γ^n. To see this we let J be a complex structure on S representing $X \in \mathcal{T}(S)$. Then the complex structure $(T_\gamma^{-n})^* J$ is a representative of $T_\gamma^n X$. Now the map $T_\gamma^n\colon (S, J) \to (S, (T_\gamma^{-n})^* J)$ is a biholomorphism, hence for any Lipschitz continuous map $h\colon S \to S$ we have $\mathscr{E}(h; (T_\gamma^{-n})^* J, \rho) = \mathscr{E}(h \circ T_\gamma^n; J, \rho)$. Noting that $h \in [\mathrm{id}]$ if and only if $h \circ T_\gamma^n \in [T_\gamma^n]$ we find that indeed

$$\mathscr{E}(T_\gamma^n X) = \inf_{h \in [\mathrm{id}]} \mathcal{E}(h; (T_\gamma^{-n})^* J, \rho) = \inf_{h' \in [T_\gamma^n]} \mathcal{E}(h'; J, \rho).$$

We will now show that the quantities $\tau^-(\cdot, \gamma)$ and $\tau^+(\cdot, \gamma)$ can be used to measure $\ell_\rho(\gamma)$.

Lemma 3.3.6. *For any $X \in \mathcal{T}(S)$ and $\gamma \subset S$ a simple closed curve we have*

$$\tau^+(X, \gamma) \leq \frac{1}{2} E_X(\gamma) \cdot \ell_\rho^2(\gamma).$$

Proof. Consider a complex structure on S that represents $X \in \mathcal{T}(S)$. For convenience we will denote S equipped with this choice of complex structure also as X.

We will find an upper bound for the quantity $\mathscr{E}(T_\gamma^n X)$. To this end we construct a Lipschitz continuous map $k_n\colon X \to (S, \rho)$ in the homotopy class of T_γ^n for which we have an explicit bound on its energy. Then the observations of Remark 3.3.5 will imply that $\mathscr{E}(T_\gamma^n X) \leq \mathcal{E}(k_n)$.

Consider the Strebel differential on X for the curve γ. Let A be the annulus in X consisting of the union of all non-singular horizontal leaves of this Strebel differential. If $M = M(A)$ is the modulus of A, then $E_X(\gamma) = 1/M$. By uniformising A we can find a conformal identification between A and the flat cylinder $[0, M] \times \mathbb{R}/\mathbb{Z}$. We use this to equip A with coordinates $(x, [y]) \in [0, M] \times \mathbb{R}/\mathbb{Z}$.

Let $\eta\colon \mathbb{R}/\mathbb{Z} \to (S, \rho)$ be a length minimising geodesic loop freely homotopic to γ (so $\ell_\rho(\gamma) = l_\rho(\eta)$). Let $0 < \epsilon < M/2$ arbitrary. By deforming the identity map of S we can find a Lipschitz continuous map $k_0\colon X \to S$ that is homotopic to the identity and on the subcylinder

$$A_\epsilon = \{(x, [y]) \mid \epsilon \le x \le M - \epsilon\}$$

is given by $k_0(x, [y]) = \eta([y])$. Let Y be the complement of A_ϵ in X. We set $C = \mathcal{E}(k_0|_Y)$ which is a constant depending only on our choice of k_0 (which in turn depends only on ϵ).

For $n \in \mathbb{N}$ we define the maps $k_n\colon X \to S$ as follows. On Y we set $k_n|_Y \equiv k_0|_Y$ and on A_ϵ we put

$$k_n(x, [y]) = \eta\left(\left[y + n \cdot \frac{x - \epsilon}{M - 2\epsilon}\right]\right).$$

The map k_n coincides with k_0 on the boundaries of A_ϵ and hence each k_n defines a Lipschitz continuous map on X. Note that each k_n is homotopic to T_γ^n.

We now calculate the energy of the maps k_n. To this end this we equip A_ϵ with the conformal flat metric obtained from the identification $A \cong [0, M] \times \mathbb{R}/\mathbb{Z}$. Using this choice of metric, we find on A_ϵ that

$$e(k_n) = \frac{1}{2}\left\{\left\|\frac{\partial k_n}{\partial x}\right\|^2 + \left\|\frac{\partial k_n}{\partial y}\right\|^2\right\} = \frac{1}{2}\left\{\left(\frac{n}{M - 2\epsilon}\right)^2 + 1\right\}\|\dot\eta\|^2.$$

Hence

$$\begin{aligned}\mathcal{E}(k_n|_{A_\epsilon}) &= \int_0^1 \int_\epsilon^{M-\epsilon} e(k_n)\,dx\,dy \\ &= \frac{1}{2}\left\{\left(\frac{n}{M - 2\epsilon}\right)^2 + 1\right\} \cdot \int_0^1 \int_\epsilon^{M-\epsilon} \|\dot\eta\|^2 dx\,dy \\ &= \frac{1}{2}\left\{\left(\frac{n}{M - 2\epsilon}\right)^2 + 1\right\} \cdot (M - 2\epsilon) \cdot \ell_\rho^2(\gamma).\end{aligned}$$

We can now estimate (cf. Remark 3.3.5)

$$\begin{aligned}\tau(X, \gamma, n) = \mathscr{E}(T_\gamma^n X) \le \mathcal{E}(k_n) &= \mathcal{E}(k_n|_{A_\epsilon}) + \mathcal{E}(k_n|_Y) \\ &= \frac{1}{2}\left\{\frac{n^2}{M - 2\epsilon} + M - 2\epsilon\right\} \cdot \ell_\rho^2(\gamma) + C.\end{aligned}$$

By dividing by n^2 and taking the limit superior for $n \to \infty$ we find

$$\tau^+(X, \gamma) \le \frac{1}{2}\frac{1}{M - 2\epsilon} \cdot \ell_\rho^2(\gamma).$$

Finally, noting that $\epsilon > 0$ was arbitrary we conclude that

$$\tau^+(X,\gamma) \leq \frac{1}{2}\frac{1}{M} \cdot \ell_\rho^2(\gamma) = \frac{1}{2}E_X(\gamma) \cdot \ell_\rho^2(\gamma).$$

□

Lemma 3.3.7. *For any $X \in \mathcal{T}(S)$ and simple closed curves $\gamma, \eta \subset S$ we have*

$$\tau^-(X,\gamma) \geq \frac{1}{2}\frac{i(\gamma,\eta)^2 \cdot \ell_\rho^2(\gamma)}{E_X(\eta)}$$

Proof. Let us again, by abuse of notation, denote by X both a point in Teichmüller space and a Riemann surface representing it. The lemma follows easily from Lemma 3.2.9 and Lemma 3.2.2. Namely, from the latter follows that a constant $C = C(\gamma,\eta) > 0$ exists such that

$$\ell_\rho(T_\gamma^n \eta) \geq n \cdot i(\gamma,\eta) \cdot \ell_\rho(\gamma) - C.$$

Any map $h\colon X \to (S,\rho)$ homotopic to T_γ^n maps η to a curve homotopic to $T_\gamma^n \eta$. Now Lemma 3.2.9 gives a lower bound on the energy of such maps. It follows that

$$\tau(X,\gamma,n) = \mathscr{E}(T_\gamma^n X) \geq \frac{1}{2}\frac{(n \cdot i(\gamma,\eta) \cdot \ell_\rho(\gamma) - C)^2}{E_X(\eta)}.$$

Dividing by n^2 and taking the limit inferior for $n \to \infty$ gives

$$\tau^-(X,\gamma) \geq \frac{1}{2}\frac{i(\gamma,\eta)^2 \cdot \ell_\rho^2(\gamma)}{E_X(\eta)}.$$

□

We now have for any $X \in \mathcal{T}(S)$ and $\gamma, \eta \subset S$ simple closed curves that

$$\frac{1}{2}\frac{i(\gamma,\eta)^2 \cdot \ell_\rho^2(\gamma)}{E_X(\eta)} \leq \tau^-(X,\gamma) \leq \tau^+(X,\gamma) \leq \frac{1}{2}E_X(\gamma) \cdot \ell_\rho^2(\gamma). \tag{3.6}$$

We observe that these bounds are close together if the quantity $E_X(\gamma)E_X(\eta)$ is close to $i(\gamma,\eta)^2$. We use Lemma 3.2.8 to finish the proof of Theorem 3.3.3.

Proof of Theorem 3.3.3. Fix a simple closed curve $\gamma \subset S$. We invoke Lemma 3.2.8 to find a simple closed curve $\eta \subset S$ with $i(\gamma,\eta) > 0$ and for every $k \in \mathbb{N}$ a $X_k \in \mathcal{T}(S)$ such that $E_{X_k}(\gamma)E_{X_k}(\eta) \leq i(\gamma,\eta)^2 + 1/k$ and $|E_{X_k}(\gamma) - 1| \leq 1/k$. Plugging these inequalities into Equation (3.6) yields

$$\frac{1}{2}\frac{i(\gamma,\eta)^2(1-1/k)}{i(\gamma,\eta)^2 + 1/k} \cdot \ell_\rho^2(\gamma) \leq \tau^-(X_k,\gamma) \leq \tau^+(X_k,\gamma) \leq \frac{1}{2}(1+1/k) \cdot \ell_\rho^2(\gamma).$$

It follows that both $\tau^-(X_k,\gamma)$ and $\tau^+(X_k,\gamma)$ converge to $\frac{1}{2} \cdot \ell_\rho^2(\gamma)$ for $k \to \infty$. We see that $\ell_\rho(\gamma)$ is entirely determined by the energy spectrum since the same holds true for the functions τ^+ and τ^-.

More precisely, if ρ' is a second non-positively curved Riemannian metric on S with equal energy spectrum, then Equation (3.6) also holds with $\ell_{\rho'}(\gamma)$ in place of $\ell_\rho(\gamma)$. We then see that

$$\frac{1}{2}\ell_{\rho'}^2(\gamma) = \lim_{k\to\infty} \tau^-(X_k, \gamma) = \lim_{k\to\infty} \tau^+(X_k, \gamma) = \frac{1}{2}\ell_\rho^2(\gamma)$$

hence $\ell_\rho(\gamma) = \ell_{\rho'}(\gamma)$. Since $\gamma \subset S$ was arbitrary, it follows that ρ and ρ' have equal simple length spectrum. □

3.4 Further comparison to the length spectra

In this section we show that the converse to the result of the previous section does not hold. Namely, the simple length spectrum does not determine the energy spectrum. Thus, we see that the energy spectrum carries more information.

Proposition 3.4.1. *For every hyperbolic metric on a surface there exists a negatively curved Riemannian metric on that surface with equal simple length spectrum but different energy spectrum.*

We will show this by proving that the energy spectrum encodes the area of a Riemannian metric on a surface, whereas the simple length spectrum does not. We will make use of the following well-known observation.

Lemma 3.4.2. *Let (S, ρ) be a surface of genus at least one equipped with a Riemannian metric. Then the energy spectrum of (S, ρ) (with $[f] = [\mathrm{id}]$) satisfies*

$$\mathscr{E}(X) \geq \mathrm{Area}(S, \rho) \textit{ for all } X \in \mathcal{T}(S).$$

If, furthermore, the metric ρ is non-positively curved, then equality is achieved if and only if X equals $[\rho] \in \mathcal{T}(S)$, the point in Teichmüller space determined by the metric ρ.

Proof. Let σ be a hyperbolic metric on S. The metrics σ and ρ determine conformal structures on S. In corresponding local conformal coordinates z resp. w on S we can write $\sigma = \sigma(z)|dz|^2$ and $\rho = \rho(w)|dw|^2$. Then the energy density of a map $h\colon (S, \sigma) \to (S, \rho)$ is given by

$$e(h; \sigma, \rho) = \frac{\rho(h(z))}{\sigma(z)} \left\{ |h_z|^2 + |h_{\overline{z}}|^2 \right\}$$

and its Jacobian is given by

$$J(h; \sigma, \rho) = \frac{\rho(h(z))}{\sigma(z)} \left\{ |h_z|^2 - |h_{\overline{z}}|^2 \right\}$$

(see [Wol89, Section 2]). Integrating over S gives

$$\mathcal{E}(h; \sigma, \rho) = \int_S e(h; \sigma, \rho) \,\mathrm{vol}_\sigma \geq \int_S J(h; \sigma, \rho) \,\mathrm{vol}_\sigma = \mathrm{Area}(S, \rho)$$

with equality if and only if h is a conformal map (i.e. $h_{\overline{z}} = 0$).

From this follows immediately that $\mathscr{E}(X) \geq \operatorname{Area}(S, \rho)$ for all $X \in \mathcal{T}(S)$. If $[\sigma] = [\rho] \in \mathcal{T}(S)$, then there exists a conformal map $h\colon (S, \sigma) \to (S, \rho)$ homotopic to the identity. For this map we see that $\mathscr{E}(h; \sigma, \rho) = \operatorname{Area}(S, \rho)$, so $\mathscr{E}([\sigma]) = \mathcal{E}(h; \sigma, \rho) = \operatorname{Area}(S, \rho)$.

Finally, suppose ρ has non-positive curvature. Assume $X = [\sigma] \in \mathcal{T}(S)$ such that $\mathscr{E}(X) = \operatorname{Area}(S, \rho)$. By [ES64] there exists a energy minimising harmonic map $h\colon (S, \sigma) \to (S, \rho)$ homotopic to the identity. Then $\mathcal{E}(h; \sigma, \rho) = \mathscr{E}(X) = \operatorname{Area}(S, \rho)$, hence h must be a conformal map. Because h has degree one, it follows from the Riemann-Hurwitz formula that it can not have branch points. We conclude that h is a biholomorphism isotopic to the identity which means that $X = [\sigma] = [\rho]$. $\square$

Proposition 3.4.1. Let ρ be any hyperbolic metric on the surface S. Let $\mathcal{G}$ be the union of all simple closed geodesics in (S, ρ). Birman and Series prove in [BS85] that this set is nowhere dense on S. In particular, there exists an open set $U \subset S$ such that $\overline{U}$ does not intersect $\overline{\mathcal{G}}$. Let $\chi\colon S \to [0,1]$ be a smooth bump function which is zero outside of U and equals one on some point in U. For $\delta > 0$ we consider the metric $\rho' = (1 + \delta \cdot \chi)\rho$. If we take δ small enough, then ρ' is still a negatively curved metric. Because $\rho = \rho'$ on an open neighbourhood of $\mathcal{G}$, it follows that the simple closed geodesics for either metric are the same. As a result their simple length spectra are equal.

Finally, on some points in U we have that $(1 + \delta \cdot \chi) > 1$ and hence $\operatorname{Area}(S, \rho') > \operatorname{Area}(S, \rho)$. Taking into consideration Lemma 3.4.2 we see (denoting the energy spectra of ρ and ρ' by $\mathscr{E}$ and $\mathscr{E}'$ respectively) that

$$\min_{X \in \mathcal{T}(S)} \mathscr{E}'(X) = \operatorname{Area}(S, \rho') > \operatorname{Area}(S, \rho) = \min_{X \in \mathcal{T}(S)} \mathscr{E}(X)$$

so $\mathscr{E} \neq \mathscr{E}'$. $\square$

We conclude that the energy spectrum is a more sensitive way to tell non-positively curved Riemannian metrics on S apart than the simple length spectrum. With this in mind, we can pose the following interesting question: how does the energy spectrum compare to the (full) marked length spectrum?

The marked length spectrum carries much more information than the simple length spectrum. Namely, Otal proved in [Ota90] that the set of negatively curved Riemannian metrics on a surface, determined up to isotopy, satisfies marked length spectrum rigidity. Furthermore, in [CFF92], it is proved that the same holds true for the set of non-positively curved Riemannian metrics under the additional assumptions that these metrics do not have conjugate points. It follows in particular that for such metrics the marked length spectrum determines the energy spectrum. A, to the author, interesting question is now whether the sensitivity of energy spectrum falls strictly between that of the simple length spectrum and full marked length spectrum or whether the energy spectrum can also distinguish between all non-positively curved Riemannian metrics.

Taking this one step further we mention that Bonahon showed in [Bon93] that when considering marked length spectrum rigidity one can not drop the assumption that the metrics under consideration are Riemannian. More precisely, for any Riemannian metric of negative curvature on S he constructed a non-Riemannian metric that has the same marked length spectrum but that is not isometric by an isometry isotopic to the identity. The notion of Dirichlet energy can be generalised to maps between manifolds with non-Riemannian metrics (see [KS93]) and hence also in this context the energy spectrum can be defined. This allows us to ask whether the energy spectrum could perhaps provide more information and distinguish between negatively or non-positively curved non-Riemannian metrics.

A similar question for harmonic maps between flat tori is considered in [Ham20]. There two non-isometric 16 dimensional flat tori are exhibited which can not be distinguished by the energy spectrum when the surface S is the two-dimensional torus.

3.5 Energy spectrum rigidity

We now consider the question whether the energy spectrum of a Riemannian metric uniquely determines that metric (up to isotopy). If $\mathcal{M}$ is a set of metrics on S, determined up to isotopy, then we can consider the map $\mathcal{M} \to C^0(\mathcal{T}(S))$ mapping a metric to its energy spectrum. We say $\mathcal{M}$ satisfies *energy spectrum rigidity* if this map is injective. In light of Theorem 3.3.3 we see that this question is closely related to the question which classes of metrics on surfaces satisfy simple length spectrum rigidity. We describe here some examples where energy spectrum rigidity does hold.

3.5.1 Hyperbolic metrics

We consider the set of hyperbolic metrics on S, defined up to isotopy. As discussed in Section 3.2.1 this is the Teichmüller space of S. The existence of the harmonic maps under consideration is in this case a consequence of [ES64].

It follows from elementary considerations on harmonic maps between surfaces that $\mathcal{T}(S)$ satisfies energy spectrum rigidity, even without invoking simple length spectrum rigidity. Namely, we see from Lemma 3.4.2 that a point in Teichmüller space can be recovered from its energy spectrum by locating the unique minimum.

Corollary 3.5.1. *The set of hyperbolic metrics on S, defined up to isotopy, satisfies energy spectrum rigidity.*

3.5.2 Singular flat metrics

As described in Section 3.2.4 a quadratic differential on a surface induces a metric on that surface. Away from the zeroes of the quadratic differential these metrics are locally flat and at the zero points they have a cone singularity of cone angle $(2+p)\pi$, $p \in \mathbb{N}$ (for more information see [DLR10]). We call such metrics *singular flat metrics* on the surface. We consider the set $\mathcal{M}$ of singular flat metrics on the surface S that are induced by quadratic differentials, up to isotopy. The space of quadratic differentials, and hence also $\mathcal{M}$, can be canonically identified with the cotangent bundle of $\mathcal{T}(S)$.

In [DLR10] Duchin, Leiniger and Rafi prove the following theorem.

Theorem 3.5.2 ([DLR10, Theorem 1])**.** *Let $\mathcal{M}_1 \subset \mathcal{M}$ be the set of singular flat metrics on S with area one, defined up to isotopy. The set $\mathcal{M}_1$ satisfies simple length spectrum rigidity.*

Combining this fact with Theorem 3.3.3 and Lemma 3.4.2 easily gives the following corollary.

Corollary 3.5.3. *The set of singular flat metrics that are induced by quadratic differentials, defined up to isotopy, satisfies energy spectrum rigidity.*

Proof. Let $\rho, \rho' \in \mathcal{M}$ be two singular flat metrics on S with equal energy spectrum. Lemma 3.4.2 gives $\operatorname{Area}(S, \rho) = \operatorname{Area}(S, \rho')$. Then the rescaled metrics $\rho / \operatorname{Area}(S, \rho)$ and $\rho' / \operatorname{Area}(S, \rho')$ lie in $\mathcal{M}_1$ and by Theorem 3.3.3 have equal simple length spectrum. It now follows from Theorem 3.5.2 that there exists an isometry between ρ and ρ' that is isotopic to the identity. □

Let us mention that also in this case the energy infimum in the definition of the energy spectrum is always realised by a harmonic map. These are however not harmonic maps in the precise sense we defined above because singular flat metrics are not actual Riemannian metrics. However, a more general notion of harmonic map, allowing for maps into metric spaces, has been developed in [KS93]. Theorem 2.7.1 of that paper yields the existence of harmonic maps into surfaces equipped with singular flat metrics. In order to apply this result we note that if S is a surface of genus at least two equipped with a singular flat metric, then its universal cover is a metric space of non-positive curvature (in the sense of Alexandrov).

3.6 Kleinian surface groups

A *Kleinian surface group* is a representation $\rho\colon \pi_1(S) \to \operatorname{PSL}(2, \mathbb{C})$ that is discrete and faithful. Because $\operatorname{PSL}(2, \mathbb{C})$ acts on $\mathbb{H}^3$ by isometries, given a Kleinian surface group ρ we can consider the hyperbolic 3-manifold $N = \mathbb{H}^3 / \rho(\pi_1(S))$. The representation ρ induces an identification between $\pi_1(S)$ and $\pi_1(N)$. As a result there is a one-to-one correspondence between the free homotopy classes of loops in S and those of loops in N. The translation length of an element $\rho(\gamma)$ ($\gamma \in \pi_1(S)$), denoted $\ell_\rho(\gamma)$, is defined to be the infimum of the lengths of loops in N that lie in the free homotopy class determined by γ. If $\rho(\gamma)$ is a parabolic element, then $\ell_\rho(\gamma) = 0$. If $\rho(\gamma)$ is an hyperbolic element, then it is conjugate to a matrix of the form

$$\begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$$

with $\lambda \in \mathbb{C}, |\lambda| > 1$. In this case

$$\ell_\rho(\gamma) = 2 \log|\lambda|. \tag{3.7}$$

The simple length spectrum of a Kleinian surface group is the vector $(\ell_\rho(\gamma))_{\gamma\in\mathcal{S}}$.

The representation ρ determines a unique homotopy class $[f]$ of maps from S to N that lift to ρ-equivariant maps $\widetilde{S} \to \mathbb{H}^3$. We define the energy spectrum of a Kleinian surface group to be the energy spectrum of the hyperbolic manifold $N = \mathbb{H}^3/\rho(\gamma)$ and the homotopy class $[f]$.

In this section we prove the following analogue to Theorem 3.3.3.

Theorem 3.6.1. *Let $\rho, \rho' \colon \Gamma \to \mathrm{PSL}(2,\mathbb{C})$ be two Kleinian surface groups. If the energy spectra of ρ and ρ' coincide, then their simple simple length spectra coincide.*

Bridgeman and Canary prove in [BC17, Theorem 1.1] that a Kleinian surface group is determined up to conjugacy by its simple length spectrum. Combining their result with Theorem 3.6.1 gives the following corollary.

Corollary 3.6.2. *If $\rho, \rho' \colon \Gamma \to \mathrm{PSL}(2,\mathbb{C})$ are Kleinian surface groups with equal energy spectrum, then ρ' is conjugate to either ρ or $\overline{\rho}$.*

The proof detailed in Section 3.3 can largely be carried over to the case of Kleinian surface groups. We do, however, need a replacement for Lemma 3.2.2. This will be provided by the following lemma.

Lemma 3.6.3. *Let $\rho \colon \Gamma \to \mathrm{PSL}(2,\mathbb{C})$ be a Kleinian surface group. Let $\gamma, \eta \subset S$ be simple closed curves with $i(\gamma,\eta) \in \{1,2\}$. Then there exists a constant $C = C(\rho,\gamma,\eta) > 0$ such that*

$$\ell_\rho(T_\gamma^n \eta) \geq n \cdot i(\gamma,\eta) \cdot \ell_\rho(\gamma) - C$$

for all $n \geq 1$.

Our proof is along similar lines as [BC17, Lemma 2.2].

Proof. We first consider the case $i(\gamma,\eta) = 2$. Let us denote $\gamma \cap \eta = \{x_0, x_1\}$. We assume that γ and η are parametrised loops starting at x_0. If we take x_0 as the basepoint of the fundamental group, then we can consider γ and η as elements of $\pi_1(S, x_0)$. We denote by γ_1 and η_1 the subarcs of γ and η respectively that connect x_0 to x_1 and we denote by γ_2 and η_2 the subarcs connecting x_1 to x_0 (see Figure 3.3).

We now find the following expression for the element $T_\gamma^n \eta \in \pi_1(S, x_0)$,

$$\begin{aligned} T_\gamma^n \eta &= \eta_2 (\gamma_2^{-1}\gamma_1^{-1})^n \eta_1 \gamma^n \\ &= \eta_2 \gamma_1 (\gamma_1^{-1}\gamma_2^{-1})^n \gamma_1^{-1} \eta_1 \gamma^n \\ &= \sigma \gamma^{-n} \nu \gamma^n \end{aligned}$$

where we put $\sigma = \eta_2\gamma_1, \nu = \gamma_1^{-1}\eta_1 \in \pi_1(S, x_0)$.

We note that if $\rho(\gamma)$ is a parabolic element, then $\ell_\rho(\gamma) = 0$ and the statement is trivial. Hence, from now on we assume $\rho(\gamma)$ is a hyperbolic element. By

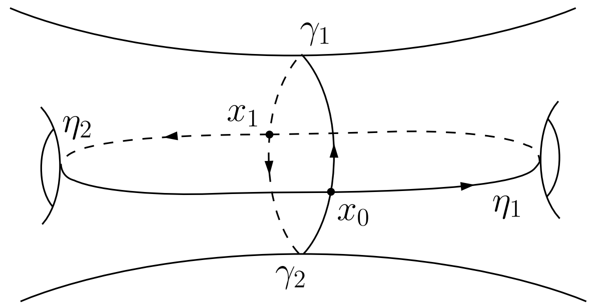

Figure 3.3: Overview of the positions of the arcs $\gamma_1, \gamma_2, \eta_1$ and η_2.

conjugating the representation ρ we can assume that, for some $\lambda \in \mathbb{C}, |\lambda| > 1$, we have

$$\rho(\gamma) = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}.$$

Note that a matrix representing an element of $\mathrm{PSL}(2, \mathbb{C})$ is only determined up to a multiplication by $\pm\,\mathrm{id}$. However, for our calculation of the translation length this does not matter.

For suitable coefficients $a, b, c, d, a', b', c', d' \in \mathbb{C}$ we can write

$$\rho(\sigma) = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \text{ and } \rho(\nu) = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}.$$

We note that coefficients of these matrices do not vanish. Namely, if a coefficient of, say, $\rho(\sigma)$ vanishes, then it maps a fixed point of $\rho(\gamma)$ to a fixed point of $\rho(\gamma)$. Then $\rho(\sigma\gamma\sigma^{-1})$ and $\rho(\gamma)$ share a fixed point which implies they must have a common power because $\rho(\Gamma)$ is discrete. Because the elements γ and $\sigma\gamma\sigma^{-1}$ do not have a common power this would contradict that the representation ρ is faithful.

A simple calculation yields that

$$\rho(T_\gamma^n \eta) = \rho(\sigma\gamma^{-n}\nu\gamma^n) = \begin{pmatrix} aa' + \lambda^2 bc' & bd' + \lambda^{-2} ab' \\ ca' + \lambda^2 dc' & dd' + \lambda^{-2} cb' \end{pmatrix}.$$

Now if $\left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right) \in \mathrm{SL}(2, \mathbb{C})$, then its eigenvalues are given by

$$\mu_\pm = \frac{\alpha + \delta}{2} \pm \frac{1}{2}\sqrt{(\alpha + \delta)^2 - 4}.$$

Applying this to $\rho(T_\gamma^n \eta)$ (that is, taking $\alpha = aa' + \lambda^2 bc'$ and $\delta = dd' + \lambda^{-2} cb'$) we find that

$$\mu_+ = \lambda^{2n}(bc' + O(|\lambda|^{-2n})).$$

Using Equation (3.7) and the fact that $bc' \neq 0$ gives

$$\begin{aligned} \ell_\rho(T_\gamma^n \eta) &= 2\log|\mu_+| = 4 \cdot n \cdot \log|\lambda| + \log(|bc' + O(|\lambda|^{-2n})|) \\ &= 2 \cdot n \cdot \ell_\rho(\gamma) + O(1) = i(\gamma, \eta) \cdot n \cdot \ell_\rho(\gamma) + O(1) \text{ as } n \to \infty \end{aligned}$$

This proves the lemma for the case $i(\gamma, \eta) = 2$. In the case $i(\gamma, \eta) = 1$ we have that $T_\gamma^n \eta = \eta\gamma^n$. The calculation of the largest eigenvalue of $\rho(\eta\gamma^n)$ is similar and is carried out in [BC17, Lemma 2.2]. Filling the formula of that lemma into Equation (3.7) immediately gives the result also in this case. $\square$

We can now give a proof of Theorem 3.6.1.

Proof of Theorem 3.6.1. The proof of Theorem 3.3.3 goes through in the present situation mostly unchanged. Let us only point the modifications that need to be made. In this proof we denote by $[f]$ the homotopy class of maps $S \to N$ that lift to a ρ-equivariant map $\widetilde{S} \to \mathbb{H}^3$.

First we consider the proof of Lemma 3.3.6. Let $\gamma \in \pi_1(S)$ be an element that corresponds to a simple closed curve. If $\rho(\gamma)$ is hyperbolic, then there exists a length minimising geodesic loop $\eta\colon \mathbb{R}/\mathbb{Z} \to N$ in the free homotopy class determined by γ. By deforming a map in $[f]$ we can construct a Lipschitz continuous map $k_0\colon S \to N$ such that $k_0 \in [f]$ and $k_0(x, [y]) = \eta([y])$ on A_ϵ (notation as in the proof of Definition 3.3.4). The maps k_n can then be constructed as before and the energy estimates also go through. We find that $\tau^+(X, \gamma) \leq \frac{1}{2} E_X(\gamma) \cdot \ell_\rho^2(\gamma)$.

If $\rho(\gamma)$ is a parabolic element, then no such geodesic loop exists. However, since $\ell_\rho(\gamma) = 0$ there exists for every $\delta > 0$ a closed loop $\eta\colon \mathbb{R}/\mathbb{Z} \to N$ with $l(\eta) \leq \delta$. If we then take a map $k_0\colon S \to N$ in the homotopy class $[f]$ with $k_0(x, [y]) = \eta([y])$ on A_ϵ and carry out the rest of the argument of the proof of proof of Lemma 3.3.6 we find

$$\tau^+(X, \gamma) \leq \frac{1}{2} E_X(\gamma) \cdot l^2(\eta) \leq \frac{1}{2} E_X(\gamma) \cdot \delta^2.$$

Since δ was arbitrary $\tau^+(X, \gamma) = \frac{1}{2} E_X(\gamma) \cdot \ell_\rho^2(\gamma) = 0$ follows.

Let us now consider the proof of Lemma 3.3.7. Suppose $\gamma, \eta \in \pi_1(S)$ correspond to simple closed curves with $i(\gamma, \eta) \in \{1, 2\}$. Any map in $[f \circ T_\gamma^n]$ maps the curve η to a curve in the free homotopy class determined by $T_\gamma^n \eta$. The results of Lemma 3.6.3 and Lemma 3.2.9 then give rise to the estimate

$$\tau^-(X, \gamma) \geq \frac{1}{2} \frac{i(\gamma, \eta)^2 \cdot \ell_\rho^2(\gamma)}{E_X(\eta)}$$

in the same way as in the proof of Lemma 3.3.7.

It follows that the estimates of Equation (3.6) are also true in the present situation whenever $i(\gamma, \eta) \in \{1, 2\}$. Because the curves γ and η constructed in Lemma 3.2.8 do satisfy this condition we see that the remainder of the proof of Theorem 3.3.3 can now be followed verbatim. $\square$

3.7 Hitchin representations

A *Hitchin representation* is a representation $\rho\colon \pi_1(S) \to \mathrm{PSL}(n, \mathbb{R})$ that lies in a particular connected component (discovered by Hitchin in [Hit92]) of the representation variety $\mathrm{Rep}(\pi_1(S), \mathrm{PSL}(n, \mathbb{R}))$. Such representations are discrete, faithful

([Lab06]) and act isometrically on the symmetric space $\mathrm{PSL}(n,\mathbb{R})/\mathrm{PSO}(n)$. It follows that their simple length spectrum and energy spectrum can be defined in the same manner as in the previous section.

As stated in the introduction our main interest is the study of the energy spectrum for Hitchin representations. Unfortunately, the methods presented here are not sufficient to conclude that a Hitchin representation is uniquely determined by its energy functional. Let us briefly describe the difficulty we encounter.

The author believes that an analogue of Lemma 3.6.3 holds also for Hitchin representations. Then the proof presented in the previous section can be carried out for Hitchin representations. Hence, it seems likely that their simple length spectrum is also determined by their energy spectrum. However, it is not known to the author whether a Hitchin representation is determined by its simple length spectrum (as we define it here).

Let us point out that very closely related results are obtain by Bridgeman, Canary and Labourie in [BCL20]. Namely, they prove that Hitchin representations are rigid for a different type of simple length spectrum[2]. Let us briefly describe the difference. If $\gamma \in \pi_1(S)$, then the $\rho(\gamma)$ is a diagonalisable matrix with real eigenvalues (which are determined up to sign). Denote these by $\lambda_1, \ldots, \lambda_n$. Then the spectral length of $\rho(\gamma)$ is $L_\rho(\gamma) = \max_{i=1,\ldots,n}|\lambda_i|$ and its trace is $|\mathrm{Tr}(\rho(\gamma))| = \sum_{i=1}^n |\lambda_i|$. In [BCL20] it is proved that a Hitchin representation is determined, up to conjugacy, by its simple (spectral) length spectrum $(L_\rho(\gamma))_{\gamma\in\mathcal{S}}$ and by its simple trace spectrum $(|\mathrm{Tr}(\rho(\gamma))|)_{\gamma\in\mathcal{S}}$. In contrast, the simple length spectrum we consider in this paper assigns to each simple closed curve γ the translation length of $\rho(\gamma)$ which is given by $\ell_\rho(\gamma) = \sqrt{\sum_{i=1}^n (\log|\lambda_i|)^2}$. So in order to finish the circle of ideas presented in this paper it remains to answer the question whether a Hitchin representation is determined, up to conjugacy, by its simple (translation) length spectrum.

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[2]Note that in [BCL20] the term ‘simple length spectrum’ is also used; however, it does not refer to the same quantity we consider here.

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Chapter 4

The energy spectrum of grafted surfaces

4.1 Introduction

In this chapter we continue the study of the energy spectrum that was introduced in Chapter 3. We will consider the energy spectrum for a particular class of metrics. Namely, the Thurston metrics on grafted surfaces. Such metrics are obtained by cutting a hyperbolic surface along a simple closed geodesic loop and gluing in a Euclidean cylinder (see Section 4.2.3).

We regard this setting as a 'toy model' for the study of the energy spectrum of Hitchin representations. We hope to gain some insight into the general properties of the energy spectrum by studying it in a simpler context. The particular question we are interested in is whether the minima of the energy spectrum are coarsely unique. More precisely, does there exists a constant $D > 0$ such that any two minimisers of the energy spectrum of a Hitchin representation are no more than D apart (measured in the Teichmüller distance). We think of this question as a coarse version of Labourie's conjecture.

With this question in mind we consider the energy spectrum of grafted surfaces. In this setting the minimum of the energy spectrum is always unique (see Lemma 4.3.3). For this reason we look at points in Teichmüller space which almost minimise the energy spectrum. Our goal is to prove that such points can not lie very far from the true minimiser.

We consider a surface S, σ a hyperbolic metric on that surface and $\gamma \subset S$ a simple closed geodesic loop. Then the grafted surface $\mathrm{Gr}_{t\cdot\gamma}(\sigma)$ is obtained by cutting S along γ and gluing in a cylinder of height t and circumference $\ell_\sigma(\gamma)$ (a more detailed definition is given in Section 4.2.3). We will examine the energy spectra of the grafted surfaces $\mathrm{Gr}_{t\cdot\gamma}(\sigma)$ for $t \geq 0$.

The results of [DK12] show that the underlying conformal structures of the family $\{\mathrm{Gr}_{t\cdot\gamma}(\sigma)\}_{t\geq 0}$ lie within bounded distance from a Teichmüller geodesic ray in $\mathcal{T}(S)$. Moreover, the length of the curve γ goes to zero as $t \to \infty$ and the amount of twisting around γ that occurs is bounded by a multiple of t. We expect that the points in Teichmüller space that almost minimise the energy spectrum of a grafted surface behave similarly. The main result of this chapter, Proposition 4.3.4, confirms that this is indeed the case.

4.2 Prerequisites

In this chapter we will follow the notation that was introduced in Chapter 3. Before we state our results we first introduce some additional concepts and notation that will be needed in our discussion.

As in Chapter 3 let S be a closed and oriented surface of genus $g \geq 2$. We denote by $\chi(S) = 2 - 2g$ its Euler characteristic. If $X \in \mathcal{T}(S)$ and $\gamma \subset S$ is a closed curve, then we denote $\ell_X(\gamma) = \ell_\sigma(\gamma)$, where σ is any hyperbolic metric on S that represents the point X in Teichmüller space. Furthermore, we will denote by $d\colon \mathcal{T}(S) \times \mathcal{T}(S) \to \mathbb{R}$ the Teichmüller distance on Teichmüller space (see [Hub06, Definition 6.4.4]).

4.2.1 Measured laminations

A reference for the material discussed in this section is [Mar16, Section 8.3].

Let S be equipped with a hyperbolic metric. A *geodesic lamination* on S is a closed subset consisting of a disjoint union of complete geodesics. A transverse measure on a geodesic lamination assigns a Borel measure to each arc that is transverse to the lamination in such a way that it is invariant under translations along the lamination. We call a geodesic lamination equipped with transverse measure a *measured lamination*.

Let $\mathcal{ML}(S)$ be the space of measured laminations. It can be equipped with a topology that does not depend on the choice of hyperbolic metric on S. By multiplying the transverse measures by positive real numbers we obtain an action of $\mathbb{R}_{>0}$ on $\mathcal{ML}(S)$. When taking the quotient of this action we obtain $\mathbb{P}\mathcal{ML}(S)$, the space of projectivised measured laminations, which is compact.

The set of weighted simple closed geodesics is dense in $\mathcal{ML}(S)$. If we define the function $\ell_X(\cdot)$ to scale as $\ell_X(s \cdot \gamma) = s \cdot \ell_X(\gamma)$, then it extends to a continuous function on $\mathcal{ML}(S)$. Similarly, $E_X(\cdot)$ also extends to a continuous function on $\mathcal{ML}(S)$ if we take the scaling to be $E_X(s \cdot \gamma) = s^2 \cdot E_X(\gamma)$.

4.2.2 Fenchel–Nielson coordinates

We now describe the *Fenchel–Nielson coordinates* on Teichmüller space which provide global coordinates for $\mathcal{T}(S)$. We begin by choosing a so-called marking of the surface S which consists of two pieces of topological data. First, let $\{\gamma_1, \ldots, \gamma_{3g-3}\}$ be a collection of pairwise disjoint oriented simple closed curves. This determines a pants decomposition of the surface S. Secondly, we pick another set $\{\eta_1, \ldots, \eta_k\}$ of simple closed curves in S, called the seams, such that the intersection of the union of seams with any pair of pants in the pants decomposition consists of three disjoint arcs connecting the boundaries of the pair of pants pairwise.

The Fenchel–Nielson coordinates associated to this choice of marking consist of $3g-3$ length parameters and $3g-3$ twist parameters. The length parameters of a point $X \in \mathcal{T}(S)$ are simply $(\ell_X(\gamma_1), \ldots, \ell_X(\gamma_{3g-3}))$. We define the twist parameter around γ_i of X by first selecting a seam η which intersects γ_i. Each pair of pants in the pants decomposition of S has a hyperbolic metric that is uniquely determined by its boundary lengths. In this metric each pair of boundary curves has a unique shortest geodesic arc connecting them. The curve η is homotopic to a loop consisting of a concatenation of these geodesics arcs and geodesic arcs that run along the curves γ_i. Let m be the signed length in

X of the geodesic arc along γ_i (signed according to whether the arc runs along or against the orientation of γ_i). The twist parameter for γ_i is now defined as $s_X(\gamma) = m/\ell_X(\gamma_i)$.

4.2.3 Complex projective structures and grafting

A *complex projective structure* on S consists of a maximal atlas of charts that take values in $\mathbb{C}\mathrm{P}^1$ and whose transition maps are restrictions of Möbius transformations. As Möbius transformations are in particular holomorphic we see that a complex projective structure determines an underlying complex structure on S.

On a complex projective surface we can define the *Thurston metric* as follows. The norm of a tangent vector $v \in TS$ in the Thurston metric is the infimum of the hyperbolic norms of tangent vectors $v' \in T\mathbb{H}^2$ for which a complex projective map $f\colon \mathbb{H}^2 \to S$ exists such that $df(v') = v$. To compare the Thurston metric with the hyperbolic metric we note that on a surface of genus at least two the hyperbolic metric coincides with the Kobayashi metric. Hence, the hyperbolic metric can be described similarly as the Thurston metric with the modification that we allow f to be any holomorphic map rather than only a projective map. It immediately follows that the hyperbolic metric is bounded from above by the Thurston metric.

Projective structures on S can be build from hyperbolic structures by cutting along a simple closed curve and gluing in a flat cylinder. This process is called *grafting*. To make this notion precise consider σ a hyperbolic metric on S and let $\gamma \subset (S, \sigma)$ be a simple closed geodesic. For $t > 0$ we look at

$$\widetilde{A}(t) = \{z = r \cdot e^{i\theta} \in \mathbb{C} \mid \theta \in [\pi/2, \pi/2 + t]\}$$

which we will consider as multisheeted if $t > 2\pi$. The projective cylinder $A(t)$ is obtained as the quotient of $\widetilde{A}(t)$ by the action of the group $\langle z \mapsto e^{\ell_\sigma(\gamma)} z\rangle \subset \mathrm{PSL}(2, \mathbb{C})$. A projective structure is now obtained by cutting (S, σ) along γ and gluing in $A(t)$ along γ.

We will denote by $\mathrm{Gr}_{t\cdot\gamma}(\sigma)$ the Thurston metric of the projective structure on the surface S that is obtained by the grafting construction. By $\mathrm{gr}_{t\cdot\gamma}(\sigma) \in \mathcal{T}(S)$ we will denote the point in Teichmüller space determined by the underlying complex structure.

The Thurston metric on the grafted surface coincides with the original hyperbolic metric σ on $S - \gamma$ and is Euclidean on the glued cylinder. In this metric the cylinder has circumference $\ell_\sigma(\gamma)$ and height t. From the jump in curvature we see that the Thurston metric can not be smooth. It is however of class $C^{1,1}$ (see [KP94, Section (5.4)]).

The collapsing map of a grafted surface $\pi\colon (S, \mathrm{Gr}_{t\cdot\gamma}(\sigma)) \to (S, \sigma)$, which collapses the grafted cylinder onto the geodesic γ, is 1-Lipschitz. From this observation the following lemma follows immediately (see also [Kim99, Lemma 1]).

Lemma 4.2.1. *For any closed curve $\eta \subset S$ we have $\ell_{\mathrm{Gr}_{t\cdot\gamma}(\sigma)}(\eta) \geq \ell_\sigma(\eta)$.*

Similarly, because the Thurston metric bounds the Kobayashi metric from above it follows that the identity map $\mathrm{id}\colon (S, \mathrm{Gr}_{t\cdot\gamma}(\sigma)) \to (S, \mathrm{gr}_{t\cdot\gamma}(\sigma))$ is also 1-Lipschitz. This observation yields the following result.

Lemma 4.2.2. *For any closed curve $\eta \subset S$ we have $\ell_{\mathrm{Gr}_{t\cdot\gamma}(\sigma)}(\eta) \geq \ell_{\mathrm{gr}_{t\cdot\gamma}(\sigma)}(\eta)$.*

4.3 Energy spectrum of grafted surfaces

We will consider the energy spectrum of a grafted surface that is equipped with the Thurston metric. Let σ be a hyperbolic metric on S and let $\gamma \subset S$ be a simple closed geodesic. These choices determine a family of projective structures $\{\mathrm{Gr}_{t\cdot\gamma}(\sigma)\}_{t\geq 0}$, which is called a *grafting ray*. For $t \geq 0$ denote by $\mathscr{E}(\cdot, t)\colon \mathcal{T}(S) \to \mathbb{R}$ the energy spectrum of the surface $(S, \mathrm{Gr}_{t\cdot\gamma}(\sigma))$, where we take $[f] = [\mathrm{id}]$ as the choice of homotopy class of maps $S \to S$ (see Section 3.3). Our aim in this chapter is to study points in Teichmüller space that are minimal points or almost minimal points of the energy spectrum.

Recall that for $[J] \in \mathcal{T}(S)$ the value $\mathscr{E}([J], t)$ is defined to be the infimum of the energies of all Lipschitz maps $(S, J) \to (S, \mathrm{Gr}_{t\cdot\gamma}(\sigma))$ that are isotopic to the identity. Let us first point out that this infimum is realised by an actual harmonic map. When considering harmonic maps between grafted surfaces one has to be careful of the fact that the Thurston metrics are not smooth. This means that standard existence results can not be applied immediately. Nevertheless, a general existence result for harmonic maps between grafted surfaces has been proved by Scannell and Wolf in [SW02, Lemma 2.3.2].

Our first observation is that, similarly to the energy spectrum of Hitchin representations, the energy spectrum of grafted surfaces is a proper function on Teichmüller space.

Proposition 4.3.1. *Fix $t \geq 0$. There exists a constant $a > 0$ depending only on $\mathrm{gr}_{t\cdot\gamma}(\sigma)$ such that for any $X \in \mathcal{T}(S)$ the following inequality holds*

$$\mathscr{E}(X, t) \geq a \cdot e^{2d(X, \mathrm{gr}_{t\cdot\gamma}(\sigma))}$$

In particular, the function $X \mapsto \mathscr{E}(X, t)$ is a proper function on Teichmüller space.

The proof of Proposition 4.3.1 and several other proofs in this chapter will rely on Lemma 3.2.9. For the readers convenience we repeat the statement of this lemma here.

Lemma 4.3.2 ([Min92, Proposition 3.1])**.** *Let X be a Riemann surface and let ρ be a Riemannian metric on S. For any map $f\colon X \to (S, \rho)$ and any simple closed curve $\gamma \subset S$ we have*

$$\mathscr{E}(f) \geq \frac{1}{2} \frac{\ell_\rho^2(f \circ \gamma)}{E_X(\gamma)}.$$

We can now give a proof of Proposition 4.3.1.

Proof of Proposition 4.3.1. We note that, for $\eta \in \mathcal{ML}(S)$, the quantity

$$\frac{\ell^2_{\mathrm{gr}_{t\cdot\gamma}(\sigma)}(\eta)}{E_{\mathrm{gr}_{t\cdot\gamma}(\sigma)}(\eta)}$$

is invariant under scaling of η and hence defines a continuous function on $\mathbb{P}\mathcal{ML}(S)$. Since $\mathbb{P}\mathcal{ML}(S)$ is compact this function has a well-defined minimum $a > 0$ which depends only on $\mathrm{gr}_{t\cdot\gamma}(\sigma)$. It follows that

$$\ell^2_{\mathrm{gr}_{t\cdot\gamma}(\sigma)}(\eta) \geq a \cdot E_{\mathrm{gr}_{t\cdot\gamma}(\sigma)}(\eta) \tag{4.1}$$

for any $\eta \in \mathcal{ML}(S)$.

We combine Equation (4.1) with Lemma 4.2.2 and Lemma 4.3.2 to find that

$$\mathscr{E}(X,t) \geq \frac{1}{2}\frac{\ell^2_{\mathrm{Gr}_{t\cdot\gamma}(\sigma)}(\eta)}{E_X(\eta)} \geq \frac{1}{2}\frac{\ell^2_{\mathrm{gr}_{t\cdot\gamma}(\sigma)}(\eta)}{E_X(\eta)} \geq \frac{a}{2}\frac{E_{\mathrm{gr}_{t\cdot\gamma}(\sigma)}(\eta)}{E_X(\eta)}, \tag{4.2}$$

for any simple closed curve $\eta \subset S$ and $X \in \mathcal{T}(S)$. Since the weighted simple closed curves lie dense in $\mathcal{ML}(S)$, it follows that this inequality holds for all $\eta \in \mathcal{ML}(S)$.

In [Ker80, Theorem 4] it is proved that, for all $X \in \mathcal{T}(S)$,

$$d(X, \mathrm{gr}_{t\cdot\gamma}(\sigma)) = \frac{1}{2}\log \sup_{\eta\in\mathcal{ML}(S)} \frac{E_{\mathrm{gr}_{t\cdot\gamma}(\sigma)}(\eta)}{E_X(\eta)}$$

Moreover, using the compactness of $\mathbb{P}\mathcal{ML}(S)$, we see that this supremum is realised by some $\eta \in \mathcal{ML}(S)$. Plugging this η into Equation (4.2) gives

$$\mathscr{E}(X,t) \geq \frac{a}{2}\cdot e^{2d(X,\mathrm{gr}_{t\cdot\gamma}(\sigma))}.$$

□

Because the energy spectrum is a proper function on Teichmüller space, it follows that it attains a global minimum. This minimum can be easily identified.

Lemma 4.3.3. *For any $X \in \mathcal{T}(S)$ and $t \geq 0$ we have*

$$\mathscr{E}(X,t) \geq 2\pi|\chi(S)| + t\cdot\ell_\sigma(\gamma).$$

Equality is achieved only at $X = \mathrm{gr}_{t\cdot\gamma}(\sigma)$.

Proof. It is proved in Lemma 3.4.2 that $\mathscr{E}(X,t) \geq \mathrm{Area}(S, \mathrm{Gr}_{t\cdot\gamma}(\sigma))$ for all $X \in \mathcal{T}(S)$ and that this minimum is achieved if and only if $X = \mathrm{gr}_{t\cdot\gamma}(\sigma)$. The existence of the harmonic map that is used in the proof of that lemma is, in the current setting, provided by [SW02, Lemma 2.3.2]. The statement of the lemma now follows from

$$\mathrm{Area}(S, \mathrm{Gr}_{t\cdot\gamma}(\sigma)) = 2\pi|\chi(S)| + t\cdot\ell_\sigma(\gamma)$$

To see this, note that, by the Gauss-Bonnet theorem, the hyperbolic parts of the surface contribute $2\pi|\chi(S)|$ to the area and that the grafted cylinder has area $t\cdot\ell_\sigma(\gamma)$. □

We now turn our attention to points in Teichmüller space that are close to minimising the energy spectrum $\mathscr{E}(\cdot, t)$. Our principal interest is the question whether such points can be far (with respect to the Teichmüller distance) from the true minimal point $\mathrm{gr}_{t\cdot\sigma}(\sigma)$. It follows from the above lemma that, as a function of t, the minimum of $\mathscr{E}(\cdot, t)$ behaves asymptotically like $t \mapsto t \cdot \ell_\sigma(\gamma)$. We will consider points in Teichmüller space for which the energy is close to this asymptotic value. Namely, we consider points $X \in \mathcal{T}(S)$ with $\mathscr{E}(X, t) \leq A \cdot t$ where $A > \ell_\sigma(\gamma)$. If we fix $t \geq 0$, then it follows from Proposition 4.3.1 that such points lie a bounded distance away from $\mathrm{gr}_{t\cdot\gamma}(\sigma)$. However, the bound we obtain is not uniform in t. Our main result is a step towards a bound that is uniform in t.

The result concerns the Fenchel–Nielson coordinates of the curve γ. In order to consider Fenchel–Nielson coordinates we first need to choose a marking of the surface S. For this we choose $3g - 2$ additional disjoint simple closed curves which together with γ determine a pair of pants decomposition of S. We also pick arbitrarily a set of seams for the pants decomposition to obtain a marking of S. Now the Fenchel–Nielson coordinates associated to the curve γ are well-defined. We recall that the length parameter is denoted by $X \mapsto \ell_X(\gamma)$ and the twist parameter is denoted by $X \mapsto s_X(\gamma)$.

Proposition 4.3.4. *For every $A > \ell_\sigma(\gamma)$ there exist constants $t_0 = t_0(A, \ell_\sigma(\gamma)) > 0$ and $c = c(A, \ell_\sigma(\gamma)) > 0$ such that if for some $t \geq t_0$ and $X \in \mathcal{T}(S)$ we have*

$$\mathscr{E}(X, t) \leq A \cdot t,$$

then $1/(c \cdot t) \leq \ell_X(\gamma) \leq c/t$ and $|s_\sigma(\gamma) - s_X(\gamma)| \leq c \cdot t$.

Remark 4.3.5. This result has a nice interpretation in terms of Minsky's product region theorem ([Min96, Theorem 6.1]). Let $S \setminus \gamma$ denote the surface S where γ has been removed and replaced by two punctures. Minsky defines a map $\Pi = (\Pi_0, \Pi_1)\colon \mathcal{T}(S) \to \mathcal{T}(S \setminus \gamma) \times \mathbb{H}^2$. The map $\Pi_1\colon \mathcal{T}(S) \to \mathbb{H}^2$ maps X to $(s_X(\gamma), 1/\ell_X(\gamma))$. The marking on S defines a marking on $S - \gamma$. The map $\Pi_0\colon \mathcal{T}(S) \to \mathcal{T}(S \setminus \gamma)$ maps X to the point in $\mathcal{T}(S \setminus \gamma)$ which has the same Fenchel-Nielson coordinates (except for the length and twist coordinates for γ). The product region theorem states that Π is a homeomorphism and if we equip $\mathcal{T}(S \setminus \gamma) \times \mathbb{H}^2$ with the supremum metric, then there exist constants $\epsilon > 0$ and $a > 0$ such that

$$|d(X, Y) - d(\Pi(X), \Pi(Y))| \leq a$$

for X, Y in the subset $\mathrm{Thin}(\epsilon, \gamma) = \{X \in \mathcal{T}(S) \mid \ell_X(\gamma) < \epsilon\}$.

Now suppose the constants $A, t_0, c > 0$ are as in Proposition 4.3.4. If, for $t \geq t_0$, a point $X \in \mathcal{T}(S)$ satisfies $\mathscr{E}(X, t) \leq A \cdot t$, then the results of Proposition 4.3.4 imply that Π_1 maps X into the region

$$R(c, t) = \{x + iy \in \mathbb{H}^2 \mid t/c \leq y \leq c \cdot t \text{ and } |x - s_\sigma(\gamma)| \leq c \cdot t\} \subset \mathbb{H}^2.$$

When t is large enough we have

$$\mathscr{E}(\mathrm{gr}_{t\cdot\gamma}(\sigma), t) = 2\pi\chi(S) + \ell_\sigma(\gamma) \cdot t \leq A \cdot t$$

and hence, by Proposition 4.3.4, also the point $\mathrm{gr}_{t\cdot\gamma}(\sigma)$ is mapped into $R(c,t)$ by Π_1. We now note that the diameter of this set in $\mathbb{H}^2$ is bounded uniformly in t. This can be easily seen by observing that the isometry $z \mapsto \frac{1}{t}\cdot(z - s_\sigma(\gamma))$ of $\mathbb{H}^2$ maps the region $R(c,t)$ into some ball of fixed radius centred around i. We find that $d(\Pi_1(X), \Pi_1(\mathrm{gr}_{t\cdot\gamma}(\sigma)))$ is bounded by a constant depending only on $\ell_\sigma(\gamma)$ and A. We conclude that almost minimisers of the energy stay, at least in the $\mathbb{H}^2$ factor, uniformly close to the true minimiser.

Our proof of Proposition 4.3.4 will consist of applying Lemma 4.3.2 to several carefully selected curves in S. The following lemma will provide the existence of one such curve.

Lemma 4.3.6. *There exists a constant $C_1 = C_1(\chi(S)) > 0$ such that for every $X \in \mathcal{T}(S)$ there is a simple closed curve $\eta \subset S$ with $i(\gamma,\eta) \in \{1,2\}$ and*

$$E_X(\eta) \leq C_1 \cdot (1 + \frac{1}{E_X(\gamma)}).$$

Proof. Consider on S the singular flat metric determined by the Strebel differential on X associated to γ. Normalise such that the area equals one. The surface S equipped with this metric can be considered as a quotient of a Euclidean cylinder with its boundary subdivided into arcs that are glued together isometrically in a pairwise fashion (see Figure 4.1). The interior of this cylinder then coincides with the union of the non-singular leaves of the horizontal foliation. If we denote the height and width of the cylinder by h and w respectively, then $E_X(\gamma) = w/h$ and $\mathrm{Area}(S) = h \cdot w = 1$. From this we find that $h = E_X(\gamma)^{-1/2}$ and $w = E_X(\gamma)^{1/2}$.

The pairs of arcs in the boundary that are glued together constitute the singular leaves of the horizontal foliation of the Strebel differential. The maximal possible number of singular leaves, we call this n, is determined entirely by the topology of S (and hence by $\chi(S)$).

We will now construct the simple closed curve $\eta \subset S$ satisfying the conditions of the lemma. We distinguish two cases. Namely, whether γ is a separating curve or not. We consider first the case where γ is a separating curve in S. Then arcs in a boundary component of the cylinder can only be glued to arcs in that same boundary component. Because there is a maximal number of arcs into which the boundary components are subdivided, there exists a pair of arcs in the top boundary component which are glued together and have length of at least $w/(2n)$. Similarly, such a pair exists in the bottom boundary component. We obtain η by connecting the midpoints of these edges by straight lines (as indicated in Figure 4.1).

Let A be the neighbourhood of points at most $w/(4n)$ away from η. This is an annulus on S with core curve η. Let $M(A)$ be the modulus of this annulus. Denote by α the homotopy class of arcs connecting the two boundary components of the annulus. It is a well known fact that the modulus of A equals the extremal length of α in A. To estimate this we note that, measured in the flat metric on the cylinder, a curve in α has length at least $w/(2n)$. Since the curve η has length at most $2\sqrt{w^2+h^2}$ it follows that the area of the annulus is at most

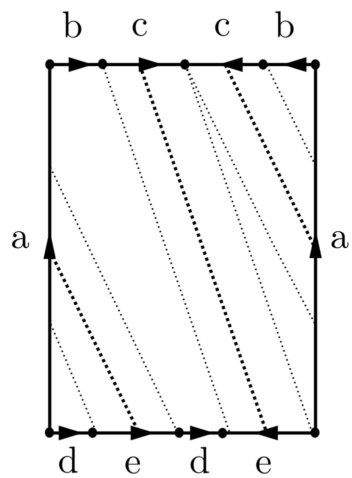

Figure 4.1: Example of a gluing of a Euclidean cylinder. The vertical sides are identified to obtain a cylinder. The arcs in the horizontal sides are identified pairwise to obtain a surface. A possible curve η is depicted with a dotted line and an annulus that contains η is depicted with finer dotted lines.

$w/n \cdot \sqrt{w^2+h^2}$. We find that (cf. Equation (3.1))

$$M(A) = E_A(\alpha) \geq \frac{\ell_A(\alpha)^2}{\mathrm{Area}(A)} \geq \frac{1}{4}\frac{w/n}{\sqrt{h^2+w^2}}.$$

This allows us to estimate $E_X(\eta)$ as follows

$$E_X(\eta)^2 \leq \frac{1}{M(A)^2} \leq 16n^2 \cdot \frac{h^2+w^2}{w^2} = 16n^2\left[1+\frac{1}{E_X(\gamma)^2}\right].$$

Now using $\sqrt{1+x^2} \leq 1+x$ for $x \geq 0$ we find

$$E_X(\eta) \leq C_1(1+\frac{1}{E_X(\gamma)})$$

with $C_1 = 4n$. Because η crosses the cylinder twice we see that $i(\gamma,\eta) = 2$.

The case where γ is not separating is similar. In this case arcs from the top boundary component can be identified with arcs in the bottom boundary component. If, nevertheless, a gluing pair in the top and a gluing pair in the bottom part of the boundary exists with lengths bounded by $w/(2n)$, then we can take η exactly as before and we obtain the same estimate on its extremal length. If two such gluing pairs do not exist, then there must exist a gluing pair consisting of one arc in the top boundary component and one in the bottom with length at least $w/(2n)$. We then obtain η by taking a single straight line connecting the midpoints of these arcs. Then η is contained in a annulus of height at least $w/(2n)$ and width not exceeding $\sqrt{w^2+h^2}$. A calculation similar to the one above shows that the extremal length of η satisfies the same bound. In this case $i(\gamma,\eta) = 1$. $\square$

A further ingredient in our proof will be the following result of Maskit which gives a comparison between extremal and hyperbolic lengths of simple closed curves on surfaces.

Lemma 4.3.7 ([Mas85]). *For any $X \in \mathcal{T}(S)$ and simple closed curve $\gamma \subset S$ we have*

$$\frac{\ell_X(\gamma)}{\pi} \leq E_X(\gamma) \leq \frac{\ell_X(\gamma)}{2} \cdot e^{\ell_X(\gamma)/2}.$$

We now begin our proof of Proposition 4.3.4.

Proof of Proposition 4.3.4. Throughout this proof we will denote $L = \ell_\sigma(\gamma)$. Let $X \in \mathcal{T}(S)$ be such that $\mathscr{E}(X,t) \leq A \cdot t$. We first prove the bounds on $\ell_X(\gamma)$.

In the Thurston metric the grafted cylinder is a Euclidean cylinder with circumference $\ell_\sigma(\gamma)$. It follows that $\ell_{\mathrm{Gr}_{t\cdot\gamma}(\sigma)}(\gamma) = \ell_\sigma(\gamma) = L$. Applying Lemma 4.3.2 to the curve γ yields

$$A \cdot t \geq \mathscr{E}(X,t) \geq \frac{1}{2}\frac{L^2}{E_X(\gamma)}.$$

It follows that $E_X(\gamma) \geq L^2/(2At)$.

To obtain an upper bound for $E_X(\gamma)$ we consider the curve $\eta \subset S$ with $i(\gamma,\eta) \in \{1,2\}$ and $E_X(\eta) \leq C_1(1 + 1/E_X(\gamma))$ that is supplied by Lemma 4.3.6. The geodesic representative of η on the grafted surface has to cross the Euclidean cylinder of height t at least once and hence $\ell_{\mathrm{Gr}_{t\cdot\gamma}(\sigma)}(\eta) \geq t$. By applying Lemma 4.3.2 to the curve η we see that

$$A \cdot t \geq \mathscr{E}(X,t) \geq \frac{1}{2}\frac{\ell^2_{\mathrm{Gr}_{t\cdot\gamma}(\sigma)}(\eta)}{E_X(\eta)} \geq \frac{1}{2C_1}\frac{t^2}{1 + 1/E_X(\gamma)}$$

which gives

$$\frac{E_X(\gamma)}{1 + E_X(\gamma)} \leq \frac{2C_1 A}{t}.$$

If we take t larger than $t_0 = 4C_1 A$, then $E_X(\gamma)/(1 + E_X(\gamma)) \leq 1/2$ from which follows that $E_X(\gamma) \leq 1$. The above inequality then becomes

$$\frac{1}{2}E_X(\gamma) \leq \frac{E_X(\gamma)}{1 + E_X(\gamma)} \leq \frac{2C_1 A}{t}$$

so $E_X(\gamma) \leq 4C_1 A/t$.

To obtain actual bounds on $\ell_X(\gamma)$ from the bounds on $E_X(\gamma)$ we use Lemma 4.3.7. The first inequality of that lemma gives $\ell_X(\gamma) \leq \pi E_X(\gamma) \leq c'/t$ when we take $c' = 4\pi AC_1$. From this bound follows that, after increasing t_0 if necessary, we can arrange that $t \geq t_0$ implies $e^{\ell_X(\gamma)/2} \leq 2$. Now the second inequality of Lemma 4.3.7 gives

$$\frac{L^2}{2At} \leq E_X(\gamma) \leq \ell_X(\gamma)/2e^{\ell_X(\gamma)/2} \leq \ell_X(\gamma).$$

Hence, after possibly increasing c' such that $1/c' \leq L^2/(2A)$, we find

$$1/(c' \cdot t) \leq \ell_X(\gamma) \leq c' \cdot t.$$

We now turn out attention to finding a bound on $|s_\sigma(\gamma) - s_X(\gamma)|$. Again, we will obtain such a bound by applying Lemma 4.3.2 to a suitable curve $\omega \subset S$. As we will see below, a good candidate for ω is such that $\ell_{\mathrm{Gr}_{t\cdot\gamma}(\sigma)}(\omega)$ is roughly comparable to $|s_\sigma(\gamma) - s_X(\gamma)|$ and $E_X(\omega)$ is roughly comparable to $1/\ell_X(\gamma)$. Such a simple closed curve will be provided by the following lemma.

Lemma 4.3.8. *There exist constants $\epsilon_1 > 0, C_2 > 0$ depending only on $\chi(S)$ and a constant $C_3 = C_3(L, \chi(S)) > 0$ such that for every $X \in \mathcal{T}(S)$ with $\ell_X(\gamma) < \epsilon_1$ there exists a simple closed curve $\omega \subset S$ with $i(\gamma, \omega) \in \{1, 2\}$ that satisfies*

$$\ell_{\mathrm{Gr}_{t\cdot\gamma}(\sigma)}(\omega) \geq L \cdot |s_\sigma(\gamma) - s_X(\gamma)| - C_3$$

and

$$E_X(\omega) \leq C_2/\ell_X(\gamma).$$

We will first finish the proof of Proposition 4.3.4 and then give a proof of this lemma.

Since we already have the bound $\ell_X(\gamma) \leq c'/t$, we can arrange, after increasing t_0, that $t \geq t_0$ implies $\ell_X(\gamma) < \epsilon_1$. Now let ω be the simple closed curve provided by the lemma. Applying Lemma 4.3.2 to ω gives

$$A \cdot t \geq \frac{1}{2} \frac{\ell^2_{\mathrm{Gr}_{t\cdot\gamma}(\sigma)}(\omega)}{E_X(\omega)} \geq \frac{1}{2} \frac{(L|s_\sigma(\gamma) - s_X(\gamma)| - C_3)^2}{C_2/\ell_X(\gamma)}.$$

Combining this with the bound $\ell_X(\gamma) \leq c'/t$ we obtain

$$(L|s_\sigma(\gamma) - s_X(\gamma)| - C_3)^2 \leq 2AC_2/c' \cdot t^2.$$

If we take $c'' = \sqrt{2AC_2/(c' \cdot L)}$, then $|s_\sigma(\gamma) - s_X(\gamma)| \leq c'' \cdot t + C_3/L$. Finally, we absorb the additive constant C_3/L into the constant c'' by taking into account that $t \geq t_0 > 0$ and increasing c''. We conclude that the stated bounds for $\ell_X(\gamma)$ and $|s_\sigma(\gamma) - s_X(\gamma)|$ hold if we take $c = \max\{c', c''\}$. $\square$

We complete our proof by proving Lemma 4.3.8. We will make use of the tools developed by Minsky in [Min96]. Let us first introduce the necessary notation and results from that paper.

We first define the so-called twisting numbers which Minsky introduces in Formula 3.2 of [Min96]. Let ρ be a hyperbolic metric on S. Let $\gamma, \beta \subset S$ be (oriented) simple closed curves and let γ^ρ, β^ρ be the geodesic representatives of these loops. Let $x \in \gamma^\rho \cap \beta^\rho$ be a point of intersection. The universal cover of (S, ρ) is isomorphic to $\mathbb{H}^2$. Let L_γ, L_β be geodesics in $\mathbb{H}^2$ that are lifts of γ^ρ and β^ρ respectively such that L_γ and L_β intersect in a point that projects to x. Let ξ_L and ξ_R be the endpoints of the geodesic L_β in $\partial_\infty \mathbb{H}^2$ which lie to the left and right of L_γ respectively. We identify L_γ isometrically with $\mathbb{R}$ in a manner that is consistent with the orientation of γ. We denote by $p_\gamma \colon \mathbb{H}^2 \cup \partial_\infty \mathbb{H}^2 \to L_\gamma$ the shortest distance projection. The twisting number $t_{\gamma,\rho}(\beta)$ is defined to be the minimum of the quantity

$$\frac{p_\gamma(\xi_R) - p_\gamma(\xi_L)}{\ell_\rho(\gamma)}$$

over all x in $\gamma^\rho \cap \beta^\rho$.

For an annulus $A \subset S$ Minsky defines a similar twisting number which measures the twisting β does inside A. The metric ρ determines a conformal structure on A. By uniformising we can identify A with a Euclidean cylinder. Let the height of this cylinder be h and its circumference w. The universal cover of A can be conformally identified with the strip $[0, h] \times \mathbb{R}$. If β' is a connected component of $\beta \cap A$, then a lift of β' to the strip $[0, h] \times \mathbb{R}$ is a curve with endpoints $(0, y_0)$ and (h, y_1). The twisting number $t_{A,\rho}(\beta)$ is defined to be the minimum of quantity $(y_1 - y_0)/w$ over all subarcs $\beta' \subset \beta \cap A$.

Another notion from [Min96] we will need is the (ϵ_0, ϵ_1)-collar decomposition of (S, ρ). Here $0 < \epsilon_1 < \epsilon_0$ are constants both smaller than the Margulis constant for $\mathbb{H}^2$. A (ϵ_0, ϵ_1)-collar is an annular neighbourhood of a geodesic of length at most ϵ_1 such that its boundaries have length ϵ_0. The (ϵ_0, ϵ_1)-collar decomposition of (S, ρ) consists of the set $\{A_i\}_i$ of all such collars and the set of hyperbolic pieces $\{P_i\}_i$ which are the closures of components of $X - \cup_i A_i$.

The definition of extremal length can be extended to surfaces with boundary and hence we can consider $E_{P,[\rho]}(\beta)$ if $P \in \{P_i\}_i$ and β is an arc in P. If β is an arc connecting the two boundary components of a collar $A \in \{A_i\}_i$, then Minksy defines ([Min96, Formula 4.3]) a quantity analogous to the extremal length as

$$E_{A,[\rho]}(\beta) = i(\beta, A)^2 \left(M(A) + \frac{t_{A,\rho}(\beta)^2}{M(A)} \right) \tag{4.3}$$

where $M(A)$ denotes the modulus of the annulus A.

Our proof of Lemma 4.3.8 will rely on the following theorem.

Theorem 4.3.9 ([Min96, Theorem 5.1])**.** *There exists a universal choice for (ϵ_0, ϵ_1) such that for any simple closed curve $\beta \subset S$ the extremal length $E_{[\rho]}(\beta)$ is proportional to the quantity*

$$\max\{\max_i E_{A_i,[\rho]}(\beta \cap A_i), \max_i E_{P_i,[\rho]}(\beta \cap P_i)\} \tag{4.4}$$

up to multiplicative constants depending only on ϵ_0, ϵ_1 and $\chi(S)$.

We can now begin our proof of Lemma 4.3.8.

Proof of Lemma 4.3.8. We look for a simple closed curve $\omega \subset S$ similar to the one constructed in Lemma 4.3.6 but with minimal twisting around γ in X. Then the twisting of ω around γ in $\mathrm{Gr}_{t\cdot\gamma}(\sigma)$ will be comparable to $|s_\sigma(\gamma) - s_X(\gamma)|$. The main difficulty is controlling both the twisting of ω around γ and its extremal length at the same time.

Let ρ be a hyperbolic metric on S that is a representative of the point $X \in \mathcal{T}(S)$. We assume $\ell_\rho(\gamma) < \epsilon_1$ which means that γ is the core curve of a unique collar $A_\gamma \in \{A_i\}_i$. A suitable simple closed curve $\omega \subset S$ is constructed in Lemma 3.3 of [Min96] (when taking, in the notation of that lemma, $t = 0$). Let us repeat the construction here. Let $P_1, P_2 \in \{P_i\}_i$ be the hyperbolic pieces that are adjacent to A_γ (possibly $P_1 = P_2$). Minsky proves there exists a constant

$r > 0$ (depending on ϵ_0, ϵ_1 and $\chi(S)$) such that for each boundary component of A_γ there exist an arc of length at most r contained in P_1 and P_2 that connects the boundary component to the other component or to itself. In the former case, if there is an arc that joins the two boundary components, then we construct ω by concatenating this arc with a straight arc crossing the annulus A_γ. Otherwise, we obtain ω by concatenating the arcs in P_1 and P_2 that join the boundary components to themselves with two parallel straight arcs in A_γ. We have either $i(\omega, A_\gamma) = 1$ or $i(\omega, A_\gamma) = 2$. Now, Lemma 3.3 of [Min96] states that the curve ω satisfies $|t_{A_\gamma,\rho}(\omega)| \leq T$ and $|t_{\gamma,\rho}(\omega)| \leq T$ (use also [Min96, Lemma 3.2]) where T is a constant that also depends only on ϵ_0, ϵ_1 and $\chi(S)$.

We will now use Theorem 4.3.9 to bound $E_{[\rho]}(\omega)$ from above. Lemma 4.3 of [Min96] states that $E_{P_i,[\rho]}(\omega \cap P_i) \leq b \cdot \ell_\sigma^2(\omega \cap P_i)$ where b is a constant depending only on ϵ_0, ϵ_1 and $\chi(S)$. Because the length of the pieces $\omega \cap P_i$ (for $i = 1, 2$) is bounded from above by r it follows that $E_{P_i,[\rho]}(\omega \cap P_i) \leq c \cdot r^2$. For the remaining pieces of ω we estimate $E_{A_\gamma,[\rho]}(\omega \cap A_\gamma)$ using Equation (4.3). We have $i(\omega, A_\gamma) \in \{1, 2\}$ and $t_{A_\gamma,\rho}(\beta)^2 \leq T^2$. For $M(A_\gamma)$ we have the formula

$$M(A_\gamma) = \frac{\pi}{\ell_\rho(\gamma)} - \frac{2}{\epsilon_0}$$

(see [Min96, Formula 2.2]) which shows that $M(A_\gamma) \leq \pi/\ell_\rho(\gamma)$ and $M(A_\gamma) \geq \pi/\epsilon_1 - 2/\epsilon_0 > 0$. If we plug these estimates into Equation (4.3) it follows that there exists a constant $B > 0$ depending only on ϵ_0, ϵ_1 and $\chi(S)$ such that

$$E_{A_\gamma,[\rho]}(\omega \cap A_\gamma) \leq 4\left(\pi/\ell_\rho(\gamma) + \frac{T^2}{\pi/\epsilon_1 - 2/\epsilon_0}\right) \leq B/\ell_\rho(\gamma).$$

Here we used that $\ell_\rho(\gamma) < \epsilon_1$ to absorb the additive constant into B.

We combine these estimates on $E_{P_i,[\rho]}(\omega \cap P_i)$ and $E_{A_\gamma,[\rho]}(\omega \cap A_\gamma)$ to find that

$$\max\{\max_i E_{A_i,[\rho]}(\omega \cap A_i), \max_i E_{P_i,[\rho]}(\omega \cap P_i)\} \leq \max\{b \cdot r^2, B/\ell_\rho(\gamma)\} \leq B/\ell_\rho(\gamma)$$

were we increased B such that $B/\ell_\rho(\gamma) \geq b \cdot r^2$ (taking into consideration that $\ell_\rho(\gamma) < \epsilon_1$). Now Theorem 4.3.9 tells us that $E_{[\rho]}(\omega)$ can be bounded from above, up to multiplicative constant, by the above expression. Hence, there is a constant $C_2 > 0$, depending only on ϵ_0, ϵ_1 and $\chi(S)$, such that $E_{[\rho]}(\omega) \leq C_2/\ell_\rho(\gamma)$.

It now remains to give a lower bound for the length of ω in $(S, \mathrm{Gr}_{t\cdot\gamma}(\sigma))$. First notice that by Lemma 4.2.1 it is enough to give a lower bound for $\ell_\sigma(\omega)$. What we will show is that in (S, σ) the number of twists the geodesic representative of ω makes around γ is roughly $|s_\sigma(\gamma) - s_\rho(\gamma)|$ and hence its length must be at least $|s_\sigma(\gamma) - s_\rho(\gamma)| \cdot L$. Recall that we denote $L = \ell_\sigma(\gamma)$.

We mimic part of the argument of the proof of [DK12, Proposition 2.1]. For simplicity we treat only the case $i(\gamma, \omega) = 1$ with the remaining case $i(\gamma, \omega) = 2$ being similar. We consider the homotopy class of arcs with endpoints on γ sliding freely that contains the loop ω. Let H be the shortest geodesic arc (with respect to σ) in this homotopy class. The loop ω is homotopic to a concatenation of H

and a geodesic arc V that lies in γ which connects the endpoints of H. Denote the length of this concatenated loop by $l_\sigma(H \cup V)$. The lift of $H \cup V$ to the universal cover of (S, σ) is a stairstep path as defined in Section 3.2.3. It follows from Lemma 3.2.4 (cf. [DS03, Lemma 5.1]) that there exists a constant $D > 0$ such that $|l_\sigma(H \cup V) - \ell_\sigma(\omega)| \leq D$.

We consider, temporarily, a new marking μ of the surface S. We choose the same pair of pants decomposition as before but we pick a new set of seams that contains ω. Denote the twisting parameter around γ with respect to this marking by $[\sigma] \mapsto s_\sigma^\mu(\gamma)$. Recalling the definition of this twist parameter we see that

$$|s_\sigma^\mu(\gamma)| = \frac{l_\sigma(V)}{\ell_\sigma(\gamma)}.$$

Minsky proves in [Min96, Lemma 3.5] that $|s_\sigma^\mu(\gamma) - t_{\gamma,\sigma}(\omega)| \leq 1$. It follows that

$$\left| \frac{l_\sigma(V)}{\ell_\sigma(\gamma)} - |t_{\gamma,\sigma}(\omega)| \right| \leq 1.$$

We use this to estimate

$$\begin{aligned}
\ell_\sigma(\omega) &\geq l_\sigma(H \cup V) - D \\
&\geq l_\sigma(V) - D \\
&\geq \ell_\sigma(\gamma) \cdot |t_{\gamma,\sigma}(\omega)| - \ell_\sigma(\gamma) - D \\
&= L \cdot |t_{\gamma,\sigma}(\omega)| - L - D
\end{aligned}$$

To estimate the quantity $|t_{\gamma,\sigma}(\omega)|$ we use [Min96, Lemma 3.5]. It states, in our notation, that

$$|(t_{\gamma,\sigma}(\omega) - t_{\gamma,\rho}(\omega)) - (s_\sigma(\gamma) - s_\rho(\gamma))| \leq 4.$$

We recall that ω satisfies $|t_{\gamma,\rho}(\omega)| \leq T$ and combine this with the above inequality to find that

$$|t_{\gamma,\sigma}(\omega)| \geq |s_\sigma(\gamma) - s_\rho(\gamma)| - 4 - T.$$

Plugging this into the estimate for $\ell_\sigma(\omega)$ above we see that if we take $C_3 = D + L(5 + T)$, then

$$\ell_\sigma(\omega) \geq L \cdot |s_\sigma(\gamma) - s_\rho(\gamma)| - C_3.$$

We conclude that

$$\ell_{\mathrm{Gr}_{t\cdot\gamma}(\sigma)}(\omega) \geq \ell_\sigma(\omega) \geq L \cdot |s_\sigma(\gamma) - s_\rho(\gamma)| - C_3$$

which finishes our proof. □

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Chapter 5

Exponential convergence rate of the harmonic heat flow

Abstract

We consider the harmonic heat flow for maps from a compact Riemannian manifold into a Riemannian manifold that is complete and of non-positive curvature. We prove that if the harmonic heat flow converges to a limiting harmonic map that is a non-degenerate critical point of the energy functional, then the rate of convergence is exponential (in the L^2 norm).

5.1 Introduction

The harmonic heat flow was introduced by Eells and Sampson in [ES64]. They used it to prove one of the first general existence results for harmonic maps between Riemannian manifolds. Since then the harmonic heat flow has been an important tool in many existence results for harmonic maps. It has also been studied much as a subject of investigation in its own right.

Suppose (M, g) and (N, h) are Riemannian manifolds and $f\colon M \to N$ a smooth map. The harmonic heat flow is an evolution equation on one-parameter families of smooth maps $(f_t\colon M \to N)_{t\in[0,\infty)}$ that continuously deforms f into a harmonic map. The parameter t is often thought of as a time parameter. The harmonic heat flow equation is

$$\begin{aligned} \frac{df_t}{dt} &= \tau(f_t) \\ f_0 &= f. \end{aligned} \tag{5.1}$$

Here $\tau(f_t)$ is the tension field of f_t (see Section 5.2). Eells and Sampson prove in [ES64] (with contributions of Hartman in [Har67]) that if M is compact and N is complete and has non-positive curvature, then a solution of Equation (5.1) exists for all $t \geq 0$. Moreover, if the images of the maps f_t stay within a compact subset of N, then the harmonic heat flow converges, for $t \to \infty$, to a harmonic map $f_\infty\colon M \to N$ that is homotopic to f.

In this note we prove that when the limiting map satisfies a certain non-degeneracy condition (which will elaborated on in Section 5.2), then the rate of convergence of the harmonic heat flow is exponential.

Theorem 5.1.1. *Let (M, g) and (N, h) be Riemannian manifolds with M compact and with N complete and of non-positive curvature. Let $(f_t)_{t\in[0,\infty)}$ be a solution to the harmonic heat flow equation. Assume that the maps f_t converge* *to a limiting harmonic map $f_\infty \colon M \to N$, as $t \to \infty$, and assume that f_∞ is a non-degenerate critical point of the Dirichlet energy functional. Then there exist constants $a, b > 0$ such that*

$$\left\| \frac{df_t}{dt} \right\|_{L^2(f_t^* TN)} \leq a \cdot e^{-b \cdot t}$$

for all $t \geq 0$. Moreover, the exponential decay rate (the constant b) depends only on f_∞.

The exponential convergence rate of the harmonic heat flow has been observed before in several different settings. For example, in [Top97] Topping proved that the harmonic heat flow for maps between 2-spheres converges exponentially fast in L^2 as $t \to \infty$. Similarly, in [Wan12] it is shown that the heat flow for mappings from the unit disk in $\mathbb{R}^2$ into closed Riemannian manifolds converges exponentially fast in H^1 when we assume that the Dirichlet energy along the heat flow is small.

Our result shows that this exponential convergence behaviour is actually present in a large class of examples. For instance, if (N, h) has negative curvature, then any harmonic map into N that does not map into the image of a geodesic is a non-degenerate critical point of the energy. Another example is provided by equivariant harmonic maps mapping into symmetric spaces of non-compact type. A result of Sunada ([Sun79]) implies that such harmonic maps are non-degenerate critical points of the energy if and only if they are unique (see [Sle20, Lemma 2.1]).

As a corollary to Theorem 5.1.1 we obtain that the Dirichlet energies along the harmonic heat flow also converge exponentially fast. For a smooth map $f \colon (M, g) \to (N, h)$ we denote by $E(f)$ its Dirichlet energy (see Section 5.2).

Corollary 5.1.2. *Let $(f_t)_{t \in [0,\infty)}$, f_∞ and $b > 0$ be as in Theorem 5.1.1. Then there exists a constant $a' > 0$ such that for all $t \geq 0$ we have*

$$|E(f_t) - E(f_\infty)| \leq a' \cdot e^{-2b \cdot t}.$$

5.2 Preliminaries

We briefly introduce the concepts related to harmonic maps that we will need in our proof. We follow mostly the presentation given in [EL83] (see also [ES64]).

Let (M, g) and (N, h) be Riemannian manifolds and assume M is compact. For any vector bundle $E \to M$ we denote by $\Gamma^k(E)$ the Banach space of k-times continuously differentiable sections of E. For any smooth map $f \colon M \to N$ let us denote by ∇ the pullback connection on $f^*TN \to M$ induced by the Levi-Civita connection of N. By taking the tensor product with the Levi-Civita connection on M we obtain an induced connection on the bundle $T^*M \otimes f^*TN$ which we will also denote by ∇.

A smooth map $f\colon (M,g)\to(N,h)$ is a *harmonic map* if it is a critical point of the Dirichlet energy

$$E(f)=\frac{1}{2}\int_M \|df\|^2 \operatorname{vol}_g .$$

Here we consider df as a section of the bundle $T^*M\otimes f^*TN$ that is equipped with the metric induced by the metrics g and h. The *tension field* of f is the smooth section of f^*TN that is defined as

$$\tau(f)=\operatorname{tr}_g \nabla df=\sum_{i=1}^m (\nabla_{e_i} df)(e_i)$$

where $(e_i)_{i=1}^m$ is any local orthonormal frame of TM and ∇ is the connection on $T^*M\otimes f^*TN$. A map $f\colon (M,g)\to(N,h)$ is harmonic if and only if its tension field vanishes identically.

The metric g on M and the metric on f^*TN induced by the metric on N give rise to the L^2 inner product

$$\langle s,s'\rangle_{L^2(f^*TN)}=\int_M \langle s(m),s'(m)\rangle \operatorname{vol}_g(m)$$

for $s,s'\in\Gamma^0(f^*TN)$. The space $L^2(f^*TN)$ is defined to be the completion of $\Gamma^0(f^*TN)$ with respect to this inner product.

The Laplace operator induced by the pullback connection ∇ on f^*TN is the operator $\Delta\colon \Gamma^2(f^*TN)\to\Gamma^0(f^*TN)$ that is given by

$$\Delta s=-\operatorname{tr}_g \nabla^2 s=-\sum_{i=1}^m (\nabla^2 s)(e_i,e_i)$$

for $s\in\Gamma^2(f^*TN)$ and any (local) orthonormal frame $(e_i)_{i=1}^m$ of TM.

Definition 5.2.1. *We define the Jacobi operator of a smooth map $f\colon M\to N$ to be the second order differential operator that acts on sections of f^*TN as*

$$\mathcal{J}_f(s)=\Delta s-\operatorname{tr}_g R^N(s,df\cdot)df\cdot=-\sum_{i=1}^m \left[(\nabla^2 s)(e_i,e_i)+R^N(s,df(e_i))df(e_i)\right]$$

where $s\in\Gamma^2(f^*TN)$, R^N is the curvature tensor[1] *of (N,h) and $(e_i)_{i=1}^m$ is any (local) orthonormal fame of TM.*

We can interpret the Jacobi operator as a densely defined operator

$$\mathcal{J}_f\colon L^2(f^*TN)\to L^2(f^*TN).$$

It is a linear elliptic and self-adjoint operator. Standard spectral theory for such operators implies the following facts.

[1]We define the curvature tensor as $R(X,Y)Z=\nabla_X\nabla_Y Z-\nabla_Y\nabla_X Z-\nabla_{[X,Y]}Z$ which differs from the convention chosen in [EL83].

Proposition 5.2.2. *The Hilbert space $L^2(f^*TN)$ splits orthogonally into eigenspaces of $\mathcal{J}_f$. These eigenspaces are finite dimensional and consist of smooth sections. The spectrum of $\mathcal{J}_f$ is discrete and consists of real numbers. If (N,h) is non-positively curved, then the eigenvalues of $\mathcal{J}_f$ are non-negative.*

Proof. See [Wel08, Chapter IV] (cf. [EL83, Section 4]). It is proved in [EL83, Proposition 1.23] that Δ is a positive operator. If (N,h) is non-positively curved, then

$$-\operatorname{tr}_g\langle R^N(s, df\cdot)df\cdot, s\rangle = -\sum_{i=1}^{m}\langle R^N(s, df(e_i))df(e_i), s\rangle \geq 0$$

for any $s \in \Gamma^0(f^*TN)$ and hence it follows that the eigenvalues of $\mathcal{J}_f$ are non-negative. $\square$

When (N,h) has non-positive curvature it follows that each $\mathcal{J}_f$ has a well-defined lowest eigenvalue which we will denote by $\lambda_1(\mathcal{J}_f) \geq 0$. This quantity is called *the spectral gap* of the operator $\mathcal{J}_f$. Using the min-max theorem the value $\lambda_1(\mathcal{J}_f)$ can alternatively be characterised as

$$\lambda_1(\mathcal{J}_t) = \min_{\substack{s\in\Gamma^2(f^*TN)\\ s\neq 0}} \frac{\langle \mathcal{J}_f s, s\rangle_{L^2(f^*TN)}}{\|s\|^2_{L^2(f^*TN)}}. \tag{5.2}$$

If f is harmonic, then the second variation of the energy at f is given by

$$\nabla^2 E(f)(s,s') = \int_M \left[\langle \nabla s, \nabla s'\rangle - \operatorname{tr}_g\langle R^n(s, df\cdot)df\cdot, s'\rangle\right]\operatorname{vol}_g = \langle \mathcal{J}_f s, s'\rangle_{L^2(f^*TN)}$$

for any $s, s' \in \Gamma^2(f^*TN)$. We stress that this equation only holds when f is harmonic. A harmonic map is a non-degenerate critical point of the energy if the bilinear form $\nabla^2 E(f)$ is non-degenerate. This happens if and only if $\ker \mathcal{J}_f = 0$. In the case that (N,h) has non-positive curvature this is equivalent to $\lambda_1(\mathcal{J}_f) > 0$.

As mentioned in the introduction, the existence of a solution to the harmonic heat flow equation is due to Eells and Sampson. We record the facts relevant to our proof here in the following theorem. We denote by $C^k(M,N)$ the Banach manifold of k-times continuously differentiable maps from M to N.

Theorem 5.2.3. *Assume (M,g) is compact and (N,h) is complete and of non-positive curvature. Let $f\colon M \to N$ be a smooth map. A solution $(f_t)_{t\in[0,\infty)}$ to the harmonic heat flow equation (Equation (5.1)) exists for all time $t \geq 0$ and the map*

$$M \times [0,\infty) \to N\colon (m,t) \mapsto f_t(m)$$

is smooth. Moreover, if the image of this map is contained in a compact subset of N, then the maps f_t converge, for $t \to \infty$, to a harmonic map f_∞ in any space $C^k(M,N)$.

The existence and smoothness of the solution is proved in [ES64] (Theorem 10.C p.154 and Proposition 6.B p.135). Note that Eells and Sampson prove these theorems under an additional assumption involving restrictions on a choice of isometric embedding $N \to \mathbb{R}^n$. Hartman proved in [Har67, Assertion (A)] that this assumption is redundant. Finally, the convergence statement for $t \to \infty$ is proved in [Har67, Assertion (B)].

5.3 Continuity of the spectral gap

Our proof of Theorem 5.1.1 will rely on the fact that if $(f_t)_{t\in[0,\infty)}$ is a solution to the harmonic heat flow equation, then the associated family of Jacobi operators $\mathcal{J}_{f_t}$ is (in a loose sense) a continuous family of differential operators. The primary difficulty here is that these operators act on sections of different vector bundles. We deal with this problem in Proposition 5.3.1 which will be the main tool in our proof.

Let us first introduce some notation. We will consider a family of smooth maps $(f_t)_{t\in[0,1]}$ and define $F\colon M \times [0,1] \to N$ as $F(m,t) = f_t(m)$. For each $t \in [0,1]$ we denote $E_t = f_t^*TN$ and $\mathcal{J}_t = \mathcal{J}_{f_t}$.

Proposition 5.3.1. *Assume $F\colon M \times [0,1] \to N$ (as above) is continuous, each $f_t\colon M \to N$ is smooth and $[0,1] \to C^3(M,N)\colon t \mapsto f_t$ is continuous. Then*

$$\liminf_{t\to 0} \lambda_1(\mathcal{J}_t) \geq \lambda_1(\mathcal{J}_0).$$

Remark 5.3.2. As we will see in the proof of this proposition, the statement is easily generalised to $\liminf_{t\to t_0} \lambda_1(\mathcal{J}_t) \geq \lambda_1(\mathcal{J}_{t_0})$ for $t_0 \in [0,1]$ (the choice of $t_0 = 0$ is in no way special). This means that the function $t \mapsto \lambda_1(\mathcal{J}_t)$ is lower semicontinuous. Because we don't need this full statement in our proof, we will restrict ourselves, for notational convenience, to $t_0 = 0$.

As mentioned before, our main difficulty is that the differential operators $\mathcal{J}_t$ do not act on sections of the same vector bundle. To address this we first construct (local) homomorphisms between E_t and E_0 which will allow us to locally identify these bundles.

Throughout this section we will consider the vector bundles $E_t = f_t^*TN$ as a subset of the larger vector bundle F^*TN by identifying M with $M \times \{t\} \subset M \times [0,1]$. Let us consider a chart U of M and a chart V of N such that for some $\epsilon > 0$ the set $U \times [0,\epsilon)$ is mapped into V by F. We will call such charts *adapted charts*. To a pair of adapted charts we will associate, for $t \in [0,\epsilon)$, homomorphisms $\psi_t\colon E_t|_U \to E_0|_U$ as follows. Let us denote by $(y^\alpha)_{\alpha=1}^n$ the coordinates of the chart $V \subset N$. First, we note that $(E_\alpha)_{\alpha=1}^n$, with $E_\alpha = F^*\frac{\partial}{\partial y^\alpha}$, is a local frame of F^*TN over $U \times [0,\epsilon)$. Furthermore, the sections $E_\alpha(\cdot,t)$ provide a frame of $E_t|_U$ for any fixed $t \in [0,\epsilon)$. If we write[2] an element $v \in E_t|_U$ as $v = v^\alpha E_\alpha(x,t)$ for some $x \in U$, then we define the map $\psi_t\colon E_t|_U \to E_0|_U$ as

$$\psi_t(v^\alpha E_\alpha(x,t)) = v^\alpha E_\alpha(x,0).$$

[2]Throughout this text we will use the Einstein summation convention.

We note that for $t = 0$ we have $\psi_0 = \mathrm{id}$ hence, by continuity, ψ_t is an isomorphism for any $t \in [0, \epsilon)$ if we take $\epsilon > 0$ small enough (after possibly shrinking U).

Because M is compact, it can be covered by a finite set of adapted charts. More precisely, there exists an $\epsilon > 0$, a finite set of charts $\{\widetilde{U}_1, \ldots, \widetilde{U}_r\}$ of M and charts $\{V_1, \ldots, V_r\}$ of N such that F maps each $\widetilde{U}_p \times [0, \epsilon)$ into V_p. Let us denote by $\psi_{t,p} \colon E_t|_{\widetilde{U}_p} \to E_0|_{\widetilde{U}_p}$ the homomorphisms associated to each pair $(\widetilde{U}_p, V_p)$ of adapted charts.

Before we proceed to the proof of Proposition 5.3.1, we will first use our choice of adapted charts to define C^k norms on the spaces $\Gamma^k(E_t)$ which will be particularly well-adjusted to our arguments. Fix a $p \in \{1, \ldots, r\}$, let $(x_i)_{i=1}^m$ be the coordinates of the chart $\widetilde{U}_p \subset M$ and let $(y^\alpha)_{\alpha=1}^n$ be the coordinates of the chart $V_p \subset N$. We set, as before, $E_\alpha = F^* \frac{\partial}{\partial y^\alpha}$. By shrinking the open sets $\widetilde{U}_p$ slightly we can find precompact open subsets $U_p \subset \widetilde{U}_p$ such that the sets $\{U_p\}_{p=1}^r$ still cover M. A section $s \in \Gamma^k(E_t)$ can, locally on $\widetilde{U}_p$, be written as $s = s^\alpha E_\alpha(\cdot, t)$. Using this notation, we define, for $k \in \mathbb{N}$ and $t \in [0, \epsilon)$, the seminorms $\|\cdot\|_{\Gamma^k(\overline{U}_p; E_t)}$ on $\Gamma^k(E_t)$ as

$$\|s\|_{\Gamma^k(\overline{U}_p; E_t)} = \sup \left\{ \left| \frac{\partial^{|\mu|}}{\partial x^\mu} s^\alpha(x) \right| : x \in \overline{U}_p, 1 \leq \alpha \leq n, |\mu| \leq k \right\}.$$

Here $\mu = (\mu_1, \ldots, \mu_m)$ is a multi-index and $\frac{\partial^{|\mu|}}{\partial x^\mu} = \frac{\partial^{\mu_1}}{\partial x_1^{\mu_1}} \cdots \frac{\partial^{\mu_m}}{\partial x_1^{\mu_m}}$. This expression is finite because $\overline{U}_p$ is compact in $\widetilde{U}_p$. We now define the norm $\|\cdot\|_{\Gamma^k(E_t)}$ on $\Gamma^k(E_t)$ as

$$\|s\|_{\Gamma^k(E_t)} = \max_{p=1,\ldots,r} \|s\|_{\Gamma^k(\overline{U}_p; E_t)}.$$

These norms induce the usual Banach space structure on the spaces $\Gamma^k(E_t)$.

For any of the sets $U_p \subset M$, with $p = 1, \ldots, r$, we will denote by $\Gamma^k(\overline{U}_p; E_t)$ the Banach space of sections of E_t over $\overline{U}_p$ that extend to k-times differentiable sections over some open set containing $\overline{U}_p$. On this space $\|\cdot\|_{\Gamma^k(\overline{U}_p; E_t)}$ defines a Banach norm.

By inspecting the definition of the (local) homomorphisms $\psi_{t,p} \colon E_t|_{\widetilde{U}_p} \to E_0|_{\widetilde{U}_p}$ and the seminorms $\|\cdot\|_{\Gamma^k(\overline{U}_p; E_t)}$ we observe the following. For all $k \in \mathbb{N}$ and $t \in [0, \epsilon)$, if $s \in \Gamma^k(\overline{U}_p; E_t)$ is a section, then

$$\|\psi_{t,p}(s)\|_{\Gamma^k(\overline{U}_p; E_0)} = \|s\|_{\Gamma^k(\overline{U}_p; E_t)}. \tag{5.3}$$

We will use this compatibility between the homomorphisms and seminorms in our proof of Proposition 5.3.1.

Proof of Proposition 5.3.1. Let the adjusted charts $(\widetilde{U}_p, V_p)$, associated homomorphisms $\psi_{t,p} \colon E_t|_{\widetilde{U}_p} \to E_0|_{\widetilde{U}_p}$ and choice of precompact opens $U_p \subset \widetilde{U}_p$ be as above.

Let us denote $\lambda = \liminf_{t \to 0} \lambda_1(\mathcal{J}_t)$. There exists a sequence $(t_u)_{u \in \mathbb{N}} \subset [0, \epsilon)$ such that $t_u \to 0$ as $u \to \infty$ and

$$\lim_{u \to \infty} \lambda_1(\mathcal{J}_{t_u}) = \lambda = \liminf_{t \to 0} \lambda_1(\mathcal{J}_t).$$

It follows from Proposition 5.2.2 that for each $u \in \mathbb{N}$ there exists a smooth eigensection $s_u \in \Gamma^\infty(E_t)$ such that $\mathcal{J}_{t_u} s_u = \lambda_1(\mathcal{J}_{t_u}) \cdot s_u$. We normalise such that $\|s_u\|_{\Gamma^0(E_t)} = 1$ for all $u \in \mathbb{N}$.

For $p = 1, \ldots, r$ we denote $\sigma_{u,p} = \psi_{t_u,p}(s_u|_{\widetilde{U}_p}) \in \Gamma^\infty(\widetilde{U}_p; E_0)$. Our proof will rely on the following two lemmas.

Lemma 5.3.3. *There exists a subsequence $(u_k)_{k\in\mathbb{N}} \subset \mathbb{N}$ such that for each $p = 1, \ldots, r$ the sequence $(\sigma_{u_k,p})_{k\in\mathbb{N}}$ converges in $\Gamma^2(\overline{U}_p; E_0)$ to a limiting section $\sigma_p \in \Gamma^2(\overline{U}_p; E_0)$. At least one of these limiting sections is not the zero section. Moreover, for all $p, q = 1, \ldots, r$ the sections σ_p and σ_q coincide on $\overline{U}_p \cap \overline{U}_q$.*

In the second lemma we consider the operator $\mathcal{J}_0$ restricted to the open sets U_p. Since the Jacobi operators $\mathcal{J}_t$ are ordinary differential operators, it follows that the value of $\mathcal{J}_t s$ at a point in M depends only on the germ of the section s at that point. Hence, we can apply $\mathcal{J}_t$ also to sections that are not globally defined.

Lemma 5.3.4. *Consider the limiting sections $\sigma_p \in \Gamma^2(\overline{U}_p; E_0)$ as defined in Lemma 5.3.3. For all $p = 1, \ldots, r$ we have on U_p that*

$$\mathcal{J}_0 \sigma_p = \lambda \cdot \sigma_p.$$

We postpone the proof of these two lemmas and first finish proof of Proposition 5.3.1.

It follows from the last statement of Lemma 5.3.3 that we can patch the limiting sections σ_p together to obtain a well-defined global limiting section $\sigma \in \Gamma^2(E_0)$. More precisely, we let $\sigma \in \Gamma^2(E_0)$ be the section that on each $\overline{U}_p \subset M$ is given by $\sigma|_{\overline{U}_p} = \sigma_p$. Note that the sets $\overline{U}_p$ cover M and that by Lemma 5.3.3 the section is well-defined on intersections $\overline{U}_p \cap \overline{U}_q$. Because at least one of the limiting sections σ_p does not vanish, it follows that σ is not the zero section.

Now Lemma 5.3.4 implies that σ is an eigensection of $\mathcal{J}_0$. Namely, we have

$$\mathcal{J}_0 \sigma = \lambda \cdot \sigma$$

because this holds on each subset $U_p \subset M$. It follows that λ is an eigenvalue of $\mathcal{J}_0$ and hence that

$$\lambda_1(\mathcal{J}_0) \le \lambda = \liminf_{t\to 0} \lambda_1(\mathcal{J}_t).$$

□

We now prove Lemma 5.3.3 and Lemma 5.3.4. The proofs of these lemmas will rely on the fact that, in suitably chosen local coordinates, the coefficients of the differential operators $\mathcal{J}_t$ depend continuously on t.

Let us first introduce the necessary notation. Let $(\widetilde{U}_p, V_p)$ be a pair of adapted charts as before, $(x^i)_{i=1}^m$ the coordinates on $\widetilde{U}_p$ and $(y^\alpha)_{\alpha=1}^n$ the coordinates on V_p. We put again $E_\alpha = F^* \frac{\partial}{\partial y^\alpha}$. The Jacobi operators $\mathcal{J}_t$ are second order differential operators. Hence, in local coordinates they can be written as

$$\mathcal{J}_t s(x) = \left\{ A^{ij,\gamma}_\alpha(x,t) \frac{\partial^2 s^\alpha}{\partial x^i x^j}(x) + B^{i,\gamma}_\alpha(x,t)\frac{\partial s^\alpha}{\partial x^i}(x) + C^\gamma_\alpha(x,t) s^\alpha(x) \right\} E_\gamma(x,t), \tag{5.4}$$

where $A^{ij,\gamma}_\alpha, B^{i,\gamma}_\alpha, C^\gamma_\alpha \colon \widetilde{U}_p \times [0,\epsilon) \to \mathbb{R}$ are suitable coefficient functions. Here we write any section s of E_t over $\widetilde{U}_p$ as $s = s^\alpha E_\alpha(\cdot, t)$.

Our proofs of Lemma 5.3.3 and Lemma 5.3.4 are based on the following observation.

Lemma 5.3.5. *Let $U' \subset \widetilde{U}_p$ be a precompact open subset. For all $i,j = 1,\dots,m$ and $\alpha,\gamma = 1,\dots,n$, we have that the maps $t \mapsto A^{ij,\gamma}_\alpha(\cdot,t), t \mapsto B^{i,\gamma}_\alpha(\cdot,t)$ and $t \mapsto C^\gamma_\alpha(\cdot,t)$ are continuous mappings from $[0,1]$ into $C^1(\overline{U'})$.*

Proof. Denote by g^{ij} the coefficients of the inverse of the metric tensor g with respect to the coordinates $(x^i)_{i=1}^m$ and by ${}^M\Gamma^k_{ij}$ the Christoffel symbols of the Levi-Civita connection of (M,g). The Jacobi operators are expressed locally as

$$\begin{aligned}\mathcal{J}_t s &= \Delta s - \operatorname{tr}_g R^N(s, df\cdot)df\cdot \\ &= -g^{ij}\left\{ \nabla_{\frac{\partial}{\partial x^i}}\nabla_{\frac{\partial}{\partial x^j}} s - {}^M\Gamma^k_{ij}\nabla_{\frac{\partial}{\partial x^k}} s + R^N\left(s, \frac{\partial f}{\partial x^i}\right)\frac{\partial f}{\partial x^j}\right\},\end{aligned}$$

with $s \in \Gamma^2(\overline{U}_p; E_t)$. Recall that ∇ is the pullback connection on the bundle $E_t = f_t^* TN$. Let us denote by ${}^N\Gamma^\gamma_{\alpha\beta}$ the Christoffel symbols of the Levi-Civita connection of (N,h) on the chart V_p. Then, for any $s = s^\alpha E_\alpha(\cdot,t) \in \Gamma^1(\widetilde{U}_p; E_t)$, we can write the pullback connection as

$$\nabla_{\frac{\partial}{\partial x^i}} s(x) = \frac{\partial s^\alpha}{\partial x^i}(x) E_\alpha(x,t) + s^\alpha(x)\frac{\partial f_t^\beta}{\partial x^i}(x)\cdot {}^N\Gamma^\gamma_{\alpha\beta}(f_t(x)) \cdot E_\gamma(x,t).$$

The coefficient functions $A^{ij,\gamma}_\alpha, B^{i,\gamma}_\alpha, C^\gamma_\alpha$ can be calculated by filling in this expression for the connection ∇ into the local expression for the Jacobi operators. It follows that these functions can be expressed entirely in terms of the quantities

$$g^{ij}, \frac{\partial f_t^\beta}{\partial x^i}, {}^M\Gamma^k_{ij}, (R^N)^\delta_{\alpha\beta\gamma}\circ f_t \text{ and } {}^N\Gamma^\gamma_{\alpha\beta}\circ f_t$$

and their first derivatives. Here $(R^N)^\delta_{\alpha\beta\gamma}$ denote the coefficients of the Riemann curvature tensor R^N in the coordinates on V_p. As a result, in the expression for the coefficient functions only spatial derivatives of the functions f_t up to second order appear. The statement of the lemma now follows immediately from our assumption that $[0,1] \to C^3(M,N) \colon t \mapsto f_t$ is a continuous mapping. $\square$

We can now prove Lemma 5.3.3.

Proof of Lemma 5.3.3. Fix a $p \in \{1, \ldots, r\}$. Let us write $s_u = s_u^\alpha E_\alpha(\cdot, t)$ on $\widetilde{U}_p$. Because each s_u is an eigensection of the Jacobi operator $\mathcal{J}_{t_u}$, we find that they satisfy

$$[\mathcal{J}_{t_u} - \lambda_1(\mathcal{J}_{t_u})]\, s_u = 0. \tag{5.5}$$

Hence, on $\widetilde{U}_p$ the coefficients $(s_u^\alpha)_{\alpha=1}^n$ satisfy a second order linear elliptic system of differential equations. We will use Schauder estimates to obtain a uniform bound on the $C^{2,\mu}$-Hölder norm of these coefficients. To this end we will apply the results of [Mor54].

The system of differential equations in Equation (5.5) is elliptic because the Jacobi operators are elliptic differential operators. The bounds on the Hölder norms of solutions to this equation that are provided by Morrey's results depend on a uniform ellipticity constant which in Morrey's paper is denoted M (defined in [Mor54, Equation 1.6]). This constant depends only on the coefficients of the second order part of the system in Equation (5.5). That is, it depends only on the coefficients $A_\alpha^{ij,\gamma}$. Because, by Lemma 5.3.5, these coefficient functions depend continuously on t, it follows that the constant M can be taken uniformly over $u \in \mathbb{N}$.

Take a precompact open $U' \subset \widetilde{U}_p$ such that $\overline{U}_p \subset U' \subset \overline{U'} \subset \widetilde{U}_p$. The coefficients of the system of differential equations in Equation (5.5) are a combination of the coefficients of $\mathcal{J}_{t_u}$ and the constant term $\lambda_1(\mathcal{J}_{t_u})$. It follows from Lemma 5.3.5 that the $C^{0,\mu}$-Hölder norms (even C^1 norms) of the coefficients of $\mathcal{J}_{t_u}$ can be bounded uniformly in u. The constant term $\lambda_1(\mathcal{J}_{t_u})$ can also be bounded uniformly in u, since the sequence $(\lambda_1(\mathcal{J}_{t_u}))_{u\in\mathbb{N}}$ is convergent. So the coefficients of the system of differential equations in Equation (5.5) have uniformly (in u) bounded $C^{0,\mu}$-Hölder norms. Moreover, because we normalised the sections s_u such that $\|s_u\|_{\Gamma^0(E_t)} = 1$, it follows that the C^0 norm (and hence also the L^2 norm) of the coefficients s_u^α is also bounded uniformly in u. We now apply [Mor54, Theorem 4.7] (with $G = U', G_1 = U_p$, in the notation of that paper) to conclude that on $\overline{U}_p$ the $C^{2,\mu}$-Hölder norms of the coefficients s_u^α are uniformly bounded in u.

We recall the notation $\sigma_{u,p} = \psi_{t_u,p}(s_u|_{\widetilde{U}_p})$. It follows from the definition of the homomorphisms $\psi_{t,p}$ that s_u and $\sigma_{u,p}$ have the same coefficients on $\widetilde{U}_p$. Namely, if we write $\sigma_{u,p} = \sigma_{u,p}^\alpha E_\alpha(\cdot, 0)$, then $s_u^\alpha = \sigma_{u,p}^\alpha$ for $\alpha = 1, \ldots, n$. Hence, also the $C^{2,\mu}$-Hölder norms of the coefficients $\sigma_{u,p}^\alpha$ are uniformly bounded. It now follows from the Arzelà-Ascoli theorem that there exists a subsequence of $(\sigma_{u,p})_{u\in\mathbb{N}}$ that converges in $\Gamma^2(\overline{U}_p; E_0)$ to a limiting section. We denote this limiting section by σ_p. By choosing subsequent refinements of the subsequence we can arrange for this to hold for each $p = 1, \ldots, r$. We denote the indices of this subsequence by $(u_k)_{k\in\mathbb{N}} \subset \mathbb{N}$.

We now prove that is it not possible that all limiting sections σ_p vanish identically. If this was the case, and all sections σ_p vanish, then this would imply $\|\sigma_{u_k,p}\|_{\Gamma^0(\overline{U}_p;E_0)} \to 0$ as $k \to \infty$ for all $p = 1, \ldots, r$. However, this contracts that for all $u \in \mathbb{N}$ we have, by Equation (5.3), that

$$\max_{p=1,\ldots,r} \|\sigma_{u,p}\|_{\Gamma^0(\overline{U}_p;E_0)} = \max_{p=1,\ldots,r} \|s_u\|_{\Gamma^0(\overline{U}_p;E_t)} = \|s_u\|_{\Gamma^0(E_t)} = 1.$$

Finally, we prove the last statement of the lemma. Let $(\widetilde{U}_p, V_p)$ and $(\widetilde{U}_q, V_q)$ be two pairs of adapted charts with corresponding local homomorphisms $\psi_{t,p}$ and $\psi_{t,q}$. Recall that the maps $\psi_{t,p}\colon E_t|_{\widetilde{U}_p} \to E_0|_{\widetilde{U}_p}$ are isomorphisms for t small enough. It can be easily seen from the definition of these homomorphisms that, on the compact set $\overline{U}_p \cap \overline{U}_q$, the maps

$$\psi_{t,q} \circ \psi_{t,p}^{-1}\colon E_0|_{\overline{U}_p \cap \overline{U}_q} \to E_0|_{\overline{U}_p \cap \overline{U}_q}$$

converge uniformly to the identity map as $t \to 0$. It follows that

$$\begin{aligned}
\sigma_p|_{\overline{U}_p \cap \overline{U}_q} &= \lim_{k\to\infty} \psi_{t_{u_k},p}(s_{u_k}|_{\overline{U}_p \cap \overline{U}_q}) \\
&= \lim_{k\to\infty} \psi_{t_{u_k},q} \circ \psi_{t_{u_k},p}^{-1} \circ \psi_{t_{u_k},p}(s_{u_k}|_{\overline{U}_p \cap \overline{U}_q}) \\
&= \lim_{k\to\infty} \psi_{t_{u_k},q}(s_{u_k}|_{\overline{U}_p \cap \overline{U}_q}) \\
&= \sigma_q|_{\overline{U}_p \cap \overline{U}_q},
\end{aligned}$$

where the limits are taken in $\Gamma^0(\overline{U}_p \cap \overline{U}_q; E_0)$. □

We finish this section with the proof of Lemma 5.3.4.

Proof of Lemma 5.3.4. Fix a $p \in \{1, \dots, p\}$. Let $(\widetilde{U}_p, V_p)$ be a pair of adapted charts and let the homomorphisms $\psi_{t,p}$ and the frame $(E_\alpha)_{\alpha=1}^n$ be as before.

We claim that

$$\|\psi_{t,p} \circ \mathcal{J}_t - \mathcal{J}_0 \circ \psi_{t,p}\|_{\mathrm{op}} \to 0 \text{ as } t \to 0. \tag{5.6}$$

Here, $\|\cdot\|_{\mathrm{op}}$ is the operator norm on the space of bounded linear operators from $\Gamma^2(\overline{U}_p; E_t)$ to $\Gamma^0(\overline{U}_p; E_0)$ (equipped with the norms $\|\cdot\|_{\Gamma^2(\overline{U}_p;E_t)}$ and $\|\cdot\|_{\Gamma^0(\overline{U}_p;E_0)}$ respectively).

We denote

$$\begin{aligned}
a_\alpha^{ij,\gamma}(x,t) &= A_\alpha^{ij,\gamma}(x,t) - A_\alpha^{ij,\gamma}(x,0) \\
b_\alpha^{i,\gamma}(x,t) &= B_\alpha^{i,\gamma}(x,t) - B_\alpha^{i,\gamma}(x,0) \\
c_\alpha^{\gamma}(x,t) &= C_\alpha^{\gamma}(x,t) - C_\alpha^{\gamma}(x,0).
\end{aligned}$$

Then, for a section $s = s^\alpha E_\alpha(\cdot, t) \in \Gamma^2(\overline{U}_p; E_t)$, we have

$$\begin{aligned}
&[\psi_{t,p} \circ \mathcal{J}_t - \mathcal{J}_0 \circ \psi_{t,p}]s(x) \\
&\qquad = \left\{ a_\alpha^{ij,\gamma}(x,t) \frac{\partial^2 s^\alpha}{\partial x^i x^j}(x) + b_\alpha^{i,\gamma}(x,t) \frac{\partial s^\alpha}{\partial x^i}(x) + c_\alpha^{\gamma}(x,t) s^\alpha(x) \right\} E_\alpha(x,0).
\end{aligned}$$

From this expression follows that

$$\|\psi_{t,p} \circ \mathcal{J}_t - \mathcal{J}_0 \circ \psi_{t,p}\|_{\mathrm{op}} \le \sum_{i,j,\alpha,\gamma} \|a_\alpha^{ij,\gamma}\|_{C^0(\overline{U}_p)} + \sum_{i,\alpha,\gamma} \|b_\alpha^{i,\gamma}\|_{C^0(\overline{U}_p)} + \sum_{\alpha,\gamma} \|c_\alpha^{\gamma}\|_{C^0(\overline{U}_p)}.$$

Our claim is now immediately implied by the results of Lemma 5.3.5.

We use the notation $(u_k)_{k\in\mathbb{N}}$ and $\sigma_{u,p}$ as in Lemma 5.3.3. From that lemma follows that $\sigma_{u_k,p} \to \sigma_p$ in $\Gamma^2(\overline{U}_p; E_0)$. We use this to find

$$\mathcal{J}_0\sigma_p = \lim_{k\to\infty} \mathcal{J}_0\sigma_{u_k,p} = \lim_{k\to\infty} \mathcal{J}_0\psi_{t_{u_k},p}(s_{u_k}|_{\overline{U}_p}).$$

From Equation (5.6) follows that

$$\mathcal{J}_0\sigma_p = \lim_{k\to\infty} \mathcal{J}_0\psi_{t_{u_k},p}(s_{u_k}|_{\overline{U}_p}) = \lim_{k\to\infty} \psi_{t_{u_k},p}(\mathcal{J}_{t_{u_k}} s_{u_k}|_{\overline{U}_p}).$$

Here we used that $\|s_{u_k}\|_{\Gamma^2(\overline{U}_p;E_t)} = \|\sigma_{u_k}\|_{\Gamma^2(\overline{U}_p;E_0)}$ remains bounded uniformly in k. Finally, using the fact that the sections s_u are eigensections of the operators $\mathcal{J}_{t_u}$ gives

$$\mathcal{J}_0\sigma_p = \lim_{k\to\infty} \psi_{t_{u_k},p}(\mathcal{J}_{t_{u_k}} s_{u_k}|_{\overline{U}_p}) = \lim_{k\to\infty} \lambda(\mathcal{J}_{t_{u_k}}) \cdot \psi_{t_{u_k},p}(s_{u_k}|_{\overline{U}_p}) = \lambda \cdot \sigma_p$$

because, by definition, $\lambda = \lim_{u\to\infty} \lambda_1(\mathcal{J}_{t_u})$. $\square$

5.4 Proof of Theorem 5.1.1

Our proof of Theorem 5.1.1 will rely on the fact that the Jacobi operator of the maps f_t appears in the evolution equation for the quantity $\tau(f_t)$. Recall the notation $E_t = f_t^*TN$.

Lemma 5.4.1. *Assume the family of maps $(f_t)_{t\in[0,\infty)}$ satisfies the harmonic heat flow equation. Then*

$$\frac{1}{2}\frac{d}{dt}\|\tau(f_t)\|^2_{L^2(E_t)} = -\langle \mathcal{J}_{f_t}\tau(f_t), \tau(f_t)\rangle_{L^2(E_t)}.$$

Proof. Assume $(x^i)_{i=1}^m$ are Riemannian normal coordinates around a point $x \in M$. In the following calculation we will consider the expression $\frac{\partial f_t}{\partial x^\alpha}$ as a local section of f_t^*TN. Because we are working in normal coordinates around x, we have that

$$\tau(f_t)|_x = \operatorname{tr}_g \nabla df|_x = \nabla_{\frac{\partial}{\partial x^i}}(\frac{\partial f}{\partial x^i})\Big|_x.$$

We use this to find that at the point x and for any $t \geq 0$ we have

$$\begin{aligned}
\nabla_{\frac{\partial}{\partial t}}\tau(f_t) &= \nabla_{\frac{\partial}{\partial t}}\left(\nabla_{\frac{\partial}{\partial x^i}}\left(\frac{\partial f}{\partial x^i}\right)\right) \\
&= R^N\left(\frac{\partial f}{\partial t}, \frac{\partial f}{\partial x^i}\right)\frac{\partial f}{\partial x^i} + \nabla_{\frac{\partial}{\partial x^i}}\nabla_{\frac{\partial}{\partial x^i}}\left(\frac{\partial f}{\partial t}\right) \\
&= -\Delta\tau(f_t) + \operatorname{tr}_g R^N(\tau(f_t), df\cdot)df\cdot = -\mathcal{J}_{f_t}\tau(f_t).
\end{aligned}$$

To get the second equality we used that $\nabla_{\frac{\partial}{\partial t}}\frac{\partial f}{\partial x^i} = \nabla_{\frac{\partial}{\partial x^i}}\frac{\partial f}{\partial t}$ (see [EL83, p.5]). Because $x \in M$ was arbitrary, we conclude that his equality holds everywhere. We use this to find that

$$\frac{1}{2}\frac{d}{dt}\|\tau(f_t)\|^2_{L^2(E_t)} = \langle\nabla_{\frac{\partial}{\partial t}}\tau(f_t), \tau(f_t)\rangle_{L^2(E_t)} = -\langle\mathcal{J}_{f_t}\tau(f_t), \tau(f_t)\rangle_{L^2(E_t)}.$$

$\square$

We can now give a proof of Theorem 5.1.1.

Proof of Theorem 5.1.1. We apply Proposition 5.3.1 to the family of maps $(f_t)_{t\in[0,\infty]}$. For this we pick some homeomorphism between $[0,\infty]$ and $[0,1]$ (mapping ∞ to 0) so we can view the heat flow as a family of maps $(f_t)_{t\in[0,1]}$ indexed by $t \in [0,1]$. It then follows from Theorem 5.2.3 that this family of maps satisfies the assumptions of Proposition 5.3.1. From this proposition follows that

$$\liminf_{t\to\infty} \lambda_1(\mathcal{J}_{f_t}) \geq \lambda_1(\mathcal{J}_{f_\infty}).$$

By assumption f_∞ is a non-degenerate critical point of the energy so $\lambda_1(\mathcal{J}_{f_\infty}) > 0$. Put $b = \lambda_1(\mathcal{J}_{f_\infty})/2 > 0$. Then, for $t \geq t_0$ large enough we have $\lambda_1(\mathcal{J}_{f_t}) \geq b$. Using Lemma 5.4.1 and Equation (5.2) we see that for such $t \geq t_0$,

$$\frac{d}{dt}\|\tau(f_t)\|^2_{L^2(E_t)} = -2\langle \mathcal{J}_{f_t}\tau(f_t), \tau(f_t)\rangle_{L^2(E_t)} \leq -2b \cdot \|\tau(f_t)\|^2_{L^2(E_t)}.$$

Grönwalls's inequality ([Gro19]) yields that

$$\|\tau(f_t)\|^2_{L^2(E_t)} \leq \|\tau(f_{t_0})\|^2_{L^2(E_t)} \cdot e^{-2b\cdot t}$$

for $t \geq t_0$. So if we pick $a > 0$ large enough, then

$$\left\|\frac{df_t}{dt}\right\|_{L^2(E_t)} = \|\tau(f_t)\|_{L^2(E_t)} \leq a \cdot e^{-b\cdot t}$$

for all $t \geq 0$. □

We end with the proof of Corollary 5.1.2.

Proof of Corollary 5.1.2. The evolution of the energy $E(f_t)$ along the harmonic heat flow is governed by the equation

$$\frac{d}{dt}E(f_t) = -\int_M \|\tau(f_t)\|^2 \operatorname{vol}_g = -\left\|\frac{df_t}{dt}\right\|^2_{L^2(E_t)}$$

(see [ES64, §6.C]). Applying the estimate of Theorem 5.1.1 gives

$$|E(f_t) - E(f_\infty)| = \int_t^\infty \left\|\frac{df_t}{dt}\right\|^2_{L^2(E_t)} dt \leq a \cdot \int_t^\infty e^{-2b\cdot t} dt \leq a' \cdot e^{-2b\cdot t}$$

with $a' = a/(2b)$. □

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Chapter 6

Equivariant barycentric maps for Hitchin representations

Abstract

We use the barycentric method as introduced by Besson, Courtois and Gallot to construct natural maps that are equivariant for Hitchin representations into $\mathrm{SL}(n, \mathbb{R})$. We prove that these maps are smooth and depend smoothly on the representation. As an application we obtain a novel parametrisation of the Hitchin component by assigning to each representation the corresponding barycentric map.

6.1 Introduction

Barycentric maps (also called natural maps) were introduced by Besson, Courtois and Gallot in [BCG95] where they built upon the work of Douady and Earle in [DE86]. These maps provide a natural way to extend a map between the boundaries of two symmetric spaces to a map between the symmetric spaces themselves. An attractive feature of barycentric maps is that they are (coarsely) area minimising in the sense that often explicit uniform bounds on their Jacobian can be found. This allowed Besson, Courtois and Gallot to use these maps in the study of the problem of minimal volume entropy rigidity (see [BCG96]). Other applications include [CF03] and [LS06].

Let $\Gamma = \pi_1(S)$ be the fundamental group of a closed surface of genus at least two. In this paper we consider Hitchin representations of Γ into $\mathrm{SL}(n, \mathbb{R})$. Such representations come equipped with a naturally associated boundary map (see Section 6.4). This raises the question whether the barycenter method can be applied to construct natural maps that are equivariant for Hitchin representations. In higher rank symmetric spaces the existence of well-defined barycentric maps is not a priori guaranteed because of the presence of flat subspaces (cf. Remark 6.5.4). Nevertheless, we will use the strong transversality properties (hyperconvexity) of the boundary maps of Hitchin representations to show that in this case the barycentric construction can be carried out.

Theorem (Theorem 6.5.2)**.** *Let $\theta\colon \Gamma \to \mathrm{SL}(2, \mathbb{R})$ be a Fuchsian representation and $\rho\colon \Gamma \to \mathrm{SL}(n, \mathbb{R})$ be a Hitchin representation. There exists a natural map $f_{\theta,\rho}\colon \mathbb{H}^2 \to \mathrm{SL}(n, \mathbb{R})/\mathrm{SO}(n)$ that intertwines the actions of θ and ρ.*

We put these equivariant natural maps forward as a geometric way to study Hitchin components. The Hitchin components, denoted Hit_n, are special connected components of the representation varieties $\mathrm{Rep}(\Gamma, \mathrm{SL}(n, \mathbb{R}))$ (see Section 6.4). Analytic methods for studying the Hitchin components are provided by the Non-Abelian Hodge correspondence (see, for example, [Li19]). The Non-Abelian Hodge correspondence provides a topological identification between the Hitchin component and a vector space of Higgs fields over a Riemann surface. An intermediate step in this identification is provided by equivariant harmonic maps (see [Li19]). These maps are minimisers of the Dirichlet energy functional and can be exhibited as solutions to a certain PDE equation. This makes harmonic maps and Higgs bundles amenable to study via analytic methods. However, studying the geometric aspects of these maps has proven to be much harder.

A geometric approach to studying Hitchin representations was initiated by Labourie in [Lab06]. The natural maps constructed here fit well into the framework introduced in that paper. In particular, they have a much more direct relation to the geometry of the Hitchin representations.

We will use the natural maps to provide a novel parametrisation of the Hitchin component. To this end we consider $\mathcal{C}$, the space of continuous maps $\mathbb{H}^2 \to X$ up to composition with an isometry of X. The compact-open topology on $C^0(\mathbb{H}^2, X)$ induces a topology on $\mathcal{C}$. The following result provides, for any fixed Fuchsian representation $\theta \colon \Gamma \to \mathrm{SL}(2,\mathbb{R})$, a parametrisation of Hit_n as a subset of $\mathcal{C}$.

Theorem (Theorem 6.9.1)**.** *The map* $\mathrm{Hit}_n \to \mathcal{C} \colon [\rho] \mapsto [f_{\theta,\rho}]$ *is a topological embedding.*

This construction is natural in the sense that if ρ is a composition of θ with the irreducible embedding $\mathrm{SL}(2,\mathbb{R}) \subset \mathrm{SL}(n,\mathbb{R})$, then $f_{\theta,\rho}$ is simply the isometric embedding of $\mathbb{H}^2$ into X (see Remark 6.5.3).

We compare this approach with the parametrisation of the Hitchin components by equivariant harmonic maps. For this we fix a complex structure on S (which is equivalent to a fixing a Fuchsian representation as above). If $\mathcal{H}$ is the space of harmonic maps from $\widetilde{S}$ to X (modulo isometries of X), then the map assigning to each representation the associated equivariant harmonic map $\widetilde{S} \to X$ is an homeomorphism between Hit_n and $\mathcal{H}$ (see [Li19, Section 2.2.6]). Both approaches assign to each representation an optimal map. Namely, a harmonic map minimises the energy functional, whereas barycentric maps should be thought of as (course) minimisers of the area functional.

A feature which sets barycentric maps apart from harmonic maps is that they have relatively explicit expressions. This makes it possible to derive properties of these maps by explicit calculation (as is evidenced by the calculation of the Jacobian in [BCG96] and [CF03]). We expect that this makes our parametrisation amenable to a more direct study in future work.

We note that both parametrisations depend on a choice of basepoint in Teichmüller space. The author proved in [Sle20] that equivariant harmonic maps depend in a real analytic fashion on the representation and the choice of complex structure on S. It follows that the parametrisation of the Hitchin component by harmonic maps depends real analytically on the choice of basepoint.

We prove an analogous result for barycentric maps.

Theorem (Theorem 6.8.1)**.** *The natural map* $f_{\theta,\rho} \colon \mathbb{H}^2 \to \mathrm{SL}(n,\mathbb{R})/\,\mathrm{SO}(n)$ *is* *smooth and depends smoothly on θ and ρ.*

As a result we see that the parametrisation of the Hitchin component provided by Theorem 6.9.1 depends smoothly on the chosen Fuchsian representation in the sense that the pointwise values of the barycentric maps and their derivatives depend smoothly on the Fuchsian representation θ.

6.2 The symmetric space $\mathrm{SL}(n,\mathbb{R})/\mathrm{SO}(n)$.

We introduce some notation and definitions regarding the Lie group $\mathrm{SL}(n,\mathbb{R})$ and the associated symmetric space. A reference for all material discussed in this section is [Ebe96] (see also [BH99]).

Throughout this paper we let $G = \mathrm{SL}(n,\mathbb{R})$. We denote $K = \mathrm{SO}(n)$ and $B \subset \mathrm{SL}(n,\mathbb{R})$ the Borel subgroup given by upper diagonal matrices. Let $X = G/K$ be the associated symmetric space. We denote $\mathfrak{g} = \mathfrak{sl}(n,\mathbb{R})$ and $\mathfrak{p} \subset \mathfrak{g}$ the subspace of symmetric matrices of trace zero. Throughout this paper we will identify $\mathfrak{p}$ with the space $T_{eK}X$. We chose $\mathfrak{a} \subset \mathfrak{p}$ to be the maximal Abelian subalgebra consisting of diagonal matrices of trace zero. We pick a positive Weyl chamber by setting

$$\mathfrak{a}^+ = \{\mathrm{diag}(\lambda_1, \ldots, \lambda_n) \mid \lambda_1 + \cdots + \lambda_n = 0 \text{ and } \lambda_1 > \ldots > \lambda_n\}.$$

The set of roots on $\mathfrak{a}$ is $\Sigma = \{\alpha_{ij}\}_{i \neq j}$ with $\alpha_{ij}(H) = H_{ii} - H_{jj}$ and the set of positive roots determined by our choice of $\mathfrak{a}^+$ is $\Sigma^+ = \{\alpha_{ij}\}_{i<j}$. The Killing form on $\mathfrak{g}$ is given by $B(X,Y) = 2n\,\mathrm{tr}(XY)$. We let $\langle \cdot, \cdot \rangle = B(\cdot, \theta \cdot)$ be the associated inner product (where θ is the Cartan involution of $\mathfrak{g}$ that fixes $\mathfrak{p}$).

The Weyl group is the group of isometries of $\mathfrak{a}$ that is generated by the orthogonal reflections through the subspaces $\ker \alpha$, $\alpha \in \Sigma$. It can be realised as $N_G(\alpha)/Z_G(\alpha)$ where $N_G(\alpha)$ and $Z_G(\alpha)$ are the normaliser and centraliser of $\mathfrak{a}$ in G respectively. The Weyl group acts on $\mathfrak{a}$ as the symmetric group (of n elements) by permutations on the entries of the diagonal matrices.

The barycenter of the Weyl chamber $\mathfrak{a}^+$ is defined by

$$b' = \sum_{\alpha \in \Sigma^+} H_\alpha \in \mathfrak{a}^+$$

where H_α is the dual vector of α with respect to $\langle \cdot, \cdot \rangle$, i.e. H_α satisfies $\alpha = \langle H_\alpha, \cdot \rangle$.

Remark 6.2.1. In general, for other Lie groups, the barycenter of the Weyl chamber is defined as

$$b' = \sum_{\alpha \in \Sigma^+} m_\alpha H_\alpha$$

where m_α is the dimension of the root space of α. For $\mathrm{SL}(n,\mathbb{R})$ we have $m_\alpha = 1$ for all $\alpha \in \Sigma$.

Lemma 6.2.2. *The barycenter $b' \in \mathfrak{a}^+$ is given by*

$$b' = \frac{1}{2n}\,\mathrm{diag}(n-1, n-3, \ldots, 3-n, 1-n).$$

Proof. We note that

$$H_{\alpha_{ij}} = \frac{1}{2n}\,\mathrm{diag}(0,\dots,0,\underbrace{1}_{\text{i-th place}},0,\dots,0,\underbrace{-1}_{\text{j-th place}},0,\dots,0).$$

Then if $b' = \sum_{i<j} H_{\alpha_{ij}}$ we see that $b'_{kk} = \frac{1}{2n}\left[(n-k)\cdot 1 + (k-1)\cdot(-1)\right] = \frac{1}{2n}(n-2k+1)$ for each $k = 1,\dots,n$. □

Since at no point the exact magnitude of b' will play a role in our arguments, we will, for convenience, work with the scaled barycenter

$$b = 2n\cdot b' = \mathrm{diag}(n-1, n-3, \dots, 3-n, 1-n).$$

We will refer to b also as the barycenter of the Weyl chamber.

6.2.1 The boundary at infinity

Two geodesic rays $\gamma_1, \gamma_2\colon [0,\infty) \to X$ are called asymptotic if $d(\gamma_1(t), \gamma_2(t))$ is bounded uniformly in t. An equivalence class of asymptotic geodesics is called a point at infinity and $\partial_\infty X$, the boundary at infinity of X, is the set of points at infinity. If $\gamma\colon [0,\infty) \to X$ is a geodesic ray, then we denote by $\gamma(\infty)$ its endpoint, i.e. the point at infinity it determines. For any tangent vector $V \in TX$ we let $V(\infty)$ be the endpoint at infinity of the geodesic ray with starting velocity V. If SX denotes the unit tangent bundle of X, then the map $E_x\colon S_xX \to \partial_\infty X\colon V \mapsto V(\infty)$ is a bijection for any $x \in X$. Using this identification $\partial_\infty X$ can be equipped with a smooth structure which is independent of the choice of x. As a manifold $\partial_\infty X$ is diffeomorphic to $S^{\dim X - 1}$.

The boundary at infinity of $\mathfrak{a}^+$ is the set $\partial_\infty \mathfrak{a}^+ = \{V(\infty) \mid V \in \mathfrak{a}^+ \subset T_{eK}X\}$. A Weyl chamber in $\partial_\infty X$ is any translate of $\partial_\infty \mathfrak{a}^+$ by an element of G. The Furstenberg boundary of X, which we will denote by $\partial_F X$, is the set of Weyl chambers in $\partial_\infty X$. The stabiliser of $\partial_\infty \mathfrak{a}^+$ for the action of G is the Borel subgroup B hence $\partial_F X = G/B$.

The barycenter $b \in \mathfrak{a}^+$ determines a point at infinity $b(\infty)$ which we call the barycenter of the Weyl chamber $\partial_\infty \mathfrak{a}^+$. Like the Weyl chamber $\partial_\infty \mathfrak{a}^+$ its stabiliser in G equals B. This allows us to identify the Furstenberg boundary with a subset of $\partial_\infty X$ via the orbit map of $b(\infty)$, i.e. $\partial_F X \cong G\cdot b(\infty) \subset \partial_\infty X$. From now on we will always consider $\partial_F X$ to be realised as subset of $\partial_\infty X$ in this way.

Remark 6.2.3. Let the group $\mathrm{SL}(2,\mathbb{R})$ act on $\mathbb{R}^2$ via the canonical action and consider the induced action on the symmetric product $S^{n-1}\mathbb{R}^2 \cong \mathbb{R}^n$ given by $g\cdot(v_1\cdots v_{n-1}) = (gv_1)\cdots(gv_{n-1})$. In this way we obtain a irreducible embedding of $\mathrm{SL}(2,\mathbb{R})$ into $\mathrm{SL}(n,\mathbb{R})$. The induced action of an element $X \in \mathfrak{sl}(2,\mathbb{R})$ on $\mathbb{R}^n$ is given by

$$X\cdot(v_1\cdots v_{n-1}) = (Xv_1)v_2\cdots v_{n-1} + v_1(Xv_2)\cdots v_{n-1} + \cdots + v_1\cdots v_{n-2}(Xv_{n-1}).$$

If (e_1, e_2) is the standard basis of $\mathbb{R}^2$, then

$$E_i = \underbrace{e_1 \cdots e_1}_{n-i \text{ times}} \cdot \underbrace{e_2 \cdots e_2}_{i-1 \text{ times}} \text{ for } i = 1, \ldots, n$$

is a basis for $S^{n-1}\mathbb{R}^2 \cong \mathbb{R}^n$. Consider now the element

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \in \mathfrak{sl}(2, \mathbb{R}).$$

To determine the image of H under the induced embedding $\mathfrak{sl}(2,\mathbb{R}) \subset \mathfrak{sl}(n,\mathbb{R})$ we calculate

$$HE_i = (n-i)(He_1)e_1 \cdots e_1 \cdot e_2 \cdots e_2 + (i-1) \cdot e_1 \cdots e_1 \cdot (He_2) \cdots e_2 = (n-2i+1)E_i.$$

We conclude that the image of H in $\mathfrak{sl}(n,\mathbb{R})$ is precisely the barycenter b.

It follows that if we consider the copy of $\mathbb{H}^2$ in X that is determined by the irreducible embedding $\mathrm{SL}(2,\mathbb{R}) \subset \mathrm{SL}(n,\mathbb{R})$, then the boundary at infinity $\partial_\infty \mathbb{H}^2$ is contained in $\partial_F X = G \cdot b(\infty)$.

A flag $F = \{F^1 \subsetneq F^2 \subsetneq \ldots \subsetneq F^k\}$ is a collection of nested subspaces of $\mathbb{R}^n$. A full flag is a flag F with $k = n$ and $\dim F^{i+1} = \dim F^i + 1$. We denote the space of full flags by $\mathrm{Flag}(\mathbb{R}^n)$. Sometimes it will be convenient to denote $F^0 = \{0\}$ and $F^n = \mathbb{R}^n$.

The boundary at infinity of X can be identified with a space of flags in the following way. To each $\xi \in \partial_\infty X$ corresponds a unique $X \in \mathfrak{p}$, $\|X\| = 1$ such that ξ is the endpoint of the geodesic ray $t \mapsto e^{tX} \cdot \mathrm{SO}(n)$. Since X is a symmetric matrix, it can be diagonalised and hence has eigenvalues $\lambda_1 > \ldots > \lambda_k$ and corresponding eigenspaces $V_1, \ldots, V_k \subset \mathbb{R}^n$ that span $\mathbb{R}^n$ $(2 \leq k \leq n)$. We denote $\lambda(\xi) = (\lambda_1, \ldots, \lambda_k)$ and define a flag $F(\xi)$ by setting $F^i(\xi) = V_1 \oplus \cdots \oplus V_i$. The pair $(F(\xi), \lambda(\xi))$ satisfies

1. $\lambda_1 > \ldots > \lambda_k$,
2. $\sum_{i=1}^k m_i \lambda_i = 0$ (since $\operatorname{tr} X = 0$) and
3. $\sum_{i=1}^k m_i \lambda_i^2 = 1$ (since $\|X\|^2 = 1$)

with $m_i = \dim F^i(\xi) - \dim F^{i-1}(\xi)$. The assignment $\xi \mapsto (F(\xi), \lambda(\xi))$ is a bijection between $\partial_\infty X$ and the set of pairs (F, λ) satisfying the above conditions. Under this identification the action of G on $\partial_\infty X$ corresponds to the usual action of G on the space of flats, i.e. $g \cdot F = (g(F^1), \ldots, g(F^k))$.

The stabiliser for the action of G on any full flag is a conjugate of the Borel subgroup B so $\mathrm{Flag}(\mathbb{R}^n) \cong G/B$. This means we can identify $\mathrm{Flag}(\mathbb{R}^n)$ with the Furstenberg boundary $\partial_F X$. The inclusion $\partial_F X = G \cdot b(\infty) \subset \partial_\infty X$ corresponds to the inclusion map $F \mapsto (F, \lambda_b)$, where

$$\lambda_b = \left(\sum_{i=1}^n (n - 2i - 1)^2 \right)^{-1/2} \cdot (n-1, n-3, \ldots, 3-n, 1-n).$$

Two points $\xi_1 = (F_1, \lambda_b), \xi_2 = (F_2, \lambda_b) \in \partial_F X$ in the Furstenberg boundary can be connected by a geodesic if and only if the flags F_1 and F_2 are transverse, i.e. $F_1^i \oplus F_2^{n-i} = \mathbb{R}^n$ for $i = 1, \ldots, n$.

The closure of a Weyl chamber Δ of $\partial_\infty X$ is a simplex that has $n-1$ vertices. If Δ corresponds to the full flag $F = (F^1, \ldots, F^n)$, then its vertices correspond to the points (V_i, λ_i) for $i = 1, \ldots, n-1$ where $V_i = \{F^i\}$ is a partial flag and the λ_i are given by

$$\lambda_i = \left(i \cdot (n-i)^2 + (n-i) \cdot i^2\right)^{-1/2} \cdot \Big(\underbrace{n-i, \ldots, n-i}_{i \text{ times}}, \underbrace{-i, \ldots, -i}_{n-i \text{ times}}\Big).$$

We will call any point that is the vertex of a Weyl chamber simply a vertex in $\partial_\infty X$. It follows from the above that the set of vertices in $\partial_\infty X$ is in one-to-one correspondence with the set of proper subspaces of $\mathbb{R}^n$, i.e. the disjoint union of Grassmanians $\sqcup_{i=1}^{n-1} \mathrm{Gr}_i(\mathbb{R}^n)$.

6.3 Barycenters of measures at infinity

Fix a basepoint $x_0 \in X$. For $\xi \in \partial_\infty X$ and $x \in X$ we define the Busemann function $B_\xi \colon X \to \mathbb{R}$ as

$$B_\xi(x) = \lim_{t \to \infty} d(x, \gamma(t)) - t$$

where γ is the unique geodesic ray with $\gamma(0) = x_0$ and $\gamma(\infty) = \xi$. It is proved in [BGS85, Section I.3.3] that the Busemann functions are well-defined functions that are convex and Lipschitz with respect to the metric on X. Furthermore, they are smooth functions on X.

Definition 6.3.1. *Let μ be a probability measure on $\partial_\infty X$. We define the weighted Busemann function as*

$$B_\mu \colon X \to \mathbb{R}$$
$$B_\mu(x) = \int_{\partial_\infty X} B_\eta(x) d\mu(\eta).$$

A smooth function $f \colon X \to \mathbb{R}$ is called (strictly) convex if for any geodesic $\gamma \colon \mathbb{R} \to X$ the function $f \circ \gamma \colon \mathbb{R} \to \mathbb{R}$ is (strictly) convex. This is equivalent to the condition that the Hessian ∇df is positive everywhere on X. If a function f is bounded from below and proper, then it attains a global minimum. Moreover, if it is also strictly convex, then this minimum is unique.

Definition 6.3.2. *Let μ be a probability measure on $\partial_\infty X$. If the weighted Busemann function B_μ is bounded from below, proper and strictly convex, then we define the barycenter of μ in X, notation $\mathrm{bar}(\mu) \in X$, to be the unique minimiser of B_μ.*

To answer the question which probability measures yield proper and strictly convex weighted Busemann functions we follows a part of the discussion found in [KLM09, paragraph 3].

Definition 6.3.3. *Let $f\colon X \to \mathbb{R}$ be a convex and Lipschitz function. For a point $\xi \in \partial_\infty X$ we define the asymptotic slope of f at ξ to be*

$$\mathrm{slope}_f(\xi) = \lim_{t\to\infty} \frac{f(\gamma(t))}{t},$$

where γ is any geodesic ray with $\gamma(\infty) = \xi$.

It follows from the convexity of f that the expression $f(\gamma(t))/t$ is non-decreasing in t. From the Lipschitz continuity follows that it is also bounded and hence the limit in the definition exists. It is routine to check it doesn't depend on the choice of geodesic ray.

If $x \in X$ and $\xi, \eta \in \partial_\infty X$ we define $\angle_x(\xi,\eta) = \angle(\dot{\gamma_\xi}(0), \dot{\gamma_\eta}(0))$ where γ_ξ and γ_η are geodesics with $\gamma_\xi(0) = \gamma_\eta(0) = x$ and $\gamma_\xi(\infty) = \xi$ and $\gamma_\eta(\infty) = \eta$. The Tits angle between ξ and η is defined as

$$\angle_\infty(\xi,\eta) = \sup_{x\in X} \angle_x(\xi,\eta).$$

The function $\angle_\infty(\cdot,\cdot)$ is a metric on $\partial_\infty X$. See [BGS85, section I.4.1] for more details.

Lemma 6.3.4. *Let $\xi, \eta \in \partial_\infty X$, then*

$$\mathrm{slope}_{B_\eta}(\xi) = -\cos\angle_\infty(\xi,\eta).$$

Proof. At a point $x \in X$ we have $\nabla B_\eta = -\dot{\gamma}_\eta$ (with γ_η as above). Hence, if ρ is any geodesic ray with $\rho(\infty) = \xi$, then

$$\frac{d}{dt} B_\eta(\rho(t)) = -\cos\angle_{\rho(t)}(\xi,\eta).$$

The lemma now follows if we combine this with the fact that $\angle_{\rho(t)}(\xi,\eta) \nearrow \angle_\infty(\xi,\eta)$ as $t \to \infty$ (see [BGS85, Lemma 4.2]). ☐

The monotone convergence theorem for integrals now gives

$$\mathrm{slope}_{B_\mu}(\xi) = \int_{\partial_\infty X} -\cos\angle_\infty(\xi,\eta)\,d\mu(\eta). \tag{6.1}$$

We write $\mathrm{slope}_\mu = \mathrm{slope}_{B_\mu}$.

Following [KLM09, Definition 3.11] we make the following definition.

Definition 6.3.5. *A probability measure μ on $\partial_\infty X$ is called stable if* $\mathrm{slope}_\mu > 0$.

Lemma 6.3.6. *If μ is a stable probability measure on $\partial_\infty X$, then B_μ is bounded from below, proper and strictly convex. In particular,* $\mathrm{bar}(\mu)$ *exists.*

A proof is given in [KLM09, Lemma 3.17] but for the convenience of the reader we also give a proof here.

Proof. First we prove that B_μ is bounded from below and proper. Assume, to the contrary, that a $C > 0$ exists such that $B_\mu((-\infty, C])$ is unbounded in X. We note that B_μ is an integral of convex functions so B_μ itself is also convex. From this follows that the set $B_\mu((-\infty, C])$ is a convex subset of X. Since this set is unbounded and convex, it must contain some geodesic ray which we will call $\gamma\colon [0, \infty) \to X$. Then

$$\operatorname{slope}_\mu(\gamma(\infty)) = \lim_{t\to\infty} B_\mu(\gamma(t))/t \leq \lim_{t\to\infty} C/t = 0$$

which contradicts $\operatorname{slope}_\mu > 0$.

Now we prove that the properness of B_μ implies it is strictly convex. We recall the following fact. If $\eta \in \partial_\infty X$ and $v \in T_x X$, then $(\nabla dB_\eta)_x(v, v) = 0$ if and only if η lies on the boundary of a flat containing v ([KLM09, Lemma 2.1]). Assume now that B_μ is not strictly convex. Because we know B_μ to be convex, this can only happen if there exists a $v \in T_x X$ such that

$$0 = (\nabla dB_\mu)_x(v, v) = \int_{\partial_\infty X} (\nabla dB_\eta)_x(v, v) d\mu(\eta).$$

Since $(\nabla dB_\eta)(v, v) \geq 0$ for each $\eta \in \partial_\infty X$, it follows that $(\nabla dB_\eta)(v, v) = 0$ for μ-almost every $\eta \in \partial_\infty X$. If $\sigma\colon \mathbb{R} \to X$ is the geodesic with $\dot\sigma(0) = v$ and $P(\sigma) \subset X$ is its parallel set, then it follows from the above that $\operatorname{supp}\mu \subset \partial_\infty P(\sigma)$. As a consequence, for μ-almost every $\eta \in \partial_\infty P(\sigma)$ and $t \in \mathbb{R}$ we have $(\nabla dB_\eta)(\dot\sigma(t), \dot\sigma(t)) = 0$ and hence $(\nabla dB_\mu)(\dot\sigma(t), \dot\sigma(t)) = 0$ for all t. We see that $t \mapsto B_\mu(\sigma(t))$ is linear which contradicts that B_μ is bounded from below and proper. □

We end this section with the following useful observation.

Lemma 6.3.7. *Let μ be a probability measure on $\partial_\infty X$ and let $\Delta \subset \partial_\infty X$ be a Weyl chamber. If* slope_μ *takes positive values on all vertices of Δ, then* slope_μ *takes positive values on the entire Weyl chamber Δ.*

Corollary 6.3.8. *If* $\operatorname{slope}_\mu > 0$ *on all vertices in $\partial_\infty X$, then μ is stable.*

Proof. Let $\{\nu_1, \ldots, \nu_{n-1}\} \subset \Delta$ be the vertices of Δ. Any other point $\xi \in \Delta$ can be written as a convex combination of the points $\{\nu_i\}_i$. Namely, if $\mathcal{F} \subset X$ is flat such that $\Delta \subset \partial_\infty \mathcal{F}$, then we can view ξ and ν_i as elements of $\partial_\infty \mathcal{F} \cong S^{n-1}$. There exist $\alpha_i \in [0, 1]$ with $\sum_{i=1}^{n-1} \alpha_i = 1$ such that

$$\xi = \frac{\sum_{i=1}^{n-1} \alpha_i \nu_i}{\|\sum_{i=1}^{n-1} \alpha_i \nu_i\|}. \tag{6.2}$$

The coefficients α_i do not depend on the chosen flat (in fact they are completely determined by the eigenvalues $\lambda(\xi)$ associated to ξ).

Now let $\eta \in \partial_\infty X$ and consider the Busemann function B_η. There exists a flat $\mathcal{F} \subset X$ such that $\partial_\infty \mathcal{F}$ contains both Δ and η. The formula of Lemma 6.3.4 gives for any $\sigma \in \partial_\infty \mathcal{F} \cong S^{n-1}$ (considering also η as element of S^{n-1}) that

$$\mathrm{slope}_{B_\eta}(\sigma) = -\langle \eta, \sigma \rangle$$

because at any point $x \in F$ we have $\angle_x(\eta, \sigma) = \angle_\infty(\eta, \sigma)$. From this expression for slope_{B_η} and Equation (6.2) follows that

$$\mathrm{slope}_{B_\eta}(\xi) = \frac{\sum_{i=1}^{n-1} \alpha_i \, \mathrm{slope}_\eta(\nu_i)}{\|\sum_{i=1}^{n-1} \alpha_i \nu_i\|}.$$

Taking the integral on both sides with respect to the measure μ we see that the same relation also holds for the function slope_μ. From this the lemma follows. □

6.4 Hitchin representations

Let $\Gamma = \pi_1(S)$ be the fundamental group of a closed and orientable surface S of genus at least two. The Teichmüller space of S, which is the space of marked hyperbolic structures on S, can be identified with a connected component of the representation variety $\mathrm{Rep}(\Gamma, \mathrm{SL}(2,\mathbb{R}))$. A representation $\theta \colon \Gamma \to \mathrm{SL}(2,\mathbb{R})$ whose conjugacy class lies in this component is called a Fuchsian representation. It is discrete and faithful and it determines a point in Teichmüller space by setting $X = \mathbb{H}^2/\theta(\Gamma)$ and considering the marking of S determined by the identification $\pi_1(S) \cong \pi_1(X)$ that is provided by θ.

Using analytic methods Hitchin (in [Hit92]) discovered analogous components in the representation varieties $\mathrm{Rep}(\Gamma, \mathrm{SL}(n,\mathbb{R}))$ (or more generally the representation varieties for split real simple Lie groups). These components are now called Hitchin components and we will denote them by Hit_n. The component Hit_n can be characterised as the unique component of $\mathrm{Rep}(\Gamma, \mathrm{SL}(n,\mathbb{R}))$ that contains the representations obtained from composing a Fuchsian representation with the irreducible embedding $\mathrm{SL}(2,\mathbb{R}) \subset \mathrm{SL}(n,\mathbb{R})$ (see Remark 6.2.3). The component Hit_2 corresponds to Teichmüller space. A representation $\rho \colon \Gamma \to \mathrm{SL}(n,\mathbb{R})$ whose conjugacy class lies in Hit_n is called a Hitchin representation.

A more geometric description of Hitchin representations was obtained by Labourie in [Lab06]. He proved that these representations are Anosov and hyperconvex (see below). In this section we describe this geometric description of Hitchin representation.

We follow the definition of Anosov representations as given in [BPS19]. Given an element $g \in \mathrm{SL}(n,\mathbb{R})$ we denote by $\sigma_1(g) \geq \ldots \geq \sigma_n(g)$ its singular values.

Definition 6.4.1. *A representation $\rho \colon \Gamma \to \mathrm{SL}(n,\mathbb{R})$ is called B-Anosov if there exist constants $C > 0$ and $\lambda > 0$ such that*

$$\frac{\sigma_{p+1}(\rho(\gamma))}{\sigma_p(\rho(\gamma))} \leq C e^{-\lambda|\gamma|} \text{ for all } \gamma \in \Gamma, 1 \leq p \leq n-1.$$

Associated to any B-Anosov representation ρ is a unique continuous flag curve $F\colon \partial_\infty\Gamma \to \mathrm{Flag}(\mathbb{R}^n)$ that satisfies the following conditions.

- Equivariance: F is equivariant for ρ which means that for all $\gamma \in \Gamma$, $t \in \partial_\infty\Gamma$ we have $F(\gamma t) = \rho(\gamma)F(t)$.
- Transversality: for $t \neq t' \in \partial_\infty\Gamma$ the flags $F(t)$ and $F(t')$ are transverse which means that $F^{n-i}(t) \oplus F^i(t) = \mathbb{R}^n$ for $i = 1, \dots, n-1$.

Definition 6.4.2. *A flag curve $F\colon \partial_\infty\Gamma \to \mathrm{Flag}(\mathbb{R}^n)$ is called hyperconvex if it satisfies the following two conditions.*

- *If $t_1, \dots, t_k$ are distinct elements of $\partial_\infty\Gamma$ and $m_1, \dots, m_k$ are positive integers satisfying $m_1 + \cdots + m_k = n$, then*
$$F^{m_1}(t_1) \oplus \cdots \oplus F^{m_k}(t_k) = \mathbb{R}^n.$$
- *If $m_1, \dots, m_k$ are positive integers satisfying $m_1 + \cdots + m_k \leq n$, then*
$$\lim_{\substack{t_1,\dots,t_k \to t\\ t_i \neq t}} F^{m_1}(t_1) \oplus \cdots \oplus F^{m_k}(t_k) = F^{m_1+\cdots+m_k}(t).$$

Definition 6.4.3. *A representation $\rho\colon \Gamma \to \mathrm{SL}(n,\mathbb{R})$ is called hyperconvex if it is B-Anosov and its flag curve is hyperconvex.*

The following geometric characterisation of Hitchin representations was proved by Labourie (in [Lab06]) and Guichard (in [Gui08]).

Theorem 6.4.4. *A representation $\rho\colon \Gamma \to \mathrm{SL}(n,\mathbb{R})$ is a Hitchin representation if and only if it is hyperconvex.*

The hyperconvexity of the flag curve of Hitchin representations will play an important role in our arguments. A further fact we will use is the following.

Lemma 6.4.5 ([Lab06, Lemma 10.1])**.** *A Hitchin representation $\rho\colon \Gamma \to \mathrm{SL}(n,\mathbb{R})$ acts irreducibly on $\mathbb{R}^n$.*

6.5 Construction of the barycentric maps

We now detail the construction of barycentric maps $\mathbb{H}^2 \to X$ that are equivariant for Hitchin representations.

Let us begin by fixing a point in Teichmüller space represented by a Fuchsian representation $\theta\colon \Gamma \to \mathrm{SL}(2,\mathbb{R})$. The orbit map $\Gamma \to \mathbb{H}^2\colon \gamma \to \theta(\gamma)z$ of any point in $z \in \mathbb{H}^2$ is a quasi-isomorphism between Γ and $\mathbb{H}^2$. This induces a θ-equivariant identification $\phi_\theta\colon \partial_\infty\Gamma \cong \partial_\infty\mathbb{H}^2 \cong S^1$ which coincides with the flag map $\partial_\infty\Gamma \to \mathrm{Flag}(\mathbb{R}^2) \cong S^1$ as described in Section 6.4.

Let $\rho\colon \Gamma \to G$ be a B-Anosov representation and let $F\colon \partial_\infty\Gamma \to \mathrm{Flag}(\mathbb{R}^n)$ be its equivariant flag curve. By composing with the identification $\phi_\theta\colon \partial_\infty\Gamma \to$

$\partial_\infty \mathbb{H}^2$ and inclusion $\mathrm{Flag}(\mathbb{R}^n) = \partial_F X \subset \partial_\infty X$ the flag curve determines a map between the boundaries of the symmetric spaces $\mathbb{H}^2$ and X. We denote this boundary map by $\xi\colon \partial_\infty \mathbb{H}^2 \to \partial_\infty X$.

We define a family of probability measures $\{\nu_z\}_{z\in\mathbb{H}^2}$ on $\partial_\infty\mathbb{H}^2$, called the visual measures, as follows. For each $z \in \mathbb{H}^2$ consider the map $E_z\colon S_z\mathbb{H}^2 \to \partial_\infty\mathbb{H}^2\colon V \mapsto V(\infty)$ as defined in Section 6.2.1 (here $S\mathbb{H}^2$ is the unit tangent bundle of $\mathbb{H}^2$). Each of the maps E_z is a diffeomorphism. We define the visual measure of z as $\nu_z = (E_z)_*(\frac{1}{2\pi}L)$ where L is the Lebesgue measure on $S_z\mathbb{H}^2$. The visual measures are the unique family of probability measures such that $g_*\nu_z = \nu_{gz}$ for all $g \in \mathrm{SL}(2,\mathbb{R})$ and $z \in \mathbb{H}^2$ and ν_0 is $1/(2\pi)$ times the Lebesgue measure on $\partial_\infty\mathbb{H}^2 \cong S^1$.

Now for every $z \in \mathbb{H}^2$ we set $\mu_z^\rho = \xi_*\nu_z$. Our main result is the following.

Theorem 6.5.1. *If ρ is a Hitchin representation, then the measure μ_z^ρ is stable for every $z \in \mathbb{H}^2$.*

Assuming this result the construction of the barycentric maps is now easily completed in the theorem below.

Theorem 6.5.2. *Let $\rho\colon \Gamma \to \mathrm{SL}(n,\mathbb{R})$ be a Hitchin representation. Then the natural map $f_{\theta,\rho}\colon \mathbb{H}^2 \to X$ defined as*

$$f_{\theta,\rho}(z) = \mathrm{bar}(\mu_z^\rho) = \mathrm{bar}(\xi_*\nu_z)$$

is a well-defined, smooth and intertwines θ and ρ.

Proof. By Theorem 6.5.1 each measure μ_z^ρ is stable and hence by Lemma 6.3.6 has a well-defined barycenter. Therefore $f_{\theta,\rho}(z) = \mathrm{bar}(\mu_z^\rho)$ is a well-defined map from $\mathbb{H}^2$ to X. We postpone the proof that $f_{\theta,\rho}$ is a smooth map to Section 6.8 where we will also prove that it depends smoothly on θ and ρ.

It remains to prove the intertwining property of $f_{\theta,\rho}$. First we observe that the assignment $\mu \mapsto \mathrm{bar}(\mu)$ is G-equivariant. Indeed, if $\eta \in \partial_\infty X$ and $g \in G$, then it is straightforward to show that $B_{g\eta} = B_\eta \circ g^{-1} - B_\eta(g^{-1}x_0)$. It follows that $B_{g_*\mu} = B_\mu \circ g^{-1} - B_\mu(g^{-1}x_0)$ and as a result we see $\mathrm{bar}(g_*\mu) = g\,\mathrm{bar}(\mu)$ if μ is a stable measure. We now find, using the equivariance of ξ, that

$$\begin{aligned} f_{\theta,\rho}(\theta(\gamma)z) &= \mathrm{bar}(\xi_*\nu_{\theta(\gamma)z}) = \mathrm{bar}(\xi_*\theta(\gamma)_*\nu_z) = \mathrm{bar}(\rho(\gamma)_*\xi_*\nu_z) \\ &= \rho(\gamma)\,\mathrm{bar}(\xi_*\nu_z) = \rho(\gamma) f_{\theta,\rho}(z). \end{aligned}$$

whenever $\gamma \in \Gamma$ and $z \in \mathbb{H}^2$. □

Remark 6.5.3. If $\rho\colon \Gamma \to \mathrm{SL}(2,\mathbb{R})$ is a Fuchsian representation, then the barycentric method for constructing $f_{\theta,\rho}\colon \mathbb{H}^2 \to \mathbb{H}^2$ coincides with the construction introduced by Douady and Earle in [DE86].

Let us also note that if $\rho = \theta$, then $\xi = \mathrm{id}_{\partial_\infty\mathbb{H}^2}$. Moreover, it is easy to see that $\mathrm{bar}(\nu_0) = \mathrm{bar}(L/2\pi) = 0$ and hence by $\mathrm{SL}(2,\mathbb{R})$-invariance $\mathrm{bar}(\nu_z) = z$ for all $z \in \mathbb{H}^2$. Thus, if $\rho = \theta$, then the natural map $f_{\theta,\rho}$ is simply the identity.

Furthermore, if $\widetilde{\rho}\colon \Gamma \to \mathrm{SL}(n,\mathbb{R})$ is the composition of a Fuchsian representation ρ with the irreducible embedding $\mathrm{SL}(2,\mathbb{R}) \subset \mathrm{SL}(n,\mathbb{R})$, then it follows from the observation in Remark 6.2.3 that $f_{\theta,\widetilde{\rho}}$ is the composition of $f_{\theta,\rho}$ with the induced isometric embedding $\mathbb{H}^2 \subset X$. In particular, if $\widetilde{\rho}$ equals the composition of θ with the embedding $\mathrm{SL}(2,\mathbb{R}) \subset \mathrm{SL}(n,\mathbb{R})$, then the natural map $f_{\theta,\widetilde{\rho}} \colon \mathbb{H}^2 \to X$ equals the isometric embedding $\mathbb{H}^2 \subset X$.

The following two sections are dedicated to the proof of Theorem 6.5.1. We prepare our proof by reducing the statement to a condition on the flag curve of ρ.

By Equation (6.1) we have that for $z \in \mathbb{H}^2$ the slope of the measure μ_z^ρ is given by

$$\mathrm{slope}_{\mu_z^\rho}(\eta) = \int_{\partial_\infty \mathbb{H}^2} -\cos \angle_\infty(\eta, \xi(t)) d\nu_z(t).$$

We first argue that $\mathrm{slope}_{\mu_z^\rho} \geq 0$. The transversality condition on the flag curve F implies that any two $\xi(t)$ and $\xi(t')$ ($t \neq t' \in \partial_\infty \mathbb{H}^2$) can be connected by a geodesic in X. This means that $\angle_\infty(\xi(t), \xi(t')) = \pi$ for all $t \neq t'$. Hence, if for two points $\eta \in \partial_\infty X, t \in \partial_\infty \mathbb{H}^2$ we have $\angle_\infty(\eta, \xi(t)) < \pi/2$, then for all $t' \neq t$ we have

$$\angle_\infty(\eta, \xi(t')) \geq \angle_\infty(\xi(t), \xi(t')) - \angle_\infty(\eta, \xi(t)) > \pi/2.$$

So for at most one $t \in \partial_\infty \mathbb{H}^2$ the inequality $\angle_\infty(\eta, \xi(t)) < \pi/2$ can hold. Therefore the integrant in the above expression for $\mathrm{slope}_{\mu_z^\rho}$ is non-negative almost everywhere (each ν_z has no atoms). Hence, $\mathrm{slope}_{\mu_z^\rho} \geq 0$.

We now use that the map $\xi \colon \partial_\infty \mathbb{H}^2 \to \partial_\infty X$ is continuous and that the function $\angle_\infty(\cdot, \cdot)$ is lower semicontinuous to observe that the function $t \mapsto -\cos \angle_\infty(\eta, \xi(t))$ is also lower semicontinuous. It follows that $\mathrm{slope}_{\mu_z^\rho}(\eta) = 0$ if and only if $\angle_\infty(\eta, \xi(t)) = \pi/2$ for all $t \in \partial_\infty \mathbb{H}^2$. Here we use that each ν_z lies in the same measure class as the Lebesgue measure.

By Corollary 6.3.8 it is enough to check $\mathrm{slope}_{\mu_z^\rho}(\eta) > 0$ for all $\eta \in \partial_\infty X$ that are vertices of Weyl chambers. Hence, as a result of the above discussion we see that it is enough to show that no 'bad' vertex $\eta \in \partial_\infty X$ exists such that $\angle_\infty(\eta, \xi(t)) = \pi/2$ for all $t \in \partial_\infty \mathbb{H}^2$. This is precisely the content of Proposition 6.7.3 below.

Remark 6.5.4. The assumption that ρ is a Hitchin representation and not merely B-Anosov is necessary for Theorem 6.5.1 to hold. We will give an example of a B-Anosov representation $\rho \colon \Gamma \to \mathrm{SL}(3,\mathbb{R})$ for which $\xi_* \nu_z$ is not stable for any $z \in \mathbb{H}^2$.

Consider the representation ρ of Γ in $\mathrm{SL}(2,\mathbb{R}) \times \mathbb{R}$ given by θ on the first factor and the map that is constant equal to zero on the second factor. The associated boundary map ξ of ρ takes values in $\partial_\infty \mathbb{H}^2 \subset \partial_\infty(\mathbb{H}^2 \times \mathbb{R})$. In particular, the image of ξ is fixed by the translations $(z,t) \mapsto (z, t+s)$. As a result we see that if ν is any measure on $\partial_\infty \mathbb{H}^2$, then the weighted Busemann function $B_{\xi_* \nu}$ on $\mathbb{H}^2 \times \mathbb{R}$ is also invariant under these translation. It follows that it can not have a unique minimum.

To make this example into a representation into $\mathrm{SL}(3,\mathbb{R})$ we can compose ρ

with the inclusion of $\mathrm{SL}(2,\mathbb{R}) \times \mathbb{R}$ into $\mathrm{SL}(3,\mathbb{R})$ given by

$$(A,t) \mapsto \begin{pmatrix} e^t \cdot A & 0 \\ 0 & e^{-2t} \end{pmatrix}.$$

It is straightforward to check that the resulting representation is B-Anosov.

6.6 Intersection types

In this section we introduce the notion of intersection type of a subspace and a flag of $\mathbb{R}^n$. This will be an important tool in the proof of Theorem 6.5.1.

Definition 6.6.1. *Let $U \in \mathrm{Gr}_k(\mathbb{R}^n)$ be a k-plane and $F \in \mathrm{Flag}(\mathbb{R}^n)$ a flag. We define the intersection type of U and F as*

$$\mathrm{type}_k(U,F) = (s_0, s_1, \ldots, s_n)$$

where $s_i = \dim(U \cap F^i)$ for $i = 0, \ldots, n$. We call a tuple of integers $(s_0, \ldots, s_n)$ that occurs in this way an intersection type.

Notice that $s_0 = 0$ and $s_n = k$ always. It is clear that a tuple $(s_0, \ldots, s_n)$ is the intersection type of some k-plane U and flag F if and only if $s_0 = 0, s_n = k$ and $s_{i+1} = s_i$ or $s_{i+1} = s_i + 1$ for all $i = 0, \ldots, n-1$. So a tuple $(s_0, \ldots, s_n)$ is an intersection type if and only if it satisfies these conditions.

As detailed in Section 6.2.1, a Weyl chamber in $\partial_\infty X$ corresponds to a full flag of $\mathbb{R}^n$ and a vertex in $\partial_\infty X$ corresponds to a proper subspace of $\mathbb{R}^n$. In the lemma below we relate the Tits angle between a vertex in $\partial_\infty X$ and the barycenter of a Weyl chamber to the intersection type of the associated subspace and flag.

Lemma 6.6.2. *Let $\eta \in \partial_\infty X$ be a vertex corresponding to a k-plane $U \in \mathrm{Gr}_k(\mathbb{R}^n)$ and let $\xi \in G \cdot b(\infty)$ be the barycenter of a Weyl chamber that corresponds to a flag $F \in \mathrm{Flag}(\mathbb{R}^n)$. Denote $\mathrm{type}_k(U,F) = (s_0, \ldots, s_n)$. Then $\angle_\infty(\eta, \xi) = \pi/2$ if and only if*

$$\sum_{i=1}^{n} s_i = \frac{(n+1)k}{2}. \tag{6.3}$$

Our proof of this lemma is along the same lines as the proof of a similar statement in [KLM09, Lemma 6.1].

Proof. There exists a flat $\mathcal{F} \subset X$ such that $\partial_\infty \mathcal{F}$ contains both η and the Weyl chamber containing ξ. We notice that acting by an isometry on both ξ and η leaves both $\angle_\infty(\xi, \eta)$ and $\mathrm{type}_k(U,F)$ invariant. Hence, without loss of generality we can, by acting by an isometry, assume $eK \in \mathcal{F}$ and $T_{eK}\mathcal{F} = \mathfrak{a}$. We note that $\angle_\infty(\xi, \eta) = \sup_x \angle_x(\xi, \eta)$ is realised at any point in $\mathcal{F}$, so in particular $\angle_{eK}(\xi, \eta) = \angle_\infty(\xi, \eta)$.

We identify $\mathfrak{a}$ with the space $\{(\lambda_1, \dots, \lambda_n) \in \mathbb{R}^n \mid \lambda_1 + \dots + \lambda_n = 0\}$. Let $(e_1, \dots, e_n)$ denote the standard basis of $\mathbb{R}^n$. We consider the vector

$$X_i = (-1, \dots, -1, \underbrace{n-1}_{\text{i-th place}}, -1, \dots, -1)$$

in $\mathfrak{a}$. The point at infinity $X_i(\infty) \in \partial_\infty \mathcal{F}$ determined by this vector is a vertex of a Weyl chamber that corresponds to $\langle e_i \rangle \in \mathrm{Gr}_1(\mathbb{R}^n)$. For $I = \{i_1, \dots, i_k\} \subset \{1, \dots, n\}$ we set $X_I = X_{i_1} + \dots + X_{i_k}$. The point $X_I(\infty) \in \partial_\infty \mathcal{F}$ is a vertex that corresponds to $\langle e_{i_1}, \dots, e_{i_k} \rangle \in \mathrm{Gr}_k(\mathbb{R}^n)$.

Since we are interested only in angles between points, we are free to choose a convenient scaling of the inner product on $\mathfrak{a}$. We scale the inner product such that $\|X_i\| = 1$ for $i = 1, \dots, n$. Because the inner product on $\mathfrak{a}$ is invariant under the action of the Weyl group, we see that $\langle X_i, X_j \rangle = \langle X_1, X_2 \rangle$ for all $1 \le i \ne j \le n$. We note that the vector $X_1 + \dots + X_n$ is fixed by the action of the Weyl group and hence must be zero. From this follows that

$$0 = \langle X_1 + \dots + X_n, X_1 + \dots + X_n \rangle = n\|X_1\|^2 + n(n-1)\langle X_1, X_2 \rangle.$$

We conclude $\langle X_i, X_j \rangle = \frac{-1}{n-1}$ for all $1 \le i \ne j \le n$.

If $I = \{i_1, \dots, i_k\}, J = \{j_1, \dots, j_l\} \subset \{1, \dots, n\}$, then

$$\langle X_I, X_J \rangle = |I||J| \cdot \frac{-1}{n-1} + |I \cap J|(1 + \frac{1}{n-1}) = \frac{n|I \cap J| - k \cdot l}{n-1}. \tag{6.4}$$

The point ξ lies in $\partial_\infty \mathcal{F}$ and, by assumption, is an element of the orbit of $b(\infty)$. Hence, by acting by an element of the Weyl group we can assume that $\xi = b(\infty)$. Furthermore, because η is a vertex in $\partial_\infty \mathcal{F}$, there exists a set $I = \{i_1, \dots, i_k\} \subset \{1, \dots, n\}$ such that U, the k-plane corresponding to η, can be written as $U = \langle e_{i_1}, \dots, e_{i_k} \rangle$. We then have that $\eta = X_I(\infty)$ and as a result $\angle_\infty(\xi, \eta) = \angle_{eK}(\xi, \eta) = \angle(b, X_I)$.

We note that $b = (n-1, n-3, \dots, 3-n, 1-n)$ can be written as

$$\frac{n}{2} b = X_1 + (X_1 + X_2) + \dots + (X_1 + X_2 + \dots + X_{n-1}).$$

Namely, if we set Y to be equal to the right hand side of the above expression, then

$$Y = \sum_{i=1}^{n-1} (n-i) X_i.$$

So, using that $(X_i)_k$ evaluates to $n-1$ if $i=k$ and -1 otherwise, we find

$$\begin{aligned}
Y_k &= (n-1)\cdot(n-k) + (-1)\cdot \sum_{i=1, i\neq k}^{n-1} (n-i) \\
&= (n-1)(n-k) - \sum_{i=1}^{n-1}(n-i) + (n-k) \\
&= n(n-k) - \frac{1}{2}n(n-1) \\
&= \frac{n}{2}(n-2k+1) = \frac{n}{2}b_k.
\end{aligned}$$

Now $\angle_\infty(\xi,\eta) = \angle(X_I, b) = \frac{\pi}{2}$ if and only if

$$0 = \frac{n}{2}\langle b, X_I\rangle = \langle Y, X_I\rangle = \sum_{i=1}^{n-1}\langle X_1 + \cdots + X_i, X_I\rangle.$$

Let $(s_0,\ldots,s_n) = \operatorname{type}_k(U,F)$. We note that $s_i = \dim(U\cap F^i) = |I\cap\{1,\ldots,i\}|$. Using Equation (6.4) we see that an individual term of the sum on the right hand side equals

$$\langle X_1 + \cdots + X_i, X_I\rangle = \frac{n\cdot s_i - k\cdot i}{n-1}.$$

So the orthogonality condition reads

$$0 = (n-1)\sum_{i=1}^{n-1}\langle X_1 + \cdots + X_i, X_I\rangle = \sum_{i=1}^{n-1}(n\cdot s_i - k\cdot i) = n\sum_{i=1}^{n-1} s_i - \frac{1}{2}kn(n-1)$$

or equivalently

$$\sum_{i=1}^{n-1} s_i = \frac{(n-1)k}{2}.$$

By adding $s_n = k$ to both sides of the equation we find that indeed $\angle_\infty(\xi,\eta) = \pi/2$ if and only if Equation (6.3) holds.

$\square$

6.7 Proof of Theorem 6.5.1

We now give a proof of Theorem 6.5.1. As discussed in Section 6.5 this amounts to excluding the existence of a certain type of bad vertex in $\partial_\infty X$. This is done in Proposition 6.7.3.

Lemma 6.7.1. *Let $\eta \in \partial_\infty X$ be a vertex corresponding to a k-plane $U \in \mathrm{Gr}_k(\mathbb{R}^n)$. Let $\xi\colon S^1 \to \partial_F X \subset \partial_\infty X$ be a boundary map corresponding to a continuous flag curve $F\colon S^1 \to \mathrm{Flag}(\mathbb{R}^n)$ such that $\angle_\infty(\eta, \xi(t)) = \pi/2$ for all $t \in S^1$. Then $\operatorname{type}_k(U, F(t))$ is constant in t.*

Lemma 6.7.2. *Let $U \subset \mathbb{R}^n$ a k-plane and $S^1 \ni t \mapsto V_t \subset \mathbb{R}^n$ be a continuous family of p-planes. Then $S^1 \ni t \mapsto \dim(U \cap V_t)$ is upper semicontinuous.*

Proof of Lemma 6.7.2. Consider $U^\perp$ the orthogonal complement of U (with respect to the usual Euclidean metric) and the orthogonal projection $\pi \colon \mathbb{R}^n \to U^\perp$. Then $\dim(\pi(V_t))$ is lower semicontinuous in t. This follows since (locally in t) we can pick a continuous family of linear maps $A_t \colon \mathbb{R}^p \to \mathbb{R}^n$ such that $V_t = \operatorname{im} A_t$. Then $\dim(\pi(V_t)) = \operatorname{rank}(\pi \circ A_t)$ and the rank of a continuous family of maps is lower semicontinuous. The lemma follows from the observation that $\dim(U \cap V_t) = p - \dim(\pi(V_t))$. □

Proof of Lemma 6.7.1. For $t \in S^1$ let us denote $\operatorname{type}_k(U, F(t)) = (s_0(t), \dots, s_n(t))$. By Lemma 6.6.2 and the assumption that $\angle_\infty(\eta, \xi(t)) = \pi/2$ we have that $(s_0(t), \dots, s_n(t))$ satisfies Equation (6.3) for each $t \in S^1$. Fix $t_0 \in S^1$. By the upper semicontinuity as proved in Lemma 6.7.2 there is an open neighbourhood O around t_0 such that $s_i(t) \le s_i(t_0)$ for all $t \in O$ and $i = 0, \dots, n$. However, the intersection type $(s_0(t), \dots, s_n(t))$ can satisfy Equation (6.3) and $s_i(t) \le s_i(t_0)$ only if $s_i(t) = s_i(t_0)$ for all i. It follows that $\operatorname{type}_k(U, F(t)) = \operatorname{type}_k(U, F(t_0))$ for $t \in O$. Hence, being of a certain intersection type is an open condition on t. Since S^1 is connected, we conclude that $\operatorname{type}_k(U, F(t))$ must be constant. □

Proposition 6.7.3. *Let $\rho \colon \Gamma \to \mathrm{SL}(n, \mathbb{R})$ be a Hitchin representation and let $F \colon \partial_\infty \Gamma \to \mathrm{Flag}(\mathbb{R}^n)$ be its hyperconvex flag curve. Let $\xi \colon S^1 \to \partial_F X \subset \partial_\infty X$ be the induced boundary map. Then there is no vertex $\eta \in \partial_\infty X$ such that $\angle_\infty(\eta, \xi(t)) = \pi/2$ for all $t \in S^1$.*

Proof. We argue by contradiction and assume a vertex $\eta \in \partial_\infty X$ exist with $\angle_\infty(\eta, \xi(t)) = \pi/2$ for all $t \in S^1$. Let $U \in \mathrm{Gr}_k(\mathbb{R}^n)$ be the corresponding k-plane. Then by Lemma 6.7.2 we have that $\operatorname{type}_k(U, F(t))$ is constant in $t \in S^1$. Denote this intersection type by $(s_0, \dots, s_n)$. We will arrive at a contradiction by dividing the possible intersection types into several cases.

Case I ($s_1 = 1$): Assume first that s_1 is one. Then $\dim(U \cap F^1(t)) = 1$ hence $F^1(t) \subset U$ for all $t \in S^1$. However, this contradicts the hyperconvexity condition

$$F^1(t_1) \oplus \cdots \oplus F^1(t_n) = \mathbb{R}^n$$

for pairwise distinct $t_1, \dots, t_n \in S^1$ since $\dim(U) < n$. We conclude that the case $s_1 = 1$ can not occur.

Case II ($s_i = k$ for $i < n$): Assume there is an $i < n$ such that $s_i = k$. This means that $\dim(U \cap F^i(t)) = k$ hence $U \subset F^i(t)$ for all $t \in S^1$. However, this implies that the set

$$W = \bigcap_{t \in S^1} F^i(t)$$

is not empty as it includes U. Also $W \neq \mathbb{R}^n$ since $\dim(F^i(t)) < n$. By ρ-equivariance of the hyperconvex curve F it is clear that W is ρ-invariant which contradicts the fact that ρ is an irreducible representation (Lemma 6.4.5). We conclude that case II can also not occur.

Case III ($\exists i, j : i + j \leq n, s_i + s_j > k$)**:** Assume there exist i, j with $i + j \leq n$ and $s_i + s_j > k$. Let $t, t' \in S^1$ be distinct elements. We have $\dim(U \cap F^i(t)) + \dim(U \cap F^j(t')) = s_i + s_j > k$ hence $(U \cap F^i(t)) \cap (U \cap F^j(t')) \neq 0$. However, this contradicts the fact that the sum of $F^i(t)$ and $F^j(t')$ is direct (first hyperconvexity condition). We conclude that case III can not occur.

Case IV: We assume that we are not in case I, II or III. Then $(s_0, \ldots, s_n)$ satisfies the following conditions:

- $s_1 = 0$ (condition i),
- $s_i < k$ for $i < n$ (condition ii) and
- $s_i + s_j \leq k$ for $i + j \leq n$ (condition iii).

We now show that there exist no intersection types that satisfy these conditions whilst simultaneously satisfying Equation (6.3).

Assume first that n is odd. We write (using that $s_n = k$ always)

$$\sum_{i=1}^{n} s_i = s_1 + s_{n-1} + s_n + \sum_{i=2}^{n-2} s_i = s_1 + s_{n-1} + k + \sum_{i=2}^{(n-1)/2} (s_i + s_{n-i}).$$

By our assumptions we have $s_1 = 0$ (condition i), $s_{n-1} < k$ (condition ii) and $(s_i + s_{n-i}) \leq k$ (condition iii) and hence

$$\sum_{i=1}^{n} s_i < 2k + \left[\frac{n-1}{2} - 1\right] k = \frac{(n+1)k}{2}.$$

We conclude that $(s_0, \ldots, s_n)$ can not satisfy Equation (6.3). Now consider the case that n is even. We observe that

$$\begin{aligned}\sum_{i=1}^{n} s_i &= s_1 + s_{n-1} + s_n + s_{n/2} + \sum_{i=2}^{n/2-1} (s_i + s_{n-i}) \\ &< k + k + s_{n/2} + \left[\frac{n}{2} - 2\right] k = \frac{n}{2} k + s_{n/2}.\end{aligned}$$

Here we use the same reasoning as in the odd case. To conclude we apply condition iii to find that $s_{n/2} + s_{n/2} \leq k$ hence $s_{n/2} \leq k/2$. We see that

$$\sum_{i=1}^{n} s_i < \frac{(n+1)k}{2}$$

and so, also in this case, $(s_0, \ldots, s_n)$ can not satisfy Equation (6.3). This concludes the proof. □

The considerations of Section 6.5 reduced the statement of Theorem 6.5.1 to the nonexistence of a bad vertex in $\partial_\infty X$. The existence of such a vertex is ruled out by Proposition 6.7.3 and so combining these results yields a proof of Theorem 6.5.1.

6.8 Smooth dependence

In this section we prove that the barycentric maps are smooth and depend smoothly on the point in Teichmüller space and the Hitchin representation they are associated to.

To consider the question of dependence on the representation it is convenient to look at smooth families of representations. Let $\{\gamma_1, \dots, \gamma_k\}$ be a finite set of generators of the group $\Gamma = \pi_1(S)$. If D is an open subset of $\mathbb{R}^N$, then we say a family of representations $(\rho_u)_{u \in D}$ of Γ in G is smooth if the mapping $D \to G \colon u \mapsto \rho_u(\gamma_i)$ is smooth for every $i = 1, \dots, k$.

Suppose $(\rho_u)_{u \in D_1}$ is a smooth family of Hitchin representations and $(\theta_v)_{v \in D_2}$ is a smooth family of Fuchsian representations. For $u \in D_1$ let us denote by $F_u \colon \partial_\infty \Gamma \to \mathrm{Flag}(\mathbb{R}^n)$ the flag curve associated to ρ_u and for $v \in D_2$ let $\phi_v \colon \partial_\infty \Gamma \xrightarrow{\cong} \partial_\infty \mathbb{H}^2$ be the flag curve of θ_v. Denote by $\xi_{u,v} = F_u \circ \phi_v^{-1} \colon \partial_\infty \mathbb{H}^2 \to \partial_F X \subset \partial_\infty X$ the induced boundary map. Finally, for $z \in \mathbb{H}^2$ we set $\mu_z^{u,v} = (\xi_{u,v})_* \nu_z$.

Theorem 6.8.1. *Let $(\rho_u)_{u \in D_1}$ be a smooth family of Hitchin representations and $(\theta_v)_{v \in D_2}$ a smooth family of Fuchsian representations. Then the map*

$$f \colon D_1 \times D_2 \times \mathbb{H}^2 \to X, f(u,v,z) = \mathrm{bar}(\mu_z^{u,v})$$

is smooth.

The existence of the barycenter points follows from Theorem 6.5.1, hence the map under consideration is well-defined. For $u \in D_1, v \in D_2$ the maps $f_{u,v}(\cdot) = f(u,v,\cdot) \colon \mathbb{H}^2 \to X$ equal the maps f_{θ_v, ρ_u} constructed in Theorem 6.5.2. The fact that these maps are smooth, as was stated in that theorem, follows from Theorem 6.8.1.

Our proof will rely on the fact that the weighted Busemann functions depend smoothly on both $z \in \mathbb{H}^2$ and $(u,v) \in D_1 \times D_2$. To show this we will need that the flag curves of B-Anosov representations depend smoothly on the representation.

To formulate this we consider the space $C^0(\partial_\infty \Gamma, \mathrm{Flag}(\mathbb{R}^n))$ of continuous maps from $\partial_\infty \Gamma$ to $\mathrm{Flag}(\mathbb{R}^n)$ equipped with the compact-open topology. This space can be equipped with the structure of a Banach manifold which allows us to talk about smoothness of mappings into this space. An alternative, but equivalent, way to characterise this is described in [BCLS15, Section 6.1].

Proposition 6.8.2. *Let $(\rho_u)_{u \in D}$ be a smooth family of B-Anosov representations. The map $D \to C^0(\partial_\infty \Gamma, \mathrm{Flag}(\mathbb{R}^n)) \colon u \mapsto F_u$ is smooth.*

This is proved in [BCLS15, Theorem 6.1] (see also [BPS19, Theorem 6.1]).

We can now give a proof of Theorem 6.8.1. We will proceed along similar lines as [BCG96, p. 636].

Proof of Theorem 6.8.1. First we note that $X \times \partial_F X \to \mathbb{R} \colon (x, \xi) \mapsto B_\xi(x)$ is a smooth function (with $\partial_F X = G \cdot b(\infty) \subset \partial_\infty X$). The smoothness in the x variable follows from the explicit description of the Busemann functions on

$X = \mathrm{SL}(n, \mathbb{R})/\mathrm{SO}(n)$ (see, for example, [BH99, Proposition II.10.67]). The joint smoothness in x and η follows from the invariance property of the Busemann function which gives $B_{g \cdot b(\infty)}(x) = B_{b(\infty)}(g^{-1}x) - B_{b(\infty)}(g^{-1}x_0)$.

A second fact we will use is that the visual measures introduced in Section 6.5 form a smooth family. We can see this by looking at an explicit description of these measures. Namely, if we consider the Poincaré disk model of $\mathbb{H}^2$, which allows us to view $\partial_\infty \mathbb{H}^2$ as $S^1 \subset \mathbb{C}$, then

$$d\nu_z(t) = \frac{1}{2\pi} \frac{1 - |z|^2}{|z-t|^2} \cdot dL(t) \text{ for } z \in \mathbb{H}^2, t \in \partial_\infty \mathbb{H}^2,$$

where L is the Lebesgue measure on S^1 (see for example [DE86, p.24]).

Thirdly, by Proposition 6.8.2 the maps F_u and ϕ_v depend smoothly on u and v respectively. It follows that $D_1 \times D_2 \to C^0(\partial_\infty \mathbb{H}^2, \partial_\infty X) \colon (u, v) \to \xi_{u,v}$ is a smooth map.

We collect the weighted Busemann functions for the measures $\mu_z^{u,v} = (\xi_{u,v})_* \nu_z$ into a single function by setting $B \colon D_1 \times D_2 \times \mathbb{H}^2 \times X \to \mathbb{R}$ to be

$$B(u, v, z, x) = \int_{\partial_\infty \mathbb{H}^2} B_{\xi_{u,v}(t)}(x) d\nu_z(t) = \int_{S^1} I(z, x, \xi_{u,v}(t)) \frac{dL(t)}{2\pi}$$

where $I \colon \mathbb{H}^2 \times X \times \partial_F X \to \mathbb{R}$ is defined as

$$I(z, x, \xi) = B_\xi(x) \cdot \frac{1 - |z|^2}{|z-t|^2}.$$

We note that I is a smooth function. Since $(u, v) \mapsto \xi_{u,v}$ is a smooth map, it follows that $(u, v, z, x) \mapsto I(z, x, \xi_{u,v}(\cdot))$ is a smooth map $D_1 \times D_2 \times \mathbb{H}^2 \times X \to C^0(\partial_\infty \mathbb{H}^2)$. All (higher order) partial derivatives of this function with respect to the variables u, v, z and x are continuous functions on $\partial_\infty \mathbb{H}^2$. Because $\partial_\infty \mathbb{H}^2$ is a compact space, all these partial derivatives are integrable functions. A standard application of the Lebesgue dominated convergence theorem yields that we can 'differentiate under the integral sign' and hence the function $B \colon D_1 \times D_2 \times \mathbb{H}^2 \times X \to \mathbb{R}$ is smooth.

By the Cartan-Hadamard theorem the space X is diffeomorphic to $\mathbb{R}^k$ ($k = \dim X$) hence it can be equipped with global coordinates $(x^1, \ldots, x^k)$. Let us denote by $D_i B$ the partial derivatives of B with respect to these coordinates. The map

$$G \colon D_1 \times D_2 \times \mathbb{H}^2 \times \mathbb{R}^k \to \mathbb{R}^k, G_i = D_i B$$

is smooth. For every $u \in D_1, v \in D_2$ and $z \in \mathbb{H}^2$ the point $x = f(u, v, z)$ is the unique minimum of $B(u, v, z, \cdot)$ and, by strict convexity, this means it is the unique point where $G(u, v, z, x) = 0$. The smooth dependence of x on u, v and z now follows from the implicit function theorem. We note, in order to apply this theorem, that the invertibility of the Jacobian matrix $(D_i G(u, v, z, \cdot))_i$ is implied by the non-degeneracy of $\nabla dB(u, v, z, \cdot)$ which was proved in Theorem 6.5.1 (also Lemma 6.3.6). $\square$

6.9 A parametrisation of the Hitchin component

We can use the equivariant natural maps to obtain a novel parametrisation of the Hitchin component. Let us consider $\mathcal{C} = C^0(\mathbb{H}^2, X)/G$ the space of maps from $\mathbb{H}^2$ to X up to composition with an isometry of X. The compact-open topology on $C^0(\mathbb{H}^2, X)$ induces a quotient topology on the space $\mathcal{C}$. We embed the Hitchin component into $\mathcal{C}$ by assigning to each conjugacy class of Hitchin representations the corresponding equivariant barycentric map. We fix some Fuchsian representation $\theta \colon \Gamma \to \mathrm{SL}(n, \mathbb{R})$.

Theorem 6.9.1. *The map $\iota \colon \mathrm{Hit}_n \to \mathcal{C} \colon [\rho] \mapsto [f_{\theta,\rho}]$ is a topological embedding.*

Proof. For ease of notation we will identify Γ with its image $\theta(\Gamma)$ in $\mathrm{SL}(2, \mathbb{R})$.

It follows from the results of Theorem 6.8.1 that ι is continuous. To prove it is an injective map we consider two points $[\rho], [\rho'] \in \mathrm{Hit}_n$ that are mapped to the same $[f] \in \mathcal{C}$. For some $z \in \mathbb{H}^2$ put $x = f(z)$. Then $\rho(\gamma)x = f(\gamma z) = \rho'(\gamma)x$ for all $\gamma \in \Gamma$. It now follows from Lemma 6.9.2 (proved below) that $[\rho] = [\rho']$.

Finally, we prove that ι is a closed map which will imply it is an embedding. Let $A \subset \mathrm{Hit}_n$ be a closed subset and let $[\rho_k] \in A$ be a sequence of Hitchin representations such that the sequence $[f_{\theta,\rho_k}]$ converges in $\mathcal{C}$. This means that $g_k \in G$ exist such that the sequence $g_k f_{\theta,\rho_k}$ converges uniformly on compacts. We conjugate the representations ρ_k by the elements g_k so that we can assume that f_{θ,ρ_n} converges to a limiting map $\widetilde{f} \colon \mathbb{H}^2 \to X$. Fix some $z \in \mathbb{H}^2$. For any $\gamma \in \Gamma$ we have

$$\begin{aligned} d(\rho_k(\gamma)\widetilde{f}(z), \widetilde{f}(\gamma z)) &\leq d(\rho_k(\gamma)\widetilde{f}(z), \rho_k(\gamma) f_{\theta,\rho_k}(z)) + d(f_{\theta,\rho_k}(\gamma z), \widetilde{f}(\gamma z)) \\ &= d(\widetilde{f}(z), f_{\theta,\rho_k}(z)) + d(f_{\theta,\rho_n}(\gamma z), \widetilde{f}(\gamma z)) \xrightarrow{k\to\infty} 0 \end{aligned}$$

and as a consequence $\rho_k(\gamma)$ is contained in the compact set

$$\{g \in G \mid d(g \cdot \widetilde{f}(z), \widetilde{f}(\gamma z)) \leq 1\} \subset G$$

for k high enough. So, by going to a subsequence, we can ensure that $\rho_k(\gamma)$ converges in G for all γ in a finite generating set of Γ. It follows ρ_k converges to a limiting representation $\widetilde{\rho}$. Because A is closed we have $[\widetilde{\rho}] \in A$ which means that $[\widetilde{f}] = [f_{\theta,\widetilde{\rho}}] \in \iota(A)$. We conclude that ι is a closed map and hence an embedding. □

Lemma 6.9.2. *Let $\rho, \rho' \colon \Gamma \to \mathrm{SL}(n, \mathbb{R})$ be Hitchin representations. If for some $x \in X$ the orbit maps $\gamma \mapsto \rho(\gamma)x$ and $\gamma \mapsto \rho'(\gamma)x$ coincide, then ρ and ρ' are conjugate.*

Proof. We will consider the Cartan projection $\mu \colon G \to \overline{\mathfrak{a}}^+$ and Lyapunov projection $\lambda \colon G \to \overline{\mathfrak{a}}^+$. For each $g \in G$ the Cartan projection $\mu(g)$ is the unique element in $\overline{\mathfrak{a}}^+$ such that $g = k \exp(\mu(g)) k'$ for some $k, k' \in K$ (see [GGKW17, Section 2.3]). The Lyapunov projection can be defined as $\lambda(g) = \lim_{n\to\infty} \frac{1}{n}\mu(g^n)$ (see [GGKW17, Section 2.4]). Moreover, if $g \in G$ is a diagonalisable matrix with real eigenvalues $\lambda_1(g), \dots, \lambda_n(g)$ (ordered such that $|\lambda_1(g)| \geq |\lambda_2(g)| \geq \cdots \geq |\lambda_n(g)|$), then $\lambda(g) = (\log|\lambda_1(g)|, \dots, \log|\lambda_n(g)|)$ (see [GGKW17, Example 2.24]).

By conjugating both ρ and ρ' by the same element we can assume that $x = eK \in X$. We note that if $g, g' \in G$ are such that $gK = g'K$, then $\mu(g) = \mu(g')$. Hence, $\rho(\gamma)x = \rho'(\gamma)x$ implies $\mu(\rho(\gamma)) = \mu(\rho'(\gamma))$ for each $\gamma \in \Gamma$. Then also

$$\lambda(\rho(\gamma)) = \lim_{n\to\infty} \frac{1}{n}\mu(\rho(\gamma)^n) = \lim_{n\to\infty} \frac{1}{n}\mu(\rho'(\gamma)^n) = \lambda(\rho'(\gamma))$$

for all $\gamma \in \Gamma$. Because ρ and ρ' are Hitchin representations, all elements $\rho(\gamma)$ and $\rho'(\gamma)$ are diagonalisable with real eigenvalues ([Lab06, Theorem 1.5]). It follows from the above observations that $|\lambda_1(\rho(\gamma))| = |\lambda_1(\rho'(\gamma))|$ for all $\gamma \in \Gamma$. Now [BCL20, Theorem 1.1] implies that ρ and ρ' are conjugate. $\square$

Our approach should be compared to a different parametrisation of the Hitchin component that is obtained by sending each representation to its corresponding unique equivariant harmonic map (see [Li19], in particular Section 2.2.6). These harmonic maps occur naturally in the study of the Hitchin component via the Non-Abelian Hodge correspondence. However, a drawback is that they, as solutions to a PDE equation, can often not be written down explicitly. This makes it difficult to study them directly. We hope that the concrete nature of the barycentric map parametrisation as we have exhibited in this paper will allow for new ways to study the Hitchin component.

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