Modeling of inhomogeneous spatio-temporal signals by least squares collocation

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Theoretical Geodesy Group
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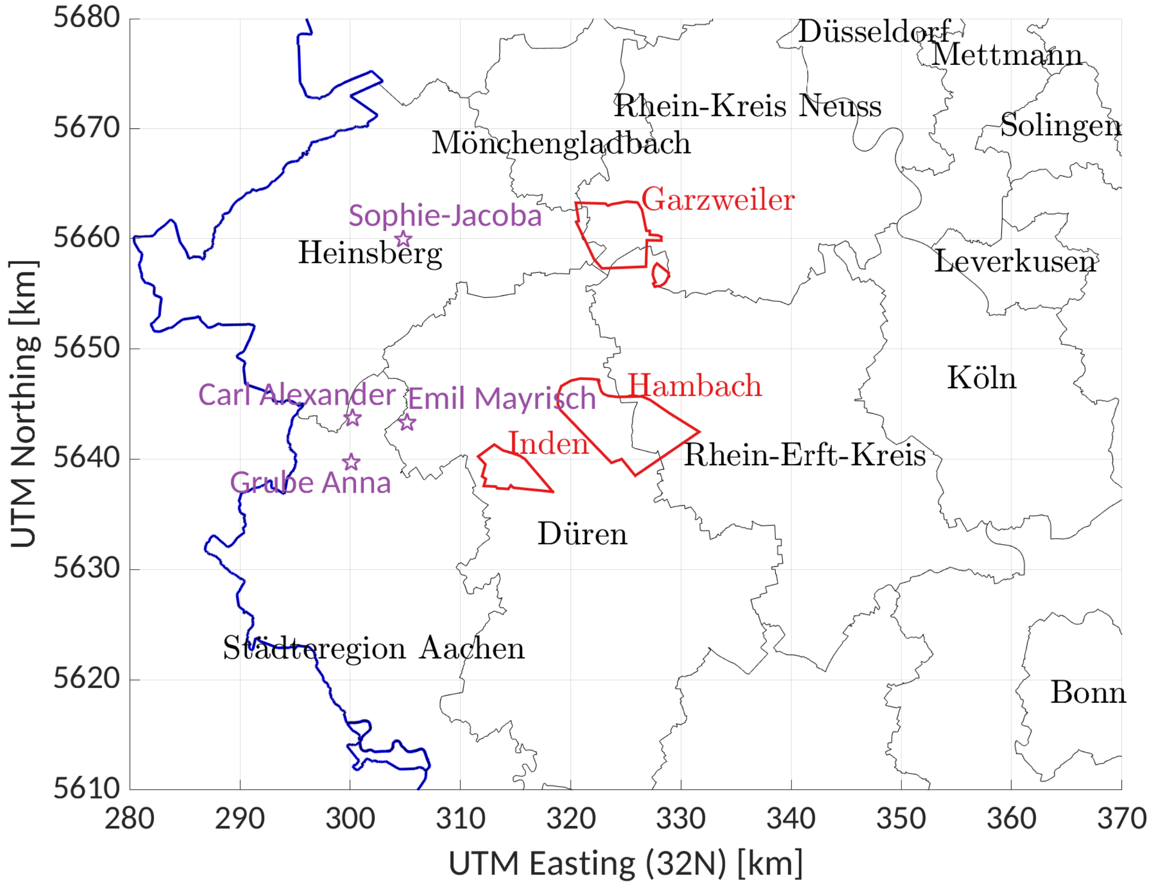
X Hotine-Marussi Symposium
Milano, June13-17, 2022

Study region: Lower-Rhine-Embayment



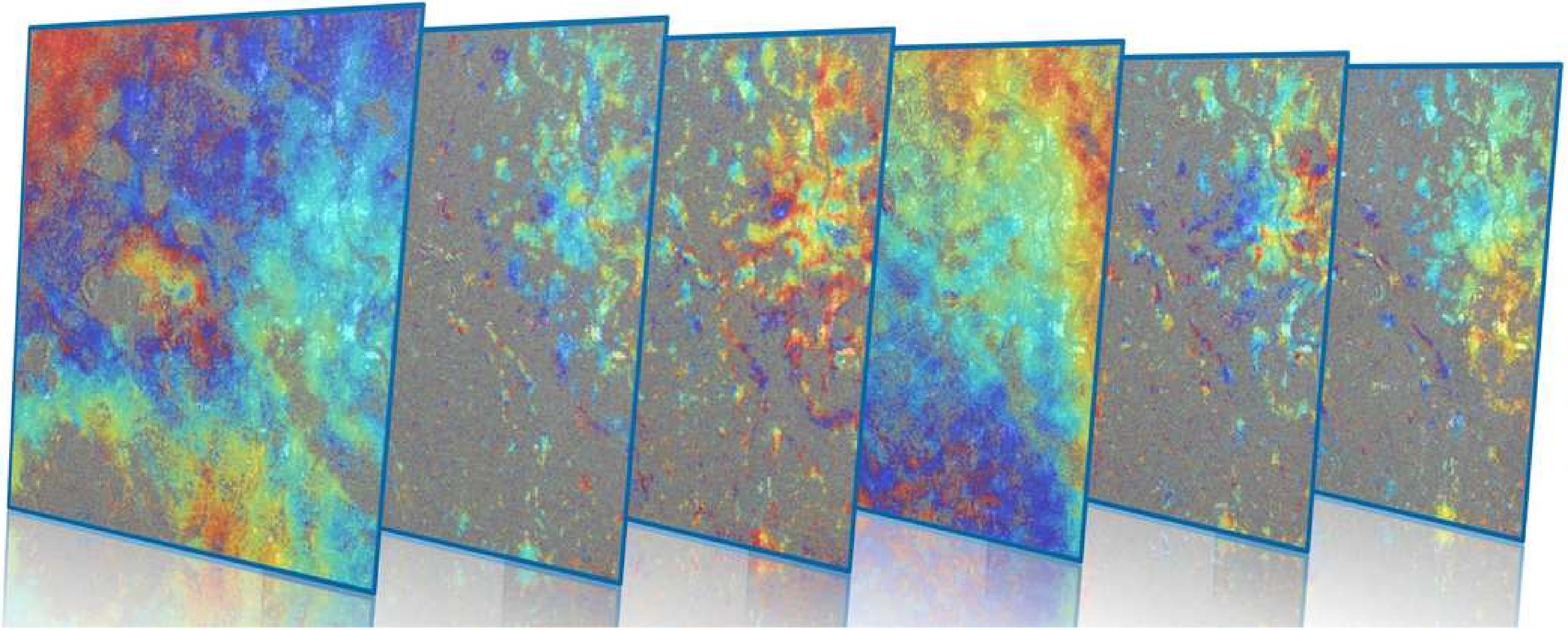
open-cast brown coal mines
Hambach, Garzweiler, Inden
still active


©R. Spekking

hard coal mines
Aachener and Erkelenzer coalfield
closed 1992

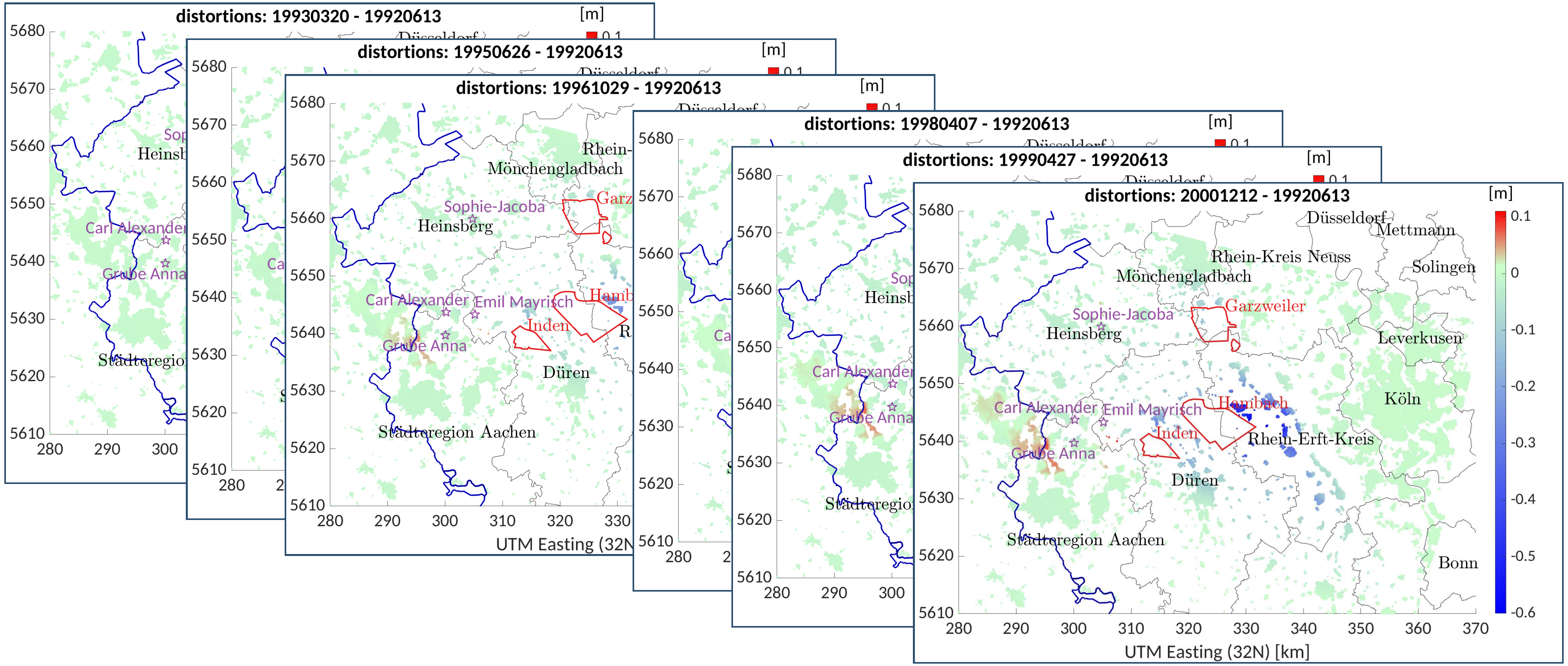

©Digit WDR

Observations: InSAR recordings of the ERS1 und ERS2 mission



DInSAR stack with 64 frames observed at irrelgular times
in the period between 1992-05-09 and 2000-12-12

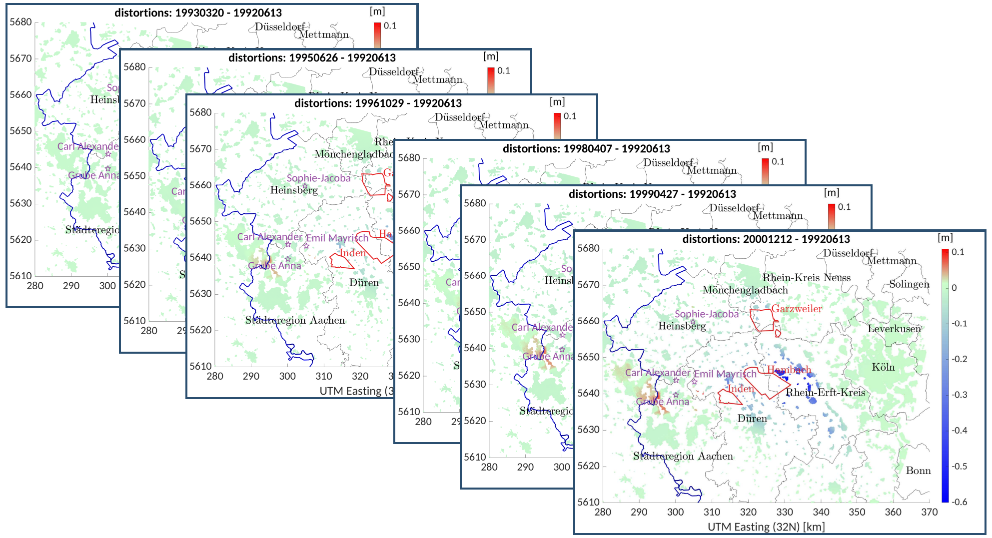
Processing: DInSAR-SBAS with RSG (Remote-Sensing-Graz) package



spatio-temporal point cloud with surface distortions

Esch et al. (2019a,b)

Objective: Analysis and Prediction



spatio-temporal point cloud (SBAS-stack):

- – huge number of data
- + identical scatterer in each time frame — 144.302 scatterer
- – irregularly distributed in space
- – clustered in urban areas
- + organized in 64 time frames
- – irregularly distributed in time

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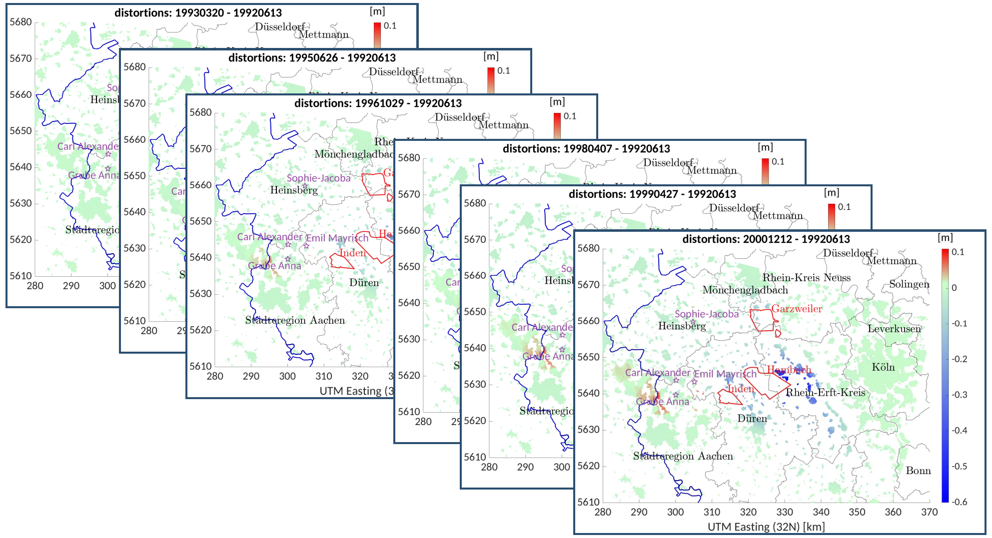

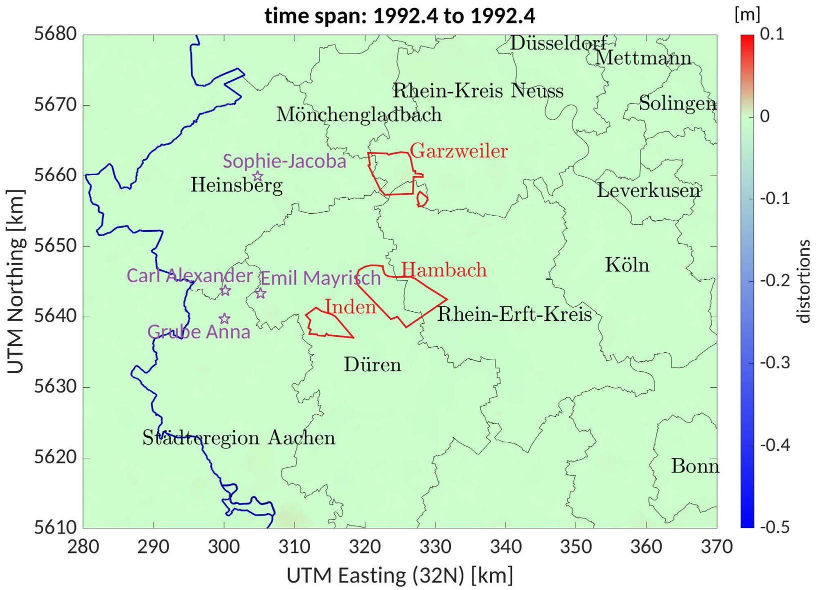

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Challenge:

design of a spatio-temporal collocation model

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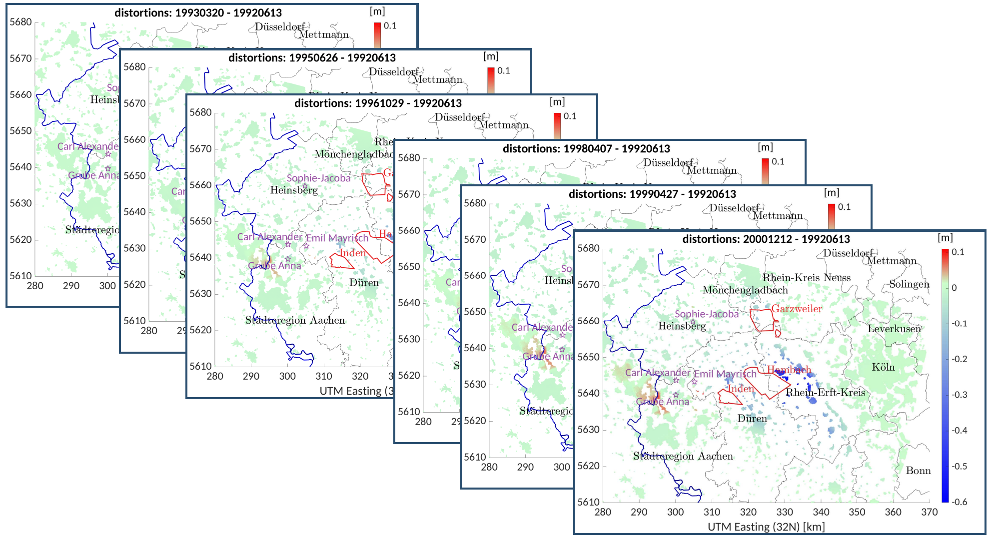

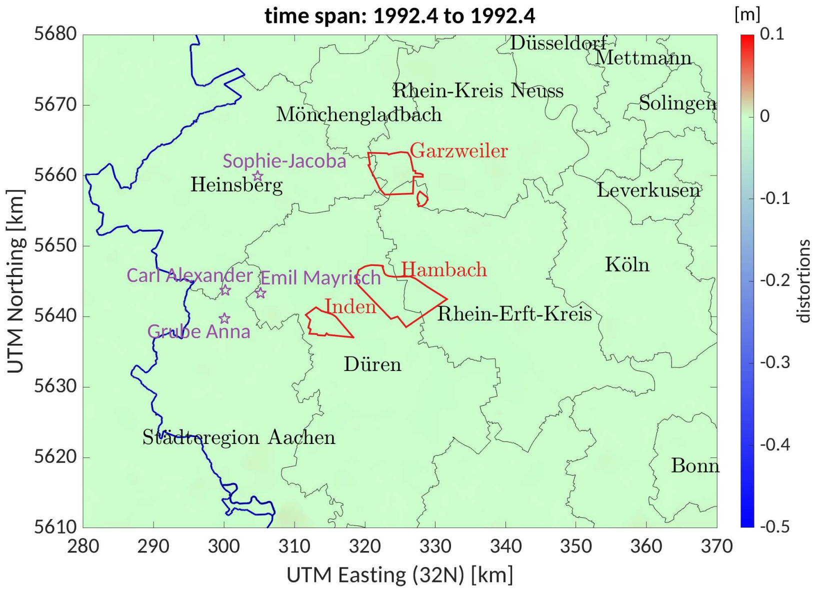

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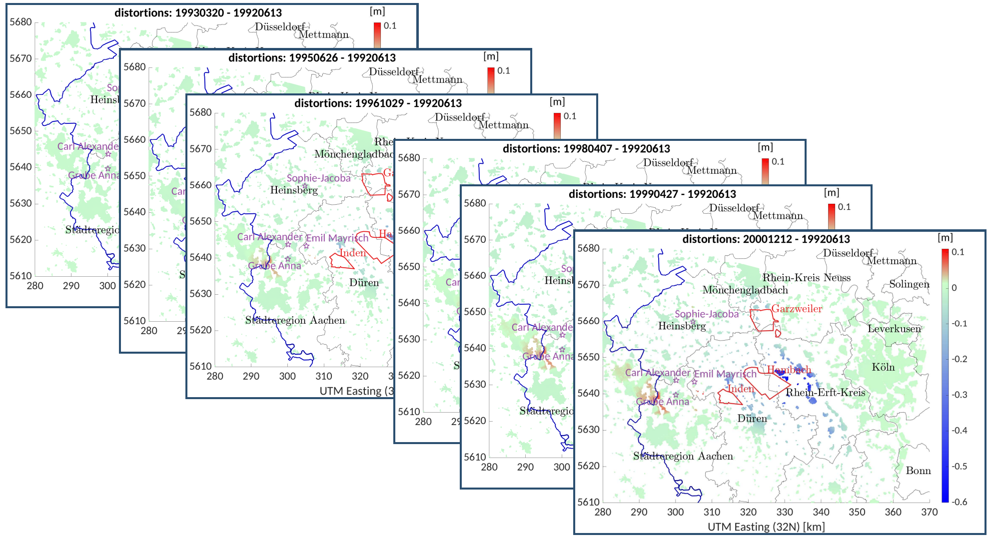

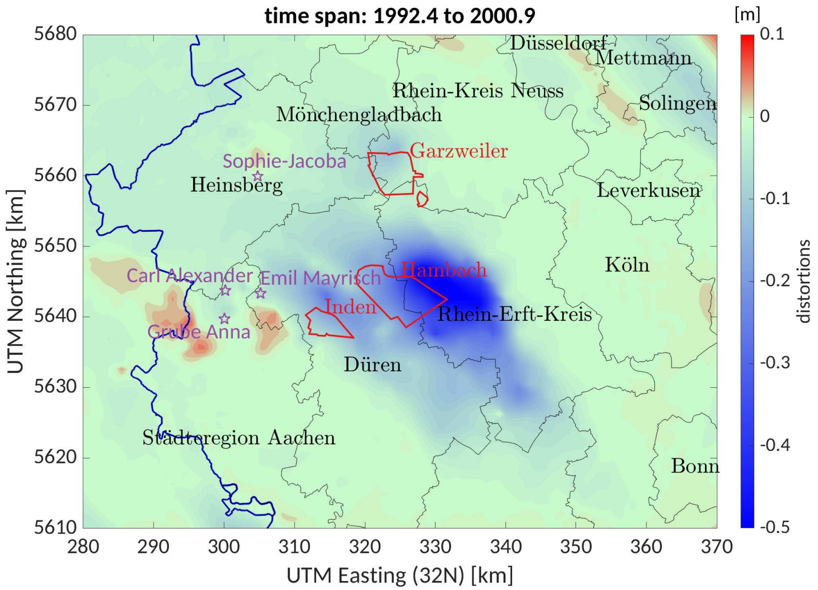

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Best Linear Unbiased Predictor (BLUP)

$$\widetilde{\boldsymbol{s}}_p = \boldsymbol{\Sigma}_{\mathcal{S}}\{\boldsymbol{x}_p, \boldsymbol{x}_o\}\Big(\boldsymbol{\Sigma}_{\mathcal{S}}\{\boldsymbol{x}_o, \boldsymbol{x}_o\} + \boldsymbol{\Sigma}_{\mathcal{N}}\{\boldsymbol{x}_o, \boldsymbol{x}_o\}\Big)^{-1}\Delta\boldsymbol{\ell}$$

$\widetilde{\boldsymbol{s}}_p$	...	best predictor
$\Delta\boldsymbol{\ell}$	...	trend reduced observations at location $\boldsymbol{x}_o$
$\boldsymbol{\Sigma}_{\mathcal{S}}\{\boldsymbol{x}_o, \boldsymbol{x}_o\}$	...	variance/covariance matrix of the signal between the observation points
$\boldsymbol{\Sigma}_{\mathcal{N}}\{\boldsymbol{x}_o, \boldsymbol{x}_o\}$	...	variance/covariance matrix of the noise between the observation points
$\boldsymbol{\Sigma}_{\mathcal{S}}\{\boldsymbol{x}_p, \boldsymbol{x}_o\}$	...	cross-covariance matrix of the signal between the points to predict and the observation points

$$\boldsymbol{\Sigma}\{\widetilde{\boldsymbol{\mathcal{E}}}_{\mathcal{S}}\} = \boldsymbol{\Sigma}_{\mathcal{S}}\{\boldsymbol{x}_p, \boldsymbol{x}_p\} - \boldsymbol{\Sigma}_{\mathcal{S}}\{\boldsymbol{x}_p, \boldsymbol{x}_o\}\Big(\boldsymbol{\Sigma}_{\mathcal{S}}\{\boldsymbol{x}_o, \boldsymbol{x}_o\} + \boldsymbol{\Sigma}_{\mathcal{N}}\{\boldsymbol{x}_o, \boldsymbol{x}_o\}\Big)^{-1}\boldsymbol{\Sigma}_{\mathcal{S}}\{\boldsymbol{x}_o, \boldsymbol{x}_p\}$$

$\boldsymbol{\Sigma}\{\widetilde{\boldsymbol{\mathcal{E}}}_{\mathcal{S}}\}$	...	estimation of the uncertainty of the predictor

„Wiener-Kolmogorov-Principle“

Crucial points in collocation

- **definition of the covariance function**

- **overcoming numerical complexity**

Crucial points in collocation

- **definition of the covariance function**

  spatio-temporal function in general: joint function

  $$\gamma(\boldsymbol{x}, \boldsymbol{t}) = \gamma(\text{space}, \text{time})$$

  in particular: separable function

  $$\gamma(\boldsymbol{x}, \boldsymbol{t}) = \gamma_{sp}(\text{space}) \cdot \gamma_t(\text{time})$$

  $\gamma_{sp}(\text{space})$ ... covariance function in space
  $\gamma_t(\text{time})$ ... covariance function in time

- **overcoming numerical complexity**

Crucial points in collocation

- **definition of the covariance function**
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  $$\gamma(\boldsymbol{x}, \boldsymbol{t}) = \gamma_{sp}(\text{space}) \cdot \gamma_t(\text{time})$$
  $$\begin{aligned} \gamma_{sp}(\text{space}) &\quad \ldots \quad \text{covariance function in space} \\ \gamma_t(\text{time}) &\quad \ldots \quad \text{covariance function in time} \end{aligned}$$

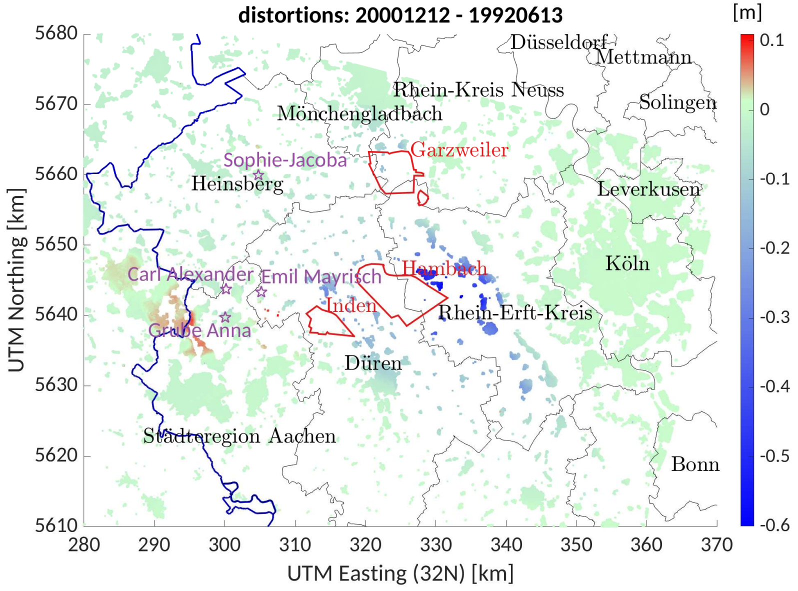
- **overcoming numerical complexity**
  very huge variance/covariance matrix:
  (here in particular: $144302 \times 64 \sim 9$ Mio.)
  - ? consider using finite covariance functions
  - ? check if a Kronecker representation of the variance/covariance matrix is possible

Modeling the covariance function in space

Modeling the covariance function in space

- **data characteristics:**
  - unevenly distributed in space
  - clustered in urban regions
  - non-stationary in time

original data set



- **work around:**
  - find a homogeneous data distribution for the whole area
    $\Longrightarrow$ divide the area in quadratic regions and select randomly the same number of points for each region
  - make the covariances in space independent from the time
    $\Longrightarrow$ consider the distortions only with respect to equal time differences

suggestions from: Leonhardt (2019)

Modeling the covariance function in space

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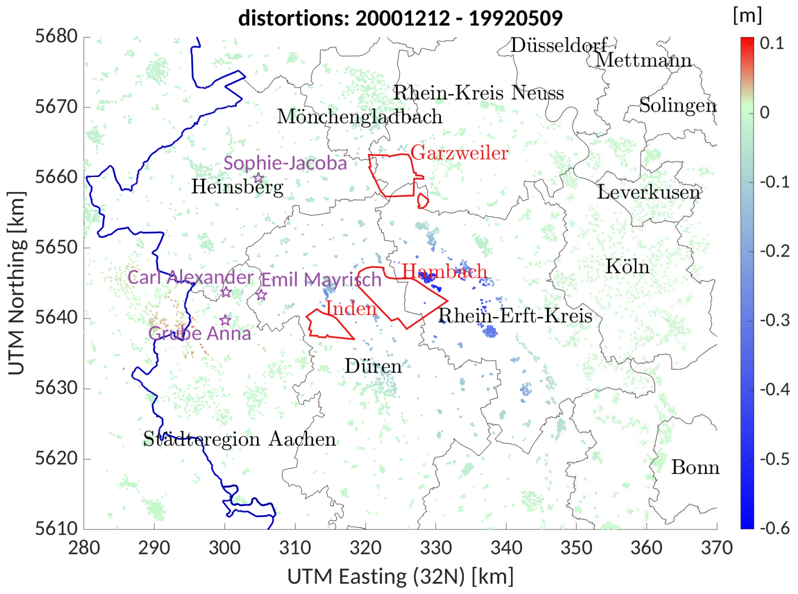
thinned data set



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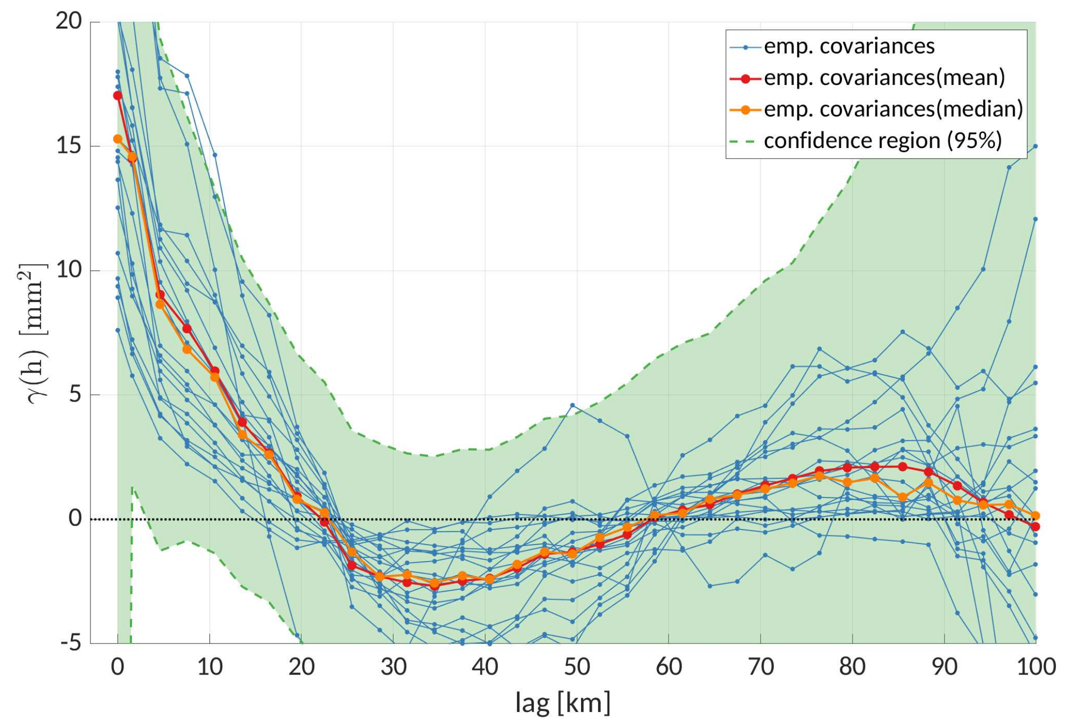
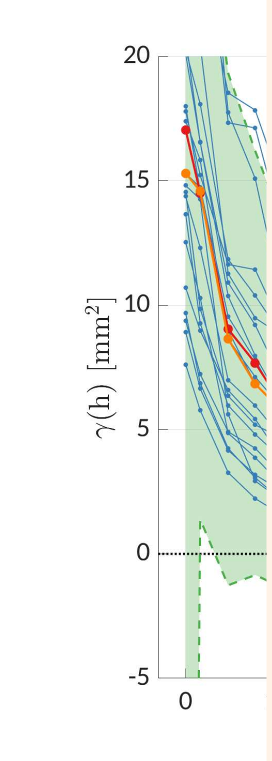
Modeling the covariance function in space

- empirical spatial covariance functions for lag Δx

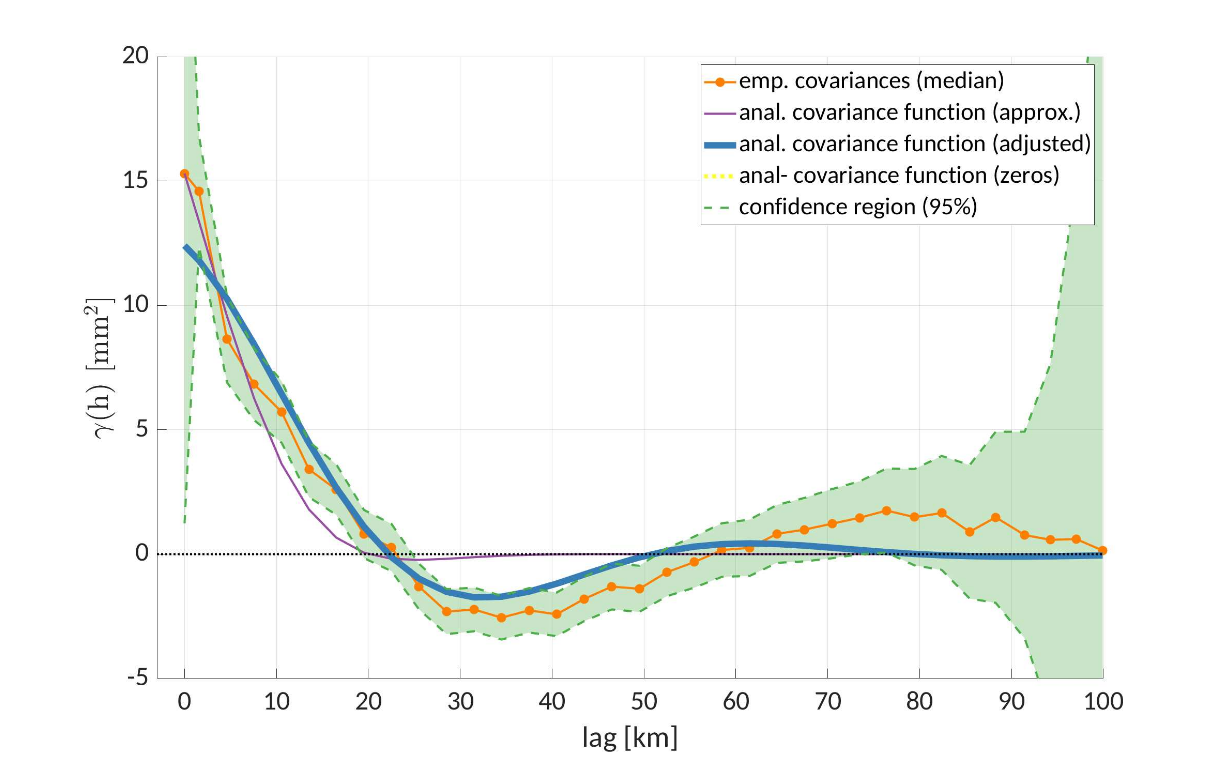


- empirica



- adapt an analytical covariance model

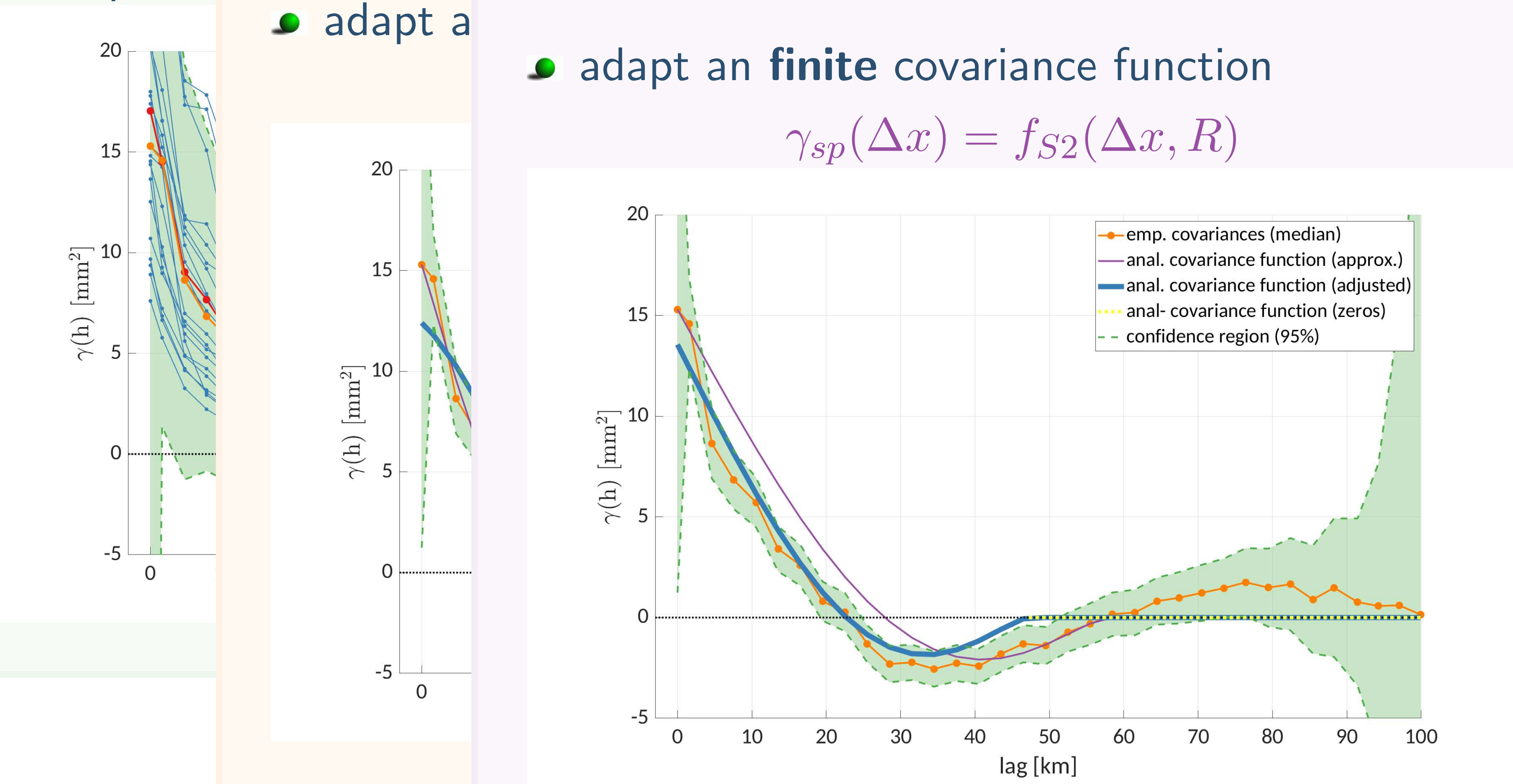
$$\gamma_{sp}(\Delta x) = a e^{-b|\Delta x|} J0(\Delta x)$$



Modeling the covariance function in space

- empirica
- adapt a
- adapt an **finite** covariance function

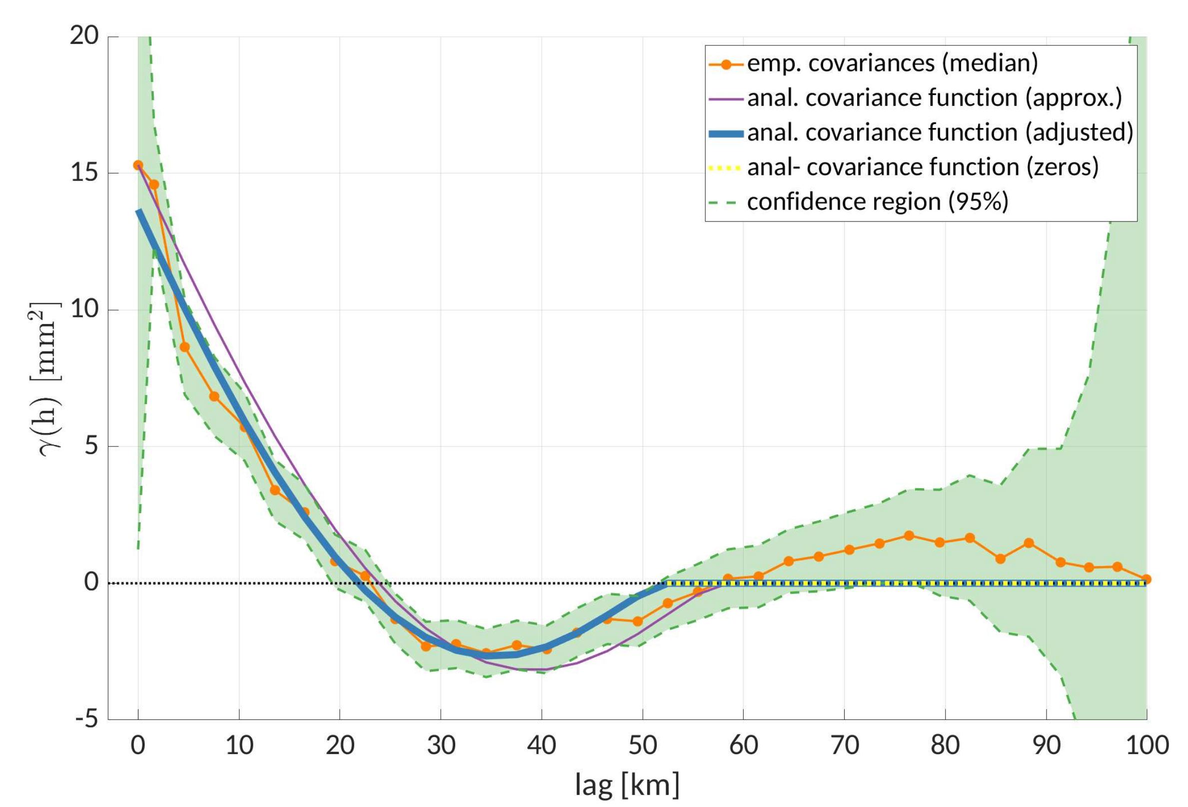
$$\gamma_{sp}(\Delta x) = f_{S2}(\Delta x, R)$$



Sansò und Schuh (1987); Schuh (2016)

Modeling the covariance function in space

- **or better:**
use the new family of **flexible finite** covariance functions constructed by autocorrelation of polynomial base functions

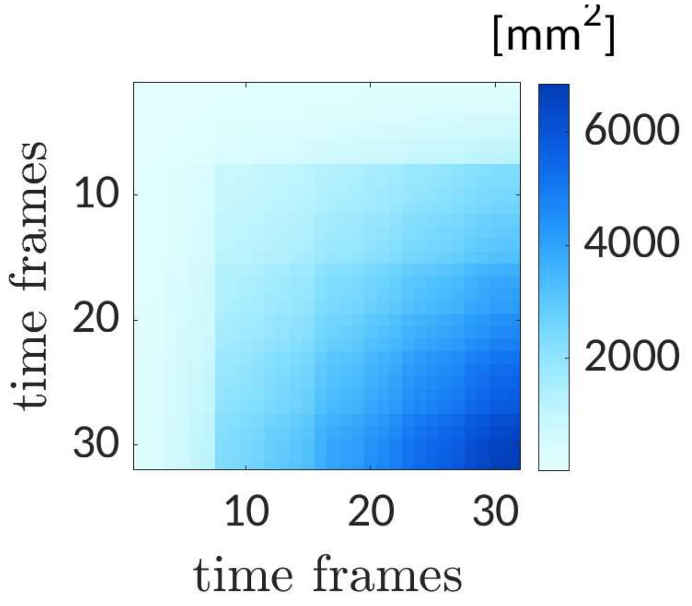


$\Longrightarrow$ see Poster today by Schubert und Schuh (2022)

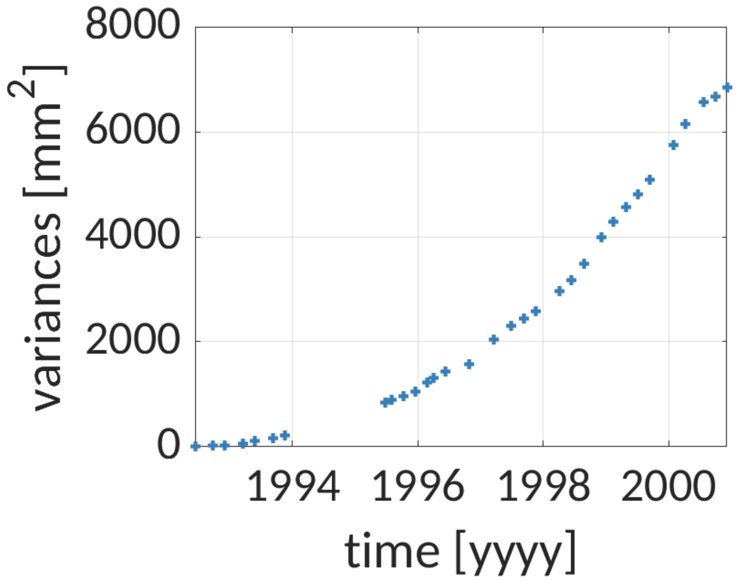
Modeling the covariance function in time

Modeling the covariance function in time

- **data characteristics:**
  - non stationary in time
  - unevenly distributed in time
  - time slices / frames
- **empirical temporal variances/covariances:**



variance/covariance matrix



variances

Modeling of the non-stationary processes with Time-Variable AutoRegressive (TVAR) processes

- **Approach:** time-variable TVAR(1) processes:

$$\mathcal{S}_t := \alpha_t\ \mathcal{S}_{t-1} + \mathcal{E}_t = \prod_{j=1}^{t} \alpha_j\ \mathcal{S}_0 + \sum_{k=1}^{t} \prod_{j=k+1}^{t} \alpha_j\ \mathcal{E}_k$$

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- Variances/covariances derived by the second central moment:

$$\begin{aligned}\gamma_t(t, \Delta t) &= \Sigma\{\mathcal{S}_t, \mathcal{S}_{t+\Delta t}\} \\ &= E\{(\mathcal{S}_t - E\{\mathcal{S}_t\})(\mathcal{S}_{t+\Delta t} - E\{\mathcal{S}_{t+\Delta t}\})\} \\ &= \prod_{n=t+1}^{t+\Delta t} \alpha_n \underbrace{\left(\prod_{j=1}^{t} \alpha_j^2 \, \sigma_{\mathcal{S}_0}^2 + \sum_{k=1}^{t} \prod_{j=k+1}^{t} \alpha_j^2 \, \sigma_{\mathcal{E}}^2\right)}_{=\gamma_t(t,0)}, \quad \Delta t > 0\end{aligned}$$

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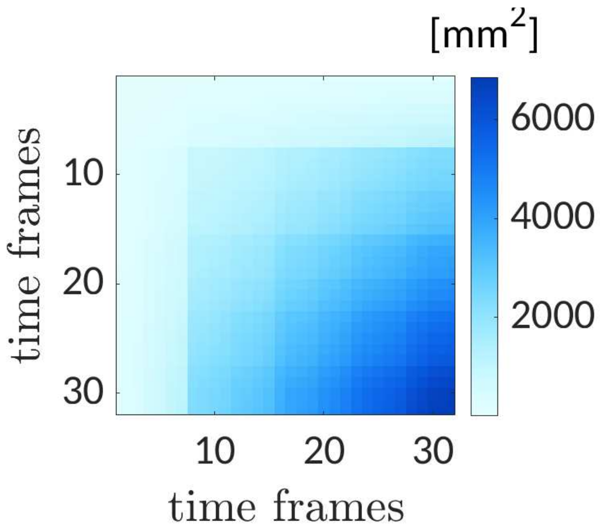
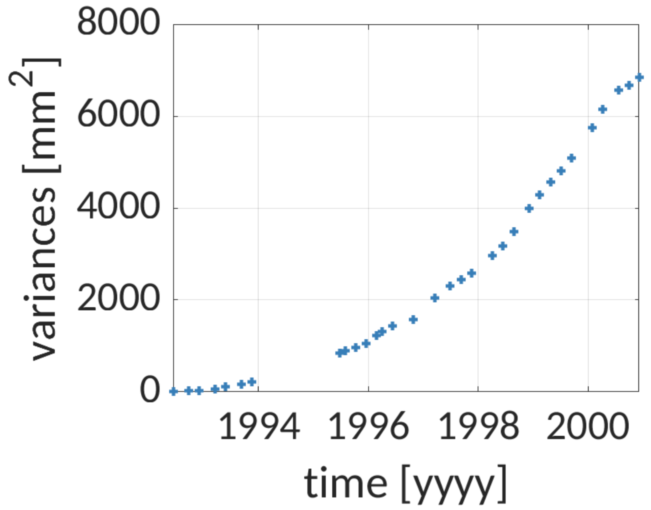
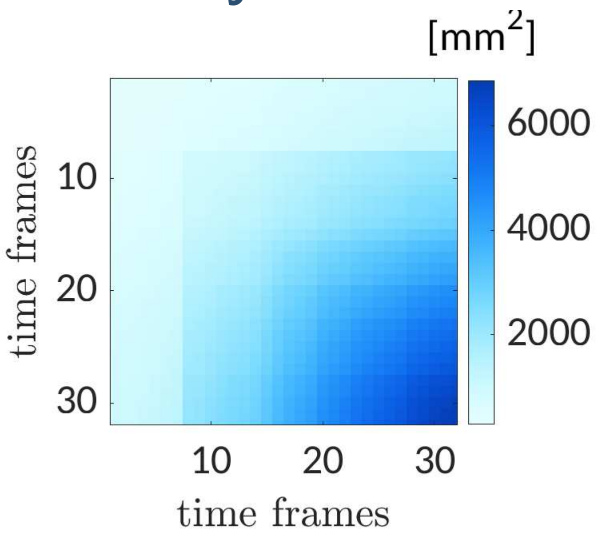
$\Longrightarrow$ oral presentation on Tuesday Korte et al. (2022) for further strategies to design time-variable AR-processes

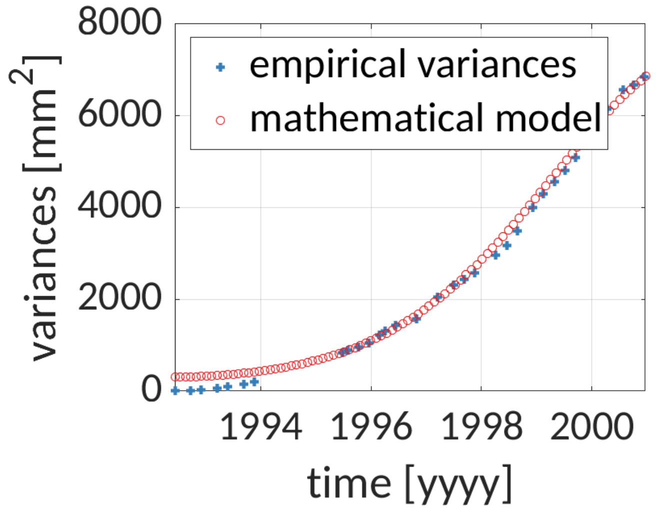
- **empirical temporal variances/covariances:**





- **analytical covariance function**





Crucial points in collocation

- **definition of the covariance function**

  spatio-temporal function in general: joint function

  $$\gamma(\boldsymbol{x}, \boldsymbol{t}) = \gamma(\text{space}, \text{time})$$

  in particular: separable function

  $$\gamma(\Delta\boldsymbol{x}, \boldsymbol{t}) = \gamma_{sp}(\Delta x) \cdot \gamma_t(t, \Delta t)$$

  $\gamma_{sp}(\Delta x)\ldots$ homogeneous covariance function in space
  $\gamma_t(t, \Delta t)\ldots$ time-variant covariance function in time

- **overcoming numerical complexity**

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  (here in particular: $144302 \times 64 \sim 9$ Mio.)

  ✓ consider using finite covariance functions

  ? check if an approach via Kronecker representation of the variance/covariance matrix is possible

Is a Kronecker representation possible?

- Covariance function is separable!

$$\gamma(\Delta x; t, \Delta t) = \gamma_{sp}(\Delta x) \cdot \gamma_t(t, \Delta t)$$

Overcome numerical complexity

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- identical scatterers $\boldsymbol{x}_o$ in each time slice $\boldsymbol{t}_o$!
  $\boldsymbol{x}_o \ldots$ observed positions, $\boldsymbol{t}_o \ldots$ observed time

Overcome numerical complexity

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- identical scatterers $\boldsymbol{x}_o$ in each time slice $\boldsymbol{t}_o$!
  $\boldsymbol{x}_o \ldots$ observed positions, $\boldsymbol{t}_o \ldots$ observed time
- variance/covariance matrix of the signal is separable!
  - variance/covariance matrix between the observed points
    $\boldsymbol{\Sigma}_{\boldsymbol{\mathcal{S}}}\{\boldsymbol{x}_o, \boldsymbol{t}_o; \boldsymbol{x}_o, \boldsymbol{t}_o\} = \boldsymbol{\Sigma}^t_{\boldsymbol{\mathcal{S}}}\{\boldsymbol{t}_o, \boldsymbol{t}_o\} \otimes \boldsymbol{\Sigma}^{sp}_{\boldsymbol{\mathcal{S}}}\{\boldsymbol{x}_o, \boldsymbol{x}_o\}$
  - covariance matrix between predicted and observed points
    $\boldsymbol{\Sigma}_{\boldsymbol{\mathcal{S}}}\{\boldsymbol{x}_p, \boldsymbol{t}_p; \boldsymbol{x}_o, \boldsymbol{t}_o\} = \boldsymbol{\Sigma}^t_{\boldsymbol{\mathcal{S}}}\{\boldsymbol{t}_p, \boldsymbol{t}_o\} \otimes \boldsymbol{\Sigma}^{sp}_{\boldsymbol{\mathcal{S}}}\{\boldsymbol{x}_p, \boldsymbol{x}_o\}$
  - covariance matrix of noise
    $\boldsymbol{\Sigma}_{\boldsymbol{\mathcal{N}}}\{\boldsymbol{x}_o, \boldsymbol{t}_o; \boldsymbol{x}_o, \boldsymbol{t}_o\} := \boldsymbol{\Sigma}_{\boldsymbol{\mathcal{N}}}$

Predictor:

$$\widetilde{s}_p = \boldsymbol{\Sigma}_{\mathcal{S}}^{t}\{\boldsymbol{t}_p, \boldsymbol{t}_o\} \otimes \boldsymbol{\Sigma}_{\mathcal{S}}^{sp}\{\boldsymbol{x}_p, \boldsymbol{x}_o\} \underbrace{\left(\boldsymbol{\Sigma}_{\mathcal{S}}^{t}\{\boldsymbol{t}_o, \boldsymbol{t}_o\} \otimes \boldsymbol{\Sigma}_{\mathcal{S}}^{sp}\{\boldsymbol{x}_o, \boldsymbol{x}_o\} + \boldsymbol{\Sigma}_{\mathcal{N}} \right)}_{\text{Kronecker representation ???}}^{-1} \Delta \boldsymbol{\ell}$$

$$\boldsymbol{\Sigma}_{\mathcal{S}+\mathcal{N}}^{t}\{\boldsymbol{t}_o, \boldsymbol{t}_o\} \otimes \boldsymbol{\Sigma}_{\mathcal{S}+\mathcal{N}}^{sp}\{\boldsymbol{x}_o, \boldsymbol{x}_o\}$$

if a Konecker representation is possible then a **compact predictor** exists!

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and the computation can be split up into a time-wise and space-wise component.

confer e.g. Blaha (1977); Rauhala (1974)

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Kronecker representation ???

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possible representations for the noise:

- $\boldsymbol{\Sigma}_{\mathcal{N}} = \boldsymbol{\Sigma}^t_{\boldsymbol{S}}\{\boldsymbol{t}_o, \boldsymbol{t}_o\} \otimes \mathbb{1}_{sp}\,\sigma^2_{sp}$ Kronecker of $(\cdot)$ exists: yes but: interpretation?
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We have to work on the third option !!!

Thoughts about separability

Thoughts: Have a look on the spectral representation of the sum

- $\boldsymbol{A} + \mathbb{1}\sigma^2 = ?$ with $\boldsymbol{A} = \boldsymbol{U_A}\,\boldsymbol{\Lambda_A}\,\boldsymbol{U_A^T}$

$\boldsymbol{U_A}$ ... eigenvectors
$\boldsymbol{\Lambda_A}$ ... eigenvalues

$$\boldsymbol{A} + \mathbb{1}\sigma^2 = \boldsymbol{U_A}\,\boldsymbol{\Lambda_A}\,\boldsymbol{U_A^T} + \boldsymbol{U_{\mathbb{1}}}\,\mathbb{1}\,\boldsymbol{U_{\mathbb{1}}^T}\,\sigma^2$$

$$= \boldsymbol{U_A}\,\boldsymbol{\Lambda_A}\,\boldsymbol{U_A^T} + \boldsymbol{U_A}\,\mathbb{1}\,\boldsymbol{U_A^T}\,\sigma^2$$

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notice:
eigenspace $\boldsymbol{E_{\mathbb{1}}}(1) \equiv \mathbb{V}_n$
$\boldsymbol{U_{\mathbb{1}}} \implies \boldsymbol{U_A}$

Thoughts about separability

Thoughts: Have a look on the spectral representation of the sum

- $A + \mathbb{1}\sigma^2 = ?$ with $A = U_A \, \Lambda_A \, U_A^T$ $U_A \dots$ eigenvectors, $\Lambda_A \dots$ eigenvalues

$$A + \mathbb{1}\sigma^2 = U_A \, \Lambda_A \, U_A^T + U_{\mathbb{1}} \, \mathbb{1} \, U_{\mathbb{1}}^T \, \sigma^2$$

$$= U_A \, \Lambda_A \, U_A^T + U_A \, \mathbb{1} \, U_A^T \, \sigma^2$$

notice:
eigenspace $E_{\mathbb{1}}(1) \equiv \mathbb{V}_n$
$U_{\mathbb{1}} \implies U_A$

$$A + \mathbb{1}\sigma^2 = U_A \, (\Lambda_A + \mathbb{1}\sigma^2) \, U_A^T$$

- $A = B \otimes C$ with $B = U_B \, \Lambda_B \, U_B^T$ and $C = U_C \, \Lambda_C \, U_C^T$

$$A = U_B \, \Lambda_B \, U_B^T \otimes U_C \, \Lambda_C \, U_C^T$$

$$A = (U_B \otimes U_C) \, (\Lambda_B \otimes \Lambda_C) \, (U_B \otimes U_C)^T$$

with $(\Lambda_B \otimes \Lambda_C)$ ... diagonal

Thoughts about separability

- $$\boldsymbol{A} + \mathbb{1}\sigma^2 = (\boldsymbol{U_B} \otimes \boldsymbol{U_C}) \underbrace{\left(\boldsymbol{\Lambda_B} \otimes \boldsymbol{\Lambda_C} + \mathbb{1}_{\boldsymbol{B}} \otimes \mathbb{1}_{\boldsymbol{C}}\, \sigma^2\right)}_{\text{diagonal matrix}} (\boldsymbol{U_B} \otimes \boldsymbol{U_C})^T$$

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  $\boldsymbol{A} + \mathbb{1}\sigma^2$ can be expressed as a Kronecker product !!!

- $$\left(\boldsymbol{A} + \mathbb{1}\sigma^2\right)^{-1} = \sum_{k=1}^{n_B} \boldsymbol{U_B}(:,k)\, \boldsymbol{U_B}(:,k)^T \otimes \boldsymbol{U_C} \left((\boldsymbol{\Lambda_B})_k \boldsymbol{\Lambda_C} + \mathbb{1}_{\boldsymbol{C}}\sigma^2\right)^{-1} \boldsymbol{U_C}^T$$

  $\left(\boldsymbol{A} + \mathbb{1}\sigma^2\right)^{-1}$ can be computed component-by-component

Predictor:

$$\widetilde{\boldsymbol{s}}_p = \boldsymbol{\Sigma}^t_{\boldsymbol{\mathcal{S}}}\{\boldsymbol{t}_p, \boldsymbol{t}_o\} \otimes \boldsymbol{\Sigma}^{sp}_{\boldsymbol{\mathcal{S}}}\{\boldsymbol{x}_p, \boldsymbol{x}_o\} \underbrace{\left(\boldsymbol{\Sigma}^t_{\boldsymbol{\mathcal{S}}}\{\boldsymbol{t}_o, \boldsymbol{t}_o\} \otimes \boldsymbol{\Sigma}^{sp}_{\boldsymbol{\mathcal{S}}}\{\boldsymbol{x}_o, \boldsymbol{x}_o\} + \mathbb{1}_t \otimes \mathbb{1}_{sp}\sigma^2_{\mathcal{N}} \right)^{-1}}_{\text{Kronecker representation } \checkmark} \Delta\boldsymbol{\ell}$$

reshaping observation/predictor vector into a matrix

columns: time slices (t)
rows: scatterer (sp)

$$\Delta\boldsymbol{L} := \text{reshape}(\Delta\boldsymbol{\ell}, n_o^{sp}, n_o^t) \qquad \widetilde{\boldsymbol{S}}_p := \text{reshape}(\widetilde{\boldsymbol{s}}_p, n_p^{sp}, n_p^t)$$

Compact Predictor:

$$\widetilde{\boldsymbol{S}}_p = \boldsymbol{\Sigma}^{sp}_{\boldsymbol{\mathcal{S}}}\{\boldsymbol{x}_p, \boldsymbol{x}_o\} \underbrace{\left(\sum_{k=1}^{n_B} \boldsymbol{U}_{sp} \left((\boldsymbol{\Lambda}_t)_k \boldsymbol{\Lambda}_{sp} + \mathbb{1}_{sp}\sigma^2_{\mathcal{N}} \right)^{-1} \boldsymbol{U}^T_{sp}\, \Delta\boldsymbol{L}\, \boldsymbol{U}_t(:,k)\, \boldsymbol{U}_t(:,k)^T \right)}_{\boldsymbol{X}} \boldsymbol{\Sigma}^t_{\boldsymbol{\mathcal{S}}}\{\boldsymbol{t}_o, \boldsymbol{t}_p\}$$

with $\boldsymbol{X}$ ... reshaped solution of $\boldsymbol{x} = \left(\boldsymbol{\Sigma}_{\boldsymbol{\mathcal{S}}} + \boldsymbol{\Sigma}_{\mathcal{N}}\right)^{-1} \Delta\boldsymbol{\ell}$

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Kronecker representation ✓

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$\Longrightarrow$ collocation can be separated into a time-wise and space-wise component

Summary and Resumé

Goal: Rigorous collocation of inhomogeneous spatio-temporal DInSAR-stack

Crucial points:

- **definition of the covariance function**
  design of separable spatio-temporal function

  $$\gamma(\Delta\boldsymbol{x}, \boldsymbol{t}) = \gamma_{sp}(\Delta x) \cdot \gamma_t(t, \Delta t)$$

  ✓ $\gamma_{sp}(\Delta x) \ldots$ homogeneous covariance function in space
  ✓ $\gamma_t(t, \Delta t) \ldots$ time-variant covariance function in time

- **overcoming numerical complexity**
  very huge variance/covariance matrix:
  (here in particular: $144.302 \times 64 \sim 9$ Mio.)

  ✓ using finite covariance function
  ✓ inclusion of a realistic noise model
  ✓ rigorous solution through Kronecker representation

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Take home message:

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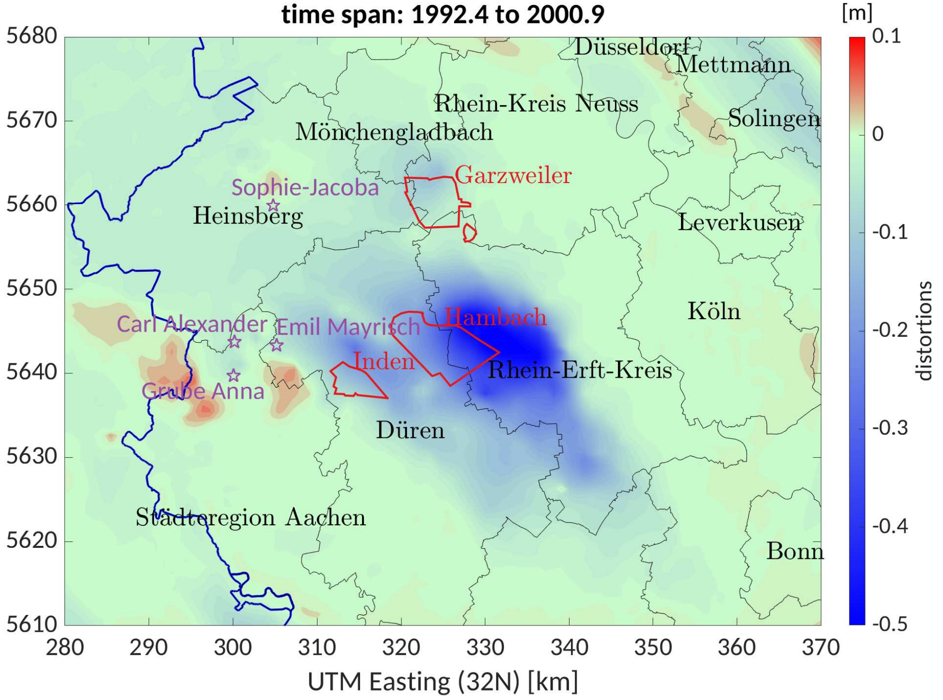
Take home message: The huge collocation problem can be rigorously solved on a notebook

Let's have a look on the results:

continous prediction in space



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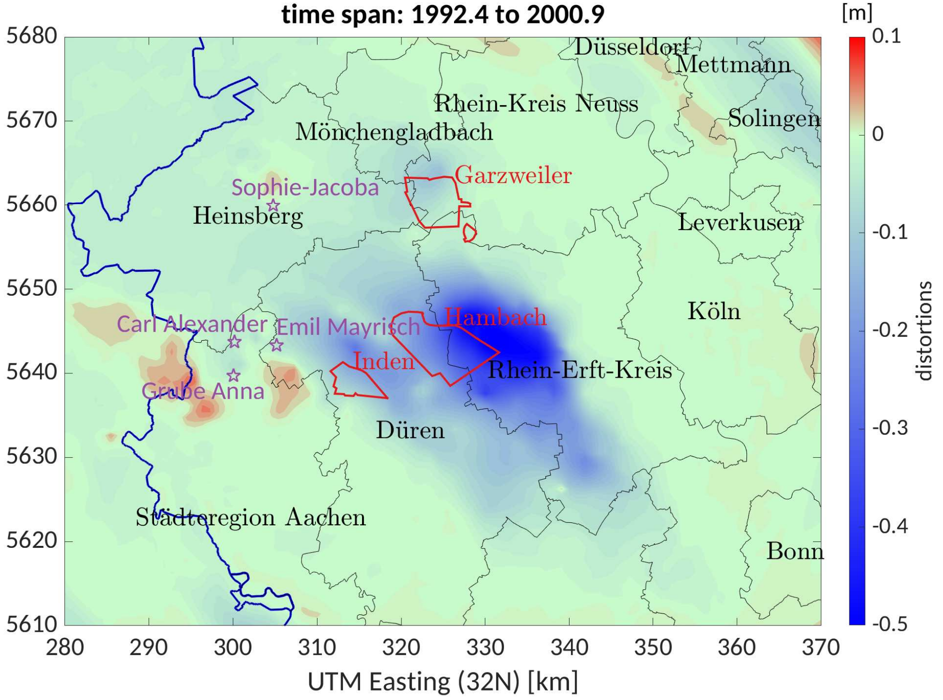
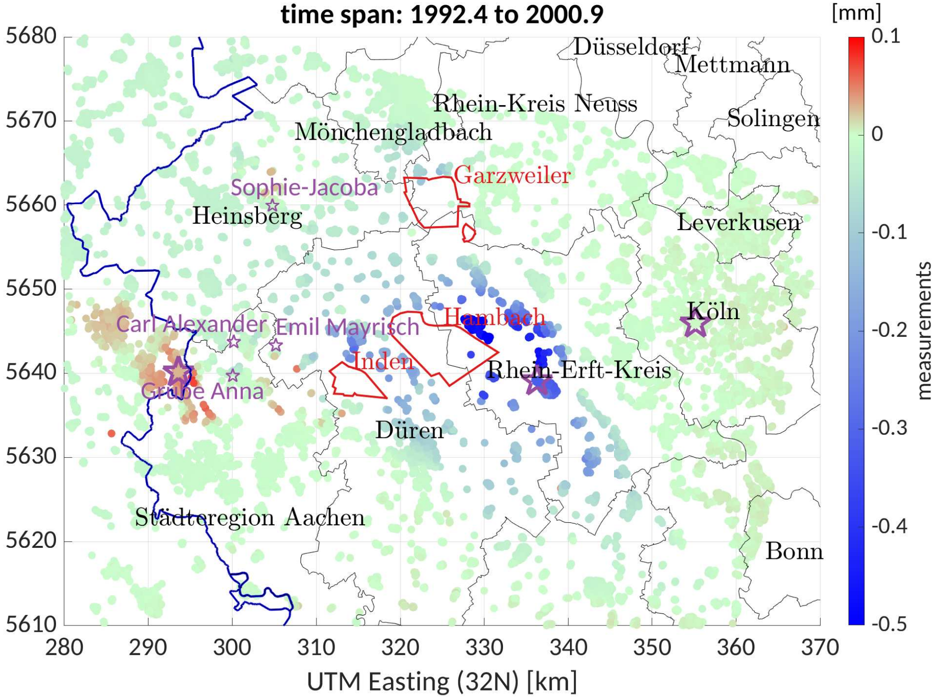


equispaced sequences in time



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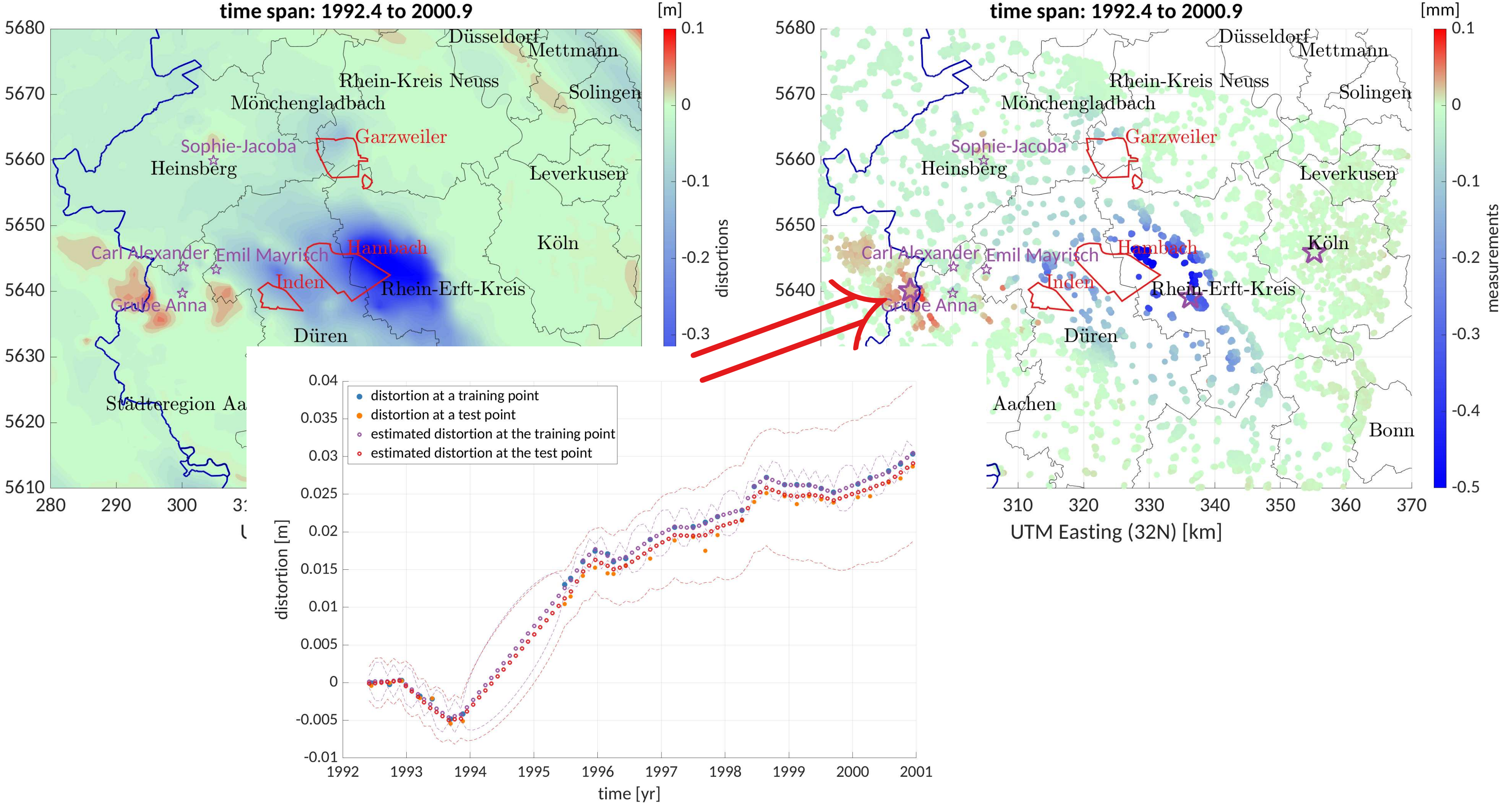
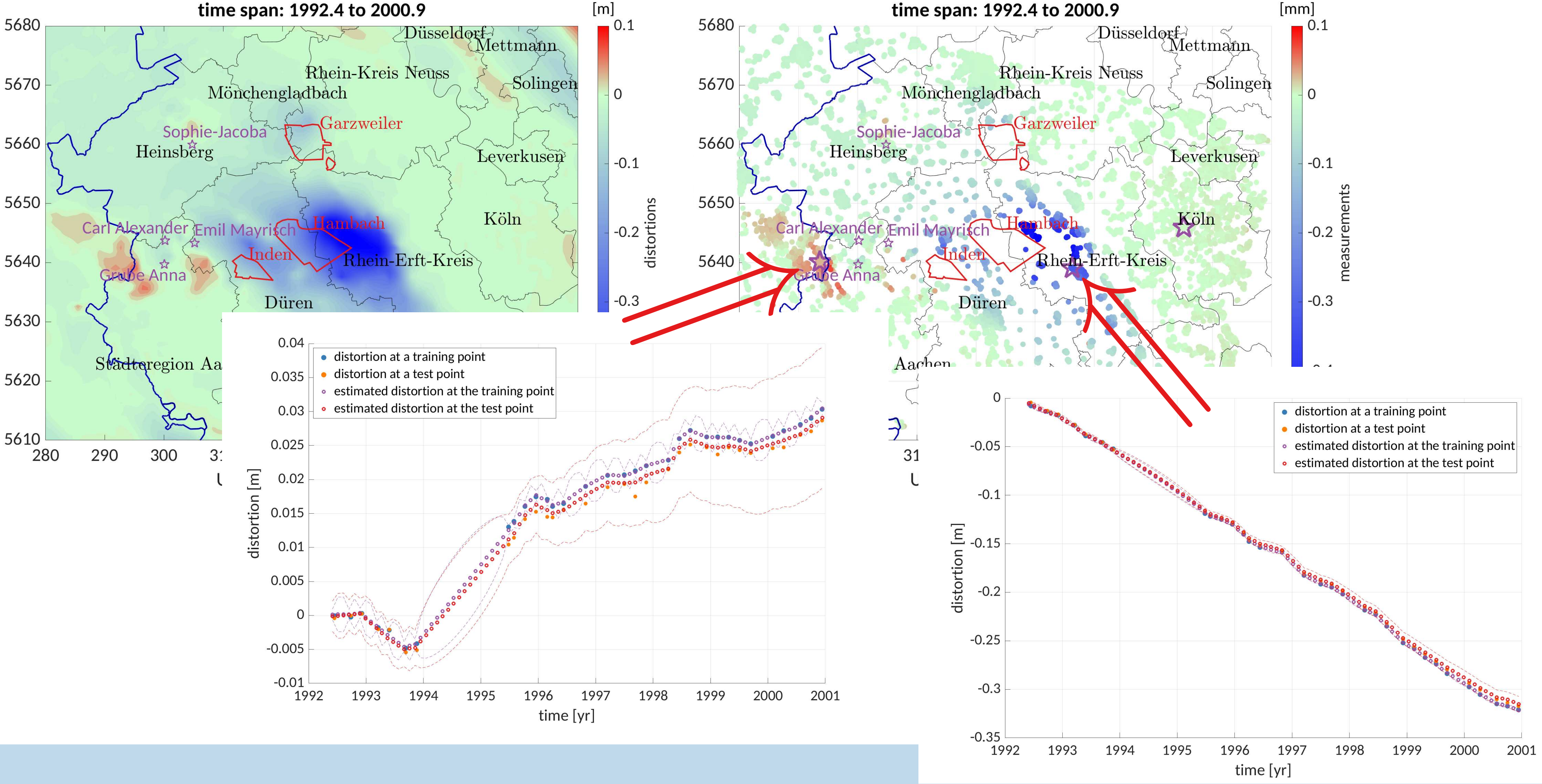


Results of the rigorous collocation of a DInSAR-stack

Let’s have a look on the results:

continous prediction in space

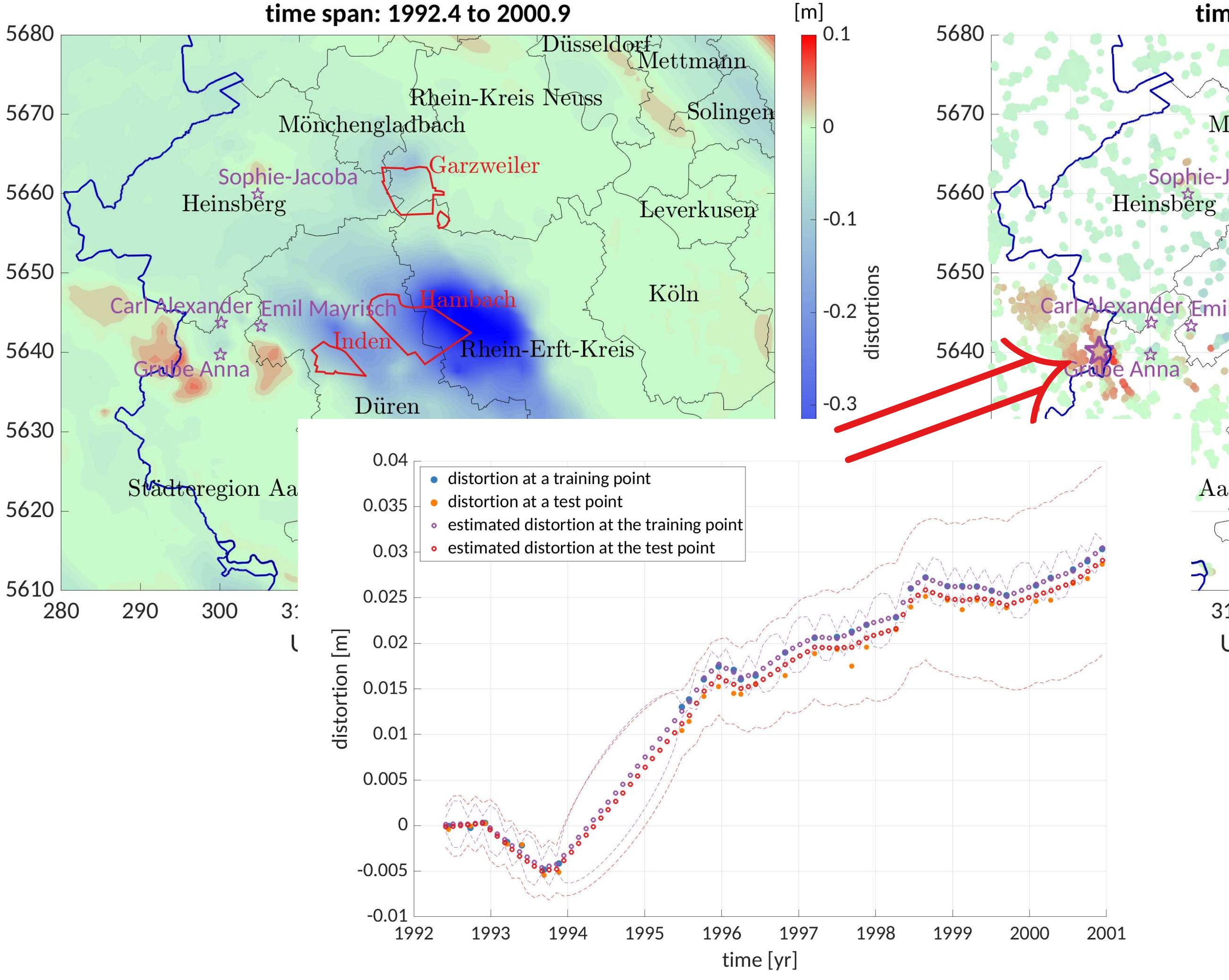
equispaced sequences in time



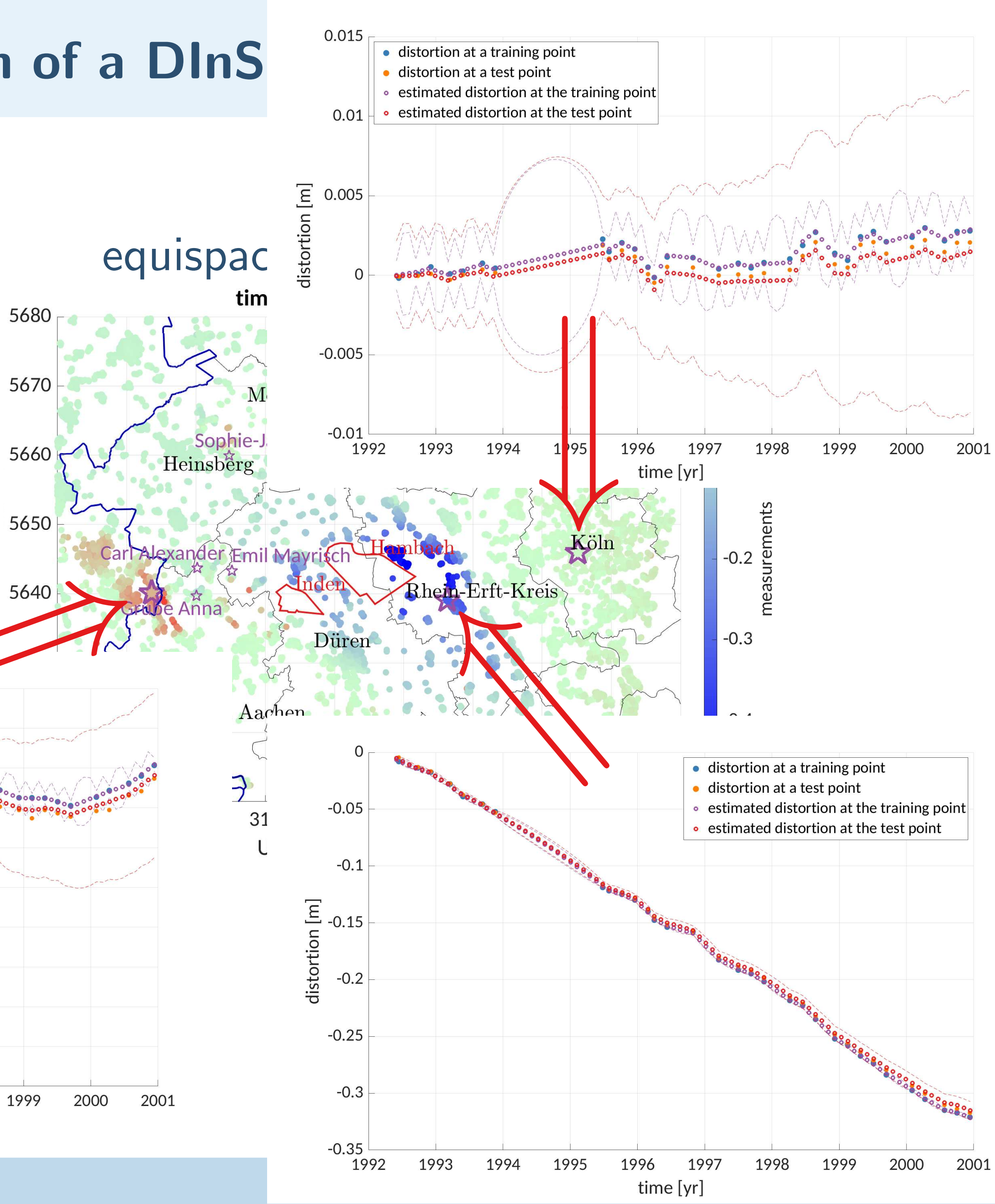
Results of the rigorous collocation of a DInS

Let's have a look on the results:

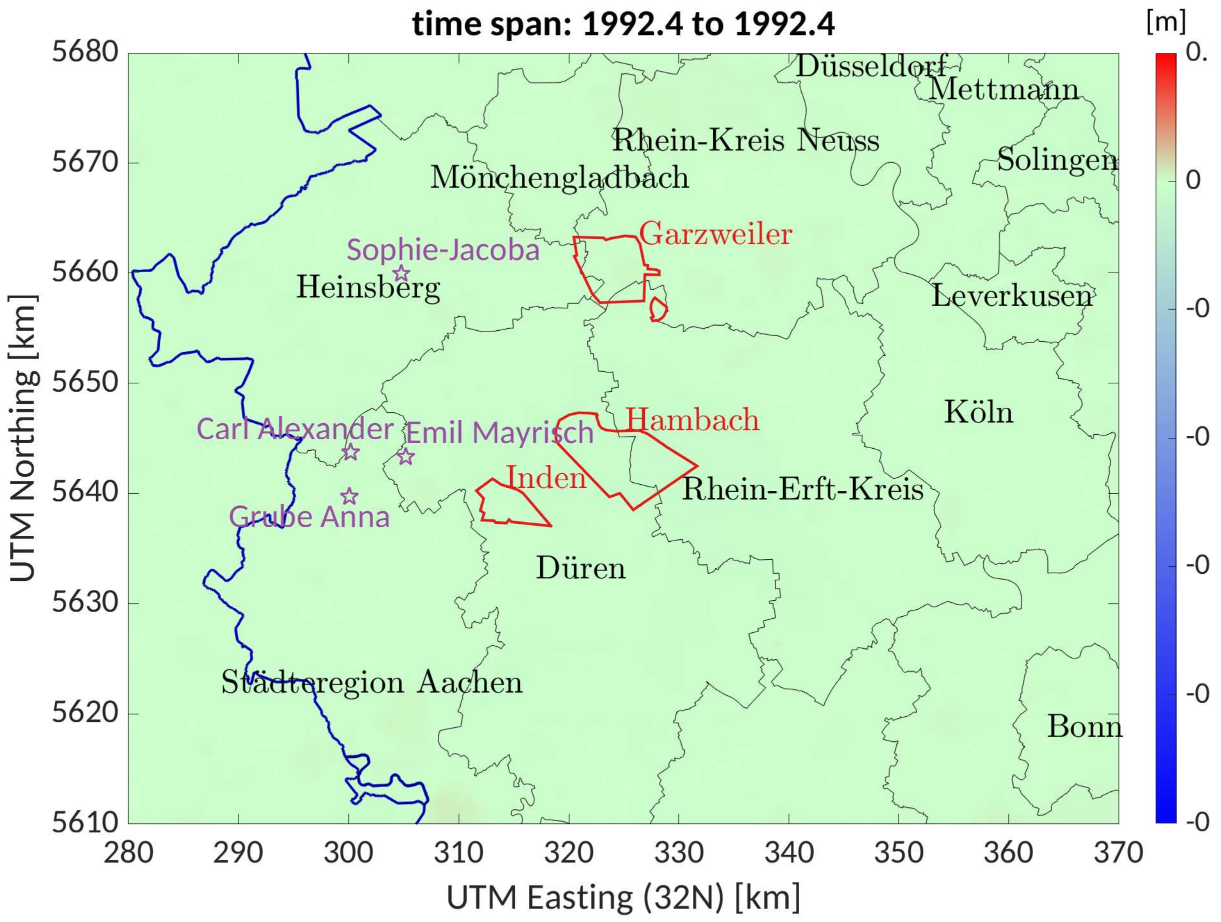
continous prediction in space



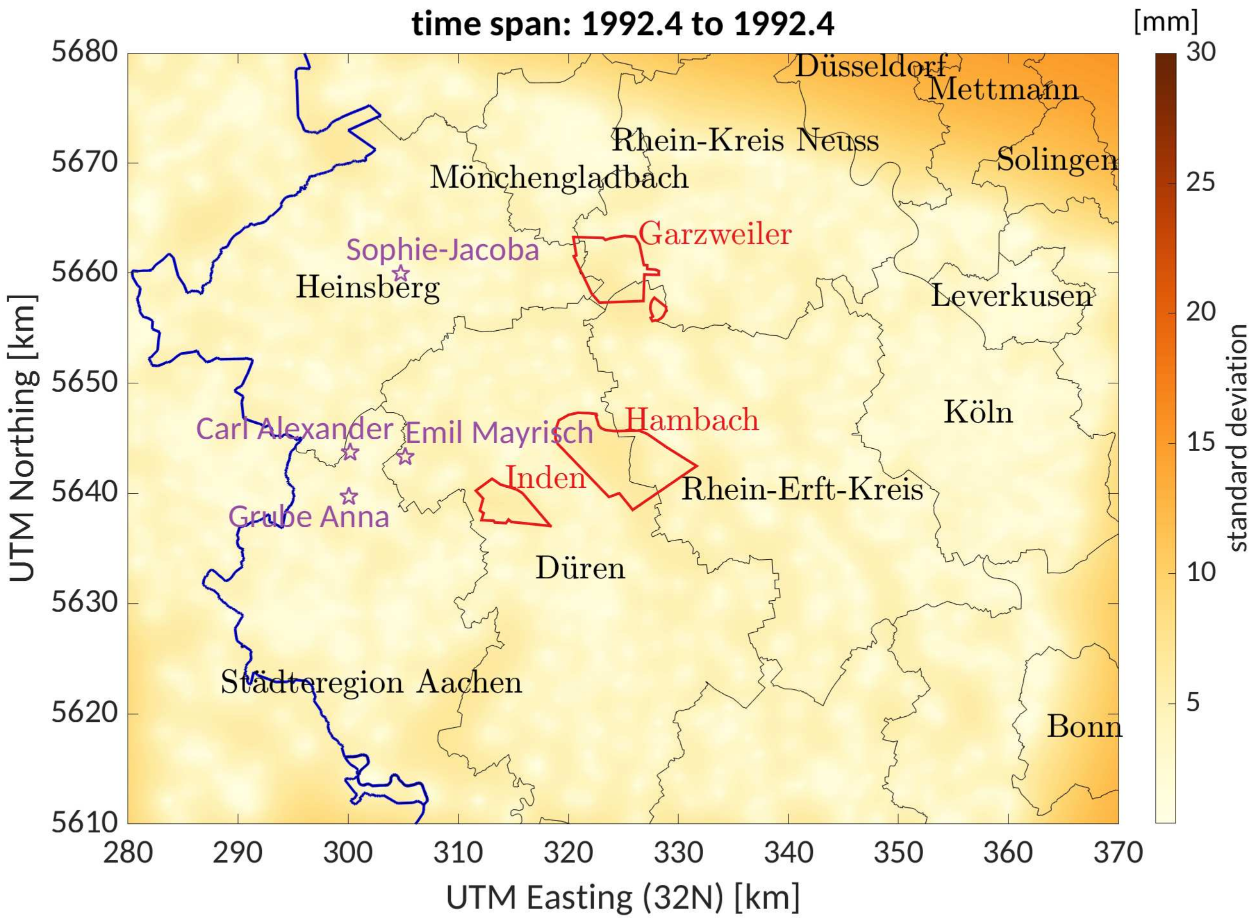
equispac



Rigorous collocation of inhomogeneous spatio-temporal signals

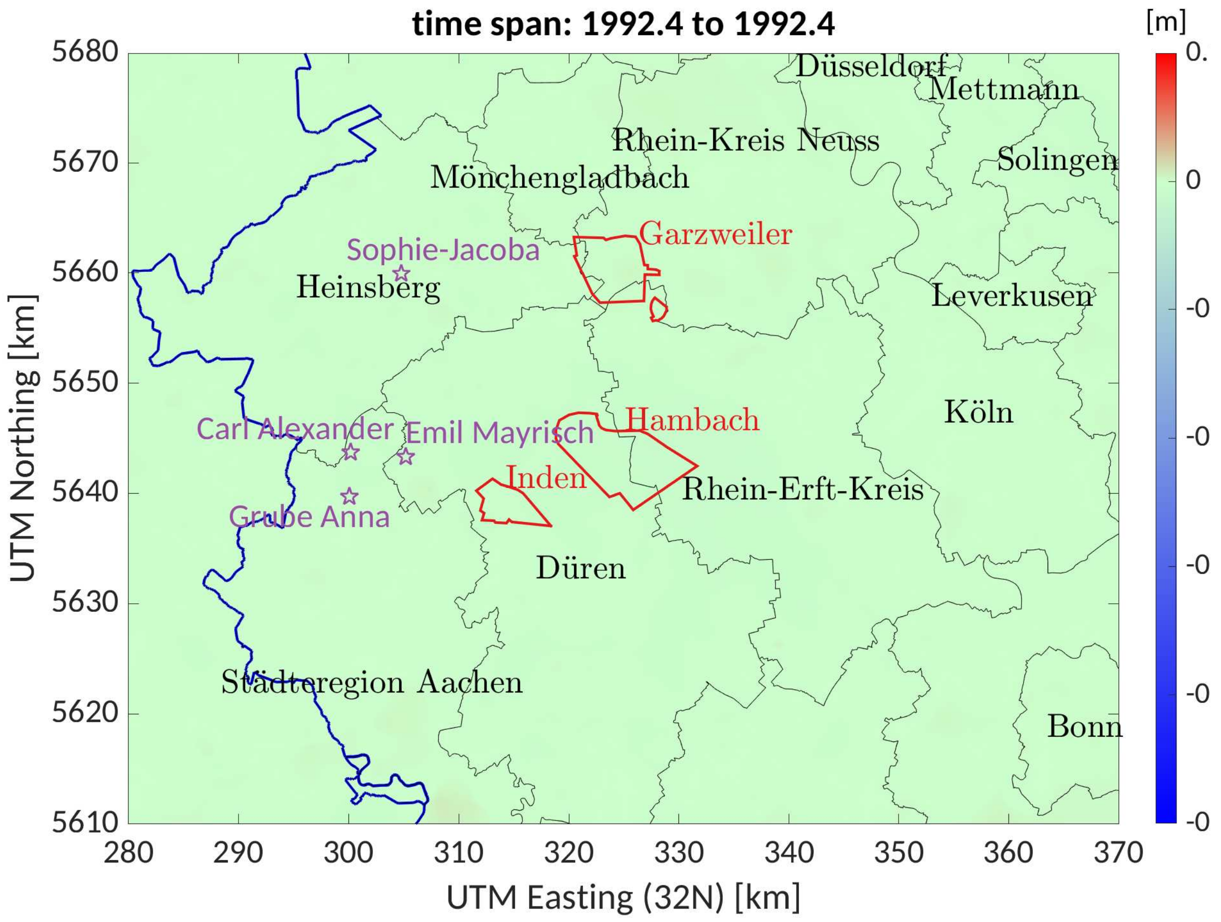


predictor

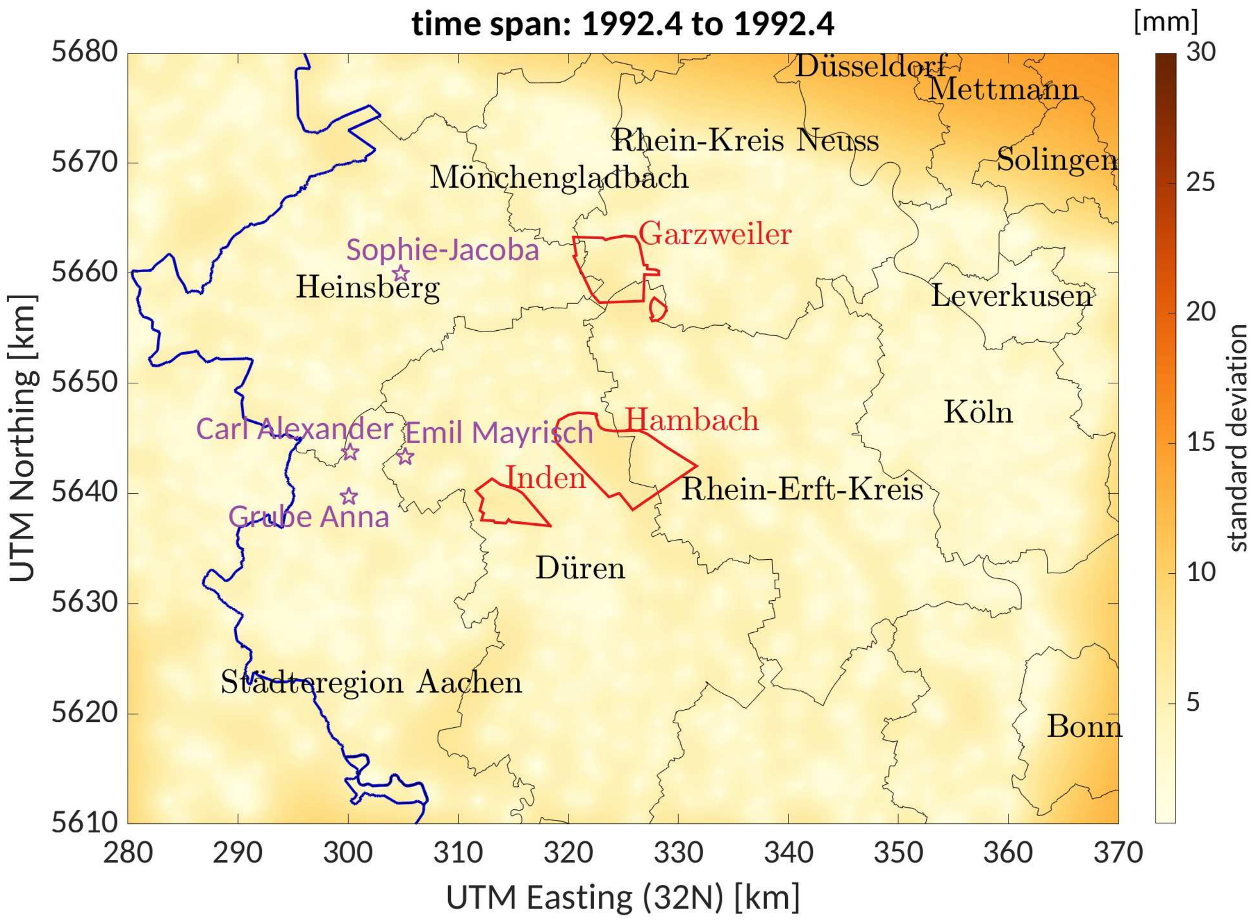


estimation error

Rigorous collocation of inhomogeneous spatio-temporal signals



predictor



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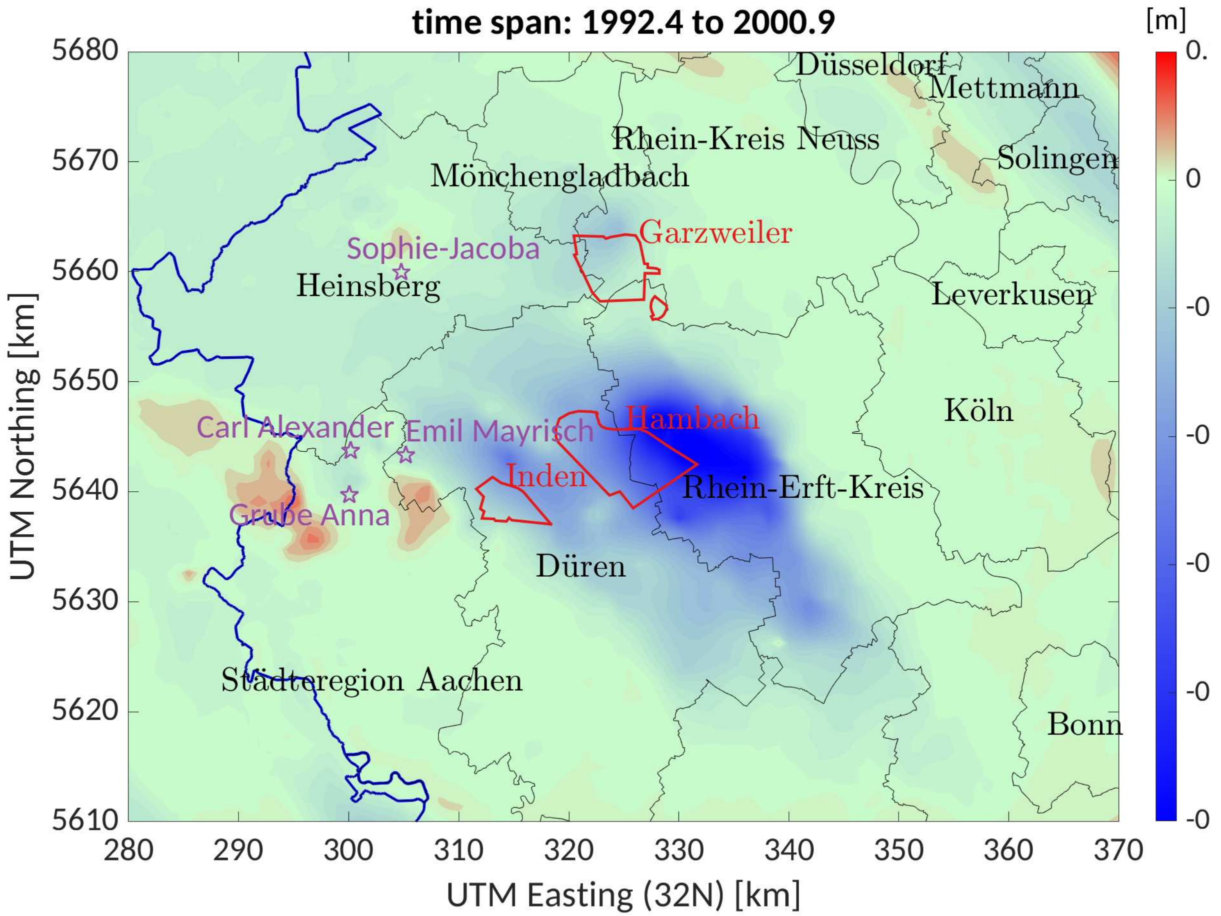
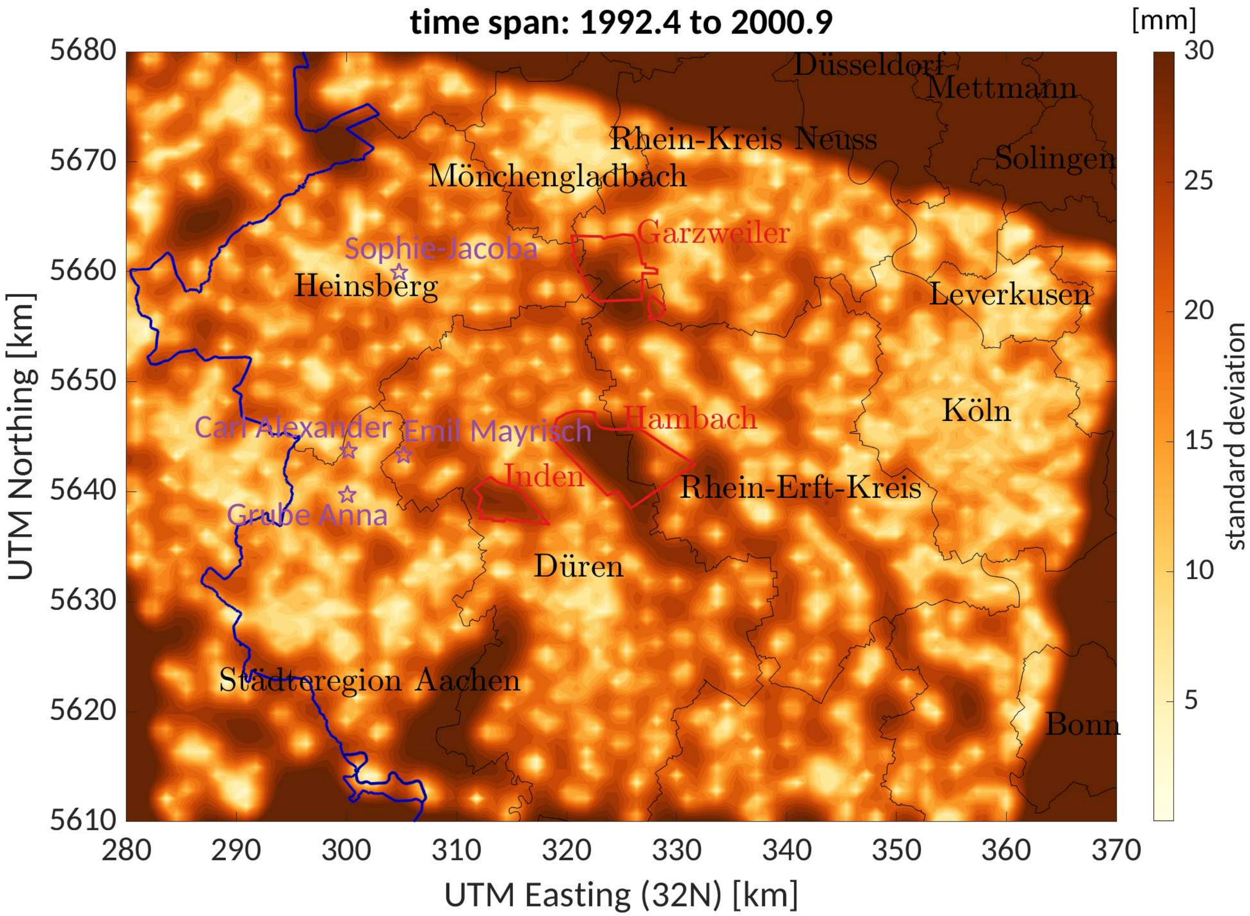
Rigorous collocation of inhomogeneous spatio-temporal signals



predictor



estimation error

Thank you for your attention!

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