Design of frequency selective filters for non-equispaced data

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IX Hotine-Marussi Symposium

Rome, June 18^{th}-22^{th}, 2018

Task: Estimation of the Surface Displacements in the Lower Rhine Embayment by RADAR interferometry

The Task

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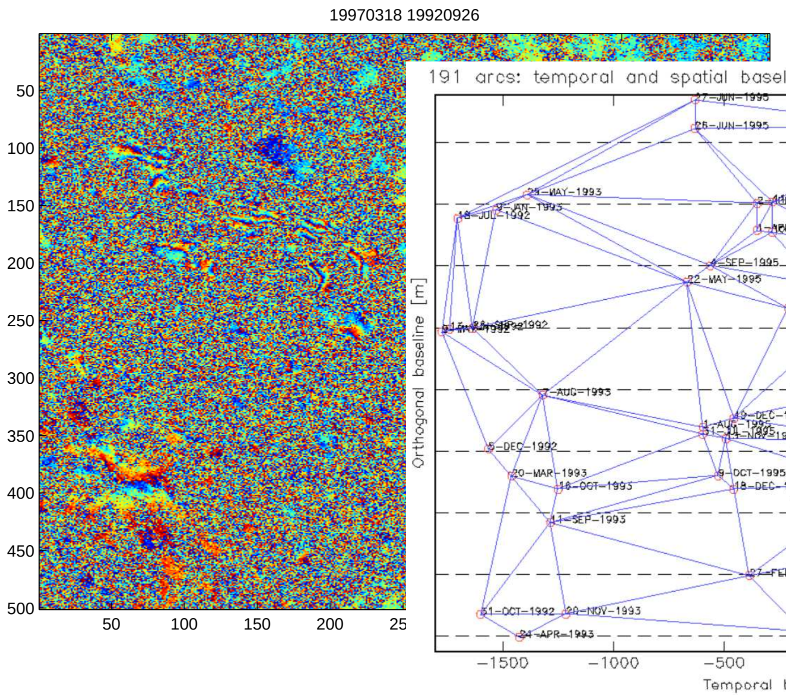
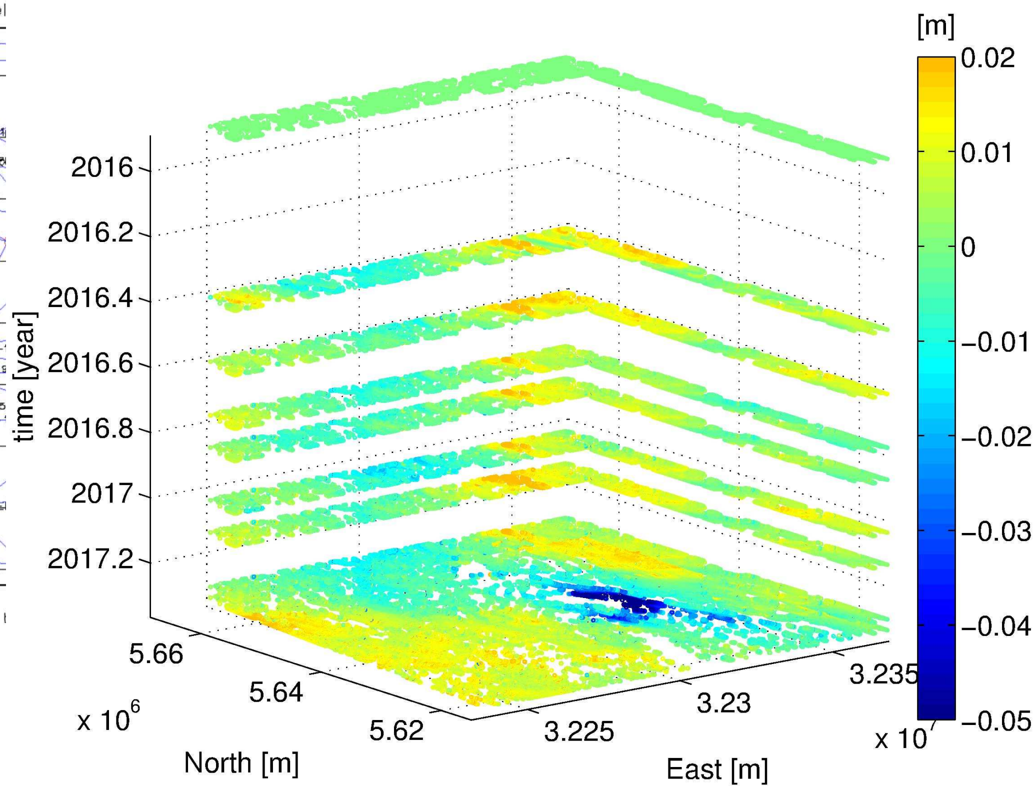
Evaluation process: SBAS (**S**mall **BA**seline **S**ubset technique) Berardino et al. (2002)





Output: image stack with surface displacements and atmospheric delay

Image stack

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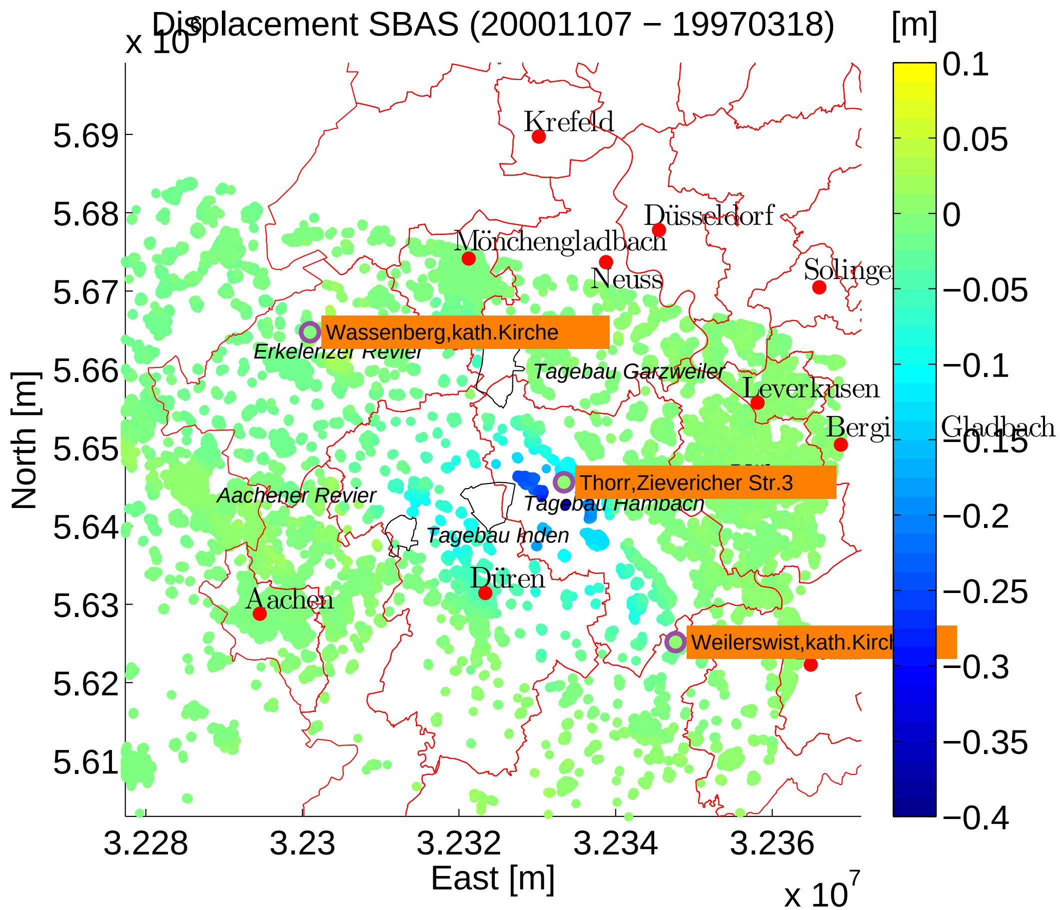
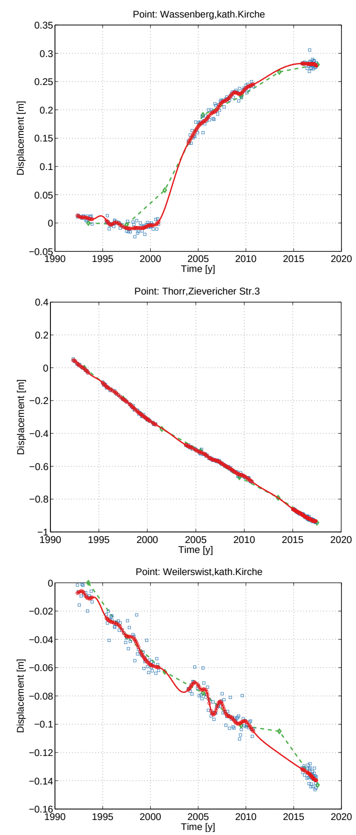
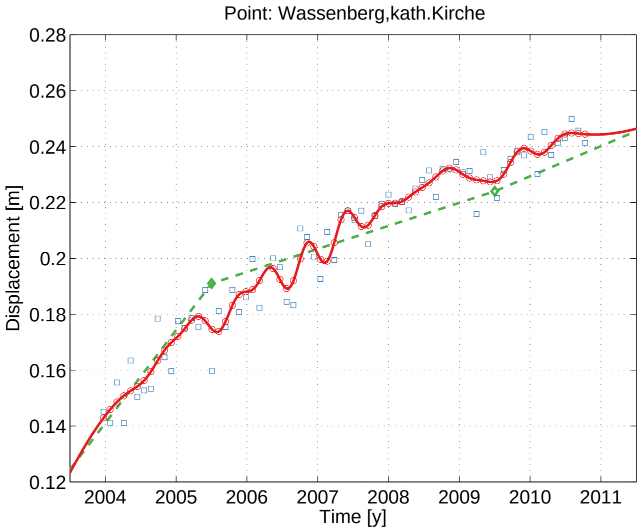


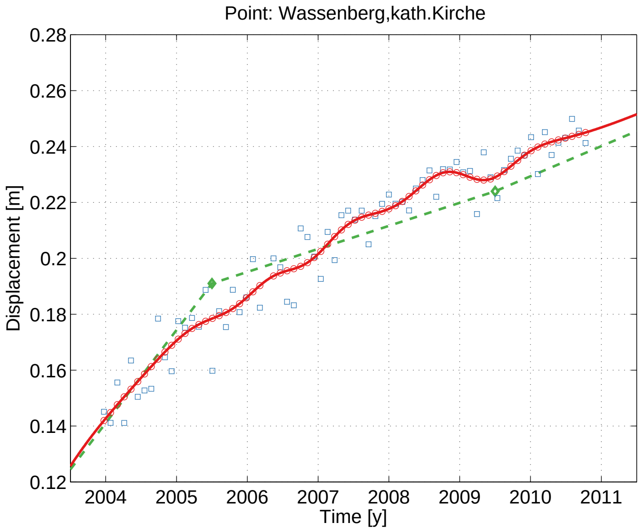


Task: separation of surface displacements (low frequent) from atmospheric disturbances (high frequent)

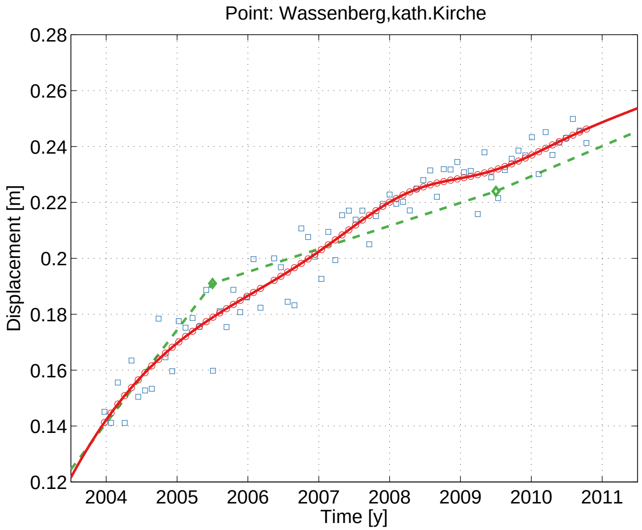
Goal: frequency selective filter for non-equispaced data



0.5 years



1 year



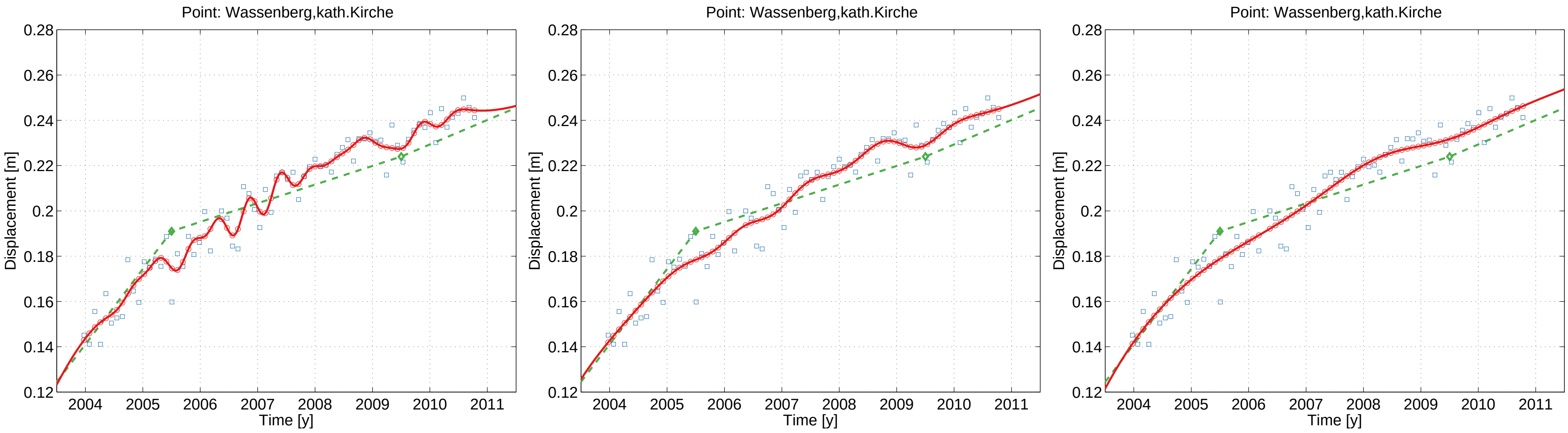
2 years

Here: low pass filter with half a year, one and two years cutoff frequency

◊ leveling □ DInSAR measurements ○ filtered surface displacements

Time series — Filtering

Goal: frequency selective filter for non-equispaced data



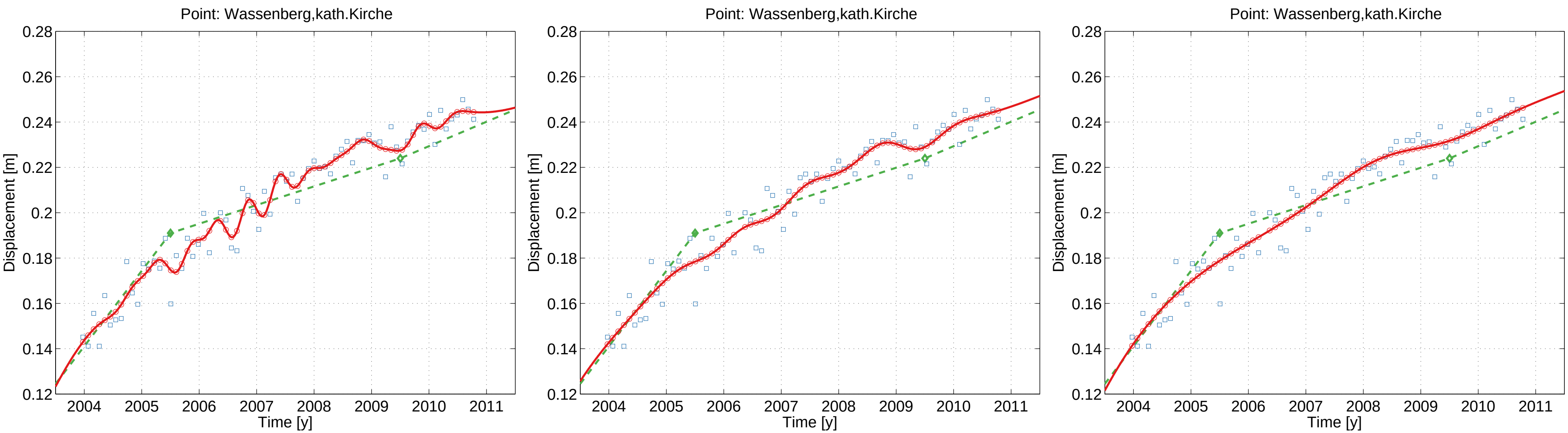
0.5 years　　1 year　　2 years

Here: low pass filter with half a year, one and two years cutoff frequency

Wanted: Filter with an exact defined frequency behaviour (transfer function)

Time series — Filtering

Goal: frequency selective filter for non-equispaced data



0.5 years 1 year 2 years

Here: low pass filter with half a year, one and two years cutoff frequency

Wanted: Filter with an exact defined frequency behaviour (transfer function)

Proposed filter strategy: **spline filter**

solution strategy — roadmap

given: $(\ell_i, x_i)\ldots$ non-equispaced samples of $f(x), i = 1, \ldots, n$

wanted: low pass filter ($\nu_{cut}\ldots$ cutoff frequency)

solution strategy — roadmap

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wanted: low pass filter ($\nu_{cut} \ldots$ cutoff frequency)

ansatz:
- equispaced B-splines $B^g(.)$ of order g ($\Delta\kappa$... interval)

$$f(x) = \sum_k \boldsymbol{a}_k \; B^g(\breve{x} - k) \qquad \text{with} \qquad \breve{x} = \frac{x - \kappa_0}{\Delta\kappa}$$

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- transition from B-splines $B^g(.)$ to Sampling splines $S^g(.)$

  reparametrization of the unapproachable coefficient $\boldsymbol{a}_k$ by equispaced function values $\boldsymbol{f}_k$

$$f(x) = \sum_k \boldsymbol{f}_k \, S^g(\breve{x} - k)$$

solution strategy — roadmap

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$$f(x) = \sum_k \boldsymbol{f}_k \; S^g(\breve{x} - k)$$

- analysis of the spectral behaviour $\widehat{f}(\nu)$ of $f(x)$

$$\widehat{f}(\nu) = \mathcal{F}\left\{\sum_k f_k \; S^g(\breve{x} - k)\right\} = \widehat{f}_k(\nu) \cdot \widehat{S}^g(\nu)$$

$\widehat{S}^g(\nu)$ . . . transfer function of the B-spline filter

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$\widehat{S}^g(\nu) \ldots$ transfer function of the B-spline filter

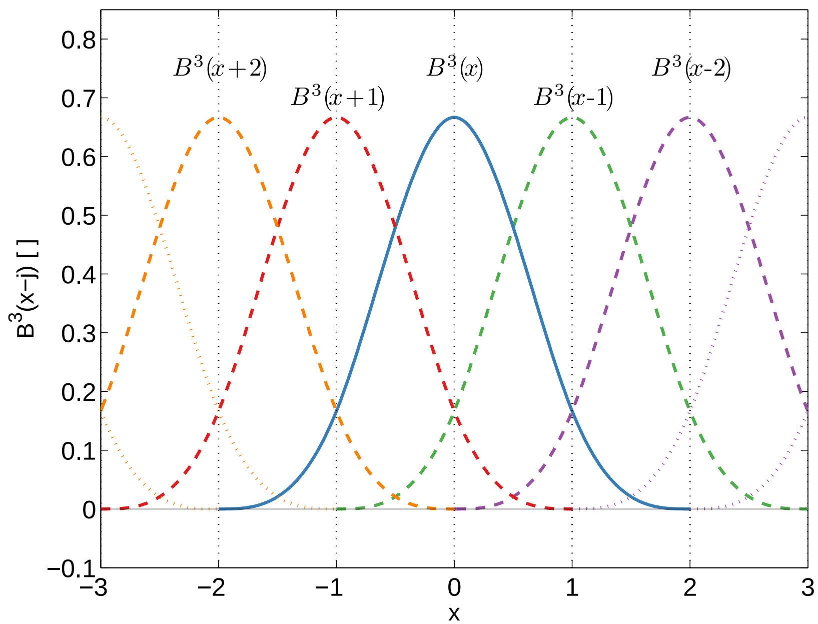
application: frequency selective spline filter with equispaced B-splines

B-spline equispaced

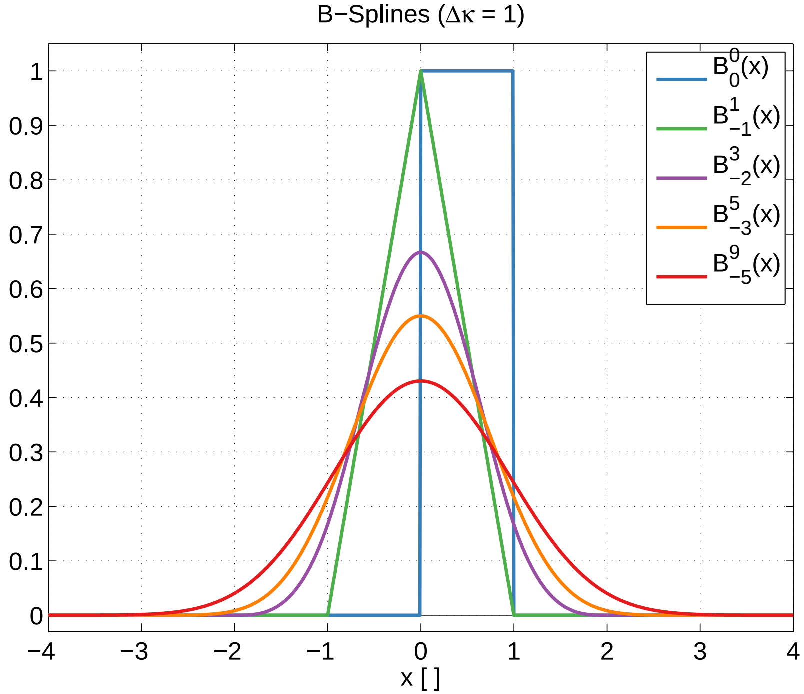
Base function: centered B-Spline in equispaced representation

$$B^g(x-j)$$

is a piecewise polynomial of order g with finite support $\left[-\frac{g+1}{2}, \frac{g+1}{2}\right]$



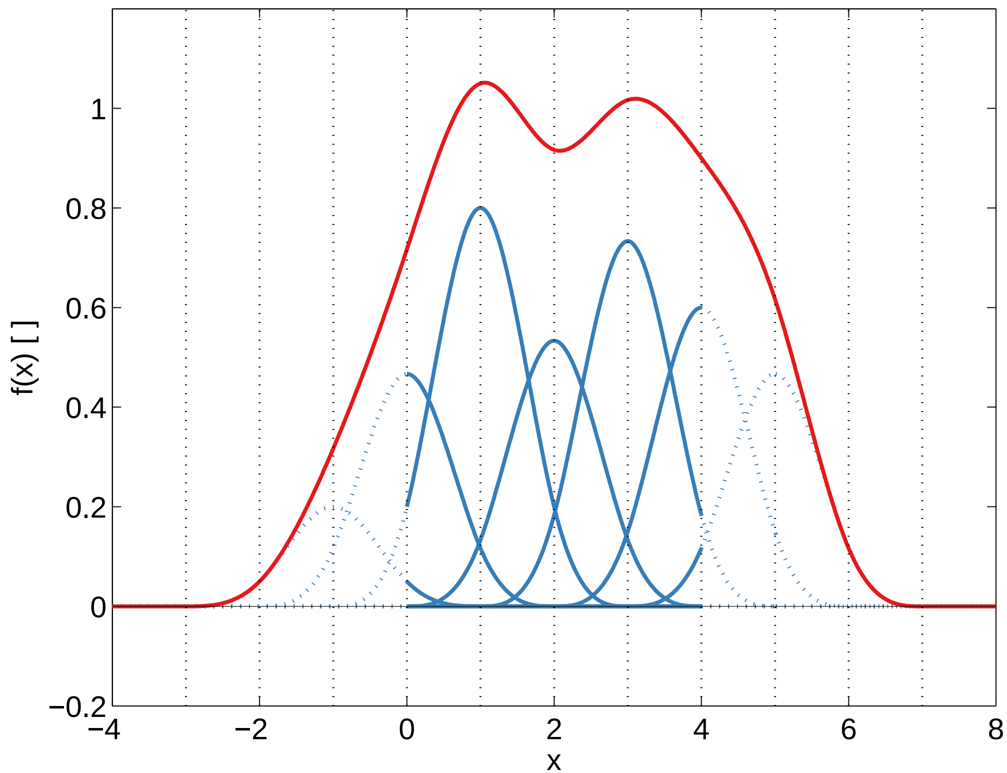
$B^3(x-j)$



$B^g(x-0)$

$f(x)$ is a linear combination of scaled and shifted B-splines $B^g(\breve{x}-k)$ multiplied by the coefficient a_k

$f(x)$ is a piecewise polynomial of order g continuous and $g-1$ times continuously differentiable, $f(x) \in \mathcal{C}^{g-1}$



$$f(x) = \sum_k a_k \; B^g(\breve{x}-k)$$

Spline approximation - B-splines

$f(x)$ is a linear combination of scaled and shifted B-splines $B^g(\breve{x} - k)$ multiplied by the coefficient a_k

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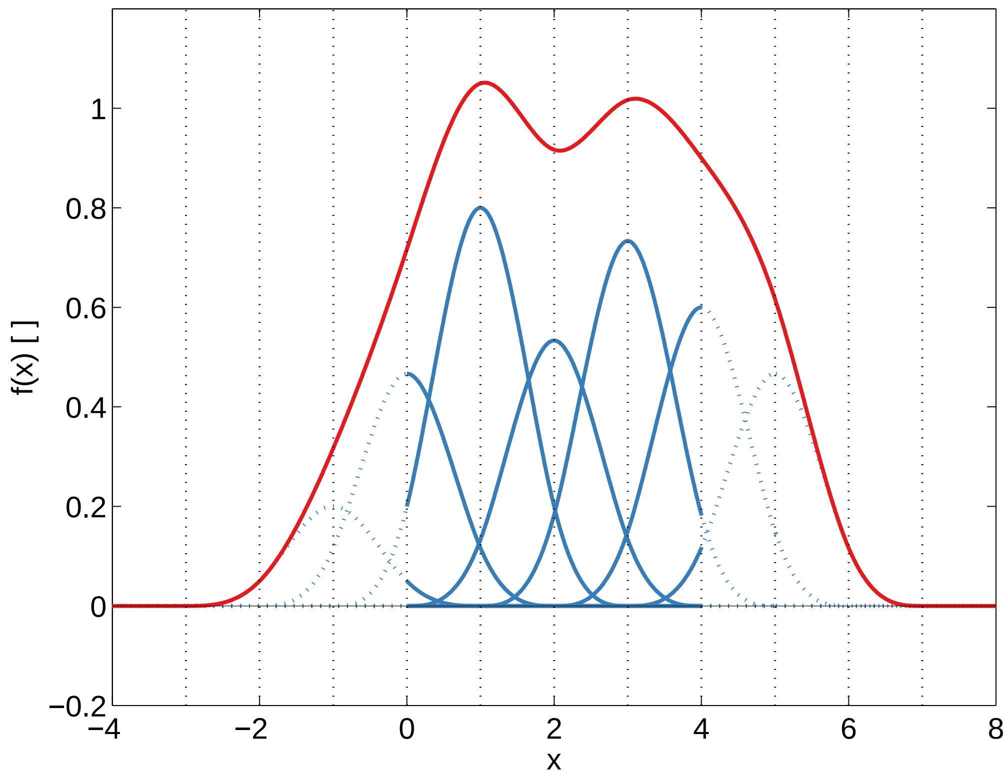


$$f(x) = \sum_k a_k \; B^g(\breve{x} - k)$$

approximation/filtering:

$$\ell_i + v_i = f(x_i)$$

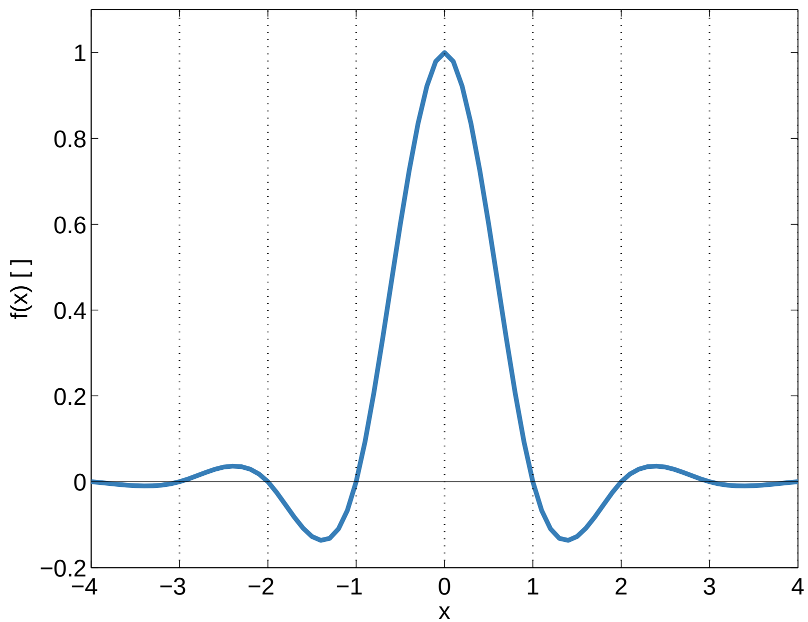
$$\arg\min_{a_k \in \mathbb{R}} \left(\sum_i v_i^2 \right)$$

Sampling Spline

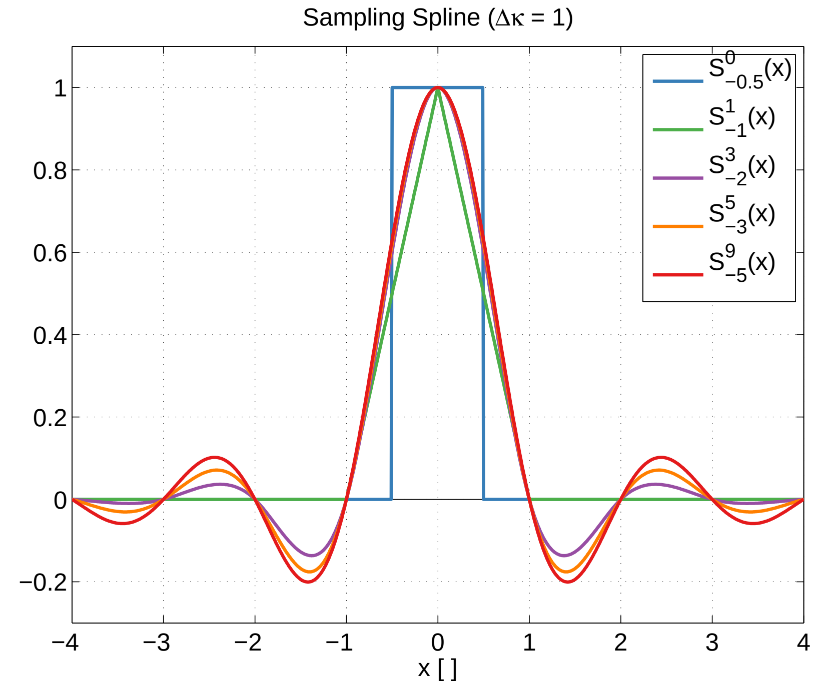
Base function: centered Sampling spline in equispaced representation

$$S^g(x)$$

is a piecewise polynomial of order g with infinte support.



$S^3(x)$



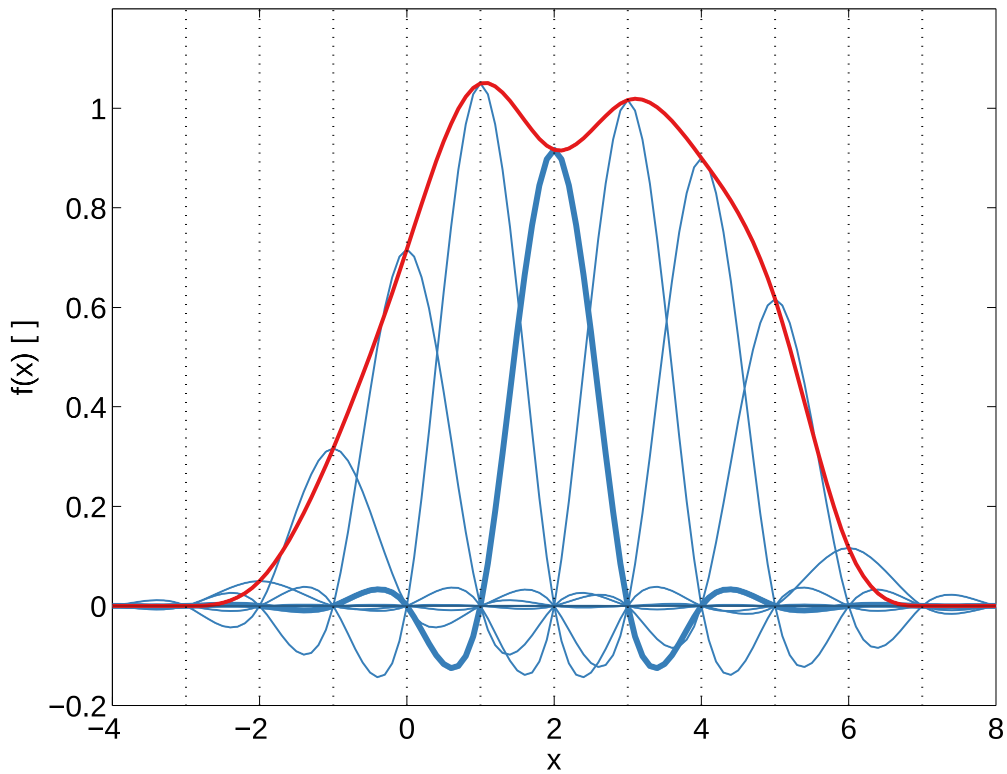
$S^g(x)$

The Sampling spline is one in the origin and zero at all knots, $S^g(x) = \delta_{x,0};\ x \in \mathbb{Z}$

Sampling-splines - Approximation

$f(x)$ is a linear combination of shifted Sampling splines $S^g(\breve{x}-k)$ multiplied by the coefficient f_k

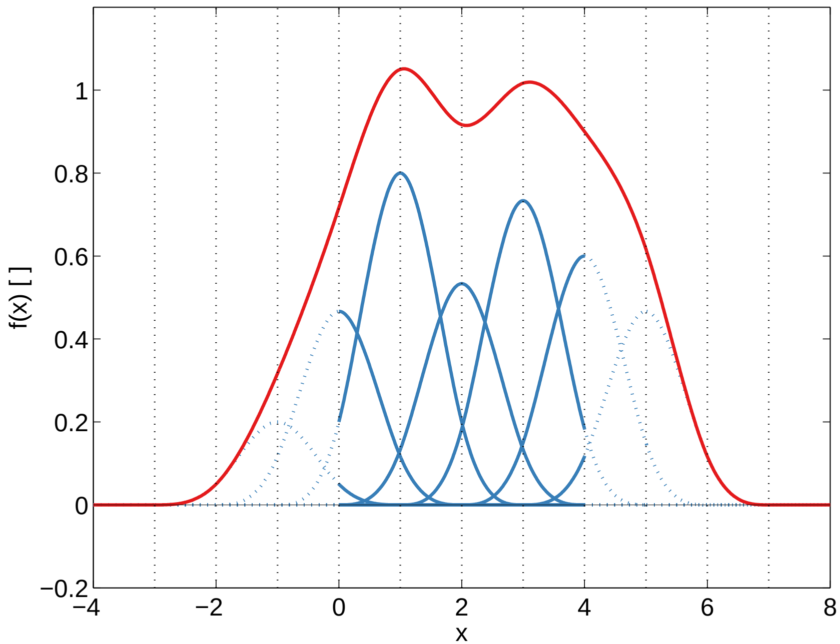
$f(x)$ is a piecewise polynomial of order g continuous and $g-1$ times continuously differentiable, $f(x) \in \mathcal{C}^{g-1}$



$$f(x) = \sum_k f_k \; S^g(\breve{x}-k)$$

Comparison - B-Spline vs. Sampling Spline

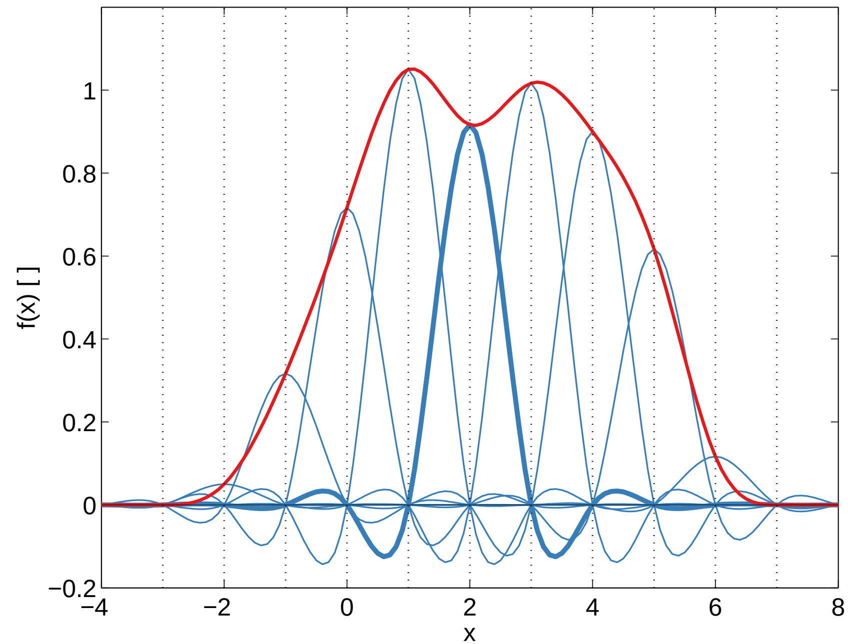
B-Spline



$$f(x) = \sum_k a_k \, B^g(\breve{x} - k)$$

$a_k \ldots$ unapproachable

Sampling Spline



$$f(x) = \sum_k f_k \, S^g(\breve{x} - k)$$

$f_k \ldots$ equispaced functions values

Construction of a Sampling Spline

$S^g(x)$ is a piecewise polynomial of order g continuous and $g-1$ times continuously differentiable, $f(x) \in \mathcal{C}^{g-1}$

$S^g(x)$ can be represented by a linear combination of B-Splines

$$S^g(\breve{x}) = \sum_{k=-\infty}^{\infty} d_k^g \; B^g(\breve{x}-k)$$

complying the restrictions

$$S^g(\breve{x}) \stackrel{!}{=} \delta_{\breve{x},0} \qquad \text{for} \qquad \breve{x} \in \mathbb{Z} \; .$$

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restrictions form an infinite Toeplitz system: $\{B^g(k)\}_{\Delta\kappa} * \{d_k^g\}_{\Delta\kappa} = \{e_k\}_{\Delta\kappa}$

$$\begin{bmatrix} \vdots \\ S^g(-2) \\ S^g(-1) \\ S^g(0) \\ S^g(1) \\ S^g(2) \\ \vdots \end{bmatrix} = \begin{bmatrix} & \ddots & \ddots & \ddots & \ddots & \ddots & & & & & \\ \cdots & B^g(2) & B^g(1) & B^g(0) & B^g(-1) & B^g(-2) & \cdots & & & & \\ & \cdots & B^g(2) & B^g(1) & B^g(0) & B^g(-1) & B^g(-2) & \cdots & & & \\ & & \cdots & B^g(2) & B^g(1) & B^g(0) & B^g(-1) & B^g(-2) & \cdots & & \\ & & & \cdots & B^g(2) & B^g(1) & B^g(0) & B^g(-1) & B^g(-2) & \cdots & \\ & & & & \cdots & B^g(2) & B^g(1) & B^g(0) & B^g(-1) & B^g(-2) & \cdots \\ & & & & & \ddots & \ddots & \ddots & \ddots & \ddots & \end{bmatrix} \begin{bmatrix} \vdots \\ d_{-2}^g \\ d_{-1}^g \\ d_0^g \\ d_1^g \\ d_2^g \\ \vdots \end{bmatrix} = \begin{bmatrix} \vdots \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ \vdots \end{bmatrix}$$

Solution of the Toeplitz system

solution of the Toeplitz system: $\{B^g(k)\}_{\Delta\kappa} * \{d_k^g\}_{\Delta\kappa} = \{e_k\}_{\Delta\kappa}$

$\{B^g(k)\}_{\Delta\kappa}$ → $\widehat{B}^g(\nu) = \sum_{k=-\infty}^{\infty} B^g(k)\; e^{-i2\pi\nu k\Delta\kappa}$ → $\widehat{B}^g(\nu)\Big|_{-\nu^N}^{\nu^N}$

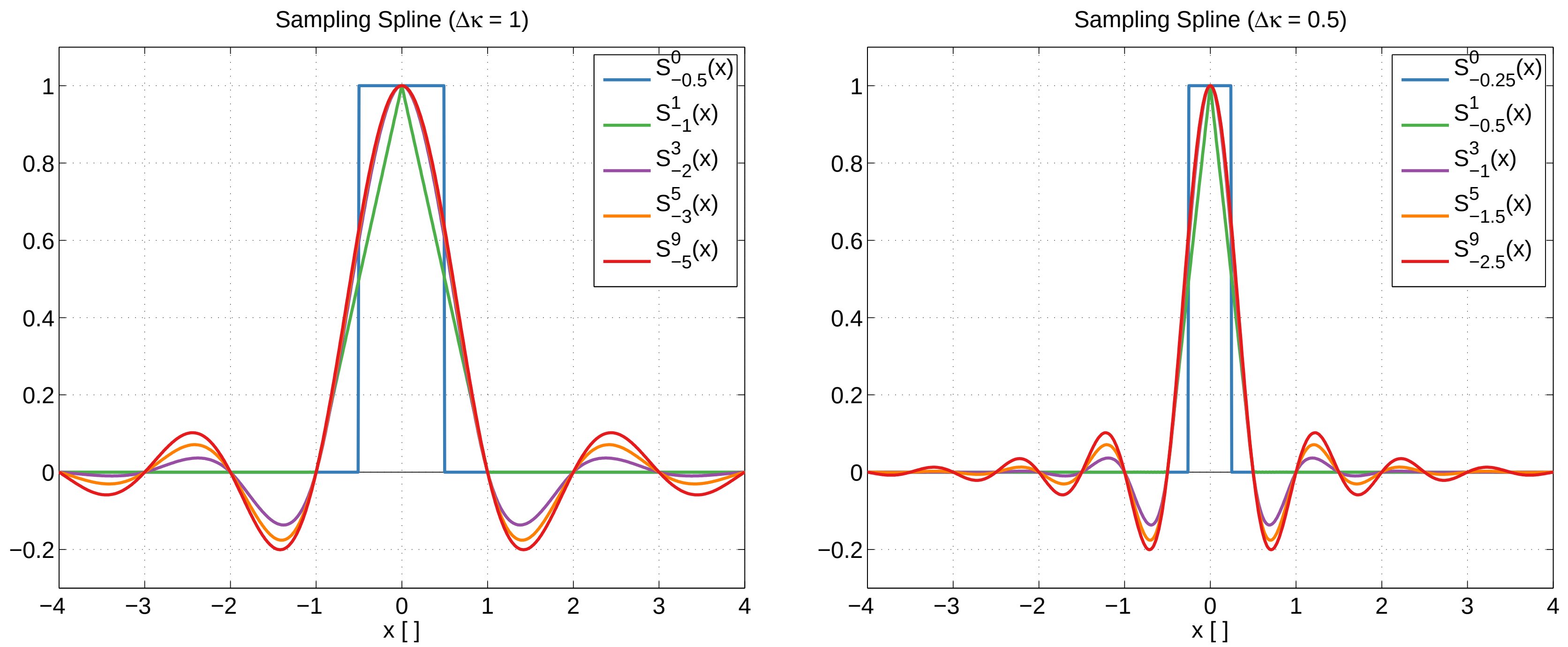
$\{e_k\}_{\Delta\kappa}$ → $\widehat{e}(\nu) = \sum_{k=-\infty}^{\infty} e_k\; e^{-i2\pi\nu k\Delta\kappa}$ → $\widehat{e}(\nu)\Big|_{-\nu^N}^{\nu^N}$

$\widehat{d}^g(\nu) = \widehat{e}(\nu)/\widehat{B}^g(\nu)$ → $\widehat{d}^g(\nu)\Big|_{-\nu^N}^{\nu^N}$

$\{d_k^g\}_{\Delta\kappa}$ ← $d_k^g = \frac{1}{2\nu^N} \int_{-\nu^N}^{\nu^N} \widehat{d}^g(\nu)\; e^{i2\pi\nu k\Delta\kappa} d\nu$

time domain (discrete) | Fourier transformation of a non-periodic sequence | spectral domain (periodic $\nu^N = \frac{1}{2\Delta\kappa}$)

given: Sampling spline $$S^g(\breve{x}) = \sum_{k=-\infty}^{\infty} \boldsymbol{d_k^g}\, B^g(\breve{x} - k)$$



$S^g(x)$ $S^g(\breve{x})$

Sampling Spline — Spectral Representation

given: Sampling spline $$S^g(\breve{x}) = \sum_{k=-\infty}^{\infty} \boldsymbol{d_k^g} \, B^g(\breve{x} - k)$$

wanted: spectral representation $\widehat{S}^g(\nu)$ of the Sampling spline $S^g(\breve{x})$

$$\widehat{S}^g(\nu) = ?$$

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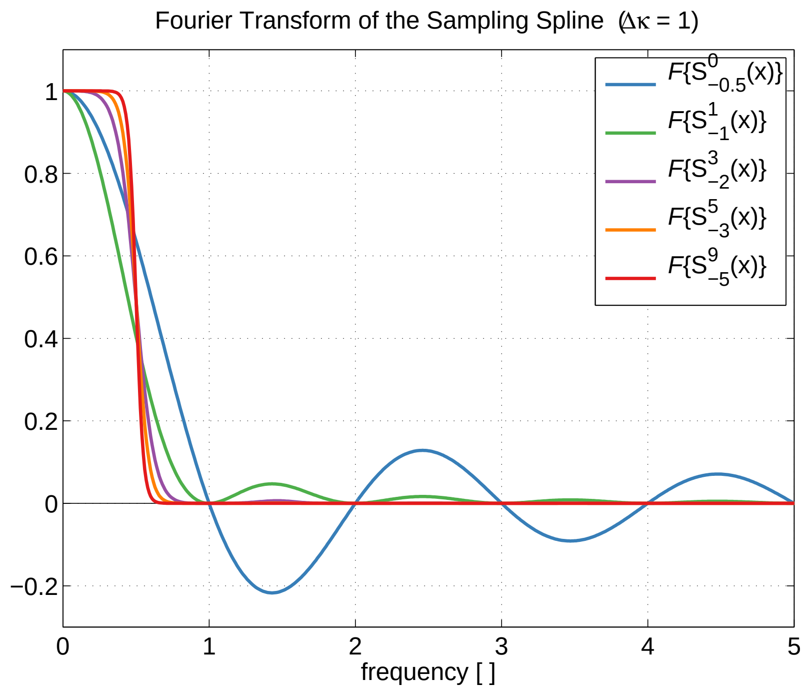
wanted: spectral representation $\widehat{S}^g(\nu)$ of the Sampling spline $S^g(\breve{x})$

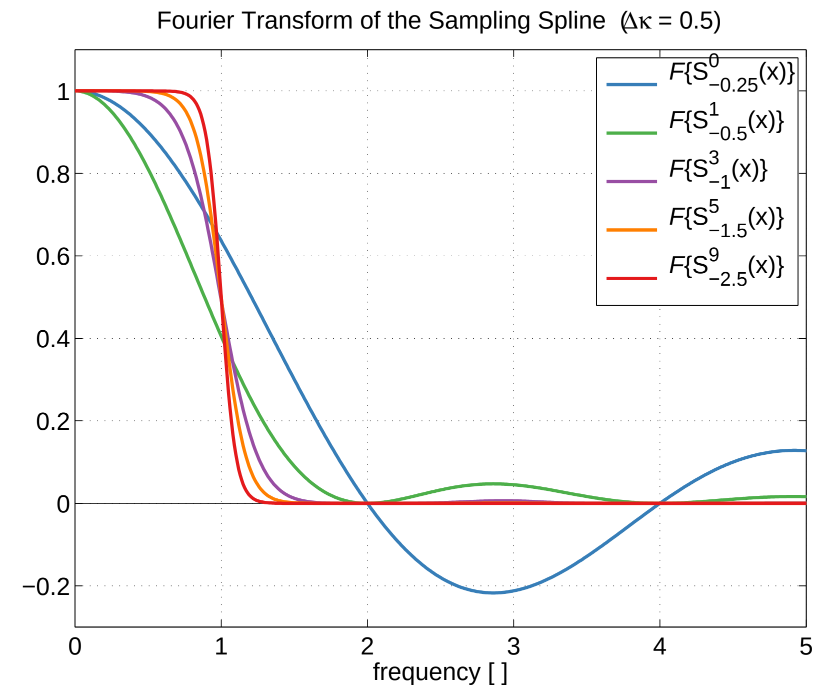
$$\begin{aligned}
\widehat{S}^g(\nu) &= \mathcal{F}\left\{ \sum_{k=-\infty}^{\infty} d_k^g\, B^g(\breve{x} - k) \right\} \\
&= \sum_{k=-\infty}^{\infty} d_k^g\, \underbrace{\mathcal{F}\{B^g(\breve{x} - k)\}} \\
&\qquad\qquad \overbrace{\qquad\qquad\qquad}^{} \quad \text{shifting theorem} \\
&= \underbrace{\sum_{k=-\infty}^{\infty} d_k^g\, e^{-i2\pi\nu k}}\; \underbrace{\widehat{B}^g(\nu)} \\
\widehat{S}^g(\nu) &= \boldsymbol{\widehat{d}^g(\nu)}\Big|_{-\nu^N}^{\nu^N} \left(\Delta\kappa \frac{\sin \pi\nu\Delta\kappa}{\pi\nu\Delta\kappa} \right)^{g+1}
\end{aligned}$$

Function — Spectral Representation

given: Spectral representation a single Sampling spline $\widehat{S}^g(\nu)$



$S^g(x)$



$S^g(\breve{x})$

$$\widehat{S}^g(\nu) = \widehat{d}^g(\nu)\Big|_{-\nu^N}^{\nu^N} \left(\Delta\kappa \frac{\sin \pi\nu\Delta\kappa}{\pi\nu\Delta\kappa}\right)^{g+1}$$

low pass characteristic with cutoff frequency: $\nu_{cut} = \frac{1}{2\Delta\kappa}$

Function — Spectral Representation

given: Spectral representation a single Sampling spline $\widehat{S}^g(\nu)$

wanted: spectral representation $\widehat{f}(\nu)$ of the function $f(x) = \sum_k f_k S^g(\breve{x} - k)$

$$\widehat{f}(\nu) = ?$$

given: Spectral representation a single Sampling spline $\widehat{S}^g(\nu)$

wanted: spectral representation $\widehat{f}(\nu)$ of the function $f(x) = \sum_k f_k S^g(\breve{x} - k)$

$$\begin{aligned}
\widehat{f}(\nu) &= \mathcal{F}\left\{\sum_k f_k S^g(\breve{x} - k)\right\} \\
&= \sum_k f_k \underbrace{\mathcal{F}\{S^g(\breve{x} - k)\}} \\
&= \underbrace{\sum_k f_k \, e^{-i2\pi\nu k}} \overbrace{\underbrace{\widehat{S}^g(\nu)}} \\
\widehat{f}(\nu) &= \widehat{f}(\nu)\Big|_{-\nu^N}^{\nu^N} \widehat{S}^g(\nu)
\end{aligned}$$

shifting theorem

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\widehat{f}(\nu) &= \underbrace{\widehat{f}(\nu)\Big|_{-\nu^N}^{\nu^N}}_{\mathcal{F}\left\{\{f_k\}_{\Delta\kappa}\right\}} \; \widehat{S}^g(\nu)
\end{aligned}
$$

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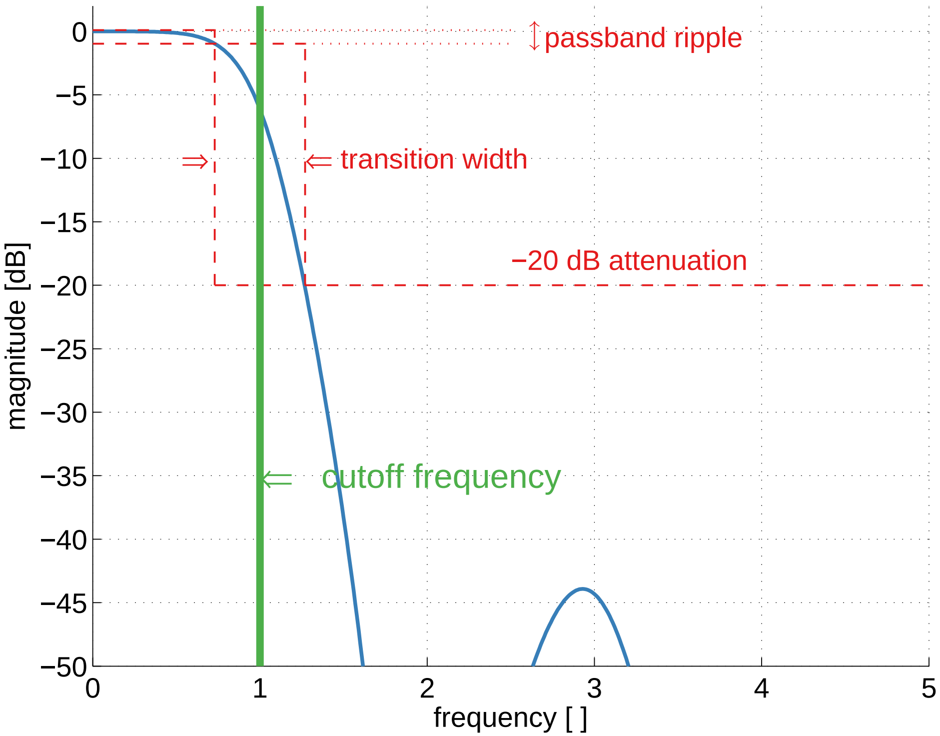
$$\widehat{f}(\nu) = \mathcal{F}\left\{\sum_k f_k S^g(\breve{x}-k)\right\}$$

$$= \sum_k f_k \underbrace{\mathcal{F}\{S^g(\breve{x}-k)\}}$$

$$= \underbrace{\sum_k f_k\, e^{-i2\pi\nu k}}\ \overbrace{\underbrace{\widehat{S}^g(\nu)}}$$ shifting theorem

$$\widehat{f}(\nu) = \underbrace{\widehat{f}(\nu)\Big|_{-\nu^N}^{\nu^N}}_{\mathcal{F}\{\{f_k\}_{\Delta\kappa}\}}\ \underbrace{\widehat{S}^g(\nu)}_{\text{transfer function}}$$

Spline Filter — Transfer Function

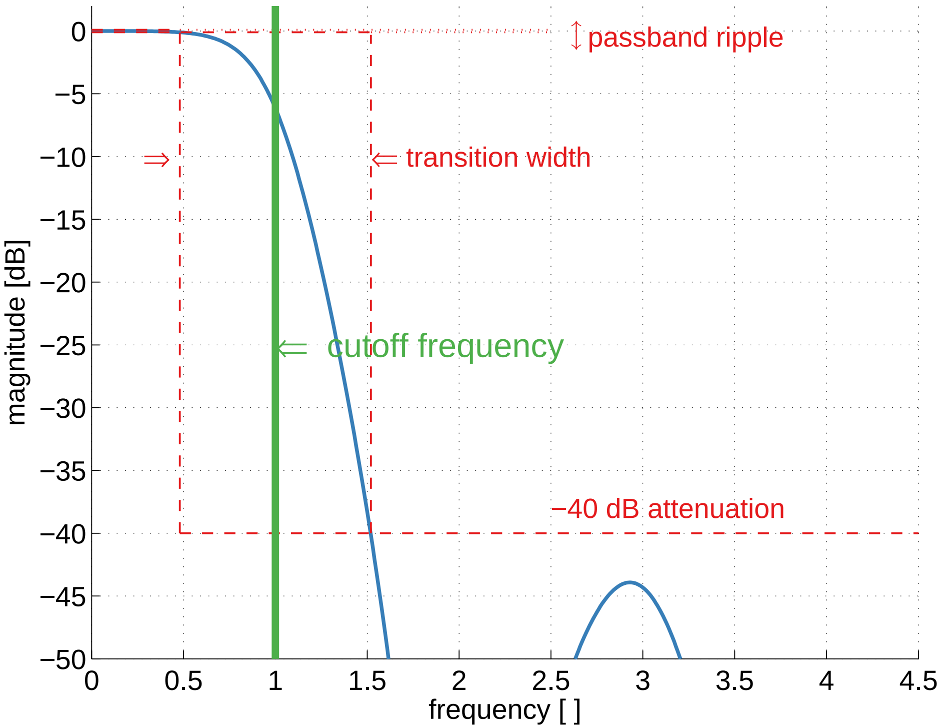


filter characteristics

B-spline	passband ripple [dB]	passband ripple [%]	transition width [Hz]
3	-0.97	-10.56	0.54
5	-0.83	-9.12	0.38
7	-0.87	-9.50	0.28
9	-0.91	-9.90	0.22
11	-0.94	-10.29	0.20
15	-0.88	-9.59	0.14
19	-1.10	-11.92	0.12

-20db (10%) stopband attenuation

spline filter: low pass filter with cutoff frequency $\nu_{cut} = \frac{1}{2\Delta\kappa}$



filter characteristics

B-spline	passband ripple [dB]	passband ripple [%]	transition width [Hz]
3	-0.103	-1.18	1.04
5	-0.084	-0.96	0.74
7	-0.087	-1.00	0.56
9	-0.080	-0.92	0.46
11	-0.086	-0.98	0.38
15	-0.095	-1.09	0.30
19	-0.068	-0.77	0.24

-40db (1%) stopband attenuation

spline filter: low pass filter with cutoff frequency $\nu_{cut} = \frac{1}{2\Delta\kappa}$

Spline Filter — Example

$$f(x) = 5.31 + t + 0.7\sin(2\pi\ 6\ t + \phi_1) + 1.0\sin(2\pi\ 8\ t + \phi_1)$$

given: f_k sampled with $100[Hz]$

wanted: filter with cutoff frequency $\nu_{cut} = 7[Hz] \quad \Longrightarrow \quad \Delta\kappa = \frac{1}{2\nu_{cut}}$

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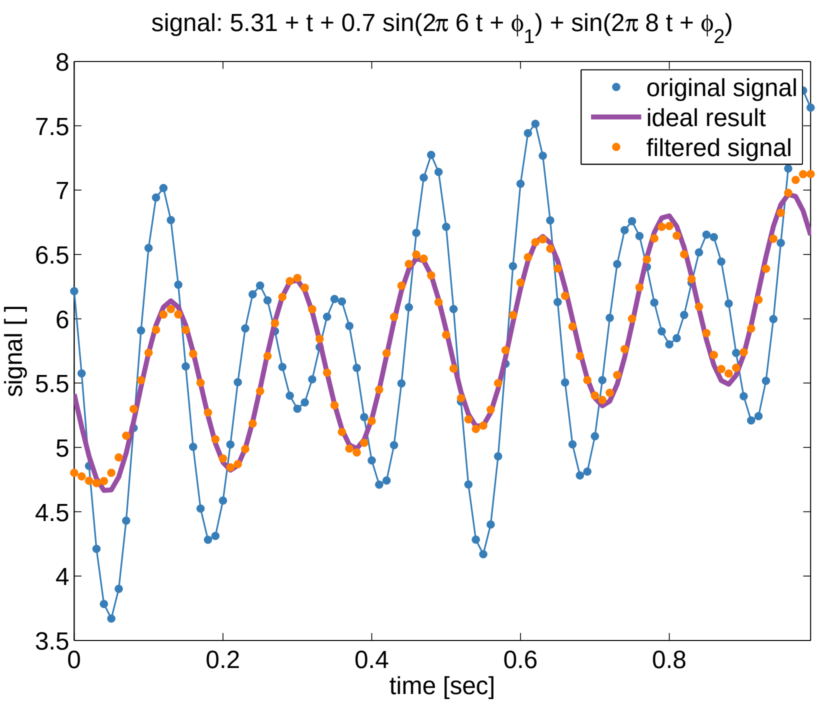
wanted: filter with cutoff frequency $\nu_{cut} = 7[Hz]$ $\Longrightarrow$ $\Delta\kappa = \frac{1}{2\nu_{cut}}$



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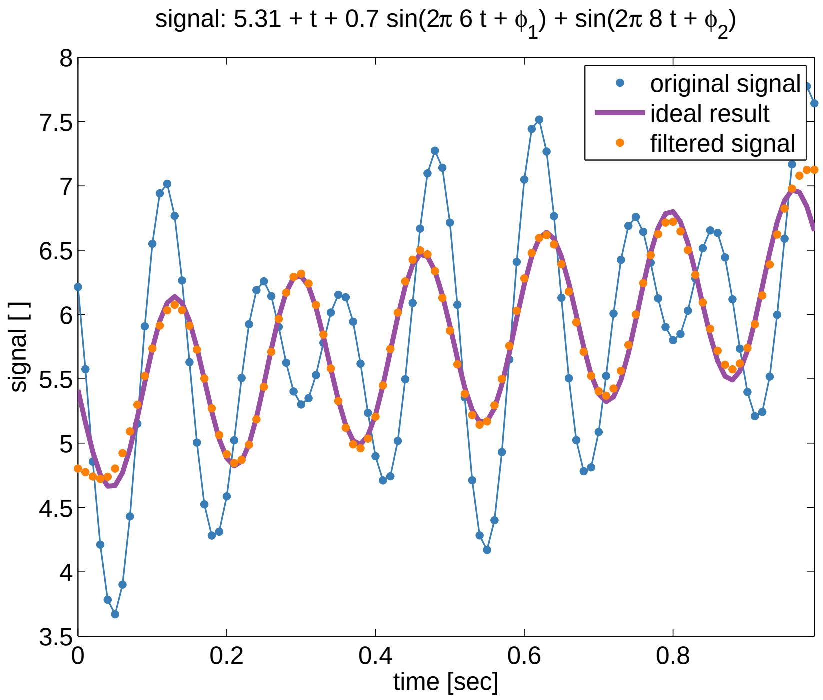
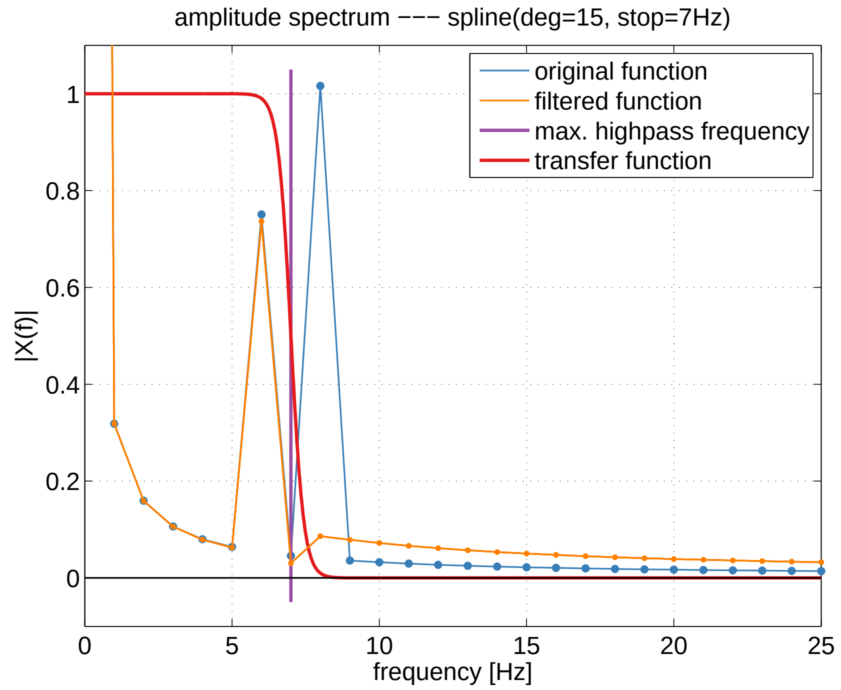
$$f(x) = 5.31 + t + 0.7\sin(2\pi\ 6\ t + \phi_1) + 1.0\sin(2\pi\ 8\ t + \phi_1)$$

given: f_k sampled with $100[Hz]$

wanted: filter with cutoff frequency $\nu_{cut} = 7[Hz] \implies \Delta\kappa = \frac{1}{2\nu_{cut}}$





Resumé - Spline Filter

Construction of frequency selective filters

- for non-equispaced data
- by B-spline approximation $B^g(\breve{x})$ of order g with knot interval $\Delta\kappa$

$$f(x) = \sum_k a_k \, B^g(\breve{x} - k) \qquad \text{with} \qquad \breve{x} = \frac{x - \kappa_0}{\Delta\kappa}$$

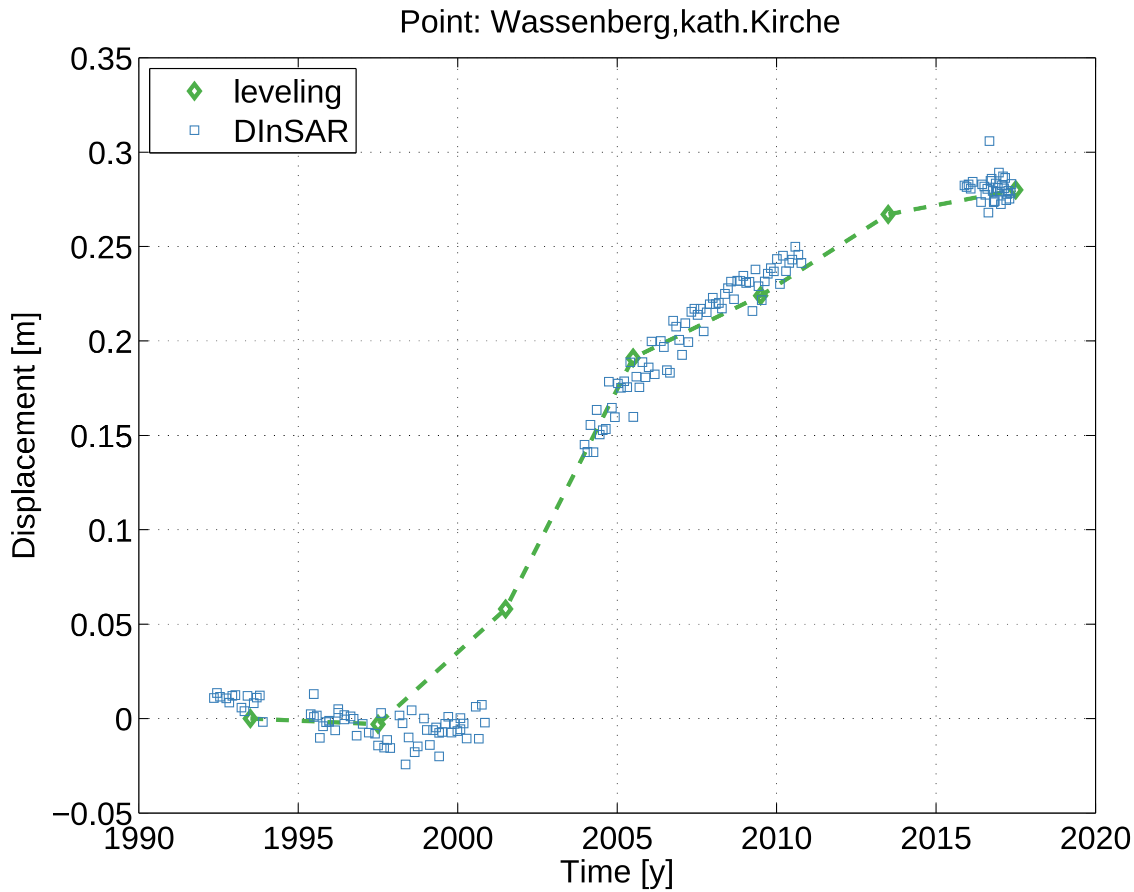
- relation between cutoff frequency ν_{cut} and knot interval $\Delta\kappa$

$$\nu_{cut} = \frac{1}{2\Delta\kappa} \qquad \text{resp.} \qquad \Delta\kappa = \frac{1}{2\nu_{cut}}$$

- transition width depends on the order g of the B-splines $B^g(\breve{x})$

larger order $g \Longrightarrow$ smaller transition width

Image stack with surface displacement and atmospheric delay

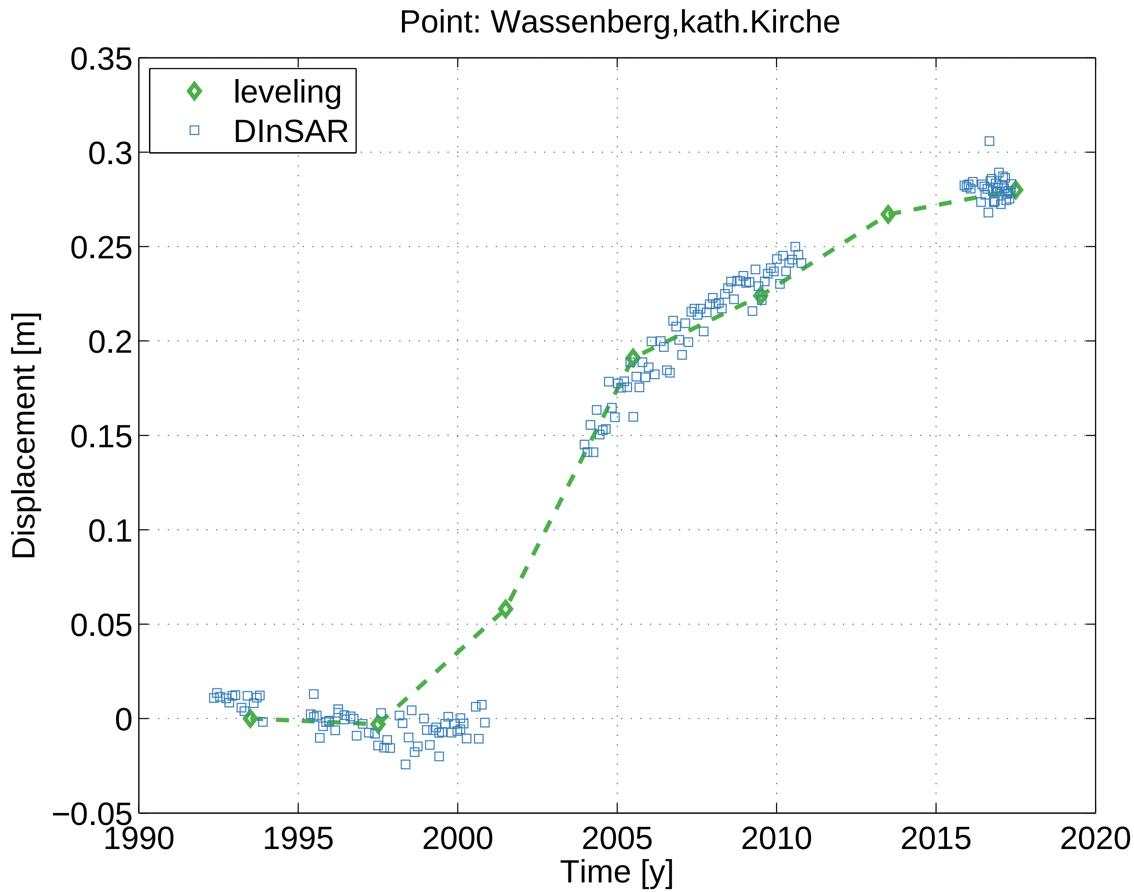


data characteristics:

- non-equispaced data
- data gaps between different missions
- data gaps during a mission

Spline Filter applied to image stack

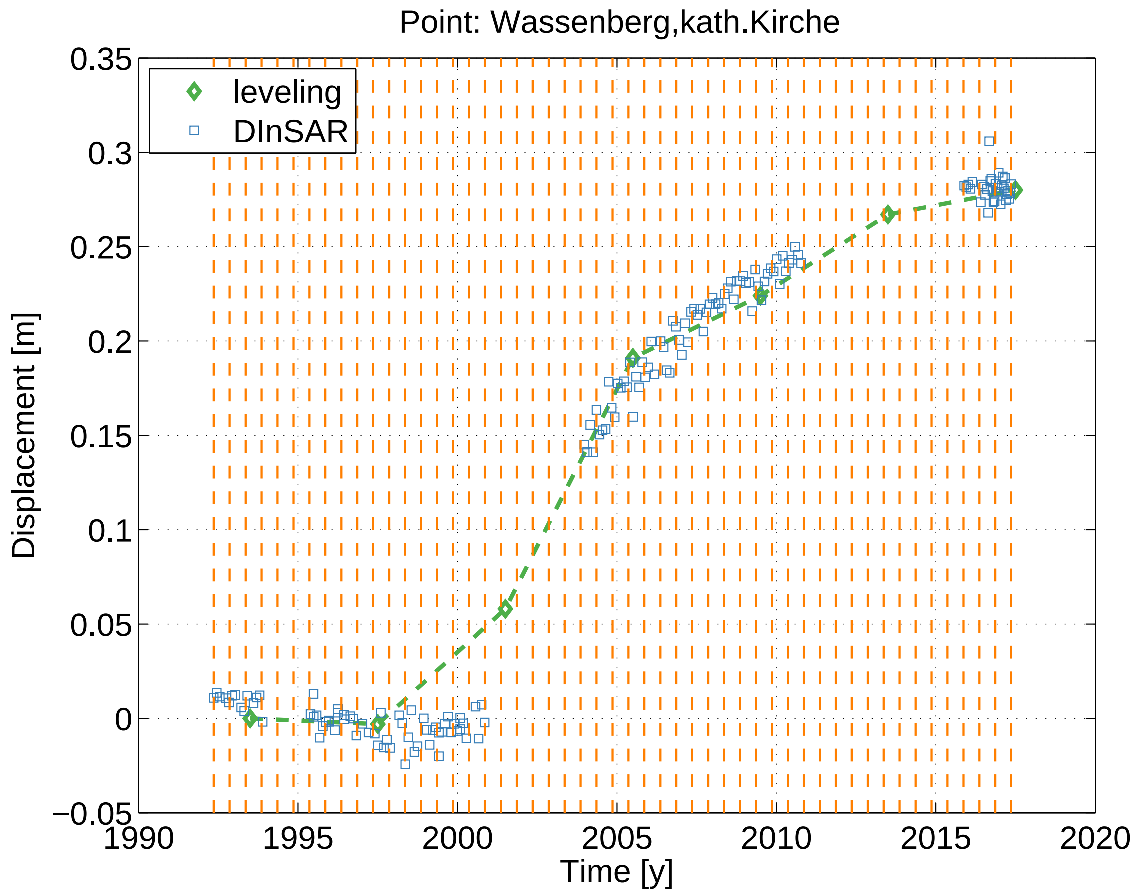
Image stack with surface displacement and atmospheric delay



frequency selective spline filter $\Longrightarrow$ equispaced knots $\Delta\kappa = \frac{1}{2\nu_{cut}}$

Spline Filter applied to image stack

Image stack with surface displacement and atmospheric delay

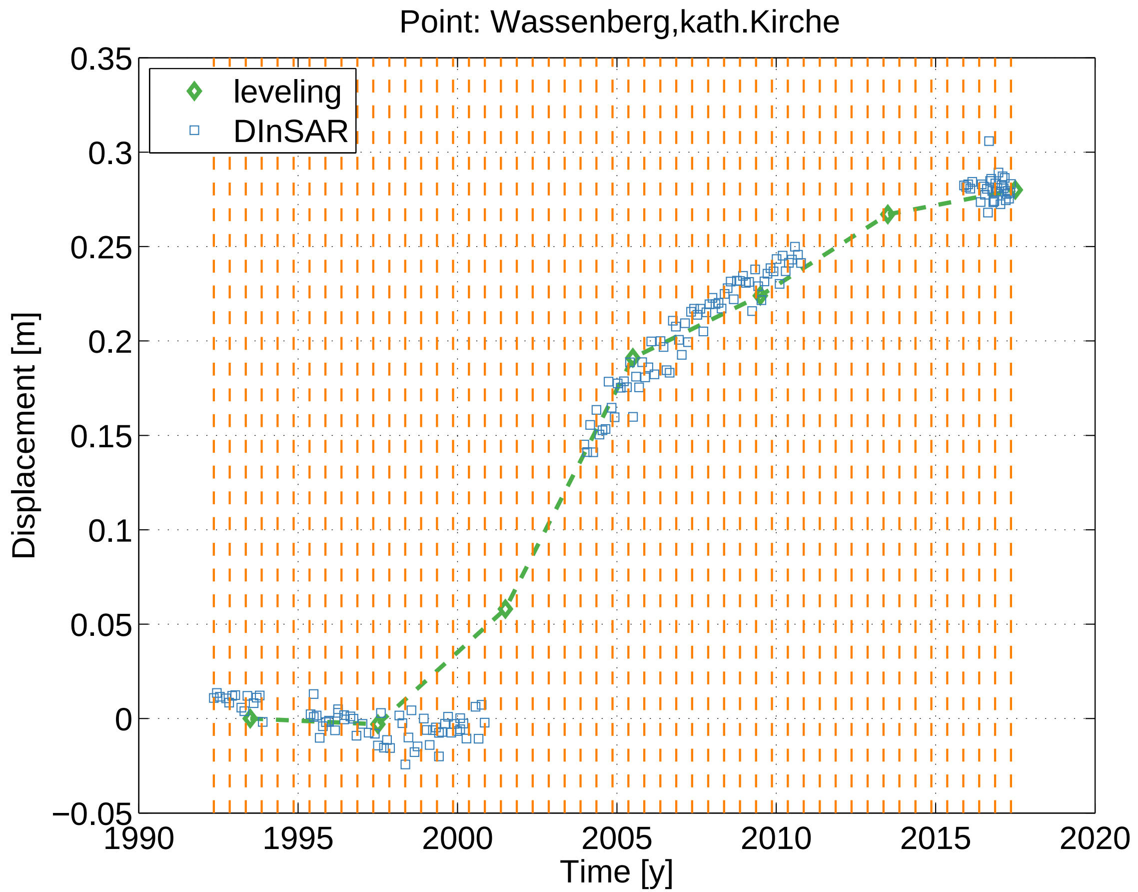


$\nu_{cut} = 1[y]$

frequency selective spline filter $\Longrightarrow$ equispaced knots $\Delta\kappa = \frac{1}{2\nu_{cut}}$

Spline Filter applied to image stack

Image stack with surface displacement and atmospheric delay



$$\nu_{cut} = 1[y]$$

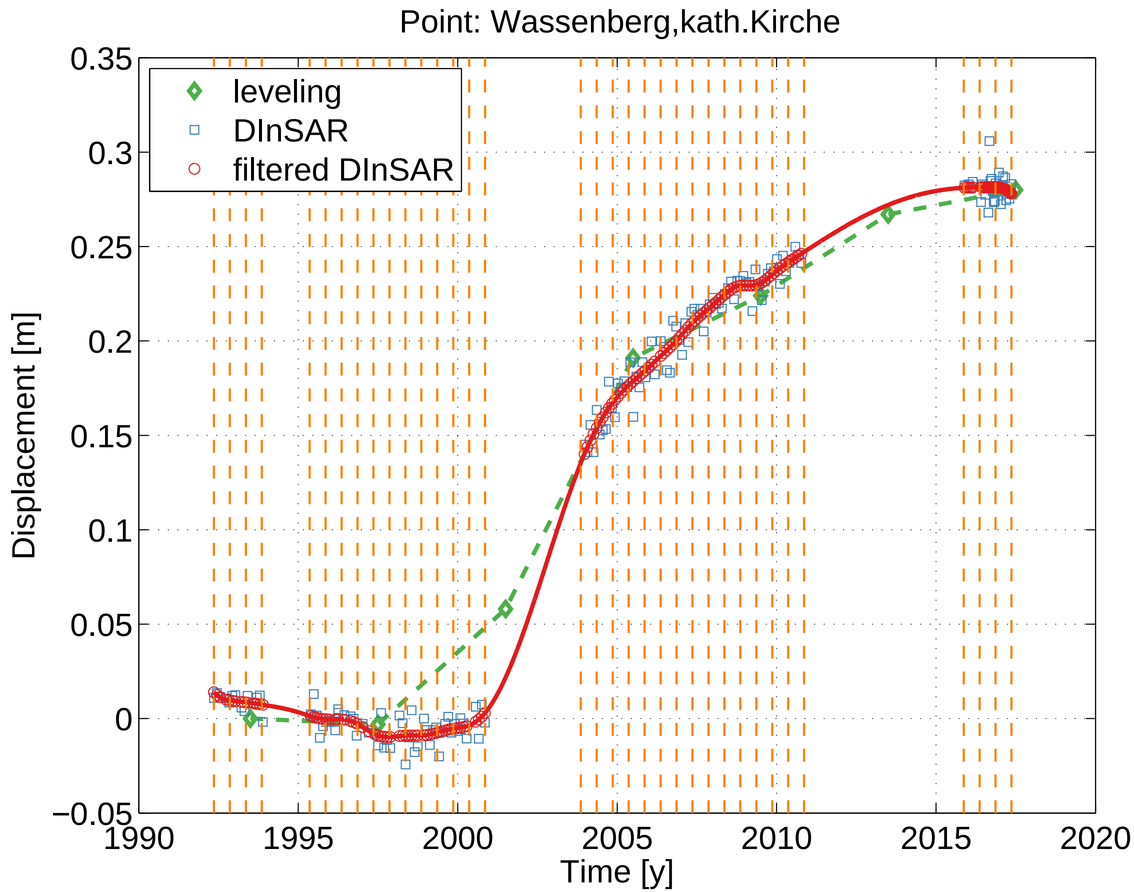
frequency selective spline filter $\Longrightarrow$ equispaced knots $\Delta\kappa = \frac{1}{2\nu_{cut}}$

data gaps lead to an underdetermined system

Possible correctives for spline filters with data gaps:

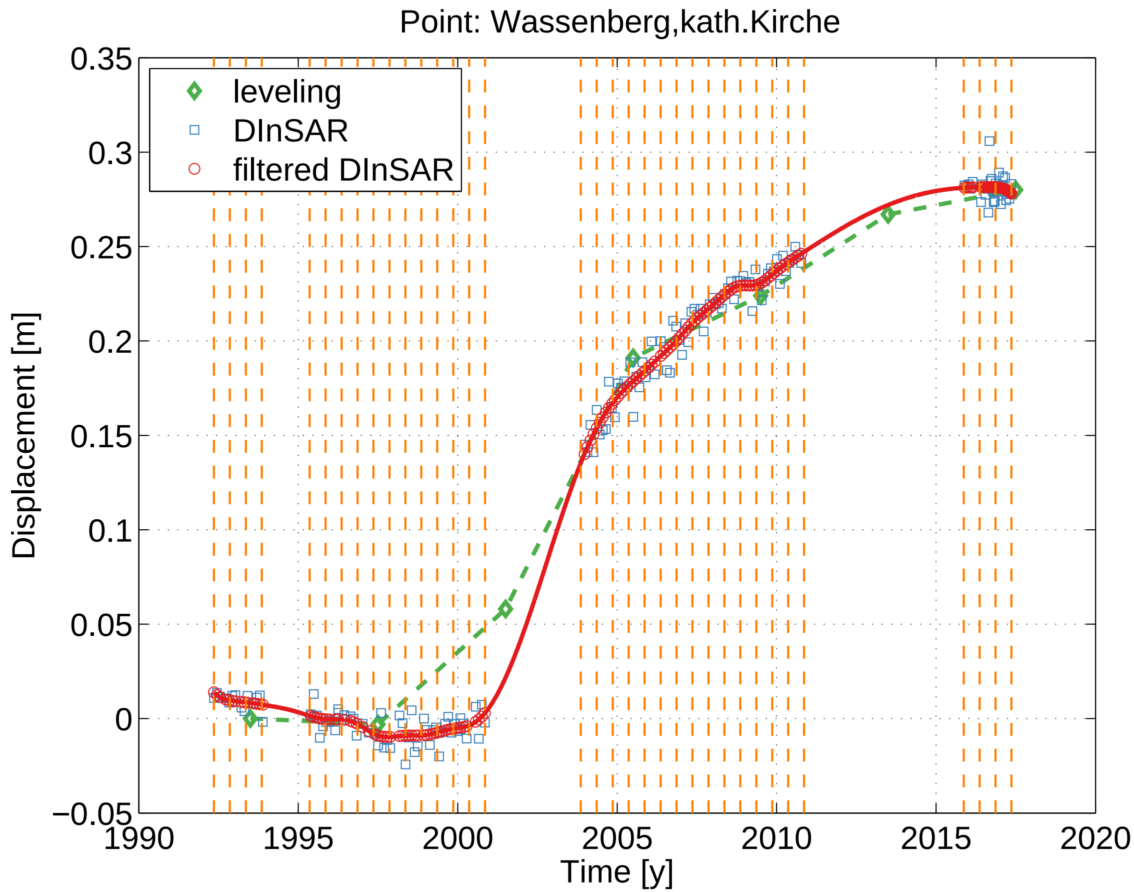
- regularization
- difference quotient approach with respect to a_k
- energy minimization approach $\int |f^{(j)}(x)|^2 dx$ to the j^{th} derivative per segment
- adaptive knot localization
- frozen derivatives of order g
- ...

Image stack with surface displacement and atmospheric delay



Result with adaptive knot localization in combination with energy minimization

Image stack with surface displacement and atmospheric delay



Thanks for your attention

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