RESTRICTION NORMS FOR SIEGEL MODULAR FORMS

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The fact that some seemingly very special functions, like the Riemann Zeta function, have such deep connections with the behavior of integers, of prime numbers, is hard to explain a priori and in depth. This is really not well understood to this day. Somehow these entities, these special analytic functions defined by infinite series, have been generalized more recently to spaces other than the plane of all complex numbers, such as to algebraic surfaces. These entities show connections between seemingly diverse notions. They also seem to show the existence (to make a metaphor stimulated by the subject itself) of another surface of reality, another Riemann surface of thought (and connections of thought) of which we are not consciously aware.

Stanisław Ulam, Adventures of a Mathematician [Ula]

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Chapter 1

Introduction

"*Can one hear the shape of a drum?*" is a famous question asked by Lipman Bers and reported in 1966 by Mark Kac in his eponymous article [Kac]. More precisely, is it possible for two different shapes of drums to make the same sound (at least for a mathematician)? This can be formalized as follows. The sound of a shape is given by the stationary waves on it. They are waves that do not move but only evolve in amplitude. These can be ordered by frequency. The first one is the fundamental tone, the pitch of the drum, and the higher ones are harmonic. They add complexity to the sound.

Mathematically, they are described by the *Laplacian*

$$\Delta = -\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)$$

on $\mathbb{R}^2$. This is a differential operator. The stationary waves are exactly the eigenfunctions of the Laplacian that vanish on the edge. That is, for a bounded shape $D \subseteq \mathbb{R}^2$, the functions f such that

$$\begin{cases} \Delta f & = \lambda f, \\ f|_{\partial D} & = 0. \end{cases}$$

Moreover, the eigenvalue λ is in that case the frequency of the wave, i.e. the pitch of the sound we hear. The set of all eigenvalues (with multiplicity) is called the *spectrum* of the Laplacian. The question of Kac can be therefore reformulated as follows:

Question. *Can we have two shapes $D_1, D_2 \subseteq \mathbb{R}^2$ with the same spectrum?*

The answer is positive and was given in 1992 [GWW]. For a bounded and nice enough shape, the spectrum looks as follows:

$$\lambda_0 = 0 \leq \lambda_1 \leq \lambda_2 \leq \cdots \to \infty.$$

More precisely, the eigenvalue zero corresponds to the constant function. All eigenvalues occur with finite multiplicity, i.e. for a fixed j, there are only finitely many k such that $\lambda_j = \lambda_k$. The λ_j do not accumulate on a point and are infinite in number. Therefore, $\lim_{j\to\infty} \lambda_j = \infty$. Various questions can be asked about the spectrum and the corresponding eigenfunctions. For example, the Weyl law [Wey] counts the eigenvalues smaller than a given constant. In particular, it implies that one can hear the area of the drum.

The Laplacian is highly generalizable. Given a nice enough surface, or more generally a Riemannian manifold M, we can associate a differential operator to it, called the *Laplace-Beltrami operator* Δ. Intuitively, it averages a function over a small neighborhood of a point. This operator also describes stationary waves on M. For a compact manifold, the spectrum looks similar to the case of a bounded shape on the plane. Moreover, there is an orthonormal basis of the space of square-integrable functions that consists of eigenfunctions of Δ in $C^\infty(M)$.

A more refined question can be asked in that general setting. It belongs to the field of ergodicity. This theory asks about the behavior in the long term of a process on a space. It can be seen as a measurement of chaos. For example, consider a gas molecule moving in a room and bumping on other molecules. After a while, one expects that the molecule has approximately an equal probability to be anywhere in the room. Equivalently, the trajectory of the molecule should fill the room uniformly. The trajectory is called *ergodic*.

In a similar way, we can define *quantum ergodicity* by replacing molecules by waves. One can think of the movement of water on a puddle after throwing a rock. In that case, we study the behavior of eigenfunctions of the Laplacian as the eigenvalue, which corresponds to the energy, goes to infinity. To see how the waves spread, we associate to an eigenfunction f_j of eigenvalue λ_j the measure $|f_j(x)|^2dx$. This describes how the mass of f_j distributes. Intuitively, the mass should spread uniformly as $\lambda_j \to \infty$. Here dx designates the canonical probabilistic measure associated to M. The following conjecture formalizes this intuition:

Conjecture (Quantum Unique Ergodicity [RS])**.** *Let M be a compact manifold with negative curvature. Then, as $\lambda_j \to \infty$,*

$$|f_j(x)|^2dx \to dx$$

in the weak sense. More precisely, for all $\phi \in C^\infty(M)$, we have*

$$\int_M \phi(x)|f_j(x)|^2dx \to \int_M \phi(x)dx.$$

Note that we restricted ourselves to manifolds with negative curvature. These are manifolds where parallel lines move away from each other. We see, at least intuitively, that it should have the effect of spreading the waves more evenly. This conjecture is really hard in general and currently out of reach. But there are special cases with more structure that gives us additional information. They are called *arithmetic manifolds*. In short, these objects have a lot of extra symmetries coming from number theory. More precisely, they are symmetric spaces equipped with a family of normal operators, called the *Hecke algebra*. It commutes with the Laplacian, and therefore we can consider eigenfunctions of both Δ and all the Hecke operators. These joint eigenfunctions have a lot of properties. They are used to say more about Quantum Unique Ergodicity and link it to other questions.

We give an example. We go back to the two-dimensional setting. The *upper half-plane* is

$$\mathbb{H} := \{x + iy \in \mathbb{C} \mid y > 0\}.$$

It is a hyperbolic space with constant curvature -1. There is a natural action of $\mathrm{SL}_2(\mathbb{R})$ on $\mathbb{H}$ given by *Möbius transforms*:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z = \frac{az+b}{cz+d}$$

for all $z \in \mathbb{H}$. Given a discrete subgroup $\Gamma \subseteq \mathrm{SL}_2(\mathbb{R})$, we get a quotient $\Gamma\backslash\mathbb{H}$ by the action. Any compact manifold M without boundary of constant negative curvature -1 can be obtained this way. In other words, $\mathbb{H}$ is the universal cover of such M. Of course, not all quotients give a compact manifold.

There are discrete subgroups of particular interest. The obvious one is $\mathrm{SL}_2(\mathbb{Z})$. More generally, we consider a subgroup Γ of finite index in $\mathrm{SL}_2(\mathbb{Z})$. The surface obtained using such a Γ is an *arithmetic surface*. It has finite volume and extra symmetries, corresponding to the Hecke operators. It might not be compact. In that case, it has *cusps* going to the boundary of $\mathbb{H}$. Smooth functions defined on such a quotient that are eigenvalues of the Laplacian and have polynomial growth at the possible cusps are called *Maass forms*. If they, moreover, vanish at the cusp, in the sense that their limit there is 0, they are called *cusp forms*.

There are other perspectives on Maass forms. In particular, they are very important in the representation theory of the group $\mathrm{GL}_2(\mathbb{R})$. From this point of view, it is natural to introduce a holomorphic analog of them. A *modular form* of weight k with respect to the congruence subgroup Γ is a holomorphic function $f : \mathbb{H} \to \mathbb{C}$ such that

1. $f\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z\right) = (cz+d)^k f(z)$ for all $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$,
2. f is holomorphic at the cusps.

That last condition just says that f stays bounded in a cusp. If, moreover, the limit of f as it goes to a cusp is 0, f is said to be a *cusp form*. The analog of Quantum Unique Ergodicity is called the *Mass Equidistribution Conjecture* [LS]. It considers the measure $|f_k(z)|^2 y^k dz$ as $k \to \infty$. The extra factor y^k makes the measure invariant under the action of Γ. In both the Maass form case and the holomorphic case, the conjectures were shown to be true for $\Gamma = \mathrm{SL}_2(\mathbb{Z})$.

Theorem 1.1 (Lindenstrauss [Lin2], Soundararajan [Sou])**.** *Quantum Unique Ergodicity is true for Maass cusp forms that are joint eigenfunctions of all Hecke operators.*

Theorem 1.2 (Holowinsky-Soundararajan [HS])**.** *The Mass Equidistribution Conjecture is true for cusp forms that are joint eigenfunctions of all Hecke operators.*

Remark. From now on, we will not distinguish between the two conjectures and use the term Arithmetic Quantum Unique Ergodicity or AQUE for both of them. We will also use QUE for the non-necessary arithmetic case.

Quantum Unique Ergodicity is a statement about equidistribution of the mass of eigenfunctions. It is possible to ask other questions in this spirit. For example, QUE tells us that the L^2-norm of a sequence of eigenfunctions converges to a constant. But what about other norms? For example, the sup-norm problem asks about the maximum of an eigenfunction on the space. It is also possible to restrict f_j to a subspace, for example a geodesic, and investigate the equidistribution on it. It leads us to our topic of interest, *restriction norms*. These are norms of functions on a space of lower dimension. In particular, the measure is different and so QUE and other statements cannot directly tell us something about them.

Restriction norms were introduced by Andre Reznikov in 2004 [Rez]. He considered restrictions of eigenfunctions along closed geodesics and circles. He obtained bounds for the L^2-norm. In particular, given a closed geodesic γ, he proved that

$$\|f|_\gamma\|_2^2 = O_\gamma(\lambda^{1/4}) \tag{1.1}$$

for an eigenfunction f on a compact hyperbolic surface with eigenvalue λ. In 2008, Peter Sarnak wrote a letter to Reznikov about his paper [Sar]. Of particular interest to us is the discussion on the link between these questions in the arithmetic case and the Lindelöf Hypothesis (in Appendix 2 of the letter). The Lindelöf Hypothesis, as described at the end of Section 1.1, gives essentially the best possible bound for the restriction norm. There were a lot of follow-up works on the question in the arithmetic setting. Ghosh, Reznikov and Sarnak [GRS] proved an optimal lower and upper bound for Maass forms, proving the best possible exponent $O(\lambda^\epsilon)$ for all $\epsilon > 0$ for Equation (1.1). The holomorphic case was considered by Blomer, Khan and Young [BKY]. It was not possible to give the optimal bound in that setting. Matthew Young gave an overview of the question of Quantum Unique Ergodicity in the restricted setting [You1]. Higher rank problems were also considered, e.g. for $\mathrm{SL}_3(\mathbb{R})$ [Mar2] or for $\mathrm{SL}_{n+1}(\mathbb{R})$ [LLY].

In the case of the Siegel upper-half plane of degree 2, a result on average was obtained by Valentin Blomer and Andrew Corbett [BC]. This article inspired our main result. We introduce that setting in the next section. Other restriction problems for Siegel modular forms were considered in [LY] and [BKY]. Finally, in the context of a general manifold, not necessarily arithmetic, Burq, Gérard and Tzvetkov showed the essentially best possible bound for an eigenvalue of the Laplacian [BGT]. It roughly corresponds to Equation (1.1) in Reznikov's case. Subsequent improvements in a non-arithmetic setting were only by a subpolynomial factor.

In Chapter 2, we start with something similar to our main result, Theorem 1.4, but in a simpler setting. Let $\Gamma_0(N) \subseteq \mathrm{SL}_2(\mathbb{Z})$ be the group of matrices $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ with $N|c$. We consider modular forms of *level* N for $\Gamma_0(N)$ and restrict them to the imaginary axis. We define the L^2-restriction norm

$$N(f) := \frac{\pi[\mathrm{SL}_2(\mathbb{Z}) : \Gamma_0(N)]}{3\|f\|^2} \int_0^\infty |f(iy)|^2 y^k \frac{dy}{y}. \tag{1.2}$$

The front factor accounts for the fact that we did not use a probability measure on the quotient $\Gamma_0(N)\backslash\mathbb{H}$. We consider $N(f)$ in two cases. First, for $N = 1$ and as $k \to \infty$. Second, for k fixed and $N \to \infty$. We restrict ourselves to prime level to avoid technicalities. In both cases, we average $N(f)$ over the space $S_k(N)$ of cusp forms of weight k and level N. More precisely, we define

$$N_{\mathrm{av}}(k,N) := \frac{1}{\dim S_k^{\mathrm{new}}(N)} \sum_{f \in B_k^{\mathrm{new}}(N)} N(f).$$

The set $B_k^{\mathrm{new}}(N)$ is a basis of *newforms*. They are forms that are not coming from lifts of lower level.

Theorem 1.3. *Let $p = 1$. If $k \to \infty$, then*

$$N_{\mathrm{av}}(k) = \frac{1}{\dim S_k(1)} \sum_{f \in B_k(1)} N(f) = 2\log(k) + C + O(k^{-1/2+\epsilon})$$

for an explicit constant C.

Let k be fixed. If $p \to \infty$ in a sequence of prime number, then

$$N_{\mathrm{av}}(k,p) = \frac{1}{\dim S_k^{\mathrm{new}}(p)} \sum_{f \in B_k^{\mathrm{new}}(p)} N(f) = 2\log(\sqrt{p}k) + O(1).$$

This theorem gives an asymptotic expansion for $N(f)$ on average. If we drop all terms but one, we get a trivial bound for $N(f)$. A better individual bound is given in [BKY]. They prove that $N(f) \ll k^{1/4+\epsilon}$ for all $\epsilon > 0$. We highlight a few ingredients of our proof. It is similar in shape to the proof of our main result, Theorem 1.4, presented in the next section. Via Parseval's identity, we get a period formula for $N(f)$ in terms of the L-function associated to f. We use an approximate functional equation and the Petersson trace formula to compute the average and get a diagonal term and an off-diagonal term. The former gives a main term and the latter gives the error term. For level $N > 1$, we need to add the old forms to apply the Petersson formula. It gives an additional error term.

1.1 Siegel modular forms and the Kitaoka formula

In this section, we present a restriction norm problem for a generalization of holomorphic modular forms, called Siegel modular forms. They are defined with respect to the symplectic group $\mathrm{Sp}_{2n}(\mathbb{R})$. Detailed definitions are given in Chapter 3. They take arguments in the *Siegel upper half-plane*

$$\mathbb{H}^{(n)} := \{Z = X + iY \in \mathrm{M}_n(\mathbb{C}) \mid Z = Z^t,\ Y > 0\}$$

where $Y > 0$ means that Y is positive-definite. On this space, the symplectic group acts by a generalization of Möbius transforms. More precisely, for a $2n$ by $2n$ matrix $\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ in the symplectic group $\mathrm{Sp}_{2n}(\mathbb{R})$ and $Z \in \mathbb{H}^{(n)}$, we define

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \cdot Z = (AZ + B)(CZ + D)^{-1}.$$

We can define Siegel modular forms and cusp forms in a natural way. A Siegel modular form of weight k is a holomorphic function $f : \mathbb{H}^{(n)} \to \mathbb{C}$ such that

$$f\left(\begin{pmatrix} A & B \\ C & D \end{pmatrix} \cdot Z\right) = \det(CZ + D)^k f(Z)$$

for all integral matrices $\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ in the symplectic group. For $n \geq 2$, we do not need to add a condition at the cusp. See also Lemma 3.2. If $n = 1$, everything reduces to the classical case of modular forms, except for the condition at the cusps. We denote by $S_k^{(n)}$ the set of cusp forms of weight k. A cusp form f has a Fourier series of the form

$$\sum_{T > 0} a_f(T) \det(T)^{\frac{k-3/2}{2}} e^{2\pi i \operatorname{tr}(TZ)},$$

where the sum is over half-integral positive definite matrices $T \in \mathcal{P}(\mathbb{Z})$. We generalize Equation (1.2) to Siegel modular forms of full level. The *L^2-restriction norm* of a Siegel modular form $f \in S_k^{(n)}$ to the imaginary axis is

$$N(f) := \frac{1}{\|f\|_2^2} \frac{\operatorname{Vol}(\operatorname{Sp}_{2n}(\mathbb{Z})\backslash\mathbb{H}^{(n)})}{\operatorname{Vol}(\operatorname{SL}_n(\mathbb{Z})\backslash \operatorname{SL}_n(\mathbb{R})/\operatorname{SO}(n))} \int_{\operatorname{SL}_n(\mathbb{Z})\backslash\{Y>0\}} |f(iY)|^2 \det(Y)^k \frac{dY}{\det(Y)^{(n+1)/2}}.$$

The ratio of volumes in front of the integral is there because we did not use probability measures. The denominator is due to the isomorphism in Equation (3.3). For $n = 2$, it is equal to $\frac{\pi^2}{90}$. In that case, $\operatorname{SL}_2(\mathbb{R})/\operatorname{SO}(2) \cong \mathbb{H}$ is the classical upper half-plane. The action of $\operatorname{SL}_n(\mathbb{Z})$ is given by $U \cdot Y := U^tYU$. It corresponds to the restriction of the action of $\operatorname{Sp}_{2n}(\mathbb{Z})$ on iY to the second generator in Equation (3.2). The space $\mathbb{H}^{(n)}$ has real dimension $n(n-1)$ and the imaginary axis has half the dimension.

It is hard to give an individual bound for $N(f)$. As often in analytic number theory, we can say something on average. From now on, we reduce to $n = 2$. It is natural to average over the space $S_k^{(2)}$ of cusp forms of weight k. As in Theorem 1.3, we consider

$$\frac{8640}{k^3} \sum_{f \in B_k^{(2)}} N(f),$$

where $B_k^{(2)}$ is a Hecke basis of $S_k^{(2)}$. The normalization is justified by the dimension formula $\dim S_k^{(2)} \sim \frac{k^3}{8640}$ [Igu]. Unfortunately, this is not good enough for our purpose so we do a further average over $k \in [K, 2K]$. For analysis purposes, we average smoothly as follows. Let $w : \mathbb{R} \to \mathbb{R}_{>0}$ be a smooth test function with support in $[1, 2]$ and $\omega = \int_1^2 w(x)x^3dx$. We consider the following average:

$$N_{\mathrm{av}}(K) = \frac{17280}{\omega K^4} \sum_{k \in 2\mathbb{N}} w\left(\frac{k}{K}\right) \sum_{f \in B_k^{(2)}} N(f). \tag{1.3}$$

The restriction to even k is for technical reasons, going back to the paper of Kitaoka [Kit] on a Petersson-like formula. It is known that this formula is also valid for odd k (see Remark 1.4 in [CKM]). Likely, our result also extends in a similar fashion. This adds a factor of 2 in the normalization. Our main result is:

Theorem 1.4 ([Fel]). *Let $\epsilon > 0$. As $K \to \infty$, we have*

$$N_{\mathrm{av}}(K) = 4\log(K) + C + O_\epsilon(K^{-1/2+\epsilon})$$

for an explicit constant C that only depends on w.

Valentin Blomer and Andrew Corbett in [BC] have a similar result for another average. They considered *Saito-Kurokawa lifts.* They are lifts of holomorphic cusp forms, via an injective map $S_{2k-2} = S_{2k-2}^{(1)} \to S_k^{(2)}$ for even k. They also are in one-to-one correspondence with some half-integral forms of weight $k - 1/2$. Blomer and Corbett considered the following average over Saito-Kurokawa lifts and again over the weight $k \in [K, 2K]$:

$$N_{\mathrm{av}}^{\mathrm{SK}}(K) = \frac{12}{\omega K^4} \sum_{k \in 2\mathbb{N}} w\left(\frac{k}{K}\right) \sum_{f \in B_k^{(2),\mathrm{SK}}} N(f).$$

Their result is similar, except for a larger error term:

$$N_{\mathrm{av}}^{\mathrm{SK}}(K) = 4\log(K) + O(1)$$

as $K \to \infty$. Our result is inspired by their article. At the center of our proof is a generalization of the Petersson trace formula, where for them, it is a relative trace formula for pairs of Heegner points. They can apply it after the following manipulation. The Fourier coefficients of a Siegel modular forms are over half-integral positive-definite matrices. For Saito-Kurokawa lifts, it is possible to factorize them with respect to the determinant. They obtain this way a Dirichlet series in the right shape for their relative trace formula. It is essentially a Rankin-Selberg product of two half-integral weight cusp forms (see also [DI]). After this initial step, our proof diverges from theirs.

It is possible to rewrite $N(f)$ in terms of some Dirichlet series. This is the first step in the proof of Theorem 1.4. We call this the *period formula*. We have an isomorphism $\mathcal{P}(\mathbb{R}) := \{Y > 0\} \to \mathbb{H}\times\mathbb{R}_{>0}$ given in Equation (3.3). For the two pieces on the right side, we have a spectral theory. For $\mathbb{H}$, it is the decomposition of $L^2(\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H})$ into Eisenstein series, cusp forms and the constant functions. We denote by Λ a set of spectral components that are also Hecke eigenforms and Λ_{ev} the subset of even forms. There is a measure $d\phi$ associated to $\phi \in \Lambda$. It is the counting measure on the discrete part and $\frac{dt}{4\pi}$ on the continuous part given by the Eisenstein spectrum $E(\cdot, 1/2 + it)$. For $\mathbb{R}_{>0}$, the spectral inversion is given by the Mellin transform. In $N(f)$, we compute the square of a function. We apply the Parseval identity for both spectral decomposition. We get the following proposition:

Proposition 1.5 ([BC], Proposition 1). *Let $f \in S_k^{(2)}$, $\phi \in \Lambda_{\mathrm{ev}}$ and $L(f \times \phi, s)$ be the Dirichlet series defined in Equation* (3.4) *and $G(f \times \phi, s)$ the corresponding gamma factor. Then*

$$N(f) = \frac{\pi^2}{2880\|f\|_2^2} \int_{-\infty}^{\infty} \int_{\Lambda_{\mathrm{ev}}} |L(f \times \phi, 1/2 + it)G(f \times \phi, 1/2 + it)|^2 d\phi\, dt.$$

More details are provided in Section 3.3. The Dirichlet series $L(f \times \phi, s)$ is of particular interest. We call it a *twisted Koecher-Maass series*. Despite what the notation might suggest, it is not a Rankin-Selberg product of the two functions. More precisely, the Dirichlet series runs over half-integral matrices T. It features the Fourier coefficients of f at T but ϕ is evaluated at the Heegner point z_T in the sum via the isomorphism in Equation (3.3). This series has an analytic extension to an entire function with a functional equation, given in Proposition 3.5. But it does not have an Euler product and is not in the Selberg class.

Theorem 1.4 fits into the context of two conjectures. First, there is Arithmetic Quantum Unique Ergodicity that we discussed above. In our case, the subspace $\mathrm{SL}_2(\mathbb{Z})\backslash\mathcal{P}(\mathbb{R})$ is of infinite measure. Suppose that AQUE is true, even when we restrict to the imaginary axis. Then the norm $N(f)$ should diverge to infinity. Our result says that, on average, it is the case. But there is more. We show in Chapter 9 that f is essentially supported in the imaginary axis on matrices with $k^{-2} \ll \det(Y) \ll k^2$. In Equation (9.1), we compute the volume of such matrices. If we conjecture that f is essentially constant there, then the main term in Theorem 1.4 and in particular the constant 4 are shown to be consistent with AQUE on the imaginary axis.

Concerning the second conjecture, the period formula leads us to the Lindelöf hypothesis. It is a consequence of the Riemann Hypothesis. Both can be generalized to L-functions. One would guess that the latter is false for the Dirichlet series $L(f \times \phi, s)$, since it does not belong in the Selberg class. Nevertheless, we still have a functional equation and some arithmetic inputs. We can hope that the Lindelöf Hypothesis holds. See also [Kim] for supporting evidence. In the simplest case, Lindelöf [Lin1] conjectured that

$$\zeta(1/2 + it) \ll_\epsilon t^\epsilon$$

for all $\epsilon > 0$. We can generalize it to other parameters of families of L-functions. For example, given a family of L-function associated to modular form f_{k_j} of weight k_j, the Lindelöf Hypothesis in the weight aspect says that

$$L(f_{k_j}, s) \ll k_j^\epsilon.$$

The Lindelöf Hypothesis is open in all instances. "Trivial" upper bounds are obtained using the Phragmén–Lindelöf principle. This is called the convexity bound. To establish it can be far from trivial in some cases. Research focuses nowadays on improving bounds beyond convexity for various families of L-functions. If we input the weight aspect bound to $L(f \times \phi, s)$, we get that $N(f) \ll k^\epsilon$. Theorem 1.4 can be seen as strong version of Lindelöf on average.

1.2 Details of the proof

In this section, we give details of the proof of Theorem 1.4. We have actually not two but three averages in the problem:

1. the sum over $k \in [K, 2K]$,
2. the sum over the basis $B_k^{(2)}$ of weight k,
3. the sum/integral over the spectrum Λ.

The last average appears in the period formula, presented in Proposition 1.5. Each average gives us additional cancellations in the various sums and integrals appearing in the problem. The sum over k is treated using oscillations of Bessel functions. The two other sums are computed using trace formulae. For the sum over the basis, the relevant formula is the Kitaoka formula that shape the whole proof. We start to explain this one. For the sum over the spectrum, the pre-trace formula for $\mathrm{SL}_2(\mathbb{Z})$ is used.

In the center of our argument is a generalization of Petersson trace formula to Siegel modular forms of degree 2. It was established by Kitaoka [Kit] to give bounds on Fourier coefficients of Siegel modular forms. Its shape is as follows. Let T, Q be two half-integral positive-definite matrices. Then we have

$$\begin{aligned} c_k \sum_{f \in B_k^{(2)}} \frac{a_f(T) a_f(Q)}{\|f\|^2} = \#\,\mathrm{Aut}(T)\delta_{T\sim Q} & \\ + \sum_{\mathrm{rk}(C)=1} KL_1(Q,T;C) J_{k-3/2}\left(\frac{4\pi\sqrt{\det(TQ)}}{c}\right) & \\ + \sum_{\mathrm{rk}(C)=2} KL_2(Q,T;C)\mathcal{J}_{k-3/2}(Q,T;C). & \qquad (1.4) \end{aligned}$$

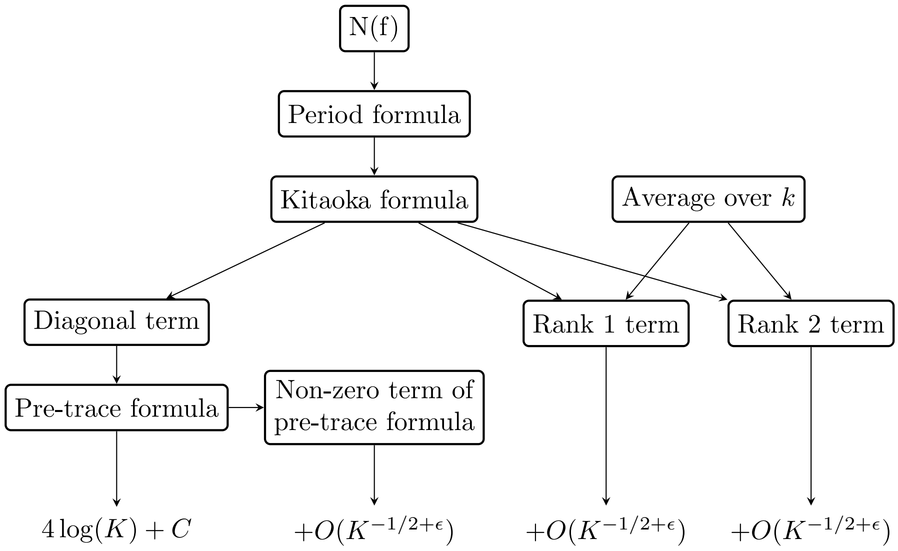
Here, $T \sim Q$ if they are related by an isomorphism and the two sums on the right-hand side feature generalizations of the Kloosterman sum and the J-Bessel function. The number c is related to the non-zero invariant factor of C. The detailed definitions are given in Section 3.4. Our main interest in this formula for us is the sum over $S_k^{(2)}$ on the left-hand side. We can take advantage of our average in $N_{\mathrm{av}}(K)$ with it. The idea is that only the diagonal term, where $T \sim Q$, should contribute to the main term as $k \to \infty$. See [KST] for a supportive statement. Let $v_T = (a_f(T))_{f \in B_k^{(2)}}$. Then Equation (1.4) says that v_T and v_Q are approximately orthogonal if $T \not\sim Q$. The left-hand side of the formula is called the *spectral side* and the right-hand side the *geometric side*. The latter has three terms, that correspond to matrices of rank respectively 0, 1 and 2.

The Kitaoka formula shapes the proof of Theorem 1.4. This is shown in Figure 1.1. We give more details. The first step in the proof is to introduce the period formula for $N(f)$. Then we can insert the Kitaoka formula in $N_{\mathrm{av}}(K)$ using an approximate formula. It gives us three terms. The diagonal one gives us the main term of our asymptotic formula. More precisely, we have in that case to use the pre-trace formula for $\mathrm{SL}_2(\mathbb{Z})$. It splits this term into two parts, one of them being the main asymptotic and the other going into the error term. The rank 1 and rank 2 terms are both giving error term. The challenge of the proof is to give error terms of the right size for every part. In the diagonal term, we do not take advantage of the average over k. In the rank 1 and the rank 2 case, it gives us a better decay for the respective Bessel function. See Lemma 5.3 and its application in Chapters 7 and 8. The rank 2 term is the most involved but the other parts all need a non-trivial argument.

The diagonal term is treated using the pre-trace formula. It gives us a main term where the distance u appearing in the formula is zero. The terms with $u \neq 0$ have to be treated using a standard counting argument. We also have to extract the even spectrum in the formula using the T_{-1} Hecke operator. After that, we treat the remaining sums and integrals. In particular, a Dirichlet series over class numbers appears and we have to compute its residue. We input a formula for the average size of class numbers. The average over k is computed at the end and does not contribute to this term.

The bound for the rank 1 term is obtained as follows. First, we use the average over k for the J-Bessel function. If we take a trivial upper bound at this point, we barely miss the desired main term. Therefore we use additional cancellation in the Bessel function. We have a sum over Fourier coefficients T, Q. The sum over $\det(T) - \det(Q)$ has oscillations that we detect using Poisson summation and a stationary phase argument. The argument is technical but essentially standard.

Figure 1.1: The shape of the proof



In the rank 2 term, we start by analyzing the cancellation in the $\mathcal{J}_{k-3/2}$ function with the average over k. Then we do a stationary phase argument for the resulting function. The question reduces to a counting problem for a matrix of the form TQ, where T and Q are half-integral positive definite matrices such that the product has close eigenvalues. We do that counting exploiting a nice identity for the distance between the eigenvalues. In the rank 1 and the rank 2 terms, we only use trivial bounds on the generalized Kloosterman sums. This is because we have short sums on C up to some negligible errors and the Kloosterman sums do not depend on k.

1.3 Higher degrees

It is also interesting to consider the generalization of the Petersson formula to Siegel modular forms of higher degrees. Little is known when the degree is greater than 2. We have a formula similar to Equation (1.4) but also with terms of rank higher than 2. They feature generalizations of Kloosterman sums and Bessel functions. For the Kloosterman sum of degree 3 and full rank, we obtain a non-trivial bound in Chapter 10.

Theorem 1.6. *Let C be an invertible matrix and Q,T be symmetric positive definite matrices. Let $K^{(3)}(Q,T;C)$ be the symplectic Kloosterman sum defined in Equation* (10.1) *for $n=3$. Suppose that $C=pI_3$ for an odd prime p. Then*

$$K^{(3)}(Q,T;pI_3)=\sum_{\substack{A,D \bmod p\\ AD=I_3 \bmod p}} e\left(\frac{\operatorname{tr}(AQ+DT)}{p}\right)\ll p^5.$$

This is inspired by a similar result of Tóth in degree 2 [Tót]. We prove it with elementary techniques, such as the evaluation of the Gauss sum and the Salié sum. When $C=pI_3$, the exponential sum is over symmetric 3 by 3 matrices modulo p with non-zero determinant. Therefore a trivial upper bound for $K^{(3)}(Q,T;pI_3)$ is p^6.

This thesis is partially based on the article *A Restriction Norm Problem for Siegel Modular Forms* [Fel]. Chapters 4 to 8 are part of the article, with only minor changes and corrections. To start, we consider the case of $\mathrm{Sp}_2(\mathbb{R})=\mathrm{GL}_2(\mathbb{R})$ in Chapter 2. We prove Theorem 1.3 for the level and the weight aspect. In Chapter 3, we introduce the theory of Siegel modular forms and give the necessary definitions. Chapter 4 is dedicated to the proof of Proposition 1.5, the insertion of the Kitaoka formula and the analysis of the cut-off function given by the approximate functional equation. Chapter 5 regroups various technical lemmas about the Gamma and Bessel functions and the stationary phase technique. Chapter 6 is dedicated to the diagonal term of the Kitaoka formula. Chapters 7 and 8 treat respectively the rank 1 and rank 2 terms. In Chapter 9, we give further comments on the proof of Theorem 1.4. In Chapter 10, we study the setting for higher degree modular forms on $\mathrm{Sp}_{2n}(\mathbb{R})$. We prove Theorem 1.6 for $n = 3$. Finally, the Appendix A gives a detailed computation of automorphisms of binary quadratic forms.

1.4 Notation and normalizations

We fix a few notations and normalize some objects in this section. What is written here is valid everywhere unless stated otherwise.

The set $\mathcal{P}(\mathbb{R})$ consists of every 2 by 2 symmetric positive-definite matrices and $\mathcal{P}(\mathbb{Z})$ is the subset of elements with integral diagonal and half-integral off-diagonal elements. In particular $i\mathcal{P}(\mathbb{R})$ corresponds to the imaginary axis of the Siegel upper-half plane $\mathbb{H}^{(2)}$. A matrix $\left(\begin{smallmatrix} a & b \\ b & c \end{smallmatrix}\right)$ in $\mathcal{P}(\mathbb{Z})$ (sometimes written $\left(\begin{smallmatrix} x & y \\ y & z \end{smallmatrix}\right)$ or $\left(\begin{smallmatrix} \alpha & \beta \\ \beta & \delta \end{smallmatrix}\right)$) corresponds to a positive-definite binary quadratic forms $ax^2 + 2bxy + dy^2$. Such a form is *weakly reduced* if $2|b| \leq a \leq d$. It is *reduced* if, moreover,

$$2|b| = a \text{ or } a = c \Longrightarrow b \geq 0.$$

We use the notation coming from matrices and do not introduce the alternative notations used for quadratic forms. In particular, we consider the determinant of the matrix but not the discriminant of the associated quadratic form.

A matrix $Y \in \mathcal{P}(\mathbb{Z})$ also corresponds to a point z_Y in $\mathbb{H}$, see Equation (3.3). The usual fundamental domain of $\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}$ is $\{z = x + iy \in \mathbb{C} \mid -1/2 \leq x \leq 1/2,\ |z| \geq 1\}$. A weakly reduced matrix Y corresponds to a point in this domain. If Y is reduced and the corresponding point is on the edge of the fundamental domain, then this point has non-positive real part.

Let $\ell = k - 3/2$ and $f \in S_k^{(2)}$ be a Siegel cusp form of weight k and degree 2. We only consider even weights. The Fourier series of f is normalized in the following way:

$$f(Z) = \sum_{T \in \mathcal{P}(\mathbb{Z})} a(T) \det(T)^{\ell/2} e(\mathrm{tr}(TZ)).$$

The set of spectral components is denoted by Λ and Λ_{ev} is the subset of even forms. The eigenvalue of $\phi \in \Lambda$ is λ_ϕ and t_ϕ is the spectral parameter, given by $\lambda_\phi = \frac{1}{4} + t_\phi^2$. The constant function has spectral parameter $i/2$. We write $\int_\Lambda$ for the integral over the spectrum with the corresponding measure, that is $\frac{dt_\phi}{4\pi}$ for the continuous part and the counting measure for the discrete part. The symbol Γ is used for the gamma function and the modular group is designated by $\mathrm{SL}_2(\mathbb{Z})$. We use the notation $e(z) = e^{2\pi i z}$ and the Vinogradov symbols $\ll$, $\gg$, $\asymp$ and $\sim$.

Finally, we always assume that k, K and p are large enough and $\epsilon > 0$ is small enough to avoid degenerate cases. We may change the value of ϵ from a display to the next, as long as the new ϵ is a constant multiple of the first one.

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Chapter 2

The case of $\mathrm{Sp}_2(\mathbb{R})$

In this chapter, we prove Theorem 1.3. It is similar to Theorem 1.4 for degree 1 Siegel modular forms, in other words, classical holomorphic modular forms. We define a restriction norm to the imaginary axis and we do an average over the space of weight k. Finally, we use Petersson trace formula to get an asymptotic formula, this time both for the weight and the level aspect. Definitions for this chapter can be found in [BGHZ] or any other introductory textbook on modular forms.

2.1 Setting

Let k be an integer. We consider modular forms attached to the congruence subgroup $\Gamma_0(p) := \{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \mid c = 0 \pmod p\}$. We consider simultaneously the cases p is prime and $p = 1$. For a Hecke newform $f \in S_k^{\mathrm{new}}(p) = S_k^{\mathrm{new}}(\Gamma_0(p))$, we set

$$f(z) = \sum_{n=1}^{\infty} a(n)(4\pi n)^{(k-1)/2} e(nz)$$

with $a(1) = 1$. We consider the restriction norm

$$N(f) := \frac{\pi[\mathrm{SL}_2(\mathbb{Z}) : \Gamma_0(p)]}{3\|f\|^2} \int_0^{\infty} |f(iy)|^2 y^k \frac{dy}{y}. \tag{2.1}$$

Here, $[\mathrm{SL}_2(\mathbb{Z}) : \Gamma_0(p)] = p + 1$ if $p > 1$ and 1 if $p = 1$ and

$$\|f\|^2 = \int_{\Gamma_0(p)\backslash\mathbb{H}} |f(z)|^2 y^k \frac{dxdy}{y^2}.$$

We have the Mellin transform

$$\int_0^{\infty} f(iy) y^{\frac{k-1}{2}+s} \frac{dy}{y} = 2^{\frac{k-1}{2}} \frac{\Gamma((k-1)/2 + s)}{(2\pi)^s} L(f, s)$$

where $L(f, s) = \sum \frac{a_n}{n^s}$. Hence Parseval's formula gives

$$N(f) = \frac{\pi(p+1)}{3\|f\|^2} \frac{1}{2\pi} \int_{-\infty}^{\infty} 2^{k-1} \left|\Lambda(f, 1/2 + it)\right|^2 dt, \tag{2.2}$$

where the L-function for f has the following completion and functional equation:

$$\Lambda(f, s) = \frac{\Gamma((k-1)/2 + s)}{(2\pi)^s} L(f, s),$$
$$p^{s/2}\Lambda(f, s) = i^k \bar{\eta} p^{(1-s)/2} \Lambda(f, 1 - s),$$

with $|\eta| = 1$.

Lemma 2.1. *Inserting an approximate functional equation, we have*

$$N(f) = \frac{p+1}{12\pi\|f\|^2}\Gamma(k-1)\sum_{m,n}\frac{a_f(m)a_f(n)}{\sqrt{mn}}V(m,n,p)$$

with the definition of V given in Equation (2.3).

We prove this lemma in Section 2.2. We consider $N(f)$ on average over the space of weight k. More precisely, we are looking for an asymptotic formula for

$$N_{\mathrm{av}}(k,p) = \frac{1}{\dim S_k^{\mathrm{new}}(p)}\sum_{f\in B_k^{\mathrm{new}}(p)} N(f),$$

where $B_k^{\mathrm{new}}(p)$ is a Hecke eigenbasis of newforms. The results are given in Theorem 1.3, where either k is fixed and $p\to\infty$ or $p=1$ and $k\to\infty$. If $p>1$, we have

$$N_{\mathrm{av}}(k,p) = \frac{1}{\dim S_k^{\mathrm{new}}(p)}\sum_{f\in B_k(p)} N(f) - \tilde{N}_{\mathrm{av}}^{\mathrm{old}}(k,p)$$

where $\tilde{N}_{\mathrm{av}}^{\mathrm{old}}(k,p)$ is a sum over forms in $S_k(1) = \mathrm{SL}_2(\mathbb{Z})$ that are lifted in two ways to $S_k(p)$. The precise definition is given in Equation (2.9). We use the following formula to take advantage of the average.

Proposition 2.2 (Petersson trace formula, [IK], Corollary 14.23)**.** *Let p, k integers, $B_k(p)$ an orthogonal basis of $S_k(p)$ consisting of Hecke eigenfunctions and $m,n\in\mathbb{N}$. Then*

$$\Gamma(k-1)\sum_{f\in B_k(p)}\frac{a_f(m)a_f(n)}{\|f\|^2} = \delta_{mn} + 2\pi i^{-k}\sum_{p|c}\frac{S(m,n,c)}{c}J_{k-1}\left(\frac{4\pi\sqrt{mn}}{c}\right).$$

2.2 Approximate functional equation

In this section, we prove Lemma 2.1. Let $f\in S_k^{\mathrm{new}}(p)$ be a new form and

$$I(f,s) = \frac{1}{2\pi i}\int_{(3)} e^{v^2}p^v\Lambda(f,v+s)\Lambda(f,v+1-s)\frac{dv}{v}.$$

Shifting the contour of the integral and using the functional equation, we get

$$\begin{aligned}|\Lambda(f,1/2+it)|^2 &= 2I(f,1/2+it)\\ &= \frac{1}{\pi}\sum_{m,n}\frac{a(m)a(n)}{\sqrt{mn}}\left(\frac{n}{m}\right)^{it}\frac{1}{2\pi i}\int_{(3)} e^{v^2}\frac{\Gamma(k/2+v+it)\Gamma(k/2+v-it)}{(2\pi)^{2v}}\left(\frac{mn}{p}\right)^{-v}\frac{dv}{v}.\end{aligned}$$

We set

$$V(m,n,N) = \int_{-\infty}^{\infty}\left(\frac{n}{m}\right)^{it}\frac{1}{2\pi i}\int_{(3)} e^{v^2}2^k\frac{\Gamma(k/2+v+it)\Gamma(k/2+v-it)}{(2\pi)^{2v}\Gamma(k-1)}\left(\frac{mn}{N}\right)^{-v}\frac{dv}{v}dt. \tag{2.3}$$

Inserting this in Equation (2.2), we obtain

$$N(f) = \frac{p+1}{12\pi\|f\|^2}\Gamma(k-1)\sum_{m,n}\frac{a(m)a(n)}{\sqrt{mn}}V(m,n,p)$$

with decay properties given in the following lemma.

Lemma 2.3. *Let m,n,N be integers and $A>0$. Then*

$$\left(\frac{m}{\sqrt{k}}\right)^{j_1}\left(\frac{n}{\sqrt{k}}\right)^{j_2}\frac{d^{j_1}}{dm^{j_1}}\frac{d^{j_2}}{dn^{j_2}}V(m,n,N) \ll_{A,j_1,j_2} k\left(1+\frac{mn}{Nk^2}\right)^{-A}\left(1+k^{1/2}|\log(m/n)|\right)^{-A}$$

This can be easily deduced from the following.

Lemma 2.4. *Let $k \geq 1$, $s = \sigma + it$ with $\sigma \geq -1$, $A > 0$ and*

$$G(k,s) := 2^k \frac{\Gamma(k/2+\sigma+it)\Gamma(k/2+\sigma-it)}{\Gamma(k-1)}.$$

We have, for $\sigma > 0$, $t \in \mathbb{R}$

$$G(k,\sigma+it) = k^{1/2+\sigma} G_{A,\sigma}(k,t) + O_A(k^{-A}),$$

where

$$k^{j_1+j_2/2} \frac{d^{j_1}}{dk^{j_1}} \frac{d^{j_2}}{dt^{j_2}} G_A(k,\sigma,t) \ll_{A,\sigma,j_1,j_2} \left(1+\frac{t^2}{k}\right)^{-A}.$$

We also have

$$2^k \frac{\Gamma(k/2+it)\Gamma(k/2-it)}{\Gamma(k-1)} = 2^{3/2}\sqrt{\pi k} e^{-2t^2/k}(1+O(k^{-1/2+\epsilon})).$$

Proof of both lemmas. We only sketch the proof of the size of V. The rest is similar to the proof of Equation (4.3). The duplication formula gives

$$2^k \frac{\Gamma(k/2+\sigma+it)\Gamma(k/2+\sigma-it)}{\Gamma(k-1)} = 4\sqrt{\pi} \frac{\Gamma(k/2+\sigma+it)\Gamma(k/2+\sigma-it)}{\Gamma((k-1)/2)\Gamma(k/2)}.$$

We apply Lemma 5.1. We see that the ratio is of size $O(k^{1/2+2\sigma})$. For V, another $\sqrt{k}$ is given by the integral over t and the decay properties are easily computed. □

2.3 Diagonal term

We apply the Petersson trace formula to $N_{\text{av}}(k,p)$ using Lemma 2.1. We split this into a diagonal term, when $\delta_{mn} = 1$, and the off-diagonal term with the sum over c. For the diagonal term, we have

$$N_{\text{av}}^{\text{diag}}(k,p) = \frac{1}{\dim S_k(p)} \frac{p+1}{12\pi} \sum_n \frac{V(n,n,p)}{n}$$

with $V(n,n,p)$ defined in (2.3). We open this definition and insert the n-sum. The v-integral gives

$$\begin{aligned}
&\int_{(3)} e^{v^2} 2^k \frac{\Gamma(k/2+v+it)\Gamma(k/2+v-it)}{(2\pi)^{2v}\Gamma(k-1)} \sum_n \frac{1}{n^{2v+1}} p^v \frac{dv}{v} \\
&= \int_{(-1+\epsilon)} e^{v^2} 2^k \frac{\Gamma(k/2+v+it)\Gamma(k/2+v-it)}{(2\pi)^{2v}\Gamma(k-1)} \zeta(2v+1) p^v \frac{dv}{v} \\
&\quad + \operatorname{Res}_{v=0} 2^k \frac{\Gamma(k/2+v+it)\Gamma(k/2+v-it)}{(2\pi)^{2v}\Gamma(k-1)} \zeta(2v+1) p^v \frac{1}{v}.
\end{aligned}$$

Concerning the middle term, we have (we can take $\epsilon = 1/2$)

$$\begin{aligned}
&\int_{-\infty}^{\infty} \int_{(-1+\epsilon)} e^{v^2} 2^k \frac{\Gamma(k/2+v+it)\Gamma(k/2+v-it)}{(2\pi)^{2v}\Gamma(k-1)} \zeta(2v+1) p^v \frac{dv}{v}\, dt \\
&\ll_\epsilon k^{-2.5+2\epsilon} p^{-1+\epsilon} \int_{-\infty}^{\infty} \left(1+\frac{t^2}{k}\right)^{-A} dt \\
&\ll_\epsilon k^{-2+\epsilon} p^{-1+\epsilon}.
\end{aligned}$$

We have $\frac{\Gamma'(z)}{\Gamma(z)} = \log(z) + O(|z|^{-1})$ for $\mathrm{Re}(z) \geq 1$. Since $t \ll k^{1/2+\epsilon}$ up to a negligible error, we have

$$\begin{aligned}
\sum_\pm \frac{\Gamma'(k/2 \pm it)}{\Gamma(k/2 \pm it)} &= \log(k^2/4+t^2) + O(k^{-1+\epsilon}) \\
&= 2\log(k) - \log(4) + O(k^{-1+2\epsilon})
\end{aligned}$$

We also have $\zeta(2v+1) = \frac{1}{2v} + \gamma + O(v)$. Therefore

$$\begin{aligned}
&\mathrm{Res}_{v=0} 2^k \frac{\Gamma(k/2+v+it)\Gamma(k/2+v-it)}{(2\pi)^{2v}\Gamma(k-1)}\zeta(2v+1)p^v \frac{1}{v} \\
&= 2^k \frac{\Gamma(k/2+it)\Gamma(k/2-it)}{\Gamma(k-1)}\left(\frac{1}{2}(2\log(k) - \log(4) - 2\log(2\pi) + \log(p)) + \gamma + O(k^{-1+\epsilon})\right).
\end{aligned}$$

Lemma 2.4 also gives

$$2^k \frac{\Gamma(k/2+it)\Gamma(k/2-it)}{\Gamma(k-1)} = 2^{3/2}\sqrt{\pi k} e^{-2t^2/k}(1 + O(k^{-1/2+\epsilon})).$$

Therefore

$$\int_{-\infty}^{\infty} 2^k \frac{\Gamma(k/2+it)\Gamma(k/2-it)}{\Gamma(k-1)} dt = 2\pi k + O(k^{1/2+\epsilon}).$$

Finally, we have $\dim S_k^{\mathrm{new}}(p) = \frac{kp}{12} + O(k+p)$ and $\dim S_k(1) = \frac{k}{12} + O(1)$ (see [Mar1]). If $p > 1$, we get

$$\begin{aligned}
N_{\mathrm{av}}^{\mathrm{diag}}(k,p) &= \frac{1}{\dim S_k(p)} \frac{p+1}{12\pi} \int_{-\infty}^{\infty} \int_{(3)} e^{v^2} 2^k \frac{\Gamma(k/2+v+it)\Gamma(k/2+v-it)}{(2\pi)^{2v}\Gamma(k-1)} \zeta(2v+1) p^v \frac{dv}{v}\, dt \\
&= \frac{1}{\pi k}\left(1 + O\left(\frac{1}{k} + \frac{1}{p}\right)\right)(2\pi k + O(k^{1/2+\epsilon}))(\log(\sqrt{p}k) + \gamma - \log(4\pi) + O(k^{-1+\epsilon})) \\
&= 2\log(\sqrt{p}k) + 2(\gamma - \log(4\pi)) + O(k^{-1/2+\epsilon}) + O(p^{-1}).
\end{aligned}$$

If $p = 1$, we get the same result with an error term of size $O(k^{-1/2+\epsilon})$. This concludes the proof of the main term.

2.4 Off-diagonal term

The off-diagonal term is

$$N_{\mathrm{av}}^{\mathrm{off}}(k,p) = \frac{1}{\dim S_k(p)} \frac{p+1}{12\pi} \sum_{m,n} \frac{V(m,n,p)}{\sqrt{mn}} 2\pi i^{-k} \sum_{p|c} \frac{S(m,n,c)}{c} J_{k-1}\left(\frac{4\pi\sqrt{mn}}{c}\right).$$

If $p > 1$, Lemma 2.3 tells us that, up to a negligible error, $mn \ll p^{1+\epsilon}k^{2+\epsilon}$. At the same time, the decay of the J-Bessel function gives $k^{1-\epsilon} \ll \frac{\sqrt{mn}}{c}$ (see Equation (5.1)). Thus, we have

$$c \ll \frac{\sqrt{mn}}{k^{1-\epsilon}} \ll p^{1/2+\epsilon}k^{2\epsilon}.$$

Since $p|c$, we see that the off-diagonal term is negligible if $p \to \infty$.

We consider the weight aspect where $p = 1$. We apply Lemma 2.3 and Equation (5.1). We fix $\epsilon > 0$ small enough. Up to a negligible error, we get

$$k^{1-\epsilon} \ll \frac{\sqrt{mn}}{c}, \qquad \sqrt{mn} \ll k^{1+\epsilon} \qquad |m-n| \ll k^{-1/2+\epsilon}m.$$

Therefore, $c \ll k^{\epsilon}$ and $k^{1-\epsilon} \ll m, n \ll k^{1+\epsilon}$. We detect cancellation in the difference between m and n. Let $d = n - m$. We know that $|d| \ll k^{-1/2+\epsilon}m$ up to a negligible error. We fix $d = d_0 \pmod{c}$ so that we can see the Kloosterman sum as constant in the d-sum. This adds a sum over $d_0 \pmod{c}$. The sum over d is

$$\begin{aligned}
&\sum_{d = d_0 \bmod c} \frac{V(m,m+d,1)}{\sqrt{m(m+d)}} J_{k-1}\left(\frac{4\pi\sqrt{m(m+d)}}{c}\right) \\
&= \frac{1}{c}\sum_{h=-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{V(m,m+t,1)}{\sqrt{m(m+t)}} J_{k-1}\left(\frac{4\pi\sqrt{m(m+t)}}{c}\right) e\left(\frac{(d_0-t)h}{c}\right) dt. \qquad (2.4)
\end{aligned}$$

We applied Poisson summation in the second line. The Bessel function is defined as

$$J_k(x) = \int_{-1/2}^{1/2} e(k\theta)e^{-ix\sin(2\pi\theta)}d\theta.$$

Let v be a positive function supported on $[-2,2]$ and equal to 1 on $[-1,1]$. In the θ-integral, we consider

$$L_k(x) := \int_{-1/2}^{1/2} e(k\theta)e^{-ix\sin(2\pi\theta)}v\left(k^{1/10}(|\theta|-1/4)\right)d\theta.$$

Then in the integral $J_{k-1}(x) - L_{k-1}(x)$, the integrand is 0 when θ is close to $\pm\frac{1}{4}$. We get

$$\frac{1}{c}\sum_{h=-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-1/2}^{1/2}\frac{V(m,m+t,1)}{\sqrt{m(m+t)}}e\left((k-1)\theta - \frac{2\sqrt{m(m+t)}}{c}\sin(2\pi\theta) + \frac{(d_0-t)h}{c}\right)$$
$$\cdot\left(1 - v\left(k^{1/10}(|\theta|-1/4)\right)\right)d\theta\, dt.$$

We do a stationary phase argument in the t-integral. More details about this technique are given in Section 5.3. We have

$$\frac{d}{dt}\left[(k-1)\theta - \frac{2\sqrt{m(m+t)}}{c}\sin(2\pi\theta) + \frac{(d_0-t)h}{c}\right] = -\frac{1}{c}\sqrt{\frac{m}{m+t}}\sin(2\pi\theta) - \frac{h}{c}.$$

We consider the case $h=0$ below. If $h \neq 0$, the stationary point is

$$t_0 = \left(\frac{\sin(2\pi\theta)^2}{h^2} - 1\right)m.$$

The function $\sin(2\pi\theta)$ is equal to 1 only at $|\theta| = \frac{1}{4}$. Since $|\theta| - \frac{1}{4} \gg k^{-1/10}$, we have $|t_0| \gg k^{-1/5}m$. The function V is negligible for such t_0. We apply Lemma 5.5 to the t-integral. Following the notations, we have

$$\begin{aligned} \beta - \alpha &\ll mk^{-1/2+\epsilon}, & R &= 1 + \frac{|h|}{c}, \\ X &= \frac{k}{m}, & U &= \frac{m}{\sqrt{k}}, \\ Y &= k^{1+\epsilon}, & Q &= m. \end{aligned} \tag{2.5}$$

Since $k^{1-\epsilon} \ll m \ll k^{1+\epsilon}$, we get that the t-integral is

$$\ll_A mk^{-1/2+\epsilon}\frac{k}{m}(1+|h|/c)^{-A}k^{-(1/2+\epsilon)A}$$

for all $A > 0$. We take trivial bounds for the θ-integral and the h-sum. We see that the contribution of this term to Equation (2.4) is $O_A(k^{-A})$.

Now, we consider the integral

$$\frac{1}{c}\sum_{h=-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{V(m,m+t,1)}{\sqrt{m(m+t)}}L_k\left(\frac{4\pi\sqrt{m(m+t)}}{c}\right)e\left(\frac{ht}{c}\right)dt.$$

We apply Lemma 5.5 to $L_k(x)$. We know that $x \gg k^{1-\epsilon}$ up to a negligible error. The derivative $2\pi k - 2\pi x\cos(2\pi\theta) \gg k$ since $|\theta| - \frac{1}{4} \ll k^{-1/10}$. We have

$$\begin{aligned} \beta - \alpha &\ll k^{-1/10}, & R &= k, \\ X &= 1, & U &= k^{-1/10}, \\ Y &= k^{1+\epsilon}, & Q &= 1. \end{aligned} \tag{2.6}$$

We get that $L_k(x) \ll_A k^{-1/10}(k^{-A/2} + k^{-9A/10})$. If $d_0 \neq h$, we integrate the t-integral by parts to get a factor h^{-2} and sum over h. Taking trivial bounds on the t-integral and the h-sum, we see that the contribution to Equation (2.4) is $O_A(k^{-A})$. We Tak trivial bounds to the rest of the sums. The contribution to $N_{\mathrm{av}}^{\mathrm{off}}(k,1)$ of the terms with $h = 0$ is negligible.

Finally, we consider the case $h = 0$. We want to estimate

$$\frac{1}{c}\int_{-\infty}^{\infty}\int_{-1/2}^{1/2} \frac{V(m,m+t,1)}{\sqrt{m(m+t)}} e\left((k-1)\theta - \frac{2\sqrt{m(m+t)}}{c}\sin(2\pi\theta)\right) d\theta\, dt.$$

We consider 3 different ranges for the couple of variables (θ, m). First, if θ is big, we can apply Lemma 5.5 to the t-integral and show that it is negligible. If θ is small and m is far from some fixed m_0, Lemma 5.5 applies to the θ-integral and shows that it is negligible. Finally, for small θ and m close to m_0, we show that the m-sum has an additional restriction and we estimate trivially the integral.

More precisely, let $\delta = 1/2 - 10\epsilon$. We consider the case $|\theta| \gg k^{-\delta}$. For this, we insert a cut-off function $v(k^\delta\theta)$ similarly to the above. We apply Lemma 5.5 to the t-integral. We have

$$\frac{d}{dt}\left[(k-1)\theta - \frac{2\sqrt{m(m+t)}}{c}\sin(2\pi\theta)\right] = \frac{1}{c}\sqrt{\frac{m}{m+t}}\sin(2\pi\theta) \gg k^{-\delta-\epsilon}.$$

Following notations of Section 5.3, we have the same constants as in Equation 2.5 except that $R = k^{-\delta-\epsilon}$. Recall that $m \gg k^{1-\epsilon}$. We get

$$\begin{aligned}
&\frac{1}{c}\int_{-\infty}^{\infty}\int_{k^{-\delta}\ll|\theta|\le 1/2} \frac{V(m,m+t,1)}{\sqrt{m(m+t)}} e\left((k-1)\theta - \frac{2\sqrt{m(m+t)}}{c}\sin(2\pi\theta)\right)\left(1 - v\left(k^\delta\theta\right)\right) d\theta\, dt \\
&\quad \ll mk^{-1/2-\epsilon}\frac{k}{m}\left((mk^{-\delta-\epsilon}k^{-1/2-\epsilon})^{-A} + (mk^{-\delta-\epsilon}/\sqrt{k})^{-A}\right) \\
&\quad \ll_A k^{-\epsilon A}.
\end{aligned}$$

Now, we consider the range $|\theta| \ll k^{-\delta}$. Let c, k, m be fixed. Suppose that

$$\left|\frac{(k-1)c}{4\pi m} - 1\right| \gg k^{-\delta}.$$

Then we can apply Lemma 5.5 to the θ-integral (recall that $t/m = o(k^{-\delta})$):

$$\begin{aligned}
\frac{d}{d\theta}\left[(k-1)\theta - \frac{2\sqrt{m(m+t)}}{c}\sin(2\pi\theta)\right] &= k - 1 - \frac{4\pi\sqrt{m(m+t)}}{c}\cos(2\pi\theta) \\
&= \left(\frac{(k-1)c}{4\pi m} - \frac{\sqrt{m(m+t)}}{m}\cos(2\pi\theta)\right)\frac{4\pi m}{c} \\
&= \left(\frac{(k-1)c}{4\pi m} - 1 + O\left(\frac{t}{m}\right) + O\left(k^{-2\delta}\right)\right)\frac{4\pi m}{c}.
\end{aligned}$$

The last expression in parenthesis is of size at least $k^{-\delta}$ since the two big O terms are smaller. Therefore, the derivative is always $\gg k^{-\delta} \cdot k^{1-\epsilon}$. We apply Lemma 5.5 to the θ-integral. We have the same constants as in Equation 2.6, except that $\beta - \alpha = k^{-\delta}$, $R = k^{1-\delta-\epsilon}$ and $U = k^{-\delta}$. We get

$$\begin{aligned}
&\frac{1}{c}\int_{-\infty}^{\infty}\int_{-1/2}^{1/2} \frac{V(m,m+t,1)}{\sqrt{m(m+t)}} e\left((k-1)\theta - \frac{2\sqrt{m(m+t)}}{c}\sin(2\pi\theta)\right) v\left(k^\delta\theta\right) d\theta\, dt \\
&\quad \ll k^{1/2+\epsilon}\cdot k^{\epsilon}\left((k^{1-\delta}/k^{1/2})^{-A} + (k^{1-2\delta-\epsilon})^{-A}\right) \\
&\quad \ll_A k^{-\epsilon A}. \qquad (2.7)
\end{aligned}$$

Applying trivial bounds to the other sums, we see that the two cases above are negligible.

Finally, if $|\theta| \ll k^{-\delta}$ and there is a m_0 such that

$$\frac{(k-1)c}{4\pi m_0} = 1 + O(k^{-\delta}). \tag{2.8}$$

This condition restricts the m-sum:

$$\frac{(k-1)c}{4\pi m} = \frac{(k-1)c}{4\pi m_0}\frac{m_0}{m} = 1 + \frac{m_0 - m}{m} + O(k^{-\delta}).$$

So $|m_0 - m| \ll m_0 k^{-\delta}$ for m still satisfying Equation 2.8. For a given m, we estimate trivially

$$\begin{aligned}\frac{1}{c}\int_{-\infty}^{\infty}\int_{|\theta|\ll k^{-\delta}} &\frac{V(m,m+t,1)}{\sqrt{m(m+t)}} e\left((k-1)\theta - \frac{2\sqrt{m(m+t)}}{c}\sin(2\pi\theta)\right) d\theta\, dt \\ &\ll k^{1/2+\epsilon}\cdot k^{-\delta}\cdot k^{\epsilon} \\ &\ll k^{1/2-\delta+2\epsilon}.\end{aligned}$$

For m not satisfying Equation 2.8, we can apply the bound in Equation 2.7. In total, we have a contribution of this term to $N_{\mathrm{av}}^{\mathrm{off}}(k,1)$ of size

$$\begin{aligned}\frac{1}{k}\sum_{|m-m_0|\ll k^{1/2+\epsilon}}\sum_{c\ll k^{\epsilon}}\sum_{d_0 \bmod c}\frac{S(m,m+d_0,c)}{c}\frac{1}{c}\int_{-\infty}^{\infty}\frac{V(m,m+t,1)}{\sqrt{m(m+t)}}J_{k-1}\left(\frac{4\pi\sqrt{m(m+t)}}{c}\right) dt \\ \ll k^{-1/2+\epsilon}.\end{aligned}$$

We conclude that

$$\begin{aligned}N_{\mathrm{av}}^{\mathrm{off}}(k,1) &= \frac{1}{\dim S_k(1)}\frac{1}{12\pi}\sum_{m,n}\frac{V(m,n,1)}{\sqrt{mn}}2\pi i^{-k}\sum_{c}\frac{S(m,n,c)}{c}J_{k-1}\left(\frac{4\pi\sqrt{mn}}{c}\right) \\ &\ll \frac{1}{k}\sum_{m\ll k^{1+\epsilon}}\sum_{c\ll k^{\epsilon}}\sum_{d_0 \bmod c}\frac{S(m,m+d_0,c)}{c} \\ &\quad\cdot\frac{1}{c}\sum_{h=-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{V(m,m+t,1)}{\sqrt{m(m+t)}}J_{k-1}\left(\frac{4\pi\sqrt{m(m+t)}}{c}\right)e\left(\frac{(d_0-t)h}{c}\right) dt \\ &\ll_{\epsilon} k^{-1/2+\epsilon}.\end{aligned}$$

This finishes the proof of Theorem 1.3 in the weight aspect. For the level aspect, we need to bound the contribution of the old forms. This is done in the next section.

2.5 Bound on old forms

In this section, we estimate the contribution of the old forms. More precisely, for prime level p and a modular form $g \in S_k(1)$, we have two lifts, $g(z)$ and $p^{k/2}g(pz)$. The two lifts are not orthogonal. We use the Gram-Schmidt process to get two orthogonal forms. Let $g \in S_k(1)$, $f_1(z) = g(z)$ and

$$f_2(z) = p^{k/2}g(pz) - \langle p^{k/2}g(pz), g(z)\rangle g(z).$$

Then the collection of f_1, f_2 for g in a Hecke basis $B_k(1)$ of $S_k(1)$ is an orthogonal basis of old forms.

Lemma 2.5 ([ILS], Lemma 2.4)**.** *The inner products between $g(z)$ and $p^{k/2}g(pz)$ have the following values:*

$$\langle g(z), p^{k/2}g(pz)\rangle = \lambda_g(p)\frac{\sqrt{p}}{p+1}\|g\|^2, \qquad \langle p^{k/2}g(pz), p^{k/2}g(pz)\rangle = \|g\|^2.$$

We conclude that $\|f_1\|^2 = \|g\|^2$ and

$$\|f_2\|^2 = \|p^{k/2}g(pz)\|^2 - \frac{\langle g(pz), g(z)\rangle^2}{\|g\|^2} = \|g\|^2\left(1 - \frac{\lambda_g(p)\sqrt{p}}{p+1}\right) \sim \|g\|.$$

We used that $|\lambda_f(p)| \ll 1$, thanks to Ramanujan conjecture, which is known in that case. Let

$$\tilde{N}(f) = \frac{p+1}{12\pi\|f\|^2}\Gamma(k-1)\sum_{m,n}\frac{a_f(m)a_f(n)}{\sqrt{mn}}V(m,n,p).$$

We want to bound

$$\tilde{N}_{\mathrm{av}}^{\mathrm{old}}(k,p) = \frac{1}{\dim S_k^{\mathrm{new}}(p)}\sum_{g\in B_k(1)}(\tilde{N}(f_1) + \tilde{N}(f_2)) \tag{2.9}$$

in the p aspect. We use Deligne's bound $a_g(n) \ll_{g,\epsilon} n^\epsilon$. Of course, $g \in B_k(1)$ does not depend on p. Therefore $\|g(z)\|^2 = [\mathrm{SL}_2(\mathbb{Z}) : \Gamma_0(p)]\|g\|^2_{\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}}$, where the last norm is taken over $\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}$ and is independent of p. For f_1, we compute

$$\begin{aligned}\tilde{N}(f_1) &= \frac{p+1}{12\pi\|f_1\|^2}\Gamma(k-1)\sum_{m,n}\frac{a_{f_1}(m)a_{f_1}(n)}{\sqrt{mn}}V(m,n,p)\\ &\ll_{k,\epsilon} \sum_{m\ll p^{1/2+\epsilon}k}\ \sum_{|m-n|\ll k^{-1/2+\epsilon}m}(mn)^{-1/2+\epsilon}\\ &\ll_{k,\epsilon} p^{1/2+2\epsilon}.\end{aligned}$$

For f_2, we have $\|f_2\|^2 \sim \|g\|^2$ and $a_{f_2}(n) \ll p^{1/2}a_g(n/p)\delta_{p|n} + a_g(n) \ll p^{1/2}n^\epsilon\delta_{p|n} + n^\epsilon$. The second term gives the same bound as above. In total, we have

$$\begin{aligned}\tilde{N}(f_2) &= \frac{p+1}{12\pi\|f_2\|^2}\Gamma(k-1)\sum_{m,n}\frac{a_{f_2}(m)a_{f_2}(n)}{\sqrt{mn}}V(m,n,p)\\ &\ll_{k,\epsilon} p\sum_{\substack{m\ll p^{1/2+\epsilon}k\\ p|m}}\ \sum_{\substack{|m-n|\ll k^{-1/2+\epsilon}m\\ p|n}}(mn)^{-1/2+\epsilon} + p^{1/2+\epsilon}\\ &\ll_{k,\epsilon} p^{1/2+2\epsilon}.\end{aligned}$$

We estimate the contribution of the old forms to be

$$\tilde{N}_{\mathrm{av}}^{\mathrm{old}}(k,p) = \frac{1}{\dim S_k^{\mathrm{new}}(p)}\sum_{g\in B_k(1)}(\tilde{N}(f_1) + \tilde{N}(f_2)) \ll_{k,\epsilon} \frac{1}{p}\cdot p^{1/2+\epsilon} \ll_k 1.$$

This concludes the proof of Theorem 1.3 in the level aspect.

Chapter 3

Siegel modular forms

In this chapter, we review the basis of the theory of Siegel modular forms. For a more general introduction, see for example [BGHZ, Kli, Fre]. In short, Siegel modular forms are a holomorphic generalization of modular forms where we replace complex numbers by complex matrices.

3.1 Definitions and first properties

Let n be an integer. We consider n by n matrices with complex coefficients $Z = X + iY \in M_n(\mathbb{C})$. If Y is symmetric and positive-definite, i.e. it has real positive eigenvalues, we write $Y > 0$. The *Siegel upper half-plane* is

$$\mathbb{H}^{(n)} := \{Z = X + iY \in \mathrm{M}_n(\mathbb{C}) \mid Z = Z^t,\ Y > 0\}.$$

There is an action of a $2n$ by $2n$ matrices group on the upper half-plane. The *symplectic group* is

$$\mathrm{Sp}_{2n}(\mathbb{R}) := \{M \in \mathrm{M}_{2n}(\mathbb{R}) \mid M^t J M = J\},$$

where $J = \left(\begin{smallmatrix} 0 & -I_n \\ I_n & 0 \end{smallmatrix}\right)$. Let $\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \mathrm{Sp}_{2n}(\mathbb{R})$ where A, B, C, D are n by n blocks and $Z \in \mathbb{H}^{(n)}$. We define

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \cdot Z := (AZ + B)(CZ + D)^{-1}.$$

Lemma 3.1. *Let $M = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \mathrm{Sp}_{2n}(\mathbb{R})$ and $Z = X + iY \in \mathbb{H}^{(n)}$. Then $\det(CZ + D) \neq 0$ and the imaginary part of $M \cdot Z$ is positive-definite. In other terms, the action above is well defined.*

We give a few more properties of $\mathrm{Sp}_{2n}(\mathbb{R})$ and $\mathrm{Sp}_{2n}(\mathbb{Z})$, the restriction of the group to integral matrices. First, writing $\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \mathrm{Sp}_{2n}(\mathbb{R})$ as above, we have the relationships

$$A^t C, B^t D \text{ symmetric}, \qquad A^t D - C^t B = I_n. \tag{3.1}$$

Actually, it is sufficient that a matrix satisfies the above equations for it to be symplectic. Second, the symplectic group over $\mathbb{R}$ is generated by the following families of matrices:

$$\begin{pmatrix} I_n & B \\ 0 & I_n \end{pmatrix},\ \begin{pmatrix} A^t & 0 \\ 0 & A^{-1} \end{pmatrix},\ J, \tag{3.2}$$

where B is symmetric and A is invertible. Finally, if $n = 1$, then $\mathrm{Sp}_2(\mathbb{R}) \cong \mathrm{SL}_2(\mathbb{R})$ so the upcoming definition of modular form is similar to the classical one in that case.

Let $f : \mathbb{H}^{(n)} \to \mathbb{C}$ a holomorphic function and $k \in \mathbb{Z}$. For $n \geq 2$, f is a *Siegel modular form* of *weight* k if

$$f\left(\begin{pmatrix} A & B \\ C & D \end{pmatrix} \cdot Z\right) = \det(CZ + D)^k f(Z)$$

for all $\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \mathrm{Sp}_{2n}(\mathbb{Z})$. Note that for $n \geq 2$, we don't give any condition at the cusps, thanks to Lemma 3.2 below.

A Siegel modular form f transforms in the following way under the action of the integral generators of Equation (3.2):

$$f\left(\begin{pmatrix} I_n & B \\ 0 & I_n \end{pmatrix} \cdot Z\right) = f(Z), \quad f\left(\begin{pmatrix} A^t & 0 \\ 0 & A^{-1} \end{pmatrix} \cdot Z\right) = \det(A)^{-k} f(Z), \quad f(J \cdot Z) = \det(Z)^k f(Z).$$

Choosing $A = -I_n$, we see that f vanishes if nk is odd. It is also possible to show that k must be non-negative and only the constant functions occur for $k = 0$. We see that f is periodic with respect to the abelian group given by the first generator in Equation (3.2) with integral B. Therefore, it admits a Fourier series of the form

$$f(Z) = \sum_T a(T) \det(T)^{\frac{k-3/2}{2}} \operatorname{etr}(TZ),$$

where $\operatorname{etr}(A) := e^{2\pi i \operatorname{tr}(A)}$. Here, T runs over half-integral symmetric matrices with integral entries. We have added a power of $\det(T)$ to the definition for normalization purposes. This moves the critical line of the associated L-function to $\frac{1}{2}$. From now on, we write ℓ for $k - 3/2$. The Fourier coefficients are given by the formula

$$a(T) = \det(T)^{-\ell/2} \int_{X \bmod 1} f(Z) \operatorname{etr}(-TZ) dX,$$

for $Z = X + iY$ and a fixed $Y > 0$. Here $dX = \prod_{1 \le i \le j \le n} x_{ij}$ is the Lebesgue measure.

Lemma 3.2 (Koecher's principle). *Let f be a Siegel modular form. Then for T not positive semi-definite, the Fourier coefficient $a(T) = 0$. Moreover f is bounded on domains of the form $\{X + iY \in \mathbb{H}^{(n)} \mid Y > cI_n\}$ for all $c > 0$.*

Proof. By contradiction, we suppose that there exists a T not positive semi-definite with $a(T) \neq 0$. Since T is not positive definite, there exist a vector v with $v^t T v < 0$. We can complete v to a unimodular matrix U. If we replace T by $U^t T U$, we can suppose that $t_{11} < 0$. Now consider the matrix

$$A = \begin{pmatrix} 1 & x & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix}.$$

Then $\operatorname{tr}(A^t T A) = t_{11} x^2 + O(x)$ and so $a(A^t T A) e^{-2\pi \operatorname{tr}(A^t T A)}$ diverges as $x \to \infty$ and so does the Fourier series at $Z = iI_n$. We conclude that $a(T) = 0$. ☐

The Koecher principle restricts the Fourier series to positive semi-definite $T \geq 0$. The function f is a *cusp form* if $a(T) = 0$ for $\det(T) = 0$ (these are the T such that $T \geq 0$ but $T \not> 0$). Given a unimodular matrix $U \in \mathrm{GL}_n(\mathbb{Z})$ with determinant ± 1, we easily see that

$$a(U^t T U) = \det(U)^k a(T).$$

There is an inner product on the space of cusp forms, given by

$$\langle f, g \rangle := \int_{\mathrm{Sp}_{2n}(\mathbb{Z}) \backslash \mathbb{H}^{(n)}} f(Z) \overline{g(Z)} \det(Y)^k \frac{dX dY}{\det(Y)^{n+1}}.$$

We define the L^2-norm of f as $\|f\|_2^2 = \langle f, f \rangle$.

Remark. We can introduce a level structure on Siegel modular forms. There are two congruence subgroups that are usually considered. The *parabolic subgroup* for $n = 2$ is

$$\tilde{\Gamma}_0^{(2)}(N) := \left\{ \begin{pmatrix} * & N* & * & * \\ * & * & * & */N \\ * & N* & * & * \\ N* & N* & N* & * \end{pmatrix} \in \mathrm{Sp}_4(\mathbb{Q}) \right\}$$

and the *Siegel subgroup* is

$$\Gamma_0^{(n)}(N) := \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{Sp}_{2n}(\mathbb{Z}) \mid C = 0 \pmod{p} \right\}.$$

We can define Siegel modular forms with respect to these subgroups in the usual way.

3.2 An isomorphism for the imaginary axis

We are interested in restricting Siegel modular forms to the imaginary axis. Let $n = 2$. There exists a morphism $\mathcal{P}(\mathbb{R}) \to \mathbb{H}$ between the imaginary axis and the upper half-plane. For a half-integral positive definite matrix $M \in \mathcal{P}(\mathbb{Z})$, it is given by sending M to the corresponding Heegner point z_M. This process loses the information on the determinant. We get an isomorphism $\mathcal{P}(\mathbb{R}) \cong \mathbb{H} \times \mathbb{R}_{>0}$ where the second component is the determinant. The converse direction goes as follows. First, we lift a point in $\mathbb{H}$ to the corresponding matrix in $\mathrm{SL}_2(\mathbb{R})$. The element in $\mathcal{P}(\mathbb{R})$ is given by the matrix times its transposed and multiplied by the determinant. More precisely,

$$\begin{aligned} \mathcal{P}(\mathbb{R}) &\xrightarrow{\sim} \mathbb{H} \times \mathbb{R}_{>0} \cong \mathrm{SO}(2)\backslash \mathrm{SL}_2(\mathbb{R}) \times \mathbb{R}_{>0}, \\ M = \begin{pmatrix} a & b \\ b & c \end{pmatrix} &\mapsto \left(\frac{-b + i\sqrt{\det(M)}}{a}, \det(M) \right), \\ r \cdot g^t g &\leftarrow\!\shortmid (\mathrm{SO}(2) \cdot g, r), \\ \sqrt{r} \begin{pmatrix} y^{-1} & -xy^{-1} \\ -xy^{-1} & y^{-1}(x^2 + y^2) \end{pmatrix} &\leftarrow\!\shortmid (x + iy, r). \end{aligned} \tag{3.3}$$

The last line is an explicit computation of the map. This isomorphism maps the measure $\frac{dY}{\det(Y)^{3/2}} \mapsto \left(\frac{dx\,dy}{y^2}, \frac{dr}{r} \right)$ and is compatible with the action of $\mathrm{SL}_2(\mathbb{Z})$ on both sides. If Y is in $\mathcal{P}(\mathbb{Z})$, the corresponding point $z_Y \in \mathbb{H}$ is called a *Heegner point*. Note that if Y is a reduced matrix, then z_Y is in the fundamental domain of $\mathbb{H}$. We summarize the discussion in the following table:

symmetric positive definite matrices	$\leftrightarrow$	positive-definite quadratic forms	$\leftrightarrow$	Heegner points (for integral forms)
$\{Y > 0\}$		$\{Q(x, y)\}$		$\mathbb{H} \times \mathbb{R}_{>0}$
$Y = \begin{pmatrix} \alpha & \beta \\ \beta & \delta \end{pmatrix}$		$Q_Y(x, y) = \alpha x^2 + 2\beta xy + \delta y^2$		$z_Y = \frac{-\beta + i\sqrt{\alpha\delta - \beta^2}}{\alpha}$, $r_Y = \alpha\delta - \beta^2$
$M^t Y M$		$Q(ax + b, cy + d)$		$\frac{az+b}{cz+d}$

Note that this isomorphism also generalizes to higher dimensions:

$$\begin{aligned} \mathrm{SO}(n)\backslash \mathrm{SL}_n(\mathbb{R}) \times \mathbb{R}_{>0} &\to \mathcal{P}^{(n)}(\mathbb{R}), \\ (\mathrm{SO}(n)g, r) &\mapsto r^{1/n} \cdot g^t g. \end{aligned}$$

3.3 Koecher-Maass series

In this section, we define two Dirichlet series associated to a Siegel cusp form $f \in S_k^{(n)}$. To this end, we consider the (shifted) Mellin transform

$$\Lambda(f, s) = \int_{Y>0} f(iY) \det(Y)^{\frac{k-1}{2}+s} \frac{dY}{\det(Y)^{(n+1)/2}}.$$

This converges for $\mathrm{Re}(s)$ large enough. The measure $\frac{dY}{\det(Y)^{(n+1)/2}}$ is invariant under multiplication by a scalar. There is an action by $\mathrm{SL}_2(\mathbb{R})$ on $\mathbb{H}^{(n)}$ via the embedding given by the second generator in Equation (3.2). It is given by $A \cdot z = A^t Z A$. Let $\mathcal{P}^{(n)}(\mathbb{R})$ be the set of symmetric positive-definite matrices in $M_n(\mathbb{R})$ and $\mathcal{P}^{(n)}(\mathbb{Z})$ be the restriction to integral coefficients. The following computations are valid for $\mathrm{Re}(s)$ large enough. This depends on a bound for $a(T)$, for example

$|a(T)| \ll_f \det(T)^k$ given in Lemma 12.1 of [Kli]. We have

$$\begin{aligned}\Lambda(f,s) &= \int_{Y>0} \sum_{T>0} a(T)\det(T)^{\ell/2} e^{-2\pi\,\mathrm{tr}(TY)} \det(Y)^{\frac{k-1}{2}+s} \frac{dY}{\det(Y)^{(n+1)/2}} \\ &= \sum_{T\in\mathcal{P}^{(n)}(\mathbb{Z})/\operatorname{PSL}_n(\mathbb{Z})} \frac{a(T)\det(T)^{\ell/2}}{\epsilon(T)} \int_{\operatorname{SL}_n(\mathbb{Z})\backslash\mathcal{P}^{(n)}(\mathbb{R})} e^{-2\pi\,\mathrm{tr}(TY)} \det(Y)^{\frac{k-1}{2}+s} \frac{dY}{\det(Y)^{(n+1)/2}} \\ &= \sum_{T\in\mathcal{P}^{(n)}(\mathbb{Z})/\operatorname{PSL}_n(\mathbb{Z})} \frac{a(T)}{\epsilon(T)\det(T)^{1/4+s}} \int_{\operatorname{SL}_n(\mathbb{Z})\backslash\mathcal{P}^{(n)}(\mathbb{R})} e^{-2\pi\,\mathrm{tr}(Y)} \det(Y)^{\frac{k-1}{2}+s} \frac{dY}{\det(Y)^{(n+1)/2}},\end{aligned}$$

where $\epsilon(T) = \#\{U \in \operatorname{PSL}_n(\mathbb{Z}) \mid U^tTU = u\}$. The last integral is computed in Lemma 6.2 of [Kli]. It is equal to

$$\begin{aligned}G_n(s) &:= \int_{\operatorname{SL}_n(\mathbb{Z})\backslash\mathcal{P}^{(n)}(\mathbb{R})} e^{-2\pi\,\mathrm{tr}(Y)} \det(Y)^{\frac{k-1}{2}+s} \frac{dY}{\det(Y)^{(n+1)/2}} \\ &= 2(2\pi)^{-ns} \prod_{r=1}^{n} \pi^{(r-1)/2} \Gamma\left(\frac{k-1}{2} + s - \frac{r-1}{2}\right).\end{aligned}$$

We define

$$L(f,s) = \sum_{T\in\mathcal{P}^{(n)}(\mathbb{Z})/\operatorname{PSL}_n(\mathbb{Z})} \frac{a(T)}{\epsilon(T)\det(T)^{1/4+s}}.$$

So $\Lambda(f,s) = L(f,s)G_n(s)$.

Proposition 3.3 ([Maa], Chapter 15)**.** *Let k be an even integer and $f \in S_k^{(n)}$. The function $L(f,s)$ has a holomorphic continuation to the whole complex plane. Moreover, it is bounded on vertical strips and it satisfies the functional equation*

$$\Lambda(f,s) = (-1)^{nk/2}\Lambda(f,1-s).$$

Kaori Imai, in her paper [Ima], gave another perspective. This was generalized by Rainer Weissauer in unpublished notes and revisited in [AMS]. For the rest of this chapter, we restrict ourselves to $n = 2$. We saw in the last section an isomorphism $\operatorname{SL}_2(\mathbb{Z})\backslash\mathcal{P}(\mathbb{Z}) \cong \operatorname{SL}_2(\mathbb{Z})\backslash\mathbb{H} \times \mathbb{R}_{>0}$. On the two terms of the right side, we have a spectral theory. Therefore, we can compute the double spectral inversion of a form $f \in S_k^{(2)}$, that is the Mellin transform and the spectral decomposition with respect to Λ. More precisely, let $iY \in \mathbb{H}^{(2)}$ be a matrix on the imaginary axis with $\det(Y) = 1$ and $f \in S_k^{(2)}$. We can see f as a function of $(z_Y, r) \in \mathbb{H} \times \mathbb{R}_{>0}$. We consider

$$\tilde{f}_s(z_Y) = \mathcal{M}(f)(z_Y, s) = \int_0^\infty f(ir^{1/2}Y) r^s \frac{dr}{r},$$

where $\mathcal{M}(f)$ is the Mellin transform with respect to the variable r. This is a function of $z_Y \in \mathbb{H}$ that is invariant under the action of $\operatorname{SL}_2(\mathbb{Z})$. It is natural to decompose it with respect to the spectral decomposition of $L^2(\operatorname{SL}_2(\mathbb{Z})\backslash\mathbb{H})$. Thus, for $\phi \in \Lambda$, i.e. a Mass cusp form, Eisenstein series or the constant function, we want to consider

$$\langle \tilde{f}_s, \phi\rangle = \langle \mathcal{M}(f)(\cdot, s), \phi\rangle.$$

We compute this inner product in Lemma 3.4. But first, we need to introduce some notations. We define the *twisted Koecher-Maass series*

$$L(f\times\phi, s) := \sum_{T\in\mathcal{P}(\mathbb{Z})/\operatorname{PSL}_2(\mathbb{Z})} \frac{a(T)}{\epsilon(T)\det(T)^{1/4+s}} \phi(z_T), \tag{3.4}$$

where $a(T)$ is the T-th Fourier coefficient of f and $\epsilon(T) = \#\{U \in \operatorname{PSL}_2(\mathbb{Z}) \mid U^tTU = T\}$ is the number of automorphisms of T. The corresponding gamma factor is

$$G(f\times\phi, s) = G(t_\phi, k, s) := 4(2\pi)^{-(k-1)-2s} \Gamma\left(\frac{\ell}{2} + s + \frac{it_\phi}{2}\right) \Gamma\left(\frac{\ell}{2} + s - \frac{it_\phi}{2}\right).$$

Recall that $\ell = k - 3/2$ and that t_ϕ is the spectral parameter of ϕ.

Lemma 3.4 ([Ima], Proposition 3.4, [Maa], pp. 85–94). *Let $f \in S_k^{(2)}$, $\phi \in \Lambda$ and $s \in \mathbb{C}$. If ϕ is even, we have*

$$\langle \mathcal{M}(f)(\cdot, (k-1)/2 + s), \phi \rangle = \frac{\sqrt{\pi}}{4} L(f \times \bar{\phi}, s) G(f \times \bar{\phi}, s).$$

If ϕ is odd, the inner product vanishes.

Proof. First, we rearrange the terms

$$\begin{aligned} \langle \mathcal{M}(f)(\cdot, (k-1)/2 + s), \phi \rangle &= \int_{\mathrm{PSL}_2(\mathbb{Z})\backslash \mathbb{H}} \int_0^\infty f(z, r) \overline{\phi(z)} r^{\frac{k-1}{2}+s} \frac{dr}{r} \, d\phi \\ &= \sum_{T \in \mathcal{P}(\mathbb{Z})} a(T) \det(T)^{\ell/2} \int_{\mathrm{PSL}_2(\mathbb{Z})\backslash \mathbb{H}} \int_0^\infty e^{-2\pi \operatorname{tr}(TY)} \overline{\phi(z)} r^{\frac{k-1}{2}+s} \frac{dr}{r} \, d\phi \\ &= \sum_{T \in \mathrm{PSL}_2(\mathbb{Z})\backslash \mathcal{P}(\mathbb{R})} \frac{a(T)}{\epsilon(T)} \det(T)^{\ell/2} \int_{\mathcal{P}(\mathbb{Z})} e^{-2\pi \operatorname{tr}(TY)} \overline{\phi(z)} \det(Y)^{\frac{k-1}{2}+s} \frac{dY}{\det(Y)^{3/2}}. \end{aligned}$$

The last integral was computed by Maass [Maa], pp. 85-94. He proved that

$$\int_{\mathcal{P}(\mathbb{R})} e^{-2\pi \operatorname{tr}(TY)} \overline{\phi(z)} \det(Y)^{\frac{k-1}{2}+s} \frac{dY}{\det(Y)^{3/2}} = \frac{\sqrt{\pi}(2\pi)^{-k+1-2s}}{\det(T)^{\frac{k-1}{2}+s}} \overline{\phi(z_T)} \prod_{\pm} \Gamma\left(\frac{\ell}{2} + s \pm \frac{it_u}{2}\right).$$

We obtain that the inner product is

$$\begin{aligned} &\sqrt{\pi} \sum_{T \in \mathrm{PSL}_2(\mathbb{Z})\backslash \mathcal{P}(\mathbb{R})} \frac{a(T)}{\epsilon(T) \det(T)^{-1/4+s}} \overline{\phi(z_T)} (2\pi)^{-k+1-2s} \prod_{\pm} \Gamma\left(\frac{\ell}{2} + s \pm \frac{it_u}{2}\right) \\ &\qquad = \frac{\sqrt{\pi}}{4} L(f \times \bar{\phi}, s) G(f \times \bar{\phi}, s). \end{aligned}$$

□

Proposition 3.5 ([Ima], Theorem 3.5). *The Dirichlet series $L(f \times \phi, s)$ extends to an entire function on $\mathbb{C}$ that is bounded in vertical strips and with the functional equation*

$$\Lambda(f \times \phi, s) := L(f \times \phi, s) G(f \times \phi, s) = L(f \times \phi, 1 - s) G(f \times \phi, 1 - s).$$

Remark. Note that this function does not have an Euler product and is not in the Selberg class.

3.4 The Kitaoka formula

The Petersson formula was generalized by Kitaoka to Siegel modular forms of degree 2 [Kit]. It is proved in the same way. We consider the Poincaré series

$$P_Q(Z) = \sum_{\gamma \in \Gamma_\infty \backslash \mathrm{Sp}_4(\mathbb{Z})} j(\gamma, Z)^{-k} \operatorname{etr}(Q\gamma Z) = \sum_{T \in \mathcal{P}(\mathbb{Z})} h_Q(T) \det(T)^{\ell/2} \operatorname{etr}(TZ),$$

where $j(\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right), Z) = \det(CZ + D)$. Klingen [Kli] showed that these functions satisfy

$$\langle f, P_Q \rangle = 8c_k \det(Q)^{-\ell/2} a_f(Q),$$

with $c_k = \frac{\sqrt{\pi}}{4}(4\pi)^{3-2k}\Gamma(k - 3/2)\Gamma(k-2)$. We compute $\langle P_T, P_Q \rangle$ in two different ways: first, using the above property and second, using an orthonormal basis $B_k^{(2)}$ of the space of cusp forms of weight k. We obtain

$$8c_k h_Q(T) \det(T)^{-\ell/2} = \langle P_Q, P_T \rangle = \sum_{f \in B_k^{(2)}} \langle P_Q, f \rangle \langle f, P_T \rangle = 64 c_k^2 \sum_{f \in B_k^{(2)}} \det(TQ)^{-\ell/2} \frac{a_f(T)\overline{a_f(Q)}}{\|f\|_2^2}.$$

The Fourier coefficients $h_Q(T)$ were computed by Kitaoka. We need to introduce a few notations. Let $C \in M_2(\mathbb{Z})$ with $\det(C) \neq 0$. The generalized Kloosterman sum of rank 2 is

$$K(Q,T;C) := \sum_{\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)} \operatorname{etr}(AC^{-1}Q + C^{-1}DT),$$

where the sum is over matrices $\left(\begin{smallmatrix} * & * \\ C & * \end{smallmatrix}\right) \in \mathrm{Sp}_{2n}(\mathbb{Z})$ in a system of representatives of $\Gamma_\infty \backslash \mathrm{Sp}_4(\mathbb{Z})/\Gamma_\infty$. This Kloosterman sum can be factorized with respect to elementary divisors of C. For $C = \left(\begin{smallmatrix} c_1 & 0 \\ 0 & c_1c_2 \end{smallmatrix}\right)$, a trivial upper bound is given by the number of representatives of the quotient, which gives

$$|K(Q,T;C)| \leq c_1^3 c_2 \leq |\det(C)|^{3/2}.$$

We also have square root cancellation for this sum. This was computed by Tóth in a mostly elementary way [Tót].

The rank 1 Kloosterman sum is defined in the following way. This setup comes from a family of representatives described using Smith normal form. We consider two integers c, s and matrices

$$U \in \left\{ \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \right\} \backslash \mathrm{GL}_2(\mathbb{Z}), \qquad V \in \mathrm{GL}_2(\mathbb{Z}) / \left\{ \begin{pmatrix} 1 & * \\ 0 & * \end{pmatrix} \right\} \tag{3.5}$$

and define

$$P = UQU^t = \begin{pmatrix} p_1 & p_2/2 \\ p_2/2 & p_4 \end{pmatrix}, \qquad S = V^{-1}TV^{-t} = \begin{pmatrix} s_1 & s_2/2 \\ s_2/2 & s_4 \end{pmatrix}.$$

Suppose that the bottom right entries of P and S are both equal to s. In that case, we define

$$H^{\pm}(P,S;c) := \delta_{p_4 = s_4} \sum_{d_1 \bmod c}^{*} \sum_{d_2 \bmod c} e\left(\frac{\bar{d}_1 s_4 d_2^2 \mp \bar{d}_1 p_2 d_2 + s_2 d_2 + \bar{d}_1 p_1 + d_1 s_1}{c} \mp \frac{p_2 s_2}{2cs_4} \right).$$

Here $\sum^*$ means that the sum is on d_1 coprime to c. We have the trivial bound $|H^{\pm}(P,S,c)| \leq c^2$.

Finally, the generalized Bessel function is defined in the following way. Let P be a diagonalizable matrix with eigenvalues equal to the squares of $s_1, s_2 > 0$. We define

$$\mathcal{J}_k(P) = \int_0^{\pi/2} J_k(4\pi s_1 \sin(\theta)) J_k(4\pi s_2 \sin(\theta)) \sin(\theta) d\theta,$$

where J_k is the Bessel function of the first kind. Combining everything, we get the following:

Theorem 3.6 ([Kit])**.** *Let $k \geq 6$ even, $B_k^{(2)}$ be an orthogonal basis for the space of Siegel modular forms of degree 2 and weight k. Then*

$$\begin{aligned} c_k \sum_{f \in \mathcal{S}_k^{(2)}} \frac{a_f(T)\overline{a_f(Q)}}{\|f\|_2^2} &= \delta_{Q \sim T} \# \operatorname{Aut}(T) \\ &+ \sum_{\pm} \sum_{c,s \geq 1} \sum_{U,V} \frac{(-1)^{k/2}\sqrt{2}\pi}{c^{3/2}s^{1/2}} H^{\pm}(UQU^t, V^{-1}TV^{-t}; c) J_\ell \left(\frac{4\pi\sqrt{\det(TQ)}}{cs} \right) \\ &+ 8\pi^2 \sum_{\det(C) \neq 0} \frac{K(Q,T;C)}{|\det(C)|^{3/2}} \mathcal{J}_\ell(TC^{-1}QC^{-t}) \end{aligned}$$

where U, V run over matrices in Equation (3.5)*.*

Remark. On the right-hand side, the three terms are called, in order, the diagonal term, the rank 1 and the rank 2 terms.

Remark. This was generalized in the level aspect to the Siegel subgroup by Chida, Katsurada and Matsumoto [CKM]. The only difference for $\Gamma_0^{(2)}(N)$ is that we restrict to $N|c$ and $N|\det(C)$ in the second and the third terms.

It is also possible to generalize the Kitaoka formula to higher degree. Unfortunately, here are only basic descriptions of the generalized Kloosterman sums and Bessel functions appearing. For the term of full rank, we get

$$\sum_{\substack{C\in \mathrm{M}_n(\mathbb{Z})\\ \det(C)\neq 0}} \frac{K^{(n)}(Q,T;C)}{|\det(C)|^k}\mathcal{J}_\ell^{(n)}(Q,T;C),$$

where

$$K^{(n)}(Q,T;C) := \sum_{\left(\begin{smallmatrix} A & * \\ C & D\end{smallmatrix}\right)\in\Gamma_\infty\backslash \mathrm{Sp}_{2n}(\mathbb{Z})/\Gamma_\infty} \mathrm{etr}(AC^{-1}Q + C^{-1}DT)$$

as above and

$$\mathcal{J}_k^{(n)}(Q,T;C) := \int_X \mathrm{etr}(-Z^{-1}C^{-1}QC^{-t} - ZT)\det(Z)^{-k}dX,$$

We discuss that topic in more detail in Chapter 10.

Chapter 4

First manipulations

In this chapter, we start the proof of Theorem 1.4. We prove Proposition 1.5, establish an approximate functional equation and insert the Kitaoka formula in $N_{\mathrm{av}}(K)$. We then prove decay properties of the cut-off function appearing in the approximate functional equation.

4.1 Spectral decomposition and Dirichlet series

We recall the discussion in Section 3.3. We consider the spectral decomposition of $L^2(\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H})$. Let Λ be a set of spectral components. For any $g \in L^2(\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H})$, we have the spectral decomposition $g(z) = \int_\Lambda \langle g, \phi\rangle \phi(z) d\phi$. We also write $\mathcal{M}(g)$ for the Mellin transform of g. We have the Parseval identities for this decomposition and for the Mellin transform:

$$\int_{\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}} |g(z)|^2 \frac{dx\,dy}{y^2} = \int_\Lambda |\langle g, \phi\rangle|^2 d\phi, \qquad \int_0^\infty |g(r)|^2 r^{2c} \frac{dr}{r} = \frac{1}{2\pi}\int_{-\infty}^{\infty} |\mathcal{M}(g)(c+it)|^2 dt,$$

with $c \in \mathbb{R}$ such that both sides of the equation make sense. We write $f(z, r)$ as a function of $z \in \mathbb{H}$ and $r > 0$. Let $\mathcal{M}(f)(z, r)$ denote the Mellin transform with respect to r. We apply the two Parseval identities. This gives

$$\begin{aligned}\int_{\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}} \int_0^\infty |f(z,r)|^2 r^k \frac{dr}{r} \frac{dx\,dy}{y^2} &= \int_{\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}} \frac{1}{2\pi}\int_{-\infty}^{\infty} |\mathcal{M}(f)(z, k/2+it)|^2 dt \frac{dx\,dy}{y^2} \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_\Lambda |\langle \mathcal{M}(f)(\cdot, k/2+it), \phi\rangle|^2 d\phi\, dt\end{aligned}$$

We consider the restriction norm

$$N(f) := \frac{\pi^2}{90\|f\|_2^2}\int_{\mathrm{SL}_2(\mathbb{Z})\backslash\mathcal{P}(\mathbb{R})} |f(iY)|^2 \det(Y)^k \frac{dY}{\det(Y)^{3/2}}.$$

Applying the Parseval identities, we get

$$N(f) = \frac{\pi^2}{90\|f\|_2^2}\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_\Lambda |\langle \mathcal{M}(f)(\cdot, k/2+it), \phi\rangle|^2 d\phi\, dt.$$

The Mellin transform can be computed explicitly. The Mellin transform of f is given in Proposition 3.4. The result is

$$\langle \mathcal{M}(f)(\cdot, (k-1)/2+s), \phi\rangle = \frac{\sqrt{\pi}}{4} L(f\times\bar{\phi}, s) G(f\times\bar{\phi}, s).$$

For odd ϕ, the scalar product inside the r-integral vanishes. For even Hecke-Maass cusp forms ϕ, we have $\bar{\phi} = \phi$ and for Eisenstein series, $|L(f\times\phi, s)| = |L(f\times\bar{\phi}, s)|$. Inserting this in the norm gives us

$$N(f) = \frac{\pi^2}{2880\|f\|_2^2}\int_{-\infty}^{\infty}\int_{\Lambda_{\mathrm{ev}}} |\Lambda(f\times\phi, 1/2+it)|^2 d\phi\, dt.$$

This concludes the proof of Proposition 1.5.

4.2 Approximate functional equation

Now, we want to evaluate the series $L(f\times\phi,s)$ on the critical line using its Dirichlet series. For this, we compute an approximate functional equation. Note that if f and ϕ are Hecke eigenfunctions, then $\overline{L(f\times\phi,s)} = L(f\times\phi,\bar{s})$ for cusp forms and the constant function and

$$\overline{L(f\times E(\cdot,1/2+i\tau),s)} = L(f\times E(\cdot,1/2-i\tau),\bar{s}) = \nu(1/2-i\tau)L(f\times E(\cdot,1/2+i\tau),\bar{s})$$

for Eisenstein series, where

$$\nu(s) = \pi^{1/2}\frac{\Gamma(s-1/2)}{\Gamma(s)}\frac{\zeta(2s-1)}{\zeta(2s)} = \frac{\pi^{-(1-s)}\Gamma(1-s)\zeta(2(1-s))}{\pi^{-s}\Gamma(s)\zeta(2s)}.$$

Let ν_ϕ be $\nu(1/2-i\tau)$ with ν as above if ϕ an Eisenstein series and 1 if ϕ is a cusp form or the constant function. Then $\nu_\phi\phi(z) = \bar{\phi}(z)$ and $\overline{L(f\times\phi,s)} = \nu_\phi L(f\times\phi,\bar{s})$. Consider

$$\begin{aligned} I(f\times\phi,s) &= \frac{1}{2\pi i}\int_{(3)} e^{v^2}\Lambda(f\times\phi,v+s)\Lambda(f\times\bar{\phi},v+1-s)\frac{dv}{v} \\ &= \frac{1}{2\pi i}\int_{(3)} e^{v^2}\nu_\phi\Lambda(f\times\phi,v+s)\Lambda(f\times\phi,v+1-s)\frac{dv}{v}. \end{aligned}$$

We take $s=1/2+it$. The integrand has no poles except for $v=0$ and decays rapidly at ∞. Moving the path of integration to $\mathrm{Re}(v)=-3$, we get

$$I(f\times\phi,1/2+it) = \frac{1}{2\pi i}\int_{(-3)} e^{v^2}\nu_\phi\Lambda(f\times\phi,v+1/2+it)\Lambda(f\times\phi,v+1/2-it)\frac{dv}{v} + |\Lambda(f\times\phi,1/2+it)|^2.$$

Using the functional equation of $\Lambda(f\times\phi,s)$, we get

$$\begin{aligned} &\frac{1}{2\pi i}\int_{(-3)} e^{v^2}\nu_\phi\Lambda(f\times\phi,v+1/2+it)\Lambda(f\times\phi,v+1/2-it)\frac{dv}{v} \\ &\quad= \frac{1}{2\pi i}\int_{(-3)} e^{v^2}\nu_\phi\Lambda(f\times\phi,-v+1/2-it)\Lambda(f\times\phi,-v+1/2+it)\frac{dv}{v} \\ &\quad= -\frac{1}{2\pi i}\int_{(3)} e^{v^2}\nu_\phi\Lambda(f\times\phi,v+1/2+it)\Lambda(f\times\phi,v+1/2-it)\frac{dv}{v} \\ &\quad= -I(f\times\phi,1/2+it). \end{aligned}$$

We conclude that $|\Lambda(f\times\phi,1/2+it)|^2 = 2I(f\times\phi,1/2+it)$. Now, we expand the Dirichlet series of $L(f\times\phi,s)$ at $s=v+\frac{1}{2}+it$:

$$\begin{aligned} I(f\times\phi,s) &= \frac{1}{2\pi i}\int_{(3)} e^{v^2}G(f\times\phi,v+1/2+it)G(f\times\phi,v+1/2-it) \\ &\quad\cdot\sum_{T,Q\in\mathcal{P}(\mathbb{Z})/\operatorname{PSL}_2(\mathbb{Z})}\frac{a(T)a(Q)}{\epsilon(T)\epsilon(Q)\det(TQ)^{1/4+v+1/2}}\left(\frac{\det(Q)}{\det(T)}\right)^{it}\phi(z_T)\bar{\phi}(z_Q)\frac{dv}{v} \\ &= \sum_{T,Q\in\mathcal{P}(\mathbb{Z})/\operatorname{PSL}_2(\mathbb{Z})}\frac{a(T)a(Q)}{\epsilon(T)\epsilon(Q)\det(TQ)^{3/4}}\phi(z_T)\bar{\phi}(z_Q) \\ &\quad\cdot\left(\frac{\det(Q)}{\det(T)}\right)^{it}\frac{1}{2\pi i}\int_{(3)} e^{v^2}G(f\times\phi,v+1/2+it)G(f\times\phi,v+1/2-it)\det(TQ)^{-v}\frac{dv}{v}. \end{aligned}$$

This gives for the norm

$$\begin{aligned} N(f) = \frac{\pi^2}{1440}&\sum_{T,Q\in\mathcal{P}(\mathbb{Z})/\operatorname{PSL}_2(\mathbb{Z})}\frac{1}{\epsilon(T)\epsilon(Q)\det(TQ)^{3/4}}\int_{\Lambda_{\mathrm{ev}}}\int_{-\infty}^{\infty}\left(\frac{\det(Q)}{\det(T)}\right)^{it} \\ &\cdot\frac{1}{2\pi i}\int_{(3)} e^{v^2}G(t_\phi,k,v+1/2+it)G(t_\phi,k,v+1/2-it)(x_1x_2)^{-v}\frac{dv}{v}\,dt\,\phi(z_T)\bar{\phi}(z_Q)d\phi \\ &\cdot\frac{a(T)\overline{a(Q)}}{\|f\|_2^2}. \end{aligned}$$

Note that only the last term depends on f (for a fixed k). We consider N_{av} as defined in Equation (1.3). This is amenable to the Kitaoka formula as stated in 3.6.

4.3 Cut-off

We define the function $V(x_1, x_2, \tau, k)$ as

$$\int_{-\infty}^{\infty} \left(\frac{x_2}{x_1}\right)^{it} \frac{1}{2\pi i} \int_{(3)} e^{v^2} c_k^{-1} G(\tau, k, v + 1/2 + it) G(\tau, k, v + 1/2 - it)(x_1 x_2)^{-v} \frac{dv}{v} dt. \tag{4.1}$$

With this definition, we rewrite the norm in a compact way:

$$N(f) = \frac{\pi^2}{1440} \sum_{T,Q \in \mathcal{P}(\mathbb{Z})/\operatorname{PSL}_2(\mathbb{Z})} \frac{1}{\epsilon(T)\epsilon(Q)\det(TQ)^{3/4}}$$
$$\cdot \int_{\Lambda_{\text{ev}}} V(\det(T), \det(Q), t_\phi, k)\phi(z_T)\bar{\phi}(z_Q) d\phi \frac{a(T)a(Q)}{\|f\|_2^2}. \tag{4.2}$$

Lemma 4.1. *Let $x_1, x_2 > 0$, $\tau \in \mathbb{C}$ such that $|\operatorname{Im}(\tau)| \le 2$, k large enough and $A > 0$. The function V satisfies the following bounds:*

$$\left(\frac{x_1}{\sqrt{k}}\right)^{j_1} \left(\frac{x_2}{\sqrt{k}}\right)^{j_2} k^{\frac{1}{2}j_3 + j_4} \frac{d^{j_1}}{dx_1^{j_1}} \frac{d^{j_2}}{dx_2^{j_2}} \frac{d^{j_3}}{d\tau^{j_3}} \frac{d^{j_4}}{dk^{j_4}} V(x_1, x_2, \tau, k)$$
$$\ll_{A,j_1,j_2,j_3,j_4} k^2 \left(1 + \frac{x_1 x_2}{k^4}\right)^{-A} \left(1 + k^{1/2}|\log(x_2/x_1)|\right)^{-A} \left(1 + \frac{|\tau|^2}{k}\right)^{-A}. \tag{4.3}$$

Remark. This lemma is similar to Equations (10.5) in [BC]. There is an error in the derivatives of x_1 and x_2. The integral over t add an extra $k^{1/2}$ term for each derivative. Note that the term is corrected when used later in Section 10.

Proof. We can bound G using the decay of the Γ function. We establish the relevant bounds in the next chapter. First, we consider the inner integral

$$V_1(x, t, \tau, k) = \frac{1}{2\pi i} \int_{(3)} e^{v^2} c_k^{-1} G(\tau, k, v + 1/2 + it) G(\tau, k, v + 1/2 - it) x^{-v} \frac{dv}{v}$$

where $x > 0$ and the other variables are as above. We prove

$$x^{j_1} k^{\frac{1}{2}j_2 + \frac{1}{2}j_3 + j_4} \frac{d^{j_1}}{dx^{j_1}} \frac{d^{j_2}}{dt^{j_2}} \frac{d^{j_3}}{d\tau^{j_3}} \frac{d^{j_4}}{dk^{j_4}} V_1(x, t, \tau, k) \ll_{A,j_1,j_2,j_3,j_4} k^{3/2} \left(1 + \frac{x}{k^4}\right)^{-A} \left(1 + \frac{t^2 + |\tau|^2}{k}\right)^{-A}. \tag{4.4}$$

This is similar to Equation (9.16) in [BC]. In Lemma 5.2, all the derivatives except the one in x are already treated. First, we move the v-integral to a large real part $\operatorname{Re}(v) = A$. Then we apply Lemma 5.2 and get

$$\begin{aligned}
x^j &\frac{d^j}{dx^j} V_1(x, t, \tau, k) \\
&= \frac{1}{2\pi i} \int_{(A)} e^{v^2} c_k^{-1} G(\tau, k, v + 1/2 + it) G(\tau, k, v + 1/2 - it) x^{-v} (-v) \dots (-v - j + 1) \frac{dv}{v} \\
&= \frac{1}{2\pi i} \int_{(A)} e^{v^2} G_A(k, t, \tau, v) x^{-v} (-v) \dots (-v - j + 1) \frac{dv}{v} + O_A(k^{-A}) \\
&\ll_A k^{3/2 + 4A} \frac{1}{2\pi i} \int_{(A)} e^{-|v|^2} \left(1 + \frac{t^2 + |\tau|^2 + \operatorname{Im}(v)^2}{k}\right)^{-A} x^{-A} |v| \dots |v + j - 1| \frac{dv}{|v|} + O_A(k^{-A}) \\
&\ll_{A,j} k^{3/2} \left(\frac{x}{k^4}\right)^{-A} \left(1 + \frac{t^2 + |\tau|^2}{k}\right)^{-A}.
\end{aligned}$$

Now, we move the v-integral to $\mathrm{Re}(v) = -\frac{1}{4}$. If $j = 0$, we pick up a pole at $v = 0$.

$$\begin{aligned}
&x^j \frac{d^j}{dx^j} V_1(x,t,\tau,k) \\
&\quad = \frac{1}{2\pi i} \int_{(-1/4)} e^{v^2} c_k^{-1} G(\tau,k,v+1/2+it) G(\tau,k,v+1/2-it) x^{-v} (-v) \dots (-v-j+1) \frac{dv}{v} \\
&\quad = \frac{1}{2\pi i} \int_{(-1/4)} e^{v^2} G_A(k,t,\tau,v) x^{-v} (-v) \dots (-v-j+1) \frac{dv}{v} + \delta_{j0} G_A(k,t,\tau,0) + O_A(k^{-A}) \\
&\quad \ll_A k^{1/2} \frac{1}{2\pi i} \int_{(-1/4)} e^{-|v|^2} \left(1 + \frac{t^2 + |\tau|^2 + \mathrm{Im}(v)^2}{k}\right)^{-A} x^{1/4} |v| \dots |v+j-1| \frac{dv}{|v|} \\
&\qquad + k^{1/2} \left(1 + \frac{t^2 + |\tau|^2 + \mathrm{Im}(v)^2}{k}\right)^{-A} + O_A(k^{-A}) \\
&\quad \ll_A k^{3/2} \left(\left(\frac{x}{k^4}\right)^{1/4} + \delta_{j0} \right) \left(1 + \frac{t^2 + |\tau|^2}{k}\right)^{-A}.
\end{aligned}$$

We conclude that Equation (4.4) holds. We consider now the t-integral. The only derivatives that we need to consider are the ones in x_1 and x_2. First, we integrate by parts.

$$\begin{aligned}
V(x_1,x_2,\tau,k) &= \int_{-\infty}^{\infty} \left(\frac{x_2}{x_1}\right)^{it} V_1(x_1x_2,t,\tau,k) dt \\
&= \int_{-\infty}^{\infty} (i \log(x_2/x_1))^{-j} \left(\frac{x_2}{x_1}\right)^{it} \frac{d^j}{dt^j} V_1(x_1x_2,t,\tau,k) dt \\
&\ll_{A,j} k^{3/2} (k^{1/2} |\log(x_2/x_1)|)^{-j} \left(1 + \frac{x_1x_2}{k^4}\right)^{-A} \int_{-\infty}^{\infty} \left(1 + \frac{t^2 + |\tau|^2}{k}\right)^{-A} dt \\
&\ll_{A,j} k^2 (k^{1/2} |\log(x_2/x_1)|)^{-j} \left(1 + \frac{x_1x_2}{k^4}\right)^{-A} \left(1 + \frac{|\tau|^2}{k}\right)^{-A}.
\end{aligned}$$

Considering $j = 0$ and $j = A$, we get the correct result for $j_1 = j_2 = 0$ in Equation (4.3). If we differentiate with respect to x_1 or x_2, we get an extra factor $\frac{1}{x_1}$ respectively $\frac{1}{x_2}$ and another factor of size either 1 or $k^{1/2}$. We conclude that the result holds. □

Chapter 5

Technical lemmas

We gather here various estimates and lemmas for the rest of the thesis. They come from Section 6 of [BC] and Section 8 of [BKY].

5.1 Gamma factors

Lemma 5.1 ([BC], Lemma 22)**.** *Let $k \geq 1$, $s = \sigma + it$ such that $k + \sigma \geq 1/2$ and $A \in \mathbb{N}$, $j_1, j_2 \in \mathbb{N}_0$. Then*

$$\frac{\Gamma(k+s)}{\Gamma(k)} = k^s G_{A,\sigma}(k,t) + O_{A,\sigma}((k+|t|)^{-A}),$$

where

$$k^{j_1+j_2/2} \frac{d^{j_1}}{dk^{j_1}} \frac{d^{j_2}}{dt^{j_2}} G_{A,\sigma}(k,t) \ll_{A,\sigma,j_1,j_2} \left(1 + \frac{t^2}{k}\right)^{-A}.$$

Moreover,

$$\frac{\Gamma(k+s)}{\Gamma(k)} = k^s \exp\left(-\frac{t^2}{2k}\right)\left(1 + O_\sigma\left(\frac{|t|}{k} + \frac{t^4}{k^3}\right)\right).$$

Lemma 5.2 ([BC], similar to Corollary 23)**.** *Let $A \geq 0$, $\sigma \geq -1/4$, $t \in \mathbb{R}$, $\tau \in \mathbb{C}$ such that $|\operatorname{Im}(\tau)| \leq 2$ and $k \in 2\mathbb{N}$. Then*

$$c_k^{-1/2} G(\tau, k, \sigma + 1/2 + it) \ll_{A,\sigma} k^{2\sigma+3/4}\left(1 + \frac{t^2 + |\tau|^2}{k}\right)^{-A}.$$

We also have, for $v \in \mathbb{C}$,

$$c_k^{-1} G(\tau, k, v + 1/2 + it) G(\tau, k, v + 1/2 - it) = G_A(k, t, \tau, v) + O_{\operatorname{Re}(v),A}(k^{-A}),$$

where

$$k^{j_1+j_2/2+j_3/2} \frac{d^{j_1}}{dk^{j_1}} \frac{d^{j_2}}{dt^{j_2}} \frac{d^{j_3}}{d\tau^{j_3}} G_A(k,t,\tau,v) \ll_{A,j_1,j_2,j_3,\operatorname{Re}(v)} k^{3/2+4\operatorname{Re}(v)}\left(1 + \frac{t^2 + |\tau|^2 + \operatorname{Im}(v)^2}{k}\right)^{-A}.$$

Moreover, for $t, \tau \ll k^{1/2+\epsilon}$, we have

$$c_k^{-1} G(\tau, k, 1/2 + it) G(\tau, k, 1/2 - it) = \frac{2}{\pi^{5/2}} k^{3/2} \exp\left(-\frac{4t^2 + \tau^2}{k}\right)\left(1 + O(k^{-1/2+\epsilon})\right).$$

Proof. Using the Legendre duplication formula, $\Gamma(z)\Gamma(z+1/2) = 2^{1-2z}\sqrt{\pi}\Gamma(2z)$, we get

$$\begin{aligned} c_k &= 2\sqrt{\pi}(4\pi)^{3-2k}\Gamma(k-3/2)\Gamma(k-2) \\ &= 2\sqrt{\pi}(4\pi)^{3-2k} 2^{2k-7/2-2}\pi^{-1}\Gamma\left(\frac{k-3/2}{2}\right)\Gamma\left(\frac{k-1/2}{2}\right)\Gamma\left(\frac{k-2}{2}\right)\Gamma\left(\frac{k-1}{2}\right) \\ &= 2^{-1}(2\pi)^{5/2-2k}\Gamma\left(\frac{k-3/2}{2}\right)\Gamma\left(\frac{k-1/2}{2}\right)\Gamma\left(\frac{k-2}{2}\right)\Gamma\left(\frac{k-1}{2}\right). \end{aligned}$$

Taking the square of the gamma factor, this gives

$$c_k^{-1}G(\tau,k,\sigma+1/2+it)^2 = \frac{2^4(2\pi)^{-2k-4\sigma-4it}}{2^{-1}(2\pi)^{5/2-2k}} \frac{\Gamma\left(\frac{k-1/2}{2}+\sigma+it+\frac{i\tau}{2}\right)^2\Gamma\left(\frac{k-1/2}{2}+\sigma+it-\frac{i\tau}{2}\right)^2}{\Gamma\left(\frac{k-3/2}{2}\right)\Gamma\left(\frac{k-1/2}{2}\right)\Gamma\left(\frac{k-2}{2}\right)\Gamma\left(\frac{k-1}{2}\right)}$$
$$= 2^5(2\pi)^{-4(\sigma+it)-5/2}\frac{\Gamma\left(\frac{k-1/2}{2}+\sigma+it+\frac{i\tau}{2}\right)^2\Gamma\left(\frac{k-1/2}{2}+\sigma+it-\frac{i\tau}{2}\right)^2}{\Gamma\left(\frac{k-3/2}{2}\right)\Gamma\left(\frac{k-1/2}{2}\right)\Gamma\left(\frac{k-2}{2}\right)\Gamma\left(\frac{k-1}{2}\right)}.$$

Applying Lemma 5.1, we get

$$\ll_{A,\sigma} (k/2)^{4\sigma+1/2+0+3/4+1/4}\left(1+\frac{(t+\tau)^2}{k}\right)^{-A}$$
$$\ll_{A,\sigma} k^{4\sigma+3/2}\left(1+\frac{t^2+|\tau|^2}{k}\right)^{-A}.$$

This gives the first formula. Similarly, for the second formula, we compute

$$c_k^{-1}G(\tau,k,v+1/2+it)G(\tau,k,v+1/2-it)$$
$$= 2^5(2\pi)^{-4v-5/2}\frac{\Gamma\left(\frac{k-1/2}{2}+v+it+\frac{i\tau}{2}\right)\Gamma\left(\frac{k-1/2}{2}+v+it-\frac{i\tau}{2}\right)}{\Gamma\left(\frac{k-3/2}{2}\right)\Gamma\left(\frac{k-1/2}{2}\right)}$$
$$\cdot\frac{\Gamma\left(\frac{k-1/2}{2}+v-it+\frac{i\tau}{2}\right)\Gamma\left(\frac{k-1/2}{2}+v-it-\frac{i\tau}{2}\right)}{\Gamma\left(\frac{k-2}{2}\right)\Gamma\left(\frac{k-1}{2}\right)}$$
$$= 2^5(2\pi)^{-4v-5/2}(k/2)^{4v+3/2}G_{A,\sigma}(k,t,\tau,\mathrm{Im}(v)) + O_{A,\sigma}\left((k+|t|+|\tau|)^{-A}\right),$$

where $G_{A,\sigma}$ is the combination of the functions $G_{A,\sigma}$ in Lemma 5.1 for the four ratios of gamma functions. We have the following properties for the $G_{A,\sigma}$ function:

$$k^{j_1+j_2/2+j_3/2}\frac{d^{j_1}}{dk^{j_1}}\frac{d^{j_2}}{dt^{j_2}}\frac{d^{j_3}}{d\tau^{j_3}}G_{A,\sigma}(k,t,\tau,\mathrm{Im}(v)) \ll_{A,\sigma,j_1,j_2,j_3} \left(1+\frac{\mathrm{Im}(v)^2+t^2+|\tau|^2}{k}\right)^{-A}.$$

We used that if $k \ll \mathrm{Im}(v)^2+t^2+|\tau|^2$, then $k \ll (\mathrm{Im}(v)\pm t\pm\tau)^2$ for one of the choices of signs. The last equation comes from the corresponding formula in Lemma 5.1. We get

$$c_k^{-1}G(\tau,k,1/2+it)G(\tau,k,1/2-it)$$
$$= 2^5(2\pi)^{-5/2}\frac{\Gamma\left(\frac{k-1/2}{2}+it+\frac{i\tau}{2}\right)\Gamma\left(\frac{k-1/2}{2}+it-\frac{i\tau}{2}\right)\Gamma\left(\frac{k-1/2}{2}-it+\frac{i\tau}{2}\right)\Gamma\left(\frac{k-1/2}{2}-it-\frac{i\tau}{2}\right)}{\Gamma\left(\frac{k-3/2}{2}\right)\Gamma\left(\frac{k-1/2}{2}\right)\Gamma\left(\frac{k-2}{2}\right)\Gamma\left(\frac{k-1}{2}\right)}$$
$$= 2^5(2\pi)^{-5/2}(k/2)^{3/2}\exp\left(-\frac{2(t+\tau/2)^2+2(t-\tau/2)^2}{k}\right)\left(1+O(k^{-1/2+\epsilon})\right)$$
$$= \frac{2k^{3/2}}{\pi^{5/2}}\exp\left(-\frac{4t^2+\tau^2}{k}\right)\left(1+O(k^{-1/2+\epsilon})\right).$$

□

5.2 The J Bessel function and the spectral integral

Concerning the J-Bessel function, we need the estimates

$$J_k(x) \ll 1, \qquad J_k(x) \ll \left(\frac{x}{k}\right)^k, \qquad J_k(x) \ll x^{-1/2}. \tag{5.1}$$

The first two are valid for $x > 0$ and $k > 2$ and the last one for $x \geq 2k$ as stated in Equations (4.1), (4.2) and (4.3) in [Blo]. They can be deduced from Equations 8.411.13, 8.411.4 and 8.451.1 in [GR]. Moreover, Equation (4.7) in [Blo], which is a correction of Equation 2.12.20 in [PBM], says that the product of two Bessel functions can be rewritten in the following way:

$$\begin{aligned} & J_k(4\pi s_1 \sin(\alpha)) J_k(4\pi s_2 \sin(\alpha)) \\ &= \frac{1}{\pi} \operatorname{Re}\left(e\left(-\frac{k+1}{4}\right) \int_0^\infty e\left((s_1^2+s_2^2)t + \frac{\sin(\alpha)^2}{t}\right) J_k(4\pi s_1 s_2 t) \frac{dt}{t}\right). \end{aligned} \tag{5.2}$$

The following lemma is used to take advantage of the average over k:

Lemma 5.3 ([BC], Lemma 20 and remark afterward)**.** *Let $x > 0$, $A > 0$, $K > 1$, $w : \mathbb{R} \to \mathbb{C}$ smooth with support in $[1, 2]$, such that $w^{(j)}(x) \ll_\epsilon K^{j\epsilon}$. Then there exist smooth functions w_0, w_- and w_+ such that, for all $j \in \mathbb{N}_0$, we have*

$$\sum_{k \text{ even}} i^k w\left(\frac{k}{K}\right) J_{k-3/2}(x) = w_0(x) + e^{ix} w_+(x) + e^{-ix} w_-(x)$$

and

$$\begin{aligned} w_0(x) &\ll_A K^{-A}, \\ \frac{d^j}{dx^j} w_\pm(x) &\ll_{j,A} \left(1 + \frac{K^2}{x}\right)^{-A} \frac{1}{x^j} \end{aligned}$$

If $x \geq 2K$, we also have $w_0, w_\pm \ll K/\sqrt{x}$. Moreover, if w depends on other parameters with control over the derivatives, so do $w_\pm$.

We state a general upper bound for the spectral integral.

Lemma 5.4. *Let $z_1, z_2 \in \mathbb{H}$ with $\operatorname{Im}(z_1), \operatorname{Im}(z_2) \gg T$ and $T \geq 1$. Then*

$$\int_{\substack{\Lambda_{\text{ev}} \\ |t_\phi| \ll T}} |\phi(z_1)\phi(z_2)| d\phi \ll_A T\sqrt{\operatorname{Im}(z_1)\operatorname{Im}(z_2)}.$$

Proof. We apply the Cauchy-Schwarz inequality. We get

$$\begin{aligned} \int_{\substack{\Lambda_{\text{ev}} \\ |t_\phi| \ll T}} |\phi(z_1)\phi(z_2)| d\phi &\ll \int_{\substack{\Lambda \\ |t_\phi| \ll T}} |\phi(z_1)\phi(z_2)| d\phi \\ &\ll \left(\int_{\substack{\Lambda \\ |t_\phi| \ll T}} |\phi(z_1)|^2 d\phi\right)^{1/2} \left(\int_{\substack{\Lambda \\ |t_\phi| \ll T}} |\phi(z_2)|^2 d\phi\right)^{1/2}. \end{aligned}$$

We bound the two terms with Proposition 15.8 in [IK]. The hypothesis give

$$\begin{aligned} &\ll \left(T^2 + T\operatorname{Im}(z_1)\right)^{1/2} \left(T^2 + T\operatorname{Im}(z_2)\right)^{1/2} \\ &\ll T\sqrt{\operatorname{Im}(z_1)\operatorname{Im}(z_2)}. \end{aligned}$$

□

5.3 Stationary phase

We state in this section two lemma from [BKY] about estimates on oscillating integral. Let w be a smooth function with support on $[\alpha, \beta]$ and h be a smooth function on $[\alpha, \beta]$. We want to bound the integral

$$I = \int_{-\infty}^{\infty} w(t) e^{ih(t)} dt.$$

This depends on the vanishing of h' in the interval $[\alpha, \beta]$.

Lemma 5.5 ([BKY], Lemma 8.1). *Let* $Y \geq 1$, $X, U, R, Q > 0$. *Suppose that*

$$\begin{aligned} w^{(j)}(t) &\ll_j XU^{-j}, && \text{for } j = 1, 2, \dots \\ |h'(t)| &\geq R, && \\ h^{(j)}(t) &\ll_j YQ^{-j}, && \text{for } j = 2, 3, \dots \end{aligned}$$

Then

$$I \ll_A (\beta - \alpha) X [(QR/\sqrt{Y})^{-A} + (RU)^{-A}].$$

Lemma 5.6 ([BKY], Proposition 8.2). *Let* $0 < \delta < 1/10$, $X, U, Y, Q > 0$, $Z = Q + X + Y + \beta - \alpha + 1$ *be such that*

$$Y \geq Z^{3\delta}, \quad \beta - \alpha \geq U \geq \frac{QZ^{\delta/2}}{\sqrt{Y}}.$$

Suppose that

$$\begin{aligned} w^{(j)}(t) &\ll_j XU^{-j}, && \text{for } j = 0, 1, \dots \\ h''(t) &\gg YQ^{-2}, && \\ h^{(j)}(t) &\ll_j YQ^{-j}, && \text{for } j = 1, 2, \dots \end{aligned}$$

and that there exists a unique $t_0 \in [\alpha, \beta]$ *such that* $h'(t_0) = 0$. *Then*

$$I \ll \frac{QX}{\sqrt{Y}}.$$

Chapter 6

Diagonal term

In this chapter, we compute the diagonal term of the Kitaoka formula, where $T \sim Q$. For this, we combine Equations (1.3) and (4.2) with the Kitaoka formula. We have in particular that $\det(T) = \det(Q)$ and $\epsilon(T) = \epsilon(Q)$. The equation for the diagonal term simplifies to

$$N_{\text{av}}^{\text{diag}}(K) = \frac{12\pi^2}{\omega K^4} \sum_{k \in 2\mathbb{N}} w\left(\frac{k}{K}\right) \sum_{T \in \mathcal{P}(\mathbb{Z}) / \operatorname{PSL}_2(\mathbb{Z})} \frac{\# \operatorname{Aut}(T)}{\epsilon(T)^2 \det(T)^{3/2}} \cdot \int_{\Lambda_{\text{ev}}} V(\det(T), \det(T), t_\phi, k) \left|\phi(z_T)\right|^2 d\phi.$$

We deal with this expression in the following steps. First, we analyze the spectral integral using the pre-trace formula. This requires a non-trivial argument and takes up an important part of this chapter. In particular, we need to count Heegner points close to the edge of the fundamental domain and take care of the restriction to the even spectrum. After that, the rest of the summations and estimations are handled. We finish by the computation of the average over k.

We fix some notations for this chapter. Let T be a reduced positive-definite matrix of determinant D. It corresponds to a Heegner z_T in the fundamental domain via Equation (3.3). We use the notation

$$T = \begin{pmatrix} \alpha & \beta \\ \beta & \delta \end{pmatrix} \longleftrightarrow z_T = \frac{-\beta + i\sqrt{D}}{\alpha}.$$

6.1 The pre-trace formula

First, we do not consider the even spectrum. We apply the pre-trace formula (see for example [Iwa], Section 10.1). It states that

$$\int_\Lambda V(\det(T), \det(T), t_\phi, k) \left|\phi(z_T)\right|^2 d\phi = \sum_{\gamma \in \operatorname{SL}_2(\mathbb{Z})} \kappa(u(z_T, \gamma z_T)).$$

The function κ is the Harish-Chandra inverse of V and it only depends on the point pair invariant $u(z_1, z_2) = \frac{|z_1 - z_2|^2}{4 \operatorname{Im}(z_1) \operatorname{Im}(z_2)}$. We keep these notations in this chapter. Recall that T is reduced, so $2|\beta| \leq \alpha \leq \delta$ and the decay properties of V give $D \ll k^{2+\epsilon}$, up to a negligible error. In particular, z_T is in the classical fundamental domain of $\Gamma \backslash \mathbb{H}$. For the edge of the domain, we pick the pieces with $\operatorname{Re}(z) \leq 0$. The goal of the following subsections is to prove the following theorem.

Theorem 6.1. *For all $\epsilon > 0$, we have*

$$\sum_{T \in \mathcal{P}(\mathbb{Z}) / \operatorname{PSL}_2(\mathbb{Z})} \frac{\# \operatorname{Aut}(T)}{\epsilon(T)^2 \det(T)^{3/2}} \int_\Lambda V(\det(T), \det(T), \tau, k) \left|\phi(z_T)\right|^2 d\phi = \sum_{T \in \mathcal{P}(\mathbb{Z}) / \operatorname{PSL}_2(\mathbb{Z})} \frac{2\# \operatorname{Aut}(T)}{\epsilon(T) \det(T)^{3/2}} \kappa(0) + O_\epsilon(k^{2.5+\epsilon}), \tag{6.1}$$

If we reduce to the even spectrum, we get

$$\sum_{T\in\mathcal{P}(\mathbb{Z})/\operatorname{PSL}_2(\mathbb{Z})} \frac{\#\operatorname{Aut}(T)}{\epsilon(T)^2\det(T)^{3/2}}\int_{\Lambda_{\mathrm{ev}}} V(\det(T),\det(T),\tau,k)\,|\phi(z_T)|^2\,d\phi$$
$$=\sum_{T\in\mathcal{P}(\mathbb{Z})/\operatorname{PSL}_2(\mathbb{Z})}\left(\frac{\#\operatorname{Aut}(T)}{\epsilon(T)}\right)^2\frac{1}{2\det(T)^{3/2}}\kappa(0)+O_\epsilon(k^{2.5+\epsilon}),$$

Remark. We will see later that the term with $\kappa(0)$ is of size $k^3\log(k)$. The two equations tell us that, on average over T, the terms on the spectral side with $u\neq 0$ are of lower size.

To get κ, we want to compute the Harish-Chandra inverse transform of the function $h(\tau) = V(\det(T),\det(T),\tau,k)$. We know that this function of τ is even, decays exponentially and is holomorphic in the strip $|\operatorname{Im}(\tau)|\leq 2$. Therefore it is suitable for the Harish-Chandra inversion. A first way to get κ in terms of h is given by Equation (1.62') in [Iwa]:

$$\kappa(u)=\frac{1}{4\pi}\int_{-\infty}^{\infty}\frac{1}{\pi}\int_0^{\pi}(2u+1+2\sqrt{u(u+1)}\cos(\theta))^{-\frac{1}{2}-i\tau}d\theta\ h(\tau)\tau\tanh(\pi\tau)d\tau.$$

At $u=0$, we get

$$\kappa(0)=\frac{1}{4\pi}\int_{-\infty}^{\infty}V(\det(T),\det(T),\tau,k)\tau\tanh(\pi\tau)d\tau. \tag{6.2}$$

For $u\ll 1$, the θ-integral is of size $\ll 1$ because $2u+1-2\sqrt{u(u+1)}$ is bounded away from 0. Using the cut-off of V given in Equation (4.3) and $\tau\tanh(\tau)=|\tau|+O(1)$, we get a trivial bound

$$\kappa(u)\ll_A k^2\left(1+\frac{\det(T)^2}{k^4}\right)^{-A}\int_{-\infty}^{\infty}|\tau|\left(1+\frac{|\tau|^2}{k}\right)^{-A}d\tau\ll_A k^3\left(1+\frac{\det(T)^2}{k^4}\right)^{-A}. \tag{6.3}$$

We also need the following lemma.

Lemma 6.2 ([Iwa], Lemma 2.11)**.** *Let $z\in\mathbb{H}$ with $\operatorname{Im}(z)\geq 1/10$ and $X>0$. We have*

$$\#\{\gamma\in\operatorname{SL}_2(\mathbb{Z})\mid u(z,\gamma z)<X\}\ll\sqrt{X(X+1)}\operatorname{Im}(z)+X+1,$$
$$\#\{\gamma\in\operatorname{SL}_2(\mathbb{Z})\mid u(z,\gamma(-\bar{z}))<X\}\ll\sqrt{X(X+1)}\operatorname{Im}(z)+X+1.$$

Remark. Note that in our case, $\operatorname{Im}(z_T)=\frac{\sqrt{D}}{\alpha}\ll k^{1+\epsilon}$ up to a negligible error.

6.2 The full spectrum

We do not restrict to the even spectrum at first. In this section, we prove a strong decay bound for $\kappa(u)$ when u is large enough.

Lemma 6.3. *Let $\epsilon>0$, $A>0$, $T\in\mathcal{P}(\mathbb{Z})$ with $\det(T)\ll k^{2+\epsilon}$ and $z_T\in\mathbb{H}$ the Heegner point corresponding to T via Equation (3.3). Then*

$$\sum_{\substack{\gamma\in\operatorname{SL}_2(\mathbb{Z})\\ u(z_T,\gamma z_T)\geq k^{-1+\epsilon}}}|\kappa(u(z_T,\gamma z_T))|\ll_{A,\epsilon}k^{-A}.$$

Proof. We apply the usual three steps to get the Harish-Chandra inverse transform (see (1.64) in [Iwa]). This gives

$$g(r)=\frac{1}{2\pi}\int_{-\infty}^{\infty}V(\det(T),\det(T),\tau,k)e^{ir\tau}d\tau,$$
$$q(v)=\frac{1}{4\pi}\int_{-\infty}^{\infty}V(\det(T),\det(T),\tau,k)(\sqrt{v+1}+\sqrt{v})^{2i\tau}d\tau,$$
$$\kappa(u)=\frac{1}{4\pi^2 i}\int_u^{\infty}\frac{1}{\sqrt{v-u}}\int_{-\infty}^{\infty}V(\det(T),\det(T),\tau,k)\frac{(\sqrt{v+1}+\sqrt{v})^{2i\tau}}{\sqrt{v(v+1)}}\tau d\tau\,dv.$$

We recall the decay property of V with respect to τ, as written in Equation (4.3):

$$\frac{d^j}{d\tau^j}V(\det(T),\det(T),\tau,k) \ll_{A,j} k^{2-j/2}\left(1+\frac{\det(T)^2}{k^4}\right)^{-A}\left(1+\frac{|\tau|^2}{k}\right)^{-A}.$$

Let $h(\tau)=V(\det(T),\det(T),\tau,k)$. We consider first $q(v)$. Since h is holomorphic in a strip, we can move the integration line to $\tau\mapsto\tau+2i$:

$$\int_{-\infty}^{\infty} h(\tau)\tau(\sqrt{v+1}+\sqrt{v})^{2i\tau}d\tau = (\sqrt{v+1}+\sqrt{v})^{-4}\int_{-\infty}^{\infty} h(\tau+2i)(\tau+2i)(\sqrt{v+1}+\sqrt{v})^{2i\tau}d\tau.$$

Integrating by parts, we get

$$\begin{aligned}4\pi q(v) &= (\sqrt{v+1}+\sqrt{v})^{-4}\int_{-\infty}^{\infty} h(\tau+2i)(\tau+2i)(\sqrt{v+1}+\sqrt{v})^{2i\tau}d\tau\\ &= (\sqrt{v+1}+\sqrt{v})^{-4}(-2i\log(\sqrt{v+1}+\sqrt{v}))^{-1}\\ &\quad\cdot\int_{-\infty}^{\infty}(h'(\tau+2i)(\tau+2i)+h(\tau+2i))(\sqrt{v+1}+\sqrt{v})^{2i\tau}d\tau\\ &= (\sqrt{v+1}+\sqrt{v})^{-4}(-2i\log(\sqrt{v+1}+\sqrt{v}))^{-j}\\ &\quad\cdot\int_{-\infty}^{\infty}(h^{(j)}(\tau+2i)(\tau+2i)+jh^{(j-1)}(\tau+2i))(\sqrt{v+1}+\sqrt{v})^{2i\tau}d\tau\\ &\ll_{A,j} (\sqrt{v+1}+\sqrt{v})^{-4}\left(\log(\sqrt{v+1}+\sqrt{v})\sqrt{k}\right)^{-j}k^2(k+j\sqrt{k})\left(1+\frac{\det(T)^2}{k^4}\right)^{-A}.\end{aligned}$$

In particular, we have a saving in k if $\log(\sqrt{v+1}+\sqrt{v})\gg k^{-1/2+\epsilon/2}$. Since $\log(\sqrt{v+1}+\sqrt{v})=\sqrt{v}+O(v^{3/2})$ for small v, this happens if v or u is $\gg k^{-1+\epsilon}$. We obtain

$$q(v)\ll_{A,j}(\sqrt{v+1}+\sqrt{v})^{-4}k^{3-j\epsilon/2}\left(1+\frac{\det(T)^2}{k^4}\right)^{-A}.$$

Then

$$\kappa(u)\ll_{A,j} k^{3-j\epsilon/2}\left(1+\frac{\det(T)^2}{k^4}\right)^{-A}\int_u^{\infty}\frac{dv}{\sqrt{v(v+1)(v-u)}(\sqrt{v+1}+\sqrt{v})^4}.$$

We split the integral in the intervals $]u,u+1[$ and $[u+1,\infty[$. We get

$$\begin{aligned}\int_u^{\infty}\frac{dv}{\sqrt{v(v+1)(v-u)}(\sqrt{v+1}+\sqrt{v})^4} &\ll \frac{1}{\sqrt{u(u+1)}(\sqrt{u+1}+\sqrt{u})^4}\int_u^{u+1}\frac{dv}{\sqrt{v-u}}+\int_{u+1}^{\infty}\frac{dv}{v^3}\\ &= \frac{2}{\sqrt{u(u+1)}(\sqrt{u+1}+\sqrt{u})^4}+\frac{1}{2(u+1)^2}.\end{aligned}$$

If $u\gg 1$, then we obtain

$$\frac{2}{\sqrt{u(u+1)}(\sqrt{u+1}+\sqrt{u})^4}+\frac{1}{2(u+1)^2}\ll\frac{1}{u^2}.$$

If $u\ll 1$, then we have $1+u\asymp 1$ and

$$\frac{2}{\sqrt{u(u+1)}(\sqrt{u+1}+\sqrt{u})^4}+\frac{1}{2(u+1)^2}\ll\frac{1}{\sqrt{u}}.$$

In summary, for $u\gg k^{-1+\epsilon}$, we computed

$$\begin{aligned}\kappa(u) &\ll_{A,j} k^{3-j\epsilon/2}\left(1+\frac{\det(T)^2}{k^4}\right)^{-A}\frac{1}{u^2} && \text{if } u\gg 1,\\ \kappa(u) &\ll_{A,j} k^{3-j\epsilon/2}\left(1+\frac{\det(T)^2}{k^4}\right)^{-A}\frac{1}{\sqrt{u}} && \text{if } u\ll 1.\end{aligned}$$

Applying Lemma 6.2, we sum over γ. For $k^{-1+\epsilon} \ll u_T \leq 1$, we have

$$\sum_{\substack{\gamma \in \mathrm{SL}_2(\mathbb{Z}) \\ k^{-1+\epsilon} \ll u(z_T, \gamma z_T) \leq 1}} |\kappa(u(z_T, \gamma z_T))| \ll_{A,j} k^{4.5 - j\epsilon/2} \left(1 + \frac{\det(T)^2}{k^4}\right)^{-A}.$$

So for j large enough, we can cancel all the powers of k. For $u \geq 1$, we split into dyadic intervals. For $X \geq 1$, we have $\sqrt{X(X+1)}\,\mathrm{Im}(z) + X + 1 \ll X k^{1+\epsilon}$. We get

$$\begin{aligned}
\sum_{\substack{\gamma \in \mathrm{SL}_2(\mathbb{Z}) \\ u(z_T, \gamma z_T) \geq 1}} |\kappa(u(z_T, \gamma z_T))| &= \sum_{n=0}^{\infty} \sum_{u \in [2^n, 2^{n+1}[} \kappa(u) \\
&\ll \sum_{n=0}^{\infty} 2^n k^{1+\epsilon} \kappa(2^n) \\
&\ll_{A,j} \sum_{n=0}^{\infty} k^{4+\epsilon - j\epsilon/2} \left(1 + \frac{\det(T)^2}{k^4}\right)^{-A} 2^{-n} \\
&\ll_{A,j} k^{4+\epsilon - j\epsilon/2} \left(1 + \frac{\det(T)^2}{k^4}\right)^{-A}.
\end{aligned}$$

We take j large enough to conclude the proof. □

Now, note that for a Heegner point $z = \frac{-\beta + i\sqrt{D}}{\alpha}$, $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ and $z \neq \gamma z$, we have

$$\begin{aligned}
u(z, \gamma z) =& \frac{|z - \gamma z|^2}{4\,\mathrm{Im}(z)\,\mathrm{Im}(\gamma z)} = \frac{|z(cz+d) - (az+b)|^2}{4\,\mathrm{Im}(z)^2} \\
=& \frac{(c\beta^2 - cD - (d-a)\alpha\beta - b\alpha^2)^2 + (-2c\beta\sqrt{D} + (d-a)\alpha\sqrt{D})^2}{4\alpha^2 D}.
\end{aligned}$$

In the last line, if the second square is non-zero, we get $u(z, \gamma z) \gg \frac{D}{\alpha^2 D} \gg \frac{1}{\alpha^2}$. If it is zero, then $2c\beta = (d-a)\alpha$. Moreover, the first square is non-zero, since $z \neq \gamma z$. The first square simplifies then to $(c\beta^2 - cD - (d-a)\alpha\beta - b\alpha^2)^2 = (-c(\beta^2 + D) - b\alpha^2)^2 = \alpha^2(-c\delta - b\alpha)^2$. Thus $u(z, \gamma z) \gg \frac{\alpha^2}{\alpha^2 D} \gg \frac{1}{D}$ in that case. Up to a negligible error, we get that for $z_T \neq \gamma z_T$,

$$u(z_T, \gamma z_T) \gg \min\left(\frac{1}{\alpha^2}, \frac{1}{D}\right) \gg k^{-2-\epsilon}.$$

It remains to deal with the z_T and γ such that $k^{-2-\epsilon} \ll u \ll k^{-1+\epsilon}$. This gives the error term in Equation (6.1). Applying Equation (6.3) and Lemma 6.2, we see that

$$\begin{aligned}
\sum_{\substack{T \in \mathcal{P}(\mathbb{Z}) / \mathrm{PSL}_2(\mathbb{Z}) \\ \det(T) \ll k^{2+\epsilon}}} \frac{\#\,\mathrm{Aut}(T)}{\epsilon(T)^2 \det(T)^{3/2}} & \sum_{\substack{\gamma \in \mathrm{SL}_2(\mathbb{Z}): \\ k^{-2-\epsilon} \ll u(z_T, \gamma z_T) \ll k^{-1+\epsilon}}} |\kappa(u(z_T, \gamma z_T))| \\
&\ll \sum_{\alpha\delta \ll k^{2+\epsilon}} \sum_{|\beta| \ll \alpha} \frac{1}{\det(T)^{3/2}} k^3 \cdot k^{-1/2+\epsilon} \frac{\sqrt{\det(T)}}{\alpha} \\
&\ll k^{2.5+\epsilon} \sum_{D' \ll k^{2+\epsilon}} \frac{1}{D'} \\
&\ll k^{2.5+2\epsilon}. \qquad (6.4)
\end{aligned}$$

This concludes the proof of Equation (6.1). We combine Lemma 6.3 and Equation (6.4) and get the correct error term. For the term $u = 0$, the set $\{\gamma \in \mathrm{SL}_2(\mathbb{Z}) \mid z_T = \gamma z_T\}$ has size $2\epsilon(T)$ since it is its lift from $\mathrm{PSL}_2(\mathbb{Z})$.

6.3 The even spectrum

In this section, we prove the second equation of Theorem 6.1. Let T_{-1} be the -1 Hecke operator acting by $T_{-1}\phi(z) = \phi(-\bar{z})$. We have

$$(\mathrm{id} + T_{-1})\,\phi(z) = \begin{cases} 2\phi(z) & \text{if } \phi \text{ is even,} \\ 0 & \text{if } \phi \text{ is odd.} \end{cases}$$

This tells us that

$$\int_{\Lambda_{\mathrm{ev}}} V(\det(T), \det(T), t_\phi, k)|\phi(z_T)|^2 d\phi$$
$$= \frac{1}{4}\int_{\Lambda} V(\det(T), \det(T), t_\phi, k)\left(|\phi(z_T)|^2 + |\phi(-\overline{z_T})|^2 + 2\operatorname{Re}(\phi(z_T)\overline{\phi(-\overline{z_T})})\right) d\phi.$$

If z is a Heegner point, then so is $-\overline{z_T}$. So we can consider Equation (6.1) when we replace $|\phi(z_T)|^2$ by $\phi(z)\overline{\phi(-\bar{z})}$. We apply the trace formula again and we consider first the term with $u = 0$.

The points z_T and $-\overline{z_T}$ are both in the classical fundamental domain. Therefore if there exists $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ such that $\gamma(-\overline{z_T}) = z$, it means that $z_T = -\overline{z_T}$ or that z_T is on the edge of the fundamental domain. In both cases, there is a γ_0 such that $\gamma_0(-\overline{z_T}) = z_T$. This gives three possibilities: $\beta = 0$ if $z_T = -\overline{z_T}$, $\beta = -\frac{1}{2}$ and $|z_T| = 1$. There γ_0 is respectively id, $\left(\begin{smallmatrix} 1 & -1 \\ 0 & 1 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$. We can post-compose with any γ such that $\gamma z_T = z_T$. Therefore the term $\phi(z_T)\phi(-\overline{z_T})$ has the same number of γ with $u = 0$ as the terms $|\phi(z_T)|^2$ and $|\phi(-\overline{z_T})|^2$. As above, the set $\{\gamma \in \mathrm{SL}_2(\mathbb{Z}) \mid z_T = \gamma z_T\}$ has size $2\epsilon(T)$. If there is such a γ_0, we get in total $\frac{8\epsilon(T)}{4} = 2\epsilon(T)$ terms for $u = 0$. If there is no γ_0 such that $\gamma_0(-\overline{z_T}) = z_T$, then the term $\phi(z_T)\phi(-\overline{z_T})$ has no term with $u = 0$ on the geometric side of the pre-trace. Therefore we only get $\epsilon(T)$. Looking at the table in Appendix A.1, we see that the ratio between $\#\operatorname{Aut}(T)$ and $\epsilon(T)$ is 4 if γ_0 exists and 2 otherwise. Thus we can write this contribution as $\frac{\#\operatorname{Aut}(T)}{2}$. If we combine this with the factor $\frac{\#\operatorname{Aut}(T)}{\epsilon(T)^2}$ in Equation (6.1), we get in total

$$\frac{1}{2}\left(\frac{\#\operatorname{Aut}(T)}{\epsilon(T)}\right)^2,$$

as in the second equation of Theorem 6.1.

We consider now the case $u \neq 0$. The only thing that matters in the error term of Equation (6.1) above is the distance $u(z_1, z_2)$ between the two points in the trace formula. If $u(z_T, \gamma(-\overline{z_T})) \gg k^{-1+\epsilon}$, then we conclude as in Lemma 6.3 using the decay of κ and Lemma 6.2. The other case is when $u(z_T, \gamma(-\overline{z_T})) \ll k^{-1+\epsilon}$. The bound in Equation (6.4) is also valid in this case. This concludes the proof of Theorem 6.1.

6.4 Main term of the pre-trace formula

We now analyze the term with $u = 0$ of the pre-trace formula. Applying Theorem 6.1, we have

$$N_{\mathrm{av}}^{\mathrm{diag}}(K) = \frac{12\pi^2}{\omega K^4}\sum_{k\in 2\mathbb{N}} w\left(\frac{k}{K}\right)\sum_{T\in\mathcal{P}(\mathbb{Z})/\,\mathrm{PSL}_2(\mathbb{Z})}\left(\frac{\#\operatorname{Aut}(T)}{\epsilon(T)}\right)^2\frac{1}{2\det(T)^{3/2}}\kappa(0) + O(K^{-1/2+\epsilon}),$$

with $\kappa(0)$ given in Equation (6.2). The error term is the combination of Theorem 6.1 and trivial estimates. In Appendix A, we calculate all the automorphisms of T in $\mathrm{GL}_2(\mathbb{Z})$. At the end of it, a table summarizes the computation. We see that $\frac{1}{2}(\frac{\#\operatorname{Aut}(T)}{\epsilon(T)})^2$ is 2 except if $T = \left(\begin{smallmatrix} \alpha & \beta \\ \beta & \delta \end{smallmatrix}\right)$ is diagonal, $\alpha = \delta$ or $\alpha = 2|\beta|$. In these cases the ratio is equal to 8. Recall the definition of V in Equation

(4.1). For $T \sim Q$, the only part that depends on T is $\det(T)^{-2v}$ for $\mathrm{Re}(v) > 0$. We consider

$$\begin{aligned}\sum_{T\in\mathcal{P}(\mathbb{Z})/\operatorname{PSL}_2(\mathbb{Z})} &\left(\frac{\#\operatorname{Aut}(T)}{\epsilon(T)}\right)^2 \frac{1}{2\det(T)^{3/2+2v}}\\ &= 2\sum_{T\in\mathcal{P}(\mathbb{Z})/\operatorname{PSL}_2(\mathbb{Z})} \frac{1}{\det(T)^{3/2+2v}} + 6\sum_{\substack{T\in\mathcal{P}(\mathbb{Z})/\operatorname{PSL}_2(\mathbb{Z})\\ \#\operatorname{Aut}(T)\neq 2\epsilon(T)}} \frac{1}{\det(T)^{3/2+2v}}\\ &=: 2L(v) + 6\tilde{L}(v).\end{aligned}$$

Lemma 6.4. *The function $\tilde{L}(v)$ converges for* $\mathrm{Re}(v) > -1/4$ *and is bounded on vertical strips.*

Proof. Let $\sigma = \mathrm{Re}(v)$. First, we consider the case of T diagonal. We have

$$\sum_{\substack{T\in\mathcal{P}(\mathbb{Z})/\operatorname{PSL}_2(\mathbb{Z})\\ T \text{ diagonal}}} \frac{1}{\det(T)^{3/2+2v}} = \sum_{0<\alpha\leq\delta} \frac{1}{(\alpha\delta)^{3/2+2v}} \ll \zeta(3/2+2\sigma)^2.$$

Therefore this sum converges for all $\sigma > -1/4$ and is bounded on vertical strips. The two other cases are similar:

$$\sum_{\substack{T\in\mathcal{P}(\mathbb{Z})/\operatorname{PSL}_2(\mathbb{Z})\\ \alpha=\delta}} \frac{1}{\det(T)^{3/2+2v}} = \sum_{\substack{0\leq 2\beta\leq\delta\\ 0<\delta}} \frac{1}{(\delta^2-\beta^2)^{3/2+2v}} \ll \sum_{0<\delta} \frac{1}{\delta^{2+4\sigma}} \ll \zeta(2+4\sigma),$$

$$\sum_{\substack{T\in\mathcal{P}(\mathbb{Z})/\operatorname{PSL}_2(\mathbb{Z})\\ \alpha=2|\beta|}} \frac{1}{\det(T)^{3/2+2v}} = \sum_{0<2\beta\leq\delta} \frac{1}{(2\beta\delta-\beta^2)^{3/2+2v}} \ll \sum_{0<\delta}\sum_{0<\beta} \frac{1}{(\beta\delta)^{3/2+2\sigma}} \ll \zeta(3/2+2\sigma)^2.$$

Note that these three cases are not disjoint. This is important if one wants to estimate the values of $\tilde{L}$ explicitly. ☐

To study $L(s)$, we need the following lemma.

Lemma 6.5 ([BC], remark after Lemma 12)**.** *Let*

$$\tilde{h}(D) := \#\{T \in \mathcal{P}(\mathbb{Z})/\operatorname{PSL}_2(\mathbb{Z}) \mid \det(T) = D\}$$

be the class number of the determinant D. We have

$$\sum_{D\leq X} \tilde{h}(D) = \frac{4\pi}{9} X^{3/2} - X + O(X^{3/4}).$$

Remark. Note that in [BC], the determinant D corresponds to the discriminant $-4D$. We have $\tilde{h}(D) = h(-4D)$. Hence X must be replaced by $4X$ between the result there and here.

Lemma 6.6. *The function $L(v)$ converges for* $\mathrm{Re}(v) > 0$ *and can be meromorphically extended to* $\mathrm{Re}(v) > -1/4$ *with a unique pole at $v = 0$ of residue $\frac{\pi}{3}$. The extension is bounded on vertical strips and away from the pole.*

Proof. We have

$$L(s) = \sum_{T\in\mathcal{P}(\mathbb{Z})/\operatorname{PSL}_2(\mathbb{Z})} \frac{1}{\det(T)^{3/2+2v}} = \sum_{0<D} \frac{\tilde{h}(D)}{D^{3/2+2v}}.$$

This converges for $\mathrm{Re}(v) > 0$. Let $X > 0$. Summing the Dirichlet series by parts, we get

$$\begin{aligned}\sum_{D\leq X} \frac{\tilde{h}(D)}{D^{3/2+2v}} &= \sum_{D\leq X} \tilde{h}(D) X^{-3/2-2v} + (3/2+2v)\int_1^X \sum_{D\leq t} \tilde{h}(D) \frac{dt}{t^{5/2+2v}}\\ &= \left(\sum_{D\leq X} \tilde{h}(D) X^{-3/2-2v} - \frac{4\pi}{9} X^{-2v}\right) + (3/2+2v)\int_1^X \left(\sum_{D\leq t} \tilde{h}(D) - \frac{4\pi}{9} t^{3/2}\right) \frac{dt}{t^{5/2+2v}}\\ &\quad + \frac{4\pi}{9} X^{-2v} + (3/2+2v)\int_1^X \frac{4\pi}{9} t^{-1-2v} dt. \qquad (6.5)\end{aligned}$$

The last integral is

$$(3/2+2v)\int_1^X \frac{4\pi}{9}t^{-1-2v}dt = -(3/2+2v)\frac{4\pi}{9}\frac{X^{-2v}-1}{2v}.$$

For $\mathrm{Re}(v) > 0$, the limit as $X \to \infty$ converges to $\frac{4\pi}{9}\frac{3/2+2v}{2v}$. Finally,

$$\lim_{X\to\infty}\left(\sum_{D\le X}\tilde{h}(D)X^{-3/2-2v} - \frac{4\pi}{9}X^{-2v}\right) = \lim_{X\to\infty}(X^{-1/2-2v} + O(X^{-3/4-2v})) = 0.$$

In total, we have that

$$(3/2+2v)\int_1^\infty \left(\sum_{D\le t}\tilde{h}(D) - \frac{4\pi}{9}t^{3/2}\right)\frac{dt}{t^{5/2+2v}} + \frac{4\pi}{9}\frac{3/2+2v}{2v}$$

converges for $\mathrm{Re}(v) > -\frac{1}{4}$ and $v \neq 0$. It is a meromorphic continuation of $L(v)$ with a unique pole of residue $\mathrm{Res}_{v=0} L(v) = \frac{\pi}{3}$ and it is bounded on vertical strips and away from $v = 0$. □

Now, we consider the T-sum combined with the v-integral of Equation (4.1):

$$\begin{aligned}
&\sum_{T\in\mathcal{P}(\mathbb{Z})/\operatorname{PSL}_2(\mathbb{Z})}\left(\frac{\#\operatorname{Aut}(T)}{\epsilon(T)}\right)^2\frac{1}{2\det(T)^{3/2}} \\
&\qquad\cdot\frac{1}{2\pi i}\int_{(3)} e^{v^2}c_k^{-1}G(\tau,k,v+1/2+it)G(\tau,k,v+1/2-it)\det(T)^{-2v}\frac{dv}{v} \\
&\qquad= \frac{1}{2\pi i}\int_{(3)} e^{v^2}c_k^{-1}G(\tau,k,v+1/2+it)G(\tau,k,v+1/2-it)(2L(v)+6\tilde{L}(v))\frac{dv}{v}.
\end{aligned}$$

Note that the integrand has a double pole at $v = 0$. We have the following Taylor expansion for the gamma factor:

$$\begin{aligned}
c_k^{-1}G(\tau,k,v+1/2+it)G(\tau,k,v+1/2-it) &= c_k^{-1}G(\tau,k,1/2+it)G(\tau,k,1/2-it) \\
&\left[1+v\left(\sum_{\pm\pm}\frac{\Gamma'}{\Gamma}\left(\frac{k-1/2}{2}\pm it\pm\frac{i\tau}{2}\right)-4\log(2\pi)\right)+O(v^2)\right].
\end{aligned}$$

Recall that according to Equation (4.3), the t- and τ-integral can be cut at $k^{1/2+\epsilon}$ up to a negligible error. Moreover, $\frac{\Gamma'}{\Gamma}(z) = \log(z) + O(|z|^{-1})$ so that for $t, \tau \ll k^{1/2+\epsilon}$,

$$\begin{aligned}
\sum_{\pm\pm}\frac{\Gamma'}{\Gamma}\left(\frac{k-1/2}{2}\pm it\pm\frac{i\tau}{2}\right)-4\log(2\pi) &= \sum_{\pm,\pm}\log((k-1/2)/2\pm it\pm i\tau/2)-4\log(2\pi)+O(k^{-1}) \\
&= 4\log(k)+C_0+O(k^{-1/2+\epsilon}),
\end{aligned}$$

for some constant $C_0 \in \mathbb{R}$. Let $C_1 = \lim_{v\to 0}(L(v) - \frac{\pi}{3v})$ be the constant term of the Laurent series of $L(v)$. In conclusion, the pole at $v = 0$ of the integrand has residue

$$c_k^{-1}G(\tau,k,1/2+it)G(\tau,k,1/2-it)\left(\frac{2\pi}{3}\left(4\log(k)+C_0+O(k^{-1/2+\epsilon})\right)+2C_1+6\tilde{L}(0)\right).$$

We define $D = \frac{2\pi}{3}C_0 + 2C_1 + 6\tilde{L}(0)$. We move the v-integral to $\mathrm{Re}(v) = -1/4+\epsilon$ for some fixed $\epsilon > 0$:

$$\begin{aligned}
&\frac{1}{2\pi i}\int_{(3)} e^{v^2}c_k^{-1}G(\tau,k,v+1/2+it)G(\tau,k,v+1/2-it)(2L(v)+6\tilde{L}(v))\frac{dv}{v} \\
&\qquad= c_k^{-1}G(\tau,k,1/2+it)G(\tau,k,1/2-it)\left(\frac{8\pi}{3}\log(k)+D+O(k^{-1/2+\epsilon})\right) \\
&\qquad+\frac{1}{2\pi i}\int_{(-1/4+\epsilon)} e^{v^2}c_k^{-1}G(\tau,k,v+1/2+it)G(\tau,k,v+1/2-it)(2L(v)+6\tilde{L}(v))\frac{dv}{v}
\end{aligned}$$

We apply the bounds of Lemma 5.2 to the second term to get

$$\begin{aligned}
&\frac{1}{2\pi i}\int_{(-1/4+\epsilon)} e^{v^2} c_k^{-1} G(\tau,k,v+1/2+it)G(\tau,k,v+1/2-it)(2L(v)+6\tilde{L}(v))\frac{dv}{v}\\
&\ll_A k^{1/2+4\epsilon}\int_{-\infty}^{\infty} e^{-w^2}\left(1+\frac{t^2+|\tau|^2+w^2}{k}\right)^{-A}(|L(-1/4+\epsilon+iw)|+|\tilde{L}(-1/4+\epsilon+iw)|)dw\\
&\ll_A k^{1/2+4\epsilon}\left(1+\frac{t^2+|\tau|^2}{k}\right)^{-A}.
\end{aligned}$$

Using Lemma 5.2 and $\tau\tanh(\pi\tau)=|\tau|+O(1)$, we get

$$\begin{aligned}
&\frac{12\pi}{\omega K^4}\sum_{k\in 2\mathbb{N}} w\left(\frac{k}{K}\right)\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{1}{2\pi i}\int_{(-1/4+\epsilon)} e^{v^2}\\
&\qquad \cdot c_k^{-1}G(\tau,k,v+1/2+it)G(\tau,k,v+1/2-it)(2L(v)+6\tilde{L}(v))\frac{dv}{v}dt\tau\tanh(\pi\tau)d\tau\\
&\qquad \ll_A K^{-3}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} K^{1/2+4\epsilon}\left(1+\frac{t^2+|\tau|^2}{k}\right)^{-A} dt\tau\tanh(\pi\tau)d\tau\\
&\qquad \ll K^{-3}\cdot K^{1/2+4\epsilon}\cdot K^{1/2}\cdot K\\
&\qquad \ll K^{-1+4\epsilon}.
\end{aligned}$$

Therefore we conclude that

$$\begin{aligned}
N_{\mathrm{av}}^{\mathrm{diag}}(K) &= \frac{12\pi^2}{\omega K^4}\sum_{k\in 2\mathbb{N}} w\left(\frac{k}{K}\right)\frac{1}{4\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} c_k^{-1}G(\tau,k,1/2+it)G(\tau,k,1/2-it)\\
&\quad\cdot\left(\frac{8\pi}{3}\log(k)+D+O(k^{-1/2+\epsilon})\right)dt\tau\tanh(\tau)d\tau+O(K^{-1/2+\epsilon})\\
&= \frac{3\pi}{\omega K^4}\sum_{k\in 2\mathbb{N}} w\left(\frac{k}{K}\right)\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} c_k^{-1}G(\tau,k,1/2+it)G(\tau,k,1/2-it)dt\tau\tanh(\tau)d\tau\\
&\quad\cdot\left(\frac{8\pi}{3}\log(k)+D+O(k^{-1/2+\epsilon})\right)+O(K^{-1/2+\epsilon}).
\end{aligned}$$

Now, we compute an approximation of the t-integral using Lemma 5.2. We also replace the gamma factors outside $t,\tau\ll k^{1/2+\epsilon}$. This gives an error of size $O_A(k^{-A})$ for all $A>0$, so it is negligible. We get

$$\begin{aligned}
&\int_{-\infty}^{\infty} c_k^{-1}G(\tau,k,1/2+it)G(\tau,k,1/2-it)dt\\
&\qquad = \frac{2}{\pi^{5/2}}k^{3/2}\int_{-\infty}^{\infty}\exp\left(-\frac{4t^2+|\tau|^2}{k}\right)\left(1+O(k^{-1/2+\epsilon})\right)dt+O_A(k^{-A})\\
&\qquad = \frac{2}{\pi^{5/2}}k^{3/2}\frac{\sqrt{\pi k}}{2}\exp\left(-\frac{|\tau|^2}{k}\right)\left(1+O(k^{-1/2+\epsilon})\right)\\
&\qquad = \frac{1}{\pi^2}k^2\exp\left(-\frac{|\tau|^2}{k}\right)+O(k^{3/2+\epsilon}).
\end{aligned}$$

We compute the τ-integral using $\tau\tanh(\tau)=|\tau|+O(1)$. This gives

$$\begin{aligned}
\int_{-\infty}^{\infty}\exp(-\tau^2/k)\tau\tanh(\tau)d\tau &= 2\int_0^{\infty}\exp(-\tau^2/k)(\tau+O(1))d\tau\\
&= -\,k\exp(-\tau^2/k)\big|_0^{\infty}+O(\sqrt{k})\\
&= k+O(\sqrt{k}).
\end{aligned}$$

We conclude that

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} c_k^{-1} G(\tau,k,1/2+it)G(\tau,k,1/2-it)dt\tau\tanh(\tau)d\tau = \frac{1}{\pi^2}k^2(k+O(\sqrt{k}))+O(k^{3/2+\epsilon})$$
$$= \frac{1}{\pi^2}k^3+O(k^{2.5+\epsilon}).$$

6.5 Sum over k

Recall that $\omega = \int_1^2 w(x)x^3dx$. We saw above that the diagonal term is

$$N_{\mathrm{av}}^{\mathrm{diag}}(f) = \frac{3\pi}{\omega K^4}\sum_{k\in 2\mathbb{N}} w\left(\frac{k}{K}\right)\frac{1}{\pi^2}k^3\left(\frac{8\pi}{3}\log(k)+D+O(k^{-1/2+\epsilon})\right)+O(K^{-1/2+\epsilon})$$
$$= \frac{8}{\omega K^4}\sum_{k\in 2\mathbb{N}} w\left(\frac{k}{K}\right)k^3\log(k)+\frac{3D}{\omega\pi K^4}\sum_{k\in 2\mathbb{N}} w\left(\frac{k}{K}\right)k^3+O(K^{-1/2+\epsilon}).$$

We deduce the main term of Theorem 1.4 by summing over k. We apply the Euler-MacLaurin formula for this.

Lemma 6.7 ([IK], Lemma 4.1)**.** *Let $a,b\in\mathbb{Z}$ and f a C^1 function on $[a,b]$. Then*

$$\sum_{\substack{n\in 2\mathbb{N}\\ a\le n\le b}} f(n) = \frac{1}{2}\int_a^b f(x)dx + O\left(\int_a^b |f'(x)|dx+|f(a)|+|f(b)|\right).$$

Let $\omega' = \int_1^2 w(x)x^3\log(x)dx$. We get

$$N_{\mathrm{av}}^{\mathrm{diag}}(f) = \frac{4}{\omega K^4}\int_K^{2K} w\left(\frac{x}{K}\right)x^3\log(x)dx+\frac{3D}{2\omega\pi K^4}\int_K^{2K} w\left(\frac{x}{K}\right)x^3dx+O(K^{-1/2+\epsilon})$$
$$+O\left(\frac{1}{K^4}\int_K^{2K}\left(\frac{1}{K}w'\left(\frac{x}{K}\right)x^3\log(x)+w\left(\frac{x}{K}\right)x^2\log(x)+w\left(\frac{x}{K}\right)x^2\right)dx\right)$$
$$= \frac{4}{\omega K^3}\int_1^2 w(x)(xK)^3\log(xK)dx$$
$$+\frac{3D}{2\omega\pi K^3}\int_1^2 w(x)(xK)^3dx+O(K^{-1/2+\epsilon})+O(K^{-1+\epsilon})$$
$$= 4\log(K)+4\frac{\omega'}{\omega}+\frac{3D}{2\pi}+O(K^{-1/2+\epsilon})$$
$$= 4\log(K)+D'+O(K^{-1/2+\epsilon}).$$

Here D' is a constant that only depends on w, $\epsilon>0$ is arbitrary and the implied constant depends only on ϵ and w.

Chapter 7

Rank 1 term

We focus now on the first off-diagonal term of the Kitaoka formula, called the rank 1 term. It comes from the combination of Equations (1.3), (4.2) and the Kitaoka formula (Theorem 3.6). Its shape is

$$\frac{12\sqrt{2}\pi^3}{\omega K^4}\sum_{k\in 2\mathbb{N}} w\left(\frac{k}{K}\right)\sum_{T,Q\in\mathcal{P}(\mathbb{Z})/\operatorname{PSL}_2(\mathbb{Z})}\frac{1}{\epsilon(T)\epsilon(Q)\det(TQ)^{3/4}}$$
$$\cdot\int_{\Lambda_{\mathrm{ev}}} V(\det(T),\det(Q),t_\phi,k)\phi(z_T)\bar{\phi}(z_Q)d\phi$$
$$\cdot\sum_{\pm}\sum_{c,s\geq 1}\sum_{U,V}\frac{(-1)^{k/2}}{c^{3/2}s^{1/2}}H^{\pm}(UQU^t,V^{-1}TV^{-t};c)J_\ell\left(\frac{4\pi\sqrt{\det(TQ)}}{cs}\right).$$

We have various sums that we need to restrict, up to a negligible error. First, we apply Lemma 5.3. The sum over k is

$$\sum_{k\in 2\mathbb{N}} w\left(\frac{k}{K}\right)(-1)^{k/2}V(\det(T)\det(Q),t_\phi,k)J_\ell\left(\frac{4\pi\sqrt{\det(TQ)}}{cs}\right).$$

We get three terms. The w_0 term is negligible because all the other sums and integral have a cut-off that gives a polynomial growth in K. The terms with w_+ and w_- have the property that $w_\pm(x)\ll_A K^2\left(1+\frac{K^2}{x}\right)^{-A}$ with $x=\frac{4\pi\sqrt{\det(TQ)}}{cs}$. They also depend on $\det(T)$, $\det(Q)$ and t_ϕ and follow the other bounds of Equation (4.3). In our case, we have

$$w_\pm(x,x_1,x_2,\tau,K)\ll_A K^2\left(1+\frac{x_1x_2}{K^4}\right)^{-A}\left(1+K^{1/2}|\log(x_2/x_1)|\right)^{-A}\left(1+\frac{|\tau|^2}{K}\right)^{-A}\left(1+\frac{K^2}{x}\right)^{-A}$$

with x as above, $x_1=\det(T)$, $x_2=\det(Q)$, $\tau=t_\phi$. We also have a control on the derivatives given by Equation (4.3) and Lemma 5.3.

7.1 First upper bound

We prove a first bound for the rank 1 term. Let $\epsilon>0$. Combining the estimate of Lemma 5.3 with Equation (4.3), we get $c^2s^2K^{4-\epsilon}\ll\det(TQ)\ll K^{4+\epsilon}$ up to a negligible error. Hence $c,s=O(K^\epsilon)$ and $K^{4-\epsilon}\ll\det(TQ)\ll K^{4+\epsilon}$. Since $\det(T)-\det(Q)\ll K^{-1/2+\epsilon}\det(T)$, we get $K^{2-\epsilon}\ll\det(T),\det(Q)\ll K^{2+\epsilon}$. Now, we look at the exponential sum $H^\pm$ defined after Theorem 3.6. It vanishes unless there are $U=\left(\begin{smallmatrix} * & * \\ u_3 & u_4\end{smallmatrix}\right)/\{\pm 1\}$ and $V=\left(\begin{smallmatrix} v_1 & * \\ v_3 & *\end{smallmatrix}\right)$ in $\mathrm{GL}_2(\mathbb{Z})$ such that

$$(UQU^t)_{22}=(V^{-1}TV^{-t})_{22}=s.$$

Let $T=\left(\begin{smallmatrix} a & b \\ b & c\end{smallmatrix}\right)$, $Q=\left(\begin{smallmatrix} x & y \\ y & z\end{smallmatrix}\right)$. Using the inequality $r^2+t^2\geq 2rt$, this gives

$$s=av_3^2-2bv_1v_3+cv_1^2\geq 2(\sqrt{ac}-|b|)|v_3v_4|,$$
$$s=xu_3^2+2yu_3u_4+zu_4^2\geq 2(\sqrt{xz}-|y|)|u_3u_4|.$$

Since T and Q are reduced, we have $2(\sqrt{ac}-|b|)\geq\sqrt{ac}\geq\sqrt{\det(T)}$ and similarly for Q. If u_3u_4 or v_1v_3 is non-zero, then we get $s\geq\sqrt{\det(T)}$ or $\sqrt{\det(Q)}$ and both are of size $\gg K^{1-\epsilon}$. Since $s=O(K^\epsilon)$ up to a negligible error, this is negligible. Otherwise we have $u_4=v_1=0$ because $c\gg\sqrt{\det(T)}\gg K^{1-\epsilon}$ and $z\gg K^{1-\epsilon}$. Since $U,V\in\mathrm{GL}_2(\mathbb{Z})$, we have the following choices of representatives for U and V:

$$U=\begin{pmatrix}0&1\\1&0\end{pmatrix},\qquad V=\pm\begin{pmatrix}0&1\\1&0\end{pmatrix}. \tag{7.1}$$

We get $s=a=x$ and in particular $x,a=O(K^\epsilon)$. Since T and Q are reduced, we also have $|y|,|b|=O(K^\epsilon)$ and $K^{2-\epsilon}\ll z\asymp c\ll K^{2+\epsilon}$. Therefore there are $O(K^{2+\epsilon}\cdot K^{3/2+\epsilon})$ choices for T and Q and $O(K^\epsilon)$ choices for c,s and U,V. We combine this with other estimates. Recall that the exponential sum is bounded by c^2. We use Equation (5.4) with $T=K^{1/2+\epsilon}$ and $K^{1-\epsilon}\ll\mathrm{Im}(z_T),\mathrm{Im}(z_Q)\ll K^{1+\epsilon}$. We get that the rank 1 term is bounded by

$$\begin{aligned}K^{-4}&\sum_{T,Q\in\mathcal{P}(\mathbb{Z})/\mathrm{PSL}_2(\mathbb{Z})}\frac{1}{\epsilon(T)\epsilon(Q)\det(TQ)^{3/4}}\sum_{c,s\geq1}\sum_{\pm}\sum_{U,V}\frac{(-1)^{k/2}}{c^{3/2}s^{1/2}}H^{\pm}(UQU^t,V^{-1}TV^{-t};c)\\&\cdot e\left(\pm\frac{2\sqrt{\det(TQ)}}{cs}\right)\int_{\Lambda_{\mathrm{ev}}}w_{\pm}\left(\frac{4\pi\sqrt{\det(TQ)}}{cs},\det(T),\det(Q),t_\phi,K\right)\phi(z_T)\bar{\phi}(z_Q)d\phi\\&\ll K^{-4}\cdot K^{3.5+\epsilon}\cdot K^{-3+\epsilon}\cdot K^{\epsilon}\cdot K^{2}\cdot K^{3/2+\epsilon}\\&\ll K^{4\epsilon}.\end{aligned}$$

7.2 Analysis of the T,Q-sum

We need to win extra cancellation somewhere. We do that in the T,Q-sum. We consider $\Delta=\det(Q)-\det(T)$. We know that $\Delta=O(K^{3/2+\epsilon})$ up to a negligible error. We can fix all the coefficients of Q except z at the cost of K^ϵ choices. The possible values of $\Delta=xz-y^2-\det(T)$ follow then an arithmetic progression as z varies. More precisely, $d:=y^2+\det(T)\equiv\Delta\pmod{x}$. Looking at the last table in Appendix A, we have $\epsilon(Q)=1$ unless $x=z$, which is a negligible case for K large enough. Similarly, we can suppose that $\epsilon(T)=1$. The T,Q-sum looks like

$$\begin{aligned}&\sum_{K^{2-\epsilon}\ll\det(T)\ll K^{2+\epsilon}}\sum_{\substack{|\Delta|\ll O(K^{3/2+\epsilon})\\ \Delta\equiv d\bmod x}}\frac{1}{(\det(T)(\det(T)+\Delta))^{3/4}}\\&\cdot\int_{\Lambda_{\mathrm{ev}}}w_{\pm}\left(\frac{4\pi\sqrt{\det(T)(\det(T)+\Delta)}}{cs},\det(T),\det(T)+\Delta,t_\phi,K\right)\phi(z_T)\bar{\phi}(z_Q)d\phi\\&\cdot H^{\pm}(UQU^t,V^{-1}TV^{-t};c)e\left(\pm\frac{2\sqrt{\det(T)(\det(T)+\Delta)}}{cs}\right).\end{aligned}\tag{7.2}$$

Recall the definition of $H^{\pm}$ just after Theorem 3.6, the representatives of U and V chosen in equation (7.1) and that $s=a=x$. We get that

$$P=UQU^t=\begin{pmatrix}z&y\\y&s\end{pmatrix},\qquad S=V^{-1}TV^{-t}=\begin{pmatrix}c&b\\b&s\end{pmatrix}.$$

Therefore, $z=p_1$ and the summand in $H^{\pm}$ is

$$\begin{aligned}&e\left(\frac{\bar{d}_1s_4d_2^2\mp\bar{d}_1p_2d_2+s_2d_2+\bar{d}_1p_1+d_1s_1}{c}\mp\frac{p_2s_2}{2cs_4}\right)\\&=e\left(\frac{\bar{d}_1z}{c}\right)e\left(\frac{\bar{d}_1s_4d_2^2\mp\bar{d}_1p_2d_2+s_2d_2+d_1s_1}{c}\mp\frac{p_2s_2}{2cs_4}\right)\\&=e\left(\frac{\bar{d}_1\Delta}{cs}\right)e\left(\frac{\bar{d}_1(\det(T)+y^2)}{cs}\right)e\left(\frac{\bar{d}_1s_4d_2^2\mp\bar{d}_1p_2d_2+s_2d_2+d_1s_1}{c}\mp\frac{p_2s_2}{2cs_4}\right).\end{aligned}\tag{7.3}$$

We fix $\Delta \pmod{cs}$, so that we can see this term as constant in the Δ-sum. This adds a sum over $d \pmod{cs}$ such that $d \equiv y^2 + \det(T) \pmod{s}$. Now, we consider the spectral integral. Let

$$\nu(s) = \pi^{1/2}\frac{\Gamma(s-1/2)}{\Gamma(s)}\frac{\zeta(2s-1)}{\zeta(2s)} = \frac{\pi^{-(1-s)}\Gamma(1-s)\zeta(2(1-s))}{\pi^{-s}\Gamma(s)\zeta(2s)}.$$

Lemma 7.1. *We have*

$$\begin{aligned}\int_{\Lambda_{\mathrm{ev}}} & w_\pm\left(\frac{4\pi\sqrt{\det(T)(\det(T)+\Delta)}}{cs}, \det(T), \det(T)+\Delta, t_\phi, K\right)\phi(z_T)\bar{\phi}(z_Q)d\phi \\ &= \tilde{w}_\pm\left(\frac{4\pi\sqrt{\det(T)(\det(T)+\Delta)}}{cs}, \det(T), \det(T)+\Delta, K\right) + O(K^{2.5+\epsilon}),\end{aligned}$$

where the function

$$\begin{aligned}\tilde{w}_\pm(x,x_1,x_2,k) := \frac{(x_1x_2)^{1/4}}{s}\int_{-\infty}^{\infty} & w_\pm(x,x_1,x_2,\tau,k) \\ &\cdot\left(\left(\frac{x_1}{x_2}\right)^{i\tau} + \left(\frac{x_2}{x_1}\right)^{i\tau} + \nu(1/2-i\tau)(x_1x_2)^{i\tau} + \nu(1/2+i\tau)(x_1x_2)^{-i\tau}\right)\frac{d\tau}{4\pi}\end{aligned}$$

(for s fixed) satisfies the following bounds:

$$\begin{aligned}x^{j_1}&\left(\frac{x_1}{k^{1/2}}\right)^{j_2}\left(\frac{x_1}{k^{1/2}}\right)^{j_3}\frac{d^{j_1}}{dx^{j_1}}\frac{d^{j_2}}{dx_1^{j_2}}\frac{d^{j_3}}{dx_2^{j_3}}\tilde{w}_\pm(x,x_1,x_2,k) \\ &\ll_{A,j_1,j_2,j_3} k^{2.5}(x_1x_2)^{1/4}\left(1+\frac{k^2}{x}\right)^{-A}\left(1+\frac{x_1x_2}{k^2}\right)^{-A}\left(1+\frac{(x_1-x_2)k^{1/2}}{x_1}\right)^{-A}.\end{aligned}$$

Proof. We have $\mathrm{Im}(z_T), \mathrm{Im}(z_Q) \gg K^{1-\epsilon}$ and the spectral parameter t_ϕ satisfies $|t_\phi| \ll K^{1/2+\epsilon}$ up to a negligible error. In that case, the cusp forms in the spectral decomposition are known to be negligible and the only terms that remain are the constant terms in the Fourier expansion of the Eisenstein series and the constant function. Details about the decay of the K-Bessel function and the Eisenstein series can be found in Lemma 3.1 of [You2]. For the Fourier coefficients of cusp forms, a polynomial bound like Equation 8.8 in [Iwa] suffices. More precisely, let

$$y_1 = \frac{\sqrt{\det(T)}}{s}, \qquad y_2 = \frac{\sqrt{\det(Q)}}{s}$$

and

$$\nu(s) = \pi^{1/2}\frac{\Gamma(s-1/2)}{\Gamma(s)}\frac{\zeta(2s-1)}{\zeta(2s)} = \frac{\pi^{-(1-s)}\Gamma(1-s)\zeta(2(1-s))}{\pi^{-s}\Gamma(s)\zeta(2s)}.$$

Then $|\nu(s)| = 1$ and the constant term of the Eisenstein series $E(x+iy,s)$ is $y^s + \nu(1-s)y^{1-s}$. We have

$$\begin{aligned}\int_{\Lambda_{\mathrm{ev}}} & w_\pm\left(\frac{4\pi\sqrt{\det(T)(\det(T)+\Delta)}}{cs}, \det(T), \det(T)+\Delta, t_\phi, K\right)\phi(z_T)\bar{\phi}(z_Q)d\phi \\ &= \int_{-\infty}^{\infty} w_\pm\left(\frac{4\pi\sqrt{\det(T)(\det(T)+\Delta)}}{cs}, \det(T), \det(T)+\Delta, t_\phi, K\right) \\ &\quad\cdot\left(y_1^{1/2+i\tau} + \nu(1/2+i\tau)y_1^{1/2-i\tau}\right)\left(y_2^{1/2-i\tau} + \nu(1/2-i\tau)y_2^{1/2+i\tau}\right)\frac{d\tau}{4\pi} \\ &\quad+ \int_{-\infty}^{\infty} V(\tau)\frac{3}{\pi}d\tau + O(e^{-cK}) \\ &= \tilde{w}_\pm\left(\frac{4\pi\sqrt{\det(T)(\det(T)+\Delta)}}{cs}, \det(T), \det(T)+\Delta, K\right) + O(K^{2.5+\epsilon}).\end{aligned}$$

Note that the bounds and control on the derivatives in the other variables of $w_\pm$ also apply to $\tilde{w}_\pm$. We use $|\tau| \ll K^{1/2+\epsilon}$, $s \geq 1$ and $|\nu(1/2 \pm i\tau)| = 1$ to get the stated bound. □

Remark. In the definition of $\tilde{w}_\pm$, it is possible to integrate by parts the τ-integral multiple times for the factors with the terms $(y_1/y_2)^{\pm it\tau}$. It gives a cut-off of the form $(k^{1/2}\log(y_2/y_1))^{-j}$ and we can get a strong decay for the other terms. We saw that $a = x = s$. Hence this is redundant information with the cut-off on $\det(Q)/\det(T)$ of Equation (4.3) in our case.

We also have

$$\frac{1}{(\det(T)+\Delta)^{3/4}} = \frac{1}{\det(T)^{3/4}} + O(K^{-2+\epsilon}),$$

up to a negligible error. Inserting this, Equation (7.3) and the result of Lemma 7.1 in Equation (7.2), we get

$$\sum_{K^{2-\epsilon}\ll\det(T)\ll K^{2+\epsilon}} \det(T)^{-3/2} \sum_{\substack{d \bmod cs\\ d\equiv y^2+\det(T) \bmod s}} H^{\pm}(P,S,c) \sum_{\substack{|\Delta|\ll K^{3/2+\epsilon}\\ \Delta\equiv d \bmod cs}} e\left(\pm\frac{2\sqrt{\det(T)(\det(T)+\Delta)}}{cs}\right)$$
$$\cdot\, \tilde{w}_\pm\left(\frac{4\pi\sqrt{\det(T)(\det(T)+\Delta)}}{cs}, \det(T), \det(T)+\Delta, k\right) + O(K^{3+\epsilon}).$$

7.3 Poisson summation and stationary phase

We apply Poisson summation formula to the Δ-sum.

$$\sum_{\Delta=d \bmod cs} \tilde{w}_\pm\left(\frac{4\pi\sqrt{\det(T)(\det(T)+\Delta)}}{cs}, \det(T), \det(T)+\Delta, K\right) e\left(\pm\frac{2\sqrt{\det(T)(\det(T)+\Delta)}}{cs}\right)$$
$$= \frac{1}{cs}\sum_{h\in\mathbb{Z}}\int_{-\infty}^{\infty} \tilde{w}_\pm\left(\frac{4\pi\sqrt{\det(T)(\det(T)+t)}}{cs}, \det(T), \det(T)+t, K\right)$$
$$\cdot\, e\left(\frac{\pm 2\sqrt{\det(T)(\det(T)+t)}+h(d-t)}{cs}\right) dt.$$

To analyze this integral, we need to compute the derivative of $\tilde{w}_\pm$ with respect to t. We have

$$\frac{d}{dt}\tilde{w}_\pm\left(\frac{4\pi\sqrt{\det(T)(\det(T)+t)}}{cs}, \det(T), \det(T)+t, K\right)$$
$$= \left(\frac{d}{dx}\tilde{w}_\pm\right)\left(\frac{4\pi\sqrt{\det(T)(\det(T)+t)}}{cs}, \det(T), \det(T)+t, K\right)\frac{4\pi}{cs}\sqrt{\frac{\det(T)}{\det(T)+t}}$$
$$+ \left(\frac{d}{dx_2}\tilde{w}_\pm\right)\left(\frac{4\pi\sqrt{\det(T)(\det(T)+t)}}{cs}, \det(T), \det(T)+t, K\right)$$
$$\ll_A \left(K^{-2+\epsilon}+K^{-3/2+\epsilon}\right)(\det(T)(\det(T)+t))^{1/4}K^{2.5}\left(1+\frac{csK^2}{\sqrt{\det(T)(\det(T)+t)}}\right)^{-A}$$
$$\cdot\left(1+\frac{\det(T)(\det(T)+t)}{K^2}\right)^{-A}\left(1+\frac{tK^{1/2}}{\det(T)}\right)^{-A}.$$

More generally, each derivative with respect to t adds a factor of size $K^{-3/2+\epsilon}$ (up to a constant depending on j). This is because it either adds a derivative in the first or the third variable of $\tilde{w}_\pm$, or it differentiates a factor of the form $(\det(T)+t)^{-r}$. All these added factors are of size $\ll K^{-3/2+\epsilon}$. If $|h| \gg K^{\epsilon}$, we integrate by parts, until we can sum over h and get a large enough power saving. Each derivative in t adds nothing in the worst case. But the h-sum can be a small as we want, so we get a strong decay. More precisely,

$$\begin{aligned}
&\sum_{|h|\gg K^\epsilon} e\left(\frac{hd}{cs}\right)\int_{-\infty}^{\infty}\tilde{w}_\pm\left(\frac{4\pi\sqrt{\det(T)(\det(T)+t)}}{cs},\det(T),\det(T)+t,k\right)\\
&\quad\cdot e\left(\frac{\pm2\sqrt{\det(T)(\det(T)+t)}-ht}{cs}\right)dt\\
&=\sum_{|h|\gg K^\epsilon} e\left(\frac{hd}{cs}\right)\left(\frac{cs}{-2\pi ih}\right)^j\int_{-\infty}^{\infty}\frac{d^j}{dt^j}\left[\tilde{w}_\pm\left(\frac{4\pi\sqrt{\det(T)(\det(T)+t)}}{cs},\det(T),\det(T)+t,k\right)\right.\\
&\quad\left.\cdot e\left(\frac{\pm2\sqrt{\det(T)(\det(T)+t)}}{cs}\right)\right]e\left(-\frac{ht}{cs}\right)dt\\
&\ll_{A,j,\epsilon} K^{1+\epsilon}\sum_{|h|\gg K^\epsilon}\frac{(cs)^j}{h^j}\int_{-\infty}^{\infty}K^{3.5}\left(1+\frac{tK^{1/2}}{x_1}\right)^{-A}dt\left(1+\frac{csK^2}{\det(T)}\right)^{-A}\left(1+\frac{\det(T)^2}{K^2}\right)^{-A}\\
&\ll_A K^{-A}\left(1+\frac{csK^2}{\det(T)}\right)^{-A}\left(1+\frac{\det(T)^2}{K^2}\right)^{-A}.
\end{aligned}$$

Using the cut-off on the other sums, we see that this term is negligible. For small h, we apply the stationary phase method. The stationary point is

$$\pm\frac{1}{cs}\sqrt{\frac{\det(T)}{\det(T)+t_0}}=\frac{h}{cs}\Longrightarrow t_0=\frac{\det(T)}{h^2}-\det(T).$$

Note that h must have the same sign as the left-hand side. There are three cases. If $h=0$, then there is no stationary point. We apply in that case Lemma 5.5. If $h=\pm1$, then $t_0=0$. We apply Lemma 5.6. Otherwise, $t_0\gg\det(T)K^{-\epsilon}$ and $\tilde{w}_\pm$ is negligible for such t. We apply again Lemma 5.5. Following notations there, we have $w=\tilde{w}_\pm$ and

$$h(t)=2\pi\left(\frac{\pm2\sqrt{\det(T)(\det(T)+t)}+h(d-t)}{cs}\right).$$

In the first and the last case, we get

$$\begin{aligned}
\alpha&=-K^{3/2+\epsilon}, & \beta&=K^{3/2+\epsilon},\\
X&=K^{3.5+\epsilon}, & U&=K^{1.5},\\
R&=K^{-\epsilon}, & &\\
Y&=K^{2+\epsilon}, & Q&=K^{2-\epsilon}.
\end{aligned}$$

Lemma 5.5 tells us that the integral is bounded by

$$\ll_A K^{3/2+\epsilon}\cdot K^{3.5+\epsilon}[(K^{2-2\epsilon}/K^{1+\epsilon})^{-A}+K^{-1.5A}].$$

Using the cut-off on the other sums, we see that these terms are negligible. If $t_0=0$, we apply Lemma 5.6. Following the notations, we get

$$\begin{aligned}
\alpha&=-K^{3/2+\epsilon}, & \beta&=K^{3/2+\epsilon},\\
X&=K^{3.5+\epsilon}, & U&=K^{1.5-\epsilon},\\
Y&=K^{2+\epsilon}, & K^{2-\epsilon}&\ll Q\ll K^{2+\epsilon}.
\end{aligned}$$

Here we mean that there exists a Q in this interval that works. Then the integral is bounded by

$$\ll\frac{QX}{\sqrt{Y}}\ll K^{5.5-1+\epsilon}=K^{4.5+\epsilon}.$$

We sum after that over T with $K^{2-\epsilon} \ll \det(T) \ll K^{2+\epsilon}$, which gives a contribution of size $K^{2+\epsilon} \cdot K^{-3+\epsilon} \ll K^{-1+2\epsilon}$. The remaining sums are the sum over the other coefficients of Q, the one over $d \pmod{cs}$ and the various $\pm, c, s, U, V$-sums for the exponential sum $H^{\pm}$. They are all of size K^{ϵ}. The rank one term is therefore bounded by

$$K^{-4} \cdot K^{-1+\epsilon} \cdot K^{\epsilon} \cdot K^{4.5+\epsilon} \ll K^{-1/2+3\epsilon}.$$

Chapter 8

Rank 2 term

In this chapter, we focus on the last error term. It comes from the combination of Equation (1.3), (4.2) and the rank 2 term of the Kitaoka formula (Theorem 3.6). Its shape is

$$\frac{96\pi^4}{\omega K^4}\sum_{k\in 2\mathbb{N}} w\left(\frac{k}{K}\right)\sum_{T,Q\in\mathcal{P}(\mathbb{Z})/\operatorname{PSL}_2(\mathbb{Z})}\frac{1}{\epsilon(T)\epsilon(Q)\det(TQ)^{3/4}}$$
$$\cdot\int_{\Lambda_{\mathrm{ev}}} V(\det(T),\det(Q),t_\phi,k)\phi(z_T)\bar{\phi}(z_Q)d\phi\sum_{\det(C)\neq 0}\frac{K(Q,T;C)}{|\det(C)|^{3/2}}\mathcal{J}_\ell(TC^{-1}QC^{-t}).$$

Using the estimate $J_k(x)\ll\left(\frac{x}{k}\right)^k$ in Equation (5.1), we get

$$\mathcal{J}_\ell(TC^{-1}QC^{-t})=\int_0^{\pi/2}J_\ell(4\pi s_1\sin(\theta))J_\ell(4\pi s_2\sin(\theta))\sin(\theta)d\theta\ll\left(\frac{s_1s_2}{k^2}\right)^k.$$

Therefore $k^{2-\epsilon}\ll s_1s_2=\det(TC^{-1}QC^{-t})^{1/2}=\frac{\det(TQ)^{1/2}}{\det(C)}\ll\frac{k^{2+\epsilon}}{\det(C)}$. The last estimate comes from Equation (4.3), up to a negligible error. Hence $\det(C)\ll k^\epsilon$ and $k^{4-\epsilon}\ll\det(T)\det(Q)\ll k^{4+\epsilon}$. Using Equation (4.3), we also have $\det(T)=\det(Q)(1+O(k^{-1/2+\epsilon}))$.

The restriction on C is a bit subtle because there exist infinitely many matrices with a fixed determinant. We prove later that actually $\|C\|_\infty\ll k^\epsilon$. Lemma 2 in [Blo] gives us already a bound

$$\|C\|^2\ll\|T\|\|Q\|. \tag{8.1}$$

This is because $s_1\gg k^{1-\epsilon}$, using Equation (5.1). Since, without loss of generality, T and Q are reduced, we have $\|C\|^2\ll\det(T)\det(Q)\ll k^{4+\epsilon}$. Recall also that the generalized Kloosterman sum $K(Q,T;C)$ is normalized by the factor $\det(C)^{3/2}$. The goal of the chapter is to prove that the C-sum is short and to detect further cancellation in the T and Q sums coming from the generalized Bessel function $\mathcal{J}_\ell$. The idea is that if s_1 and s_2 are far from each other, $\mathcal{J}_\ell$ should be small. This is made more precise in Section 8.2.

8.1 Summing over k

First, we use Lemma 5.3 to take advantage of the average over k. Let $s_1\geq s_2>0$ be the square root of the two eigenvalues of $TC^{-1}QC^{-t}$. We want to analyze the sum

$$\sum_{k\in 2\mathbb{N}}\tilde{w}(k)\mathcal{J}_\ell(TC^{-1}QC^{-t})=\sum_{k\in 2\mathbb{N}}\tilde{w}(k)\int_0^{\pi/2}J_\ell(4\pi s_1\sin(\alpha))J_\ell(4\pi s_2\sin(\alpha))\sin(\alpha)d\alpha,$$

where

$$\tilde{w}(k)=\tilde{w}(k,\det(T),\det(Q),t_\phi)=w\left(\frac{k}{K}\right)V(\det(T),\det(Q),t_\phi,k)$$

and we temporarily drop the other dependencies. Applying Equation (5.2) gives

$$\begin{aligned}&\sum_{k\in 2\mathbb{N}} \tilde{w}(k)\mathcal{J}_\ell(TC^{-1}QC^{-t})\\&= \operatorname{Re}\left(\frac{1}{\pi}e\left(-\frac{k-1/2}{4}\right)\int_0^{\pi/2}\int_0^\infty e\left((s_1^2+s_2^2)t+\frac{\sin(\alpha)^2}{t}\right)\sum_{k\in 2\mathbb{N}}\tilde{w}(k)J_\ell(4\pi s_1 s_2 t)\frac{dt}{t}\sin(\alpha)d\alpha\right).\end{aligned}$$

We apply Lemma 5.3 to the sum over k. Using $w_0(x) \ll_A \min\{k^{-A}, x^{-1/2}\}$, we see that the term with w_0 is negligible. For the two other terms, we get

$$\operatorname{Re}\left(\frac{e(1/8)}{\pi}\int_0^{\pi/2}\int_0^\infty e\left((s_1^2+s_2^2)t+\frac{\sin(\alpha)^2}{t}\pm 2s_1s_2t\right)w_\pm(4\pi s_1s_2t)\frac{dt}{t}\sin(\alpha)d\alpha\right).$$

We forget about the real part and bound what is inside. We show first a trivial bound for this integral. We use the bounds on $w_\pm$ of Lemma 5.3 and the last Equation of (5.1). As stated in the remark after Lemma 20 of [BC], this is also valid for $w_\pm$. We get

$$\begin{aligned}I := &\int_0^{\pi/2}\int_0^\infty e\left((s_1^2+s_2^2)t+\frac{\sin(\alpha)^2}{t}\pm 2s_1s_2t\right)w_\pm(4\pi s_1s_2t)\frac{dt}{t}\sin(\alpha)d\alpha\\ \ll &\int_0^\infty |w_\pm(4\pi s_1s_2t)|\frac{dt}{t}\\ \ll &K^2\left(\int_0^1\left(1+\frac{K^2}{s_1s_2t}\right)^{-1}\left(1+\frac{K^2}{s_1s_2t}\right)^{-\epsilon}\frac{dt}{t}+\int_1^{K^2}\frac{dt}{t}+K\int_{K^2}^\infty\frac{dt}{t^{3/2}}\right)\\ \ll &K^{2+\epsilon}\end{aligned}\tag{8.2}$$

8.2 Analysis of the integral and distance between eigenvalues

Lemma 8.1. *Let $w_\pm$ as above, $a > 0$, $0 < b \ll 1$ and $K^{2-\epsilon} \ll c \ll K^{2+\epsilon}$. If $a \gg K^{6\epsilon}$, then*

$$\int_0^\infty e\left(at+\frac{b}{t}\right)w_\pm(ct)\frac{dt}{t} \ll_A K^{-A}.$$

Proof. Lemma 5.3 says that $\frac{d^j}{dt^j}w_\pm(ct) \ll_{A,j} t^{-j}K^2(1+K^{-\epsilon}/t)^{-A}$ for all $A > 0$. By induction, we have that

$$\frac{d^j}{dt^j}\frac{w_\pm(ct)}{t} \ll_{A,j} t^{-(j+1)}K^2\left(1+\frac{1}{K^\epsilon t}\right)^{-A}.$$

This is because each derivative add either a derivative on $w_\pm(ct)$ or a $\frac{1}{t}$ factor. We apply Lemma 5.5. Following the notations there, we have

$$\begin{aligned}h(t) &= 2\pi\left(at+\frac{b}{t}\right), & h'(t) &= 2\pi\left(a-\frac{b}{t^2}\right),\\ h^{(j)}(t) &= (-1)^j j!\frac{2\pi b}{t^{j+1}} \quad \text{for } j\geq 2, & &\\ w(t) &= \frac{w_\pm(ct)}{t}, & w^{(j)}(t) &\ll_{A,j} t^{-(j+1)}K^2\left(1+\frac{1}{K^\epsilon t}\right)^{-A}.\end{aligned}$$

The only stationary point t_0 is such that $0 = h'(t_0) = a - \frac{b}{t_0^2}$, that is $t_0 = \sqrt{\frac{b}{a}}$ (it only exists if $a \neq 0$). Let suppose that $a \gg K^{6\epsilon}$. Then in particular $t_0 \ll K^{-3\epsilon}$ since $b \ll 1$. But in that part of the t-integral, the function $w_\pm$ is negligible. For $t \leq K^{-2\epsilon}$, we use the bound on $w_\pm$:

$$\int_0^{K^{-2\epsilon}} e\left(at+\frac{b}{t}\right)w_\pm(ct)\frac{dt}{t} \ll_A K^2\int_0^{K^{-2\epsilon}}\left(1+\frac{1}{K^\epsilon t}\right)^{-A}\frac{dt}{t} \ll_A K^{2-\epsilon A}.$$

For $t \geq K^{-2\epsilon}$, we apply Lemma 5.5. We split everything into dyadic intervals $[\alpha, 2\alpha]$ with $\alpha \geq K^{-2\epsilon}$. We use the constants

$$X = \frac{K^2}{\alpha}, \qquad U = \alpha,$$
$$Y = 1 \gg \frac{b}{\alpha}, \qquad Q = \alpha,$$
$$R = \pi a \leq 2\pi \left(a - \frac{b}{\alpha^2}\right).$$

Lemma 5.5 gives us

$$\int_{\alpha}^{2\alpha} e\left(at + \frac{b}{t}\right) w_{\pm}(ct)\frac{dt}{t} \ll_A K^2(\pi a\alpha)^{-A} \ll_A K^2 \cdot K^{-6\epsilon A}\alpha^{-A}.$$

This can be summed for a dyadic decomposition of $[K^{-2\epsilon}, \infty[$ to get

$$\int_{K^{-2\epsilon}}^{\infty} e\left(at + \frac{b}{t}\right) w_{\pm}(ct)\frac{dt}{t} \ll_A K^{2-6\epsilon A}\sum_{j=0}^{\infty}(2^j K^{-2\epsilon})^{-A} \ll_A K^{2+(2\epsilon-6\epsilon)A}.$$

Combining both estimates, we have

$$\int_0^{\infty} e\left(at + \frac{b}{t}\right) w_{\pm}(ct)\frac{dt}{t} \ll_A K^{2-\epsilon A}.$$

□

Remark. In our case, we have $a = (s_1 \pm s_2)^2$, $b = \sin(\alpha)^2$ and $c = 4\pi s_1 s_2$. Using the various estimates coming from Equations (4.3) and (8.1), we see that, up to a negligible error, $a \ll K^{3\epsilon}$. Note that $(s_1 + s_2)^2 \geq 4s_1s_2 \gg K^{2-\epsilon}$. So the term with this sign is always negligible.

8.3 Size of the T, Q and C sums

We are left to analyze the case $a = (s_1 - s_2)^2 \ll K^{\epsilon}$. We changed the value of ϵ here. In this section and the next ones, we may change again the value of ϵ from one display to the other. We only do this if the new ϵ is only a constant multiple of the old one.

The goal of this section is to see which T, Q and C satisfy the bound $(s_1 - s_2) \ll K^{\epsilon}$. Note first that if $\lambda_1 \geq \lambda_2$ are the two eigenvalues of $M = TC^{-1}QC^{-t}$, then

$$\lambda_1 - \lambda_2 = s_1^2 - s_2^2 = (s_1 - s_2)(s_1 + s_2) \ll K^{1+2\epsilon}.$$

This comes from the fact that $K^{2-\epsilon} \ll s_1s_2 \ll K^{2+\epsilon}$, so that $K^{1-\epsilon} \ll s_1 + s_2 \asymp s_1 \asymp s_2 \ll K^{1+\epsilon}$. We fix some notations for this section:

$$T = \begin{pmatrix} a & b \\ b & c \end{pmatrix}, \qquad Q = \begin{pmatrix} x & y \\ y & z \end{pmatrix}, \qquad C^{-1} = (c_{ij}),$$
$$\tilde{Q} = C^{-1}QC^{-t} = \begin{pmatrix} \tilde{x} & \tilde{y} \\ \tilde{y} & \tilde{z} \end{pmatrix}, \qquad M = T\tilde{Q} = TC^{-1}QC^{-t}, \qquad M = (m_{ij}).$$

Note that all numbers are integers or half-integers except for $c_{ij} \in \frac{1}{\det(C)}\mathbb{Z}$ and for $\tilde{x}, \tilde{y}, \tilde{z} \in \frac{1}{2\det(C)^2}\mathbb{Z}$. But since $|\det(C)| \ll K^{\epsilon}$, this only creates a negligible difference in terms of estimates for distances between coordinates. So we treat them as if they were integers in the rest of the argument and point out where the difference occurs. Recall also that T and Q are reduced, so $2|b| \leq a \leq c$ and $2|y| \leq x \leq z$. In particular, $K^{2-\epsilon} \ll ac \asymp \det(T) \ll K^{2+\epsilon}$ and similarly for Q. Consider $(\lambda_1 - \lambda_1)^2$ the square of the difference between the two eigenvalues of M. By the quadratic formula, this corresponds to the discriminant of the characteristic polynomial of M. We have

$$K^{2+\epsilon} \gg (\lambda_1 - \lambda_2)^2 = \operatorname{tr}(M)^2 - 4\det(M) = (m_{11} - m_{22})^2 + 4m_{12}m_{21}.$$

Inserting the values of the product $T\tilde{Q}$, we get

$$\Delta = (a\tilde{x} - c\tilde{z})^2 + 4(a\tilde{y} + b\tilde{z})(b\tilde{x} + c\tilde{y}).$$

We rearrange the second term. Completing the squares with respect to $\tilde{y}$, we have

$$\begin{aligned} 4(a\tilde{y} + b\tilde{z})(b\tilde{x} + c\tilde{y}) &= 4ac\tilde{y}^2 + 4b\tilde{y}(a\tilde{x} + c\tilde{z}) + 4b^2\tilde{x}\tilde{z} \\ &= \left(2\sqrt{ac}\tilde{y} + b\frac{a\tilde{x} + c\tilde{z}}{\sqrt{ac}}\right)^2 - b^2\frac{(a\tilde{x} + c\tilde{z})^2}{ac} + 4b^2\tilde{x}\tilde{z} \\ &= \frac{1}{ac}(2ac\tilde{y} + b(a\tilde{x} + c\tilde{z}))^2 - \frac{b^2}{ac}(a\tilde{x} - c\tilde{z})^2. \end{aligned}$$

We can do a similar computation by completing the square on b. We get

$$\Delta = (a\tilde{x} - c\tilde{z})^2\left(1 - \frac{b^2}{ac}\right) + \frac{1}{ac}(2ac\tilde{y} + b(a\tilde{x} + c\tilde{z}))^2, \tag{8.3}$$

$$\Delta = (a\tilde{x} - c\tilde{z})^2\left(1 - \frac{\tilde{y}^2}{\tilde{x}\tilde{z}}\right) + \frac{1}{\tilde{x}\tilde{z}}(2\tilde{x}\tilde{z}b + \tilde{y}(a\tilde{x} + c\tilde{z}))^2. \tag{8.4}$$

Since T is reduced, $\frac{b^2}{ac} \leq \frac{1}{4}$. We also have $\tilde{x}\tilde{z} > \tilde{y}^2$ because $\det(\tilde{Q}) > 0$. So all the squares in Equations (8.3) and (8.4) must be bounded by $K^{2+\epsilon}$. For the first square, we get

$$a\tilde{x} - c\tilde{z} \ll K^{1+\epsilon}. \tag{8.5}$$

In particular $K^{2-\epsilon} \ll a\tilde{x} \sim c\tilde{z} \ll K^{2+\epsilon}$, i.e. $a\tilde{x}$ and $c\tilde{z}$ are of the same size. This is because the product of the terms is of size $\det(T\tilde{Q}) \gg K^{4-\epsilon}$ (by Equation (4.3) and considerations at the beginning of this chapter). Using that $a \leq c$ and $K^{2-\epsilon} \ll ac \ll K^{2+\epsilon}$, this equation also gives

$$\tilde{z} = \frac{a}{c}\tilde{x} + O\left(\frac{K^{1+\epsilon}}{c}\right) \leq \tilde{x} + O(K^{2\epsilon}).$$

We introduce the notation: $\tilde{z} \lesssim \tilde{x} \Leftrightarrow \tilde{z} \leq \tilde{x} + O(K^\epsilon)$ as $K \to \infty$. Equation (8.5) allows us to rearrange the right square:

$$K^{2+\epsilon} \gg 2ac\tilde{y} + b(a\tilde{x} + c\tilde{z}) \sim 2c(a\tilde{y} + b\tilde{z}) \sim 2a(c\tilde{y} + b\tilde{x}).$$

This gives the two other relations

$$a\tilde{y} + b\tilde{z} \ll \frac{K^{2+\epsilon}}{c}, \tag{8.6}$$

$$c\tilde{y} + b\tilde{x} \ll \frac{K^{2+\epsilon}}{a}. \tag{8.7}$$

In particular, we have $\tilde{y} = -\tilde{z}\frac{b}{a} + O(K^\epsilon)$. Using the relation $2|b| \leq a$, we get $2|\tilde{y}| \lesssim \tilde{z} \lesssim \tilde{x}$. So $\tilde{Q}$ is almost in a "reversed" reduced form and in particular $K^{2-\epsilon} \ll \tilde{x}\tilde{z} \asymp \det(\tilde{Q}) \asymp \frac{xz}{\det(C^2)} \ll K^{2+\epsilon}$.

Lemma 8.2. *Let $\epsilon > 0$ and $K \in 2\mathbb{N}$. Let $T, Q \in \mathcal{P}(\mathbb{Z})$ such that $K^{4-\epsilon} \ll \det(TQ) \ll K^{4+\epsilon}$ and $\det(T) - \det(Q) \ll K^{3/2+\epsilon}$, and $C \in M_2(\mathbb{Z})$ such that $0 \neq \det(C) \ll K^\epsilon$ and $\|C\| \ll K^{2+\epsilon}$. If*

$$\|C\| \gg K^{2\epsilon}$$

then the integral I in Equation (8.2) satisfies

$$I \ll_A K^{-A},$$

i.e. $\|C\| \ll K^{2\epsilon}$ up to a negligible error.

Proof. Following the hypothesis, we see that C^{-1} has coefficients in $\frac{1}{\det(C)}\mathbb{Z}$ and $|\det(C^{-1})| \ll 1$. Therefore $\det(C)\|C^{-1}\| = \|C\|$ for the ∞-norm on $M_2(\mathbb{R})$. So it is equivalent to prove that $\|C^{-1}\| \ll K^\epsilon$. We can use the results of this subsection and the last.

The proof relies on the numbers of non-zero entries in C^{-1}. Because $\det(C) \neq 0$, there are at most two zeros and in that last case, C^{-1} is diagonal or anti-diagonal. Since $\det(C^{-1}) \ll 1$, the result is obvious in this case. Computing the product $\tilde{Q} = C^{-1}QC^{-t}$, we have

$$\tilde{x} = xc_{11}^2 + 2yc_{11}c_{12} + zc_{12}^2$$
$$\tilde{z} = xc_{21}^2 + 2yc_{21}c_{22} + zc_{22}^2.$$

The matrix Q is reduced. Therefore we have $2|yc_{11}c_{12}| \leq |y|(c_{11}^2 + c_{12}^2) \leq \frac{1}{2}xc_{11}^2 + \frac{1}{2}zc_{12}^2$ and the same for the second equation. We get

$$\tilde{x} \asymp xc_{11}^2 + zc_{12}^2,$$
$$\tilde{z} \asymp xc_{21}^2 + zc_{22}^2.$$

If $c_{12}c_{22} \neq 0$, then we have $\tilde{x}, \tilde{z} \gg z \geq x$. Therefore $xz \leq z^2 \ll \tilde{x}\tilde{z} \asymp \frac{xz}{\det(C)^2} \ll xz$. We deduce that $x \asymp z$ and we must have $\tilde{x} \asymp \|C^{-1}\|^2 x$ or $\tilde{z} \asymp \|C^{-1}\|^2 x$. Then $\frac{xz}{\det(C)^2} \asymp \tilde{x}\tilde{z} \gg \|C^{-1}\|^2 xz$ and so $\|C^{-1}\|^2 \ll \det(C)^{-2} \ll 1$.

If $c_{12} = 0$, then we have $\tilde{x} \gtrsim \tilde{z} \gg z \geq x$. So $\frac{xz}{\det(C)^2} \asymp \tilde{x}\tilde{z} \gg z(z + O(K^\epsilon))$ and $z \geq x \gg z + O(K^\epsilon)$. Therefore $z \asymp x$ and we can finish as above.

The last case is $c_{22} = 0$. We have $\frac{xz}{\det(C)^2} \asymp \tilde{x}\tilde{z} \gg c_{12}^2c_{21}^2xz$ so $c_{12}, c_{21} \ll 1$ and $\tilde{z} \asymp xc_{21}^2 \asymp x$. For $\tilde{y}$, we have

$$\tilde{y} = c_{11}c_{21}x + (c_{11}c_{22} + c_{12}c_{21})y + c_{12}c_{22}z = c_{21}(c_{11}x + c_{12}y).$$

Let suppose that $c_{11} \gg K^{2\epsilon}$, so that

$$\tilde{z} + O(K^\epsilon) \gg |\tilde{y}| \asymp c_{21}c_{11}x \gg K^{2\epsilon}c_{21}x \asymp K^{2\epsilon}\tilde{z}.$$

This is a contradiction. Therefore $\|C\| \ll K^{2\epsilon}$. □

The next lemma is a way to decouple the relationship between the variables.

Lemma 8.3. *Let $\epsilon > 0$ and $K \in 2\mathbb{N}$. Let $T, Q \in \mathcal{P}(\mathbb{Z})$ such that $K^{4-\epsilon} \ll \det(TQ) \ll K^{4+\epsilon}$ and $\det(T) - \det(Q) \ll K^{3/2+\epsilon}$, and $C \in M_2(\mathbb{Z})$ such that $\det(C) \neq 0$ and $\|C\| \ll K^\epsilon$. If*

$$ac - \det(C)^2\tilde{x}\tilde{z} \gg K^{3/2+\epsilon}$$

or

$$a - \det(C)\tilde{z} \gg K^{-1/2+\epsilon}\tilde{z}$$

then the integral I in Equation (8.2) satisfies

$$I \ll_A K^{-A},$$

that is, up to a negligible error,

$$ac = \det(C)^2\tilde{x}\tilde{z} + O(K^{3/2+\epsilon}), \tag{8.8}$$
$$a = \det(C)\tilde{z} + O(K^{-1/2+\epsilon}\tilde{z}). \tag{8.9}$$

Proof. We know that

$$\det(T) = \det(Q) + O(K^{3/2+\epsilon})$$
$$\Leftrightarrow ac - b^2 = \det(C)^2(\tilde{x}\tilde{z} - \tilde{y}^2) + O(K^{3/2+\epsilon}). \tag{8.10}$$

We want to simplify this using the other equations in this section. We multiply the Equations (8.6) and (8.7) together.

$$\begin{aligned}(ac)^2\tilde{y}^2 &= (c\tilde{z}b + O(K^{2+\epsilon}))(a\tilde{x}b + O(K^{2+\epsilon}))\\ &= ac\tilde{x}\tilde{z}b^2 + O(bK^{2+\epsilon}(a\tilde{x}+c\tilde{z}) + K^{4+2\epsilon}),\\ \Rightarrow \tilde{y}^2 &= \frac{\tilde{x}\tilde{z}}{ac}b^2 + O(bK^{4\epsilon}+K^{2\epsilon}).\end{aligned}$$

We simplified the big O term using Equation (8.5) that tells us that $K^{2-\epsilon} \ll a\tilde{x} \asymp c\tilde{z} \ll K^{2+\epsilon}$ and $ac \gg K^{2-\epsilon}$. Inserting this result in Equation (8.10), we get

$$\begin{aligned}ac-b^2 &= \det(C)^2(\tilde{x}\tilde{z}-\tilde{y}^2) + O(K^{3/2+\epsilon})\\ &= \det(C)^2\left(\tilde{x}\tilde{z} - \frac{\tilde{x}\tilde{z}}{ac}b^2 + O(bK^\epsilon + K^\epsilon)\right) + O(K^{3/2+\epsilon})\\ &= \det(C)^2\frac{\tilde{x}\tilde{z}}{ac}(ac-b^2) + O(K^{3/2+\epsilon}).\end{aligned}$$

We have $0 \neq ac - b^2 \asymp ac$. Multiplying by $\frac{ac}{ac-b^2}$, we get

$$ac = \det(C)^2\tilde{x}\tilde{z} + O(K^{3/2+\epsilon}).$$

By subtraction of Equation (8.10), we also get $b^2 = \det(C)^2y^2 + O(K^{3/2+\epsilon})$. Combining Equations (8.5) and (8.8) and recalling that $ac, c\tilde{z} \gg K^{2-\epsilon}$ (Equations (4.3) and (8.5)), we have

$$\begin{aligned}&\det(C)^2\tilde{z}(a\tilde{x}-c\tilde{z}) \ll K^{1+\epsilon}\det(C)^2\tilde{z},\\ +\quad &a(ac-\det(C)^2\tilde{x}\tilde{z}) \ll K^{3/2+\epsilon}a,\\ \hline &ca^2 - \det(C)^2c\tilde{z}^2 \ll K^{1+\epsilon}\det(C)^2\tilde{z} + K^{3/2+\epsilon}a,\\ &a^2 - \det(C)^2\tilde{z}^2 \ll K^{1+3\epsilon}\frac{\tilde{z}}{c} + K^{3/2+\epsilon}\frac{a}{c} \ll K^{-1+4\epsilon}\tilde{z}^2 + K^{-1/2+2\epsilon}a^2,\\ &a - \det(C)\tilde{z} \ll K^{-1/2+2\epsilon}a \asymp K^{-1/2+3\epsilon}\tilde{z}.\end{aligned}$$

The last equation comes from the observation that, on the line above, the two bounds on the right are smaller than a term on the left. Therefore the two terms must be of the same size. We can then factorize the left-hand side and simplify. □

Remark. Similarly, we can prove that $c - \det(C)\tilde{x} \ll K^{-1/2+\epsilon}\tilde{x}$ up to a negligible error.

8.4 Estimate of the rank 2 term

We now gather the results of the chapter and give the bound for the rank 2 term. First, we analyze the sum coming from the Fourier series and the spectral integral. Each term in the T, Q-sum has the following shape. We suppose for the following argument that C is fixed.

$$\begin{aligned}&\frac{1}{\epsilon(T)\epsilon(Q)\det(TQ)^{3/4}}\int_0^{\pi/2}\int_0^\infty e\left((s_1-s_2)^2t + \frac{\sin(\alpha)^2}{t}\right)\\ &\qquad\cdot\int_{\Lambda_{\text{ev}}} w_-(\pi s_1 s_2 t)\phi(z_T)\bar{\phi}(z_Q)d\phi\,\frac{dt}{t}\sin(\alpha)d\alpha\\ &\qquad\ll \det(TQ)^{-3/4}\cdot K^\epsilon\cdot K^2\cdot K^{1/2+\epsilon}\frac{\det(TQ)^{1/4}}{\sqrt{ax}}\\ &\qquad\ll K^{1/2+3\epsilon}\frac{1}{\sqrt{ax}}\end{aligned}$$

We used the following estimates. Recall that $\det(T), \det(Q) \gg K^{2-\epsilon}$ and $\epsilon(T), \epsilon(Q) \ll 1$ up to a negligible error. The integrals over t and α are of size K^ϵ, as seen in Equation (8.2) (with K^2 being the size of $w_\pm$). The spectral integral is bounded using Lemma 5.4. Now, we count the number of

T and Q using the cut-off we computed. First, we fix Q. This also fix $\tilde{Q}$ since C is fixed. We fix a, c and b in this order. Equations (8.9), (8.5) and (8.6) give respectively

$$a = \det(C)\tilde{z} + O\left(\frac{\tilde{z}}{K^{1/2-\epsilon}}\right), \qquad c = \frac{a\tilde{x}}{\tilde{z}} + O\left(\frac{K^{1+\epsilon}}{\tilde{z}}\right), \qquad b = \frac{a\tilde{y}}{\tilde{z}} + O\left(\frac{K^{2+\epsilon}}{c\tilde{z}}\right).$$

Note that in the big O of the first equation, the fraction can be smaller than 1. We also know that $c\tilde{z} \gg K^{2-\epsilon}$. Therefore to fix T, we have

$$O\left(\left(\frac{\tilde{z}}{K^{1/2-\epsilon}} + 1\right) \cdot \frac{K^{1+\epsilon}}{\tilde{z}} \cdot \frac{K^{2+\epsilon}}{c\tilde{z}}\right) = O\left(\left(K^{-1/2+\epsilon} + \frac{1}{\tilde{z}}\right) K^{1+3\epsilon}\right)$$

possible choices. We also see that $a \sim \frac{K^{2+\epsilon}}{\tilde{x}}$. We use the divisor bound, Equation (8.9), $\tilde{z} \ll K^{1+\epsilon}$, $x \gg 1$ and $\tilde{x}\tilde{z} \ll K^{2+\epsilon}$ to get that the T, Q-sum is bounded by

$$\begin{aligned}
&\sum_{K^{2-\epsilon} \ll \tilde{x}\tilde{z} \ll K^{2+\epsilon}} \sum_{2|\tilde{y}| \ll \tilde{z} + O(K^\epsilon)} \left(K^{-1/2+\epsilon} + \frac{1}{\tilde{z}}\right) K^{1+\epsilon} \cdot K^{1/2+\epsilon} \frac{\sqrt{\tilde{x}}}{K^{1-\epsilon}} \\
&\qquad \ll \sum_{K^{2-\epsilon} \ll \tilde{x}\tilde{z} \ll K^{2+\epsilon}} \left(\frac{\tilde{z}\sqrt{\tilde{x}}}{K^{1/2-\epsilon}} + \sqrt{\tilde{x}}\right) K^{1/2+4\epsilon} \\
&\qquad \ll \sum_{K^{2-\epsilon} \ll \tilde{x}\tilde{z} \ll K^{2+\epsilon}} \left(K^{1/2+2\epsilon}\sqrt{\tilde{z}} + \sqrt{\tilde{x}}\right) K^{1/2+4\epsilon} \\
&\qquad \ll K^{2+\epsilon} \cdot \left(K^{1+3\epsilon} + K^{1+\epsilon}\right) \cdot K^{1/2+4\epsilon} \\
&\qquad \ll K^{3.5+8\epsilon}.
\end{aligned}$$

We have estimated the T, Q-sum, as always up to a negligible error. Now we combine this with other estimates to bound the term of rank 2. Note that there are $O(K^\epsilon)$ choices for C by Lemma 8.2. We get

$$\begin{aligned}
&\frac{96\pi^3}{\omega K^4} \sum_{T,Q \in \mathcal{P}(\mathbb{Z})/\operatorname{PSL}_2(\mathbb{Z})} \frac{1}{\epsilon(T)\epsilon(Q)\det(TQ)^{3/4}} \sum_{\substack{\|C\| \ll 1 \\ \det(C) \neq 0}} |\det(C)|^{3/2} \\
&\quad \cdot \int_0^{\pi/2} \int_0^\infty e\left((s_1 - s_2)^2 t + \frac{\sin(\alpha)^2}{t}\right) \int_{\Lambda_{\mathrm{ev}}} w_-(\pi s_1 s_2 t) \phi(z_T) \bar{\phi}(z_Q) d\phi \, \frac{dt}{t} \sin(\alpha) d\alpha \\
&\ll K^{-4} \cdot K^\epsilon \cdot K^{3.5+\epsilon} \\
&\ll K^{-1/2+2\epsilon}.
\end{aligned}$$

This proves the bound on the rank 2 term. Together with the results of Chapters 6 and 7, it concludes the proof of Theorem 1.4.

Chapter 9

Comments on the proof

This chapter is devoted to additional comments on the proof of Theorem 1.4. It also generalizes some lemmas that are seen in the course of the proof.

9.1 Complement to Lemma 5.3

The decay of the functions in Lemma 5.3 for $x \geq 2K$ can be obtained from the proof in [BC], Lemma 20. Following notations there, we have $w_\pm = W_2^\pm$, where

$$W_2^+ = \frac{e(3/8)}{2} \int_{-\infty}^{\infty} w(y) \int_{-\infty}^{\infty} v(\theta K^{-1/10}) e^{i\phi(\theta;x,y)} d\theta\, dy,$$

where w and v have support respectively in $[1, 2]$ and $[-2, 2]$. The phase function is

$$\phi(\theta; x, y) = -\frac{3\pi\theta}{K} + 2\pi\theta y + x\left(\cos\left(2\pi\frac{\theta}{K}\right) - 1\right).$$

So we have $1 \leq y \leq 2$ and $\theta \in [-2K^{1/10}, 2K^{1/10}]$. The derivative of ϕ is

$$\frac{d}{d\theta}\phi(\theta; x, y) = -\frac{3\pi}{K} + 2\pi y + 2\pi\frac{x}{K}\sin\left(2\pi\frac{\theta}{K}\right).$$

We use the Taylor expansions $\sin(t) = t + O(t^3)$. We have a stationary point θ_0 where

$$\sin(2\pi\theta_0/K) = \left(\frac{3}{2K} - y\right)\frac{K}{x},$$

i.e. $\theta_0 \asymp -\frac{K^2}{x}$. We apply Lemma 5.6 to the θ-integral. Following notations there, we have

$$\begin{aligned} \alpha, \beta &= \pm 2K^{1/10}, \\ X &= 1, & U &= K^{1/10}, \\ Y &= x, & Q &= K. \end{aligned}$$

We see that the integral is bounded by

$$\ll \frac{QX}{\sqrt{Y}} = \frac{K}{\sqrt{x}}.$$

This is also valid for w_- by a similar argument. The last inequality in Equation (5.1) gives the same bound for $\sum_{k \text{ even}} i^k w(k/K) J_{k-3/2}(x)$ so this is also valid for $w_0(x)$.

9.2 Generalization of Lemma 6.3.

The following is a generalization of the Lemma on the Harish-Chandra inverse transform. It shows how the decay property of a function on the spectral side of the pre-trace formula transfers to the geometric side. The proof is basically the same.

Lemma 9.1. *Let $h(\tau)$ be a function satisfying the condition for the Harish-Chandra inversion, that is holomorphic in the strip* $\mathrm{Im}(\tau) \leq 2$ *and such that*

$$T^j \frac{d^j}{d\tau^j} h(u) \ll_{A,j} \left(1 + \frac{|\tau|}{T}\right)^{-A}.$$

Let $\kappa(u)$ be the Harish-Chandra inverse of h and let $\epsilon > 0$, $A > 0$, $T \geq 1$. Then the geometric side of the pre-trace formula is negligible for u much larger than T^{-2}. More precisely

$$\sum_{\substack{\gamma \in \mathrm{SL}_2(\mathbb{Z}) \\ u(z,\gamma z) \geq T^{-2+\epsilon}}} |\kappa(u(z, \gamma z))| \ll_{A,\epsilon} T^{-A}(\mathrm{Im}(z) + 1).$$

Proof. We have (see (1.64) in [Iwa]):

$$\begin{aligned} g(r) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ir\tau} h(\tau) d\tau, \\ q(v) &= \frac{1}{4\pi} \int_{-\infty}^{\infty} h(\tau)(\sqrt{v+1} + \sqrt{v})^{2i\tau} d\tau, \\ \kappa(u) &= \frac{1}{4\pi^2 i} \int_u^{\infty} \frac{1}{\sqrt{v-u}} \int_{-\infty}^{\infty} h(\tau) \frac{(\sqrt{v+1} + \sqrt{v})^{2i\tau}}{\sqrt{v(v+1)}} \tau d\tau \, dv. \end{aligned}$$

We consider first $q(v)$. Since h is holomorphic in a strip, we can move the integration line to $\tau \mapsto \tau + 2i$. Integrating by parts, we get

$$\begin{aligned} 4\pi q(v) &= (\sqrt{v+1} + \sqrt{v})^{-4} (-2i \log(\sqrt{v+1} + \sqrt{v}))^{-j} \\ &\quad \cdot \int_{-\infty}^{\infty} (h^{(j)}(\tau + 2i)(\tau + 2i) + j h^{(j-1)}(\tau + 2i))(\sqrt{v+1} + \sqrt{v})^{2i\tau} d\tau \\ &\ll_{A,j} (\sqrt{v+1} + \sqrt{v})^{-4} \left(\log(\sqrt{v+1} + \sqrt{v}) T\right)^{-j} T. \end{aligned}$$

In particular, we have a saving if $u \gg T^{-2+\epsilon}$. We obtain

$$\kappa(u) \ll_{A,j} T^{1-j\epsilon} \int_u^{\infty} \frac{dv}{\sqrt{v(v+1)(v-u)}(\sqrt{v+1} + \sqrt{v})^4}.$$

We split the integral in the intervals $]u, u+1[$ and $[u+1, \infty[$. We get

$$\int_u^{\infty} \frac{dv}{\sqrt{v(v+1)(v-u)}(\sqrt{v+1} + \sqrt{v})^4} \ll \frac{2}{\sqrt{u(u+1)}(\sqrt{u+1} + \sqrt{u})^4} + \frac{1}{2(u+1)^2}.$$

We have either $u \gg 1$ or $1 + u \asymp u$. For $u \gg T^{-2+\epsilon}$, we obtain respectively

$$\begin{aligned} \kappa(u) &\ll_{A,j} T^{1-j\epsilon} \frac{1}{u^2} && \text{if } u \gg 1, \\ \kappa(u) &\ll_{A,j} T^{2-j\epsilon} && \text{if } u \ll 1. \end{aligned}$$

We sum over γ. For $T^{-2+\epsilon} \ll u \ll 1$, we apply Lemma 6.2 directly and for $1 \ll u$, we split into dyadic intervals. We compute that

$$\sum_{\substack{\gamma \in \mathrm{SL}_2(\mathbb{Z}) \\ u(z,\gamma z) \gg T^{-2+\epsilon}}} |\kappa(u(z, \gamma z)| \ll_{j,\epsilon} T^{2-j\epsilon}(\mathrm{Im}(z) + 1).$$

We take j large enough to conclude the proof. □

Remark. We can improve the conditions on $h(\tau)$. If we ask for it to be holomorphic in the strip $\mathrm{Im}(\tau) \leq 1 + \delta$, $\delta > 0$, the proof works again but the bound blows up as $\delta \to 0$. Conversely, if we have a larger strip, we gain nothing in the bound with this proof.

9.3 Orbits of Heegner points

The estimate in Equation (6.4) can also be computed directly. For this, we analyze the distribution of Heegner points and their orbits. Recall that they lie in the classical fundamental domain for a corresponding reduced matrix. We use the notation of Chapter 6. We obtain the following result:

Lemma 9.2. *Let z_T be the Heegner point corresponding to the matrix T. Then*

$$\sum_{\substack{T\in\mathcal{P}(\mathbb{Z})/\operatorname{PSL}_2(\mathbb{Z})\\ \det(T)\ll k^{2+\epsilon}}} \frac{\#\operatorname{Aut}(T)}{\epsilon(T)^2\det(T)^{3/2}} \sum_{\substack{\gamma\in\operatorname{SL}_2(\mathbb{Z}):\\ k^{-2-\epsilon}\ll u(z_T,\gamma z_T)\ll k^{-1+\epsilon}}} |\kappa(u(z_T,\gamma z_T))| \ll_\epsilon k^{2.5+\epsilon}.$$

Proof. Since $u(z_T,\gamma z_T)$ is smaller than 1, we have that $\kappa(u(z,\gamma z)) \ll k^3$ by Equation (6.3). Combining this with $\frac{\#\operatorname{Aut}(T)}{\epsilon(T)^2} \ll 1$, we see that we only have to show that

$$\sum_{\substack{T\in\mathcal{P}(\mathbb{Z})/\operatorname{PSL}_2(\mathbb{Z})\\ \det(T)\ll k^{2+\epsilon}}} \frac{1}{\det(T)^{3/2}} \sum_{\substack{\gamma\in\operatorname{SL}_2(\mathbb{Z}):\\ k^{-2-\epsilon}\ll u(z_T,\gamma z_T)\ll k^{-1+\epsilon}}} 1 \ll k^{-1/2+\epsilon}$$

to get a bound of size $O(k^{2.5+\epsilon})$ for Equation (6.1). Geometrically, it is clear that z_T must be close to an edge of the fundamental domain if we want it to be close to another point in its orbit. We make this more precise. There are two types of Heegner points such that $u(z_T,\gamma z_T)$ is small. First, suppose z_T has large imaginary part, say $\operatorname{Im}(z_T) > 10$. For a translation $\gamma z = z_T + n$, $n\in\mathbb{Z}$, we have

$$u(z_T, z_T+n) = \frac{n^2}{4\operatorname{Im}(z_T)^2} = \frac{n^2\alpha^2}{4D}.$$

Therefore if $u(z_T, z_T+n) \ll k^{-1+\epsilon}$, we get $n^2 \ll \frac{D}{\alpha^2 k^{1-\epsilon}}$. To compute a bound, we sum over $D' = \alpha\delta \asymp D \ll k^{2+\epsilon}$. For a fixed D', there are $\ll (D')^{\epsilon/4}$ choices for α and δ by the divisor bound and there are α choices for β, so that

$$\begin{aligned}
\sum_{\substack{T\in\mathcal{P}(\mathbb{Z})/\operatorname{PSL}_2(\mathbb{Z})\\ \det(T)\ll k^{2+\epsilon}}} \frac{1}{\det(T)^{3/2}} \sum_{\substack{\gamma\in\operatorname{SL}_2(\mathbb{Z})\text{ translation}\\ k^{-2-\epsilon}\ll u(z_T,\gamma z_T)\ll k^{-1+\epsilon}}} 1 &\ll \sum_{D'\ll k^{2+\epsilon}} (D')^{-3/2} \sum_{\alpha\delta=D'} \sum_{|\beta|\le\alpha/2} \sum_{n^2\ll\frac{D'}{\alpha^2k^{1-\epsilon}}} 1\\
&\ll \sum_{D'\ll k^{2+\epsilon}} (D')^{-3/2+\epsilon/4} \sum_{\alpha\delta=D'} \alpha\sqrt{\frac{D'}{\alpha^2k^{1-\epsilon}}}\\
&\ll_\epsilon k^{-1/2+\epsilon/2} \sum_{D'\ll k^{2+\epsilon}} (D')^{-1+\epsilon/4}\\
&\ll_\epsilon k^{-1/2+\epsilon}
\end{aligned}$$

If $\operatorname{Im}(z_T) > 10$ and γ is not a translation, then $|z_T| \asymp \operatorname{Im}(z_T)$ and $\operatorname{Im}(\gamma z_T) \le 1$. Therefore $|z_T - \gamma z_T| \gg \operatorname{Im}(z_T) - \operatorname{Im}(\gamma z_T) \asymp |z_T|$ and

$$u(z_T,\gamma z_T) = \frac{|z_T-\gamma z_T|^2}{4\operatorname{Im}(z_T)\operatorname{Im}(\gamma z_T)} \gg \frac{|z_T|^2}{\operatorname{Im}(z_T)} \gg |z_T| \gg 1.$$

So there is no such γ with $u(z_T,\gamma z_T) \ll k^{-1+\epsilon}$. Now, we analyze low-lying Heegner points where $\operatorname{Im}(z_T) \le 10$. Note that we have $|z_T|^2 = \frac{\delta}{\alpha} \asymp 1$. If γ is a translation, then the computation above shows that $u(z_T,\gamma z_T) \gg 1$. Since the Heegner points are in the fundamental domain and we ruled out the case where γ is a translation, $\operatorname{Im}(\gamma z_T) \le 1$. Hence

$$u(z_T,\gamma z_T) = \frac{|z_T-\gamma z_T|^2}{4\operatorname{Im}(z_T)\operatorname{Im}(\gamma z_T)} \gg |z_T-\gamma z_T|^2.$$

Any point in the orbit of z_T is at least as far as the point $z_T/|z_T|$ (in the Euclidean distance), as one can see by growing a circle around z_T. Suppose that $|z_T| - 1 \gg k^{(-1+\epsilon)/2}$, then we have

$$|z_T - z_T/|z_T||^2 = \frac{|z_T|^2}{|z_T|^2}\,||z_T|-1|^2 \gg k^{-1+\epsilon},$$

so this rules out this case. Now, if $|z_T| - 1 \ll k^{(-1+\epsilon)/2}$, that means

$$k^{(-1+\epsilon)/2} \gg |z_T| - 1 \asymp (|z_T| - 1)(|z_T| + 1) = |z_T|^2 - 1 = \frac{\delta - \alpha}{\alpha}.$$

So that $\delta - \alpha \ll k^{(-1+\epsilon)/2}\alpha$. In particular, $\alpha \asymp \delta$ and $\delta^2 \asymp \det(T) \ll k^{2+\epsilon}$. Clearly, for such a z_T, there is a finite number of γ such that $u(z_T, \gamma z_T) \ll k^{-1+\epsilon}$ (at most 12 when z_T is close to $\frac{\pm 1 + i\sqrt{3}}{2}$, for k large enough). The sum over such T and γ gives

$$\begin{aligned}\sum_{\substack{T\in\mathcal{P}(\mathbb{Z})/\operatorname{PSL}_2(\mathbb{Z})\\ \det(T)\ll k^{2+\epsilon}}} \frac{1}{\det(T)^{3/2}} \sum_{\substack{\text{such } \gamma\in\operatorname{SL}_2(\mathbb{Z})\\ k^{-2-\epsilon}\ll u(z_T,\gamma z_T)\ll k^{-1+\epsilon}}} 1 &\ll \sum_{\delta\ll k^{1+\epsilon}} \delta^{-3} \sum_{\substack{\alpha\\ \delta-\alpha\ll k^{(-1+\epsilon)/2}\delta}} \sum_{|\beta|\leq\alpha/2} 1 \\ &\ll \sum_{\delta\ll k^{1+\epsilon}} \delta^{-3} \sum_{\substack{\alpha\\ \delta-\alpha\ll k^{(-1+\epsilon)/2}\delta}} \alpha \\ &\ll \sum_{\delta\ll k^{1+\epsilon}} \delta^{-1} k^{(-1+\epsilon)/2} \\ &\ll k^{-1/2+\epsilon}\end{aligned}$$

□

Remark. It is also possible to do a similar computation on z_T and γ such that

$$k^{-2-\epsilon} \ll u(z_T, \gamma(-\overline{z_T})) \ll k^{-1+\epsilon}.$$

This situation appears when restricting to the even spectrum. In that case, we also have to consider the points z_T close to the imaginary axis.

9.4 Volume computation

We give a heuristic argument supporting that Theorem 1.4 is consistent with Quantum Unique Ergodicity on the imaginary axis. This is based on Appendix C in [BC]. A cusp form $f \in S_k^{(2)}$ is essentially supported on $iY \in i\mathcal{P}(\mathbb{R})$ such that there exists a $T \in \mathcal{P}(\mathbb{Z})$ with TY having eigenvalues λ_1, λ_2 of size $k/4\pi + O(k^{1+\epsilon})$. We make this more precise. We use the following estimate:

Proposition 9.3 ([DK], Lemma 4.1). *Let $f \in S_k^{(2)}$, $k \geq 6$ and $\alpha = 13/4$, $\beta = -3/4$. Then*

$$\frac{|a_f(T)|}{\|f\|_2} \ll \frac{(4\pi)^k k^\alpha \det(T)^{3/4-\beta}}{\Gamma(k)}.$$

Let $Y \in \mathcal{P}(\mathbb{R})$ with $\det(Y) \geq 1$. We have

$$\begin{aligned}F(Y) := \frac{f(iY)|\det(Y)^{k/2}}{\|f\|_2} &\ll \sum_{T>0} \frac{|a(T)|}{\|f\|} \det(TY)^{k/2} \det(T)^{-3/4} e^{-2\pi \operatorname{tr}(TY)} \\ &\ll \sum_{T>0} \frac{1}{\det(T)^2} \frac{(4\pi)^k k^\alpha \det(TY)^{k/2+2-\beta} e^{-2\pi \operatorname{tr}(TY)}}{\Gamma(k)}\end{aligned}$$

We consider the function

$$Y \mapsto \frac{(4\pi)^k k^\alpha \det(TY)^{k/2+2-\beta} e^{-2\pi \operatorname{tr}(TY)}}{\Gamma(k)}.$$

Note that the rest of the T-sum converges. Recall the Stirling formula $\log(\Gamma(k)) = k\log(k) - k + O(\log(k))$. Let $\lambda_i = \frac{k}{4\pi}(1 + E_i)$ for $i = 1, 2$ be the eigenvalues of TY. We estimate the logarithm of the function above:

$$\log\left[\frac{(4\pi)^k k^\alpha \det(TY)^{k/2+2-\beta}e^{-2\pi\operatorname{tr}(TY)}}{\Gamma(k)}\right] = k\log\left(\frac{4\pi\sqrt{\lambda_1\lambda_2}}{k}\right) + k - 2\pi(\lambda_1+\lambda_2) + O(\log(k))$$
$$= \frac{k}{2}\log((1+E_1)(1+E_2)) - \frac{k}{2}(E_1+E_2) + O(\log(k))$$
$$= -k\frac{E_1^2+E_2^2}{4} + O(k(E_1^3+E_2^3)) + O(\log(k)).$$

So if we do not have $|E_i| \ll k^{-1/2}\log(k)$ for $i=1$ and $i=2$, we get an exponential decay in k. Note that if $E_1 \gg 1$ or $E_2 \gg 1$, we already have an exponential decay in the second line of the equation above. Clearly $\det(T) \gg 1$. We conclude that, up to a negligible error, $\det(Y) \ll k^2$. Since $|F(Y)|$ is invariant under $Y \mapsto Y^{-1}$, we can also suppose that $\det(Y) \gg k^{-2}$. We compute the volume of such Y:

$$\int_{\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}}\int_{k^{-2}\ll r\ll k^2}\frac{dr}{r}\frac{dx\,dy}{y^2} = \mathrm{Vol}(\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H})4\log(k) + O(1). \tag{9.1}$$

We see that the constant 4 is consistent with QUE.

9.5 Other remarks

Concerning Lemma 6.5, the literature gives for the average of the class number of *primitive* discriminants the formula [CI]

$$\sum_{-D\le X} h_{\mathrm{prim}}(D) = \frac{\pi}{18\zeta(3)}X^{3/2} - \frac{2}{\pi^2}X + O(X^{3/4}).$$

The formula at the end of Section 5.1 in [BC] is for *all* discriminants. We recall it:

$$\sum_{-D\le X} h(D) = \frac{\pi}{18}X^{3/2} - \frac{1}{4}X + O(X^{3/4}).$$

It is easy to go from one formula to the other. We just notice the following. Let $ax^2+bxy+cy^2$ be a primitive quadratic form of discriminant $D<0$ with $(a,b,c)=1$. It gives rise to a non-primitive one of the form $nax^2+nbxy+ncy^2$ for any integer n. Its discriminant is n^2D. Hence we have

$$\sum_{-D\le X} h(D) = \sum_{-D\le X} h_{\mathrm{prim}}(D) + \sum_{-4D\le X} h_{\mathrm{prim}}(D) + \sum_{-9D\le X} h_{\mathrm{prim}}(D) + \dots$$
$$= \frac{\pi}{18\zeta(3)}X^{3/2}\sum_{n\le\sqrt{X}}\frac{1}{n^3} - \frac{2}{\pi^2}X\sum_{n\le\sqrt{x}}\frac{1}{n^2} + O\left(X^{3/4}\sum_{n\le\sqrt{X}}\frac{1}{n^{3/2}}\right)$$
$$= \frac{\pi}{18}X^{3/2} - \frac{1}{4}X + O(X^{3/4}).$$

The opposite way between the two formulae goes by Möbius inversion.

A natural variation of Theorem 1.4 is to consider the level aspect. For Siegel modular forms, there are two possibilities: the Siegel subgroup and the paramodular subgroup. We consider the first one. The Siegel congruence subgroup is

$$\Gamma_0^{(2)}(N) := \left\{\begin{pmatrix}A & B\\ C & D\end{pmatrix} \mid C = 0 \pmod N\right\}.$$

We fix a weight K and consider $N\to\infty$. The restriction of the action on the imaginary axis to this subgroup is again $\mathrm{SL}_2(\mathbb{Z})$. We obtain the twisted Koecher-Maass series in a similar way. The Kitaoka formula is modified, with $N|c$ in the rank 1 term and $N|\det(C)$ in the rank 2 term. We do a discussion similar to the start of Chapter 8 to see how large the C-sum is in the rank 2 term. The conductor of the series is N^2k^4. Therefore, the approximate functional equation gives a cut-off

$$\det(TQ) \ll (N^2k^4)^{1+\epsilon}.$$

The generalized Bessel function $\mathcal{J}_\ell$ gives us a cut-off

$$k^{1-\epsilon} \ll \frac{\sqrt{\det(TQ)}}{\det(C)}.$$

So that

$$N \leq \det(C) \ll \frac{\sqrt{\det(TQ)}}{k^{1-\epsilon}} \ll k^{\epsilon}N^{1+\epsilon}.$$

We see that the C-sum is short, but not zero, in contrast to Chapter 2. A challenging part in this problem is to understand the old forms. It is possible to lift forms from each level $M|N$ to level N. These cusp forms appear in the Kitaoka formula, but they are not well behaved for the Koecher-Maass series. To make the problem approachable, we can reduce to square-free level or prime level. Unfortunately, this makes it difficult to average over the level as we did over $k \in [K, 2K]$ in the weight aspect. Therefore, we conjecture that our method to prove Theorem 1.4 does not give an asymptotic in the level aspect if we do not average over N.

Chapter 10

Higher degrees

In this section, we discuss the higher degree analog of the Kitaoka formula for modular forms on $\mathrm{Sp}_{2n}(\mathbb{R})$. The shape of the formula is similar but it has more terms, corresponding to matrices C of rank 0 to n. Of central interest are the Kloosterman sum and the Bessel function of "full rank". They are the objects that appear in the rank n term of the generalized Kitaoka formula. The generalized Kloosterman sum is

$$K^{(n)}(Q,T;C) := \sum_{\left(\begin{smallmatrix} A & * \\ C & D \end{smallmatrix}\right) \in \Gamma_\infty \backslash \mathrm{Sp}_{2n}(\mathbb{Z})/\Gamma_\infty} \mathrm{etr}(AC^{-1}Q + C^{-1}DT). \tag{10.1}$$

Here for a matrix M, $\mathrm{etr}(M) = e^{2\pi i \,\mathrm{tr}(M)}$, $\Gamma_\infty = \{\left(\begin{smallmatrix} I_n & X \\ 0 & I_n \end{smallmatrix}\right) \mid X = X^t\}$ and the sum is over a set of representatives $X(C)$ of matrices in $\Gamma_\infty \backslash \mathrm{Sp}_{2n}(\mathbb{Z})/\Gamma_\infty$ with bottom-left block equal to C. The generalized Bessel function is

$$\mathcal{J}_k^{(n)}(Q,T;C) := \int_X \mathrm{etr}(-Z^{-1}C^{-1}QC^{-t} - ZT)\det(Z)^{-k}dX,$$

where $Z = X + iY \in \mathbb{H}^{(n)}$, $\mathrm{Im}(Z) > 0$ is fixed and X runs over symmetric matrices. In the next section, we show how to factorize $K^{(n)}$ with respect to C. In the last section, we prove a non-trivial bound in the case $n = 3$ and $C = pI_3$.

10.1 Factorization and Smith normal form

Lemma 10.1 (Smith normal form)**.** *Let $A \in M_{m,n}(\mathbb{Z})$. There exist matrices $U \in \mathrm{GL}_m(\mathbb{Z}), V \in \mathrm{GL}_n(\mathbb{Z})$ such that $UCV = D$ is a $m \times n$ diagonal matrix with the elements on the diagonal satisfying $d_{ii} | d_{i+1,i+1}$ and $d_{ii} = 0$ for $i > \mathrm{rk}(A)$. The d_{ii} are unique up to a unit and called* invariant factors.

Notation. Let A, B be two matrices. We set $A[B] := B^t AB$.

Lemma 10.2. *For $U, V \in \mathrm{GL}_n(\mathbb{Z})$, we have*

$$K^{(n)}(Q[U], T[V]; U^t CV) = K(Q,T;C).$$

Proof. Let $X(C)$ be as above. We have

$$\begin{pmatrix} U^{-1} & \\ & U^t \end{pmatrix} X(C) \begin{pmatrix} V & \\ & V^{-t} \end{pmatrix} = X(U^t CV).$$

This can be deduced from the identity

$$\begin{pmatrix} U^{-1} & \\ & U^t \end{pmatrix} \begin{pmatrix} A & * \\ C & D \end{pmatrix} \begin{pmatrix} V & \\ & V^{-t} \end{pmatrix} = \begin{pmatrix} U^{-1}AV & * \\ U^t CV & U^t DV^{-t} \end{pmatrix}.$$

The matrix on the right-hand side is also in $\mathrm{Sp}_{2n}(\mathbb{Z})$ and the identity can be reversed. Moreover, matrices of the form $\left(\begin{smallmatrix} U & \\ & U^{-t} \end{smallmatrix}\right)$ normalize Γ_∞. So we have a bijection between $X(C)$ and $X(U^t CV)$.

The lemma is then established by invariance of the trace under conjugation. More precisely,

$$\begin{aligned}
K^{(n)}(Q[U],T[V];U^tCV) &= \sum_{\left(\begin{smallmatrix} A & * \\ C & D\end{smallmatrix}\right)\in X(U^tCV)} \operatorname{etr}(AC^{-1}Q[U]+C^{-1}DT[V]) \\
&= \sum_{\left(\begin{smallmatrix} A & * \\ C & D\end{smallmatrix}\right)\in X(C)} \operatorname{etr}((U^{-1}AV)(U^tCV)^{-1}Q[U]+(U^tCV)^{-1}(U^tDV^{-t})T[V]) \\
&= \sum_{\left(\begin{smallmatrix} A & * \\ C & D\end{smallmatrix}\right)\in X(C)} \operatorname{etr}(U^{-t}AC^{-1}U^{-t}Q[U]+V^{-t}C^{-1}XV^{-t}T[V]) \\
&= K^{(n)}(Q,T;C).
\end{aligned}$$

□

Lemma 10.3. *Let F,H,C integral diagonal matrices with $FH=C$, $f_{ii}|f_{i+1,i+1}$, $h_{ii}|h_{i+1,i+1}$ and $(f_{nn},g_{nn})=1$. Let $r,s\in\mathbb{Z}$ such that $rf_{nn}+sh_{nn}=1$. Then*

$$K^{(n)}(Q[\bar{H}],T;F)\cdot K^{(n)}(Q[\bar{F}],T;H)=K^{(n)}(Q,T;C),$$

where

$$\bar{F}=rf_{nn}F^{-1}, \qquad \bar{H}=sh_{nn}H^{-1}.$$

In particular, $\bar{F}F+\bar{H}H=I_n$.

Proof. This is done in Lemmas 1, 2 and 3 of Kitaoka's article [Kit]. The proof applies to larger n without substantial change. First, we have $\left(\begin{smallmatrix} A & B \\ C & D\end{smallmatrix}\right)\in \mathrm{Sp}_{2n}(\mathbb{Z})$ if and only if

$$\begin{pmatrix} HA & HB-\bar{F}A^tD \\ F & \bar{H}D\end{pmatrix}, \begin{pmatrix} FA & FB-\bar{H}A^tD \\ H & \bar{F}D\end{pmatrix}\in \mathrm{Sp}_{2n}(\mathbb{Z}).$$

We use the characterization of symplectic matrices given in Equation (3.1). If $\left(\begin{smallmatrix} A & B \\ C & D\end{smallmatrix}\right)\in \mathrm{Sp}_{2n}(\mathbb{R})$, the two other matrices are as well. Conversely, we have

$$\begin{aligned}
&A=\bar{F}HA+\bar{F}FA, \\
&B=2\bar{F}\bar{H}A^tD+\bar{F}(FB-\bar{H}A^tD)+\bar{H}(HB-\bar{F}A^tD), \\
&D=H\bar{H}D+FX_1D.
\end{aligned}$$

We get

$$\begin{aligned}
\operatorname{tr}(HAF^{-1}&Q[\bar{H}]+F^{-1}\bar{H}DT)+\operatorname{tr}(FAH^{-1}Q[\bar{F}]+H^{-1}\bar{F}DT) \\
&=\operatorname{tr}(\bar{H}HAF^{-1}\bar{H}Q+\bar{F}FAH^{-1}\bar{F}Q)+\operatorname{tr}((F^{-1}\bar{H}+H^{-1}\bar{F})DT) \\
&=\operatorname{tr}(A(sh_{nn}F^{-1}\bar{H}+rf_{nn}H^{-1}\bar{F})Q)+\operatorname{tr}(C^{-1}DT) \\
&=\operatorname{tr}((A((sh_{nn})^2+(rf_{nn})^2)C^{-1}Q)+\operatorname{tr}(C^{-1}DT).
\end{aligned}$$

We used that $\bar{H}H$ and $\bar{F}F$ are scalar matrices. Finally, notice that $sh_{nn}rf_{nn}C^{-1}$ has integral entries, so

$$\operatorname{etr}(A((sh_{nn})^2+(rf_{nn})^2)C^{-1}Q)=\operatorname{etr}(A(sh_{nn}+rf_{nn})^2C^{-1}Q)=\operatorname{etr}(AC^{-1}Q).$$

It remains to show that the map

$$\begin{pmatrix} A & B \\ C & D\end{pmatrix}\mapsto\left(\begin{pmatrix} HA & HB-\bar{F}A^tD \\ F & \bar{H}D\end{pmatrix}, \begin{pmatrix} FA & FB-\bar{H}A^tD \\ H & \bar{F}D\end{pmatrix}\right)$$

is an isomorphism between $X(C)$ and $X(F)\times X(H)$. Kitaoka showed in Lemma 2 of [Kit] that the map and its inverse factor through the double quotient. It can be explicitly verified or using that the block D (mod $C\Lambda$), where Λ is the set of integral symmetric matrices, characterizes an element in the quotient for a given C. □

In view of the above lemmas, we can suppose that C is an integral diagonal matrix with coefficients that are prime powers, i.e. $C = \operatorname{diag}(p^{\alpha_1}, p^{\alpha_1+\alpha_2}, \ldots, p^{\alpha_1+\cdots+\alpha_n})$. There are four different cases. First, there are the cases where $\alpha_1 = 1$. Second, if $C = p^\alpha I_n$ for $\alpha > 1$, we can apply Salié's estimate as done by Tóth [Wil, Tót] to get square root cancellation. The most technical case is $C = pI_n$. Finally, if there is an n with $\alpha_n > 1$, we can make a Taylor series argument to get some cancellation. We consider the third case for $n = 3$ and $p \geq 3$ in the next section. We do not develop the other cases in this thesis.

10.2 The case $C = pI_3$

In this section, we prove Theorem 1.6. Let $C = pI_3$ for an odd prime p. We can rewrite $K^{(3)}(Q, T; C)$ as

$$\sum_{AD=I_3 \bmod p} etr\left(\frac{AQ + DT}{p}\right) = \sum_{\substack{A \bmod p \\ k\det(A)=1 \bmod p}} etr\left(\frac{AQ + \bar{k}A^*T}{p}\right)$$

where $k = \det(A) \pmod p$. The sum is over integral matrices $A \pmod p$ such that $(p, \det(A)) = 1$ and A^* designates the adjugate matrix of A, with $A^* = \det(A)A^{-1}$. Suppose that a_{11} and $m = a_{11}a_{22} - a_{12}^2$ are non-zero. We do a change of variable with respect to the diagonal minor $m = A(3|3)$. More precisely

$$\begin{aligned} m = \begin{vmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{vmatrix} &= a_{11}a_{22} - a_{12}^2 \Leftrightarrow a_{22} = \bar{a}_{11}(m + a_{12}^2), \\ k = \det(A) &= a_{13}\begin{vmatrix} a_{12} & a_{22} \\ a_{13} & a_{23} \end{vmatrix} - a_{23}\begin{vmatrix} a_{11} & a_{12} \\ a_{13} & a_{23} \end{vmatrix} + a_{33}\begin{vmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{vmatrix} \\ \Leftrightarrow a_{33} = \bar{m}&\left(k - a_{13}\begin{vmatrix} a_{12} & \bar{a}_{11}(m + a_{12}^2) \\ a_{13} & a_{23} \end{vmatrix} + a_{23}\begin{vmatrix} a_{11} & a_{12} \\ a_{13} & a_{23} \end{vmatrix}\right). \end{aligned}$$

Therefore $a_{22} \mapsto m,\ a_{33} \mapsto k$ is a valid change of variable on matrices A with $a_{11} \neq 0$ and $m \neq 0$. We go from 7 variables with one equation $(A = (a_{ij}), k$ with $k = \det(A))$ to 6 variables without relationships. The computation of the change of variable was done on Sage. We obtain

$$\begin{aligned} &\operatorname{tr}(AQ + \bar{k}A^*T) \\ &\quad = ((a_{12}^2 + m)(((a_{12}^2 + m)a_{13}\overline{a_{11}} - a_{12}a_{23})a_{13} - (a_{12}a_{13} - a_{11}a_{23})a_{23} + k)\overline{a_{11}}\bar{m} - a_{23}^2)\bar{k}t_{11} \\ &\quad - 2\,((((a_{12}^2 + m)a_{13}\overline{a_{11}} - a_{12}a_{23})a_{13} - (a_{12}a_{13} - a_{11}a_{23})a_{23} + k)a_{12}\bar{m} - a_{13}a_{23})\bar{k}t_{12} \\ &\quad - 2\,((a_{12}^2 + m)a_{13}\overline{a_{11}} - a_{12}a_{23})\bar{k}t_{13} \\ &\quad + ((((a_{12}^2 + m)a_{13}\overline{a_{11}} - a_{12}a_{23})a_{13} - (a_{12}a_{13} - a_{11}a_{23})a_{23} + k)a_{11}\bar{m} - a_{13}^2)\bar{k}t_{22} \\ &\quad + 2\,(a_{12}a_{13} - a_{11}a_{23})\bar{k}t_{23} + ((a_{12}^2 + m)a_{11}\overline{a_{11}} - a_{12}^2)\bar{k}t_{33} + (a_{12}^2 + m)\overline{a_{11}}q_{22} \\ &\quad + (((a_{12}^2 + m)a_{13}\overline{a_{11}} - a_{12}a_{23})a_{13} - (a_{12}a_{13} - a_{11}a_{23})a_{23} + k)\bar{m}q_{33} \\ &\quad + a_{11}q_{11} + 2\,a_{12}q_{12} + 2\,a_{13}q_{13} + 2\,a_{23}q_{23}. \end{aligned}$$

This (very long) expression has inverses in a_{11}, m and k. The degree of the different variables are summarized in the table below.

Coef.	a_{11}	$\overline{a_{11}}$	a_{12}	a_{13}	a_{23}	m	$\bar{m}$	k	$\bar{k}$
Degree	2	2	4	2	2	1	1	1	1

(10.2)

We sum $\operatorname{tr}(AQ + \bar{k}A^*T)$ over the above variable with the extra condition that $a_{11}mk \neq 0 \pmod p$, so that we can invert them. If $a_{11} = 0$, we take a trivial sum over the other variables and get a bound of size $O(p^5)$. If $a_{11} \neq 0$ and $m = 0$, we get a formula for a_{22}. We take again a trivial bound on the rest of the variables. Otherwise, we consider the sums in a_{23} and m. In number theory, there are essentially two exponential sums that can be computed exactly, the Gauss sum and the Salié sum. We give their exact values in the next two lemmas.

Lemma 10.4 (Gauss sum, [IK], Equation (3.38)). *Let p be an odd prime and α, β be two integers. The Gauss sum has the following value. If $p|\alpha, \beta$, the sum has value p. If $(p,\alpha) = 1$, it is below and the sum vanishes otherwise.*

$$G(\alpha, \beta; p) = \sum_{n \bmod p} e\left(\frac{\alpha n^2 + 2\beta n}{p}\right) = \sum_{n \bmod p} e\left(\frac{\alpha(n+\bar{\alpha}\beta)^2 - \bar{\alpha}\beta^2}{p}\right) = \epsilon_p \sqrt{p}\left(\frac{\alpha}{p}\right) e\left(\frac{-\bar{\alpha}\beta^2}{p}\right),$$

where $\epsilon_p = 1$ if $p = 1 \pmod 4$ and i if $p = 3 \pmod 4$.

Lemma 10.5 (Salié sum, [IK], Lemma 12.4). *Let p be an odd prime and α, β be two integers such that $(p, \alpha\beta) = 1$. Then*

$$S(\alpha, \beta; p) = \sideset{}{^*}\sum_{n \bmod p} \left(\frac{n}{p}\right) e\left(\frac{\alpha\bar{n} + \beta n}{p}\right) = \epsilon_p \sqrt{p}\left(\frac{\beta}{p}\right) \sum_{m^2 = \alpha\beta \bmod p} e\left(\frac{2m}{p}\right)$$

where ϵ_p is as above. If $p|\alpha$ or $p|\beta$ but not both, this is a Gauss sum of size $\sqrt{p}$. If $p|\alpha, \beta$, the sum is 0.

Looking at the table in Equation (10.2), we see that we have a Gauss sum in a_{23} and a Kloosterman sum in m. We consider the Gauss sum in a_{23}. The coefficients in a_{23} are

$$\begin{aligned}\alpha a_{23}^2 + 2\beta a_{23} = &\left[\begin{pmatrix} a_{12} & -a_{11}\end{pmatrix}\begin{pmatrix} t_{11} & t_{12} \\ t_{12} & t_{22}\end{pmatrix}\begin{pmatrix} a_{12} \\ -a_{11}\end{pmatrix}\bar{k} + a_{11}q_{33}\right]\bar{m}a_{23}^2 \\ &+ \big[-(a_{12}^2\bar{m}+1)a_{12}a_{13}\overline{a_{11}}\bar{k}t_{11} + (2a_{12}^2\bar{m}+1)a_{13}\bar{k}t_{12} - a_{11}a_{12}a_{13}\bar{m}\bar{k}t_{22} \\ &+ a_{12}\bar{k}t_{13} - a_{11}\bar{k}t_{23} + q_{23} - a_{12}a_{13}\bar{m}q_{33}\big]\, 2a_{23}.\end{aligned}$$

We rewrite this as

$$\alpha_1\bar{m}a_{23}^2 + (\beta_1\bar{m} + \beta_2)2a_{23}$$

with $\alpha_1, \beta_1, \beta_2$ independent of m. Note that $\bar{m}$ is factorized in the coefficient in front of a_{23}^2. We apply Lemma 10.4. If $p|\alpha$, since T is positive definite and $a_{11} \neq 0$, we get a relationship for k. In that case, a trivial bound gives the result, since we only have 5 variables left. If $(p, \alpha) = 1$, then we get the following:

$$G(\alpha, \beta; p) = \epsilon_p\sqrt{p}\left(\frac{\alpha_1}{p}\right)\left(\frac{\bar{m}}{p}\right) e\left(\frac{-\overline{\alpha_1}m(\beta_1^2\bar{m}^2 + 2\beta_1\beta_2\bar{m} + \beta_2^2)}{p}\right).$$

We are only interested in the variable m. We get a Legendre symbol and two new terms which are constant or of degree 1 in m and $\bar{m}$. The coefficient in m and $\bar{m}$ that do not depend on a_{23} were unaffected by the computation of the Gauss sum. They are the following:

$$\begin{aligned}\gamma_1 m + \gamma_2\bar{m} = &\left[\begin{pmatrix}\overline{a_{11}}a_{13} & -1\end{pmatrix}\begin{pmatrix} t_{11} & t_{13} \\ t_{13} & t_{33}\end{pmatrix}\begin{pmatrix}\overline{a_{11}}a_{13} \\ -1\end{pmatrix}\bar{k} + \overline{a_{11}}q_{22}\right] m \\ &+ \left[\begin{pmatrix}\overline{a_{11}}a_{12} & -1\end{pmatrix}\begin{pmatrix} t_{11} & t_{12} \\ t_{12} & t_{22}\end{pmatrix}\begin{pmatrix}\overline{a_{11}}a_{12} \\ -1\end{pmatrix}(a_{12}^2a_{13}^2\bar{k} + a_{11}) + a_{12}^2a_{13}^2\overline{a_{11}}q_{33} + kq_{33}\right]\bar{m}.\end{aligned}$$

We obtain the following Salié sum in m:

$$\sideset{}{^*}\sum_{m \bmod p}\left(\frac{\bar{m}}{p}\right) e\left(\frac{\left(-\overline{\alpha_1}\beta_2^2 + \gamma_1\right)m + \left(-\overline{\alpha_1}\beta_1^2 + \gamma_2\right)\bar{m}}{p}\right).$$

We apply Lemma 10.5. In all cases, we obtain a result of size $O(\sqrt{p})$ or 0. Finally, we use a trivial bound for the other sums in a_{11}, a_{12}, a_{13} and k. In total, we obtain

$$|K^{(3)}(Q, T; pI_3)| \ll p \sum_{a_{12}, a_{13} \bmod p} \sideset{}{^*}\sum_{a_{11}, k \bmod p} 1 + p^5 \ll p^5.$$

This concludes the proof of Theorem 1.6.

Appendix A

Automorphisms of binary quadratic forms

The goal of this appendix is to compute all the automorphisms in $\mathrm{GL}_2(\mathbb{Z})$ of a binary quadratic form. We set

$$Q = \begin{pmatrix} x & y \\ y & z \end{pmatrix} \qquad\qquad M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Here Q is a (weakly) reduced integral quadratic form, that is $x, z \neq 0$, $2|y| \leq x \leq z$, $2y, x, z \in \mathbb{Z}$ and $\det(Q) > 0$, and $M \in \mathrm{GL}_2(\mathbb{Z})$. We are looking for the couples (Q, M) such that

$$Q = M^t Q M.$$

Note first that if we replace M by $-M$, we get the same result. Therefore we only consider matrices up to multiplication by ± 1. The computation gives

$$0 = M^t Q M - Q = \begin{pmatrix} a^2x + 2acy + c^2z - x & abx + (ad+bc)y + cdz - y \\ abx + (ad+bc)y + cdz - y & b^2x + 2bdy + d^2z - z \end{pmatrix}. \tag{A.1}$$

We consider the first entry. Using the identity $u^2 + v^2 \geq 2|uv|$, we have

$$0 = a^2x + 2acy + c^2z - x \geq 2|ac|(\sqrt{xz} - |y|) - x \geq |ac|x - x.$$

Therefore we have $|ac| \leq 1$. We have to work a bit more for the other entries. Suppose that $|d| \geq 2$. Then $d^2 - 1 \geq \frac{3}{4}d^2$ and so

$$0 = b^2x + 2bdy + (d^2-1)z \geq 2|b|\sqrt{d^2-1}\sqrt{xz} - 2|bdy| \geq 2|bd|(\sqrt{3/4}\sqrt{xz} - |y|).$$

Since $\sqrt{3/4} > 1/2$, we have $b = 0$. Therefore we have two cases: $|d| \leq 1$ or $b = 0$.

A.1 Diagonal and antidiagonal M

We begin with the two easy cases of diagonal and antidiagonal M. There are four possibilities up to multiplication by -1:

$$M = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

The identity is an automorphism for any matrix Q. Looking at Equation (A.1), we get respectively for the other three matrices

$$0 = \begin{pmatrix} 0 & -2y \\ -2y & 0 \end{pmatrix}, \begin{pmatrix} z-x & -2y \\ -2y & x-z \end{pmatrix}, \begin{pmatrix} x-z & 0 \\ 0 & z-x \end{pmatrix}.$$

Therefore the conditions on Q are respectively $y = 0$, $x = z \wedge y = 0$ and $x = z$.

A.2 Diagonal Q

We quickly consider the case $y = 0$, so we can rule out this later. Equation (A.1) rewrites as

$$0 = \begin{pmatrix} a^2x + c^2z - x & abx + cdz \\ abx + cdz & b^2x + d^2z - z \end{pmatrix}.$$

First, if $a = 0$, then $b, c = \pm 1$ since the determinant is $bc = \pm 1$. The first entry gives $x = z$ and the second entry gives $d = 0$. If $c = 0$, then $a, d = \pm 1$ and the diagonal entries vanish. The second entry gives $b = 0$. In both cases, we are back to a diagonal or antidiagonal M. Otherwise, if $ac = \pm 1$, then the first entry gives $z = 0$ which is a contradiction. So all these cases fit in the last section. From now, we suppose that $y \neq 0$.

A.3 The case $ac = 0$

If $c = 0$, then automatically a and d equal ± 1 since the determinant is ad. That gives the matrices

$$M = \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & n \\ 0 & -1 \end{pmatrix}$$

for n a non-zero integer. The other cases can be obtained by multiplying by -1. Looking at Equation (A.1), we have

$$0 = \begin{pmatrix} 0 & anx + (ad-1)y \\ anx + (ad-1)y & n^2x + 2dny \end{pmatrix}.$$

So if $ad = 1$ like in the first case, then $x = 0$ and there is no such Q. In the second case, $ad = -1$ and we get $nx = 2y$ or $nx + 2y = 0$. Since $x \geq 2|y|$, we get $n = \operatorname{sgn}(y)$ and $x = 2|y|$. Now, if $a = 0$ then $bc = \pm 1$ and we have the matrices

$$M = \begin{pmatrix} 0 & 1 \\ 1 & n \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & n \end{pmatrix}.$$

Equation (A.1) rewrite as

$$0 = \begin{pmatrix} z - x & (bc-1)y + cnz \\ (bc-1)y + cnz & x + 2bny + (n^2-1)z \end{pmatrix}.$$

If $bc = 1$, then $z = 0$ and there is no such matrix. Otherwise, $x = z$ and we get the two equations $nx = 2y$ and $nx + 2y = 0$. Again, $x \geq 2|y|$ so $n = -\operatorname{sgn}(y)$ and $x = 2|y|$.

A.4 The case $ac = 1$

We have $a = c = \pm 1$, without loss of generality say $a = c = 1$. Therefore the first entry of the matrix is $2y + z = 0$. Since $2|y| \leq x \leq z$, we get $-2y = x = z$. Equation (A.1) rewrites as

$$0 = \begin{pmatrix} 0 & -by - dy - y \\ -by - dy - y & -2b^2y + 2bdy - 2(d^2-1)y \end{pmatrix}.$$

If $b = 0$, then the second entry gives $d + 1 = 0$ so $d = -1$ and this is compatible with the last entry. If $b \neq 0$, then we have two cases. If $d = 0$, then the second equation gives $b = -1$. This is compatible with the last entry. If $d = \pm 1$, then the last entry is $-2b^2y + 2bdy = 0$, so that $b = d$. There is no such matrix with determinant ± 1 and it is also incompatible with the second entry.

A.5 The case $ac = -1$

We have $a = -c = \pm 1$, without loss of generality say $a = -c = 1$. So the first entry of Equation (A.1) gives $2y = z$. Since $2|y| \leq x \leq z$, we have $2y = x = z$. The full matrix rewrites

$$0 = \begin{pmatrix} 0 & by - dy - y \\ by - dy - y & 2b^2y + 2bdy + 2d^2y - 2y \end{pmatrix}.$$

If $b = 0$, then the second entry gives $d = -1$ and is compatible with the last. If $b \neq 0$, then $d = 0$ gives $b = 1$ for both equations. If $d = \pm 1$, then the last entry is $2b^2y + 2bdy = 0$ so $b = -d$. This is incompatible with the second entry that says $b = d + 1$ (for integral b and d).

A.6 Summary

We summarize the result in the table below. The first column indicates the sign of the determinant of M. For each matrix M, there is the matrix $-M$ that has the same action on Q. Note that except for the fourth entry, y is always supposed to be non-zero.

det(M)	M	Q
+	$\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}$	Any
−	$\begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}$	$\begin{pmatrix}x & 0\\0 & z\end{pmatrix}$
+	$\begin{pmatrix}0 & 1\\-1 & 0\end{pmatrix}$	$\begin{pmatrix}x & 0\\0 & x\end{pmatrix}$
−	$\begin{pmatrix}0 & 1\\1 & 0\end{pmatrix}$	$\begin{pmatrix}x & y\\y & x\end{pmatrix}$
−	$\begin{pmatrix}1 & 1\\0 & -1\end{pmatrix}$	$\begin{pmatrix}2y & y\\y & z\end{pmatrix}$
−	$\begin{pmatrix}1 & -1\\0 & -1\end{pmatrix}$	$\begin{pmatrix}2y & -y\\-y & z\end{pmatrix}$
+	$\begin{pmatrix}0 & 1\\-1 & 1\end{pmatrix}$	$\begin{pmatrix}2y & y\\y & 2y\end{pmatrix}$
+	$\begin{pmatrix}0 & 1\\-1 & -1\end{pmatrix}$	$\begin{pmatrix}2y & -y\\-y & 2y\end{pmatrix}$
−	$\begin{pmatrix}1 & 0\\1 & -1\end{pmatrix}$	$\begin{pmatrix}2y & -y\\-y & 2y\end{pmatrix}$
+	$\begin{pmatrix}1 & -1\\1 & 0\end{pmatrix}$	$\begin{pmatrix}2y & -y\\-y & 2y\end{pmatrix}$
−	$\begin{pmatrix}1 & 0\\-1 & -1\end{pmatrix}$	$\begin{pmatrix}2y & y\\y & 2y\end{pmatrix}$
+	$\begin{pmatrix}1 & 1\\-1 & 0\end{pmatrix}$	$\begin{pmatrix}2y & y\\y & 2y\end{pmatrix}$

We rewrite the above table in terms of Q. The second column lists all the automorphisms of Q (modulo $\pm id$). The three following columns indicate respectively the number of automorphisms in $\mathrm{SL}_2(\mathbb{Z})$, in $\mathrm{GL}_2(\mathbb{Z})$ and the ratio between the two. The number of automorphisms $\epsilon(Q)$ in $\mathrm{PSL}_2(\mathbb{Z})$ is just half of the number in $\mathrm{SL}_2(\mathbb{Z})$. The last column gives the corresponding Heegner point $z = \frac{-y+i\sqrt{xz-y^2}}{x}$. Here $y \neq 0$ everywhere and $y > 0$ except in the third row. Recall that if Q is reduced and $x = z$ or $x = 2|y|$, we can, furthermore, suppose that $y > 0$. This removes the fifth and the seventh rows.

Q	M	$\mathrm{SL}_2(\mathbb{Z})$	$\mathrm{GL}_2(\mathbb{Z})$	Ratio	Heegner pt
$\begin{pmatrix} x & 0 \\ 0 & z \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$	2	4	2	$i\sqrt{\frac{z}{x}}$
$\begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$	4	8	2	i
$\begin{pmatrix} x & y \\ y & x \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$	2	4	2	$\frac{-y+i\sqrt{x^2-y^2}}{x}$
$\begin{pmatrix} 2y & y \\ y & z \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$	2	4	2	$\frac{-1}{2} + i\frac{\sqrt{2z-y}}{2\sqrt{y}}$
$\begin{pmatrix} 2y & -y \\ -y & z \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & -1 \\ 0 & -1 \end{pmatrix}$	2	4	2	$\frac{1}{2} + i\frac{\sqrt{2z-y}}{2\sqrt{y}}$
$\begin{pmatrix} 2y & y \\ y & 2y \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ -1 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}$	6	12	2	$\frac{-1+i\sqrt{3}}{2}$
$\begin{pmatrix} 2y & -y \\ -y & 2y \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & -1 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix}, \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}$	6	12	2	$\frac{1+i\sqrt{3}}{2}$
Other	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	2	2	1	$\frac{-y+i\sqrt{xz-y^2}}{x}$

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