Modularity of strings on F-theory backgrounds

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Abstract

This thesis regards topological string theory on elliptically and genus one fibered Calabi-Yau compactifications to study counting indexes over states in a dual six-, five-, and four-dimensional theory realized by F-theory.

The first part focuses on studying several quantum gravity constraints that should fulfill a six-dimensional theory with minimal supersymmetry. We exploit the modular properties of the topological string partition function, equivalently the elliptic genera of six-dimensional strings, to argue the consistency of such constraints—including the absence of quantum anomalies. In particular, we manage to prove the sublattice non-Abelian weak gravity conjecture for F-/M-theory compactified on elliptically and genus one fibered Calabi-Yau 3-folds admitting a $K3$ fibration.

In the next part, we focus on a subclass of $K3$ fibered geometries to analyze their $K3$ fiber reduced Gromov-Witten theory, equivalently their Noether-Lefschetz's theory: We find a correspondence between the elliptic genera of Heterotic strings and the enumerative geometry in the reduced $K3$ fibers. Through Noether-Lefschetz symmetries, we derive that this information encodes the massless spectrum of six-dimensional supergravity theories, the data necessary to compute conjectured refined BPS invariants in compact geometries, and a supersymmetric index for four-dimensional theories with $\mathcal{N} = 2$ supersymmetry. Using the latter result, we examine Heterotic theories with a CHL orbifold construction that possess a 5d M-theory dual interpretation compactified on a genus one fibered Calabi-Yau admitting a $K3$ fibration. We conclude that the CHL-Heterotic strings elliptic genera must be meromorphic vector-valued lattice Jacobi forms.

In the last part, we turn to study elliptically fibered Calabi-Yau 4-folds and the modularity of their quantum periods. By turning on gauge fluxes, we achieve F-theory compactifications leading to four-dimensional theories with minimal supersymmetry. Moreover, the quantum periods of gauge fluxes interpret as elliptic genera of four-dimensional strings, described by quasi-Jacobi forms. We exemplify these objects and realize that they follow conjectural holomorphic anomaly equations.

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Contents

CHAPTER 1

Introduction

String theory is the only known consistent framework, up to date, that describes both gravitational and non-gravitational interactions as a quantum physics theory. Although its validity is yet in dispute and its experimental testing is a challenge, it allows us to probe questions for an ultimate theory of everything. Therefore, it is natural to ask how much we can learn about quantum gravity and quantum field theory by exploiting string theory as a theoretical laboratory? With this in mind, we aim to search in this work for features or mathematical principles that a consistent quantum gravity theory should have.

In our quest, we center our study on the concept of modularity in the following sense: functions undergoing transformations that preserve the shape of tori. Its appearance is ubiquitous in string theory in several different forms. Our subject is to interpret these subtle differences in terms of physical information corresponding to a particular physics model.

At first sight, this task seems daunting as there are five string theories to consider: Type IIA, Type IIB, Heterotic $E_8 \times E_8$, Heterotic $SO(32)$, and Type I. These are 10d theories connected by a web of dualities through a conjectural 11d theory, M-theory. To achieve a lower viable physics model, we compactify some of the extra dimensions into a small compact manifold. This way, the shape, size, and topological properties of such an internal manifold manifest as parameters determining the lower dimensional effective theory. In this work, we will restrict to compactifications that have a torus fibration. The modularity we pursue will descend from the torus fiber of those geometries.

Having fixed our working setup, let us expose its physics motivation. We now elaborate on a few ingredients present in this thesis.

There are two types of quantum fields distinguished by their statistical behavior in our universe: bosons and fermions. For instance, in the standard model—the most successful quantum field theory agreeing with experiments to this day—, matter building blocks are fermions, while force carriers are bosons. A proposal to extend the standard model is supersymmetry, a symmetry that exchanges bosons with fermions and vice-versa. For future notation, we refer to physics theories with multiple supersymmetries, or $\mathcal{N}$ supersymmetries, as an "$\mathcal{N}$ theory" with $\mathcal{N} \in \mathbb{N}$. In particular, potential models of phenomenological interest are those with $\mathcal{N} = 1$ supersymmetry in 4d. Now, suppose a form of supersymmetry realizes in nature. In that case, it must be spontaneously broken at some energy scale that we do not access through our current experiments.

String theories are quantum gravity theories with varying degrees of supersymmetry. Upon compactification, a given string theory results in an $\mathcal{N}$ theory coupled to gravity in d-dimensions, where $\mathcal{N}$ depends on the internal manifold's dimension, among other properties. This last point incites us to consider Calabi-Yau manifolds compactifications, as they give rise to theories with $\mathcal{N} = 1, 2$. We remark that it is possible to consider other compactifications leading to theories with no supersymmetry, e.g., Spin(7) manifolds. However, those cases remain virtually unexplored, and their study goes beyond the scope of this work. It is important to remember that the more supersymmetry one has, the more symmetric and easier to solve a theory is, and thus, the more unrealistic it is. A modest prior step is to understand string compactifications with minimal supersymmetry. Having said this, let us explain why we are interested in torus fibrations.

One of the most promising formulations in string theory is F-theory: a non-perturbative extension of Type IIB that demands torus fibered Calabi-Yau manifolds. The advantage it offers is that it translates plausible physics models into geometry and vice-versa, including realistic 4d $\mathcal{N} = 1$ theories. The idea is to engineer desired physics properties, such as gauge groups (non-gravitational interactions), matter content, Yukawa's interactions, in a single unifying framework encapsulated by the geometry of torus fibrations. In general, we achieve physics of minimal supersymmetry in even dimensions through this approach. A subject of discussion for us will be theories in six and four dimensions with minimal supersymmetry. We discuss next the relevance of the former type of theories.

The more we decrease the number of dimensions via F-theory, the more the complexity of physics phenomena increases, and so does the structure of the internal geometry we require. However, many of the features we look for in lower dimensions—say 4d—are already present in higher dimensional setups. Therefore, an intermediate strategy is studying 6d $N = (1, 0)$ theories, where we use the notation $N = (1, 0)$ instead of $\mathcal{N} = 1$ due to literature conventional reasons. In particular, 6d $N = (1, 0)$ theories are exciting since they exhibit diverse matter fields charged under gauge groups.

An attractive aspect of F-theory is that it renders consistency of the theory into topological data. By this, we mean that the theories are free of quantum anomalies, both gauge and gravitational. It is the absence of quantum anomalies that provides strong constraints in the realm of seemingly viable quantum gravity theories. Thus, F-theory provides a large subclass of models within the more extensive set of string compactifications leading to a low energy effective field theory. Each of these theories represents a choice of vacuum in a given string compactification; the vast but finite range of possibilities for these vacua is called the *string landscape*. Therefore, it is essential to ask what further criteria can constrain, even more, such string vacua to an actual description close to our reality. F-theory provides an excellent framework for this task, as it accounts for both *perturbative* and *non-perturbative* effects in string theory.

By construction, F-theory includes branes, dynamical objects that extend in spatial dimensions, i.e., a generalization of strings and particles. We can think of these as the underlying quantum microscopic degrees of freedom of uncompactified string theory. Unlike strings, in general, they do not possess a perturbative description; in the case of the strings, there is a string coupling parameter g_s for strings that allow for calculations of the type

$$\mathcal{O} = \sum_{n=0}^{\infty} O_n g_s^{2n} \,, \quad O_n \in \mathbb{C} \,, \tag{1.1}$$

where $\mathcal{O}$ is an observable in the perturbative regime $g_s \ll 1$. However, when considering string theory compactifications, branes wrap or curl along cycles or loops in the internal geometry. This way, parameters characterizing the structure of the internal manifold encode dynamical information that branes dictate. Thus, we can access theories and computations through geometry that otherwise we cannot via standard perturbative methods.

When considering torus fibered compactifications, we can also analyze limits when gravity decouples, giving, in turn, valid quantum field theories. As a result, we can compute some observables in a quantum field theory non-perturbative regime that points out the reminiscence of branes. By taking this systematic path, we say we geometrically engineer quantum field theories. It turns out that a large class of geometrically engineered theories is such that the effective theories have a Lagrangian description. However, once again, we can describe them and make computations through geometry. This phenomenon poses a challenge to our understanding of quantum field theory as a framework since it does not have a final rigorous mathematical foundation. Improving this status is essential to formulate further models, which we might require for physics beyond the standard model. In this sense, F-theory is also a testing ground that pushes the limits we know of quantum field theory.

Note that, so far, we have talked exclusively about F-theory since it employs torus fibered Calabi-Yau manifolds, and those are the geometries of our interest in this work. However, another vantage of this language is that it allows us to enhance our understanding of M-theory, as we can relate both descriptions by string dualities. The same goes for the converse. For this reason, our strategy relies on studying generalizations of *modular forms* with an M-theory interpretation. Namely, we consider the topological string partition function and periods of Calabi-Yau manifolds, as they capture counting information about geometry—enumerative geometry invariants.

By modular forms here, we mean functions on the upper-half complex plane $\mathbb{H} \subset \mathbb{C}$ that transform in a certain way under a modular group Γ, a discrete subgroup of $\mathrm{SL}(2, \mathbb{R})$. Each value $\tau \in \mathbb{H}$ represents a torus, whereas its modular transformation $\tau' = \gamma \cdot \tau$ by $\gamma \in \Gamma$ is such that both tori, corresponding to τ and τ', have the same shape. Thus, the philosophy is that modular forms or generalizations thereof should relate under reparametrizations of tori or other geometrical parameters.

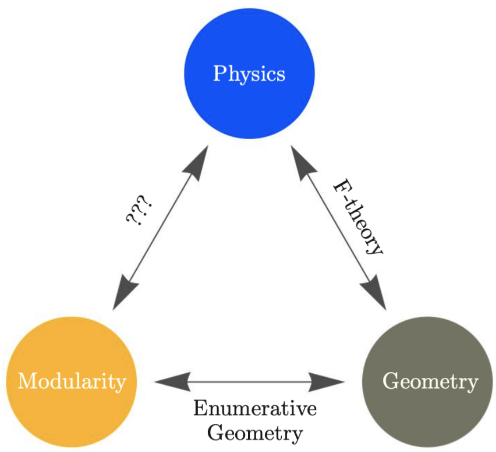


Figure 1.1: Sketch for triality correspondence among physics, geometry, and modularity.

In addition, modularity is a powerful tool that allows us to classify geometries. For instance, every elliptic curve over $\mathbb{Q}$ can be parametrized by modular functions. Remarkably, the latter result was crucial to prove Fermat's last theorem in the 20th century, which states that no three integers a, b, and c satisfy $a^n + b^n = c^n$ for $n > 2$. However, this is another notion of modularity that we do not address here; instead, we restrict ourselves to elliptic curves over $\mathbb{C}$ (tori) and complex geometries. Nevertheless, the critical message here is a modular $\leftrightarrow$ algebraic-geometric interplay that suggests another possibility, which establishes a correspondence between modularity and physics. Based on previous discussions concerning F-theory and enumerative geometry, we illustrate this hypothesis through Figure 1.1. Along the course of this thesis, we will explore this connection.

This thesis consists of six further chapters within three main parts. In part I (Chapter 2 & 3), we develop the background necessary to unfold the core of this work. Chapter 2 describes our main techniques for computations of physics observables through geometrical invariants of Calabi-Yau manifolds. Namely, we include an overview of topological string theory and mirror symmetry; we explain the toric geometry constructions of Calabi-Yau manifolds specialized for torus fibrations. Our end here is to connect topological string computations with information about the M-theory spectrum. Chapter 3 introduces our Physics arena, F-/M-theory. There we discuss in detail the physics-geometrical dictionary of torus fibrations.

In part II (Chapters 4, 5, & 6), we expose an extended discussion on results based on our publications [1–3]. We follow the next anachronous order:

- **Modularity and Quantum Gravity consistency of 6d $N = (1, 0)$ theories:** Over the recent years, a string theory research line focuses on determining underlying quantum gravity principles to constraint models, which goes beyond the absence of quantum anomalies. For instance, in our everyday experience, we observe that gravity is the weakest force. Therefore, a low energy effective theory should resemble this behavior. This principle is known as the weak gravity conjecture. In general, we refer to swampland conjectures to the set of criteria distinguishing which theories have a consistent ultraviolet completion. Up to date, the strongest refinement for the weak gravity conjecture is the non-Abelian sublattice weak gravity conjecture. Given our context of 6d $N = (1, 0)$ theories realized by Calabi-Yau 3-folds, we invoke modularity of counting functions over strings propagating in 6d to prove such a conjecture, at least in a weakly coupled limit. Furthermore, we argue that modularity provides a working tool to verify other swampland conjectures, in addition to the absence of anomalies.

- **Noether-Lefschetz theory and F-/M-theory compactifications:** As we study 6d theories in the F-theory language, we notice that a Heterotic dual description is possible whenever the Calabi-Yau 3-fold inner space admits a $K3$ fibration. Note that this is a further requirement besides the torus fibration structure. Once we ensure this, we can define yet another type of invariants associated with the $K3$ fibers, Noether-Lefschetz numbers. These numbers possess another type of modularity. We show that the latter has a one-to-one correspondence with the counting of 6d strings we study. This result leads to three separate applications: F-theory spectrum $\leftrightarrow$ modularity correspondence, the calculation of conjectured reined BPS invariants in 5d, and an index for 4d $\mathcal{N} = 2$ dual theories. We exploit this last point to understand the modularity of 5d theories with a CHL Heterotic orbifold realization.

- **Modularity on elliptically fibered Calabi-Yau 4-folds:** To achieve the most realistic physics models from F-theory, we need compactifications on Calabi-Yau 4-folds. Furthermore, we require turned-on gauge fluxes on such a geometry. This way, we can achieve 4d $\mathcal{N}$=1 physics. However, in these cases, the study of geometrical invariants is more subtle and consequently their associated modular counterpart. To encapsulate these objects, we explore the theory of *quasi-Jacobi forms*. We explain the correspondence of gauge fluxes with this kind of modularity, which leads to the counting of strings propagating in 4d.

Finally, in chapter 7, we elaborate on our conclusions based on the results of this thesis. There, we also include prospects for future work.

To avoid interrupting flow of our exposition, we include the complementary Appendix A, which gives a crash overview on Lie algebras and representation theory. Moreover, we also include a catalog in Appendix B, which reviews the modular objects we utilize and modular expressions we regard in our calculations. Lastly, we include Appendix C, which covers Noether-Lefschetz theory and geometrical invariants of $K3$ fibrations.

We excuse the reader for not being utterly self-contained since we do not have space to review all topics from scratch. For this reason, we will assume a minimum in complex and Kähler geometry that is available in several sources. We recommend [4] (Chapters 2, 3, 4, & 5) for this matter. Also, as we advance throughout the thesis, we will assume basic notions in string theory without reference, as it is material taught in graduate string theory courses. A comprehensive introductory string theory book that could serve as guidance is the reference [5].

CHAPTER 2

Mirror symmetry and topological string theory

One outstanding feature of string theories is that they can relate via dualities. In general, we say that a string theory $\mathcal{S}_A(M)$ on a manifold M is dual to another string theory $\mathcal{S}_{\mathrm{B}}(W)$ with another compactification W if $\mathcal{S}_A(M) = \mathcal{S}_{\mathrm{B}}(W)$. In particular, *mirror symmetry* is a duality that originates from two equivalent choices for a topological realization of string theory, the A-model, and the B-model. Topological here means no metric dependence on worldsheet Riemann surfaces Σ in the string fields $\Phi : \Sigma \to X$ with target space X. Thus, the A-model and B-model are dual descriptions whose target spaces are Calabi-Yau manifolds. The latter possess non-trivial equivalences despite having different topologies; we say that these geometries are a mirror dual pair.

In this Chapter, we will review the basics of topological string theory. Followed by that, we will summarize briefly the basics on toric geometry. After that, we explain the Batyrev-Borisov construction for mirror dual pairs of Calabi-Yau manifolds. Then, we explain the basics of the B-model approach for computing string amplitudes, as well as homological mirror symmetry. Lastly, we explain how to relate string amplitudes with M-theory BPS invariants and F-theory strings excitations.

2.1 Topological String Theory

In this section we review several aspects of topological string theory, based on the references [4, 6, 7]. Our focus here is grasp some of the underlying physics behind mirror symmetry. We cover next some of the ingredients necessary to describe a topological string theory, at the level of 2d conformal field theory (CFT).

$N = 2$ superconformal field theories: In the 2d CFT language [8], an $N = 2$ superconformal algebra is generated by the energy momentum-tensor $T(z)$, two anti-commutating currents $G^{\pm}(z)$, and a $U(1)$ current $J(z)$; their conformal weights read respectively $h_T = 2$, $h_{\pm} = \pm 3/2$, and $h_J = 1$. These (quasi-)primary fields possess Fourier modes expansions $T(z) = \sum_n L_n z^{-n-2}$, $G^{\pm}(z) = \sum_n G^{\pm}_{n\pm s} z^{-n\mp s-3/2}$, and $J(z) = \sum_n J_n z^{-n-1}$, whose Fourier modes satisfy the following superconformal algebra commutator $[\cdot\,,\cdot]$ relations:

$$
\begin{aligned}
[L_m, L_n] &= (m-n)L_{m+n} + \frac{\mathrm{c}}{12}m(m^2-1)\delta_{m+n,0} \\
[J_m, J_n] &= \frac{\mathrm{c}}{3}m\delta_{m+n,0} \\
[L_n, J_m] &= -mJ_{m+n} \\
[L_n, G^{\pm}_{m\pm s}] &= \left(\frac{n}{2} - (m \pm s)\right) G^{\pm}_{m+n\pm a} \\
[J_n, G^{\pm}_{m\pm s}] &= \pm G^{\pm}_{n+m\pm s} \\
\{G^{+}_{n+s}, G^{-}_{m-s}\} &= 2L_{m+n} + (n-m+2s)J_{n+m} + \frac{\mathrm{c}}{3}\left((n+s)^2 - \frac{1}{4}\right)\delta_{m+n,0}
\end{aligned} \tag{2.1}
$$

Here s is a continuous parameter with range $0 \leq s < 1$ and c is the CFT central charge. Every pair of values for the parameter s yields a pair of isomorphic algebras. Such isomorphisms induce a continuous operation on the spectrum, which we refer to the *spectral flow*, that provides an equivalence between the Ramond (R) sector ($s = 0$) and the Neveu-Schwarz (NS) sector ($s = 1/2$). This map furnishes symmetry between spacetime bosons and spacetime fermions, i.e., the spectral flow induces spacetime supersymmetry.

The chiral and anti-chiral rings: In the representation theory of a $N = 2$ superconformal algebra, there arises a substructure with ring properties that will be relevant to us. But first, let us define the highest weight states $\{|\phi\rangle\}$ as those states that satisfy

$$
\begin{array}{cccc}
L_n\,|\phi\rangle = 0, \qquad & G^{\pm}_{r}\,|\phi\rangle = 0, \qquad & J_m\,|\phi\rangle = 0, \qquad & n, r, m > 0 \\
L_0\,|\phi\rangle = h_\phi\,|\phi\rangle\,, \qquad & J_0\,|\phi\rangle = q_\phi\,|\phi\rangle & & .
\end{array} \tag{2.2a}
$$

Here h_ϕ and q_ϕ are the eigenvalues of the zero index modes L_0 and J_0 respectively, i.e., conformal weight and $U(1)$ charge. A *chiral primary* field state is a highest weight state $|\phi\rangle$ that satisfies the additional condition

$$
G^{+}_{-\frac{1}{2}}\,|\phi\rangle = 0, \tag{2.3}
$$

Similarly, we refer to *anti-chiral* primaries as those highest weight states annihilated by $G^{-}_{-1/2}$. The requirement of a unitary theory, together with the superconformal algebra, imposes on the (anti-)chiral primary field states the constraints

$$
h_\phi \geq \frac{q_\phi}{2}, \qquad h_\phi \leq \frac{\mathrm{c}}{6} \tag{2.4}
$$

Recall the operator-state 1:1 correspondence $\phi \leftrightarrow |\phi\rangle$. From the constraint $h_\phi \geq q_\phi/2$ and the $U(1)$-charge conservation, the operator product expansion (OPE) of two (anti-)chiral primary fields ϕ_i and ϕ_j has the form

$$
\phi_i(z) \cdot \phi_j(w) = \sum_p C^p_{ij}\psi_p(w)\,, \tag{2.5}
$$

where each ψ_p is a (anti-)chiral primary field of conformal weight $h_p = h_i + h_j$ and $U(1)$-charge $q_p = q_i + q_j$. The product (2.5) defines a ring for the set of (anti-) chiral primary fields, which we refer to the *(anti-)chiral ring*. In an $N = (2, 2)$ superconformal field theory, we can combine the (anti-)chiral ring in the holomorphic sector with the (anti-)chiral ring of the anti-holomorphic sector. We denote all possible combinations by $(c,c),(a,c),(a,a)$ and (c,a), where c stands for chiral ring, a for the anti-chiral ring, and the pairing of sectors goes as (holomorphic, anti-holomorphic).

Topological field theories: A non-linear σ-model (NLSM) is a geometrical realization of a 2d $N=(2,2)$ superconformal field theory. It is a field theory of bosons ϕ and fermions ψ,χ related by supersymmetry. On the one hand, the bosonic fields are maps $\phi:\Sigma_g\to M$, where Σ_g denotes a Riemann surface of genus g, whereas M is a Kähler manifold of complex dimension n that we call *target space*. On the other hand, the fermions are sections of the following type:

$$\begin{aligned}
\psi^i &\in \Gamma\left(\Sigma_g, K^{\frac{1}{2}}\otimes\phi^*(T^{1,0}M)\right)\,, \quad \psi^{\bar{\imath}} \in \Gamma\left(\Sigma_g, K^{\frac{1}{2}}\otimes\phi^*(T^{0,1}M)\right)\,, \\
\chi^i &\in \Gamma\left(\Sigma_g, \bar{K}^{\frac{1}{2}}\otimes\phi^*(T^{1,0}M)\right)\,, \quad \chi^{\bar{\imath}} \in \Gamma\left(\Sigma_g, \bar{K}^{\frac{1}{2}}\otimes\phi^*(T^{0,1}M)\right)\,.
\end{aligned} \tag{2.6}$$

Here K and $\bar{K}$ are the canonical and anti-canonical line bundles over Σ_g, while $K^{\frac{1}{2}}$ and $\bar{K}^{\frac{1}{2}}$ their square roots. We say that a NLSM is topological σ-model of the cohomological type, if there exists a Grassmann nilpotent charge operator $\mathcal{Q}$ and a $\mathcal{Q}$-exact energy-momentum tensor T^{top} such that

$$T^{\mathrm{top}}_{\alpha\beta} = \{\mathcal{Q}, G_{\alpha\beta}\}\,. \tag{2.7}$$

Here $\{\cdot\,,\cdot\}$ is the anti-commutator operator, and $G_{\alpha\beta}$ is some symmetric local fermionic operator. In a topological field theory of the cohomological type, physical observables $\mathcal{O}_i$ are closed under the action of the $\mathcal{Q}$-operator, i.e. $\{\mathcal{Q},\mathcal{O}_i\}=0$. A direct consequence of (2.7) is that correlation functions do not depend on the worldsheet metric h of Σ, as

$$\langle T^{\mathrm{top}}_{\alpha\beta}\mathcal{O}_1\cdots\mathcal{O}_k\rangle \propto \frac{\delta}{\delta h_{\alpha\beta}}\langle\mathcal{O}_1\cdots\mathcal{O}_k\rangle = 0\,. \tag{2.8}$$

Topological twisting: Restricting an $N=(2,2)$ superconformal field theory to its (anti-)chiral rings yields a topological field theory. A construction for such a statement goes as follows. For the (a,c) or (c,c) rings, we define—respectively—the Q_A and Q_B cohomology operators

$$\begin{aligned}
(a,c):\ \ Q_{\mathrm{A}} &:= G^-_{-\frac{1}{2}} + \overline{G}^+_{-\frac{1}{2}}\,, \\
(c,c):\ \ Q_{\mathrm{B}} &:= G^+_{-\frac{1}{2}} + \overline{G}^+_{-\frac{1}{2}}\,.
\end{aligned} \tag{2.9}$$

Here we denote by $\bar{L}_n$, $\overline{G}^{\pm}_r$, and $\bar{J}_n$ the Fourier modes in the anti-holomorphic sector that satisfy the same $N=2$ superconformal algebra (2.1). The operators (2.9) fulfill the nilpotence property with fields in the associated (anti-)chiral rings closed under $\mathcal{Q}$-action. However, the respective energy-momentum tensors are not $\mathcal{Q}$-exact under any of the choices (2.9). To overcome this problem, we introduce the so-called *topological twist* [9]. The result amounts to redefine the energy-momentum tensors as in Table 2.1, where $\overline{T}$ and $\bar{J}$ denote energy-momentum tensor and $U(1)$-current generator in the anti-holomorphic sector. Thus, the resulting theories are topological string theories. The topological twist on the left side of Table 2.1 gives rise to the A-model, whose target space is a Calabi-Yau n-fold M and its (a,c)-ring is identified with elements of de Rham Cohomology $H^*_{DR}(M)$. On the other hand, the right side topological twist in 2.1 gives the B-model, which has a Calabi-Yau

A	B
$T \to T + \frac{1}{2}\partial J$	$T \to T - \frac{1}{2}\partial J$
$\overline{T} \to \overline{T} - \frac{1}{2}\bar{\partial}\bar{J}$	$\overline{T} \to \overline{T} - \frac{1}{2}\bar{\partial}\bar{J}$
(a, c)	(c, c)

Table 2.1: Topological twists.

n-fold target space W and its (c, c)-ring corresponds to the Dolbeault cohomology of W. We summarize such equivalence of rings as follows:

$$\text{Top. string theory:} \begin{cases} \text{A-model:} \quad \phi_{\mathcal{A}} \in (a, c) \quad \xleftrightarrow{1:1} \quad \mathcal{A} \in \bigoplus_{k=1}^{n} H^{k,k}(M) \\ \\ \text{B-model:} \quad \phi_{\mathcal{B}} \in (c, c) \quad \xleftrightarrow{1:1} \quad \mathcal{B} \in \bigoplus_{p,q=0}^{n} H^{0,p}(W, \wedge^q TW) \end{cases} \tag{2.10}$$

Mirror symmetry: We can perform a deformation of a topological string theory such that it gives another superconformal field theory (SCFT) of the same type. From the SCFT perspective, we do this by perturbing the action through the so-called *marginal operators*. This way, smooth deformations sweep over a deformation family of SCFTs $\mathcal{M}$. Geometrically, in the A-model, such deformations are captured by the complexified Kähler moduli

$$t^k = \int_{C_k} \mathrm{B} + i\mathcal{J}\,, \quad k = 1, \dots, h^{1,1}(M)\,, \tag{2.11}$$

where $C_k \in H_2(M)$, $\mathrm{B} \in H^2(M, \mathbb{C})/H^2(M, \mathbb{Z})$ is the Neveu-Schwarz B-field, and $\mathcal{J}$ is the Kähler form of M. Accordingly, the A-model deformation family is the Kähler moduli space $\mathcal{M}_{\text{ks}}(M)$. On the B-model side, the complex structure moduli of W parametrize deformations, which means that the complex structure moduli space $\mathcal{M}_{\text{cs}}(W)$ encodes the B-model deformation family. Now, let us note the equivalence of A-model and B-model as a mere choice of sign for the holomorphic $U(1)$-current, i.e., $J \mapsto -J$ exchanges both descriptions. As a consequence of this, both deformation families are the same and, in fact, mirror symmetry asserts that $\mathcal{M}_{\text{ks}}(M)$ and $\mathcal{M}_{\text{cs}}(W)$ are isomorphic. In that case, we say that the pair of Calabi-Yau manifolds (M, W) are mirror dual to each other, which is a non-trivial property since it relates two geometrical that seem different. In particular, the A-model correlation functions receive quantum corrections upon deformations, while their B-model counterparts do not. In the A-model geometrical picture, this translates into a counting problem over holomorphic maps $\Phi : \Sigma \to C_\kappa \subset M$, where C_κ is a curve of class $\kappa \in H_2(M, \mathbb{Z})$. In contrast to that, Hodge structure variations on W control the B-model deformations, including its correlation functions.

To appreciate the power of mirror symmetry, let us consider in the A-model the genus g amplitude F_g which has the form [4]

$$F_g(\boldsymbol{t}) = \sum_{\kappa \in H_2(M,\mathbb{Z})} N_{g,\kappa} \mathrm{e}^{2\pi i \int_\kappa \mathrm{B} + i\mathcal{J}}\,, \quad \text{where} \quad N_{g,\kappa} = \int_{[\overline{\mathcal{M}}_g(M,\kappa)]^{\text{vir}}} 1 \in \mathbb{Q}\,. \tag{2.12}$$

Here $\overline{\mathcal{M}}_g(M, \kappa)$ is the moduli space of stable maps from connected genus g curves of degree

$\kappa \in H_2(M,\mathbb{Z})$ to M, and $[-]^{\mathrm{vir}}$ is its virtual fundamental class. The numbers $N_{g,\kappa}$ are called Gromov-Witten invariants and their study is a subject itself in the realm of enumerative geometry. In general, these invariants are hard to compute, but their calculation becomes accessible by invoking mirror symmetry. On the B-model side, we obtain the same topological amplitudes by solving a set of differential equations, the Picard-Fuchs system, together with a series of recursive differential equations [10]. We will come back to counting invariants in section 2.5. For the moment, we focus now on the properties of Calabi-Yau geometries and methods to construct them.

2.2 Calabi-Yau geometry

In the forthcoming, we will consider Calabi-Yau manifolds to describe the internal space of string theories. Note that we do not restrict the discussion here to the case of topological string theory. With this in mind, we state our working definition next:

(a) **Definition:** Let M be a Kähler manifold of complex dimension n. We say that M is Calabi-Yau manifold if its first Chern class is trivial, i.e. $c_1(TM) = 0$.

In the literature, it is common to encounter partly equivalent definitions for Calabi-Yau manifolds. Depending on the context, a given formulation can be more convenient. In any case, a Kähler manifold M that is Calabi-Yau is also characterized by the properties:

(b) For each Kähler class of M, there exists a unique Kähler metric g whose Ricci tensor vanishes, i.e., $R_{i\bar{j}}(g) = 0$.

(c) The canonical class K_M is trivial.

(d) There exists a unique nowhere vanishing holomorphic $(n,0)$-form, up to a constant, which we denote by Ω.

(e) The holonomy group $\mathrm{Hol}(g)$ of M is a subgroup of $SU(n)$.

(f) M admits a pair of covariantly constant spinors $\epsilon, \bar{\epsilon}$, which are globally defined. The latter have opposite (same) chirality when n is odd (even).

Strictly speaking, only the definitions (a), (b), (c), and (d) are equivalent, whereas the equivalence of the latter with (e) and (f) is ensured by assuming that M is simply connected and not of a product form. Proves for the equivalence of these definitions have been exposed in the survey [11].

As a historical remark, Eugenio Calabi formulated first his famous conjecture, which we state as follows:

CY : Let M be a compact Kähler manifold with Kähler metric g, Kähler form ω, and Ricci form $\mathcal{R}$. To every closed $(1,1)$-form $\mathcal{R}'$ that represents the first Chern class, i.e., $[\mathcal{R}] = [\mathcal{R}'] = 2\pi c_1(TM)$, there exists a unique metric Kähler metric g' with associated Kähler form ω', such that $[\omega'] = [\omega] \in H^2(M,\mathbb{R})$, and the Ricci tensor of g' is $\mathcal{R}'$.

	T^k	$K3$ (CY 2-fold)	CY 3-fold	CY 4-fold
SUSY	1	1/2	1/4	1/8

Table 2.2: Reduction factor of supercharges.

This conjecture was proved by Shing-Tung Yau in [12]. In particular, a vanishing first Chern-class, as in definition (a), implies the existence of a unique Ricci flat metric. See definition (b).

For the physicist interests, one approach to achieve appealing physics models from string theory is via a compactification ansatz $\mathbb{R}^{1,D-1} \to \mathbb{R}^{1,D-2n-1} \times M$, where we take M to be a Calabi-Yau n-fold. In doing this, we choose a given string theory on $\mathbb{R}^{1,D-1}$ and note that its spinors representations decompose as follows

$$\mathfrak{so}(1, D-1) \to \mathfrak{so}(1, D-2n-1) \oplus \mathfrak{so}(2n)\,. \tag{2.13}$$

Here, on the right hand side, the first entry corresponds to representations of the effective theory and the second entry to those of the internal space M. The holonomy group $H(g)$ of M acts on the $\mathfrak{so}(2n)$-representations and the remaining amount of supercharges derives from the number of covariantly constant spinor components on M. We summarize in Table 2.2 the reduction factor of supercharges for the compactifcation manifolds we will consider in this work [13].

As a last remark, note that Calabi-Yau manifolds generalize to non-compact spaces. As we will see, this type of Calabi-Yau will also be of physics interest. Namely, when geometric engineering quantum field theories, which arise by decoupling gravity in certain string theory settings [14]. Having said this, we proceed to explain our methods to construct Calabi-Yau geometries explicitly.

2.3 Toric geometry

The most systematic approach for constructing Calabi-Yau manifolds, up to date, is through subspaces defined by one or more algebraic equations in a toric ambient space. Toric varieties are a class of algebraic varieties encoded entirely by combinatorial data, e.g., weighted projective spaces and their products. Moreover, toric methods provide tools that allow for the classification and characterization of explicit physics models realized by string theory. Furthermore, mirror symmetry has developed next to Batyrev-Borisov toric mirror constructions, a computational tool of our concern.

To explain our geometric and mirror symmetry setups, we will present in this section a summary of the basics of toric geometry. For this, we take as references the sources [4, 15, 16]. Let us start by presenting the definition of a toric variety:

Definition: A toric variety X is a complex algebraic variety that contains an algebraic torus $\mathsf{T} = (\mathbb{C}^*)^r$ as a Zariski open subset, together with an action of T on X whose restriction to $\mathsf{T} \subset X$ is the usual multiplication on T.

By a dense subset, we mean that $\overline{\mathsf{T}} = X$, where $\overline{\mathsf{T}}$ is the closure of T. There are two main approaches for constructing toric varieties:

1. Employing a *fan* Σ.

2. Utilizing lattice points in a *polytope* Δ.

Both approaches turn out to be relevant for constructing mirror manifolds. In the following, we explain the recipe for realizing the torus T in both constructions.

An essential feature of fans and polytopes is that of a lattice $\mathsf{N} \simeq \mathbb{Z}^r$ of rank r, together with its dual lattice $\mathsf{M} = \mathrm{Hom}(\mathsf{N}, \mathbb{Z})$. These lattices have a canonical pairing via an intersection form $\langle \cdot , \cdot \rangle : \mathsf{M} \times \mathsf{N} \to \mathbb{Z}$. On the one hand, the lattice N defines an algebraic torus $\mathsf{T}_\mathsf{N} \equiv \mathsf{N} \otimes_\mathbb{Z} \mathbb{C}$ via the canonical isomorphism

$$\mathsf{N} \otimes_\mathbb{Z} \mathbb{C}^* \simeq (\mathbb{C}^*)^r \,, \quad \boldsymbol{v} \otimes t \mapsto (t^{v_1}, \ldots t^{v_r}) \,, \tag{2.14}$$

where $\boldsymbol{v} \in \mathsf{N}$ with $\boldsymbol{v} = (v_1 \,, \ldots \,, v_r)$ in a given $\mathbb{Z}$-basis and $t \in \mathbb{C}^*$. On the other hand, the dual lattice M describes rational functions on the torus T_N. To see this, let us consider a $\boldsymbol{m} \in \mathsf{M}$ with $\boldsymbol{m} = (m_1 \,, \ldots \,, m_r)$ in a given $\mathbb{Z}$-basis. Then, the canonical pairing $\langle \cdot , \cdot \rangle$ induces a natural morphism $\chi^{\boldsymbol{m}} : \mathsf{T}_\mathsf{N} \to \mathbb{C}^*$, which is defined as the following set

$$\chi^{\boldsymbol{m}} (t_1 \,, \ldots \,, t_r) = t_1^{m_1} \cdots t_r^{m_r} \in \mathbb{C}[t_1^{\pm 1} \,, \ldots \,, t_r^{\pm 1}] \,. \tag{2.15}$$

We call the elements $\chi^{\boldsymbol{m}} \in \mathrm{Hom}(\mathsf{T}_\mathsf{N}, \mathbb{C}^*) \simeq \mathsf{M}$ the *characters* of the torus T_N. Their relevance will become clear in the upcoming sections.

2.3.1 Fans

Before proceeding to define fans, we need to introduce some of their building blocks that we define next:

- A *convex polyhedral cone* (or simply cone) σ in $\mathsf{N}_\mathbb{R} \equiv \mathsf{N} \otimes_\mathbb{Z} \mathbb{R}$ is defined by a finite set of generators $\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_k\} \subset \mathsf{N}_\mathbb{R}$ as follows

$$\sigma = \mathrm{Cone}\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_k\} = \left\{ \sum_{i=1}^k \lambda_i \boldsymbol{v}_i \;\mid \lambda_i \in \mathbb{R}_{\geq 0} \right\} . \tag{2.16}$$

- We say that a cone σ is rational if $\sigma = \mathrm{Cone}\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_k\}$ with $\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_k\} \subset \mathsf{N}$.

- We say that a cone σ is a *strongly convex* whenever $\sigma \cap (-\sigma) = \{0\}$.

- To every cone σ we define its *dual cone* σ^* as the following set

$$\sigma^* = \{\boldsymbol{u} \in \mathsf{N}_\mathbb{R}^* \;\mid\; \langle \boldsymbol{u}, \boldsymbol{v} \rangle \geq 0 \quad \forall \boldsymbol{v} \in \sigma\} \,. \tag{2.17}$$

- A face τ of a cone σ is the set

$$\tau = \sigma \cap \boldsymbol{u}^\perp = \{\boldsymbol{v} \in \sigma \;\mid\; \langle \boldsymbol{u}, \boldsymbol{v} \rangle = 0\} \,, \text{ for some } \boldsymbol{u} \in \sigma^* \,. \tag{2.18}$$

With this information at hand, we now introduce the concept of fans:

Definition: A fan Σ is a finite collection of strongly convex rational polyhedral cones in $\mathsf{N}_{\mathbb{R}}$ such that:

1. Each face τ of a cone $\sigma \in \Sigma$ is contained in Σ.
2. The intersection of two cones in Σ is a face of each.

To appreciate how a fan gives rise to a toric variety, let us consider a given $\sigma \in \Sigma$. Then, we focus on the restriction of its dual σ^* to the lattice points in M, which we denote by $\mathsf{S}_\sigma \equiv \sigma^* \cap \mathsf{M}$. Thus, each lattice point $\boldsymbol{u} \in \mathsf{S}_\sigma$ defines a torus character of T_N. This way, the set of lattice points S_σ defines the $\mathbb{C}$-vector space of Laurent polynomials

$$\mathbb{C}[\mathsf{S}_\sigma] = \left\{ \sum_{\boldsymbol{u} \in \mathsf{S}_\sigma} s_{\boldsymbol{u}} \chi^{\boldsymbol{u}} \;\middle|\; s_{\boldsymbol{u}} \in \mathbb{C} \right\} . \tag{2.19}$$

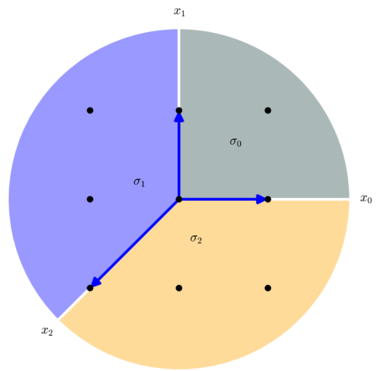
Due to the strongly convex property of the cone $\sigma \in \Sigma$, the affine irreducible variety $U_\sigma \equiv \mathrm{Spec}\left(\mathbb{C}\left[\mathsf{S}_\sigma\right]\right)$ is a normal toric variety of dimension $r = \dim(\mathsf{N})$, whose algebraic torus is $\mathsf{T}_\mathsf{N} \simeq (\mathbb{C}^*)^r$.[1] Now, the idea behind a fan Σ is that it encodes the information to glue affine toric varieties $\{U_\sigma\}_{\sigma\in\Sigma}$ to create an *abstract toric variety* X_Σ:[2]

$$X_\Sigma = \bigcup_{\sigma\in\Sigma} U_\sigma \,. \tag{2.20}$$

Before we proceed to discuss an example, we highlight three properties of the fan Σ that encode relevant information about the toric variety X_Σ:

1. X_Σ is compact iff Σ is *complete*, i.e. $|\Sigma| = \bigcup_{\sigma\in\Sigma} \sigma = \mathsf{N}_\mathbb{R}$.
2. X_Σ is smooth iff every $\sigma \in \Sigma$ is generated by a subset of a $\mathbb{Z}$-basis of N.
3. X_Σ has only finite quotient singularities, i.e. is an orbifold, iff Σ is *simplicial*. In this case the minimal generators of every cone $\sigma \in \Sigma$ are linearly independent over $\mathbb{R}$.

Example: The toric variety $X_\Sigma = \mathbb{P}^2$



[1] This assertion follows from theorem 1.2.18 in [16].

[2] A collection of data $(\{V_\alpha\}_\alpha, \{V_{\alpha,\beta}\}_{\alpha,\beta}, \{g_{\alpha\beta}\}_{\alpha\beta})$ with V_α an affine variety, each $V_{\alpha\beta} \subset V_\alpha$ is Zariski open, and isomorphisms $g_{\alpha\beta} : V_{\alpha\beta} \to V_{\beta\alpha}$ such that $g_{\alpha\alpha} = 1_{V_\alpha}$ $\forall\alpha$ and $g_{\beta\gamma}|_{V_{\beta\alpha}\cap V_{\beta\gamma}} \circ g_{\alpha\beta}|_{V_{\alpha\beta}\cap V_{\alpha\gamma}} = g_{\alpha\gamma}|_{V_{\alpha\beta}\cap V_{\alpha\gamma}}$ is required. This way, an abstract variety X can be obtained by *gluing* the V_α along the $V_{\alpha\beta}$ with the $g_{\alpha\beta}$. In this case gluing works, because for a pair $\sigma, \sigma' \in \Sigma$ we have that $U_{\sigma\cap\sigma'} = U_\sigma \cap U_{\sigma'} \simeq U_{\sigma'\cap\sigma}$.

Figure 2.1: The fan Σ for $\mathbb{P}^2$.

We consider the fan in $\mathsf{N} = \mathbb{Z}^2$ spanned by vectors $\{\boldsymbol{v}_0, \boldsymbol{v}_1, \boldsymbol{v}_2\} \subset \mathbb{Z}^2$, where

$$\boldsymbol{v}_0 = (1\,,0)\ , \quad \boldsymbol{v}_1 = (0\,,1)\ , \quad \boldsymbol{v}_2 = (-1\,,-1)\ . \tag{2.21}$$

These vectors are indicated by the blue arrows in Figure 2.3.1. Thus, the full fan Σ is the set $\Sigma = \{\{0\}, \tau_0, \tau_1, \tau_2, \sigma_0, \sigma_1, \sigma_2\}$, where $\tau_a = \text{Cone}\{\boldsymbol{v}_a\}$ with $a = 0, 1, 2$, and

$$\sigma_0 = \text{Cone}\{\boldsymbol{v}_0, \boldsymbol{v}_1\}\,, \quad \sigma_1 = \text{Cone}\{\boldsymbol{v}_1, \boldsymbol{v}_2\}\,, \quad \sigma_2 = \text{Cone}\{\boldsymbol{v}_0, \boldsymbol{v}_2\}\,. \tag{2.22}$$

A straighforward computation reveals that the dual cones to (2.22) read

$$\sigma_0^* = \text{Cone}\{\boldsymbol{v}_0, \boldsymbol{v}_1\}\,, \quad \sigma_1^* = \text{Cone}\{-\boldsymbol{v}_1, -\boldsymbol{v}_1 + \boldsymbol{v}_2\}\,, \quad \sigma_2^* = \text{Cone}\{\boldsymbol{v}_1 - \boldsymbol{v}_2, -\boldsymbol{v}_2\}\,. \tag{2.23}$$

Consequently, we obtain the associated affine toric varieties that read

$$\begin{aligned} U_{\sigma_0} &= \text{Spec}\left(\mathbb{C}[\mathsf{S}_{\sigma_0}]\right) \simeq \text{Spec}\left(\mathbb{C}[X, Y]\right) \simeq \mathbb{C}^2\,, \\ U_{\sigma_1} &= \text{Spec}\left(\mathbb{C}[\mathsf{S}_{\sigma_1}]\right) \simeq \text{Spec}\left(\mathbb{C}[X^{-1}, X^{-1}Y]\right) \simeq \mathbb{C}^2\,, \\ U_{\sigma_2} &= \text{Spec}\left(\mathbb{C}[\mathsf{S}_{\sigma_1}]\right) \simeq \text{Spec}\left(\mathbb{C}[XY^{-1}, Y^{-1}]\right) \simeq \mathbb{C}^2\,. \end{aligned} \tag{2.24}$$

By checking how these toric varieties fit along $U_{\sigma_i \cap \sigma_j}$, we get the construction for $\mathbb{P}^2$. Morever, $\mathbb{P}^2$ is compact, smooth, and all cones in its associated fan Σ are simplicial.

There is an alternative description of an abstract toric variety X_Σ, which is more convenient for us in practice. We call this approach the *quotient construction*. The recipe goes as follows. Let $\Sigma(1) \subset \Sigma$ be the set of one dimensional cones in Σ. For each $\rho \in \Sigma(1)$, we denote by $\boldsymbol{v}_\rho \in \rho \cap \mathsf{N}$ the corresponding generators that span the fan Σ. To each of these vectors we associate the coordinates x_ρ in the polynomial ring $\mathbb{C}\left[x_\rho : \rho \in \Sigma(1)\right]$. Then the toric variety (2.20) is isomorphic to the following quotient space [16]

$$X_\Sigma \simeq \left(\mathbb{C}^{\Sigma(1)} - Z\left(\Sigma\right)\right)/\mathsf{G}\,, \tag{2.25}$$

where $\mathsf{G} = \text{Hom}_{\mathbb{Z}}\left(A_{r-1}\left(X_\Sigma\right), \mathbb{C}^*\right)$, and $Z(\Sigma)$ is the exceptional set that tells us where the homogenous coordinates $\{x_\rho\}$ are not allowed to vanish simultaneously. More precisely, for each cone $\sigma \in \Sigma$, there is a monomial $x_{\hat{\sigma}} := \prod_{\rho \not\subset \sigma} x_\rho$; the exceptional subset is defined by

$$Z(\Sigma) = \{x_{\hat{\sigma}} = 0 \mid \sigma \in \Sigma\}\,. \tag{2.26}$$

We omit here the details for the derivation for the result (2.25), but refer to [16]. Instead, we elaborate more about the reductive group G.

The torus action on X_Σ defines a finite set of orbits that have a 1:1 correspondence with cones in Σ; for each cone $\sigma \in \Sigma$ we have that $\dim(\sigma) + \dim(O) = r$, where O is the associated torus orbit to the cone σ [16]. In particular, each ray $\rho \in \Sigma(1)$ defines a codimension-1 irreducible subvariety that reads [17]

$$D_\rho = \{x_\rho = 0\} \subset X_\Sigma\,. \tag{2.27}$$

This means that D_ρ is a T-invariant divisor in X_Σ, which we refer to *toric divisor*. Recall that each $\boldsymbol{u} \in \mathsf{M}$ gives a torus character $\chi^{\boldsymbol{u}} : \mathsf{T}_\mathsf{N} \to \mathbb{C}^*$, which is a rational map on X_Σ. It turns out that we can express the divisor of $\chi^{\boldsymbol{u}}$ in the following way [15]

$$\operatorname{div}(\chi^{\boldsymbol{u}}) = \sum_{\rho\in\Sigma(1)} \langle \boldsymbol{u}, \boldsymbol{v}_\rho\rangle D_\rho\,. \tag{2.28}$$

Moreover, we can obtain the Chow group $A_{r-1}(X_\Sigma)$ from the set of toric divisors $\{D_\rho\}_{\rho\in\Sigma(1)}$ via the exact sequence [15]

$$\mathsf{M} \xrightarrow{\ f\ } \bigoplus_{\rho\in\Sigma(1)} \mathbb{Z}\cdot D_\rho \xrightarrow{\ g\ } A_{r-1}\left(X_\Sigma\right) \longrightarrow 0\,, \tag{2.29}$$

where $f_1 : \boldsymbol{u} \mapsto \operatorname{div}(\chi^{\boldsymbol{u}})$, and f_2 is the map that takes a Weil divisor to its class in the Chow group. Furthermore, an application $\operatorname{Hom}_{\mathbb{Z}}(\bullet, \mathbb{C}^*)$ to the exact sequence (2.29) gives the result

$$1 \longrightarrow G \longrightarrow (\mathbb{C}^*)^{\Sigma(1)} \longrightarrow \mathsf{T}_{\mathsf{N}}\,, \tag{2.30}$$

which is also an exact sequence. Here we recall that $\mathsf{T}_\mathsf{N} = \mathsf{N}\otimes_{\mathbb{Z}}\mathbb{C}^* = \operatorname{Hom}_{\mathbb{Z}}(\mathsf{M}, \mathbb{C}^*)$. Thus, we obtain that $\mathsf{T}_\mathsf{N} \simeq (\mathbb{C}^*)^{\Sigma(1)}/G$. Accordingly, the quotient group is of the form $G \simeq (\mathbb{C}^*)^s \times H$, where $s = |\Sigma(1)| - r$, and H is a finite abelian group that appears when $A_{r-1}(X_\Sigma)$ has torsion. It is this first factor that leads to identify points in the quotient (2.25) via the action

$$x_\rho \mapsto \lambda_a^{l_\rho^{(a)}} x_\rho\,, \quad \lambda_a \in \mathbb{C}^*\,, \quad a = 1, \dots, s\,, \tag{2.31}$$

where $l_\rho^{(a)}$ are integer values that we specify in the upcoming section.

2.3.2 The Mori cones of toric varieties

As an interlude, we introduce a set that will be relevant for our calculations. Again, we do not intend to be rigorous here, but to provide useful results.

Mori cone: Let X_Σ be simplicial and complete. In this case the dual to the Kähler cone is the Mori cone $\overline{\mathrm{NE}}(X_\Sigma)$, which is the cone of effective 1-cycles in $A_1(X_\Sigma)\otimes\mathbb{R} \simeq H_2(X_\Sigma, \mathbb{R})$. We recall the torus orbit 1:1 correspondence with cones in Σ, in which we associate each cone $\sigma \in \Sigma(r-1)$ with a torus invariant curve C_σ. The *Toric cone theorem* states that the set of curves $C_\sigma \subset X_\Sigma$ span $\overline{\mathrm{NE}}(X_\Sigma)$ as follows [18]

$$\overline{\mathrm{NE}}(X_\Sigma) = \sum_{\sigma\in\Sigma(r-1)} \mathbb{R}_{\geq 0}[C_\sigma]\,, \tag{2.32}$$

where $[C]$ denotes a class of irreducible complete curves in X_Σ that are *numerically equivalent*. The latter condition means that $C, C' \in [C]$ if $(C - C')\cdot D = 0$ for all Cartier divisors D in X_Σ.

Let us introduce the following space of linear relations among ray generators

$$\mathsf{L}_\mathbb{R} = \left\{ \boldsymbol{l} \in \mathbb{R}^{\Sigma(1)} \mid \sum_{\rho\in\Sigma(1)} l_\rho \boldsymbol{v}_\rho = 0 \right\}\,, \tag{2.33}$$

which follows the equivalence

$$\mathsf{L}_\mathbb{R} \simeq A_1(X_\Sigma)\otimes\mathbb{R}\,. \tag{2.34}$$

Let us consider a basis $\{[C_a]\}_{a=1,\dots s}$ of $A_1(X_\Sigma)$. When X_Σ is smooth, then the intersection numbers

$$l_\rho^{(a)} = D_\rho \cdot C_a\,, \quad a = 1,\dots s\,, \tag{2.35}$$

are relatively prime integers that give rise to the linear relations [17]

$$\sum_{\rho\in\Sigma(1)} l_\rho^{(a)} \boldsymbol{v}_\rho = 0\,. \tag{2.36}$$

Thus, given the appropriate conditions for X_Σ, the linear relations (2.36) provide us the precise form of the $(\mathbb{C}^*)^s$-action in (2.31).

Let us denote the set of ray generators by $\{\boldsymbol{v}_1,\dots,\boldsymbol{v}_{r+s}\}$. In the literature, it is a common practice to introduce the matrices

$$P \mid Q = \left(\begin{array}{ccc|ccc} \leftarrow & \boldsymbol{v}_1 & \rightarrow & \uparrow & & \uparrow \\ & \vdots & & \boldsymbol{l}^{(1)} & \dots & \boldsymbol{l}^{(s)} \\ \leftarrow & \boldsymbol{v}_{r+s} & \rightarrow & \downarrow & & \downarrow \end{array}\right)\,. \tag{2.37}$$

Here the P entries encode the information about the fan Σ, while Q gives the data to reconstruct the $(\mathbb{C}^*)^s$-action and the Mori cone. The entries $Q_{i,a}$ defining the matrix Q have an interpretation as charges of chiral fields of a 2d $\mathcal{N}=(2,2)$ *gauge linear sigma model* [19, 20] with $U(1)^s$ gauge symmetry. We will not be interested in describing these theories, but only refer to them for conventional reasons.

2.3.3 Polytopes

Yet another necessary ingredient for Calabi-Yau mirror pairs construction is polytopes. A polytope in $\mathbb{R}^r$ is simply the convex hull of a finite set of points in $\mathbb{R}^n$. More useful for us is the definition next:

Definition: A *lattice polytope* $\Delta \subset \mathsf{N}_\mathbb{R}^*$ is the convex hull of a finite set of points in M.

Note that the dimension of a lattice polytope Δ is that of the smallest subspace of $\mathsf{M}_\mathbb{R} \simeq \mathbb{R}^r$ that contains Δ. For concreteness, we stick here to r-dimensional polytopes.

An alternative way to describe a polytope is in terms of its faces. A *proper face* F of a lattice polytope Δ is the intersection of Δ with a supporting affine hyperplane. In other words, we have that

$$\mathsf{F} = \{\boldsymbol{u}\in\Delta \mid \langle \boldsymbol{u}, \boldsymbol{v}_\mathsf{F}\rangle = -a_\mathsf{F}\}\,, \tag{2.38}$$

where $\boldsymbol{v}_\mathsf{F}$ are some normal non-zero vectors in the dual lattice N, and $a_\mathsf{F}\in\mathbb{Z}$. We can obtain all faces from *facets*, which are codimension-1 faces, because every proper face F of Δ is the intersection of facets that contain F. In particular, a *vertex* is a codimension-0 face (where r facets meet). Now, let us introduce the closed half-space F^+ associated with a face $\mathsf{F}\subset\Delta$ that reads

$$\mathsf{F}^+ = \{\boldsymbol{u}\in\mathsf{M}_\mathbb{R} \mid \langle \boldsymbol{u}, \boldsymbol{v}_\mathsf{F}\rangle \geq -a_\mathsf{F}\}\,. \tag{2.39}$$

From this, we have two representations for the lattice polytope Δ:

$$\Delta = \bigcap_{\text{All facets }\mathsf{F}} \mathsf{F}^+ = \mathrm{Conv}(\text{vertices})\,. \tag{2.40}$$

Now, we establish the connection between lattice polytopes and toric varieties. We note that the normal vectors $\{\boldsymbol{v}_\mathsf{F}\}$, which define the affine hyperplane of a face F, span a strongly convex rational cone σ_F. Each element $\boldsymbol{v}_\mathsf{f} \in \{\boldsymbol{v}_\mathsf{F}\}$ corresponds to a facet $\mathsf{f} \supset \mathsf{F}$. This way, we generate the so-called *normal fan* Σ_Δ associated with the lattice polytope Δ, where

$$\Sigma_\Delta = \{\sigma_\mathsf{F} \mid \mathsf{F} \text{ is a face of } \Delta\} \, . \tag{2.41}$$

We denote by X_Δ the abstract toric variety that results by gluing affine toric varieties $\{U_{\sigma_\mathsf{F}}\}_{\sigma_\mathsf{F}\in\Sigma_\Delta}$. Moreover, we have the following correspondence:

$$\begin{cases} \Sigma_\Delta(1) & \longleftrightarrow & \{\text{Facets of } \Delta\} \\ \Sigma_\Delta(r) & \longleftrightarrow & \{\text{Vertices of } \Delta\} \end{cases} . \tag{2.42}$$

Here the notation $\Sigma(a)$ indicates the subset of a-dimensional cones in a fan Σ. This implies there is a toric divisor on X_Δ that corresponds to a facet of Δ. In fact, each lattice polytope Δ_D determines a Cartier divisor D that reads

$$D = \sum_{\rho\in\Sigma_\Delta(1)} a_\rho D_\rho \quad \longleftrightarrow \quad \Delta = \{\boldsymbol{u} \in \mathsf{N}^*_\mathbb{R} \mid \langle \boldsymbol{u}, \boldsymbol{v}_\rho\rangle \geq -a_\rho \ \forall\ \rho \in \Sigma_\Delta(1)\} \, . \tag{2.43}$$

Here by the two-sides arrow we mean the converse proposition: when a toric variety X_Σ is complete and $D = \sum_\rho a_\rho D_\rho$ is Cartier, then Δ_D is a polytope. Notice that Δ_D in (2.43) follows the description (2.40) in terms of its facets. Lastly, the global sections of the line bundle $\mathcal{O}_{X_\Delta}(D)$ are [15]

$$\Gamma\left(X_\Delta, \mathcal{O}_{X_\Delta}(D)\right) = \bigoplus_{\boldsymbol{u}\in\Delta\cap\mathsf{M}} \mathbb{C}\cdot\chi^{\boldsymbol{u}} \, , \tag{2.44}$$

which means that the lattice points in $\Delta \cap \mathsf{M}$ determine the rational functions in X_Δ.

2.3.4 Toric construction of mirror pairs

In the following, we will briefly review the Batyrev construction of Calabi-Yau n-fold mirror pairs (M, W) as hypersurfaces in toric ambient spaces [21]. From now on, we will restrict our discussions to this type of toric Calabi-Yau. More generally, this construction extends to complete intersection Calabi-Yau (CICY) in toric ambient spaces [22].

The data of the mirror pair is encoded in an $(n + 1)$-dimensional reflexive lattice polytope $\Delta \subset \mathsf{N}_\mathbb{R}$ with $\mathbf{0} \in \Delta$ and the choice of a regular star triangulation of Δ and the polar polytope

$$\Delta^* = \{\boldsymbol{\mu} \in \mathsf{N}^*_\mathbb{R} \mid \langle \boldsymbol{\nu}, \boldsymbol{\mu}\rangle \geq -1 \, , \ \forall\ \boldsymbol{\nu} \in \Delta\,\} \subset \mathsf{M} \, . \tag{2.45}$$

The triangulation of Δ^* leads to the fan Σ_Δ by taking the cones over the facets that in turn is associated to the ambient space $\mathbb{P}_\Delta$. The family M of Calabi-Yau n-folds is given by the vanishing loci of sections $P_\Delta \in \mathcal{O}(K_{\Delta^*})$

$$P_\Delta = \sum_{\boldsymbol{\nu}\in\Delta\cap\mathsf{N}} \prod_{\boldsymbol{\nu}^*\in\Delta^*\cap\mathsf{N}^*} s_{\boldsymbol{\nu}} x_{\boldsymbol{\nu}^*}^{\langle\boldsymbol{\nu},\boldsymbol{\nu}^*\rangle+1} = 0 \, . \tag{2.46}$$

The mirror family W is obtained by exchanging $\Delta \leftrightarrow \Delta^*$.

For the family of mirror pairs (M, W), there are formulae that relate their Hodge numbers in terms of combinatorial data of the reflexive pair (Δ, Δ^*). This result reads [21]

$$\begin{aligned} h^{1,1}(M) &= h^{n-1,1}(W) \\ &= l(\Delta^*) - (n+2) - \sum_{\mathrm{codim}(\mathsf{F})=1} l'(\mathsf{F}) + \sum_{\mathrm{codim}(\mathsf{F})=2} l'(\mathsf{F}^*) l'(\mathsf{F}) \,, \\ h^{n-1,1}(M) &= h^{1,1}(W) \\ &= l(\Delta) - (n+2) - \sum_{\mathrm{codim}(\mathsf{F}^*)=1} l'(\mathsf{F}^*) + \sum_{\mathrm{codim}(\mathsf{F}^*)=2} l'(\mathsf{F}) l'(\mathsf{F}^*) \,. \end{aligned} \tag{2.47}$$

Here F and F^* is a dual pair of faces of Δ and Δ^*. Moreover, $l(\mathsf{F})$ is the number of lattice points of a face F and $l'(\mathsf{F})$ is the number of its interior lattice points.

2.3.5 Toric geometry of elliptic and genus one fibrations

For F-theory we need Calabi-Yau manifolds that are elliptically fibered, or more generally genus one fibered [23]. One way to construct these is by taking a torically fibered ambient space such that the hypersurface constraint cuts out a genus one curve from the fiber [24]. Toric fibrations can be understood in terms of toric morphisms. A toric morphism $\phi:\, X_\Sigma \to X_{\Sigma_B}$ in turn is encoded in a lattice morphisms

$$\overline{\phi}:\, \mathsf{N} \to \mathsf{N}_B \,, \tag{2.48}$$

such that the image of every cone in Σ is completely contained inside a cone of Σ_B. We obtain a fibration with the fan of the generic fiber given by $\Sigma_F \in \mathsf{N}_F$ if the morphism $\overline{\phi} : \mathsf{N} \to \mathsf{N}_B$ is surjective and the sequence

$$0 \to \mathsf{N}_F \hookrightarrow \mathsf{N} \xrightarrow{\overline{\phi}} \mathsf{N}_B \to 0 \,, \tag{2.49}$$

is exact. To obtain a fiber bundle over X_{Σ_B} with fibers X_{Σ_F}, we further require that there exists a subfan $\widetilde{\Sigma} \subseteq \Sigma$ such that:

- The map $\overline{\phi}$ defines a bijection $\widetilde{\Sigma} \simeq \Sigma_B$.
- Every cone $\sigma \in \Sigma$ is of the form $\sigma = \tilde{\sigma} + \sigma_f$, where $\tilde{\sigma} \in \widetilde{\Sigma}$ and $\sigma_f \in \Sigma_F$.

If this is the case, then the fibration $\phi : X_\Sigma \to X_{\Sigma_B}$ is locally a product space $\phi^{-1}(U) \simeq X_{\Sigma_F} \times U$, where U is an affine open subset of X_{Σ_B}.

We can now obtain elliptically fibered (genus one fibered) Calabi-Yau n-fold M from the following construction in terms of polytopes. First we combine a $(n-1)$-dimensional base polytope Δ_B and a 2-dimensional reflexive fiber polytope Δ_F and embed them into a $(n+1)$-dimensional polytope Δ as follows:

$$\boldsymbol{\nu}^* \in \Delta^* \left\{ \left| \begin{array}{c:c} \Delta_B^* & \begin{array}{c} \bullet \\ \vdots \\ \bullet \end{array} \\ \hdashline 0 & \Delta_F^* \end{array} \right| \right. \tag{2.50}$$

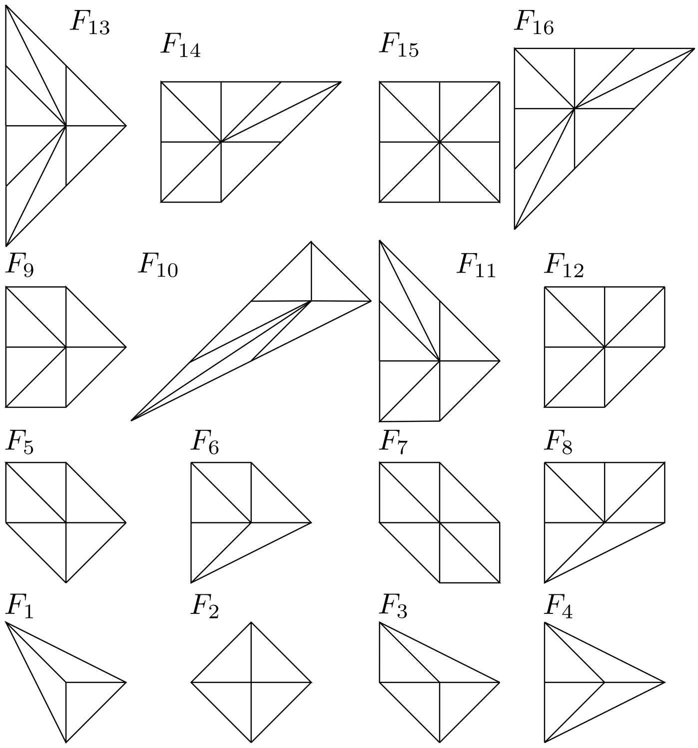


Figure 2.2: The 16 2-dimensional reflexive polytopes. In this notation the polytope F_i and F_{17-i} are dual to each other, i.e. $F_i^* = F_{17-i}$, but the F_i for $i = 7, \ldots, 10$ are self-dual, i.e. $F_i^* = F_i$.The image is taken from [25].

Here the bullets $\bullet$ denote points in Δ_F that can be repeated. In doing this we must ensure that the Calabi-Yau condition holds. In that case the fan associated to $\mathbb{P}_\Delta$ is spanned by the ray vectors that read from the rows in the matrix (2.63) . Thus, using the Batyrev construction one gets an n-fold M from the locus given by (2.46) on the ambient space $\mathbb{P}_\Delta$ i. As mentioned above, M inherits a fibration structure from the ambient spaces $\mathbb{P}_\Delta \to \mathbb{P}_{\Delta_B}$ and we can identify a map

$$M = \{\boldsymbol{x} \in \mathbb{P}_\Delta | P_\Delta(\boldsymbol{x}) = 0\} \xrightarrow{\pi} B = \mathbb{P}_{\Delta_B} \,. \tag{2.51}$$

2.3.6 Fiber based approach construction

To achieve a desired elliptic (genus one) fibration structure, we follow the method by the authors [25]. The advantage of this approach is that it allows us to engineer specific properties for the resulting physics theories, e.g., gauge group, matter content, Higgsing chains. Moreover, many of these properties are base-independent, which means we can obtain a large class of models for both Calabi-Yau 3-folds and 4-folds compactifications.

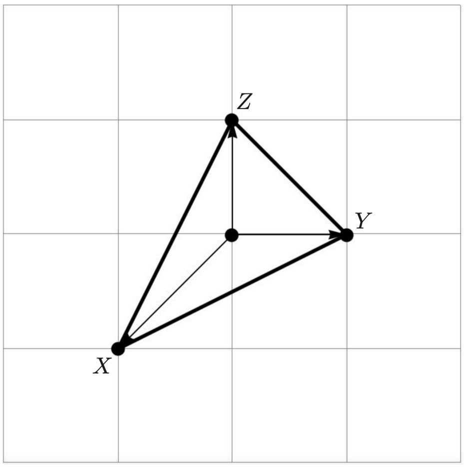
The starting point is to consider a toric hypersurface in a 2-dimensional toric variety that is almost Fano. There are 16 reflexive polytopes F_i that give rise to such kind of toric surfaces $\mathbb{P}_{F_i}$. See Figure 2.2. Thus, the wanted hypersurface is the genus one curve

$$\mathcal{C}_{F_i} = \{\boldsymbol{c} \in \mathbb{P}_{F_i} \mid P_{F_i}(\boldsymbol{c}) = 0\} \,. \tag{2.52}$$

Here P_{F_i} is a section of the anti-canonical bundle $\mathcal{O}(K_{F_i^*})$ that has the form (2.46). Then, the idea is to promote the P_{F_i}-polynomial coefficients to suitable line bundles over the base B. This way, the promoted hypersurface equation cuts through an ambient space with a fibration structure

$$\begin{tikzcd} \mathbb{P}_{F_i} \arrow[r, hook] & \mathbb{P}^B_{F_i}(\mathcal{S}_7, \mathcal{S}_9) \arrow[d] \\ & B \end{tikzcd} . \tag{2.53}$$

Here the total space fibration $\mathbb{P}^{F_i}_B(\mathcal{S}_7, \mathcal{S}_9)$ is parametrized by two divisors $\mathcal{S}_7$ and $\mathcal{S}_9$ in the base B.[3] To see this, note that the toric divisors associated with the coordinate ring in $\mathbb{P}_{F_i}$ lift to non-trivial divisors over the base B. However, we can take the $(\mathbb{C}^*)^s$-action (2.31) to gauge s coordinates—out of $s+2$ coordinates in the coordinate ring in $\mathbb{P}_{F_i}$—such that they turn into sections of the trivial line bundle over B. Thus, we end up with only two coordinates that become sections of non-trivial line bundles over the base B. Followed by this, we must demand the Calabi-Yau condition for the hypersurface on the fibration ambient space. In other words, P_{F_i} must lift into a section of the anti-canonical bundle of $\mathbb{P}^B_{F_i}(\mathcal{S}_7, \mathcal{S}_9)$. Consequently, the promoted coefficients for P_{F_i} are constrained by an appropriate choice of base divisors $(\mathcal{S}_7, \mathcal{S}_9)$.



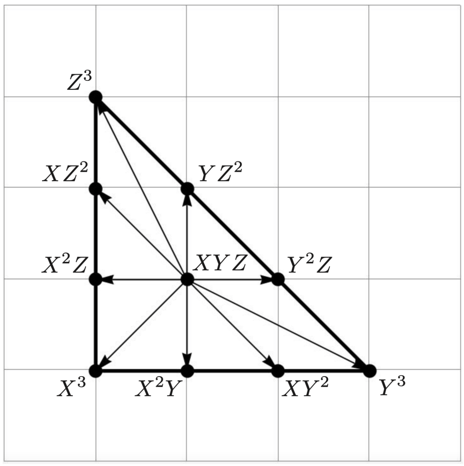


Figure 2.3: The left image shows the polytope F_1. The right image shows the polytope F_{16}.

It is illustrative to consider the case of a generic fibration with fiber polytope F_1. Note that the fiber ambient space is $\mathbb{P}_{F_1} = \mathbb{P}^2$. The hypersurface equation (2.46) for the genus one curve $\mathcal{C}_{F_1}$ reads

$$\begin{aligned} P_{F_1} = s_1 X^3 + s_2 X^2 Y + s_3 X Y^2 + s_4 Y^3 + s_5 x_1^2 Z \\ + s_6 XYZ + s_7 Y^2 Z + s_8 Y Z^2 + s_9 Y Z^2 + s_{10} Z^3 \,. \end{aligned} \tag{2.54}$$

At the fiber level, we have that $s_i \in \mathbb{C}$ with $i = 1, \dots, 10$. The latter coefficients parametrize redundantly the complex structure of $\mathcal{C}_{F_1} \subset \mathbb{P}^2$. By taking automorphisms on $\mathbb{P}^2$, we can reduce such set of parameters to a unique one. Now, in the context of the fibration $\mathbb{P}^{F_1}_B$, the coefficients s_i are nonequivalent sections over the base B. Similarly, for the subset of

[3] This notation is a convention of the references [25, 26].

homogenous coordinates $[X:Y:Z]$ in $\mathbb{P}^{F_1}_B$. Using the $\mathbb{C}^*$-action in $\mathbb{P}^2$, which reads

$$[X:Y:Z] \sim [\lambda X:\lambda Y:\lambda Z] \quad \forall\, \lambda \in \mathbb{C}^*\,, \tag{2.55}$$

we can fix Z to be a section of the trivial bundle over B. Then, the remaining coordinates, X, and Y are sections of non-trivial line bundles over B. Following the conventions of [25], we pick a choice of two base divisors $(\mathcal{S}_7\,,\mathcal{S}_9)$, such that the toric divisor classes follow the form

$$[X] = H + \mathcal{S}_9 - c_1(B)\,, \quad [Y] = H + \mathcal{S}_9 - \mathcal{S}_7\,, \quad [Z] = H\,. \tag{2.56}$$

Here H denotes the hyperplane class in the $\mathbb{P}^2$ fiber. As a next step, we must ensure that the Calabi-Yau condition holds. To do this, we use the adjunction formula to calculate the anti-canonical bundle divisor class on the fibration ambient space, which yields

$$\left[K^{-1}(\mathbb{P}^B_{F_i}(\mathcal{S}_7,\mathcal{S}_9))\right] = c_1(B) + c_1(\mathbb{P}^2) = c_1(B) + [X] + [Y] + [Z] = 3H + 2\mathcal{S}_9 - \mathcal{S}_7\,. \tag{2.57}$$

As outlined before, we must impose that P_{F_1} lifts into a section of $K^{-1}(\mathbb{P}^B_{F_i})$. By using the hypersurface equation (2.46), together with the result (2.57), we calculate the divisor classes $[s_{\boldsymbol{\mu}}]$ for the sections $s_{\boldsymbol{\mu}}$ such that the Calabi-Yau constraint holds. We obtain that

$$[s_{\boldsymbol{\mu}}] = \left[K^{-1}(\mathbb{P}^B_{F_i})\right] - \sum_{\rho\in\Sigma_F(1)} \left(\langle \boldsymbol{\mu}, \boldsymbol{\nu}_\rho\rangle + 1\right)[D_\rho]_B\,, \quad \boldsymbol{\mu} \in \Delta_{F_1^*} \cap \mathsf{M}\,. \tag{2.58}$$

Here D_ρ is the divisor that corresponds to the ray generator $\rho \in \Sigma_F(1)$, where Σ_F the fan of the fiber $\mathbb{P}^2$. See Figures 2.3.1 and 2.3. Moreover, $[D_\rho]_B$ denotes the base divisors contributions to a divisor class $[D_\rho]$, which we take from (2.56). Thus, using the polytopes information shown in the Figure 2.3 with the appropriate labelings of (2.54), we obtain that the sections $s_{\boldsymbol{\mu}}$ must be of the form:

Section	Divisor class
s_1	$3c_1(B) - \mathcal{S}_7 - \mathcal{S}_9$
s_2	$2c_1(B) - \mathcal{S}_9$
s_3	$c_1(B) + \mathcal{S}_7 - \mathcal{S}_9$
s_4	$2\mathcal{S}_7 - \mathcal{S}_9$
s_5	$2c_1(B) - \mathcal{S}_7$
s_6	$c_1(B)$
s_7	$\mathcal{S}_7$
s_8	$c_1(B) + \mathcal{S}_9 - \mathcal{S}_7$
s_9	$\mathcal{S}_9$
s_{10}	$2\mathcal{S}_9 - \mathcal{S}_7$

Table 2.3: Sections of consistent line bundles over B. Notice that $[s_7] = \mathcal{S}_7$ and $[s_9] = \mathcal{S}_9$, which determine the nomenclature of the references [25, 26].

The analysis for the rest of the fiber choices F_i is similar. The major replacement comes from the hypersurface polynomial equation that we denote by P_{F_i}. However, as explained in [25], there is a shortcut. We can obtain the toric ambient space $\mathbb{P}_{F_i}$ from $\mathbb{P}_{F_1}$ by performing a number a of successive blowups, i.e.

$$\mathbb{P}_{F_i} = \mathrm{Bl}_{P_1,\ldots,P_a}\mathbb{P}_{F_1}\,. \tag{2.59}$$

For each blowup we obtain a new coordinate E_i, but also a new linear relation that results into a $(\mathbb{C}^*)^{a+1}$-action in $\mathbb{P}_{F_i}$. Thus, we follow the same parametrization in (2.56) for $[X]_B$ and $[Y]_B$, while letting Z and the new coordinates E_i transform in the trivial line bundle over B. Note that adding rays to the polytope F_1 removes rays in its dual $F_1^* = F_{16}$. By each blowup we perform to obtain $\mathbb{P}_{F_i}$, we remove one monomial in P_{F_1} to obtain the polynomial P_{F_i}.[4] Nevertheless, for the remaining sections $s_{\boldsymbol{\mu}}$ that we observe in P_{F_i}, the assignments in Table 2.3 holds. The exception to this rule are the fiber polytopes F_2 and F_4, since we cannot obtain them via blowups (2.59).

Lastly, we comment on further constraints that the choice of basis B sets up. For the purposes of this work we will assume that B is a smooth toric variety. Consider a basis $\{D^{\mathrm{b}}_\alpha\}_{\alpha=1,\ldots,h^{1,1}(B)}$ of effective divisors in B, whose classes we assume to be linearly independent. In this manner, we express the choice of divisors $\mathcal{S}_7, \mathcal{S}_9 \in H^2(B,\mathbb{Z})$ as

$$\mathcal{S}_7 = \sum_{\alpha=1}^{h^{1,1}(B)} p_\alpha D^{\mathrm{b}}_\alpha\,, \qquad \mathcal{S}_9 = \sum_{\alpha=1}^{h^{1,1}(B)} q_\alpha D^{\mathrm{b}}_\alpha\,. \tag{2.60}$$

To assure the appearance of a section $s_{\boldsymbol{\mu}}$ in the hypersurface equation p_{F_i}, we must impose effectiveness condition on all their associated divisor classes $[s_{\boldsymbol{\mu}}]$. This gives further constraints on the allowed values for the vectors $\boldsymbol{p} = (p_1, \ldots, p_k)$ and $\boldsymbol{q} = (q_1, \ldots, q_k)$. We illustrate these techniques with an explicit example next.

Example: $\mathcal{C}_{F_1}$ fibration over $B = \mathbb{P}^2$

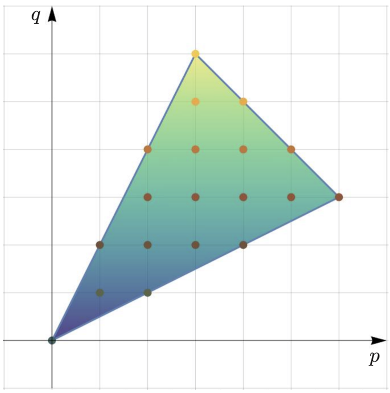


Figure 2.4: Region for consistent choices of line bundles over $\mathbb{P}^2$, where $\mathcal{S}_7 = pH_B$ and $\mathcal{S}_9 = qH_B$.

Let us consider the base $B = \mathbb{P}^2$. To avoid confusion we denote by $[u : v : w]$ the homogeneous coordinates in B. The cone of effective divisors in B is generated by the hyperplane class, which we denote by H_B. Then, the first Chern class reads $c_1(B) = 3H_B$. We set the divisors that parametrize the fibration geometry by $\mathcal{S}_7 = pH_B$ and $\mathcal{S}_9 = qH_B$. This way, we proceed to impose the condition $[s_{\boldsymbol{\mu}}] \geq 0$ for all $\boldsymbol{\mu} \in \Delta_{F_1^*}$. For this, we use the information of Table 2.3 to obtain a set of inequalities for the allowed values of $(p,q) \in \mathbb{Z}^2$. We depict the result in Figure 2.3.6.

[4] Up to correct assignment of sections wrt the fiber coordinates. However, this is truth upon blowdown of all exception coordinates E_i. In any case, the coefficients $s_{\boldsymbol{\mu}}$ lift to pure line bundles over B.

	X	Y	Z
Q_B	-2	0	0

Table 2.4: Charges of the homogeneous coordinates $[X:Y:Z]$ wrt the basis of effective divisors on $B=\mathbb{P}^2$ for the choice $\mathcal{S}_7=\mathcal{S}_9=H_B$.

Obtaining the fibration polytope: With the information we have from the fibration ambient space, we want to obtain its associated fiber polytope Δ. For definiteness we look into the case in which $\mathcal{S}_7=\mathcal{S}_9=H_B$. The torus action in the fibration ambient space is of the form

$$[X:Y:Z:u:v:w]\sim[\lambda^{l_1^{(a)}}X:\lambda^{l_2^{(a)}}Y:\lambda^{l_3^{(a)}}Z:\lambda^{l_4^{(a)}}u:\lambda^{l_5^{(a)}}v:\lambda^{l_6^{(a)}}w]\,, \tag{2.61}$$

for all $\lambda\in\mathbb{C}^*$. By using the scaling relations in (2.55), together with the analogous relation for the base coordinates $[u:v:w]\sim[\lambda u:\lambda v:\lambda w]$ with $\lambda\in\mathbb{C}^*$, we deduce the vectors $\boldsymbol{l}^{(a)}$ associated to the Mori cone generators:

$$\begin{aligned}\boldsymbol{l}^{(1)}&=(-2,0,0,1,1,1)\,,\\ \boldsymbol{l}^{(2)}&=(\quad 1,1,1,0,0,0)\,.\end{aligned} \tag{2.62}$$

Here the last three entries of $\boldsymbol{l}^{(1)}$ correspond to the equivalence relation of the $\mathbb{C}^*$-action in $B=\mathbb{P}^2$, whereas the first three entries correspond to the charges $Q_B=C_B\cdot D_\rho|_B$ with C_B the curve that spans $H_2(B,\mathbb{Z})$. We obtain the latter values from (2.56). See Table 2.3.6. The entries of $\boldsymbol{l}^{(2)}$ are simply the scaling relations in the fiber, i.e. (2.55).

As a next step, we take the result (2.62) to reconstruct the polytope that yields the ambient space $\mathbb{P}_\Delta$. To this end, we consider the matrix Q in (2.37) with entries $Q_{a\rho}=l_\rho^{(a)}$, together with the linear relations (2.36), and notice that the nullspace $V=\ker(Q^t)$ gives the linear span of the ray generators $\{\boldsymbol{v}_\rho\}_{\rho\in\Sigma}$. Then, we look for an integral basis for $V\cap\mathsf{N}$, where $\mathsf{N}=\mathbb{Z}^6$ in our current case, such that it matches the form (2.63). In our example determined by (2.62), we obtain the following data

$$\boldsymbol{\nu}\in\Delta\left\{\begin{array}{|cc:cc|}1&0&0&1\\0&1&0&0\\-1&-1&0&1\\ \hdashline 0&0&1&0\\0&0&0&1\\0&0&-1&-1\\0&0&0&0\end{array}\right.\,. \tag{2.63}$$

At first glance, from Figure 2.3.6, we obtain 19 possible geometries with the desired fibration structure. However, many of them are topologically equivalent. We can see this by using the T.C. Wall theorem [27][5]. Using computer programs such as SAGE, we can obtain these geometries' intersection data. A careful inspection reveals that we get five distinct geometries with the following Euler characteristics

$$\chi(M):-216,-186,-180,-168,-162\,. \tag{2.64}$$

[5] Two Calabi-Yau 3-folds M and M' are topologically equivalent if: (a) their fundamental groups are the same, (b) their fundamental groups are the same, (c) there is a choice of basis $\{\omega_i\}_{i=1,\ldots,h^{1,1}(M)}$ for $H^{1,1}(M)$, and similar for the M' case, such that they have the same intersection numbers $c_{ijk}=\int_M\omega_i\wedge\omega_j\wedge\omega_k$ and $c_i=\int_M c_2(TM)\wedge\omega_i$.

Let us note that the geometry with $\chi(M) = -162$ is the Calabi-Yau hypersurface denoted by $X_{(3|3)}(1,1,1 \mid 1,1,1)$ in [28], whose ambient space is $\mathbb{P}^2 \times \mathbb{P}^2$. Consequently, it exhibits a symmetry between base and fiber, explored by the Master thesis work [29].

2.4 Mirror symmetry

In this section, we elaborate on the mirror symmetry computations induced by the so-called *mirror map*. As a starting point, we consider the B-model, which relies on complex structure deformations that are captured by period integrals of a given Calabi-Yau W. To calculate those periods, we must first choose a local region of the complex structure moduli space. Once we achieve this, we can define a local mirror map that exchanges the B-model periods into *quantum periods* for the corresponding mirror dual Calabi-Yau M. As a preamble, these objects furnish some observables of our interest for the subsequent sections in this thesis. Namely, observables determined by genus zero Gromov-Witten invariants.

At the end of this section, we will introduce a sophisticated approach to calculate quantum periods. This formulation is called homological mirror symmetry, a conjecture that exchanges topological branes in the A-model with those in the B-model and vice-versa.

2.4.1 Picard-Fuchs operators

In the B-model, the main objects to compute are the periods of the holomorphic n-form on a Calabi-Yau n-fold W. To obtain them, we solve a set of differential equations defined by operators that annihilate periods. Such a set of operators is called the Picard-Fuchs system. For Calabi-Yau varieties constructed as hypersurfaces in a toric ambient space it is easy to write down differential equations for which the solution set is in general larger than that spanned by the periods. However, in many cases the solution sets are equal and it is sufficient to study the so-called GKZ system. For the purposes of this work, we will restrict to such cases.

The GKZ system is a generalized $\mathcal{A}$-*system of hypergeometric differential equations* introduced by Gel'fand, Kapranov, and Zelevinskỹ in [30, 31]. Such system is described by a set $\mathcal{A} = \{\boldsymbol{v}_1, \dots, \boldsymbol{v}_k\} \subset \mathbb{Z}^{r+1}$ in which $k > r+1$, a complex variable s_i for each $\boldsymbol{v}_i \in \mathcal{A}$, and a vector $\hat{\boldsymbol{\beta}} \in \mathbb{C}^{r+1}$. By construction there are linear relations among the vectors in $\mathcal{A}$ that define the non-empty set

$$\mathscr{L} = \left\{ \boldsymbol{\ell} = (\ell_1, \dots, \ell_k) \in \mathbb{Z}^k \mid \sum_{i=k} \ell_i \boldsymbol{v_i} = 0 \right\} . \tag{2.65}$$

Having said this, the GKZ system consists of a system of differential equations

$$\text{GKZ} : \{\mathcal{D}_{\boldsymbol{\ell}} \Phi = 0\}_{\boldsymbol{\ell} \in \mathscr{L}} \cup \{\mathcal{Z}_j \Phi = 0\}_{j \in \{1,\dots,r+1\}} , \tag{2.66}$$

where the differential operators follow the form

$$\mathcal{D}_{\boldsymbol{\ell}} = \prod_{\ell_i > 0} \left(\frac{\partial}{\partial s_i} \right)^{\ell_i} - \prod_{\ell_i < 0} \left(\frac{\partial}{\partial s_i} \right)^{-\ell_i} , \quad \mathcal{Z}_j = \sum_{i=1}^{k} v_i^j s_i \frac{\partial}{\partial s_i} - \hat{\beta}^j . \tag{2.67}$$

Here $\boldsymbol{v}_i = (v_i^1, \dots, v_i^{r+1})$, and $\hat{\boldsymbol{\beta}} = (\hat{\beta}^1, \dots \hat{\beta}^{r+1})$.

In the case of Batyrev-Borisov mirror constructions for toric hypersurfaces, we identify the GKZ system with the data

$$\mathcal{A} = \{1\} \times (\Delta^* \cap \mathsf{N}) \subset \mathbb{Z} \times \mathsf{N} \simeq \mathbb{Z}^{r+1}\,, \quad \hat{\boldsymbol{\beta}} = (-1, 0, \dots, 0)\,. \tag{2.68}$$

This enforces us to have that $\sum_i \ell_i(1, \boldsymbol{v}_i) = 0$ with $\boldsymbol{v}_i \in \Sigma(1) \cap \mathsf{N}$, which implies that $\sum_i \ell_i = 0$. From this, we define the extended Mori vectors

$$\boldsymbol{\ell}^{(a)} \equiv (-l_0^{(a)}, \boldsymbol{l}^{(a)})\,, \quad l_0^{(a)} = -\sum_{\rho \in \Sigma(1)} l_\rho^{(a)}\,, \tag{2.69}$$

where $\{\boldsymbol{l}^{(a)}\}$ is the set of Mori vector generators that we introduced in (2.36). Moreover, we take the coefficients $s_{\boldsymbol{\nu}^*}$ in P_{Δ^*} as complex variables of the GKZ data. We recall that such coefficients parametrize redundantly the complex structure moduli of W. However, the set of vectors $\{\boldsymbol{\ell}^{(a)}\}$ define the Batyrev coordinates

$$z^a = (-1)^{\ell_0^{(a)}} s_0^{\ell_0^{(a)}} \cdots s_k^{\ell_k^{(a)}}\,, \tag{2.70}$$

which determines the point of *maximal unipotent monodromy* (MUM) in the complex structure moduli space of the Calabi-Yau W. With this in mind, we introduce the operators GKZ operators $\mathcal{L}_a \equiv \mathcal{D}_{\boldsymbol{\ell}^{(a)}}$ in terms of Batyrev coordinates. Each Mori vector $\boldsymbol{l}^{(a)}$ yields an operator

$$\begin{aligned} \mathcal{L}_a = & \prod_{\ell_j^{(a)}>0} \left(\prod_{i=0}^{\ell_j^{(a)}-1} (\Theta_j - i) \right) \\ & - \prod_{i=1}^{|\ell_0^{(a)}|} \left(i - |\ell_0^{(k)}| - \Theta_0 \right) \prod_{\substack{\ell_j^{(a)}<0 \\ j\neq 0}} \left(\prod_{i=0}^{|\ell_j^{(a)}|-1} \left(\Theta_j + |\ell_j^{(a)}| - i \right) \right) z^a\,, \end{aligned} \tag{2.71}$$

where we used the logarithmic derivatives that read

$$\Theta_j = \sum_{a=1}^{s} \ell_j^{(a)} \theta_a\,, \qquad \theta_a = z^a \frac{\partial}{\partial z^a}\,. \tag{2.72}$$

Although we only discuss here toric hypersurfaces, a straightforward generalization for the GKZ system of toric CICY is discussed in the work [32].

Once we obtain the GKZ differential operators, we have to implement further a factorization to extract the Picard-Fuchs system. We do this in such a way that the number of solutions matches with the number of period integrals over the $h^n_{\text{hor}}(W)$ horizontal cycles restricted to a basis of $H_n(W, \mathbb{Z})$. When this is possible, we obtain h differential operators of the form

$$L_a = p_a(\boldsymbol{\theta}) + \mathcal{O}(\boldsymbol{z}) q_a(\boldsymbol{\theta}, \boldsymbol{z})\,, \quad a = 1, \dots, h\,, \tag{2.73}$$

where $p_a(\boldsymbol{\theta})$ is a homogenous polynomial in $\mathbb{C}[\theta^1, \dots \theta^s]$ whose degree determine the order of the differential operator L_a. At the end of the day, the differential operators (2.73) generate a left ideal of rank $h^n_{\text{hor}}(W)$ that annihilate the periods, i.e. the Picard Fuchs ideal $\mathcal{I}_{PF}$.

To solve the Picard-Fuchs equations, we apply the Frobenius method. This consists of a power series Ansatz of the form

$$\varpi(\boldsymbol{z},\boldsymbol{\rho}) = \sum_{\boldsymbol{n}\in\mathbb{Z}^s_{\geq 0}} c(\boldsymbol{n},\boldsymbol{\rho})\boldsymbol{z}^{\boldsymbol{n}+\boldsymbol{\rho}}\,, \qquad s = h_{n-1,1}(W)\,. \tag{2.74}$$

Here $\boldsymbol{\rho} = (\rho_1, \ldots \rho_s)$ are the indicial parameters. For the choice of the Batyrev variables $\boldsymbol{z}$, the recursion relation around the MUM point $\boldsymbol{z} = 0$ gives the expression

$$c(\boldsymbol{n},\boldsymbol{\rho}) = \sum_{\boldsymbol{n}\in\mathbb{Z}^s_{\geq 0}} \frac{\Gamma\left(1-\ell_0^{(\boldsymbol{a})}\cdot(\boldsymbol{n}+\boldsymbol{\rho})\right)}{\prod_{i>0}\Gamma\left(1+\ell_i^{(\boldsymbol{a})}\cdot(\boldsymbol{n}+\boldsymbol{\rho})\right)} \frac{\prod_{i>0}\Gamma\left(1+\ell_i^{(\boldsymbol{a})}\cdot\boldsymbol{\rho}\right)}{\Gamma\left(1-\ell_0^{(\boldsymbol{a})}\cdot\boldsymbol{\rho}\right)} \boldsymbol{z}^{\boldsymbol{n}+\boldsymbol{\rho}}\,, \tag{2.75}$$

where we denote by $\ell_*^{(\boldsymbol{a})}$ the vector with entries $(\ell_*^{(1)}, \ldots, \ell_*^{(s)})$. With this choice of coordinates, the indicial equation results $\boldsymbol{\rho}^{2s+2} = 0$; we also obtain logarithmic solutions by applying a derivative on ϖ wrt ρ_a. Thus, we have two essential solutions of the Picard-Fuchs system that read

$$X^0(\boldsymbol{z}) = \varpi(\boldsymbol{z},\boldsymbol{\rho})|_{\boldsymbol{\rho}=0}\,, \qquad X^a(\boldsymbol{z}) = \frac{1}{2\pi i}\partial_{\rho_a}\varpi(\boldsymbol{z},\boldsymbol{\rho})|_{\boldsymbol{\rho}=0}\,, \qquad a = 1,\ldots,s\,, \tag{2.76}$$

where the former is referred to the *fundamental period* and the latter to the *regular periods*. More generally, we construct solutions from functions we obtain by applying k succesive derivatives wrt indicial parameter, as

$$\begin{aligned} \varpi_{a_1,\ldots,a_k}(\boldsymbol{z}) &\equiv \left(\frac{1}{2\pi i}\right)^k \partial_{\rho_{a_1}}\cdots\partial_{\rho_{a_k}}\varpi(\boldsymbol{z},\boldsymbol{\rho})\Big|_{\boldsymbol{\rho}=0} \\ &= \left(\frac{1}{2\pi i}\right)^k X^0(\boldsymbol{z})\log(z^{a_1})\cdots\log(z^{a_k}) + \mathcal{O}(\boldsymbol{z})\,. \end{aligned} \tag{2.77}$$

To assure these functions are solutions to the Picard-Fuchs system, we must ensure that the leading part $p_a(\boldsymbol{\theta})$ in each Picard-Fuchs operator (2.73) annihilates the k-logarithmic contribution to $\varpi_{a_1,\ldots,a_k}(\boldsymbol{z})$.

2.4.2 The mirror map

Our next step in the mirror symmetry program is to translate B-model information into A-model objects. To this end, we employ the mirror map

$$t_*^{-1} : \mathcal{M}_{\text{cs}}(W) \to \mathcal{M}_{\text{ks}}(M)\,, \tag{2.78}$$

which is taken over a patch $U_* \subset \mathcal{M}_{\text{cs}}(W)$ that maps into a corresponding patch $V_* \subset \mathcal{M}_{\text{ks}}(M)$; it is a series inversion of a covergent power series—depending on complex structure moduli—with center at a given point $\boldsymbol{z}_* \in \mathcal{M}_{\text{cs}}(W)$. Note that we can take these maps patchwise over a cover for $\mathcal{M}_{\text{cs}}(W)$, as performed in [33]. For our interests, we will focus only on the MUM point since its associated mirror map leads to a Gromov-Witten theory counting.[6]

[6] There is an orbifold Gromov-Witten theory for points in $\mathcal{M}_{\text{ks}}(M)$ such that M is locally an orbifold [34], but we do not treat such cases here.

Recall that at the MUM point a canonical choice of coordinates for the complex structure moduli is the set of Batyrev coordinates $\{z^a\}_{a=1,\dots,h^{n-1,1}(W)}$. With this in mind, we introduce the *flat coordinates* as the rational maps given by

$$t^a(\boldsymbol{z}) = \frac{X^a(\boldsymbol{z})}{X^0(\boldsymbol{z})}\,, \qquad a = 1,\dots,h^{n-1,1}(W)\,, \tag{2.79}$$

where X^0 and X^a are respectively the fundamental and regular periods. We emphasize the role of flat coordinates in mirror symmetry as those identified with Kähler structure moduli via the mirror map. In this case, we have a series inversion of the form

$$\mathcal{O}(\boldsymbol{z}) + \log z^a = t^a(\boldsymbol{z}) \in \mathcal{M}_{\mathrm{cs}}(W) \xrightarrow{\text{mirror map}} \mathcal{M}_{\mathrm{ks}}(M) \ni z^a(\boldsymbol{t}) = q^a + \mathcal{O}\left(\boldsymbol{q}\right)\ , \tag{2.80}$$

where $q^a = \exp(2\pi i t^a)$. On the A-model side, the coordinates $t^a = \int_{\mathcal{C}^a}\omega$ are complex volumes of curves $\mathcal{C}^a$ in the Mori cone of M, where ω is the complexified Kähler class in M. To illustrate the computation of the mirror map, we look into a simple example now.

Example: The elliptic curve $\boldsymbol{\mathcal{C}_{F_1}}$ To make a first contact with modular forms, we examine the periods of the Calabi-Yau 1-fold $\mathcal{C}_{F_1}$, or elliptic curve, defined by the locus (2.54) in $\mathbb{P}^2$. Considering $\Delta^* = F_1$, the toric data reads below

$$\begin{array}{c c} & \begin{array}{ccc} & & C \end{array} \\ \begin{array}{c} \\ X \\ Y \\ Z \end{array} & \left(\begin{array}{cc|c} 0 & 0 & -3 \\ -1 & -1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{array}\right) \end{array} \tag{2.81}$$

We obtain the complex structure coordinate

$$z = \frac{s_1 s_2 s_3}{s_0^3}\,, \tag{2.82}$$

which parametrizes the family of elliptic curves

$$\mathcal{C}_{F_1} : X^3 + Y^3 + Z^3 - z^{\frac{1}{3}} XYZ = 0\,. \tag{2.83}$$

Using the information of the Mori cone vector $\mathcal{C}_{\mathrm{e}}$ in (2.81), together with the GKZ-operator formula (2.71), we obtain the Picard-Fuchs equation for $\mathcal{C}_{F_1}$ that reads below

$$\mathcal{L}_{\mathrm{e}} = \theta_z^2 - 3z(3\theta_z + 2)(3\theta_z + 1)\,. \tag{2.84}$$

Using the Frobenius method, we obtain directly the following periods

$$X^0(z) = \sum_{n\geq 0} \frac{(3n)!}{(n!)^3} z^n\,, \quad X^1(z) = X^0(z)\log z + \mathcal{O}(z) \tag{2.85}$$

Note that the fundamental period can be easily identified with the *Gauss-hypergeometric* *function* ${}_2F_1$, as expected from the GKZ-system. This reads

$$X^0(z) = {}_2F_1\left(\frac{1}{3}, \frac{2}{3}; 1; 27z\right) . \tag{2.86}$$

We introduce the flat coordinate τ which is given by the ratio of the periods in (2.85). Note that τ is the complex structure of a torus with complexified symplectic form $\tau \mathrm{d}x \wedge \mathrm{d}y$. We apply the mirror map (2.80) and obtain the following inversion

$$\tau = \frac{X^1(z)}{X^0(z)} \xrightarrow{\text{mirror map}} z(\tau) = q - 15q^2 + 171q^3 - 1679q^4 + 15054q^5 - \mathcal{O}\left(q^6\right) \;, \tag{2.87}$$

where $q = \exp(2\pi i\tau)$. We identify this expression with a modular function with modular group $\Gamma_1(3)$. More precisely, we have that

$$z(\tau) = \frac{\eta^{12}(3\tau)}{27\eta^{12}(3\tau) + \eta^{12}(\tau)} . \tag{2.88}$$

In the A-model side, the fundamental period reads

$$X^0\left(z(\tau)\right) = \Theta(\tau) = 1 + 6q + 6q^3 + 6q^4 + 12q^7 + 6q^9 + 6q^{12} + 12q^{13} + \mathcal{O}(q^{15}) \,, \tag{2.89}$$

where we identified $X^0(\tau)$ with the modular form of weight 1 on $\Gamma_1(3)$ that reads

$$\Theta(\tau) = \sum_{n,m\in\mathbb{Z}} q^{n^2+nm+m^2} \in M_1\left(\Gamma_1(3), \varepsilon^3\right) . \tag{2.90}$$

Let us give a remark about the observed modular form of weight 1, which we call Θ in (2.89). There is a beautiful connection between modular forms and differential equations, exposed in the proposition A of Appendix B.1. In our example, Θ has an associated differential equation of order 2 for the function $\Phi(z)$ given by the Gauss-hyergeometric function ${}_2F_1$ such that we have locally $\Theta(\tau) = \Phi(z(\tau))$ with $z(\tau)$ as modular function generator for $\Gamma_1(3)$ (2.88). Such an equation is precisely the Picard-Fuchs equation for the family of elliptic curves (2.83).

Of course we are interested in studying Calabi-Yau manifolds that determine more realistic physics, i.e., Calabi-Yau 3-folds and 4-folds. In particular, periods of elliptic and genus one fibrations exhibit modular properties as first observed by [35] and more general modular objects as solutions [36]. To proceed with these motivations, we employ some Holomogical mirror symmetry techniques to fix periods solutions bases for Calabi-Yau n-folds.

2.4.3 Elements of homological mirror symmetry

The homological mirror symmetry conjecture proposes the isomorphism of categories

$$\mathcal{F}(W) \simeq D^b(M) \,, \tag{2.91}$$

if and only if (M, W) are Calabi-Yau mirror dual pairs [37]. Here $\mathcal{F}(W)$ is the Fukaya's A_∞ category, whereas $D^b(M)$ is the bounded derived category of coherent sheaves. Before we explain the objects of each category, let us establish their connection with our motivations. Homological mirror symmetry provides a powerful method that furnishes an integral basis of periods by considering central charges of branes. The upshot is that periods Π_I represent a basis for the central charge lattice of D-branes Λ_{D}. The central charge of a D-brane follows from the formula $Z(\boldsymbol{Q}) = \sum_I Q^I \Pi_I$, where $Q^I \in \mathbb{Z}$; a D-brane is subjected to the BPS mass bound $M \geq |Z(\boldsymbol{Q})|$. When branes saturate the BPS bound, they are said to be topological. There are two types of topological D-branes, which exchange under mirror duality, corresponding to objects in the categories $D^b(M)$ and $\mathcal{F}(W)$. We describe such objects in detail next.

A-branes: A topological A-brane is an object L in $\mathcal{F}(W)$ where L is a special Lagrangian submanifold of W with a flat $U(1)$ bundle on it. By special Lagrangian, we mean that L fulfills the conditions

$$\begin{aligned} \omega|_L &= 0\,, \\ \mathrm{Re}\left(\mathrm{e}^{i\theta}\Omega|_L\right) &= 0\,. \end{aligned} \tag{2.92}$$

Here ω and Ω is the complexified Kähler class and the holomorphic $(n,0)$-form in W, whereas θ is an arbitrary phase. Then, the central charge of topological A-branes reads from the period integral [38]

$$Z_A(L) = \int_L \Omega\,. \tag{2.93}$$

B-branes: As proposed in [37], the topological B-branes $\mathcal{E}^\bullet$ on M are objects in the derived category of bounded complexes of coherent sheaves $D^b(M)$. These are described as bounded complexes of locally free sheaves

$$\mathcal{E}^\bullet : 0 \to \cdots \to \mathcal{E}^{i-1} \to \mathcal{E}^i \to \mathcal{E}^{i+1} \to \cdots \to 0\,. \tag{2.94}$$

Each locally free sheaf $\mathcal{E}^i$ is equivalent to a vector bundle E^i. Physically, we interpret $\mathcal{E}^{2i}$ as coincident branes, while $\mathcal{E}^{2i+1}$ as coincident anti-branes [38, 39]. The object that measures the RR charges of this configuration of branes is the K-theory group $K(M)$ [40]. The latter is defined by the set of complex vector bundles (E,F), which are subjected to an equivalence relation $(E,F) \sim (E \oplus H, F \oplus H)$ for any vector bundle H. Let us introduce the Chern character ring homomorphism $\mathrm{ch} : K(M) \to H^{2*}(M,\mathbb{Q})$, which acts on the associated vector bundles

$$\mathrm{ch}(\mathcal{E}^\bullet) = \cdots - \mathrm{ch}\left(E^{2i-1}\right) + \mathrm{ch}\left(E^{2i}\right) - \mathrm{ch}\left(E^{2i+1}\right) + \cdots \tag{2.95}$$

Moreover, given a E vector bundle of rank k with decomposition $E = L_1 \oplus \cdots \oplus L_k$, where every L_i is a complex line bundle, the following formula holds

$$\mathrm{ch}(E) = \prod_{i=1}^{k} \mathrm{e}^{c_1(L_i)}\,. \tag{2.96}$$

With this in mind, we assign to each B-brane a central charge that receive quantum corrections. The leading terms thereof read from the formula

$$Z_B\left(\mathcal{E}^\bullet\right) = \int_M \mathrm{e}^{\omega}\Gamma_{\mathbb{C}}(M)\mathrm{ch}\left(\mathcal{E}^\bullet\right)^\vee + \text{quantum corrections}\,, \tag{2.97}$$

where the Gamma class for Calabi-Yau manifolds reads

$$\Gamma_{\mathbb{C}}(M) = 1 + \frac{1}{24}c_2(M) + \frac{\zeta(3)}{(2\pi i)^3}c_3(M) + \frac{7c_2^2(M) - 4c_4(M)}{5760} + \mathcal{O}(5)\,, \tag{2.98}$$

and the operator $\vee : H^{2*}(M) \to H^{2*}(M)$ acts as $\gamma^\vee \mapsto (-1)^k\gamma$ for $\gamma \in H^{k,k}(M)$.

Given a general Calabi-Yau n-fold M, a general basis for $2n$, $2(n-1)$, 2, and 0 branes was constructed in [41]. We list such a basis next:

- The top $2n$-brane is defined by the structure sheaf $\mathcal{O}_M$. Its corresponding asymptotic central charge is given by

$$Z_B(\mathcal{O}_M)\Big|_{\text{class.}} = \int_M \mathrm{e}^\omega \Gamma_{\mathbb{C}}(M)\,. \tag{2.99}$$

- A set of $2(n-1)$ branes $\mathcal{E}_a^\bullet$ is defined by the complex

$$\mathcal{E}_a^\bullet : 0 \to \mathcal{O}_M(-D_a) \to \mathcal{O}_M \to 0\,, \tag{2.100}$$

where D_a is the associated Kähler cone divisor D_a. Their central charges read

$$Z_B(\mathcal{E}_a^\bullet)\Big|_{\text{class.}} = \int_M \mathrm{e}^\omega \Gamma_{\mathbb{C}}(M)\left(1 - \mathrm{ch}\left(\mathcal{O}_M(D_a)\right)\right)\,. \tag{2.101}$$

- Let C_a be a mori cone curve dual to the Kähler cone divisor D_a. A set of 2-branes $\mathcal{C}_a^\bullet$ is defined as follows

$$\mathcal{C}_a^\bullet = \iota_! \mathcal{O}_{C_a}\left(K_{C_a}^{1/2}\right)\,, \tag{2.102}$$

where $\mathcal{O}_{C_a}(K_{C_a}^{1/2})$ is the structure sheaf twisted by the spin structure $K_{C_a}^{1/2}$ of C_a, and ι is the embedding $C_a \hookrightarrow M$ with K-theoretic pushforward $\iota_! : K(C_a) \to K(M)$. The image $\mathrm{ch}(\mathcal{C}_a^\bullet)$ results into the curve class $[C_a]$ [41], which leads to the following result

$$Z_B(\mathcal{C}_a^\bullet)\Big|_{\text{class.}} = (-1)^{n-1}t^a\,, \tag{2.103}$$

where $t^a = \omega \cdot [C_a]$.

- The 0-brane is defined by the skyscraper sheaf $\mathcal{O}_p$, which is located at a point $p \hookrightarrow M$. Its asymptotic central charge reads

$$Z_B(\mathcal{O}_p)\Big|_{\text{class.}} = (-1)^n\,. \tag{2.104}$$

Let us consider the case of Calabi-Yau threefolds M in this segment. In this case, the construction of even branes by [41] furnishes a complete integral basis of periods. A calculation in terms of intersection data of M yields the asymptotic expressions

$$\begin{aligned} Z_B(\mathcal{O}_M) &= \frac{1}{6}c_{abc}t^a t^b t^c + c_a t^a + \frac{\zeta(3)}{(2\pi i)^3}\chi(M) + \mathcal{O}(\mathrm{e}^{2\pi i \boldsymbol{t}})\,, \\ Z_B(\mathcal{E}_a) &= -\frac{1}{2}c_{abc}t^b t^c - \frac{1}{2}c_{aab}t^b - \frac{1}{6}c_{aaa} - c_a + \mathcal{O}(\mathrm{e}^{2\pi i \boldsymbol{t}})\,, \\ Z_B(\mathcal{C}_a) &= t^a + \mathcal{O}(\mathrm{e}^{2\pi i \boldsymbol{t}})\,, \\ Z_B(\mathcal{O}_p) &= -1 + \mathcal{O}(\mathrm{e}^{2\pi i \boldsymbol{t}})\,. \end{aligned} \tag{2.105}$$

Here $\chi(M)$ is the Euler characteristic of the Calabi-Yau threefold, while c_{abc} and c_a are given by the intersection numbers

$$c_{abc} = \int_M \omega_a \wedge \omega_b \wedge \omega_c\,, \qquad c_a = \frac{1}{24}\int_M c_2(M) \wedge \omega_a\,. \tag{2.106}$$

Note that we imply in (2.105) the basis choice of quantum periods

$$\vec{\Pi}(\boldsymbol{t}) = \Big(Z_B(\mathcal{O}_M), Z_B(\mathcal{E}_a), Z_B(\mathcal{C}_a), Z_B(\mathcal{O}_p)\Big)\,. \tag{2.107}$$

By equivalence of B-branes in M with A-branes in W, the leading terms in (2.105) fix a basis of holomorphic periods in W. Here we consider a region of the Kähler moduli space that is near the large volume, i.e., $\mathrm{Im}(t^a) \to \infty$ for all $a = 1, \dots, h^{1,1}(M)$, which corresponds to the maximal unipotent monodromy point on the mirror side. Then, solving the Picard-Fuchs equations for W through the B-model, we obtain the quantum corrections of (2.105) when performing the mirror map back towards M. In general, we can implement this strategy for Calabi-Yau n-folds.

The intersection of two A-branes $L_1 \cap L_2$ equals the Hierzebruch-Riemann-Roch pairing that defines an intersection of open strings for two B-branes $\mathcal{E}^\bullet, \mathcal{F}^\bullet \in D^b(M)$, as

$$\chi(\mathcal{E}^\bullet, \mathcal{F}^\bullet) = \int_M \mathrm{Td}(M)\mathrm{ch}(\mathcal{E}^\bullet)^\vee \mathrm{ch}(\mathcal{F}^\bullet)\,, \tag{2.108}$$

where the Todd class $\mathrm{Td}(M)$ follows the expansion over Chern classes

$$\begin{aligned} \mathrm{Td}(M) = 1 &+ \frac{1}{2}c_1(M) + \frac{1}{12}\left(c_1^2(M) + c_2(M)\right) + \frac{1}{24}c_1(M)c_2(M) \\ &+ \frac{1}{720}\left(-c_1^4(M) + 4c_1^2(M)c_2(M) + c_1(M)c_3(M) + 3c_2^2(M) - c_4(M)\right) + \mathcal{O}(5)\,. \end{aligned} \tag{2.109}$$

Taking the intersection form $\chi(\mathcal{E}^\bullet, \mathcal{F}^\bullet)$ with the respective basis of quantum periods (2.107), when M is a Calabi-Yau 3-fold, we obtain two times the prepotential $\mathcal{F} = (X^0)^2 F_0(\boldsymbol{t})$, where X^0 is the fundamental period, and $F_0(\boldsymbol{t})$ reads

$$F_0(\boldsymbol{t}) = \frac{c_{abc}}{6}t^a t^b t^c + \frac{c_{aab}}{2}t^a t^b + \frac{c_a}{24}t^a + \frac{\zeta(3)}{2(2\pi i)^3} + \sum_{m\in\mathbb{N}}\sum_{\kappa\in H_2(M,\mathbb{Z})} \frac{n^0_\kappa}{m^3}\mathrm{e}^{m\kappa\cdot\boldsymbol{t}}\,. \tag{2.110}$$

Alternatively, we can express $F_0(\boldsymbol{t}) = \text{classical terms} + \sum_{\kappa>0} N_{0,\kappa}\mathrm{e}^{\kappa\cdot\boldsymbol{t}}$, where $N_{0,\kappa}$ are the genus zero Gromov-Witten invariants, introduced in (2.12). Thus, the genus zero Gromov-Witten theory for M is captured by the enumerative geometry invariants n^0_κ, whose physical interpretation will be discussed in the upcoming chapter. Before moving on, let us discuss how modularity manifests in Calabi-Yau geometries that possess an elliptic or genus one fibration.

Modularity from monodromies: The integral periods in the B-model undergo monodromy transformations when enclosing a non-contractible loop in the complex structure moduli space. In the A-model, the monodromies lift to auto-equivalences $\Phi_{\mathcal{E}} : D^b(M) \to D^b(M)$ by homological mirror symmetry [37], where $\Phi_{\mathcal{E}}$ is a Fourier-Mukai transform [42]. We do not address the latter type of transformations here but only remark that elliptically fibered Calabi-Yau manifolds exhibit auto-equivalences identifying with transformations of the full modular group $\mathrm{SL}(2,\mathbb{Z})$ [1, 43–45], and in genus one fibrations, congruence subgroups $\Gamma \subset \mathrm{SL}(2,\mathbb{Z})$ [2, 46]. Such auto-equivalences induce an action on the B-brane central charges that imply modular transformations for the Kähler moduli parameters. Moreover, the modular group action extends to objects like (2.110), more generally the topological string amplitudes F_g, pointing out their automorphic forms behavior. We include in Appendix B the automorphic forms objects that we consider in this thesis.

2.5 M-theory

M-theory is conjectured to be a non-perturbative extension to type IIA string theory [47]. This underlying theory manifests when analyzing the degrees of freedom in uncompactified Type IIA, such that we take a strong string coupling limit $g_s \to \infty$. For instance, a bound state of N D0-branes has mass $N/g_s\sqrt{\alpha'}$, which is precisely that associated to a KK mode of a KK circle with radius $g_s\sqrt{\alpha'}$. The configuration $g_s \to \infty$ suggests that Type IIA theory unwraps a KK circle into another spatial direction of a higher dimensional spacetime. More generally, we can realize all Type IIA dynamical objects via KK reduction of a putative 11d quantum gravity theory. In such an 11d theory, the microscopic degrees of freedom are 2-branes and 5-branes, which we respectively call M2-branes and M5-branes We summarize in Table 2.5 how to obtain the rest of Type IIA objects by KK reduction of M-theory [13]. Despite the lack of a fundamental formulation of M-theory, we can describe its low energy limit in terms of 11d supergravity. We elaborate on that theory next.

M-theory:	M2	M2	M5	M5	
$\downarrow S^1$	$\downarrow S^1$	$\downarrow$	$\downarrow S^1$	$\downarrow$	
Type IIA:	F1	D2	D4	NS5	D6 $\longleftrightarrow$ KK-monopole

Table 2.5: Kaluza-Klein reduction of M-theory. We indicate with S^1 those objects that arise by wrapping a KK circle.

A striking fact is the existence of a unique 11d supergravity theory [48]. Besides that, $d = 11$ is the maximal dimension that allows the graviton to sit in a supersymmetry representation with spin $J \leq 2$ [49]. This theory possesses $\mathcal{N} = 1$ supersymmetry and its low energy action for the bosonic fields reads

$$S_{11d} = \frac{1}{2\kappa_{11}} \int \sqrt{g} \left(R - \frac{1}{2}|F_4|^2 \right) - \frac{1}{12\kappa_{11}} \int C_3 \wedge F_4 \wedge F_4 \,. \tag{2.111}$$

Here g stands for the 11d metric with corresponding curvature R, whereas C_3 is a 3-form field with corresponding field strength $F_4 = \mathrm{d}C_3$. The 3-form C_3 is the field to which a

2-dimensional extended object couples electrically, i.e. an M2-brane. On the other hand, M5-branes couple to the magnetic dual 6-form C_6, where $\mathrm{d}C_6$ yields the Hodge dual form $*F_4$. As expected, KK reduction of (2.111) furnishes the low energy effective action of Type IIA string theory.

As explained in section 2.2, the Calabi-Yau compactification program extends to M-theory. However, neither M-theory nor its 11d supergravity low energy limit possesses a perturbative description. Nevertheless, we argue now how to compute certain M-theory observables by employing topological string theory techniques, which we introduced in section 2.4.

2.5.1 Gopakumar-Vafa invariants

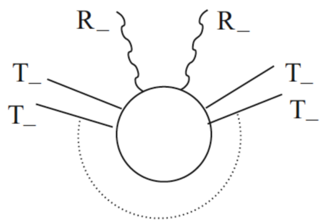


Figure 2.5: 1-loop integral contribution due to BPS saturated states. Image taken from [7].

A beautiful application for the topological string amplitudes F_g is that they count BPS states in Type IIA/M-theory. This crucial result derives by identifying F_g with gravitational couplings in a 4d $\mathcal{N}=2$ supergravity theory, obtained via Type IIA on a Calabi-Yau 3-fold M [50]. More precisely, a non-trivial set of F-terms—a subsector of the effective action $\mathcal{I} \subset S$—capture the gravitational couplings as follows

$$\mathcal{I} = -i \sum_{g=0}^{\infty} \int_{M_4} \mathrm{d}^4\, x\, \mathrm{d}^4\, \theta F_g\left(\boldsymbol{t}, \bar{\boldsymbol{t}}\right) \mathcal{W}^{2g} \,. \tag{2.112}$$

Here $\mathcal{W}$ is the chiral superfield of the supergravity multiplet whose bottom component is the anti-self-dual part of the graviphoton field-strength T_-. In the works [51, 52], Gopakumar and Vafa proposed that the effective action is computable by adapting the *Schwinger calculation*, i.e., a 1-loop effective action due to a charged particle in an external electromagnetic field with constant field-strength [53]. To this end, the authors consider the topological limit $F_g(\boldsymbol{t}) = \lim_{\bar{\boldsymbol{t}} \to i\infty} F(\boldsymbol{t}, \bar{\boldsymbol{t}})$, as well as a constant vacuum expectation value for $T_- = \langle T_- \rangle = \lambda$. With this in mind, and taking R_- as the anti-self-dual part of the Riemann tensor, the integrand within (2.112) reduces to a vertex interaction $R_-^2 F_g \lambda^{2g-2}$ which is 1-loop exact and only receives contributions from BPS particles. See Figure 2.5. This way, the Schwinger calculation leads to the expression

$$\mathcal{I} = \textstyle\int_{M_4} F(\boldsymbol{t}, \lambda) R_-^2 \xrightarrow{\text{Schwinger 1-loop}} F(\boldsymbol{t}, \lambda) = \int_{\epsilon}^{\infty} \frac{ds}{s} \frac{\mathrm{Tr}(-1)^{\mathrm{F}} \mathrm{e}^{-sm} \mathrm{e}^{-2sJ_L\lambda}}{\left(2\sin\left(s\frac{\lambda}{2}\right)\right)^2} \,. \tag{2.113}$$

Here F is the total fermion number, and J_L is the Cartan generator of $\mathfrak{su}(2)_L \subset \mathfrak{su}(2)_L \oplus \mathfrak{su}(2)_R \simeq \mathfrak{so}(4)$ that we associate with representations of the Lorentz group $SO(4)$. The relevant point to discuss here is that we can reinterpret the topological free energy as a counting over Type IIA/M-theory degrees of freedom.

Now, we direct our attention to the internal degrees of freedom appearing in the Gopakumar-Vafa calculation. The BPS masses considered in the formula (2.113) are of the form

$$m(\kappa, k) = \frac{1}{\lambda} 2\pi i \kappa \cdot \boldsymbol{t} + 2\pi i k\,, \quad \kappa \in H_2(M, \mathbb{Z})\,, \quad k \in \mathbb{Z}\,. \tag{2.114}$$

In the M-theory picture, the mass (2.114) corresponds to a BPS state that arises from an M2-brane that wraps a curve $\kappa \in H_2(M, \mathbb{Z})$ with KK momentum k along the M-theory circle. Upon circle reduction to Type IIA, we obtain a bound state of (D2-D0)-branes in which the D2-brane wraps the curve C_κ and k denotes the number of D0-branes. The states running in the 1-loop integral are off shell quantum fields that transform under the group $\mathrm{Spin}(4) \simeq SU(2)_L \times SU(2)_R$, i.e. the double cover of the Lorentz group $SO(4)$. It turns out that $\mathcal{N} = 2$ BPS states which fall in $SU(2)_L \times SU(2)_R$ representations have the form [13]

$$\left(\left[\frac{1}{2}\,, 0\right] \oplus 2\,[0\,,0] \right) \otimes [j_L\,, j_R]\,, \tag{2.115}$$

where $[j_L, j_R]$ denotes am $SU(2)_L \times SU(2)_R$-representation with $j_L, j_R \in \mathbb{Z}/2$.

Back to the Schwinger calculation, the integrand R_-^2 in (2.113) takes into account the first factor in the representations product (2.115), whereas the second entries correspond to representations of BPS states that are integrated out in $F(\boldsymbol{t}, \lambda)$. The fields R_- and T_- couple only to the $SU(2)_L$ part, while the representations in the $SU(2)_R$ factor become irrelevant for expressions (2.113). Thus, by considering BPS masses and j_L spins, the topological free energy turns into the so-called *multicovering formula* that reads

$$F(\boldsymbol{t}, \lambda) = \sum_{\kappa \in H_2(M,\mathbb{Z})} \sum_{g=0}^{\infty} \sum_{\ell=1}^{\infty} \frac{n_\kappa^g}{\ell} \left(2 \sin\left(\frac{\ell\lambda}{2}\right)\right)^{2g-2} \mathrm{e}^{2\pi i \ell \kappa \cdot \boldsymbol{t}}\,. \tag{2.116}$$

Here $n_\kappa^g \in \mathbb{Z}$ is the *Gopakumar-Vafa invariant* associated to a curve C_κ of class $\kappa \in H_2(M, \mathbb{Z})$ and genus g. These numbers relate to multiplicities $N^\kappa_{j_L, j_R}$ of 5d BPS states with representation (2.115) that arise from M2-branes wrapping curves C_κ. The precise relation reads from the weighted right spin sum

$$\sum_{j_L, j_R \in \frac{1}{2}\mathbb{Z}_{\geq 0}} (-1)^{2j_R}(2j_R + 1) N^\kappa_{j_L, j_R}\,[j_L] = \sum_{g \in \mathbb{Z}_{\geq 0}} n_\kappa^g I_g\,, \text{ where } I_g = \left(\left[\frac{1}{2}\right] + 2\,[0] \right)^g\,. \tag{2.117}$$

Note that the basis exchange $[j_L] \mapsto I_g$ relates the left spin to the genus g of the curve C_κ.

Let us remark that the topological free energies $F_g(\boldsymbol{t})$ have pure dependence on Kähler structure moduli, which are captured by the topological string A-model on M. On the other hand, the string coupling reads $g_s = \exp(\Phi_{\mathrm{IIA}})$, where Φ_{IIA} is the dilaton which is in a hypermultiplet parametrized by the complex strcture moduli space $\mathcal{M}_{\mathrm{cs}}(M)$. Thus, regardless of either regime $g_s \ll 1$ (perturbative Type IIA) or $g_s \gg 1$ (M-theory), the information captured by $\mathcal{I}$ and Gopakumar-Vafa invariants is exact.

2.5.2 Refined BPS invariants and the Nekrasov partition function

As we argued in the previous section, the Gopakumar-Vafa invariants correspond to a weighted sum of multiplicities of BPS states. In general, the number of BPS states with a particular mass and spin jump across lines of marginal stability in the complex structure moduli space. Thus, the actual numbers $N^{\kappa}_{j_L,j_R}$ themselves are not invariant. However, in the presence of an additional S^1 isometry in the internal geometry—besides the M-theory circle one—we can refine the topological string partition function [54]. By refinement, here we mean to encode the actual number of BPS states for a given set of quantum numbers.

It is possible to geometrize a deformation of the topological free energy by means of the so-called Ω-background. We can realize this background from a 6d $N = (1, 0)$ theory with topology $\mathbb{R}^4 \times T^2$ and metric [55]

$$\mathrm{d}s^2 = \left(\mathrm{d}x^\mu + \Omega^\mu \mathrm{d}z + \bar{\Omega}^\mu \mathrm{d}\bar{z}\right)^2 + \mathrm{d}z\mathrm{d}\bar{z}\,, \tag{2.118}$$

where $(z, \bar{z})$ are coordinates on T^2 and the Ω^μ entries in (2.118) fulfil the equation

$$T = \mathrm{d}\Omega = \epsilon_1 \mathrm{d}x^1 \wedge \mathrm{d}x^2 + \epsilon_2 \mathrm{d}x^3 \wedge \mathrm{d}x^4\,. \tag{2.119}$$

Here the linear combination $\epsilon_\pm = \frac{1}{2}(\epsilon_1 \pm \epsilon_2)$ define equivariant rotation parameters of $\mathbb{C}^2 \cong \mathbb{R}^4$, which can be expressed in the spinor notation as

$$\epsilon_-^2 = -\det\left(T_{\alpha\beta}\right), \quad \epsilon_+^2 = -\det\left(T_{\dot\alpha\dot\beta}\right), \tag{2.120}$$

where $\alpha, \beta, \dot\alpha, \dot\beta = 1, 2$ are spinor indices for $SU(2)_L \times SU(2)_R$. In addition to the non-trivial metric (2.118), another $SU(2)_I$ R-symmetry is necessary to preserve the amount of supersymmetry. The effect of the latter symmetry is to compensate metric induced holonomies on the spinors (2.120), which must be covariantly constant. Once this is done, the strategy is to take the limit $\mathrm{Vol}(T^2) \to 0$, while keeping the parameters $\epsilon_\pm$ finite [56]. In this way we obtain a deformed 4d theory, such that the antiself-dual part ϵ_- and self-dual part ϵ_+ of the field strength T couple to the left spin and right spin of the BPS particles respectively.

If gravity can be decoupled, a further $SU(2)_I$ R-symmetry emerges. In this case the degeneracies of BPS states are protected. Consequently, it makes sense to introduce the BPS supertrace [57]

$$\mathcal{Z}_{\mathrm{Nek}}(\epsilon_-, \epsilon_+, \boldsymbol{t}) = \mathrm{Tr}_{\mathrm{BPS}}\,(-1)^{2(J_L+J_R)}\,\mathrm{e}^{-2\epsilon_- J_L}\mathrm{e}^{-2\epsilon_+(J_R+J_I)}\mathrm{e}^{-\beta H}\,, \tag{2.121}$$

where J_* is the Cartan generators of each $SU(2)_*$ symmetry group. Moreover, the combination $J_R' \equiv J_R + J_I$ enables a twist of $SU(2)_R$ such that the BPS degeneracies $N^{\beta}_{j_L,j_R'}$ are invariant [54]. From now on, we will denote by j_R the spin of the twisted J_R' generator. Furthermore, the refined Schwinger loop calculation yields the multicovering formula for the refined topological free energy $\mathcal{F}(\epsilon_-, \epsilon_+, \boldsymbol{t}) = \log \mathcal{Z}_{\mathrm{Nek}}(\epsilon_-, \epsilon_+, \boldsymbol{t})$ [54, 58]:

$$\mathcal{F}(\epsilon_1, \epsilon_2, \boldsymbol{t}) = \sum_{\kappa \in H_2(M,\mathbb{Z})} \sum_{j_L, j_R \in \frac{1}{2}\mathbb{Z}_{\geq 0}} \sum_{m=1}^{\infty} (-1)^{2j_L+2j_R} \frac{N^{\kappa}_{j_L,j_R}}{m} \frac{[j_L]_{x^m}\,[j_R]_{y^m}}{(\mathsf{X}^{\frac{m}{2}} - \mathsf{X}^{-\frac{m}{2}})(\mathsf{Y}^{\frac{m}{2}} - \mathsf{Y}^{-\frac{m}{2}})} \mathrm{e}^{2\pi i m \boldsymbol{t}\cdot\kappa}\,. \tag{2.122}$$

where

$$\mathsf{X} = \exp\left(2\pi i \epsilon_1\right) = xy\,, \quad \mathsf{Y} = \exp\left(2\pi i \epsilon_2\right) = \frac{y}{x}\,, \tag{2.123}$$

and for $j \in \frac{1}{2}\mathbb{Z}_{\geq 0}$

$$[j]_u \equiv u^{-2j} + u^{-2j+2} + \cdots + u^{2j-2} + u^{2j}\,. \tag{2.124}$$

2.5.3 The modular bootstrap and the HKK conjecture

On a compact Calabi-Yau 3-fold that exhibits an elliptic fibration, the most efficient method to solve the topological string partition function—up to date—is the modular bootstrap [59]. Here we expose the idea behind this program that applies to backgrounds for F-theory compactifications. Concretely, we consider Calabi-Yau 3-folds M that enjoy a fibration structure $\pi : M \to B$, where the fibers are genus one curves $\mathcal{C}$. See for instance the fibrations discussed in sections 2.3.5 and 2.3.6. With this in mind, we discuss now the modular behaviour of the toplogical string partition function $Z_{\rm top} = \exp\Big(\sum_{g\geq 0} \lambda^{2g-2} F_g\Big)$. The crucial point is that it follows a Fourier expansion over the base Kähler moduli

$$Z_{\rm top}(\lambda, \tau, \boldsymbol{z}, \boldsymbol{t}_{\rm b}) = Z_0(\lambda, \tau, \boldsymbol{z}) \left(1 + \sum_{\beta\in H_2(B,\mathbb{Z})} Z_\beta(\lambda, \tau, \boldsymbol{z}) Q_\beta\right) , \tag{2.125}$$

where the coefficients $Z_\beta(\tau, \lambda, \boldsymbol{z})$ are meromorphic lattice Jacobi forms of weight zero for a congruence subgroup $\Gamma_1(N) \subset \mathrm{SL}(2,\mathbb{Z})$. As usual, λ is the topological string coupling parameter, while $\{\tau, \boldsymbol{z}, \boldsymbol{t}_{\rm b}\}$ are Kähler moduli that we organize by using the Shioda-Tate-Wazir theorem for elliptic and genus one fibrations [23, 60]. Moreover, $N\tau$ denotes the complex volume of the fibers $\mathcal{C}$, $\boldsymbol{z}$ are complexified volumes of fibral curves, and $\boldsymbol{t}_{\rm b}$ are shifted volume of curves in the base B. To prevent the flow of our exposition, we include in the Appendix B.5 the recipe for calculating the base coefficients $Z_\beta(\lambda, \tau, \boldsymbol{z})$. Instead, we focus now on explaining the underlying physics of elliptic and genus one fibrations.

From topological strings to elliptic genera of strings: So far, we have seen, the topological string partition function counts information of 5d BPS particles, realized by M2-branes wrapping curves on a Calabi-Yau 3-fold M. It turns out that we can describe the same information by considering a parental 6d $N = (1, 0)$ theory reduced along a circle. In the 6d counterpart, the dynamical objects to consider are non-BPS solitonic strings that, upon Kaluza-Klein reduction, give rise to 5d BPS states. This case realizes in F-theory on an elliptic fibration $\pi : M \to B$, in which 6d strings descend from D3-branes wrapping curves $C_\beta \subset B$ with class $\beta \in H_2(B, \mathbb{Z})$. By the duality F-theory on $M \times S^1 \longleftrightarrow$ M-theory on M, a string wrapped along S^1, with wrapping number w and Kaluza-Klein momentum k, maps to a particle whose M2-brane origin derives from a curve $wC_\beta + k\mathcal{E}$ [61, 62], where $\mathcal{E}$ is the elliptic fiber.

When we consider the trace of a 6d string compactified along S^1 while further taking time-periodic boundary conditions, we can define the so-called *elliptic genus* of a string furnished by $C_\beta \subset B$ that reads [61, 63–65]

$$Z_\beta(\tau, \lambda, \boldsymbol{z}) = \mathrm{Tr}_{RR}\left[(-1)^{\rm F} \mathrm{F}^2 q^{H_L} \bar{q}^{H_R} y^{2J_-} \prod_{a=1}^{n_V} (\zeta^a)^{J_a}\right] . \tag{2.126}$$

Here F denotes fermion number, while $q \equiv \mathrm{e}^{2\pi i\tau}$, $y \equiv \mathrm{e}^{2\pi i\lambda}$, and $\zeta^a = \mathrm{e}^{2\pi i z^a}$ are weighting parameters that we explain next. This index interprets as counting over excited string states propagating in 6d, where the parameter τ plays the role of modular parameter for a stringy worldsheet torus. The z^a's, with $a = 1, \ldots, n_V$, are fugacity parameters that weigh over $U(1)_a$ Cartan (non-Cartan) gauge group factors, measured by the respective generators J_a, which turn into global symmetries at the worldsheet level. Similarly, λ weighs over a subgroup factor $SU(2)_-$ of the transverse Lorentz group $SO(4)_\perp$—towards the propagating string—, with generator J_-. Note that, on purpose, we utilized the same notation for the elliptic genus (2.126) as for the topological string coefficients (2.125). This is because an enormous amount of works noticed the equivalence between these two objects.

For cases when M is a local Calabi-Yau, a description on the Ω-background is possible [54], and the elliptic genera allow a refinement. We denote the refined topological string partition by $\mathcal{Z}_{\text{top}}$ and its base Fourier coefficients by $\mathcal{Z}_\beta(\tau, \epsilon_+, \epsilon_-, \boldsymbol{z})$ in these cases. The most systematic and efficient method to solve for $\mathcal{Z}_{\text{top}}$ is the so-called blow-up equations [66–69], where $\epsilon_\pm$ are the parameters introduced in (2.120). However, we do not consider such cases in this work. Instead, we will consider compact geometries, consequently, the unrefined topological string partition function Z_{top}, for which the modular bootstrap is the most effective method to this day.

Finally, we remark that we kept the discussion for elliptic genera to elliptic fibrations geometries. Further extensions to more general genus one fibrations are a subject of Chapter 5, also explored in the works [46, 70, 71]. Nevertheless, in the upcoming Chapter, we intend to give a more detailed explanation about the physics of elliptic and genus one fibrations, i.e., F-theory.

CHAPTER 3

The physics of torus fibrations

3.1 F-theory

F-theory is a beautiful framework that ties together geometry and fundamental physics. It describes many models with a geometric realization in the string landscape, including those with a non-perturbative character. Moreover, its versatility in applications ranges from phenomenology in particle physics to the study of formal quantum field theory and non-perturbative methods thereof. In practical terms, we think of F-theory as a dictionary that translates the geometry of torus fibrations into physics theories characterized by properties such as gauge groups, matter spectra content, Yukawa's interactions, to name a few. With such an understanding, we can systematically engineer both supergravity and quantum field theories with minimal supersymmetry.

There are two main approaches to understand F-theory. On the one hand, we have F-theory's formal definition as a non-perturbative extension of Type IIB string theory (Type IIB for shortness), which we cover in Section 3.1.1. On the other hand, we have the duality equivalence with M-theory through a torus fibration, which becomes an auxiliary object encoding all physics phenomena—see Section 3.1.2. The latter approach will be more interesting for us, as it connects with enumerative geometry invariants explained in Chapter 2. Having said this, we will focus on explaining in the subsequent sections the physics interpretation of properties that classify torus fibrations. We will base our exposition mainly on the references [72, 73].

3.1.1 Type IIB and 7-branes

F-theory is a non-perturbative extension of Type IIB that incorporates 7-branes that back-react on spacetime geometry. In this framework a primary object is a D7-brane which is a 7d extended object magnetic dual to the RR axion C_0 appearing in the axion-dilaton field

$$\tau = C_0 + i\mathrm{e}^{\Phi_{\mathrm{IIB}}}\,, \tag{3.1}$$

where Φ_{IIB} is the dilaton that determines the string coupling $g_s = \exp\{\Phi_{\mathrm{IIB}}\}$ in Type IIB. A relevant property to us of D7-branes is that they preserve 16 supercharges in flat spacetime $\mathbb{R}^{1,9}$. Moreover, when enclosing a loop around the transverse directions to a D7-brane, this induces a monodromy $\mathrm{T} : \tau \mapsto \tau + 1$ on the axio-dialaton, where T is a generator of the modular group $\mathrm{SL}(2,\mathbb{Z})$. See Figure 3.1.

More generally, $\mathrm{SL}(2,\mathbb{Z})$ transformations manifest in Type IIB as follows. Let us consider

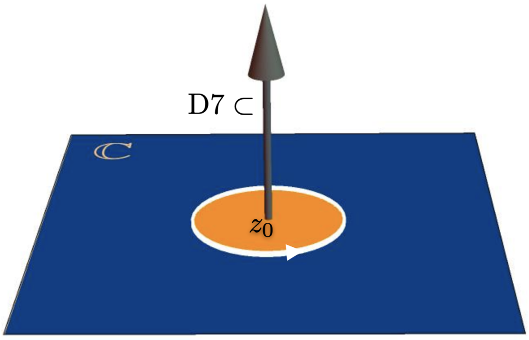


Figure 3.1: D7-brane located inside $\{z_0\} \times \mathbb{R}^{1,7} \subset \mathbb{C} \times \mathbb{R}^{1,7} \simeq \mathbb{R}^{1,9}$. As we encircle z_0, the D7-brane induces a T-monodromy transformation on the axio-dilaton τ.

the Type IIB effective action in the Einstein frame

$$S_{\mathrm{IIB}} = 2\pi \int \mathrm{d}^{10}x\sqrt{-g}\left(R - \frac{\partial_\mu \tau \partial^\mu \bar{\tau}}{2(\mathrm{Im}\tau)^2} - \frac{1}{2}\frac{|G_3|^2}{\mathrm{Im}\tau} - \frac{1}{4}|F_5|^2\right) - \frac{i\pi}{2}\int \frac{1}{\mathrm{Im}\tau} C_4 + G_3 \wedge \bar{G}_3\,, \quad (3.2)$$

where C_{2p} are the RR $2p$-form fields, B is the NS 2-form field, $G_3 = \mathrm{d}C_2 - \tau \mathrm{dB}$, and $F_5 = \mathrm{d}C_4 - C_2 \wedge \mathrm{dB}/2 + \mathrm{B} \wedge \mathrm{d}C_2/2$. The action S_{IIB} follows invariance under $\mathrm{SL}(2,\mathbb{Z})$ transformations determined by the group action

$$\tau \mapsto \frac{a\tau + b}{c\tau + d}\,, \qquad \begin{pmatrix} C_2 \\ \mathrm{B} \end{pmatrix} \mapsto \gamma \begin{pmatrix} C_2 \\ \mathrm{B} \end{pmatrix}, \qquad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{Z})\,, \quad (3.3)$$

while the $\mathrm{SL}(2,\mathbb{Z})$ action on C_4 and $g_{\mu\nu}$ is trivial. In fact, the action S_{IIB} is invariant under $\mathrm{SL}(2,\mathbb{R})$ transformations that follow the replacement rules (3.3). However, it is expected that the full non-perturbative Type IIB only preserves invariance under the subgroup $\mathrm{SL}(2,\mathbb{Z}) \subset \mathrm{SL}(2,\mathbb{R})$. For instance, D$(-1)$ instanton effects involve terms of the form $\mathrm{e}^{2\pi i \tau}$ that restrict τ-shift symmetries into $\mathrm{SL}(2,\mathbb{Z})$. As we will see, 7-branes are the objects responsible for the $\mathrm{SL}(2,\mathbb{Z})$ monodromies on τ, e.g., we associate D7-branes with T-monodromies.

(p,q) strings & 7-branes: A (p,q) string is a bound state of p fundmanetal strings with q D1-branes—strings that couple electrically to C_2—such that p and q are coprime integers. Consequently, (p,q) strings couple electrically to $p\mathrm{B} + qC_2$ and attach to 7-branes that are called (p,q) 7-branes. Note that a (1,0) 7-brane is a D7-brane and we can think of every (p,q) 7-brane, in a local frame, as a D7-brane. To see this, we take a (p,q) string into a $(1,0)$ string via an appropriate transformation $\gamma_{p,q}^{-1} \in \mathrm{SL}(2,\mathbb{Z})$. Thus, a (p,q) 7-brane induces a non-trivial monodromy transformation $\mathrm{T}_{p,q} \equiv \gamma_{p,q}\mathrm{T}\gamma_{p,q}^{-1}$ on the τ profile, which occurs when encircling the (p,q) 7-brane in its normal direction. This is analogous to the situation for D7-branes depicted in Figure 3.1. Nonetheless, we remark that a pair of 7-branes—say a (p,q) 7-brane and a (p',q') 7-brane—, in general, cannot turn into D7-branes simultaneously by performing an $\mathrm{SL}(2,\mathbb{Z})$ transformation. In this case, we say that the 7-branes are *mutually non-local* and this happens iff $[\mathrm{T}_{p,q}, \mathrm{T}_{p',q'}] \neq 0$.

F-theory compactifications: When we consider an F-theory compactification

$$\mathbb{R}^{1,9} \to \mathbb{R}^{1,9-2(n-1)} \times B\,, \tag{3.4}$$

such that B is a compact manifold of complex dimension $n-1$, we also include 7-branes that wrap a codimension-1 cycle $\mathcal{S}^{\mathrm{b}} \subset B$. To preserve supersymmetry in this setup, we need to ensure the requirements:

$$\begin{cases} \text{(a)} : B \text{ is a complex Kähler manifold}\,, \\ \text{(b)} : \text{7-brane cycle } \mathcal{S}^{\mathrm{b}} \text{ is holomorphic}\,, \\ \text{(c)} : \tau \text{ profile varies holomorphically over } B\,. \end{cases} \tag{3.5}$$

Taking into account these conditions, we find out that supersymmetry relates the curvature of B to the variation of the dilaton Φ_{IIB}. More precisely, Einstein's equations lead to [74]

$$R_{a\bar{b}} = \nabla_a \nabla_{\bar{b}} \Phi_{\mathrm{IIB}} \neq 0\,, \tag{3.6}$$

where $R_{a\bar{b}}$ are the components of the hermitian Ricci 2-form $\mathcal{R}$ associated with B, and ∇_a are complex covariant derivatives in B. We interpret this result as a consequence of the back-reaction of 7-branes that lead to curve the geometry B. In other words, B is not locally Ricci flat and hence cannot be a Calabi-Yau manifold, as $c_1(B) = [\mathcal{R}]$.

The condition (c) in (3.5), together with the $\mathrm{SL}(2,\mathbb{Z})$ transformations acting on τ, allows to define a holomorphic line bundle $\mathcal{L}$ over B such that a section $h \in \Gamma(B, \mathcal{L})$ transforms from patch to patch as

$$h \to (c\tau + d)h\,, \quad \text{where} \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{Z})\,, \quad \text{and} \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \tau = \frac{a\tau + b}{c\tau + d}\,. \tag{3.7}$$

Let us discuss the significance of this bundle. In each open set $U \subset B$ a local $\mathrm{SL}(2,\mathbb{Z})$ frame defines the supergravity fields, which are complex functions that we associate with sections of $\mathcal{L}$. In another open set $V \subset B$, different 7-branes subject the supergravity fields to change into another $\mathrm{SL}(2,\mathbb{Z})$ frame.

It is possible to show that Einstein's equations (3.6) are equivalent to [74]

$$c_1(B) = c_1(\mathcal{L})\,. \tag{3.8}$$

The crucial point is that we can geometrize the $\mathrm{SL}(2,\mathbb{Z})$-bundle $\mathcal{L}$ by means of an elliptic fibration over B, where the varying axio-dilaton τ translates into the complex structure of an elliptic curve fiber $\mathcal{E}_\tau \simeq \mathbb{C}/(\tau\mathbb{Z} \oplus \mathbb{Z})$. To achieve this equivalence, we note that there is a one-to-one correspondence between the holomorphic line bundle $\mathcal{L}$ and a fibration structure

$$\begin{tikzcd} \mathcal{E}_\tau \arrow[r, hook] & M \arrow[d, "\pi"] \\ & B \end{tikzcd} \,. \tag{3.9}$$

Details about this construction will be discussed in section 3.1.3. In this treatment we will find out that Einstein's equations and supersymmetry requirement imply that M is an elliptically fibered Calabi-Yau manifold. Before we elaborate more on elliptic fibrations and the physical information they encode, we discuss F-theory duality with M-theory first.

3.1.2 M-theory/F-theory duality

So far we found an auxilary device that enables us to keep track of Type IIB compactifications with 7-branes. However, we now argue that we can realize a duality with M-theory in which the ellipically fibered Calabi-Yau n-fold M acquires physical significance. This argument involves a chain of dualities that we summarize next: [75]

1. Compactify M-theory on $T^2 \simeq S^1_M \times S^1_T$ with complex structure τ. Here S^1_M and S^1_T are circles of radius R_M and R_F respectively.

2. Take small radius R_M for M-theory circle $S^1_M \Longrightarrow$ weakly coupled Type IIA on S^1_T.

3. Perform T-duality along $S^1_T \Longrightarrow$ Type IIB on S^1_F. Here S^1_F is the T-dual circle to S^1_T and the Einstein frame metric reads

$$ds^2_{\text{IIB}} = ds^2_{\mathbb{R}^{1,8}} + \frac{\ell_s^2}{V} dx^2 \,, \tag{3.10}$$

where V is the volume of T^2 and x is the coordinate along S^1_F.

4. Take the limit $V \to 0$ while ℓ_s finite $\Rightarrow$ uncompactified Type IIB with axio-dilaton τ.

5. Promote this duality limit, fiberwise, for M-theory on an elliptically fibered Calabi-Yau n-fold M.

There are two types of F-/M-theory duality that are commonly invoked in the literature. On the one hand, the third step establishes directly the duality

$$\text{M-theory on } M \;\longleftrightarrow\; \text{F-theory on } M \times S^1_F \sim \text{Type IIB on } B \times S^1_F \quad , \tag{3.11}$$

where we emphasize that F-theory on the elliptic fibration $\pi : M \to B$ means Type IIB on B with 7-branes and $\mathrm{SL}(2,\mathbb{Z})$-bundle. The other type of duality, which we call *F-theory limit*, refers to taking the fiberwise version of the fourth step, which yields the projection

$$\pi : \text{M-theory on } M \xrightarrow{\mathrm{Vol}(\mathcal{E}_\tau)\to 0} \text{Type IIB on } B \text{ with 7-branes} \quad . \tag{3.12}$$

In any case, each elliptic fiber $\mathcal{E}_\tau$ in an elliptic fibration takes the role of internal space on the M-theory side. As such, it possesses finite volume and complex structure τ. In the F-theory limit the volume of $\mathcal{E}_\tau$ becomes irrelevant, but its complex structure τ descends into the axio-dilaton that manifests modularity in non-perturbative Type IIB. Having said this, we examine now in detail the physics and geometry of elliptic fibrations.

3.1.3 Elliptic fibrations and non-Abelian gauge symmetries

Every elliptic fibration $\pi : M \to B$ is birationally equivalent to a Weierstrass model [76]. The latter is described by the locus $P_W(x, y, z) = 0$ in a weighted projective bundle $\mathbb{P}^{(2,3,1)}(\mathcal{L}^2 \oplus \mathcal{L}^3 \oplus \mathcal{O}_B)$ over B, where $\mathcal{L}$ is a line bundle over B, $\mathcal{O}_B$ is the trivial line bundle over B, and the Weierstrass form polynomial reads

$$P_W(x, y, z) \equiv y^2 - \left(x^3 + fxz^4 + gz^6\right) \,. \tag{3.13}$$

Here, when we consider the Weierstrass model associated with the $\mathrm{SL}(2,\mathbb{Z})$-bundle $\mathcal{L}$, the homogeneous coordinates $[x:y:z]$ in the weighted projective space $\mathbb{P}^{(2,3,1)}$ promote to sections $x \in \Gamma(B,\mathcal{L}^2)$, $y \in \Gamma(B,\mathcal{L}^3)$, and $z \in \Gamma(B,\mathcal{O}_B)$, while f and g are sections of the form $f \in \Gamma(B,\mathcal{L}^4)$, $g \in \Gamma(B,\mathcal{L}^6)$. Furthermore, through the adjunction formulae we can compute that

$$c_1(M) = c_1(B) - c_1(\mathcal{L}) \stackrel{!}{=} 0\,. \tag{3.14}$$

In order to relate an elliptic fibration $\pi : M \to B$ with an F-theory compactification, we must fulfil the supersymmetry and Einstein's equations constraint (3.8). Thus, this implies that M must be a Calabi-Yau n-fold and we can fix that $\mathcal{L} = K_B^{-1}$, where we denote by K_B the canonical bundle in B.

A feature of Weierstrass models is the zero section $s_0 : p \mapsto [1:1:0]$, where p is any point in B and $[x:y:z] = [1:1:0]$ is a marking point in the elliptic fiber. This map defines the zero section divisor

$$S_0 : \{z = 0\}\,. \tag{3.15}$$

Moreover, the base contains a set of divisors $\{D^{\mathrm{b}}_\alpha\}_{\alpha=1,\dots h^{1,1}(B)}$ that span $H^{1,1}(B,\mathbb{Z})$, and their pullbacks $\pi^* D^{\mathrm{b}}_\alpha$ embed into $H^{1,1}(M,\mathbb{Z})$. Now, when we compactify M-theory on M, the M-theory harmonic 3-form C_3 expands in terms of harmonic forms in $H^{1,1}(M,\mathbb{Z})$, i.e., $C_3 = \sum_i A_i \wedge D_i$ with $D_i \in H^{1,1}(M,\mathbb{Z})$.[1] This leads to a set of vector fields $\{A_i\}_{i=1,\dots h^{1,1}(M)}$ that identify with $U(1)$ symmetries in the effective theory action. We now discuss the vector fields realized by the zero section and base divisors in M.

On the one hand, F-theory on B realizes tensor fields $\{b_\alpha\}_{\alpha=1,\dots h^{1,1}(B)}$ through the Type IIB 4-form C_4 that couples with classes in $H^{1,1}(B)$. Upon F-/M-theory duality reduction over S^1_F, as in (3.11), the terms in C_4 expand as

$$C_4 = \textstyle\sum_{\alpha=1}^{h^{1,1}(B)} b_\alpha \wedge D^{\mathrm{b}}_\alpha \xrightarrow{S^1_F} \textstyle\sum_{\alpha=1}^{h^{1,1}(B)} A^{\mathrm{b}}_\alpha \wedge \mathrm{e}_\alpha \wedge \pi^* D^{\mathrm{b}}_\alpha\,, \tag{3.16}$$

where e_α is a 1-form along S^1_F. On the other hand, when performing F-/M-theory duality, a Kaluza-Klein reduction along the circle S^1_F furnishes a Kaluza-Klein $U(1)$ symmetry, which is embodied by the vector field $\tilde{A}_0$ that couples with the divisor class $\tilde{S}_0 = S_0 - \pi^* c_1(B)/2$ [77, 78]. Let us emphasize that one motivation to consider the F-theory language is obtaining gauge symmetries.

As observed so far, not every vector field A_i manages to have a gauge symmetry origin in its F-theory uplift. In order to achieve this, a divisor $D_{\mathfrak{g}} \in H^{1,1}(M)$—dual to a vector field component $A_{\mathfrak{g}}$—must fulfil the transversality conditions [72]

$$\begin{cases} (\mathrm{a}) : \; D_{\mathfrak{g}} \cdot S_0 \cdot \pi^*(\gamma) = 0 & \text{for every } \gamma \in H^{2n-2}(B)\,, \\ (\mathrm{b}) : \; D_{\mathfrak{g}} \cdot \pi^*(\alpha) = 0 & \text{for every } \alpha \in H^{2n}(B)\,. \end{cases} \tag{3.17}$$

Here the condition (a) ensures that no curve in the base intersects $D_{\mathfrak{g}}$, as $S_0 \cdot \pi^* D^{\mathrm{b}}_{\alpha_1} \cdots \pi^* D^{\mathrm{b}}_{\alpha_{n-1}}$ yields a curve class $\beta \in H_2(B)$. The condition (b) implies vanishing intersection with the generic fiber $\mathcal{E}$ since $\pi^* D^{\mathrm{b}}_{\alpha_1} \cdots \pi^* D^{\mathrm{b}}_{\alpha_n} \propto \mathcal{E} \in H_2(M)$. In what follows, we explain how to realize gauge symmetries in F-theory via special divisors of elliptic fibrations that fulfill these constraints.

[1] Strictly speaking, we should use the symbol $[\cdot]$ to refer to cohomology classes and similar for homology classes, e.g., $[C_3]$ instead of C_3. From this point, we will make abuse of notation by dropping the $[\cdot]$ symbol for the latter type of classes.

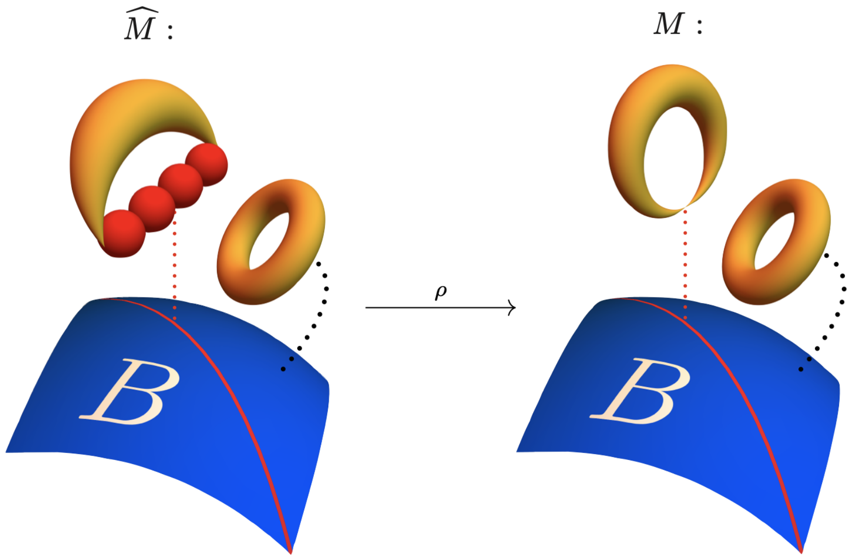


Figure 3.2: Schematic representation of the resolution morphism $\rho : \widehat{M} \to M$.
Right: Elliptic fibration where the fiber degenerates at a curve $\mathcal{S}^{\mathrm{b}}_{\mathfrak{g}_I} \subset B$, depicted by a red curve.
Left: Resolved elliptic fibration. In this picture the fiber acquires the topology of the affine Dynkin diagram associated with $\mathfrak{g}_I = A_4$, whenever it localizes at $\mathcal{S}^{\mathrm{b}}_{\mathfrak{g}_I} \subset B$.

Codimension-1 singularties: A crucial object for the Weierstrass model is the discriminant $\Delta \equiv 4f^3 + 27g^2$. When this object vanishes, it indicates that the elliptic fiber $\mathcal{E}$ associated to the Weiestress model degenerates. Physically, we intepret this locus $\{\Delta = 0\} \subset B$ as the location of 7-branes that back-react on the axio-dilation τ. It turns out that degenerations over codimension one loci in B have been classified by Kodaira and Nerón, which take as input data the vanishing order of non-generic polynomials (f, g, Δ) over $4K_B^{-1}$, $6K_B^{-1}$ and $12K_B^{-1}$ respectively. In this classification one considers a resolution morphism

$$\rho : \widehat{M} \to M \,, \tag{3.18}$$

where $\widehat{M}$ is smooth and isomorphic to M away from a divisor defined by Δ. See Figure 3.2. In a generic case, the discriminant factorizes as $\Delta = \Delta_0 \cdot (\Delta_1)^{p_1} \cdots (\Delta_N)^{p_N}$, where Δ_0 is an irreducible polynomial of vanishing order 1 (Kodaira I_1) and Δ_I is an irreducible polynomial of multiplicty p_I. Moreover, each factor Δ_I defines an irreducible divisor of the form

$$\mathcal{S}^{\mathrm{b}}_{\mathfrak{g}_I} \equiv \{\Delta_I = 0\} \subset B \,. \tag{3.19}$$

When resolving M, the fiber $\mathcal{E}$ at $\mathcal{S}^{\mathrm{b}}_{\mathfrak{g}_I}$ splits into a set of rational curves $\widetilde{C}_{a_I} \cong \mathbb{P}^1$ that form the topology of the affine Dynkin diagram associated to an ADE Lie algebra $\widetilde{\mathfrak{g}}_I$, where $a_I = 0, \ldots, \mathrm{rk}(\widetilde{\mathfrak{g}}_I)$. See, for instance, Figure 3.2. If the monodromies along $\mathcal{S}^{\mathrm{b}}_{\mathfrak{g}_I}$ act on such fibral curves, their invariant orbits result in a folded affine Dynkin diagram associated to a non-simply laced Lie algebra $\mathfrak{g}_I$; otherwise $\mathfrak{g}_I = \widetilde{\mathfrak{g}}_I$. Fibering each monodromy invariant orbit C_{a_I} over $\mathcal{S}^{\mathrm{b}}_{\mathfrak{g}_I}$ yields a set of fibral divisors $\{E_{i_I}\}_{i_I=1,\ldots \mathrm{rk}(\mathfrak{g}_I)}$. With this data, F-theory provides a dictionary that establishes the correspondence:

$$\begin{aligned} C_{i_I} &\quad\longleftrightarrow\quad \text{Simple roots } -\boldsymbol{\alpha}_{i_I} \text{ of } \mathfrak{g}_I\,, \\ E_{i_I} &\quad\longleftrightarrow\quad \text{Simple coroots } \boldsymbol{\alpha}^\vee_{i_I} \text{ of } \mathfrak{g}_I\,. \end{aligned} \tag{3.20}$$

This identification follows from the intersections relations that encode group theoretical information: [72]

$$\begin{aligned} E_{i_I} \cdot E_{j_J} \cdot \pi^*(\gamma) &= -\delta_{IJ} \left(\boldsymbol{\alpha}^\vee_{i_I}, \boldsymbol{\alpha}^\vee_{j_J}\right)_{\mathfrak{g}_J} \mathcal{S}^{\mathrm{b}}_{\mathfrak{g}_I} \cap \gamma \quad \text{for all} \quad \gamma \in H^{2n-2}(B)\,, \\ E_{i_I} \cdot C_{j_J} &= -\delta_{IJ} \left(\boldsymbol{\alpha}^\vee_{i_I}, \boldsymbol{\alpha}_{j_J}\right)_{\mathfrak{g}_J}\,, \\ S_0 \cdot C_{i_I} &= 0\,. \end{aligned} \tag{3.21}$$

Here we introduced the symbol $\cap$ to refer to intersections in B, while we reserve the symbol $\cdot$ for intersections in M. Moreover, $(\cdot\,,\cdot)_{\mathfrak{g}_I}$ denotes the Killing form associated to the Lie algebra $\mathfrak{g}_I$. Equivalently, $\mathrm{C}_{i_I j_I} \equiv (\boldsymbol{\alpha}^\vee_i, \boldsymbol{\alpha}_j)_{\mathfrak{g}_I}$ is the Cartan matrix of $\mathfrak{g}_I$, whereas $\mathfrak{C}_{i_I j_I} \equiv (\boldsymbol{\alpha}^\vee_i, \boldsymbol{\alpha}^\vee_j)_{\mathfrak{g}_I}$ defines the metric for the coroot lattice $L^\vee(\mathfrak{g}_I)$. We include Appendix A for more details about Lie algebras and representation theory.

Let us discuss now the origin of gauge fields in F-theory. For this, we consider M-theory compactified on a smooth elliptic fibration $\widehat{M}$ that results by resolving Kodaira-Nerón singularities. Upon this reduction, Abelian vector fields A_i emerge from the M-theory harmonic 3-form C_3 via the expansion

$$C_3 = \sum_{i=1}^{h^{1,1}(\widehat{M})} A_i \wedge D_i\,, \tag{3.22}$$

where we introduced a basis of divisors $\{D_i\}$ in $H^{1,1}(\widehat{M})$. In particular, a vector field that couples to a fibral divisor E_{i_I} gives rise to a gauge field A_{i_I} associated with the Cartan subalgebra $\mathfrak{h}_I \subset \mathfrak{g}_I$. Recall that an M2-brane wrapping a holomorphic curve C results into a BPS particle of mass $|m(C)| = \mathrm{Vol}_\omega(C)$, where we introduced the notation $\mathrm{Vol}_\omega(C) \equiv \int_C \omega$ with ω the complexified Kähler form of M. Then, a curve C_{i_I} associated with a simple root $-\boldsymbol{\alpha}_{i_I}$ gives rise to a particle with charge $\mathsf{q}_{-\boldsymbol{\alpha}_{i_I}}$ and mass z_{i_I}, as

$$\int_{C_{i_I}} C_3 = \mathsf{q}_{-\boldsymbol{\alpha}_{i_I}} A_{j_J} \delta_{IJ}\,, \quad \text{where} \quad \mathsf{q}_{-\boldsymbol{\alpha}_{i_I}} \equiv \int_{C_{i_I}} E_{j_I} = -\mathrm{C}_{i_I j_I}\,, \quad z_{i_I} \equiv \mathrm{Vol}_\omega(C_{i_I})\,. \tag{3.23}$$

Similarly, anti-M2-branes wrapping the same curve yields the particle associated with the simple root $\boldsymbol{\alpha}_{i_I}$ that has charge $\mathsf{q}_{\boldsymbol{\alpha}_{i_I}} = -\mathsf{q}_{-\boldsymbol{\alpha}_{i_I}}$. Thus, by taking appropriate linear combinations of curves C^I_i wrapped by M2- and anti-M2-branes, we generate massive particles in the adjoint representation $\mathbf{adj}(\mathfrak{g}_I)$. Since these particles enter as massive fields in the M-theory effective action, they break the gauge symmetry $\mathfrak{g}_I$ to its Cartan subgroup $\mathfrak{h}_I$. But taking the masless limit for all $z_{i_I} \to 0$ corresponds to gauge enhancement of $\mathfrak{h}_I \to \mathfrak{g}_I$. This way, the resolution map (3.18) realizes full gauge enhancement $\oplus_{I=1}^N \mathfrak{h}_I \to \mathfrak{g}$ in the effective theory compactified on M. Note that, when we taking the F-theory limit $\mathrm{Vol}(\mathcal{E}_\tau) \to 0$, we obtain full non-Abelian gauge group $\mathfrak{g}$ in the effective Type IIB compactified on B, which contains 7-branes charged under $\mathfrak{g}_I$ at each $\mathcal{S}^{\mathrm{b}}_{\mathfrak{g}_I} \subset B$. Typically we will work with smooth elliptically fibered Calabi-Yau compactifications, which should be interpreted as the resulting manifolds after resolution—unless differntly stated.

Codimension-2 singularities: At a given a codimension-2 loci $\mathcal{S}^{\mathrm{b}}_{IJ} \equiv \mathcal{S}^{\mathrm{b}}_{\mathfrak{g}_I} \cap \mathcal{S}^{\mathrm{b}}_{\mathfrak{g}_J} \subset B$, the polynomials (f, g, Δ) of the Weierstrass model increase their vanishing order. This represents an intersection of 7-branes with stretched (p,q)-strings that results into localized massless matter. To see this, let us separate the codimension-2 divisor $\mathcal{S}^{\mathrm{b}}_{IJ}$ into a set of irreducible components $\mathcal{S}^{\mathrm{b}}_{IJ} = \cup_\ell C^\ell_{IJ}$, which are points when $\dim(B) = 2$, irreducible curves when $\dim(B) = 3$, etc. By resolving the codimension-2 singularity at a component C^ℓ_{IJ}, some of the fiber curves C_{i_I} associated to the divisor $\mathcal{S}^{\mathrm{b}}_{\mathfrak{g}_I}$ split into several rational curves C^ℓ_{sp}. The relative Mori cone $\mathrm{NE}(C^\ell_{IJ})$ contains such splitting curves $\{C^\ell_{\mathrm{sp}}\}$, as well as the curves $\{C_{i_I}, C_{j_J}\}$ that arise via codimension-1 enhancements at the loci $\mathcal{S}^{\mathrm{b}}_{\mathfrak{g}_I}$ and $\mathcal{S}^{\mathrm{b}}_{\mathfrak{g}_J}$; here $\mathrm{NE}(C^\ell_{IJ})$ is the set of numerically effective curve classes in the fiber at C^ℓ_{IJ}, upon resolution, that do not intersect the zero section S_0. The crucial point is that a splitting curve $C_{\boldsymbol{\lambda}} \in \mathrm{NE}(S^\ell_{IJ})$ intersects with a fibral divisor E^I_i as

$$\lambda_{i_I} = C_{\boldsymbol{\lambda}} \cdot E_{i_I} = \left(\boldsymbol{\lambda}, \boldsymbol{\alpha}^\vee_{i_I}\right)_{\mathfrak{g}_I} \in \mathbb{Z}\,. \tag{3.24}$$

This is precisely the Dynkin label of a weight $\boldsymbol{\lambda} \in L_{\mathrm{w}}(\mathfrak{g}_I \oplus \mathfrak{g}_J)$ that defines a weight space $V_{\boldsymbol{\lambda}}$ in a highest weight module $V_{\mathbf{R}}$, which has an associated irreducible representation $\mathbf{R}$ of $\mathfrak{g}_I \oplus \mathfrak{g}_J$. By taking suitable linear combinations, we can generate all weights associated to $\mathbf{R}$ and establish the geometric-representation correspondence:

$$C_{\boldsymbol{\omega}} = C_{\boldsymbol{\lambda}} + \sum_{i_I=1}^{\mathrm{rk}\mathfrak{g}_I} n_{i_I} C_{i_I} + \sum_{j_J=1}^{\mathrm{rk}\mathfrak{g}_J} \tilde{n}_{j_J} C_{j_J} \quad \longleftrightarrow \quad \text{weights } \boldsymbol{\omega} \text{ in } V_{\mathbf{R}} = \bigoplus_{\boldsymbol{\omega}} V_{\boldsymbol{\omega}}\,. \tag{3.25}$$

Here the entries n_{i_I} and $\tilde{n}_{j_J}$ are appropriate integers numbers, such that it realizes the Dynkin labels associated with $\boldsymbol{\omega} \in L_{\mathrm{w}}(\mathfrak{g}_I \oplus \mathfrak{g}_J)$. This way an M2-brane that wraps a curve $C_{\boldsymbol{\omega}}$ realizes a BPS state with weight $\boldsymbol{\omega}$ in the representation $\mathbf{R}$, i.e. in the weight space $V_{\boldsymbol{\omega}} \subset V_{\mathbf{R}}$. Again, an anti-M2-brane wrapping the same curve yields a BPS state with negative weight $-\boldsymbol{\omega}$ and hence in the conjugate representation $\bar{\mathbf{R}}$. Moreover, the volume of the curve $C_{\boldsymbol{\omega}}$ parametrizes the mass of such BPS states, which implies that in the F-theory limit we obtain massless mater states localized at $C^\ell_{IJ} \subset \mathcal{S}^{\mathrm{b}}_{\mathfrak{g}_I} \cap \mathcal{S}^{\mathrm{b}}_{\mathfrak{g}_J} \subset B$.

Although it is possible to consider higher codimension singularities for a given Weierstrass model, these will not be relevant to our exposition. On the other hand, so far, we only discussed the manifestation of non-Abelian gauge groups in F-theory. In the next section we will consider non-Cartan abelian gauge symmetries of the type $U(1)^r$ with $r \in \mathbb{N}$, as well as the geometrical object that incarnates them.

3.1.4 The Mordell Weil group of rational sections

In the previous section, we stated that an elliptic fibration $\pi : M \to B$ always possesses a zero section s_0. More generally, an elliptically fibered Calabi-Yau M can contain additional rational sections s_Q, which are rational maps $s_Q : B \to M$ that mark one point in each elliptic fiber, i.e., $\pi \circ s_Q = \mathrm{id}_B$. By rational map, we mean a meromorphic function f in the function field of the base $K(B)$, where $f = \frac{p_1}{p_2}$ is a quotient of polynomials p_1, p_2 that depend on local coordinates of B. Thus, a section is a rational solution $x_Q, y_Q, z_Q \in K(B)$ that defines a point $Q = [x_Q : y_Q : z_Q]$ such that $P_W(x_Q, y_Q, z_Q) = 0$.

Points in an elliptic curve $\mathcal{E}$ over a field K form an abelian group $\mathcal{E}(K)$ with composition law $\boxplus : \mathcal{E}\times\mathcal{E} \to \mathcal{E}$, where $(P,Q) \mapsto P\boxplus Q \in \mathcal{E}(K)$ [79]. The identity is defined by the marked point $O = [1:1:0]$, where we use the convention in (3.13) for the Weierstrass form in a weighted projective space $\mathbb{P}^{(2,3,1)}$. In the context of elliptic fibrations $\pi : M \to B$, the case of interest is the function field $K(B)$. There, the rational sections follow the group composition law $s_P(p) \boxplus s_Q(p) = (s_P \boxplus s_Q)(p)$ that defines the Mordell-Weil group $\mathrm{MW}(M) = \mathcal{E}(K(B))$. The Mordell-Weil theorem for function fields states that $\mathrm{MW}(M)$ is a finitely generated abelian group of the form

$$\mathrm{MW}(M) \simeq \mathbb{Z}^{\oplus r} \oplus \mathrm{MW}(M)_{\mathrm{tor}}. \tag{3.26}$$

Here the factor $\mathbb{Z}^{\oplus r}$ gives the number of independent rational sections, besides the zero section s_0, and $r = \mathrm{rk}(\mathrm{MW})$ is the rank of the Mordell-Weil group. Moreover, $\mathrm{MW}(M)_{\mathrm{tor}}$ denotes the torsional part of the Mordell-Weil group.

Analogous to the zero section divisor S_0 in (3.15), there is a divisor $S_Q \equiv \mathrm{div}(s_Q)$ associated to each rational section in $\mathrm{MW}(M)$. A property of section divisors is that they intersect the elliptic fiber $\mathcal{E}$ once, i.e., $S_0 \cdot \mathcal{E} = S_Q \cdot \mathcal{E} = 1$. In addition to this, the section divisors fulfil the following intersection relations: [72]

$$\begin{aligned} S \cdot \pi^* D_1^{\mathrm{b}} \cdots \pi^* D_{n-1}^{\mathrm{b}} &= D_1^{\mathrm{b}} \cap \cdots \cap D_{n-1}^{\mathrm{b}}\,, \\ S \cdot S' \cdot \pi^* D_1^{\mathrm{b}} \cdots \pi^* D_{n-2}^{\mathrm{b}} &= \pi_* \left(S \cdot S'\right) \cap D_1^{\mathrm{b}} \cap \cdots \cap D_{n-2}^{\mathrm{b}}\,, \\ S \cdot S \cdot \pi^* D_1^{\mathrm{b}} \cdots \pi^* D_{n-2}^{\mathrm{b}} &= -c_1(B) \cap D_1^{\mathrm{b}} \cap \cdots \cap D_{n-2}^{\mathrm{b}}\,, \\ S \cdot E_i^I \cdot \pi^* D_1^{\mathrm{b}} \cdots \pi^* D_{n-2}^{\mathrm{b}} &= \left(S \cdot C_i^I\right) \mathcal{S}^{\mathrm{b}}_{\mathfrak{g}_I} \cap D_1^{\mathrm{b}} \cap \cdots \cap D_{n-2}^{\mathrm{b}}\,. \end{aligned} \tag{3.27}$$

Here $S, S' \in \{S_0, S_{Q_1}, \dots, S_{Q_m}\}$, where S_{Q_i} is the divisor of a rational section in $\mathrm{MW}(M)$. However, the rational section divisors $\{S_{Q_i}\}$ do not follow the group composition law $\boxplus$ of $\mathrm{MW}(M)$. Instead, an adequate set of divisors follows from the Shioda map[2]

$$\sigma : \mathrm{MW}(M) \to \mathrm{NS}(M) \otimes \mathbb{Q}\,, \tag{3.28}$$

which is a group homomorphism that fulfills the property $\sigma(s_P \boxplus s_Q) = \sigma(s_P) + \sigma(s_Q)$. Concretely, this map has an explicit expression in terms of intersection data, which reads

$$\sigma(s_Q) = S_Q - S_0 - \pi^{-1}\left(\pi_*\left((S_Q - S_0)\cdot S_0\right)\right) + \sum_{I=1}^{N} \left(S_Q \cdot C_{i_I}\right) \mathrm{C}^{i_I j_I} E_{j_I}\,, \tag{3.29}$$

where $\mathrm{C}^{i_I j_I}$ is the inverse Cartan matrix of the Lie algebra $\mathfrak{g}_I$.

The physical relevance of the Shioda map is that it furnishes $U(1)$ abelian gauge fields. To appreciate this, we recall that the reduction of the M-theory 3-form $C_3 = \sum_i A_i \wedge D_i$ with $D_i \in H^{1,1}(M)$ leads to massless vector fields in the F-theory limit. Thus, an expansion of the form $C_3 = A_Q \wedge \sigma(s_Q) + \cdots$ results into consistent $U(1)$ gauge fields A_Q, as each divisor $\sigma(s_Q)$ satisfies the F-theory gauge constraints[3]

$$\begin{cases} (\mathrm{a}) : \sigma(s_Q)\cdot\mathcal{E} & = 0\,, \\ (\mathrm{b}) : \sigma(s_Q)\cdot\pi^* C_{\mathrm{b}} & = 0\,, \\ (\mathrm{c}) : \sigma(s_Q)\cdot C_i^I & = 0\,, \end{cases} \tag{3.30}$$

[2] Here $\mathrm{NS}(X)$ is the Nerón-Severi group, which is the set of Weil divisors modulo algebraic equivalence. For our case of interest, a compact Kähler manifold X, we have that $\mathrm{NS}(X) = H^{1,1}(X,\mathbb{Z})$ [80].

[3] By definition, the divisor $\sigma(s_Q)$ fulfils the transversality conditions (3.30)

where $C_{\rm b} \in H_2(B)$. We already mentioned the conditions (a) and (b) in (3.17). The new ingredient here is (c), which means that non-Abelian gauge bosons have no $U(1)$-charges associated to $\sigma(s_Q)$.

Similarly as in the non-Abelian gauge case, $U(1)$-charged matter arise from M2-branes wrapping fibral curves $C_{\pm}$ with non-vanishing intersection. Such fibral curves $C_{\pm}$ appear, upon resolution, at codimension-2 loci over B. Thus, the charges for their associated particles read

$$\mathsf{q}_{\pm} = \sigma(s_Q) \cdot C_{\pm}\,, \tag{3.31}$$

giving rise to massless matter, in the F-theory limit, or through the image of the resolution morphism.

Lastly, let us consider the case when Mordell-Weil group $\mathrm{MW}(M)$ has torsion. Say there is one element $s_T \in \mathrm{MW}(M)$ such that $s_T \boxplus \cdots \boxplus s_T = s_0$, where we $\boxplus$-summed over s_T k-times. Then, the overall effect of such k-torsional section translates into a gauge group of the form $G' = G/\mathbb{Z}_k$, i.e., we obtain a non-simply connected gauge group with $\pi_1(G') = \mathbb{Z}_k$. Since we do not cover these cases in this work, we will skip the details for this type of geometries.

3.1.5 Genus one fibrations

Up to now, we discussed how elliptic fibrations realize a gauge group G corresponding to any semisimple Lie algebra $\mathfrak{g}$. However, it is desirable to include discrete gauge symmetries in our geometric program, as some physics models require them, e.g. to prevent proton decays. Of course, one way to achieve these symmetries is via a Higgsing $U(1) \to \mathbb{Z}_N$ [81]. Remarkably, F-theory enables us to geometrize this process through a more general type of torus fibrations, *genus one fibrations* [23, 82].

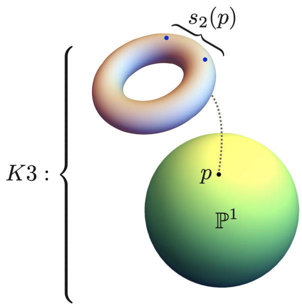


Figure 3.3: Schematic representation of a genus one fibered $K3$ surface with 2-section.

Let us consider a fibration $\pi_N : M \to B$ with N-section, such that the fiber $\pi^{-1}(p) \simeq \mathcal{C}$ at each $p \in B$ is a genus one curve, i.e. an elliptic curve or torus, and M is Calabi-Yau. By N-section, we mean a multivalued function $s_N : B \to M$, such that it marks N points in the fiber that are mapped into each other by monodromies on the base B. If the fibration $\pi_N : M \to B$ has at least a 1-section, i.e. a section, we say that M is elliptically fibered.[4] On the other hand, we reserve the term genus one fibration for those torus fibrations that have no section but only N-sections with $N > 1$.

[4] This is the case we discussed in previous sections. Namely, the zero section and rational sections in a Weierstrass model.

With this nomenclature in mind, each torus fiber $\mathcal{C}$ in a genus one fibration defines a τ profile that is identical to that of an actual elliptic fibration, which is the *Jacobian fibration* $J(M)$. To construct the elliptic fibration $\pi : J(M) \to B$ associated to a genus one fibration $\pi_N : M \to B$, we take as fibers the Jacobian $\pi^{-1}(p) = J(\mathcal{C})$, where $J(\mathcal{C})$ is the group of zero degree line bundles on $\pi_N^{-1}(p) = \mathcal{C}$ with $p \in B$. Then each fiber $J(\mathcal{C})$ has a distinguished point defined by the trivial line bundle, which defines a section for $J(M)$. The resulting fibration $J(M)$ not only has a Weierstrass model but also shares the same discriminant locus $\Delta \subset B$ associated with M [23]. This fact implies that both fibrations M and $J(M)$ encode the same 7-branes configuration in F-theory and consequently the same physics.

There is a drawback to work with Jacobian fibrations. They have codimension-2 singularities that do not admit resolution, together with torsional cohomology [23]. The latter type of group signals the presence of a discrete symmetry that relates with the *Tate-Shafarevich group* $Ш(J(M))$, which is essentially the set of torus fibered Calabi-Yau manifolds with same Jacobian fibration $J(M)$—endowed with an Abelian group action. We will omit here an explicit construction of $Ш(J(M))$, but instead refer to the Appendix A of the work [83]. Moreover, for the case of Calabi-Yau 3-folds, there is mathematical proof for [84]

$$\mathrm{Tor}\left(H^3(J(M), \mathbb{Z})\right) \simeq Ш(J(M)) \simeq \mathbb{Z}_N \,. \tag{3.32}$$

However, the same is expected to hold for Calabi-Yau n-folds with $n \geq 4$. In the F-theory limit, the geometries in $Ш(J(M))$ yield the same physics interpretation, but their M-theory duals lead to nonequivalent vacua. In the upcoming we discuss the physics of those distinct fibrations in $Ш(J(M))$ at the level of M-theory compactifications.

First, we consider the M-theory compactification on a Jacobian fibration $J(M)$. Having torsional cohomology (3.32) implies that there is a pair of non-harmonic 2- and 3-forms $(\omega_{\mathrm{tor}}, \alpha_{\mathrm{tor}})$, such that $\mathrm{d}\omega_{\mathrm{tor}} = N\alpha_{\mathrm{tor}}$ [85], where $[\alpha_{\mathrm{tor}}] \in \mathrm{Tor}H^3(J(M), \mathbb{Z})$. As usual, we regard an ansatz in which the M-theory 3-form expands as $C_3 = A \wedge \omega_{\mathrm{tor}} + c\alpha_{\mathrm{tor}} + \cdots$, where A is a 1-form, c is a scalar field, and the rest of terms $\cdots$ are irrelevant for the discussion next. Then the M-theory effective action yields a term

$$S_{\text{Stück}} = \int_{\mathbb{R}^{1,10-2n}} (\mathrm{d}c + NA) \wedge *\, (\mathrm{d}c + NA) \;, \tag{3.33}$$

which descends from the 11d kinetic term $|\mathrm{d}C_3|^2$ and is invariant under simultaneous gauge transformations

$$A \to A + \mathrm{d}\chi \,, \quad c \to c - N\chi \,. \tag{3.34}$$

Here we identify the action term $S_{\text{Stück}}$ with that of an axion c of $U(1)$ charge N which fulfils an additional shift symmetry $c \sim c + 2\pi$. Thus, by fixing the choice $\chi = (c + 2\pi k)/N$ with $k \in \mathbb{Z}_N$ the axion is gauged away, giving rise to a massive gauge potential that breaks the $U(1)$ symmetry. Nonetheless, a remnant $\mathbb{Z}_N \subset U(1)$ symmetry survives via the choices $k \in \mathbb{Z}_N$. This process is known as the *Stückelberg mechanism*. At the end of the day, by F/M-theory duality, this discrete symmetry $\mathbb{Z}_N$ uplifts to F-theory compactified on B.

Let us consider the F-theory compactification leading to a theory with a $U(1)$ gauge group. Then, we can turn on a flux along S^1_F for a $U(1)$ vector field component A_F as

$$\xi = \oint_{S^1_F} A_F \,. \tag{3.35}$$

By Higgsing $U(1) \to \mathbb{Z}_N$, the Wilson line parameter ξ takes only discrete values in $\mathbb{Z}_N$, up to normalization factors. When $\xi = 0$ the F-theory Higgsing $U(1) \to \mathbb{Z}_N$ descends simply to $\mathbb{Z}_N \times U(1)_{\mathrm{KK}}$ in the M-theory side, where $U(1)_{\mathrm{KK}}$ is the Kaluza-Klein $U(1)$ symmetry group. This transition is equivalent to M-theory on a Jacobian fibration, whose non-trivial torsional cohomology element and zero section divisor correspond to $\mathbb{Z}_N$ and $U(1)_{\mathrm{KK}}$ respectively. In contrast to the fluxless case, the discrete choices $\xi \neq 0$ induce in the M-theory effective action to mix kinetic terms associated with a massive $U(1)$ vector field and the Kaluza-Klein vector field $\tilde{A}_0$ corresponding to $U(1)_{\mathrm{KK}}$ [83, 86]. This results into a single effective $U(1)_E$ symmetry group generated by $U(1)_E \equiv kU(1) - NU(1)_{\mathrm{KK}}$ with $k \in \mathbb{Z}_N$. The associated $U(1)_E$ vector field is identified with an N-section divisor $S_N \equiv \mathrm{div}(s_N)$ of a genus one fibration [83, 86]. Note that a genus one fibration does not possess torsional cohomology. Hence its M-theory compactification lacks a discrete symmetry.

To summarize, we can realize geometrically F-theory/M-theory vacua resulting from a Higgsing $U(1) \to \mathbb{Z}_N$ as follows. For a given pair of fibrations M and $J(M)$ in $\text{Ш}(M)$, there is a third elliptic fibration $M_{U(1)}$, such that it has a Mordell-Weil group of rank one, and it relates to the former fibrations via distinct conifold transitions: [82]

$$\begin{tikzcd} M_{U(1)} \arrow[r, "\xi\neq 0"] \arrow[dr, "\xi=0"'] & M \\ & J(M) \end{tikzcd} \quad . \tag{3.36}$$

Here parameter ξ are discrete choices of fluxes, which give rise to nonequivalent M-theory vacua determined by elements of $\text{Ш}(J(M))$. However, all geometries in $\text{Ш}(J(M))$ yield the same F-theory description. Explicit constructions for these assertions have been realized in the works [46, 71, 83, 87, 88], which investigated N-sections up to $N = 5$. One topic of concern for the works [2, 46] was to investigate the topological string partition function on genus on fibrations backgrounds, which we include in Appendix B.5.

3.2 6d $N = (1, 0)$ theories

A remarkable feature of F-theory is its versatility to geometrically engineer consistent quantum gravity theories and quantum field theories in a non-perturbative regime. As already explained, in practice, we think of a dictionary that translates essential information of our desired physics theory into a torus fibration. Keep in mind that, a priori, what we do is a top-down approach to derive effective field theories. For this reason, we aim now to look into more detail the resulting physical degrees of freedom, but from a field theoretical point of view. In this section we consider an F-theory compactification on an elliptic fibration $\pi : M \to B$ with $\dim(B) = 2$, so that we obtain 6d $N = (1, 0)$ theories.

In particular, this geometrization enables us to study strongly coupled phenomena and theories with no known Lagrangian description. For concreteness, let us mention the case of 6d $N = (1, 0)$ superconformal field theories (SCFTs). We emphasize that $D = 6$ is the maximal spacetime dimension for a superconformal algebra to exist [49], and thereby for a SCFT. Besides that, a conjecture states that all possible 6d SCFTs are generated by gluing minimal building blocks theories via F-theory compactifications. This program is known as the atomic classification of 6d SCFTs [89, 90].

We will base our exposition on the sources [91, 92].

3.2.1 Effective strings in 6d

As a starting point, we summarize the massless spectrum content in a 6d supergravity theory. This is given by the following $N = (1, 0)$ supermultiplets: [91]

- **Gravity multiplet** $\left(g_{\mu\nu}\,, \psi^{+}_{\mu}\,, B^{+}_{\mu\nu}\right)$**:** Here $g_{\mu\nu}$ is the metric tensor, ψ^{+}_{μ} is the gravitino, and $B^{+}_{\mu\nu}$ is a self-dual 2-form gauge field.

- **Tensor multiplets** $\left(t\,, B^{-}_{\mu\nu}\,, \chi^{-}\right)$ **:** Here t is a scalar field, $B^{-}_{\mu\nu}$ is an anti self-dual 2-form gauge field, and χ^{-} is a spin $\frac{1}{2}$ Majorana-Weyl fermion of negative chirality.

- **Vector multiplets** $(A_{\mu}\,, \lambda^{+})$ **:** Here A_{μ} is a gauge field corresponding to the gauge algebra $\mathfrak{g}$, and λ^{+} is a spin $\frac{1}{2}$ adjoint-valued symplectic Majorana-Weyl fermion of positive chirality.

- **Hyper multiplets** $(4\varphi\,, \psi^{-})$**:** Here 4φ denotes a pair of complex bosons or four real bosons, and ψ^{-} is a spin $\frac{1}{2}$ Weyl fermion of negative chirality.

We consider 6d $N = (1, 0)$ supergravity theories that contain one gravity multiplet, n_T tensor multiplets, n_V vector multiplets, and n_H hypermultiplets; where $n_T, n_V, n_H \in \mathbb{N}$. These type of theories have a gauge group of the form $G = \prod_{a=1}^{N} G_a \times U(1)^r/\Gamma$, where G_a are simple non-Abelian gauge group factors and Γ is a discrete group. Accordingly, the matter content, i.e. hyper- and vector multiplets, transforms under representations of G. Let us remark that, whenever $n_T \geq 2$, such a theory has no Lagrangian description [91].

In section 3.1.3, we explained the F-theoretical origin of hyper- and vector multiplets by invoking F-/M-theory duality. Now, we regard the tensor multiplets. In an elliptic or genus one fibration, the base B has a Kähler form $J = t^{\alpha}\omega_{\alpha}$ that we expand in terms of a basis $\{\omega_{\alpha}\}_{\alpha=1,\ldots,h^{1,1}(B)}$ for $H^{1,1}(B)$. In this basis, the Type IIB Ramond-Ramond form reduces as $C_4 = B^{\alpha} \wedge \omega_{\alpha}$, which gives rise to a set of 2-form fields B^{α}, i.e. the $h^{1,1}(B) = n_T + 1$ tensor fields of the massless spectrum. The n_T scalar fields sitting in the tensor multiplets form an $SO(1, n_T)$ unit-norm vector that we associate with the Kähler moduli t^{α}. Thus, they parametrize the tensor moduli space with $SO(1, n_T)$ symmetry. We discuss now the role of these moduli in their assigned theory.

The tensor branch of the moduli space: In a 6d $N = (1, 0)$ gauge supersymmetric theory, scalar fields have two possible origins. Either they arise from the hypermultiplets, or they do it from the tensor multiplets. Thus, the moduli space splits into two branches: The *Higgs branch*, and the *tensor branch*. The Higgs branch case corresponds to the hypermultiplets, whereas the tensor branch to the tensor multiplets. In any case, each of them parametrizes the VEVs acquired by scalars. Non-trivial vacua are those given by *non-critial strings theories*, a class of quantum field theories located at singular points of the tensor branch. As we will see, these theories are associated with tensionless strings decoupled from gravity [93].

Decompactification of gravity: In a 6d F-theory compactification on an elliptic fibration $\pi : M \to B$, a simple calculation—by reducing (3.2) on B—shows that the 6d Newton's constant G_{6d} is set by the volume of the base B as follows

$$\frac{4\pi \mathrm{Vol}_J\,(B)}{\ell_s^8} = \frac{1}{G_{6d}}\,. \tag{3.37}$$

Here $\mathrm{Vol}_J(B)$ is the volume form of B with respect to J and ℓ_s denotes the string length scale. To decouple gravity means to take the limit when the size of B is much larger than the scale ℓ_s. To achieve this, we regard the limit when the Kähler manifold B becomes *non-compact* while keeping degrees of freedom localized on compact subspaces of B.

	$\Sigma \subset \mathbb{R}^{1,5}$		$\mathbb{R}^{1,5}$				$C \subset B$		B	
$\mathbb{R}^{5,1} \times B$	X^0	X^1	X^2	X^3	X^4	X^5	X^6	X^7	X^8	X^9
D3	$\times$	$\times$	-	-	-	-	$\times$	$\times$	-	-

Table 3.1: Here X^μ are some local coordinates in $\mathbb{R}^{1,5} \times B$. In these coordinates, $\times$ indicates the spacetime directions filled by the D3 brane leading to an effective string in 6D. Similarly, - indicates the transverse directions to the D3 brane. We denote the worldvolume of the effective string by Σ.

Effective strings: Effective strings in 6d are realized by reducing D3-branes wrapped on curves $C \subset B$. We illustrate this construction in Table 3.1. Note that these strings enjoy a 2d $\mathcal{N} = (0,4)$ worldsheet theory [94, 95]. Let us now emphasize that the Kähler moduli t^α control physical quantities, which characterize the effective strings. On the one hand, the tension T of an effective string due to a D3-brane wrapping $C \subset B$—measured in the Type IIB Einstein-frame—is set as follows

$$T = \frac{2\pi}{\ell_s^4}\mathrm{Vol}_J(C)\,, \tag{3.38}$$

where $\mathrm{Vol}_J(C) \equiv \int_C J$. On the other hand, the gauge coupling of a 7-brane, located at $\{\Delta_K = 0\} \subset B$, that wraps $\mathcal{S}^{\mathrm{b}}_{\mathfrak{g}_K} \subset B$ is given by

$$\frac{1}{g^2_{\mathrm{YM}}} = \frac{1}{2\pi\ell_s^8}\mathrm{Vol}_J(\mathcal{S}^{\mathrm{b}}_{\mathfrak{g}_K})\,. \tag{3.39}$$

The gravity decompactification limit $M^2_{\mathrm{Pl};6d} \to \infty$ results into a non-compact base $\hat{B}$, where there remain a subset of compact curves $\{C_{\hat{\alpha}}\}_{\hat{\alpha}=1,\ldots\hat{n}_T}$. Consequently, there is a reduced set of Kähler moduli $\{t^{\hat{\alpha}}\}_{\hat{\alpha}=1,\ldots\hat{n}_T}$ that parametrize $\hat{n}_T$ tensor multiplets. It is precisely the origin of the tensor branch, i.e. $t^{\hat{\alpha}} = 0 \;\forall\; \hat{\alpha} = 1\,,\ldots\,,\hat{n}_T$, where the effective 6d theory becomes strongly coupled with infinitely many tensionless strings as degrees of freedom [93]. This is referred to the *conformal fixed point*[5], which is evident from equations (3.38) and (3.39).

3.2.2 Anomalies in 6d supergravity

In a six-dimensional F-theory vacuum, the gauge symmetries themselves are anomalous, and the anomalies have to be canceled by a generalized Green-Schwarz mechanism [96, 97]. Let us introduce the intersection matrix $\Omega_{\alpha\beta}$ given by the topological intersection pairing

$$\Omega_{\alpha\beta} = \int_B \omega_\alpha \wedge \omega_\beta\,. \tag{3.40}$$

[5] It is believed that due to the absence of a mass scale, we have reached a non-trivial 6d SCFT.

Then, the Green-Schwarz counterterm takes the form

$$S_{\rm GS} = -\frac{1}{2}\int_{M_6} \Omega_{\alpha\beta} B^\alpha \wedge X_4^\beta\,, \tag{3.41}$$

where

$$X_4^\alpha = \frac{1}{2} a^\alpha \mathrm{tr} R\wedge R + 2\sum_I \frac{b_I^\alpha}{\lambda_I}\mathrm{tr} F_I \wedge F_I + 2\sum_{a,b} b_{ab}^\alpha F^a\wedge F^b\,. \tag{3.42}$$

Here R is the gravitational field strength, F_I is the field strength associated to a non-Abelian gauge algebra $\mathfrak{g}_I$, and F^a is the Abelian field strength associated to the section s_{Q_a}. The anomaly coefficients a^α, b_I^α and b_{ab}^α are given by

$$a^\alpha = c_1(B)\cdot\omega^\alpha\,,\quad b_I^\alpha = \mathcal{S}^{\rm b}_{\mathfrak{g}_I}\cdot\omega^\alpha\,,\quad b_{ab}^\alpha = -\pi_*(\sigma(s_{Q_a})\cdot\sigma(s_{Q_b}))\cdot\omega^\alpha\,, \tag{3.43}$$

where $\omega^\alpha = (\Omega^{-1})^{\alpha\beta}\omega_\beta$ and $b_{ab} = -\pi_*(\sigma(s_{Q_a})\cdot\sigma(s_{Q_b}))$ is also called the height pairing. Here λ_I is a group theoretical normalization constant determined by Cartan generators $\{T_i\}_{i=1,\ldots,\mathrm{rk}\mathfrak{g}_I}$—that span the coroot lattice—such that $\mathrm{tr}T_iT_j = \lambda_I\mathfrak{C}^I_{ij}$, where $\mathfrak{C}^I_{ij}$ is the coroot matrix of $\mathfrak{g}_I$.

The anomalies are canceled via the counterterm (3.41) if the 1-loop anomaly polynomial I_8 factorizes as

$$I_8 = -\frac{1}{2}\Omega_{\alpha\beta}X_4^\alpha\wedge X_4^\beta\,. \tag{3.44}$$

This cancellation is equivalent to the so-called *anomaly equations*, which impose non-trivial relations among the multiplicities of representations and massless spectrum information. Geometrically, we interpret the anomaly equations as intersection data relations of elliptically fibered Calabi-Yau 3-folds [77].

3.2.3 The anomaly inflow polynomial and index of elliptic genera

As pointed out in previous setion, factorization of the 1-loop anomaly polynomial I_8 entails cancellation of anomalies in 6d F-theory vacua. However, in the presence of a solitonic string source, further contributions need to be cancelled via the so-called *anomaly inflow mechanism* [98, 99]. For an effective string realized by a curve C_β with class $\beta\in H_2(B,\mathbb{Z})$, its associated 2d $\mathcal{N}=(0,4)$ worldsheet theory yields an anomaly that reads [64, 95, 99, 100]

$$\begin{aligned}\mathcal{A}_4 = &-\frac{1}{4}(c_1(B)\cdot\beta)p_1(T_\Sigma) - \frac{1}{2}\sum_I (b_I\cdot\beta)\frac{1}{\lambda_I}\mathrm{tr}F_I\wedge F_I - \frac{1}{2}(b_{ab}\cdot\beta)F^a\wedge F^b \\ &+\frac{1}{4}\beta\cdot(c_1(B)+\beta)\mathrm{tr}F_+\wedge F_+ + \frac{1}{4}\beta\cdot(c_1(B)-\beta)\mathrm{tr}F_-\wedge F_- + \frac{1}{2}\mathrm{tr}F_{\mathcal{R}}\wedge F_{\mathcal{R}}\,.\end{aligned} \tag{3.45}$$

Here T_Σ is the tangent bundle of the worldsheet Σ associated to the string source and $p_1(T_\Sigma)$ is its first Pontrjagin class. Moreover, $F_\pm$ is the field strength associated with the factor $SU(2)_\pm$ of the transverse rotation group $SO(4)_{\Sigma_\perp}\simeq SU(2)_+\times SU(2)_-$. Furthermore, $F_{\mathcal{R}}$ is a field strength due to an $SU(2)$ R-symmetry inherited from the 6d $N=(1,0)$ theory.

For the moment, let us assume effective strings on the Ω-background $\mathbb{R}^4_{\epsilon_1,\epsilon_2}\times T^2$, so that the elliptic genus $\mathcal{Z}_\beta$—of a string descending from a D3-brane wrapping a curve C_β—becomes equivalent to a base coefficient for the refined topological string partition function. It turns out that the anomaly inflow polynomial (3.45) encodes the overall index bilinear form

M_β of a meromorphic Weyl invariant lattice Jacobi form, which determines $\mathcal{Z}_\beta$ via the modular ansatz. The index M_β appears in the modular anomaly equation, given by a second Eisenstein series E_2 derivative on $\mathcal{Z}_\beta$:

$$\left(\partial_{E_2} + \frac{1}{12} M_\beta(\boldsymbol{\mu})\right) \mathcal{Z}_\beta(\tau, \boldsymbol{\mu}) = 0 \,, \tag{3.46}$$

where we denote by $\boldsymbol{\mu}$ all elliptic parameters for $\mathcal{Z}_\beta$. To understand the identification $\mathcal{A}_4 \leftrightarrow M_\beta$, let us consider the generalized supersymmetric Casimir energy proposed in [101].

In a generic SCFT in d-dimensions, the supersymmetric partition function on $S^1 \times M_{d-1}$ factorizes as follows $Z = \mathrm{e}^{-RE} I$, where I is a superconformal index, R is the radius of S^1, and E is the proposed supersymmetric Casimir energy. The idea is that the limit $E = -\lim_{R\to\infty} \partial_R \log Z$ provides an observable analogous to the Casimir Energy of a 2d CFT on a cylinder [8], which can be obtained from a 2d CFT partition function on $S^1 \times S^1$ in the infinite radius limit along one circle. The authors [101] conjectured that such an observable results from the equivariant integration in $\mathbb{R}^d$

$$E(\boldsymbol{\mu}) = \int_{\mathrm{eq}} \mathcal{A}_{d+2} \,, \tag{3.47}$$

where $\mathcal{A}_{d+2}$ is an anomaly polynomial in d-dimensions and here $\boldsymbol{\mu}$ are chemical potentials of the supersymmetric partition.

In our case of discussion, we consider an S-transformation $\mathrm{S} : \tau \mapsto -1/\tau$ of the elliptic genus. Let us assume the modular transformation properties for the refined topological string partition function. Then, the S-transformed elliptic genus

$$\mathrm{S} : \mathcal{Z}_\beta(\tau, \boldsymbol{\mu}) \mapsto \mathrm{e}^{-\frac{2\pi i}{\tau} M_\beta(\boldsymbol{\mu})} \mathcal{Z}_\beta(\tau, \boldsymbol{\mu}) \tag{3.48}$$

has the factorization indicated by [101]. Note that we identify the radius R parameter with $-2\pi i/\tau$. In other words, when we calculate that the supersymmetric Casimir energy for this case

$$E(\boldsymbol{\mu}) = -\frac{1}{2\pi i} \lim_{\tau\to 0} \tau^2 \frac{\partial}{\partial \tau} \log \mathcal{Z}_\beta \left(-\frac{1}{\tau}, \frac{\boldsymbol{\mu}}{\tau}\right) = M_\beta(\boldsymbol{\mu}) \,, \tag{3.49}$$

we find it is precisely the lattice index of Jacobi forms. With this observation in mind, we can obtain the index M_β explicitly by integrating the anomaly polynomial (3.45) as in (3.47). The result amounts to follow the replacement rules: [64, 100, 102–104]

$$M_\beta(\boldsymbol{\mu}) = \int_{\mathrm{eq}} \mathcal{A}_4 \Longrightarrow \begin{cases} p_1(T_\Sigma) & \mapsto 0 \\ \frac{1}{2}\frac{1}{\lambda_I} \mathrm{tr} F_I^2 & \mapsto -(\boldsymbol{z}_I, \boldsymbol{z}_I)_{\mathfrak{g}_I} \\ \frac{1}{2} F^a \wedge F^b & \mapsto -z^a z^b \\ \frac{1}{2} \mathrm{tr} F_-^2 & \mapsto -\epsilon_-^2 \\ \frac{1}{2} \mathrm{tr} F_+^2 & \mapsto -\epsilon_+^2 \\ \frac{1}{2} \mathrm{tr} F_{\mathcal{R}}^2 & \mapsto -\epsilon_+^2 \end{cases} . \tag{3.50}$$

Here we identify $\mu = (\epsilon_+, \epsilon_-; \boldsymbol{z})$, where $\epsilon_\pm$ denote the equivariant rotation paramaters of the Ω-bakground (2.120), $\boldsymbol{z}_I \in L^\vee(\mathfrak{g}_I) \otimes \mathbb{C}$ are elliptic parameters identified with volumes of reducible fiber components in elliptic fibrations and similar for z^a which are $U(1)$ elliptic parameters. When working with the unrefined topological string partition function, the treatment is completely analogous; the only exception is that $\epsilon_+ \mapsto 0$.

CHAPTER 4

Modularity and Quantum Gravity consistency in 6d $N = (1, 0)$ theories

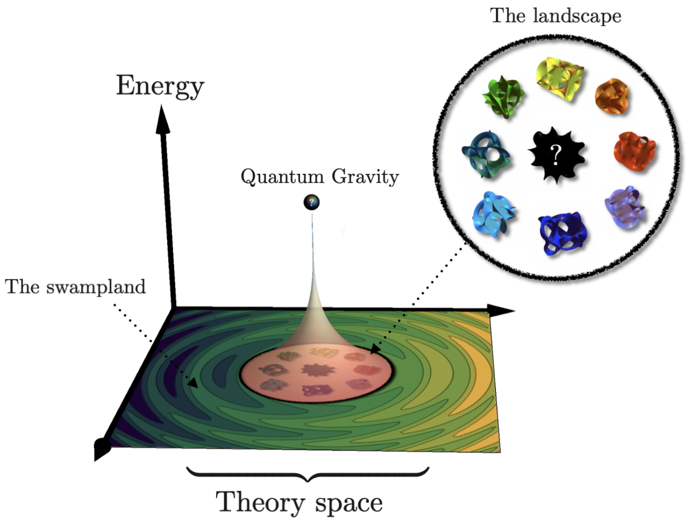


Figure 4.1: Schematic representation of the swampland and the landscape of low-energy effective theories. The swampland constraints become stronger as the energy increases and we get closer to the quantum gravity scale Λ_{QG}.

Over the recent years, there has emerged in the literature interest in finding underlying principles that quantum gravity should fulfill. It turns out that the absence of anomalies is not enough to assure the consistency of a quantum gravity theory. For this reason, we need to determine further constraints required for quantum gravity. Those apparently consistent low energy theories that cannot embed in an ultraviolet consistent quantum gravity are said to be in the *swampland*, while its complement is the *landscape*. See Figure 5.1. The subject of the swampland program is to distinguish the border between these two types of theories. There is a set of proposals to take over this task, referred to as swampland conjectures. In most cases, these bottom-up criteria are based on black hole physics arguments or universal features encountered in string theory examples.

In this section, we will first review some of the most backed-up swampland conjectures. In particular, we will elaborate on sharper variants of the weak gravity conjecture, which states that gravity is the weakest force [105]; one of the strongest versions is the non-Abelian sublattice weak gravity conjecture (nAsLWGC) [106]. After that, we will show that the nAsLWGC, for F-theory on elliptically fibered Calabi-Yau 3-folds, derives as a consequence of the properties of lattice Jacobi forms that encode the elliptic genera of effective strings propagating in 6d. Finally, we will sketch how the elliptic genera relates to other quantum gravity constraints. Namely the absence of gauge anomalies and the completeness hypothesis.

4.1 Summary of various swampland conjectures

As preparation for our upcoming discussions, we include a brief summary of the swampland program. Our exposition will be based on the reviews [107–109] and will not be exhaustive but focused on our immediate needs, which will be proving the non-Abelian sublattice weak gravity conjecture in 6d/5d F-theoretic backgrounds.

A) No global symmetries in quantum gravity: *A theory with a finite number of states, coupled to gravity, cannot have exact global symmetries* [81].

One heuristic argument in favor of this conjecture is that it avoids the so-called *remnants*. These are black holes that result at the final stage of Hawking evaporation and have a minimal mass of the order M_{Pl}, together with a finite amount of global charge due to a global symmetry group. While Hawking radiation is blind to the presence of a global symmetry group, i.e. global charges do not radiate, it is the remaining global charges that stabilize these types of black holes. Furthermore, the latter can occur for any representation of the global symmetry group, and in a quantum theory of gravity, this leads to an infinite number of remnants. There are arguments pointing out that these infinite species of remnants derive into a thermodynamic catastrophe [110], but rigorous proof of such a statement is not available. Namely, because of the breakdown of semi-classical gravity when we deal with the regime of strongly coupled gravity at M_{Pl}.

B) The completeness hypothesis: *A gauge field theory coupled to gravity must contain physical states with all possible gauge charges consistent with Dirac quantization* [111].

Since black holes can have any gauge charge, we expect in a quantum theory of gravity a physical state populating each entry in the lattice of gauge charges, i.e. a *complete spectrum*. On the other hand, a mechanism that prevents global symmetries is adding charged matter states that leads to break the former into a smaller subgroup. Thus, in a gauge theory, a complete spectrum destroys any possible global symmetry, which implies a tight connection between the conjectures **A)** and **B)**. Recent efforts managed to establish this link, in a precise manner, at the level of quantum field theory: *In a gauge theory with compact gauge group G its spectrum is complete iff there are no topologial Gukov-Witten operators in the theory* [112, 113]. Here completeness of the spectrum means that there exist states transforming in all possible representations of G. Moreover, the Gukov-Witten operators [114, 115] are codimension-2 topological operators that, in general, are non-invertible and have an associated 1-form electric global symmetry.[1]

[1] If G is abelian, then the Gukov-Witten operators are invertible and generate a global symmetry that is isomorphic to G.

C) The infinite distance conjecture: *In a theory coupled to gravity with a moduli space $\mathcal{M}$ that is parametrized by massless scalar fields ϕ^i, the following happens:* [116]

1. *For any point $P \in \mathcal{M}$ there exists another point $Q \in \mathcal{M}$ such that the geodesic distance between the two points P and Q, denoted by $d(P, Q)$, is infinite.*

2. *There is an infinite tower of states that become exponentially light at any infinite distance limit*

$$m(Q) \sim m(P)\mathrm{e}^{-\alpha d(P,Q)} \quad \text{when} \quad d(P,Q) \to \infty\,, \tag{4.1}$$

where $m(R)$ denotes the mass scale at $R \in \mathcal{M}$ and α is an unspecified positive constant.

Let us remark that the origin of the constant α is unknown, i.e. a derivation thereof from first principles, so far, is not available. In order to avoid interference with the exponential behaviour, it is conjectured to be of the order $\mathcal{O}(1)$. Moreover, at the infinite distance limit, the infinite tower of states signals the breakdown of the EFT.

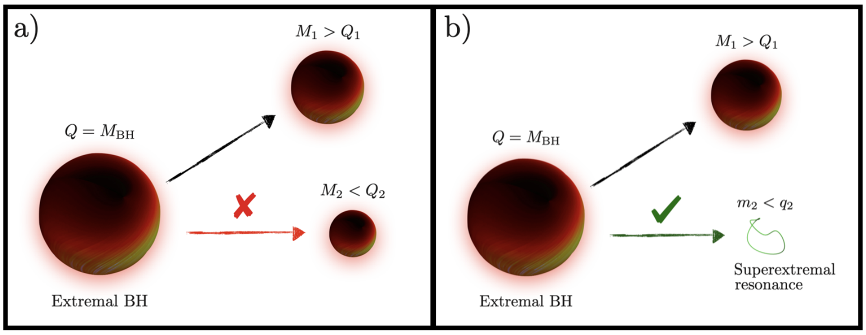


Figure 4.2: Decay of an extremal black hole.

D) The weak gravity conjecture (WGC): *In any d-dimensional $U(1)$ gauge theory coupled to gravity, the following should hold:* [105]

1. **The electric WGC:** *Given an extremal black hole of mass $M_{\rm BH}$ and charge Q, there must exist a particle p of charge q and mass m_{p} such that*

$$\frac{|\mathsf{q}|}{m_{\mathsf{p}}} \geq \left.\frac{|\mathsf{Q}|}{M_{\rm BH}}\right|_{\rm ext}\,, \tag{4.2}$$

2. **The magnetic WGC:** *The EFT cut-off Λ is bounded from above by the gauge coupling g associated to $U(1)$*

$$\Lambda \lesssim g M_{\mathrm{Pl};d}^{\frac{d-2}{2}}\,, \tag{4.3}$$

where $M_{\mathrm{Pl};d}$ is the d-dimensional Planck scale.

A particle which fulfils the WGC bound (4.10) is called *superextremal*. The motivation to include these kinds of particles in a quantum theory of gravity—again—arises from black hole physics. Due to the weak cosmic censorship hypothesis, i.e. to avoid naked singularities not hidden by a horizon, charged black holes that are Reissner-Nordström solutions must fulfill an extremality bound $M_{\rm BH} \geq |\mathsf{Q}|$. When $M_{\rm BH} = |\mathsf{Q}|$ we say that the black hole is extremal. Now, the principle behind the electric WGC is that extremal black holes should be able to decay. To see this, let us consider a decay of an extremal black hole into several decay products. By energy and charge conservation, one realizes there must be at least one decay product that follows the WGC bound (4.10), which implies it violates the black hole extremality bound. Thus, such an object must be a superextremal particle and not a black hole. Otherwise, the initial extremal black hole cannot decay. For illustration of this process, see Figure 4.2.

A remarkable point of the WGC is its affinity with the absence of global symmetries. This fact is a consequence of the magnetic WGC, which derives by considering as superextremal particle a monopole with mass of the order Λ/g^2. When taking the coupling limit $g \to 0$, we realize effectively an $U(1)$ global symmetry, but the cut-off $\Lambda \to 0$ signals an effective field theory breakdown.

Although the WGC formulation has an origin in 4d [105], there is no obstruction to generalize this conjecture to d-dimensions as performed in [108]. Even more general is including extended objects, as they commonly appear in string theory. To this end, it is convenient to consider, as low energy effective action, a generic d-dimensional Einstein-Maxwell-dilaton theory for a p-form gauge field—ignoring any particular string theory origin. In this case, the weak gravity conjecture invokes $(p-1)$-branes, in turn, to allow extremal black branes to decay. This way, we obtain an analogous WGC bound that reads [117][2]

$$\mathrm{WGC}_{p;d}: \quad e_{p;d}^2 \mathsf{q}^2 M_{\mathrm{Pl};d}^{d-2} \geq \left[\frac{\alpha_{p;d}^2}{2} + \frac{p(d-p-2)}{d-2}\right] T_p^2 \,, \tag{4.4}$$

where $e_{p;d}$ is the p-form gauge coupling, $\alpha_{p;d}$ is the coupling of a massless dilaton to the gauge theory field strength, while q and T_p are respectively the charge and tension of the superextremal brane. The important lesson here is that the generalized WGC bound (4.4) is invariant under KK reduction, up to additional $U(1)_{\mathrm{KK}}$-charges that emerge upon circle compactification [117, 118].

Up to now, we only discussed an extension for the WGC wrt a single $U(1)$ gauge group factor in a gauge theory. In practice, however, we encounter gauge theories that contain multiple $U(1)$s, whose charges we must take into account simultaneously. A criterion that ensures the existence of superextremal states for every rational direction in the charge lattice—here rational direction means a spot in the lattice—is the *convex hull condition* [119]. It states that given the spectrum of states with charge-to-mass ratio $\boldsymbol{\zeta} = \mathbf{q}/m$, where $\mathbf{q} = (\mathsf{q}_1, \ldots \mathsf{q}_r)$ is a $U(1)^r$ charge, the convex hull of all vectors $\{\boldsymbol{\zeta}\}$ must contain the unit ball measured by the quadratic form defined by the black hole extremality bound.

Let us now consider a $(d+1)$-dimensional $U(1)^r$ gauge theory, coupled to gravity, and its KK reduction into a d-dimensional one. If the WGC holds in the $(d+1)$-theory, it is not necessarily the case for the d-dimensional one. To see this, note that the KK reduced theory contains another $U(1)_{\mathrm{KK}}$ gauge group, whose charges mix into the extremality bound. Then, generally speaking, the convex hull condition does not hold for the $U(1)^{r+1}$ charged spectrum. For this reason, efforts lead to formulating stronger versions for a WGC that we quote next:

[2] Here we have assumed a single $U(1)$ field, but the argument can be generalized

D-i) The sublattice weak gravity conjecture (sLWGC): *For a theory with charge lattice Γ, there exists a sublattice $\Gamma_{\text{ext}} \subseteq \Gamma$ of finite coarseness such that for each $\mathbf{q} \in \Gamma_{\text{ext}}$, there is a (possibly unstable) superextremal particle of charge $\mathbf{q}$* [120].

D-ii) The non-Abelian sublattice weak gravity conjecture (nAsLWGC): *For any quantum gravity in $d \geq 5$ dimensions with zero cosmological constant and unbroken gauge group G, there is a finite-index Weyl-invariant sublattice Γ of the weight lattice $L_{\text{w}}(G)$ such that for every dominant weight $\boldsymbol{\lambda}_R \in \Gamma$ there is a superextremal resonance transforming in the G irreducible representation R with highest weight $\boldsymbol{\lambda}_R$* [106].

Note that for purely Abelian gauge groups this reduces to the ordinary sLWGC. In particular, the condition that a resonance is superextremal means that its charge to mass ratio is greater than that of a large, extremal Reissner-Nordström black hole with parallel charge vector.

This stronger WGC version has a tight connection with the so-called *emergence proposal*, which suggests that both weakly coupled gauge theories and gravity emerge in the infrared upon integrating out a tower of massive charged states below an ultraviolet cutoff—lower than the Planck scale. Let us denote by Λ_{gauge} the scale at which the gauge theory loop expansion in D dimensions breaks down, and by Λ_{QG} the scale at which gravity becomes strongly coupled. To capture the size of 1-loop contributions, the authors in [106] introduced the parametric scale dependent functions:

$$\lambda_{\text{gauge}}(E) := g^2_{\text{YM}} E^{D-4} \sum_{i:m_i<E} I(R_i)\,, \quad \lambda_{\text{grav}}(E) := G_N E^{D-2} \sum_{i:m_i<E} \dim(R_i)\,. \tag{4.5}$$

Here g_{YM} denotes the respective gauge theory coupling, while G_N the D-dimensional Newton's gravitational constant. Moreover, the sums (4.5) run over a set $\{i\}_{i\in \mathsf{I}}$ of particles, where the i-th particle has mass $m_i < E$ and transforms under the representation R_i of the gauge group G; $I(R_i)$ denotes the Dynkin index of R_i. In particular, the aforementioned scales manifest implicitely at the values $\lambda_{\text{gauge}}(\Lambda_{\text{gauge}}) = 1$ and $\lambda_{\text{grav}}(\Lambda_{\text{QG}}) = 1$. Now, if the nAsLWGC holds, its representation theory counting—over the infinite tower of superextremal states populating the sublattice $\Gamma \subseteq L_{\text{w}}(G)$—assures that $\lambda_{\text{gauge}}(E) \sim \lambda_{\text{grav}}(E)$. The latter assertion implies a common upper bound to both cutoff scales, up to order one factors, and therefore $\Lambda_{\text{gauge}} \approx \Lambda_{\text{QG}}$.

Despite the promising implications of the nAsLWGC, it is necessary to perform tests to the check consistency of this conjecture. The first evidence of this appeared in [106], where the authors consider perturbative Heterotic string theory and tori/toroidal compactifications thereof. There, modular invariance of the worldsheet CFT partition function suffices to prove the stronger condition that demands a subset of superextremal resonances to transform under appropriate representations. One of the goals of our work [3] is to extend the sLWGC evidence via elliptically fibered Calabi-Yau 3-folds in F-theory compactifications [64, 65]. But now we consider the representations constraints for non-Abelian gauge backgrounds, i.e., probe the nAsLWGC in 6d (1,0) theories realized by general F-theoretic backgrounds. In what follows, we explain the geometrical configuration by [64, 65], where self-vanishing intersection curves furnish nearly tensionless strings accounting for the infinite tower of superextremal resonances.

4.2 6d nearly tensionless strings and the weak gravity conjecture

As a preparation for our mathematical proves concerning the nAsLWGC, we review the geometrical setup that realizes the physical theories which we will examine, namely, 6d $N = (1,0)$ supergravities coupled to a gauge group G. We will base our exposition on the references [64, 65].

The starting point is to consider what happens when we approach a region in the moduli space, such that $g_{\mathrm{YM}} \to 0$ while keeping gravity dynamic. As discussed in the previous section, the censorship of global symmetries states that there should exist a mechanism preventing us from taking this limit. To attain this configuration, let us recall that an F-theory compactification with 2-complex dimensional base B yields a 6d $N = (1,0)$ supergravity theory in which

$$M_{\mathrm{Pl}}^4 = 4\pi \mathrm{Vol}_J(B)\,, \quad \frac{1}{g_{\mathrm{YM}}^2} = \frac{1}{2\pi}\mathrm{Vol}_J(C)\,. \tag{4.6}$$

Here the curve C denotes either a discriminant factor $\mathcal{S}^{\mathrm{b}}_{\mathfrak{g}_I}$ associated with a non-abelian gauge algebra $\mathfrak{g}_I$ or the height pairing due to a rational section associated with a $\mathfrak{u}(1)$ gauge algebra. More generally, the fibration can admit multiple rational sections $\{S_{Q_a}\}$ and instead of a curve C we have a height-pairing matrix $C_{ab} \equiv -\pi_*(\sigma(S_{Q_a}) \cdot \sigma(S_{Q_b}))$. In this case, a gauge kinetic matrix f_{ab} due to a $U(1)^r$ gauge group replaces the gauge coupling g_{YM}. With this in mind, the desired limit translates into

$$\mathrm{Vol}_J(C) \to \infty \quad \text{with} \quad \mathrm{Vol}_J(B) \text{ finite}\,. \tag{4.7}$$

The crucial point to realize this critical limit is that B must contain a rational curve C_0 [64], such that it asymptotically vanishes as $\mathrm{Vol}_J(C) \to \infty$; the curve C_0 has the properties:

$$C \cdot C_0 \neq 0\,, \quad C_0 \cdot c_1(B) = 2\,, \quad \text{and} \quad C_0^2 = 0\,. \tag{4.8}$$

Moreover, C_0 is unique up to a multiplicative factor and the existence of the limit (4.7) implies that M admits a $K3$ fibration with $K3$ fiber class $\pi^{-1}(C_0)$.

As explained in section (3.2.1), a D3 brane that wraps the curve C_0 gives rise to an effective string propagating in 6d, whose tension reads

$$T = \frac{2\pi}{\ell_s^4}\mathrm{Vol}_J(C_0)\,. \tag{4.9}$$

Note that the critical limit (4.7) implies that the effective strings, realized by the curve C_0, have a tensionless limit behaviour. The effective 2d worldsheet theory of such effective strings can be described by a 4d $N =$ SYM compactified on $\mathbb{R}^{1,1} \times C_0$, together with a topological duality twist along C_0 [94, 95]. The massless spectrum of the resulting worldsheet theory contains 3-7 string modes, i.e. strings charged under a 7-brane gauge group, that localize at the non-trivial intersection $C \cap C_0 \subset B$. It turns out that these strings identifiy with Heterotic strings in a dual frame, whose geometrical realization is an elliptically fibered $K3$ surface—not necessarily related with the class $\pi^{-1}(C_0)$.

Let us remark that the Heterotic dual side does not necessarily have a perturbative CFT description. When the base B is a Hierzebruch surface $\mathrm{p} : \mathbb{F}_n \to \mathbb{P}^1_{\mathrm{b}}$, where $\mathbb{P}^1_{\mathrm{b}} \simeq \mathbb{P}^1$, the p-fibers F identify with C_0 and parametrize the Heterotic string coupling as $g_{\mathrm{het}}^2 = \mathrm{Vol}_J(F)/\mathrm{Vol}_J(\mathbb{P}^1_{\mathrm{b}})$. Thus, in the critical limit (4.7), the effective strings associated with F translate into weakly coupled Heterotic strings with a presumptive perturbative framework. However, in cases with more tensor multiplets, the presence of NS5-branes give rise to non-perturbative effects.

Tensionless strings limit and the sLWGC: One of the main findings of [64, 65] is the test for the sLWGC in a theory with several abelian gauge group factors $U(1)_a$, where $a = 1\,, \dots\,, n_V$ and n_V denotes the number of vector multiplets. As we recall, this conjecture states that a sublattice of the charge lattice is populated by physical states whose charge-to-mass ratio exceeds an extremality bound. For the case of an $U(1)^{n_V}$ Einstein-Maxwell-dilaton theory with multiple scalar fields $\boldsymbol{\phi} \equiv (\phi_1, \dots, \phi_{n_s})$, the extremality bound reads [65, 121]

$$\frac{(\mathbf{q},\mathbf{q})_{f(\boldsymbol{\phi})}}{m_{\mathsf{q}}^2} \geq \frac{(\mathbf{Q},\mathbf{Q})_{f(\boldsymbol{\phi}_0)}}{M_{\mathrm{ADM}}^2} = \mu \frac{1}{M_{\mathrm{Pl};d}^{d-2}}\,, \tag{4.10}$$

where $(\cdot\,,\cdot)_{f(\boldsymbol{\phi})}$ denotes a quadratic form characterized by the gauge kinetic terms of the dilatonic theory. The left-hand side of (4.10) describes the charge-to-mass ratio of a state with charges q_a and mass m_{q}. The right hand of the bound (4.10) expresses the charge-to-mass ratio of an extremal dilatonic Reissner-Nordström black hole with charges Q_a and ADM mass M_{ADM}. The μ value can be obtained from the solution of an extremal dilatonic Reissner-Nordström black hole. Let $\vec{\alpha}$ be a dilaton coupling, then μ reads

$$\mu = \frac{d-3}{d-2} + \frac{1}{4}\vec{\alpha}^2\,. \tag{4.11}$$

In our current case of discussion, 6d (1,0) effective supergravity theories with abelian gauge symmetry $U(1)^{n_V}$, we realize the extremality bound as follows (4.10).

First, we consider the 6d (1,0) effective pseudo-action [78]. In general, such an action has a dependence on $h^{1,1}(B) = n_T + 1$ scalar fields $\{t^\alpha\}_{\alpha=0,\dots,n_T}$ that transform under $SO(1, n_T)$—besides gauge fields, curvature, etc. It turns out that the critical limit (4.7) corresponds to a dominant real scalar $x \to \infty$ that parametrizes the scalar fields t^α, and in this case, the 6d (1,0) effective pseudo-action reduces to [65]

$$S_{6d}\Big|_{\mathrm{Abelian}} = \int_{M_6} \frac{M_{\mathrm{Pl}}^4}{2}\sqrt{-g}R - \frac{M_{\mathrm{Pl}}^4}{2}\mathrm{d}x \wedge *\mathrm{d}x - \frac{1}{2}\frac{m_{ab}}{|\mathrm{const}|}\mathrm{e}^x F^a \wedge *F^b \quad \text{as } x \to \infty\,, \tag{4.12}$$

Here g is the spacetime metric, R is the curvature, F^a is a $U(1)$ strength field tensor, and the factor $f_{ab} \propto m_{ab}e^x$ is the gauge kinetic matrix determined by $m_{ab} = \frac{1}{2}C_{ab} \cdot C_0$. In other words, what we obtain is effectively an Einstein-Maxwell-dilaton theory in 6d with value (4.11) $\mu = 1$. This way, one obtains Reissner-Nordström black hole solutions in 6d, in particular extremal black holes, for which the extremality bound (4.10) makes physical sense.

Second, we need to verify the existence for a set of superextremal states fulfilling the bound (4.10). To this end, in the next section, we will argue the existance of an even lattice L such that

$$\Delta_J \equiv 4n - (\mathbf{q},\mathbf{q}) \stackrel{!}{=} 0 \text{ for all } \mathbf{q} \in L \subset L^*\,, \quad \text{i.e.} \quad n(\mathbf{q}) = \frac{1}{4}m^{ab}\mathsf{q}_a\mathsf{q}_b \in \mathbb{Z}_{\geq 0}\,. \tag{4.13}$$

In this treatment, we associate the dual lattice L^* with the charge lattice and m^{ab} is the matrix form of the bilinear product $(\cdot\,,\cdot) : L^* \times L^* \to \mathbb{Q}$. The object Δ_J is the discriminant of a lattice Jacobi form determined by the elliptic genus associated with the nearly tensionless strings, which are geometrized by the curve $C_0 \subset B$. According to the Heterotic frame, the entries $\mathbf{q} \in L$ correspond to level $n(\mathbf{q})$ excitated string modes of mass $M^2_{n(\mathbf{q})} = 4T(n-1)$. A series of geometrical relations, together with the charge-integer correspondence (4.13) of charges in L, gives the expression [64, 65]

$$(\mathbf{q}, \mathbf{q})_f = \frac{M^2_{n(\mathbf{q})}}{M^4_{\text{Pl}}} + |\text{const}| \cdot \mathrm{e}^{-x} \quad \text{as} \quad x \to \infty \quad \Rightarrow \quad \text{WGC: } \frac{(\mathbf{q}, \mathbf{q})_f}{M^2_{n(\mathbf{q})}} > \mu \frac{1}{M^4_{\text{Pl}}} \quad \checkmark \,. \tag{4.14}$$

To summarize, the elliptic genus counts a subsector of the charged spectrum, whose charges populate the sublattice $L \subset L^*$, and every state there fulfils superextremality condition. Thus, the sLWGC holds in the limit of nearly tensionless Heterotic strings.

Now, let us include in the 6d (1,0) effective pseudo-action a field-strength 4-form due to a non-Abelian gauge symmetry $\mathfrak{g}_I$. To make sense of the weak gravity conjecture bound (4.11), we need to compactify on a circle and turn on a Wilson line, which effectively breaks $\mathfrak{g}_I$ into its Cartan subalgebra $\mathfrak{h}_I$ [117]. This way, the 6d effective action contains a term that reduces as [78]

$$\mathfrak{g}_I \, : \; (j \cdot b_I) \, \tfrac{1}{\lambda_I} \text{tr} F_I \wedge * F_I \xrightarrow{\text{Wilson line}} \mathfrak{h}_I \, : \; \tfrac{1}{\sqrt{2\mathcal{V}}} \left(J \cdot \mathcal{S}^{\text{b}}_{\mathfrak{g}_I} \right) \mathfrak{C}_{ij} F^i \wedge * F^j \,. \tag{4.15}$$

Here $\mathfrak{C}_{ij}$ denotes the coroot intersection matrix of the Lie algebra $\mathfrak{g}_I$ and F^i denotes an Abelian gauge field-strength with $i = 1, \ldots, \text{rk}(\mathfrak{g}_I)$. We remind that the extremality bounds (4.11) preserves under circle compactification, up to KK charges contributions [117, 118]. Thus, it is equivalent to study the 6d (1,0) non-Abelian gauge theory, restricted to its Cartan subalgebra, as the sLWGC/nAsLWGC in this Einstein-Maxwell-dilaton theory implies that of its actual physical 5d KK reduction [106, 117, 120].

4.3 The non-Abelian sLWGC and lattice Jacobi forms

Let us now consider this proposal in the context of F-theory on a Calabi-Yau 3-fold such that the effective theory admits a limit in which the gauge coupling would go to zero. It follows from the general results in [64] that the base of the elliptic fibration contains a curve of self-intersection zero and geometrically the limit amounts to this curve shrinking to zero volume. The string that arises from a D3-brane that wraps this curve becomes tensionless and is dual to a, not necessarily perturbative, critical Heterotic string. More precisely, the string is weakly coupled if the base of the fibration is a Hirzebruch surface. In general additional tensor multiplets signal the presence of NS5-branes and therefore of non-perturbative effects. Nevertheless, in each case we can calculate the elliptic genus and this will be a modular or, in the non-perturbative case, a quasi-modular Jacobi form. Let us stress that the non-perturbative effects due to the presence of NS5-branes are different from those that lead to a non-perturbative gauge group. In those cases where the gauge group is not perturbative but the base of the Calabi-Yau 3-fold on the F-theory side is a Hirzebruch surface, the elliptic genus of the dual Heterotic string will still be a modular Jacobi form.

The elliptic genus encodes a subset of the particle-like string excitations via a trace over the Ramond-Ramond sector along a torus

$$Z(\tau, \boldsymbol{z}) = \text{Tr}_{RR} (-1)^F F^2 q^{H_L} \bar{q}^{H_R} \prod_{a=1}^{n_V} (\zeta^a)^{J_a} \,, \tag{4.16}$$

where F is the Fermion number, $H_{L/R}$ are the left- and right-moving Hamiltonians and $q = \exp(2\pi i\tau)$ is the modular parameter of the torus. Moreover, the rank of the gauge group is n_V and it acts as a global symmetry with generators J_a on the worldsheet theory. The corresponding fugacities are denoted by $\zeta^a = \exp(2\pi i z^a)$. In terms of modular objects the elliptic genus takes the general form

$$Z(\tau, \boldsymbol{z}) = \frac{\Phi_{10,\boldsymbol{m}}(\tau, \boldsymbol{z})}{\eta^{24}(\tau)}\,, \tag{4.17}$$

where $\Phi_{10,\boldsymbol{m}}(\tau, \boldsymbol{z})$ is a holomorphic, Weyl invariant lattice Jacobi form of weight 10 with index matrix $\boldsymbol{m}$. It therefore admits an expansions (B.26)

$$\Phi_{10,\boldsymbol{m}}(\tau, \boldsymbol{z}) = \sum_{\boldsymbol{\mu}\in L^*/L} h_{\boldsymbol{\mu}}(\tau)\vartheta_{\underline{L},\boldsymbol{\mu}}(\tau, \boldsymbol{z})\,, \tag{4.18}$$

with $\vartheta_{\underline{L},\boldsymbol{\lambda}}(\tau, \boldsymbol{z})$ being Jacobi theta functions associated to the, in general twisted, coroot lattice $\underline{L}$ of the gauge group.

The elliptic genus essentially encodes only the left-moving excitations and the states have to be paired with right moving excitations, taking into account the level matching condition, to lead to actual states of the theory. In particular, the numerator $\Phi_{10,\boldsymbol{m}}$ in (4.17) always contains a non-zero constant term that arises from the left-moving tachyon. This implies that there is always a contribution[3]

$$\Phi_{10,\boldsymbol{m}} = h_{\boldsymbol{0}}(\tau)\vartheta_{\underline{L},\boldsymbol{0}}(\tau, \boldsymbol{z}) + \dots\,, \tag{4.19}$$

with $h_0(\tau) = -2 + \mathcal{O}(q)$ [122]. It was shown in [65] that the states from this sector, with the corresponding Jacobi theta function given by

$$\vartheta_{\underline{L},\boldsymbol{0}}(\tau, \boldsymbol{z}) = \sum_{\substack{\boldsymbol{\lambda}\in L^* \\ \boldsymbol{\lambda}\equiv\boldsymbol{0} \bmod L}} q^{\frac{1}{2}(\boldsymbol{\lambda},\boldsymbol{\lambda})} \exp\left(2\pi i(\boldsymbol{\lambda}, \boldsymbol{z})\right)\,, \tag{4.20}$$

are superextremal with respect to a dilatonic Reissner-Nordström of the corresponding Cartan $U(1)$ charges. Moreover, they form a sublattice of the $U(1)$ charge lattice and therefore are sufficient to satisfy the original sublattice weak gravity conjecture.

We will now argue that, at least for simple gauge groups, this sector in fact satisfies the stronger claim of the non-Abelian sLWGC. To this end we need to see how the states encoded in (4.20) arrange into representations of the gauge group G with Lie algebra $\mathfrak{g}$. More precisely, we will rewrite the right-hand side in terms of Weyl characters. Let us first recall some definitions.[4] Let $(R, V_{\boldsymbol{\lambda}})$ be an irreducible, finite-dimensional representation of a complex semisimple Lie algebra $\mathfrak{g}$ with Cartan subalgebra $\mathfrak{h}$. Here $V_{\boldsymbol{\lambda}}$ is a highest weight module with highest weight $\boldsymbol{\lambda}$, and R is the $\mathfrak{g}$-representation associated to $V_{\boldsymbol{\lambda}}$. The Weyl character of $(R, V_{\boldsymbol{\lambda}})$ is the function $\chi_{\boldsymbol{\lambda}} : \mathfrak{h} \to \mathbb{C}$ with

$$\chi_{\boldsymbol{\lambda}}(\boldsymbol{z}) = \mathrm{tr}_{V_{\boldsymbol{\lambda}}}\left(\exp\left(2\pi i R(\boldsymbol{z})\right)\right)\,. \tag{4.21}$$

From now on we assume that the gauge group is a simple Lie group with algebra $\mathfrak{g}$. However, we remark that the upcoming derivations can be generalized for semisimple Lie

[3] We thank Timo Weigand for explaining this point to us.
[4] For further guidance on Lie algebras and representation theory, we refer the reader to Appendix A.

algebras via the product formula (B.34). Thus, for a simple Lie algebra the index $\boldsymbol{m}$ of the elliptic genus as a lattice Jacobi form will then be m times the negative of the coroot lattice intersection form, where m is some positive integer. The numerator of the elliptic genus admits an expansion in terms of Weyl invariant lattice theta functions

$$\Phi_{10,m}(\tau,\boldsymbol{z}) = \sum_{\boldsymbol{\lambda}\in L_{\mathrm{w}}(\mathfrak{g})/mL^\vee(\mathfrak{g})} h_{\boldsymbol{\lambda}}(\tau)\vartheta^{\mathfrak{g}}_{m,\boldsymbol{\lambda}}(\tau,\boldsymbol{z})\,, \tag{4.22}$$

with the states encoded in $-2\cdot\vartheta^{\mathfrak{g}}_{m,\boldsymbol{0}}(\tau,\boldsymbol{z})$ again forming the superextremal sublattice. The expansion

$$\vartheta^{\mathfrak{g}}_{m,\boldsymbol{0}}(\tau,\boldsymbol{z}) = \sum_{w\in W(\mathfrak{g})}\sum_{\substack{w\cdot\boldsymbol{\lambda}\in mL^\vee(\mathfrak{g})\\ \boldsymbol{\lambda}\in P_+(\mathfrak{g})}} \mathrm{sign}(w) q^{\frac{1}{2m}(w\cdot\boldsymbol{\lambda},w\cdot\boldsymbol{\lambda})}\chi_{\boldsymbol{\lambda}}(\boldsymbol{z})\,, \tag{4.23}$$

follows from a more general relation that we are going to prove in the next section. Here $W(\mathfrak{g})$ is the Weyl group and the shifted Weyl reflection

$$\cdot : W(\mathfrak{g})\times L_{\mathrm{w}}(\mathfrak{g}) \to L_{\mathrm{w}}(\mathfrak{g})\,, \tag{4.24}$$

acts as $w\cdot\boldsymbol{\lambda} = w(\boldsymbol{\lambda}+\boldsymbol{\rho})-\boldsymbol{\rho}$. Moreover, $P_+(\mathfrak{g}) = L_{\mathrm{w}}(\mathfrak{g})\cap\mathcal{W}(\mathfrak{g})$ is the set of dominant weights, i.e. those in the fundamental Weyl chamber $\mathcal{W}(\mathfrak{g})$.

We can further decompose the expansion into

$$\begin{aligned}\vartheta^{\mathfrak{g}}_{m,\boldsymbol{0}}(\tau,\boldsymbol{z}) = &\sum_{\boldsymbol{\lambda}\in mL^\vee(\mathfrak{g})\cap P_+(\mathfrak{g})} q^{\frac{1}{2m}(\boldsymbol{\lambda},\boldsymbol{\lambda})}\chi_{\boldsymbol{\lambda}}(\boldsymbol{z})\\ &+\sum_{\substack{w\in W(\mathfrak{g})\\ w\neq\mathrm{id}}}\sum_{\substack{w\cdot\boldsymbol{\lambda}\in mL^\vee(\mathfrak{g})\\ \boldsymbol{\lambda}\in P_+(\mathfrak{g})}} \mathrm{sign}(w) q^{\frac{1}{2m}(w\cdot\boldsymbol{\lambda},w\cdot\boldsymbol{\lambda})}\chi_{\boldsymbol{\lambda}}(\boldsymbol{z})\,,\end{aligned} \tag{4.25}$$

It is easy to show that $(w\cdot\boldsymbol{\lambda},w\cdot\boldsymbol{\lambda}) > (\boldsymbol{\lambda},\boldsymbol{\lambda})$ for $\boldsymbol{\lambda}\in P_+(\mathfrak{g})$ if $w\neq\mathrm{id}$. This follows from the inequality

$$\begin{aligned}(w\cdot\boldsymbol{\lambda},w\cdot\boldsymbol{\lambda}) &= (w(\boldsymbol{\lambda}+\boldsymbol{\rho})-\boldsymbol{\rho},w(\boldsymbol{\lambda}+\boldsymbol{\rho})-\boldsymbol{\rho})\\ &= (\boldsymbol{\lambda},\boldsymbol{\lambda}) + \underbrace{\left(\boldsymbol{\lambda}+\boldsymbol{\rho},\boldsymbol{\rho}-w^{-1}(\boldsymbol{\rho})\right)}_{\geq 0} + \underbrace{\left(\boldsymbol{\rho},\boldsymbol{\lambda}+\boldsymbol{\rho}-w^{-1}(\boldsymbol{\lambda}+\boldsymbol{\rho})\right)}_{\geq 0} \geq (\boldsymbol{\lambda},\boldsymbol{\lambda})\,,\end{aligned} \tag{4.26}$$

where $\boldsymbol{\rho}$ is the Weyl vector

$$\boldsymbol{\rho} = \frac{1}{2}\sum_{\boldsymbol{\alpha}\in\Phi^+(\mathfrak{g})}\boldsymbol{\alpha}\,, \tag{4.27}$$

with $\Phi^+(\mathfrak{g})$ being the set of positive roots. The inequality is in turn a consequence of proposition 2.4 and note 4.14 of [123]: *If* $\boldsymbol{\gamma}\in\mathcal{W}(\mathfrak{g})$ *and* $w\in W(\mathfrak{g})$, *then* $(\boldsymbol{\gamma}-w(\boldsymbol{\gamma}),\boldsymbol{\tau})\geq 0$ *for every* $\boldsymbol{\tau}\in\mathrm{Int}\,(\mathcal{W}(\mathfrak{g}))$; $\boldsymbol{\gamma}+\boldsymbol{\rho}\in\mathrm{Int}\,(\mathcal{W}(\mathfrak{g}))$ *iff* $\boldsymbol{\gamma}\in\mathcal{W}(\mathfrak{g})$. Here $\mathrm{Int}\,(\mathcal{W}(\mathfrak{g}))$ denotes the interior of $\mathcal{W}(\mathfrak{g})$.

The second line of (4.26) ensures no cancelations for the first line in (4.25). This implies there is a superextremal resonance transforming in the irreducible representation associated to every dominant weight $\boldsymbol{\lambda}$ in the sublattice $mL^\vee(\mathfrak{g})\subset L_{\mathrm{w}}(\mathfrak{g})$.

4.4 Weyl invariant character sums over dominant weights

We now introduce a generalized version of the identity (4.20) and prove it:

Claim I: Let $\mathfrak{g}$ be a complex simple Lie algebra. For any positive integer $m \in \mathbb{N}$ and Weyl invariant subset $\{\boldsymbol{\mu}_{i,\, i=1,\dots,k}\} \subseteq L_{\rm w}(\mathfrak{g})/mL^\vee(\mathfrak{g})$ we can define the lattice subset

$$K = \left(\boldsymbol{\mu}_1 + mL^\vee(\mathfrak{g})\right) \oplus \cdots \oplus \left(\boldsymbol{\mu}_k + mL^\vee(\mathfrak{g})\right) \,, \tag{4.28}$$

and the corresponding sum of theta functions satisfies the relation

$$\vartheta^{\mathfrak{g}}_{m,\boldsymbol{\mu}_1}(\tau,\boldsymbol{z}) + \cdots + \vartheta^{\mathfrak{g}}_{m,\boldsymbol{\mu}_k}(\tau,\boldsymbol{z}) = \sum_{w\in W(\mathfrak{g})} \sum_{\substack{w\cdot\boldsymbol{\lambda}\in K \\ \boldsymbol{\lambda}\in P_+(\mathfrak{g})}} \mathrm{sign}(w) q^{\frac{1}{2m}(w\cdot\boldsymbol{\lambda}, w\cdot\boldsymbol{\lambda})} \chi_{\boldsymbol{\lambda}}(\boldsymbol{z}) \,. \tag{4.29}$$

In order to prove the **Claim I**, we need to discuss the *Weyl character formula.* The trace (4.21) that defines the Weyl character $\chi_{\boldsymbol{\lambda}}(\boldsymbol{z})$ results into a weighted sum over the highest weight module $V_{\boldsymbol{\lambda}}$. We recall that the latter decomposes as a direct sum of weight spaces, i.e. $V_{\boldsymbol{\lambda}} = \oplus_{\boldsymbol{\omega}} V_{\boldsymbol{\omega}}$. This means that the Weyl character $\chi_{\boldsymbol{\lambda}}(\boldsymbol{z})$ admits an expansion of the form

$$\chi_{\boldsymbol{\lambda}}(\boldsymbol{z}) = \sum_{\boldsymbol{\omega}\in V_{\boldsymbol{\lambda}}} m_{\boldsymbol{\omega}} \mathrm{e}^{2\pi i(\boldsymbol{\omega},\boldsymbol{z})} \,, \tag{4.30}$$

where $m_{\boldsymbol{\omega}} \in \mathbb{N}$ is the multiplicity of each weight space $V_{\boldsymbol{\omega}} \subset V_{\boldsymbol{\lambda}}$ associated to a weight $\boldsymbol{\omega}$. Alternatively, we can expand the characters using the famous Weyl character formula [124]

$$\chi_{\boldsymbol{\lambda}}(\boldsymbol{z}) = \frac{1}{\Delta_W(\boldsymbol{z})} \sum_{w\in W(\mathfrak{g})} \mathrm{sign}(w) \exp\left[2\pi i \left(w(\boldsymbol{\lambda}+\boldsymbol{\rho}), \boldsymbol{z}\right)\right] \,, \tag{4.31}$$

where $\boldsymbol{\rho}$ is again the Weyl vector and $\Delta_W(\boldsymbol{z})$ is defined by

$$\Delta_W(\boldsymbol{z}) \equiv \prod_{\boldsymbol{\alpha}\in\Phi^+(\mathfrak{g})} \left(\mathrm{e}^{\pi i(\boldsymbol{\alpha},\boldsymbol{z})} - \mathrm{e}^{-\pi i(\boldsymbol{\alpha},\boldsymbol{z})}\right) = \sum_{w\in W(\mathfrak{g})} \mathrm{sign}(w) \exp\left[2\pi i \left(w(\boldsymbol{\rho}), \boldsymbol{z}\right)\right] \,. \tag{4.32}$$

Here $\Phi^+(\mathfrak{g})$ denotes the set of positive roots in $\mathfrak{g}$. Note that $w(\boldsymbol{\lambda})$ is the Weyl reflection w applied to the weight $\boldsymbol{\lambda}$ and should not be confused with the shifted Weyl reflection $w\cdot\boldsymbol{\lambda}$ defined in (4.24). With this information at hand, we proceed to prove **Claim I**.

Proof of Claim I: First, we prove the relation

$$\sum_{\boldsymbol{\omega}\in W_{\boldsymbol{\lambda}}} \mathrm{e}^{2\pi i(\boldsymbol{\omega},\boldsymbol{z})} = \sum_{\boldsymbol{\omega}\in W_{\boldsymbol{\lambda}}} \chi_{\boldsymbol{\omega}}(\boldsymbol{z}) \,, \tag{4.33}$$

where we introduced $W_{\boldsymbol{\lambda}} \equiv \{w(\boldsymbol{\lambda})\}_{w\in W(\mathfrak{g})}$ for a given $\boldsymbol{\lambda} \in L_{\rm w}(\mathfrak{g})$. This is equivalent to showing that

$$\sum_{\boldsymbol{\omega}\in W_{\boldsymbol{\lambda}}} \Delta_W(\boldsymbol{z}) \cdot \mathrm{e}^{2\pi i(\boldsymbol{\omega},\boldsymbol{z})} = \sum_{\boldsymbol{\omega}\in W_{\boldsymbol{\lambda}}} A_{\boldsymbol{\omega}+\boldsymbol{\rho}}(\boldsymbol{z}) \,, \tag{4.34}$$

where $A_{\boldsymbol{\omega}+\boldsymbol{\rho}}$ is the numerator of the Weyl character formula (4.31), i.e. $\chi_{\boldsymbol{\omega}}(\boldsymbol{z}) = \Delta_W^{-1} A_{\boldsymbol{\omega}+\boldsymbol{\rho}}(\boldsymbol{z})$. It directly follows from expanding the left-hand side of the equation (4.34)

$$
\begin{aligned}
\sum_{\boldsymbol{\omega}\in W_{\boldsymbol{\lambda}}} \Delta_W(\boldsymbol{z}) \cdot \mathrm{e}^{2\pi i(\boldsymbol{\omega},\boldsymbol{z})} &= \sum_{w'\in W(\mathfrak{g})} \sum_{w\in W(\mathfrak{g})} \mathrm{sign}(w')\mathrm{e}^{2\pi i (w'(\boldsymbol{\rho})+w(\boldsymbol{\lambda})),\boldsymbol{z})} \\
&= \sum_{w\in W(\mathfrak{g})} A_{w(\boldsymbol{\lambda})+\boldsymbol{\rho}}(\boldsymbol{z}) \\
&= \sum_{\boldsymbol{\omega}\in W_{\boldsymbol{\lambda}}} A_{\boldsymbol{\omega}+\boldsymbol{\rho}}(\boldsymbol{z}) \,,
\end{aligned}
\tag{4.35}
$$

where we have used the fact

$$
\sum_{w\in W(\mathfrak{g})} \mathrm{e}^{2\pi i(w(\boldsymbol{\lambda}),\boldsymbol{z})} = \sum_{w\in W(\mathfrak{g})} \mathrm{e}^{2\pi i(w'w(\boldsymbol{\lambda}),\boldsymbol{z})} \,, \quad w' \in W(\mathfrak{g}) \,. \tag{4.36}
$$

Now we proceed to prove the formula (4.29). Recall that the left-hand side reads

$$
\vartheta_K(\tau,\boldsymbol{z}) \equiv \vartheta^{\mathfrak{g}}_{m,\boldsymbol{\mu}_1}(\tau,\boldsymbol{z}) + \cdots + \vartheta^{\mathfrak{g}}_{m,\boldsymbol{\mu}_k}(\tau,\boldsymbol{z}) \,. \tag{4.37}
$$

Using the identity (4.33) this can be rewritten as

$$
\begin{aligned}
\vartheta_K(\tau,\boldsymbol{z}) &= \sum_{\boldsymbol{\omega}\in K} q^{\frac{1}{2m}(\boldsymbol{\omega},\boldsymbol{\omega})} \mathrm{e}^{2\pi i(\boldsymbol{\omega},\boldsymbol{z})} \\
&= \sum_{\boldsymbol{\lambda}\in K\cap \mathcal{W}(\mathfrak{g})} \sum_{\boldsymbol{\omega}\in W_{\boldsymbol{\lambda}}} q^{\frac{1}{2m}(\boldsymbol{\omega},\boldsymbol{\omega})} \mathrm{e}^{2\pi i(\boldsymbol{\omega},\boldsymbol{z})} \\
&= \sum_{\boldsymbol{\omega}\in K} q^{\frac{1}{2m}(\boldsymbol{\omega},\boldsymbol{\omega})} \chi_{\boldsymbol{\omega}}(\boldsymbol{z}) \\
&= \sum_{w\in W(\mathfrak{g})} \sum_{\substack{w\cdot\boldsymbol{\lambda}\in K \\ \boldsymbol{\lambda}\in P_+(\mathfrak{g})}} q^{\frac{1}{2m}(w\cdot\boldsymbol{\lambda},w\cdot\boldsymbol{\lambda})} \chi_{w\cdot\boldsymbol{\lambda}}(\boldsymbol{z}) \,.
\end{aligned}
\tag{4.38}
$$

where we used the fact the $K \subset L_{\mathrm{w}}(\mathfrak{g})$ is Weyl invariant.

Moreover, a simple calculation reveals the following identity

$$
\begin{aligned}
\chi_{w\cdot\boldsymbol{\lambda}}(\boldsymbol{z}) &= \frac{1}{\Delta_W(\boldsymbol{z})} \sum_{w'\in W(\mathfrak{g})} \mathrm{sign}(w')\mathrm{e}^{2\pi i(w'(w\cdot\boldsymbol{\lambda}+\boldsymbol{\rho}),\boldsymbol{z})} \\
&= \frac{\mathrm{sign}\left(w^{-1}\right)}{\Delta_W(\boldsymbol{z})} \sum_{w'\in W(\mathfrak{g})} \mathrm{sign}(w'w)\mathrm{e}^{2\pi i(w'w(\boldsymbol{\lambda}+\boldsymbol{\rho}),\boldsymbol{z})} \\
&= \mathrm{sign}(w)\chi_{\boldsymbol{\lambda}}(\boldsymbol{z}) \,.
\end{aligned}
\tag{4.39}
$$

Inserting the identity (4.39) into the last expression in (4.38) gives the conjectured expression (4.29).

To prove the non-Abelian sublattice weak gravity conjecture, we were particularly interested in the Weyl invariant theta function $\vartheta_{m,\boldsymbol{0}}(\tau,z) = \vartheta_{mL^\vee(\mathfrak{g})}(\tau,\boldsymbol{z})$ which corresponds to the sublattice $mL^\vee(\mathfrak{g}) \subset L_{\mathrm{w}}(\mathfrak{g})$ [106]. However, our **Claim I** is more general. In the following section we make use of this fact to argue the completeness hypothesis for the spectrum of a specific set of theories with non-Abelian gauge symmetry.

4.5 From cancellation of anomalies to the completeness hypothesis

In this section we consider the cancellation of gauge anomalies via modular properties of Jacobi forms. For simplicity we restrict to examples that do have an heterotic perturbative description, but the generalization to non-perturbative backgrounds is straightforward [125]. We consider a few examples with gauge symmetry $\mathfrak{g} = A_n$ [126] and find that modularity also implies the completeness hypothesis, at least in these cases.

The elliptic genus not only contains information about the nAsLWGC but also encodes the massless spectrum and the generalized Green-Schwarz mechanism, which we exposed in section 3.2.2. To see this, let us note that the 1-loop anomaly polynomial takes the form

$$I_8 = \sum_{\boldsymbol{\lambda},s} n_s(\mathbf{R}_{\boldsymbol{\lambda}}) I_s(\mathbf{R}_{\boldsymbol{\lambda}})\,. \tag{4.40}$$

Here we sum over all massless fields in the spectrum that are characterized by multiplicity $n_s(\mathbf{R})$, a spin s, and a representation $\mathbf{R}_{\boldsymbol{\lambda}}$ with associated highest weight $\boldsymbol{\lambda}$. Moreover, $I_s(\mathbf{R}_{\boldsymbol{\lambda}})$ is given by products of traces of the gauge field strengths and the curvature 2-form R. For concreteness, we focus on the anomaly contribution associated to a gauge algebra factor $\mathfrak{g}_I$ that reads

$$I_8\Big|_{\mathfrak{g}_I} = A^{(6)}\mathrm{tr}F_I^4\,, \tag{4.41}$$

where F_I is the $\mathfrak{g}_I$ field-strength, "tr" denotes the trace in the fundamental representation, and [77]

$$A^{(6)} = \frac{1}{4!}\left(C_{\mathbf{adj}(\mathfrak{g}_I)} - \sum_{\boldsymbol{\lambda}} n_H(\mathbf{R}_{\boldsymbol{\lambda}})\cdot C_{\mathbf{R}_{\boldsymbol{\lambda}}}\right)\,. \tag{4.42}$$

Here $n_H(\mathbf{R})$ denotes the multiplicity of hypermultiplets that transform under a given representation $\mathbf{R}$ for the gauge algebra $\mathfrak{g}_I$, while $C_{\mathbf{R}}$ are group theoretical constants that are defined by the relations

$$\mathrm{tr}_{\mathbf{R}}F_I^4 = B_{\mathbf{R}}\mathrm{tr}F_I^4 + C_{\mathbf{R}}\left(\mathrm{tr}F_I^2\right)^2\,, \tag{4.43}$$

where $B_{\mathbf{R}}$ is another group theoretical constant that will not be relevant to us for the rest of this discussion. In perturbative string theories, the cancellation of anomalies has been proved via modular properties of elliptic genera [127, 128]. For 6d perturbative Heterotic string theory settings, we have that: [127–130]

1. The anomaly coefficient $A^{(6)}$ is given by the coefficient $(\boldsymbol{z},\boldsymbol{z})^2_{\mathfrak{g}_I}$ of the elliptic genus at q^0, which we compute by

$$A^{(6)} = \frac{1}{4!}L^2_{\mathfrak{g}_I}Z(\tau,\boldsymbol{z})\Big|_{\boldsymbol{z}=0,q^0}\,,\quad \text{where}\quad L_{\mathfrak{g}_I} \equiv \frac{1}{\mathrm{rk}(\mathfrak{g}_I)}\mathfrak{C}^{i_I j_J}\partial_{z_i}\partial_{z_j}\,. \tag{4.44}$$

   Here $\mathfrak{C}^{i_I j_I}$ denotes the inverse of the coroot matrix associated to $\mathfrak{g}_I$. Morever, here we have that $\boldsymbol{z} \in L^\vee(\mathfrak{g}_I)\otimes\mathbb{C}$.

2. The pure gauge contribution to the Green-Schwarz counterterm $S_{\mathrm{GS}} = -\frac{1}{2}\int B\wedge X_4$

reads from

$$A^{(6)}_{\mathrm{GS}} \equiv X_4\Big|_{\mathrm{tr}F_I^2} = -\frac{1}{2!}\left[L_{\mathfrak{g}_I}\frac{1}{16\pi}\int_{\mathcal{F}} \mathrm{d}\tau \mathrm{e}^{\frac{\pi^2}{3}m(\boldsymbol{z},\boldsymbol{z})_{\mathfrak{g}_I}E_2(\tau)} Z(\tau,\boldsymbol{z})\right]_{\boldsymbol{z}=0} \\ = m^{-1}A^{(6)} . \tag{4.45}$$

Here m is the index of the elliptic genus $Z(\tau, \boldsymbol{z})$.

In [125], the authors revisited the calculation for the Green-Schwarz counter term factor $A^{(6)}_{\mathrm{GS}}$ for the case of a $U(1)$ gauge group. Nevertheless, the same calculation applies straightforwardly to our non-Abelian case $\mathfrak{g}_I$, leading to the result (4.45). Moreover, the authors [125] discussed extensions for cancellation of anomalies in non-perturbative Heterotic strings cases, where one has to consider an extension for the ring of Jacobi forms into a graded module over the ring of quasimodular forms. For simplicity, we keep the discussion to perturbative cases.

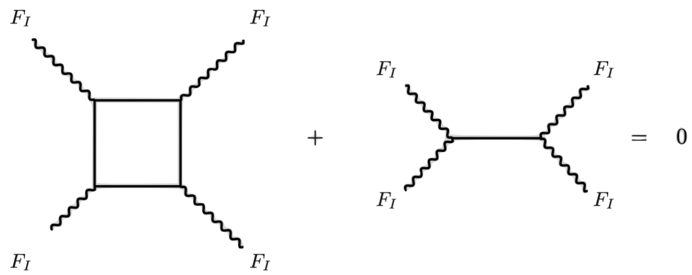


Figure 4.3: Green-Schwarz cancellation. The factor $A^{(6)}$ corresponds to the 1-loop anomaly on the left. The factor $A^{(6)}_{\mathrm{GS}}$ corresponds to the Green-Schwarz counterterm on the right.

Let us consider a 6d $N = (1,0)$ supergravity theory with non-Abelian gauge symmetry $\mathfrak{g}$, such that $\mathfrak{g}$ is a simple non-Abelian gauge algebra. We formulate the ansatz

$$Z(\tau,\boldsymbol{z})\Big|_{q^0} = 2(g-1)\chi_{\boldsymbol{\theta}}(\boldsymbol{z}) + \sum_{\boldsymbol{\lambda}} n_H(\mathbf{R}_{\boldsymbol{\lambda}})\chi_{\boldsymbol{\lambda}}(\boldsymbol{z}) - \chi(M) + 2\mathrm{rk}(\mathfrak{g}_I)(1-\mathfrak{g}_I) . \tag{4.46}$$

Here g is the genus of the gauge curve $\mathcal{S}^{\mathrm{b}}_{\mathfrak{g}} \subset B$ and $\boldsymbol{\theta}$ denotes the highest coroot of $\mathfrak{g}$, the highest weight of the adjoint reperesentation, i.e., $\mathbf{R}_{\boldsymbol{\theta}} = \mathbf{adj}(\mathfrak{g})$. We observe that our ansatz always matches with the anomaly factor (4.42) when performing the differential operation (4.44). It is non-trivial that the 4th order Taylor expansion series for (4.46) have a factorization of the type $(\boldsymbol{z},\boldsymbol{z})^2_{\mathfrak{g}}$. We interpret this fact as a consequence of the factorization (3.44) for the 1-loop anomaly polynomial I_8. The modularity of Jacobi forms implies that such a factorization always occurs, which is evident from the modular anomaly equation (3.46). In section 5.2 we will justify our ansatz (4.46) through Noether-Lefschetz theory. However, first, we regard a few examples and the connection of our ansatz (4.46) with the completeness hypothesis.

$\mathfrak{g} = A_n$ examples: We consider now the set of geometries described in [126], which posses a gauge symmetry $\mathfrak{g} = A_n$ with $1 \leq n \leq 4$. They are elliptic fibrations over the Hierzebruch surface base $B = \mathbb{F}_2$ and possess a perturbative Heterotic dual description [131]. We provide the massless spectrum data for these geometries and their Euler characteristic in Table 4.1.

n	masless spectrum	$C_{\mathbf{R}}$ values	$\chi(M)$
1	$282\mathbf{2} \oplus 282\overline{\mathbf{2}} \oplus 1911$	$C_{\mathbf{adj}(A_1)} = 8, \quad C_{\mathbf{2}} = 1/2$	-372
2	$303\mathbf{3} \oplus 303\overline{\mathbf{3}} \oplus 1621$	$C_{\mathbf{adj}(A_2)} = 9, \quad C_{\mathbf{3}} = 1/2$	-312
3	$4\mathbf{6} \oplus 24\mathbf{4} \oplus 24\overline{\mathbf{4}} \oplus 1391$	$C_{\mathbf{adj}(A_2)} = 6, \quad C_{\mathbf{4}} = 0, \quad C_{\mathbf{6}} = 3$	-264
4	$22\mathbf{5} \oplus 22\overline{\mathbf{5}} \oplus 4\mathbf{10} \oplus 4\overline{\mathbf{10}} \oplus 1181$	$C_{\mathbf{adj}(A_2)} = 6, \quad C_{\mathbf{5}} = 0, \quad C_{\mathbf{10}} = 3$	-220

Table 4.1: Massless spectrum data for $\mathfrak{g} = A_n$ geometries with $1 \leq n \leq 4$ [126].

The elliptic genera associated with the Hirzebruch fiber curve read, in each A_n case,

$$Z_{A_n}(\tau, \boldsymbol{z}) = \frac{1}{\eta^{24}(\tau)} \left(2E_4 E_6 \phi_{0,1}^{A_n} - \frac{7}{72} E_4^3 \phi_{-2,1}^{A_n} - \frac{5}{72} E_6^2 \phi_{-2,1}^{A_n} \right) . \tag{4.47}$$

Here the Weyl invariant Jacobi forms $\phi_{-2,1}^{A_n}$ and $\phi_{0,1}^{A_n}$ read from (B.31). On the one hand, we can use genus zero Gopakumar-Vafa data to fix expression (4.47). Conversely, the massless spectrum information suffices to determine the elliptic genera (4.47) through (4.46). By taking the differential operation (4.44) on Z_{A_n}, we obtain the Anomaly factor (4.42), which we can double-check by using data in Table 4.1. Note that the reduction map [132]

$$\phi_\bullet^{A_n} \xrightarrow{z_n \to 0} \phi_\bullet^{A_{n-1}} \xrightarrow{z_{n-1} \to 0} \cdots \xrightarrow{z_2 \to 0} \phi_\bullet^{A_1} , \tag{4.48}$$

where $\phi_\bullet^{A_n} \in \{\phi_{-2,1}^{A_n}, \phi_{0,1}^{A_n}\}$, implies that

$$Z_{A_4} \xrightarrow{z_4 \to 0} Z_{A_3} \xrightarrow{z_3 \to 0} Z_{A_2} \xrightarrow{z_2 \to 0} Z_{A_1} , \tag{4.49}$$

which corresponds to the geometric transitions described in [126].

The completeness hypothesis: We recall that the completeness hypothesis for a non-Abelian gauge theory with gauge symmetry $\mathfrak{g}$: all $\mathfrak{g}$-representations should occur in the spectrum. In what follows, we test this conjecture for the examples above with $\mathfrak{g} = A_n$. We note that such elliptic genera Z_{A_n} follow an expansion

$$\begin{aligned} Z_{A_n}(\tau, \boldsymbol{z}) = &\left(-2q^{-1} - \chi(M) + \mathcal{O}(q)\right) \vartheta_{L(A_n)}(\tau, \boldsymbol{z}) \\ &+ \sum_{\{\boldsymbol{\omega}_i\}} \left(n_H(\mathbf{R}_{\boldsymbol{\omega}_i}) + \mathcal{O}(q)\right) \vartheta_{[\boldsymbol{\omega}_i]}(\tau, \boldsymbol{z}) , \end{aligned} \tag{4.50}$$

where $\{\boldsymbol{\omega}_i\}_{i=1,\ldots,n}$ are the fundamental weights and we introduced the notation $[\boldsymbol{\omega}_i] \equiv \boldsymbol{\omega}_i + L(A_n)$. Here we decompose $L_{\mathrm{w}}(A_n) \simeq L(A_n) \oplus_{\{\boldsymbol{\omega}_i\}} [\boldsymbol{\omega}_i]$ and recall that $L_{\mathrm{w}}(A_n)/L(A_n) \simeq \mathbb{Z}_n$. We focus now on the term contribution

$$\begin{aligned} Z_{A_n}(\tau, \boldsymbol{z})\Big|_{\boldsymbol{\omega}_i} &= \left(n_H(\mathbf{R}_{\boldsymbol{\omega}_i}) + \mathcal{O}(q)\right) \vartheta_{[\boldsymbol{\omega}_i]}(\tau, \boldsymbol{z}) \\ &= \left(n_H(\mathbf{R}_{\boldsymbol{\omega}_i}) + \mathcal{O}(q)\right) \sum_{w \in W(A_n)} \sum_{\substack{w \cdot \boldsymbol{\lambda} \in [\boldsymbol{\omega}_i] \\ \boldsymbol{\lambda} \in P_+(\mathfrak{g})}} \mathrm{sign}(w) q^{\frac{1}{2m}(w \cdot \boldsymbol{\lambda}, w \cdot \boldsymbol{\lambda})} \chi_{\boldsymbol{\lambda}}(\boldsymbol{z}) \\ &= n_H(\mathbf{R}_{\boldsymbol{\omega}_i}) \sum_{\boldsymbol{\lambda} \in P_+(A_n) \cap [\boldsymbol{\omega}_i]} q^{\frac{(\boldsymbol{\lambda}, \boldsymbol{\lambda})}{2}} \chi_{\boldsymbol{\lambda}}(\boldsymbol{z}) + \cdots \end{aligned} . \tag{4.51}$$

Here we used the fact that $W(A_n) \cdot [\boldsymbol{\omega}_i] = [\boldsymbol{\omega}_i]$, which allows us to use **Claim I**; the argument presented in equations (4.25) and (4.26) also holds for the term $Z_{A_n}|_{\boldsymbol{\omega}_i}$, leading to a non-trivial set of terms indicated in the last line of (4.51). In these cases, we have that $w \cdot \boldsymbol{\lambda} \in [\boldsymbol{\omega}_i]$ for every $\boldsymbol{\lambda} \in [\boldsymbol{\omega}_i] \cap P_+(A_n)$. Thus, the counting over representations of $Z_{A_n}|_{\boldsymbol{\omega}_i}$ and $Z_{A_n}|_{\boldsymbol{\omega}_j}$ does not mix, where $\boldsymbol{\omega}_i \neq \boldsymbol{\omega}_j$. Taking all $Z_{A_n}|_{\boldsymbol{\omega}_i}$, together with the non-trivial contribution due to the nAsLWGC, we find a non-trivial counting for every A_n-representation. Thus, the spectrum for these A_n theories is complete.

CHAPTER 5

Noether-Lefschetz theory and F-/M-theory compactifications

The previous chapter discussed F-theory compactifications on elliptically fibered Calabi-Yau 3-folds admitting a $K3$ fibration, consequently theories with a Heterotic dual description. This chapter aims to deepen the enumerative geometry reduced on $K3$ fibers since it corresponds with Noether-Lefschetz theory [133–136], and we want to realize this connection in terms of physical objects. However, these geometrical configurations can have different physics realizations, each of them having a distinct Heterotic dual interpretation. To guide the reader among the dualities we implicitly use in the discussion, we provide a diagram in Table 5.1. Depending on the context, a given type of theory is more convenient, and on occasions, we interchange them.

$$
\begin{tikzcd}
\textbf{6d } \boldsymbol{N=(1,0)} \textbf{ theories:} & \text{Heterotic on } K3 \arrow[r, leftrightarrow] \arrow[d, "S^1"] & \text{F-theory on } M \arrow[d, "S^1"] \\
\textbf{5d } \boldsymbol{\mathcal{N}=2} \textbf{ theories:} & \text{Heterotic on } (K3\times S^1)/\mathbb{Z}_N \arrow[r, leftrightarrow] \arrow[d, "S^1"] & \text{M-theory on } M \arrow[d, "S^1"] \\
\textbf{4d } \boldsymbol{\mathcal{N}=2} \textbf{ theories:} & \text{Heterotic on } (K3\times T^2)/\mathbb{Z}_N \arrow[r, leftrightarrow] & \text{Type IIA on } M
\end{tikzcd}
$$

Table 5.1: Web of dualities for theories with eight supercharges. Cases with a CHL orbifolding by a $\mathbb{Z}_N$ action on the Heterotic side, with $N > 1$, corresponds to a Type IIA/M-theory dual on a genus one fibration M.

More generally, we also consider genus one fibered Calabi-Yau 3-folds that admit a $K3$ fibration. A class of such cases lead to CHL-Heterotic orbifolds that we discuss in Section 5.3. Before that, we keep the discussion to elliptically fibered Calabi-Yau 3-folds and for definitiveness with base $B = \mathbb{F}_k$, i.e., a Hirzebruch surface with $k \in \mathbb{N}$. Having said this, we start by discussing the duality among 4d $\mathcal{N} = 2$ theories.

5.1 Heterotic/Type IIA duality

Conjectural Heterotic/Type IIA dual pairs in 4d with $\mathcal{N} = 2$ supersymmetry involve the Heterotic string on $K3 \times T^2$ and the Type IIA string on a $K3$ fibered Calabi-Yau 3-fold M. The gravitational coupling (2.112) can be calculated at least with partial moduli dependence on both sides and serve as an important check of the duality. In the perturbative regime of the Heterotic string $g_{\text{het}} \to 0$, all F_g can be calculated by a BPS saturated 1-loop amplitude, which depends in general on all vector multiplet moduli—excluding the photograviton modulus—and the Heterotic dilaton

$$S_{\text{het}} = \frac{4\pi}{g_{\text{het}}^2} + i\theta \,. \tag{5.1}$$

Typical moduli from Abelian Heterotic vector multiplets are the Kähler structure T_{het} and the complex structure U_{het} of the two torus T^2. Depending on the gauge bundle configurations on the Heterotic side, one can have up to 15 more perturbative Abelian vector multiplets $\boldsymbol{V} = (V_1, \ldots, V_r)$ from the unbroken gauge group (we consider mainly the $E_8 \times E_8$ version) $G \subset E_8 \times E_8$, where $r = \text{rk}(G)$, whose moduli correspond to Wilson lines along the cycles of T^2. The Heterotic 1-loop computation involves an integration of the worldsheet complex structure over the fundamental domain that can be solved by the unfolding trick [137], which is more systematically implemented by the lattice reduction method of Borcherds [138, 139]. Examples of these calculations can be found in [139–142], as well as recent extensions to CHL-Heterotic orbifolds on $(K3 \times T^2)/\mathbb{Z}_N$ [143–145]. In the perturbative Heterotic limit, the Borcherds lift calculation implies that $F_g(\boldsymbol{t}, \bar{\boldsymbol{t}})$ are automorphic forms under the T-duality group $SO(2+r, 2; \mathbb{Z})$. The latter group acts on the combined moduli space

$$\mathcal{M}_{T,U,\boldsymbol{V}} = \frac{SO(2, 2+r)}{SO(2) \times SO(2+r)} \Big/ SO(2+r, 2; \mathbb{Z}) \,, \tag{5.2}$$

which is spanned by the moduli $(T, U, \boldsymbol{V})$ [146, 147].

A strong non-trivial check of Heterotic $\leftrightarrow$ Type II duality, with respective compactifications $K3 \times T^2 \leftrightarrow M$ in which M is $K3$ fibered, is comparing the Heterotic 1-loop amplitude with the topological free energy (2.116). This is done by choosing an appropriate basis of Kähler moduli in M, such that $\boldsymbol{t} = (S_{\text{het}}, T_{\text{het}}, U_{\text{het}}, \boldsymbol{V})$. In particular, we identify $S_{\text{het}} = \text{Vol}_\omega\left(\mathbb{P}^1_{\text{b}}\right)$, where $\mathbb{P}^1_{\text{b}}$ is the base of the $K3$ fibration. Thus, in the holomorphic limit, the assertion of Heterotic-Type II duality reduces in the perturbative regime to

$$\lim_{S_{\text{het}} \to \infty} F(\lambda, S_{\text{het}}, T_{\text{het}}, U_{\text{het}}, \boldsymbol{V}) = F^{\text{1-loop}}_{\text{het}}(\lambda, T_{\text{het}}, U_{\text{het}}, \boldsymbol{V}) \,. \tag{5.3}$$

The Heterotic 1-loop calculation from the worldsheet point has been first described in [148] and concretely evaluated for the STU-model [149] in [139]. The topological string zero sector for the corresponding STU Calabi-Yau 3-fold M, the standard elliptic fibration over the Hirzebruch surface $\mathbb{F}_2$ realised as resolved hypersurface of degree 24 in the weighted projective space $\mathbb{P}^4(1, 1, 2, 8, 12)$, has been solved by mirror symmetry in [28].

In the next section we argue that a simple calculation in mirror symmetry provides the necessary information to reconstruct the automorphic forms for $SO(2, 2+r; \mathbb{Z})$ in (5.3). There, the computations rely purely on the geometrical information provided by the $K3$ fibrations or the elliptic genera of 6d $N = (1, 0)$ strings, when M is elliptically fibered.

5.2 Elliptic genera and Noether-Lefschetz theory

In the case that a $K3$-fibered Calabi-Yau 3-fold admits an elliptic fibration, there is also a correspondence between Noether-Lefschetz theory and lattice Jacobi forms defined by the Λ-polarized lattice of $K3$ fibrations. To make this assertion concrete, we will consider $K3$

fibrations with polarization lattice Λ of the form

$$\Lambda = U \oplus L_1(-m_1) \oplus \cdots \oplus L_{\tilde{N}}(-m_{\tilde{N}})\,, \tag{5.4}$$

where U is the $(1,1)$-hyperbolic lattice $U \simeq \mathbb{Z}e \oplus \mathbb{Z}f$ with quadratic form defined by $e^2 = f^2 = 0$ and $(e,f)_U = 1$; each lattice $L_I(-m_I)$ is the coroot lattice of a simple Lie algebra $\mathfrak{g}_I$ with a twist determined by a number $m_I \in \mathbb{N}$. [1] Having said this, our objective here is to establish the connection between the 6d Heterotic strings elliptic genera with lattice index $L \equiv \oplus_I L_I(m_I)$ (or twists of L) and Noether-Lefschetz theory. After all, the former objects encode the Gromov-Witten invariants of $K3$ fibers.

To obtain the Λ lattice (5.4) we first regard Calabi-Yau 3-folds M that admit an elliptic fibration

$$\begin{tikzcd} \mathcal{E} \arrow[r, hook] & M \arrow[d, "\pi"] \\ & \mathbb{F}_k \end{tikzcd}\,, \tag{5.5}$$

which develops Kodaira singularities over curves in the base given by the Hirzebruch surface $B = \mathbb{F}_k$. Recall that the latter is a $\mathbb{P}^1$-fibration $\mathrm{p} : \mathbb{F}_k \to \mathbb{P}^1_{\mathrm{b}}$ such that the fiber $F \cong \mathbb{P}^1$ has vanishing self-intersection, while $\mathbb{P}^1_{\mathrm{b}}$ has self-intersection $-k$. It is the property $F^2 = 0$ that enables us to find an elliptic $K3$ surface S with fibration structure

$$\begin{tikzcd} \mathcal{E} \arrow[r, hook] & S \arrow[d, "\varpi"] \\ & F \end{tikzcd}\,. \tag{5.6}$$

This way the Calabi-Yau 3-fold M has also a $K3$ fibration $\pi' : M \to \mathbb{P}^1_{\mathrm{b}}$ with fibers given by S in (5.6). With this construction in mind, let us comment now on the appearance of the Λ-polarized lattice (5.4).

To discuss the Noether-Lefschetz theory for the lattice configuration (5.4), it is convenient to choose the basis of curves on the $K3$ fiber S. For this, we recall that a $K3$ surface admits an elliptic fibration whenever there is an embedding $U \hookrightarrow \mathrm{Pic}(S)$ [150]. At the $K3$ level, we have that $F \cdot_{K3} \mathcal{E} = 1$, $\mathcal{E} \cdot_{K3} \mathcal{E} = 0$, and $F \cdot_{K3} F = -2$, where we denote by $\cdot_{K3}$ the intersections on the $K3$ fiber. Taking curves $C_U \equiv F + \mathcal{E}$ and $C_T \equiv \mathcal{E}$ as a basis for the U lattice we get an off-diagonal intersection form with $C_U \cdot_{K3} C_T = 1$ and $C_U \cdot_{K3} C_U = C_T \cdot_{K3} C_T = 0$. All other elements of $\mathrm{Pic}(S)$ are negative self-intersection curves C_{ext}, such that $C_{\mathrm{ext}} \cdot \mathcal{E} = C_{\mathrm{ext}} \cdot F = 0$. See the transversality conditions (3.17). Thus, in an F-theoretic construction, we identify the latter type of curves as fibral divisors $\{E_{i_I}\}_{i_I=1,\ldots,\mathfrak{g}_I;I=1,\ldots,a}$ in $H^{1,1}(M,\mathbb{Z})$ restricted to the $K3$ fiber S. This way, all divisors in M, excluding the one that determines $S_{\mathrm{het}} = \mathrm{Vol}_\omega(\mathbb{P}^1_{\mathrm{b}})$, span the Λ-polarized lattice (5.4). In particular, we identify the bilinear form for the $\mathfrak{g}_I$-coroot lattice $L_I(-m_I)$ via the intersections

$$E_{i_I} \cdot E_{i_J} \cdot \pi^*(F) = -m_I \mathfrak{C}_{i_I j_J} \delta_{IJ}\,, \quad \text{with} \quad m_I = F \cdot_B \mathcal{S}^{\mathrm{b}}_{\mathfrak{g}_I}\,. \tag{5.7}$$

[1] Here the twist $\mathrm{L}(m)$ of a lattice L is obtained by changing the intersection $(\cdot\,,\cdot)$ of L by $m \in \mathbb{Z}$ as $(\cdot\,,\cdot)_{\mathrm{L}(m)} \equiv m(\cdot\,,\cdot)$.

When we project the degree of a curve $\varphi \in H_2(M,\mathbb{Z})^\pi$, it decomposes as

$$\varphi = \ell C_U + nC_T + \sum_{I=1}^{a}\sum_{i_I=1}^{\mathrm{rk}(\mathfrak{g}_I)} \lambda_I^{i_I} C_{i_I} \mapsto (\ell, n, \boldsymbol{\lambda}_1, \ldots, \boldsymbol{\lambda}_a)\ , \tag{5.8}$$

where $\{C_{i_I}\}$ is the set of curves dual to the fibral divisors $\{E_{i_I}\}$ that follow the intersection relations

$$C_{i_I}\cdot C_{i_J} = -m_I^{-1}\left(\boldsymbol{\omega}_{i_I}, \boldsymbol{\omega}_{j_J}\right)\delta_{IJ}\ , \tag{5.9}$$

Here the vectors $\{\boldsymbol{\omega}_{i_I}\}$ are fundamental weights of $\mathfrak{g}_I$, which are dual to $\{\boldsymbol{\alpha}_{i_I}^\vee\}$, the simple coroots of $\mathfrak{g}_I$. Hence, we identify $\boldsymbol{\lambda}_I$ with an element in the weight lattice $L_{\mathrm{w}}(\mathfrak{g}_I)$ with twist m_I^{-1}. For shortness, we will introduce the notation $\boldsymbol{\lambda} \equiv (\boldsymbol{\lambda}_1, \ldots, \boldsymbol{\lambda}_a)$.

Multi-wrapping of 6d $\boldsymbol{N = (1,0)}$ Heterotic strings: Let us discuss how topological string theory encodes the enumerative geometry of $K3$ fibrations. For instance, the genus zero reduced topological string free energy—when $\mathrm{Vol}_\omega(\mathbb{P}^1_{\mathrm{b}}) \to \infty$—reads

$$F_0(p,q,\boldsymbol{\zeta})\Big|_{\mathrm{Het.}} = F_0(u,\tau,\boldsymbol{z})\Big|_{\mathrm{class.}} + \sum_{(\ell,n,\boldsymbol{\lambda})>0} n^0_{(\ell,n,\boldsymbol{\lambda})}\mathrm{Li}_3(p^\ell q^n \boldsymbol{\zeta}^{\boldsymbol{\lambda}})\ , \tag{5.10}$$

where $n^0_{(\ell,n,\boldsymbol{\lambda})}$ is a genus zero Gopakumar-Vafa invariant associated with a class $(\ell,n,\boldsymbol{\lambda})$ of positive degree. Here we chose a basis such that $(\ell,n)\in U$ and $\boldsymbol{\lambda}\in L^*$, with

$$\mathrm{Vol}_\omega(C_U) = u\ ,\quad \mathrm{Vol}_\omega(\mathcal{E}) = \tau\ ,\quad \mathrm{Vol}_\omega(C_{i_I}) = z_{i_I}\ , \tag{5.11}$$

and

$$p = \mathrm{e}^{2\pi i u}\ ,\quad q = \mathrm{e}^{2\pi i\tau}\ ,\quad \boldsymbol{\zeta}^{\boldsymbol{\lambda}} = \prod_{I=1}^{a}\prod_{i_I=1}^{\mathrm{rk}(\mathfrak{g}_I)} \exp\bigl\{\bigl(2\pi i\lambda^{i_I} z_{i_I}\bigr)\bigr\}. \tag{5.12}$$

When we consider the topological string partition function, the lowest entry giving rise to an elliptic genus for a Heterotic string reads

$$Z_F(\tau,\boldsymbol{z},\lambda) = \frac{\Phi(\tau,\boldsymbol{z})}{\eta^{24}(\tau)\phi_{-2,1}(\tau,\lambda)} = \sum_{g=0}^{\infty}(2\pi i\lambda)^{2g-2}\varphi_{2g-2}(\tau,\boldsymbol{z})\ , \tag{5.13}$$

where the first equality represents the modular bootstrap due to the curve $F\subset\mathbb{F}_n$. On the other hand, the second equality is merely its Taylor expansion over the topological string coupling λ. Thus, the essential information we need to compute this object is the holomorphic Jacobi form $\Phi\in J_{10,L}$, which furnishes the meromorphic quasi-Jacobi forms φ_{2g-2} of weight $2g-2$. It turns out that Hecke operator lifts determine the genus g reduced topological free energies, as [151]

$$F_g(u,\tau,\boldsymbol{z})\Big|_{\mathrm{Het.}} = \sum_{\ell=0}^{\infty} p^\ell \varphi_{2g-2}\Big|_{V_\ell}(\tau,\boldsymbol{z})\ . \tag{5.14}$$

where the Hecke operator V_ℓ action on φ_{2g-2} for $\ell \in \mathbb{N}$ is defined as

$$\varphi_{2g-2}\Big|_{V_\ell}(\tau, \boldsymbol{z}) = \ell^{k-1} \sum_{\substack{ad=\ell \\ a>0}} \sum_{b=0}^{d-1} d^{-k} \phi\left(\frac{a\tau + b}{d}, a\boldsymbol{z}\right) . \tag{5.15}$$

Physically, the latter object gives a refined counting over 6d Heterotic strings, which realize in the F-theory picture from D3-branes wrapping $\ell F \subset \mathbb{F}_n$ with $\ell > 1$ [63, 64, 152]. Note that the all genera sum over (5.14) reproduces the 1-loop Heterotic amplitude (5.3). In the following, we relate the Jacobi form generator Φ with Noether-Lefschetz theory.

Noether-Lefschetz theory and VVMF: Given our Λ-polarized lattice configuration (5.4), we can compute Noether-Lefschetz numbers for these geometries through the coefficients of a vector-valued modular form

$$\Phi^\pi(\tau) = \sum_{\boldsymbol{\mu} \in \Lambda^*/\Lambda} \Phi^\pi_{\boldsymbol{\mu}}(\tau) \mathrm{e}_{\boldsymbol{\mu}} \in M_{11-\mathrm{rk}(\Lambda)/2}(\rho^*_\Lambda) . \tag{5.16}$$

Here the Noether-Lefschetz numbers $NL^\pi_{h,(n,l,\boldsymbol{\lambda})}$ read from

$$NL^\pi_{h,(n,l,\boldsymbol{\lambda})} = \mathrm{Coeff}\left(\Phi^\pi_{\boldsymbol{\mu}}, q^{\Delta_{NL}}\right) , \tag{5.17}$$

where $\boldsymbol{\mu} = [(\ell, n, \boldsymbol{\lambda})] \in \Lambda^*/\Lambda$. For more details on Noether-Lefschetz numbers, see Appendix C.1. Using the lattice data (5.4) with (C.8), we obtain

$$\Delta_{NL}(h, \ell, n, \boldsymbol{\lambda}) = n\ell - \sum_{I=1}^{a} \frac{1}{2m_I} (\boldsymbol{\lambda}_I, \boldsymbol{\lambda}_I)_{\mathfrak{g}_I} + 1 - h . \tag{5.18}$$

As a reminder, we recall the GW-NL correspondence theorem, which states that

$$n^g_{(\ell,n,\boldsymbol{\lambda})}(\Delta_J) = \sum_{h=0}^{\infty} r^g_h \cdot NL^\pi_{h,(\ell,n,\boldsymbol{\lambda})} . \tag{5.19}$$

Here $n^g_{(\ell,n,\boldsymbol{\lambda})}$ is the Gopakumar-Vafa invariant associated with the curve of positive degree $(\ell, n, \boldsymbol{\lambda})$ and r^g_h the coefficients of the KKV formula (C.15). Thus, the lattice L Jacobi form discriminant $\Delta_J(\ell n, \boldsymbol{\lambda}) \equiv \Delta_{NL}(h, \ell, n, \boldsymbol{\lambda}) - 1 + h$ and the degree class $[(\ell, n, \boldsymbol{\lambda})] \in \Lambda^*/\Lambda$ uniquely determine the numbers (5.19). In particular, for elliptically fibered cases, we have that $\Lambda^*/\Lambda \simeq L^*/L$. With this observation in mind, we can relate the Noether-Lefschetz vector-valued modular form with the elliptic genera generator (5.13). For this, we consider the genus zero contributions to (5.13) and its expansion in terms of vector valued-modular forms that reads:

$$
\begin{aligned}
Z_F(\tau,\boldsymbol{z},\lambda)\Big|_{\lambda^{-2}} &= \frac{\Phi(\tau,\boldsymbol{z})}{\eta^{24}(\tau)} \\
&= \left(\sum_{h=0}^{\infty} r_h^0 q^{h-1}\right)\left(\sum_{n=0}^{\infty}\sum_{\substack{\boldsymbol{\lambda}\in L^*\\ \Delta_J\geq 0}} c(n,\boldsymbol{\lambda}) q^n \boldsymbol{\zeta}^{\boldsymbol{\lambda}}\right) \\
&= \left(\sum_{h=0}^{\infty} r_h^0 q^{h-1}\right)\left(\sum_{\boldsymbol{\mu}\in L^*/L}\sum_{\substack{\Delta_J\in\mathbb{Z}-(\boldsymbol{\mu},\boldsymbol{\mu})\\ \Delta_J\geq 0}} c_{\boldsymbol{\mu}}(\Delta_J) q^{\Delta_J}\vartheta_{L,\boldsymbol{\mu}}(\tau,\boldsymbol{z})\right) \\
&= \sum_{\boldsymbol{\mu}\in L^*/L}\left(\sum_{h=0}^{\infty}\sum_{\substack{\Delta_{NL}\in\mathbb{Z}-(\boldsymbol{\mu},\boldsymbol{\mu})\\ \Delta_{NL}\geq 0}} r_h^0\cdot c_{\boldsymbol{\mu}}(\Delta_J+1-h) q^{\Delta_J}\right)\vartheta_{L,\boldsymbol{\mu}}(\tau,\boldsymbol{z}) \\
&= \sum_{\boldsymbol{\mu}\in L^*/L}\left[\sum_{\substack{\Delta_J\in\mathbb{Z}-(\boldsymbol{\mu},\boldsymbol{\mu})\\ \Delta_J\geq -1}}\left(\sum_{h=0}^{\infty} r_h^0 c_{\boldsymbol{\mu}}(\Delta_{NL})\right) q^{\Delta_J}\right]\vartheta_{L,\boldsymbol{\mu}}(\tau,\boldsymbol{z})
\end{aligned}
\tag{5.20}
$$

Here we introduced in the first line the Fourier expansion for the holomorphic Jacobi form $\Phi(\tau,\boldsymbol{z})$, where $c(n,\boldsymbol{\lambda}) = c_{\boldsymbol{\mu}}(\Delta_J)$ are its Fourier coefficients that depend on a Δ_J-value and a class $\boldsymbol{\mu}\in L^*/L$. Moreover, $\vartheta_{L,\boldsymbol{\mu}}(\tau,\boldsymbol{z})$ are the theta functions introduced in (B.24). The crucial point is that, in the last term, the factor under parenthesis furnishes $n^0_{(1,n,\boldsymbol{\lambda})}(\Delta_J)$. Thus, we find the equivalence between Jacobi forms and vector-valued modular forms

$$
h: J_{10,L}\to M_{10-\mathrm{rk}(L)/2}(\rho_L^*)\,,\quad h:\Phi(\tau,\boldsymbol{z})\mapsto\Phi^{\pi}(\tau)=\sum_{\boldsymbol{\mu}\in L^*/L} h_{\boldsymbol{\mu}}(\tau)\mathrm{e}_{\boldsymbol{\mu}}\,,
\tag{5.21}
$$

where $\Phi(\tau,\boldsymbol{z})=\sum_{\boldsymbol{\mu}\in L^*/L} h_{\boldsymbol{\mu}}(\tau)\vartheta_{L,\boldsymbol{\mu}}$. Note that the h-map projection yields an isomorphism between modules. We discuss now three applications that we can exploit through this correspondence.

GV-spectroscopy and elliptic genera: With the established connection between Noether-Lefschetz theory and elliptic genera, we argue that the latter modular objects encode the massless spectrum information of their corresponding 6d $N=(1,0)$ theory. To see this, let us point out the works [46, 88, 153], where the authors realize the massless matter content of these theories by considering the Gopakumar-Vafa invariants at base degree zero. By this we mean fibral curves C_f that satisfy $C_f\cdot\pi^{-1}(C_{\mathrm{b}})$ for any $C_{\mathrm{b}}\in H_2(B)$. Namely, there are three types of fibral curves furnishing such invariants: [46, 88, 153]

1. Isolated fibral curves that occur over points $\mathcal{S}^{\mathrm{b}}_{\mathfrak{g}_I}\cap\mathcal{S}^{\mathrm{b}}_{\mathfrak{g}_J}\subset B$. M2-branes wrapping these curves give rise to charged hypermultiplets that transform under a representation $\mathbf{R}$ of $\mathfrak{g}_I\oplus\mathfrak{g}_J$. The Gopakumar-Vafa invariant of an isolated curve C_1 is the number of curves within the same class. Thus, we have that $n^0_{C_1}=n_H(\mathbf{R})$.

2. A fibral curve component C_2 of the generic fiber over a curve $\mathcal{S}^{\mathrm{b}}_{\mathfrak{g}_I}\subset B$ of genus g_I. Such a curve gives $n^0_{C_2}=-\chi(\mathcal{S}^{\mathrm{b}}_{\mathfrak{g}_I})=2(g_I-1)$.

3. A generic fiber curve, i.e. the elliptic fiber $C_3 = \mathcal{E}$, yields $n^0_{C_3} = -\chi(M)$.

As observed in [2, 46, 88], all fibral curves follow the periodicity $n^0_{C_f} = n^0_{C_f+\mathcal{E}}$. In the $K3$ fibrations we consider, such a property follows from Noether-Lefschetz symmetry, equivalently Jacobi forms. To see this, let us note that fibral curves give rise to the prepotential contribution

$$\sum_{(0,n,\boldsymbol{\lambda})>0} n^0_{(0,n,\boldsymbol{\lambda})} \mathrm{Li}_3(q^n \boldsymbol{\zeta}^{\boldsymbol{\lambda}}) \subset F_{g=0}(\tau, u, \boldsymbol{z})\Big|_{\mathrm{Het.}} . \tag{5.22}$$

Then, by inspecting equation (5.19), we have that $n^0_{(0,n,\boldsymbol{\lambda})} = n^0_{(0,m,\boldsymbol{\lambda})}$ as $\Delta_J(0{\cdot}n, \boldsymbol{\lambda}) = \Delta_J(0{\cdot}m, \boldsymbol{\lambda})$, where $n, m \geq 0$. We can exploit this symmetry further and note that $n^0_{(0,1,\boldsymbol{\lambda})} = n^0_{(1,0,\boldsymbol{\lambda})}$. As the coefficients of the elliptic genus $Z(\tau, \boldsymbol{z}) \equiv Z_F(\tau, \boldsymbol{z}, \lambda)|_{\lambda^{-2}}$[2] determine these numbers, this observation implies that

$$\begin{aligned} Z(\tau, \boldsymbol{z})\Big|_{q^0} = 2\sum_{I=1}^{a} (g_I - 1)\chi_{\boldsymbol{\theta}_I}(\boldsymbol{z}) + \sum_{I \geq J} \sum_{\boldsymbol{\lambda} \in P_+(\mathfrak{g}_I \oplus \mathfrak{g}_J)} n_H(\mathbf{R}_{\boldsymbol{\lambda}}) \chi_{\boldsymbol{\lambda}}(\boldsymbol{z}) \\ - \chi(M) + 2\sum_{I=1}^{a} \mathrm{rk}(\mathfrak{g}_I)(1 - g_I) + \mathrm{const}\,. \end{aligned} \tag{5.23}$$

Here $\boldsymbol{\theta}_I$ is the highest coroot associated with the gauge algebra $\mathfrak{g}_I$ determined by each curve $\mathcal{S}^{\mathrm{b}}_{\mathfrak{g}_I}$. Note that the Weyl characters $\chi_{\boldsymbol{\lambda}}(\boldsymbol{z})$ carry the correct gauge charges due to M2-branes wrapping fibral curves, which we discussed in (3.23) and (3.24). Moreover, we introduced a few constants in the last line of (5.23) to compensate for the neutral hypermultiplets required to complete each adjoint representation. An additional constant term is necessary for correcting zero weights terms when roots and weights coincide [153]. For an ADE semisimple Lie algebra $\mathfrak{g} = \oplus_I \mathfrak{g}_I$, this is simply const $= 0$.

This result provides several constraints, which suffice to fix the elliptic genus in all examples discussed in this work. See also those discussed in [3]. Moreover, the modular properties of the elliptic genus enforce the cancellation of anomalies, as discussed in the references [127–130] and exemplified in section 4.5. With this observation in mind, let us discuss our next application.

Refined P/NL correspondence: If the constraints (5.23) suffice, we can use the GV-spectroscopy data to fix the Jacobi form coefficients determining the elliptic genus $Z(\tau, \boldsymbol{z})$. Then, we perform the h-map projection to obtain the Noether-Lefschetz generator Φ^π. This way, we find an interesting method to compute conjectural refined BPS invariants [136]—in compact geometries—by taking only data determining a 6d $N = (1, 0)$ supergravity theory [77]. We include in Appendix C.2 the proposal of [136] to refine the counting function for BPS states in M-theory compactifications on $K3$ fibered Calabi-Yau 3-folds by using Noether-Lefschetz theory in the $K3$ fiber. To our knowledge all examples for these proposed refined BPS invariants are consistent. See for instance the examples in [3, 136, 154]. For completeness, in Appendix C.3, we include the refined BPS invariants for the geometries discussed in 4.5 by using the proposal of [136]. Using the same method of computation to obatin Φ^π, we have yet another application that we discuss next. Lastly, using the same computation method to obtain Φ^π, we discuss another application next.

[2] This is precisely the object we considered in (4.17).

4d $\mathcal{N}=2$ supersymmetric index: Let us finally relate the Noether-Lefshetz generators to the new supersymmetric index [155]

$$\mathcal{Z}(\tau,\bar{\tau}) = \frac{1}{\eta^2(\tau)}\mathrm{Tr}_R(-1)^F F q^{L_0-\frac{c}{24}}\bar{q}^{\bar{L}_0-\frac{\bar{c}}{24}} \ . \tag{5.24}$$

which corresponds to the BPS index of a 4d $\mathcal{N}=2$ dual Heterotic theory on $K3\times T^2$ [156]. In [142] it was argued that

$$\mathcal{Z}(\tau,\bar{\tau}) = \sum_{\gamma\in G_\Lambda}\theta_{\Lambda'(-1)+\gamma}(\tau,\bar{\tau})\frac{\Phi^{\pi}_{\gamma}(\tau)}{\eta^{24}(\tau)}\,, \tag{5.25}$$

where $\theta_{\Lambda'(-1)+\gamma}$ are the vector components of the Siegel theta function (B.48) for the extended polarization lattice $\Lambda' = U(-1)\oplus\Lambda$, where $U(-1)\cong H^0(S,\mathbb{Z})\oplus H^4(S,\mathbb{Z})$. In our configuration (5.4) the Siegel theta function follows the form

$$\theta_{\Lambda'(-1)+\boldsymbol{\mu}}(\tau,\bar{\tau}) = \sum_{\boldsymbol{\lambda}_1\in L_1(m_1)+\boldsymbol{\mu}_1}\cdots\sum_{\boldsymbol{\lambda}_a\in L_a(m_a)+\boldsymbol{\mu}_a}\ \sum_{(k_0,w_0)\in U}\ \sum_{(k,w)\in U(-1)} q^{\frac{p_L^2(\boldsymbol{v})}{2}}\bar{q}^{\frac{p_R^2(\boldsymbol{v})}{2}}\,, \tag{5.26}$$

where $\boldsymbol{v} = (\boldsymbol{\lambda}_1,\dots\boldsymbol{\lambda}_a;k_0,w_0;k,w)$. Replacing $\mathrm{e}_{\boldsymbol{\mu}}\mapsto\theta_{\Lambda'(-1)+\boldsymbol{\mu}}(\tau,\bar{\tau})$ in (C.12) maps the Noether-Lefschetz generator into the new supersymmetric index of the Heterotic string (5.24). Note that in both (5.26) and (5.24) we have suppressed the dependence on the four-dimensional vector moduli $T_{\text{het}}, U_{\text{het}}, \boldsymbol{V}$, which are encoded in the left-right moving momenta $p_{L/R}$. We will discuss the generalization of this map for $K3$ fibrations that are not elliptic but only exhibit a genus one fibration in Section 5.4.

5.3 Sublattice conjectures for M-theory on genus one fibrations

If we compactify on a circle to five dimensions, the excitations of the wrapped strings turn into BPS particles and the same argument about the sublattice holds. However, we also have the freedom to perform an orbifolding on the Heterotic side of the duality and consider compactifications on $(K3\times S^1)/\mathbb{Z}_N$. These are dual to M-theory compactified on genus one fibered Calabi-Yau 3-folds that do not have a section but only N-sections [2, 144, 157]. In this situation the elliptic genus of the Heterotic strings is replaced by the twisted elliptic genera and the arguments for the sublattice conjectures have to be modified.

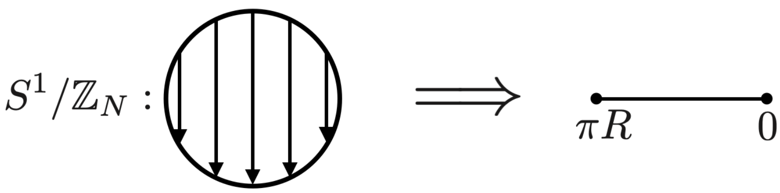


Figure 5.1: Schematic representation of the CHL circle orbifolding due to the δ-shift action (5.27). In this picture we have that $N=2$.

As a starting point, let us consider the 5d Heterotic strings on $(K3\times S^1)/\mathbb{Z}_N$ with no

Wilson lines turned on. The $\mathbb{Z}_N$ orbifold group acts on the circle via the shift

$$\delta : x \mapsto x + \frac{2\pi}{N} R \,, \tag{5.27}$$

and on the $K3$ via some non-trivial order N automorphism. We choose the momentum and winding numbers (k, w) along the circle to be quantized in $\mathbb{Z} \oplus \mathbb{Z}$ such that the left- and right-moving momenta satisfy

$$(p_L, p_R) = \frac{1}{\sqrt{2}} \left(\frac{k}{R} - \frac{wR}{N}, \frac{k}{R} + \frac{wR}{N} \right) \,, \tag{5.28}$$

With this in mind, the momentum lattice $\Gamma_N^{1,1}$ is equipped with the quadratic form

$$p_L^2 - p_R^2 = \frac{2kw}{N} \,. \tag{5.29}$$

Note that a state with momentum number k will have eigenvalue $\exp(2\pi i k/N)$ under the action of δ. Moreover, with $\psi_{r,k}$ we denote a state in the r-twisted sector on $K3/\mathbb{Z}_N$ with eigenvalue $g = \exp(-2\pi i k/N)$, where g acts as a $\mathbb{Z}_N$ automorphism on the CFT associated with the $K3$ [158]. Together the invariant states in the spectrum of the compactification on $(K3 \times S^1)/\mathbb{Z}_N$ are then of the form [159, 160]

$$|k, w, \psi_{r,k}\rangle \,, \quad r \equiv w \bmod N \,. \tag{5.30}$$

Now we turn back to the M-theory compactification on a genus one fibered Calabi-Yau 3-fold $\pi_N : M \to B$ with N-section. We want the Calabi-Yau to also exhibit a $K3$ fibration $\pi : M \to \mathbb{P}^1_{\mathrm{b}}$ such that in the limit of large $\mathbb{P}^1_{\mathrm{b}}$ we have a dual description in terms of weakly coupled Heterotic strings. Recall that a $K3$ fiber with polarization lattice Λ of rank two has an intersection form

$$I_\Lambda \Big|_{\mathrm{rk}(\Lambda)=2} = \begin{pmatrix} 2a & b \\ b & 2c \end{pmatrix} \,, \quad a, b, c \in \mathbb{Z} \,, \quad 4ac - b^2 < 0 \,. \tag{5.31}$$

An algebraic $K3$ surface with Λ-polarized lattice of rank two admits a genus one fibration with N-section iff $b^2 - 4ac = N^2$ [161–163].

For elliptic fibrations, i.e. when $N = 1$, five-dimensional light BPS states arise in the limit $\mathrm{Vol}_{\mathbb{C}}(\mathbb{P}^1_{\mathrm{b}}) \to \infty$, where the dual Heterotic string is weakly coupled, from M2 branes that wrap curves of the type [164]

$$C_{\mathrm{M2}} = k\mathcal{E} + w(F + \mathcal{E}) \,, \tag{5.32}$$

with F being the base of the $K3$ fiber and $\mathcal{E}$ the generic fiber of the genus one fibration. The shift of F by $\mathcal{E}$ is fixed by demanding that the self-intersection of C_{M2} inside the $K3$ fiber matches the quadratic form on the momentum lattice. However, on a genus one fibration with N-section we find components of reducible fibers that intersect the N-section only once so that the analogous expansion reads

$$C_{\mathrm{M2}} = k\frac{\mathcal{E}}{N} + w(F + \mathcal{E}) \,, \tag{5.33}$$

with self-intersection given by (5.29), i.e.

$$C_{\mathrm{M2}}^2 = \frac{2kw}{N}\,. \tag{5.34}$$

A crucial implication is, that the coefficients $Z_{n\cdot F}$, $n = 1, \ldots, N-1$ of the topological string partition function (2.125) encode twisted sectors of the dual Heterotic strings, while the contributions from untwisted sectors appear in $Z_{n\cdot F}$, $n = 0 \bmod N$.

As a next step we extend the construction and turn on Heterotic Wilson lines. On the M-theory side this amounts to considering $K3$ fibrations with Λ-polarized lattices of the form

$$\Lambda = U(N) \oplus L(-m)\,, \tag{5.35}$$

where $m \in \mathbb{N}$ is some twist that signals a non-perturbative realization of the Heterotic gauge group. So far the situation is completely analogous to that of elliptically fibered $K3$ surfaces which we described in section 5.2. The new feature here is that the discriminant group G_Λ associated with the lattice (5.35) decomposes into two factors $G_\Lambda \cong G_{U(N)} \oplus G_{L(-m)}$. Consequently, the Noether-Lefschetz generators of these geometries will be vector-valued modular forms that transform under the dual of the Weil representation $\rho_\Lambda^* : \mathrm{Mp}(2,\mathbb{Z}) \to \mathrm{End}\left(\mathbb{C}[G_\Lambda]\right)$, where $\mathbb{C}[G_\Lambda] \cong \mathbb{C}[G_{U(N)}] \otimes_{\mathbb{C}} \mathbb{C}[G_{L(-m)}]$.

If we recall that for Heterotic strings on $K3 \times T^2$ the elliptic genus could be obtained from the Noether-Lefschetz generator by inverting the map (B.27), this leads to the question what happens when the latter transforms non-trivially with respect to

$$G_{U(N)} \cong \left(\mathbb{Z}/N\mathbb{Z}\right)^2\,. \tag{5.36}$$

It turns out that the objects we obtain are the twisted twined elliptic genera

$$Z^{(r,s)}(\tau, \boldsymbol{z}) = \mathrm{Tr}_{\mathcal{H}_r} g^s (-1)^F F^2 q^{H_L} \bar{q}^{H_R} \prod_{a=1}^{n_V} (\zeta^a)^{J_a}\,, \tag{5.37}$$

that have been introduced by [165] and further studied in [143–145, 158, 166]. Here we use the same notation that we already defined for the ordinary elliptic genus in (4.16). The only difference here is that the trace is taken over the r-twisted Ramond-Ramond sector $\mathcal{H}_r$, whereas the insertion g^s projects multiples of the phase eigenvalue of states (5.30). Moreover, the twisted twined elliptic genera are lattice Jacobi forms for $\Gamma_1(N)$, while the vector $\left(Z^{(r,s)}(\tau, \boldsymbol{z})\right)_{r,s\in\mathbb{Z}_N}$ transforms as vector-valued lattice Jacobi form under $\mathrm{SL}(2,\mathbb{Z})$. In particular, the relevant object to extend the discussion of the sublattice weak gravity conjecture to M-theory on genus one fibrations is the untwisted, untwined elliptic genus $Z^{(0,0)}(\tau, \boldsymbol{z})$. Contrary to the elliptic case this receives contributions not only from base degree one invariants of the topological string partition function but from the first N degrees with respect to the base of the $K3$ fiber. We discuss now how exactly the twisted twined elliptic genera can be obtained from the Noether-Lefschetz generator and also how the latter relates to the new supersymmetric index.

5.4 Twisted elliptic genera and Noether-Lefschetz theory

We will now extend the discussion of the Noether-Lefschetz generator, the new supersymmetric index and the elliptic genera to Heterotic strings on $(K3 \times S^1)/\mathbb{Z}_N$, where we in fact have to consider the twisted twined elliptic genera (5.43). For simplicity we again restrict to the case where the Heterotic gauge algebra consists of a single simple factor $\mathfrak{g}$. The dual $K3$ fibered Calabi-Yau 3-folds $S \hookrightarrow M \xrightarrow{\pi} \mathbb{P}^1_{\mathrm{b}}$ then exhibit a polarization lattice

$$\Lambda = U(N) \oplus L(-m)\,, \quad \Lambda' = U(-1) \oplus \Lambda\,, \tag{5.38}$$

where Λ' is the extended lattice defined by $U(-1) \cong H^0(S,\mathbb{Z}) \oplus H^4(S,\mathbb{Z})$. Moreover, L is a rank $\mathrm{rk}(\mathfrak{g})$ lattice associated with the Wilson lines parameters, whose intersection form is given by $-m(\cdot\,,\cdot)$ with $(\cdot\,,\cdot)$ the Killing form of $\mathfrak{g}$ for some twist $m \in \mathbb{N}$. If the gauge group is realized perturbatively, the twist is $m = 1$.

In [166] the authors introduced the shifted Siegel theta function

$$\Gamma^{(r,s)}_{\boldsymbol{\mu}} = \sum_{\boldsymbol{\lambda}\in L(m)+\boldsymbol{\mu}} \sum_{k_0,w_0,k\in\mathbb{Z}} \sum_{w\in\mathbb{Z}+\frac{r}{N}} \mathrm{e}^{-2\pi i \frac{ks}{N}} q^{\frac{p_L^2}{2}} \bar{q}^{\frac{p_R^2}{2}}\,, \tag{5.39}$$

where they considered the case $L = L^\vee(A_1)$ and $\boldsymbol{\mu} \in \mathbb{Z}/2\mathbb{Z}$. The left- and right moving momenta can be obtained from the relations

$$\begin{aligned} \frac{1}{2}(p_L^2 - p_R^2) &= \frac{1}{2}m\,(\boldsymbol{\lambda},\boldsymbol{\lambda}) - kw + k_0 w_0\,, \\ \frac{1}{2}p_R^2 &= -\frac{1}{2\langle \mathrm{Im}(\boldsymbol{y}), \mathrm{Im}(\boldsymbol{y})\rangle}\left|\boldsymbol{\lambda}\cdot\boldsymbol{V} + kU_{\mathrm{het}} + wT_{\mathrm{het}} + k_0 - w_0\frac{\langle \boldsymbol{y},\boldsymbol{y}\rangle}{2}\right|^2, \end{aligned} \tag{5.40}$$

where $\boldsymbol{y}$ denotes the $(T,U,\boldsymbol{V})$ Heterotic moduli with intersection pairing

$$\langle \boldsymbol{y},\boldsymbol{y}\rangle \equiv -2T_{\mathrm{het}}U_{\mathrm{het}} + m(\boldsymbol{V},\boldsymbol{V})\,. \tag{5.41}$$

Note that here we choose the signs to match the conventions for the quadratic form of [142, 156]. Based on the results from [143–145, 158] we observe that the new supersymmetric index for Heterotic strings on $(K3 \times T^2)/\mathbb{Z}_N$ factorizes as

$$\mathcal{Z}(\tau,\bar{\tau}) = \frac{1}{N}\sum_{\boldsymbol{\mu}\in G_L} \sum_{r,s\in\mathbb{Z}/N\mathbb{Z}} \Gamma^{(r,s)}_{\boldsymbol{\mu}}(\tau,\bar{\tau}) Z^{(r,s)}_{\boldsymbol{\mu}}(\tau)\,, \tag{5.42}$$

where we denote by $G_{L(m)}$ the discriminant group of $L(m)$ and $Z^{(r,s)}_{\boldsymbol{\mu}}$ are the components of a meromorphic vector-valued modular form

$$Z(\tau) = \sum_{\boldsymbol{\mu}\in G_{L(m)}} \sum_{r,s\in\mathbb{Z}/N\mathbb{Z}} Z^{(r,s)}_{\boldsymbol{\mu}}(\tau)\mathrm{e}_{\boldsymbol{\mu}} \otimes \mathrm{e}_{(r,s)}\,, \quad Z^{(r,s)}_{\boldsymbol{\mu}}(\tau) = \frac{h^{(r,s)}_{\boldsymbol{\mu}}(\tau)}{\eta^{24}(\tau)}\,. \tag{5.43}$$

We claim that $(Z^{(r,s)}_{\boldsymbol{\mu}})_{\boldsymbol{\mu}\in G_{L(m)}}$ is nothing but the *h-map projection* of the twisted twined elliptic genus (5.43). In other words, the vector of twisted twined elliptic genera, which form a vector-valued Jacobi form under $\mathrm{SL}(2,\mathbb{Z})$ transformations, can be obtained by replacing

$$\mathrm{e}_{\boldsymbol{\mu}} \mapsto \vartheta_{L(m),\boldsymbol{\mu}}(\tau,\boldsymbol{z})\,. \tag{5.44}$$

To make the connection with the Noether-Lefschetz generator, we expand the new supersymmetric index into

$$
\begin{aligned}
\mathcal{Z}(\tau,\bar\tau) &= \sum_{\boldsymbol{\mu}\in G_{L(m)}} \sum_{r,s\in\mathbb{Z}/N\mathbb{Z}} \sum_{k_0,w_0\in\mathbb{Z}} \sum_{\substack{w\in\mathbb{Z}+\frac{r}{N}\\ k\in\mathbb{Z}}} q^{\frac{p_L^2}{2}}\bar q^{\frac{p_R^2}{2}} \frac{\mathrm{e}^{-2\pi i\frac{ks}{N}}}{N} Z_{\boldsymbol{\mu}}^{(r,s)}(\tau) \\
&= \sum_{\boldsymbol{\mu}\in G_{L(m)}} \sum_{r,\ell\in\mathbb{Z}/N\mathbb{Z}} \sum_{k_0,w_0\in\mathbb{Z}} \sum_{\substack{w\in\mathbb{Z}+\frac{r}{N}\\ k\in N\mathbb{Z}+\ell}} q^{\frac{p_L^2}{2}}\bar q^{\frac{p_R^2}{2}} \widehat{Z}_{\boldsymbol{\mu}}^{(r,\ell)}(\tau) \\
&= \sum_{\boldsymbol{\mu}\in G_{L(m)}} \sum_{(k_0,w_0)\in U} \sum_{(r,\ell)\in G_\delta} \sum_{(k,w)\in L_\delta^{\mathrm{unt}}+(r,\ell)} q^{\frac{p_L^2}{2}}\bar q^{\frac{p_R^2}{2}} \widehat{Z}_{\boldsymbol{\mu}}^{(r,\ell)}(\tau) \\
&= \sum_{\boldsymbol{\mu}\in G_{L(m)}} \sum_{(r,\ell)\in G_\delta} \theta_{\Lambda'(-1)+\boldsymbol{\mu}\oplus(r,\ell)}(\tau,\bar\tau)\widehat{Z}_{\boldsymbol{\mu}}^{(r,\ell)}(\tau)\,,
\end{aligned}
\tag{5.45}
$$

where we used several new definitions. First, we introduced the Fourier discrete transform of a $\mathbb{C}\left[\mathbb{Z}/N\mathbb{Z}\times\mathbb{Z}/N\mathbb{Z}\right]$-valued function F on $\mathbb{H}$, whose component read [167]

$$
\widehat{F}_{(r,\ell)} \equiv \frac{1}{N}\sum_{s\in\mathbb{Z}/N\mathbb{Z}} \mathrm{e}^{-2\pi i\frac{s\ell}{N}} F_{(r,s)}\,. \tag{5.46}
$$

Second, following the discussion of section 2.2 from [160], we introduced the CHL lattice $L_\delta = N^{-1}\mathbb{Z}\oplus\mathbb{Z}$ spanned by the vectors $\delta = \mathrm{e}_1/N = (1/N,0)$ and $\mathrm{e}_2=(0,1)$, such that

$$
\delta^2 = \mathrm{e}_2^2 = 0\,,\quad \delta\cdot\mathrm{e}_2 = \frac{1}{N}\,, \tag{5.47}
$$

whereas the untwisted δ-invariant sublattice $L_\delta^{\mathrm{unt}}\subset U = \mathrm{Span}_{\mathbb{Z}}\{\mathrm{e}_1,\mathrm{e}_2\}\cong\mathbb{Z}\oplus\mathbb{Z}$ is defined as

$$
L_\delta^{\mathrm{unt}} = \{(w,k)\in U \mid (w,k)\cdot\delta\in\mathbb{Z}\} \cong \mathbb{Z}\oplus N\mathbb{Z}\,. \tag{5.48}
$$

Note that by definition $(L_\delta^{\mathrm{unt}})^* = L_\delta$ and the intersection form of L_δ^{unt} is that of $U(N)$. Moreover, the CHL discriminant group is given by

$$
G_\delta = L_\delta/L_\delta^{\mathrm{unt}} \cong G_{U(N)} \cong (\mathbb{Z}/N\mathbb{Z})^2\,. \tag{5.49}
$$

Third, we introduced the components of the *Siegel theta function* $\Theta_{\Lambda'(-1)}(\tau,\bar\tau)$ associated with the lattice $\Lambda'(-1)$, which we describe in more detail in (B.48) in Appendix B.4. With this in mind, let us now recall the claim of the authors [142]

$$
\mathcal{Z}(\tau,\bar\tau) = \sum_{\gamma\in G_\Lambda} \theta_{\Lambda'(-1)+\gamma}(\tau,\bar\tau)\frac{\Phi_\gamma^\pi(\tau)}{\eta^{24}(\tau)}\,, \tag{5.50}
$$

where in our case $G_\Lambda = G_{L(m)}\oplus G_{U(N)}$. By comparing (5.45) and (5.43) we find that the Noether-Lefschetz generator is given by

$$
\Phi_\gamma^\pi(\tau)\Big|_{\gamma=\boldsymbol{\mu}\oplus(r,\ell)} = \widehat{h}_{\boldsymbol{\mu}}^{(r,\ell)}(\tau)\,. \tag{5.51}
$$

Together with the NL-GW correspondence theorem (C.14) we have thus found a straightforward recipe to obtain the twisted twined elliptic genera directly from the topological string partition function of a $K3$ fibered Calabi-Yau 3-fold that also exhibits a compatible genus one fibration:

1. Calculate the genus zero Gromov-Witten invariants, for example via mirror symmetry, and use (C.14) to determine the Noether-Lefschetz generator (5.51).

2. Invert the discrete Fourier transformation to recover (5.43).

3. Apply the replacement (5.44) to obtain the twisted twined elliptic genera.

We extend next the discussion of the sublattice weak gravity conjecture to genus one fibrations.

5.5 The sublattice weak gravity conjectures for genus one fibrations

We will now discuss how the arguments for the sublattice and the non-Abelian sublattice weak gravity conjecture apply to M-theory on a genus one fibered Calabi-Yau 3-fold M with N-sections. The existence of a limit in which the gauge coupling goes to zero while the Planck mass remains finite still implies that the base is a Hirzebruch surface or a blowup thereof [64]. Moreover, in the same limit a dual Heterotic string, that is now compactified on $(K3 \times S^1)/\mathbb{Z}_N$, becomes weakly coupled and, using the prescription that we worked out in the previous subsection, one can calculate the twisted twined elliptic genera $Z^{(r,s)}(\tau, \boldsymbol{z})$ (5.43) which are of the form

$$Z^{(r,s)}(\tau, \boldsymbol{z}) = \mathrm{Tr}_{\mathcal{H}_r} g^s (-1)^F F^2 q^{H_L} \bar{q}^{H_R} \prod_{a=1}^{n_V} (\zeta^a)^{J_a} = \frac{\Phi^{(r,s)}(\tau, \boldsymbol{z})}{\eta(\tau)^{24}}\,, \tag{5.52}$$

with $\Phi^{(r,s)}(\tau, \boldsymbol{z})$ being holomorphic, Weyl invariant lattice Jacobi forms for $\Gamma_1(N)$. We can therefore again perform an expansion (4.18)

$$\Phi^{(r,s)}(\tau, \boldsymbol{z}) = \sum_{\boldsymbol{\mu} \in L^*/L} h_{\boldsymbol{\mu}}^{(r,s)}(\tau) \vartheta_{\underline{L}, \boldsymbol{\mu}}(\tau, \boldsymbol{z})\,, \tag{5.53}$$

where $\vartheta_{\underline{L}, \boldsymbol{\mu}}(\tau, \boldsymbol{z})$ are Jacobi theta functions associated with the, in general twisted, coroot lattice $\underline{L}$ of the gauge group and $(h_{\boldsymbol{\mu}}^{(r,s)})_{(r,s)\oplus\boldsymbol{\mu}}$ are vector-valued modular or, in the presence of NS5-branes, vector-valued quasi-modular forms that transform under the dual of the Weil representation $\rho^* : \mathrm{Mp}(2, \mathbb{Z}) \to \mathbb{C}\left[\mathbb{Z}/N\mathbb{Z} \times \mathbb{Z}/N\mathbb{Z}\right] \otimes \mathbb{C}\left[L^*/L\right]$.

The Tachyon is part of the untwisted sector and appears in $Z^{(0,0)}(\tau, \boldsymbol{z})$. We can therefore apply the same arguments that held for elliptic fibrations to conclude that there is a sublattice of superextremal states which satisfies the sublattice weak gravity conjecture and, in the case of a single simple factor of the Heterotic gauge group, also the non-Abelian generalization. In fact, the untwisted untwined elliptic genus $Z^{(0,0)}(\tau, \boldsymbol{z})$ is identical to the elliptic genus of the Heterotic string on $K3$. The crucial difference to the elliptic case is now that it is not identical to the genus zero contribution to the topological string partition function $Z_{\beta=F}(\tau, \lambda)$ but receives contributions from $Z_{\beta=n\cdot F}(\tau, \lambda)$ for $n = 1, \ldots, N$.

Example: We consider the genus one fibered Calabi-Yau 3-fold

$$M = (F_4 \to \mathbb{F}_1)\,[SU(2)\times\mathbb{Z}_2]_4^{-144}$$

that has been discussed in [2]. Here the polarization lattice is of the form $\Lambda \simeq U(2)\oplus L^\vee(A_1)(-2)$. The twisted elliptic genera (5.52) can be fixed via the procedure outlined in Section 5.4 and we obtain the following result:

$$\begin{aligned}
\Phi^{(0,0)}(\tau,z) &= -\frac{11}{12}E_6(\tau)E_{4,2}(\tau,z) - \frac{13}{12}E_4(\tau)E_{6,2}(\tau,z)\,,\\
\Phi^{(0,1)}(\tau,z) &= -\frac{1}{2592}\left(E_4(\tau)-4\mathcal{E}_2^2(\tau)\right)^2\Big[2\phi_{-2,1}^2(\tau,z)\mathcal{E}_2(\tau)\left(7\mathcal{E}_2^2(\tau)-5E_4(\tau)\right)\\
&\qquad + \phi_{-2,1}(\tau,z)\phi_{0,1}(\tau,z)\left(E_4(\tau)-9\mathcal{E}_2^2(\tau)\right)+4\phi_{0,1}^2(\tau,z)\mathcal{E}_2(\tau)\Big]\,,\\
\Phi^{(1,0)}(\tau,z) &= -\frac{1}{2592}\left(E_4(\tau)-\mathcal{E}_2^2\left(\frac{\tau}{2}\right)\right)^2\left[\phi_{-2,1}^2(\tau,z)\mathcal{E}_2\left(\frac{\tau}{2}\right)\left(5E_4(\tau)-\frac{7}{4}\mathcal{E}_2^2\left(\frac{\tau}{2}\right)\right)\right.\\
&\qquad \left.+ \phi_{-2,1}(\tau,z)\phi_{0,1}(\tau,z)\left(E_4(\tau)-\frac{9}{4}\mathcal{E}_2^2\left(\frac{\tau}{2}\right)\right)-2\phi_{0,1}^2(\tau,z)\mathcal{E}_2\left(\frac{\tau}{2}\right)\right]\,,\\
\Phi^{(1,1)}(\tau,z) &= -\frac{1}{2592}\left(E_4(\tau)-\mathcal{E}_2^2\left(\frac{\tau+1}{2}\right)\right)^2\left[\phi_{-2,1}^2(\tau,z)\mathcal{E}_2\left(\frac{\tau+1}{2}\right)\left(5E_4(\tau)-\frac{7}{4}\mathcal{E}_2^2\left(\frac{\tau+1}{2}\right)\right)\right.\\
&\qquad \left.+ \phi_{-2,1}(\tau,z)\phi_{0,1}(\tau,z)\left(E_4(\tau)-\frac{9}{4}\mathcal{E}_2^2\left(\frac{\tau+1}{2}\right)\right)-2\phi_{0,1}^2(\tau,z)\mathcal{E}_2\left(\frac{\tau+1}{2}\right)\right]\,.
\end{aligned} \tag{5.54}$$

Here $\mathcal{E}_2(\tau)\equiv E_2(2\tau)-E_2(\tau)$. Indeed, the untwined untwisted elliptic genus is identical to the one for the Heterotic strings dual to elliptic fibrations. Morever, we find out the following exchange transformation under $\mathrm{S}:\tau\mapsto -1/\tau$, and $\mathrm{T}:\tau\mapsto\tau+1$ with $\mathrm{S},\mathrm{T}\in\mathrm{SL}(2,\mathbb{Z})$:

$$\mathrm{T}\circlearrowright\Phi^{(0,1)}\xleftrightarrow{\mathrm{S}}\Phi^{(1,0)}\xleftrightarrow{\mathrm{T}}\Phi^{(1,1)}\circlearrowleft\mathrm{S}\,. \tag{5.55}$$

This observation derives from the trivial relations $\Phi^{(1,0)}(\tau+1,z)=\Phi^{(1,1)}(\tau,z)$ and $\Phi^{(0,1)}(\tau+1,z)=\Phi^{(0,1)}(\tau,z)$, whereas the exchange under S transformations follows from the identities

$$\mathcal{E}_2\left(-\frac{1}{2\tau}\right)=-2\tau^2\mathcal{E}_2(\tau)\,,\quad \mathcal{E}_2\left(-\frac{1}{2\tau}+\frac{1}{2}\right)=\tau^2\mathcal{E}_2\left(\frac{\tau+1}{2}\right)\,. \tag{5.56}$$

CHAPTER 6

Modularity of elliptically fibered Calabi-Yau 4-folds

In this section, we expose some findings of our paper [1] that postdate the pioneering work by [36]. There, we study the modularity of topological string amplitudes on elliptically fibered Calabi-Yau 4-folds. These results provide non-trivial evidence for a more general conjecture of elliptically fibered Calabi-Yau n-folds, which predicts Gromov-Witten potentials—in our case topological string amplitudes—are meromorphic quasi-Jacobi forms that satisfy holomorphic anomaly equations [168, 169].

Extensions of this program are those by [122, 125], where the authors considered elliptically fibered Calabi-Yau 4-folds with Mordell-Weil group of rank one. Our standpoint here is to explain the essential features for the modularity of strings propagating in 4d, at least those aspects differing from their 6d counterpart. Surprisingly, evidence shows a tight interplay between elliptic genera, cancellation of anomalies in 4d $\mathcal{N} = 1$ theories, the sublattice weak gravity conjecture, and the theory of quasi-Jacobi forms [125, 170, 171]. Hence the importance of extending our catalog of modular objects, for which we introduce a brief review on quasi-Jacobi forms in the Appendix B.3.

6.1 Enumerative geometry on elliptically fibered Calabi-Yau 4-folds

Let us start by introducing the basics of enumerative geometry on Calabi-Yau 4-folds. Although it is possible to define genus one BPS invariants on Calabi-Yau 4-folds, we will only analyze here genus zero invariants. The latter turns to be of significant relevance for the physics realized by F-theory models, which yield 4d $\mathcal{N} = 1$ effective theories. These types of compactifications are elliptic fibrations whose Gromov-Witten theory corresponds with the theory of quasi-Jacobi forms. We also briefly review the latter topic.

Genus zero invariants and string amplitudes: The topological string A-model encodes Gromov-Witten invariants that count holomorphic maps

$$f : \Sigma_{g,\bar{p}} \to M \,, \tag{6.1}$$

from pointed curves $\Sigma_{g,\bar{p}}$ of genus g into M. The general formula for the virtual dimension of the moduli stack of stable maps[1] into a Calabi-Yau M is given by

$$\mathrm{vir\,dim}\,\bar{M}_{g,n}(M,\beta) = (\dim M - 3)(1-g) + n\,, \tag{6.2}$$

where n is the number of marked points $\bar{p} = (p_1, \dots p_n)$ in $\Sigma_{g,\bar{p}}$; we require $f_*[\Sigma] = \kappa \in H_2(M)$ for $f \in M_{g,n}(M,\kappa)$ and Σ the domain of f. Given $\gamma_1, \dots, \gamma_n \in H^{2,2}(M,\mathbb{Z})$, we obtain the Gromov-Witten invariants

$$N_{0,\kappa}(\gamma_1,\dots,\gamma_n) = \int_{[\bar{M}_{g,n}(M,\kappa)]^{\mathrm{virt.}}} \prod_{i=1}^{n} \mathrm{ev}_i^*(\gamma_i)\,, \tag{6.3}$$

where $[\,\cdot\,]^{\mathrm{virt.}}$ denotes virtual fundamental class, and we introduced the evaluation map

$$\mathrm{ev}_i : \bar{M}_{g,n}(M,\kappa) \to M\,, \tag{6.4}$$

which is the i-th point evaluation map given by $\mathrm{ev}_i = f(p_i)$ for p_i in $\bar{p}$.

In what follows, we will be interested in the non-trivial case for a Calabi-Yau 4-fold M when $\mathrm{vir\,dim}\,\bar{M}_{0,1} = 2$. From the topological string theory perspective, the genus zero Gromov-Witten invariants (6.3) are encoded in the instanton part of the normalized double-logarithmic quantum periods

$$F_\gamma^{(0)} = \text{classical} + \sum_{\kappa \in H_2(M,\mathbb{Z})} N_{0,\kappa}(\gamma) \mathrm{e}^{\boldsymbol{t}\cdot\kappa}\,. \tag{6.5}$$

In particular, we can compute the classical terms corresponding to $F_\gamma^{(0)}$ via the asymptotic B-brane central charges $Z_B(\mathcal{O}_\gamma)|_{\mathrm{class.}}$ [1], which we explained in section 2.4.3. While the Gromov-Witten invariants are in general rational numbers, they are conjecturally related to integral *instanton numbers* $n_{0,\kappa}(\gamma)$ via [172]

$$\sum_{\kappa \in H_2(M,\mathbb{Z})} N_{0,\kappa}(\gamma)\mathrm{e}^{\boldsymbol{t}\cdot\kappa} = \sum_{\kappa \in H_2(M,\mathbb{Z})} n_{0,\kappa}(\gamma) \sum_{d=1}^{\infty} \frac{\mathrm{e}^{m\boldsymbol{t}\cdot\kappa}}{m^2}\,. \tag{6.6}$$

Having introduced genus zero Gromov-Witten potentials, we now discuss the modularity they exhibit when M admits an elliptic fibration.

Modular amplitudes and holomorphic anomaly equations: As a sepecialization for Calabi-Yau 4-folds M, we consider those that enjoy a fibration structure $\pi : M \to B$. Here B is a complex-3 Kähler manifold and M is a non-singular projective variety. Given this setup, the authors conjectured that the π-relative Gromov-Witten classes are cycled valued quasi-Jacobi forms [168, 169]. To our case of interest, such conjecture translates into

$$\mathrm{Coeff}\left(F_\gamma^{(0)}, Q_\beta\right) \in \frac{1}{\eta^{12c_1(B)\cdot\beta}} \mathrm{QJac}_L\,, \tag{6.7}$$

where $\beta \in H_2(B,\mathbb{Z})$. Consequently, properties of quasi-Jacobi forms indicate that $\partial_{E_2} F_\gamma^{(0)} \neq 0$. However, the authors also conjectured a general recursive formula for Gromov-Witten

[1] A map is stable if it has at most a finite number of non-trivial automorphisms that preserve marked and nodal points.

potentials that predicts in our case an equation of the type $\partial_{E_2} F^{(0)}_\gamma = \mathrm{RHS}(F^{(0)}_{\gamma_1}, \dots F^{(0)}_{\gamma_k})$, where $\gamma_1, \dots \gamma_k \in H^*(M)$. We illustrate now such a *modular anomaly equation* with a set of examples, which we studied in [1].

We now consider elliptically fibered 4-folds with at most I_1 singularities. As they only have two types of divisors, the zero-section, and vertical divisors, the lattice L becomes trivial. Thus, the base coefficients of string amplitudes $F^{(0)}_\gamma$, according to the conjecture (6.7), should be meromorphic quasi-modular forms. Indeed, via monodromies transformations on the quantum periods, we argued in [1] that they follow an Ansatz of the form (6.7), whereas the authors first observed their quasi-modular behavior [36]. For Calabi-Yau 4-folds, a basis of middle-dimensional cycles has to be specified as well. For these cases, a basis for such 4-cycles reads

$$H_k = S_0 \wedge \tilde{D}_k\,, \quad H^k = \pi^{-1}\tilde{C}^k\,, \tag{6.8}$$

with $\tilde{D}_k \equiv \pi^* D^{\mathrm{b}}_k$, where $\{D^{\mathrm{b}}_k\}_{k=1,\dots,h^{1,1}(B)}$ is a basis of base divisors, $\tilde{C}^k = S_0 \cdot \pi^{-1} C^k$ with $\{\tilde{C}^k\}_{k=1,\dots,h^{1,1}(B)}$ generators of the Mori cone of B, and

$$H_i \cdot H_j = -a_{ij}\,, \quad H_i \cdot H^j = \delta_i^j\,, \quad H^i \cdot H^j = 0\,. \tag{6.9}$$

Here we introduced the base topological data

$$a^k = \int_{C^k} c_1(B)\,, \quad a_{ij} = c_1(B) \cap D^{\mathrm{b}}_i \cap D^{\mathrm{b}}_k\,,. \tag{6.10}$$

We call the 4-cycles $H^k = \pi^{-1}\tilde{C}'^k$, $k = 1, \dots, h^{1,1}(B)$ that result from lifting a curve in the base to a 4-cycle in M the π-vertical 4-cycles. We define a "modular basis" by taking the 4-cycles,

$$H^i = a^{ij} a^k \tilde{D}_j \wedge \tilde{D}_k\,, \quad H^\circ_i = \left(S_0 + \frac{1}{2}\pi^* c_1(B)\right) \wedge \tilde{D}_i\,, \tag{6.11}$$

where a^{ij} is the inverse matrix of a_{ij}. Thus, we obtain the intersection relations

$$H^i \cdot H^\circ_j = \delta^i_j\,, \quad H^i \cdot H^j = 0\,, \quad H^\circ_i \cdot H^\circ_j = 0\,. \tag{6.12}$$

Note that $H^i \in H_4(M, \mathbb{Z})$ while in general $H^\circ_i \notin H_4(M, \mathbb{Z})$. Let $\ell \in H^2(B)$ such that $\beta \cap \ell \neq 0$. Then for a given $\gamma \in H^{2,2}(M, \mathbb{C})$ conjecture B of [168, 169] implies a modular anomaly equation for $F^{(0)}_\gamma$, which in the modular basis (6.11) reads[2]

$$\begin{aligned} \frac{\partial F^{(0)}_{\gamma,\beta}}{\partial E_2} = -\frac{1}{12}\Bigg[&\sum_{\beta=\beta'+\beta''} \left(\beta'_i F^{(0)}_{\gamma,\beta'} F^{(0)}_{H^i,\beta''} - \frac{(\beta'\cap\ell)^2 \beta''\cap\pi_*\gamma + (\beta''\cap\ell)^2\beta'\cap\pi_*\gamma}{(\beta\cap\ell)^2} F^{(0)}_{H^i,\beta'} F^{(0)}_{H^\circ_i,\beta''} \right) \\ &+ \frac{2}{\beta\cap\ell} F^{(0)}_{\pi^*(\pi_*(\gamma)\cup\ell),\beta} - \frac{\pi_*\gamma\cap\beta}{(\beta\cap\ell)^2} F^{(0)}_{\pi^*\ell^2,\beta} \Bigg]\,. \end{aligned} \tag{6.13}$$

From the properties of the Gysin morphisms it follows that $\pi_* H^\circ_i = D^{\mathrm{b}}_i$ and $\pi_* H^i = 0$.

Example: To illustrateto the holomorphic anomaly equations (6.13), we consider generic degree 24 hypersurface X_{24} in $\mathbb{P}(1,1,1,1,8,12)$ that has been used by [1, 36, 172, 173], which is an elliptic fibration with base $B = \mathbb{P}^3$, i.e., with $h^{1,1}(B) = 1$. We provide the toric

[2] We thank Georg Oberdieck for explaining this point to us.

data for this geometry next:

$$
\begin{array}{cc|rrrrr|rr|}
div. & coord. & & & \nu_i^* & & & C_e & C_b \\
K_M & x_0 & 0 & 0 & 0 & 0 & 0 & -6 & 0 \\
D_1 & x & -1 & 0 & 0 & 0 & 0 & 2 & 0 \\
D_2 & y & 0 & -1 & 0 & 0 & 0 & 3 & 0 \\
S_0 & z & 2 & 3 & 0 & 0 & 0 & 1 & -4 \\
L & u_1 & 2 & 3 & 1 & 1 & 1 & 0 & 1 \\
L & u_2 & 2 & 3 & -1 & 0 & 0 & 0 & 1 \\
L & u_3 & 2 & 3 & 0 & -1 & 0 & 0 & 1 \\
L & u_4 & 2 & 3 & 0 & 0 & -1 & 0 & 1
\end{array} \, . \tag{6.14}
$$

We use the Sage to calculate the intersections given by the divisors $\tilde{D}_b = L$ and $\tilde{D}_e = L + 4S_0$. This way, we determine the constants (6.10) and obtain that $a^b = 4$, and $a_{bb} = 4$. We can determine a basis $\{H^b, H_b\}$ of 4-cycles on X_{24} given by $H^b = \tilde{D}_b^2$ and $H_b = S_0 \cdot \tilde{D}_b$. By introducing $H_b^\circ \equiv H_b + 2H^b$, the intersection matrix of 4-cycles in the pure modular basis takes the form

$$
\eta^{(2)} = \begin{array}{c} \begin{array}{cc} H^b & H_b \end{array} \\ \begin{pmatrix} 0 & 1 \\ 1 & -4 \end{pmatrix} \end{array} \begin{array}{c} \\ H^b \\ H_b \end{array} \quad \longrightarrow \quad \eta'^{(2)} = \begin{array}{c} \begin{array}{cc} H^b & H_b^\circ \end{array} \\ \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \end{array} \begin{array}{c} \\ H^b \\ H_b^\circ \end{array} \, .
$$

In this basis the amplitudes $F^{(0)}_{H^b}$ and $F^{(0)}_{H_b^\circ}$ have base Fourier-coefficients that are meromorphic quasimodular forms of weights $k = -2$ and weight $k = 0$ respectively. We find that the $F^{(0)}_{H^b}$ satisfies the relation

$$
\frac{\partial F^{(0)}_{H^b,d}}{\partial E_2} = -\frac{1}{12} \sum_{s=1}^{d-1} s F^{(0)}_{H^b,d-s} F^{(0)}_{H^b,s} \, , \tag{6.15}
$$

while the base Fourier-coefficients of $F^{(0)}_{H_b^\circ}$ follow

$$
\frac{\partial F^{(0)}_{H_b^\circ,d}}{\partial E_2} = -\frac{1}{12d} \left(\sum_{s=1}^{d-1} s^2 F^{(0)}_{H_b^\circ,s} F^{(0)}_{H^b,d-s} + F^{(0)}_{H^b,d} \right) . \tag{6.16}
$$

These recursive relations turn to be useful when solving for the elliptic genera of strings, which we discuss in the upcoming sections.

6.2 Gauge fluxes

It is our interest to connect the enumerative geometry of Calabi-Yau 4-folds with the physics of M-/F-theory compactifications. It turns out that M-theory/F-theory vacua in 3d/4d require a choice of background for the M-theory 3-form C_3 and its field strength $G_4 = \mathrm{d}C_3$, which must be quantized as [174]

$$
G_4 + \frac{1}{2} c_2(M) \in H^4(M, \mathbb{Z}) \, , \tag{6.17}
$$

where $c_2(M)$ is the second Chern class of M. Moreover, supersymmetry imposes the constraints

$$\begin{cases} G_4 \in H^{2,2}(M,\mathbb{R}) \cap H^4(M,\mathbb{Z}/2) \\ \omega \wedge G_4 = 0 \end{cases} . \tag{6.18}$$

Here ω denotes the Kähler form of M. Let us note that $H^{2,2}(M)$ is rather subtle and needs further discussion.

Horizontal and vertical fluxes: We are intersted in describing elements of the cohomology $H^{2,2}(M)$, which follows an orthogonal decomposition [175]

$$H^{2,2}(M,\mathbb{C}) = H^{2,2}_H(M,\mathbb{C}) \oplus H^{2,2}_V(M,\mathbb{C}) \oplus H^{2,2}_{RM}(M,\mathbb{C}) . \tag{6.19}$$

We elaborate briefly on each of these subspaces next:

1. The horizontal piece $H^{2,2}_H(M,\mathbb{C}) = H^{2,2}(M,\mathbb{C}) \cap H^4_H(M,\mathbb{C})$ is the subspace of the horizontal cohomology $H^4_H(M,\mathbb{C}) \subset H^4(M,\mathbb{C})$, which is the space spanned by derivatives $\partial_{z_{i_1}} \cdots \partial_{z_{i_n}} \Omega_4$, where z_{i_*} is a complex structure local coordinate, and Ω_4 is the unique no-where vanishing holomorphic $(4,0)$-form in M.

2. The vertical piece $H^{2,2}_V(M)$ is the subspace that follows the form

$$H^{2,2}_V(M,\mathbb{C}) = \langle H^{1,1}(M,\mathbb{C}) \wedge H^{1,1}(M,\mathbb{C}) \rangle . \tag{6.20}$$

3. The remainder subspace $H^{2,2}_{RM}(M,\mathbb{C})$ is the orthogonal compliment to $H^{2,2}_H(M,\mathbb{C}) \oplus H^{2,2}_V(M,\mathbb{C})$.

As seen in an example in [41], the naive expectation that mirror symmetry maps vertical into horizontal classes and vice versa, while the remaining component maps into themselves, does not hold. However, we can avoid this subtlety when phrasing the problem in terms of branes and homological mirror symmetry.

F-theory fluxes: Besides the physical constraints for a 4-flux G_4, Poincaré invariance in the F-theory vacuum requires the transversality condition [72]

$$G_4 \cdot \pi^* D^{\mathrm{b}}_i \cdot \pi^* D^{\mathrm{b}}_j = 0 , \qquad G_4 \cdot S_0 \cdot \pi^* D^{\mathrm{b}}_i = 0 , \quad \forall\, \pi^* D^{\mathrm{b}}_i \in H^{1,1}(B) . \tag{6.21}$$

Here S_0 is the zero-section of the elliptically fibered Calabi-Yau 4-fold M. The simplest type of fluxes satisfying this constraint that we can construct are vertical 4-cycles of the type

$$\text{Gauge flux:} \begin{cases} \text{non-Abelian:} & G_4|_{i_I} = \pi^*\alpha \wedge E_{i_I} \\ \text{Abelian:} & G_4|_Q = \pi^*\alpha \wedge \sigma(s_Q) \end{cases} , \qquad \alpha \in H^{1,1}(B) . \tag{6.22}$$

Here E_{i_I} is a fibral divisor associated with a Cartan generator of a gauge algebra $\mathfrak{g}_I$, whereas $\sigma(s_Q)$ is the Shioda map of a rational section S_Q that gives rise to a $U(1)_Q$ non-Cartan flux. We note from conditions (3.17) that indeed the gauge fluxes (6.22) satisfy the F-theory uplift conditions (6.21).

In what follows, we will discuss the case of a single $U(1)$ gauge 4-flux turned on. This case arises when the elliptically fibered Calabi-Yau 4-fold M has a Mordell-Weil group of rank one. Our aim now is to inquire into the physics behind the modular amplitudes of gauge fluxes, as performed in [122, 125]. Before, let us motivate this incursion by noting that

$$N_{0,\kappa(\mathsf{q})}(G_4|_Q) = \int_\xi \mathrm{ev}^* G_4|_Q = \int_{S_\mathsf{q}} G_4|_Q \,, \tag{6.23}$$

where we introduced the fibral curve class $\kappa(\mathsf{q}) = \mathsf{q}C_+$ in which C_+ is a splitting curve such that $\sigma(s_Q) \cdot C_+ = 1$, and ξ is the virtual class of the moduli space $\bar{M}_{0,1}$ associated with the curve class $\kappa(\mathsf{q})$. Moreover, S_q is the fibration of $\kappa(\mathsf{q})$ over $C_{\text{matter}} \subset B$ [122], the base curve where the splitting for C_+ occurs. Here the crucial observation is that one identifies the moduli space of the curve $\kappa(\mathsf{q})$ with S_q [122]. In the F-theory dictionary, S_q is the so-called matter surface, which represents massless matter fields with $U(1)$-charge q that localize at C_{matter}. Its integration with the 4-form flux $G_4|_Q$ generates a non-trivial chiral index for massless charged matter [72]

$$\chi_{G_4}(\mathsf{q}) \equiv n_\mathsf{q}^+ - n_\mathsf{q}^- = \int_{S_{0,\mathsf{q}}} G_4|_Q = N_{0,\kappa(\mathsf{q})}(G_4|_Q) \,, \tag{6.24}$$

where n_q^+ is the number of matter fields with charge $+\mathsf{q}$, while n_q^- is the number of matter fields with charge $-\mathsf{q}$. Hence, the enumerative geometry determined by gauge fluxes encodes relevant data for the spectrum of F-theory compactifications. An extension for this observation is to connect gauge fluxes invariants with the elliptic genera of 4d $\mathcal{N} = 1$ effective strings, which we cover next.

6.3 Elliptic genera of 4d $\mathcal{N} = 1$ solitonic strings

In this section we will be interested in computing the elliptic genera of 4d $\mathcal{N} = 1$ solitonic strings. This kind of object arises when a D3-brane wraps a curve $C_\text{b} \subset B$. The effective string has a worldsheet theory description along its worldvolume Σ with $N = (0, 2)$ supersymmetry [95]. The elliptic genus of such an effective string is a charge weighted trace over the Ramond-Ramond sector of its worldsheet theory. More precisely, in a $U(1)$ gauge theory, the elliptic genus of a 4d $\mathcal{N} = 1$ solitonic string reads from the following expression

$$Z_\beta(q\,,\zeta) = \mathrm{Tr}_R\left[(-1)^{F_R} F_R q^{H_L} \bar{q}^{H_R} \zeta^Q\right] \,, \qquad q = \mathrm{e}^{2\pi i\tau}, \quad \zeta = \mathrm{e}^{2\pi i z} \,. \tag{6.25}$$

Here τ is considered to be the complex structure of the torus $\Sigma = T^2$, z is the fugacity parameter associated with the $U(1)$ gauge background field, and Q is the generator of the $\mathfrak{a}_1$ algebra. In the following, we summarize the main conjectures established in the work [125]:

Conjecture 1: Consider the Gromov-Witten potential base Fourier-coefficient $F^{(0)}_{\mathbf{G}}$ at $Q_\beta = \exp(2\pi i \boldsymbol{t}_\text{b} \cdot \beta)$, where $\boldsymbol{t}_\text{b}$ are base Kähler moduli, $\beta \in H_2(B, \mathbb{Z})$, and $\mathbf{G} \equiv G_4|_Q$ is a 4-flux as in (6.22). Then it relates to the elliptic genus (6.25) in the following way:

$$Z_\beta(q\,,\zeta) = -q^{\frac{1}{2}\beta\cdot c_1(B)} \mathrm{Coeff}\left(F^{(0)}_{\mathbf{G}}, Q_\beta\right) \,. \tag{6.26}$$

Moreover, the relative BPS invariants $n_{0,(\beta,n,\mathsf{q})}(\mathbf{G})$ match with the degeneracies $d_\beta(n, \mathsf{q})$ of 4d $\mathcal{N} = 1$ solitonic strings excitations at level n and charge q.

Conjecture 2: The elliptic genus (6.25) is a meromorphic quasi-Jacobi form of weight $k=-1$ and fugacity index $m=\frac{1}{2}b\cdot C_{\mathrm{b}}$, where $b=-\pi_*\left(\sigma(s_Q)\cdot\sigma(s_Q)\right)$ is the height-paring determined by the corresponding elliptically fibered geometry. Then, we can express the elliptic genus through the ansatz

$$\begin{aligned}Z_\beta(\tau,z)&=g^M Z_{-1,m}(\tau,z)+g^{QM}Z^{QM}_{-1,m}(\tau,z)\\&\quad+g^M_\partial\partial_z Z^M_{-2,L}(\tau,z)+g^{QM}_\partial\partial_z Z^{QM}_{-2,m}(\tau,z)\,.\end{aligned}\tag{6.27}$$

Here $g^M, g^{QM}, g^M_\partial, g^{QM}_\partial$ are flux-dependent coefficients and

$$Z^{\bullet}_{-1,m}(\tau,z)=q^{\frac{1}{2}\beta\cdot c_1(B)}\frac{\Phi^{\bullet}_{k,z}(\tau,z)}{\eta^{12\beta\cdot c_1(B)}(\tau)}\,,\quad Z^{\bullet}_{-2,m}(\tau,z)=q^{\frac{1}{2}\beta\cdot c_1(B)}\frac{\Phi^{\bullet}_{k-1,z}(\tau,z)}{\eta^{12\beta\cdot c_1(B)}(\tau)}\,.\tag{6.28}$$

Here the bullet $\bullet$ denotes the entries M or QM, which mean $\Phi^M_{*,m}\in J_{*,m}$ and $\Phi^{QM}_{*,L}\in J^{QM}_{*,m}$. See appendix B.3. Moreover, here the weight number reads $k=-1+6\beta\cdot c_1(B)$.

We explain now the strategy to fix modular expressions for the elliptic genera $Z_\beta(\tau,z)$. As was noted in [1, 36, 122, 176], the 4-cycles $H^{2,2}_V(M)$ decompose orthogonally into three different spaces

$$H^{2,2}_V(M)=H^{2,2}_{(-2)}(M)\oplus H^{2,2}_{(-1)}(M)\oplus H^{2,2}_{(0)}(M)\,,\tag{6.29}$$

which follow the form

$$\begin{aligned}H^{2,2}_{(-2)}(M)&=\left\langle\pi^*D^{\mathrm{b}}_i\wedge\pi^*D^{\mathrm{b}}_j\right\rangle\,,\\H^{2,2}_{(-1)}(M)&=\left\langle\sigma(s_Q)\wedge\pi^*D^{\mathrm{b}}_i\right\rangle\,,\\H^{2,2}_{(0)}(M)&=\left\langle\left(S_0+\frac{1}{2}\pi^*c_1(B)\right)\wedge\pi^*D^{\mathrm{b}}_i\right\rangle\,.\end{aligned}\tag{6.30}$$

Here each Fourier coefficient $F^{(0)}_{\gamma,\beta}$, with $\gamma\in H^{2,2}_{(k)}(M)$ and $\beta\in H_2(B,\mathbb{Z})$, transforms as a quasi-Jacobi form of weight k. In particular, the space of significant physical relevance is $H^{2,2}_{(-1)}(M)$, as we associate this with a $U(1)_Q$ non-Cartan gauge flux $\mathbf{G}$. Its associated Gromov-Witten potential yields elliptic genera of strings propagating in 4d. However, to fix the latter, we utilize the enumerative geometrical information given by subspace $H^{2,2}_{(-2)}(M)$, as observed in the ansatz (6.27).

6.4 Example: Elliptically fibered Calabi-Yau 4-fold with two sections

In this section we consider the Calabi-Yau 4-fold with two sections $M=\left(F_6\to\mathbb{P}^3\right)[U(1)]^{9504}_3$. Let us explain this notation. First, we have that $\chi(M)=9504$ and $h^{1,1}(M)=3$. Moreover, to realize this fibration we use the methods of [25], which we explained in Section 2.3.5. Here the base is $B=\mathbb{P}^3$ and the elliptic fiber follows from the toric hypersurface F_6, whose corresponding polyhedron is shown in Figure 6.1. Furthermore, M possesses an additional non-torsional rational section that translates into a $U(1)$ gauge symmetry for its associated F-theory compactification. As pointed out in [125], M is an extension of the Calabi-Yau 4-fold X_{24} which connects to M by a Higgs transition. Having said this, we show the toric

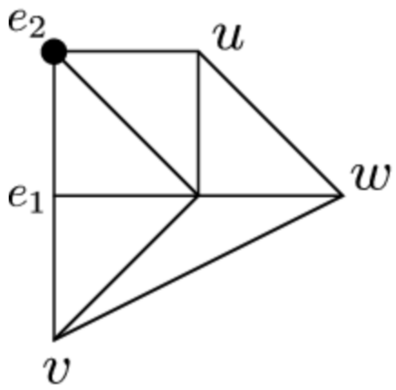


Figure 6.1: Polyhedron F_6. Image taken from [25].

data of M below:

$$
\begin{array}{rcl}
 & \begin{array}{rrrrrrrr} & & & & & C_1 & C_2 & C_3 \end{array} & \\
\begin{array}{r} x_1 \\ x_2 \\ x_3 \\ x_4 \\ e_2 \\ u \\ v \\ w \\ \end{array} &
\left(\begin{array}{rrrrr|rrr}
1 & 0 & 0 & -1 & 1 & 0 & 0 & 1 \\
0 & 1 & 0 & -1 & 1 & 0 & 0 & 1 \\
0 & 0 & 1 & -1 & 1 & 0 & 0 & 1 \\
-1 & -1 & -1 & -1 & 1 & 0 & 0 & 1 \\
0 & 0 & 0 & -1 & 1 & 1 & -1 & -4 \\
0 & 0 & 0 & 0 & 1 & -1 & 2 & 0 \\
0 & 0 & 0 & -1 & -1 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & -2 & 0
\end{array}\right) &
\begin{array}{l} \\ \\ \\ \leftarrow \text{vertical divisor } \pi^* D^{\mathrm{b}} \\ \leftarrow \text{holomorphic section } S_0 \\ \leftarrow \text{rational section } S_Q \\ \\ \\ \\ \end{array}
\end{array} . \tag{6.31}
$$

The intersection ring of M reads

$$
\begin{aligned}
\mathcal{R}(M) = & 1984 J_1^4 + 1024 J_2 J_1^3 + 240 J_3 J_1^3 + 512 J_2^2 J_1^2 + 28 J_3^2 J_1^2 + 128 J_2 J_3 J_1^2 \\
& + 256 J_2^3 J_1 + 3 J_3^3 J_1 + 16 J_2 J_3^2 J_1 + 64 J_2^2 J_3 J_1 + 128 J_2^4 + 2 J_2 J_3^3 \\
& + 8 J_2^2 J_3^2 + 32 J_2^3 J_3 .
\end{aligned} \tag{6.32}
$$

By Shioda-Tate-Wazir theorem, the Kähler form ω of M can be expanded in the following form

$$
\begin{aligned}
\omega &= S_0 \tau + \sigma(s_Q) z + \pi^* D^{\mathrm{b}} t_{\mathrm{b}} \\
&= (J_1 - J_2 - 4J_3)\tau + (3J_2 - 2J_1) z + J_3 t_{\mathrm{b}} .
\end{aligned} \tag{6.33}
$$

We find out that the Kähler moduli parameters in the "modular basis" read

$$
\tau = 2t^1 + 3t^2 , \qquad z = t^1 + t^2 , \qquad t_{\mathrm{b}} = t^3 . \tag{6.34}
$$

Thus, we identify the curve classes of the elliptic fiber $\mathcal{E}$, the isolated fibral curve C_f, and the base curve C_{b} as follows

$$
\mathcal{E} = 3C_1 + 2C_2 , \qquad C_f = C_1 + C_2 , \qquad C_{\mathrm{b}} = C_3 . \tag{6.35}
$$

Besides that we must choose a basis of vertical fluxes. Following (6.30), we obtain that

$$\begin{aligned} \mathbf{G} &= \sigma(s_Q) \wedge \pi^* D^{\mathrm{b}} = (3J_2 - 2J_1)\, J_3\,, \\ H^b &= \pi^* D^{\mathrm{b}} \wedge \pi^* D^{\mathrm{b}} = J_3^2\,, \\ H_b^\circ &= \left(S_0 + \frac{1}{2}\pi^* c_1(B)\right) \wedge \pi^* D^{\mathrm{b}} = (J_1 - J_2 - 2J_3)\, J_3\,. \end{aligned} \tag{6.36}$$

Note that by definition $\mathbf{G}$ is a transversal flux. Moreover, the corresponding intersection form for the 4-cycles in (6.36) read

$$\eta^{(2)} = \begin{array}{cc} \begin{array}{ccc} G_4|_{U(1)} & H^b|_{U(1)} & H_b^\circ|_{U(1)} \end{array} & \\ \begin{pmatrix} -8 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} & \begin{array}{l} G_4|_{U(1)} \\ H^b|_{U(1)} \\ H_b^\circ|_{U(1)} \end{array} \end{array} . \tag{6.37}$$

We proceed to discuss the Gromov-Witten potentials associated with the 4-cycles (6.36). Our aim is to obtain elliptic genera of strings arising from D3-branes wrapping curves of class $\beta \in H_2(B, \mathbb{Z})$, as exposed in Section 6.3. For this, we must consider both Gromov-Witten potentials in the sub-cohomologies generated by the cycles H^b and $\mathbf{G}$. Moreover, we consider an irreducible curve $C_{\mathrm{b}} = D^{\mathrm{b}} \cap D^{\mathrm{b}}$, where D^{b} is a representative for the hyperplane class divisor in $B = \mathbb{P}^3$. In the following, we compute the base Fourier-coefficients $F^{(0)}_{\beta,\gamma}$ associated with the curve classes $\beta = \ell[C_{\mathrm{b}}]$ with values $\ell \in \{1, 2\}$. By abuse of notation, from now on, we drop the symbol $[\cdot]$.

First, we consider the Gromov-Witten potential $F^{(0)}(H^b)$. For the curve class $\beta = C_{\mathrm{b}}$, we obtain the expression

$$\begin{aligned} F^{(0)}_{H^b,C_{\mathrm{b}}}(\tau, z) =& \frac{1}{2^{11}3^7} \frac{q^2}{\eta^{48}} \Big[\phi_{-2,1}^3 \phi_{0,1} E_4 \left(2173 E_4^6 + 12406 E_4^3 E_6^2 + 2701 E_6^4\right) \\ & - 2\phi_{-2,1}^4 \left(835 E_4^6 E_6 + 1252 E_4^3 E_6^3 + 73 E_6^5\right) - 72 \phi_{-2,1}^2 \phi_{0,1}^2 E_4^2 E_6 \left(191 E_4^3 + 169 E_6^2\right) \\ & + \phi_{-2,1} \phi_{0,1}^3 \left(2281 E_4^6 + 13342 E_4^3 E_6^2 + 1657 E_6^4\right) - 60 \phi_{0,1}^4 E_4 E_6 \left(35 E_4^3 + 37 E_6^2\right)\Big] \end{aligned} . \tag{6.38}$$

A mirror symmetry computation, together with a modular bootstrap ansatz, yields for the case $\beta = 2C_{\mathrm{b}}$ a meromorphic quasimodular Jacobi form that reads

$$F^{(0)}_{H^b,2C_{\mathrm{b}}}(\tau, z) = \frac{q^4}{\eta^{96}(\tau)} \Phi^{(-2)}_{46,8}(\tau, z) - \frac{1}{12} E_2(\tau) \left(F^{(0)}_{H^b,C_{\mathrm{b}}}(\tau, z)\right)^2 . \tag{6.39}$$

Here $\Phi^{(-2)}_{46,8}$ is a weak Jacobi form of weight $k = 46$ and index $m = 8$ that we specify in the Appendix B.6. Moreover, the expression (6.39) satisfies the modular anomaly equation

$$\frac{\partial}{\partial E_2} F^{(0)}_{H^b,2C_{\mathrm{b}}}(\tau, z) + \frac{1}{12} \left(F^{(0)}_{H^b,C_{\mathrm{b}}}(\tau, z)\right)^2 = 0\,. \tag{6.40}$$

Note that our computations $F^{(0)}_{H^b,\ell C_{\mathrm{b}}}(\tau, z)$, at the special point $z = 0$, reproduce exactly the quasimodular forms expressions for the Gromov-Witten potential relative to the cycle $H^b \in H^{2,2}_V(X_{24})$ of X_{24}, whose corresponding holomorphic anomaly equation (6.40) at $z = 0$ was first observed in [36]. In general, by unHiggsing the geometry X_{24} to M, the holomorphic anomaly equations (6.15) and(6.16) remain valid [170]. This was further revised by the authors [170] through a BCOV theory approach [10]. Now, recall that the 4-cycle H^b gives rise to a non-transversal flux. It was conjectured in [125] that

$$\widetilde{Z}_\beta(\tau,z) = \mathrm{Coeff}\left(F^{(0)}(H^b)\,,Q^\beta\right)\,, \tag{6.41}$$

is an elliptic genus associated to the non-Calabi-Yau elliptic 3-fold, defined by $M_3 = M \cap \pi^* D^{\mathrm{b}}$.

We consider now elliptic genera of 4d $\mathcal{N}=1$ effective strings, as introduced by **Conjectures 1 & 2** in Section 6.3. The elliptic genus for $\beta = C_{\mathrm{b}}$ was computed in [125], and it reads

$$Z_{C_{\mathrm{b}}}(\tau,z) = \frac{q^2}{\eta^{48}(\tau)}\phi_{-1,2}(\tau,z)\Phi^{(-1)}_{24,4}(\tau,z) + g^{QJ}\partial_m \widetilde{Z}_{C_{\mathrm{b}}}(\tau,z)\,, \tag{6.42}$$

where $g^{QJ}=-1$, and

$$\Phi^{(-1)}_{24,4}(\tau,m) = \frac{1}{72}\eta^{24}(\tau)\left[\phi^2_{-2,1}E_4\left(163E_4^3+29E_6^2\right) - 384\phi_{-2,1}\phi_{0,1}E_4^2E_6 + \phi^2_{0,1}\left(13E_4^3+179E_6^2\right)\right]\,. \tag{6.43}$$

Following the ansatz (6.27) in **Conjecture 2**, together with the holomorphic anomaly equations of **Conjecture B** in [169], we fix the elliptic genus due to a D3-brane wrapping $\beta = 2C_{\mathrm{b}}$ via the quasi-Jacobi form ansatz

$$\begin{aligned} Z_{2C_{\mathrm{b}}}(\tau,z) =& \frac{q^4}{\eta^{96}(\tau)}\phi_{-1,2}(\tau,z)\Phi^{(-1)}_{48,6}(\tau,z) + g^{QM}E_2(\tau)\widetilde{Z}_{C_{\mathrm{b}}}(\tau,z)Z_{C_{\mathrm{b}}}(\tau,z) \\ &+ g_1^{QJ}\partial_z\widetilde{Z}_{C_{\mathrm{b}}}(\tau,z) + g_2^{QJ}E_2(\tau)\partial_z\left(\widetilde{Z}_{C_{\mathrm{b}}}(\tau,z)\right)^2\,. \end{aligned} \tag{6.44}$$

Here the values $g^{QM}=-\frac{1}{12}$, $g_1^{QJ}=-\frac{1}{2}$, $g_2^{QJ}=-\frac{1}{24}$, and $\Phi^{(-1)}_{48,6}$ in (B.59) reproduce the relative BPS invariants of $Z_{2C_{\mathrm{b}}}(\tau,z)$.

Lastly, we calculate the modular expression for the Gromov-Witten potential of the 4-cycle H_b° at base degree $b = C_{\mathrm{b}}$. This result into the quasi-Jacobi form

$$\begin{aligned} F^{(0)}_{H_b^\circ,C_{\mathrm{b}}}(\tau,z) =& \frac{q^2}{\eta^{48}(\tau)}\Phi^{(0)}_{24,4}(\tau,z) + g^{QM}E_2(\tau)F^{(0)}_{H^b,C_{\mathrm{b}}}(\tau,z) \\ &+ g_1^{QJ}\partial_z Z_{C_{\mathrm{b}}}(\tau,z) + g_2^{QJ}\partial_m^2 F^{(0)}_{H^b,C_{\mathrm{b}}}(\tau,z)\,. \end{aligned} \tag{6.45}$$

Here $g^{QM}=-\frac{1}{2}$, $g_1^{QJ}=-\frac{1}{2}$, $g_2^{QJ}=-\frac{1}{4}$, and $\Phi^{(0)}_{24,4}(\tau,m)$ in (B.60).

CHAPTER 7

Conclusions and outlook

For the sake of refreshing our goals in this thesis, let us recapitulate the introductory chapters—including the background ones. First, we proposed modularity as a mathematical principle to aid our understanding of quantum gravity. Second, we stated as a central hypotheses the interconnection between physics, geometry, and modularity. To this end, we reviewed, on the one hand, topological string theory with a focus on the topological string partition function. By duality, this object establishes a bridge between the counting of BPS particles in M-theory and the elliptic genera of strings in F-theory backgrounds. On the other hand, we introduced the F-theory dictionary for torus fibered Calabi-Yau manifolds, which translates geometry into minimal supersymmetric physics. Having established our objects of study, we now elaborate on how we address our hypotheses within our core chapters. We start by discussing the second point.

As we advanced through the main chapters, we analyzed a set of models systematically determined by a given F-theory compactification with a torus fibered Calabi-Yau $\pi : M \to B$. It is worth making a parallel comparison between the geometry and modularity we encountered for each class of theories.

Kodaira singularities of the torus fibration signal the presence of non-Abelian gauge symmetries; in particular, massless matter, transforming under group representations, localize at codimension-2 singularities in the base B. In Chapters 4 & 5, we studied these setups and realized that the elliptic genera are meromorphic lattice Jacobi forms manifesting Weyl invariance, which allows a count of gauge group representations [3], signaling, in turn, the presence of charged matter. This observation extends for Abelian gauge symmetries, but the Mordell-Weil group generates these symmetries instead, while their elliptic genera counterpart are purely meromorphic lattice Jacobi forms.

Also, in Chapters 4 & 5, we considered extensively elliptic fibrations admitting a $K3$ fibration, allowing an F-theory Heterotic dual interpretation. In contrast, we have a Hecke operator lift of Heterotic strings like elliptic genera on the modular side. When these geometries have a tensor multiplet $n_T > 1$, their effective theories have no longer a Lagrangian, and their non-perturbative behavior signals the presence of Heterotic 5-branes. In these cases, the lattice Jacobi forms describing the elliptic genera of Heterotic strings have a bi-graded extension due to quasi-modular forms.

At the end of Chapter 5, we considered geometries with a genus one fibration, which possess no section but an N-section and lead to non-trivial M-theory vacua. On the one hand, we found out that the elliptic genera of genus one fibered Calabi-Yau geometries are meromorphic Jacobi forms that transform under $\Gamma_1(N)$ [2]. On the other hand, for some cases, when genus one fibered Calabi-Yau 3-folds admit a $K3$ fibration, we managed to establish a CHL-Heterotic dual picture. With this, we realized that the elliptic genera of CHL-Heterotic strings behave as vector-valued lattice Jacobi forms.

Lastly, in Chapter 6, we worked out in detail elliptically fibered Calabi-Yau 4-folds. There, we need to introduce gauge fluxes fulfilling constraints to have an F-theory interpretation that leads to 4d $\mathcal{N} = 1$ theory. With this in mind, we exemplified that the elliptic genera of strings in 4d require odd weight meromorphic quasi-Jacobi forms that do not occur in Calabi-Yau 3-folds.

Overall, this collection of phenomena suggests a dictionary between modularity and physics; in the same way, F-theory is a dictionary between geometry and physics. To deepen into this reflection, let us examine one of our results in Chapter 5.

Two types of data characterize 6d theories with minimal supersymmetry: anomaly coefficients and the massless spectrum. We can realize this information through topological data, together with constraints of the anomaly equations. In view of this, in Chapter 5, we studied $K3$ fibered geometries that lead to 6d $N = (1, 0)$ theories with one tensor multiplet. Namely, when the base is a Hirzebruch surface $B = \mathbb{F}_n$. In these cases, applications of Noether-Lefschetz theory for counting holomorphic curves in the $K3$ fibers are possible. Then, combining the GV-spectroscopy of fibral genus zero Gromov-Witten invariants with Noether-Lefschetz symmetries, we realized that the elliptic genera of 6d Heterotic strings encode the massless spectrum and intersection data required for the anomaly coefficients. For a large class of examples [3], we observed that this information suffices to fix the elliptic genera of such 6d strings and consequently the reduced $K3$ fiber Gromov-Witten theory—or equivalently its Noether-Lefschetz theory. Schockingly, with this data we provided automatically a solution for conjectured refined BPS invariants in compact geometries [177, 178]. It would be exciting to verify if there is a one-to-one correspondence between 6d theory data $\leftrightarrow$ GW/NL theory. Note that in our treatment we restricted to one tensor multiplet, i.e., $n_T = 1$. In principle, we can also define Noether-Lefschetz theory for toric cases with $n_T > 1$, but we leave such an extension for future work.

Now, we recall our proposition for studying modularity to provide some understanding of quantum gravity. This endeavor was central in Chapter 4, where we discussed consistency of gravity beyond anomaly equations for 6d $N = (1, 0)$ theories. For this, we examined some of the main swampland conjectures in a moduli space region for a base B, leading to a weak gauge coupling $g_{\text{YM}} \to 0$. Surprisingly, we found out that the Weyl invariant lattice Jacobi forms determining the elliptic genera served as a single unifying structure that accounts for congruency of quantum gravity, at least for all swampland constraints we exposed. In particular, we managed to prove the non-Abelian sublattice weak gravity conjecture, which led us to find a novel relation between theta functions and Weyl characters [3]. The latter finding establishes a connection between modularity and representation theory, similar to the Weyl-Kač character formula in terms of theta functions of affine Lie algebras [179], but non-equivalent. This representation counting contains the massless spectrum information regarded in Chapter 5. In addition to this, we notice that this same object encodes the generalized Green-Schwarz mechanism for canceling anomalies. Moreover, we verified the completeness of the spectrum for concrete examples. A prospect for future work is to prove the completeness of the spectrum through modularity in a more general setting. Steps towards proving completeness of the spectrum of F-theory compactification have been recently initiated in [180]. Let us remark that if the spectrum is complete in a 6d $N = (1, 0)$ supergravity theory, then the absence of global symmetries should hold.

Finally, we developed a few steps in Chapter 6 to include some previously discussed results, but in a 4d $\mathcal{N} = 1$ setting. On this path, we considered compactifications on elliptically fibered Calabi-Yau 4-folds and their enumerative geometry. The increase in complexity gives rise to absent features in Calabi-Yau 3-folds and consequently in 6d $N = (1, 0)$ theories. For instance, genus zero Gromov-Witten invariants have a dependence on a choice of 4-cycles in $H^*(M)$, but Gromov-Witten theory for genus $g \geq 2$ becomes trivial. These cycle-valued invariants are captured by the quantum periods of Calabi-Yau 4-folds; for elliptic fibration cases, they transform as quasi-Jacobi forms and follow conjectured holomorphic anomaly equations, which we encountered in [1]. When fulfilling appropriate F-theory constraints, these objects are the fluxes required to achieve 4d $\mathcal{N} = 1$ physics. The same F-theory flux Gromov-Witten potentials yield the elliptic genera of 4d $\mathcal{N} = 1$ solitonic strings, allowing for a more careful treatment of the spectrum, as it gives a counting over string-like excitations besides massless states. Thus, by invoking automorphic forms associated with Calabi-Yau 4-folds, one can follow strategies to argue consistency of 4d $\mathcal{N} = 1$ string compactifications similar to those of 6d $N = (1, 0)$ theories. For instance in the work [171], the modular properties of quasi-Jacobi forms were crucial to ensure that the sublattice weak gravity conjecture holds. Yet there remains work to be done that we leave for the future. For instance, this begs for a further study of modularity with non-Abelian gauge fluxes, which are particularly interesting to formulate physics beyond the standard model. Also, we noticed that chiral index of massless matter states acquire a similar role to that of matter multiplicities in the GV-spectroscopy of Calabi-Yau 3-folds, with presumably a generalization to study non-Abelian charged matter spectra in 4d $\mathcal{N} = 1$ theories. Finally, we also raise the question if a Noether-Lefschetz on $K3$ fibered Calabi-Yau 4-folds is possible, or similarly, whether we can perform a Hecke operator lift for the elliptic genera of non-perturbative 4d $\mathcal{N} = 1$ Heterotic strings.

APPENDIX A

Lie algebras and representation theory

To guide the reader with notions of Lie algebras and representation theory, we include a summary thereof. We stick mainly to the conventions and notation of [181].

- A Lie algebra $\mathfrak{g}$ is a vector space equipped with a commutator operation $[\cdot\,,\cdot] : \mathfrak{g}\times\mathfrak{g} \to \mathfrak{g}$, that is bilinear, antisymmetric, and satisfies the Jacobi identity

$$[X,[Y,Z]] + [Z,[X,Y]] + [Y,[Z,X]] = 0 \quad \forall\ X,Y,Z \in \mathfrak{g}\,. \tag{A.1}$$

- We say that a Lie algebra $\mathfrak{g}$ is *abelian* if $[\mathfrak{g},\mathfrak{g}] = \{0\}$. We say that a Lie algebra is *simple* if it is not abelian and it contain no proper ideals, i.e., there is no non-trivial subset $\mathfrak{k} \subset \mathfrak{g}$ such that $[\mathfrak{k},\mathfrak{g}] \subset \mathfrak{k}$. A direct sum of simple Lie algebras is called *semisimple*.

- For a complex simple Lie algebra $\mathfrak{g}$, the Cartan-Weyl basis consists of the decomposition

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\boldsymbol{\alpha}\in\Phi(\mathfrak{g})} \mathfrak{g}_{\boldsymbol{\alpha}}\,. \tag{A.2}$$

  Here $\mathfrak{h}$ denotes the Cartan subalgebra of $\mathfrak{g}$, which is spanned my a maximal set of commuting Hermitian generators $\{H^i\}_{i=1,\ldots,\mathrm{rk}(\mathfrak{g})}$, where $\mathrm{rk}(\mathfrak{g}) \equiv \dim(\mathfrak{h})$. Moreover, $\Phi(\mathfrak{g})$ is the root system, which is a set of vectors $\boldsymbol{\alpha} = (\alpha^1,\ldots,\alpha^{\mathrm{rk}(\mathfrak{g})}) \in \mathbb{R}^{\mathrm{rk}(\mathfrak{g})}$ called *roots*. Furthermore, Each $\mathfrak{g}_{\boldsymbol{\alpha}}$ is the linear span over $\mathbb{C}$ of a generator $E^{\boldsymbol{\alpha}}$ in $\mathfrak{g}$. Putting all together, the Cartan-Weyl basis obeys the following commutator relations

$$[H^i,H^j] = 0\,,\quad [H^i,E^{\boldsymbol{\alpha}}] = \alpha^i E^{\boldsymbol{\alpha}}\,,\quad [E^{\boldsymbol{\alpha}},E^{\boldsymbol{\beta}}] = \begin{cases} \mathrm{const}\cdot E^{\boldsymbol{\alpha}+\boldsymbol{\beta}} & \text{if } \boldsymbol{\alpha}+\boldsymbol{\beta}\in\Phi(\mathfrak{g}) \\ \frac{2}{(\boldsymbol{\alpha},\boldsymbol{\alpha})}\boldsymbol{\alpha}\cdot\boldsymbol{H} & \text{if } \boldsymbol{\alpha}+\boldsymbol{\beta}=0 \\ 0 & \text{otherwise} \end{cases}\,. \tag{A.3}$$

- The Killing form on $\mathfrak{g}$ is the symmetric bilinear map $(\cdot\,,\cdot) : \mathfrak{g}\times\mathfrak{g} \to \mathbb{C}$ given by

$$(X,Y) = \frac{1}{2g}\mathrm{tr}\left(\mathrm{ad}_X \circ \mathrm{ad}_Y\right)\,, \tag{A.4}$$

  where $\mathrm{ad}_X : \mathfrak{g} \to \mathfrak{g}$ is the linear map defined by $\mathrm{ad}_X(Y) = [X,Y]$, and g is a normalization constant that will not be relevant for us. A finite dimensional Lie algebra $\mathfrak{g}$ is semisimple if and only if $(\cdot\,,\cdot)$ is non-degenerate.

- Through the linear map $\boldsymbol{\alpha}(H^i) = \alpha^i$, we associate roots $\boldsymbol{\alpha} \in \Phi(\mathfrak{g})$ with elements in the dual vector space $\mathfrak{h}^*$. In fact $\mathrm{Span}_{\mathbb{C}}(\Phi(\mathfrak{g})) = \mathfrak{h}^*$. Thus, we can transfer the Killing form for elements in the Cartan subalgebra $\mathfrak{h}$ as follows

$$(\boldsymbol{\alpha},\boldsymbol{\beta}) = \left(H^{\boldsymbol{\alpha}}, H^{\boldsymbol{\beta}}\right)\,, \quad H^{\boldsymbol{\gamma}} \equiv \sum_{i=1}^{\mathrm{rk}(\mathfrak{g})} \gamma^i H^i \text{ for } \boldsymbol{\gamma} \in \Phi(\mathfrak{g})\,. \tag{A.5}$$

- For each root $\boldsymbol{\alpha} \in \Phi(\mathfrak{g})$, then $-\boldsymbol{\alpha} \in \Phi(\mathfrak{g})$. It is possible find an hyperplane $\mathcal{H}$ that contains no roots, such that it divides the root system into two disjoint half-spaces $\mathcal{H}_{\pm}$. We can declare the set of *positive roots* $\Phi^+(\mathfrak{g}) \subset \Phi(\mathfrak{g})$ to be those roots lying on $\mathcal{H}_+$. A *simple root* $\boldsymbol{\alpha}_i$ is a root that cannot be written as a sum of positive roots, where $i = 1, \ldots, \mathrm{rk}(\mathfrak{g})$. There are $\mathrm{rk}(\mathfrak{g})$ simple roots that provide a basis for $\mathfrak{h}^*$.

- To each root $\boldsymbol{\alpha} \in \Phi(\mathfrak{g})$ there is an associated coroot $\boldsymbol{\alpha}^\vee$ defined by the normalization

$$\boldsymbol{\alpha}^\vee = \frac{2\boldsymbol{\alpha}}{(\boldsymbol{\alpha},\boldsymbol{\alpha})}\,. \tag{A.6}$$

- The set of simple roots $\{\boldsymbol{\alpha}_i\}_{i=1,\ldots,\mathrm{rk}(\mathfrak{g})}$ with associated simple coroots $\{\boldsymbol{\alpha}_i^\vee\}_{i=1,\ldots,\mathrm{rk}(\mathfrak{g})}$ define the Cartan matrix via the intersections

$$\mathrm{C}_{ij} = \left(\boldsymbol{\alpha}_i, \boldsymbol{\alpha}_j^\vee\right)\,. \tag{A.7}$$

- Given a complex semisimple Lie algebra $\mathfrak{g}$, every $\mathfrak{g}$-module V has a basis in which the Cartan subalgebra $\mathfrak{h}$ acts diagonally. Thus, there is a decomposition

$$V = \bigoplus_{\boldsymbol{\lambda}} V_{\boldsymbol{\lambda}}\,, \tag{A.8}$$

into weight spaces $V_{\boldsymbol{\lambda}}$, such that $R(H^i)\,|\boldsymbol{\lambda}\rangle = \lambda^i\,|\boldsymbol{\lambda}\rangle$ for all $|\boldsymbol{\lambda}\rangle \in V_{\boldsymbol{\lambda}}$. Here $R : \mathfrak{g} \to \mathrm{End}(V)$ is a representation of the Lie algebra $\mathfrak{g}$. The vectors $\boldsymbol{\lambda} = (\lambda^1, \ldots, \lambda^{\mathrm{rk}(\mathfrak{g})})$ are called *weights* of the module V.

- We associate the weights with elements in $\mathfrak{h}^*$ via the linear maps $\boldsymbol{\lambda}(H^i) = \lambda^i$. It is convenient to introduce the basis of *fundamental weights* $\{\boldsymbol{\omega}_i\}_{i=1,\ldots,\mathrm{rk}(\mathfrak{g})}$ defined by

$$\left(\boldsymbol{\omega}_i, \boldsymbol{\alpha}_j^\vee\right) = \delta_{ij}\,. \tag{A.9}$$

In this basis, each weight $\boldsymbol{\lambda}$ has an expansion

$$\boldsymbol{\lambda} = \sum_{i=1}^{\mathrm{rk}(\mathfrak{g})} \lambda^i \boldsymbol{\omega}_i\,, \quad \Longleftrightarrow \quad \lambda^i = \left(\boldsymbol{\lambda}, \boldsymbol{\alpha}_i^\vee\right)\,, \tag{A.10}$$

where each λ^i is the eigenvalue assoiated to the representation $R(H^i)$, also called *Dynkin label*. Moreover, $\boldsymbol{\lambda}$ is a weight of a finite dimensional module if and only if its Dynkin labels $\lambda^i \in \mathbb{Z}$.

- Given two weights $\boldsymbol{\lambda} = \sum_{i=1}^{\mathrm{rk}(\mathfrak{g})} \lambda^i \boldsymbol{\omega}_i$ and $\boldsymbol{\mu} = \sum_{i=1}^{\mathrm{rk}(\mathfrak{g})} \mu^i \boldsymbol{\omega}_i$, their intersection pairing reads

$$(\boldsymbol{\lambda}, \boldsymbol{\mu}) = \sum_{i,j} Q_{ij} \lambda^i \mu^j\,, \quad Q_{ij} \equiv (\boldsymbol{\omega}_i, \boldsymbol{\omega}_j)\,, \tag{A.11}$$

where we identify Q_{ij} with the weight space metric.

- There are three lattices relevant for Lie algebras. These are the weight lattice, the root lattice and the coroot lattice

$$L_{\mathrm{w}}(\mathfrak{g}) \equiv \mathrm{Span}_{\mathbb{Z}}\left\{\boldsymbol{\omega}_i\right\}\,, \quad L(\mathfrak{g}) \equiv \mathrm{Span}_{\mathbb{Z}}\left\{\boldsymbol{\alpha}_i\right\}\,, \quad L^{\vee}(\mathfrak{g}) \equiv \mathrm{Span}_{\mathbb{Z}}\left\{\boldsymbol{\alpha}_i^{\vee}\right\}\,. \tag{A.12}$$

Note that the weight lattice is the dual lattice over $\mathbb{Z}$ to the coroot lattice, i.e.,

$$L_{\mathrm{w}}(\mathfrak{g}) = \left(L^{\vee}(\mathfrak{g})\right)^* = \left\{\boldsymbol{\lambda} \mid \left(\boldsymbol{\lambda}, \boldsymbol{\alpha}^{\vee}\right) \in \mathbb{Z}\ \forall\ \boldsymbol{\alpha}^{\vee} \in L^{\vee}(\mathfrak{g})\right\}\,. \tag{A.13}$$

Throughout this paper we use the upperscript $\vee$ to denote coroots, whereas we reserve the notation $(\,\cdot\,)^*$ for dual vector spaces.

- Another integral space of interest is the set of *dominant weights* defined by

$$P_+(\mathfrak{g}) \equiv \mathbb{Z}_{\geq 0}\boldsymbol{\omega}_1 + \cdots + \mathbb{Z}_{\geq 0}\boldsymbol{\omega}_{\mathrm{rk}(\mathfrak{g})}\,. \tag{A.14}$$

- For every finite-dimensional $\mathfrak{g}$-module V there exists a maximal weight $\boldsymbol{\lambda}_R \in P_+(\mathfrak{g})$, such that

$$R(E^{\boldsymbol{\alpha}})\,|\boldsymbol{\lambda}_R\rangle = 0\,,\ \forall\ \boldsymbol{\alpha} \in \Phi^+(\mathfrak{g})\,, \tag{A.15}$$

If the $\mathfrak{g}$-module V is irreducible, then there is exactly one weight with the property (A.15). In this case, we call $\boldsymbol{\lambda}_R$ the *highest weight* of the *irreducible highest weight module* $V_{\boldsymbol{\lambda}_R}$.

- The highest weight theorem states that for any dominant weight $\boldsymbol{\lambda} \in P_+(\mathfrak{g})$ there exists a unique, irreducible, finite-dimensional representation $R_{\boldsymbol{\lambda}}$ of $\mathfrak{g}$ with highest weight $\boldsymbol{\lambda}$. Here $R_{\boldsymbol{\lambda}}$ acts on an irreducible highest weight module $V_{\boldsymbol{\lambda}}$. A proof of this theorem can be found in [124].

- For each root $\boldsymbol{\alpha} \in \Phi(\mathfrak{g})$, there is a map $w_{\boldsymbol{\alpha}} : \Phi(\mathfrak{g}) \to \Phi(\mathfrak{g})$ whose action reads

$$w_{\boldsymbol{\alpha}} : \boldsymbol{\beta} \mapsto \boldsymbol{\beta} - \left(\boldsymbol{\alpha}^{\vee}, \boldsymbol{\beta}\right)\boldsymbol{\alpha}\,. \tag{A.16}$$

The Weyl group $W(\mathfrak{g})$ is the group generated by all the $w_{\boldsymbol{\alpha}}$'s with $\boldsymbol{\alpha} \in \Phi(\mathfrak{g})$. The action of the Weyl group extends to the weight space of $\mathfrak{g}$.

- We say that a connected component of the complement of the union of the hyperplanes

$$\left\{\boldsymbol{\lambda} \in \mathbb{R}^{\mathrm{rk}(\mathfrak{g})} \mid (\boldsymbol{\lambda}, \boldsymbol{\alpha}) = 0 \quad \forall\ \boldsymbol{\alpha} \in \Phi(\mathfrak{g})\right\}\,, \tag{A.17}$$

is an *open Weyl chamber*. We call *Weyl chamber* the closure of an open Weyl chamber. The Weyl group acts simply transitively on the set of Weyl chambers, i.e. for every pair of Weyl chambers $\mathcal{C}, \mathcal{C}'$ there is exactly one $w \in W(\mathfrak{g})$ such that $w\mathcal{C} = \mathcal{C}'$. We define the fundamental Weyl chamber $\mathcal{W}(\mathfrak{g})$ by the set

$$\mathcal{W}(\mathfrak{g}) = \left\{\boldsymbol{\lambda} \in \mathbb{R}^{\mathrm{rk}(\mathfrak{g})} \mid (\boldsymbol{\lambda}, \boldsymbol{\alpha}_i) \geq 0\,, \quad i = 1, \ldots, \mathrm{rk}(\mathfrak{g})\right\}\,. \tag{A.18}$$

APPENDIX B

Modular Appendix

B.1 Modular forms

Here we present the basics about modular forms, based on the reference [182].

- The uper-half plane is the complex subset $\mathbb{H} = \{\tau \in \mathbb{C} \mid \mathrm{Im}(\tau) > 0\}$.
- An elliptic curve $\mathcal{E}$ over $\mathbb{C}$ is analytically isomorphic to a torus $\mathbb{C}/\Lambda$ with complex structure τ, where $\Lambda = \mathbb{Z}\tau + \mathbb{Z}$. Thus, we can express $\mathcal{E}$ as

$$\mathcal{E} \simeq \{z \in \mathbb{C} \mid z \sim z + (n + m\tau)\}\,, \quad n, m \in \mathbb{Z}\,, \quad \tau \in \mathbb{H}\,. \tag{B.1}$$

- The set of transformations

$$\mathrm{SL}(2,\mathbb{Z}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a, b, c, d \in \mathbb{Z} \text{ and } ad - bc = 1 \right\}\,, \tag{B.2}$$

  that act on $\mathbb{H}$ as

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} : \mathbb{H} \to \mathbb{H}\,, \quad \gamma : \tau \mapsto \frac{a\tau + b}{c\tau + d}\,, \tag{B.3}$$

  leave the lattice $\Lambda = \mathbb{Z}\tau + \mathbb{Z}$ and hence the shape of the torus $\mathbb{C}/\Lambda$ invariant.

- The group $\mathrm{SL}(2,\mathbb{Z})$ is called *the full modular group*. Given a $N \in \mathbb{N}$, we define the following congruence subgroups of $\mathrm{SL}(2,\mathbb{Z})$ as

$$\begin{aligned}
\Gamma_0(N) &= \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{Z}) \mid c \equiv 0 \ (\mathrm{mod}\ N) \right\}\,, \\
\Gamma_1(N) &= \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{Z}) \mid c \equiv 0 \ (\mathrm{mod}\ N) \text{ and } a \equiv d \equiv 1 \ (\mathrm{mod}\ N) \right\}\,, \\
\Gamma(N) &= \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{Z}) \mid b \equiv c \equiv 0 \ (\mathrm{mod}\ N) \text{ and } a \equiv d \equiv 1 \ (\mathrm{mod}\ N) \right\}\,.
\end{aligned} \tag{B.4}$$

- A modular form $f(\tau)$ of weight k on $\Gamma \subseteq \mathrm{SL}(2,\mathbb{Z})$ is a holomorphic function on $\mathbb{H}$, which transforms

$$f\left(\frac{a\tau + b}{c\tau + d}\right) = (c\tau + d)^k f(\tau)\,, \quad \forall \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma\,. \tag{B.5}$$

Moreover, a modular form f is periodic, as $f(\tau+1) = f(\tau)$, and has a Fourier expansion

$$f(\tau) = \sum_{n\in\mathbb{Z}} a(n) q^n\,, \quad q \equiv \mathrm{e}^{2\pi i\tau}\,, \tag{B.6}$$

that is bounded as $\mathrm{Im}(\tau) \to \infty$. If $a(0) = 0$, then the modular form vanishes at $\mathrm{Im}(\tau) \to \infty$ and is called a *cusp form.*

- We denote by $M_k(\Gamma)$ the vector space of modular forms of weight k on $\Gamma \subseteq \mathrm{SL}(2,\mathbb{Z})$. The space $M_k(\Gamma)$ is finite-dimensional for all k—and zero for $k < 0$—, and the graded ring $M_*(\Gamma) \equiv \bigoplus_k M_k(\Gamma)$ of all modular forms on Γ is finitely generated over $\mathbb{C}$.

- The Eisenstein series $E_{2k}(\tau)$, with $k > 1$, are modular forms on $\Gamma = \mathrm{SL}(2,\mathbb{Z})$ of weight $2k$ and can be written as

$$E_{2k}(\tau) = 1 + \frac{2}{\zeta(1-2k)} \sum_{n=1}^{\infty} \frac{n^{2k-1} q^n}{1-q^n}\,, \tag{B.7}$$

where $\zeta(s)$ is the Riemann zeta function.

- The Dedekind η-function

$$\eta(\tau) = q^{\frac{1}{24}} \prod_{i=1}^{\infty} (1-q^i)\,, \tag{B.8}$$

is also not quite modular but satisfies

$$\eta(\tau+1) = e^{\frac{\pi i}{12}} \eta(\tau)\,, \quad \eta(-1/\tau) = \sqrt{-i\tau}\,\eta(\tau)\,. \tag{B.9}$$

However, $\Delta_{12}(\tau) \equiv \eta^{24}(\tau)$ is a cusp form of weight 12 on $\Gamma = \mathrm{SL}(2,\mathbb{Z})$ that vanishes as $\mathrm{Im}(\tau) \to \infty$.

- If we consider the case $k = 1$ in (B.7), we obtain the second Eisenstein series E_2. However, E_2 transforms under a $\mathrm{SL}(2,\mathbb{Z})$ transformation as

$$E_2\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^2 E_2(\tau) - \frac{6}{\pi} ic(c\tau+d)\,, \quad \forall \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{Z})\,, \tag{B.10}$$

which is not quite modular. Nevertheless, the object

$$\widehat{E}_2(\tau) \equiv E_2(\tau) - \frac{3}{\pi \mathrm{Im}(\tau)}\,, \tag{B.11}$$

follows the transformation behavior (B.6) with weight $k = 2$, at the expense of holomorphicity.

- An almost holomorphic modular form F on $\Gamma \subseteq \mathrm{SL}(2,\mathbb{Z})$ of weight k and depth p is defined as a polynomial over $(\mathrm{Im}(\tau))^{-1}$ of the form

$$F(\tau) = \sum_{r=0}^{p} f_r(\tau) \left(-4\pi \mathrm{Im}(\tau)\right)^{-r}\,, \tag{B.12}$$

where each coefficient $f_r(\tau)$ is an holomorphic function with similar growth conditions as modular forms. Every $f_r(\tau)$ transforms in a certain way under Γ such that

$$F\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^k F(\tau)\,, \quad \forall \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma\,. \tag{B.13}$$

Unlike modular forms, F is non-holomorphic. On the other hand, we say that the holomorphic term $f_0(\tau)$ is a *quasimodular form* on Γ of weight k.

- $\widehat{E}_2$ is an almost holomorphic modular form on $\mathrm{SL}(2,\mathbb{Z})$ of weight 2 and depth 1, while E_2 is a quasimodular form on $\mathrm{SL}(2,\mathbb{Z})$ of weight 2.
- We denote by $\widehat{M}_k^{(p)}(\Gamma)$ the space of almost holomorphic modular forms on $\Gamma \subseteq \mathrm{SL}(2,\mathbb{Z})$ of weight k and depth p; $\widehat{M}_k(\Gamma) = \cup_p \widehat{M}_k^{(p)}(\Gamma)$ is the space of all almost holomorphic modular forms on Γ of weight k; $\widehat{M}_*(\Gamma) \equiv \bigoplus_k \widehat{M}_k(\Gamma)$ is the graded ring of all almost holomorphic modular forms on Γ. Restricting all $F \in \widehat{M}_*(\Gamma)$, to their quasimodular forms terms, we obtain the graded ring of quasimodular forms $\widetilde{M}_*(\Gamma)$. The map $F \mapsto f_0$ yields an isomorphism $\widehat{M}_*(\Gamma) \simeq \widetilde{M}_*(\Gamma)$.
- Every quasimodular form on Γ is a polynomial over a quasimodular form $\phi \in \widetilde{M}_2(\Gamma)$ with coefficients that are modular forms on Γ, i.e., $\widetilde{M}_*(\Gamma) = M_*(\Gamma) \otimes \mathbb{C}[\phi]$.
- Let us introduce the following holomorphic modular form on $\Gamma = \Gamma_1(N)$ of weight 2 with $N \in \{2,3,4\}$
$$\mathcal{E}_N(\tau) \equiv -\frac{1}{N-1}\frac{\partial_\tau}{2\pi i}\log\left(\frac{\eta(\tau)}{\eta(N\tau)}\right)\,. \tag{B.14}$$
- We can generate the rings of even weight modular forms as follows
$$\begin{aligned} M_{2*}(\mathrm{SL}(2,\mathbb{Z})) &= \langle E_4(\tau), E_6(\tau)\rangle\,, \\ M_{2*}(\Gamma_1(2)) &= \langle \mathcal{E}_2(\tau), E_4(\tau)\rangle\,, \\ M_{2*}(\Gamma_1(3)) &= \langle \mathcal{E}_3(\tau), E_4(\tau)\rangle\,, \\ M_{2*}(\Gamma_1(4)) &= \langle \mathcal{E}_2(\tau), \mathcal{E}_4(\tau)\rangle\,. \end{aligned} \tag{B.15}$$
- To use the modular bootstrap of genus one fibered Calabi-Yau 3-folds with N-section, where $N \in \{2,3,4\}$, we introduce the following modular forms
$$\begin{aligned} \Delta_4(\tau) &\equiv \frac{1}{192}\left(E_4(\tau) - \mathcal{E}_2^2(\tau)\right)\,, \\ \Delta_6(\tau) &\equiv \frac{1}{2^4\cdot 3^6}\left(7\mathcal{E}_3^3 - 5\mathcal{E}_3 E_4 - 2E_6\right)\,, \\ \Delta_8(\tau) &\equiv \frac{1}{2^{17}\cdot 3^2\cdot 17}\left(187\mathcal{E}_2^4 - 144\mathcal{E}_4^4 - 33E_4^2 - E_6(154\mathcal{E}_2 - 144\mathcal{E}_4)\right)\,. \end{aligned} \tag{B.16}$$
- **Proposition A:** Let $f(\tau)$ be a meromorphic modular form on $\Gamma \subseteq \mathrm{SL}(2,\mathbb{Z})$ of positive weight k and $t(z)$ a modular function with respect to Γ. If we can express locally $f(\tau)$ as $\Phi(z(\tau))$, then the function $\Phi(z)$ satisfies a linear differential equation of order $k+1$ with algebraic coefficients, or with polynomial coefficients if $\Gamma/\mathbb{H}$ has genus zero and $z(\tau)$ generates the field of modular functions on Γ.

B.2 Jacobi forms of lattice index

Let L denote an integral lattice L equipped with a symmetric non-degenerate bilinear form $(\cdot\,,\cdot) : L \times L \to \mathbb{Z}$. A Jacobi form of weight k and index L is a holomorphic function $\phi(\tau, \boldsymbol{z})$ of variables $\tau \in \mathbb{H}$ and $\boldsymbol{z} \in L \otimes \mathbb{C}$ which satisfies the following properties:

- **Modular transformation:** For all $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{Z})$ it satisfies

$$\phi\left(\frac{a\tau+b}{c\tau+d}, \frac{\boldsymbol{z}}{c\tau+d}\right) = (c\tau+d)^k \mathrm{e}^{\frac{2\pi i c(\boldsymbol{z},\boldsymbol{z})}{c\tau+d}} \phi(\tau, \boldsymbol{z})\,. \tag{B.17}$$

- **Elliptic transformation:** For all $\boldsymbol{\lambda}, \boldsymbol{\mu} \in L$ it satisfies

$$\phi(\tau, \boldsymbol{z} + \boldsymbol{\lambda}\tau + \boldsymbol{\mu}) = \mathrm{e}^{-2\pi i\left(\frac{1}{2}(\boldsymbol{\lambda},\boldsymbol{\lambda}) + (\boldsymbol{\lambda},\boldsymbol{z})\right)} \phi(\tau, \boldsymbol{z})\,. \tag{B.18}$$

- **Fourier expansion:** The Fourier expansion of ϕ is of the form

$$\phi(\tau, \boldsymbol{z}) = \sum_{n \geq n_0} \sum_{\boldsymbol{\lambda} \in L^*} c(n, \boldsymbol{\lambda}) q^n \mathrm{e}^{2\pi i (\boldsymbol{\lambda}, \boldsymbol{z})}\,, \tag{B.19}$$

where $q = \exp(2\pi i \tau)$, $n_0 \in \mathbb{Z}$ and L^* is the dual lattice of L.

If the Fourier coefficients $c(n, \boldsymbol{\lambda})$ of ϕ vanish for $n < 0$, we say that ϕ is a weak Jacobi form. If on the other hand $c(n, \boldsymbol{\lambda})$ vanishes unless $2n - (\boldsymbol{\lambda}\,, \boldsymbol{\lambda}) \geq 0$ ($2n - (\boldsymbol{\lambda}\,, \boldsymbol{\lambda}) > 0$), we say that ϕ is a holomorphic Jacobi form (cusp Jacobi form). Otherwise, we say that ϕ is a weakly holomorphic Jacobi form. We respectively denote by

$$J^!_{k,L} \supseteq J^w_{k,L} \supseteq J_{k,L} \supseteq J^{\mathrm{cusp}}_{k,L}\,. \tag{B.20}$$

the vector spaces of weakly holomorphic, weak, holomorphic and cusp Jacobi forms of weight k and index L. It is possible to extend this definition of Jacobi forms by including characters or by considering odd lattices. We refer the reader to the reference [183] for a broader view of the theory of lattice index Jacobi forms.

One variable Jacobi forms: The ring of weak Jacobi forms of a single variable and integer index is freely generated over the ring of modular froms by the two generators

$$\begin{aligned} \phi_{-2,1}(\tau, z) =& -\frac{\theta_1(\tau,z)^2}{\eta(\tau)^6} \\ \phi_{0,1}(\tau, z) =& 4\left[\frac{\theta_2(\tau,z)^2}{\theta_2(\tau,0)^2} + \frac{\theta_3(\tau,z)^2}{\theta_3(\tau,0)^2} + \frac{\theta_4(\tau,z)^2}{\theta_4(\tau,0)^2}\right]\,, \end{aligned} \tag{B.21}$$

of index one and respective weight -2 and 0. Here the expressions are written in terms of

Jacobi theta functions, which are defined as

$$\begin{aligned} \theta_1(\tau,z) =& \vartheta_{\frac{1}{2}\frac{1}{2}}(\tau,z)\,, \quad \theta_2(\tau,z) = \vartheta_{\frac{1}{2}0}(\tau,z)\,, \\ \theta_3(\tau,z) =& \vartheta_{00}(\tau,z)\,, \quad \theta_4(\tau,z) = \vartheta_{0\frac{1}{2}}(\tau,z)\,, \\ \vartheta_{ab}(\tau,z) =& \sum_{n=-\infty}^{\infty} e^{\pi i(n+a)^2\tau+2\pi iz(n+a)+2\pi ib(n+a)}\,. \end{aligned} \tag{B.22}$$

Another useful single variable Jacobi form is

$$\phi_{-1,\frac{1}{2}}(\tau,z) = i\frac{\theta_1(\tau,z)}{\eta(\tau)^3}\,, \tag{B.23}$$

which has weight -1 and index $1/2$.

The theta expansion of lattice Jacobi forms The Jacobi theta functions associated to an integral lattice L are defined as

$$\vartheta_{L,\boldsymbol{\mu}}(\tau,\boldsymbol{z}) = \sum_{\substack{\boldsymbol{\lambda}\in L^* \\ \boldsymbol{\lambda}\equiv\boldsymbol{\mu} \bmod L}} q^{\frac{1}{2}(\boldsymbol{\lambda},\boldsymbol{\lambda})} \exp\left(2\pi i(\boldsymbol{\lambda},\boldsymbol{z})\right)\,, \tag{B.24}$$

and they span a $\mathbb{C}$-vector space

$$\Theta(L) = \mathrm{Span}_{\mathbb{C}}\left\{\vartheta_{L,\boldsymbol{\mu}} \;\middle|\; \boldsymbol{\mu}\in L^*/L\right\}\,, \tag{B.25}$$

which has dimension $|L^*/L|$. Every lattice index Jacobi form $\phi \in J_{k,L}$ has a theta expansion of the following form [183, 184]

$$\phi(\tau,\boldsymbol{z}) = \sum_{\boldsymbol{\mu}\in L^*/L} h_{\boldsymbol{\mu}}(\tau)\vartheta_{L,\boldsymbol{\mu}}(\tau,\boldsymbol{z})\,. \tag{B.26}$$

Moreover, the space of Jacobi theta functions $\Theta(L)$ follows invariance under the metaplectic group $\mathrm{Mp}(2,\mathbb{Z})$ [184]. This enables to define a projection of the theta expansions coefficients (B.26), as [183, 185]

$$h : J_{k,L} \to M_{k-\frac{r}{2}}(\rho_L^*)\,, \quad \phi(\tau,\boldsymbol{z}) \mapsto h(\phi) \equiv \sum_{\boldsymbol{\mu}\in L^*/L} h_{\boldsymbol{\mu}}(\tau)\mathrm{e}_{\boldsymbol{\mu}}\,. \tag{B.27}$$

which yields an isomorphism with between Jacobi forms and vector-valued modular forms [176, 184, 185].

Weyl invariant Jacobi forms: Let us consider the special case when the lattice index is $L = \bigoplus_{I=1}^{a} L^\vee(\mathfrak{g}_I)(m_I)$, where each lattice $L^\vee(\mathfrak{g}_I)(m_I)$ denotes the coroot lattice with m_I-twist that is associated to a simple Lie algebra $\mathfrak{g}_I$. Here we take as bilinear form for $L^\vee(\mathfrak{g}_I)$ the Killing form on $\mathfrak{g}_I$[1], which we deonte by $(\cdot,\cdot)_{\mathfrak{g}_I}$. Consequently, $L^\vee(\mathfrak{g}_I)(m_I)$ has bilinear form $m_I\,(\cdot,\cdot)_{\mathfrak{g}_I}$. In this treatment, we have a semisimple Lie algebra $\mathfrak{g} = \bigoplus_{I=1}^{a}\mathfrak{g}_I$ with Weyl group $W(\mathfrak{g})$. We say that a Jacobi form $\phi \in J_{k,L}$ is a $W(\mathfrak{g})$ *Weyl invariant Jacobi form*, if in addition to the properties (B.17), (B.18), and (B.19), it fulfils the symmetry:

[1] Here we take a normalization, such that the highest root θ in $\mathfrak{g}_I$ has norm $(\theta,\theta)_{\mathfrak{g}_I} = 2$.

- **Weyl symmetry:** For all $w \in W(\mathfrak{g})$, $\phi \in J_{k,L}$ satisfies

$$\phi\left(\tau, w(\boldsymbol{z})\right) = \phi(\tau, \boldsymbol{z})\,. \tag{B.28}$$

***A_n* Jacobi forms:** Let us introduce the differential operator for an A_n Lie algebra

$$\mathcal{L} = \frac{1}{2\pi i}\left(\sum_{i=1}^{n+1}\frac{\partial}{\partial x_i} + \frac{\pi^2}{3}E_2\sum_{i=1}^{n+1}x_i\right), \tag{B.29}$$

and the product form

$$\Phi^{A_n} = \prod_{i=1}^{n+1}\phi_{-1,\frac{1}{2}}(\tau, x_i)\,. \tag{B.30}$$

Then the ring of A_n Weyl invariant weak Jacobi forms is generated by [186, 187]

$$\phi^{A_n}_{-k,1} = \left(\mathcal{L}^{n+1-k}\Phi^{A_n}\right)\Big|_{\sum x_i=0}, \qquad k = 0, 2, 3, \ldots, n+1\,, \tag{B.31}$$

where the condition $\sum x_i = 0$ is imposed after acting by the operator $\mathcal{L}$.

The discriminant group of $L^\vee(\mathfrak{g}_I)(m_I)$ can be defined by the following quotients [188]

$$(L^\vee(\mathfrak{g}_I)(m_I))^*/L^\vee(\mathfrak{g}_I)(m_I) \cong (L_{\mathrm{w}}(\mathfrak{g}_I)/\sqrt{m_I})/(\sqrt{m_I}L^\vee(\mathfrak{g}_I)) \cong L_{\mathrm{w}}(\mathfrak{g}_I)/m_I L^\vee(\mathfrak{g})\,. \tag{B.32}$$

It is convenient to use the last quotient form to express the theta functions (B.24) for L in terms of the weight lattices $L_{\mathrm{w}}(\mathfrak{g}_I)$ and the m_I-scaled coroot lattices $m_I L^\vee(\mathfrak{g}_I)$. By using (B.24), we arrive to the following expression for each theta function of $L^\vee(\mathfrak{g}_I)(m_I)$ [179, 189]

$$\vartheta^{\mathfrak{g}_I}_{m_I,\boldsymbol{\mu}_I}(\tau, \boldsymbol{z}_I) = \sum_{\substack{\boldsymbol{\omega}\in L_{\mathrm{w}}(\mathfrak{g}_I)\\ \boldsymbol{\omega}\equiv\boldsymbol{\mu}_I \bmod m_I L^\vee(\mathfrak{g}_I)}} q^{\frac{1}{2m_I}(\boldsymbol{\omega},\boldsymbol{\omega})_{\mathfrak{g}_I}}\,\mathrm{e}^{2\pi i(\boldsymbol{\omega},\boldsymbol{z})_{\mathfrak{g}_I}}\,. \tag{B.33}$$

The extension for a semisimple Lie algebra $\mathfrak{g} = \bigoplus_{I=1}^{a}\mathfrak{g}_I$ generalizes as follows

$$\vartheta^{\mathfrak{g}}_{\boldsymbol{m},\boldsymbol{\mu}}(\tau, \boldsymbol{z}) = \prod_{I=1}^{a}\vartheta^{\mathfrak{g}_I}_{m_I,\boldsymbol{\mu}_I}(\tau, \boldsymbol{z}_I)\,, \tag{B.34}$$

where $\boldsymbol{m} = (m_1, \ldots, m_a)$, $\boldsymbol{z} = (\boldsymbol{z}_1; \ldots; \boldsymbol{z}_a)$, $\boldsymbol{z}_I \in L^\vee(\mathfrak{g}_I)\otimes\mathbb{C}$, and

$$\boldsymbol{\mu} = (\boldsymbol{\mu}_1; \ldots; \boldsymbol{\mu}_a) \in L_{\mathrm{w}}(\mathfrak{g}_1)/m_1L^\vee(\mathfrak{g}_1)\oplus\cdots\oplus L_{\mathrm{w}}(\mathfrak{g}_a)/m_aL^\vee(\mathfrak{g}_a)\,. \tag{B.35}$$

B.3 Quasi-Jacobi forms

Here we define quasi-Jacobi forms and related objects briefly [169].

Consider $\boldsymbol{z} = (z_1, \ldots, z_n) \in \mathbb{C}^n$. We define the real analytic functions $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_n)$ in the following way:

$$\alpha_i(\tau, \boldsymbol{z}) = \frac{z_i - \bar{z}_i}{\tau - \bar{\tau}_i} = \frac{\mathrm{Im}\,(z_i)}{\mathrm{Im}\,(\tau_i)}\,. \tag{B.36}$$

The object (B.36) satisfies the following behaviour under modular transformations

$$\boldsymbol{\alpha}\left(\frac{a\tau+b}{c\tau+d},\frac{\boldsymbol{z}}{c\tau+d}\right)=(c\tau+d)\boldsymbol{\alpha}(\tau,\boldsymbol{z})-c\boldsymbol{z}\quad\forall\quad\begin{pmatrix}a&b\\c&d\end{pmatrix}\in\mathrm{SL}(2,\mathbb{Z})\,. \tag{B.37}$$

Moreover, an elliptic transformation on $\boldsymbol{\alpha}(\tau,\boldsymbol{z})$ reads

$$\boldsymbol{\alpha}(\tau,\boldsymbol{z}+\boldsymbol{\lambda}\tau+\boldsymbol{\mu})=\boldsymbol{\alpha}(\tau,\boldsymbol{z})+\boldsymbol{\lambda}\quad\forall\quad\boldsymbol{\lambda},\boldsymbol{\mu}\in\mathbb{Z}^n\,. \tag{B.38}$$

Consider the almost holomorphic function on $\mathbb{H}\times\mathbb{C}^n$ of the following form

$$\widehat{\psi}(\tau,\boldsymbol{z})=\sum_{i,j_1,\dots,j_n\geq0}\psi_{i,j_1,\dots,j_n}(\tau,\boldsymbol{z})\left(\frac{1}{\mathrm{Im}\tau}\right)^i\alpha_1^j\cdots\alpha_n^j\,, \tag{B.39}$$

where each $\psi_{i,j_1,\dots,j_n}$ is holomorphic with Fourier expansion convergent in $|q|<1$. We say that $\widehat{\psi}(\tau\,,\boldsymbol{z})$ is an *almost holomorphic weak Jacobi form* of wieght k and index L if it follows the transformation law of an ordinary Jacobi form.

A *quasi-Jacobi form* $\psi(\tau,\boldsymbol{z})$ of weight k and index L is a function on $\mathbb{H}\times\mathbb{C}^n$ such that there exists an almost holomorphic weak Jacobi form $\widehat{\psi}(\tau,\boldsymbol{z})$ of weight k and index L with $\psi_{0,0,\dots,0}(\tau,\boldsymbol{z})=\psi(\tau,\boldsymbol{z})$. Let us denote by $\mathrm{AHJ}_{k,L}$ ($\mathrm{QJac}_{k,L}$) the vector space of almost holomorphic weak Jacobi forms (quasi-Jacobi forms) of weight k and index L. In fact, the constant map given by $\widehat{\psi}(\tau,\boldsymbol{z})\mapsto\psi(\tau,\boldsymbol{z})$ establishes the isomorphism $\mathrm{AHJ}_{k,L}\simeq\mathrm{QJac}_{k,L}$.

For the case of a single elliptic parameter, we define the ring of *quasimodular Jacobi forms* $J^{QM}_{*,*}$ as the space spanned by the following set

$$J^{QM}_{*,*}=\{E_2,E_4,E_6,\phi_{0,1},\phi_{-2,1},\phi_{-1,2}\}\subset\mathrm{QJac}_{*,*}\,. \tag{B.40}$$

B.4 Vector-valued modular forms

In this section, we review the theory of vector-valued modular forms based on the literature [185, 190, 191]. For more details on the subject, we encourage the reader to consult the latter references.

- The metaplectic group $\mathrm{Mp}(2,\mathbb{Z})$ is a double cover of $\mathrm{SL}(2,\mathbb{Z})$. Its elements are pairs of the form

$$\left(\gamma=\begin{pmatrix}a&b\\c&d\end{pmatrix},\pm\sqrt{c\tau+d}\right)\,, \tag{B.41}$$

  where $\gamma\in\mathrm{SL}(2,\mathbb{Z})$ and $w_\gamma(\tau)\equiv\sqrt{c\tau+d}$ is a holomorphic function on $\mathbb{H}$ whose square is $c\tau+d$. The multiplication in $\mathrm{Mp}(2,\mathbb{Z})$ follows the composition law $(\gamma_1,w_{\gamma_1}(\tau))\cdot(\gamma_2,w_{\gamma_2}(\tau))=(\gamma_1\gamma_2,w_{\gamma_1}(\gamma_2\tau)w_{\gamma_2}(\tau))$.

- Let V be an r-dimensional vector space and let $\rho:\mathrm{Mp}(2,\mathbb{Z})\to\mathrm{End}(V)$ be a representation of $\mathrm{Mp}(2,\mathbb{Z})$. A vector-valued modular form of weight k and type ρ, where $k\in\frac{1}{2}\mathbb{Z}$, is a real analytical function $\Phi:\mathbb{H}\to V$ that transforms as follows

$$\Phi(\gamma\tau)=w_\gamma(\tau)^{2k}\rho\left(\gamma,w_\gamma(\tau)\right)\Phi(\tau)\,,\ \text{for all}\ \left(\gamma,w_\gamma(\tau)\right)\in\mathrm{Mp}(2,\mathbb{Z})\,. \tag{B.42}$$

  We denote the space of such vector-valued modular forms by $M_k(\rho)$. An element

$\Phi \in M_k(\rho)$ has a Fourier expansion at the cusp at infinity of the form

$$\Phi(\tau) = \sum_{\ell=1}^{r} \sum_{n\in\mathbb{Q}} c_\ell(n) q^n \mathrm{e}_\ell \,, \tag{B.43}$$

where $\{\mathrm{e}_\ell\}$ is a basis of V.

- Let $G = L^*/L$ be the discriminant group endowed with a quadratic form $P : L^*/L \to \mathbb{Q}/\mathbb{Z}$ defined by $\boldsymbol{\lambda} + L \mapsto \frac{1}{2}(\boldsymbol{\lambda}, \boldsymbol{\lambda}) \bmod \mathbb{Z}$, where L is an even lattice with quadratic form $(\cdot\,,\cdot)$. A canonical choice for a representation $\rho : \mathrm{Mp}(2,\mathbb{Z}) \to \mathrm{End}\,(\mathbb{C}[G])$ is the *Weil representation*. The latter is defined by the following action on the standard basis $\{\mathrm{e}_{\boldsymbol{\lambda}}\}_{\boldsymbol{\lambda}\in G}$ of $\mathbb{C}[G]$:

$$\begin{aligned} \rho(T)(\mathrm{e}_{\boldsymbol{\lambda}}) &= \exp\left(2\pi i P(\boldsymbol{\lambda})\right) \mathrm{e}_{\boldsymbol{\lambda}} \,, \\ \rho(S)(\mathrm{e}_{\boldsymbol{\lambda}}) &= \frac{(-i)^{\frac{\mathrm{sign}(L)}{2}}}{\sqrt{|L^*/L|}} \sum_{\boldsymbol{\mu}\in L^*/L} \mathrm{e}^{2\pi i\left(P(\boldsymbol{\lambda})+P(\boldsymbol{\mu})-P(\boldsymbol{\lambda}+\boldsymbol{\mu})\right)} \mathrm{e}_{\boldsymbol{\mu}} \,, \end{aligned} \tag{B.44}$$

  where $\mathrm{sign}(L)$ denotes the signature of the lattice L and $T, S \in \mathrm{Mp}(2,\mathbb{Z})$ read

$$T = \left(\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, 1 \right) , \quad S = \left(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \sqrt{\tau} \right) . \tag{B.45}$$

  Moreover, ρ^* denotes the inverse transpose of ρ.

- A relevant normal subgroup of $\mathrm{Mp}(2,\mathbb{Z})$ is the following one

$$\Gamma(4N)^* = \left\{ \left(\gamma\,, \frac{\theta(\gamma\tau)}{\theta(\tau)}\right) \in \mathrm{Mp}(2,\mathbb{Z}) \,\middle|\, \gamma \in \Gamma(4N) \right\} , \tag{B.46}$$

  where $\theta(\tau) = \sum_{n\in\mathbb{Z}} q^{n^2}$ and $\Gamma(4N)$ is a principal congruence subgroup of $\mathrm{SL}(2,\mathbb{Z})$ defined by

$$\Gamma(4N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{Z}) \,\middle|\, b \equiv c \equiv 0 \pmod N \text{ and } a \equiv d \equiv 1 \pmod N \right\} . \tag{B.47}$$

- Let Γ be an even lattice with signature (b^+, b^-), dual lattice Γ^*, and an isometry $P : \Gamma \otimes \mathbb{R} \to \mathbb{R}^{b^+,b^-}$. The isometry P defines projections on $\mathbb{R}^{b^+,0}, \mathbb{R}^{0,b^-}$ written as $P_+(p) = p_R$, $P_-(p) = p_L$ respectively. The Siegel Theta function Θ_Γ of Γ is defined by

$$\Theta_\Gamma(\tau,\bar\tau) = \sum_{\gamma\in\Gamma^*/\Gamma} \theta_{\Gamma+\gamma}(\tau,\bar\tau) \mathrm{e}_\gamma \,, \quad \theta_{\Gamma+\gamma}(\tau,\bar\tau) = \sum_{p\in\Gamma+\gamma} q^{\frac{p_L^2}{2}} \bar{q}^{\frac{p_R^2}{2}} \,, \tag{B.48}$$

  where $\tau \in \mathbb{H}$. Further generalizations of Siegel theta functions can be found in [138].

B.5 The modular bootstrap of elliptic and genus one fibrations

Here we summarize our main findings in our work [2]. There, we determine how to formulate an ansatz for each coefficient $Z_\beta(\lambda,\tau,\boldsymbol{z})$ in (2.125). After exposing our results, we motivate our ansatz by considering three ingredients that determine automorphic properties of Z_{top}:

1. Using the action of monodromies on the periods of the mirror dual W, it is possible to find a $\Gamma_1(N) \subset \mathrm{SL}(2,\mathbb{Z})$ group that embeds into $\mathrm{Sp}(b_3(W),\mathbb{Z})$. In the A-model on M, the monodromies correspond to autoquivalences in $D^b(M)$ that yield directly an action for $\Gamma_1(N)$ on the Kähler moduli $(\tau, \boldsymbol{z})$. With these transformations we argue that $Z_\beta(\tau, \boldsymbol{z}, \lambda)$ fulfills the quasi-periodicity property on the moduli $\boldsymbol{z}$, which reveals the matrix index $\boldsymbol{m}_\beta$ associated to the elliptic parameters $\boldsymbol{z}$.

2. Witten's wave function equation [192] determines the index m_β^λ of Z_β, which we associate with the topological string coupling λ.

3. In contrast to the case of elliptic fibrations, for a genus one fibered Calabi-Yau 3-fold M with N-section, we must consider an additional automorphic form factor in the modular ansatz. More precisely, it is a rational power of the modular form Δ_{2N} given in (B.16). We interpret this factor as a reminder of the topological string partition function of $M_{U(1)}$, where the latter geometry is an elliptic fibration that descends into M through a conifold transition $M_{U(1)} \to M$ [82].

We introduce now the ansatz for a genus one fibered Calabi-Yau threefold M with N-sections for $N \in \{2,3,4\}$. To this end, let us start by introducing the geometrical data we need. Firstly, by Shioda-Tate-Wazir theorem generalized to genus one fibered Calabi-Yau threefolds [23, 60], we expand the Kähler form as

$$\omega = \tau \cdot (S_0 + D) + \sum_{i=1}^{r} z_i \cdot \sigma(S_i) + \sum_{i=r+1}^{\mathrm{rk}(\mathfrak{g})} z_i \cdot E_{i-r} + \sum_{i=1}^{b_2(B)} \tilde{t}_i \cdot D_i' \,, \tag{B.49}$$

where $\{E_i\}_{i=1,\dots,\mathrm{rk}(\mathfrak{g})}$ denotes a basis of fibral divisors and $\{S_i\}_{i=0,\dots,r}$ is a set of independent N-section divisors. Here S_0 is the "zero N-section" and the vertical divisors $\{D_i'\}_{i=1,\dots,b_2(B)}$ are dual to the curves $C_i = S_0 \cdot D_i$, $i = 1, \dots, b_2(B)$ with $D_i = \pi^{-1} D_i^{\mathrm{b}}$ in the sense that

$$D_i' \cdot C_j = N \cdot \delta_{ij} \,. \tag{B.50}$$

Note that the zero N-section S_0 in (B.49) is shifted by the unique vertical divisor D, such that $\tilde{S}_0 = S_0 + D$ is orthogonal to all curves C_i, $i = 1, \dots, b_2(B)$. Having said this, we define the shifted Kähler parameters t_i, $i = 1, \dots, h^{1,1}(B)$ as

$$t_i = \tilde{t}_i + \frac{\tilde{a}_i}{2N}\tau \,, \quad \text{with} \quad \tilde{a}_i = \int_M \tilde{S}_0^2 \cdot D_i \,. \tag{B.51}$$

These are the coordinates we consider for the base expansion in (2.125), i.e. $Q_\beta := \prod_{i=1}^{b_2(B)} \mathrm{e}^{2\pi i t_i \beta^i}$ with $\beta \in H_2(B,\mathbb{Z})$. Furthermore, we will also assume that there are no fibral divisors at a generic point of the complex structure moduli space of M.

Our objective is to determine the coefficients Z_β of the topological string partition function in (2.125), as meromorphic lattice Jacobi forms. The computation consists of a modular ansatz, which we fix by using Gopakumar-Vafa invariants as input data. For genus one fibered Calabi-Yau 3-folds, we formulate the modular bootstrap as follows:

1. For the coefficients in (2.125), we consider an ansatz of the form

$$Z_\beta(\tau, \boldsymbol{z}, \lambda) = \frac{1}{\eta(N\tau)^{12 \cdot c_1(B) \cdot \beta}} \frac{\phi_\beta(\tau, \boldsymbol{z}, \lambda)}{\prod_{l=1}^{b_2(B)} \prod_{s=1}^{\beta_l} \phi_{-2,1}(N\tau, s\lambda)} \,, \tag{B.52}$$

where the numerator $\phi_\beta(\tau,\lambda)$ is an element

$$\phi_\beta(\tau,\boldsymbol{z},\lambda) \in M_*(N)[\phi_{-2,1}(N\tau,\bullet),\phi_{0,1}(N\tau,\bullet)]\cdot\Delta_{2N}(\tau)^{1-\frac{r_\beta}{N}\text{ mod }1}\,, \tag{B.53}$$

where $\bullet$ stands for any elliptic parameter, i.e. $\bullet\in\{\lambda,\boldsymbol{z}\}$. Here $M_*(N)$ stands for the ring of even modular forms that transform under the congruence subgroup $\Gamma_1(N)$.

2. The exponent of $\Delta_{2N}(\tau)$ is determined by the congruence relation

$$1-\frac{r_\beta}{N}\equiv\frac{1}{2}\left[Nc_1(B)-\frac{\tilde{a}}{N}\right]\cdot\beta\ \text{ mod }\ 1\,. \tag{B.54}$$

3. The weight w_β of ϕ_β reads

$$w_\beta = 6c_1(B)\cdot\beta-\sum_l\beta_l\,. \tag{B.55}$$

4. The index m_β^λ with respect to the topological string coupling λ reads

$$m_\beta^\lambda=\frac{1}{2N}\beta\cdot(\beta-c_1(B))\,. \tag{B.56}$$

5. The index matrix with respect to the geometric elliptic parameters m_i, $i=1,...,\mathrm{rk}(G)$ reads

$$m_{ij}^\beta=\frac{1}{N}\cdot\left\{\begin{array}{cl} -\frac{1}{2}\pi_*\left(\sigma(S_i)\cdot\sigma(S_j)\right)\cdot\beta & \text{for } 1\leq i,j\leq r\\ -\frac{1}{2}\pi_*\left(E_i\cdot E_j\right)\cdot\beta & \text{for } r<i,j\leq \mathrm{rk}(G)\\ 0 & \text{otherwise}\end{array}\right.\,. \tag{B.57}$$

B.6 Modular expressions

$$\begin{aligned}\Phi^{(-2)}_{48,8}=&\frac{1}{2^{31}3^{15}}\Big[-2\phi^8_{-2,1}E_4^2E_6\Big(418217155E_4^{12}+6453885176E_4^9E_6^2+14430898662E_4^6E_6^4+6590367296E_4^3E_6^6\\&+473479711E_6^8\Big)+\phi^7_{-2,1}\phi_{0,1}\Big(1166575057E_4^{15}+51091761628E_4^{12}E_6^2+209321974054E_4^9E_6^4\\&+167568698684E_4^6E_6^6+24511632785E_4^3E_6^8+208925792E_6^{10}\Big)\\&-96\phi^2_{-2,1}\phi^6_{0,1}E_4^2E_6\left(1075402613E_4^9+7411457817E_4^6E_6^2+7120965999E_4^3E_6^4+939501571E_6^6\right)\\&-12\phi^6_{-2,1}\phi^2_{0,1}E_4E_6\Big(4961121163E_4^{12}+44271226044E_4^9E_6^2+64359934026E_4^6E_6^4+18212268812E_4^3E_6^6\\&+574073955E_6^8\Big)+\phi^5_{-2,1}\phi^3_{0,1}E_4^2\Big(18582697771E_4^{12}+561002956436E_4^9E_6^2+1637785240386E_4^6E_6^4\\&+887075195156E_4^3E_6^6+72640886251E_6^8\Big)-2\phi^4_{-2,1}\phi^4_{0,1}E_6\Big(112524942247E_4^{12}\\&+815939553020E_4^9E_6^2+891359100258E_4^6E_6^4+163891827116E_4^3E_6^6+1963937359E_6^8\Big)\\&+\phi^3_{-2,1}\phi^5_{0,1}E_4\Big(24673110263E_4^{12}+692315952484E_4^9E_6^2+1725339487242E_4^6E_6^4+703454544868E_4^3E_6^6\\&+31303881143E_6^8\Big)-120\phi^8_{0,1}E_4E_6\left(29908007E_4^9+207234483E_4^6E_6^2+208392741E_4^3E_6^4+27245569E_6^6\right)\\&+\phi_{-2,1}\phi^7_{0,1}\Big(3730882093E_4^{12}+102706594316E_4^9E_6^2+251407831758E_4^6E_6^4\\&+93709966220E_4^3E_6^6+2314293613E_6^8\Big)\Big]\end{aligned} \tag{B.58}$$

$$
\begin{aligned}
\Phi^{(-1)}_{48,6} =& \frac{1}{2^{30}3^{14}} \left(E_4^3 - E_6^2\right) \Big[-9216\phi_{-2,1}\phi_{0,1}^5 E_4^2 E_6 \left(787639E_4^6 + 4725442E_4^3E_6^2 + 2408071E_6^4\right) \\
& - 24\phi^5_{-2,1}\phi_{0,1}E_4E_6 \left(563474779E_4^9 + 1720939119E_4^6E_6^2 + 727580433E_4^3E_6^4 + 29728037E_6^6\right) \\
& + 3\phi^4_{-2,1}\phi^2_{0,1}E_4^2 \left(1867928987E_4^9 + 26622425391E_4^6E_6^2 + 28473146577E_4^3E_6^4 + 3870946405E_6^6\right) \\
& - 40\phi^3_{-2,1}\phi^3_{0,1}E_6 \left(963239627E_4^9 + 3685106847E_4^6E_6^2 + 1399118433E_4^3E_6^4 + 35979829E_6^6\right) \\
& + 27\phi^2_{-2,1}\phi^4_{0,1}E_4 \left(118872393E_4^9 + 2632343077E_4^6E_6^2 + 3638134747E_4^3E_6^4 + 370032823E_6^6\right) \\
& \phi^6_{-2,1} \left(413474065E_4^{12} + 4900784877E_4^9E_6^2 + 5874684627E_4^6E_6^4 + 971117903E_4^3E_6^6 + 6828000E_6^8\right) \\
& + \phi^6_{0,1} \left(95773579E_4^9 + 3344820063E_4^6E_6^2 + 7540782753E_4^3E_6^4 + 1185513077E_6^6\right) \Big]
\end{aligned}
\tag{B.59}
$$

$$
\begin{aligned}
\Phi^{(0)}_{24,4} =& \frac{1}{2^{14}3^6} \Big[\phi^4_{-2,1}E_4^2 \left(3151E_4^6 + 51346E_4^3E_6^2 + 31903E_6^4\right) - 48\phi^3_{-2,1}\phi_{0,1}E_6 \left(2665E_4^6 + 4282E_4^3E_6^2 + 253E_6^4\right) \\
& + 10\phi^2_{-2,1}\phi^2_{0,1}E_4 \left(5753E_4^6 + 37598E_4^3E_6^2 + 8489E_6^4\right) - 192\phi_{-2,1}\phi^3_{0,1}E_4^2E_6 \left(907E_4^3 + 893E_6^2\right) \\
& + 9\phi^4_{0,1} \left(1247E_4^6 + 7154E_4^3E_6^2 + 1199E_6^4\right) \Big]
\end{aligned}
\tag{B.60}
$$

APPENDIX C

Geometric Appendix

C.1 Noether-Lefschetz theory

In this section we review some aspects of Noether-Lefschetz theory. In our presentation we closely follow the references [133–135].

Λ-polarized lattices Let S be a $K3$ surface. We say that a primitive class $L \in \mathrm{Pic}(S) = H^2(S,\mathbb{Z}) \cap H^{1,1}(S,\mathbb{C})$ is a quasi-polarization if

$$\int_S L^2 > 0\,, \qquad \int_C L \geq 0 \text{ for all } C \in H_2(S,\mathbb{Z})\,. \tag{C.1}$$

Let Λ be a rank r lattice of signature $(1\,, r-1)$ with a torsion-free embedding of the following form

$$\Lambda \hookrightarrow U \oplus U \oplus U \oplus E_8(-1) \oplus E_8(-1) \cong H^2(S,\mathbb{Z})\,. \tag{C.2}$$

Here U is the hyperbolic lattice of rank 2 and signature $(1\,,1)$, whereas $E_8(-1)$ is the E_8 lattice with intersection form defined by the negative Cartan matrix of the exceptional Lie group E_8. We say that S is a Λ-polarized $K3$ surface if there is a torsion-free embedding

$$\jmath : \Lambda \hookrightarrow \mathrm{Pic}(S)\,, \tag{C.3}$$

such that

- The embeddings of Λ in $U^3 \oplus E_8(-1)^3$ and $H^2(S,\mathbb{Z})$ are isomorphic via an isometry that restricts to the identity on Λ.
- $\jmath(\Lambda)$ contains a quasi-polarization.

The polarization lattice Λ has an intersection form $I_\Lambda : \Lambda \times \Lambda \to \mathbb{Z}$. Let $\{v_1, \ldots, v_r\}$ be an integral basis of Λ. It is useful to consider I_Λ as the matrix with entries ${(I_\Lambda)^i}_j = I_\Lambda(v_i\,, v_j)$. In this fashion, the discriminant of Λ reads from $\Delta(\Lambda) = |\det(I_\Lambda)|$.

We will denote the moduli space of Λ-polarized $K3$ surfaces by $\mathcal{M}_\Lambda$, see [134] and [193] for a math and [194] for a physics review.

Polarized $K3$ surfaces from $K3$ fibrations Our main interest will be Λ-polarized $K3$ surfaces that arise from $K3$ fibrations

$$\begin{tikzcd} S \arrow[r, hook] & M \arrow[d, "\pi"] \\ & \mathbb{P}^1_{\mathrm{b}} \end{tikzcd} , \tag{C.4}$$

equipped with holomorphic line bundles $(L_1\,, \ldots\,, L_r)$. The tuple $(M\,, L_1\,, \ldots\,, L_r\,, \pi)$ is a one-parameter family of Λ-polarized $K3$ surfaces if

- The fibers $(S_p, L_{1,p}, \ldots, L_{r,p})$ of $(M, L_1, \ldots, L_r)$ at $p \in \mathbb{P}^1_{\mathrm{b}}$ define Λ-polarized $K3$ surfaces via the replacement $v_i \mapsto L_{i,p}$ for every $p \in \mathbb{P}^1_{\mathrm{b}}$.
- There is a quasi-polarization $\lambda^\pi_p = \sum_{i=1}^r \lambda^\pi_i L_{i,p} \in \Lambda$ that satisfies (C.1) with respect to the $K3$ fiber S_p for all $p \in \mathbb{P}^1_{\mathrm{b}}$.

The quasi-polarization vector λ^π defines a notion of positivity. We say that a vector $(d_1, \ldots, d_r) \in \mathbb{Z}^r$ is positive if $\sum_{i=1}^r \lambda^\pi_i d_i > 0$.

We are interested in the enumerative geometry of curves $\varphi \in H_2(M, \mathbb{Z})$ that live in the $K3$ fibers of M, i.e. that project to points in $\mathbb{P}^1_{\mathrm{b}}$. These classes of curves are furnished by the π-vertical cohomology $H_2(M, \mathbb{Z})^\pi$ which is defined by the following short exact sequence

$$0 \longrightarrow H_2(M, \mathbb{Z})^\pi \longrightarrow H_2(M, \mathbb{Z}) \xrightarrow{\pi_*} H_2(\mathbb{P}^1_{\mathrm{b}}, \mathbb{Z}) \longrightarrow 0\,. \tag{C.5}$$

For a set of divisors $(L_1, \ldots L_r)$ the degree of $\varphi \in H_2(M, \mathbb{Z})^\pi$ is obtained via the projection

$$\varphi \mapsto \left(\int_\varphi L_1, \ldots, \int_\varphi L_r \right) = (d_1, \ldots, d_r)\,. \tag{C.6}$$

Noether-Lefschetz divisors Let $\mathbb{L}$ be a rank $r+1$ lattice with an even symmetric bilinear form $\langle \cdot\,, \cdot \rangle$ and a primitive embedding $\iota : \Lambda \hookrightarrow \mathbb{L}$. Consequently, the lattice $\mathbb{L}$ has an additive struture of the following form

$$\mathbb{L} = \iota\,(\Lambda) \oplus \mathbb{Z} v\,, \qquad v \in \mathbb{L}\,. \tag{C.7}$$

Consider the extended basis $\{v_1, \ldots, v_r, v\}$, where $\Lambda = \mathrm{Span}_{\mathbb{Z}}\{v_1, \ldots, v_r\}$. The bilinear pairing $\langle \cdot\,, \cdot \rangle$ of $\mathbb{L}$ is encoded in the following matrix

$$\mathbb{L}_{h,d_1,\ldots,d_r} = \begin{pmatrix} \langle v_1\,, v_1 \rangle & \ldots & \langle v_1\,, v_r \rangle & d_1 \\ \vdots & \ddots & \vdots & \vdots \\ \langle v_r\,, v_1 \rangle & \ldots & \langle v_r\,, v_r \rangle & d_r \\ d_1 & \ldots & d_r & 2h-2 \end{pmatrix}. \tag{C.8}$$

There are two invariants for the data $(\mathbb{L}, \iota)$, namely

- the discriminant $\Delta(h, d_1, \ldots, d_r) = (-1)^r \det(\mathbb{L}_{h,d_1,\ldots,d_r})$,
- the coset, defined as the image of map δ_v represented by $v_i \mapsto d_i$, is denoted $\delta(h, d_1, \ldots d_r) \in G_\Lambda/\pm$, where $G_\Lambda = \Lambda^*/\Lambda$ is the discriminant group, an abelian group of order $\Delta(\Lambda)$.

r_h^g	$h=0$	1	2	3	4	5
$g=0$	1	24	324	3200	25650	176256
1		-2	-54	-800	-8550	-73440
2			3	88	1401	15960
3				-4	-126	-2136
4					5	168
5						-6

Table C.1: Non-vanishing BPS invariants r_h^g for $K3$ surfaces with $h \leq 5$. For a physical interpretation of these numbers, we refer the reader to [196].

Generically a Noether-Lefschetz divisor $[P_{\Delta,\delta}] \subset \mathcal{M}_\Lambda$ is the closure of the locus of Λ-polarized $K3$ surfaces where $(\mathrm{Pic}(S), \jmath)$ has rank $r+1$, discriminant Δ, and coset δ. We defined the Noether-Lefschetz divisor $[D_{h,(d_1,\ldots,d_r)}]$ as

$$\left[D_{h,(d_1,\ldots,d_r)}\right] = \sum_{\Delta,\delta} m\left(h, d_1, \ldots, d_r \mid \Delta, \delta\right) \cdot [P_{\Delta,\delta}] \subset \mathcal{M}_\Lambda \,. \tag{C.9}$$

Here $m(h, d_1, \ldots d_r \mid \Delta, \delta)$ denotes the number of elements $\varphi \in \mathbb{L}$ of type (Δ, δ) such that

$$\langle \varphi, \varphi \rangle = 2h - 2 \,, \qquad \langle \varphi, v_i \rangle = d_i \,. \tag{C.10}$$

Noether-Lefschetz numbers The Noether-Lefschetz number $NL^\pi_{h,(d_1,\ldots,d_r)}$ is defined as the classical intersection of the Noether-Lefschetz divisor (C.9) with the image of $\mathbb{P}^1_\mathrm{b}$ in $\mathcal{M}_\Lambda$, i.e.

$$NL^\pi_{h,(d_1,\ldots,d_r)} = \int_{\mathbb{P}^1_\mathrm{b}} \imath^*_\pi \left[D_{h,(d_1,\ldots,d_r)}\right] \,, \tag{C.11}$$

where $\imath_\pi : \mathbb{P}^1_\mathrm{b} \to \mathcal{M}_\Lambda$ is the morphism associated to a one-parameter family of Λ-polarized $K3$ surfaces $(M, L_1, \ldots, L_r, \pi)$.

The key fact of Noether-Lefschetz theory in this context observed is that the generator of Noether-Lefschetz numbers is a vector-valued modular form of weight $k = 11 - \frac{r}{2}$ and type ρ^*_Λ of the form [195],

$$\Phi^\pi(q) = \sum_{\gamma \in G} \Phi^\pi_\gamma(q) \mathrm{e}_\gamma \in \mathbb{C}\left[\left[q^{\frac{1}{2\Delta(\Lambda)}}\right]\right] \otimes \mathbb{C}\left[G_\Lambda\right] \,, \tag{C.12}$$

where $\{\mathrm{e}_\gamma\}$ is a formal basis in $\mathbb{C}[G_\Lambda]$. The coefficients of $\Phi^\pi_\gamma(q)$ are determined by the Noether-Lefschetz numbers via

$$\mathrm{Coeff}\left(\Phi^\pi_\gamma, q^{\Delta_{NL}}\right) = NL^\pi_{h,(d_1,\ldots,d_r)} \,, \text{ where } \Delta_{NL} = \frac{\Delta(h, d_1, \ldots, d_r)}{2\Delta(\Lambda)} \,. \tag{C.13}$$

For a short review on vector-valued modular forms and the definition of the representation ρ^*_Λ we refer the reader to Appendix B.4.

The GW-NL correspondence theorem [133, 135] *For degrees* $(d_1, \ldots, d_r)$ *positive with respect to a quasi-polarization* λ^π,

$$n^g_{(d_1,\ldots,d_r)} = \sum_{h=0}^{\infty} r^g_h \cdot NL^\pi_{h,(d_1,\ldots,d_r)} \,. \tag{C.14}$$

Here $n^g_{(d_1,\ldots,d_r)}$ is the Gopakumar-Vafa invariant associated to a curve class $\varphi \in H_2(M,\mathbb{Z})^\pi$ with positive degree $(d_1,\ldots,d_r)$. Moreover, r^g_h are the coefficients of the *KKV formula* expansion that reads [196]

$$\sum_{h=0}^{\infty}\sum_{g=0}^{\infty}(-1)^g r^g_h \left(y^{\frac{1}{2}} - y^{-\frac{1}{2}}\right)^{2g-2} q^{h-1} = \frac{1}{\eta^{24}(\tau)\phi_{-2,1}(\tau,\lambda)}\,, \tag{C.15}$$

with $q = \exp(2\pi i\tau)$, and $y = \exp(2\pi i\lambda)$. For concreteness we show some r^g_h numbers in Table C.1.

C.2 The KKP conjecture

Following the formulation of [136], we introduce refinements for the Noether-Lefschetz numbers (C.11), which are representations of $SU(2)_L \times SU(2)_R$ lying in the space

$$\mathbb{Z}_{\geq 0}\,[0\,,0] \oplus \mathbb{Z}_{\geq 0}\left[0\,,\frac{1}{2}\right]\,. \tag{C.16}$$

There are two kind of refined Noether-Lefschetz numbers to be considered: $\mathsf{RNL}^{\pi,\circ}_{h,(d_1,\ldots,d_n)}$ and $\mathsf{RNL}^{\pi,\diamond}_{h,(d_1,\ldots,d_n)}$. The former ones are defined as follows

$$\mathsf{RNL}^{\pi,\circ}_{h,(d_1,\ldots,d_n)} = NL^\pi_{h,(d_1,\ldots,d_n)} \cdot [0\,,0]\,. \tag{C.17}$$

The other Noether-Lefschetz refinement requires more explanation. Let an effective divisor $\left[D_{h,(d_1,\ldots,d_n)}\right]$ be divided in two components

$$D_{h,(d_1,\ldots,d_n)} = S_\imath + T_\imath\,, \tag{C.18}$$

where $S_\imath$ is the sum of divisors not containing $\imath_\pi(\mathbb{P}^1_{\mathrm{b}})$ and $T_\imath$ is the sum of divisors containing $\imath_\pi(\mathbb{P}^1_{\mathrm{b}})$. Having said this, we define the refinement $\mathsf{RNL}^{\pi,\diamond}_{h,(d_1,\ldots d_n)}$ in the following way

- If $\Delta(h, d_1, \ldots, d_n) < 0$, then $\mathsf{RNL}^{\pi,\diamond}_{h,(d_1,\ldots d_n)} = 0$.
- If $\Delta(h, d_1, \ldots, d_n) = 0$, then $\mathsf{RNL}^{\pi,\diamond}_{h,(d_1,\ldots d_n)} = \left[0\,,\frac{1}{2}\right]$.
- If $\Delta(h, d_1, \ldots, d_n) > 0$, then

$$\mathsf{RNL}^{\pi,\diamond}_{h,(d_1,\ldots d_n)} = \int_{\mathbb{P}^1_{\mathrm{b}}} \imath^*_\pi S_\imath \cdot [0\,,0] - \frac{1}{2}\int_{\mathbb{P}^1_{\mathrm{b}}} \imath^*_\pi T_\imath \cdot \left[0\,,\frac{1}{2}\right]\,. \tag{C.19}$$

As we will see, the KKP conjecture is a refined version of the GW-NL correspondence theorem (C.14). Therefore, an important ingredient will be the refinement of the KKV formula (C.15), which we introduce next.

$\mathsf{R}^0_{j_L,j_R}$	$2j_+=0$
$2j_-=0$	1

$\mathsf{R}^1_{j_L,j_R}$	$2j_+=0$	1
$2j_-=0$	20	
1		1

$\mathsf{R}^2_{j_L,j_R}$	$2j_+=0$	1	2
$2j_-=0$	231		
1		21	
2			1

$\mathsf{R}^3_{j_L,j_R}$	$2j_+=0$	1	2	3
$2j_-=0$	1981		1	
1		252		
2	1		21	
3				1

$\mathsf{R}^4_{j_L,j_R}$	$2j_+=0$	1	2	3	4
$2j_-=0$	13938		21		
1		2233		1	
2	21		253		
3		1		21	
4					1

Table C.2: Non-vanishing refined BPS invariants $\mathsf{R}^h_{j_L,j_R}$ for $K3$ surfaces with $h \leq 4$.

The refined BPS invariants $\mathsf{R}^h_{j_L,j_R}$ for $K3$ surfaces, where $j_L, j_R \in \frac{1}{2}\mathbb{Z}_{\geq 0}$, were defined in [136] via the refined KKV formula:

$$\sum_{h=0}^{\infty} \sum_{j_L,j_R \in \frac{1}{2}\mathbb{Z}_{\geq 0}} \mathsf{R}^h_{j_L,j_R} \frac{[j_L]_y}{X^{\frac{1}{2}} - X^{-\frac{1}{2}}} \frac{[j_R]_x}{Y^{\frac{1}{2}} - Y^{-\frac{1}{2}}} q^{h-1} = \frac{1}{\eta^{24}(\tau)} \frac{1}{\phi_{-1,\frac{1}{2}}(\tau,\epsilon_1)\phi_{-1,\frac{1}{2}}(\tau,\epsilon_2)} \,. \tag{C.20}$$

For illustration, we show a few BPS invariants $\mathsf{R}^h_{j_L,j_R}$ in table C.2. Let the $K3$ invariants $\mathsf{R}^h_{j_L,j_R}$ be divided into two parts

$$\mathsf{R}^h_{j_L,j_R} = \mathsf{R}^{h,\circ}_{j_L,j_R} + \mathsf{R}^{h,\diamond}_{j_L,j_R} \,, \tag{C.21}$$

where the $\mathsf{R}^{h,\diamond}_{j_L,j_R}$ contributions read from the formula

$$\sum_{h=0}^{\infty} \sum_{j_L,j_R \in \frac{1}{2}\mathbb{Z}_{\geq 0}} \mathsf{R}^{h,\diamond}_{j_L,j_R} \frac{[j_L]_y}{X^{\frac{1}{2}} - X^{-\frac{1}{2}}} \frac{[j_R]_x}{Y^{\frac{1}{2}} - Y^{-\frac{1}{2}}} q^{h-1} = \frac{q^{-\frac{5}{6}}}{\eta^{4}(\tau)} \frac{1}{\phi_{-1,\frac{1}{2}}(\tau,\epsilon_1)\phi_{-1,\frac{1}{2}}(\tau,\epsilon_2)} \,. \tag{C.22}$$

With this information at hand, we can compute refined BPS numbers along $K3$ fibers via the KKP conjecture, which we state now.

Refined P-NL correspondence conjecture: [136] *A 1-parameter family of Λ-polzarized $K3$ surfaces*

$$\pi : M \to \mathbb{P}^1_{\mathrm{b}} \,, \tag{C.23}$$

with Calabi-Yau total space determines a division (C.21) satisfying the following property. For degrees $(d_1, \cdots, d_r)$ positive with respect to the quasi-polarization λ^π,

$$\begin{aligned} \sum_{j_L,j_R} \mathsf{N}^{(d_1,\cdots,d_n)}_{j_L,j_R}[j_L,j_R] = & \sum_{j_L,j_R} \sum_{h=0}^{\infty} \mathsf{R}^{h,\circ}_{j_L,j_R} \otimes \mathsf{RNL}^{\pi,\circ}_{h,(d_1,\cdots,d_n)} \\ & + \sum_{j_L,j_R} \sum_{h=0}^{\infty} \mathsf{R}^{h,\diamond}_{j_L,j_R} \otimes \mathsf{RNL}^{\pi,\diamond}_{h,(d_1,\cdots,d_n)} \,. \end{aligned} \tag{C.24}$$

We give a few remarks about the numbers $\mathsf{N}^{(d_1,\cdots,d_n)}_{j_L,j_R}$ that we obtain from (C.24). First, recall the 5d Hilbert space of BPS states due to M2-branes wrapping a curve $\kappa \in H_2(M,\mathbb{Z})$, whose structure reads

$$\sum_{j_L,j_R\in\frac{1}{2}\mathbb{Z}_{\geq 0}} N^{\beta}_{j_L,j_R}\left(\left[\frac{1}{2},0\right]\oplus 2\,[0,0]\right)\otimes[j_L\,,j_R]\,. \tag{C.25}$$

Here $N^{\kappa}_{j_L,j_R}$ count the multiplicities of the $SU(2)_L\times SU(2)_R$ representations $[j_L,j_R]$. This physical constraint implies that $N^{\kappa}_{j_L,j_R}\in\mathbb{Z}_{\geq 0}$. For a positive degree $(d_1,\dots,d_n)\in H_2(M,\mathbb{Z})^{\pi}$, the refined BPS number $N^{(d_1,\dots d_r)}_{j_L,j_R}$ can be calculated via the stable pairs of moduli spaces of M [54, 136]. Our computations should fulfil the additional constraint

$$\sum_{j_L,j_R\in\frac{1}{2}\mathbb{Z}_{\geq 0}}(-1)^{2j_R}(2j_R+1)\mathsf{N}^{(d_1,\dots,d_n)}_{j_L,j_R}\,[j_L]=\sum_{g\in\mathbb{Z}_{\geq 0}} n^{g}_{(d_1,\dots,d_n)}I^{g}_{L}\,, \tag{C.26}$$

where $n^{g}_{(d_1,\dots,d_n)}$ are the unrefined Gopakumar-Vafa invariants in (C.14) and I^g_L denotes the $SU(2)_L$ representation that reads

$$I^g_L=\left(2\,[0]+\left[\frac{1}{2}\right]\right)^{\otimes g}\,. \tag{C.27}$$

C.3 Refined BPS invariants

Here we provide a few refined BPS numbers corresponding to the example geometries [126] in Section 4.5, by using the *KKP conjecture* (C.24) and the methods introduced in Section 5.2. These geometries are complete intersection Calabi-Yau varieties in a weighted projective space $\mathbb{P}^n(w_1,\dots,w_{n+1})$ defined by loci of transversal quasihomogeneous polynomials of degree d_i such that $\sum_i d_i=\sum_j w_j$. We denote each Calabi-Yau M by

$$\left(\mathbb{P}^n(w_1,\dots,w_{n+1})[d_1,\dots,d_m]\right)^{h^{1,1}(M),h^{2,1}(M)}_{\chi(M)}\,, \tag{C.28}$$

where $h^{i,j}(M)$ are its hodge numbers, and $\chi(M)$ its Euler characteristic.

Following the reference [141], we introduce the notation $[\frac{\Delta_J}{2}]$ and $N^{[\Delta_j/2]}_{J_L,J_R}$ to denote the class of refined BPS multiplicities $N^{(n,\ell,\boldsymbol{\lambda})}_{j_L,j_R}$ in which their positive degree curves $(n,\ell,\boldsymbol{\lambda})$ have the same value $\Delta_J=2n\ell-(\boldsymbol{\lambda},\boldsymbol{\lambda})_{A_n}$, with $(\cdot,\cdot)_{A_n}$ the Killing form for the A_n Lie algebra, as described in Section 5.2. Note that using the weighted trace computation (C.26) on Tables C.4, C.5, and C.6, we obtain—respectively—the unrefined invariants in Tables C.13, C.10, and C.7 in [141]. In contrast to our methods, the author [141] used the Borcherds-Harvey-Moore lattice reduction Heterotic approach [139]. As the Gopakumar-Vafa invariants for the geometry with associated Lie Algebra A_4 have no previous appearance in the literature, for completeness, we include Table C.3. We show the corresponding refined invariants in Table C.7.

$g\backslash\frac{\Delta_J}{2}$	-1	$-\frac{3}{5}$	$-\frac{2}{5}$	0	$\frac{2}{5}$	$\frac{3}{5}$	1	$\frac{7}{5}$	$\frac{8}{5}$	2
0	-2	4	22	220	4530	12518	61248	304716	639420	2359220
1	0	0	0	4	-8	-44	-428	-9084	-25168	-123800
2	0	0	0	0	0	0	-6	12	66	628
3	0	0	0	0	0	0	0	0	0	8

Table C.3: $n^g_{[\Delta_J/2]}$ GV invariants of the $K3$ fiber for $\left(\mathbb{P}^5(1,1,2,5,7,9)[14,11]\right)^{7,117}_{-220}$.

$N^{[-1]}_{j_-j_+}$	$2j_+=0$	1
$2j_-=0$		1

$N^{[-\frac{1}{4}]}_{j_-j_+}$	$2j_+=0$
$2j_-=0$	56

$N^{[0]}_{j_-j_+}$	$2j_+=0$	1	2
$2j_-=0$	380		
1	1		1

$N^{[\frac{3}{4}]}_{j_-j_+}$	$2j_+=0$	1
$2j_-=0$	53728	
1		56

$N^{[1]}_{j_-j_+}$	$2j_+=0$	1	2	3
$2j_-=0$	172748	1		
1	1	380	1	
2		1		1

$N^{[\frac{7}{4}]}_{j_-j_+}$	$2j_+=0$	1	2
$2j_-=0$	3522096		
1		53784	
2			56

$N^{[2]}_{j_-j_+}$	$2j_+=0$	1	2	3	4
$2j_-=0$	8538644	2		1	
1	2	173128	2		
2		2	380	1	
3			1		1

Table C.4: Refined BPS invariants $N^{[\Delta_J/2]}_{j_-,j_+}$ of the geometry $\left(\mathbb{P}^4(1,1,2,6,10)[20]\right)^{4,190}_{-372}$.

$N^{[-1]}_{j_-j_+}$	$2j_+=0$	1
$2j_-=0$		1

$N^{[-\frac{1}{3}]}_{j_-j_+}$	$2j_+=0$
$2j_-=0$	30

$N^{[0]}_{j_-j_+}$	$2j_+=0$	1	2
$2j_-=0$	320		
1	1		1

$N^{[\frac{2}{3}]}_{j_-j_+}$	$2j_+=0$	1
$2j_-=0$	26544	
1		30

$N^{[1]}_{j_-j_+}$	$2j_+=0$	1	2	3
$2j_-=0$	119600	1		
1	1	320	1	
2		1		1

$N^{[\frac{5}{3}]}_{j_-j_+}$	$2j_+=0$	1	2
$2j_-=0$	1641420		
1		26574	
2			30

$N^{[2]}_{j_-j_+}$	$2j_+=0$	1	2	3	4
$2j_-=0$	5202720	2		1	
1	2	119920	2		
2		2	320	1	
3			1		1

Table C.5: Refined BPS invariants $N^{[\Delta_J/2]}_{j_-,j_+}$ of the geometry $\left(\mathbb{P}^4(1,1,2,6,8)[18]\right)^{5,161}_{-312}$.

$N^{[-1]}_{j_-j_+}$	$2j_+ = 0$	1
$2j_- = 0$		1

$N^{[-\frac{1}{2}]}_{j_-j_+}$	$2j_+ = 0$
$2j_- = 0$	8

$N^{[-\frac{3}{8}]}_{j_-j_+}$	$2j_+ = 0$
$2j_- = 0$	24

$N^{[0]}_{j_-j_+}$	$2j_+ = 0$	1	2
$2j_- = 0$	272		
1	1		1

$N^{[\frac{1}{2}]}_{j_-j_+}$	$2j_+ = 0$	1
$2j_- = 0$	9072	
1		8

$N^{[\frac{5}{8}]}_{j_-j_+}$	$2j_+ = 0$	1
$2j_- = 0$	17176	
1		24

$N^{[1]}_{j_-j_+}$	$2j_+ = 0$	1	2	3
$2j_- = 0$	85232	1		
1	1	272	1	
2		1		1

$N^{[\frac{3}{2}]}_{j_-j_+}$	$2j_+ = 0$	1	2
$2j_- = 0$	598072		
1		9080	
2			8

$N^{[\frac{13}{8}]}_{j_-j_+}$	$2j_+ = 0$	1	2
$2j_- = 0$	940920		
1		17200	
2			24

$N^{[2]}_{j_-j_+}$	$2j_+ = 0$	1	2	3	4
$2j_- = 0$	3302736	2		1	
1	2	85504	2		
2		2	272	1	
3			1		1

Table C.6: Refined BPS invariants $N^{[\Delta_J/2]}_{j_-,j_+}$ of the geometry $\left(\mathbb{P}^4(1,1,2,6,8,10)[16,12]\right)^{6,138}_{-264}$.

$N^{[-1]}_{j_-j_+}$	$2j_+ = 0$	1
$2j_- = 0$		1

$N^{[-\frac{3}{5}]}_{j_-j_+}$	$2j_+ = 0$
$2j_- = 0$	4

$N^{[-\frac{2}{5}]}_{j_-j_+}$	$2j_+ = 0$
$2j_- = 0$	22

$N^{[0]}_{j_-j_+}$	$2j_+ = 0$	1	2
$2j_- = 0$	228		
1	1		1

$N^{[\frac{2}{5}]}_{j_-j_+}$	$2j_+ = 0$	1
$2j_- = 0$	4514	
1		4

$N^{[\frac{3}{5}]}_{j_-j_+}$	$2j_+ = 0$	1
$2j_- = 0$	12430	
1		22

$N^{[1]}_{j_-j_+}$	$2j_+ = 0$	1	2	3
$2j_- = 0$	60364	1		
1	1	228	1	
2		1		1

$N^{[\frac{7}{5}]}_{j_-j_+}$	$2j_+ = 0$	1	2
$2j_- = 0$	286608		
1		4518	
2			4

$N^{[\frac{13}{8}]}_{j_-j_+}$	$2j_+ = 0$	1	2
$2j_- = 0$	589414		
1		12452	
2			22

$N^{[2]}_{j_-j_+}$	$2j_+ = 0$	1	2	3	4
$2j_- = 0$	2114880	2		1	
1	2	60592	2		
2		2	228	1	
3			1		1

Table C.7: Refined BPS invariants $N^{[\Delta_J/2]}_{j_-,j_+}$ of the geometry $\left(\mathbb{P}^5(1,1,2,5,7,9)[14,11]\right)^{7,117}_{-220}$.

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