Mathematical theory for multi-dimensional coagulation models

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SUMMARY

In this thesis, we consider problems that appear in the study of coagulation models. Coagulation equations describe the evolution in time of a system of particles that are characterized by their volume. They are a set of non-linear, integro-differential equations and are used predominantly in the study of blood coagulation, polymerization, and aerosol physics. A general introduction on coagulation equations can be found in Chapter 1 of this manuscript. The objective of this thesis consists in the analysis of models which incorporate additional information about the system of particles, as well as the observation of the new physical phenomena which may arise due to the modifications of the model.

A first approach is the addition of variables which give information about the shape of the particles. We thus study the long-time asymptotics of a coagulation model which describes the evolution of a system of particles characterized by their volume and surface area. The aggregation mechanism takes place in two stages: collision and fusion of particles. During the collision stage, the two particles merge at a contact point. After collision, the fusion phase begins and during it the geometry of the interacting particles is modified in such a way that the volume of the total system is preserved and the surface area is reduced. We prove existence of self-similar profiles for some choices of the functions describing the fusion rate for which the particles have a shape that is close to spherical. On the other hand, for other fusion mechanisms and suitable choices of initial data, we show that the particle distribution describes a system of ramified-like particles. The result has been obtained together with J. J. L. Velázquez [CV23] (arXiv preprint). Chapter 2 of this thesis contains general aspects about the model and presents the main results. Appendix A is the actual paper.

The long-time behavior of the model introduced in [CV23] depends strongly on the competition between fusion and collision. We further study this competition with the aim of finding connections between the solutions of our two-dimensional model in [CV23] and the standard coagulation equation. On one hand, if the fusion process occurs at a faster rate than the collision of particles, the interacting particles become spheres almost immediately. As such, we prove that we can reduce our model to

the standard one-dimensional model in this case. On the other hand, if the fusion term is negligible compared to the collision term, the total area and the volume stay conserved in the process and the particles attach to each other at a contact point. As such, we recover a two-dimensional coagulation model in which fusion does not occur. More details can be found in Chapter 3 and Appendix B. Chapter 3 introduces the results of Appendix B. The contents of Appendix B can be found in [CV24] and were published in ***Journal de Mathématiques Pures et Appliquées***. The results were obtained in collaboration with J. J. L. Velázquez.

Another possibility to incorporate more information about the system of particles is the addition of a term which represents an external source that injects particles into the system. For this model, only the regime where the transport of mass toward infinity is due to the collision between particles of comparable sizes has been studied. We study the long-time behavior of solutions for a range of exponents for which the transport of mass toward infinity is driven by collisions between particles of different sizes. This leads to an additional transport term in the coagulation equation that approximates the solution of the original coagulation equation with injection for large times. The paper is available online [CFFV23] and was published in ***Archive for Rational Mechanics and Analysis***. A short overview of the paper can be found in Chapter 4, while the paper is contained in Appendix C. This is a joint work with M. A. Ferreira, E. Franco, and J. J. L. Velázquez.

Additional variables to the standard coagulation models can also be used to describe the position in space of the particles. In [CNV24b], we study a two-dimensional coagulation equation which is inhomogeneous in the spatial variable and contains a transport term in space modeling the sedimentation of clusters. We prove local existence of mass-conserving solutions for a class of coagulation kernels for which in the space homogeneous case instantaneous loss of mass occurs. Our result holds true in particular for sum-type kernels of homogeneity greater than one, for which solutions do not exist at all in the spatially homogeneous case. Moreover, our result covers kernels that in addition vanish on the diagonal, which have been used to describe the onset of rain and the behavior of air bubbles in water. Instantaneous loss of mass for the latter type of kernel has been proven in [CNV24a], but this paper is not included in the thesis. The paper [CNV24b] (arXiv preprint) is contained in Appendix D and an overview of the results can be found in Chapter 5. The results were obtained together with B. Niethammer and J. J. L. Velázquez.

Lastly, we discuss in more detail the methods needed to prove our results and present possible future research directions. This is the content of Chapter 6.

CONTENTS

NOTATION

Chapter 1

- Let $d \in \mathbb{N}$. We denote by $\mathbb{R}^d_{\geq 0} := [0,\infty)^d$ and by $\mathbb{R}^d_{>0} =: \mathbb{R}^d_{\geq 0} \setminus \{O\}$, where $O := \{(0,\dots,0)\} \in \mathbb{R}^d$.
- For $n \in \mathbb{N}$, we use the notation $i = \overline{1,n}$ to mean $i \in \mathbb{N}$, $1 \leq i \leq n$.
- For $I \subseteq \mathbb{R}^d_{\geq 0}$, for some $d \in \mathbb{N}$, $\mathscr{M}_+(I)$ will denote the space of non-negative Radon measures, while $\mathrm{L}^1_+(I)$ will be the space of non-negative Lebesgue functions.
- For a function $f : (0,\infty) \to [0,\infty)$, we use the notation $M_l(f) := \int_{(0,\infty)} v^l f(v)\mathrm{d}v$, for some $l \in \mathbb{R}$. We call $M_l(f)$ the l-th order moment of the function f.

Chapter 2 and Chapter 3

- We denote $\mathbb{R}^d_{>0} := (0,\infty)^d$, for $d \in \mathbb{N}$. We use the notation $\eta := (a,v)$. We will use interchangeably both notations for convenience.
- We keep the notation $f(a,v)\mathrm{d}v\mathrm{d}a$ or $f(\eta)\mathrm{d}\eta$ for Radon measures, independently of the fact that the measure may not be absolutely continuous with respect to the Lebesgue measure.
- $M_{k,l}(f) := \int_{(0,\infty)^2} a^k v^l f(a,v)\mathrm{d}v\mathrm{d}a$, for some $k,l \in \mathbb{R}$.
- We use $f \approx g$ to mean that there exists a constant $C > 1$ such that $\frac{1}{C} \leq \frac{f}{g} \leq C$.

Chapter 4

- We use the notation $f \sim g$ as $x \to x_0$ to indicate that $\lim_{x\to x_0} \frac{f(x)}{g(x)} = 1$, while we use the notation $f \approx g$ to say that there exists a constant $M > 1$ such that $\frac{1}{M} \leq \frac{f}{g} \leq M$.

- ☞ We use the notation $x \gg 1$ for large cluster sizes x. Additionally, for cluster sizes x, y, we denote $x \gg y$ or $y \ll x$ to mean that x is much larger than y.
- ☞ We keep the notation $f(v)\mathrm{d}v$ for Radon measures, independently of the fact that the measure may not be absolutely continuous with respect to the Lebesgue measure.

Chapter 5

- ☞ We use the notation $f \lesssim g$, for two functions f, g, to mean that there exists a constant $C > 0$ such that $f \leq Cg$ and we keep the notation $f \approx g$ as in the previous chapters.

Chapter 6

- ☞ We keep the notation from the previous chapters. Moreover, for two terms, A and B, we use the notation $A \simeq B$ to mean informally that A can be approximated in terms of B in the region under consideration in the respective formula.

CHAPTER 1

INTRODUCTION

Coagulation (or more generally coagulation-fragmentation) models are used to describe the evolution in time of the size distribution of a system of particles. They play a fundamental role in a vast variety of applications such as aerosol physics, polymerization, blood agglomeration or phytoplankton aggregation. As such, the equations have received increasing interest in the last two decades, but the first work dates back to Marian von Smoluchowski in 1916, see [Smo16], in which he introduced an infinite set of ordinary differential equations to describe the evolution of a system of particles which allows interactions as a consequence of Brownian motion.

Modifications to the equation have been made depending on the described underlying process. The most predominant models used in the mathematical literature are the discrete and continuous models which we explain in more detail below.

Continuous version - coagulation equations

We denote by $f(v,t)$ the number of particles of volume v at time t and wish to describe the evolution in time of f keeping in mind the following assumptions

- ☞ only binary collisions are allowed, i.e., only two particles can interact with each other;
- ☞ the process does not depend on the order of interaction between particles;
- ☞ once two particles collide, they form a new particle with volume equal to the sum of the volumes of the two interacting particles.

The continuous version of the coagulation equation dates back to [Mü28] and is a non-linear, integro-differential equation with the form

$$\begin{aligned}\overset{\substack{\text{density}\\ \text{of particles}\\ \downarrow}}{\partial_t f(v,t)} &= \underbrace{\frac{1}{2}\int_{(0,v)} K(v-v',v')f(v',t)f(v-v',t)\mathrm{d}v'}_{\substack{\text{gain mass}\\ \uparrow\\ (v-v')+(v')=(v)}} - \underbrace{\int_{(0,\infty)} \overbrace{K(v,v')}^{\text{collision operator}} \quad f(v,t)f(v',t)\mathrm{d}v'}_{\substack{\text{lose mass}\\ \uparrow\\ \text{collision with other particles}}} \\ &=: \mathbb{K}[f](v,t),\end{aligned} \tag{1.0.1}$$

where $K(v,v')$, for $v,v' \in (0,\infty)$, represents the collision speed between a particle of volume v and a particle of volume v'. The derivation of this equation can be understood as follows. On one side, if two particles of smaller volume interact, such that the sum of their volumes equals v, then the total number of particles of volume v increases. The first term in (1.0.1) keeps track of this type of interactions and will be termed as the *gain term*. On the other side, if the particle of volume v interacts with a particle of any given volume, then its volume increases and the total number of particles of volume v decreases. The second term in (1.0.1) is used to count this type of interactions and will be termed from now on as the *loss term*.

From the physical assumptions that the model needs to satisfy, we can derive conditions for f and K as follows

☞ f represents the number of particles and as such $f \geq 0$;

☞ the order of interactions does not matter, so K is a symmetric function.

Discrete version - coagulation equations

We assume that we are interested in clusters that are composed of identical fundamental units, which we call *monomers*. This implies that the mass of a cluster is a multiple of the mass of the fundamental unit and, as a direct consequence, it suffices to analyze only discrete values of the mass. It is worthwhile to mention here that the mass can be replaced by equivalent quantities such as polymer length. This leads us to the following discrete model for the function f_n at time t

$$\partial_t f_n(t) = \frac{1}{2}\sum_{i=1}^{n-1} K(n-i,i)f_i(t)f_{n-i}(t) - \sum_{i=1}^{\infty} K(n,i)f_n(t)f_i(t) =: \mathbb{K}[f](n,t), \quad n \geq 2, \tag{1.0.2}$$

where $K(i,j)$, for $i,j \in \mathbb{N}$, is the collision speed between a cluster of size i and a cluster of size j. The assumptions and conditions made for the continuous version are the same for the discrete case, the only difference is that our functions f and K now take discrete values.

The model in (1.0.2) gives a straightforward interpretation of how the model is derived. The gain term is obtained by counting the number of all the possible interactions that can occur between clusters of size smaller than n that lead to the formation of a cluster of size n. The $\frac{1}{2}$ in front of the gain term appears in order to avoid the double counting of interactions. In the same manner, the loss term is obtained by counting all possible interactions between clusters of size n and clusters of any different size, only in this case double counting cannot happen.

A first observation to the discrete model however is related to the interaction between clusters of the same size as it does not respect the counting rule described above. In [Col04] the following model is discussed

$$\begin{aligned}\partial_t f_n(t) =& \frac{1}{2}\sum_{i=1, i\neq \frac{n}{2}}^{n-1} K(n-i,i) f_i(t) f_{n-i}(t) + K\left(\frac{n}{2},\frac{n}{2}\right)\left(f_{\frac{n}{2}}(t)\right)^2 \\ &- \sum_{i=1, i\neq n}^{\infty} K(n,i) f_n(t) f_i(t) - 2K(n,n)(f_n(t))^2,\end{aligned}$$

for when n is even, and

$$\partial_t f_n(t) = \frac{1}{2}\sum_{i=1}^{n-1} K(n-i,i) f_i(t) f_{n-i}(t) - \sum_{i=1, i\neq n}^{\infty} K(n,i) f_n(t) f_i(t) - 2K(n,n)(f_n(t))^2,$$

for n odd, as in this case terms of the form $K(i,i)$, for $i \in \mathbb{N}$, cannot appear in the gain term. However, the simplified form in (1.0.2) is the standard discrete version which is analyzed in both the physical and mathematical literature as it offers a sufficiently good approximation for applications.

1.1 Assumptions on the coagulation kernel

We gather here some standard assumptions for the coagulation kernel in (1.0.1) and offer some examples that will be of importance later on. As mentioned before, because of the underlying physical setting, K needs to be a symmetric, non-negative function. Thus, we assume that

$$K(v,v') \geq 0, \text{ for all } v,v' \in (0,\infty), \tag{1.1.1}$$

and that

$$K(v,v') = K(v',v), \text{ for all } v,v' \in (0,\infty). \tag{1.1.2}$$

The most common examples of coagulation kernels found in applications have the form of power laws. The existing results for coagulation processes treat kernels which can be estimated from above by such power laws, namely

$$K(v,v') \leq K_1(v^{\alpha} v'^{\beta} + v^{\beta} v'^{\alpha}), \tag{1.1.3}$$

for some $\alpha, \beta \in \mathbb{R}$ and some constant $K_1 > 0$. Sometimes a similar bound from below is also necessary. We will mention this condition explicitly when we encounter such cases.

We now offer some specific examples of coagulation kernels that can be found in applications.

The constant kernel

The most standard example of collision rate is the case when the interaction of particles does not depend on the cluster size, i.e., we work with a constant coagulation kernel. While a simple example, this type of kernel is rather rich in features and was used as a motivation for many of the existing conjectures in coagulation equations, such as the long-time behavior of solutions, which will be explained in more detail in Subsection 1.5. It is thus understandable that particular attention should be assigned to the analysis of solutions in this case. We first present some general properties of coagulation equations that would make this analysis easier to follow. Thus, we only mention the constant kernel here and will further discuss its properties in Subsection 1.5.

Example of coagulation kernels used in the study of polymers

We work here with the discrete-sized version of the model as it is more often encountered in the study of polymerization. The following assumptions are taken from [BLL19a, Section 2.2.3], which follows the discussion in [vDE84]. We suppose that

$\boxed{1}$ Each polymer in the system is composed of identical monomers, with an n-mer comprising of n monomers.

$\boxed{2}$ Each monomer carries an identical number of fundamental groups, some of which can form chemical bonds with fundamental groups carried by monomers in other polymers.

$\boxed{3}$ Intramolecular reactions leading to cyclic structures do not occur.

We consider the case when a monomer is formed of $m \in \mathbb{N}$ identical fundamental A groups and thus we can only have chemical bonds of the form $A - A$. The coagulation rate $K(n, j)$ is proportional to the number of bonds that can occur between an n-mer and a j-mer. For this setting, an n-mer will have $2(n-1)$ reacted A groups and $n(m-2)+2$ unreacted A groups. We refer to Figure 1.1 for more details. Thus, we can deduce that

$$K(n, j) \text{ is proportional to } [n(m-2)+2] \times [j(m-2)+2].$$

Since m is a constant which represents the number of fundamental groups forming a monomer, the resulting kernel is of order nj up to multiplication by some constant. As such, product kernels $K(n, j) = nj$ play an important role in the study of polymerization. Similar computations can be made for more complicated structures, also leading to product-type kernels. We refer to [BLL19a, Section 2.2.3] and [vDE84] for further details.

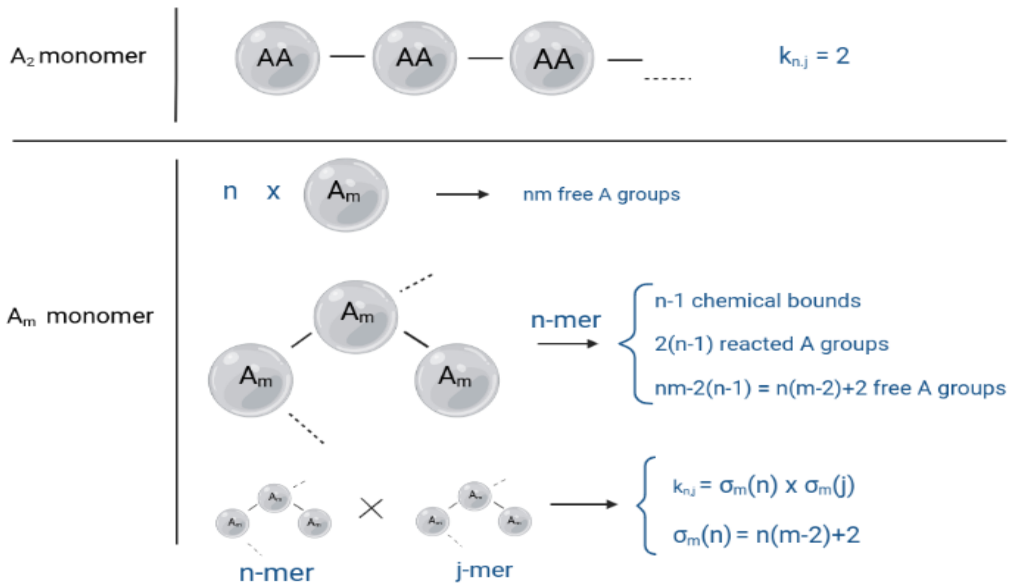

Figure 1.1: Derivation of the kernel when monomers carry identical fundamental A groups

Example of coagulation kernels used to describe rain formation

We consider the continuous version of the coagulation model and follow the discussion in [HNS07]. This choice of kernel is motivated as follows. We assume that spherical particles can move on the vertical line (due to gravitation or buoyancy) and merge upon touching. The cross-section of interaction between two particles of radii r and r', and volume v and v', respectively, that merged upon touching is given by then $\pi(r+r')^2$, which can be described in terms of volume as $(v^{\frac{1}{3}}+v'^{\frac{1}{3}})^2$. We omit the constants here as the relevant contribution is given by the dependence on the volume of the coagulation rate.

Additionally, the velocity is taken to be approximately $v^{\frac{2}{3}}$, which represents the Stokes velocity of a rigid sphere with no slip boundary condition, and the collision rate between two particles is taken to be proportional to their relative velocities $|v^{\frac{2}{3}}-v'^{\frac{2}{3}}|$.

This leads to a sum-type kernel which vanishes on the diagonal, also referred to as a *differential sedimentation kernel* in the physics literature, which has the following form

$$K(v,v') = |v^{\frac{2}{3}}-v'^{\frac{2}{3}}|(v^{\frac{1}{3}}+v'^{\frac{1}{3}})^2. \tag{1.1.4}$$

The model is used to describe the behavior of air bubbles in water which move due to buoyancy and it is also valid for water droplets. We refer to [CGW78, FSV06, PK97] for more details. However, the assumption that the particles move in space requires a different model than the one in (1.0.1), namely an additional variable to the system. We omit further details here as this will be the subject of study of Subsection 1.6.2.

1.2 Coagulation-fragmentation models

For completeness, we also present some versions of coagulation-fragmentation models. However, the focus of this thesis will be on pure coagulation processes. It is useful to remark that in the case of coagulation, only binary collisions are allowed and multiple collisions lead to significantly more complicated models. In the case of fragmentation however, particles can breakup into two or multiple smaller particles. We term the former process as *binary fragmentation*, while the latter is called *multiple fragmentation*.

Discrete coagulation-fragmentation

When we work with discrete values and in the case of multiple fragmentation, we arrive to the following model, which has been studied for example in [BT45, Lau02, Zif92],

$$\partial_t f_n(t) = \mathbb{K}[f](n,t) - a_n f_n(t) + \sum_{i=n+1}^{\infty} a_i b_{n,i} f_i(t), \text{ for } n \geq 2. \tag{1.2.1}$$

The term $\mathbb{K}[f](n,t)$ is the coagulation operator introduced in (1.0.2). The new terms appearing in the model (1.2.1) compared to the model (1.0.2) can be interpreted as follows

- ☞ a_n is the fragmentation rate of an n-mer.
- ☞ $b_{n,i}$ is known as *the daughter distribution function* and should be thought of as the number of n-mers produced by the fragmentation of an i-mer.

Since monomers cannot fragment further and cannot be created by the interaction of smaller entities, the discrete version of the coagulation-fragmentation equation for $n = 1$ can be written as

$$\partial_t f_1(t) = \sum_{i=2}^{\infty} a_i b_{1,i} f_i(t) - \sum_{i=1}^{\infty} K(1,i) f_1(t) f_i(t). \tag{1.2.2}$$

Further assumptions can be made when considering the physical interpretation of the daughter distribution function $b_{n,i}$. For example, since smaller polymers cannot produce larger polymers upon fragmentation, it holds that

$$b_{j,i} = 0 \text{ when } j \geq i. \tag{1.2.3}$$

Moreover, assuming that no mass loss should occur in the system upon fragmentation, then the daughter distribution function should satisfy an equality of the form

$$\sum_{j=1}^{n-1} j b_{j,n} = n, \text{ for all } n \geq 2. \tag{1.2.4}$$

Discrete binary fragmentation has also been considered for example in [BC90], or in [BD35], for a type of kernel which allows only single monomers to interact with clusters. We will discuss more details about the connection between binary and multiple fragmentation when we present the continuous version of the model (1.2.1).

Continuous coagulation-fragmentation

The continuous version of (1.2.1) can be found in [MZ87, VZ89] and takes the following form

$$\partial_t f(v,t) = \mathbb{K}[f](v,t) - a(v)f(v,t) + \int_{(v,\infty)} a(v')b(v,v')f(v',t)\mathrm{d}v', \tag{1.2.5}$$

where $\mathbb{K}$ is the coagulation operator introduced in (1.0.1). We remind that we are in the case of multiple fragmentation. The terms a and b in (1.2.5) are the continuous counterpart of the same terms in (1.2.1) and have the same physical interpretation. As such, the daughter distribution function b satisfies the same properties as in the discrete case, namely

$$b(v',v) = 0 \text{ when } v' > v \text{ and } \int_{(0,v)} v' b(v',v)\mathrm{d}v' = v. \tag{1.2.6}$$

There exist other equivalent [Mel57] or closely related [Fil61] formulations of (1.2.5) in the mathematical literature. For example, the following coagulation-fragmentation model is used to describe *binary fragmentation*, cf. [Ste89, Ste90],

$$\begin{aligned}\partial_t f(v,t) &= \mathbb{K}[f](v,t) - \frac{1}{2}\int_{(0,v)} F(v',v-v')f(v,t)\mathrm{d}v' + \int_{(v,\infty)} F(v,v'-v)f(v',t)\mathrm{d}v' \qquad (1.2.7)\\ &= \mathbb{K}[f](v,t) - \frac{1}{2}\int_{(0,v)} F(v',v-v')f(v,t)\mathrm{d}v' + \int_{(0,\infty)} F(v,v')f(v+v',t)\mathrm{d}v'.\end{aligned}$$

The terms appearing due to fragmentation in (1.2.7) can be understood as follows

- ☞ $F(v',v)$ represents the fragmentation rate of a particle of volume $v+v'$ into a particle of volume v' and a particle of volume v. Since a particle can only decompose into two particles, F needs to be symmetric.

- ☞ $\int_{(v,\infty)} F(v,v'-v)f(v',t)\mathrm{d}v'$ represents the production of particles of volume v, due to the fragmentation of larger particles.

- ☞ $-\frac{1}{2}\int_{(0,v)} F(v',v-v')f(v,t)\mathrm{d}v'$ represents the loss of particles of volume v, due to their fragmentation into a particle of volume v' and a particle of volume $v-v'$, with $v' < v$.

Equation (1.2.7) is equivalent to equation (1.2.5) in the case when particles can only fragment into two smaller particles. Indeed, assuming in the model (1.2.5) that only binary fragmentation can occur and remembering that the daughter distribution function $b(v', v)$ represents the expected number of particles of volume v' produced by a particle of volume v, then b satisfies the following conditions

$$\int_{(0,v)} b(v', v)\mathrm{d}v' = 2 \text{ and } b(v', v) = b(v - v', v) \text{ with } v' < v. \tag{1.2.8}$$

Define now the fragmentation rate F to be

$$F(v, v') := a(v + v')b(v, v + v').$$

Notice that F is indeed symmetric since due to (1.2.8) it holds that

$$F(v, v') = a(v + v')b(v, v + v') = a(v + v')b(v', v + v') = F(v', v).$$

With this definition of F, we arrive to equation (1.2.7) by noticing that due to (1.2.8) we have that

$$\frac{1}{2}\int_{(0,v)} F(v', v - v')f(v, t)\mathrm{d}v' = \frac{a(v)f(v, t)}{2}\int_{(0,v)} b(v', v)\mathrm{d}v' = a(v)f(v, t).$$

Conversely, we can obtain (1.2.5) with a daughter distribution function satisfying (1.2.8) from (1.2.7) by choosing the functions a and b in an appropriate manner. We refer to [BLL19a, Subsection 2.2.2] for more details.

1.3 The weak formulation of the coagulation model

Most of the results discussed in this thesis hold for weak solutions of some coagulation models. As such, we show now how one can derive the weak formulation of (1.0.1). We test (1.0.1) with a suitable test function $\varphi(v)$ and arrive to the equation

$$\begin{aligned}\partial_t \int_{(0,\infty)} f(v, t)\varphi(v)\mathrm{d}v =& \frac{1}{2}\int_{(0,\infty)}\int_{(0,v)} K(v - v', v')f(v', t)f(v - v', t)\varphi(v)\mathrm{d}v'\mathrm{d}v \\ &- \int_{(0,\infty)}\int_{(0,\infty)} K(v, v')f(v, t)f(v', t)\varphi(v)\mathrm{d}v'\mathrm{d}v.\end{aligned} \tag{1.3.1}$$

For the gain term, we use Fubini's theorem and then make the change of variable $z = v - v'$ in order to deduce that

$$\begin{aligned}&\int_{(0,\infty)}\int_{(0,v)} K(v - v', v')f(v', t)f(v - v', t)\varphi(v)\mathrm{d}v'\mathrm{d}v \\ &\qquad = \int_{(0,\infty)}\int_{(v',\infty)} K(v - v', v')f(v', t)f(v - v', t)\varphi(v)\mathrm{d}v\mathrm{d}v' \\ &\qquad = \int_{(0,\infty)}\int_{(0,\infty)} K(z, v')f(v', t)f(z, t)\varphi(z + v')\mathrm{d}z\mathrm{d}v'.\end{aligned} \tag{1.3.2}$$

From (1.3.2) and then using the symmetry of the coagulation kernel in the loss term, we can rewrite (1.3.1) as

$$\partial_t \int_{(0,\infty)} f(v,t)\varphi(v)\mathrm{d}v = \frac{1}{2}\int_{(0,\infty)}\int_{(0,\infty)} K(v,v')f(v,t)f(v',t)\chi_{\varphi}(v,v')\mathrm{d}v'\mathrm{d}v, \tag{1.3.3}$$

where we denoted by

$$\chi_{\varphi}(v,v') := \varphi(v+v') - \varphi(v) - \varphi(v'). \tag{1.3.4}$$

Equation (1.3.3) is known as the *weak formulation* of the coagulation equation. Intuitively, the underlying coagulation process is now incorporated within the test function φ. More precisely, the term χ_{φ} in (1.3.4) is understood as follows: for the formation of a particle of volume $v+v'$, a particle of volume v and a particle of volume v' are needed.

1.4 Mass-conserving solutions and gelation in coagulation equations

Since the volume of a new particle is always the sum of the volumes of other two interacting particles, the solutions of (1.3.3) should in principle conserve the total mass of the system. This can be noticed at a formal level by testing in (1.3.3) with $\varphi \equiv v$, from which we deduce that

$$\partial_t \int_{(0,\infty)} f(v,t)\varphi(v)\mathrm{d}v = 0 \text{ and thus } \int_{(0,\infty)} f(v,t)\varphi(v)\mathrm{d}v = \int_{(0,\infty)} f(v,0)\varphi(v)\mathrm{d}v, \text{ for all } t \geq 0. \tag{1.4.1}$$

However, $\varphi \equiv v$ is not a valid test function when looking for example for solutions f of (1.3.3) which belong to $\mathrm{C}([0,\infty);\mathrm{L}^1_+(0,\infty))$ or $\mathrm{C}([0,\infty);\mathscr{M}_+(0,\infty))$ and significant research has been conducted toward finding the exact conditions under which solutions of coagulation equations conserve mass.

More precisely, when the coagulation rate K has the form of a power law of homogeneity γ (or is bounded from above by such a kernel), it is known that global in time existence of mass-conserving solutions can be proven when $\gamma < 1$. This is relevant as most of the kernels used in applications are power laws or similar kernels, as seen in Subsection 1.1. For the continuous version of the coagulation equation, existence results can be traced back to [Gal77], while for the continuous coagulation-fragmentation equation for unbounded kernels, the first existence result was due to [Ste89].

However, one can consider for example the case of polymerization. Polymers can form links between them which leads to larger polymers. If we deal with fast interactions, the linking of larger polymers will lead to the formation of a macroscopic molecule, point in which the viscosity of the system will increase. This phenomenon is known as *gelation* (or *gel formation*) and it can be also explained mathematically.

When the coagulation kernel is a power law (or bounded from above and below by a power law) of homogeneity $\gamma > 1$, solutions to (1.3.3) do not conserve mass. A general proof of this can be found in [EMP02]. As in the case of polymerization, this can be explained as follows. Fast interactions for large particles will lead to the formation of a molecule with *infinite* mass. However, our equation does not take infinity into account and mathematically this can be observed by a loss of mass. The mass that was lost is termed as the *gel part* and can be thought as the part of the solution that has become a gel, while the leftover mass is the part that has not yet been converted. In order to explain in more detail the different situations that can occur, we define T_{gel} as the point in time when the mass loss occurs, namely

$$T_{\text{gel}} := \sup\left\{t \geq 0 : \int_{(0,\infty)} vf(v,t)\mathrm{d}v = \int_{(0,\infty)} vf(v,0)\mathrm{d}v\right\}. \tag{1.4.2}$$

T_{gel} will be termed from now on as the *gel point*. The following tables are taken from [BLL19b, Section 9] and portray the existing results for two of the main types of coagulation rates.

$K(v,v') = (vv')^{\frac{\gamma}{2}}$	
$\gamma \in [0,1]$	mass-conserving solutions
$\gamma \in (1,2]$	gelation in finite time
$\gamma > 2$	instantaneous gelation

$K(v,v') = v^{\gamma} + v'^{\gamma}$	
$\gamma \in [0,1]$	mass-conserving solutions
$\gamma > 1$	non-existence in L^1

For product-type kernels, i.e., $K(v,v') = (vv')^{\frac{\gamma}{2}}$, it has been proven that gelation happens in finite time when $\gamma \in (1,2]$ and that $T_{\text{gel}} = 0$ when $\gamma > 2$, phenomenon also known as *instantaneous gelation*. For sum-type kernels, i.e., $K(v,v') = v^{\gamma} + v'^{\gamma}$, solutions of (1.3.3) belonging to L^1 do not exist. This follows by contradiction from the following properties of solutions in this case. More precisely, due to the form of the coagulation kernel, it has been proven in [BC90] in the discrete case that, if solutions exist, they must be mass-conserving solutions. On the other hand, for this type of kernels and in the discrete case, instantaneous gelation has been proven in [CdC92]. Thus, if solutions exist, they need to lose mass in zero time, as well as to conserve the mass, from where the needed contradiction is deduced. Adaptations of this proof to the continuous case can be found in [BLL19b].

It is worthwhile to mention that it is rather straightforward to notice that gelation occurs in the particular case when $K(v,v') = vv'$. We test in (1.3.3) with $\varphi(v) \equiv 1$ and consider $t \leq T_{\text{gel}}$, where T_{gel} is as in (1.4.2). It follows that

$$\begin{aligned}\int_{(0,\infty)} f(v,t)\mathrm{d}v - \int_{(0,\infty)} f(v,0)\mathrm{d}v &= -\frac{1}{2}\int_0^t \left(\int_{(0,\infty)} vf(v,s)\mathrm{d}v\right)^2 \mathrm{d}s \\ &\overset{t\leq T_{\text{gel}}}{=} -\frac{1}{2}\left(\int_{(0,\infty)} vf(v,0)\mathrm{d}v\right)^2 t.\end{aligned}$$

Moreover, by the non-negativity of f, we further deduce that

$$0 \leq \int_{(0,\infty)} f(v,t)\mathrm{d}v = \int_{(0,\infty)} f(v,0)\mathrm{d}v - \frac{1}{2}\left(\int_{(0,\infty)} vf(v,0)\mathrm{d}v\right)^2 t$$

and thus we obtain that

$$t \leq \frac{\int_{(0,\infty)} f(v,0)\mathrm{d}v}{\frac{1}{2}\Big(\int_{(0,\infty)} v f(v,0)\mathrm{d}v\Big)^2} < \infty,$$

for all $t \leq T_{\mathrm{gel}}$. In other words, $T_{\mathrm{gel}} < \infty$.

While the proof is short, similar computations can be adapted to hold for more general classes of kernels. Moreover, the product kernel plays a significant role in applications to polymerization, as explained in Subsection 1.1. The fact that gelation occurs in the case of the product kernel can be traced back to [LT81] in the case of coagulation equations and to [Lau00] for coagulation-fragmentation models.

1.5 Self-similar profiles

As explained before, since the constant kernel was used as a basis for the intuition behind self-similar profiles, we first show some formal computations describing solutions for the constant kernel.

The constant kernel revised

For computational considerations, let us choose $K(v,v') \equiv 2$, for all $v, v' \in (0,\infty)$, and we consider an initial condition of the form $f(v,0) = e^{-v}$. We test with $\varphi(v) \equiv 1$ in (1.3.3) and (1.3.3) becomes in this case

$$\partial_t M_0(f(t)) = -M_0^2(f(t)), \tag{1.5.1}$$

which leads us to

$$M_0(f(t)) = \frac{M_0(f(0))}{tM_0(f(0)) + 1} = \frac{1}{t+1}. \tag{1.5.2}$$

We remember that due to mass conservation, see (1.4.1), we also have

$$M_1(f(t)) = M_1(f(0)) = 1. \tag{1.5.3}$$

Using now the form of the zero-th order moment (1.5.2) in (1.0.1), we obtain that in the case of the constant kernel our solution f solves the equation

$$\partial_t f(v,t) + \frac{2f(v,t)}{t+1} = \int_{(0,v)} f(v-v',t)f(v',t)\mathrm{d}v'. \tag{1.5.4}$$

Equivalently, by multiplying with $(1+t)^2$ in (1.5.4), we obtain the following formulation

$$\partial_t\Big((t+1)^2 f(v,t)\Big) = (t+1)^2 \int_{(0,v)} f(v-v',t)f(v',t)\mathrm{d}v'. \tag{1.5.5}$$

Thus, by taking

$$h(v,\tau) := (t+1)^2 f(v,t), \text{ where } \tau = \frac{1}{t+1}, \tag{1.5.6}$$

we arrive to a simplified form of equation (1.5.5) for h, namely

$$\partial_\tau h(v,\tau) = -\int_{(0,v)} h(v-v',\tau)h(v',\tau)\mathrm{d}v', \text{ with } h(v,1) = f(v,0), \tag{1.5.7}$$

where $\tau \in [0,1)$. In 1940, Schumann [Sch40] noticed that a solution to (1.5.7) in this case is given by $h(v,\tau) = e^{-v\tau}$, which by (1.5.6) leads us to a solution of (1.5.4)

$$f(v,t) = \frac{1}{(t+1)^2} e^{-\frac{v}{t+1}}. \tag{1.5.8}$$

For more general initial conditions, Melzak [Mel53] used Laplace transform methods in order to compute solutions to (1.5.7). For simplicity of notation, we restrict ourselves to the case when $f(v,0) = e^{-v}$ and we will obtain the same solution as above. This is since our focus is to show the underlying methods which can be used to obtain solutions to the coagulation equations in this case, rather than to find explicit formulas given various initial data.

For a function H, let us denote the Laplace transform of H by

$$\hat{H}(\xi) := \int_{(0,\infty)} e^{-v\xi} H(v)\mathrm{d}v. \tag{1.5.9}$$

Then, by performing in (1.5.7) similar computations with the ones in (1.3.2), we arrive to the following equation for the Laplace transform of h, which was defined in (1.5.6),

$$\partial_\tau \hat{h}(\xi,\tau) = -[\hat{h}(\xi,\tau)]^2. \tag{1.5.10}$$

We further deduce that

$$\hat{h}(\xi,\tau) = \frac{1}{\tau - 1 + \frac{1}{\hat{f}(\xi,0)}}. \tag{1.5.11}$$

In the particular case when $f(v,0) = e^{-v}$, direct computations show that $\hat{h}(\xi,1) = \hat{f}(\xi,0) = \frac{1}{\xi+1}$. This implies by (1.5.11) that $\hat{h}(\xi,\tau) = \frac{1}{\tau+\xi}$. We can then conclude by noticing that this is the Laplace transform of $h(v,\tau) = e^{-v\tau}$.

For more general initial conditions, since $\hat{h}(\xi,\tau) = (t+1)^2\hat{f}(\xi,t)$ and $\tau = \frac{1}{t+1}$, we obtain by (1.5.11) that

$$\hat{f}(\xi,t) = \frac{\hat{f}(\xi,0)}{(t+1)^2}\Big(1 - \frac{t\hat{f}(\xi,0)}{t+1}\Big)^{-1}. \tag{1.5.12}$$

By using the power series decomposition $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$, for $x = \frac{t\hat{f}(\xi,0)}{t+1}$, and applying the inverse Laplace transform, which we denote by $\mathcal{L}^{-1}$, term by term, we obtain that

$$f(v,t) = \frac{1}{(t+1)^2} \sum_{n=0}^{\infty} \left(\frac{t}{t+1}\right)^n \mathcal{L}^{-1}\Big(\hat{f}(\xi,0)^{n+1}\Big)(v,t). \tag{1.5.13}$$

We can apply the formula (1.5.13) when $f(v,0) = e^{-v}$. Since $\hat{f}(\xi,0)^{n+1} = \frac{1}{(\xi+1)^{n+1}}$ and $\mathcal{L}^{-1}\Big(\frac{1}{(\xi+1)^{n+1}}\Big) = \frac{v^n e^{-v}}{n!}$, it follows that

$$f(v,t) = \frac{1}{(t+1)^2} \sum_{n=0}^{\infty} \left(\frac{vt}{t+1}\right)^n \frac{e^{-v}}{n!} = \frac{1}{(t+1)^2} e^{-\frac{v}{t+1}}. \tag{1.5.14}$$

We notice that the solution in (1.5.14) corresponds to the one in (1.5.8), while keeping in mind that the methods using the Laplace transform have the advantage that they can be used to obtain solutions of (1.5.4) for more general initial data.

Self-similar profiles for general types of kernels

One can ask what is the expected long-time behavior of solutions to (1.0.1). Since the particles coagulate and form larger particles, it is expected that, for pure coagulation processes, all the particles have merged after sufficiently long time has passed. It was proven rigorously in [BLL19b, Section 10.2] that

$$\lim_{t\to\infty} M_0(f(t)) := \lim_{t\to\infty} \int_{(0,\infty)} f(v,t)\mathrm{d}v = 0 \tag{1.5.15}$$

by only requiring some positivity assumption for the coagulation kernel K. However, this result does not offer much information about the qualitative behavior of solutions and other methods for the analysis of the behavior of solutions to (1.0.1) are expected.

Further intuition on this matter can be derived from explicitly solvable kernels, such as the case of the constant kernel $K \equiv 2$. As noticed in (1.5.8), the system (1.0.1) with $K \equiv 2$ and $f(v,0) = e^{-v}$ has an explicit solution of the form

$$f(v,t) = \frac{1}{(t+1)^2} f\Big(\frac{v}{t+1}, 0\Big), \text{ for } t \in [0,\infty), v \in (0,\infty). \tag{1.5.16}$$

We denote by $\sigma(t) := 1+t$ and direct computations show that the zero-th order moment in this case is given by $M_0(f(t)) = \frac{1}{t+1}$, where $M_0(f(t))$ is as in (1.5.15), while the first order moment $M_1(f(t)) := \int_{(0,\infty)} vf(v,t)\mathrm{d}v = 1$. Thus, it holds that

$$\sigma(t) = \frac{M_1(f(t))}{M_0(f(t))} \tag{1.5.17}$$

and as such $\sigma(t)$ can be regarded as the average particle size at time t.

For the initial condition $f(v,0) = \delta(v-1)$ in the case of the constant kernel $K \equiv 2$, it was shown in [Sch40] that the solution to (1.0.1) approaches the solution in (1.5.16). It was also proven in [KP94] that the solution in (1.5.16) attracts all solutions with $M_1(f(0)) = 1$ which decay exponentially fast as $v \to \infty$.

Following the approach performed in the case of the constant kernel, the so-called scaling hypothesis suggests that the long-time behavior of solutions is described by *self-similar profiles* of the form

$$f(v,t) = \frac{1}{\sigma(t)^d} g\Big(\frac{v}{\sigma(t)}\Big), \text{ for some } d \in \mathbb{R}. \tag{1.5.18}$$

Furthermore, we deduce from (1.5.15) and (1.5.17) that $\sigma(t)$ should go to infinity as $t \to \infty$. We assume without loss of generality that $\sigma(0) = 1$.

Available results about self-similar profiles assume that the coagulation kernels are homogeneous. Thus, we work with kernels which satisfy the property that

$$K(\lambda v, \lambda v') = \lambda^\gamma K(v,v'), \tag{1.5.19}$$

for any $\lambda \in (0,\infty)$, $v, v' \in (0,\infty)$, and for some $\gamma \in \mathbb{R}$. This includes most of the coagulation kernels used in applications, as discussed in Subsection 1.1. We restrict our analysis to the case of mass-conserving solutions. More precisely, we need to choose $\gamma < 1$ in (1.5.19), as shown in Subsection 1.4. Direct computations show that we need to take $d = 2$ in (1.5.18) in order to have mass-conserving solutions since

$$\int_{(0,\infty)} v g(v) \mathrm{d}v = \int_{(0,\infty)} v f(v,0) \mathrm{d}v = \int_{(0,\infty)} v f(v,t) \mathrm{d}v = \int_{(0,\infty)} \frac{v}{\sigma(t)^d} g\Big(\frac{v}{\sigma(t)}\Big) \mathrm{d}v = \frac{1}{\sigma(t)^{d-2}} \int_{(0,\infty)} \tilde{v} g(\tilde{v}) \mathrm{d}\tilde{v}.$$

Plugging (1.5.18) for $d = 2$ into (1.0.1), we find that g solves the equation:

$$\sigma(t)^{-\gamma} \partial_t \sigma(t) \Big(-v\partial_v g(v) - 2g(v) \Big) = \mathbb{K}[g](v). \tag{1.5.20}$$

Separating the variables, we obtain that

$$\sigma(t)^{-\gamma} \partial_t \sigma(t) = c, \tag{1.5.21}$$

for some $c > 0$ and that

$$c\Big(-v\partial_v g(v) - 2g(v) \Big) = \mathbb{K}[g](v). \tag{1.5.22}$$

Solving (1.5.21) under the condition that $\gamma < 1$ and that $\sigma(0) = 1$, we obtain that

$$\sigma(t) = \Big(1 + c(1-\gamma)t\Big)^{\frac{1}{1-\gamma}}, \tag{1.5.23}$$

which is in accordance with the fact that we expect $\sigma(t) \to \infty$ as $t \to \infty$.

We notice that if g is a solution to (1.5.22) with parameter $c > 0$ and mass $M := M_1(g)$, then $G := kg(lv)$ is a solution to (1.5.22) with parameter $ckl^{-\gamma-1}$ and mass $M_1(G) = kl^{-2}M$. Since $\gamma < 1$, we can use this rescaling argument to find appropriate k and l in order to assume without loss of generality that

$$M_1(g) = 1 \text{ and } c = \frac{1}{1-\gamma}, \tag{1.5.24}$$

so that σ in (1.5.23) becomes

$$\sigma(t) = (1+t)^{\frac{1}{1-\gamma}}. \tag{1.5.25}$$

We thus arrive to the following equation for self-similar profiles

$$-v\partial_v g(v) - 2g(v) = (1-\gamma)\mathbb{K}[g](v) \text{ with } M_1(g) = 1. \tag{1.5.26}$$

The first step into the study of self-similar profiles is to rigorously prove the existence of solutions to (1.5.26). This has been proven for example in [FL05] for finite mass solutions or in [NV12] for solutions with infinite mass. However, there exist very few rigorous results proving that self-similar profiles describe the long-time behavior of solutions to (1.0.1) in the literature. We refer here to [MP04] for a detailed analysis of the long-time behavior of solutions to (1.0.1) for explicitly solvable kernels.

In order to prove existence of solutions to (1.5.26) in the case of homogeneous kernels for which solutions cannot be explicitly computed, the standard approach is to use fixed point methods. More precisely, we look at time-dependent problem

$$\partial_t g(v,t) = v\partial_v g(v,t) + 2g(v,t) + (1-\gamma)\mathbb{K}[g](v,t) \tag{1.5.27}$$

and notice that a solution to (1.5.26) is a stationary solution in time of (1.5.27). Moreover, we can correlate solutions to (1.0.1) with solutions to (1.5.27) via

$$g(v,t) := e^{2t} f(ve^t, e^{(1-\gamma)t} - 1), \tag{1.5.28}$$

or conversely

$$f(v,t) = (1+t)^{-\frac{2}{1-\gamma}} g\Big(\frac{v}{(1+t)^{\frac{1}{1-\gamma}}}, \frac{\ln(1+t)}{1-\gamma}\Big), \tag{1.5.29}$$

for all $v \in (0,\infty)$ and $t \in [0,\infty)$. Existence of solutions to (1.5.27) thus follows directly from the existence of solutions to (1.0.1).

1.6 Modified coagulation models

While the models in (1.0.1), (1.0.2) offer sufficient information for application purposes, it is often challenging to incorporate additional information about the system about quantities which cannot be expressed in terms of the volume. In recent years, a multitude of works about modified models of (1.0.1) has emerged. This can be achieved for example by adding additional terms to the equation and here we refer to the addition of an external source term which injects particles into the system. Another approach is the addition of other variables which can play an important role in the description of the system such as information about the chemical structure of the monomers, about their shape or about their position in space. We discuss these examples below as they will be the main focus of this thesis. We present coagulation models with an added source term in Subsection 1.6.1 and multi-dimensional coagulation equations in Subsection 1.6.2.

1.6.1 Coagulation equations with source

In addition to allowing coagulation of particles, we can assume that there exists a source term that injects particles into the system, which we can incorporate into the model (1.0.1) as follows

$$\partial_t f(v,t) = \frac{1}{2}\int_{(0,v)} K(v-v',v')f(v',t)f(v-v',t)\mathrm{d}v' - \int_{(0,\infty)} K(v,v')f(v,t)f(v',t)\mathrm{d}v' + \underset{\substack{\uparrow \\ \text{source}}}{\eta(v)}. \tag{1.6.1}$$

In addition to self-similar profiles, stationary solutions provide another set of solutions which may describe the long-time behavior of solutions to (1.6.1). The physical interpretation of this type of solutions is that the source term can inject particles into the system at the same rate at which particles coagulate to form larger clusters. More precisely, one looks for functions $f(v)$ which satisfy

$$0 = \mathbb{K}[f](v) + \eta(v). \tag{1.6.2}$$

The well-posedness of stationary solutions in the case of bounded kernels has been obtained in [Dub94]. The existence of solutions of the time-dependent problem (1.6.1) in the case when the portion of mass contained in the zero size particles is positive and under certain conditions for the coagulation kernel has been obtained in [EM06]. More recently, precise conditions for the existence and non-existence of stationary solutions in the case of a compactly supported source term for a general class of coagulation kernels have been obtained in [FLNV21b].

Flux of mass

In the rest of this subsection we discuss some properties of the stationary solutions for the coagulation equation with source as in (1.6.2) in relation to the transport of mass from smaller clusters to clusters of larger size. Multiplying in (1.0.1) with $\varphi(v) = v\mathbb{1}_{(0,z]}$, we obtain that

$$\partial_t \int_{(0,z]} vf(v,t)\mathrm{d}v = -J(z;f), \tag{1.6.3}$$

where

$$J(z;f) := \int_{(0,z]} \int_{[z-v,\infty)} vK(v,v')f(v,t)f(v',t)\mathrm{d}v'\mathrm{d}v. \tag{1.6.4}$$

$J(z;f)$ is termed in the literature as *the flux of mass* from sizes smaller than z to sizes larger than z. Solutions for the standard coagulation equation (1.0.1) which have in addition the property that $J(z;f)$ is constant are known as *constant flux solutions*.

It is clear that in the case of coagulation equations with source one does not have mass-conservation due to the presence of the source term. However, as in (1.6.3), we can rewrite (1.6.1) by means of the flux of mass and thus we obtain that

$$\partial_t \int_{(0,z]} vf(v,t)\mathrm{d}v = -J(z;f) + \int_{(0,z]} v\eta(v)\mathrm{d}v. \tag{1.6.5}$$

If $\int_{(0,\infty)} v\eta(v)\mathrm{d}v < \infty$ and we can pass to the limit as $z \to \infty$ in (1.6.5), this would imply that the total mass of particles increases linearly in time.

Similarly with (1.6.3), by multiplying with v in equation (1.6.2), we deduce that in the case of stationary solutions of coagulation equations with source it holds that

$$\partial_v J(v;f) = v\eta(v).$$

In the case of a compactly supported source term η, this implies that $J(v;f)$ is constant for large particles and thus stationary solutions of equation (1.6.1) are constant flux solutions for large particles.

We conclude this subsection by mentioning that $J(v;f)$ also plays a role in the study of self-similar profiles introduced in Subsection 1.5. Another representation of equation (1.5.26) is more convenient for some particular choices of coagulation kernels. More precisely, we multiply formally in (1.5.26) with $v\varphi(v)$ and then use the symmetry of the coagulation kernel to deduce that

$$\begin{aligned}\int_{(0,\infty)} \varphi(v)\partial_v(v^2g(v))\mathrm{d}v &= -(1-\gamma)\int_{(0,\infty)}\int_{(0,\infty)} vK(v,v')g(v)g(v')[\varphi(v+v')-\varphi(v)]\mathrm{d}v'\mathrm{d}v \\ &= -(1-\gamma)\int_{(0,\infty)}\int_{(0,\infty)} vK(v,v')g(v)g(v')\int_{[v,v+v']}\partial_z\varphi(z)\mathrm{d}z\mathrm{d}v'\mathrm{d}v.\end{aligned}$$

Applying Fubini's theorem, we further obtain that

$$\int_{(0,\infty)} v^2g(v)\partial_v\varphi(v)\mathrm{d}v = (1-\gamma)\int_{(0,\infty)}\partial_z\varphi(z)\int_{(0,z]}\int_{[z-v,\infty)} vK(v,v')g(v)g(v')\mathrm{d}v'\mathrm{d}v\mathrm{d}z,$$

which leads to the following reformulation of (1.5.26) assuming a suitable behavior of the function $v^2g(v)$

$$v^2g(v) = (1-\gamma)\int_{(0,v]}\int_{[v-v',\infty)} v'K(z,v')g(z)g(v')\mathrm{d}z\mathrm{d}v' = J(v;g). \tag{1.6.6}$$

1.6.2 Multi-dimensional coagulation equations

Multicomponent coagulation systems used to describe the chemical structure of monomers

Allowing the situation in which the clusters are formed of different types of monomers and keeping track of the composition of clusters plays a significant role in atmospheric science. For example, the resulting effects on the composition of clusters when the interacting particles are sulfuric acid and ammonia (acid-base interactions) are discussed in [OKMO+13].

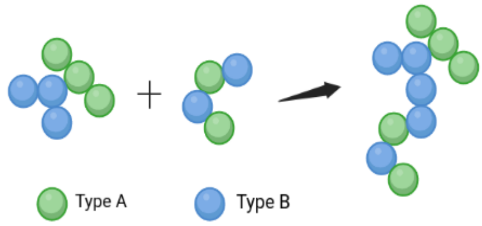

Figure 1.2: Clusters of different chemical composition

In the mathematical literature, existence of solutions for these types of models was discussed in [FLNV23]. More precisely, for some $d \in \mathbb{N}$, we now consider $v \in \mathbb{R}^d_+ \setminus \{O\}$ with components $v = (v_1, \ldots, v_d)$. For two elements $v, v' \in \mathbb{R}^d_+ \setminus \{O\}$, we say that $v' < v$ if $v'_i \leq v_i$, for all $i = \overline{1, d}$ and $v' \neq v$. Following the logic behind the derivation of its one-dimensional counterpart, we are lead to the following model

$$\begin{aligned}\partial_t f(v,t) =& \frac{1}{2}\int_{\{0<v'<v\}} K(v-v', v') f(v-v', t) f(v', t) \mathrm{d}v' \\ &- \int_{\mathbb{R}^d_+\setminus\{O\}} K(v, v') f(v,t) f(v', t) \mathrm{d}v' + \eta(v),\end{aligned} \tag{1.6.7}$$

where η represents the source term as described in Subsection 1.6.1 and we allow $\eta \equiv 0$ in the case of pure coagulation processes.

For pure coagulation processes (i.e., $\eta \equiv 0$), we have conservation of the total mass of each monomer type. More precisely, if we denote by $|v| := \sum_{i=1}^d v_i$, one can obtain formally that

$$\partial_t \Big(\int_{\mathbb{R}^d_+\setminus\{O\}} v f(v,t) \mathrm{d}v \Big) = 0 \text{ and } \partial_t \Big(\int_{\mathbb{R}^d_+\setminus\{O\}} |v| f(v,t) \mathrm{d}v \Big) = 0, \tag{1.6.8}$$

where the first equality in (1.6.8) is to be understood as a vector equality.

An interesting property of solutions to (1.6.7) in the case of $\eta \equiv 0$ that does not occur in the case of one-dimensional coagulation models is the *localization property*. In other words, large particles tend to concentrate along a line whose orientation is uniquely determined by the cluster composition of the initial data. The same feature arises in the case of stationary solutions for non-trivial source terms and the direction in this case depends on the first order moment of the source term. The localization property has been proven for a general class of kernels in [FLNV21a, FLNV22].

A consequence of the localization property is that it can be used to compute self-similar profiles of (1.6.7) in the case when $\eta \equiv 0$ in terms of self-similar profiles of the one-dimensional model (1.0.1), thus establishing a clear connection between standard one-dimensional models and their multicomponent counterpart.

Multi-dimensional coagulation models describing particles with different geometrical shapes

Assume that at initial time we have a system of spherical particles. We deal with a coagulation model describing the evolution of this particle system in which we characterize the particles by their volume and surface area, namely

$$\overset{\text{density of particles}}{\overset{\downarrow}{\partial_t f(a,v,t)}} + \partial_a\Big(\overbrace{\underset{\substack{\uparrow\\ \text{fusion kernel}}}{r(a,v)} \ (\underset{\substack{\uparrow\\ \text{area of the sphere}}}{c_0 v^{\frac{2}{3}}} - a) f(a,v,t)}^{\text{finite fusion time} \Rightarrow \text{changes in the area}} \Big) = \underbrace{\mathbb{K}[f](a,v,t)}_{\text{collision operator}}, \tag{1.6.9}$$

where $c_0 := (36\pi)^{\frac{1}{3}}$ represents the surface area of a sphere of volume one and where $\mathbb{K}$ is the two-dimensional counterpart of the one-dimensional coagulation operator introduced in (1.0.1), namely

$$\begin{aligned} \mathbb{K}[f](a,v,t) :=& \frac{1}{2}\int_{(0,a)\times(0,v)} K(a-a',v-v',a',v')f(a',v',t)f(a-a',v-v',t)\mathrm{d}v'\mathrm{d}a' \\ &- \int_{(0,\infty)^2} K(a,v,a',v')f(a,v,t)f(a',v',t)\mathrm{d}v'\mathrm{d}a'. \end{aligned} \tag{1.6.10}$$

Compared to the standard one-dimensional models, this model incorporates information about the geometry of the particles in addition to information about their volume and has been used in the physical literature in the study of aerosol flame reactors, see [Fri00]. We describe the coagulation process as a combination between collision (moment in time in which the particles attach each other at a contact point) and fusion (the process that begins after the collision of particles up until the newly-formed particle becomes a sphere). We allow the fusion time to be positive, i.e., the two interacting particles do not become a sphere immediately after collision. Thus, the operator $\mathbb{K}$ in (1.6.10) is now used to describe the collision of particles. The two-dimensionality of the model together with the second term on the left-hand side of (1.6.9) offer information about the changes in shape of the particles during the fusion process and hence we call this term the *fusion term*. We omit further details here as we will discuss this model in a different section.

Multi-dimensional coagulation models describing rain formation

A space-dependent coagulation equation has been used to model rain formation or the behavior of air bubbles in water, cf. [Bur83, CD95, CD97, Dub90a, Dub90b, Gal77, Gal85, Gal87, HNS07]. Here spherical particles of volume v move in space vertically, for example due to gravitation, and merge when their trajectories cross. This leads to the following spatially inhomogeneous coagulation equation for the density f of particles of size v at the point x:

$$\begin{aligned}\partial_t f(x,v,t) + v^{\alpha}\partial_x f(x,v,t) = \frac{1}{2}\int_{(0,v)} K(v-v',v')f(x,v-v',t)f(x,v',t)\mathrm{d}v' \\ - \int_{(0,\infty)} K(v,v')f(x,v,t)f(x,v',t)\mathrm{d}v', \end{aligned} \tag{1.6.11}$$

where $\alpha \in (0,1)$, with a so called differential sedimentation kernel (sum kernel which vanishes on the diagonal), namely

$$K(v,v') = |v^{\frac{2}{3}} - v'^{\frac{2}{3}}|(v^{\frac{1}{3}} + v'^{\frac{1}{3}})^2. \tag{1.6.12}$$

The derivation of the coagulation kernel (1.6.12) was discussed in Subsection 1.1. We only present the model here as we will discuss further details in a different section.

1.7 Overview of the results

In this section we give an overview of the results that will be presented in this thesis.

Multi-dimensional coagulation models describing particles with different geometrical shapes

We wish to study the long-time behavior of the model introduced in (1.6.9). More precisely, we study the existence of self-similar profiles for the model (1.6.9). The rescaling properties of the self-similar profiles are chosen in a manner such that the spherical geometry of the particles is preserved during the coagulation process. We remember that coagulation is considered a combination of collision and fusion in this case, as explained in Subsection 1.6.2.

We notice that the long-time asymptotics of the model depend quite heavily on the competition between the collision term and the fusion term. On one hand, we prove existence of self-similar profiles if the fusion of particles occurs at a faster rate than the collision process for particles of large size. Moreover, we formulate precise conditions that determine when the fusion process occurs faster than collision. On the other hand, if the fusion term is negligible compared to the collision term, we show that particles attach each other at a contact point, forming a *fractal-like* system in time. This is since the total volume and area of the system are conserved, while the total number of particles decreases in time. We refer to Chapter 2 for more details.

We then further study this competition with the aim of finding connections between the solutions of our two-dimensional model (1.6.9) and the standard coagulation equation (1.0.1). More precisely, we analyze the model

$$\partial_t f_\epsilon(a,v,t) + \frac{1}{\epsilon}\partial_a[r(a,v)(c_0 v^{\frac{2}{3}} - a) f_\epsilon(a,v,t)] = \mathbb{K}[f_\epsilon](a,v,t)$$

as $\epsilon \to 0$ and as $\epsilon \to \infty$, where c_0 and $\mathbb{K}[f]$ are as in (1.6.9).

We first prove that as $\epsilon \to 0$ solutions concentrate their mass around the isoperimetric line $\{a = c_0 v^{\frac{2}{3}}\}$ and tend in an appropriate sense to a measure which can be computed by solving a suitable one-dimensional coagulation equation. This is since the fusion process takes place much faster as $\epsilon \to 0$ than the collision process. We then study the model as $\epsilon \to \infty$. In this case, the effect of the fusion term becomes negligible and thus only the collision term contributes to the formation of particles. As such, we recover a two-dimensional coagulation model in which fusion does not occur as $\epsilon \to \infty$. We refer to Chapter 3 for the precise results.

Coagulation equations with source

As mentioned in Subsection 1.6.1, the precise conditions for existence and non-existence of stationary solutions as in (1.6.2) in the case of a compactly supported source term have been obtained in [FLNV21b]. The main idea in order to prove that solutions do not exist is based on the fact that the transfer of clusters of size of order one toward very large cluster sizes is so fast that the concentration of clusters with size of order one would become zero. In the case when stationary solutions do not exist, we cannot expect them to describe the long-time behavior of solutions to (1.6.1).

As such, we derive a new coagulation model in the case when the main contribution to the long-time behavior of solutions is given by the interaction of particles of different sizes using formal arguments. This leads to an additional transport term in the coagulation equation that approximates the solution of the original coagulation equation with injection for large times.

We establish exact conditions for existence and non-existence of self-similar profiles for the newly-derived model. In addition, if self-similar profiles exist, they exhibit an interesting property, namely they vanish on an interval near the origin that we can compute explicitly. This vanishing property of solutions is a remarkable feature that, to our knowledge, has not been observed in the literature on self-similar solutions for the coagulation equations. For an overview of the results we refer to Chapter 4.

Multi-dimensional coagulation models describing rain formation

We prove local in time existence of mass-conserving solutions for the model (1.6.11) for a class of coagulation kernels for which instantaneous gelation would otherwise occur. As observed in Section

1.4, for coagulation kernels of homogeneity $\gamma > 1$, solutions do not conserve mass. The kernels that constitute our main object of study for the model (1.6.11) are sum-type kernels of homogeneity $\gamma > 1$, i.e., $K(v, v') = v^{\gamma} + v'^{\gamma}$, and the differential sedimentation kernel introduced in (1.1.4).

As mentioned in Section 1.4, solutions in L^1 of the standard one-dimensional coagulation equation do not exist at all for sum-type kernels of homogeneity $\gamma > 1$. This is to our knowledge the first result of existence of solutions involving sum kernels of homogeneity $\gamma > 1$, regardless of whether one considers one-dimensional or multi-dimensional coagulation models.

On the other hand, the differential sedimentation kernel, $K(v, v') = |v^{\frac{2}{3}} - v'^{\frac{2}{3}}|(v^{\frac{1}{3}} + v'^{\frac{1}{3}})^2$, whose derivation was explained in Section 1.1, is relevant due to its physical applications. The model in (5.1.1) with $\alpha = \frac{2}{3}$ and with a differential sedimentation kernel is used in the physical literature in order to describe the behavior of air bubbles in water that move due to buoyancy and it is also a valid model for the description of the onset of rain. In addition, it was proven in [CNV24a] that instantaneous gelation occurs for this type of kernel in the case of the standard one-dimensional coagulation model (1.0.1). For further details on the matter, we refer to Chapter 5.

For possible future research directions in relation to all the models presented in this section we refer to Chapter 6.

CHAPTER 2

INTRODUCTION TO COAGULATION EQUATIONS FOR NON-SPHERICAL CLUSTERS

Description

In this chapter we will give an overview of the results of Appendix A. The paper is available online [CV23] and is a joint work with J. J. L. Velázquez.

In this part we study the possible long-time behavior of the solutions to the multi-dimensional coagulation model introduced in (1.6.9). We notice that the long-time asymptotics of the model depend quite heavily on the competition between the collision term and the fusion term.

2.1 Physical motivation of the model

We wish to analyze a coagulation model which offers information about both the volume and the shape of the interacting particles. Thus, we deal with a two-dimensional coagulation model which is used to describe the evolution in time of the number of particles $f(a, v, t)$, where a represents the total surface area of the cluster while v represents the volume. Such a model was introduced in [Fri00, KF90] in order to study aerosol flame reactors. Indeed, in aerosols, changes of the temperature or the addition of impurities to the system lead to changes in the interaction speed of the particles.

We assume the underlying interaction process to be as follows. At initial time we have a finite system of spherical particles. Two particles first attach each other at a contact point and we term this phenomenon as *collision* of particles. Particles then start to merge into one another until they become a bigger spherical particle, with volume equal to the sum of the two interacting particles. We call this process the *fusion* of particles.

It is worthwhile to notice that the collision process describes the same interaction of particles as the one-dimensional coagulation model (1.0.1). Thus, it is expected that the collision operator $\mathbb{K}[f](a,v,t)$ will be very similar to its one-dimensional counterpart. However, as soon as the particles start to merge, depending on the temperature of the system, the fusion process can take place at a slower or faster rate and the shape of the particles starts playing an important role. If fusion happens very fast, the interacting particles become a new sphere almost immediately. On the other hand, fusion can take place slowly leading to dumbbell-shaped particles. We refer to Figure 2.1 for an analysis of the resulting shapes of the particles depending on the fusion time.

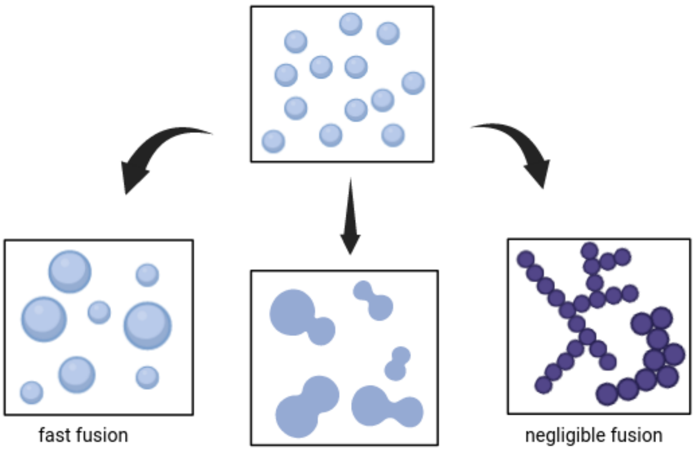

Figure 2.1: Possible outcomes depending on the fusion time

More precisely, we look at the following model

$$\partial_t f(a,v,t) + \partial_a[r(a,v)(c_0 v^{\frac{2}{3}} - a)f(a,v,t)] = \mathbb{K}[f](a,v,t), \; c_0 := (36\pi)^{\frac{1}{3}}, \tag{2.1.1}$$

where $\mathbb{K}$ is the two-dimensional counterpart of the one-dimensional coagulation operator, namely

$$\begin{aligned}\mathbb{K}[f](a,v,t) :=& \frac{1}{2}\int_{(0,a)\times(0,v)} K(a-a',v-v',a',v')f(a',v',t)f(a-a',v-v',t)\mathrm{d}v'\mathrm{d}a' \\ &- \int_{(0,\infty)^2} K(a,v,a',v')f(a,v,t)f(a',v',t)\mathrm{d}v'\mathrm{d}a'.\end{aligned} \tag{2.1.2}$$

As most of our results hold for the weak formulation of the problem, we mention here that by using similar computations as in Section 1.3, we obtain the following weak formulation of (2.1.1) for a suitable test function φ

$$\begin{aligned}\partial_t \int_{(0,\infty)^2} f(\eta,t)\varphi(\eta)\mathrm{d}\eta =& \int_{(0,\infty)^2} r(a,v)(c_0 v^{\frac{2}{3}} - a)f(\eta,t)\partial_a\varphi(\eta)\mathrm{d}\eta \\ &+ \frac{1}{2}\int_{(0,\infty)^2}\int_{(0,\infty)^2} K(\eta,\eta')f(\eta,t)f(\eta',t)[\varphi(\eta+\eta')-\varphi(\eta)-\varphi(\eta')]\mathrm{d}\eta'\mathrm{d}\eta,\end{aligned} \tag{2.1.3}$$

where we used the notation

$$\eta = (a, v),\ \eta' = (a', v'),\ \eta + \eta' = (a + a', v + v').$$

2.1.1 Setting and assumptions on the kernels

As discussed in Section 1.5, the long-time behavior of solutions of coagulation equations is often associated with the existence of self-similar profiles. As such, we focus on studying the existence of self-similar profiles for the equation (2.1.1).

Moreover, we wish that the underlying process preserves the geometry of the particles. In other words, if the particle volume is scaled by a factor λ, then the diameter is scaled by a factor $\lambda^{\frac{1}{3}}$ and the area scales like $\lambda^{\frac{2}{3}}$. It thus becomes natural to look for self-similar profiles of the form

$$f(a, v, t) = \frac{1}{(1+t)^{\frac{8}{3}\xi}} g\Big(\frac{a}{(1+t)^{\frac{2}{3}\xi}}, \frac{v}{(1+t)^{\xi}}\Big) \text{ for } \xi = \frac{1}{1-\gamma}, \tag{2.1.4}$$

where $\gamma < 1$. The choice of γ is consistent with scaling used in the study of existence of mass-conserving self-similar solutions for one-dimensional coagulation models as explained in Section 1.4.

Assumptions on the collision kernel

As mentioned above, the collision kernel describes how fast two particles attach to each other at a contact point. During the collision process, the total area and volume of particles are conserved. This can also be seen at the level of the equation. In the case when fusion does not occur in (2.1.3) (i.e., $r \equiv 0$), we obtain the following version of the equation

$$\partial_t \int_{(0,\infty)^2} f(\eta, t)\varphi(\eta) \mathrm{d}\eta = \langle \mathbb{K}[f](t), \varphi \rangle. \tag{2.1.5}$$

We test formally with $\varphi \equiv a$ and $\varphi \equiv v$ in (2.1.5) and notice that in this case

$$\partial_t \int_{(0,\infty)^2} a f(\eta, t) \mathrm{d}\eta = 0 \text{ and } \partial_t \int_{(0,\infty)^2} v f(\eta, t) \mathrm{d}\eta = 0. \tag{2.1.6}$$

Our collision kernel K satisfies the expected assumptions arising from the physical interpretation of the model. As such, K is a symmetric function which preserves the spherical geometry of the particles. Moreover, our kernel also satisfies the technical conditions that appear in the study of one-dimensional models and which remain valid in the multi-dimensional case, such as homogeneity and conditions needed for mass-conservation. We gather below the main assumptions on the collision kernel together with a short explanation of the conditions.

$K(a,v,a',v') = K(a',v',a,v)$	$\rightsquigarrow$	symmetry of interactions
$K(\lambda^{\frac{2}{3}}a, \lambda v, \lambda^{\frac{2}{3}}a', \lambda v') = \lambda^{\gamma} K(a,v,a',v'), \lambda > 0$	$\rightsquigarrow$	similar aggregation properties for particles with the same geometry but different size
$K_1(v^{-\alpha}v'^{\beta} + v'^{-\alpha}v^{\beta}) \leq K(a,v,a',v') \leq K_0(v^{-\alpha}v'^{\beta} + v'^{-\alpha}v^{\beta})$	$\rightsquigarrow$	area has small effect on the collision mechanism
$\gamma := \beta - \alpha, \gamma \in (0,1), \alpha \in (0,1), \beta \in (0,1)$	$\rightsquigarrow$	gelation does not occur (i.e., solutions conserve mass)

Table 2.1: Assumptions on the collision kernel

We notice that, while the area has little effect on the collision kernel, the collision kernel may depend on the surface area of the particles. However, it will be worth to study models involving a more explicit dependence on the area of the collision kernel. We will discuss more details about this in Chapter 6. Moreover, the condition $\gamma > 0$ implies that the collision rate increases with the particle size. While our paper [CV23] contains some results for the cases when $\gamma = 0$ or $\alpha = 0$, we restrict ourselves here to positive values of α and γ in order to better focus on the physical intuition behind our results rather than the technicalities appearing when dealing with different values for the parameters α and γ. We leave a more detailed analysis of the coefficients α and γ to Chapter 6, as well as to the actual paper that is contained in Appendix A.

Assumptions on the fusion kernel

The fusion term $\partial_a[r(a,v)(c_0 v^{\frac{2}{3}} - a) f(a,v,t)]$ describes an evolution of the particles toward a spherical shape, while the fusion kernel $r(a,v)$ keeps track of the speed of the fusion process. For example, if $r \equiv 0$ then fusion does not occur and particles attach each other at a contact point. On the other hand, for sufficiently large values of r, the newly-formed particles become spherical almost immediately, cf. Figure 2.2. A distinction between these two cases is not possible when dealing with the standard one-dimensional coagulation model.

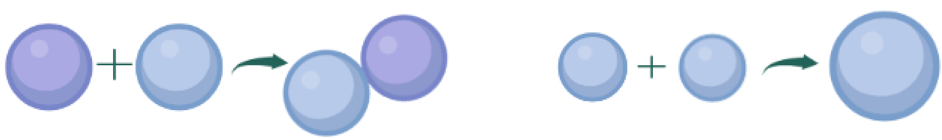

Figure 2.2: Absence of fusion (left). Instantaneous fusion (right)

In addition, since we work with physically relevant particles described by both area and volume, it is natural to assume that particles satisfy the isoperimetric inequality, i.e., $\{a \geq c_0 v^{\frac{2}{3}}\}$. Thus, we work with Radon measures which vanish in the region where $\{a < c_0 v^{\frac{2}{3}}\}$. More precisely, $f(\cdot,\cdot,t)$ in (2.1.1) belongs to the space

$$\mathscr{M}_+^I(\mathbb{R}^2_{>0}) := \{h \in \mathscr{M}_+(\mathbb{R}^2_{>0}) \mid h(\{a < c_0 v^{\frac{2}{3}}\}) = 0\}. \tag{2.1.7}$$

While it is straightforward to notice that the evolution generated by the fusion term preserves the total number of particles and the total volume of the system, the total surface area of the system is however decreasing under these assumptions. Indeed, by testing formally with $\varphi \equiv a$ in (2.1.3), we obtain that

$$\partial_t \int_{(0,\infty)^2} a f(\eta,t)\mathrm{d}\eta = \int_{(0,\infty)^2} r(a,v)[c_0 v^{\frac{2}{3}} - a] f(\eta,t)\mathrm{d}\eta \leq 0$$

since f is supported outside the region where $\{a < c_0 v^{\frac{2}{3}}\}$.

While looking for self-similar profiles of the form (2.1.4) for (2.1.1) and assuming the fusion kernel has a power law behavior as often found in applications (see [KF90]), we need to impose additional assumptions in order for the fusion term and the collision term in (2.1.1) to rescale in a similar manner. We gather these conditions below.

$R_0 a^\mu v^\sigma \leq r(a,v) \leq R_1 a^\mu v^\sigma$	$\rightsquigarrow$	power law behavior
$r(\lambda^{\frac{2}{3}} a, \lambda v) = \lambda^{\gamma-1} r(a,v), \lambda > 0$	$\rightsquigarrow$	keeps self-similar structure
$\frac{2}{3}\mu + \sigma = \gamma - 1$	$\rightsquigarrow$	keeps self-similar structure

Table 2.2: Assumptions on the fusion kernel

There is one additional technical assumption that appears when trying to prove the existence of such profiles, namely

$$\begin{cases} [\partial_a r(a,v) - \mu a^{-1} r(a,v)](a - c_0 v^{\frac{2}{3}}) + r(a,v) \geq 0, \text{ and } \partial_a r(a,v) \leq B a^{-1} r(a,v), & \text{if } \mu > 0; \\ \partial_a r(a,v)(a - c_0 v^{\frac{2}{3}}) + r(a,v) \geq 0, \text{ and } \partial_a r(a,v) \leq B a^{-1} r(a,v), & \text{if } \mu \leq 0, \end{cases} \tag{2.1.8}$$

for all $(a,v) \in (0,\infty)^2$, with $a \geq c_0 v^{\frac{2}{3}}$ and for some constant $B > 0$.

In the established context, self-similar profiles g of the form (2.1.4) for the equation (2.1.1) need to satisfy the following equation

$$0 = \frac{8}{3} g(\eta) + \frac{2}{3} a\partial_a g(\eta) + v\partial_v g(\eta) + (1-\gamma)\partial_a[r(\eta)(a - c_0 v^{\frac{2}{3}})g(\eta)] + (1-\gamma)\mathbb{K}[g](\eta). \tag{2.1.9}$$

Moreover, we prove existence for such profiles in the context of Radon measures and for physically relevant particles, i.e., $g \in \mathscr{M}_+^I(\mathbb{R}^2_{>0})$, where $\mathscr{M}_+^I(\mathbb{R}^2_{>0})$ was defined in (2.1.7). A precise definition of self-similar profiles is then as follows.

Definition 2.1.1 (Definition A.2.2)**.** Assume $\alpha > 0$. We will say that a measure $g \in \mathscr{M}_+^I(\mathbb{R}^2_{>0})$ is a self-similar profile for the two-dimensional coagulation equation (2.1.1) if

$$\int_{(0,\infty)^2} (v^{-\alpha} + v^{\beta}) g(a, v) \mathrm{d}v \mathrm{d}a < \infty \tag{2.1.10}$$

and for every $\varphi \in \mathrm{C}^1_c(\mathbb{R}^2_{>0})$ the following equality is satisfied:

$$\begin{aligned}&\int_{(0,\infty)^2} g(\eta)\varphi(\eta)\mathrm{d}\eta - \frac{2}{3}\int_{(0,\infty)^2} g(\eta) a \partial_a \varphi(\eta)\mathrm{d}\eta - \int_{(0,\infty)^2} g(\eta) v \partial_v \varphi(\eta)\mathrm{d}\eta \\ &+(1-\gamma)\langle \mathbb{K}[g], \varphi\rangle + (1-\gamma)\int_{(0,\infty)^2} r(\eta)(c_0 v^{\frac{2}{3}} - a) g(\eta)\partial_a\varphi(\eta)\mathrm{d}\eta = 0.\end{aligned} \tag{2.1.11}$$

As explained in Section 1.5, we can think of self-similar profiles as fixed points in time for the time-dependent formulation of (2.1.11), cf. (1.5.27)-(1.5.29). As many of the properties of self-similar profiles are a result of the analysis of this time-dependent version, it is useful to mention its form here.

Definition 2.1.2 (Definition A.2.1)**.** Assume $\alpha > 0$. Let $g \in \mathrm{C}([0,\infty); \mathscr{M}_+^I(\mathbb{R}^2_{>0}))$. We say that g is a self-similar solution for the weak version of the time-dependent two-dimensional coagulation equation (2.1.1) if, for every $T > 0$,

$$\sup_{t\in[0,T]} \int_{(0,\infty)^2} (v^{-\alpha} + v^{\beta}) g(a, v, t) \mathrm{d}v \mathrm{d}a < \infty$$

and, for all $\varphi \in \mathrm{C}^1_c([0,\infty); \mathrm{C}^1_c(\mathbb{R}^2_{>0}))$ and $t \in [0,\infty)$

$$\begin{aligned}&\int_{(0,\infty)^2} g(\eta, t)\varphi(\eta, t)\mathrm{d}\eta - \int_{(0,\infty)^2} g(\eta, 0)\varphi(\eta, 0)\mathrm{d}\eta - \int_0^t \int_{(0,\infty)^2} g(\eta, s)\partial_s\varphi(\eta, s)\mathrm{d}\eta\mathrm{d}s = \\ &\int_0^t \int_{(0,\infty)^2} g(\eta, s)\varphi(\eta, s)\mathrm{d}\eta\mathrm{d}s - \frac{2}{3}\int_0^t \int_{(0,\infty)^2} g(\eta, s) a\partial_a\varphi(\eta, s)\mathrm{d}\eta\mathrm{d}s - \int_0^t \int_{(0,\infty)^2} g(\eta, s) v\partial_v\varphi(\eta, s)\mathrm{d}\eta\mathrm{d}s \\ &+(1-\gamma)\int_0^t \langle \mathbb{K}[g](s), \varphi(s)\rangle \mathrm{d}s + (1-\gamma)\int_0^t \int_{(0,\infty)^2} r(\eta)(c_0 v^{\frac{2}{3}} - a) g(\eta, s)\partial_a\varphi(\eta, s)\mathrm{d}\eta\mathrm{d}s.\end{aligned} \tag{2.1.12}$$

2.2 Main results and their physical interpretation

In all the following we will assume that K is a continuous, non-negative kernel satisfying the assumptions presented in Table 2.1 and that $r \in \mathrm{C}^1(\mathbb{R}^2_{>0})$ satisfies the assumptions presented in Table 2.2 and (2.1.8). Moreover, in order to explain the physical interpretation of our results, it is useful to introduce the following notation. Given $H = H(a, v)$, we write

$$\langle H\rangle(t) = \frac{\int_{(0,\infty)^2} H(a, v) f(a, v, t)\mathrm{d}v\mathrm{d}a}{\int_{(0,\infty)^2} f(a, v, t)\mathrm{d}v\mathrm{d}a}, \text{ for any time } t \geq 0,$$

where f is as in (2.1.1). As noticed in Figure 2.1, the long-time behavior of solutions depends on the competition between the fusion term and the collision term in (2.1.1). More precisely, if the fusion term is comparable to the collision term for large particles, then the particles stay close to spherical and we can expect self-similar profiles to exist in this case. For power-law fusion kernels r as in Table 2.2, the fusion kernel overtakes the collision kernel for large particles when $\mu > 0$. Thus, in this situation we have that

$$\frac{\langle a\rangle(t)}{(\langle v\rangle(t))^{\frac{2}{3}}} \approx 1 \text{ as } t \to \infty$$

and we can prove existence of self-similar profiles since we are in a similar situation to the one-dimensional case. More precisely, we prove that

Theorem 2.2.1 (Theorem A.2.3)**.** *Let $\mu, \alpha > 0$ and $v_0 > 0$. Then there exists a self-similar profile for the two-dimensional coagulation equation* (2.1.1) *in the sense of Definition 2.1.1 with* $\int_{(0,\infty)^2} vg(a,v)\mathrm{d}v\mathrm{d}a = v_0$. *In addition, g satisfies $M_{n,k}(g) < \infty$, for all $n, k \in \mathbb{R}$.*

In the case when $\mu < 0$, the fusion term is negligible compared to the collision term and the first expected result is that particles attach each other at a contact point since fusion takes place at a slow rate for large particles. We term this phenomenon the *ramification* of particles. This can be observed at an intuitive level since in the case when the fusion is absent from our model, as in (2.1.5), the total volume and area of the system are conserved, see (2.1.6).

More precisely, for $\mu < 0$, we prove that if we start with an initial distribution for which particles differ from spheres, namely if there exists a sufficiently large $\lambda_0 > 0$ such that

$$\frac{\langle a\rangle(0)}{(\langle v\rangle(0))^{\frac{2}{3}}} \geq \lambda_0,$$

then we obtain the following behavior of particles

$$\frac{\langle a\rangle(t)}{(\langle v\rangle(t))^{\frac{2}{3}}} \to \infty \text{ as } t \to \infty. \tag{2.2.1}$$

We will actually prove that there exists some constant $c > 0$ such that

$$\frac{c}{\langle v\rangle(0)} \leq \frac{\langle a\rangle(t)}{\langle v\rangle(t)} \leq \frac{\langle a\rangle(0)}{\langle v\rangle(0)} \tag{2.2.2}$$

and (2.2.2) implies (2.2.1) since $\langle v\rangle(t) \to \infty$ as $t \to \infty$. Thus, we are in a situation where the total surface area is comparable to the total volume of particles and the total number of particles decreases in time. It is thus expected that the resulting particles will differ significantly from spherical particles and will have a fractal-like aspect.

A precise mathematical formulation of the ramification phenomenon is as follows. For clarity of the presentation, we omit some technical assumptions which appear due to the fact that we want to test in (2.1.12) with functions which are not necessarily compactly supported. The content of the theorem below can also be found in Theorem A.2.12 and we refer to it for the exact conditions under which our result is valid.

Theorem 2.2.2. *Assume $\alpha > 0$ and $\mu < 0$. Let $g \in \mathrm{C}([0,\infty); \mathscr{M}_+^I(\mathbb{R}^2_{>0}))$ be a self-similar solution of the time-dependent two-dimensional coagulation equation (2.1.1) as in Definition 2.1.2 with total volume of particles equal to v_0 and satisfying some sufficiently strong moment estimates as presented later in (A.2.17) and (A.2.18). Then the following holds: there exists a constant $C_2(v_0) > 0$ such that, if*

$$R_1 \leq v_0 \tag{2.2.3}$$

and

$$\int_{(0,\infty)^2} a g_{\mathrm{in}}(a,v)\mathrm{d}v\mathrm{d}a \geq C_2(v_0), \tag{2.2.4}$$

then

$$\int_{(0,\infty)^2} g(a,v,t) a\mathrm{d}v\mathrm{d}a \geq C_\mu C_2(v_0)\mathrm{e}^{\frac{1}{3}t}, \tag{2.2.5}$$

for some C_μ depending on μ.

Due to the assumed rescaling properties, cf. (2.1.4), it holds that (2.2.5) together with the fact that the total volume of particles stays constant implies (2.2.2).

However, if the fusion kernel r is sufficiently large when a and v are of order one, a regime similar to the one where fusion overtakes coagulation occurs and thus self-similar profiles exist in this case too. In other words, in the case when $\mu < 0$, the solutions can exhibit different behaviors depending on the initial data and fusion speed. For fusion kernels of order one when a and v are of order one and sufficiently large initial surface area, ramification occurs as presented in Theorem 2.2.2. However, there exist fusion kernels for which we have existence of self-similar profiles also in the case $\mu < 0$. More precisely, the following holds.

Theorem 2.2.3 (Theorem A.2.7)**.** *Let $\mu \leq 0$ and $\alpha > 0$. Then, there exists $\lambda > 1$, depending only on K_0, K_1 and γ, such that for any $v_0 > 0$, if*

$$R_0 \geq \lambda v_0, \tag{2.2.6}$$

then there exists a self-similar profile g for the two-dimensional coagulation equation (2.1.1), in the sense of Definition 2.1.1, with total volume $\int_{(0,\infty)^2} v g(a,v)\mathrm{d}v\mathrm{d}a = v_0$.

2.3 Multi-dimensional coagulation equations in the mathematical literature

Coagulation equations including also transport terms have been considered in the mathematical literature in relation to different contexts. For instance, in [AD03], the well-posedness of a model for the evolution of the cluster size of phytoplankton, which has been introduced in [AF97], has been studied. A model including also fragmentation effects of the phythoplakton clusters has been analyzed in [BL09], where results on existence and uniqueness of solutions for some classes of kernels have been discussed.

In these papers the transport term is related to the growth of the cluster due to cell division within it, while in the model (2.1.1) the transport term describes the fusion of particles. This allows the observation of the changes in the geometry of the particles with no changes to the volume.

Multi-dimensional coagulation equations have not been as extensively studied in the mathematical literature as the one-dimensional coagulation model. Several discrete multi-component coagulation problems which are relevant in aerosol physics have been mentioned in [Wat06b]. A discrete version of the model in (2.1.1) has been studied in [Wat06a]. The model considered in there includes coagulation of particles and an effect similar to the fusion of particles in (2.1.1), which has been termed compaction. The diameter of the particles is restricted by the total number of monomers as well as by the isoperimetric inequality. The coagulation and the fusion rates are assumed to be constant. Due to this, the model considered in [Wat06a] is explicitly solvable using generating functions. The long-time behavior of the solutions which depends on the ratio between the fusion and coagulation kernels has been then analyzed using the explicit formulas of the solutions.

In [FLNV21a, FLNV22, FLNV23], the mathematical properties of some classes of coagulation equations describing clusters that are composed of several types of monomers with different chemical composition are analyzed.

The main differences between the models studied in these papers and our model are the following:

- ☞ The two variables used to describe the particles in our model rescale in a different manner. Additionally, we consider coagulation kernels that do not have a strong dependence on the area variable. As a consequence, the variables describing the clusters appear in a less symmetric manner.

- ☞ The proof in [FLNV21a, FLNV22, FLNV23] relies on the conservation of mass of each monomer type. Due to the presence of the fusion term, the solutions of (2.1.1) do not have two conserved quantities, but only the volume is conserved.

Contribution by the author of the thesis

All the results presented in this chapter were obtained in collaboration with J. J. L. Velázquez. The physical understanding of the model (2.1.1) has started during the master's thesis of the author of this thesis and the natural continuation of looking at self-similar profiles of the model began when the author started her Ph.D. studies. All the results are obtained and shared equally between the authors.

CHAPTER 3

INTRODUCTION TO FAST FUSION IN A TWO-DIMENSIONAL MODEL

Description

In this chapter we present the results of Appendix B. The results were obtained in collaboration with J. J. L. Velázquez. The contents of Appendix B can be found in [CV24] and were published in

Journal de Mathématiques Pures et Appliquées, 184:91–117, 2024.

As we have seen in Chapter 2, the long-time behavior of the model introduced in (2.1.1) depends strongly on the competition between the fusion term and the collision term. In this chapter we further study this competition with the aim of finding connections between the solutions of our two-dimensional model (2.1.1) and the standard coagulation equation (1.0.1).

3.1 The model

As explained in Chapter 2, if we let $r \equiv 0$ in (2.1.1), then particles attach each other at a contact point forming a ramified-like system. On the other hand, if the fusion of particles occurs at a faster rate than the collision of particles, the interacting particles become spheres almost immediately. Therefore, it should be possible in this case to approximate the solutions of (2.1.1) by means of solutions of a coagulation model in only one variable.

More precisely, we analyze the following model as $\epsilon \to 0$ and as $\epsilon \to \infty$:

$$\partial_t f_\epsilon(a, v, t) + \frac{1}{\epsilon}\partial_a[r(a, v)(c_0 v^{\frac{2}{3}} - a) f_\epsilon(a, v, t)] = \mathbb{K}[f_\epsilon](a, v, t). \tag{3.1.1}$$

The setting for equation (3.1.1) follows the assumptions on the collision and the fusion kernel mentioned in Chapter 2. However, since in this case we are not interested in the long-time behavior of solutions, we do not assume the kernels to be homogeneous. Moreover, in Chapter 2, we focused on kernels yielding similar aggregation properties for particles with the same geometry but different size. In this work, the focus lies on the analysis of the competition between the two interaction rates, namely the fusion and collision of particles, independently of the rescaling properties of the kernels. Thus, our results hold for more general classes of kernels. We gather below the precise conditions needed for our study.

$K(a,v,a',v') = K(a',v',a,v)$	$\rightsquigarrow$	symmetry of interactions
$K_1(v^{\alpha}v'^{\beta} + v'^{-\alpha}v^{\beta}) \leq K(a,v,a',v') \leq K_0(v^{\alpha}v'^{\beta} + v'^{-\alpha}v^{\beta})$	$\rightsquigarrow$	area has small effect on the collision mechanism
$\gamma := \beta - \alpha, \gamma \in (0,1), \alpha \in (0,1), \beta \in (0,1)$	$\rightsquigarrow$	gelation does not occur (i.e., solutions conserve mass)

Table 3.1: Assumptions on the collision kernel

For the fusion kernel, we only assume that $r \in C^1(\mathbb{R}^2_{>0})$ has a power law behavior. More precisely, there exist two constants $R_0, R_1 > 0$ such that

$$R_0 a^{\mu} v^{\sigma} \leq r(a,v) \leq R_1 a^{\mu} v^{\sigma}, \tag{3.1.2}$$

for some $\sigma, \mu \in \mathbb{R}$. In addition, in order to control the mass of the solutions when $|a - c_0 v^{\frac{2}{3}}| > 0$ in (3.1.1) if ϵ is small, we require the following technical assumptions on the fusion kernel r:

$$\begin{cases} \partial_a r(a,v) - \mu a^{-1} r(a,v) \geq 0, \text{ and } \partial_a r(a,v) \leq B a^{-1} r(a,v), & \text{if } \mu > 0; \\ \partial_a r(a,v)(a - c_0 v^{\frac{2}{3}}) + r(a,v) \geq 0, \text{ and } \partial_a r(a,v) \leq B a^{-1} r(a,v), & \text{if } \mu \leq 0, \end{cases} \tag{3.1.3}$$

for all $(a,v) \in (0,\infty)^2$, with $a \geq c_0 v^{\frac{2}{3}}$, and for some constant $B > 0$.

Our results are valid for solutions of the weak formulation of (3.1.1) and so we state the exact definition of solutions here.

Definition 3.1.1 (Definition B.1.1)**.** Fix $\epsilon > 0$. Let $K : (0,\infty)^4 \to [0,\infty)$ be a continuous kernel satisfying the assumptions in Table 3.1. Assume the fusion kernel $r \in C^1(\mathbb{R}^2_{>0})$ satisfies (3.1.2) and (3.1.3). Let $f_\epsilon \in C([0,\infty); \mathscr{M}^I_+(\mathbb{R}^2_{>0}))$, with $\mathscr{M}^I_+(\mathbb{R}^2_{>0})$ defined in (2.1.7). We say that f_ϵ is a solution for the weak version of the time-dependent ϵ-fusion (3.1.1) problem if, for every $T > 0$,

$$\sup_{t \in [0,T]} \int_{(0,\infty)^2} (v^{-\alpha} + v^{\beta}) f_\epsilon(a,v,t) \mathrm{d}v \mathrm{d}a < \infty$$

and, for all $\varphi \in C^1_c([0,\infty); C^1_c(\mathbb{R}^2_{>0}))$ and $t \in [0,\infty)$

$$\begin{aligned}\int_{(0,\infty)^2} f_\epsilon(\eta,t)\varphi(\eta,t)\mathrm{d}\eta - \int_{(0,\infty)^2} f_{\mathrm{in}}(\eta)\varphi(\eta,0)\mathrm{d}\eta - \int_0^t \int_{(0,\infty)^2} f_\epsilon(\eta,s)\partial_s\varphi(\eta,s)\mathrm{d}\eta\mathrm{d}s \\ = \int_0^t \langle \mathbb{K}[f_\epsilon(s)], \varphi(s)\rangle \mathrm{d}s + \frac{1}{\epsilon}\int_0^t \int_{(0,\infty)^2} r(\eta)(c_0 v^{\frac{2}{3}} - a) f_\epsilon(\eta,s)\partial_a\varphi(\eta,s)\mathrm{d}\eta\mathrm{d}s. \qquad (3.1.4)\end{aligned}$$

3.2 The case of fast fusion

We will assume in all the following that $K : (0,\infty)^4 \to [0,\infty)$ is a continuous kernel satisfying the assumptions in Table 3.1 and that the fusion kernel $r \in C^1(\mathbb{R}^2_{>0})$ satisfies (3.1.2) and (3.1.3). Moreover, our results hold for general $\mu \in \mathbb{R}$. However, since the required moment estimates differ depending on the choice of μ, we restrict ourselves to the case when $\mu > 0$ for simplicity of notation. We refer to Appendix B for the precise setting and results.

Our main goal is to prove that all solutions of equation (3.1.1) which satisfy some very general moment estimates concentrate their mass around the isoperimetric line $\{a = c_0 v^{\frac{2}{3}}\}$ as $\epsilon \to 0$ and tend in an appropriate sense to a measure which can be computed by solving a suitable one-dimensional coagulation equation. The reason we can reduce the evolution equation for the two-dimensional system to a one-dimensional one is that, as $\epsilon \to 0$, the fusion process takes place much faster than the collision process. As such, the particles are transported close to the isoperimetric line $\{a = c_0 v^{\frac{2}{3}}\}$ almost instantaneously. We state the exact conditions and results below.

Theorem 3.2.1 (Theorem B.1.4)**.** *Let $\mu > 0$. Assume in addition that $\int_{(0,\infty)^2}(v^{-\mu-3} + v^{\mu+3} + v^{\sigma(\mu+3)} + a^{\mu+3}) f_{\mathrm{in}}(a,v)\mathrm{d}v\mathrm{d}a < \infty$. Let $T > 0$. Then we can construct f_ϵ as in Definition 3.1.1, for every $\epsilon \in (0,1)$. For this sequence, we have that there exists a constant $C(T) > 0$, which is independent of $\epsilon \in (0,1)$, such that*

$$\sup_{t\in[0,T]} \int_{(0,\infty)^2} (v^{-\mu-3} + v^{\mu+3} + v^{\sigma(\mu+3)} + a^{\mu+3}) f_\epsilon(a,v,t)\mathrm{d}v\mathrm{d}a \leq C(T). \qquad (3.2.1)$$

Moreover, there exists a subsequence (which we do not relabel) and $\overline{f} \in C((0,T]; \mathscr{M}^I_+(\mathbb{R}^2_{>0}))$ such that $f_\epsilon(t) \to \overline{f}(t)$ as $\epsilon \to 0$ in the sense of measures, for every $t \in [\overline{\sigma}, T]$ and every $\overline{\sigma} > 0$.

Theorem B.1.4 states in principle that a limit exists for any positive time. We then show that, independently of the initial condition, the limit behaves like a Dirac measure in the area variable. More precisely, the limit has the form $\overline{f}(a,v,t) = F(v,t)\delta(a - c_0 v^{\frac{2}{3}})$, with F satisfying the standard one-dimensional coagulation equation for a suitably chosen collision kernel.

Lemma 3.2.2 (Lemma B.1.7)**.** *Let $T > 0$. Let f_ϵ and $\overline{f}$ as in Theorem 3.2.1. Then $\overline{f}$ has the form*

$$\overline{f}(a,v,t) = F(v,t)\delta(a - c_0 v^{\frac{2}{3}}) \qquad (3.2.2)$$

in the sense of measures. More precisely, there exists $F \in \mathrm{C}((0,T];\mathscr{M}_+(\mathbb{R}_{>0}))$ *such that*

$$\int_{(0,\infty)^2} \overline{f}(a,v,t)\varphi(a,v)\mathrm{d}v\mathrm{d}a = \int_{(0,\infty)^2} F(v,t)\varphi(a,v)\delta(a - c_0 v^{\frac{2}{3}})\mathrm{d}a\mathrm{d}v,$$

for every $\varphi \in \mathrm{C}_0(\mathbb{R}^2_{>0})$.

Theorem 3.2.3 (Theorem B.1.10)**.** *Let F as in (3.2.2). We set the following initial value*

$$\int_{(0,\infty)} F(v,0)\varphi(v)\mathrm{d}v := \int_{(0,\infty)^2} f_{\mathrm{in}}(a,v)\varphi(v)\mathrm{d}a\mathrm{d}v, \tag{3.2.3}$$

for every $\varphi \in \mathrm{C}_0(\mathbb{R}_{>0})$. *If F has the initial value as in* (3.2.3)*, then* $F \in \mathrm{C}([0,T];\mathscr{M}_+(\mathbb{R}_{>0}))$ *and F satisfies the standard one-dimensional coagulation equation, namely, for every* $t \in [0,T]$ *and every* $\varphi \in \mathrm{C}_\mathrm{c}(\mathbb{R}_{>0})$*, the following holds*

$$\begin{aligned}&\int_{(0,\infty)} F(v,t)\varphi(v)\mathrm{d}v - \int_{(0,\infty)} F(v,0)\varphi(v)\mathrm{d}v\\ &= \int_0^t \int_{(0,\infty)} \int_{(0,\infty)} K(c_0 v^{\frac{2}{3}}, v, c_0 v'^{\frac{2}{3}}, v')F(v,s)F(v',s)[\varphi(v+v') - \varphi(v) - \varphi(v')]\mathrm{d}v'\mathrm{d}v\mathrm{d}s.\end{aligned}$$

Lemma 3.2.2 states that there exists a subsequence of the solutions of (3.1.4) which converges in the case of fast fusion in a suitable sense to a measure that can be computed by solving the standard one-dimensional coagulation equation. However, uniqueness of solutions for standard coagulation equations is not known with the exception of some choices of kernels. As a consequence, there are multiple possibilities for this measure as the only information we have is that it solves a one-dimensional coagulation equation. Uniqueness results for the one-dimensional coagulation equation would imply that the limit whose existence has been proven in Theorem 3.2.1 is uniquely determined and therefore the convergence would be independent of the subsequence. As such, the limit for the sequence $\{f_\epsilon\}$ as $\epsilon \to 0$ would exist.

Another observation is the connection with the localization property introduced in Subsection 1.6.2 for the multicomponent model (1.6.7) which was studied in [FLNV21a, FLNV22, FLNV23]. The solutions in [FLNV21a, FLNV22, FLNV23] concentrate along a line with the orientation fixed by the initial distribution of the cluster composition or the source term. Thus, the solutions in [FLNV21a, FLNV22, FLNV23] can concentrate along different lines depending on the initial distribution of particles. On the contrary, in our case, the solutions concentrate always near the isoperimetric line, independently of the initial data. This is since when $\epsilon \to 0$ in (3.1.1), we have that the collision operator transports particles away from the isoperimetric line but these are transported extremely fast toward the isoperimetric region due to the fusion term.

3.3 The case of negligible fusion

As $\epsilon \to \infty$ in (3.1.1), the effect of the fusion term becomes negligible and thus only the collision term contributes to the formation of particles. However, the underlying process described by the collision term for two particles characterized by the values (a, v) and (a', v') is that of addition, namely

$$(a', v') + (a, v) \longrightarrow (a + a', v + v'). \tag{3.3.1}$$

Thus, the total area and the volume stay conserved in the process and the particles attach to each other at a contact point, forming a ramified-like system in time. As such, the natural result is that if we let $\epsilon \to \infty$ in equation (3.1.1), we recover a two-dimensional coagulation model in which fusion does not occur as in (2.1.5).

Theorem 3.3.1 (Theorem B.1.12)**.** *Let $\mu > 0$. Assume in addition that $\int_{(0,\infty)^2}(v^{-2}+v^2+a^2)f_{\text{in}}(a,v)\mathrm{d}v\mathrm{d}a < \infty$. Let $T > 0$. Then we can construct f_ϵ as in Definition 3.1.1 satisfying equation (3.1.4), for every $\epsilon > 1$. For this sequence, we have that there exists a constant $C(T) > 0$, which is independent of $\epsilon > 1$, such that*

$$\sup_{t\in[0,T]} \int_{(0,\infty)^2} (v^{-2} + v^2 + a^2) f_\epsilon(a, v, t)\mathrm{d}v\mathrm{d}a \leq C(T) \tag{3.3.2}$$

and that there exists a subsequence (which we do not relabel) and $\underline{f} \in \mathrm{C}([0, T]; \mathscr{M}_+^I(\mathbb{R}^2_{>0}))$ such that

$$\int_{(0,\infty)^2} f_\epsilon(\eta, t)\varphi(\eta)\mathrm{d}\eta \to \int_{(0,\infty)^2} \underline{f}(\eta, t)\varphi(\eta)\mathrm{d}\eta,$$

as $\epsilon \to \infty$, for every $t \in [0, T]$ and every $\varphi \in \mathrm{C}_0(\mathbb{R}^2_{>0})$. Additionally, we have that $\underline{f}$ satisfies a standard two-dimensional coagulation equation, namely, for every $t \in [0, T]$ and every $\varphi \in \mathrm{C}_c(\mathbb{R}^2_{>0})$, the following holds

$$\begin{aligned} &\int_{(0,\infty)^2} \underline{f}(\eta, t)\varphi(\eta)\mathrm{d}\eta - \int_{(0,\infty)^2} f_{\text{in}}(\eta)\varphi(\eta)\mathrm{d}\eta \\ &= \int_0^t \int_{(0,\infty)^2} \int_{(0,\infty)^2} K(\eta, \eta')\underline{f}(\eta, s)\underline{f}(\eta', s)[\varphi(\eta + \eta') - \varphi(\eta) - \varphi(\eta')]\mathrm{d}\eta'\mathrm{d}\eta\mathrm{d}s. \end{aligned} \tag{3.3.3}$$

The advantage of the two-dimensional system in (3.3.3) is that unlike in the standard one-dimensional case (1.0.1), we have additional information on the geometry of the particles in this case. More exactly, we know that a newly-formed particle has area and volume as in (3.3.1) and thus the resulting particle is formed by two particles attaching to each other at a contact point as in Figure 3.1. We are not able to obtain this information by only observing the evolution of the volume of the interacting particles.

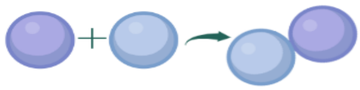

Figure 3.1: Particles attach each other at a contact point

Contribution by the author of the thesis

All the results presented in this chapter were obtained in collaboration with J. J. L. Velázquez. The discussions about the project began as a natural continuation of the results presented in Chapter 2. They were originally intended to be included within one paper together with the results of Chapter 2. However, the project was later separated into two papers as one of the papers deals with the long-time behavior of solutions of (2.1.1), while the other focuses on the analysis of the competition between the fusion term and the collision term. All the results are obtained and shared equally between the authors.

CHAPTER 4

INTRODUCTION TO LONG-TIME ASYMPTOTICS FOR COAGULATION EQUATIONS WITH INJECTION THAT DO NOT HAVE STATIONARY SOLUTIONS

Description

In this chapter we will give an overview of the results of Appendix C. This is a joint work with M. A. Ferreira, E. Franco, and J. J. L. Velázquez. The paper is available online [CFFV23] and was published in

Archive for Rational Mechanics and Analysis, 247(6), 2023.

In this part we study a class of coagulation equations including a source term that injects in the system clusters of size of order one as introduced in Subsection 1.6.1. We restrict the analysis to a range of exponents for which the transport of mass toward infinity is driven by collisions between particles of different sizes. This is different from other cases studied in literature such as [FLNV21b] where the transport of mass toward infinity is due to the collision between particles of comparable sizes. In our case, the interaction between particles of different sizes leads to an additional transport term in the coagulation equation that approximates the solution of the original coagulation equation with injection for large times.

4.1 Derivation of the model

Our goal is to study the long-time asymptotics of the coagulation equation with injection which was introduced in Subsection 1.6.1

$$\partial_t f(v,t) = \mathbb{K}[f](v,t) + \eta(v), \tag{4.1.1}$$

where

$$\mathbb{K}[f](v,t) := \frac{1}{2}\int_0^v K(v-v',v')\, f(v-v',t)\, f(v',t)\,\mathrm{d}v' - \int_0^\infty K(v,v')\, f(v,t)\, f(v',t)\,\mathrm{d}v' \tag{4.1.2}$$

for some suitably chosen $\eta \geq 0$. More precisely, $\eta(v) \not\equiv 0$ will be either compactly supported or it will decay sufficiently fast as $v \to \infty$ and we work with a coagulation kernel which satisfies standard assumptions used in the study of coagulation equations such as continuity, symmetry, homogeneity, and which has a power law form

$$c_2\left[v^{\gamma+\lambda}v'^{-\lambda} + v'^{\gamma+\lambda}v^{-\lambda}\right] \leq K(v,v') \leq c_1\left[v^{\gamma+\lambda}v'^{-\lambda} + v'^{\gamma+\lambda}v^{-\lambda}\right], \text{ with } \gamma + 2\lambda \geq 0. \tag{4.1.3}$$

Kernels of this form appear in atmospheric science applications, cf. [Fri00].

Due to the presence of the source we can expect the long-time behavior of solutions to (4.1.1) to be given by a stationary non-equilibrium solution, i.e., by a solution of

$$\mathbb{K}[f](v) + \eta(v) = 0. \tag{4.1.4}$$

In the case of general power law kernels and source terms η decreasing sufficiently fast, it has been proven in [FLNV21b] that the solutions of (4.1.4) exist if and only if $\gamma + 2\lambda < 1$. Moreover, the solutions of (4.1.4) behave like $v^{-\frac{3+\gamma}{2}}$ for large values of v. The main idea in order to prove that solutions do not exist when $\gamma + 2\lambda \geq 1$ is based on the fact that the transfer of clusters of size v of order one toward very large cluster sizes is so fast that the concentration of clusters with size of order one would become zero.

Since in the case $\gamma + 2\lambda \geq 1$ stationary solutions do not exist, we cannot expect the solutions to (4.1.1) to behave as a stationary solution as $t \to \infty$ for v of order one. It is then natural to ask what is the long-time asymptotic behavior of the solutions to (4.1.1). Using formal arguments, we arrive to a new coagulation model with a transport term, namely

$$\begin{tikzcd}
\underline{\textbf{Coagulation equation with source:}} \arrow[r] \arrow[d, squiggly, "\gamma+2\lambda>1"] & \partial_t f(v,t) = \mathbb{K}[f](v,t) + \eta(v) \arrow[d, squiggly, "\gamma+2\lambda>1"] \\
\underline{\textbf{New model for } v \gg 1\textbf{:}} \arrow[r] & \partial_t f(v,t) + \underbrace{\frac{\partial_v\left(v^{\gamma+\lambda} f(v,t)\right)}{\int_0^\infty z^{\gamma+\lambda} f(z,t)\,\mathrm{d}z}}_{v\approx 1 \to \text{ new transport term}} = \overbrace{\mathbb{K}[f](v,t)}^{v\gg 1}.
\end{tikzcd}$$

We explain here briefly the motivation behind the derivation of this model and we refer to Appendix C for a more detailed analysis. We derive the long-time behavior of $f(v,t)$ separately for v of order one and for large values of v. In order to make a clear distinction, we denote our solution by f_{inner} when we work with particles of size of order one and by f_{outer} when we work with particles of size $v \gg 1$.

In other words, we write $f(v,t)$ as

$$f(v,t) = f_{inner}(v,t) + f_{outer}(v,t), \tag{4.1.5}$$

where $f_{inner}(v,t) := f(v,t)\mathbb{1}_{v\approx 1}$ and $f_{outer}(v,t) := f(v,t)\mathbb{1}_{v\gg 1}$. We mention here that strictly speaking there should be a region in between particles of order one and very large particles. However, for our purpose of analyzing the long-time behavior of solutions, the decomposition (4.1.5) offers a sufficiently good approximation since we expect injected particles to be instantaneously transferred to large cluster sizes.

We start by deriving an equation for $f_{inner}(v,t)$ and thus we assume that v is of order one. When the order of v' is much larger than the one of v, due to the form of our coagulation kernel in (4.1.2), we have that the kernel K behaves like

$$K(v,v') \approx v'^{\gamma+\lambda} v^{-\lambda}. \tag{4.1.6}$$

As such, for v of order one and since we expect that the gain term of the coagulation term in (4.1.1) converges to zero as $t\to\infty$ since $f(v,t)\to 0$ as $t\to\infty$, we obtain the following approximation of the coagulation model with injection (4.1.1) for large times and particles of order one

$$\partial_t f_{inner}(v,t) = -v^{-\lambda} f_{inner}(v,t) \int_0^\infty v'^{\gamma+\lambda} f_{outer}(v',t)\,\mathrm{d}v' + \eta(v). \tag{4.1.7}$$

For $\gamma+2\lambda>1$, we expect to have $f(v,t)\to 0$ as $t\to\infty$ and $M_{\gamma+\lambda} := \int_0^\infty v^{\gamma+\lambda} f(v,t)\mathrm{d}v \to \infty$ as $t\to\infty$. Thus, by using (4.1.7), we further deduce that

$$f_{inner}(v,t) \sim \frac{v^\lambda \eta(v)}{\int_0^\infty v'^{\gamma+\lambda} f_{outer}(v',t)\,\mathrm{d}v'} \quad \text{as} \quad t\to\infty. \tag{4.1.8}$$

We now derive an equation for f_{outer}. In order to analyze the long-time behavior of solutions for large particles $v\gg 1$, we divide the coagulation operator into two parts. The first part offers information about the interaction of particles of different size and the second part provides insight about the interaction of particles of similar size. More precisely, we take $L\ll v$ and we consider which interactions of particles create particles of order $v\gg 1$. Using the decomposition (4.1.5) and the fact that $v\gg L$, we obtain that the contribution of the coagulation operator which creates particles of large size is

$$\left[\int_0^L K(v-v',v') f_{outer}(v-v',t) f_{inner}(v',t)\,\mathrm{d}v' - \int_0^L K(v,v') f_{outer}(v,t) f_{inner}(v',t)\,\mathrm{d}v'\right]$$
$$+\left[\frac{1}{2}\int_L^{v-L} K(v-v',v') f_{outer}(v-v',t) f_{outer}(v',t)\,\mathrm{d}v' - \int_L^\infty K(v,v') f_{outer}(v,t) f_{outer}(v',t)\,\mathrm{d}v'\right].$$

We are interested in the part of the coagulation operator which offers information about the interaction of particles of different size, as this is what we expect will give the main contribution to the long-time behavior of our solutions. Using the approximation in (4.1.6) together with the fact that

$$\int_0^\infty v^{1-\lambda} f_{inner}(v,t)\,\mathrm{d}v \sim \frac{\int_0^\infty v\eta(v)\,\mathrm{d}v}{\int_0^\infty v'^{\gamma+\lambda} f_{outer}(v',t)\,\mathrm{d}v'} \quad \text{as} \quad t\to\infty, \tag{4.1.9}$$

we arrive to the following formula

$$\int_0^L [K(v-v',v') f_{outer}(v-v',t) - K(v,v') f_{outer}(v,t)] f_{inner}(v',t)\,\mathrm{d}v'$$
$$= -\frac{\int_0^\infty v'\eta(v')\,\mathrm{d}v'}{\int_0^\infty v'^{\gamma+\lambda} f_{outer}(v',t)\,\mathrm{d}v'} \frac{\partial}{\partial v}\left[v^{\gamma+\lambda} f_{outer}(v,t)\right].$$

This leads us to the following model for particles of volume $v \gg 1$

$$\partial_t f_{outer}(v,t) + \frac{\int_0^\infty v'\eta(v')\,\mathrm{d}v'}{\int_0^\infty v'^{\gamma+\lambda} f_{outer}(v',t)\,\mathrm{d}v'} \frac{\partial}{\partial v}\left[v^{\gamma+\lambda} f_{outer}(v,t)\right] = \mathbb{K}[f_{outer}](v,t). \tag{4.1.10}$$

For completeness, it is worth to remark that the long-time behavior of large particles in the case when $\gamma+2\lambda=1$ is described by the same model as in (4.1.10). However, different methods need to be employed in order to derive the model in this case. This is since in the case $\gamma+2\lambda=1$ it is expected that the $M_{\gamma+\lambda}$ moment will stay constant in time instead of converging to infinity as in the case $\gamma+2\lambda>1$. We refer to the actual paper in Appendix C and to Chapter 6 for more details about the derivation of the model in the case when $\gamma+2\lambda=1$.

We also observe that the transport term on the left-hand side of (4.1.10) offers information about the way in which the injection of particles of size v of order one affects the outer distribution of clusters f_{outer}. Indeed, multiplying (4.1.10) by v and integrating we obtain

$$\partial_t\left(\int_0^\infty v f_{outer}(v,t)\,\mathrm{d}v\right) = \frac{\int_0^\infty v'\eta(v')\,\mathrm{d}v'}{\int_0^\infty v'^{\gamma+\lambda} f_{outer}(v',t)\,\mathrm{d}v'} \int_0^\infty v^{\gamma+\lambda} f_{outer}(v,t)\,\mathrm{d}v = \int_0^\infty v'\eta(v')\,\mathrm{d}v'. \tag{4.1.11}$$

The identity (4.1.11) states that the total mass of the clusters in the outer region is equal to the injection rate, which is self-consistent with our assumption that the injected particles are transferred instantaneously to large cluster sizes.

4.2 Existence and non-existence of self-similar profiles

As the assumptions on the coagulation kernel are the standard conditions used in the study of coagulation equations, we do not refer to them in order to avoid technicalities. We remember that we work with kernels of the form (4.1.3) and that we are in the case when $\gamma+2\lambda\geq 1$. As a natural continuation of the derivation of the model (4.1.10), which describes the behavior of particles of large size for the coagulation equation with injection, we are interested in the exact conditions for existence and non-existence of self-similar profiles of the model in (4.1.10). This is also motivated by the fact that, as previously mentioned, when $\gamma + 2\lambda \geq 1$, stationary solutions as in (4.1.4) do not exist and as such we cannot expect these type of solutions to give the long-time behavior of solutions to (4.1.1).

The scaling hypothesis which was discussed in Section 1.5 suggests that the mass of the particle distribution $f(v,t)$ is concentrated in cluster sizes v of order t^p for a suitable exponent p that would be determined from dimensional considerations, which take into account the way in which the mass rescales in time. For our framework, it was suggested in [DKW99] that the long-time behavior of the solutions of (4.1.1) is given by self-similar solutions Φ of the form

$$f(v,t) = \frac{1}{t^{\frac{3+\gamma}{1-\gamma}}}\Phi(\xi) \ , \ \ \xi = \frac{v}{t^{\frac{2}{1-\gamma}}}, \tag{4.2.1}$$

scaling which is compatible with our model in (4.1.10). Plugging (4.2.1) into (4.1.10), we obtain the following formulation for self-similar profiles for the coagulation equation with injection in the case when the interaction of particles of different size gives the main contribution to the long-time behavior of our solutions.

Definition 4.2.1 (Definition C.3.1)**.** Let $\gamma < 1$, $\gamma + \lambda < 1$, and $\gamma + 2\lambda \geq 1$. A self-similar profile of equation (4.1.10) with respect to the kernel K is a measure $\Phi \in \mathcal{M}_+((0,\infty))$ such that

$$0 < \int_0^\infty \xi^{\gamma+\lambda}\Phi(\xi)\mathrm{d}\xi < \infty \text{ and } \int_0^1 \xi^{1-\lambda}\Phi(\xi)\mathrm{d}\xi < \infty \tag{4.2.2}$$

and such that it satisfies the following equation

$$\begin{aligned} &\int_0^\infty \varphi'(\xi)\left[\frac{2}{1-\gamma}\xi - \frac{\xi^{\gamma+\lambda}}{\int_0^\infty z^{\gamma+\lambda}\Phi(z)\mathrm{d}z}\right]\Phi(\xi)\mathrm{d}\xi - \frac{1+\gamma}{1-\gamma}\int_0^\infty \varphi(\xi)\Phi(\xi)\mathrm{d}\xi \\ &= \frac{1}{2}\int_0^\infty\int_0^\infty K(\xi,\xi')\left[\varphi(\xi+\xi') - \varphi(\xi) - \varphi(\xi')\right]\Phi(\xi)\Phi(\xi')\mathrm{d}\xi'\mathrm{d}\xi \end{aligned} \tag{4.2.3}$$

for every test function $\varphi \in C_c^1((0,\infty))$.

The conditions $\gamma + \lambda < 1$ and $\gamma < 1$ are standard assumptions in the study of coagulation equations in order to avoid the loss of mass of solutions as explained in Section 1.4. In the following, we present the main results in the case when $\gamma + 2\lambda > 1$. While the case $\gamma + 2\lambda = 1$ is similar, the results are less precise and we only present here the case $\gamma + 2\lambda > 1$ for a clearer overview. We refer to Appendix C for the exact statements in the case $\gamma + 2\lambda = 1$. The following theorem is a compilation of Theorem C.3.3 and Theorem C.3.5 for the case $\gamma + 2\lambda > 1$.

Theorem 4.2.2. *Let $\gamma < 1$, $\gamma + \lambda < 1$, and $\gamma + 2\lambda > 1$.*

- ✍ *If $\gamma > -1$, there exists at least one self-similar profile as in Definition 4.2.1.*

✍ *If $\gamma \leq -1$, no self-similar profiles as in Definition 4.2.1 exist.*

If self-similar profiles exist, they exhibit an interesting property, namely they vanish on an interval near the origin that we can compute explicitly. More precisely, they vanish identically for $0 < \xi < \rho(M_{\gamma+\lambda}) := \left(\frac{1-\gamma}{2M_{\gamma+\lambda}}\right)^{\frac{1}{1-\gamma-\lambda}}$, where

$$M_{\gamma+\lambda} := \int_0^\infty \xi^{\gamma+\lambda}\Phi(\xi)\mathrm{d}\xi.$$

This vanishing property of solutions Φ of (4.2.3) on an interval $(0, \rho(M_{\gamma+\lambda}))$ is transferred to solutions of the equation (4.1.10) via the scaling (4.2.1). It means that for large times t, the injected particles are transferred almost instantaneously to clusters with sizes $v \geq \rho(M_{\gamma+\lambda})t^{\frac{2}{1-\gamma}}$. Moreover, the fraction of clusters with sizes $v < \rho(M_{\gamma+\lambda})t^{\frac{2}{1-\gamma}}$ becomes negligible for very long times. The existence of this "minimal" cluster size for large times is a remarkable feature that, to our knowledge, has not been observed in the literature on self-similar solutions for the coagulation equations.

Theorem 4.2.3 (Properties of the self-similar profiles, Theorem C.3.3)**.** *Let $\gamma < 1$ and $\gamma + \lambda < 1$ such that*

$$-1 < \gamma, \quad \gamma + 2\lambda > 1.$$

Then there exists a self-similar profile Φ as in Definition 4.2.1. Moreover, Φ is such that $\Phi((0, \rho(M_{\gamma+\lambda})))$ $= 0$ for

$$\rho(M_{\gamma+\lambda}) := \left(\frac{1-\gamma}{2\int_0^\infty \xi^{\gamma+\lambda}\Phi(\xi)\mathrm{d}\xi}\right)^{\frac{1}{1-\gamma-\lambda}}. \tag{4.2.4}$$

Since we do not claim uniqueness of the self-similar profiles, the question of whether the vanishing property from Theorem 4.2.3 holds for each self-similar profile arises. In the case when $\gamma + 2\lambda > 1$, we proved that this property indeed holds for all self-similar profiles as in Definition 4.2.1. The situation in the case $\gamma + 2\lambda = 1$ is however different and we refer to Appendix C for the full picture.

Contribution by the author of the thesis

All the results presented in this chapter were obtained in collaboration with M. A. Ferreira, E. Franco, and J. J. L. Velázquez. The idea of the paper started from a previous work of M. A. Ferreira, J. Lukkarinen, A. Nota, and J. J. L. Velázquez (see [FLNV21b]) in which there was a desire to derive a new model for the case in which solutions of (4.1.4) do not exist in order to better study the long-time asymptotics of (4.1.1). The discussions on the subject started online. M. A. Ferreira, E. Franco were employed at the time at University of Helsinki. The author of the thesis together with J. J. L. Velázquez were on a research stay for one week at University of Helsinki and E. Franco stayed one week at the University of Bonn in order to organize the details of the paper. All the results are obtained and shared equally between the authors. We are also grateful to A. Nota for many interesting mathematical discussions that led to several ideas contained in this paper.

CHAPTER 5

INTRODUCTION TO AN INHOMOGENEOUS COAGULATION MODEL WITH A DIFFERENTIAL SEDIMENTATION KERNEL

Description

In this chapter we will give an overview of the results of Appendix D. The paper is available online [CNV24b] and is a joint work with B. Niethammer and J. J. L. Velázquez.

In this part we study a multi-dimensional coagulation model that allows particles to move in space on the vertical line as introduced in (1.6.11). The model (1.6.11) was introduced in [HNS07] in order to describe rain formation or the behavior of air bubbles in water which move due to buoyancy. It is an inhomogeneous model in the spatial variable with a transport term.

5.1 The model in comparison with standard coagulation models

We work with a coagulation model that is inhomogeneous in the spatial variable and contains a transport term in space as introduced in Subsection 1.6.2, namely

$$\partial_t f(x,v,t) + v^{\alpha}\partial_x f(x,v,t) = \frac{1}{2}\int_{(0,v)} K(v-v',v')f(x,v-v',t)f(x,v',t)\mathrm{d}v' - \int_{(0,\infty)} K(v,v')f(x,v,t)f(x,v',t)\mathrm{d}v'. \tag{5.1.1}$$

Our main goal is to prove local in time existence of mass-conserving solutions for a class of coagulation kernels for which instantaneous gelation would otherwise occur. As discussed in Section 1.4, for coagulation kernels of homogeneity $\gamma > 1$, some of the mass is lost in finite time. For these types of kernels, depending on the behavior of the kernel near the origin, mass loss can happen at a positive time or at time zero. The latter phenomenon is known as *instantaneous gelation*. The kernels that constitute our main object of study for the model (5.1.1) are sum-type kernels of homogeneity $\gamma > 1$ and the differential sedimentation kernel introduced in (1.1.4).

The sum-type kernels we work with are further restricted by an upper bound on the homogeneity which appears so that the transport term controls the contribution given by the non-linear coagulation term. More precisely, we work with coagulation kernels of the form

$$K(v, v') = v^{\gamma} + v'^{\gamma}, \text{ with } \gamma \in (1, 1 + \alpha). \tag{5.1.2}$$

General theory of existence for the standard coagulation equations states that mass-conserving solutions exist for kernels that behave like power laws of homogeneity $\gamma < 1$. In addition, solutions (which may not conserve the mass) exist for product-type kernels, i.e., $K(v, v') = (vv')^{\frac{\gamma}{2}}$, independently of their homogeneity. On the other hand, solutions in L^1 of the standard one-dimensional coagulation equation do not exist at all for kernels as in (5.1.2), see [BC90, BLL19b, CdC92, vD87]. This is to our knowledge the first result of existence of solutions involving sum kernels of homogeneity $\gamma > 1$, regardless of whether one considers one-dimensional or multi-dimensional coagulation models.

On the other hand, the differential sedimentation kernel, $K(v, v') = |v^{\frac{2}{3}} - v'^{\frac{2}{3}}|(v^{\frac{1}{3}} + v'^{\frac{1}{3}})^2$, whose derivation was explained in Section 1.1, is relevant due to its physical applications. The model in (5.1.1) with $\alpha = \frac{2}{3}$ and with a differential sedimentation kernel is used in the physical literature in order to describe the behavior of air bubbles in water that move due to buoyancy. It is also a valid model for the description of the onset of rain. However, slip-flow corrections for water droplets are discussed in [HNS07] and it is mentioned that changes in the power of the volume for the velocity may be needed. This motivates our choice of values of $\alpha \in (0, 1)$ in (5.1.1).

5.2 Free merging versus forced locality regime

Once the model (5.1.1) is derived, a first step is to determine whether the model allows stationary power law solutions of non-zero flux. We first verify whether solutions of the form $f(x, v, t) \equiv v^i$, for some $i \in \mathbb{R}$, satisfy (5.1.1). We then determine in Subsection 5.2.1 if these solutions are "admissible" in the sense that the coagulation operator in (5.1.1) is finite.

Let K be a coagulation kernel of homogeneity $\gamma > 1$ and let $\alpha \in (0, 1)$. We follow the analysis made in [HNS07] and work with the following reformulation of (5.1.1).

$$\begin{aligned}
&\partial_t f(x,v,t) + v^\alpha \partial_x f(x,v,t)\\
=&\frac{1}{2}\int_{(0,\infty)} \mathrm{d}v_1 \int_{(0,\infty)} \mathrm{d}v_2 K(v_1,v_2) f(x,v_1,t) f(x,v_2,t)\delta(v-v_1-v_2)\\
&-\frac{1}{2}\int_{(0,\infty)} \mathrm{d}v_1 \int_{(0,\infty)} \mathrm{d}v_2 K(v,v_2) f(x,v,t) f(x,v_2,t)\delta(v_1-v-v_2)\\
&-\frac{1}{2}\int_{(0,\infty)} \mathrm{d}v_1 \int_{(0,\infty)} \mathrm{d}v_2 K(v,v_1) f(x,v,t) f(x,v_1,t)\delta(v_2-v-v_1)\\
&= I_1 - I_2 - I_3.
\end{aligned} \tag{5.2.1}$$

We thus look for steady states (i.e., time and space independent) solutions of the form $f(x,v,t) \equiv v^i$. This is in accordance with standard theory of coagulation equations as, for example, stationary solutions for coagulation models with source behave like power laws. We refer to Chapter 4 for more details. One of the techniques available to find this type of solutions is the so-called *Zakharov transform* which we will use below. For the first term, we make the change of variables $v_1 = vv_1'$ and $v_2 = vv_2'$ in order to obtain

$$I_1 = \frac{v^{\gamma+2i+1}}{2}\int_{(0,\infty)} \mathrm{d}v_1' \int_{(0,\infty)} \mathrm{d}v_2' K(v_1',v_2') v_1'^i v_2'^i \delta(1-v_1'-v_2').$$

For the remaining two terms, I_2 and I_3, we first make the change of variables $(v_1, v_2) = (vv_1', vv_2')$, followed by the *Zakharov transform*

$$v_1' = \frac{1}{\overline{v}_1} \text{ and } v_2' = \frac{\overline{v}_2}{\overline{v}_1} \text{ for } I_2 \text{ and } v_2' = \frac{1}{\overline{v}_2} \text{ and } v_1' = \frac{\overline{v}_1}{\overline{v}_2} \text{ for } I_3.$$

Thus, we can rewrite the terms I_2 and I_3 as follows

$$I_2 = \frac{v^{\gamma+2i+1}\overline{v}_1^{-\gamma-2i-2}}{2}\int_{(0,\infty)} \mathrm{d}\overline{v}_1 \int_{(0,\infty)} \mathrm{d}\overline{v}_2 K(\overline{v}_1,\overline{v}_2)\overline{v}_1^i \overline{v}_2^i \delta(1-\overline{v}_1-\overline{v}_2)$$

and

$$I_3 = \frac{v^{\gamma+2i+1}\overline{v}_2^{-\gamma-2i-2}}{2}\int_{(0,\infty)} \mathrm{d}\overline{v}_1 \int_{(0,\infty)} \mathrm{d}\overline{v}_2 K(\overline{v}_1,\overline{v}_2)\overline{v}_1^i \overline{v}_2^i \delta(1-\overline{v}_1-\overline{v}_2).$$

In other words, it holds that

$$I_1 - I_2 - I_3 = \frac{v^{\gamma+2i+1}}{2}\int_{(0,\infty)} \mathrm{d}v_1 \int_{(0,\infty)} \mathrm{d}v_2 K(v_1,v_2) v_1^i v_2^i \Big(1 - v_1^{-\gamma-2i-2} - v_2^{-\gamma-2i-2}\Big)\delta(1-v_1-v_2).$$

Thus, we have that $I_1 - I_2 - I_3 = 0$ if $v_j^{-\gamma-2i-2} = v_j$, for $j \in \{1,2\}$, so if $i = -\frac{\gamma+3}{2}$. It follows that $f(x,v,t) = v^{-\frac{\gamma+3}{2}}$ solves (5.2.1). It however remains to check if the coagulation term is finite in this case. As we will see in Section 5.2.1, this is not the case and hence $f(x,v,t) = v^{-\frac{\gamma+3}{2}}$ is not an "admissible" solution of our model. This further motivates us to look at functions with more complicated structure when searching for solutions, such as solutions that incorporate information about the spatial variable.

Another approach in order for the integral containing the coagulation kernel to converge is to introduce a cut-off in the kernel. The model in (5.2.1) is referred to as the *free merging regime* in [HNS07], since particles merge when their trajectories cross. The *forced locality regime* in which only particles of similar sizes can interact with one another was also studied in [HNS07]. More precisely, for the *forced locality regime* a cut-off in the coagulation kernel is introduced that makes the kernel vanish outside the region where $\frac{1}{q} < \frac{v'}{v} < q$, for some $q > 1$. In this case, stationary solutions of non-zero flux for the model can be rigorously found due to the fact that the integral containing the coagulation kernel converges when $f = v^{-\frac{\gamma+3}{2}}$. However, as mentioned above, for the *free merging regime* a more detailed analysis is required when searching for solutions.

5.2.1 Convergence of the coagulation term

We now check whether we can rigorously prove that $f(x, v, t) = v^{-\frac{\gamma+3}{2}}$ is a solution for the model in (5.2.1). This means that we need to check whether the coagulation term in (5.2.1) is finite in the case when $f(x, v, t) = v^{-\frac{\gamma+3}{2}}$. If this holds true, this means that the evolution of particles is dominated by interactions of similar size and this is known as *locality*. Conversely, if the interactions between particles of very different sizes dominate, this is known as *non-locality*. We follow the computations in [HNS07] in order to determine which power-law solutions of the form $f(x, v, t) = v^i$ are “admissible” solutions.

Let $K(v, v') = v^\gamma + v'^\gamma$, for $\gamma > 1$. We can rewrite the model in (5.2.1) in the following manner

$$\begin{aligned}\partial_t f(x, v, t) + v^\alpha \partial_x f(x, v, t) = &\int_{(0,\frac{v}{2})} [K(v - v', v')f(x, v - v', t) - K(v, v')f(x, v, t)]f(x, v', t)\mathrm{d}v' \\ &- \int_{[\frac{v}{2},\infty)} K(v, v')f(x, v, t)f(x, v', t)\mathrm{d}v'. \end{aligned} \tag{5.2.2}$$

We approximate the two terms appearing on the right-hand side of (5.2.2) as

$$T_1 := \int_{[\frac{v}{2},\infty)} K(v, v')f(x, v, t)f(x, v', t)\mathrm{d}v \approx f(x, v, t) \int_{[\frac{v}{2},\infty)} v'^\gamma f(x, v', t)\mathrm{d}v'$$

and

$$\begin{aligned} T_2 := &\int_{(0,\frac{v}{2})} [K(v - v', v')f(x, v - v', t) - K(v, v')f(x, v, t)]f(x, v', t)\mathrm{d}v' \\ \approx &\int_{(0,\frac{v}{2})} v' f(x, v', t)\mathrm{d}v' \partial_v(v^\gamma f(x, v, t)). \end{aligned}$$

Assuming our solution is of the form $f(x, v, t) = v^i$, we need to check if the integrals $\int_{(0,\frac{v}{2})} v'^{1+i}\mathrm{d}v'$ and $\int_{[\frac{v}{2},\infty)} v'^{\gamma+i}\mathrm{d}v'$ are finite. We obtain the following cases.

	$i < -\gamma - 1$	$-\gamma - 1 \leq i \leq -2$	$-2 < i$
T_1	Local	Non-local	Non-local
T_2	Non-local	Non-local	Local

Table 5.1: Locality of interactions

From Table 5.1, it immediately follows that neither T_1 nor T_2 is finite when $f(x, v, t) = v^{-\frac{\gamma+3}{2}}$ if $\gamma > 1$. It is also worth mentioning that similar computations can be performed to prove that Table 5.1 also holds in the case of the differential sedimentation kernel (1.1.4).

5.3 Local in time existence

We now state the precise setting under which our result holds true. As we discuss these conditions in Section 5.4 below, we only state our result here without additional mentions to it.

Assumption 5.3.1 (Assumption D.1.1)**.** *We assume that* $K\colon [0, \infty) \times [0, \infty) \to [0, \infty)$ *is a symmetric and continuous function that satisfies*

$$0 \leq K(v, v') \leq K_1(v^\gamma + v'^\gamma), \text{ with } \gamma \in [0, 1 + \alpha) \tag{5.3.1}$$

for some constant $K_1 > 0$ *and*

$$K(v - v', v') \leq K(v, v'), \text{ when } v' \in [0, \tfrac{v}{2}]. \tag{5.3.2}$$

We assume from now on that the coagulation kernel K satisfies Assumption 5.3.1 and define the type of solutions we work with.

Definition 5.3.2 (Mild solutions, Definition D.1.2)**.** Let $\alpha \in (0, 1)$, $\gamma \in [0, 1 + \alpha)$ and $m > \frac{\gamma+1}{\alpha}$. Let $T > 0$. We say that a non-negative function $f \in \mathrm{C}([0, T] \times \mathbb{R} \times (0, \infty))$ such that

$$\sup_{t\in[0,T]} \int_{(0,\infty)} (1 + v^\gamma) f(x, v, t) \mathrm{d}v \leq \frac{C_T}{\max\left\{1, |x|^{m-\frac{\gamma+1}{\alpha}}\right\}}, \text{ for } x \in \mathbb{R}, \tag{5.3.3}$$

is a mild solution of equation (5.1.1) if

$$\begin{aligned} &f(x, v, t) - f(x - v^\alpha t, v, 0) S[f](x, v, 0, t) = \\ &\frac{1}{2} \int_0^t \int_{(0,v)} S[f](x, v, s, t) K(v - v', v') f(x - (t - s)v^\alpha, v - v', s) f(x - (t - s)v^\alpha, v', s) \mathrm{d}s, \end{aligned} \tag{5.3.4}$$

for all $t \in [0, T]$, $v \in (0, \infty)$ and $x \in \mathbb{R}$, where

$$S[f](x, v, s, t) := \mathrm{e}^{-\int_s^t a[f](x - v^\alpha(t-\tau), v, \tau) \mathrm{d}\tau}, \tag{5.3.5}$$

with

$$a[f](x, v, t) := \int_{(0,\infty)} K(v, v') f(x, v', t) \mathrm{d}v'. \tag{5.3.6}$$

In order to obtain existence of strong solutions for the model (5.1.1) we would need better regularity of solutions. However, our main goal is to obtain existence of solutions for our model, rather than studying properties of the obtained solutions. Thus, for our purpose it suffices to work with mild solutions as in Definition 5.3.2. We mention that we call $f \in \mathrm{C}([0,T] \times \mathbb{R} \times (0,\infty))$ a mass-conserving solution of equation (5.1.1) if f is as in Definition 5.3.2 and in addition conserves the total volume of particles. Our result is as follows.

Theorem 5.3.3 (Local existence of solutions, Theorem D.1.4)**.** *Let $\alpha \in (0,1)$, $\gamma \in [0, 1+\alpha)$, $m \in \mathbb{N}$ even, and $p = \alpha m$ with $m > \max\{\frac{2\gamma+1}{\alpha}, \frac{2}{\alpha} + 3\}$. Let $T > 0$ be sufficiently small. Let $f_{\mathrm{in}} \in \mathrm{C}^1(\mathbb{R} \times (0,\infty))$ such that*

$$f_{\mathrm{in}}(x,v) \leq \frac{C_0}{1+|x|^m + v^p},$$

for some $C_0 > 0$ and all $x \in \mathbb{R}$, $v \in (0,\infty)$. Then there exists a mass-conserving solution f of (5.1.1) that satisfies

$$f(x,v,t) \leq \frac{C}{1+|x|^m + v^p}, \tag{5.3.7}$$

for all $t \in [0,T]$, for some $C > 0$.

5.4 Approximation of the model and discussion on the conditions for existence

Our approach to prove local in time existence of a solution to (5.1.1) is based on constructing a suitable supersolution by approximating the coagulation term for large particles by a transport term. We present this approximation here in order to provide more insight into the conditions imposed in Section 5.3. We refer to Appendix D and to [HNS07, Section 4 and Appendix 1] for more details. It is worthwhile to mention that the following computations are not rigorous and their purpose is to provide some insight into the type of solutions we are searching for.

We start with the model in (5.1.1). Since our strategy relies on finding a suitable supersolution, it suffices to find a lower bound for (5.1.1). It follows that

$$\begin{aligned}
&-\int_{(0,\frac{v}{2})} K(v-v',v') f(x,v-v',t) f(x,v',t) \mathrm{d}v' + \int_{(0,\infty)} K(v,v') f(x,v,t) f(x,v',t) \mathrm{d}v' \\
&\geq -\int_{(0,\frac{v}{2})} [K(v-v',v') f(x,v-v',t) - K(v,v') f(x,v,t)] f(x,v',t) \mathrm{d}v' \\
&\quad + \int_{[\frac{v}{2},\infty)} K(v,v') f(x,v,t) f(x,v',t) \mathrm{d}v' \\
&\geq -\int_{(0,\frac{v}{2})} [K(v-v',v') f(x,v-v',t) - K(v,v') f(x,v,t)] f(x,v',t) \mathrm{d}v'.
\end{aligned} \tag{5.4.1}$$

We now use the condition that $K(v-v',v') \leq K(v,v')$ when $v' \in [0, \frac{v}{2}]$ in (5.3.2) and obtain that

$$\begin{aligned}
&\partial_t f(x,v,t) + v^\alpha \partial_x f(x,v,t) \\
&\quad - \int_{(0,\frac{v}{2})} [K(v-v',v') f(x,v-v',t) - K(v,v') f(x,v,t)] f(x,v',t) \mathrm{d}v' \\
&\quad \geq \partial_t f(x,v,t) + v^\alpha \partial_x f(x,v,t) - \int_{(0,\frac{v}{2})} K(v,v')[f(x,v-v',t) - f(x,v,t)] f(x,v',t) \mathrm{d}v'. \qquad (5.4.2)
\end{aligned}$$

Condition (5.3.2) is a rather standard assumption in the study of coagulation equations, see for example [LM02], and most of the kernels used in applications satisfy this condition, in particular, kernels of the form $K(v,v') = v^\gamma + v'^\gamma$ or (1.1.4).

We further make use of the fact that the coagulation kernel K is bounded from above by $v'^\gamma + v^\gamma$ in condition (5.3.1) and, since $v' \in [0, \frac{v}{2}]$, we further deduce by making abstraction of some constants that

$$\begin{aligned}
&\partial_t f(x,v,t) + v^\alpha \partial_x f(x,v,t) - \int_{(0,\frac{v}{2})} K(v,v')[f(x,v-v',t) - f(x,v,t)] f(x,v',t) \mathrm{d}v' \\
&\gtrsim \partial_t f(x,v,t) + v^\alpha \partial_x f(x,v,t) - \int_{(0,\frac{v}{2})} v^\gamma [f(x,v-v',t) - f(x,v,t)] f(x,v',t) \mathrm{d}v' \\
&= \partial_t f(x,v,t) + v^\alpha \partial_x f(x,v,t) - v^\gamma \int_{(0,\frac{v}{2})} \int_{v-v'}^{v} \partial_w f(x,w,t) \mathrm{d}w f(x,v',t) \mathrm{d}v'. \qquad (5.4.3)
\end{aligned}$$

Assume that $\partial_w f(x,w,t)$ behaves similarly for $w \in [\frac{v}{2}, v]$. In order to prove this rigorously, we will make use of the fact that we chose the time to be sufficiently small in Theorem 5.3.3 and of the decay for f in (5.3.7). Thus we arrive to the following formula

$$\partial_t f(x,v,t) + v^\alpha \partial_x f(x,v,t) + v^\gamma \partial_v f(x,v,t) \int_{(0,\frac{v}{2})} v' f(x,v',t) \mathrm{d}v'. \qquad (5.4.4)$$

We denote by $M_1(x,t) := \int_{(0,\infty)} v' f(x,v',t) \mathrm{d}v'$ the first moment in v of f. We consider only large values of v so that we can safely assume that $\int_{(0,\frac{v}{2})} v' f(x,v',t) \mathrm{d}v'$ contains most of the mass. In this manner, we can further approximate (5.4.4) by

$$\partial_t f(x,v,t) + v^\alpha \partial_x f(x,v,t) + v^\gamma \partial_v f(x,v,t) M_1(x,t). \qquad (5.4.5)$$

Suppose now that $M_1(x,t)$ decays sufficiently fast for large values of x as assumed in Definition (5.3.3). Actually, this will follow from the decay of our solution f in (5.3.7). That is, assume there is a sufficiently large $\overline{m}$ and a constant $L > 0$ such that

$$M_1(x,t) \leq \frac{L}{1 + |x|^{\overline{m}}}. \qquad (5.4.6)$$

Combining (5.4.6) with (5.4.5), we obtain that f should behave formally like the solution of the equation

$$\partial_t f(x,v,t) + v^{\alpha}\partial_x f(x,v,t) + \frac{Lv^{\gamma}\partial_v f(x,v,t)}{1+|x|^{\overline{m}}} = 0. \tag{5.4.7}$$

The analysis of the equation (5.4.7) will play an important role in our proof of local existence.

Contribution by the author of the thesis

All the results presented in this chapter were obtained in collaboration with B. Niethammer and J. J. L. Velázquez. The authors started by reading the paper [HNS07] in order to obtain a physical understanding of the problem. Afterwards, they continued by proving that the space homogeneous version of the model (5.1.1) does not admit mass-conserving solutions, providing a mathematical motivation for the further study of the inhomogeneous model. The focus then switched to trying to prove existence for the model (5.1.1). The authors had weekly meetings in which they discussed possible approaches in order to prove the existence of solutions. When meetings were not possible, they would exchange e-mails. All the results are obtained and shared equally between the authors.

CHAPTER 6

SUMMARY AND OUTLOOK

Description

In this chapter we give an overview of the results presented so far and dive further into the methods used to obtain them. Moreover, we discuss possible further research directions in relation to each of the papers that stem from the properties of the models or from the methods used to tackle them.

6.1 Coagulation equations for non-spherical clusters - Future directions

In Chapter 2, we studied the model

$$\partial_t f(a,v,t) + \partial_a[r(a,v)(c_0 v^{\frac{2}{3}} - a) f(a,v,t)] = \mathbb{K}[f](a,v,t) \tag{6.1.1}$$

and presented the results obtained under the assumptions that both the collision K and the fusion kernel r behave like power laws. We refer to Table 2.1 for the exact conditions used for the collision kernel and to Table 2.2 together with (2.1.8) for the exact conditions used for the fusion kernel. More precisely, we assumed that the collision kernel is bounded from below and above by

$$K_1(v^{-\alpha}v'^{\beta} + v'^{-\alpha}v^{\beta}) \leq K(a,v,a',v') \leq K_0(v^{-\alpha}v'^{\beta} + v'^{-\alpha}v^{\beta}), \text{ for some constants } K_0, K_1 > 0, \tag{6.1.2}$$

and that the fusion kernel has the form

$$R_0 a^{\mu} v^{\sigma} \leq r(a,v) \leq R_1 a^{\mu} v^{\sigma}, \text{ for some constants } R_0, R_1 > 0. \tag{6.1.3}$$

In this section, we will discuss open problems which appear when dealing with different values of the parameters α and μ. For the clarity of the presentation of Chapter 2, we only stated our results for collision kernels of the form (6.1.2) in the case when $\alpha > 0$. As the case $\alpha = 0$ is interesting in itself, we summarize in Subsection 6.1.1 the existing results for $\alpha = 0$ and present some possible extensions. Concerning the fusion kernel, most of the results presented in Chapter 2 hold for values of $\mu \neq 0$ in (6.1.3). This is since it is not clear in the case $\mu = 0$ if the fusion term is comparable to the collision term in (6.1.1). In Subsection 6.1.2, we explain the difficulty of dealing with the case $\mu = 0$ and state some conjectures in this case.

6.1.1 Discussion on the choice of coefficients for the collision kernel

In [CV23], we proved the existence of self-similar profiles for the model (6.1.1) when the coefficients in (6.1.2) satisfy

$$\alpha = 0 \text{ and } \gamma = \beta \in \left(0; \frac{2}{3}\right). \tag{6.1.4}$$

We will use the following definition of self-similar profiles for collision kernels satisfying (6.1.4).

Definition 6.1.1 (Definition A.2.4)**.** Assume $\alpha = 0$. We will say that a measure $g \in \mathscr{M}_+^I(\mathbb{R}^2_{>0})$ is a self-similar profile for the model (6.1.1) if

$$\int_{(0,\infty)^2} (a + v^{\beta+1} + v^{\beta}) g(a, v) \mathrm{d}v \mathrm{d}a < \infty \tag{6.1.5}$$

and for every $\varphi \in \mathrm{C}^1_\mathrm{c}(\mathbb{R}^2_{>0})$ the following equality is satisfied:

$$\begin{aligned} \frac{2}{3} \int_{(0,\infty)^2} g(\eta) a v \partial_a \varphi(\eta) \mathrm{d}\eta + \int_{(0,\infty)^2} g(\eta) v^2 \partial_v \varphi(\eta) \mathrm{d}\eta - (1-\gamma) \int_{(0,\infty)^2} v r(\eta)(c_0 v^{\frac{2}{3}} - a) g(\eta) \partial_a \varphi(\eta) \mathrm{d}\eta \\ = (1-\gamma) \int_{(0,\infty)^2} \int_{(0,\infty)^2} K(\eta, \eta') g(\eta) g(\eta') v [\varphi(\eta + \eta') - \varphi(\eta)] \mathrm{d}\eta' \mathrm{d}\eta. \end{aligned} \tag{6.1.6}$$

Notice that we obtain equation (6.1.6) by replacing the test function φ with a test function of the form $v\varphi$ in the equation satisfied by the self-similar profile in the case when $\alpha > 0$, namely in (2.1.11). The new form of the equation is chosen since in the case $\alpha = 0$ we expect the self-similar profiles to be singular for values of v near zero and for this reason not all the terms in Definition 2.1.1 are well-defined. Under this new formulation of the definition of self-similar profiles, we obtain the following result.

Theorem 6.1.2 (Theorem A.2.5)**.** *Let $\alpha = 0$. Assume K is a continuous, non-negative kernel satisfying the conditions in Table 2.1 with coefficients as in (6.1.4). Suppose that $r \in \mathrm{C}^1(\mathbb{R}^2_{>0})$ satisfies the conditions in Table 2.2 and (2.1.8). If $\mu > 0$, there exists a self-similar profile for the model (6.1.1), in the sense of Definition 6.1.1, satisfying $M_{n,k}(g) < \infty$, for all $n \geq 0$ and $k \geq \gamma$.*

Theorem 6.1.2 above and the existence of self-similar profiles for the model (6.1.1) in the case when $\alpha > 0$ presented in Theorem 2.2.1, Chapter 2, illustrate that a similar situation to the standard one-dimensional coagulation equation occurs. More precisely, for the standard one-dimensional coagulation model

$$\partial_t f(v,t) = \frac{1}{2}\int_{(0,v)} K(v-v',v')f(v',t)f(v-v',t)\mathrm{d}v' - \int_{(0,\infty)} K(v,v')f(v,t)f(v',t)\mathrm{d}v' \tag{6.1.7}$$

with a coagulation kernel of the form

$$K(v,v') \approx v'^{-\alpha}v^{\beta} + v'^{\beta}v^{-\alpha}, \tag{6.1.8}$$

it is known that there exist self-similar profiles for which all the moments with negative powers of v are bounded if $\alpha > 0$. On the contrary, for the one-dimensional coagulation equation (6.1.7) with a coagulation kernel of the form $K(v,v') \approx v^{\gamma} + v'^{\gamma}$, i.e., $\alpha = 0$ in (6.1.8), the self-similar profiles can be singular for small values of v and we can expect to have boundedness only for the moments containing powers of the form v^d, with $d \geq \gamma$.

We present below the precise statement of this result in the one-dimensional case and we refer to [BLL19b, Chapter 10.2.4, Theorem 10.2.17] and [FL05] for more details. For $m \in \mathbb{R}$, we denote by

$$X_m := L^1((0,\infty); v^m \mathrm{d}v).$$

Moreover, let

1. $k_1(v,v') := (v^b + v'^b)(v^{-a} + v'^{-a})$, for some $b \in [0,1)$ and $a \in (0,\infty)$, and as such $\gamma := b - a < 1$.
2. $k_2(v,v') := (v^a + v'^a)^b$, for some $a \in (0,\infty)$ and $b \in (0, \frac{1}{a})$, and as such $\gamma := b + a \in (0,1)$.

Theorem 6.1.3 ([BLL19b], Chapter 10.2.4, Theorem 10.2.17)**.** *Assume that there is $i \in \{1,2\}$ such that the coagulation kernel k is given by $k = k_i$. Then there exists at least one mass-conserving self-similar profile φ to the standard one-dimensional coagulation equation* (6.1.7) *which satisfies the following additional properties.*

1. *If $k = k_1$, then $\varphi \in L^{\infty}(0,\infty)$ and $\varphi \in X_m$ for all $m \in \mathbb{R}$.*
2. *If $k = k_2$, then $\varphi \in L^p((0,\infty); v^{\gamma-1+2p}\mathrm{d}v)$ for all $p \in (1,\gamma)$ and $\varphi \in X_m$ for all $m \geq \gamma$.*

Theorem 2.2.1 (case $\alpha > 0$ in (6.1.2)) is analogous to Case 1 of Theorem 6.1.3. In this case, we have that since the coagulation rate is very large for small particles, we can expect self-similar profiles to be bounded (and small) for small values of v. In particular, for the self-similar profile of the model (6.1.1) obtained when $\alpha > 0$, we have $M_{0,d}(g) := \int_{(0,\infty)^2} v^d g(a,v)\mathrm{d}a\mathrm{d}v < \infty$, for all $d \in \mathbb{R}$.

On the other hand, Theorem 6.1.2 is the two-dimensional equivalent of Case 2 of Theorem 6.1.3. The difference is that Theorem 6.1.2 is only stated for values of $\gamma \in (0, \frac{2}{3})$, while Case 2 of Theorem 6.1.3 holds for all values of $\gamma \in (0,1)$. This is since the competition between the fusion term and the collision term in (6.1.1) might lead to significantly different results from the one-dimensional case for this range of parameters.

We thus study the possible scenarios that can occur in the case when $\alpha = 0$ and $\gamma \geq \frac{2}{3}$. As such, let us consider a collision kernel of the form

$$K(a, v, a', v') = v^{\gamma} + v'^{\gamma}, \text{ with } \gamma \geq \frac{2}{3}.$$

There are two possibilities.

$\boxed{\textbf{Case } 1.}$ If fusion overtakes collision and self-similar profiles of the model (6.1.1) exist, then the total surface area of such a profile is bounded. The isoperimetric inequality would imply that we have moments of order smaller than γ of the self-similar profile which are bounded. More precisely, it holds that

$$c_0 \int_{(0,\infty)^2} v^{\frac{2}{3}} g(a, v) \mathrm{d}v \mathrm{d}a \leq \int_{(0,\infty)^2} a g(a, v) \mathrm{d}v \mathrm{d}a < \infty.$$

If $\gamma > \frac{2}{3}$, then this would imply that all the moments $M_{0,d}$, with $d \in [\frac{2}{3}, \gamma]$, are bounded. This situation does not happen in the one-dimensional case for this choice of coefficients. As presented in Theorem 6.1.3, Case 2, only moments of self-similar profiles of order greater or equal than γ are bounded in the one-dimensional case.

$\boxed{\textbf{Case } 2.}$ Alternatively, we do not exclude the possibility that collision can overtake fusion and particles attach at a contact point as in Theorem 2.2.2.

Independently, both results would imply that we are in a very different situation compared to what happens in the one-dimensional case.

6.1.2 Discussion on the choice of coefficients for the fusion kernel

In Chapter 2, we noticed that our results depend strongly on the competition between the fusion and the collision term in (6.1.1) and that this competition is influenced by the value of μ in (6.1.3). In the following, we explain the idea behind the proof of the results in Chapter 2 if we work with fusion kernels as in (6.1.3) in the cases when $\mu > 0$ and $\mu < 0$ in order to illustrate the difficulties which appear when dealing with the case $\mu = 0$.

The case of $\mu > 0$

In the case when $\mu > 0$ in (6.1.3) we have that the fusion term overtakes the collision term and we are able to prove existence of self-similar profiles. Due to our choice of collision speed that has little dependence on the area variable, we adapt existing estimates used in the study of one-dimensional coagulation models in order to bound moments involving only powers of the volume. As such, the main new ingredient in this case is the control of the total surface area, which is done by making use of the presence of the fusion term. We refer to Theorem 2.2.1 for the precise framework and statement of the result.

In order to better understand the stated results, we give some heuristic arguments. Assume for simplicity that $r(a, v) = a^{\mu}$ and that $\mu > 0$. As mentioned above, the most important part is to estimate the total surface area. This is since the moments in the v variable can be bounded using standard arguments used in the study of coagulation equations. For the same reason, we can assume without loss of generality that $a \geq 2c_0 v^{\frac{2}{3}}$ since the region $\{c_0 v^{\frac{2}{3}} \leq a \leq 2c_0 v^{\frac{2}{3}}\}$ can be bounded using uniform moment estimates in the v variable. The rigorous proof of Theorem 2.2.1 will be a generalization of the following idea.

Denote by $A(t) := \int_{(0,\infty)^2} a g(\eta, t) \mathrm{d}\eta$. We test formally in (2.1.12) with $\varphi \equiv a$. Since $a \geq 2c_0 v^{\frac{2}{3}}$, equation (2.1.12) becomes

$$\begin{aligned} \partial_t A(t) &= \frac{1}{3} A(t) + \int_{(0,\infty)^2} a^{\mu} (c_0 v^{\frac{2}{3}} - a) g(\eta, t) \mathrm{d}\eta \\ &\leq \frac{1}{3} A(t) - \frac{1}{2} \int_{(0,\infty)^2} a^{\mu+1} g(\eta, t) \mathrm{d}\eta. \end{aligned} \tag{6.1.9}$$

Since $\mu > 0$, by Young's inequality, we have that there exists some sufficiently large constant $C > 0$ such that

$$\frac{4}{3} a \leq \frac{a^{\mu+1}}{2} + C. \tag{6.1.10}$$

Assuming $\int_{(0,\infty)^2} g(\eta, t) \mathrm{d}\eta \leq C$, (6.1.9) then becomes

$$\partial_t A(t) \leq -A(t) + C. \tag{6.1.11}$$

Inequality (6.1.11) implies that the region $\{ \int_{(0,\infty)^2} a g(\eta, t) \mathrm{d}\eta \leq C\}$ is invariant in time and as such we obtain the desired estimate for the total surface area.

The case of $\mu < 0$

In the case when $\mu < 0$ in (6.1.3) we discover that we can have two different types of behaviors depending on the choice of initial data and on the value of the fusion speed when the area and volume are of order one. In particular, for some suitable choice of initial data, the fusion term plays a negligible role compared to the collision operator and we can prove that the total surface area and the total volume of solutions f to (6.1.1) remain almost constant in time, phenomenon which we term *ramification*. Due to the choice of scaling of self-similar profiles in (2.1.4), this is equivalent to saying that the total surface area of a self-similar profile behaves like $\mathrm{e}^{\frac{1}{3}t}$. We refer to Theorem 2.2.2 for more details.

Theorem 2.2.2 can be understood in the following manner. We can assume without loss of generality that the total volume of particles is one by means of a rescaling argument. We refer to Appendix A for more details about this. With this rescaling, Theorem 2.2.2 states that if we have $R_1 \leq 1$ in (6.1.3) and $\int_{(0,\infty)^2} a g_{\mathrm{in}}(a, v) \mathrm{d}v \mathrm{d}a \geq C$, for some sufficiently large constant $C > 0$, then ramification occurs.

We now provide some heuristic explanation in order to justify the validity of Theorem 2.2.2. We explain below the motivation behind the condition of large initial datum.

Assume for simplicity *that $r(a,v)=a^{\mu}$ and that $\mu \le -1$*. In other words, in order to be consistent with the condition that the fusion kernel keeps the self-similar structure from Table 2.2, we have $r(a,v)=a^{\frac{3(\gamma-1)}{2}}$. Test formally in (2.1.12) with $\varphi \equiv a$. We have $\langle \mathbb{K}[g],\varphi\rangle = 0$. Denote by $A(t) := \int_{(0,\infty)^2} a g(\eta,t)\mathrm{d}\eta$. Equation (2.1.12) becomes

$$\begin{aligned}\partial_t A(t) &= \frac{1}{3}A(t) + \int_{(0,\infty)^2} a^{\mu}(c_0 v^{\frac{2}{3}} - a) g(\eta,t)\mathrm{d}\eta \\ &\ge \frac{1}{3}A(t) - \int_{(0,\infty)^2} a^{\mu+1} g(\eta,t)\mathrm{d}\eta. \end{aligned} \tag{6.1.12}$$

Since $\mu \le -1$ and since our measures are supported in the region where $\{c_0 v^{\frac{2}{3}} \le a\}$ due to the isoperimetric inequality, (6.1.12) then becomes

$$\partial_t A(t) \ge \frac{1}{3}A(t) - c_0^{\mu+1}\int_{(0,\infty)^2} v^{\gamma-\frac{1}{3}} g(\eta,t)\mathrm{d}\eta. \tag{6.1.13}$$

It turns out that we can prove

$$\int_{(0,\infty)^2} v^{\gamma-\frac{1}{3}} g(\eta,t)\mathrm{d}\eta \le \max\left\{\int_{(0,\infty)^2} v^{\gamma-\frac{1}{3}} g_{\mathrm{in}}(\eta)\mathrm{d}\eta, C\right\}, \tag{6.1.14}$$

for some fixed constant $C>0$. The proof of this result is made in a similar manner as the proof of the analogous estimate for the one-dimensional coagulation equations. Thus, (6.1.13) becomes

$$\partial_t A(t) \ge \frac{1}{3}A(t) - C, \tag{6.1.15}$$

for some constant $C>0$ depending only on g_{in}. From (6.1.15), we deduce that if $A(0)$ is sufficiently large, then $A(t)$ behaves like $\mathrm{e}^{\frac{1}{3}t}$. This is the content of the ramification result of Theorem 2.2.2.

If $r(a,v)=a^{\mu}$ with $\mu \in (-1,0)$, by Young's inequality, for every $\epsilon \in (0,1)$, we have that there exists some sufficiently large constant $C_{\epsilon}>0$, which depends on ϵ, such that

$$a^{\mu+1} \le \epsilon a + C_{\epsilon}. \tag{6.1.16}$$

(6.1.12) then becomes

$$\partial_t A(t) \ge \left(\frac{1}{3}-\epsilon\right)A(t) - C_{\epsilon}. \tag{6.1.17}$$

(6.1.17) gives a lower bound of order $\mathrm{e}^{\left(\frac{1}{3}-\epsilon\right)t}$ if we start with sufficiently large total surface area. We can improve this bound to one of order $\mathrm{e}^{\frac{1}{3}t}$ by letting ϵ decrease to zero as $t\to\infty$. However, we only focus here on illustrating the main steps of the proof and omit further technical details as our goal in this subsection is to illustrate the discrepancy of the results when $\mu<0$ and $\mu>0$.

The case of $\mu = 0$

The intriguing case is thus when the collision term and the fusion term are comparable - which corresponds to the case when $\mu = 0$ in (6.1.3) - and this provides a future research direction. We notice that in the case $\mu = 0$ the characteristic equations associated to the fusion term can be explicitly solved. More precisely, we can look at the system

$$\begin{cases} \partial_t A(a_0, v_0, t) = c_0 V^{\frac{2}{3}}(a_0, v_0, t) - A(a_0, v_0, t); \\ \partial_t V(a_0, v_0, t) = V(a_0, v_0, t), \end{cases} \tag{6.1.18}$$

with initial conditions

$$\begin{cases} A(a_0, v_0, 0) = & a_0; \\ V(a_0, v_0, 0) = & v_0. \end{cases} \tag{6.1.19}$$

Then $A(a_0, v_0, t) = e^{-t}(a_0 - c_0 v_0^{\frac{2}{3}}) + c_0 v_0^{\frac{2}{3}}$ is a solution of this system and we can use this form in order to obtain more information about the solutions to (6.1.1). Existence of self-similar profiles if the fusion kernel is sufficiently large when the area and the volume are of order one has been obtained in the case when $\mu = 0$ as stated in Theorem 2.2.3. However, a complete picture of this case is yet to be obtained.

Final remark

As a final mention, our main interest in Chapter 2 was to discover how much of the mathematical theory for the one-dimensional coagulation equation can be carried on to the two-dimensional case and to observe the new mathematical phenomena that this model leads us to. As such, all the results presented so far involve a collision kernel similar to the one used in the study of one-dimensional coagulation models, cf. (6.1.2). It would be interesting to understand the behavior of the solutions of (6.1.1) when the collision kernel K contains an explicit dependence on the area.

The main issue in trying to tackle this problem is the lack of examples coming from physical literature of such kernels. A first step in dealing with this problem is thus to find both the shape and the interaction speed of particles that could give physically interesting models, as well as models for which existence of self-similar profiles can be proven.

6.2 Long-time asymptotics for coagulation equations with injection that do not have stationary solutions - Future directions

In Chapter 4, we discussed the long-time asymptotics of the model

$$\partial_t f(v, t) = \mathbb{K}[f](v, t) + \eta(v) \tag{6.2.1}$$

for coagulation kernels of the form

$$c_2 \left[v^{\gamma+\lambda} v'^{-\lambda} + v'^{\gamma+\lambda} v^{-\lambda} \right] \leq K(v, v') \leq c_1 \left[v^{\gamma+\lambda} v'^{-\lambda} + v'^{\gamma+\lambda} v^{-\lambda} \right], \text{ with } \gamma + 2\lambda \geq 1. \tag{6.2.2}$$

Starting from the assumption that the injected particles are transferred instantaneously to large cluster sizes, we arrived to a new coagulation model for particles of large size using formal asymptotics, namely

$$\partial_t f_{outer}(v,t) + \frac{\int_0^\infty v'\eta(v')\,\mathrm{d}v'}{\int_0^\infty v'^{\gamma+\lambda} f_{outer}(v',t)\,\mathrm{d}v'} \frac{\partial}{\partial v}\left[v^{\gamma+\lambda} f_{outer}(v,t)\right] = \mathbb{K}\left[f_{outer}\right](v,t), \tag{6.2.3}$$

where we denoted by $f_{outer}(v,t) := f(v,t)\mathbb{1}_{v\gg 1}$.

As mentioned in Chapter 4, we arrive to the model (6.2.3) in the case when $\gamma + 2\lambda > 1$, as well as when $\gamma+2\lambda = 1$. However, the derivation in the two cases differs, despite leading to the same equation. In Chapter 4, we focused on the case when $\gamma + 2\lambda > 1$ for the clarity of the presentation. We switch our attention in Subsection 6.2.1 to the situation when $\gamma + 2\lambda = 1$ as the results obtained are more rudimentary in this case, opening many directions of study.

In addition, when $\gamma + 2\lambda > 1$, we proved existence of self-similar profiles for the newly-derived model in (6.2.3) if and only if $\gamma > -1$. We present the current available results about the long-time asymptotics of coagulation equations with injection (6.2.1) in Subsection 6.2.2 together with some open problems.

6.2.1 The case $\gamma + 2\lambda = 1$

The difference between the case $\gamma + 2\lambda > 1$ and $\gamma + 2\lambda = 1$ lies in the approximation of the concentrations of clusters $f(v,t)$ in the region where v is of order one. The reason is that when $\gamma+2\lambda = 1$, we cannot expect the moment $M_{\gamma+\lambda} := \int_0^\infty v^{\gamma+\lambda} f(v,t)\mathrm{d}v$ to converge to infinity as $t \to \infty$. In [FLNV21b], it has been proven that when $\gamma + 2\lambda < 1$, stationary solutions to (6.2.1) behave like $v^{-\frac{3+\gamma}{2}}$ for large values of v. Assuming this behavior for $\gamma + 2\lambda = 1$, it would imply that the $M_{\gamma+\lambda}$ moment is constant.

Indeed, suppose that most of the mass of the monomers is distributed in a characteristic length $L(t)$ that increases as $t \to \infty$. Then, if we denote as M_0 and M_1 the moments of f of order zero and one (i.e., $\int_0^\infty f(v,t)\mathrm{d}v$ and $\int_0^\infty v f(v,t)\mathrm{d}v$, respectively), we have that $M_1 = M_0 L(t)$ and $M_{\gamma+\lambda} = M_0 L(t)^{\gamma+\lambda}$.

Assuming that $\int_0^\infty v\eta(v)\mathrm{d}v = 1$ and hence that $M_1 \simeq t$ as $t \to \infty$, we deduce that $M_0 L(t) = t$. The rescaling properties of the model (6.2.1) suggest that $\frac{M_0}{t} \simeq (M_0)^2 L(t)^\gamma$. Therefore, plugging the identity $M_0 = \frac{t}{L(t)}$ in this formula we obtain $L(t) = t^{\frac{2}{1-\gamma}}$. Hence

$$M_{\gamma+\lambda} = M_0 L(t)^{\gamma+\lambda} = tL(t)^{\gamma+\lambda-1} = t^{1+\frac{2}{1-\gamma}(\gamma+\lambda-1)} = 1. \tag{6.2.4}$$

A first consequence of (6.2.4) is that we cannot approximate the model (6.2.1) for clusters of order one by means of (4.1.8). Instead, we analyze in more detail the evolution equation for $f_{inner}(v,t) := f(v,t)\mathbb{1}_{v\approx 1}$. Remembering the decomposition used in (4.1.5), it holds that

$$\partial_t f_{inner}(v,t) = \mathbb{K}\left[f_{inner}\right](v,t) - f_{inner}(v,t)\int_0^\infty K(v,v') f_{outer}(v',t)\mathrm{d}v' + \eta(v). \tag{6.2.5}$$

As in (4.1.6), when $v \approx 1$ and $v' \gg 1$, the coagulation kernel behaves like

$$K(v, v') \approx v'^{\gamma+\lambda} v^{-\lambda}$$

and (6.2.5) becomes

$$\partial_t f_{inner}(v,t) = \mathbb{K}\left[f_{inner}\right](v,t) - f_{inner}(v,t)v^{-\lambda} \int_0^\infty v'^{\gamma+\lambda} f_{outer}(v',t)\mathrm{d}v' + \eta(v). \tag{6.2.6}$$

(6.2.4) suggests that $\int_0^\infty v'^{\gamma+\lambda} f_{outer}(v',t)\mathrm{d}v'$ approaches to a positive constant as $t \to \infty$ when $\gamma + 2\lambda = 1$. Taking $M_{\gamma+\lambda}$ to be a constant, we can write (6.2.6) as

$$\partial_t f_{inner}(v,t) = \mathbb{K}\left[f_{inner}\right](v,t) - M_{\gamma+\lambda} f_{inner}(v,t)v^{-\lambda} + \eta(v). \tag{6.2.7}$$

Assuming then that the function f_{inner} approaches a stationary solution for large times, we then obtain the following equation which would be expected to describe the long-time behavior of solutions to (6.2.1) for values of v of order one

$$\mathbb{K}\left[f_{inner}\right](v) - M_{\gamma+\lambda} f_{inner}(v)v^{-\lambda} + \eta(v) = 0. \tag{6.2.8}$$

We also notice that by multiplying (6.2.8) by v and integrating in $(0,\infty)$, we obtain that

$$M_{\gamma+\lambda} \int_0^\infty v^{1-\lambda} f_{inner}(v)\mathrm{d}v = \int_0^\infty v\eta(v)\mathrm{d}v.$$

Notice that the above equality is the same as in (4.1.9), which was the condition needed when $\gamma + 2\lambda > 1$ in order to derive the evolution equation in time of f_{outer}. As such, we can reproduce the computations made in Chapter 4 to arrive to the same equation for f_{outer} in this case too, namely equation (4.1.10).

The whole asymptotic behavior derived here relies on the existence of solutions for the equation (6.2.8). Equation (6.2.8) can then be interpreted as a stationary point in time for a coagulation equation with source η and a removal term $-M_{\gamma+\lambda} f_{inner}(v,t)v^{-\lambda}$, cf. (6.2.7). We know from [FLNV21b] that no solutions of (6.2.8) exist if $M_{\gamma+\lambda} = 0$ and $\gamma + 2\lambda = 1$. However, when $M_{\gamma+\lambda} > 0$, the non-existence result may change due to the presence of the removal term. As such, the existence/non-existence of stationary solutions for equation (6.2.8) remains an open problem.

6.2.2 The case $\gamma \leq -1$

We remember that our goal is to completely understand the long-time asymptotics of solutions to (6.2.1) for coagulation kernels of the form

$$c_2\left[v^{\gamma+\lambda} v'^{-\lambda} + v'^{\gamma+\lambda} v^{-\lambda}\right] \leq K(v,v') \leq c_1\left[v^{\gamma+\lambda} v'^{-\lambda} + v'^{\gamma+\lambda} v^{-\lambda}\right].$$

Table 6.1 contains a summary of the results present in the literature.

Model	Assumptions	Parameters	Results
[FLNV21b]	-	$\lvert\gamma + 2\lambda\rvert < 1$	Existence if and only if
[CFFV23]	$\gamma + 2\lambda > 1, \gamma < 1, \gamma + \lambda < 1$	$\gamma > -1$	Existence if and only if
[FFL+23]	$\gamma + 2\lambda > 1, \gamma + \lambda < 1$	$\gamma \leq -1$	Scaling determined

Table 6.1: Results so far

More precisely, the following situation holds.

$\boxed{\text{Case } \lvert\gamma + 2\lambda\rvert < 1.}$

The expected long-time behavior of solutions to (6.2.1) is given by stationary solutions of the form $\mathbb{K}[f](v) + \eta(v) = 0$ and $\lvert\gamma + 2\lambda\rvert < 1$ is a necessary and sufficient condition for the existence of such stationary solutions.

$\boxed{\text{Case } \gamma + 2\lambda > 1, \gamma > -1.}$

Since stationary solutions of (6.2.1) do not exist in this case, we derived a new model describing the behavior of the concentration of clusters, cf. (6.2.3). The natural assumptions $\gamma + \lambda < 1$ and $\gamma < 1$ assure mass-conservation. The expected long-time behavior in this case is given by self-similar profiles of the form (4.2.1) and $\gamma > -1$ is a necessary and sufficient condition for their existence.

$\boxed{\text{Case } \gamma + 2\lambda > 1, \gamma \leq -1.}$

In order to better understand the behavior of coagulation equations with source, the scaling of the self-similar profiles in the case when $\gamma + 2\lambda > 1, \gamma + \lambda < 1$ and $\gamma \leq -1$ has been studied in [FFL+23] using dimensional analysis and formal arguments. A rigorous analysis for this case is yet to be made. The difficulty comes from the fact that, as shown in [FFL+23], there is a rich range of different possible long-time behaviors in this case.

6.3 On an inhomogeneous coagulation model with a differential sedimentation kernel - Future directions

As mentioned in Chapter 5, we proved existence of solutions for short times for the model

$$\begin{aligned}\partial_t f(x,v,t) + v^\alpha \partial_x f(x,v,t) = \frac{1}{2}\int_{(0,v)} &K(v-v',v')f(x,v-v',t)f(x,v',t)\mathrm{d}v' \\ - \int_{(0,\infty)} &K(v,v')f(x,v,t)f(x,v',t)\mathrm{d}v' \end{aligned} \tag{6.3.1}$$

when the coagulation kernel has the form of a differential sedimentation kernel as in (1.1.4), as well as for sum kernels of homogeneity greater than one, as in (5.1.2).

Other multi-dimensional coagulation models [CV23, FLNV21a] and in particular inhomogeneous coagulation models [AB79, CDF08, CDF10, LM02] have already been the subject of study in the mathematical literature. In addition to the physical relevance of the model in (6.3.1), the main novelty in our paper [CNV24b] is the existence of solutions for sum kernels of homogeneity greater than one. For the homogeneous case, such solutions do not exist.

Our aim is to verify if the strategy used for the local existence result can be extended to hold for larger times. This will be the content of Section 6.4. As such, it is useful to first emphasize the steps of the proof of Theorem 5.3.3 (i.e., the theorem which states local existence of solutions). The idea in order to prove existence is as follows. We work with a linear version of the model obtained by defining an inductive sequence of solutions. For this linear version, we are able to find a suitable supersolution. This offers us sufficiently nice upper bounds for the solutions of the linear problem that allow us to pass in the limit back to our original model. We explain below this strategy in more detail. We omit some technicalities and only focus on the heuristic ideas behind our result. From now on, we assume that we work with a coagulation kernel of the form

$$K(v,v') \approx v^{\gamma} + v'^{\gamma}, \text{ for some } \gamma > 1. \tag{6.3.2}$$

Strategy:

$\boxed{\textbf{Step } 1}$ (*The contribution of the sedimentation term*)

We assume that the contribution of the coagulation operator is negligible for small times and try to obtain a supersolution by only using the sedimentation term. Towards this goal, we notice that the solution to

$$\partial_t f_0(x,v,t) + v^{\alpha}\partial_x f_0(x,v,t) = 0 \text{ with } f_0(x,v,0) = \frac{C_0}{1+|x|^m+v^p}, \text{ for some } C_0 > 0, \tag{6.3.3}$$

is given by $f_0(x,v,t) = \frac{C_0}{1+|x-v^{\alpha}t|^m+v^p}$.

$\boxed{\textbf{Step } 2}$ (*The approximated model*)

We make use of the approximation in (5.4.7) derived in Section 5.4 and analyze the properties of the solution to

$$\partial_t G(x,v,t) + v^{\alpha}\partial_x G(x,v,t) + \frac{Lv^{\gamma}}{1+|x|^{m-d}}\partial_v G(x,v,t) = 0, \tag{6.3.4}$$

for some suitably chosen $d \in \mathbb{R}$ and some sufficiently large constant $L > 0$. We assume the initial condition has sufficiently fast decay for large values of x and v, namely $G(x,v,0) = \frac{C_0}{1+|x|^m+v^p}$, for some constant $C_0 > 0$. We look at the system

$$\begin{cases} \partial_t X(x,v,t) = -V^{\alpha}(x,v,t), & X(x,v,0) = x\,, \\ \partial_t V(x,v,t) = -\dfrac{LV^{\gamma}(x,v,t)}{1+|X|^{m-d}(x,v,t)}, & V(x,v,0) = v. \end{cases} \tag{6.3.5}$$

Step 1 offers intuition on how we would like our solutions to (6.3.1) to behave. Thus, we study the system (6.3.5) and prove that $G(x,v,t) = \frac{C_0}{1+|X|^m+V^p}$, where $X(x,v,t) - x$ behaves like $v^{\alpha}t$ and $V(x,v,t)$ behaves like v.

$\boxed{\textbf{Step } 3}$ (*Inductive sequence*)

We work with a linear version of the model (6.3.1) as follows.

$$\partial_t f_n(x,v,t) + v^\alpha \partial_x f_n(x,v,t) = \mathbb{K}[f_{n-1}, f_n](x,v,t) \text{ and } f_n(x,v,0) = f(x,v,0) \le \frac{C_0}{1+|x|^m + v^p}, \tag{6.3.6}$$

where we denoted by

$$\begin{aligned} \mathbb{K}[f_{n-1}, f_n](x,v,t) := & \int_{(0,\frac{v}{2})} K(v-v', v') f_n(x, v-v', t) f_{n-1}(x, v', t) \mathrm{d}v' \\ & - \int_{(0,\infty)} K(v,v') f_n(x,v,t) f_{n-1}(x,v',t) \mathrm{d}v'. \end{aligned} \tag{6.3.7}$$

By Step 2, we assume that G defined in (6.3.4) behaves like $\frac{C_0}{1+|x-v^\alpha t|^m + v^p}$ and try to prove that G (up to modifications) is a supersolution for the model (6.3.6). More precisely, we would like to prove that

$$\partial_t G(x,v,t) + v^\alpha \partial_x G(x,v,t) - \mathbb{K}[f_{n-1}, G](x,v,t) \ge 0. \tag{6.3.8}$$

This will offer sufficiently nice upper estimates for the solutions f_n of (6.3.6) that will allow us to pass to the limit as $n \to \infty$. As an immediate observation, the function $\frac{C_0}{1+|x-v^\alpha t|^m + v^p}$ attains a local maximum at $(v_{\max})^{\frac{2}{3}} \approx x t^{\frac{1}{m-1}}$ if $t \in [0,1]$ and is increasing for $v \le v_{\max}$. As such, we make the following modification to the function G.

$$H(x,v,t) = \begin{cases} G(x,v,t), & \text{if } \partial_v G(x,v,t) \le 0; \\ G(x, v_{\max}, t), & \text{otherwise.} \end{cases} \tag{6.3.9}$$

By its definition, the function H is decreasing in the v variable.

$\boxed{\textbf{Step } 4}$ (*H is a supersolution of the linear model*)

We now try to prove that (6.3.8) holds for H. We assume by induction that $f_{n-1} \le H$ and wish to prove that

$$(*) := \partial_t H(x,v,t) + v^\alpha \partial_x H(x,v,t) - \int_0^{\frac{v}{2}} v^\gamma [H(x, v-v', t) - H(x,v,t)] H(x,v',t) \mathrm{d}v' \ge 0. \tag{6.3.10}$$

We do this by proving that

$$(*) \ge \partial_t H(x,v,t) + v^\alpha \partial_x H(x,v,t) + \frac{L v^\gamma}{1+|x|^{m-d}} \partial_v H(x,v,t) \tag{6.3.11}$$

and that

$$\partial_t H(x,v,t) + v^\alpha \partial_x H(x,v,t) + \frac{L v^\gamma}{1+|x|^{m-d}} \partial_v H(x,v,t) \ge 0. \tag{6.3.12}$$

(6.3.12) will follow from (6.3.4) and (6.3.9) and as such we focus on proving that (6.3.11) holds. We analyze the term $H(x, v-v', t) - H(x,v,t)$. Since $v' \in [0, \frac{v}{2}]$, it holds that $v - v' \in [\frac{v}{2}, v]$. From the definition of H in (6.3.9), we then notice we have three cases depending on the value of v, namely

$\{v \leq v_{\max}\}$, $\{v_{\max} \leq v \leq 2v_{\max}\}$, and $\{v \geq 2v_{\max}\}$. In the case $\{v_{\max} \leq v \leq 2v_{\max}\}$, we need to further split the integral in (6.3.10) into the region where $v' < v - v_{\max}$ and $v' \geq v - v_{\max}$ since $H(x, v - v', t) = H(x, v_{\max}, t)$ in the latter region.

We then prove suitable estimates for H in order to conclude that H satisfies (6.3.11). We remember here that we omit some technicalities for the sake of clarity. More precisely, the function H requires further modifications in this case in order for (6.3.11) to hold.

Conclusion

As noticed above, the main ingredient of the proof of existence of solutions for the inhomogeneous coagulation model (6.3.1) relies on the fact that the contribution given by the coagulation operator is negligible for small times. Thus, it is worth to study the behavior of solutions for the following modified model

$$\partial_t f_\epsilon(x, v, t) + v^\alpha \partial_x f_\epsilon(x, v, t) = \epsilon \mathbb{K}[f_\epsilon](x, v, t), \tag{6.3.13}$$

for some sufficiently small $\epsilon \in (0, 1)$. We expect that this modification of our model (6.3.1) will offer clues on how to prove global existence of solutions. Alternatively, a small contribution coming from the coagulation term is equivalent to a large contribution from the sedimentation term. Indeed, assume f_ϵ satisfies (6.3.13) and take

$$F_\epsilon(x, v, t) = f_\epsilon\Big(x, v, \frac{t}{\epsilon}\Big).$$

Then F_ϵ satisfies the following model

$$\partial_t F_\epsilon(x, v, t) + \frac{1}{\epsilon} v^\alpha \partial_x F_\epsilon(x, v, t) = \mathbb{K}[F_\epsilon](x, v, t). \tag{6.3.14}$$

In the rest of this chapter, we discuss possible ways which could be used to prove the existence of solutions of the models (6.3.13) and (6.3.14). It will turn out that the steps explained above to tackle local in time existence cannot be reproduced in order to prove global existence of solutions (or even existence for times of order one) for the models (6.3.13) and (6.3.14). This will be the content of Section 6.4. In a desire to derive more information about possible solutions to (6.3.13) and (6.3.14), we derive estimates for the coagulation operator in (6.3.13) assuming different behaviors of solutions (Section 6.5). We then conclude this chapter by presenting an alternative model (Section 6.6), whose analysis might prove useful to tackle equations such as (6.3.13) and (6.3.14).

6.4 Discussion on global in time existence

We begin by checking if the strategy presented in Section 6.3 (and we refer here also to [CNV24b]) can be adapted to prove the existence of solutions for the model

$$\partial_t F(x, v, t) + \lambda v^{\frac{2}{3}} \partial_x F(x, v, t) = \mathbb{K}[F](x, v, t), \tag{6.4.1}$$

for some $\lambda > 1$ sufficiently large, with a coagulation kernel of the form

$$K(v, v') \approx v^{\frac{4}{3}} + v'^{\frac{4}{3}}.$$

We try to prove a similar result as the one in Theorem 5.3.3. More precisely, we examine if by extending the techniques in [CNV24b] we can prove a result of the following form. There exists some sufficiently large $\lambda_0 > 0$ such that if $F(x, v, 0) \leq \frac{C}{1+|x|^m+v^p}$, for some constant $C > 0$, then there exists a global in time solution to (6.4.1) if $\lambda \geq \lambda_0$.

Notice that the model (6.4.1) is the same as the one in (6.3.14), where we used the notation with λ instead of $\frac{1}{\epsilon}$ and where we restricted our attention to the case $\alpha = \frac{2}{3}$ for simplicity of notation in later computations. This implies in particular that in this case $p = \frac{2}{3}m$ in order to be able to reproduce the computations needed to prove Theorem 5.3.3.

As in the proof of local existence, the plan is to find a suitable supersolution to a linear version of the model (6.4.1). We check if we can reproduce the steps discussed in Section 6.3. We omit certain constants in the following as our goal is only to discuss possible approaches to extend the existence result to larger times rather than to show rigorous computations.

Step 1 & 2 (*The contribution of the sedimentation term* & *The approximated model*)

If we assume that the contribution of the coagulation operator in (6.4.1) is negligible for all times, as in Step 1 from Section 6.3, then our solution has the form $G(x, v, t) = \frac{1}{1+|x-\lambda v^{\frac{2}{3}} t|^m+v^p}$ and this will be our first candidate when trying to find a supersolution for (6.4.1).

Step 3 (*Inductive sequence*)

As in Step 3 from Section 6.3, we check the zero points of $\partial_v G(x, v, t)$. We have that $\partial_v G(x, v, t) = -\frac{\partial_v\left[1+(x-\lambda v^{\frac{2}{3}} t)^m+v^p\right]}{(1+|x-\lambda v^{\frac{2}{3}} t|^m+v^p)^2}$. Remembering that m is even and that $p = \frac{2}{3}m$, we obtain that

$$\partial_v[1 + (x - \lambda v^{\frac{2}{3}} t)^m + v^p] = -p(\lambda t)v^{-\frac{1}{3}}(x - \lambda v^{\frac{2}{3}} t)^{m-1} + pv^{p-1}.$$

We then have that

$$\partial_v[1 + (x - \lambda v^{\frac{2}{3}} t)^m + v^p] = 0 \text{ if and only if } v^{\frac{2}{3}} = \frac{(\lambda t)^{\frac{1}{m-1}} x}{1 + (\lambda t)^{\frac{m}{m-1}}}.$$

It is worthwhile to notice that $v_{\max} := \left(\frac{(\lambda t)^{\frac{1}{m-1}} x}{1+(\lambda t)^{\frac{m}{m-1}}}\right)^{\frac{3}{2}}$ is bounded and converges to zero as $\lambda t \to 0$ and as $\lambda t \to \infty$. Moreover, for large values of λt, we have that $(v_{\max})^{\frac{2}{3}} \approx \frac{x}{\lambda t}$. We then define as before

$$H(x, v, t) = \begin{cases} G(x, v, t), & \text{if } \partial_v G(x, v, t) \leq 0; \\ G(x, v_{\max}, t), & \text{otherwise.} \end{cases} \tag{6.4.2}$$

Our aim is to prove that H is a supersolution for the iterated version of the model (6.4.1), namely

$$\partial_t F_n(x, v, t) + \lambda v^{\frac{2}{3}} \partial_x F_n(x, v, t) = \mathbb{K}[F_{n-1}, F_n](x, v, t), \tag{6.4.3}$$

$$F_n(x, v, 0) = F(x, v, 0) \leq \frac{C}{1 + |x|^m + v^p}, \text{ for some } \lambda \text{ sufficiently large,} \tag{6.4.4}$$

where $\mathbb{K}[F_{n-1}, F_n](x, v, t)$ is as in (6.3.7) and where we assume that $F_{n-1} \leq \frac{1}{1+|x-\lambda v^{\frac{2}{3}} t|^m + v^p}$. We denote by

$$\tau := \lambda t \text{ and } \sigma := \frac{\tau^{\frac{1}{m-1}}}{1+\tau^{\frac{m}{m-1}}} \text{ such that } (v_{\max})^{\frac{2}{3}} = \sigma x \approx \frac{x}{\tau} \text{ for large values of } \tau.$$

Step 4 (*H is a supersolution of the linear model*)

Following Step 4 from Section 6.3, we wish to analyze the term

$$T := \int_0^{\frac{v}{2}} v^{\frac{4}{3}} [H(x, v - v', t) - H(x, v, t)] H(x, v', t) \mathrm{d}v', \tag{6.4.5}$$

for H as in (6.4.2). We separate the regions $\{v \leq v_{\max}\}$, $\{v_{\max} \leq v \leq 2v_{\max}\}$, and $\{v \geq 2v_{\max}\}$, where $(v_{\max})^{\frac{2}{3}} \approx \frac{x}{\tau}$ in this case. It holds that $T = 0$ in the region where $\{v \leq v_{\max}\}$. We continue by analyzing the region $\{v_{\max} \leq v \leq 2v_{\max}\}$. We have that

$$\begin{aligned} T =& v^{\frac{4}{3}} H(x, v_{\max}, t) \int_0^{v - v_{\max}} [H(x, v - v', t) - H(x, v, t)] \mathrm{d}v' \\ &+ v^{\frac{4}{3}} H(x, v_{\max}, t) \int_{v - v_{\max}}^{\frac{v}{2}} [H(x, v_{\max}, t) - H(x, v, t)] \mathrm{d}v'. \end{aligned} \tag{6.4.6}$$

We are interested in estimating the term T for large values of τ. Let us assume for the moment that $H(x, v_{\max}, t) \approx H(x, v, t)$ and proceed with estimating T. This can be motivated by the fact that we are in the region $\{v_{\max} \leq v \leq 2v_{\max}\}$. We use the rather rough approximations

$$v^{\frac{2}{3}} \approx v_{\max}^{\frac{2}{3}} \approx \frac{x}{\tau}, \; H(x, v_{\max}, t) \lesssim \frac{1}{1 + (\frac{x}{\tau})^m} \text{ and } H(x, v, t) \approx \frac{1}{1 + (\frac{x}{\tau})^m + (\tau v^{\frac{2}{3}} - x)^m}. \tag{6.4.7}$$

By the way we defined H in (6.4.2), it holds that $H(x, v_{\max}, t) \geq H(x, v, t)$, when $v \geq v_{\max}$. Using (6.4.7), the last term in (6.4.6) becomes

$$\begin{aligned} v^{\frac{4}{3}} H(x, v_{\max}, t) \int_{v - v_{\max}}^{\frac{v}{2}} [H(x, v_{\max}, t) - H(x, v, t)] \mathrm{d}v' &\lesssim \left(\frac{x}{\tau}\right)^{2+\frac{3}{2}} H(x, v_{\max}, t) [H(x, v_{\max}, t) - H(x, v, t)] \\ &\lesssim \frac{(\frac{x}{\tau})^{2+\frac{3}{2}}}{1 + (\frac{x}{\tau})^m} [H(x, v_{\max}, t) - H(x, v, t)]. \end{aligned}$$

We further obtain that

$$\frac{(\frac{x}{\tau})^{2+\frac{3}{2}}}{1 + (\frac{x}{\tau})^m} [H(x, v_{\max}, t) - H(x, v, t)] \lesssim \frac{(\frac{x}{\tau})^{2+\frac{3}{2}} (\tau v^{\frac{2}{3}} - x)^m}{[1 + (\frac{x}{\tau})^m]^2} H(x, v, t) \approx \boxed{\frac{\tau^m (\frac{x}{\tau})^{2+\frac{3}{2}+m}}{[1 + (\frac{x}{\tau})^m]^2} H(x, v, t).} \tag{6.4.8}$$

Following the proof for small values of t, we wish to bound (6.4.8) from above by a term of order $-\frac{v^{\frac{4}{3}} \partial_v H(x,v,t)}{1+|x|^{m-d}}$ as in (6.3.11). We thus check the order of the contribution of $-\frac{v^{\frac{4}{3}} \partial_v H(x,v,t)}{1+|x|^{m-d}}$ in the region

$\{v_{\max} \leq v \leq 2v_{\max}\}$. Since $v \approx v_{\max} \approx (\frac{x}{\tau})^{\frac{3}{2}}$, we have that

$$-\partial_v H(x,v,t) \approx \frac{\tau v^{-\frac{1}{3}}(\tau v^{\frac{2}{3}} - x)^{m-1}}{[1 + (\frac{x}{\tau})^m + (\tau v^{\frac{2}{3}} - x)^m]^2} \approx \frac{\tau^m (\frac{x}{\tau})^{m-1-\frac{1}{2}}}{1 + (\frac{x}{\tau})^m + |x|^m} H(x,v,t)$$

and as such it holds that

$$-\frac{v^{\frac{4}{3}}\partial_v H(x,v,t)}{1 + |x|^{m-d}} \approx \boxed{\frac{\tau^m (\frac{x}{\tau})^{m-1+\frac{3}{2}}}{1 + (\frac{x}{\tau})^m + |x|^m} \frac{H(x,v,t)}{1 + |x|^{m-d}}}. \tag{6.4.9}$$

We compare the boxed term in (6.4.8) with the one in (6.4.9), namely

$$\frac{\tau^m (\frac{x}{\tau})^{2+\frac{3}{2}+m}}{[1 + (\frac{x}{\tau})^m]^2} H(x,v,t) \quad \Bigg| \quad \frac{\tau^m (\frac{x}{\tau})^{m-1+\frac{3}{2}}}{1 + (\frac{x}{\tau})^m + |x|^m} \frac{H(x,v,t)}{1 + |x|^{m-d}}.$$

This is equivalent to comparing the terms

$$I_1 := \frac{(\frac{x}{\tau})^3}{[1 + (\frac{x}{\tau})^m]^2} \quad \Bigg| \quad I_2 := \frac{1}{(1 + (\frac{x}{\tau})^m + |x|^m)(1 + |x|^{m-d})}.$$

However, for large values of τ and x, the term I_2 may become smaller than I_1 and as such this approach may not lead us to the desired result.

While the approximations in (6.4.8) and (6.4.9) are not necessarily optimal, the computations show that the proof of existence for the model (6.4.1) for larger values of t may prove more complex than expected.

6.5 Dirac measure solutions

We have seen in Section 6.4 that the techniques from Section 6.3 cannot be reproduced in order to extend the existence result to hold for larger times. We now try to obtain further insight into the matter by assuming different behavior of solutions such as Dirac measures.

6.5.1 Some computations for when the solution has the form of a Dirac measure

Using the time scale $\tau = \lambda t$ in (6.4.1), we can analyze instead the equation

$$\partial_\tau F(x,v,\tau) + v^{\frac{2}{3}}\partial_x F(x,v,\tau) = \frac{1}{\lambda}\mathbb{K}[F](x,v,\tau), \tag{6.5.1}$$

for some sufficiently large $\lambda > 1$. Assume now that the solutions behave like

$$F(x,v,\tau) = \delta(x - \tau v^{\frac{2}{3}}) = \frac{3}{2\tau}\left(\frac{x}{\tau}\right)^{\frac{1}{2}} \delta\left(v - \left(\frac{x}{\tau}\right)^{\frac{3}{2}}\right).$$

Our main goal is to estimate the contribution coming from the coagulation operator in (6.5.1). We thus omit constants from the computations and try to approximate separately the gain term and the loss term of the coagulation operator in (6.5.1). It is thus useful to establish the following notation which will be used throughout this section.

$$\begin{aligned} \text{We denote by } F &\rightsquigarrow \text{ solutions to equation (6.5.1)}, && (6.5.2)\\ \text{by } F_G &\rightsquigarrow \text{ solutions to } \partial_\tau F_G(x,v,\tau) + v^{\frac{2}{3}}\partial_x F_G(x,v,\tau) = \mathbb{K}_G[F_G](x,v,\tau), && (6.5.3)\\ \text{and by } F_L &\rightsquigarrow \text{ solutions to } \partial_\tau F_L(x,v,\tau) + v^{\frac{2}{3}}\partial_x F_L(x,v,\tau) = \mathbb{K}_L[F_L](x,v,\tau), && (6.5.4) \end{aligned}$$

where

$$\mathbb{K}_G[F_G](x,v,\tau) := \frac{1}{\lambda}\int_0^{\frac{v}{2}} K(v-v',v')F_G(x,v-v',\tau)F_G(x,v',\tau)\mathrm{d}v';$$
$$\mathbb{K}_L[F_L](x,v,\tau) := -\frac{1}{\lambda}\int_{(0,\infty)} K(v,v')F_L(x,v,\tau)F_L(x,v',\tau)\mathrm{d}v'.$$

We assume that $F \approx F_G \approx F_L$. Since $K(v,v') \approx v^{\frac{4}{3}} + v'^{\frac{4}{3}}$, it follows that

$$\lambda\mathbb{K}_G[F_G](x,v,\tau) \approx \frac{v^{\frac{4}{3}}}{\tau^2}\frac{x}{\tau}\int_0^{\frac{v}{2}} \delta\Big(v - v' - \Big(\frac{x}{\tau}\Big)^{\frac{3}{2}}\Big)\delta\Big(v' - \Big(\frac{x}{\tau}\Big)^{\frac{3}{2}}\Big)\mathrm{d}v' = \frac{v^{\frac{4}{3}}}{\tau^2}\frac{x}{\tau}\delta\Big(v - 2\Big(\frac{x}{\tau}\Big)^{\frac{3}{2}}\Big).$$

We examine if we can estimate directly the contribution due to the gain term, ignoring for the moment the loss term in the coagulation operator. We thus look at the equation

$$\partial_\tau F_G(x,v,\tau) + \partial_x\Big(v^{\frac{2}{3}}F_G(x,v,\tau)\Big) = \frac{v^{\frac{4}{3}}}{\lambda\tau^2}\frac{x}{\tau}\delta\Big(v - 2\Big(\frac{x}{\tau}\Big)^{\frac{3}{2}}\Big). \tag{6.5.5}$$

We integrate (6.5.5) along characteristics and obtain that

$$F_G(x,v,\tau) = \frac{v^2}{\lambda}\int_1^\tau \frac{1}{s^2}\delta\Big(v - 2\Big(\frac{x - v^{\frac{2}{3}}(\tau-s)}{s}\Big)^{\frac{3}{2}}\Big)\mathrm{d}s.$$

We have that $v = 2\Big(\frac{x - v^{\frac{2}{3}}(\tau-s)}{s}\Big)^{\frac{3}{2}}$ if and only if $s = \frac{\tau v^{\frac{2}{3}} - x}{(1-2^{-\frac{2}{3}})v^{\frac{2}{3}}}$. Since we look at values of s between $1 \leq s \leq \tau$, this implies that we obtain a non-trivial contribution if $2^{-\frac{2}{3}}v^{\frac{2}{3}}\tau \leq x \leq v^{\frac{2}{3}}[\tau - (1-2^{-\frac{2}{3}})]$.

6.5.2 Some computations for when the solution has the form of a Dirac measure and decays for large values of the volume

In order to obtain a measure more similar to the initial one, it seems convenient to allow some decay for large values of v. We thus look for solutions with initial condition $F(x,v,0) = \frac{1}{1+v^m}\delta(x)$ and then by solving only the transport equation we obtain a behavior of the solutions of the form $F(x,v,\tau) = \frac{1}{1+v^m}\delta(x - v^{\frac{2}{3}}\tau)$. Let us thus assume in the following that

$$F_G(x,v,\tau) \approx F_L(x,v,\tau) \approx F(x,v,\tau) = \frac{1}{1+v^m}\delta(x - v^{\frac{2}{3}}\tau), \tag{6.5.6}$$

with F, F_G, and F_L as in (6.5.2) - (6.5.4). We analyze separately the contribution of the gain term and of the loss term in the coagulation operator.

The contribution of the gain term

We start with the gain term. We thus look at the equation

$$\partial_\tau F_G(x,v,\tau) + v^{\frac{2}{3}}\partial_x F_G(x,v,\tau) = \mathbb{K}_G[F_G](x,v,\tau), \tag{6.5.7}$$

or, alternatively, we have that

$$F_G(x,v,\tau) = \frac{1}{1+v^m}\delta(x - v^{\frac{2}{3}}\tau) + \int_0^\tau \frac{v^{\frac{4}{3}}}{\lambda}\int_0^{\frac{v}{2}} F_G(x - v^{\frac{2}{3}}(\tau - s), v - v', s) F_G(x - v^{\frac{2}{3}}(\tau - s), v', s)\mathrm{d}v'\mathrm{d}s.$$

We can try to obtain a space of invariant measures under this operator. We thus look to estimate from above the term

$$\begin{aligned} &\int_0^\tau \frac{v^{\frac{4}{3}}}{\lambda}\int_0^{\frac{v}{2}} \frac{1}{1+(v-v')^m}\frac{1}{1+v'^m}\delta(x - v^{\frac{2}{3}}(\tau-s) - (v-v')s)\delta(x - v^{\frac{2}{3}}(\tau-s) - v's)\mathrm{d}v'\mathrm{d}s \\ &\approx \frac{v^{\frac{4}{3}}}{\lambda(1+v^m)}\int_0^{\frac{v}{2}} \frac{1}{1+v'^m}\int_0^\tau \delta(x - v^{\frac{2}{3}}(\tau-s) - (v-v')^{\frac{2}{3}}s)\delta(x - v^{\frac{2}{3}}(\tau-s) - v'^{\frac{2}{3}}s)\mathrm{d}v'\mathrm{d}s. \end{aligned} \tag{6.5.8}$$

It holds that

$$\delta(x - v^{\frac{2}{3}}(\tau - s) - v'^{\frac{2}{3}}s) = \delta(x - v^{\frac{2}{3}}\tau + [v^{\frac{2}{3}} - v'^{\frac{2}{3}}]s) = \frac{1}{v^{\frac{2}{3}} - v'^{\frac{2}{3}}}\delta\Big(s - \frac{v^{\frac{2}{3}}\tau - x}{v^{\frac{2}{3}} - v'^{\frac{2}{3}}}\Big). \tag{6.5.9}$$

Since $0 \le s \le \tau$, (6.5.9) gives a non-trivial contribution when $v'^{\frac{2}{3}}\tau \le x \le v^{\frac{2}{3}}\tau$. Then, using that $v' \le \frac{v}{2}$ and (6.5.9), (6.5.8) becomes

$$\begin{aligned} &\frac{v^{\frac{2}{3}}}{\lambda(1+v^m)}\int_0^{\frac{v}{2}} \frac{1}{1+v'^m}\delta\Big(x - v^{\frac{2}{3}}\tau - \frac{v^{\frac{2}{3}} - (v-v')^{\frac{2}{3}}}{v^{\frac{2}{3}} - v'^{\frac{2}{3}}}(x - v^{\frac{2}{3}}\tau)\Big)\mathrm{d}v' \\ &= \frac{v^{\frac{2}{3}}}{\lambda(1+v^m)}\int_0^{\frac{v}{2}} \frac{1}{1+v'^m}\delta\Big((x - v^{\frac{2}{3}}\tau)\Big[1 - \frac{v^{\frac{2}{3}} - (v-v')^{\frac{2}{3}}}{v^{\frac{2}{3}} - v'^{\frac{2}{3}}}\Big]\Big)\mathrm{d}v' \\ &= \frac{v^{\frac{2}{3}}}{\lambda(1+v^m)}\int_0^{\frac{v}{2}} \frac{1}{1+v'^m}\delta\Big((x - v^{\frac{2}{3}}\tau)\frac{(v-v')^{\frac{2}{3}} - v'^{\frac{2}{3}}}{v^{\frac{2}{3}} - v'^{\frac{2}{3}}}\Big)\mathrm{d}v'. \end{aligned}$$

The contribution to the Dirac measure is due to $v' = \frac{v}{2}$ that belongs to the region of integration. Then, we can estimate the solution of (6.5.7) from above up to some constants by

$$F_G(x,v,\tau) \lesssim \frac{v^{\frac{5}{3}}}{\lambda(1+v^m)^2}\frac{1}{|x - v^{\frac{2}{3}}\tau|} \text{ in the region } \{2^{-\frac{2}{3}}v^{\frac{2}{3}}\tau \le x \le v^{\frac{2}{3}}\tau\}.$$

As such, we obtain a faster decay for the solution of (6.5.7). This is related however to the Dirac mass structure.

The contribution of the loss term

We continue by estimating the contribution of the loss term in the coagulation operator. As before, we assume F_L to behave like in (6.5.6) and look at the equation

$$\partial_\tau F_L(x,v,\tau) + v^{\frac{2}{3}}\partial_x F_L(x,v,\tau) = \mathbb{K}_L[F_L](x,v,\tau). \tag{6.5.10}$$

We try to obtain estimates for the right-hand side of (6.5.10). We have that

$$\begin{aligned}\int_0^\infty (v^{\frac{4}{3}} + v'^{\frac{4}{3}})F_L(x,v',\tau)\mathrm{d}v' &\approx \int_0^\infty \frac{v^{\frac{4}{3}} + v'^{\frac{4}{3}}}{1+v'^m}\delta(x - v'^{\frac{2}{3}}\tau)\mathrm{d}v' = \int_0^\infty \frac{3(v^{\frac{4}{3}} + v'^{\frac{4}{3}})v'^{\frac{1}{3}}}{2(1+v'^m)\tau}\delta\Big(v' - \Big(\frac{x}{\tau}\Big)^{\frac{3}{2}}\Big)\mathrm{d}v' \\ &\approx \frac{(v^{\frac{4}{3}} + (\frac{x}{\tau})^2)(\frac{x}{\tau})^{\frac{1}{2}}}{\Big(1 + (\frac{x}{\tau})^{\frac{3m}{2}}\Big)\tau}.\end{aligned}$$

We remember that $F_L(x,v,\tau)$ behaves like $\frac{1}{1+v^m}\delta(x - v^{\frac{2}{3}}\tau)$ and as such equation (6.5.10) becomes

$$\partial_\tau F_L(x,v,\tau) + v^{\frac{2}{3}}\partial_x F_L(x,v,\tau) \approx -\frac{1}{\lambda}\frac{(v^{\frac{4}{3}} + (\frac{x}{\tau})^2)(\frac{x}{\tau})^{\frac{1}{2}}}{\Big(1 + (\frac{x}{\tau})^{\frac{3m}{2}}\Big)\tau}F_L(x,v,\tau) \approx -\frac{1}{\lambda}\frac{v^{\frac{5}{3}}}{1+v^m}\frac{F_L(x,v,\tau)}{\tau}.$$

This indicates that $F_L \to 0$ as $\tau \to \infty$. The next natural step is thus to look for self-similar profiles of the form

$$F(x,v,\tau) = \frac{1}{t}S(y,v) \text{ with } y = \frac{x}{t}, \tag{6.5.11}$$

where F solves

$$\partial_\tau F(x,v,\tau) + v^{\frac{2}{3}}\partial_x F(x,v,\tau) = \frac{1}{\lambda}\mathbb{K}[F](x,v,\tau). \tag{6.5.12}$$

If F solves (6.5.12) and we take S as in (6.5.11), then S satisfies the equation

$$\partial_y((y - v^{\frac{2}{3}})S(y,v)) = \frac{1}{\lambda}\mathbb{K}[S](y,v). \tag{6.5.13}$$

It is thus worthwhile to study in more detail the time-dependent version of equation (6.5.13), namely

$$\partial_t S(y,v,t) + \partial_y((y - v^{\frac{2}{3}})S(y,v,t)) = \frac{1}{\lambda}\mathbb{K}[S](y,v,t) \tag{6.5.14}$$

and we will observe in the following some rather intriguing properties of this model.

6.6 Derivation of the diagonal kernel

In a current work [CNV24c], we further analyzed the model (6.5.14) in order to obtain some insight into how we could extend the existence result for the equation (6.3.1). We noticed that for the model

$$\partial_t h_\epsilon(x,v,t) + \frac{1}{\epsilon}\partial_x[(v^\alpha - x)h_\epsilon(x,v,t)] = \mathbb{K}[h_\epsilon](x,v,t), \tag{6.6.1}$$

we can formally obtain as $\epsilon \to 0$ a coagulation equation in which only particles of the same size interact (i.e., a coagulation equation with a diagonal kernel). More precisely, if we denote the limit (in an appropriate sense) of the sequence $\{h_\epsilon\}$ as $\epsilon \to 0$ by h, then h should have the form $h(x, v, t) = \overline{h}(v, t)\delta(x - v^\alpha)$, with $\overline{h}$ satisfying the equation

$$\partial_t \overline{h}(v, t) = \frac{1}{\alpha}\left[\frac{1}{4}K\left(\frac{v}{2}, \frac{v}{2}\right)\left(\frac{v}{2}\right)^{1-\alpha}\left(\overline{h}\left(\frac{v}{2}, t\right)\right)^2 - K(v, v)v^{1-\alpha}(\overline{h}(v, t))^2\right]. \tag{6.6.2}$$

Despite the fact that one-dimensional coagulation equations with a diagonal collision kernel have already been the subject of study in the mathematical literature (see [Ley06, LNV18, NV11]), a rigorous motivation of how this type of kernels could appear has not yet been made.

6.6.1 Connection with the case of fast fusion

We already proved in [CV24] (we also refer here to Chapter 3) that for the model

$$\partial_t P_\epsilon(a, v, t) + \frac{1}{\epsilon}\partial_a[r(a, v)(c_0 v^{\frac{2}{3}} - a)P_\epsilon(a, v, t)] = \mathbb{K}[P_\epsilon](a, v, t) \tag{6.6.3}$$

we have that as $\epsilon \to 0$, the limit of the sequence $\{P_\epsilon\}$ is of the form $P = \overline{P}(v, t)\delta(a - c_0 v^{\frac{2}{3}})$, with $\overline{P}$ satisfying the weak formulation of the standard one-dimensional coagulation equation

$$\begin{aligned}
&\int_{(0,\infty)} \overline{P}(v, t)\varphi(v)\mathrm{d}v - \int_{(0,\infty)} \overline{P}(v, 0)\varphi(v)\mathrm{d}v \\
&= \int_0^t \int_{(0,\infty)} \int_{(0,\infty)} K(c_0 v^{\frac{2}{3}}, v, c_0 v'^{\frac{2}{3}}, v')\overline{P}(v, s)\overline{P}(v', s)[\varphi(v + v') - \varphi(v) - \varphi(v')]\mathrm{d}v'\mathrm{d}v\mathrm{d}s,
\end{aligned} \tag{6.6.4}$$

for all appropriately chosen test functions φ.

A clear similarity can be noticed between the model in (6.6.3) and the equation (6.6.1). The presence of the diagonal kernel in (6.6.2), as opposed to the model in (6.6.4), can be explained as follows. Assume that the limit h of the sequence $\{h_\epsilon\}$ in (6.6.1) behaves indeed like a Dirac measure in the space variable. This means that as we pass to the limit as $\epsilon \to 0$ in (6.6.1), only particles having the same size can interact due to the space inhomogeneity of the model (we remember here that in the derivation of the model (5.1.1) we assumed that particles on the same vertical line interact with each other). Moreover, while there exist some similarities between the models in (6.6.3) and (6.6.1), since in the latter case we deal with a spatially inhomogeneous model, the same methods as in [CV24] cannot be reproduced. One issue stems from the fact that our results in relation to the model (6.6.3) are valid for Radon measure solutions. However, in order to rigorously arrive the the equation (6.6.2), we need $\overline{h}^2$ to be well-defined.

6.6.2 Motivation behind existence of solutions for equation (6.6.1)

We remember that one relevant aspect of the existence result in Theorem 5.3.3, Chapter 5, is that it holds true for sum-type kernels of homogeneity $\gamma > 1$. Our main goal is to show rigorously that if $h_\epsilon \to h$ in some sense, where h_ϵ solves (6.6.1), then h can be computed by solving a suitable one-dimensional coagulation equation. However, solutions do not exist for the standard one-dimensional coagulation equation for sum-type kernels of homogeneity $\gamma > 1$, as seen in Section 1.4.

Moreover, let f_ϵ be the solutions of (6.3.13). We know from Theorem 5.3.3, Chapter 5, that there exists a sufficiently small time $T > 0$ such that f_ϵ exists. Then we can pass from equation (6.3.13) to (6.6.1) via the change of variables

$$h_\epsilon(x, v, t) = e^{\frac{t}{\epsilon}} f_\epsilon(e^{\frac{t}{\epsilon}} x, v, e^{\frac{t}{\epsilon}} - 1)$$

or alternatively

$$f_\epsilon(x, v, t) = \frac{1}{t+1} h_\epsilon\Big(\frac{x}{t+1}, v, \epsilon \ln(t+1)\Big).$$

As such, we have the following existence time for the function h_ϵ.

f_ϵ satisfying (6.3.13)	h_ϵ satisfying (6.6.1)
existence in $[0, T]$, with $T \ll 1$	existence in $[0, \epsilon \ln(T+1)]$, with $T \ll 1$

Table 6.2: Connection between equation (6.3.13) and (6.6.1)

From Table 6.2, we notice that the existence time for equation (6.6.1) goes to zero as $\epsilon \to 0$. As such, we do not have a uniform existence time for the sequence $\{h_\epsilon\}$ which is independent of $\epsilon > 0$. It is thus important to understand which role the fact that we arrive to a diagonal coagulation kernel in the limit equation plays for the existence of solutions. Assume thus that our kernel is of the form

$$K(v, v') = v^\gamma + v'^\gamma, \text{ for } 1 < \gamma < 1 + \alpha,$$

and that the limit h of the sequence $\{h_\epsilon\}$ in (6.6.1) behaves like a Dirac measure in the space variable, i.e.,

$$h(x, v, t) = \overline{h}(v, t)\delta(x - v^\alpha). \tag{6.6.5}$$

We assume (6.6.5) and we integrate the coagulation operator in the x variable. For the loss term, we have

$$\begin{aligned}\int_{\mathbb{R}} \int_{(0,\infty)} K(v, v')\overline{h}(v, t)\delta(x - v^\alpha)\overline{h}(v', t)\delta(x - v'^\alpha)\mathrm{d}v'\mathrm{d}x &= \int_{(0,\infty)} K(v, v')\overline{h}(v, t)\overline{h}(v', t)\delta(v'^\alpha - v^\alpha)\mathrm{d}v' \\ &= \frac{v^{1-\alpha}}{\alpha} \int_{(0,\infty)} K(v, v')\overline{h}(v, t)\overline{h}(v', t)\delta(v' - v)\mathrm{d}v' \\ &= \frac{1}{\alpha} K(v, v)v^{1-\alpha}(\overline{h}(v, t))^2.\end{aligned}$$

For the gain term, it holds that

$$
\begin{aligned}
&\frac{1}{2}\int_{\mathbb{R}}\int_{(0,v)} K(v-v',v')\overline{h}(v-v',t)\delta(x-(v-v')^{\alpha})\overline{h}(v',t)\delta(x-v'^{\alpha})\mathrm{d}v'\mathrm{d}x \\
&= \frac{1}{2}\int_{(0,v)} K(v-v',v')\overline{h}(v-v',t)\overline{h}(v',t)\delta((v-v')^{\alpha}-v'^{\alpha})\mathrm{d}v' \\
&= \frac{1}{4\alpha}\Big(\frac{v}{2}\Big)^{1-\alpha}\int_{(0,v)} K(v-v',v')\overline{h}(v-v',t)\overline{h}(v',t)\delta\Big(v'-\frac{v}{2}\Big)\mathrm{d}v' = \frac{1}{4\alpha}K\Big(\frac{v}{2},\frac{v}{2}\Big)\Big(\frac{v}{2}\Big)^{1-\alpha}\Big(\overline{h}\Big(\frac{v}{2},t\Big)\Big)^{2}.
\end{aligned}
$$

Thus, $\overline{h}$ satisfies indeed the equation (6.6.2). Since $K(v,v') = v^{\gamma}+v'^{\gamma}$, for $1<\gamma<1+\alpha$, then $\overline{h}$ satisfies a one-dimensional coagulation equation in which the coagulation kernel is a diagonal kernel of the form $v^{\gamma+1-\alpha}\delta(v-v')$ multiplied by some constant. Notice that this means the kernel has homogeneity $\gamma-\alpha<1$. Existence of solutions to coagulation equations with a diagonal kernel of homogeneity smaller than one has been proven in [Ley06, NV11]. For small values of ϵ we expect that h_{ϵ} behaves like a product between a Dirac measure in the space variable and a function which solves the equation (6.6.2). This motivates a possible extension of the existence result for equation (6.6.1) to larger times than the ones from Table 6.2. Moreover, let us assume we are able to rigorously prove existence of solutions h_{ϵ} to (6.6.1) up to times of order one. From Table 6.2, we conclude that solutions f_{ϵ} to (6.3.13) exist up to times of order $e^{\frac{1}{\epsilon}}-1\to\infty$ as $\epsilon\to 0$.

APPENDIX A

COAGULATION EQUATIONS FOR NON-SPHERICAL CLUSTERS

$\boxed{\text{ABSTRACT.}}$ In this work, we study the long time asymptotics of a coagulation model which describes the evolution of a system of particles characterized by their volume and surface area. The aggregation mechanism takes place in two stages: collision and fusion of particles. During the collision stage, the two particles merge at a contact point. The newly formed particle has volume and area equal to the sum of the respective quantities of the two colliding particles. After collision, the fusion phase begins and during it the geometry of the interacting particles is modified in such a way that the volume of the total system is preserved and the surface area is reduced. During their evolution, the particles must satisfy the isoperimetric inequality. Therefore, the distribution of particles in the volume and area space is supported in the region where $\{a \geq (36\pi)^{\frac{1}{3}} v^{\frac{2}{3}}\}$. We assume the coagulation kernel has a weak dependence on the area variable. We prove existence of self-similar profiles for some choices of the functions describing the fusion rate for which the particles have a shape that is close to spherical. On the other hand, for other fusion mechanisms and suitable choices of initial data, we show that the particle distribution describes a system of ramified-like particles.

A.1 Introduction

Most of the works on coagulation equations assume that the particles are characterized by a single variable, usually the particle volume (or equivalent quantities like polymer length), see for instance [ELMP03, Nor99, Smo16, Ste89]. Nevertheless, other parameters that might provide insight about the geometry or other features of the particle are usually omitted. In this paper, we study the mathematical properties of a class of coagulation equations in which the aggregating particles are characterized by two degrees of freedom, namely the volume v and the surface area a. This type of models was introduced in [KF90] (see also [Fri00] for a detailed discussion about its properties). More precisely, the model that we consider in this paper is the following:

$$\partial_t f(a,v,t) + \partial_a[r(a,v)(c_0 v^{\frac{2}{3}} - a) f(a,v,t)] = \mathbb{K}[f](a,v,t),\ c_0 := (36\pi)^{\frac{1}{3}}, \tag{A.1.1}$$

where

$$\begin{aligned}\mathbb{K}[f](a,v,t) :=& \frac{1}{2}\int_{(0,a)\times(0,v)} K(a-a',v-v',a',v') f(a',v',t) f(a-a',v-v',t) \mathrm{d}v' \mathrm{d}a' \\ &- \int_{(0,\infty)^2} K(a,v,a',v') f(a,v,t) f(a',v',t) \mathrm{d}v' \mathrm{d}a'.\end{aligned}$$

In this model, f is the density of the particles in the space of area and volume for any given time $t \geq 0$. The coagulation operator $\mathbb{K}[f]$ is the classical coagulation operator that was introduced by Smoluchowski (see [Smo16]). However, a difference is the fact that this operator now describes the evolution of particles characterized by both volume and surface area. Notice that the coagulation operator gives the coagulation rate of particles which evolve according to the following mechanism:

$$(a_1, v_1) + (a_2, v_2) \longrightarrow (a_1 + a_2, v_1 + v_2).$$

It is assumed that the particles attach to each other at their contact point and therefore in this way both the total area and volume of the particles involved in the process are preserved (see Figure A.1).

The main difference between (A.1.1) and the standard one-dimensional coagulation model is the presence of the term $\partial_a[r(a,v)(c_0 v^{\frac{2}{3}} - a) f(a,v,t)]$. We call $\partial_a[r(a,v)(c_0 v^{\frac{2}{3}} - a) f(a,v,t)]$ the "fusion term". This describes an evolution of the particles towards a spherical shape (see Figure A.1). The dynamics generated by this term preserves the total number and volume of the particles. The term $c_0 v^{\frac{2}{3}} - a$ indicates that the area of the particles tends to be reduced as long as it is larger than that of a sphere $c_0 v^{\frac{2}{3}}$. In particular, spherical particles remain spherical and they do not evolve at all due to the fusion term. This fusion mechanism holds, for example, for the merging of droplets consisting of highly viscous fluids (see [KF90]).

Figure A.1: Coagulation mechanism (left). Fusion mechanism (right)

Additionally, $r(a,v)$ will indicate the fusion rate and describes how quickly the particles evolve towards the spherical shape and thus has units of the inverse of the fusion time. In the particular case when $r \equiv 0$, fusion does not occur and particles attach at contact points forming a ramified-like system in time. On the contrary, if the fusion rate is much faster than the coagulation rate, the particles tend to become spherical immediately after colliding (see Figure A.2). A distinction between these two cases is not possible in the standard one-dimensional model.

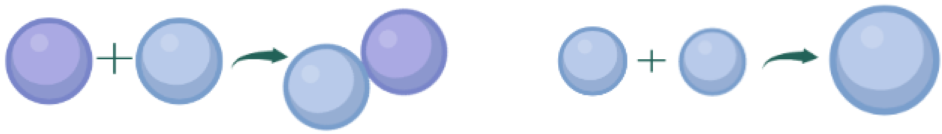

Figure A.2: Absence of fusion (left). Instantaneous fusion (right)

As stated in [KF90], in aerosols, changes of temperature or adding impurities to the system can lead to different fusion rates, showing that the non-spherical shape of the particles plays a significant role. The main goal of this paper is to see how much of the mathematical theory for the one-dimensional coagulation equation can be carried on to the two-dimensional case and to observe the new mathematical phenomena that this model leads us to.

We remark that the particles must satisfy the isoperimetric inequality, therefore the density f should be supported in the region where $\{a \geq c_0 v^{\frac{2}{3}}\}$. Moreover, the evolution generated by (A.1.1) has the property that it preserves the set of measures supported in this region. For simplicity, we define the set

$$\tilde{\mathrm{S}} := \{(a, v) \in (0, \infty)^2, a \geq c_0 v^{\frac{2}{3}}\}. \tag{A.1.2}$$

To obtain a better understanding of how fusion affects interactions between particles, we can check that it gives a decrease in the total surface area. We multiply by a in (A.1.1) and integrate formally, obtaining

$$\partial_t \int_{(0,\infty)^2} af(a, v, t)\mathrm{d}a\mathrm{d}v = \int_{(0,\infty)^2} r(a, v)[c_0 v^{\frac{2}{3}} - a]f(a, v, t)\mathrm{d}a\mathrm{d}v \leq 0, \tag{A.1.3}$$

since f is supported in the region where the isoperimetric inequality is satisfied.

We assume $r(a, v)$ behaves like a power law of a and v. This covers the case of coalescence of viscous liquid spheres (see [KF90]), where the fusion time depends on the diameter. For the coagulation kernel K, we assume that it has a weak dependence on the surface area of the interacting particles, but it can have a power law behaviour in the volume of the coalescing particles.

It is well-known that the solutions of coagulation equations behave often as self-similar solutions. Using the fact that the solutions of (A.1.1) preserve the total volume of the particles, it is natural to look for solutions of (A.1.1) of the form:

$$f(a, v, t) = \frac{1}{(1+t)^{\frac{8}{3}\xi}} g\left(\frac{a}{(1+t)^{\frac{2}{3}\xi}}, \frac{v}{(1+t)^{\xi}}\right) \text{ for } \xi = \frac{1}{1-\gamma}. \tag{A.1.4}$$

Notice that the total surface area of solutions of the form (A.1.4) is not preserved, as it can be expected due to the presence of the fusion term in (A.1.1).

Assumptions on the coagulation kernel

The reason for the self-similar behavior in the case of the one-dimensional coagulation equation is that the coagulation rate scales like a power law of the particle size. In the case of particles characterized by volume and area, if the particle volume is scaled by a factor λ (without modifying the geometry), then the diameter is scaled with a factor $\lambda^{\frac{1}{3}}$ and the area scales like $\lambda^{\frac{2}{3}}$. This suggests the following assumptions for the coagulation kernel:

$$K(\lambda^{\frac{2}{3}}a, \lambda v, \lambda^{\frac{2}{3}}a', \lambda v') = \lambda^{\gamma} K(a, v, a', v'), \tag{A.1.5}$$

for all $(a, v, a', v') \in (0, \infty)^4$, $\lambda \in (0, \infty)$ and $\gamma \in [0, 1)$. We assume $\gamma < 1$ in order to avoid gelation and to obtain volume-conserving solutions. $\gamma > 0$ means physically that the coagulation rate increases with the particle size.

Since collision does not change if we permute the colliding particles, i.e. $(a, v) \leftrightarrow (a', v')$, the coagulation kernel must satisfy the following symmetry property:

$$K(a, v, a', v') = K(a', v', a, v), \tag{A.1.6}$$

for all $(a, v, a', v') \in (0, \infty)^4$.

We work with non-negative continuous kernels on $(0, \infty)^4$ that, in addition to the properties already stated, i.e. (A.1.5) and (A.1.6), have the following bounds:

$$K_1(v^{-\alpha}v'^{\beta} + v'^{-\alpha}v^{\beta}) \leq K(a, v, a', v') \leq K_0(v^{-\alpha}v'^{\beta} + v'^{-\alpha}v^{\beta}), \tag{A.1.7}$$

for some $K_1, K_0 > 0$, for all a, v, a', v' and for the following coefficients:

$$\alpha > 0 \text{ and } \gamma = \beta - \alpha \in [0, 1) \text{ and } \beta \in (0, 1). \tag{A.1.8}$$

Notice that condition (A.1.7) implies that the kernel has a weak dependence on the area variable, but K is not necessarily independent of the area variable.

Most of the results of the paper are obtained for coagulation kernels K with bounds (A.1.7) with $\alpha > 0$. In that case, since the coagulation rate is very large for small particles, we can expect g (defined in (A.1.4)) to be bounded (and small) for small values of v. In particular, for the self-similar profiles g obtained when $\alpha > 0$, we have $M_{0,d}(g) := \int_{(0,\infty)^2} v^d g(a, v) \mathrm{d}a\mathrm{d}v < \infty$, for all $d \in \mathbb{R}$. This is analogous to what happens in the one-dimensional case for the standard coagulation model, where it is known that there exists self-similar profiles for which all the moments with negative powers of v are bounded if $\alpha > 0$. For details, see, for example [BLL19b, Chapter 10.2.4, Theorem 10.2.17] or [FL05]. On the contrary, for the one-dimensional coagulation equation, for coagulation kernels satisfying (A.1.7) with $\alpha = 0$, the self-similar profiles can be singular for small values of v and we can expect to have boundedness only for the moments containing powers of the form v^d, with $d \geq \gamma$, cf. the previously stated result in [BLL19b]. An analogous situation takes place for the two-dimensional coagulation model considered in this paper. To illustrate this situation, we show some results concerning self-similar profiles of (A.1.1) for coagulation kernels K satisfying (A.1.7) with $\alpha = 0$. Specifically, we restrict ourselves to the case when

$$\alpha = 0 \text{ and } \gamma = \beta \in (0; \frac{2}{3}). \tag{A.1.9}$$

The reason to restrict ourselves to the case when $\gamma < \frac{2}{3}$ is because for this range of values it will be easier to obtain estimates for $M_{1,0}(g) := \int_{(0,\infty)^2} ag(a,v)\mathrm{d}a\mathrm{d}v$. Due to the isoperimetric inequality, this estimate implies an estimate for the moment $M_{0,\frac{2}{3}}$. Since we expect to have estimates only for moments $M_{0,d}$, with $d \geq \gamma$, it is natural to assume $\gamma < \frac{2}{3}$.

Assumptions on the fusion kernel

Concerning the fusion kernel r, we assume that $r \in \mathrm{C}^1(\mathbb{R}^2_{>0})$ and that there exist constants $R_0, R_1 > 0$ such that:

$$R_0 a^{\mu} v^{\sigma} \leq r(a,v) \leq R_1 a^{\mu} v^{\sigma}, \tag{A.1.10}$$

for all $(a,v) \in (0,\infty)^2$ and some coefficients $\mu, \sigma \in \mathbb{R}$.

In order to keep the self-similar structure, in other words, to have solutions of (A.1.1) with the particular form (A.1.4), we require in addition:

$$r(\lambda^{\frac{2}{3}} a, \lambda v) = \lambda^{\gamma-1} r(a,v), \tag{A.1.11}$$

for all $\lambda \in (0,\infty)$ and

$$\frac{2}{3}\mu + \sigma = \gamma - 1. \tag{A.1.12}$$

The condition (A.1.12) means that the fusion term and the coagulation term in (A.1.1) rescale in a similar manner as the particle sizes are rescaled (keeping the geometry property).

The following technical assumption on the kernel r is needed for the existence of self-similar profiles:

$$\begin{cases} [\partial_a r(a,v) - \mu a^{-1} r(a,v)](a - c_0 v^{\frac{2}{3}}) + r(a,v) \geq 0, \text{ and } \partial_a r(a,v) \leq B a^{-1} r(a,v), & \text{if } \mu > 0; \\ \partial_a r(a,v)(a - c_0 v^{\frac{2}{3}}) + r(a,v) \geq 0, \text{ and } \partial_a r(a,v) \leq B a^{-1} r(a,v), & \text{if } \mu \leq 0, \end{cases} \tag{A.1.13}$$

for all $(a,v) \in (0,\infty)^2$, with $a \geq c_0 v^{\frac{2}{3}}$ and for some constant $B > 0$. A particular case used in applications that satisfies the above mentioned properties is when $r(a,v) = a^{\mu} v^{\sigma}$, with $\mu \geq -1$ and σ satisfying (A.1.12).

Physical interpretation of the results

The main result that we prove in this article is that depending on the choice of the exponents μ and σ, we can have different behaviors for the solutions of the described model.

For $\mu > 0$, there exist volume-conserving self-similar solutions with the form (A.1.4). For these solutions, the fusion term is comparable to the coagulation term for particles of large sizes. On the other hand, we will obtain in the case when $\mu < 0$ that we can have two different behaviors depending on the choice of initial data, see Figure A.4. In particular, for some suitable initial data, the fusion term plays a negligible role compared to the coagulation operator $\mathbb{K}[f]$ in (A.1.1). We will term the long-time behavior of the particle distribution f in this case as ramification.

In order to explain why we use this terminology, it is convenient to introduce the following notation. Given $H = H(a, v)$, we write

$$\langle H\rangle(t) := \frac{\int_{(0,\infty)^2} H(a, v) f(a, v, t) \mathrm{d}v\mathrm{d}a}{\int_{(0,\infty)^2} f(a, v, t) \mathrm{d}v\mathrm{d}a}, \text{ for any time } t \geq 0.$$

More precisely, for $\mu < 0$ and keeping in mind that the fusion does not change the total volume, if we start with a distribution of particles for which

$$\frac{\langle a\rangle(0)}{(\langle v\rangle(0))^{\frac{2}{3}}} \geq \lambda_0,$$

for λ_0 sufficiently large, then we obtain the following behavior

$$\frac{\langle a\rangle(t)}{(\langle v\rangle(t))^{\frac{2}{3}}} \to \infty \text{ as } t \to \infty. \tag{A.1.14}$$

Notice that (A.1.14) implies that for most of the particles the surface area is much larger than the area of a sphere with the same volume. It is relevant to notice that in the case of self-similar solutions of (A.1.1) with the form (A.1.4), we have

$$\frac{\langle a\rangle(t)}{(\langle v\rangle(t))^{\frac{2}{3}}} \approx 1 \text{ as } t \to \infty.$$

Actually, we will obtain a result stronger than (A.1.14). Namely, in the case $\mu < 0$, we obtain in addition

$$\frac{c}{\langle v\rangle(0)} \leq \frac{\langle a\rangle(t)}{\langle v\rangle(t)} \leq \frac{\langle a\rangle(0)}{\langle v\rangle(0)}, \tag{A.1.15}$$

for some constant $c > 0$. Notice that (A.1.15) implies immediately (A.1.14) since, due to the coagulation of the particles, $\langle v\rangle(t) \to \infty$ as $t \to \infty$. We remark that (A.1.15) suggests that for most of the particles the surface area is comparable to the volume $a \approx v$, while (A.1.14) suggests that $a >> v^{\frac{2}{3}}$ as $t \to \infty$. In particular, particles satisfying $a >> v^{\frac{2}{3}}$ differ very much from spherical particles and they

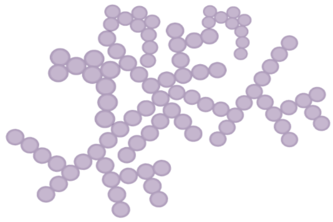

Figure A.3: Ramification

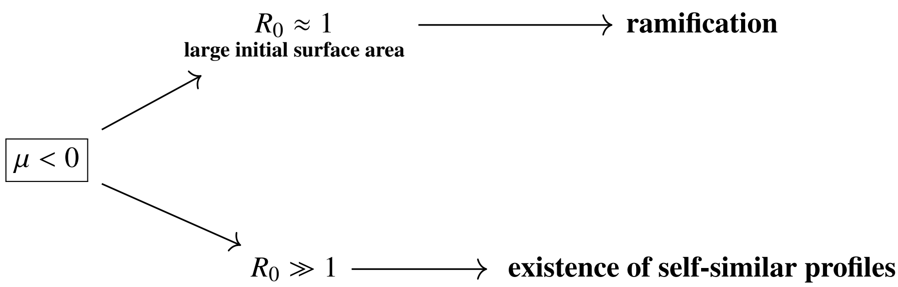

Figure A.4: Different scenarios in the case $\mu < 0$

have a fractal-like, *ramified* aspect, see Figure A.3.

However, if the fusion kernel r is sufficiently large when a and v are of order one, we can obtain existence of self-similar profiles in the case $\mu < 0$. Actually, the proof covers in addition the case when $\mu = 0$, which corresponds to the case of fusion kernels considered in [KF90]. Thus, in the case $\mu < 0$ we find two possible scenarios, see Figure A.4.

Multi-dimensional coagulation equations in the mathematical literature

One can imagine situations in which the collision kernel K and the fusion kernel r do not rescale in the same manner as the particle size. In such situations we can expect to have one of the terms (fusion or coagulation) to be dominant for small size particles, and the other term to be the dominant one for large particles. The analysis of such type of models is also interesting from the point of view of applications to material science, (see [Fri00]).

If the fusion kernel r is very large compared with the coagulation rate, we expect that the particles become spherical in very short times. Therefore, it is possible to approximate the solutions of (A.1.1) by means of solutions of a coagulation model depending only on the variable v, i.e. an one-dimensional coagulation equation. The rigorous proof of this result is presented in [CV24].

Coagulation equations for particle distributions characterized by a single variable have been extensively studied. In particular, the long-time behavior for coagulation equations for which solutions can be explicitly computed has been studied in detail in [MP04, MP06, MP08]. The existence of self-similar solutions for general classes of kernels has been obtained in [EMR05, FL05]. Coagulation models including drift terms have been studied in several contexts. One example is the classical Lifshitz-Slyozov-Wagner equation with encounters that was introduced in [LS61]. A rigorous analysis of the self-similar profiles for this model was studied in [HLN09, HNV09, Lau01]. Models combining the effect of coagulation and particle growth have been studied extensively in the physical literature, cf. [Fri00, Chapter 11] and [LK00]. Rigorous mathematical results for these models can be found in [Gaj83].

Multi-dimensional coagulation equations have not been as extensively studied in the mathematical literature as the one-dimensional coagulation model. Several discrete multi-component coagulation problems which are relevant in aerosol physics have been mentioned in [Wat06b]. A discrete version of the model in (A.1.1) has been studied in [Wat06a]. The model considered in there includes coagulation of particles and an effect similar to the fusion of particles in (A.1.1), which has been termed compaction. The diameter of the particles is restricted by the total number of monomers as well as by the isoperimetric inequality. The coagulation and the fusion rates are assumed to be constant. Due to this, the model considered in [Wat06a] is explicitly solvable using generating functions. The long-time behavior of the solutions which depends on the ratio between the fusion and coagulation kernels has been then analysed using the explicit formulas of the solutions.

In [FLNV21a, FLNV22, FLNV23], the mathematical properties of some classes of coagulation equations describing clusters that are composed of several types of monomers with different chemical composition are analysed. More recently, uniqueness of the solutions for the models of multi-component coagulation equations considered in [FLNV21a, FLNV22, FLNV23] has been studied in [Thr23].

The main differences between the models studied in these papers and our model are the following:

- The two variables used to describe the particles in this paper rescale in a different manner. Additionally, we consider coagulation kernels that do not have a strong dependence on the area variable. As a consequence, the variables describing the clusters appear in a less symmetric manner;
- The proof in [FLNV21a, FLNV22, FLNV23] relies on the conservation of mass for each of the types of monomers. Due to the presence of the fusion term, the solutions of (A.1.1) do not have two conserved quantities, but only the volume is conserved.

A.1.1 Notations and plan of the paper

For $I \subset [0,\infty)^2$, we denote by $\mathrm{C}_c(I)$ and $\mathrm{C}_0(I)$ the space of continuous functions on I with compact support and the space of continuous functions on I which vanish at infinity, respectively, both endowed with the supremum norm. $\mathscr{M}_+(I)$ will denote the space of non-negative Radon measures, while $\mathscr{M}_{+,\mathrm{b}}(I)$ will be the space of non-negative, bounded Radon measures, which we endow with the weak-* topology. We denote $\mathbb{R}^2_{>0} := (0,\infty)^2$.

We make in addition the following simplifications:

- We use the notation $\eta := (a,v)$. We will use interchangeably both notations for convenience.
- We keep the notation $f(a,v)\mathrm{d}v\mathrm{d}a$ or $f(\eta)\mathrm{d}\eta$ for Radon measures, independently of the fact the measure may not be absolutely continuous with respect to the Lebesgue measure.
- $M_{k,l}(f) := \int_{(0,\infty)^2} a^k v^l f(a,v)\mathrm{d}v\mathrm{d}a$, for some $k,l \in \mathbb{R}$.
- For a suitably chosen $\varphi : \mathbb{R}^2_{>0} \to \mathbb{R}$ and for $(a,v,a',v') \in (0,\infty)^4$, we will denote:

$$\chi_\varphi(a,v,a',v') := \varphi(a+a',v+v') - \varphi(a,v) - \varphi(a',v');$$
$$\langle \mathbb{K}[f],\varphi\rangle := \frac{1}{2}\int_{(0,\infty)^2}\int_{(0,\infty)^2} K(a,v,a',v')\chi_\varphi(a,v,a',v')f(a',v')f(a,v)\mathrm{d}v'\mathrm{d}a'\mathrm{d}v\mathrm{d}a.$$

- We use C to denote a generic constant which may differ from line to line and depends only on the parameters characterizing the kernels K and r.
- We use the symbols $\lesssim$ and $\gtrsim$ when the inequalities hold up to a constant, i.e. $f \lesssim g$ if and only if $f \leq Cg$. In addition, for some $N \in \mathbb{N}$, we use the notation $i = \overline{1,N}$ to mean $i \in \mathbb{N}$, $1 \leq i \leq N$.

Structure of the paper

The structure of the paper is as follows. In Section A.2, we establish the setting and state the main definitions and results.

In Section A.3, we prove the existence of a self-similar solution when $\mu > 0$. To this end, we first need to prove well-posedness for the time-dependent problem with a truncated kernel. It turns out that it does not seem feasible to obtain uniform estimates for large values of a for the distribution f if we use approximations of solutions that are compactly supported. In order to avoid this difficulty, we work with a space where large values of the area a are controlled. Since in this space the fusion term will not be well defined, we work with a truncated version of the fusion term which increases linearly at infinity. In order to prove existence of a self-similar solution for the original problem, we need to obtain moment estimates that are uniform in the truncation parameters. The relevant moments to be estimated contain powers of v and a. The moments containing only powers of v can be estimated following the ideas in [EMR05, EM06] due to the fact that the fusion term does not affect the volume of particles. Nevertheless, the adaptation of the estimates in these papers is possible in spite of the fact that we have a coagulation model with two variables due to the choice of the space of functions non-compactly supported in the variable a described before. The total area can be controlled making use of the contribution given by the fusion. This will prove to be enough to obtain existence of a self-similar profile. In Section A.3, we derive in addition estimates for higher order moments using an iterative argument.

In Section A.4, we show that we can have different behaviors for the solutions in the case when $\mu < 0$, namely the existence of self-similar profiles as well as ramification. Ramification will be obtained by deriving estimates for the moments of the solutions of the time-dependent problem. On the other hand, in order to prove the existence of self-similar profiles in this case we cannot use the methods described in Section A.3. This is because of the fact that the estimates for large values of a for positive μ are a consequence of the fast growth of the fusion ratio $r(\eta)$, which does not take place now. We will be able in this case to replace the fast growth of $r(\eta)$ with the presence of a sufficiently large constant in front of the fusion term.

A.2 Setting and main results

From the scaling (A.1.4), self-similar profiles with fusion satisfy formally the equation

$$0 = \frac{8}{3}g(\eta) + \frac{2}{3}a\partial_a g(\eta) + v\partial_v g(\eta) + (1-\gamma)\partial_a[r(\eta)(a - c_0 v^{\frac{2}{3}})g(\eta)] + (1-\gamma)\mathbb{K}[g](\eta). \tag{A.2.1}$$

In particular, if g solves (A.2.1) and f satisfies (A.1.4), then f solves (A.1.1).

Since we work with physically relevant particles, i.e. the particles for which the isoperimetric inequality is satisfied, it is helpful to define the following space

$$\mathscr{M}_+^I(\mathbb{R}^2_{>0}) := \{h \in \mathscr{M}_+(\mathbb{R}^2_{>0}) \mid h(\{a < c_0 v^{\frac{2}{3}}\}) = 0\}. \tag{A.2.2}$$

The superscript I stands for isoperimetric. We endow the newly-defined space with the weak-* topology on $\mathscr{M}_+(\mathbb{R}^2_{>0})$. Similarly, we denote

$$\mathscr{M}_{+,b}^I(\mathbb{R}^2_{>0}) := \{h \in \mathscr{M}_{+,b}(\mathbb{R}^2_{>0}) \mid h(\{a < c_0 v^{\frac{2}{3}}\}) = 0\}. \tag{A.2.3}$$

In order to study the long-time behavior for the equation (A.1.1), we analyse the time-dependent version of equation (A.2.1). We will use the following concept of weak solutions for the time-dependent problem.

Definition A.2.1. Assume $\alpha > 0$. Let $g \in \mathrm{C}([0,\infty); \mathscr{M}_+^I(\mathbb{R}^2_{>0}))$. We say that g is a solution for the weak version of the time-dependent fusion problem in self-similar variables if, for every $T > 0$,

$$\sup_{t\in[0,T]} \int_{(0,\infty)^2} (v^{-\alpha} + v^{\beta}) g(a,v,t) \mathrm{d}v\mathrm{d}a < \infty$$

and, for all $\varphi \in \mathrm{C}^1_{\mathrm{c}}([0,\infty);\mathrm{C}^1_{\mathrm{c}}(\mathbb{R}^2_{>0}))$ and $t \in [0,\infty)$

$$\int_{(0,\infty)^2} g(\eta,t)\varphi(\eta,t)\mathrm{d}\eta - \int_{(0,\infty)^2} g(\eta,0)\varphi(\eta,0)\mathrm{d}\eta - \int_0^t \int_{(0,\infty)^2} g(\eta,s)\partial_s\varphi(\eta,s)\mathrm{d}\eta\mathrm{d}s =$$
$$\int_0^t \int_{(0,\infty)^2} g(\eta,s)\varphi(\eta,s)\mathrm{d}\eta\mathrm{d}s - \frac{2}{3}\int_0^t \int_{(0,\infty)^2} g(\eta,s)a\partial_a\varphi(\eta,s)\mathrm{d}\eta\mathrm{d}s - \int_0^t \int_{(0,\infty)^2} g(\eta,s)v\partial_v\varphi(\eta,s)\mathrm{d}\eta\mathrm{d}s$$
$$+(1-\gamma)\int_0^t \langle \mathbb{K}[g](s),\varphi(s)\rangle\mathrm{d}s + (1-\gamma)\int_0^t \int_{(0,\infty)^2} r(\eta)(c_0v^{\frac{2}{3}}-a)g(\eta,s)\partial_a\varphi(\eta,s)\mathrm{d}\eta\mathrm{d}s. \tag{A.2.4}$$

Well-posedness of solutions of the form (A.2.4) has been studied in [Cri21]. In this paper, we focus on proving the existence of self-similar profiles and long-time behavior for solutions of equation (A.1.1). We now give a precise meaning for (A.2.1).

Definition A.2.2. Assume $\alpha > 0$. We will say that a measure $g \in \mathscr{M}^I_+(\mathbb{R}^2_{>0})$ is a self-similar profile for the two-dimensional coagulation-equation if

$$\int_{(0,\infty)^2} (v^{-\alpha}+v^{\beta})g(a,v)\mathrm{d}v\mathrm{d}a < \infty \tag{A.2.5}$$

and for every $\varphi \in \mathrm{C}^1_{\mathrm{c}}(\mathbb{R}^2_{>0})$ the following equality is satisfied:

$$\int_{(0,\infty)^2} g(\eta)\varphi(\eta)\mathrm{d}\eta - \frac{2}{3}\int_{(0,\infty)^2} g(\eta)a\partial_a\varphi(\eta)\mathrm{d}\eta - \int_{(0,\infty)^2} g(\eta)v\partial_v\varphi(\eta)\mathrm{d}\eta$$
$$+(1-\gamma)\langle\mathbb{K}[g],\varphi\rangle + (1-\gamma)\int_{(0,\infty)^2} r(\eta)(c_0v^{\frac{2}{3}}-a)g(\eta)\partial_a\varphi(\eta)\mathrm{d}\eta = 0. \tag{A.2.6}$$

A.2.1 The case $\mu > 0$

The following result states the existence of self-similar profiles in the case when $\alpha > 0$.

Theorem A.2.3. *Let $\mu, \alpha > 0$ and $v_0 > 0$. Assume K is a continuous, non-negative kernel satisfying (A.1.5), (A.1.6), (A.1.7) and (A.1.8) and suppose that $r(a,v)$ satisfies (A.1.10), (A.1.11), (A.1.12) and (A.1.13). Then there exists a self-similar profile for the two-dimensional coagulation-equation in the sense of Definition A.2.2 with $\int_{(0,\infty)^2} vg(a,v)\mathrm{d}v\mathrm{d}a = v_0$. In addition, g satisfies $M_{n,k}(g) < \infty$, for all $n, k \in \mathbb{R}$.*

The existence of self-similar profiles in the case $\mu > 0$ can be explained since, in this regime, fusion overtakes coagulation for large values of a. Therefore, the fusion term keeps the particles with a shape that does not differ too much from that of spheres, and thus we can expect a not to be too far away from $c_0v^{\frac{2}{3}}$.

We will use the following definition for self-similar profiles for coagulation kernels satisfying (A.1.9)

Definition A.2.4. Assume $\alpha = 0$. We will say that a measure $g \in \mathscr{M}_+^I(\mathbb{R}^2_{>0})$ is a self-similar profile for the two-dimensional coagulation-equation if

$$\int_{(0,\infty)^2} (a + v^{\beta+1} + v^{\beta}) g(a, v) \mathrm{d}v \mathrm{d}a < \infty \tag{A.2.7}$$

and for every $\varphi \in \mathrm{C}^1_c(\mathbb{R}^2_{>0})$ the following equality is satisfied:

$$\begin{aligned}\frac{2}{3} \int_{(0,\infty)^2} g(\eta) a v \partial_a \varphi(\eta) \mathrm{d}\eta + \int_{(0,\infty)^2} g(\eta) v^2 \partial_v \varphi(\eta) \mathrm{d}\eta - (1-\gamma) \int_{(0,\infty)^2} v r(\eta)(c_0 v^{\frac{2}{3}} - a) g(\eta) \partial_a \varphi(\eta) \mathrm{d}\eta \\ = (1-\gamma) \int_{(0,\infty)^2} \int_{(0,\infty)^2} K(\eta, \eta') g(\eta) g(\eta') v [\varphi(\eta + \eta') - \varphi(\eta)] \mathrm{d}\eta' \mathrm{d}\eta. \end{aligned} \tag{A.2.8}$$

Notice that we obtain equation (A.2.8) by replacing in (A.2.6) the test function φ with a test function of the form $v\varphi$. The new form of the equation is chosen since in the case $\alpha = 0$ we expect the self-similar profiles to be singular for values of v near zero and for that reason not all the terms in Definition A.2.2 are well-defined. This also justifies why we are assuming condition (A.2.7) instead of (A.2.5).

Theorem A.2.5. *Let $\alpha = 0$. Assume K is a continuous, non-negative kernel satisfying (A.1.5), (A.1.6), (A.1.7) and (A.1.9). Suppose that $r(a, v)$ satisfies (A.1.10), (A.1.11), (A.1.12) and (A.1.13). If $\mu > 0$, there exists a self-similar profile for the two-dimensional coagulation-equation, in the sense of Definition A.2.4, satisfying $M_{n,k}(g) < \infty$, for all $n \geq 0$ and $k \geq \gamma$.*

Remark A.2.6. We observe that the moment estimates obtained in Theorem A.2.5 imply estimates for additional moments due to the fact that the self-similar profiles are supported in the isoperimetric region $\{a \geq c_0 v^{\frac{2}{3}}\}$, namely $M_{-n,\gamma+\frac{2}{3}n}(g) \leq c_0^{-n} M_{0,\gamma}(g) < \infty$.

A.2.2 The case $\mu \leq 0$

When μ is negative, fusion takes place at a slower pace for particles with large area. We have two different behaviors depending on the fusion rate.

If we start with a sufficiently large fusion rate, a regime similar to the one where fusion overtakes coagulation occurs and thus self-similar profiles exist in this case too.

Theorem A.2.7 (Self-similarity in the case of slow fusion)**.** *Let $\mu \leq 0$ and $\alpha > 0$. Assume K is a continuous, non-negative kernel satisfying (A.1.5), (A.1.6), (A.1.7) and (A.1.8). Suppose that $r(a, v)$ satisfies (A.1.10), (A.1.11), (A.1.12) and (A.1.13). Then, there exists $\lambda > 1$, depending only on K_0, K_1 and γ, such that for any $v_0 > 0$, if $r(a, v)$ satisfies (A.1.10) with $R_0 \geq \lambda v_0$, then there exists a self-similar profile g for the two-dimensional coagulation-equation, in the sense of Definition A.2.2, with total volume $\int_{(0,\infty)^2} v g(a, v) \mathrm{d}v \mathrm{d}a = v_0$.*

Remark A.2.8. Notice that Theorem A.2.7 applies also in the case when $\mu = 0$, which corresponds to the type of fusion kernels considered in [KF90].

In order to better understand the stated results, we give some heuristic arguments. We can explain the condition needed for Theorem A.2.7, namely that R_0 in (A.1.10) needs to be sufficiently large, in the following manner stated below.

First we can assume without loss of generality that $v_0 = 1$ by means of a rescaling argument (see Appendix A.5). With this rescaling, the theorem states that if $R_0 \geq \lambda$, for some sufficiently large constant $\lambda > 0$, then there exists a self-similar profile.

Assume for simplicity that $r(a, v) = R_0 a^{\mu}$ and that $\mu < 0$. In other words $r(a, v) = R_0 a^{\frac{3(\gamma-1)}{2}}$ in order to be consistent with condition (A.1.10). Additionally, we work with a coagulation kernel $K \equiv 1$. The most important part is to estimate the total surface area. This is since the moments in the v variable can be bounded using standard arguments used in the study of coagulation equations.

Assume without loss of generality that $a \geq 2c_0 v^{\frac{2}{3}}$ since the region $\{c_0 v^{\frac{2}{3}} \leq a \leq 2c_0 v^{\frac{2}{3}}\}$ is bounded using uniform moment estimates in the v variable. The rigorous proof of Theorem A.2.7 will be a generalization of the following idea.

Denote by $A(t) := \int_{(0,\infty)^2} a g(\eta, t) \mathrm{d}\eta$. We test formally in (A.2.4) with $\varphi \equiv a$. Equation (A.2.4) becomes

$$\begin{aligned} \partial_t A(t) &= \frac{1}{3} A(t) + R_0 \int_{(0,\infty)^2} a^{\mu} (c_0 v^{\frac{2}{3}} - a) g(\eta, t) \mathrm{d}\eta \\ &\leq \frac{1}{3} A(t) - \frac{R_0}{2} \int_{(0,\infty)^2} a^{\mu+1} g(\eta, t) \mathrm{d}\eta. \end{aligned} \tag{A.2.9}$$

Fix $\epsilon \in (0, 1)$. Since $\mu < 0$, by Young's inequality, we have that there exists some $\overline{\lambda}_{\epsilon} > 0$, depending on ϵ, sufficiently large such that

$$\frac{4}{3} a \leq \overline{\lambda}_{\epsilon} a^{\mu+1} + \epsilon a^2. \tag{A.2.10}$$

Choosing $R_0 \geq 2\overline{\lambda}_{\epsilon}$, (A.2.9) becomes

$$\partial_t A(t) \leq -A(t) + \epsilon \int_{(0,\infty)^2} a^2 g(\eta, t) \mathrm{d}\eta. \tag{A.2.11}$$

We then analyse the moment $M_{2,0}$. To this end, we test (A.2.9) with $\varphi \equiv a^2$. Since $K \equiv 1$ and the fusion term is non-positive, we deduce

$$\partial_t \int_{(0,\infty)^2} a^2 g(\eta, t) \mathrm{d}\eta \leq -\frac{1}{3} \int_{(0,\infty)^2} a^2 g(\eta, t) \mathrm{d}\eta + (1-\gamma) \Big(\int_{(0,\infty)^2} a g(\eta, t) \mathrm{d}\eta \Big)^2. \tag{A.2.12}$$

Combining (A.2.11) and (A.2.12) and choosing ϵ to be sufficiently small, it follows that

$$\partial_t \Big(A(t) + \int_{(0,\infty)^2} a^2 g(\eta, t) \mathrm{d}\eta \Big) \leq -\frac{1}{6} \int_{(0,\infty)^2} a^2 g(\eta, t) \mathrm{d}\eta - A(t) + (1-\gamma)(A(t))^2. \tag{A.2.13}$$

In other words, using the notation $D(t) := \int_{(0,\infty)^2} a^2 g(\eta, t) d\eta + A(t)$, we obtain

$$\partial_t D(t) \leq -\frac{1}{6} D(t) + (1-\gamma)(D(t))^2. \tag{A.2.14}$$

If we take an initial condition g_{in} such that $D(0) \leq \frac{1}{12(1-\gamma)}$, we will have by (A.2.14) that $D(t) \leq \frac{1}{12(1-\gamma)}$ is an invariant region in time. This enables us to use standard methods used in the study of coagulation equations to conclude that there exists a self-similar profile.

Notice that we use in the argument that R_0 is large enough.

In order to prove the ramification result, we first prove the existence of a weak solution for the time-dependent fusion problem in self-similar variables which satisfies some suitable moment estimates.

Proposition A.2.9. *Assume $\alpha > 0$ and $\mu < 0$. Assume K is a continuous, non-negative kernel satisfying (A.1.5), (A.1.6), (A.1.7) and (A.1.8). Suppose that $r(a, v)$ satisfies (A.1.10), (A.1.11), (A.1.12) and (A.1.13). Assume $g_{\text{in}} \in \mathscr{M}_+^I(\mathbb{R}_{>0}^2)$ with $\int_{(0,\infty)^2} v g_{\text{in}} dv da = v_0$. There exists a constant $C_1(v_0) > 0$, depending on v_0, K_0, K_1, and γ, cf. (A.1.5) and (A.1.7), such that, if*

$$\int_{(0,\infty)^2} v^{x_1} g_{\text{in}}(a, v) dv da \leq C_1(v_0), \tag{A.2.15}$$

where $x_1 \in \{\frac{\sigma}{|\mu|}, \gamma - \frac{1}{3}\}$ and

$$\int_{(0,\infty)^2} (a^\mu + a^2)(v^{-\alpha-1} + v^{\max\{1+\tilde{\epsilon}, \sigma+\frac{2}{3}\}}) g_{\text{in}}(a, v) dv da < \infty, \tag{A.2.16}$$

for some $\tilde{\epsilon} \in (0, 1)$, then there exists a weak solution of the time-dependent fusion problem g as in Definition A.2.1 with $g \in \mathrm{C}([0, \infty); \mathscr{M}_+^I(\mathbb{R}_{>0}^2))$, which in addition satisfies the following moment estimates.

$$\sup_{t \in [0,\infty)} \int_{(0,\infty)^2} v^{x_1} g(a, v, t) dv da \leq C_1(v_0) \tag{A.2.17}$$

and, for every $T > 0$,

$$\sup_{t \in [0,T]} \int_{(0,\infty)^2} (a^\mu + a^2)(v^{-\alpha-1} + v^{\max\{1+\tilde{\epsilon}, \sigma+\frac{2}{3}\}}) g(a, v, t) dv da < \infty. \tag{A.2.18}$$

Remark A.2.10. We have that $C_1(v_0) = \overline{C}_1 v_0^{\xi(x_1-\gamma)}$, where $\overline{C}_1 > 0$ depends only on K_0, K_1, and γ, cf. (A.1.5) and (A.1.7), and ξ is as in (A.1.4). For more details explaining the rescaling properties of g, see Appendix A.5.

Remark A.2.11. The moment estimates in (A.2.18) are needed in order to prove that we are able to test the equation (A.2.4) with $\varphi \equiv a$. Actually, we do not need estimates for moments of the form $M_{2,0}$ in order to prove this, but we keep the form (A.2.18) in order to emphasize that more general moment estimates can be obtained.

If $\mu < 0$ and we start with sufficiently large surface area, we can expect a fast growth in the area in self-similar variables since coagulation overtakes fusion for large particles. Notice that, in equation (A.1.1), if we ignore the fusion term, we obtain a particle distribution f for which the area stays constant in time. The exponential growth stated in the next theorem is equivalent to a lower estimate for the total area associated to the distribution f, cf. (A.1.15).

Theorem A.2.12. *Assume $\alpha > 0$ and $\mu < 0$. Assume K is a continuous, non-negative kernel satisfying (A.1.5), (A.1.6), (A.1.7) and (A.1.8). Suppose that $r(a, v)$ satisfies (A.1.10), (A.1.11), (A.1.12) and (A.1.13). Let $g \in \mathrm{C}([0,\infty); \mathscr{M}_+^I(\mathbb{R}^2_{>0}))$ be a solution of the time-dependent fusion problem as in Definition A.2.1 with total volume of particles equal to v_0 and satisfying (A.2.17) and (A.2.18). Then the following holds: there exists a constant $C_2(v_0) > 0$, depending on v_0, K_0, K_1, γ, μ and σ , cf. (A.1.5), (A.1.7) and (A.1.10), such that, if*

$$R_1 \leq v_0, \tag{A.2.19}$$

where R_1 is as in (A.1.10) and

$$\int_{(0,\infty)^2} a g_{\mathrm{in}}(a, v)\mathrm{d}v\mathrm{d}a \geq C_2(v_0), \tag{A.2.20}$$

then

$$\int_{(0,\infty)^2} g(a, v, t) a\mathrm{d}v\mathrm{d}a \geq C_\mu C_2(v_0)\mathrm{e}^{\frac{1}{3}t}, \tag{A.2.21}$$

for some C_μ depending on μ.

Remark A.2.13. We have that $C_2(v_0) = \overline{C}_2 v_0^{\xi(\frac{2}{3}-\gamma)}$, for ξ as in (A.1.4), where $\overline{C}_2 > 0$ depends only on K_0, K_1, γ, μ and σ , cf. (A.1.5), (A.1.7) and (A.1.10). For more details about the rescaling properties of g, see Appendix A.5.

Notice that Theorem A.2.12 holds for any weak solution in the sense of Definition A.2.1 that satisfies in addition (A.2.17), (A.2.18) and (A.2.20). On the other hand, combining Proposition A.2.9 and Theorem A.2.12, we obtain the following

Corollary A.2.14. *Suppose that $g_{\mathrm{in}} \in \mathscr{M}_+^I(\mathbb{R}^2_{>0})$ and satisfies (A.2.15), (A.2.16), (A.2.19) and (A.2.20), with $C_1(v_0)$ and $C_2(v_0)$ as in Proposition A.2.9 and Theorem A.2.12, respectively. Then there exists a weak solution for the time-dependent fusion problem in the sense of Definition A.2.1 which satisfies (A.2.21).*

Theorem A.2.12 can be understood in the following manner. Identically as before, we can assume without loss of generality that $v_0 = 1$ by means of a rescaling argument (see Appendix A.5). With this rescaling, is we have $R_1 \leq 1$ and $\int_{(0,\infty)^2} a g_{\text{in}}(a,v)\mathrm{d}v\mathrm{d}a \geq C$, for some sufficiently large constant $C > 0$, then ramification occurs.

We now provide some heuristic explanation in order to justify the validity of Theorem A.2.12. We explain below the condition needed for Theorem A.2.12, namely condition (A.2.20).

Assume for simplicity that $r(a,v) = a^{\mu}$ and that $\mu \leq -1$. In other words, in order to be consistent with the condition (A.1.10), $r(a,v) = a^{\frac{3(\gamma-1)}{2}}$. The rigorous proof of Theorem A.2.12 will be a generalization of the following idea.

Test formally in (A.2.4) with $\varphi \equiv a$. We have $\langle \mathbb{K}[g], \varphi \rangle = 0$. Denote by $A(t) := \int_{(0,\infty)^2} a g(\eta,t)\mathrm{d}\eta$. Equation (A.2.4) becomes

$$\begin{aligned}
\partial_t A(t) &= \frac{1}{3}A(t) + \int_{(0,\infty)^2} a^{\mu}(c_0 v^{\frac{2}{3}} - a) g(\eta,t)\mathrm{d}\eta \\
&\geq \frac{1}{3}A(t) - \int_{(0,\infty)^2} a^{\mu+1} g(\eta,t)\mathrm{d}\eta \\
&\geq \frac{1}{3}A(t) - c_0^{\mu+1}\int_{(0,\infty)^2} v^{\gamma-\frac{1}{3}} g(\eta,t)\mathrm{d}\eta.
\end{aligned} \tag{A.2.22}$$

It turns out that we can prove

$$\int_{(0,\infty)^2} v^{\gamma-\frac{1}{3}} g(\eta,t)\mathrm{d}\eta \leq \max\{\int_{(0,\infty)^2} v^{\gamma-\frac{1}{3}} g_{\text{in}}(\eta)\mathrm{d}\eta, C\}, \tag{A.2.23}$$

for some fixed constant $C > 0$. The proof of this result is made in a similar manner as the proof of the analogous estimate for the one-dimensional coagulation equations. Thus, (A.2.22) becomes

$$\partial_t A(t) \geq \frac{1}{3}A(t) - C, \tag{A.2.24}$$

for some constant $C > 0$ depending only on g_{in}.

From (A.2.24), we deduce that if $A(0)$ is sufficiently large, then $A(t)$ behaves like $\mathrm{e}^{\frac{1}{3}t}$. This is the content of the ramification result in Theorem A.2.12.

A.3 Existence of self-similar solutions when $\mu > 0$

The strategy for the proof of Theorem A.2.3 (and Theorem A.2.5) follows the approach of obtaining self-similar profiles as a fixed point of a truncated version of the time-dependent problem by showing invariance in time of a compact set. It is convenient to work with truncated versions of the coagulation kernel, as well as a modified fusion rate. This is done in order to avoid singular behavior and unbounded terms. Notice that, since g is supported in the region $\{a \geq c_0 v^{\frac{2}{3}}\}$, information about one of the variables implies some information over the other. Estimates for moments depending only on v follow then in the same manner as for the one-dimensional coagulation equation, due to the particular form of the coagulation kernel.

In order to define the truncated problem, we introduce the following functions. For $\epsilon \in (0,1)$ and $R > 1$, we define a truncation $K_{\epsilon,R} : (0,\infty)^4 \to [0,\infty)$ for the coagulation kernel to be a continuous function such that:

$$K_{\epsilon,R}(a,v,a',v') = \min\{K(a,v,a',v'), 2^{1+\beta}K_0\epsilon^{-\alpha}R^{\beta}\}, \tag{A.3.1}$$

where K satisfies (A.1.7) and take $\xi_R : \mathbb{R}_{>0} \to [0,1]$ to be continuous and defined in the following manner:

$$\xi_R(v) = 0, \qquad \text{when } v \geq 2R, \tag{A.3.2}$$

$$\xi_R(v) = 1, \qquad \text{on } (0,R]. \tag{A.3.3}$$

We first discuss on how the existence of strong solutions for the truncated version of (A.2.1) will be proven. We take a truncation for the linear transport terms, namely we take $\Theta_\epsilon : (0,\infty) \to \mathbb{R}$ to be a smooth monotonically increasing function such that:

$$\Theta_\epsilon(v) = \begin{cases} 1, & v > 2\epsilon, \\ 0, & v \leq \epsilon. \end{cases} \tag{A.3.4}$$

The main issue is to find a suitable subset of $\mathscr{M}_+^I(\mathbb{R}^2_{>0})$ in which we can obtain uniform estimates in time. Thus, we first take a cut-off near the origin and show that the support of g remains in this region. As mentioned before, information about the behavior of v near the origin is enough to control a near the origin. However, this is not the case for large values of a and thus we have to deal with the fusion rate.

We replace the fusion term by terms linearly increasing in the area variable in order to avoid the characteristics to arrive from infinity in finite time. The linear growth in a will enable us to test with functions that are not compactly supported in the area variable. So, for $\varphi \in C_0^1(\mathbb{R}^2_{>0})$, we analyse the following regularized version of (A.2.1):

$$\begin{aligned}
&\int_{(0,\infty)^2} g(\eta,t)\Theta_\epsilon(v)\varphi(\eta)\mathrm{d}\eta - \frac{2}{3}\int_{(0,\infty)^2} g(\eta,t)\Theta_\epsilon(v)a\partial_a\varphi(\eta)\mathrm{d}\eta - \int_{(0,\infty)^2} g(\eta,t)\Theta_\epsilon(v)v\partial_v\varphi(\eta)\mathrm{d}\eta \\
&+(1-\gamma)\langle \mathbb{K}_{\epsilon,R}[g],\varphi\rangle + (1-\gamma)\int_{(0,\infty)^2} r_\delta(\eta)(c_0v^{\frac{2}{3}} - a)g(\eta,t)\partial_a\varphi(\eta)\mathrm{d}\eta = \partial_t\int_{(0,\infty)^2} g(\eta,t)\varphi(\eta)\mathrm{d}\eta,
\end{aligned} \tag{A.3.5}$$

where

$$\langle \mathbb{K}_{\epsilon,R}[g],\varphi\rangle := \frac{1}{2}\int_{(0,\infty)^2}\int_{(0,\infty)^2} K_{\epsilon,R}(a,v,a',v')\xi_R(v+v')\chi_\varphi(\eta,\eta')\mathrm{d}\eta'\mathrm{d}\eta. \tag{A.3.6}$$

We have replaced the fusion term $r(a,v)$ by

$$r_\delta(a,v) := \frac{r(\eta)\max\{v^\sigma, L\delta\}}{v^\sigma(1+\delta a^\mu)}, \tag{A.3.7}$$

for $\delta \in (0,1)$ and where we denoted

$$L := \frac{12}{R_0(1-\gamma)}, \tag{A.3.8}$$

where R_0 is as in (A.1.10). Notice that $r_\delta(\eta) \to r(\eta)$ for fixed η as $\delta \to 0$. L was chosen in such a way to derive uniform estimates for the total area of solutions. This means it has to be sufficiently large in order to compensate for the linear transport term appearing due to the coagulation kernel.

For some $\tilde{g} \in \mathscr{M}_+^I(\mathbb{R}^2_{>0})$ such that

$$\int_{(0,\infty)^2} (1+a)\tilde{g}(a,v)\mathrm{d}v\mathrm{d}a < \infty, \tag{A.3.9}$$

we define the space

$$U_{\epsilon,R} := \{\tilde{g} \in \mathscr{M}_+^I(\mathbb{R}^2_{>0}), \tilde{g}\Big(\mathbb{R}^2_{>0} \setminus [c_0\epsilon^{\frac{2}{3}},\infty) \times [\epsilon, 2R)\Big) = 0, \tilde{g} \text{ satisfies (A.3.9)}\}. \tag{A.3.10}$$

In this section we will prove the following technical results.

Proposition A.3.1. *Take $K_{\epsilon,R}$ as above, i.e. (A.3.1) holds, and assume it satisfies (A.1.6). Let $g_{\text{in},R} \in \mathscr{M}_{+,\text{b}}^I(\mathbb{R}^2_{>0}) \cap U_{\epsilon,R}$. There exists a unique solution $g_{\epsilon,R,\delta} \in \mathrm{C}^1([0,\infty); \mathscr{M}_+^I(\mathbb{R}^2_{>0}))$, $g_{\epsilon,R,\delta}(t) \in U_{\epsilon,R}$, that satisfies*

$$\sup_{t\in[0,T]} \int_{(0,\infty)^2} (1+a) g_{\epsilon,R,\delta}(a,v,t)\mathrm{d}v\mathrm{d}a < \infty,$$

for all times $T \in [0,\infty)$, for the weak formulation of the coagulation equation (A.3.5) with initial datum $g_{\text{in},R}$.

The proof of this proposition will be the content of Subsection A.3.1.

Let $K_{\epsilon,R}$ defined as in (A.3.1) and assume K and r satisfy the conditions stated in Theorem A.2.3 or in Theorem A.2.5. Let $T > 0$. We define the map $S(t) : U_{\epsilon,R} \to U_{\epsilon,R}$ in the following way:

$$S(t) g_{\text{in},R} = g_{\epsilon,R,\delta}(\cdot,\cdot,t), \tag{A.3.11}$$

for all $t \in [0,T]$, where $g_{\epsilon,R,\delta}$ is the unique solution of the weak formulation of the coagulation equation with coagulation kernel $K_{\epsilon,R}$ found Proposition A.3.1.

In order to prove Theorem A.2.3 (and Theorem A.2.5), the next lemma will be useful:

Proposition A.3.2. *Let $\epsilon \in (0,1), R > 1, \delta \in (0,1)$. Let $K_{\epsilon,R}$ defined as in (A.3.1) and r_δ as in (A.3.7). Assume K and r satisfy the conditions stated in Theorem A.2.3 or in Theorem A.2.5. Let $T > 0$ and $S(t) : U_{\epsilon,R} \to U_{\epsilon,R}$ as in (A.3.11), for $t \in [0,T]$.*

Let $\mu > 0$ in (A.1.10). Then there exist constants $c_{0,-\alpha-\tilde{\epsilon}}$, $c_{0,\tilde{m}}$, $c_{1,0} > 0$, where $\tilde{\epsilon} \in (0,1)$ and $\tilde{m} > \max\{1, \frac{|\sigma|}{\mu}\}$, with the property that the set $\omega(\epsilon, R, \delta)$, defined as

$$\omega(\epsilon,R,\delta) :=\{M_{0,1}(g_{\epsilon,R,\delta}) = 1; M_{0,-\alpha-\tilde{\epsilon}}(g_{\epsilon,R,\delta}) \leq c_{0,-\alpha-\tilde{\epsilon}}; M_{0,\tilde{m}}(g_{\epsilon,R,\delta}) \leq c_{0,\tilde{m}}; M_{1,0}(g_{\epsilon,R,\delta}) \leq c_{1,0}\} \tag{A.3.12}$$

if $\alpha > 0$ and

$$\omega(\epsilon,R,\delta) :=\{M_{0,1}(g_{\epsilon,R,\delta}) = 1; M_{0,\gamma}(g_{\epsilon,R,\delta}) \leq c_{0,\gamma}; M_{0,\tilde{m}}(g_{\epsilon,R,\delta}) \leq c_{0,\tilde{m}}; M_{1,0}(g_{\epsilon,R,\delta}) \leq c_{1,0}\} \tag{A.3.13}$$

if $\alpha = 0$,

is preserved in time uniformly in ϵ, R, δ under equation (A.3.5), i.e. $S(t)\omega(\epsilon,R,\delta) \subseteq \omega(\epsilon,R,\delta)$, for all $t \in [0,T]$.

The proof of this proposition will be given in Subsection A.3.2.

To get volume-conserving solutions, we need to control the total area. In order to obtain this, we need to assume that an additional moment is bounded and this is why the moment $M_{0,\frac{|\sigma|}{\mu}}$ appears in Proposition A.3.2.

A.3.1 Existence of solutions for the truncated time-dependent problem

We define the functions $A, V : \mathbb{R}^2_{>0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}$ in the following manner, by looking at the characteristic equations:

$$\begin{cases} \partial_t A(a_0, v_0, t) = (1-\gamma) r_\delta(A,V)(c_0 V^{\frac{2}{3}} - A) - \dfrac{2}{3}\Theta_\epsilon(V)A; \\ \partial_t V(a_0, v_0, t) = -\Theta_\epsilon(V)V, \end{cases} \tag{A.3.14}$$

with initial conditions

$$\begin{cases} A(a_0,v_0,0) = & a_0; \\ V(a_0,v_0,0) = & v_0. \end{cases} \tag{A.3.15}$$

By (A.1.13), we have the following inequality, that we write for future reference

$$\begin{aligned} \partial_A[r_\delta(A,V)(A - c_0V^{\frac{2}{3}})] &= \max\{V^\sigma, L\delta\}\frac{[\partial_A r(A,V) - \frac{\mu\delta A^{\mu-1}}{1+\delta A^\mu} r(A,V)](A - c_0V^{\frac{2}{3}}) + r(A,V)}{V^\sigma(1+\delta A^\mu)} \\ &\geq \max\{V^\sigma, L\delta\}\frac{[\partial_A r(A,V) - \mu A^{-1} r(A,V)](A - c_0V^{\frac{2}{3}}) + r(A,V)}{V^\sigma(1+\delta A^\mu)} \geq 0. \end{aligned}$$

Fix $t \geq 0$. We denote the pair $(A(a_0,v_0,t), V(a_0,v_0,t)) =: \phi_t(a_0,v_0)$. Observe that, due to the form of the equations in (A.3.14), the function V is independent of a_0. In particular, there exists a family of functions $\{y_t\}_{t\geq 0} : \mathbb{R}_{>0} \to \mathbb{R}_{>0}$ such that $y_t(v_0) := V(a_0,v_0,t)$. In the same manner, we fix v_0 and we define $x_{t,v_0} : \mathbb{R}_{>0} \to \mathbb{R}$ by $x_{t,v_0}(a_0) := A(a_0,v_0,t)$.

We gather in the following proposition a list of properties for the solutions of the system (A.3.14) that will be used throughout the paper.

Proposition A.3.3 (Properties of the characteristics)**.** *Let A, V be as in (A.3.14) with initial conditions (A.3.15). Then*

(1) $V(a_0, v_0, t) \equiv v_0$, when $V \leq \epsilon$;

(2) $V(a_0, v_0, t) = v_0 \mathrm{e}^{-t}$, when $V > 2\epsilon$.

We have in addition that

(3) Let $P(v_0, t)$, with $(v_0, t) \in (0, \infty) \times [0, \infty)$ be a solution of $\partial_t P = -\Theta_\epsilon(P)P$. Define then $H = c_0 P^{\frac{2}{3}}$, for all $t \geq 0$ and all $v_0 \in (0, \infty)$. Then H solves the first equation in (A.3.14). In particular, we have that, if $(a_0, v_0) \in \partial\tilde{\mathrm{S}}$, then $(A, V) \in \partial\tilde{\mathrm{S}}$, where $\tilde{\mathrm{S}}$ was defined in (A.1.2);

(4) The mapping ϕ_t sends $\tilde{\mathrm{S}}$ to $\tilde{\mathrm{S}}$, i.e. $\phi_t(\tilde{\mathrm{S}}) \subseteq \tilde{\mathrm{S}}$, where $\tilde{\mathrm{S}}$ was defined in (A.1.2);

(5) If $(a_0, v_0) \in \tilde{\mathrm{S}}$, we have that $\partial_{a_0} x_{t,v_0} \geq 0$ and $\partial_{v_0} y_t \geq 0$.

Proof. The fact that $A(a_0, v_0, t)$ and $V(a_0, v_0, t)$ are well-defined for $(a_0, v_0) \in \tilde{\mathrm{S}}$, with $\tilde{\mathrm{S}}$ as in (A.1.2), and for $t \geq 0$ follows from standard ODE theory, as well as the choice of truncation r_δ in (A.3.7), which avoids blow-up in finite time. Statement (1) follows from the fact that $\Theta_\epsilon(v) = 0$, when $v \leq \epsilon$. Statement (2) follows from the fact that $\Theta_\epsilon(v) = 1$, when $v \geq 2\epsilon$.

In order to prove Statement (3), take $H = c_0 P^{\frac{2}{3}}$. Notice on one hand that

$$\partial_t H = \partial_t\Big(c_0 P^{\frac{2}{3}}\Big) = \frac{2c_0}{3} P^{-\frac{1}{3}} \partial_t P = -\frac{2c_0}{3} \Theta_\epsilon(P) P^{\frac{2}{3}}.$$

On the other hand, it follows that

$$(1-\gamma) r_\delta(H, P)(c_0 P^{\frac{2}{3}} - H) - \frac{2}{3}\Theta_\epsilon(P) H = -\frac{2}{3}\Theta_\epsilon(P) H = -\frac{2c_0}{3}\Theta_\epsilon(P) P^{\frac{2}{3}}.$$

Thus, $H = c_0 P^{\frac{2}{3}}$ solves the first equation in (A.3.14). The statement then follows from the uniqueness theory of ODE's.

Statement (4) is a consequence of Statement (3) and also of uniqueness theory of ODE's.

Finally, for Statement (5), we will only prove that $\partial_{v_0} y_t \geq 0$ since the proof of the fact that $\partial_{a_0} x_{t,v_0} \geq 0$ follows using a similar argument. We have that

$$\begin{cases} \partial_t \partial_{v_0} V(a_0, v_0, t) = -\partial_V(\Theta_\epsilon(V)V)\partial_{v_0} V; \\ \quad \partial_{v_0} V(a_0, v_0, t) = 1. \end{cases}$$

Thus

$$\partial_{v_0} V(a_0, v_0, t) = \mathrm{e}^{-\int_0^t \partial_V\left(\Theta_\epsilon(V(s))V(s)\right)\mathrm{d}s} \geq 0.$$

□

Definition A.3.4. We define the pair $(l_1, l_2) \in \tilde{\mathrm{S}}$, with $\tilde{\mathrm{S}}$ defined in (A.1.2), to be the solution of

$$\begin{cases} y_t(l_2(v_0, v_0', t)) & = y_t(v_0) + y_t(v_0'); \\ x_{t, l_2(v_0, v_0', t)}(l_1(a_0, a_0', l_2(v_0, v_0', t), t)) & = x_{t, v_0}(a_0) + x_{t, v_0'}(a_0'), \end{cases} \tag{A.3.16}$$

where $(a_0, v_0) \in \tilde{\mathrm{S}}$, $(a_0', v_0') \in \tilde{\mathrm{S}}$ and $t \in [0, \infty)$.

We make the following notation: $\phi_t^{-1}(A, V) := (x^{-1}_{t, y_t^{-1}(V)}(A), y_t^{-1}(V))$.

For further use, we define

$$h_\epsilon(v_0, t) = \int_0^t \Theta_\epsilon(y_s(v_0)) \mathrm{d}s. \tag{A.3.17}$$

Most of the results in this subsection hold true if, instead of working with $K_{\epsilon,R}$ defined in (A.3.1), we work with a function $K_R : (0, \infty)^4 \to [0, \infty)$, which is defined in the following manner:

$$K_R(a, v, a', v') = \min\{K(a, v, a', v'), R\}, \tag{A.3.18}$$

where K is as in (A.1.7). Thus, in order to simplify computations (and without loss of generality), we will interchange between K_R and $K_{\epsilon,R}$ throughout this subsection. We will also write $\mathbb{K}_R$ instead of $\mathbb{K}_{\epsilon,R}$ (which was defined in (A.3.6)) when we work with K_R. The same notation will be used in Subsection A.3.3 and Appendix A.6. Notice that we will need to work with $K_{\epsilon,R}$ in Subsection A.3.2 in order to obtain suitable moment estimates.

We first start by proving there exists $G \in \mathrm{C}([0, \infty); \mathscr{M}_+^I(\mathbb{R}^2_{>0}))$ which satisfies the following equation:

$$\partial_t \int_{\mathbb{R}^2_{>0}} G(A, V, t)\varphi(A, V) \mathrm{d}V \mathrm{d}A = \frac{1-\gamma}{2} \int_{\mathbb{R}^2_{>0}} \int_{\mathbb{R}^2_{>0}} K_R(\phi_t(A, V), \phi_t(A', V'))\xi_R(y_t(V) + y_t(V'))G(A', V', t)$$
$$G(A, V, t)[\varphi(\phi_t^{-1}(\phi_t(A, V) + \phi_t(A', V')))\Phi(V, V', t) - \varphi(A, V)\mathrm{e}^{h_\epsilon(V', t)} - \varphi(A', V')\mathrm{e}^{h_\epsilon(V, t)}] \mathrm{d}V' \mathrm{d}A' \mathrm{d}V \mathrm{d}A, \tag{A.3.19}$$

for every $\varphi \in \mathrm{C_c}(\mathbb{R}^2_{>0})$ and

$$\Phi(V, V', t) := \mathrm{e}^{-h_\epsilon(l_2(V, V', t), t) + h_\epsilon(V, t) + h_\epsilon(V', t)}.$$

Notice that the operator K_R on the right-hand side of equation (A.3.19) encodes information about the fusion process and not only about coagulation through the function $\phi_t(A, V)$ and that (A.3.19) is a reformulation of (A.3.5) using characteristics.

Let $T > 0$. Take $M = 2\int_{(0,\infty)^2}(1 + a)g_{\mathrm{in},R}(a, v)\mathrm{d}v\mathrm{d}a + 1$, for some $g_{\mathrm{in},R}$ as in Proposition A.3.1. Choose $\tau < T$ such that

$$(M + 2(1-\gamma)M + 1)\|K_R\|_\infty(\mathrm{e}^\tau - 1) < \frac{1}{2}; \tag{A.3.20}$$

$$2(1-\gamma)M^2\|K_R\|_\infty \tau < 1 \text{ and } \tau \leq \ln 2. \tag{A.3.21}$$

We will use the following auxiliary metric space

$$\begin{aligned}\mathrm{Y}_{\epsilon,\tau} =&\{G \in \mathrm{C}([0,\tau];\mathscr{M}^I_{+,\mathrm{b}}(\mathbb{R}^2_{>0})) : \|G\| = \sup_{0\leq t\leq\tau}\left|\int_{(0,\infty)^2} \mathrm{e}^{h_\epsilon(V,t)}(1+x_{t,V}(A))G(A,V,t)\mathrm{d}V\mathrm{d}A\right| \leq 2M;\\ &G\Big(\mathbb{R}^2_{>0}\setminus[c_0\epsilon^{\frac{2}{3}},\infty)\times[\epsilon,2R\mathrm{e}^t),t\Big)=0,\ \text{for every } t\in[0,\tau]\}. \end{aligned} \tag{A.3.22}$$

Given $G \in \mathrm{Y}_{\epsilon,\tau}$, we define

$$a[G](A,V,t) := \int_{\{A'\geq V'^{\frac{2}{3}}\}} K_R(\phi_t(A,V),\phi_t(A',V'))\xi_R(y_t(V)+y_t(V'))G(A',V',t)\mathrm{e}^{h_\epsilon(V,t)}\mathrm{d}V'\mathrm{d}A',$$

which is well-defined since K_R is bounded. For $g_{\mathrm{in},R}$ as in Proposition A.3.1, we analyse the properties of the map $J : \mathrm{Y}_{\epsilon,\tau} \to \mathrm{C}([0,\tau];\mathscr{M}(\mathbb{R}^2_{>0}))$, defined by

$$\int_{\mathbb{R}^2_{>0}} J[G](A,V,t)\varphi(A,V)\mathrm{d}V\mathrm{d}A = J_1(G,t)+J_2(G,t), \tag{A.3.23}$$

where:

$$J_1(G,t) := \int_{\mathbb{R}^2_{>0}} g_{\mathrm{in},R}(A,V)e^{-\int_0^t a[G](A,V,\xi)\mathrm{d}\xi}\varphi(A,V)\mathrm{d}V\mathrm{d}A$$

and

$$\begin{aligned}J_2(G,t) :=& \frac{1-\gamma}{2}\int_0^t\int_{\mathbb{R}^2_{>0}}\int_{\mathbb{R}^2_{>0}} e^{-\int_s^t a[G](A,V,\xi)\mathrm{d}\xi}K_R(\phi_s(A,V),\phi_s(A',V'))\xi_R(y_s(V)+y_s(V'))\times\\ &G(A,V,s)G(A',V',s)\varphi(\phi_s^{-1}(\phi_s(A,V)+\phi_s(A',V')))\Phi(V,V',s)\mathrm{d}V'\mathrm{d}A'\mathrm{d}V\mathrm{d}A\mathrm{d}s,\end{aligned}$$

for every $\varphi \in \mathrm{C_c}(\mathbb{R}^2_{>0})$.

Since G is non-negative, then $J[G] \in \mathrm{C}([0,\tau];\mathscr{M}_+(\mathbb{R}^2_{>0}))$. Our plan is to use Banach fixed-point theorem for the map J. The reason is that, as explained below, a fixed point of the operator J will give the desired solution. In other words, we use a similar approach as the one used to prove well-posedness for pure coagulation equations with bounded kernels, which has been repeatedly used in literature (see, for example, [BLL19b]).

Proposition A.3.5. *Assume $g_{\mathrm{in},R}$ is as in Proposition A.3.1 and K_R as in (A.3.18). Assume $G \in \mathrm{Y}_{\epsilon,\tau}$. Then $J[G]$ is supported in the same domain as the measures in $\mathrm{Y}_{\epsilon,\tau}$, namely $J[G] \in \mathrm{C}([0,\tau];\mathscr{M}^I_+(\mathbb{R}^2_{>0}))$ and $J[G]\Big(\mathbb{R}^2_{>0}\setminus[c_0\epsilon^{\frac{2}{3}},\infty)\times[\epsilon,2R\mathrm{e}^t),t\Big)=0$, for every $t\in[0,\tau]$.*

Proof. Let $\varphi \in \mathrm{C}_0(\mathbb{R}^2_{>0})$ such that $\mathrm{supp}\varphi \subseteq \{A < c_0V^{\frac{2}{3}}\}$. Since $g_{\mathrm{in},R} \in \mathrm{Y}_{\epsilon,\tau}$, then the term $J_1(G,t)$ defined in (A.3.23) vanishes. Due to the support of G, we have $A \geq c_0V^{\frac{2}{3}}$ and $A' \geq c_0V'^{\frac{2}{3}}$. By Proposition A.3.3, Statement (4), $x_{t,V}(A) \geq c_0y_t(V)^{\frac{2}{3}}$ and $x_{t,V'}(A') \geq c_0y_t(V')^{\frac{2}{3}}$. This implies:

$$x_{t,V}(A)+x_{t,V'}(A') \geq c_0y_t(V)^{\frac{2}{3}}+c_0y_t(V')^{\frac{2}{3}} \geq c_0(y_t(V)+y_t(V'))^{\frac{2}{3}},$$

which implies that $l_1(A, A', l_2(V, V', t)) \geq c_0 l_2(V, V', t)^{\frac{2}{3}}$ by Proposition A.3.3 and using the notation from Definition A.3.4. Thus the term $J_2(G, t)$ in (A.3.23) vanishes as the support of φ gives that we have to work in the set where $l_1(A, A', l_2(V, V', t)) < c_0 l_2(V, V', t)^{\frac{2}{3}}$.

For the sets $(0, \infty)^2 \setminus (0, \infty) \times (0, 2Re^t]$ and $(0, \infty)^2 \setminus [c_0 \epsilon^{\frac{2}{3}}, \infty) \times [\epsilon, \infty)$, the proof is done in the same manner taking note of the fact that the kernel is chosen to vanish on these sets. For example, if $l_2(V, V', t) > 2Re^t$, then $y_t(V) + y_t(V') \geq 2R$ and $\xi_R(y_t(V) + y_t(V'))$ vanishes on this set. If $l_2(V, V', t) < \epsilon$, then $y_t(V) + y_t(V') < \epsilon$, meaning $V, V' < \epsilon$ and G vanishes on these sets. Lastly, we deal with the set $(0, \infty)^2 \setminus [c_0 \epsilon^{\frac{2}{3}}, \infty) \times (0, \infty)$ by making use of the isoperimetric inequality $c_0 l_2(V, V', t)^{\frac{2}{3}} \leq x^{-1}_{t, l_2(V, V', t)}(x_{t,V}(A) + x_{t,V'}(A')) < c_0 \epsilon^{\frac{2}{3}}$. Thus, $l_2(V, V', t) < \epsilon$ and we proceed as before. □

Proposition A.3.6. *Let $g_{\mathrm{in},R}$ as stated in Proposition A.3.1 and K_R as in (A.3.18). Assume $F, G \in \mathrm{Y}_{\epsilon,\tau}$ with τ as in (A.3.20) and (A.3.21). Then*

1. $\|J[G]\| \leq 2M$;
2. *The map J is contractive, more explicitly,* $\|J[F] - J[G]\| \leq \frac{1}{2}\|F - G\|$.

In particular, Proposition A.3.5 and Proposition A.3.6 imply that $J[G] \in \mathrm{Y}_{\epsilon,\tau}$.

The proof consists in a combination of standard methods used in the study of coagulation equations and some of the properties proven in Proposition A.3.3. A detailed proof is given in Appendix A.6.

Later on, we will prove moment estimates for higher order powers of a. For this, it is useful to keep in mind that the above computations can be done in a more general case.

We now use Banach fixed point theorem to conclude that there exists a fixed point in the space $\mathrm{Y}_{\epsilon,\tau}$ for the map J, which we will denote by $G_{\epsilon,R,\delta}$. We will extend the solution to arbitrary times. To do so, we show that the previous computations can be done if we replace $g_{\mathrm{in},R}$ with $G_{\epsilon,R,\delta}(\cdot, \cdot, \tau)$ and then use induction.

Proposition A.3.7. *Let $G_{\epsilon,R,\delta}$ be the found fixed point for (A.3.23) up to time τ defined as in (A.3.20), (A.3.21) and with initial datum $g_{\mathrm{in},R}$ taken as in Proposition A.3.1. For any $T > 0$, there exists a unique solution, for which we keep the notation $G_{\epsilon,R,\delta} \in \mathrm{C}^1([0, T]; \mathscr{M}^I_{+,\mathrm{b}}(\mathbb{R}^2_{>0}))$ that satisfies (A.3.19).*

As before, the proof consists in a combination of standard methods used in the study of coagulation equations and some of the properties proven in Proposition A.3.3. A detailed proof of this proposition is given in Appendix A.6.

Passage to the initial equation and properties of the semigroup

Definition A.3.8. Let $T > 0$. Let $G_{\epsilon,R,\delta}$ be as in Proposition A.3.7, that is $G_{\epsilon,R,\delta} \in \mathrm{C}^1([0, T]; \mathscr{M}^I_{+}(\mathbb{R}^2_{>0}))$ with $G_{\epsilon,R,\delta}\left(\mathbb{R}^2_{>0} \setminus [c_0 \epsilon^{\frac{2}{3}}, \infty) \times [\epsilon, 2Re^t), t\right) = 0$, for every $t \in [0, T]$. We define $g_{\epsilon,R,\delta} \in$

$\mathrm{C}^1([0,T];\mathscr{M}_+(\mathbb{R}^2_{>0}))$ in the following manner:

$$\int_{(0,\infty)^2} g_{\epsilon,R,\delta}(a,v,t)\varphi(a,v)\mathrm{d}v\mathrm{d}a = \int_{(0,\infty)^2} \mathrm{e}^{h_\epsilon(V,t)} G_{\epsilon,R,\delta}(A,V,t)\varphi(\phi_t(A,V))\mathrm{d}V\mathrm{d}A, \tag{A.3.24}$$

for every $\varphi \in \mathrm{C}_0(\mathbb{R}^2_{>0})$ and every $t \in [0,T]$.

Proof of Proposition A.3.1. Let $g_{\epsilon,R,\delta}$ as in Definition A.3.8. We will prove that $g_{\epsilon,R,\delta} \in \mathrm{C}^1([0,T];\mathscr{M}^I_+(\mathbb{R}^2_{>0}))$ and $g_{\epsilon,R,\delta}$ satisfies equation (A.3.5) with coagulation kernel K_R and initial value $g_{\mathrm{in},R}$. Moreover, $g_{\epsilon,R,\delta}\Big(\mathbb{R}^2_{>0} \setminus [c_0\epsilon^{\frac{2}{3}},\infty)\times[\epsilon,2R),t\Big) = 0$, for every $t \in [0,T]$.

Firstly, we prove that $g_{\epsilon,R,\delta}$ has the stated support. Assume $\varphi \in \mathrm{C}_0(\mathbb{R}^2_{>0})$ is supported in $\{a < c_0 v^{\frac{2}{3}}\}$. The right-hand side of (A.3.24) implies $x_{t,V}(A) < c_0 y_t(V)^{\frac{2}{3}}$. By Proposition A.3.3, we have $A < c_0 V^{\frac{2}{3}}$. Since $G_{\epsilon,R,\delta}(t) \in \mathscr{M}^I_+(\mathbb{R}^2_{>0})$, then:

$$\int_{(0,\infty)^2} g_{\epsilon,R,\delta}(a,v,t)\varphi(a,v)\mathrm{d}v\mathrm{d}a = 0.$$

A similar argument can be used to prove $g_{\epsilon,R,\delta}\Big(\mathbb{R}^2_{>0} \setminus [c_0\epsilon^{\frac{2}{3}},\infty)\times[\epsilon,2R),t\Big) = 0$, for every $t \in [0,T]$.

We now prove that $g_{\epsilon,R,\delta}$ satisfies (A.3.5). We make use of (A.3.14) and (A.3.24).

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_{(0,\infty)^2} g_{\epsilon,R,\delta}(a,v,t)\varphi(a,v,t)\mathrm{d}v\mathrm{d}a &= \frac{\mathrm{d}}{\mathrm{d}t}\int_{(0,\infty)^2}\int_{(0,\infty)^2} \mathrm{e}^{h_\epsilon(V,t)} G_{\epsilon,R,\delta}(A,V,t)\varphi(\phi_t(A,V))\mathrm{d}V\mathrm{d}A\\
= \int_{(0,\infty)^2} \frac{\mathrm{d}}{\mathrm{d}t}\Big(\mathrm{e}^{h_\epsilon(V,t)}\varphi(\phi_t(A,V))\Big)G_{\epsilon,R,\delta}(A,V,t)\mathrm{d}V\mathrm{d}A &+ \int_{(0,\infty)^2} \mathrm{e}^{h_\epsilon(V,t)}\varphi(\phi_t(A,V))\frac{\mathrm{d}}{\mathrm{d}t}G_{\epsilon,R,\delta}(A,V,t)\mathrm{d}V\mathrm{d}A.\end{aligned}$$

The first term becomes:

$$\begin{aligned}&\int_{(0,\infty)^2} \frac{\mathrm{d}}{\mathrm{d}t}\Big(\mathrm{e}^{h_\epsilon(V,t)}\varphi(\phi_t(A,V))\Big)G_{\epsilon,R,\delta}(A,V,t)\mathrm{d}V\mathrm{d}A\\
&\quad= \int_{(0,\infty)^2}\Big((1-\gamma)r_\delta(\phi_t(A,V))(c_0 y_t(V)^{\frac{2}{3}} - x_{t,V}(A))\partial_1\varphi(\phi_t(A,V))\\
&\quad- \frac{2}{3}\Theta_\epsilon(y_t(V))x_{t,V}(A)\partial_1\varphi(\phi_t(A,V)) - \Theta_\epsilon(y_t(V))y_t(V)\partial_2\varphi(\phi_t(A,V))\\
&\quad+ \Theta_\epsilon(y_t(V))\varphi(\phi_t(A,V))\Big)\mathrm{e}^{h_\epsilon(V,t)}G_{\epsilon,R,\delta}(A,V,t)\mathrm{d}V\mathrm{d}A,\end{aligned}$$

where we used (A.3.17). For the second term, we have:

$$\begin{aligned}&\int_{(0,\infty)^2} \mathrm{e}^{h_\epsilon(V,t)}\varphi(\phi_t(A,V))\frac{\mathrm{d}}{\mathrm{d}t}G_{\epsilon,R,\delta}(A,V,t)\mathrm{d}V\mathrm{d}A =\\
&\quad\frac{1-\gamma}{2}\int_{(0,\infty)^2}\int_{(0,\infty)^2} K_R(\phi_t(A,V),\phi_t(A',V'))\xi_R(y_t(V)+y_t(V'))G_{\epsilon,R,\delta}(A',V',t)G_{\epsilon,R,\delta}(A,V,t)\\
&\quad\mathrm{e}^{h_\epsilon(V,t)+h_\epsilon(V',t)}[\varphi(\phi_t(A,V)+\phi_t(A',V')) - \varphi(\phi_t(A,V)) - \varphi(\phi_t(A',V'))]\mathrm{d}V'\mathrm{d}A'\mathrm{d}V\mathrm{d}A.\end{aligned}$$

Using now the definition in (A.3.24), we see that $g_{\epsilon,R,\delta}$ satisfies (A.3.5). ◻

We now focus on proving the continuity in the weak topology of the semigroup defined in (A.3.11). This will be useful in order to show that there exists a fixed point in time for equation (A.3.5). In order to prove Propositions A.3.10 - A.3.13, we need better regularity for the coagulation kernel. We solve the adjoint problem for a mollified version of the coagulation kernel. A similar approach can be found in [NV12]. We then show that the difference between the terms containing the two coagulation kernels can be made small due to the uniform estimates for the total surface area.

We first define the rectangles

$$V_{\epsilon,R} := (c_0\Big(\frac{\epsilon}{2}\Big)^{\frac{2}{3}}, \infty) \times (\frac{\epsilon}{2}, 4R) \subset \mathbb{R}^2_{>0} \text{ and } \tilde{V}_{\epsilon,R} := [c_0\Big(\frac{\epsilon}{4}\Big)^{\frac{2}{3}}, \infty) \times [\frac{\epsilon}{4}, 8R] \subset \mathbb{R}^2_{>0}.$$

Assumption A.3.9. Let then $K^n_{\epsilon,R} \in \mathrm{C}^1(V_{\epsilon,R} \times V_{\epsilon,R})$ be a mollified version of $K_{\epsilon,R}\mathbb{1}_{\tilde{V}_{\epsilon,R}\times\tilde{V}_{\epsilon,R}}(\eta,\eta')$ chosen in such a way that

$$\sup_{(\eta,\eta')\in K} |K_{\epsilon,R}(\eta,\eta') - K^n_{\epsilon,R}(\eta,\eta')| \leq \frac{1}{n},$$

for some $n \in \mathbb{N}$ sufficiently large, to be fixed later, and some fixed compact set $K \subset V_{\epsilon,R} \times V_{\epsilon,R}$.

Proposition A.3.10. *Let $K^n_{\epsilon,R}$ be as in Assumption A.3.9. Let g_1, g_2 be two solutions of (A.3.5) with initial values $g_{\mathrm{in},1}, g_{\mathrm{in},2}$, respectively. Assume both initial conditions satisfy the assumptions in Proposition A.3.1. Let $T > 0$. We work with functions on the space*

$$\mathrm{W}_T := \{\varphi \in \mathrm{C}^1([0,T], \mathrm{C}^1_{\mathrm{b}}(\tilde{\mathrm{S}})) \mid \varphi(\eta, t) = 0 \text{ if } v \notin [\frac{\epsilon}{2}, 4R], \text{ for every } t \leq T\},$$

where $\tilde{\mathrm{S}}$ was defined in (A.1.2). Let $\chi(\eta)$ be an arbitrary function in $\mathrm{C}^1_{\mathrm{b}}(\tilde{S})$ that is zero when $v \notin [\frac{\epsilon}{2}, 4R]$. Then, for every $T > 0$, there exists a unique solution $\varphi \in \mathrm{W}_T$, with $\varphi(\cdot, T) = \chi(\cdot)$, which solves the following equation:

$$\partial_t\varphi(\eta,t) + \Theta_\epsilon(v)\Big(\varphi(\eta,t) - v\partial_v\varphi(\eta,t) - \frac{2}{3}a\partial_a\varphi(\eta,t)\Big) + (1-\gamma)r_\delta(a,v)(c_0v^{\frac{2}{3}} - a)\partial_a\varphi(\eta,t) + \mathbb{L}(\varphi)(\eta,t) = 0,$$

where

$$\mathbb{L}(\varphi(\eta,t))(\eta,t) := \frac{1-\gamma}{2}\int_{(0,\infty)^2} K^n_{\epsilon,R}(\eta,\eta')\xi_R(v+v')\chi_\varphi(\eta,\eta',t)(g_1(\eta',t) + g_2(\eta',t))\mathrm{d}\eta'.$$

Proof. Let us start by integrating along the characteristics. This means that it is enough to prove that there exists a function $\tilde{\varphi}$ that satisfies the equation:

$$\partial_t\tilde{\varphi}(\eta,t) + \Theta_\epsilon(y_t(v))\tilde{\varphi}(\eta,t) + \mathbb{L}(\tilde{\varphi}(\eta,t))(\phi_t(\eta),t) = 0,$$

where $\phi_t(\eta)$ is defined as in (A.3.14).

We consider a modified version of the operator $\mathbb{L}(\varphi)$, which is possible due to the way the kernel was truncated, as well as the support of g_1, g_2. Let $\chi_{\epsilon,R}(v)$ be a continuous function which is equal to zero when $v \leq \frac{\epsilon}{2}$ and when $v \geq 4R$ and is equal to 1, when $v \in [\epsilon, 2R)$. Instead of working with $\mathbb{L}(\varphi)$, we work with:

$$\overline{\mathbb{L}}(\varphi(\eta,t))(\eta,t) := \chi_{\epsilon,R}(v)\mathbb{L}(\varphi(\eta,t))(\eta,t).$$

We observe that, if φ is continuous and compactly supported in the v variable, then $\overline{\mathbb{L}}(\varphi)(\eta,t)$ is continuous and compactly supported in the v variable. We emphasize the fact that, since g_1 and g_2 are supported in the region where $\{a \geq c_0 v^{\frac{2}{3}}\}$, the space of functions in $\mathrm{C}([0,T], \mathrm{C}^1_{\mathrm{b}}(\tilde{\mathrm{S}}))$ will be enough to prove continuity of the semigroup in the weak-* topology later on. We thus prove that there exists a $\tilde{\varphi}$ that satisfies the equation

$$\partial_t\tilde{\varphi}(\eta,t) + \Theta_\epsilon(y_t(v))\tilde{\varphi}(\eta,t) + \overline{\mathbb{L}}(\tilde{\varphi}(\eta,t))(\phi_t(\eta),t) = 0. \tag{A.3.25}$$

This will impose no problems when we prove the continuity in the weak-* topology of the semigroup, since g_1 and g_2 are supported outside the region where $\overline{\mathbb{L}}(\varphi) \neq \mathbb{L}(\varphi)$. We prove the existence of a $\tilde{\varphi}$ that satisfies (A.3.25) via a fixed-point argument in the space

$$\tilde{W}_T := \{\tilde{\varphi} \in \mathrm{C}^1([0,T], \mathrm{C}^1_{\mathrm{b}}(\tilde{\mathrm{S}})) | \tilde{\varphi}(\eta,t) = 0 \text{ if } v \leq \frac{\epsilon}{2} \text{ or } v \geq 4R\mathrm{e}^t, \text{ for every } t \leq T\}.$$

We use the fact that the kernel is bounded and that

$$\int_{(0,\infty)^2} [g_1(\eta',t) + g_2(\eta',t)]\mathrm{d}\eta' \leq \int_{(0,\infty)^2} [g_{\mathrm{in},1}(\eta') + g_{\mathrm{in},2}(\eta')]\mathrm{d}\eta'$$

in order to prove contractivity. To prove that $\varphi(\cdot,t) \in \mathrm{C}^1_{\mathrm{b}}(\tilde{\mathrm{S}})$, for $t \in [0,T]$, we first prove that there exists a constant $C(t) > 0$ such that

$$C(t) \leq \partial_a x_{t,v}(a) \leq 1. \tag{A.3.26}$$

This is since

$$0 \leq \partial_a[r_\delta(a,v)(a - c_0 v^{\frac{2}{3}})] = \partial_a r_\delta(a,v)(a - c_0 v^{\frac{2}{3}}) + r_\delta(a,v). \tag{A.3.27}$$

By (A.1.10) and (A.1.13), there exists a constant $C_{\epsilon,R,\delta} > 0$ such that

$$\begin{aligned}
\partial_a r_\delta(a,v)(a - c_0 v^{\frac{2}{3}}) &= \Big[\frac{\partial_a r(a,v)}{1+\delta a^\mu} - \frac{r(a,v)\frac{\mu\delta a^{\mu-1}}{1+\delta a^\mu}}{1+\delta a^\mu}\Big]\frac{\max\{v^\sigma, L\delta\}}{v^\sigma}(a - c_0 v^{\frac{2}{3}}) \\
&\leq \frac{\partial_a r(a,v)}{1+\delta a^\mu}\frac{\max\{v^\sigma, L\delta\}}{v^\sigma}(a - c_0 v^{\frac{2}{3}}) \\
&\leq \Big[\frac{Ba^{-1}r(a,v)}{1+\delta a^\mu}(a - c_0 v^{\frac{2}{3}})\Big]\frac{\max\{v^\sigma, L\delta\}}{v^\sigma} \\
&\leq \Big[\frac{BR_1 a^{\mu-1}v^\sigma}{1+\delta a^\mu}(a - c_0 v^{\frac{2}{3}})\Big]\frac{\max\{v^\sigma, L\delta\}}{v^\sigma} \\
&\leq \frac{BR_1 a^\mu v^\sigma}{1+\delta a^\mu}\frac{\max\{v^\sigma, L\delta\}}{v^\sigma} \leq \frac{BR_1}{\delta}\max\{v^\sigma, L\delta\} \leq C_{\epsilon,R,\delta}
\end{aligned} \tag{A.3.28}$$

and

$$r_\delta(a,v) = \frac{r(a,v)}{1+\delta a^\mu}\frac{\max\{v^\sigma, L\delta\}}{v^\sigma} \leq \frac{R_1 a^\mu v^\sigma}{1+\delta a^\mu}\frac{\max\{v^\sigma, L\delta\}}{v^\sigma} \leq C_{\epsilon,R,\delta}. \tag{A.3.29}$$

Combining (A.3.27)-(A.3.29) and making use of (A.3.14), we obtain (A.3.26). Using (A.3.25) and (A.3.26), we prove the desired bound for the derivative of φ. □

Remark A.3.11. Let g_1, g_2 be two solutions of (A.3.5) with initial values $g_{\text{in},1}, g_{\text{in},2} \in U_{\epsilon,R}$, respectively. Let $T > 0$. Let $\varphi \in \mathrm{C}^1([0,T], \mathrm{C}^1(\tilde{\mathrm{S}}))$ such that $\varphi(\eta, T) = \chi(\eta)$ be the function found in Proposition A.3.10. Assume, in addition, that $\sup_{\eta\in\tilde{\mathrm{S}}} |\chi(\eta)| \leq 1$. Then there exists a constant $C_{\epsilon,R}(T)$, which is independent of the choice of $n \in \mathbb{N}$ in Assumption A.3.9 and depends only on the norm of g_1 and g_2, T, ϵ and R, such that

$$\sup_{s\in[0,T]} \sup_{\eta\in\tilde{\mathrm{S}}} |\varphi(\eta, s)| \leq C_{\epsilon,R}(T). \tag{A.3.30}$$

We now prove that the solution φ that we have found in Proposition A.3.10 is Lipschitz continuous. This will provide a suitable compactness property that will allow us to prove the desired weak continuity of the mapping $S(t) : U_{\epsilon,R} \to U_{\epsilon,R}$ defined in (A.3.11).

Proposition A.3.12. *Let g_1, g_2 be two solutions of (A.3.5) with initial values $g_{\text{in},1}, g_{\text{in},2} \in U_{\epsilon,R}$, respectively. Let $T > 0$. Let $\varphi \in \mathrm{C}^1([0,T], \mathrm{C}^1(\tilde{\mathrm{S}}))$ such that $\varphi(\eta, T) = \chi(\eta)$ be the function found in Proposition A.3.10. Assume, in addition, that $\sup_{\eta\in\tilde{\mathrm{S}}} |\chi(\eta)| \leq 1$ and that $\chi(\eta)$ is Lipschitz. Then φ is Lipschitz continuous, in the sense that, for every $t \in [0,T]$, there exists $C(t) > 0$ such that*

$$\sup_{s\in[0,t]} |\varphi(\eta, s) - \varphi(\tilde{\eta}, s)| \leq C(t)|\eta - \tilde{\eta}|,$$

for every $\eta, \tilde{\eta} \in \tilde{\mathrm{S}}$. Moreover, $C(t)$ may depend on the norm of g_1 and g_2, but is otherwise independent of the choice of g_1 and g_2.

Proof. Notice first that, since $\sup_{\eta\in\tilde{\mathrm{S}}} |\chi(\eta)| \leq 1$, then $\sup_{s\in[0,t],\eta\in\tilde{\mathrm{S}}} |\varphi(\eta, s)| \leq C_{\epsilon,R}(T)$ by (A.3.30). We use Grönwall in (A.3.25):

$$\begin{aligned}|\tilde{\varphi}(\eta, t) - \tilde{\varphi}(\tilde{\eta}, t)| \leq |\chi(\eta) - \chi(\tilde{\eta})| &+ \sup_{z\in[0,t],\eta\in\tilde{\mathrm{S}}} |\tilde{\varphi}(\eta, z)| \int_t^T |\Theta_\epsilon(y_s(v)) - \Theta_\epsilon(y_s(\tilde{v}))| \mathrm{d}s \\ &+ \int_t^T |\tilde{\varphi}(\eta, s) - \tilde{\varphi}(\tilde{\eta}, s)| \mathrm{d}s + \int_t^T |\overline{\mathbb{L}}(\tilde{\varphi}(\eta, s) - \tilde{\varphi}(\tilde{\eta}, s))(\phi_s(\eta), s)| \mathrm{d}s \\ &+ \int_t^T |\overline{\mathbb{L}}(\tilde{\varphi}(\eta, s))(\phi_s(\eta) - \phi_s(\tilde{\eta}), s)| \mathrm{d}s.\end{aligned}$$

By the definition of $K^n_{\epsilon,R}$ in Assumption A.3.9, we have its first order derivatives are bounded from above. Moreover, $\phi_s(\eta)$ is Lipschitz continuous. Thus, there exists a constant $C > 0$, which can depend on ϵ, R, δ and the norms of g_1 and g_2 such that

$$|\tilde{\varphi}(\eta, t) - \tilde{\varphi}(\tilde{\eta}, t)| \leq C \int_t^T |\tilde{\varphi}(\eta, s) - \tilde{\varphi}(\tilde{\eta}, s)| + C|\eta - \tilde{\eta}|.$$

We use Grönwall and obtain $|\tilde{\varphi}(\eta, t) - \tilde{\varphi}(\tilde{\eta}, t)| \leq C(t)|\eta - \tilde{\eta}|$. Combining this with (A.3.26), we conclude that $|\varphi(\eta, t) - \varphi(\tilde{\eta}, t)| \leq C(t)|\eta - \tilde{\eta}|$. □

This is enough to enable us to prove that the semigroup defined in (A.3.11) is continuous in the weak-* topology.

Proposition A.3.13. *Let $K^n_{\epsilon,R}$ be as in Assumption A.3.9. Let $t \geq 0$ and $S(t) : U_{\epsilon,R} \to U_{\epsilon,R}$ be the mapping defined in (A.3.11). Then, we have that $S(t)$ is continuous in the weak-* topology.*

Proof. As before, assume that g_1, g_2 are two solutions of (A.3.5) with initial conditions $g_{\text{in},1}, g_{\text{in},2} \in U_{\epsilon,R}$, respectively. Assume furthermore that there exists a constant $C > 0$ such that

$$\int_{(0,\infty)^2} (1 + a)(g_{\text{in},1}(\eta) + g_{\text{in},2}(\eta))\mathrm{d}\eta \leq C.$$

Let $\phi \in \mathrm{C}^1_0(\mathbb{R}^2_{>0})$ with $\phi(\eta) = 0$ if $(a, v) \notin [c_0(\frac{\epsilon}{2})^{\frac{2}{3}}, L] \times [\frac{\epsilon}{2}, 4R]$, where ϵ, R are as in Proposition A.3.1 and for some $L > 0$. Assume in addition that $\|\phi\|_\infty \leq 1$. Let $\tilde{\delta} < 1$. Our goal is to prove that, if for every $\chi \in \mathrm{C}_0(\mathbb{R}^2_{>0})$, with $\|\chi\|_\infty \leq 1$,

$$\int_{(0,\infty)^2} [g_{\text{in},2}(\eta) - g_{\text{in},1}(\eta)]\chi(\eta)\mathrm{d}\eta \text{ is sufficiently small,}$$

then for every $\chi \in \mathrm{C}_0(\mathbb{R}^2_{>0})$, with $\|\chi\|_\infty \leq 1$,

$$\int_{(0,\infty)^2} [g_2(\eta, t) - g_1(\eta, t)]\chi(\eta)\mathrm{d}\eta < \tilde{\delta}.$$

We make the following notation:

$$C_{\text{norm}} := \sup_{s\in[0,t]} \int_{(0,\infty)^2} (1 + a)(g_1(\eta, s) + g_2(\eta, s))\mathrm{d}\eta < \infty.$$

Notice that C_{norm} may depend on the time t, but is independent of the choice of $g_1(\eta, s), g_2(\eta, s)$, for $s \in [0, t]$. This is since we can bound

$$\sup_{s\in[0,t]} \int_{(0,\infty)^2} (1 + a)(g_1(\eta, s) + g_2(\eta, s))\mathrm{d}\eta \leq C(t) \int_{(0,\infty)^2} (1 + a)(g_{\text{in},1}(\eta) + g_{\text{in},2}(\eta))\mathrm{d}\eta$$

by using similar arguments to the ones used in Proposition A.3.7.

We fix $n \in \mathbb{N}$ in Assumption A.3.9 such that $\frac{3tC_{\epsilon,R}(t)C^2_{\text{norm}}}{n} \leq \frac{\tilde{\delta}}{4}$, where $C_{\epsilon,R}(t)$ is defined as in (A.3.30).

Let φ be the function found in Proposition A.3.10 associated to the coagulation kernel $K^n_{\epsilon,R}$, for $n \in \mathbb{N}$ as above, and with $\varphi(\eta, t) = \phi(\eta)$. Since $\phi \in \mathrm{C}^1_c(\mathbb{R}^2_{>0})$, then by Proposition A.3.12 the function

φ is Lipschitz continuous. Then there exists $\tilde{C}(\epsilon, R, \delta, t, C_{\text{norm}}) > 0$ such that $|\varphi(\eta, s) - \varphi(\tilde{\eta}, s)| \leq \tilde{C}(\epsilon, R, \delta, t, C_{\text{norm}})|\eta - \tilde{\eta}|$, for every $\eta, \tilde{\eta} \in \tilde{\mathrm{S}}$ and $s \in [0, t]$.

Assume $M > 1$ is a fixed constant whose value will be determined later. We take $\chi_M : \mathbb{R}_{>0} \to [0, 1]$, continuous, to be

$$\chi_M(a) = \begin{cases} 1, & \text{when } a \in [c_0\epsilon^{\frac{2}{3}}, M], \\ 0, & \text{when } a \notin [c_0(\frac{\epsilon}{2})^{\frac{2}{3}}, 2M]. \end{cases} \tag{A.3.31}$$

Then there exists $C(\epsilon, R, \delta, t, C_{\text{norm}}) > 0$ such that $|\varphi(\eta, s)\chi_M(a) - \varphi(\tilde{\eta}, s)\chi_M(\tilde{a})| \leq C(\epsilon, R, \delta, t, C_{\text{norm}})$ $|\eta - \tilde{\eta}|$, for every $\eta, \tilde{\eta} \in \tilde{\mathrm{S}}$ and $s \in [0, t]$.

We look at the set

$$K = \{\tilde{\varphi} \in \mathrm{C}(\tilde{S})|\ \tilde{\varphi}(\eta) = 0 \text{ if } \eta \notin [c_0(\frac{\epsilon}{2})^{\frac{2}{3}}, 2M] \times [\frac{\epsilon}{2}, 4R];\ |\tilde{\varphi}(\eta) - \tilde{\varphi}(\eta')| \leq C(\epsilon, R, \delta, t, C_{\text{norm}})|\eta - \eta'|, \\ \text{for all } \eta, \eta' \in \tilde{S}\}.$$

As the set is totally bounded, there exist $N \in \mathbb{N}$ and $\psi_1, \ldots, \psi_N \in K$ with the property that $K \subseteq \cup_{i=1}^{N} B(\psi_i, \frac{\tilde{\delta}}{4C_{\text{norm}}})$. As $\varphi(\cdot, 0)\chi_M \in K$, then $\min_{i=\overline{1,N}} \sup_{\eta \in \tilde{S}} |\varphi(\eta, 0)\chi_M(a) - \psi_i(\eta)| \leq \frac{\tilde{\delta}}{4C_{\text{norm}}}$. Assume that, for every $\chi \in \mathrm{C}_0(\mathbb{R}^2_{>0})$, with $||\chi||_\infty \leq 1$,

$$\int_{(0,\infty)^2} [g_{\text{in},2}(\eta) - g_{\text{in},1}(\eta)]\chi(\eta)\mathrm{d}\eta < \frac{\tilde{\delta}}{8N}.$$

We then have

$$\begin{aligned}
&\int_{(0,\infty)^2} [g_2 - g_1](\eta, t)\phi(\eta)\mathrm{d}\eta = \int_{(0,\infty)^2} [g_2 - g_1](\eta, t)\varphi(\eta, t)\mathrm{d}\eta \\
&\quad = \int_{(0,\infty)^2} [g_{2,\text{in}} - g_{1,\text{in}}](\eta)\varphi(\eta, 0)\mathrm{d}\eta + \frac{1-\gamma}{2}\int_0^t \int_{(0,\infty)^2} \int_{(0,\infty)^2} (K_{\epsilon,R}(\eta, \eta') - K^n_{\epsilon,R}(\eta, \eta'))\xi_R(v + v') \\
&\quad \chi_\varphi(\eta, \eta', s)(g_1(\eta', s) + g_2(\eta', s))(g_1(\eta', s) - g_2(\eta', s))\mathrm{d}\eta'\mathrm{d}\eta\mathrm{d}s \\
&\quad \leq \int_{(0,\infty)^2} [g_{2,\text{in}} - g_{1,\text{in}}](\eta)\varphi(\eta, 0)\mathrm{d}\eta + \int_0^t \int_{V_{\epsilon,R,M}} \int_{V_{\epsilon,R,M}} |(K_{\epsilon,R}(\eta, \eta') - K^n_{\epsilon,R}(\eta, \eta'))\xi_R(v + v') \\
&\quad \chi_\varphi(\eta, \eta', s)(g_1(\eta', s) + g_2(\eta', s))(g_1(\eta', s) - g_2(\eta', s))|\mathrm{d}\eta'\mathrm{d}\eta\mathrm{d}s \\
&\quad + \int_0^t \int_{(M,\infty)\times[\epsilon,2R]} \int_{(0,\infty)^2} |(K_{\epsilon,R}(\eta, \eta') - K^n_{\epsilon,R}(\eta, \eta'))\xi_R(v + v')\chi_\varphi(\eta, \eta', s)(g_1(\eta', s) + g_2(\eta', s)) \\
&\quad (g_1(\eta', s) - g_2(\eta', s))|\mathrm{d}\eta'\mathrm{d}\eta\mathrm{d}s \\
&\quad = I_1 + I_2 + I_3,
\end{aligned} \tag{A.3.32}$$

where $V_{\epsilon,R,M} := [c_0\epsilon^{\frac{2}{3}}, M] \times [\epsilon, 2R]$, for some sufficiently large M as in (A.3.31).

We bound the term I_1 in (A.3.32) by:

$$\begin{aligned}
&\int_{(0,\infty)^2}[g_{2,\text{in}}-g_{1,\text{in}}](\eta)\varphi(\eta,0)\mathrm{d}\eta\\
&\quad=\int_{(0,\infty)^2}[g_{2,\text{in}}-g_{1,\text{in}}](\eta)\varphi(\eta,0)\chi_M(a)\mathrm{d}\eta+\int_{(0,\infty)^2}[g_{2,\text{in}}-g_{1,\text{in}}](\eta)\varphi(\eta,0)[1-\chi_M(a)]\mathrm{d}\eta\\
&\quad\leq\int_{(0,\infty)^2}[g_{2,\text{in}}-g_{1,\text{in}}](\eta)\varphi(\eta,0)\chi_M(a)\mathrm{d}\eta+2\|\varphi(\eta,0)\|_\infty\int_{a>M}[g_{2,\text{in}}+g_{1,\text{in}}](\eta)\mathrm{d}\eta\\
&\quad\leq\min_{i=\overline{1,N}}\int_{(0,\infty)^2}|[g_{2,\text{in}}-g_{1,\text{in}}](\eta)||\varphi(\eta,0)\chi_M(a)-\psi_i(\eta)|\mathrm{d}\eta\\
&\quad+\max_{i=\overline{1,N}}\int_{(0,\infty)^2}[g_{2,\text{in}}-g_{1,\text{in}}](\eta)\psi_i(\eta)\mathrm{d}\eta+2M^{-1}C_{\text{norm}}\|\varphi(\eta,0)\|_\infty.
\end{aligned}\tag{A.3.33}$$

Notice that K was chosen independently of g_1,g_2 since it depends only on $C(\epsilon,R,\delta,t,C_{\text{norm}})$. Thus we can bound $\max_{i=\overline{1,N}}\int_{(0,\infty)^2}[g_{2,\text{in}}-g_{1,\text{in}}](\eta)\psi_i(\eta)\mathrm{d}\eta$ in (A.3.33) by $\frac{\tilde{\delta}}{8}$ independently of the choice of g_1,g_2, as $\psi_i\in K$, $i=\overline{1,N}$. As $\|\phi(\eta)\|_\infty\leq 1$, then $\sup_{s\in[0,t]}\|\varphi(\cdot,s)\|_\infty\leq C_{\epsilon,R}(t)$. We can then choose $M>\frac{16C_{\text{norm}}C_{\epsilon,R}(t)}{\tilde{\delta}}$ in order to bound $2M^{-1}C_{\text{norm}}\|\varphi(\eta,0)\|_\infty$. This means that

$$\int_{(0,\infty)^2}[g_{2,\text{in}}-g_{1,\text{in}}](\eta)\varphi(\eta,0)\mathrm{d}\eta\leq\frac{\tilde{\delta}}{4C_{\text{norm}}}\int_{(0,\infty)^2}[g_{2,\text{in}}+g_{1,\text{in}}](\eta)\mathrm{d}\eta+\frac{\tilde{\delta}}{4}\leq\frac{\tilde{\delta}}{2}.\tag{A.3.34}$$

In order to bound I_2 in (A.3.32), we have that on $([c_0\epsilon^{\frac{2}{3}},M]\times[\epsilon,2R])^2$

$$\sup_{(\eta,\eta')\in([c_0\epsilon^{\frac{2}{3}},M]\times[\epsilon,2R])^2}|K^n_{\epsilon,R}(\eta,\eta')-K_{\epsilon,R}(\eta,\eta')|\leq\frac{1}{n}$$

by Assumption A.3.9. Thus $I_2\leq\frac{3t\|\varphi\|_\infty C^2_{\text{norm}}}{n}\leq\frac{\tilde{\delta}}{4}$.

For I_3 in (A.3.32), we take $M\geq 6C_{\epsilon,R}(t)C^2_{\text{norm}}t\|K_{\epsilon,R}\|_\infty\frac{4}{\tilde{\delta}}$. This is because I_3 can be estimated from above by

$$M^{-1}3tC_{\epsilon,R}(t)(\|K^n_{\epsilon,R}\|_\infty+\|K_{\epsilon,R}\|_\infty)\sup_{s\in[0,t]}\int_{\{a\geq M\}}a(g_1(\eta,s)+g_2(\eta,s))\mathrm{d}\eta\int_{(0,\infty)^2}(g_1(\eta,s)+g_2(\eta,s))\mathrm{d}\eta'.$$

We then combine this with (A.3.34) and our choice of $n\in\mathbb{N}$ in (A.3.32) in order to conclude the argument.

To extend this argument to all functions $\phi\in C_0(\mathbb{R}^2_{>0})$ we use again that $(M_{0,1}+M_{0,0})(g_1(s)+g_2(s))<\infty$, for all $s\in[0,t]$. □

For $g\in U_{\epsilon,R}$, it is worthwhile to observe that the map $S(\cdot)g:[0,T]\to U_{\epsilon,R}$ defined in (A.3.11) is also continuous in time.

Proposition A.3.14. *Let $g\in U_{\epsilon,R}$, with $U_{\epsilon,R}$ defined in (A.3.10) and $T>0$. Let $S(\cdot)g:[0,T]\to U_{\epsilon,R}$ be as in (A.3.11). Then $S(\cdot)g$ is continuous in time.*

The proof consists of standard methods used in the study of coagulation equations. A proof of it can be found in Appendix A.7.

A.3.2 Existence of self-similar profiles

Assume $\mu > 0$. We focus on proving Proposition A.3.2. The following moment estimates involving the v variable (Propositions A.3.16-A.3.18) are an adaptation to the two-dimensional case of [EM06, Lemma 3.3, Lemma 3.4]. There was an additional need to control the escape to infinity of the area, which has been dealt with by choosing an appropriate norm in $\mathrm{Y}_{\epsilon,\tau}$ defined in (A.3.22), as well as a specific truncation for the coagulation kernel and fusion term. We then continue with proving an additional uniform moment estimate involving the area which will be needed for removing the truncation in (A.3.5).

Remark A.3.15. Assume $K_{\epsilon,R}$ satisfies (A.3.1). Then $K_{\epsilon,R}(a,v,a',v') \geq K_1(v^{\beta}v'^{-\alpha} + v^{-\alpha}v'^{\beta})$ if $v, v' > \epsilon$ and $v + v' < 2R$, for all $(a,a') \in (0,\infty)^2$.

We first mention that due to the choice of $\mathrm{Y}_{\epsilon,\tau}$ and r_δ, we can test (A.3.5) with functions that do not necessarily have compact support. More precisely, we can test with functions of the form $a^c v^d$, $d \in \mathbb{R}$, as long as $c \leq 1$.

As previously mentioned, the proof of Propositions A.3.16-A.3.18 is an adaptation of methods used in [EM06] to our setting. We thus only state the estimates here and move their proof to Appendix A.7.

Proposition A.3.16. *Let $g_{\mathrm{in},R} \in \mathscr{M}_+^I(\mathbb{R}^2_{>0}) \cap \omega(\epsilon,R,\delta)$. Let $g_{\epsilon,R,\delta}$ be the solution found in Proposition A.3.1 with kernel $K_{\epsilon,R}$ as in (A.3.1). Then*

$$\sup_{t\geq 0} M_{0,1}(g_{\epsilon,R,\delta}(\cdot,\cdot,t)) = M_{0,1}(g_{\mathrm{in},R}). \tag{A.3.35}$$

Moreover, there exists a constant $C_{0,\gamma} > 0$ such that:

$$\sup_{t\geq 0} M_{0,\gamma}(g_{\epsilon,R,\delta}(\cdot,\cdot,t)) \leq \max\{M_{0,\gamma}(g_{\mathrm{in},R}); C_{0,\gamma}\}, \tag{A.3.36}$$

uniformly in ϵ, R and δ.

Proposition A.3.17. *Let $g_{\mathrm{in},R}$ and $g_{\epsilon,R,\delta}$ (which we will denote by g_{in} and g, respectively) be as in Proposition A.3.16. Then, for any $m > 1$, there exists a constant $C_{0,m} > 0$, independent of ϵ, R and δ, such that:*

$$\sup_{t\geq 0} M_{0,m}(g_{\epsilon,R,\delta}(\cdot,\cdot,t)) \leq \max\{M_{0,m}(g_{\mathrm{in}}), C_{0,m}\}. \tag{A.3.37}$$

We can also obtain bounds independent of time for moments of the form $M_{0,-l}$, with $l > 0$, if we require in addition that $\alpha > 0$.

Proposition A.3.18. *Let $g_{\text{in},R}$ and $g_{\epsilon,R,\delta}$ (which we will denote by g_{in} and g, respectively) be as in Proposition A.3.16. Assume $\alpha > 0$. Let $l > 0$. Then there exists a constant $C_{0,-l} > 0$ for which the following estimate holds:*

$$\sup_{t\geq 0} M_{0,-l}(g_{\epsilon,R,\delta}(t)) \leq \max\{M_{0,-l}(g_{\text{in}}), C_{0,-l}\}, \tag{A.3.38}$$

uniformly in ϵ, R and δ.

We now find uniform estimates for the total surface area of $g_{\epsilon,R,\delta}$. The uniform estimates for the surface area are a consequence of the fusion term overtaking the coagulation operator in the case $\mu > 0$. We prove a more general statement which will be needed later on for the improvement of moment estimates.

Proposition A.3.19. *Let $g_{\text{in},R}$ and $g_{\epsilon,R,\delta}$ (which we will denote by g_{in} and g, respectively) be as in Proposition A.3.16. Then*

$$\sup_{t\geq 0} M_{1,0}(g(\cdot,\cdot,t)) \leq \max\{M_{1,0}(g_{\text{in}}), C_{1,0}\}. \tag{A.3.39}$$

Moreover, if (A.1.8) holds and if $M_{1,-\alpha}(g_{\text{in}}) < \infty$, there exists $C_{1,-\alpha} > 0$ such that:

$$\sup_{t\geq 0} M_{1,-\alpha}(g(\cdot,\cdot,t)) \leq \max\{M_{1,-\alpha}(g_{\text{in}}), C_{1,-\alpha}\}. \tag{A.3.40}$$

If (A.1.9) holds, there exists $\delta_1 > 0$, for which $\gamma \leq \frac{2}{3} - \delta_1$, and there exists $C_{1,-\delta_1} > 0$ such that:

$$\sup_{t\geq 0} M_{1,-\delta_1}(g(\cdot,\cdot,t)) \leq \max\{M_{1,-\delta_1}(g_{\text{in}}), C_{1,-\delta_1}\}. \tag{A.3.41}$$

Proof. We test equation (A.3.5) with $\varphi(a,v) = av^l$, with $l \leq 0$. This is possible due to our choice of the space $\mathrm{Y}_{\epsilon,\tau}$, which allows us to test with functions that do not have compact support if they are bounded from above by a function of the form $a^c v^d$, with $c \leq 1$ and $d \in \mathbb{R}$. Since $l \leq 0$, we have:

$$K_{\epsilon,R}(a,v,a',v')[(a+a')(v+v')^l - av^l - a'v'^l] \leq 0.$$

Equation (A.3.5) thus becomes:

$$\frac{\mathrm{d}}{\mathrm{d}t} M_{1,l}(g(t)) \leq (\frac{1}{3} + |l|) M_{1,l}(g(t)) + (1-\gamma) R_0 \int_{(0,\infty)^2} \frac{a^\mu \max\{v^\sigma, L\delta\}}{1+\delta a^\mu}(c_0 v^{\frac{2}{3}} - a) g(a,v,t) v^l \mathrm{d}v\mathrm{d}a. \tag{A.3.42}$$

We need some control over the drift term:

$$\begin{aligned}\int_{\{a\geq c_0 v^{\frac{2}{3}}\}} \frac{a^\mu \max\{v^\sigma, L\delta\}}{1+\delta a^\mu}(c_0 v^{\frac{2}{3}} - a) g(\eta,t) v^l \mathrm{d}\eta &\leq \int_{\{a\geq 2c_0 v^{\frac{2}{3}}\}} \frac{a^\mu \max\{v^\sigma, L\delta\}}{1+\delta a^\mu}(c_0 v^{\frac{2}{3}} - a) g(\eta,t) v^l \mathrm{d}\eta \\ &\leq -\frac{1}{2}\int_{\{a\geq 2c_0 v^{\frac{2}{3}}\}} \frac{a^\mu \max\{v^\sigma, L\delta\}}{1+\delta a^\mu} a v^{-|l|} g(\eta,t) \mathrm{d}\eta.\end{aligned} \tag{A.3.43}$$

This integral can now be estimated as follows:

$$\begin{aligned}
&-\frac{1}{2}\int_{\{a\geq 2c_0v^{\frac{2}{3}},a^\mu\geq\frac{1}{\delta}\}}\frac{a^\mu\max\{v^\sigma,L\delta\}}{1+\delta a^\mu}av^{-|l|}g(\eta,t)\mathrm{d}\eta-\frac{1}{2}\int_{\{a\geq 2c_0v^{\frac{2}{3}},a^\mu<\frac{1}{\delta}\}}\frac{a^\mu\max\{v^\sigma,L\delta\}}{1+\delta a^\mu}av^{-|l|}g(\eta,t)\mathrm{d}\eta\\
&\leq -\frac{1}{4\delta}\int_{\{a\geq 2c_0v^{\frac{2}{3}},a^\mu\geq\frac{1}{\delta}\}}\max\{v^\sigma,L\delta\}av^{-|l|}g(a,v,t)\mathrm{d}v\mathrm{d}a-\frac{1}{4}\int_{\{a\geq 2c_0v^{\frac{2}{3}},a^\mu<\frac{1}{\delta}\}}a^{\mu+1}v^{\sigma-|l|}g(a,v,t)\mathrm{d}v\mathrm{d}a\\
&\leq -\frac{L}{4}\int_{\{a\geq 2c_0v^{\frac{2}{3}},a^\mu\geq\frac{1}{\delta}\}}av^{-|l|}g(a,v,t)\mathrm{d}v\mathrm{d}a-\frac{1}{4}\int_{\{a\geq 2c_0v^{\frac{2}{3}},a^\mu<\frac{1}{\delta}\}}a^{\mu+1}v^{\sigma-|l|}g(a,v,t)\mathrm{d}v\mathrm{d}a. \qquad (A.3.44)
\end{aligned}$$

In order to estimate the second term on the right-hand side of (A.3.44), we apply Young's inequality. This is possible since $\mu+1>1$. Thus, we obtain:

$$av^l = av^{\frac{\sigma-|l|}{\mu+1}}v^{l-\frac{\sigma-|l|}{\mu+1}}\leq\frac{\epsilon}{p}(av^{\frac{\sigma-|l|}{\mu+1}})^{\mu+1}+\frac{\epsilon^{-\frac{q}{p}}}{q}v^{(l-\frac{\sigma-|l|}{\mu+1})q},$$

where $p=\mu+1$ and $\frac{1}{p}+\frac{1}{q}=1$. Thus, there exists a constant $C_\epsilon>0$ depending on ϵ for which $av^l\lesssim\epsilon a^{\mu+1}v^{\sigma-|l|}+\epsilon C_\epsilon v^{l-\frac{\sigma}{\mu}}$, or equivalently:

$$-a^{\mu+1}v^{\sigma-|l|}\lesssim-\frac{1}{\epsilon}av^l+C_\epsilon v^{l-\frac{\sigma}{\mu}}$$

and this implies:

$$-\int_{\{a\geq 2c_0v^{\frac{2}{3}},a^\mu<\frac{1}{\delta}\}}a^{\mu+1}v^{\sigma-|l|}g(a,v,t)\mathrm{d}v\mathrm{d}a\lesssim-\frac{1}{\epsilon}\int_{\{a\geq 2c_0v^{\frac{2}{3}},a^\mu<\frac{1}{\delta}\}}av^lg(a,v,t)\mathrm{d}v\mathrm{d}a+C_\epsilon M_{0,l-\frac{\sigma}{\mu}}(g(t)). \qquad (A.3.45)$$

From Propositions A.3.16-A.3.18, we have that $M_{0,l-\frac{\sigma}{\mu}}(g(t))$ is uniformly bounded from above. We combine (A.3.43), (A.3.44) and (A.3.45) and then we choose ϵ sufficiently small in order to obtain that

$$\int_{\{a\geq c_0v^{\frac{2}{3}}\}}\frac{a^\mu\max\{v^\sigma,L\delta\}}{1+\delta a^\mu}(c_0v^{\frac{2}{3}}-a)g(\eta,t)v^l\mathrm{d}\eta\lesssim-\frac{L}{4}\int_{\{a\geq 2c_0v^{\frac{2}{3}}\}}av^{-|l|}g(a,v,t)\mathrm{d}v\mathrm{d}a+C_\epsilon. \qquad (A.3.46)$$

Moreover, in order to pass back to the region $\{a\geq c_0v^{\frac{2}{3}}\}$ in the integral term on the right-hand side of (A.3.46), we notice that:

$$\begin{aligned}
-\int_{\{a\geq 2c_0v^{\frac{2}{3}}\}}av^lg(a,v,t)\mathrm{d}v\mathrm{d}a &= -\int_{\{a\geq c_0v^{\frac{2}{3}}\}}av^lg(a,v,t)\mathrm{d}v\mathrm{d}a+\int_{\{c_0v^{\frac{2}{3}}\leq a<2c_0v^{\frac{2}{3}}\}}av^lg(a,v,t)\mathrm{d}v\mathrm{d}a\\
&\leq -\int_{\{a\geq c_0v^{\frac{2}{3}}\}}av^lg(a,v,t)\mathrm{d}v\mathrm{d}a+2c_0\int_{(0,\infty)^2}v^{\frac{2}{3}+l}g(a,v,t)\mathrm{d}v\mathrm{d}a. \qquad (A.3.47)
\end{aligned}$$

Combining (A.3.42), (A.3.46) and (A.3.47), we deduce that

$$\frac{\mathrm{d}}{\mathrm{d}t}M_{1,l}(g(t))\lesssim(\frac{1}{3}+|l|)M_{1,l}(g(t))-(1-\gamma)R_0\frac{L}{4}M_{1,l}(g(t))+C_\epsilon. \qquad (A.3.48)$$

Assume $\alpha>0$. If $l=0$ we conclude using a comparison argument since $(1-\gamma)R_0\frac{L}{4}=3$ cf. (A.3.8). The proof remains valid if we choose $l=-\alpha$.

If (A.1.9) holds and we choose $l=-\delta_1$ to be such that $\frac{2}{3}-\delta_1\geq\gamma$, we have that $M_{0,\frac{2}{3}+l}$ and $M_{0,\frac{|\sigma|}{\mu}+l}$ are uniformly bounded from above and we can conclude in a similar manner. □

Proof of Proposition A.3.2. Let $T > 0$. Let $S(t) : U_{\epsilon,R} \to U_{\epsilon,R}$, for $t \in [0,T]$, be as in (A.3.11). Proposition A.3.1 guarantees that the semigroup is well-defined. Proposition A.3.14 gives us continuity in time of the mapping $S(\cdot)g : [0,T] \to U_{\epsilon,R}$, for $g \in U_{\epsilon,R}$. Propositions A.3.16 - A.3.19 prove that $S(t)\omega(\epsilon,R,\delta) \subseteq \omega(\epsilon,R,\delta)$ if we choose the constants $c_{0,-\alpha-\tilde{\epsilon}}$, $c_{0,\tilde{m}}$, $c_{1,0} > 0$ in Proposition A.3.2 to correspond to the constants found in Propositions A.3.16 - A.3.19. □

We are now in a position to prove Theorem A.2.3.

Proof of Theorem A.2.3, case $\alpha > 0$. Existence of self-similar profiles. Assume first $\alpha > 0$. Let $T > 0$. Let $S(t) : U_{\epsilon,R} \to U_{\epsilon,R}$, for $t \in [0,T]$, be as in (A.3.11). For fixed $t \in [0,T]$, Proposition A.3.13 assures that the mapping $S(t) : U_{\epsilon,R} \to U_{\epsilon,R}$ is continuous in the weak-* topology. Proposition A.3.2 gives us that $S(t)\omega(\epsilon,R,\delta) \subseteq \omega(\epsilon,R,\delta)$.

Using a variant of Tykonov's fixed point theorem (see, for instance, [EMR05, Theorem 1.2]), we can find a stationary solution in time, which we denote by $\tilde{g}_{\epsilon,R,\delta}(\eta)$. As $\tilde{g}_{\epsilon,R,\delta} \in \omega(\epsilon,R,\delta)$, we can find a subsequence $(\tilde{g}_{\epsilon_k,R_k,\delta_k})_{k\geq 1}$ that converges to some g in the weak-* topology as $k \to \infty$. This argument has been used extensively to prove the existence of self-similar profiles for coagulation equations and therefore we do not give more details.

We need to prove that this g satisfies (A.2.6). Fix $\varphi \in \mathrm{C}^1_\mathrm{c}(\mathbb{R}^2_{>0})$.

As $\tilde{g}_{\epsilon_k,R_k,\delta_k} \rightharpoonup g$, as $k \to \infty$, the linear terms containing $\tilde{g}_{\epsilon_k,R_k,\delta_k}$ will converge to the linear terms in (A.2.6). We deal with the fusion term in the following manner:

$$\begin{aligned}&\left|\int_{(0,\infty)^2} r_\delta(\eta)(c_0 v^{\frac{2}{3}} - a)\partial_a\varphi(\eta)\tilde{g}_{\epsilon_k,R_k,\delta_k}(\mathrm{d}\eta) - \int_{(0,\infty)^2} r(\eta)(c_0 v^{\frac{2}{3}} - a)\partial_a\varphi(\eta)g(\mathrm{d}\eta)\right| \\ &\leq C\int_{(0,\infty)^2} |r_\delta(\eta) - r(\eta)||\partial_a\varphi(\eta)|\tilde{g}_{\epsilon_k,R_k,\delta_k}(\mathrm{d}\eta) + \left|\int_{(0,\infty)^2} r(\eta)(c_0 v^{\frac{2}{3}} - a)\partial_a\varphi(\eta)[g - \tilde{g}_{\epsilon_k,R_k,\delta_k}](\mathrm{d}\eta)\right|\end{aligned} \tag{A.3.49}$$

As stated before, $r_\delta(\eta) \to r(\eta)$ for fixed η as $\delta \to 0$. Due to the compact support of φ and since $\tilde{g}_{\epsilon_k,R_k,\delta_k} \in \omega(\epsilon,R,\delta)$, the first term in (A.3.49) goes to zero as $k \to \infty$. The second term in (A.3.49) goes to zero as $k \to \infty$ since $\tilde{g}_{\epsilon_k,R_k,\delta_k} \rightharpoonup g$ as $k \to \infty$.

We now examine the coagulation term

$$\int_{(0,\infty)^2}\int_{(0,\infty)^2} K_{\epsilon,R}(\eta,\eta')\xi_R(v+v')[\varphi(\eta+\eta') - \varphi(\eta) - \varphi(\eta')]\tilde{g}_{\epsilon_k,R_k,\delta_k}(\eta)\tilde{g}_{\epsilon_k,R_k,\delta_k}(\eta')\mathrm{d}\eta'\mathrm{d}\eta. \tag{A.3.50}$$

Notice first that the support of $\tilde{g}_{\epsilon_k,R_k,\delta_k}$ is contained in the strip $v \in [\epsilon, 2R)$. We can then replace $K_{\epsilon,R}(\eta,\eta')$ by $K(\eta,\eta')$. On the other hand,

$$\int_{(0,\infty)^2}\int_{(0,\infty)^2} K(\eta,\eta')|\xi_R(v+v') - 1|[\varphi(\eta+\eta') - \varphi(\eta) - \varphi(\eta')]\tilde{g}_{\epsilon_k,R_k,\delta_k}(\eta)\tilde{g}_{\epsilon_k,R_k,\delta_k}(\eta')\mathrm{d}\eta'\mathrm{d}\eta$$

will go uniformly to zero as $R \to \infty$ using only the uniform moment estimates in v, namely $M_{0,1}$ and $M_{0,-\alpha}$. Take a continuous function $p : \mathbb{R}_+ \to [0,1]$ such that $p(x) = 1$, when $x \leq 1$ and $p(x) = 0$, when

$x \geq 2$. Define then $p_M(a,v,a',v') = p(\frac{a}{M})p(\frac{v}{M})p(\frac{a'}{M})p(\frac{v'}{M})p(\frac{1}{aM})p(\frac{1}{vM})p(\frac{1}{a'M})p(\frac{1}{v'M})$, for $M > 1$ large enough. We have that

$$\int_{(0,\infty)^2}\int_{(0,\infty)^2} K(\eta,\eta')[\varphi(\eta+\eta')-\varphi(\eta)-\varphi(\eta')]p_M(\eta,\eta')\tilde{g}_{\epsilon_k,R_k,\delta_k}(\eta)\tilde{g}_{\epsilon_k,R_k,\delta_k}(\eta')\mathrm{d}\eta'\mathrm{d}\eta$$

converges to

$$\int_{(0,\infty)^2}\int_{(0,\infty)^2} K(\eta,\eta')[\varphi(\eta+\eta')-\varphi(\eta)-\varphi(\eta')]p_M(\eta,\eta')g(\eta)g(\eta')\mathrm{d}\eta'\mathrm{d}\eta,$$

as $k \to \infty$ since the support of the integral is now on a compact set. An argument closely related can be found in [FLNV21b, Proof of Theorem 2.3]. It remains to estimate the contribution outside the set $[\frac{1}{M}, M]^4$. When $v < \frac{1}{M}$ or $v > M$, the estimates are similar to the one-dimensional case as the kernel is bounded by functions depending only on v. We can prove that the integral converges to zero as $M \to \infty$ in the region $\{v < \frac{1}{M}\}$, using the uniform estimates for $M_{0,-\alpha-\tilde{\epsilon}}$ and $M_{0,\beta}$. When $v > M$, we use that $M_{0,-\alpha}$ and $M_{0,1}$ are uniformly bounded to obtain the desired convergence. The regions where $\{v' < M\}$ and $\{v' > \frac{1}{M}\}$ are treated in the same manner by symmetry. We can treat the region where $\{a < \frac{1}{M}\}$ in the same manner as the region where $\{v < (\frac{1}{c_0 M})^{\frac{3}{2}}\}$ due to the isoperimetric inequality $a \geq c_0 v^{\frac{2}{3}}$. For $a > M$, the main point is to control

$$\int_{(0,\infty)^2\cap\{a>M\}} (v^{-\alpha}+v^{\beta})\tilde{g}_{\epsilon,R,\delta}(\eta).$$

We split the region into three parts and use the fact that for large and small values of v, we can control this region using only moments in v as before. More precisely, let $\tilde{C} > 1$ large, then

$$\begin{aligned}\int_{\{a>M\}} (v^{-\alpha}+v^{\beta})\tilde{g}_{\epsilon,R,\delta}(\eta)\mathrm{d}\eta &\leq 2\int_{(0,\infty)^2\cap\{v<\frac{1}{\tilde{C}}\}} v^{-\alpha}\tilde{g}_{\epsilon,R,\delta}(\eta)\mathrm{d}\eta + \int_{\{a>M, v\in[\frac{1}{\tilde{C}},\tilde{C}]\}} (v^{-\alpha}+v^{\beta})\tilde{g}_{\epsilon,R,\delta}(\eta)\mathrm{d}\eta \\ &\quad + 2\int_{(0,\infty)^2\cap\{v>\tilde{C}\}} v^{\beta}\tilde{g}_{\epsilon,R,\delta}(\eta)\mathrm{d}\eta \\ &\leq 2\tilde{C}^{-\tilde{\epsilon}}\int_{(0,\infty)^2\cap\{v<\frac{1}{\tilde{C}}\}} v^{-\alpha-\tilde{\epsilon}}\tilde{g}_{\epsilon,R,\delta}(\eta)\mathrm{d}\eta + 2\tilde{C}M^{-1}\int_{\{a>M, v\in[\frac{1}{\tilde{C}},\tilde{C}]\}} a\tilde{g}_{\epsilon,R,\delta}(\eta)\mathrm{d}\eta \\ &\quad + 2\tilde{C}^{\beta-1}\int_{(0,\infty)^2\cap\{v>\tilde{C}\}} v\tilde{g}_{\epsilon,R,\delta}(\eta)\mathrm{d}\eta \\ &\leq 2\tilde{C}^{-\tilde{\epsilon}}c_{0,-\alpha-\tilde{\epsilon}} + 2\tilde{C}M^{-1}c_{1,0} + 2\tilde{C}^{\beta-1},\end{aligned} \tag{A.3.51}$$

where $\tilde{\epsilon}$ is chosen as in Proposition A.3.2 and $c_{1,0}, c_{0,-\alpha-\tilde{\epsilon}}$ are the constants found in Proposition A.3.2. For given $\tilde{\delta} \in (0,1)$, we first take $\tilde{C}$ to be such that $2\tilde{C}^{-\tilde{\epsilon}}c_{0,-\alpha-\tilde{\epsilon}} + 2\tilde{C}^{\beta-1} \leq \frac{\tilde{\delta}}{2}$ and then M such that $2\tilde{C}M^{-1}c_{1,0} \leq \frac{\tilde{\delta}}{2}$. This proves our desired convergence of $\int_{(0,\infty)^2\cap\{a>M\}}(v^{-\alpha}+v^{\beta})\tilde{g}_{\epsilon,R,\delta}(\eta)$ to zero as $M \to \infty$, thus concluding our proof. □

Proof of Theorem A.2.5, case $\alpha = 0$. Existence of self-similar profiles. We keep the notation used in the Proof of Theorem A.2.3, case $\alpha > 0$. The proof is done in the same manner as in the case $\alpha > 0$.

The difference is that we derived uniform bounds for moments of the form $M_{0,c}$, with $c \geq \gamma$ and for the moment $M_{1,0}$, but we have no information about moments of the form $M_{0,c}$, with $c < \gamma$. This implies that we now have to control the contribution of the regions $\{v \leq \frac{1}{M}\}$ and $\{v' \leq \frac{1}{M}\}$, for a sufficiently large M, in the term containing the coagulation kernel

$$\int_{(0,\infty)^2}\int_{(0,\infty)^2} K_{\epsilon,R}(\eta,\eta')\xi_R(v+v')v[\varphi(\eta+\eta')-\varphi(\eta)]\tilde{g}_{\epsilon_k,R_k,\delta_k}(\eta)\tilde{g}_{\epsilon_k,R_k,\delta_k}(\eta')\mathrm{d}\eta'\mathrm{d}\eta. \tag{A.3.52}$$

For the region $\{v < \frac{1}{M}\}$, we have that:

$$\begin{aligned} &\left|\int_{\{v<\frac{1}{M}\}}\int_{(0,\infty)^2} K_{\epsilon,R}(\eta,\eta')\xi_R(v+v')v[\varphi(\eta+\eta')-\varphi(\eta)]\tilde{g}_{\epsilon_k,R_k,\delta_k}(\eta)\tilde{g}_{\epsilon_k,R_k,\delta_k}(\eta')\mathrm{d}\eta'\mathrm{d}\eta\right| \qquad \text{(A.3.53)} \\ &\lesssim \int_{\{v<\frac{1}{M}\}}\int_{(0,\infty)^2} (v^{\beta+1}+vv'^{\beta})|\varphi(\eta+\eta')-\varphi(\eta)|\tilde{g}_{\epsilon_k,R_k,\delta_k}(\eta)\tilde{g}_{\epsilon_k,R_k,\delta_k}(\eta')\mathrm{d}\eta'\mathrm{d}\eta \\ &\lesssim 2\|\varphi\|_\infty M^{\beta-1}M_{0,\beta}(\tilde{g}_{\epsilon_k,R_k,\delta_k})^2 + M^{-\beta}\int_{\{v<\frac{1}{M}\}}\int_{(0,\infty)^2} v|\varphi(\eta+\eta')-\varphi(\eta)|\tilde{g}_{\epsilon_k,R_k,\delta_k}(\eta)\tilde{g}_{\epsilon_k,R_k,\delta_k}(\eta')\mathrm{d}\eta'\mathrm{d}\eta. \end{aligned}$$

Thus, in order to show that the term in (A.3.53) goes to zero as $M \to \infty$, it suffices to show that the integral above is bounded.

$$\begin{aligned} &\int_{\{v<\frac{1}{M}\}}\int_{(0,\infty)^2} v|\varphi(\eta+\eta')-\varphi(\eta)|\tilde{g}_{\epsilon_k,R_k,\delta_k}(\eta)\tilde{g}_{\epsilon_k,R_k,\delta_k}(\eta')\mathrm{d}\eta'\mathrm{d}\eta \\ &\leq \|\nabla\varphi\|_\infty M_{0,1}(\tilde{g}_{\epsilon_k,R_k,\delta_k})\int_{(0,\infty)^2}|\eta'|\tilde{g}_{\epsilon_k,R_k,\delta_k}(\eta')\mathrm{d}\eta' \leq C. \end{aligned}$$

We now deal with the region where $\{v' < \frac{1}{M}\}$:

$$\begin{aligned} &\left|\int_{(0,\infty)^2}\int_{\{v'<\frac{1}{M}\}} K_{\epsilon,R}(\eta,\eta')\xi_R(v+v')v[\varphi(\eta+\eta')-\varphi(\eta)]\tilde{g}_{\epsilon_k,R_k,\delta_k}(\eta)\tilde{g}_{\epsilon_k,R_k,\delta_k}(\eta')\mathrm{d}\eta'\mathrm{d}\eta\right| \\ &\lesssim \int_{(0,\infty)^2}\int_{\{v'<\frac{1}{M}\}} (v^{\beta+1}+vv'^{\beta})|\varphi(\eta+\eta')-\varphi(\eta)|\tilde{g}_{\epsilon_k,R_k,\delta_k}(\eta)\tilde{g}_{\epsilon_k,R_k,\delta_k}(\eta')\mathrm{d}\eta'\mathrm{d}\eta. \qquad \text{(A.3.54)} \end{aligned}$$

For the first term in (A.3.54), we have:

$$\begin{aligned} &\int_{(0,\infty)^2}\int_{\{v'<\frac{1}{M}\}} v^{\beta+1}|\varphi(\eta+\eta')-\varphi(\eta)|\tilde{g}_{\epsilon_k,R_k,\delta_k}(\eta)\tilde{g}_{\epsilon_k,R_k,\delta_k}(\eta')\mathrm{d}\eta'\mathrm{d}\eta \\ &\leq \|\nabla\varphi\|_\infty\int_{(0,\infty)^2}\int_{\{v'<\frac{1}{M}\}} v^{\beta+1}(a'+v')\tilde{g}_{\epsilon_k,R_k,\delta_k}(\eta)\tilde{g}_{\epsilon_k,R_k,\delta_k}(\eta')\mathrm{d}\eta'\mathrm{d}\eta \\ &\leq M^{\beta-1}\|\nabla\varphi\|_\infty M_{0,\beta+1}(\tilde{g}_{\epsilon_k,R_k,\delta_k})M_{0,\beta}(\tilde{g}_{\epsilon_k,R_k,\delta_k}) + M^{-\delta_1}\|\nabla\varphi\|_\infty M_{0,\beta+1}(\tilde{g}_{\epsilon_k,R_k,\delta_k})M_{1,-\delta_1}(\tilde{g}_{\epsilon_k,R_k,\delta_k}). \end{aligned}$$

For the second term in (A.3.54), we have:

$$\begin{aligned} &\int_{(0,\infty)^2}\int_{\{v'<\frac{1}{M}\}} vv'^{\beta}|\varphi(\eta+\eta')-\varphi(\eta)|\tilde{g}_{\epsilon_k,R_k,\delta_k}(\eta)\tilde{g}_{\epsilon_k,R_k,\delta_k}(\eta')\mathrm{d}\eta'\mathrm{d}\eta \\ &\leq M^{-\beta}\int_{(0,\infty)^2}\int_{\{v'<\frac{1}{M}\}} v|\varphi(\eta+\eta')-\varphi(\eta)|\tilde{g}_{\epsilon_k,R_k,\delta_k}(\eta)\tilde{g}_{\epsilon_k,R_k,\delta_k}(\eta')\mathrm{d}\eta'\mathrm{d}\eta \\ &\leq M^{-\beta}\|\nabla\varphi\|_\infty[M_{0,1}^2(\tilde{g}_{\epsilon_k,R_k,\delta_k}) + M_{0,1}(\tilde{g}_{\epsilon_k,R_k,\delta_k})M_{1,0}(\tilde{g}_{\epsilon_k,R_k,\delta_k})]. \end{aligned}$$

We prove that the term in (A.3.52) converges to zero as $M \to \infty$ in the regions $\{v > M\}, \{v' > M\}$, $\{a < \frac{1}{M}\}$ and $\{a' < \frac{1}{M}\}$ in a similar manner. We deal with the regions where $\{a > M\}$ and $\{a' > M\}$ as in (A.3.51). This concludes the proof. □

A.3.3 Moment estimates for arbitrary powers of area and volume

We construct a self-similar profile $\tilde{g}$ which satisfies $M_{n,k}(\tilde{g}) < \infty$, for $n, k \in \mathbb{R}$, if (A.1.8) holds, and $M_{n,k}(\tilde{g}) < \infty$, for $n \in \mathbb{N}$, $k \geq \gamma$, if (A.1.9) holds. In order to do so, we apply the strategy used to find a self-similar profile in Theorem A.2.3. In order to estimate the moments $M_{n,0}(\tilde{g})$, $n \in \mathbb{N}$, we notice that we can improve the estimates found in Proposition A.3.19, while to derive bounds for $M_{-n,0}(\tilde{g})$, $n \in \mathbb{N}$, we use the fact that they can be estimated in terms of $M_{0,-\frac{2}{3}n}(\tilde{g})$.

Remark A.3.20. Let $n > 0$ and $k \in \mathbb{R}$. In order to bound $M_{-n,k}(\tilde{g})$, it is enough to use $a \geq c_0 v^{\frac{2}{3}}$ to obtain $\int_{\{a \geq c_0 v^{\frac{2}{3}}\}} a^{-n} v^k \tilde{g}(a,v) \mathrm{d}v\mathrm{d}a \leq c_0^{-n} \int_{(0,\infty)^2} v^{k-\frac{2}{3}n} \tilde{g}(a,v)\mathrm{d}v\mathrm{d}a$ if $M_{0,k-\frac{2}{3}n}(\tilde{g}) < \infty$.

In order to find uniform bounds for the moments $M_{n,0}(\tilde{g})$, we need to be able to test (A.3.5) with higher powers of a.

Let $N \in \mathbb{N}, N \geq 2$. For some $\tilde{g} \in \mathscr{M}_+^I(\mathbb{R}_{>0}^2)$ such that

$$\int_{(0,\infty)^2} (1 + a^N)\tilde{g}(a,v)\mathrm{d}v\mathrm{d}a < \infty, \tag{A.3.55}$$

we define the space

$$U_{\epsilon,R,N} := \{\tilde{g} \in \mathscr{M}_+^I(\mathbb{R}_{>0}^2), \tilde{g}\Big(\mathbb{R}_{>0}^2 \setminus [c_0\epsilon^{\frac{2}{3}}, \infty) \times [\epsilon, 2R)\Big) = 0, \tilde{g} \text{ satisfies (A.3.55)}\}. \tag{A.3.56}$$

Proposition A.3.21. *Let $K_{\epsilon,R}$ be the kernel defined in (A.3.1). Let $N \in \mathbb{N}$, $N \geq 2$. Let $g_{\text{in},R} \in \mathscr{M}_{+,\mathrm{b}}^I(\mathbb{R}_{>0}^2) \cap U_{\epsilon,R,N}$. There exists a unique solution $g_{\epsilon,R,\delta} \in \mathrm{C}^1([0,\infty); \mathscr{M}_{+,\mathrm{b}}^I(\mathbb{R}_{>0}^2))$, $g_{\epsilon,R,\delta}(t) \in U_{\epsilon,R,N}$, that satisfies*

$$\sup_{t\in[0,T]} \int_{(0,\infty)^2} (1 + a^N) g_{\epsilon,R,\delta}(a,v,t)\mathrm{d}v\mathrm{d}a < \infty,$$

for every $T > 0$, for the weak formulation of the coagulation equation (A.3.5) with initial datum $g_{\text{in},R}$.

Proof. The proof is done via a standard fixed-point argument as in the proof of Proposition A.3.1. The space $U_{\epsilon,R,N}$ was chosen since we would like to later test with higher powers of the area a and we need to control the escape to infinity of a. We prove existence in the same manner as in Proposition A.3.1 with the aid of Propositions A.3.22 and A.3.24, which will be stated below. □

Proposition A.3.21 states that we can obtain better moment estimates if we assume moment estimates for higher order powers for the initial datum, namely with (A.3.55) instead of (A.3.9).

As before, we use K_R, see (A.3.18), and $K_{\epsilon,R}$, see (A.3.1), interchangeably.

We define the space

$$
\begin{aligned}
\mathrm{Y}_N =&\{G \in \mathrm{C}([0,\tau_N];\mathscr{M}^I_{+,\mathrm{b}}(\mathbb{R}^2_{>0})) : \|G\|_N = \sup_{0\le t\le \tau_N}\left|\int_{(0,\infty)^2} \mathrm{e}^{h_\epsilon(V,t)}(1+x^N_{t,V}(A))G(A,V,t)\mathrm{d}V\mathrm{d}A\right| \le 2M_N;\\
&G(\mathbb{R}^2_{>0}\setminus[c_0\epsilon^{\frac{2}{3}},\infty)\times[\epsilon,2R\mathrm{e}^t),t)=0,\ t\in[0,\tau_N]\}
\end{aligned}
$$

instead of $Y_{\epsilon,\tau}$ in (A.3.22), where $M_N = 2\int_{(0,\infty)^2}(1+a^N)g_{\mathrm{in},R}(a,v)\mathrm{d}v\mathrm{d}a + 1$, $g_{\mathrm{in},R}$ is as in Proposition A.3.21 and we fix a time τ_N instead of τ such that

$$
(2^N(1-\gamma)M_N + M_N + 1)\|K_R\|_\infty(\mathrm{e}^{\tau_N}-1) < \frac{1}{2}; \tag{A.3.57}
$$

$$
2^N(1-\gamma)M_N^2\|K_R\|_\infty\tau_N < 1 \text{ and } \tau_N \le \ln 2. \tag{A.3.58}
$$

Proposition A.3.22. *Let $N\in\mathbb{N}, N\ge 2$. Let $K_{\epsilon,R}$ defined as in (A.3.1) and assume K and r satisfy the conditions stated in Theorem A.2.3 or in Theorem A.2.5. Let $g_{\mathrm{in},R}$ as stated in Proposition A.3.21. Let J be as in (A.3.23). Assume $F,G\in \mathrm{Y}_N$ with τ_N as in (A.3.57) and (A.3.58). Then*

1. *$\|J[G]\| \le 2M_N$;*

2. *The map J is contractive, more explicitly, $\|J[F]-J[G]\|_N \le \frac{1}{2}\|F-G\|_N$.*

Proof. We follow the proof of Proposition A.3.6, working with Y_N, M_N and τ_N instead of $\mathrm{Y}_{\epsilon,\tau}$, M and τ, respectively. We make use in addition of the inequality

$$
1+(x_{t,V}(A)+x_{t,V'}(A'))^N \le 2^{N-1}(1+x^N_{t,V}(A))(1+x^N_{t,V'}(A')).
$$

For more details, see Proof of Proposition A.3.6 in Appendix A.6, where similar arguments were used. □

We make use in addition of the following inequality provided in [BGP04, Lemma 2].

Lemma A.3.23. *Assume that $p>1$ and let $k_p := [\frac{p+1}{2}]$. Then, for all $x,y>0$, the following inequalities hold:*

$$
\sum_{k=1}^{k_p-1}\binom{p}{k}(x^ky^{p-k}+x^{p-k}y^k) \le (x+y)^p - x^p - y^p \le \sum_{k=1}^{k_p}\binom{p}{k}(x^ky^{p-k}+x^{p-k}y^k).
$$

Proposition A.3.24. *Let $N\in\mathbb{N}$, $N\ge 2$, N fixed. Let $G_{\epsilon,R,\delta}$ be the found fixed point for (A.3.23) up to time τ_N defined as in (A.3.57), (A.3.58) and with initial datum $g_{\mathrm{in},R}$ taken as in Proposition A.3.21. For any $T>0$, there exists a unique solution, for which we keep the notation $G_{\epsilon,R,\delta}\in \mathrm{C}^1([0,T];\mathscr{M}^I_{+,\mathrm{b}}(\mathbb{R}^2_{>0}))$ that satisfies (A.3.19).*

The proof of this proposition is based on an iterative argument and is given in Appendix A.6.

Let $K_{\epsilon,R}$ defined as in (A.3.1) and assume K and r satisfy the conditions stated in Theorem A.2.3 or in Theorem A.2.5. Let $T > 0$. Define $S(t) : U_{\epsilon,R,N} \to U_{\epsilon,R,N}$ in the following way:

$$S(t)g_{\text{in},R} = g_{\epsilon,R,\delta}(\cdot,\cdot,t) \tag{A.3.59}$$

for all $t \in [0,T]$, where $g_{\epsilon,R,\delta}$ is the unique solution of the weak formulation of the coagulation equation with coagulation kernel $K_{\epsilon,R}$ found Proposition A.3.21.

Proposition A.3.25. *Assume $g_{\epsilon,R,\delta}$ is the solution found in Proposition A.3.21 for some $N \in \mathbb{N}$, $N \geq 2$, fixed, with kernel $K_{\epsilon,R}$ taken as in (A.3.1). Let $\mu > 0$.*

If we are in the case (A.1.8), for $k \in \mathbb{R}, n \in [1,N]$, there exist constants $c_{0,k}, c_{1,-\alpha}, c_{1,1}, c_{n,0} > 0$ for which the set

$$\omega = \{M_{0,1}(g_{\epsilon,R,\delta}(t)) = 1, M_{0,k}(g_{\epsilon,R,\delta}(t)) \leq c_{0,k}, M_{1,-\alpha}(g_{\epsilon,R,\delta}(t)) \leq c_{1,-\alpha}, M_{1,1}(g_{\epsilon,R,\delta}(t)) \leq c_{1,1},$$
$$M_{n,0}(g_{\epsilon,R,\delta}(t)) \leq c_{n,0}\}$$

is preserved for all $t \in [0,T]$, that means $S(t)\omega \subseteq \omega$.

If we are in the case (A.1.9), for $k \geq \gamma, n \in [1,N]$, there exist constants $c_{0,k}, c_{1,1}, c_{n,0} > 0$ for which the set

$$\omega = \{M_{0,1}(g_{\epsilon,R,\delta}(t)) = 1, M_{0,k}(g_{\epsilon,R,\delta}(t)) \leq c_{0,k}, M_{1,1}(g_{\epsilon,R,\delta}(t)) \leq c_{1,1}, M_{n,0}(g_{\epsilon,R,\delta}(t)) \leq c_{n,0}\}$$

is preserved for all $t \in [0,T]$, that means $S(t)\omega \subseteq \omega$.

We prove this proposition after proving first some uniform moment estimates. To bound moments of the form $M_{0,c}$, $c \in \mathbb{R}$ and $M_{1,-\alpha}$, we use the same method that we used in Section A.3.2, namely Propositions A.3.16 - A.3.19. In the next propositions we prove uniform bounds for $M_{1,1}$ and $M_{n,0}$, for $n \in [2,N]$.

We prove uniform bounds for $M_{1,m}(g_{\epsilon,R,\delta})$, $m \geq 1$, instead of only proving uniform bounds for the moment $M_{1,1}$ as the proof is done in a similar manner.

Proposition A.3.26. *Let $g_{\text{in},R} \in \omega$ and $g_{\epsilon,R,\delta}$ (which we will denote by g_{in} and g, respectively) be as in Proposition A.3.21. Let $m \geq 1$. Assume in addition that $\int_{(0,\infty)^2} av^m g_{\text{in},R}(a,v)\mathrm{d}v\mathrm{d}a < \infty$. Then there exists $C_{1,m} > 0$, which does not depend on ϵ, R, δ, such that:*

$$\sup_{t\geq 0} M_{1,m}(g(\cdot,\cdot,t)) \leq \max\{M_{1,m}(g_{\text{in}}), C_{1,m}\}. \tag{A.3.60}$$

Proof. The proof relies on the fact that the coagulation operator does not contribute with too fast interactions, using in addition the uniform estimate for the total surface area, which we were able to derive making use of the fusion term.

We use

$$-\int_{(0,\infty)^2} av^m g(a,v,t)\Theta_\epsilon(v)\mathrm{d}v\mathrm{d}a \leq -M_{1,m}(g(t)) + 2^m \int_{(0,\infty)^2} ag(a,v,t)\mathrm{d}v\mathrm{d}a$$

and, as $M_{1,0}(g)$ is uniformly bounded from above, equation (A.3.5) becomes:

$$\frac{\mathrm{d}}{\mathrm{d}t}M_{1,m}(g(t)) \leq (1-\frac{2}{3}-m)M_{1,m}(g(t)) + C + (1-\gamma)R_0\int_{(0,\infty)^2}\frac{a^\mu \max\{v^\sigma, L\delta\}}{1+\delta a^\mu}(c_0 v^{\frac{2}{3}} - a)g(a,v,t)v^m\mathrm{d}\eta$$
$$+\frac{1-\gamma}{2}\int_{(0,\infty)^2}\int_{(0,\infty)^2} K(a,v,a',v')[(a+a')(v+v')^m - av^m - a'v'^m]g(a,v,t)g(a',v',t)\mathrm{d}\eta'\mathrm{d}\eta.$$

The term with the fusion kernel is non-positive and can be omitted. For the term with the coagulation kernel, we combine

$$(a+a')(v+v')^m - av^m - a'v'^m = a[(v+v')^m - v^m] + a'[(v+v')^m - v'^m]$$

with the inequality:

$$(v+v')^m - v^m = m\int_v^{v+v'} s^{m-1}\mathrm{d}s \leq m(v+v')^{m-1}v' \leq C_m(v'^m + v^{m-1}v'),$$

and obtain

$$K(\eta,\eta')[(a+a')(v+v')^m - av^m - a'v'^m] \lesssim C_m K(a,v,a',v')(av^{m-1}v' + av'^m + a'v'^{m-1}v + a'v^m)$$
$$\lesssim K_0 C_m(v^{-\alpha}v'^\beta + v'^{-\alpha}v^\beta)(av^{m-1}v' + av'^m + a'v'^{m-1}v + a'v^m).$$

The moments $M_{0,c}$, $c \geq \gamma$, are bounded from above. The term $M_{1,-\alpha}$ is uniformly bounded from above. This was proven in Proposition A.3.19. If $m > 1$, then $m+\beta-1, \beta, m-\alpha-1 \in (-\alpha, m)$ and we use Hölder's inequality and conclude using a comparison argument. If $m = 1$, we only have to bound av^β. As $\beta \in (-\alpha, m)$, we conclude in the same manner as before. □

Proposition A.3.27. *Let $g_{\mathrm{in},R} \in \omega$ and $g_{\epsilon,R,\delta}$ (which we will denote by g_{in} and g, respectively) be as in Proposition A.3.21, for $N \in \mathbb{N}$, $N \geq 2$, fixed. Then, for any $n \in [2,N]$, there exists $C_{n,0} > 0$, independent of ϵ, R, δ, N, but dependent on n, such that:*

$$\sup_{t\geq 0} M_{n,0}(g(\cdot,\cdot,t)) \leq \max\{M_{n,0}(g_{\mathrm{in}}), C_{n,0}\}. \tag{A.3.61}$$

Proof. This estimate follows from the fact that fusion overtakes coagulation in the case $\mu > 0$.

Notice first that due to the choice of the space Y_N, we can test equation (A.3.5) with a^n, for $n \in [2,N]$. We have that there exists $C_n > 0$ such that:

$$K_{\epsilon,R}(a,v,a',v')[(a+a')^n - a^n - a'^n] \leq K(a,v,a',v')\sum_{k=1}^{k_n}\binom{n}{k}(a^k a'^{n-k} + a^{n-k}a'^k)$$
$$\leq C_n(v^{-\alpha}v'^\beta + v^\beta v'^{-\alpha})(a^{n-1}a' + aa'^{n-1}),$$

where k_n is taken as in Lemma A.3.23. We have a uniform upper bound for $\int_{(0,\infty)^2}(av^{-\alpha}+av^{\beta})g(\eta,t)\mathrm{d}\eta$ due to Proposition A.3.26.

If $\alpha > 0$, we use the fact that there exists $C_{n,\epsilon_1} > 0$ for which:

$$\int_{(0,\infty)^2} a^{n-1}v^{-\alpha}g(a,v,t)\mathrm{d}v\mathrm{d}a \leq \epsilon_1 \int_{(0,\infty)^2} a^n g(a,v,t)\mathrm{d}v\mathrm{d}a + C_{n,\epsilon_1}\int_{(0,\infty)^2} v^{-n\alpha}g(a,v,t)\mathrm{d}v\mathrm{d}a$$

and

$$\int_{(0,\infty)^2} a^{n-1}v^{\beta}g(a,v,t)\mathrm{d}v\mathrm{d}a \leq \epsilon_1 \int_{(0,\infty)^2} a^n g(a,v,t)\mathrm{d}v\mathrm{d}a + C_{n,\epsilon_1}\int_{(0,\infty)^2} v^{n\beta}g(a,v,t)\mathrm{d}v\mathrm{d}a.$$

If $\alpha = 0$ and $n = 2$, we use that $M_{1,0}$ and $M_{1,\beta}$ are uniformly bounded from above. If $\alpha = 0$ and $n \geq 3$, then

$$\int_{(0,\infty)^2} a^{n-1}g(a,v,t)\mathrm{d}v\mathrm{d}a \leq \epsilon_1 \int_{(0,\infty)^2} a^n g(a,v,t)\mathrm{d}v\mathrm{d}a + C_{n,\epsilon_1}\int_{(0,\infty)^2} ag(a,v,t)\mathrm{d}v\mathrm{d}a.$$

As the linear term is non-positive, equation (A.3.5) becomes:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}M_{n,0}(g(t)) \leq (1-\gamma)nR_0\int_{(0,\infty)^2}\frac{a^{\mu}\max\{v^{\sigma},L\delta\}}{1+\delta a^{\mu}}(c_0v^{\frac{2}{3}}-a)g(\eta,t)a^{n-1}\mathrm{d}\eta \\ + C_{(-\alpha,n,\beta)}\epsilon_1 M_{n,0}(g(t)) + C_{(-\alpha,n,\beta)}.\end{aligned}$$

We now use the same argument as in Proposition A.3.19, modifying the Young's inequality part. As $\mu + n > n$, we can apply Young's inequality to obtain:

$$a^n = a^n v^{\frac{n\sigma}{\mu+n}}v^{-\frac{n\sigma}{\mu+n}} \leq \frac{\epsilon_2}{p}(av^{\frac{\sigma}{\mu+n}})^{\mu+n} + \frac{\epsilon_2^{-\frac{q}{p}}}{q}v^{-\frac{n\sigma}{\mu+n}q},$$

where $p = \frac{\mu+n}{n}$ and $\frac{1}{p}+\frac{1}{q} = 1$. So there exists a constant $C_{\epsilon_2} > 0$, depending on ϵ_2, for which:

$$a^n \lesssim \epsilon_2 a^{\mu+n}v^{\sigma} + C_{\epsilon_2}v^{-n\frac{\sigma}{\mu}}$$

and this implies:

$$-\int_{\{a\geq 2c_0v^{\frac{2}{3}}\}} a^{\mu+n}v^{\sigma}g(a,v,t)\mathrm{d}v\mathrm{d}a \lesssim -\frac{1}{\epsilon_2}\int_{\{a\geq 2c_0v^{\frac{2}{3}}\}} a^n g(a,v,t)\mathrm{d}v\mathrm{d}a + C_{\epsilon_2}\int_{(0,\infty)^2} v^{-n\frac{\sigma}{\mu}}g(a,v,t)\mathrm{d}v\mathrm{d}a.$$

As moments of the form $M_{0,c}$ are bounded from above, we choose ϵ_2 sufficiently small in order to be able to repeat the argument in (A.3.46) and then we recover the region where $\{a \geq c_0v^{\frac{2}{3}}\}$:

$$\begin{aligned}-\int_{\{a\geq 2c_0v^{\frac{2}{3}}\}} a^n g(a,v,t)\mathrm{d}v\mathrm{d}a &= -\int_{\{a\geq c_0v^{\frac{2}{3}}\}} a^n g(a,v,t)\mathrm{d}v\mathrm{d}a + \int_{\{c_0v^{\frac{2}{3}}\leq a<2c_0v^{\frac{2}{3}}\}} a^n g(a,v,t)\mathrm{d}v\mathrm{d}a \\ &\leq -\int_{\{a\geq c_0v^{\frac{2}{3}}\}} a^n g(a,v,t)\mathrm{d}v\mathrm{d}a + (2c_0)^n\int_{(0,\infty)^2} v^{\frac{2}{3}n}g(a,v,t)\mathrm{d}v\mathrm{d}a.\end{aligned}$$

We can choose $\epsilon_1 > 0$ sufficiently small so that equation (A.3.5) becomes:

$$\frac{\mathrm{d}}{\mathrm{d}t}M_{n,0}(g(t)) \lesssim -n\frac{L(1-\gamma)}{4}R_0M_{n,0}(g(t)) + 3M_{n,0}(g(t)) + C_{\epsilon_1,\epsilon_2,n} \leq -M_{n,0}(g(t)) + C_{\epsilon_1,\epsilon_2,n}$$

and we conclude using a comparison argument. □

Proof of Proposition A.3.25. Let $T > 0$. Let $N \in \mathbb{N}$, $N \geq 2$, fixed. Let $S(t) : U_{\epsilon,R,N} \to U_{\epsilon,R,N}$, for $t \in [0,T]$, be as in (A.3.59). Proposition A.3.21 guarantees that the semigroup is well-defined. Propositions A.3.26 - A.3.27 together with Propositions A.3.16 - A.3.19 prove that $S(t)\omega \subseteq \omega$ if we choose the constants $c_{0,k}, c_{1,-\alpha}, c_{1,1}, c_{n,0} > 0$ in Proposition A.3.25 to correspond to the constants found in Propositions A.3.26 - A.3.27. □

We now prove the stated moment bounds for the self-similar profiles found in Theorem A.2.3 and Theorem A.2.5.

Proof of Theorem A.2.3 and Theorem A.2.5. Moment bounds. The main point is to control positive powers of the area. Following the proof of Theorem A.2.3 and with the help of Proposition A.3.21 and Proposition A.3.25, we can prove that there exists a self-similar profile g_N for the two-dimensional coagulation equation satisfying

$$\int_{(0,\infty)^2} (1 + a^n) g_N(a, v) \mathrm{d}v\mathrm{d}a \leq c_{n,0}, \tag{A.3.62}$$

for every $n \in [1, N]$. Moreover, if $\alpha > 0$, we have that $\int_{(0,\infty)^2} (v^{-\alpha-\tilde{\epsilon}} + v + a) g_N(a, v)\mathrm{d}v\mathrm{d}a$ is bounded uniformly independently of N, for $\tilde{\epsilon}$ as in Proposition A.3.2. If $\alpha = 0$, we have that $\int_{(0,\infty)^2} (v^{\gamma} + v^{\gamma+1} + a) g_N(a, v)\mathrm{d}v\mathrm{d}a$ is bounded uniformly independently of N. We can thus find a subsequence of $\{g_N\}_{N\geq 2}$ that converges to some $\tilde{g}$ in the weak-* topology and $\tilde{g}$ is a self-similar profile for the two-dimensional coagulation equation. This is since these are the moment estimates needed to prove the existence of a self-similar profile.

As the constants $c_{n,0}$ in (A.3.62) are independent of N, for $n \leq N$, we can conclude that there exist some constants $c_d > 0$ such that $\int_{(0,\infty)^2} a^d \tilde{g}(a, v)\mathrm{d}v\mathrm{d}a < c_d$, for all $d \in [1, \infty)$.

To control negative powers of a, we use Remark A.3.20. For combinations of positive powers of a and powers of v, we use Young's inequality. Take $d > 0$ and $k \in \mathbb{R}$. Then:

$$\int_{(0,\infty)^2} a^d v^k \tilde{g}(a, v)\mathrm{d}v\mathrm{d}a \leq \frac{d}{d+1} \int_{(0,\infty)^2} a^{d+1} \tilde{g}(a, v)\mathrm{d}v\mathrm{d}a + \frac{1}{d+1} \int_{(0,\infty)^2} v^{k(d+1)} \tilde{g}(a, v)\mathrm{d}v\mathrm{d}a.$$

□

A.4 Different asymptotic behaviors for $\mu < 0$

A.4.1 Ramification theory

First, we prove existence of a weak solution for the time-dependent fusion problem as in Definition A.2.1 that satisfies the moment bounds (A.2.17) and (A.2.18) and with initial value $g_{\text{in}} \in \mathscr{M}_+^I(\mathbb{R}^2_{>0})$. We initially prove well-posedness for a truncated version of the time-dependent problem. Since μ is negative, we can prove well-posedness using the truncation for the coagulation kernel, while the cut-off in the fusion rate is not needed. So we look at the equation:

$$\int_{(0,\infty)^2} g(\eta,t)\Theta_\epsilon(v)\varphi(\eta)\mathrm{d}\eta - \frac{2}{3}\int_{(0,\infty)^2} g(\eta,t)\Theta_\epsilon(v)a\partial_a\varphi(\eta)\mathrm{d}\eta - \int_{(0,\infty)^2} g(\eta,t)\Theta_\epsilon(v)v\partial_v\varphi(\eta)\mathrm{d}\eta$$
$$+(1-\gamma)\langle\mathbb{K}_R[g],\varphi\rangle + (1-\gamma)\int_{(0,\infty)^2} r(\eta)(c_0 v^{\frac{2}{3}} - a)g(\eta,t)\partial_a\varphi(\eta)\mathrm{d}\eta = \partial_t\int_{(0,\infty)^2} g(\eta,t)\varphi(\eta)\mathrm{d}\eta \quad \text{(A.4.1)}$$

and assume the initial value is $g_{\mathrm{in},R} = g_{\mathrm{in}}\mathbb{1}_{\{[c_0\epsilon^{\frac{2}{3}},\infty)\times[\epsilon,2R)\}}$. For this form of the equation, we make use of Proposition A.3.21 for the case $N = 2$ and obtain a sequence of solutions $\{g_{\epsilon,R}\}_{\{\epsilon\in(0,1),R>1\}}$. Denote by $\alpha_1 = \min\{-\alpha-\tilde{\epsilon},\gamma-\frac{1}{3}\}$ and let $\tilde{m}$ as in Proposition A.3.2. Using the same proof as for Proposition A.3.2, we can prove that the set

$$\omega = \omega(\epsilon,R) := \{\frac{1}{2} \leq M_{0,1}(g_{\epsilon,R}) \leq 1; M_{0,\alpha_1}(g_{\epsilon,R}) \leq c_{0,\alpha_1}; M_{0,\tilde{m}}(g_{\epsilon,R}) \leq c_{0,\tilde{m}}, M_{0,\frac{\sigma}{|\mu|}}(g_{\epsilon,R}) \leq c_{0,\frac{\sigma}{|\mu|}}\} \quad \text{(A.4.2)}$$

stays preserved in time uniformly in ϵ, R.

In order to prove uniform moment bounds for negative powers of v, i.e. in order to prove $M_{0,c}$, with $c \leq 0$, we need a lower bound for the $M_{0,1}$ moment. Thus, we need to prove that the lower bound on $M_{0,1}$ is preserved in time, independently of ϵ, R and of the initial condition $g_{\mathrm{in},R}$.

To do this, we test equation (A.4.1) with $\varphi(a,v) = v$ and make use of the following technical proposition.

Proposition A.4.1. *Let $g_{\epsilon,R}$ be the solution of equation (A.4.1) with initial value $g_{\mathrm{in},R}(a,v) = g_{\mathrm{in}}\mathbb{1}_{\{[c_0\epsilon^{\frac{2}{3}},\infty)\times[\epsilon,2R)\}}$ and assume in addition that $g_{\mathrm{in},R} \in \omega$, where ω is as in (A.4.2). Then there exists $M > 0$ such that:*

$$\frac{M_{0,1}(g_{\mathrm{in}})}{2} \leq M_{0,1}(g_{\epsilon,R}(\cdot,\cdot,t)) \leq M_{0,1}(g_{\mathrm{in}}), \quad \text{(A.4.3)}$$

for every $t \geq 0$ and uniformly in $\epsilon < \frac{1}{M}, R > M$.

Proof. Fix $M > 0$, sufficiently large, such that

$$M^{\gamma-1}\int_{(0,\infty)^2} v^\gamma g_{\mathrm{in}}(a,v)\mathrm{d}v\mathrm{d}a + M^{1-\tilde{m}}\int_{(0,\infty)^2} v^{\tilde{m}} g_{\mathrm{in}}(a,v)\mathrm{d}v\mathrm{d}a + M^{-\frac{2}{3}}\int_{(0,\infty)^2} v^{\frac{1}{3}} g_{\mathrm{in}}(a,v)\mathrm{d}v\mathrm{d}a \leq \frac{M_{0,1}(g_{\mathrm{in}})}{2},$$

where $\tilde{m}$ is as in Proposition A.3.2 and β as in (A.1.7).

Testing in (A.4.1) with $\varphi(a,v) = v$, we obtain that

$$\int_{(0,\infty)^2} v g_{\epsilon,R}(a,v,t)\mathrm{d}v\mathrm{d}a = \int_{(0,\infty)^2} v g_{\mathrm{in},R}(a,v)\mathrm{d}v\mathrm{d}a. \quad \text{(A.4.4)}$$

Take $\epsilon < \frac{1}{M}$ and $R > M$. Since $a \geq c_0 v^{\frac{2}{3}}$, we have that

$$\begin{aligned}\int_{(0,\infty)^2} v g_{\mathrm{in},R}(a,v)\mathrm{d}v\mathrm{d}a &\geq \int_{(c_0 M^{-\frac{2}{3}},\infty)\times(\frac{1}{M},M)} v g_{\mathrm{in}}(a,v)\mathrm{d}v\mathrm{d}a \\ &\geq M_{0,1}(g_{\mathrm{in}}) - M^{\gamma-1}\int_{(0,\infty)^2} v^{\gamma} g_{\mathrm{in}}(a,v)\mathrm{d}v\mathrm{d}a - M^{1-\tilde{m}}\int_{(0,\infty)^2} v^{\tilde{m}} g_{\mathrm{in}}(a,v)\mathrm{d}v\mathrm{d}a \\ &- M^{-\frac{2}{3}}\int_{(0,\infty)^2} v^{\frac{1}{3}} g_{\mathrm{in}}(a,v)\mathrm{d}v\mathrm{d}a \geq \frac{M_{0,1}(g_{\mathrm{in}})}{2}. \end{aligned} \tag{A.4.5}$$

We combine (A.4.4) and (A.4.5) in order to conclude our proof. □

The estimates for the remaining moments follow as in the proof of Proposition A.3.2. In particular, we obtain that there exists a function $g \in C([0,\infty); \mathscr{M}_+(\mathbb{R}^2_{>0})) \cap \omega$ and a subsequence of $\{g_{\epsilon,R}\}$ such that $g_{\epsilon,R}(t)$ converges to $g(t)$ in the weak-* topology, for every $t \geq 0$. We now want to prove that g satisfies equation (A.4.1). For this, we follow the steps used in the proof of Theorem A.2.3. The difference is that now we work with an equation that depends on time instead of trying to prove the existence of a self-similar profile. Therefore, unlike Theorem A.2.3, we do not need to obtain an estimate for $M_{1,0}$ independent of time, but it is enough to derive an estimate of the form $M_{1,0}(t) \leq m(T)$, for $0 \leq t \leq T$, for some function $m : \mathbb{R}_+ \to \mathbb{R}_+$, increasing, but finite for any fixed time. Notice that we can expect to have $M_{1,0}$ growing as $t \to \infty$ since this is the predicted behavior for $M_{1,0}$ in Theorem A.2.12.

Proposition A.4.2. *Let $\mu < 0$. There exists a solution for the weak version of the time-dependent fusion problem as in Definition A.2.1.*

Proof. We test equation (A.4.1) with $\varphi(\eta) = a$. The fusion term is non-positive due to the isoperimetric inequality and we have that $\langle \mathbb{K}_R[g_{\epsilon,R}], a\rangle = 0$. Thus, we obtain:

$$\int_{(0,\infty)^2} g_{\epsilon,R}(\eta,t) a \mathrm{d}\eta \leq \int_{(0,\infty)^2} g_{\mathrm{in}}(\eta) a \mathrm{d}\eta + \frac{1}{3}\int_0^t \int_{(0,\infty)^2} g_{\epsilon,R}(\eta,s) a \mathrm{d}\eta \mathrm{d}s.$$

By Grönwall's inequality, this implies that $M_{1,0}(g_{\epsilon,R}(t))$ is bounded for every $t \geq 0$. Thus, we can redo the argument used in Theorem A.2.3 to prove that g satisfies equation (A.2.4). □

We now wish to prove that g satisfies the estimates in (A.2.18) in order to prove Proposition A.2.9.

Proposition A.4.3. *The solutions $g_{\epsilon,R}$ of the truncated version of the time-dependent fusion problem (A.4.1) satisfy (A.2.18) uniformly in ϵ, R.*

Proof. In order to prove the bounds in (A.2.18), we prove that the estimates hold uniformly in ϵ, R for $g_{\epsilon,R}$ and then pass to the limit. As the ideas for different moment estimates are similar, we focus on proving $M_{2,0}$ is uniformly bounded. Since it may not be possible to prove bounds for $M_{2,0}$ which are independent of time, we restrict ourselves to proving that, for some fixed $T > 0$, we have that $M_{2,0}(t)$ is bounded for all $t \leq T$.

Let $T > 0$, $t \leq T$. Due to Proposition A.3.21 in the case $N = 2$, we are able to test (A.4.1) with $\varphi(a, v) = a^2$. Since the fusion term is non-positive due to the isoperimetric inequality, the equation becomes:

$$\begin{aligned}\int_{(0,\infty)^2} g_{\epsilon,R}(\eta, t)a^2 \mathrm{d}\eta &\leq \int_{(0,\infty)^2} g_{\text{in}}(\eta)a^2 \mathrm{d}\eta + (1 - \frac{4}{3}) \int_0^t \int_{(0,\infty)^2} g_{\epsilon,R}(\eta, s)\Theta_\epsilon(v)a^2 \mathrm{d}\eta \mathrm{d}s \\ &+ C \int_0^t \int_{(0,\infty)^2} av^{-\alpha} g_{\epsilon,R}(\eta, s)\mathrm{d}\eta \int_{(0,\infty)^2} a'v'^{\beta} g_{\epsilon,R}(\eta', s)\mathrm{d}\eta' \mathrm{d}s \\ &\leq \int_{(0,\infty)^2} g_{\text{in}}(\eta)a^2 \mathrm{d}\eta + C \int_0^t \int_{(0,\infty)^2} av^{-\alpha} g_{\epsilon,R}(\eta, s)\mathrm{d}\eta \int_{(0,\infty)^2} a'v'^{\beta} g_{\epsilon,R}(\eta', s)\mathrm{d}\eta' \mathrm{d}s\end{aligned}$$

Thus, if $M_{1,-\alpha}$ and $M_{1,\beta}$ are bounded, we obtain that:

$$\int_{(0,\infty)^2} g_{\epsilon,R}(\eta, t)a^2 \mathrm{d}\eta \leq \int_{(0,\infty)^2} g_{\text{in}}(\eta)a^2 \mathrm{d}\eta + C(t) \leq \int_{(0,\infty)^2} g_{\text{in}}(\eta)a^2 \mathrm{d}\eta + C(T)$$

and we can conclude using Grönwall's inequality.

$M_{1,-\alpha}$ is bounded in the same manner as $M_{1,0}$ in Proposition A.4.2.

We now estimate $M_{1,\beta}$. The contribution of the fusion term is non-positive. We estimate the term containing the coagulation kernel using:

$$\begin{aligned}&\int_{(0,\infty)^2} K_{\epsilon,R}(a, v, a', v')[(a + a')(v + v')^{\beta} - av^{\beta} - a'v'^{\beta}]g_{\epsilon,R}(\eta, t)g_{\epsilon,R}(\eta', t')\mathrm{d}\eta' \mathrm{d}\eta \\ &\quad\leq \int_{(0,\infty)^2} K_{\epsilon,R}(a, v, a', v')[av'^{\beta} + a'v^{\beta}]g_{\epsilon,R}(\eta, t)g_{\epsilon,R}(\eta', t')\mathrm{d}\eta' \mathrm{d}\eta \\ &\quad\leq C \int_{(0,\infty)^2} av^{\beta} g(\eta, t)\mathrm{d}\eta \int_{(0,\infty)^2} v'^{\beta-\alpha} g(\eta', t')\mathrm{d}\eta' + C \int_{(0,\infty)^2} av^{-\alpha} g(\eta, t)\mathrm{d}\eta \int_{(0,\infty)^2} v'^{2\beta} g(\eta', t')\mathrm{d}\eta'.\end{aligned}$$

Since we can bound from above the moment estimates of the form $M_{0,d}$, with $d = 2\beta$ or $d = \beta - \alpha$, when testing (A.4.1) with av^{β}, the equation becomes

$$\begin{aligned}\partial_t M_{1,\beta}(g_{\epsilon,R}(t)) &\leq (1 - \frac{2}{3} - \beta) \int_{(0,\infty)^2} g_{\epsilon,R}(\eta, t)\Theta_\epsilon(v)av^{\beta} \mathrm{d}\eta + C_1 M_{1,\beta}(g_{\epsilon,R}(t)) + C_2 M_{1,-\alpha}(g_{\epsilon,R}(t)) \\ &\leq \overline{C}_1 M_{1,\beta}(g_{\epsilon,R}(t)) + \overline{C}_2(t),\end{aligned}$$

for some constants $C_1, C_2, \overline{C}_1, \overline{C}_2(t) > 0$, where $\overline{C}_2$ is a function $\overline{C}_2 : \mathbb{R}_+ \to \mathbb{R}_+$, increasing, but finite for any fixed time. Thus, for $t \in [0, T]$, we have

$$\partial_t M_{1,\beta}(g_{\epsilon,R}(t)) \leq \overline{C}_1 M_{1,\beta}(g_{\epsilon,R}(t)) + \overline{C}_2(T).$$

We can conclude again using Grönwall's inequality.

The rest of the moments in (A.2.18) are estimated using the same methods. The main idea is to use an iterative argument that allows to reduce the exponents of the powers of a and v to lower order terms, for which we already obtained uniform bounds.

For example, we analyse the moment $M_{2,m}$, when $m > 1$. The fusion term is non-positive and the linear terms in (A.4.1) satisfy $(1 - \frac{4}{3} - m) \int_{(0,\infty)^2} g_{\epsilon,R}(\eta,t)\Theta_\epsilon(v)a^2v^m \mathrm{d}\eta \leq 0$. We thus obtain:

$$\begin{aligned}\partial_t M_{2,m}(g_{\epsilon,R}(t)) &\lesssim \int_{(0,\infty)^2}\int_{(0,\infty)^2} K_{\epsilon,R}(\eta,\eta')[(a+a')^2(v+v')^m - a^2v^m - a'^2v'^m]g_{\epsilon,R}(\eta,t)g_{\epsilon,R}(\eta',t)\mathrm{d}\eta'\mathrm{d}\eta \\ &\leq \int_{(0,\infty)^2}\int_{(0,\infty)^2} K(\eta,\eta')[(a+a')^2(v+v')^m - a^2v^m - a'^2v'^m]g_{\epsilon,R}(\eta,t)g_{\epsilon,R}(\eta',t)\mathrm{d}\eta'\mathrm{d}\eta\end{aligned} \tag{A.4.6}$$

and use the fact that

$$\begin{aligned}(a+a')^2(v+v')^m - a^2v^m - a'^2v'^m &\leq [(a+a')^2 - a^2 - a'^2](v+v')^m + (a^2+a'^2)(v+v')^m - a^2v^m - a'^2v'^m \\ &\leq 2aa'(v+v')^m + (a^2+a'^2)[(v+v')^m - v^m - v'^m] \\ &+ (a^2+a'^2)(v^m+v'^m) - a^2v^m - a'^2v'^m \\ &\leq 2aa'(v+v')^m + \sum_{k=1}^{k_m}\binom{m}{k}(a^2+a'^2)(v^kv'^{m-k} + v^{m-k}v'^k) \\ &+ a^2v'^m + a'^2v^m.\end{aligned}$$

The terms containing powers of a of lower order, such as av^k, $k \in [0,m]$, are bounded iteratively. The rest of the terms are of the form a^2v^b, with $b-\alpha, b+\beta \in (-\alpha, m)$ or with $b - \alpha = -\alpha$. For $b-\alpha, b+\beta \in (-\alpha,m)$, we can use Hölder's inequality for the integral containing the coagulation kernel:

$$\int_{(0,\infty)^2} g_{\epsilon,R}(\eta,t)a^2v^{b+\beta}\mathrm{d}\eta \leq \Big(\int_{(0,\infty)^2} g_{\epsilon,R}(\eta,t)a^2v^m\mathrm{d}\eta\Big)^\theta\Big(\int_{(0,\infty)^2} g_{\epsilon,R}(\eta,t)a^2v^{-\alpha}\mathrm{d}\eta\Big)^{1-\theta}, \tag{A.4.7}$$

for some $\theta \in (0,1)$. We can estimate $M_{2,-\alpha}$ in a completely analogous manner as $M_{2,0}$. The moments containing only powers of v are bounded from above.

Thus, we combine (A.4.6) and (A.4.7) to deduce that

$$\begin{aligned}\partial_t M_{2,m}(g_{\epsilon,R}(t)) &\lesssim C_1(t)\Big(\int_{(0,\infty)^2} g_{\epsilon,R}(\eta,t)a^2v^m\mathrm{d}\eta\Big)^\theta + C_2(t) \\ &\leq C_1(T)\Big(\int_{(0,\infty)^2} g_{\epsilon,R}(\eta,t)a^2v^m\mathrm{d}\eta\Big)^\theta + C_2(T),\end{aligned}$$

for some $C_1, C_2 : \mathbb{R}_+ \to \mathbb{R}_+$, increasing, but finite for any fixed time. We conclude using Grönwall's inequality. □

In Proposition A.4.3, we derived uniform estimates in ϵ, R for the moments in (A.2.18) of $g_{\epsilon,R}$. This means that the limit g will satisfy the same moment estimates.

Remark A.4.4. g satisfies the estimates in (A.2.18).

Proof of Proposition A.2.9. Proposition A.2.9 follows by combining Propositions A.4.1 - A.4.3 and Remark A.4.4. ◻

We now prove a technical proposition which shows that we can test (A.2.4) with $\varphi \equiv a$.

Proposition A.4.5. *Let $T > 0$. Let $\mu < 0$. Then equation (A.2.4) holds for every $\varphi \in \mathrm{C}^1(\mathbb{R}^2_{>0})$ with $\sup_{\eta\in\mathbb{R}^2_{>0}} |\varphi(\eta) + a\partial_a\varphi(\eta)| \leq Ca$ and $\sup_{\eta\in\mathbb{R}^2_{>0}} |\partial_v\varphi(\eta, t)| \leq C$ if (A.2.18) holds.*

Proof. Let $T > 0$, $t \in [0, T]$. We construct a sequence of functions $\{\zeta_n\}_{n\in\mathbb{N}} \subset \mathrm{C}^1_c(\mathbb{R}^2_{>0})$ such that $\zeta_n(\eta) = 1$ when $\eta \in [\frac{1}{n}, n]^2$ and $\zeta_n(\eta) = 0$ when $\eta \notin [\frac{1}{2n}, 2n]^2$. The idea is to use Lebesgue's dominated convergence theorem in (A.2.4) for the functions $\varphi_n = \zeta_n\varphi$. We thus show below only the needed estimates for the proof. The term with the coagulation kernel in (A.2.4) can be bounded directly by

$$\left|\langle\mathbb{K}[g](t), \varphi_n\rangle\right| \leq C \sup_{s\in[0,T]} M_{1,-\alpha}(g(s)) \sup_{s\in[0,T]} M_{0,\beta}(g(s)) + C \sup_{s\in[0,T]} M_{0,-\alpha}(g(s)) \sup_{s\in[0,T]} M_{1,\beta}(g(s)) \leq C(T).$$

We now wish to control the fusion term in (A.2.4), namely

$$\int_{(0,\infty)^2} |r(\eta)(c_0 v^{\frac{2}{3}} - a)\partial_a\varphi_n(\eta)| g(\eta, s)\mathrm{d}\eta.$$

Notice that we can construct ζ_n such that $|a\partial_a\zeta_n(\eta)| \leq C$, for some constant independent of $n \in \mathbb{N}$. Moreover, we know that the fusion kernel satisfies (A.1.10) and that $a \geq c_0 v^{\frac{2}{3}}$. Thus

$$|r(a, v)(c_0 v^{\frac{2}{3}} - a)| \lesssim a^{\mu+1}v^{\sigma} + a^{\mu}v^{\sigma+\frac{2}{3}} \leq 2a^{\mu+1}v^{\sigma} \leq 2c_0^{\mu} a v^{\gamma-1}. \tag{A.4.8}$$

Using the above inequality, we find that the following upper bound holds

$$|r(\eta)(c_0 v^{\frac{2}{3}} - a)\partial_a\varphi_n(\eta)| \lesssim a v^{\gamma-1}|\partial_a\varphi(\eta)| + a v^{\gamma-1}|\varphi(\eta)\partial_a\zeta_n(\eta)|$$

up to a multiplicative constant. We then use Young's inequality to deduce that

$$a v^{\gamma-1}|\partial_a\varphi(\eta)| \lesssim a v^{\beta} + a v^{-\alpha-1}$$

and

$$a v^{\gamma-1}|\varphi(\eta)\partial_a\zeta_n(\eta)| \leq C a v^{\gamma-1} \lesssim a v^{\beta} + a v^{-\alpha-1}.$$

Since we have that

$$\int_{(0,\infty)^2} (a v^{\beta} + a v^{-\alpha-1}) g(\eta, s)\mathrm{d}\eta \leq \sup_{s\in[0,T]} M_{1,\beta}(g(s)) + \sup_{s\in[0,T]} M_{1,-\alpha-1}(g(s)) < \infty,$$

the moment estimates in (A.2.18) suffice in order to conclude our proof. ◻

We now have all the necessary parts in order to prove Theorem A.2.12. We first prove that the total area goes to infinity as $t \to \infty$. More precisely, we will obtain a lower bound for the total surface area that increases exponentially in time.

Proof of Theorem A.2.12. From Proposition A.4.5, we notice that the estimates (A.2.18) on g allow us to test with $\varphi(a, v, t) = a$.

Since $\langle \mathbb{K}[g], a\rangle = 0$ and using (A.1.10), equation (A.2.4) becomes

$$\begin{aligned}\partial_t \int_{(0,\infty)^2} g(\eta, t) a \mathrm{d}\eta &= \int_{(0,\infty)^2} g(\eta, t) a \mathrm{d}\eta - \frac{2}{3}\int_{(0,\infty)^2} g(\eta, t) a \mathrm{d}\eta + (1-\gamma)\int_{(0,\infty)^2} r(\eta)(c_0 v^{\frac{2}{3}} - a) g(\eta, t)\mathrm{d}\eta \\ &\geq \frac{1}{3}\int_{(0,\infty)^2} g(\eta, t) a \mathrm{d}\eta - (1-\gamma)R_1 \int_{(0,\infty)^2} a^{-|\mu|+1} v^{\sigma} g(\eta, t)\mathrm{d}\eta.\end{aligned}$$

We distinguish now two cases:

- Case 1:

  If $\mu \in (-1, 0)$, we can apply Young's inequality to obtain:

$$a^{1+\mu} v^{\sigma} \leq \frac{\epsilon}{p} a + \frac{\epsilon^{-\frac{q}{p}}}{q} v^{\frac{\sigma}{|\mu|}}, \tag{A.4.9}$$

  where $p = \frac{1}{1-|\mu|}$ and $q = \frac{1}{|\mu|}$. Thus, there exists a constant $C_\epsilon > 0$ for which:

$$-\int_{(0,\infty)^2} a^{\mu+1} v^{\sigma} g(a, v, t)\mathrm{d}v\mathrm{d}a \gtrsim -\frac{\epsilon}{p}\int_{(0,\infty)^2} a g(a, v, t)\mathrm{d}v\mathrm{d}a - C_\epsilon \int_{(0,\infty)^2} v^{-\frac{\sigma}{\mu}} g(a, v, t)\mathrm{d}v\mathrm{d}a.$$

  Choose $\epsilon(R_1)$ such that $(1-\gamma)R_1 \frac{\epsilon(R_1)}{p} < \tilde{\delta}$, for some $\tilde{\delta} \in (0, \frac{1}{3})$. Then, condition (*A.2.17*) implies:

$$\partial_t \int_{\mathbb{R}^2_{>0}} g(\eta, t) a \mathrm{d}\eta \geq \left(\frac{1}{3} - \tilde{\delta}\right)\int_{\mathbb{R}^2_{>0}} g(\eta, t) a \mathrm{d}\eta - \tilde{C}_\epsilon, \tag{A.4.10}$$

  where $\tilde{C}_\epsilon = (1-\gamma)R_1 \frac{\epsilon(R_1)^{-\frac{q}{p}}}{q} \sup_{t\geq 0} M_{0,\frac{\sigma}{|\mu|}}(g(t))$.

- Case 2:

  If $\mu \leq -1$, using (A.1.12), we have that $a^{1+\mu} v^{\sigma} \leq c_0^{1+\mu} v^{\gamma - \frac{1}{3}}$ and the equation for g becomes:

$$\partial_t \int_{\mathbb{R}^2_{>0}} g(\eta, t) a \mathrm{d}\eta \geq \frac{1}{3}\int_{\mathbb{R}^2_{>0}} g(\eta, t) a \mathrm{d}\eta - C \int_{(0,\infty)^2} v^{\gamma - \frac{1}{3}} g(\eta, t)\mathrm{d}\eta.$$

  The moment $M_{0,\gamma-\frac{1}{3}}$ is bounded from above, cf. (A.4.2), so we are in the same situation as in Case 1.

We define $C(R_1) := \frac{(1-\gamma)R_1}{\frac{1}{3}-\tilde{\delta}} \frac{\epsilon(R_1)^{-\frac{q}{p}}}{q} \sup_{t\geq 0} M_{0,\frac{\sigma}{|\mu|}}(g(t))$, where $\tilde{\delta}$ is as in (A.4.10), when $\mu \in (-1,0)$ and $C(R_1) := 3(1-\gamma)R_1 c_0^{1+\mu} \sup_{t\geq 0} M_{0,\gamma-\frac{1}{3}}(g(t))$, when $\mu \leq -1$.

We have that, if $\mu \in (-1,0)$,

$$\partial_t \int_{\mathbb{R}^2_{>0}} g(\eta,t) a \mathrm{d}\eta \geq (\frac{1}{3}-\tilde{\delta}) \int_{\mathbb{R}^2_{>0}} g(\eta,t) a \mathrm{d}\eta - (\frac{1}{3}-\tilde{\delta}) C(R_1).$$

If $\mu \leq -1$, we have

$$\partial_t \int_{\mathbb{R}^2_{>0}} g(\eta,t) a \mathrm{d}\eta \geq \frac{1}{3} \int_{\mathbb{R}^2_{>0}} g(\eta,t) a \mathrm{d}\eta - \frac{1}{3} C(R_1).$$

Thus, if we choose an initial condition such that

$$\int_{(0,\infty)^2} a g_{\mathrm{in}}(a,v) \mathrm{d}v \mathrm{d}a \geq 2C(R_1)$$

then

$$\int_{(0,\infty)^2} a g(a,v,t) \mathrm{d}v \mathrm{d}a \geq e^{(\frac{1}{3}-\tilde{\delta})t} C(R_1), \tag{A.4.11}$$

if $\mu \in (-1,0)$ and

$$\int_{(0,\infty)^2} a g(a,v,t) \mathrm{d}v \mathrm{d}a \geq e^{\frac{1}{3}t} C(R_1),$$

when $\mu \leq -1$.

We now prove that we can improve the lower bound found in (A.4.11) by removing the $\tilde{\delta}$ in the exponential. This will conclude the proof of Theorem A.2.12.

In the following, we do not write explicitly the constants that are not necessarily relevant for the proof. We want to improve the lower bound in (A.4.9) in the case when $\mu \in (-1,0)$. For $\mu \in (-1,0)$, we let ϵ to decrease to 0 as $t \to \infty$ in (A.4.9).

More precisely, we take a function $\epsilon(t) := \epsilon_1 \mathrm{e}^{-\epsilon_1 \frac{|\mu|}{1-|\mu|} t} \in (0, \epsilon_1]$, for some constant

$$\epsilon_1 \in (0, \frac{1}{12}). \tag{A.4.12}$$

This means that $\epsilon(t)^{-\frac{q}{p}} = \epsilon_1^{-\frac{q}{p}} \mathrm{e}^{\epsilon_1 t}$ in (A.4.9) and thus:

$$a^{1+\mu} v^{\sigma} \leq \frac{\epsilon(t)}{p} a + C_{\epsilon_1,\mu} \frac{\mathrm{e}^{\epsilon_1 t}}{q} v^{\frac{\sigma}{|\mu|}},$$

where $C_{\epsilon_1,\mu}$ is a constant depending on ϵ_1 and μ. We denote $A(t) := \int_{\mathbb{R}^2_{>0}} g(\eta,t) a \mathrm{d}\eta$ and equation (A.2.4) then becomes:

$$\partial_t A(t) \geq \Big(\frac{1}{3} - \epsilon(t)\Big) A(t) - \tilde{C} \mathrm{e}^{\epsilon_1 t}, \tag{A.4.13}$$

where $\tilde{C}$ is a constant depending on ϵ_1, μ and $M_{0,\frac{\sigma}{|\mu|}}(g(t))$. We note that $\int_0^t \epsilon(s)\mathrm{d}s = \frac{1-|\mu|}{|\mu|}\left(1 - \mathrm{e}^{-\epsilon_1 \frac{|\mu|}{1-|\mu|}t}\right)$.

Since $\epsilon(t) \leq \epsilon_1$, for every $t \geq 0$, we have that $\int_0^t \epsilon(s)\mathrm{d}s \leq \epsilon_1 t$. Multiplying (A.4.13) with $\mathrm{e}^{-\frac{1}{3}t+\int_0^t \epsilon(s)\mathrm{d}s}$, we obtain:

$$\mathrm{e}^{-\frac{1}{3}t+\int_0^t \epsilon(s)\mathrm{d}s}\left[\partial_t A(t) - (\frac{1}{3} - \epsilon(t))A(t)\right] \geq -\tilde{C}\mathrm{e}^{\epsilon_1 t}\mathrm{e}^{-\frac{1}{3}t+\int_0^t \epsilon(s)\mathrm{d}s} \geq -\tilde{C}\mathrm{e}^{2\epsilon_1 t}\mathrm{e}^{-\frac{1}{3}t} \geq -\tilde{C}\mathrm{e}^{-\frac{1}{6}t}, \tag{A.4.14}$$

since ϵ_1 satisfies (A.4.12). Then we integrate in time from 0 to t:

$$\mathrm{e}^{-\frac{1}{3}t+\int_0^t \epsilon(s)\mathrm{d}s}A(t) - A(0) \geq 6\tilde{C}\mathrm{e}^{-\frac{1}{6}t} - 6\tilde{C}$$

and thus

$$A(t) \geq A(0)\mathrm{e}^{\frac{1}{3}t-\int_0^t \epsilon(s)\mathrm{d}s} - 6\tilde{C}\mathrm{e}^{\frac{1}{3}t-\int_0^t \epsilon(s)\mathrm{d}s} + 6\tilde{C}\mathrm{e}^{\frac{1}{6}t-\int_0^t \epsilon(s)\mathrm{d}s} \geq A(0)\mathrm{e}^{\frac{1}{3}t-\int_0^t \epsilon(s)\mathrm{d}s} - 6\tilde{C}\mathrm{e}^{\frac{1}{3}t-\int_0^t \epsilon(s)\mathrm{d}s}.$$

Using the definition of $A(t)$, this can be written as

$$\int_{\mathbb{R}^2_{>0}} g(\eta,t)a\mathrm{d}\eta \geq \left(\int_{\mathbb{R}^2_{>0}} g(\eta,0)a\mathrm{d}\eta - 6\tilde{C}\right)\mathrm{e}^{\frac{1}{3}t-\int_0^t \epsilon(s)\mathrm{d}s}.$$

This implies that, if we start with sufficiently large total surface area, $\int_{\mathbb{R}^2_{>0}} g(\eta,0)a\mathrm{d}\eta \geq 12\tilde{C}$, we have:

$$\int_{\mathbb{R}^2_{>0}} g(\eta,t)a\mathrm{d}\eta \geq 6\tilde{C}\mathrm{e}^{\frac{1}{3}t-\int_0^t \epsilon(s)\mathrm{d}s} \gtrsim C\mathrm{e}^{\frac{1}{3}t},$$

where we used that $1 - \mathrm{e}^{-\epsilon_1\frac{|\mu|}{1-|\mu|}t} \in [0,1]$ and thus $\mathrm{e}^{-\int_0^t \epsilon(s)\mathrm{d}s} \geq \mathrm{e}^{-\frac{1-|\mu|}{|\mu|}}$.

This concludes the proof of Theorem A.2.12. □

A.4.2 Self-similarity in the case of fast fusion

In this subsection we prove Theorem A.2.7.

We look at the following truncated version of equation (A.2.4):

$$\begin{aligned}&\int_{(0,\infty)^2} g(\eta,t)\Theta_\epsilon(v)\varphi(\eta)\mathrm{d}\eta - \frac{2}{3}\int_{(0,\infty)^2} g(\eta,t)\tilde{\Theta}_\epsilon(a,v)a\partial_a\varphi(\eta)\mathrm{d}\eta - \int_{(0,\infty)^2} g(\eta,t)\Theta_\epsilon(v)v\partial_v\varphi(\eta)\mathrm{d}\eta\\ &+(1-\gamma)\langle\mathbb{K}_{\epsilon,R}[g],\varphi\rangle + (1-\gamma)\int_{(0,\infty)^2} r(\eta)(c_0 v^{\frac{2}{3}} - a)g(\eta,t)\partial_a\varphi(\eta)\mathrm{d}\eta = \partial_t\int_{(0,\infty)^2} g(\eta,t)\varphi(\eta)\mathrm{d}\eta,\end{aligned} \tag{A.4.15}$$

where $\mathbb{K}_{\epsilon,R}$ is the term in (A.3.6) with coagulation kernel $K_{\epsilon,R}$ defined in (A.3.1), Θ_ϵ was defined in (A.3.4) and $\varphi \in \mathrm{C}^1_c(\mathbb{R}^2_{>0})$. We define a continuous function $\tilde{\Theta}_\epsilon : \mathbb{R}^2_{>0} \to [0,1]$ such that:

$$\tilde{\Theta}_\epsilon(a,v) = \begin{cases}0 & a \leq 1-\epsilon \text{ and } v \leq \epsilon;\\ \Theta_\epsilon(v) & a = c_0 v^{\frac{2}{3}};\\ 1 & a > 1 \text{ or } v \geq 2\epsilon,\end{cases} \tag{A.4.16}$$

which satisfies $\tilde{\Theta}_\epsilon(a, v) \geq \Theta_\epsilon(v)$. The choice in (A.4.16) was made in order for (A.4.15) to preserve the isoperimetric inequality in time. This means that if g solves (A.4.15) and $g(0) \in \mathscr{M}_+^I(\mathbb{R}^2_{>0})$, then $g(t) \in \mathscr{M}_+^I(\mathbb{R}^2_{>0})$, for any $t \geq 0$. Additionally, $\tilde{\Theta}_\epsilon$ does not vanish in the region $\{a > 1\}$. This form will simplify the proof of the moment estimates.

Notice that, differently from Section A.3, we do not truncate the fusion rate $r(\eta)$ since $r(\eta)(c_0 v^{\frac{2}{3}} - a)$ grows at most linearly.

The statement of Proposition A.3.21 holds if we replace equation (A.3.5) with (A.4.15). This is obtained using the same proof and the growth rate of $r(\eta)(c_0 v^{\frac{2}{3}} - a)$ in (A.4.15). Thus, we prove the existence of solutions of (A.4.15) as in Proposition A.3.21 with $N = 2$, which we will denote by $g_{\epsilon,R}$. The choice $N = 2$ was made since we want to test equation (A.4.15) with a^2 in order to obtain uniform moment bounds later in the proof.

To prove Theorem A.2.7 we will follow the strategy of Theorem A.2.3. The main point is to find an invariant set that allows us to pass to the limit as $\epsilon \to 0$ and $R \to \infty$ in the two-dimensional coagulation equation. In this case, it will be harder to estimate the moment $M_{1,0}$ than in the case when $\mu > 0$. This is because of the fact that, in the case of (A.3.5), the moment estimates are a consequence of the fast growth of the fusion ratio $r(\eta)$. This fast growth does not take place in (A.4.15). We will be able to replace the fast growth of $r(\eta)$ with the choice of a sufficiently large constant λ.

We first prove some elementary inequalities that will be useful for the proof of Theorem A.2.7.

Proposition A.4.6. *Let $\mu \leq 0$. For $\delta_1 \in (0, 1)$, there exist λ sufficiently large (depending on δ_1) and some constants $m_1 = \alpha(\mu - 2) - \sigma$, $m_2 = \beta(2 - \mu) - \sigma$ such that:*

$$av^{-\alpha} \leq \frac{1-\gamma}{6}\lambda a^{\mu+1} v^{\sigma-\alpha} + \delta_1 a^2 + v^{m_1}; \tag{A.4.17}$$

$$av^{\beta} \leq \frac{1-\gamma}{2(K_0 c_{0,\beta-\alpha} + 2)}\lambda a^{\mu+1} v^{\sigma+\beta} + \delta_1 a^2 + v^{m_2}, \tag{A.4.18}$$

where $c_{0,\beta-\alpha}$ is a constant that will be fixed in Lemma A.4.7 and K_0 is as in (A.1.7).

Proof. We prove (A.4.17) in detail. The strategy to prove (A.4.17) and (A.4.18) is the same. The proof is as follows.

For any $\delta_2, \delta_3 \in (0, 1)$, there exist $C_{\delta_2} > 0$ and $C_{\delta_3} > 0$ such that

$$av^{-\alpha} \leq C_{\delta_2} a^{\frac{\mu+1}{2}} v^{-\alpha\frac{3-\mu}{2}} + \delta_2 a^2$$

and

$$a^{\frac{\mu+1}{2}} v^{-\alpha\frac{3-\mu}{2}} \leq C_{\delta_3} a^{\mu+1} v^{\sigma-\alpha} + \delta_3 v^{m_1},$$

with $m_1 = \alpha(\mu - 2) - \sigma$. Combining the two inequalities, we obtain

$$av^{-\alpha} \leq C_{\delta_2} C_{\delta_3} a^{\mu+1} v^{\sigma-\alpha} + C_{\delta_2}\delta_3 v^{m_1} + \delta_2 a^2.$$

To conclude the argument, we choose λ sufficiently large such that $\frac{1-\gamma}{6}\lambda \geq C_{\delta_2}C_{\delta_3}$ and choose δ_3 such that $C_{\delta_2}\delta_3 \leq 1$. □

We now make use of Proposition A.4.6 to prove that we have uniform bounds in ϵ and R for the solutions of equation (A.4.15).

Lemma A.4.7. *Let $K_{\epsilon,R}$ be as in (A.3.1) and assume r satisfies the assumptions of Theorem A.2.7. Let $T > 0$. We define a semigroup $S(t) : U_{\epsilon,R,2} \to U_{\epsilon,R,2}$ in the following way: $S(t)g_{\text{in},R} = g_{\epsilon,R}(\cdot,\cdot,t)$, for all $t \in [0,T]$ and with $U_{\epsilon,R,2}$ defined as in (A.3.56) with $N = 2$. Take $\mu \leq 0$, fix $\tilde{\epsilon} \in (0,1)$, let $m_i \in \mathbb{R}$, $i = \overline{1,2}$, to be the coefficients found in Proposition A.4.6 and denote $\tilde{\alpha} := \alpha + \tilde{\epsilon}$. Then there exist constants $c_{0,-\tilde{\alpha}}, c_{0,m_i}, c_{2,0}, c_{1,-\alpha}, c_{1,\beta} > 0$, $i \in \{1,2\}$, for which the set*

$$\begin{aligned}\tilde{\omega} =&\{M_{0,1}(g_{\epsilon,R}) = 1; M_{0,-\tilde{\alpha}}(g_{\epsilon,R}) \leq c_{0,-\tilde{\alpha}}; M_{0,2}(g_{\epsilon,R}) \leq c_{0,2}; M_{0,m_i}(g_{\epsilon,R}) \leq c_{0,m_i};\\ &M_{2,0}(g_{\epsilon,R}) \leq c_{2,0}; M_{1,-\alpha}(g_{\epsilon,R}) \leq c_{1,-\alpha}; M_{1,\beta}(g_{\epsilon,R}) \leq c_{1,\beta}\},\end{aligned}$$

$i \in \{1,2\}$, is preserved in time uniformly in ϵ, R under equation (A.4.15), or equivalently $S(t)\tilde{\omega} \subseteq \tilde{\omega}$, for all $t \in [0,T]$.

Proof. Uniform bounds for moments of the form $M_{0,d}$, with $d \in \mathbb{R}$, are derived as in Propositions A.3.16 - A.3.18. We now find uniform bounds for $M_{2,0}(g)$, $M_{1,-\alpha}(g)$ and $M_{1,\beta}(g)$. For $\varphi \in \mathrm{C}^1(\mathbb{R}^2_{>0})$, whose growth in a is of order $1 + a^2$ and with $\partial_a\varphi(a,v) \geq 0$, we have that:

$$\begin{aligned}&\partial_t \int_{(0,\infty)^2} g(\eta,t)\varphi(\eta)\mathrm{d}\eta \leq \int_{(0,\infty)^2} g(\eta,t)\Theta_\epsilon(v)\varphi(\eta)\mathrm{d}\eta - \frac{2}{3}\int_{(0,\infty)^2} g(\eta,t)\tilde{\Theta}_\epsilon(a,v)a\partial_a\varphi(\eta)\mathrm{d}\eta\\ &- \int_{(0,\infty)^2} g(\eta,t)\Theta_\epsilon(v)v\partial_v\varphi(\eta)\mathrm{d}\eta + (1-\gamma)\langle\mathbb{K}_{\epsilon,R}[g],\varphi\rangle + \lambda(1-\gamma)\int_{(0,\infty)^2} a^\mu v^\sigma(c_0v^{\frac{2}{3}} - a)g(\eta,t)\partial_a\varphi(\eta)\mathrm{d}\eta.\end{aligned}$$

Thus we obtain:

$$\begin{aligned}\partial_t M_{2,0}(g(t)) &\leq -\frac{1}{3}\int_{(0,\infty)^2} g(\eta,t)a^2\mathrm{d}\eta + \frac{4}{3}\int_{\{a\leq 1\}} g(\eta,t)a^2\mathrm{d}\eta\\ &\quad + 2K_0(1-\gamma)\int_{(0,\infty)^2} g(\eta,t)av^{-\alpha}\mathrm{d}\eta\int_{(0,\infty)^2} g(\eta',t)a'v'^{\beta}\mathrm{d}\eta'.\\ \partial_t M_{1,-\alpha}(g(t)) &\leq (\frac{1}{3}+\alpha)\int_{(0,\infty)^2} g(\eta,t)\Theta_\epsilon(v)av^{-\alpha}\mathrm{d}\eta + (1-\gamma)\lambda\int_{(0,\infty)^2} a^\mu v^{\sigma-\alpha}(c_0v^{\frac{2}{3}} - a)g(\eta,t)\mathrm{d}\eta.\\ \partial_t M_{1,\beta}(g(t)) &\leq (\frac{1}{3}-\beta)\int_{(0,\infty)^2} g(\eta,t)\Theta_\epsilon(v)av^{\beta}\mathrm{d}\eta + (1-\gamma)\lambda\int_{(0,\infty)^2} a^\mu v^{\sigma+\beta}(c_0v^{\frac{2}{3}} - a)g(\eta,t)\mathrm{d}\eta\\ &\quad + K_0(1-\gamma)\int_{(0,\infty)^2} g(\eta,t)av^{\beta}\mathrm{d}\eta\int_{(0,\infty)^2} g(\eta',t)v'^{\beta-\alpha}\mathrm{d}\eta'\\ &\quad + K_0(1-\gamma)\int_{(0,\infty)^2} g(\eta,t)av^{-\alpha}\mathrm{d}\eta\int_{(0,\infty)^2} g(\eta',t)v'^{2\beta}\mathrm{d}\eta'.\end{aligned} \tag{A.4.19}$$

Notice that in (A.4.19) the truncation on the kernel does not create problems: for $M_{1,-\alpha}$ the term with the coagulation kernel is non-positive and for $M_{2,0}$, $M_{1,\beta}$ we used $K_{\epsilon,R} \leq K$.

Making use of (A.4.17) and (A.4.18), (A.4.19) becomes:

$$\begin{aligned}
\partial_t M_{2,0}(g(t)) &\leq -\frac{1}{3} M_{2,0}(g(t)) + \frac{4}{3} M_{0,0}(g(t)) + 2K_0(1-\gamma) M_{1,-\alpha}(g(t)) M_{1,\beta}(g(t)).\\
\partial_t M_{1,-\alpha}(g(t)) &\leq (\frac{1}{3} + \alpha) M_{1,-\alpha}(g(t)) - \frac{(1-\gamma)\lambda}{2} \int_{\{a \geq 2c_0 v^{\frac{2}{3}}\}} a^{\mu+1} v^{\sigma-\alpha} g(\eta,t) \mathrm{d}\eta\\
&\leq -M_{1,-\alpha}(g(t)) + 6c_0 M_{0,\frac{2}{3}-\alpha}(g(t))\\
&+ 3 \int_{\{a \geq 2c_0 v^{\frac{2}{3}}\}} a v^{-\alpha} g(\eta,t) \mathrm{d}\eta - \frac{(1-\gamma)\lambda}{2} \int_{\{a \geq 2c_0 v^{\frac{2}{3}}\}} a^{\mu+1} v^{\sigma-\alpha} g(\eta,t) \mathrm{d}\eta\\
&\leq -M_{1,-\alpha}(g(t)) + 3\delta_1 M_{2,0}(g(t)) + 3M_{0,m_1}(g(t)) + 6c_0 M_{0,\frac{2}{3}-\alpha}(g(t)),
\end{aligned} \tag{A.4.20}$$

where we have used that $\alpha < 1$ (see (A.1.8)) and thus $2c_0(\alpha + \frac{4}{3}) \leq 6c_0$ and $\alpha + \frac{4}{3} \leq 3$, and

$$\begin{aligned}
\partial_t M_{1,\beta}(g(t)) &\leq -M_{1,\beta}(g(t)) + 2M_{1,\beta}(g(t)) - \frac{(1-\gamma)\lambda}{2} \int_{\{a \geq 2c_0 v^{\frac{2}{3}}\}} a^{\mu+1} v^{\sigma+\beta} g(\eta,t) \mathrm{d}\eta\\
&+ K_0(1-\gamma) c_{0,\beta-\alpha} \int_{(0,\infty)^2} g(\eta,t) a v^{\beta} \mathrm{d}\eta + K_0(1-\gamma) c_{0,2\beta} \int_{(0,\infty)^2} g(\eta,t) a v^{-\alpha} \mathrm{d}\eta\\
&\leq -M_{1,\beta}(g(t)) - \frac{(1-\gamma)\lambda}{2} \int_{\{a \geq 2c_0 v^{\frac{2}{3}}\}} a^{\mu+1} v^{\sigma+\beta} g(\eta,t) \mathrm{d}\eta + K_0 c_{0,2\beta} M_{1,-\alpha}(g(t))\\
&+ (K_0 c_{0,\beta-\alpha} + 2) M_{1,\beta}(g(t))\\
&\leq -M_{1,\beta}(g(t)) + K_0 c_{0,2\beta} M_{1,-\alpha}(g(t)) + 2C(c_{0,\beta-\alpha}) c_0 M_{0,\frac{2}{3}+\beta}(g(t)) + C(c_{0,\beta-\alpha}) \delta_1 M_{2,0}(g(t))\\
&+ C(c_{0,\beta-\alpha}) M_{0,m_2}(g(t)),
\end{aligned} \tag{A.4.21}$$

where $C(c_{0,\beta-\alpha}) = K_0 c_{0,\beta-\alpha} + 2$. As we have that $M_{0,0}$, $M_{0,\frac{2}{3}-\alpha}$, M_{0,m_1}, M_{0,m_2}, $M_{0,\frac{2}{3}+\beta}$ are uniformly bounded from above, (A.4.20) and (A.4.21) become:

$$\begin{aligned}
\partial_t M_{2,0}(g(t)) &\leq -\frac{1}{3} M_{2,0}(g(t)) + 2K_0(1-\gamma) M_{1,-\alpha}(g(t)) M_{1,\beta}(g(t)) + C;\\
\partial_t M_{1,-\alpha}(g(t)) &\leq -M_{1,-\alpha}(g(t)) + 3\delta_1 M_{2,0}(g(t)) + C;\\
\partial_t M_{1,\beta}(g(t)) &\leq -M_{1,\beta}(g(t)) + K_0 c_{0,2\beta} M_{1,-\alpha}(g(t)) + C(c_{0,\beta-\alpha}) \delta_1 M_{2,0}(g(t)) + C,
\end{aligned} \tag{A.4.22}$$

where $C > 0$ is a constant depending only on γ, c_0 and the upper bound of some moments of the form $M_{0,d}$, with $d \in \mathbb{R}$. Using (A.4.22) we are able to find a region that stays invariant in time.

In order to give some insight about the proof, we consider first the case $\delta_1 = 0$, since the result for $\delta_1 > 0$, sufficiently small, follows by a perturbative argument. (A.4.22) becomes

$$\begin{aligned}
\partial_t M_{2,0}(g(t)) &\leq -\frac{1}{3} M_{2,0}(g(t)) + 2K_0(1-\gamma) M_{1,-\alpha}(g(t)) M_{1,\beta}(g(t)) + C;\\
\partial_t M_{1,-\alpha}(g(t)) &\leq -M_{1,-\alpha}(g(t)) + C;\\
\partial_t M_{1,\beta}(g(t)) &\leq -M_{1,\beta}(g(t)) + K_0 c_{0,2\beta} M_{1,-\alpha}(g(t)) + C.
\end{aligned} \tag{A.4.23}$$

Notice that from (A.4.23) we deduce that the set $\{M_{1,-\alpha}(g) \leq C, M_{1,\beta}(g) \leq C(K_0 c_{0,2\beta}+1), M_{2,0}(g) \leq 3[C + 2(1-\gamma)K_0C^2(K_0c_{0,2\beta}+1)]\}$ stays invariant in time.

Let $\delta_1 > 0$. We prove that the region defined by $\{M_{1,-\alpha}(g) \leq 2C, M_{1,\beta}(g) \leq 2C(K_0c_{0,2\beta}+1), M_{2,0}(g) \leq 3[C + 32(1-\gamma)K_0C^2(K_0c_{0,2\beta}+1)]\}$ is invariant in time.

Assume that $M_{1,-\alpha}(g(0)) \leq 2C$, $M_{1,\beta}(g(0)) \leq 2C(K_0c_{0,2\beta}+1)$ and $M_{2,0}(g(0)) \leq 3[C + 32(1-\gamma)K_0C^2(K_0c_{0,2\beta}+1)]$. By continuity in time, for all s sufficiently small, we obtain that $M_{1,-\alpha}(g(s)) \leq 4C$, $M_{1,\beta}(g(s)) \leq 4C(K_0c_{0,2\beta}+1)$. Plugging this in (A.4.22), we obtain that

$$\partial_s M_{2,0}(g(s)) \leq -\frac{1}{3}M_{2,0}(g(s)) + 32K_0(1-\gamma)C^2(K_0c_{0,2\beta}+1) + C, \tag{A.4.24}$$

for all times s small enough. Thus, the region $M_{2,0}(g(s)) \leq 3[C + 32(1-\gamma)K_0C^2(K_0c_{0,2\beta}+1)]$ is invariant for small enough times. We now make use again of (A.4.22) and the newly obtained bound for $M_{2,0}$, to deduce that

$$\begin{aligned}
\partial_s M_{1,-\alpha}(g(s)) &\leq -M_{1,-\alpha}(g(s)) + 9\delta_1[C + 32(1-\gamma)K_0C^2(K_0c_{0,2\beta}+1)] + C;\\
\partial_s M_{1,\beta}(g(s)) &\leq -M_{1,\beta}(g(s)) + K_0c_{0,2\beta}M_{1,-\alpha}(g(s))\\
&\quad + 3C(c_{0,\beta-\alpha})\delta_1[C + 32(1-\gamma)K_0C^2(K_0c_{0,2\beta}+1)] + C,
\end{aligned} \tag{A.4.25}$$

for sufficiently small times. Choosing δ_1 sufficiently small so that $9\delta_1[C+32(1-\gamma)K_0C^2(K_0c_{0,2\beta}+1)] \leq C$ and $3C(c_{0,\beta-\alpha})\delta_1[C+32(1-\gamma)K_0C^2(K_0c_{0,2\beta}+1)] \leq C$, we obtain that the regions $M_{1,-\alpha}(g(s)) \leq 2C$ and $M_{1,\beta}(g(s)) \leq 2C(K_0c_{0,2\beta}+1)$ are invariant in time for all s sufficiently small. We are now able to iterate the argument, extending it to all times. These bounds are independent of ϵ, R. □

We now have all the necessary parts to conclude the proof of Theorem A.2.7.

Proof of Theorem A.2.7. We can uniformly bound $M_{1,0}(g_{\epsilon,R}(t)) \leq M_{1,-\alpha}(g_{\epsilon,R}(t)) + M_{1,\beta}(g_{\epsilon,R}(t))$, since $M_{1,-\alpha}(g_{\epsilon,R}(t))$ and $M_{1,\beta}(g_{\epsilon,R}(t))$ are bounded. This is enough to enable us to follow the steps of the proof of Theorem A.2.3 to conclude that there exists a self-similar profile for the two-dimensional coagulation equation in this case. This finishes the proof of Theorem A.2.7. □

A.5 Formal rescaling properties

It is worthwhile to mention that the equation (A.1.1) has some useful rescaling properties. More explicitly, if f satisfies (A.1.1) and $\int_{(0,\infty)^2} vf(a,v,t)\mathrm{d}v\mathrm{d}a = v_0$, then we can define a set of functions $\tilde{f}_k$

$$f(a,v,t) = \frac{v_0}{k^{\frac{8}{3}}}\tilde{f}_k(\frac{a}{k^{\frac{2}{3}}}, \frac{v}{k}, k^{\gamma-1}v_0t). \tag{A.5.1}$$

The rescaling (A.5.1) can be used to remove the dependence on v_0, the problem being reduced to one where the total volume of particles is equal to 1. We have that $\tilde{f}_k$ solves (A.1.1) with the fusion kernel

r replaced by $\frac{r}{v_0}$ and

$$\int_{(0,\infty)^2} v\tilde{f}_k(a,v,t)\mathrm{d}v\mathrm{d}a = 1.$$

Notice that, if f satisfies (A.1.1) and g is defined as

$$f(a,v,t) = \frac{1}{t^{\frac{8}{3}\xi}} g\left(\frac{a}{t^{\frac{2}{3}\xi}}, \frac{v}{t^{\xi}}, \xi\log(t)\right), \tag{A.5.2}$$

up to a translation in time, then g satisfies (A.2.1). Combining (A.5.1) and (A.5.2), we obtain that $\tilde{g}_k$, which can be expressed in terms of g in the following manner

$$\tilde{g}_k(\tilde{A},\tilde{V},\tilde{\tau}) = v_0^{\frac{8}{3}\xi-1} g(v_0^{\frac{2}{3}\xi}\tilde{A}, v_0^{\xi}\tilde{V}, \tau), \tag{A.5.3}$$

where $\tau = \xi\log(t)$ and $\tilde{\tau} = \xi\log(k^{\gamma-1}v_0 t)$, satisfies (A.2.1) with total volume equal to 1. In particular, if we choose $k = v_0^{\xi}$, we have $\tilde{\tau} = \tau$.

We obtain the following rescaling property for $\tilde{g}_k$ in terms of g:

$$M_{y_1,y_2}(\tilde{g}_k(\tilde{\tau})) = v_0^{\frac{\gamma-\frac{2}{3}y_1-y_2}{1-\gamma}} M_{y_1,y_2}(g(\tau)), \tag{A.5.4}$$

for $y_1, y_2 \in \mathbb{R}$, which we make use of in Proposition A.2.9 and Theorem A.2.12. Notice that in the relation between the moments of g and the moments of $\tilde{g}_k$ the dependence on the variable k vanishes and the reason we choose to fix $k = v_0^{\xi}$ is in order to avoid the time translation.

The existence of a function g_{in} which satisfies (A.2.15), (A.2.20) and has volume v_0 is equivalent by means of the scaling (A.5.3) to finding a function $\tilde{g}_{\mathrm{in}}(\tilde{A},\tilde{V})$ such that all these inequalities hold with $v_0 = 1$. The existence of such a function can be easily seen if we choose its support in an appropriate region contained in $\tilde{V} \in [\frac{1}{2}, 1]$ and with $\tilde{A}$ sufficiently large.

Thus, we can assume without loss of generality that the total volume of the particles is equal to 1, keeping in mind that the constants R_0 and R_1 in (A.1.10) change by a factor v_0.

A.6 Proof of some results used to obtain the existence of solutions for the truncated time-dependent problem

Proof of Proposition A.3.6. The following remark is used in order to prove the stated properties:

$$\Phi(V,V',t)\mathrm{e}^{h_\epsilon(l_2(V,V',t),t)} = \mathrm{e}^{h_\epsilon(V,t)+h_\epsilon(V',t)}. \tag{A.6.1}$$

We make use of (A.3.20), (A.3.21), the kernel bound and the inequality $|e^{-x_1} - e^{-x_2}| \leq |x_1 - x_2|$, for $x_1, x_2 \geq 0$.

We first prove that the map stays in the ball of radius $2M$. In order to bound $J_1[G]$ in (A.3.23), we use that $x_{t,V}(A) \leq A$, for every $t \in [0,\tau]$ and the assumption $\tau \leq \ln 2$. For the term $J_2[G]$, we notice that $1 + x_{t,V}(A) + x_{t,V'}(A') \leq (1 + x_{t,V}(A))(1 + x_{t,V'}(A'))$. Thus, for every $\varphi \in C_c(\mathbb{R}^2_{>0})$ with $\|\varphi\|_\infty \leq 1$, we have:

$$\begin{aligned}\int_{(0,\infty)^2} e^{h_\epsilon(V,t)}(1 + x_{t,V}(A))J[G](A, V, t)\varphi(A, V)dVdA &\leq 2\|\varphi\|_\infty \int_{(0,\infty)^2} (1 + a)g_{in,R}(a, v)dvda \\ &+ \frac{1-\gamma}{2}\|K_R\|_\infty\|G\|^2\|\varphi\|_\infty\tau \\ &\leq M + (1-\gamma)2M^2\|K_R\|_\infty\tau \leq M + 1 \leq 2M.\end{aligned}$$

We now prove that the map J, which was defined in (A.3.23), is contractive:

$$\begin{aligned}&\left|\int_{(0,\infty)^2} e^{h_\epsilon(V,t)}(1 + x_{t,V}(A))[J[F](A, V, t) - J[G](A, V, t)]\varphi(A, V)dVdA\right| \\ &\leq 2\int_{(0,\infty)^2} (1 + a)g_{in,R}(a, v)dvda\|K_R\|_\infty\|\varphi\|_\infty\|F - G\|(e^\tau - 1) \\ &+ \frac{1-\gamma}{2}\|K_R\|_\infty(\|F\| + \|G\|)\|\varphi\|_\infty\|F - G\|\tau + \frac{1-\gamma}{2}\|K_R\|^2_\infty\|G\|^2\|\varphi\|_\infty\|F - G\|\tau(e^\tau - 1) \\ &\leq (M\|K_R\|_\infty + 2M(1-\gamma)\|K_R\|_\infty + \|K_R\|_\infty)(e^\tau - 1)\|F - G\| < \frac{1}{2}\|F - G\|.\end{aligned}$$

□

Proof of Proposition A.3.7. By testing with $e^{h_\epsilon(V,t)}x_{t,v}$, for $t \in [0, \tau]$, and using (A.6.1), we obtain that:

$$\begin{aligned}\int_{\mathbb{R}^2_{>0}} G_{\epsilon,R,\delta}(\eta, t)e^{h_\epsilon(V,t)}x_{t,V}(A)d\eta \leq 2\int_{\mathbb{R}^2_{>0}} g_{in,R}(\eta)Ad\eta + \frac{1-\gamma}{2}\int_0^t\int_{\mathbb{R}^2_{>0}}\int_{\mathbb{R}^2_{>0}} K_R(\phi_s(\eta), \phi_s(\eta'))G_{\epsilon,R,\delta}(d\eta, s) \\ G_{\epsilon,R,\delta}(d\eta', s)e^{h_\epsilon(V,s)+h_\epsilon(V',s)}[(x_{s,V}(A) + x_{s,V'}(A')) - x_{s,V}(A) - x_{s,V'}(A')]ds.\end{aligned}$$

Thus:

$$\int_{\mathbb{R}^2_{>0}} G_{\epsilon,R,\delta}(A, V, t)e^{h_\epsilon(V,t)}x_{t,V}(A)dVdA \leq 2\int_{\mathbb{R}^2_{>0}} g_{in,R}(A, V)AdVdA,$$

which is bounded by assumption. Similarly, testing with $e^{h_\epsilon(V,t)}$, we get:

$$\int_{\mathbb{R}^2_{>0}} G_{\epsilon,R,\delta}(A, V, t)e^{h_\epsilon(V,t)}dVdA \leq 2\int_{\mathbb{R}^2_{>0}} g_{in,R}(A, V)dVdA.$$

The condition imposed on $g_{in,R}$ in order to prove Proposition A.3.6 was $2\int_{(0,\infty)^2}(1+a)g_{in,R}(a, v)dvda \leq M$. Thus, we can replace $g_{in,R}$ by $G_{\epsilon,R,\delta}(\cdot, \cdot, \tau)$ and then iterate the argument to extend the solution to all times. □

Proof of Proposition A.3.14. Let $s, t \in [0, T]$. We denote by $g(\cdot, t) := S(t)g(\cdot)$. Let $n \in \mathbb{N}$ be sufficiently large. Let $\varphi \in \mathrm{C_c}(\mathbb{R}^2_{>0})$ and $\varphi_n \in \mathrm{C}^1_\mathrm{c}(\mathbb{R}^2_{>0})$ be such that $\sup_{(a,v)\in\mathbb{R}^2_{>0}} |\varphi_n(a,v) - \varphi(a,v)| \leq \frac{1}{n}$. Then

$$
\begin{aligned}
\Big|\int_{(0,\infty)^2} [S(t)g(\eta) - S(s)g(\eta)]\varphi(\eta)\mathrm{d}\eta\Big| &\leq \\
&\leq \Big|\int_{(0,\infty)^2} [g(\eta,t) - g(\eta,s)]\varphi_n(\eta)\mathrm{d}\eta\Big| + \Big|\int_{(0,\infty)^2} [g(\eta,t) - g(\eta,s)][\varphi_n(\eta) - \varphi(\eta)]\mathrm{d}\eta\Big| \\
&\leq \Big|\int_{(0,\infty)^2} [g(\eta,t) - g(\eta,s)]\varphi_n(\eta)\mathrm{d}\eta\Big| + \frac{2}{n}\sup_{r\in[0,T]}\int_{(0,\infty)^2} g(\eta,r)\mathrm{d}\eta \\
&\leq \Big|\int_{(0,\infty)^2} [g(\eta,t) - g(\eta,s)]\varphi_n(\eta)\mathrm{d}\eta\Big| + \frac{C}{n},
\end{aligned} \tag{A.6.2}
$$

where we used that we can bound $\sup_{r\in[0,T]}\int_{(0,\infty)^2} g(\eta,r)\mathrm{d}\eta$ from above. On the other hand,

$$
\begin{aligned}
\Big|\int_{(0,\infty)^2} [g(\eta,t) - g(\eta,s)]\varphi_n(\eta)\mathrm{d}\eta\Big| \leq \int_s^t\int_{(0,\infty)^2} |g(\eta,r)\varphi_n(\eta)|\mathrm{d}\eta\mathrm{d}r + \frac{2}{3}\int_s^t\int_{(0,\infty)^2} |g(\eta,r)a\partial_a\varphi_n(\eta)|\mathrm{d}\eta\mathrm{d}r \\
+ \int_s^t\int_{(0,\infty)^2} |g(\eta,r)v\partial_v\varphi_n(\eta)|\mathrm{d}\eta\mathrm{d}r + (1-\gamma)\int_s^t |\langle\mathbb{K}_{\epsilon,R}[g](r),\varphi_n\rangle|\mathrm{d}r \\
+ (1-\gamma)\int_s^t\int_{(0,\infty)^2} r_\delta(\eta)|c_0v^{\frac{2}{3}} - a|g(\eta,r)|\partial_a\varphi_n(\eta)|\mathrm{d}\eta\mathrm{d}r.
\end{aligned}
$$

As $K_{\epsilon,R}$ is bounded and $g_{\epsilon,R,\delta}$ has compact support in the v variable, we can find a constant $C(\epsilon,R,\delta)$, which depends on ϵ, R, δ such that

$$
\begin{aligned}
\Big|\int_{(0,\infty)^2} [g(\eta,t) - g(\eta,s)]\varphi_n(\eta)\mathrm{d}\eta\Big| \leq \frac{2}{3}\int_s^t\int_{(0,\infty)^2} |g(\eta,r)a\partial_a\varphi_n(\eta)|\mathrm{d}\eta\mathrm{d}r + C(\epsilon,R,\delta)|t-s| \\
+ \frac{1}{\delta}C(\epsilon,R,\delta)\int_s^t\int_{(0,\infty)^2} |1+a|g(\eta,r)|\partial_a\varphi_n(\eta)|\mathrm{d}\eta\mathrm{d}r.
\end{aligned}
$$

As $\sup_{r\in[0,T]}\int_{(0,\infty)^2}(1+a)g_{\epsilon,R,\delta}(\eta,r)\mathrm{d}\eta$ is bounded from above, we find that there exists $C(\epsilon,R,\delta)$ depending on the written parameters, such that

$$
\Big|\int_{(0,\infty)^2} [g(\eta,t) - g(\eta,s)]\varphi_n(\eta)\mathrm{d}\eta\Big| \leq C(\epsilon,R,\delta)|t-s|. \tag{A.6.3}
$$

Combining (A.6.3) with (A.6.2), we obtain the continuity in time of the map $t \to \int_{(0,\infty)^2} g(\eta,t)\varphi(\eta)\mathrm{d}\eta$, if $\varphi \in \mathrm{C_c}(\mathbb{R}^2_{>0})$. We now extend the argument to all functions $\varphi \in \mathrm{C}_0(\mathbb{R}^2_{>0})$. Let $\varphi \in \mathrm{C}_0(\mathbb{R}^2_{>0})$, with $||\varphi||_\infty \leq 1$. Due to the support of $g \in U_{\epsilon,R}$, it is enough to cut the function φ for large values of a in order to make it compactly supported. Let $\chi_M(a)$ be as in (A.3.31). Then

$$
\begin{aligned}
\Big|\int_{(0,\infty)^2} [g(\eta,t) - g(\eta,s)]\varphi(\eta)\mathrm{d}\eta\Big| &\leq \\
&\leq \Big|\int_{(0,\infty)^2} [g(\eta,t) - g(\eta,s)]\varphi(\eta)\chi_M(a)\mathrm{d}\eta\Big| + 2\int_{(M,\infty)\times[\epsilon,2R]} \Big|[g(\eta,t) - g(\eta,s)]\varphi(\eta)\Big|\mathrm{d}\eta \\
&\leq C|t-s| + 2M^{-1}\int_{(M,\infty)\times[\epsilon,2R]} a[g(\eta,t) + g(\eta,s)]\mathrm{d}\eta \\
&\leq C|t-s| + c(t)M^{-1}.
\end{aligned}
$$

We conclude the argument by taking M sufficiently large. □

Proof of Proposition A.3.24. Let $n \in \mathbb{N}$. By Lemma A.3.23, we have that:

$$\int_{\mathbb{R}^2_{>0}} G_{\epsilon,R,\delta}(\eta,t)\mathrm{e}^{h_\epsilon(V,t)}x_{t,V}(A)^n \mathrm{d}\eta \leq$$

$$\leq 2\int_{\mathbb{R}^2_{>0}} g_{\text{in},R}(\eta)A^n \mathrm{d}\eta + \frac{1-\gamma}{2}\int_0^t \int_{\mathbb{R}^2_{>0}}\int_{\mathbb{R}^2_{>0}} K_R(\phi_s(A,V),\phi_s(A',V'))G_{\epsilon,R,\delta}(\eta',s)G_{\epsilon,R,\delta}(\eta,s)$$
$$\mathrm{e}^{h_\epsilon(V,s)+h_\epsilon(V',s)}[(x_{s,V}(A)+x_{s,V'}(A'))^n - x_{s,V}(A)^n - x_{s,V'}(A')^n]\mathrm{d}\eta'\mathrm{d}\eta\mathrm{d}s$$
$$\leq 2\int_{\mathbb{R}^2_{>0}} g_{\text{in},R}(\eta)A^n \mathrm{d}\eta + (1-\gamma)\|K_R\|_\infty \sum_{k=1}^{k_n}\binom{n}{k}\int_0^t\int_{\mathbb{R}^2_{>0}} \mathrm{e}^{h_\epsilon(V,s)}x^k_{s,V}(A)G_{\epsilon,R,\delta}(\eta,s)\mathrm{d}\eta$$
$$\int_{\mathbb{R}^2_{>0}} \mathrm{e}^{h_\epsilon(V',s)}x^{n-k}_{s,V'}(A')G_{\epsilon,R,\delta}(\eta',s)\mathrm{d}\eta'\mathrm{d}s,$$

where k_n is taken as in Lemma A.3.23. The above computations show that, in order to find an upper bound for $\int_{\mathbb{R}^2_{>0}} G_{\epsilon,R,\delta}(\eta,t)\mathrm{e}^{h_\epsilon(V,t)}x_{t,V}(A)^n\mathrm{d}\eta$, it is enough to estimate $\int_{\mathbb{R}^2_{>0}} G_{\epsilon,R,\delta}(\eta,t)\mathrm{e}^{h_\epsilon(V,t)}x_{t,V}(A)^k\mathrm{d}\eta$, where $k \in [1, n-1]$. As such, we can use an iteration argument for the exponents of $x_{t,V}(A)$. We then use that

$$\int_{(0,\infty)^2} G_{\epsilon,R,\delta}(A,V,t,)e^{h_\epsilon(V,t)}x_{t,V}(A)\mathrm{d}V\mathrm{d}A \leq 2\int_{(0,\infty)^2} a g_{\text{in},R}(a,v)\mathrm{d}v\mathrm{d}a$$

as in the proof of Proposition A.3.7, which can be found in this appendix. In this manner, we derive suitable moment estimates which allow to extend the obtained solution to all times by iterating the argument. □

A.7 Some technical results used to prove the existence of self-similar profiles

Remark A.7.1. In order to simplify the notation we will write g and g_{in} instead of $g_{\epsilon,R,\delta}$ and $g_{\text{in},R}$, respectively, in the following computations. It is relevant to take into account that $g_{\epsilon,R,\delta} := g$ is supported in the region $(a,v) \in [c_0\epsilon^{\frac{2}{3}},\infty)\times[\epsilon,2R)$.

Proof of Proposition A.3.16. We obtain (A.3.35) by testing (A.3.5) with $\varphi(a,v) = v$. In order to derive (A.3.36), we test (A.3.5) with $\varphi(a,v) = v^\gamma$.

By Remark A.3.15 and since g is supported in the region $(a,v) \in [c_0\epsilon^{\frac{2}{3}},\infty)\times[\epsilon,2R)$, we can ignore the dependence of $K_{\epsilon,R}$ on R:

$$\begin{aligned}&\frac{(1-\gamma)}{2}\int_{(0,\infty)^2}\int_{(0,\infty)^2}K_{\epsilon,R}(a,v,a',v')\xi_R(v+v')g(a,v,t)g(a',v',t)[(v+v')^\gamma-v^\gamma-v'^\gamma]\mathrm{d}v'\mathrm{d}a'\mathrm{d}v\mathrm{d}a\\&\leq\frac{K_1(1-\gamma)}{2}\int_{(0,\infty)^2}\int_{(0,\infty)^2}(v^{-\alpha}v'^\beta+v^\beta v'^{-\alpha})\xi_R(v+v')g(a,v,t)g(a',v',t)[(v+v')^\gamma-v^\gamma-v'^\gamma]\mathrm{d}v'\mathrm{d}a'\mathrm{d}v\mathrm{d}a\\&\leq\frac{K_1(1-\gamma)}{2}\int_{\{v<\frac{R}{2}\}}\int_{\{v'<\frac{R}{2}\}}(v^{-\alpha}v'^\beta+v^\beta v'^{-\alpha})g(a,v,t)g(a',v',t)[(v+v')^\gamma-v^\gamma-v'^\gamma]\mathrm{d}v'\mathrm{d}a'\mathrm{d}v\mathrm{d}a.\end{aligned}$$

Suppose $v\geq v'$ and take $z:=\frac{v'}{v}\in(0,1]$. We have:

$$\begin{aligned}(v^{-\alpha}v'^\beta+v^\beta v'^{-\alpha})((v+v')^\gamma-v^\gamma-v'^\gamma)&\leq v^\gamma(z^\beta+z^{-\alpha})v^\gamma((1+z)^\gamma-1-z^\gamma)\\&\leq-(1-\gamma)v^{2\gamma}z^\gamma=-(1-\gamma)v^\gamma v'^\gamma,\end{aligned}$$

since $(1+z)^\gamma-1-z^\gamma\leq 0$ and $z^{-\alpha}\geq 1$. We also used that:

$$(1+z)^\gamma-1=\gamma\int_1^{1+z}s^{\gamma-1}\mathrm{d}s\leq\gamma z\leq\gamma z^\gamma.$$

Equation (A.3.5) now becomes:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}M_{0,\gamma}(g(t))&\leq(1-\gamma)\int_{(0,\infty)^2}v^\gamma\Theta_\epsilon(v)g(a,v,t)\mathrm{d}v\mathrm{d}a-\frac{(1-\gamma)^2K_1}{2}\Big(\int_{\{v<\frac{R}{2}\}}v^\gamma g(a,v,t)\mathrm{d}v\mathrm{d}a\Big)^2\\&\leq(1-\gamma)M_{0,\gamma}(g(t))-\frac{(1-\gamma)^2K_1}{4}M_{0,\gamma}(g(t))^2+\frac{(1-\gamma)^2K_1}{2}\Big(\int_{\{v>\frac{R}{2}\}}v^\gamma g(a,v,t)\mathrm{d}v\mathrm{d}a\Big)^2.\end{aligned}$$

We can control the region where $v>\frac{R}{2}$ using (A.3.35), namely:

$$\int_{\{v>\frac{R}{2}\}}v^\gamma g(a,v,t)\mathrm{d}v\mathrm{d}a\leq\Big(\frac{R}{2}\Big)^{\gamma-1}\int_{(0,\infty)^2}vg(a,v,t)\mathrm{d}v\mathrm{d}a\leq\Big(\frac{R}{2}\Big)^{\gamma-1}M_{0,1}(g_{\mathrm{in}}),$$

thus finding a constant $C>0$, independent of $\epsilon\in(0,1), R>1$ and $\delta\in(0,1)$, for which

$$\frac{\mathrm{d}}{\mathrm{d}t}M_{0,\gamma}(g(t))\leq(1-\gamma)M_{0,\gamma}(g(t))-\frac{(1-\gamma)^2K_1}{4}M_{0,\gamma}(g(t))^2+C.$$

We then conclude that there exists a constant $C_{0,\gamma}$, independent of ϵ, R and δ, for which the region $M_{0,\gamma}(g(t))\leq C_{0,\gamma}$ is invariant in time. ◻

Proof of Proposition A.3.17. We want to bound

$$K(a,v,a',v')[(v+v')^m-v^m-v'^m].$$

Assume $v\geq v'$. Denote $z:=\frac{v'}{v}\in(0,1]$ and observe that:

$$\begin{aligned}K(a,v,a',v')[(v+v')^m-v^m-v'^m]\leq&K(a,v,a',v')v^m[(1+z)^m-1-z^m]\leq 2C_mK(a,v,a',v')v^mz\\\leq&2C_mK_0v^\gamma(z^\beta+z^{-\alpha})v^mz\leq 4C_mK_0v^{\gamma+m}z^{-\alpha+1}\\\leq&4C_mK_0(v^{\beta+m-1}v'^{-\alpha+1}+v'^{\beta+m-1}v^{-\alpha+1}),\end{aligned}\tag{A.7.1}$$

where we used that

$$(1+z)^m - 1 = m\int_1^{1+z} s^{m-1}\mathrm{d}s \le m(1+z)^{m-1}z \le C_m(1+z^{m-1})z,$$

since $m > 1$ and $z \in (0,1]$. By symmetry, (A.7.1) holds for all $(v,v') \in (0,\infty)^2$.

Equation (A.3.5) becomes:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}M_{0,m}(g(t)) &\le -(m-1)M_{0,m}(g(t))\\ &\quad + (m-1)\int_{\{v\le 2\epsilon\}} v^m g(\eta,t)\mathrm{d}\eta + 2(1-\gamma)C_m K_0 M_{0,\beta+m-1}(g)M_{0,-\alpha+1}(g)\\ &\le -(m-1)M_{0,m}(g(t)) + (m-1)2^{m-1}M_{0,1}(g_{\text{in}}) + CM_{0,\beta+m-1}(g(t))M_{0,-\alpha+1}(g(t)).\end{aligned} \tag{A.7.2}$$

By (A.3.35) and (A.3.36), we obtain that $M_{0,-\alpha+1}(g(t))$ is uniformly bounded as $-\alpha+1 \in (\gamma,1]$.

Additionally, we have that $m+\beta-1 \in (\gamma,m)$. Let $C_{0,\gamma}$ be the constant found in Proposition A.3.16, then there exists $\theta \in (0,1)$ such that:

$$M_{0,m+\beta-1}(g(t)) \le M_{0,\gamma}(g(t))^{1-\theta}M_{0,m}(g(t))^{\theta} \le C_{0,\gamma}^{1-\theta}M_{0,m}(g(t))^{\theta}. \tag{A.7.3}$$

Hence, combining (A.7.2) and (A.7.3), we find an invariant region in time for the moment $M_{0,m}$. □

Proof of Proposition A.3.18. Let $C_{0,\tilde{m}}$ be the constant found in (A.3.37), with $\tilde{m} > 1$ as in Proposition A.3.2, and let $\beta < 1$ as in (A.1.7). For all $t \ge 0$:

$$\begin{aligned}1 = M_{0,1}(g(t)) &\le N^{1-\beta}\int_{(0,\infty)\times(0,N)} v^{\beta}g(a,v,t)\mathrm{d}v\mathrm{d}a + N^{1-\tilde{m}}\int_{(0,\infty)\times(N,\infty)} v^{\tilde{m}}g(a,v,t)\mathrm{d}v\mathrm{d}a\\ &\le N^{1-\beta}M_{0,\beta}(g(t)) + N^{1-\tilde{m}}\max\{M_{0,\tilde{m}}(g_{\text{in}}), C_{0,\tilde{m}}\}.\end{aligned} \tag{A.7.4}$$

Thus, for $N > 0$ sufficiently large, we obtain

$$M_{0,\beta}(g(t)) \ge \frac{1}{2N^{1-\beta}}. \tag{A.7.5}$$

We analyse the term

$$\int_{(0,\infty)^2}\int_{(0,\infty)^2} K_{\epsilon,R}(a,v,a',v')\xi_R(v+v')g(a,v,t)g(a',v',t)[(v+v')^{-l} - v^{-l} - v'^{-l}]\mathrm{d}v'\mathrm{d}a'\mathrm{d}v\mathrm{d}a.$$

We make use of Remark A.3.15: From the definition of $K_{\epsilon,R}$ in (A.3.1) and the support of g, we have that $K_{\epsilon,R} = K$ when $v,v' \in [\epsilon,2R)^2$. Since $(v+v')^{-l} - v^{-l} - v'^{-l} \le 0$, we can use the lower bound for K in (A.1.7). We obtain:

$$\begin{aligned}&\int_{(0,\infty)^2}\int_{(0,\infty)^2} K_{\epsilon,R}(a,v,a',v')\xi_R(v+v')g(a,v,t)g(a',v',t)[(v+v')^{-l} - v^{-l} - v'^{-l}]\mathrm{d}v'\mathrm{d}a'\mathrm{d}v\mathrm{d}a\\ \le &K_1\int_{\{v<\frac{R}{2}\}}\int_{\{v'<\frac{R}{2}\}}(v^{-\alpha}v'^{\beta} + v^{\beta}v'^{-\alpha})g(a,v,t)g(a',v',t)[(v+v')^{-l} - v^{-l} - v'^{-l}]\mathrm{d}v'\mathrm{d}a'\mathrm{d}v\mathrm{d}a\\ \le &K_1\int_{\{v<\frac{R}{2}\}}\int_{\{v'<\frac{R}{2}\}}(-v^{-\alpha-l}v'^{\beta} - v'^{-\alpha-l}v^{\beta})g(a,v,t)g(a',v',t)\mathrm{d}v'\mathrm{d}a'\mathrm{d}v\mathrm{d}a\end{aligned}$$

and (A.3.5) thus becomes:

$$\frac{\mathrm{d}}{\mathrm{d}t} M_{0,-l}(g(t)) \leq (1+l) M_{0,-l}(g(t)) - (1-\gamma) K_1 \int_{\{v<\frac{R}{2}\}} v^{-\alpha-l} g(a,v,t) \mathrm{d}v \mathrm{d}a \int_{\{v<\frac{R}{2}\}} v^{\beta} g(a,v,t) \mathrm{d}v \mathrm{d}a. \tag{A.7.6}$$

By (A.7.5)

$$\int_{\{v<\frac{R}{2}\}} v^{\beta} g(a,v,t) \mathrm{d}v \mathrm{d}a \geq \frac{1}{2N^{1-\beta}} - \left(\frac{R}{2}\right)^{\beta-1} \geq \frac{1}{4N^{1-\beta}}, \tag{A.7.7}$$

for all $R > 1$ that are sufficiently large.

As $-l \in (-\alpha - l, 1)$, there exists $\theta \in (0,1)$ for which $M_{0,-l}(g(t)) \leq M_{0,1}(g(t))^{1-\theta} M_{0,-\alpha-l}(g(t))^{\theta} \leq M_{0,1}(g_{\text{in}})^{1-\theta} M_{0,-\alpha-l}(g(t))^{\theta}$. Combining this with (A.7.6) and (A.7.7), there exists a constant $C > 0$ such that:

$$\frac{\mathrm{d}}{\mathrm{d}t} M_{0,-l}(g(t)) \leq (1+l) M_{0,-l}(g(t)) - C \Big(\int_{\{v<\frac{R}{2}\}} v^{-l} g(a,v,t) \mathrm{d}v \mathrm{d}a \Big)^{\frac{1}{\theta}}.$$

We use $x^{\frac{1}{\theta}} \geq 2^{1-\frac{1}{\theta}} (x+y)^{\frac{1}{\theta}} - y^{\frac{1}{\theta}}$ with $x = \int_{\{v<\frac{R}{2}\}} v^{-l} g(a,v,t) \mathrm{d}v \mathrm{d}a$ and $y = \int_{\{v>\frac{R}{2}\}} v^{-l} g(a,v,t) \mathrm{d}v \mathrm{d}a$. We can bound $\int_{\{v>\frac{R}{2}\}} v^{-l} g(a,v,t) \mathrm{d}v \mathrm{d}a$ from above because $R > 1$ and $M_{0,1}(g(t))$ is uniformly bounded. Thus

$$\frac{\mathrm{d}}{\mathrm{d}t} M_{0,-l}(g(t)) \leq (1+l) M_{0,-l}(g(t)) - C_1 M_{0,-l}(g(t))^{\frac{1}{\theta}} + 2^{l+1} C M_{0,1}(g_{\text{in}})$$

and we conclude using the uniform bound on $M_{0,1}$ and then a comparison argument. □

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APPENDIX B

FAST FUSION IN A TWO-DIMENSIONAL COAGULATION MODEL

ABSTRACT. In this work, we study a particular system of coagulation equations characterized by two values, namely volume v and surface area a. Compared to the standard one-dimensional models, this model incorporates additional information about the geometry of the particles. We describe the coagulation process as a combination between collision and fusion of particles. We prove that we are able to recover the standard one-dimensional coagulation model when fusion happens quickly and that we are able to recover an equation in which particles interact and form a ramified like system in time when fusion happens slowly.

B.1 Introduction

Most of the works on coagulation equations assume that the particles are characterized by a single variable, usually the particle volume (or equivalent quantities like polymer length), see for instance [Nor99, NV12, Smo16, Ste89]. Nevertheless, other parameters that might provide insight about the geometry or other features of the particles are usually omitted. In a previous work (see [CV23]), we study the mathematical properties of a class of coagulation equations in which the aggregating particles are characterized by two degrees of freedom, namely the volume v and the surface area a. This type of models was introduced in [Fri00, KF90]. More precisely, the model considered is the following:

$$\partial_t f(a,v,t) + \partial_a[r(a,v)(c_0 v^{\frac{2}{3}} - a) f(a,v,t)] = \mathbb{K}[f](a,v,t),\ c_0 := (36\pi)^{\frac{1}{3}}, \tag{B.1.1}$$

where

$$\begin{aligned}\mathbb{K}[f](a,v,t) :=& \frac{1}{2}\int_{(0,a)\times(0,v)} K(a-a',v-v',a',v')f(a',v',t)f(a-a',v-v',t)\mathrm{d}v'\mathrm{d}a' \\ &- \int_{(0,\infty)^2} K(a,v,a',v')f(a,v,t)f(a',v',t)\mathrm{d}v'\mathrm{d}a'.\end{aligned}$$

In this model, f is the density of particles in the space of area and volume for any given time $t \geq 0$. The coagulation operator $\mathbb{K}[f]$ is the classical coagulation operator that was introduced by

Smoluchowski (see [Smo16]) and gives the coagulation rate of particles which evolve according to the following mechanism:

$$(a_1, v_1) + (a_2, v_2) \longrightarrow (a_1 + a_2, v_1 + v_2).$$

It is assumed that the particles attach to each other at their contact point and therefore in this way both the total area and volume of the particles involved in the process are preserved.

On the other hand, the fusion term $\partial_a[r(a,v)(c_0 v^{\frac{2}{3}} - a) f(a,v,t)]$ describes an evolution of the particles towards a spherical shape. The dynamics generated by this term preserves the total number and volume of the particles. The term $c_0 v^{\frac{2}{3}} - a$ indicates that the area of the particles tends to be reduced as long as it is larger than that of a sphere $c_0 v^{\frac{2}{3}}$ (see Figure B.1 for a description of the complete coagulation process assumed in (B.1.1)).

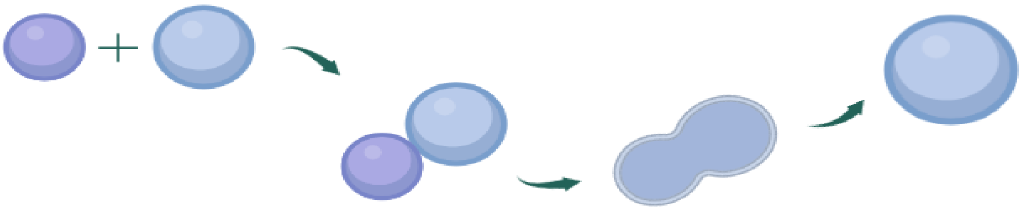

Figure B.1: Coagulation process: collision of particles followed by fusion

In addition, $r(a,v)$ will indicate the fusion rate and describes how quickly the particles evolve towards the spherical shape and thus has units of the inverse of the fusion time. If the fusion kernel r is very large compared to the coagulation rate, we expect that the particles become spherical in very short times. On the contrary, in the particular case when $r \equiv 0$, fusion does not occur and particles attach at contact points forming a ramified-like system in time.

The main advantage of this model is that the distribution f contains more information about the geometry of the particles than a standard coagulation model, where only information about the volume of the particles is taken into account. In particular, this allows to include in the coagulation rate information about the particle shape, which enables us to distinguish between the case when two particles attach at a contact point and the case when they merge upon collision as explained above. This distinction is not possible in the case where only volume is of importance. We refer to [Fri00, KF90] for details about the derivation of the model, and to [CV23] for additional information about the mathematical properties of this model.

Therefore, it should be possible to approximate the solutions of (B.1.1) by means of solutions of a coagulation model depending on only the variable v, i.e. an one-dimensional coagulation equation, in the case when fusion overtakes coagulation. On the other hand, when r is very small compared with the coagulation rate, we can approximate the solutions of (B.1.1) by means of solutions of a two-dimensional coagulation model without fusion depending on two variables, a and v.

More precisely, we analyse the following model as $\Lambda \to 0$ and as $\Lambda \to \infty$:

$$\partial_t f_\Lambda(a, v, t) + \frac{1}{\Lambda}\partial_a[r(a, v)(c_0 v^{\frac{2}{3}} - a) f_\Lambda(a, v, t)] = \mathbb{K}[f_\Lambda](a, v, t). \tag{B.1.2}$$

We remark that the particles must satisfy the isoperimetric inequality, therefore the density f_Λ should be supported in the region where $\{a \geq c_0 v^{\frac{2}{3}}\}$. Moreover, the evolution generated by (B.1.1) (or by (B.1.2)) has the property that it preserves the set of measures supported in this region.

We assume $r(a, v)$ behaves like a power law of a and v. For the coagulation kernel K, we assume that it has a weak dependence on the surface area of the interacting particles, but it can have a power law behavior in the volume of the coalescing particles.

Since collision does not change if we permute the colliding particles, i.e. $(a, v) \leftrightarrow (a', v')$, the coagulation kernel must satisfy the following symmetry property:

$$K(a, v, a', v') = K(a', v', a, v), \tag{B.1.3}$$

for all $(a, v, a', v') \in (0, \infty)^4$.

Concerning the fusion kernel r, we assume that $r \in \mathrm{C}^1(\mathbb{R}^2_{>0})$ and that there exist constants $R_0, R_1 > 0$ such that:

$$R_0 a^\mu v^\sigma \leq r(a, v) \leq R_1 a^\mu v^\sigma, \tag{B.1.4}$$

for all $(a, v) \in (0, \infty)^2$ and some coefficients $\mu, \sigma \in \mathbb{R}$.

In order to control the mass of the solutions when $|a - c_0 v^{\frac{2}{3}}| > 0$ in (B.1.2) if Λ is small, we require the following technical assumptions on the fusion kernel r:

$$\begin{cases} \partial_a r(a, v) - \mu a^{-1} r(a, v) \geq 0, \text{ and } \partial_a r(a, v) \leq B a^{-1} r(a, v), & \text{if } \mu > 0; \\ \partial_a r(a, v)(a - c_0 v^{\frac{2}{3}}) + r(a, v) \geq 0, \text{ and } \partial_a r(a, v) \leq B a^{-1} r(a, v), & \text{if } \mu \leq 0, \end{cases} \tag{B.1.5}$$

for all $(a, v) \in (0, \infty)^2$, with $a \geq c_0 v^{\frac{2}{3}}$, and for some constant $B > 0$. A particular case used in applications that satisfies the above mentioned properties is when $r(a, v) = R a^\mu v^\sigma$, with $\mu \geq -1$ and $R \in [R_0, R_1]$. The condition (B.1.5) is not optimal and it would be possible to impose weaker conditions on the fusion kernel. However, this would require more involved arguments in the proofs later on. We impose the stronger condition (B.1.5) as our main goal is that the statements of our theorems hold for fusion kernels that behave as power laws, a case which is included in condition (B.1.5).

In comparison to [CV23], since in this paper we are not interested in the long-time behavior of solutions, we do not assume homogeneity of neither the fusion nor coagulation kernel and we simply assume that they behave like power laws, see (B.1.4) and (B.1.6). In [CV23], we restricted the analysis to coagulation and fusion kernels which rescale in a similar manner when the size of the particles is changed without modifying their geometry. This was needed for the study of the long-time behavior of particles as the fusion term was chosen in a desire for particles to form a spherical shape in time. In particular, it meant that, if the particle volume is scaled by a factor λ, then the diameter is scaled with a factor $\lambda^{\frac{1}{3}}$ and the area scales like $\lambda^{\frac{2}{3}}$ and that we needed to impose the additional assumption that $\frac{2}{3}\mu + \sigma = \gamma - 1$, where γ is the homogeneity of the coagulation kernel and μ, σ are as in (B.1.4). In this work, we treat the more general case of arbitrary $\mu, \sigma \in \mathbb{R}$. In order to deal with this case, we will need to obtain additional moment estimates.

Our main goal for this paper is to prove that all solutions of equation (B.1.2) which satisfy some very general moment estimates concentrate their mass around the isoperimetric line $\{a = c_0 v^{\frac{2}{3}}\}$ as $\Lambda \to 0$ and tend in an appropriate sense to a measure which can be computed by solving a suitable one-dimensional coagulation equation. Moreover, we prove that solutions of (B.1.2) satisfying these moment estimates do exist. The limit measure f of the sequence $\{f_\Lambda\}$ acts like a Dirac-like measure in the area variable, namely $f(v,t) = \delta(a - c_0 v^{\frac{2}{3}})F(v,t)$, with F satisfying the standard one-dimensional coagulation equation. We can then use the known results for the one-dimensional coagulation equations (for example, it was proven mathematically in [BLL19b, Proposition 10.2.1] for solutions in $\mathrm{C}([0,\infty);\mathrm{L}^1(\mathbb{R}_{>0}))$) to prove that the average volume of the particles increases in time.

The reason we can reduce the evolution equation for the two-dimensional system to a one-dimensional one is that, as $\Lambda \to 0$, the fusion process takes place much faster than the collision process and then the particles are transported close to the isoperimetric line $\{a = c_0 v^{\frac{2}{3}}\}$ almost instantaneously. Equivalently, fusion happens immediately after collision.

On the other hand, if we let $\Lambda \to \infty$ in equation (B.1.2), we recover a two-dimensional coagulation model, in which particles attach to each other at a contact point, forming a ramified-like system in time. A physical interpretation of this is that, as $\Lambda \to \infty$, the effect of the fusion term becomes negligible (see Figure B.2).

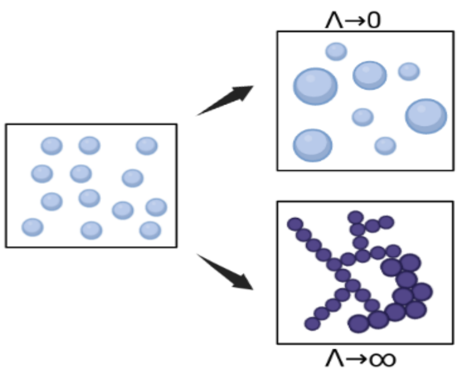

Figure B.2: System of particles under different fusion times

The evolution of a system of coagulation equations which can be described by area and volume and where the particles undergo a fusion process after they come in contact has been described in [Fri00, Chapter 12]. The specific problem under consideration was the study of aerosol flame reactors. A heuristic analysis of the shapes for the resulting particles for different values of the ratio between the average fusion time and the average collision time can be found in there. This ratio is given by the parameter Λ in our model in (B.1.2). In the types of models considered in [Fri00, Chapter 12], it is seen that, for small particles or high temperatures, the parameter Λ is small. On the contrary, for sufficiently large particles or after the gas has been cooled, we must assume that Λ is very large. The results in this paper provide a precise mathematical formulation of the behavior that has been suggested in [Fri00, Chapter 12]. The results in this work are complementary to those in [CV23], in which we consider a particular form of the fusion term for which $\Lambda \approx 1$ for arbitrary times.

Coagulation equations for particle distributions characterized by a single variable have been extensively studied. In particular, the long-time behavior for coagulation equations for which solutions can be explicitly computed has been studied in [MP04]. The existence of self-similar solutions for general classes of kernels has been obtained in [EMR05, FL05].

Coagulation equations including also transport terms have been considered in the mathematical literature in relation to different contexts. For instance, in [AD03], the well-posedness of a model for the evolution of the cluster size of phytoplankton, which has been introduced in [AF97], has been studied. A model including also fragmentation effects of the phythoplakton clusters has been analysed in [BL09], where results on existence and uniqueness of solutions for some classes of kernels has been studied.

In these papers the transport term is related to growth of the particles, while in the model (B.1.1), (B.1.2) the transport term describes the fusion of particles. This allows the observation of the changes in the geometry of the particles with no changes to the volume. Due to the multi-dimensionality of our model, the techniques required to tackle the two problems vary, with the difficulty coming from the fact that we need to control large values of the additional variable, we refer to [CV23] for more details. The main focus of this paper consists in finding connections between the two-dimensional and the one-dimensional model.

Other coagulation models combining the effect of coagulation and particle growth have been studied in the physical literature, cf. [Fri00, Chapter 11] and [LK00], with rigorous mathematical results for the last paper obtained in [Gaj83]. Moreover, coagulation equations including drift terms have been studied in several contexts. A relevant example is the Lifshitz-Slyozov-Wagner equation with encounters that was introduced in [LS61]. A rigorous analysis of the self-similar profiles for this model was done in [HLN09, HNV09, Lau01].

Multi-dimensional coagulation equations have not been studied as much in the mathematical literature as their one-dimensional counterpart. Several discrete multi-component coagulation problems which are relevant in aerosol physics have been mentioned in [Wat06b]. A discrete version of the model in (B.1.1) has been studied in [Wat06a]. The model considered in there includes coagulation of particles and an effect similar to the fusion of particles in (B.1.1), which has been termed compaction. The diameter of the particles is restricted by the total number of monomers as well as by the isoperimetric inequality. The coagulation and the fusion rates are assumed to be constant. Due to this, the model considered in [Wat06a] is explicitly solvable using generating functions. The long-time behavior of the solutions which depends on the ratio between the fusion and coagulation kernels has been then analysed using the explicit formulas of the solutions.

In [FLNV21a, FLNV22, FLNV23], the mathematical properties of some classes of coagulation equations describing clusters that are composed of several types of monomers with different chemical composition are analysed. More recently, uniqueness of the solutions for the models of multi-component coagulation equations considered in [FLNV21a, FLNV22, FLNV23] has been studied in [Thr23].

More precisely, it has been proven in [FLNV21a, FLNV22] that time-dependent solutions for the multi-dimensional coagulation equation concentrate along a line in the space of cluster concentrations for long times for coagulation kernels for which the scaling properties of each of the components are the same for all the species that compose the system. However, as the surface area and volume appear in a less symmetric manner in our model, it does not seem feasible to adapt the proof in [FLNV21a, FLNV22] to obtain our result, even in the absence of the fusion term.

Another difference between our model and the one in [FLNV21a, FLNV22] is that the proof in the latter relies on the conservation of mass for each of the types of monomers. Due to the fusion term, we do not have two conserved quantities for (B.1.2), but only the volume is conserved. In addition, the fusion term in (B.1.1), (B.1.2) yields a non-trivial evolution of the distribution of particles. The solutions in [FLNV21a, FLNV22, FLNV23] concentrate along a line with the orientation fixed by the initial distribution of cluster compositions or the source term. Thus, the solutions in [FLNV21a, FLNV22, FLNV23] can concentrate along different lines depending on the initial distribution of particles. On the contrary, in one of the situations considered in this paper, the solutions concentrate always near the isoperimetric line, independently of the initial data. When $\Lambda \to 0$ in (B.1.2), we have that the coagulation operator transports particles away from the isoperimetric line but these are transported extremely fast towards the isoperimetric region due to the fusion term.

B.1.1 Notations and plan of the paper

For $I \subset [0,\infty)^2$, we denote by $\mathrm{C_c}(I)$ and $\mathrm{C_0}(I)$ the space of continuous functions on I with compact support and the space of continuous functions on I which vanish at infinity, respectively, both endowed with the supremum norm. $\mathscr{M}_+(I)$ will denote the space of non-negative Radon measures, while $\mathscr{M}_{+,\mathrm{b}}(I)$ will be the space of non-negative, bounded Radon measures, which we endow with the weak-* topology.

We make in addition the following simplifications:

- We use the notation $\eta := (a, v)$. We will use interchangeably both notations for convenience.
- We will use the notation $f(a, v)\mathrm{d}v\mathrm{d}a$ or $f(\eta)\mathrm{d}\eta$ for Radon measures, independently of the fact the measure may not be absolutely continuous with respect to the Lebesgue measure.
- $M_{k,l}(f) := \int_{(0,\infty)^2} a^k v^l f(a, v)\mathrm{d}v\mathrm{d}a$, for some $k, l \in \mathbb{R}$.
- For a suitably chosen $\varphi : \mathbb{R}^2_{>0} \to \mathbb{R}$ and for $(a, v, a', v') \in (0, \infty)^4$, we will denote:
$$\chi_\varphi(a, v, a', v') := \varphi(a + a', v + v') - \varphi(a, v) - \varphi(a', v');$$
$$\langle \mathbb{K}[f], \varphi\rangle := \frac{1}{2}\int_{(0,\infty)^2}\int_{(0,\infty)^2} K(\eta, \eta')\chi_\varphi(\eta, \eta')f(\eta')f(\eta)\mathrm{d}\eta'\mathrm{d}\eta.$$
- We use C to denote a generic constant which may differ from line to line and depends only on the parameters characterizing the kernels K and r.
- We use the symbols $\lesssim$ and $\gtrsim$ when the inequalities hold up to a constant, i.e. $f \lesssim g$ if and only if $f \leq Cg$, for some $C > 0$.

The structure of the paper is as follows. In the rest of this section, we establish the setting and state the main definitions and results.

In Section B.2, we prove that there exists a limit for the sequence of solutions of equation (B.1.2) as $\Lambda \to 0$. To this end, we first prove that the mass of solutions concentrates around the isoperimetric line. This is done by looking at the adjoint equation of (B.1.2). The fact that the measures take small values if we are at a positive distance from the line $\{a = c_0 v^{\frac{2}{3}}\}$ together with the fact that we can control large values of the area a suffices to prove the equicontinuity in time of solutions and conclude that a limit of the sequence exists. We then prove that the found limit is a solution for the standard one-dimensional coagulation equation. This is since now the a variable acts like $c_0 v^{\frac{2}{3}}$ and we can omit the fusion term by testing (B.1.2) with functions only depending on the v variable.

In Section B.3, we deal with the case when $\Lambda \to \infty$ in equation (B.1.2). We prove that a limit exists as $\Lambda \to \infty$ and that the limit satisfies a two-dimensional coagulation equation where the interaction of particles consists of particles which attach at a contact point. The proof of this result is straightforward after obtaining suitable moment estimates for the solutions, which are independent of the value of Λ.

B.1.2 Setting and main results

We work with non-negative continuous kernels on $(0, \infty)^4$ that, in addition to the properties already stated, i.e. (B.1.3), have the following bounds:

$$K_1(v^{-\alpha}v'^{\beta} + v'^{-\alpha}v^{\beta}) \leq K(a, v, a', v') \leq K_0(v^{-\alpha}v'^{\beta} + v'^{-\alpha}v^{\beta}), \tag{B.1.6}$$

for some $K_1, K_0 > 0$, for all a, v, a', v' and for the following coefficients:

$$\alpha > 0 \text{ and } \beta \in (0,1) \text{ such that } \beta - \alpha \in (0,1). \tag{B.1.7}$$

Notice that condition (B.1.6) implies that the kernel has a weak dependence on the area variable, although K is not necessarily independent of the area variable. Moreover, we focus on the study of kernels for which gelation (i.e., solutions do not preserve mass) does not take place. For more details on gelation theory, we refer to [CdC92, EMP02, Lau00, vD87] and [BLL19b, Section 9]. It would be interesting to check if the results of this paper can be extended to some classes of kernels which exhibit gelation. Notice that this opens several questions. For instance, in the gelation case there might be several continuations of solutions depending on how is the interaction between particles with finite size and the *gel* part. It would be relevant to know if the limit as $\Lambda \to 0$ of $\{f_\Lambda\}$ in (B.1.2) selects some particular solution.

Since we work with physically relevant particles, i.e. the particles for which the isoperimetric inequality is satisfied, it is helpful to define the following space

$$\mathscr{M}_+^I(\mathbb{R}^2_{>0}) := \{h \in \mathscr{M}_+(\mathbb{R}^2_{>0}) \mid h(\{a < c_0 v^{\frac{2}{3}}\}) = 0\}. \tag{B.1.8}$$

The superscript I stands for isoperimetric. We endow the newly-defined space with the weak-* topology on $\mathscr{M}_+(\mathbb{R}^2_{>0})$. Similarly, we denote

$$\mathscr{M}_{+,\mathrm{b}}^I(\mathbb{R}^2_{>0}) := \{h \in \mathscr{M}_{+,\mathrm{b}}(\mathbb{R}^2_{>0}) \mid h(\{a < c_0 v^{\frac{2}{3}}\}) = 0\}. \tag{B.1.9}$$

Definition B.1.1. Fix $\Lambda > 0$. Let $K : (0,\infty)^4 \to [0,\infty)$ be a continuous kernel satisfying (B.1.3), (B.1.6) and (B.1.7). Assume the fusion kernel $r \in \mathrm{C}^1(\mathbb{R}^2_{>0})$ satisfies (B.1.4) and (B.1.5). Let $f_\Lambda \in \mathrm{C}([0,\infty); \mathscr{M}_+^I(\mathbb{R}^2_{>0}))$. We say that f_Λ is a solution for the weak version of the time-dependent Λ-fusion problem if, for every $T > 0$,

$$\sup_{t\in[0,T]} \int_{(0,\infty)^2} (v^{-\alpha} + v^{\beta}) f_\Lambda(a,v,t)\mathrm{d}v\mathrm{d}a < \infty$$

and, for all $\varphi \in \mathrm{C}^1_\mathrm{c}([0,\infty); \mathrm{C}^1_\mathrm{c}(\mathbb{R}^2_{>0}))$ and $t \in [0,\infty)$

$$\begin{aligned}\int_{(0,\infty)^2} f_\Lambda(\eta,t)\varphi(\eta,t)\mathrm{d}\eta - \int_{(0,\infty)^2} f_{\mathrm{in}}(\eta)\varphi(\eta,0)\mathrm{d}\eta - \int_0^t \int_{(0,\infty)^2} f_\Lambda(\eta,s)\partial_s\varphi(\eta,s)\mathrm{d}\eta\mathrm{d}s \\ = \int_0^t \langle \mathbb{K}[f_\Lambda(s)], \varphi(s)\rangle \mathrm{d}s + \frac{1}{\Lambda}\int_0^t \int_{(0,\infty)^2} r(\eta)(c_0 v^{\frac{2}{3}} - a) f_\Lambda(\eta,s)\partial_a\varphi(\eta,s)\mathrm{d}\eta\mathrm{d}s.\end{aligned} \tag{B.1.10}$$

Remark B.1.2. Functions f_Λ as in Definition B.1.1 exist and the methods to prove their existence are similar to the ones used to prove existence of self-similar solutions in [CV23] (in order to derive some moment estimates in [CV23], ideas from the one-dimensional case in [EMR05, EM06] were adapted). A sketch for proving their existence will be shown in Proposition B.2.1.

The case of fast fusion

Remark B.1.3. From now on, in order to simplify the notation, we replace Λ by ϵ in (B.1.2) when we consider the case $\Lambda \to 0$ and we replace Λ by $\frac{1}{\epsilon}$ in (B.1.2) when we consider the case $\Lambda \to \infty$.

Theorem B.1.4. *Let $K : (0,\infty)^4 \to [0,\infty)$ be a continuous kernel satisfying (B.1.3), (B.1.6) and (B.1.7). Assume the fusion kernel $r \in \mathrm{C}^1(\mathbb{R}^2_{>0})$ satisfies (B.1.4) and (B.1.5) with $\mu > 0$. Assume in addition that $\int_{(0,\infty)^2}(v^{-\mu-3} + v^{\mu+3} + v^{\sigma(\mu+3)} + a^{\mu+3})f_{\text{in}}(a,v)\mathrm{d}v\mathrm{d}a < \infty$. Let $T > 0$. Then we can construct f_ϵ as in Definition B.1.1, for every $\epsilon \in (0,1)$. For this sequence, we have that there exists a constant $C(T) > 0$, which is independent of $\epsilon \in (0,1)$, such that*

$$\sup_{t\in[0,T]} \int_{(0,\infty)^2} (v^{-\mu-3} + v^{\mu+3} + v^{\sigma(\mu+3)} + a^{\mu+3})f_\epsilon(a,v,t)\mathrm{d}v\mathrm{d}a \leq C(T). \tag{B.1.11}$$

Moreover, there exists a subsequence (which we do not relabel) and $\overline{f} \in \mathrm{C}((0,T];\mathscr{M}^I_+(\mathbb{R}^2_{>0}))$ such that $f_\epsilon(t) \to \overline{f}(t)$ as $\epsilon \to 0$ in the sense of measures, for every $t \in [\overline{\sigma},T]$ and every $\overline{\sigma} > 0$.

Remark B.1.5. Theorem B.1.4 holds true also in the case $\mu \leq 0$ if we assume instead that $\int_{(0,\infty)^2}(v^{-1} + v^2 + a)f_{\text{in}}(a,v)\mathrm{d}v\mathrm{d}a < \infty$ (plus some additional moment bound of the form $M_{0,d}$, with d depending on σ, which does not offer much qualitative information), which in turn will imply that there exists a constant $C(T) > 0$, which is independent of $\epsilon \in (0,1)$, such that

$$\sup_{t\in[0,T]} \int_{(0,\infty)^2} (v^{-1} + v^2 + a)f_\epsilon(a,v,t)\mathrm{d}v\mathrm{d}a \leq C(T).$$

The methods to prove the two cases, $\mu > 0$ and $\mu \leq 0$, are similar up to minor technicalities and thus we restrict our attention to the case $\mu > 0$ for simplicity of notation.

Remark B.1.6. The reason we want (B.1.11) to hold is mostly technical, namely because we will have to test in (B.1.10) with functions that do not necessarily have compact support and we will use these estimates in order to prove that (B.1.10) also holds when testing with functions in C^1_b using Lebesgue's dominated convergence theorem. While the extension to functions that do not necessarily have compact support would require less assumptions in the case of the standard one-dimensional coagulation equation, in our case we need the assumption (B.1.11) in order to control the contribution coming from fusion term.

Lemma B.1.7. *Let $K : (0,\infty)^4 \to [0,\infty)$ be a continuous kernel satisfying (B.1.3), (B.1.6) and (B.1.7). Assume the fusion kernel $r \in \mathrm{C}^1(\mathbb{R}^2_{>0})$ satisfies (B.1.4) and (B.1.5). Let $T > 0$. Let f_ϵ and $\overline{f}$ as in Theorem B.1.4. Then $\overline{f}$ has the form*

$$\overline{f}(a,v,t) = F(v,t)\delta(a - c_0 v^{\frac{2}{3}}) \tag{B.1.12}$$

in the sense of measures. More precisely, there exists $F \in \mathrm{C}((0,T];\mathscr{M}_+(\mathbb{R}_{>0}))$ *such that*

$$\int_{(0,\infty)^2} \overline{f}(a,v,t)\varphi(a,v)\mathrm{d}v\mathrm{d}a = \int_{(0,\infty)^2} F(v,t)\varphi(a,v)\delta(a-c_0v^{\frac{2}{3}})\mathrm{d}a\mathrm{d}v,$$

for every $\varphi \in \mathrm{C}_0(\mathbb{R}^2_{>0})$.

Definition B.1.8. Let F be as in (B.1.12) and f_{in} as in Theorem B.1.4. We define the following initial value

$$\int_{(0,\infty)} F(v,0)\varphi(v)\mathrm{d}v := \int_{(0,\infty)^2} f_{\mathrm{in}}(a,v)\varphi(v)\mathrm{d}a\mathrm{d}v, \tag{B.1.13}$$

for every $\varphi \in \mathrm{C}_0(\mathbb{R}_{>0})$.

Definition B.1.9. Let $\epsilon \in (0,1)$, fix $T > 0$ and let f_ϵ be as in Theorem B.1.4. Define $F_\epsilon \in \mathrm{C}([0,T];\mathscr{M}_+(\mathbb{R}_{>0}))$ as

$$\int_{(0,\infty)} F_\epsilon(v,t)\varphi(v)\mathrm{d}v := \int_{(0,\infty)^2} f_\epsilon(a,v,t)\chi_\epsilon(a)\varphi(v)\mathrm{d}v\mathrm{d}a, \tag{B.1.14}$$

for every $t \in [0,T]$ and $\varphi \in \mathrm{C}_0(\mathbb{R}_{>0})$ and where $\chi_\epsilon : (0,\infty) \to [0,1]$ is a continuous function such that $\chi(a) = 1$ on $(0, \frac{1}{\epsilon^2}]$ and $\chi(a) = 0$ on $[\frac{2}{\epsilon^2}, \infty)$.

Theorem B.1.10. *Let* $\{F_\epsilon\}_{\epsilon\in(0,1)}$ *be as in Definition B.1.9 and* F *as in (B.1.12) with initial value as in Definition B.1.8. We then have that there exists a subsequence (which we do not relabel) such that*

$$\int_{(0,\infty)} F_\epsilon(v,t)\varphi(v)\mathrm{d}v \to \int_{(0,\infty)} F(v,t)\varphi(v)\mathrm{d}v,$$

as $\epsilon \to 0$, *for every* $t \in [0,T]$ *and every* $\varphi \in \mathrm{C}_0(\mathbb{R}_{>0})$. *Moreover,* $F \in \mathrm{C}([0,T];\mathscr{M}_+(\mathbb{R}_{>0}))$ *and* F *satisfies the standard one-dimensional coagulation equation, namely, for every* $t \in [0,T]$ *and every* $\varphi \in \mathrm{C_c}(\mathbb{R}_{>0})$, *the following holds*

$$\begin{aligned}&\int_{(0,\infty)} F(v,t)\varphi(v)\mathrm{d}v - \int_{(0,\infty)} F(v,0)\varphi(v)\mathrm{d}v\\ &= \int_0^t\int_{(0,\infty)}\int_{(0,\infty)} K(c_0v^{\frac{2}{3}},v,c_0v'^{\frac{2}{3}},v')F(v,s)F(v',s)[\varphi(v+v')-\varphi(v)-\varphi(v')]\mathrm{d}v'\mathrm{d}v\mathrm{d}s.\end{aligned}$$

The case of negligible fusion

Remark B.1.11. In order to simplify the notation, in the case when $\Lambda \to \infty$, we replace Λ by $\frac{1}{\epsilon}$, for $\epsilon > 0$. Thus, for $T > 0$ and $t \in [0,T]$, we look at the equation

$$\begin{aligned}&\int_{(0,\infty)^2} f_\epsilon(\eta,t)\varphi(\eta,t)\mathrm{d}\eta - \int_{(0,\infty)^2} f_{\mathrm{in}}(\eta)\varphi(\eta,0)\mathrm{d}\eta - \int_0^t\int_{(0,\infty)^2} f_\epsilon(\eta,s)\partial_s\varphi(\eta,s)\mathrm{d}\eta\mathrm{d}s\\ &= \int_0^t \langle \mathbb{K}[f_\epsilon(s)],\varphi(s)\rangle\mathrm{d}s + \epsilon\int_0^t\int_{(0,\infty)^2} r(\eta)(c_0v^{\frac{2}{3}}-a)f_\epsilon(\eta,s)\partial_a\varphi(\eta,s)\mathrm{d}\eta\mathrm{d}s,\end{aligned} \tag{B.1.15}$$

for $\epsilon \in (0,1)$, $\varphi \in \mathrm{C}^1_0(\mathbb{R}^2_{>0})$ and with $f_{\mathrm{in}} \in \mathscr{M}^I_+(\mathbb{R}^2_{>0})$.

Theorem B.1.12. *Let $K : (0,\infty)^4 \to [0,\infty)$ be a continuous kernel satisfying (B.1.3), (B.1.6) and (B.1.7). Assume the fusion kernel $r \in \mathrm{C}^1(\mathbb{R}^2_{>0})$ satisfies (B.1.4) and (B.1.5) with $\mu > 0$. Assume in addition that $\int_{(0,\infty)^2}(v^{-2} + v^2 + a^2) f_{\mathrm{in}}(a,v)\mathrm{d}v\mathrm{d}a < \infty$. Let $T > 0$. Then we can construct f_ϵ as in Definition B.1.1 satisfying equation (B.1.15), for every $\epsilon \in (0,1)$. For this sequence, we have that there exists a constant $C(T) > 0$, which is independent of $\epsilon \in (0,1)$, such that*

$$\sup_{t\in[0,T]} \int_{(0,\infty)^2} (v^{-2} + v^2 + a^2) f_\epsilon(a,v,t)\mathrm{d}v\mathrm{d}a \leq C(T) \tag{B.1.16}$$

and that there exists a subsequence (which we do not relabel) and $\underline{f} \in \mathrm{C}([0,T]; \mathscr{M}_+^I(\mathbb{R}^2_{>0}))$ such that

$$\int_{(0,\infty)^2} f_\epsilon(\eta,t)\varphi(\eta)\mathrm{d}\eta \to \int_{(0,\infty)^2} \underline{f}(\eta,t)\varphi(\eta)\mathrm{d}\eta,$$

as $\epsilon \to 0$, for every $t \in [0,T]$ and every $\varphi \in \mathrm{C}_0(\mathbb{R}^2_{>0})$. Additionally, we have that $\underline{f}$ satisfies a standard two-dimensional coagulation equation, namely, for every $t \in [0,T]$ and every $\varphi \in \mathrm{C}_c(\mathbb{R}^2_{>0})$, the following holds

$$\begin{aligned}&\int_{(0,\infty)^2} \underline{f}(\eta,t)\varphi(\eta)\mathrm{d}\eta - \int_{(0,\infty)^2} f_{\mathrm{in}}(\eta)\varphi(\eta)\mathrm{d}\eta \\ &= \int_0^t \int_{(0,\infty)^2}\int_{(0,\infty)^2} K(\eta,\eta')\underline{f}(\eta,s)\underline{f}(\eta',s)[\varphi(\eta+\eta') - \varphi(\eta) - \varphi(\eta')]\mathrm{d}\eta'\mathrm{d}\eta\mathrm{d}s.\end{aligned} \tag{B.1.17}$$

Remark B.1.13. The function $\underline{f}$ found in Theorem B.1.12 satisfies

$$\int_{(0,\infty)^2} a\underline{f}(\eta,t)\mathrm{d}\eta = \int_{(0,\infty)^2} a f_{\mathrm{in}}(\eta)\mathrm{d}\eta \text{ and } \int_{(0,\infty)^2} v\underline{f}(\eta,t)\mathrm{d}\eta = \int_{(0,\infty)^2} v f_{\mathrm{in}}(\eta)\mathrm{d}\eta$$

for every $t \in [0,T]$.

Remark B.1.14. The reason Theorem B.1.12 holds under the weaker condition (B.1.16) instead of (B.1.11) as in Theorem B.1.4 is due to the fact that the proof in this case involves more straightforward arguments. We will still require some moment estimates in order to prove that we can test with $\varphi(\eta) = a$ or $\varphi(\eta) = v$ in (B.1.17) so that Remark B.1.13 holds true. However, (B.1.17) does not contain the fusion term, so weaker moment estimates suffice in this case.

B.1.3 Short discussion on uniqueness of solutions

Remark B.1.15. Theorem B.1.4 says that we can construct a sequence of functions $\{f_\epsilon\}_{\epsilon\in(0,1)}$ such that a limit exists. However, Theorem B.1.4 is valid for any sequence of functions $\{f_\epsilon\}_{\epsilon\in(0,1)}$ for which (B.1.11) holds. The same holds for all the other results stated in this paper, with the mention that the case of negligible fusion (Theorem B.1.12) holds with its associated moment estimates, namely (B.1.16). Note that uniqueness of coagulation equations is, with the exception of some particular choices of coagulation kernels, still an open problem. Condition (B.1.11) is however a rather strong condition and an upper bound for the moments for the initial condition is a sufficient condition to construct a sequence for which (B.1.11) is true. We expect (B.1.11) to hold under weaker assumptions than the ones stated. This will be considered in a future work.

Lemma B.1.7 states that there exists a subsequence of the solutions of (B.1.10) which converges in the case of fast fusion in a suitable sense to a measure that can be computed by solving the standard one-dimensional coagulation equation. However, without uniqueness, there would be multiple possibilities of this measure as the only information we have is that it solves a one-dimensional coagulation equation.

Nevertheless, uniqueness results for the one-dimensional coagulation equation would imply that the limit whose existence has been proven in Theorem B.1.4 is uniquely determined and therefore the convergence would be independent of the subsequence. As such, the limit for the sequence $\{f_\epsilon\}$ as $\epsilon \to 0$ would exist.

Thus, it is worthwhile to mention that uniqueness of one-dimensional coagulation equations has been proven for some classes of kernels. We refer to [Mel57, McL64] for pioneering works on this matter for the constant and product coagulation kernel, [FL06b, MP04, Nor99] for additional classes of kernels in the Radon measures context, and to [Ste90] for other classes of kernels in the L^1 context.

The same situation is valid for Theorem B.1.12, which states that the limit is a solution of a standard two-dimensional coagulation equation in the case of negligible fusion. In the case of multi-dimensional coagulation equations, uniqueness results for some classes of kernels can be found in [Thr23].

B.2 The case of fast fusion

B.2.1 Existence of a limit of solutions of coagulation equations with fast fusion

This subsection is dedicated to proving Theorem B.1.4 and Lemma B.1.7. We first prove that if we are at a positive distance from the line $a = c_0 v^{\frac{2}{3}}$, the measures f_ϵ take values close to zero (Proposition B.2.7). We then make use of this property to show that the sequence $\{f_\epsilon\}_{\epsilon\in(0,1)}$ is equicontinuous in time for positive times (Proposition B.2.9). We are then able to conclude that there exists a subsequence of $\{f_\epsilon\}$ such that a limit exists. Moreover, we will need to make use of some properties of the solution of a simplified form for the adjoint problem of (B.1.10) for the proof of Proposition B.2.7 (see Proposition B.2.3 and Proposition B.2.4).

Let $T > 0$ and $t \in [0, T]$. We consider the equation

$$\int_{(0,\infty)^2} f_\epsilon(\eta, t)\varphi(\eta, t)\mathrm{d}\eta - \int_{(0,\infty)^2} f_{\mathrm{in}}(\eta)\varphi(\eta, 0)\mathrm{d}\eta - \int_0^t \int_{(0,\infty)^2} f_\epsilon(\eta, s)\partial_s\varphi(\eta, s)\mathrm{d}\eta\mathrm{d}s$$
$$= \int_0^t \langle \mathbb{K}[f_\epsilon(s)], \varphi(s)\rangle \mathrm{d}s + \frac{1}{\epsilon}\int_0^t \int_{(0,\infty)^2} r(\eta)(c_0 v^{\frac{2}{3}} - a) f_\epsilon(\eta, s)\partial_a\varphi(\eta, s)\mathrm{d}\eta\mathrm{d}s, \tag{B.2.1}$$

for $\epsilon \in (0,1)$, $\varphi \in \mathrm{C}_0^1(\mathbb{R}_{>0}^2)$ and with $f_{\text{in}} \in \mathscr{M}_+^I(\mathbb{R}_{>0}^2)$.

For completeness, we have to prove that solutions of (B.2.1) as in Definition B.1.1 exist. We begin by remembering the truncated functions used to prove the existence of solutions for truncated versions of coagulation equations allowing fusion of particles, which is done using a fixed point argument. More details can be found in [CV23].

We define $K_R : (0,\infty)^4 \to [0,\infty)$ to be a continuous function such that:

$$K_R(a,v,a',v') = \min\{K(a,v,a',v'), R\}, \tag{B.2.2}$$

where K satisfies the upper bound in (B.1.6) and take $\xi_R : \mathbb{R}_{>0} \to [0,\infty)$ to be continuous and defined in the following manner:

$$\xi_R(v) = 0, \qquad \text{when } v \geq 2R, \tag{B.2.3}$$
$$\xi_R(v) = 1, \qquad \text{on } (0,R]. \tag{B.2.4}$$

Then, for $\varphi \in \mathrm{C}_0^1(\mathbb{R}_{>0}^2)$, we denote by

$$\langle \mathbb{K}_R[f], \varphi \rangle := \frac{1}{2} \int_{(0,\infty)^2} \int_{(0,\infty)^2} K_R(\eta,\eta')\xi_R(v+v')[\varphi(\eta+\eta') - \varphi(\eta) - \varphi(\eta')] f(\mathrm{d}\eta) f(\mathrm{d}\eta'). \tag{B.2.5}$$

For the fusion term, we use the following truncation:

$$r_\delta(a,v) := \frac{r(\eta)\max\{v^\sigma, L\delta\}}{v^\sigma(1+\delta a^\mu)}, \tag{B.2.6}$$

for $\delta \in (0,1)$ and some fixed $L > 0$. L was chosen in [CV23] to be $L := \frac{12}{R_0(1-\gamma)}$, where R_0 is as in (B.1.4), in order to obtain existence of self-similar profiles for equation (B.1.1).

For functions f satisfying $f \in \mathrm{C}^1([0,\infty); \mathscr{M}_+^I(\mathbb{R}_{>0}^2))$ with

$$f\Big(\mathbb{R}_{>0}^2 \setminus [c_0\overline{\epsilon}^{\frac{2}{3}}, \infty) \times [\overline{\epsilon}, 2R), t\Big) = 0 \tag{B.2.7}$$

and such that

$$\sup_{t\in[0,T]} \int_{(0,\infty)^2} (1+a) f_{\overline{\epsilon},R,\delta}(a,v,t)\mathrm{d}v\mathrm{d}a < \infty, \tag{B.2.8}$$

for all times $T \in [0,\infty)$, we define the space

$$U_{\overline{\epsilon},R} := \{f \in \mathrm{C}^1([0,\infty); \mathscr{M}_+^I(\mathbb{R}_{>0}^2)), f \text{ satisfies } (B.2.7) \text{ and } (B.2.8)\}. \tag{B.2.9}$$

We now prove the following result.

Proposition B.2.1 (Existence of solutions). *Assume* $f_{\text{in}} \in \mathscr{M}_+^I(\mathbb{R}_{>0}^2)$ *and*

$$\int_{(0,\infty)^2} (v^{-1} + v^2 + a) f_{\text{in}}(\eta)\mathrm{d}\eta < \infty. \tag{B.2.10}$$

Let $T>0$ and fix $\epsilon\in(0,1)$. Then there exists an $f_\epsilon\in \mathrm{C}([0,T];\mathscr{M}_+^I(\mathbb{R}^2_{>0}))$ as in Definition B.1.1 with

$$\sup_{t\in[0,T]}\int_{(0,\infty)^2}(v^{-1}+v^2+a)f_\epsilon(\eta,t)\mathrm{d}\eta\leq C(T). \tag{B.2.11}$$

Proof. We study the following truncated version of equation (B.2.1):

$$\partial_t\int_{(0,\infty)^2}f_{\epsilon,R}(\eta,t)\varphi(\eta)\mathrm{d}\eta=\langle\mathbb{K}_R[f_{\epsilon,R}(t)],\varphi\rangle+\frac{1}{\epsilon}\int_{(0,\infty)^2}\tilde r(\eta)(c_0v^{\frac{2}{3}}-a)f_{\epsilon,R}(\eta,t)\partial_a\varphi(\eta)\mathrm{d}\eta, \tag{B.2.12}$$

with

$$\tilde r(\eta)=\begin{cases} r_\delta(\eta), & \mu>0,\\ r(\eta), & \mu\leq 0,\end{cases}$$

where r_δ is as in (B.2.6) and $\mathbb{K}_R$ as in (B.2.5).

We prove existence and uniqueness of $f_{\epsilon,R}$ in the space $U_{\tilde\epsilon,R}$ defined as in (B.2.9), for some $\tilde\epsilon\in(0,1)$. We keep the notation $f_{\epsilon,R}$ for convenience. Notice however that $f_{\epsilon,R}$ depends on $\epsilon,R,\tilde\epsilon,\delta$ when $\mu>0$ and on $\epsilon,R,\tilde\epsilon$ when $\mu\leq 0$.

We look at the system of characteristic equations:

$$\begin{cases}\partial_tA(a_0,v_0,t)=\dfrac{1}{\epsilon}\tilde r(A,V)(c_0V^{\frac{2}{3}}-A), & A(a_0,v_0,0)=a_0;\\ \partial_tV(a_0,v_0,t)=0, & V(a_0,v_0,0)=v_0;\\ \partial_tc(a_0,v_0,t)=\dfrac{1}{\epsilon}\partial_A[\tilde r(A,V)(A-c_0V^{\frac{2}{3}})]c(a_0,v_0,t), & c(a_0,v_0,0)=1.\end{cases} \tag{B.2.13}$$

Remark B.2.2. We remark that $V(a_0,v_0,t)\equiv v_0$. Fix $t\geq 0$, we denote the pair $(A(a_0,v_0,t),V(a_0,v_0,t))=:\phi_{t,\epsilon}(a_0,v_0)$. We fix v_0, we define $x^\epsilon_{t,v_0}:\mathbb{R}_{>0}\to\mathbb{R}$ by $x^\epsilon_{t,v_0}(a_0):=A(a_0,v_0,t)$.

We first prove the existence and uniqueness of functions $\overline{F}_{\epsilon,R}\in\mathrm{C}([0,T];\mathscr{M}^I_{+,\mathrm{b}}(\mathbb{R}^2_{>0}))$ satisfying

$$\begin{aligned}\partial_t\int_{\mathbb{R}^2_{>0}}\overline{F}_{\epsilon,R}(\eta,t)\varphi(\eta)\mathrm{d}\eta=\frac{1}{2}\int_{\mathbb{R}^2_{>0}}\int_{\mathbb{R}^2_{>0}}&K_R(x^\epsilon_{t,V}(A),V,x^\epsilon_{t,V'}(A'),V')\xi_R(V+V')\overline{F}_{\epsilon,R}(A',V',t)\\ &\overline{F}_{\epsilon,R}(A,V,t)[\varphi(\phi^{-1}_{t,\epsilon}(\phi_{t,\epsilon}(A,V)+\phi_{t,\epsilon}(A',V')))-\varphi(A,V)-\varphi(A',V')]\mathrm{d}V'\mathrm{d}A'\mathrm{d}V\mathrm{d}A,\end{aligned}$$

for every $\varphi\in\mathrm{C_c}(\mathbb{R}^2_{>0})$. For this, we repeat the arguments used in [CV23, Proposition 3.1].

We then define $f_{\epsilon,R}\in\mathrm{C}^1([0,\infty);\mathscr{M}^I_+(\mathbb{R}^2_{>0}))$ as

$$\int_{(0,\infty)^2}f_{\epsilon,R}(a,v,t)\varphi(a,v)\mathrm{d}\eta=\int_{(0,\infty)^2}\overline{F}_{\epsilon,R}(a,v,t)\varphi(\phi_{t,\epsilon}(a,v))\mathrm{d}\eta, \tag{B.2.14}$$

for every $\varphi\in\mathrm{C}_0(\mathbb{R}^2_{>0})$. Notice that the functions $f_{\epsilon,R}$ defined in this manner will satisfy equation (B.2.12). For more details, see [CV23, Proposition 3.1].

We are now left to prove uniform estimates for $f_{\epsilon,R}$ in order to finish the proof. Due to the choice of the space $U_{\tilde{\epsilon},R}$, we can test (B.2.12) with $\varphi(a,v) = a$ and $\varphi(a,v) = v^d$, with $d \in \mathbb{R}$. When $d \leq 1$, we obtain:

$$\int_{(0,\infty)^2} (v^d + a) f_{\epsilon,R}(\eta,t)\mathrm{d}\eta \leq \int_{(0,\infty)^2} (v^d + a) f_{\mathrm{in}}(\eta)\mathrm{d}\eta.$$

For $d = 2$, since $M_{0,1-\alpha}(f_{\epsilon,R})$ is now uniformly bounded, we have

$$\begin{aligned}\int_{(0,\infty)^2} v^2 f_{\epsilon,R}(\eta,t)\mathrm{d}\eta &\leq \int_{(0,\infty)^2} v^2 f_{\mathrm{in}}(\eta)\mathrm{d}\eta + C\int_0^t M_{0,1-\alpha}(f_{\epsilon,R}(s))M_{0,1+\beta}(f_{\epsilon,R}(s))\mathrm{d}s \\ &\leq \int_{(0,\infty)^2} v^2 f_{\mathrm{in}}(\eta)\mathrm{d}\eta + C\int_0^t M_{0,2}(f_{\epsilon,R}(s))\mathrm{d}s + C.\end{aligned}$$

Thus, there exists $C(t) > 0$ such that

$$\int_{(0,\infty)^2} v^2 f_{\epsilon,R}(\eta,t)\mathrm{d}\eta \leq C(t)\int_{(0,\infty)^2} v^2 f_{\mathrm{in}}(\eta)\mathrm{d}\eta.$$

Equicontinuity

Fix $\varphi \in \mathrm{C}_c^1(\mathbb{R}^2_{>0})$. Let $s,t \in [0,T]$. Assume without loss of generality $s \leq t$. Then

$$\begin{aligned}&\left|\int_{(0,\infty)^2} [f_{\epsilon,R}(\eta,t) - f_{\epsilon,R}(\eta,s)]\varphi(\eta)\mathrm{d}\eta\right| \\ &\lesssim \|\varphi\|_\infty \int_s^t M_{0,-\alpha}(f_{\epsilon,R}(z))M_{0,\beta}(f_{\epsilon,R}(z))\mathrm{d}z + \frac{1}{\epsilon}\int_s^t\int_{(0,\infty)^2} a^\mu v^\sigma(a + c_0 v^{\frac{2}{3}})f_{\epsilon,R}(\eta,z)|\partial_a\varphi(\eta)|\mathrm{d}\eta\mathrm{d}z \\ &\leq \|\varphi\|_\infty \int_s^t M_{0,-\alpha}(f_{\epsilon,R}(z))M_{0,\beta}(f_{\epsilon,R}(z))\mathrm{d}z + C\|\partial_a\varphi\|_\infty \int_s^t\int_{(0,\infty)^2} f_{\epsilon,R}(\eta,z)\mathrm{d}\eta\mathrm{d}z \\ &\leq C|t-s|. \end{aligned} \tag{B.2.15}$$

Using the fact that $f_{\epsilon,R} \in U_{\tilde{\epsilon},R}$, we can extend (B.2.15) to hold for functions $\varphi \in \mathrm{C}_c(\mathbb{R}^2_{>0})$ and then for all $\varphi \in \mathrm{C}_0(\mathbb{R}^2_{>0})$. For details, see [CV23, Proposition 3.14].

Combining the found equicontinuity in (B.2.15) with the uniform moment estimates, which are independent of $\tilde{\epsilon}, R$ and δ, we conclude using Arzelà–Ascoli theorem that there exists a subsequence of $\{f_{\epsilon,R}\}$, which we do not relabel, and an $f_\epsilon \in \mathrm{C}([0,T];\mathscr{M}_+^I(\mathbb{R}^2_{>0}))$, such that $f_{\epsilon,R}(t)$ converge to $f_\epsilon(t)$ in the weak-* topology as $\tilde{\epsilon} \to 0$, $R \to \infty$ and $\delta \to 0$, for every $t \in [0,T]$.

Thus, we can use standard arguments found in the study of coagulation equations in order to pass to the limit as $\tilde{\epsilon} \to 0$ and $R \to \infty$ (and $\delta \to 0$ if $\mu > 0$) in (B.2.12). □

In order to understand the reasoning behind the statement that our measures will concentrate along the line $\{a = c_0 v^{\frac{2}{3}}\}$, it is helpful to study some properties of the solution of the characteristic ODE associated with (B.2.1). This is the content of the following proposition.

Proposition B.2.3. *Let $\mu > 0$. We look at the system*

$$\partial_t x^{\epsilon}_{A,V}(t) = \frac{1}{\epsilon} r(x^{\epsilon}_{A,V}(t), V)(c_0 V^{\frac{2}{3}} - x^{\epsilon}_{A,V}(t)), \qquad x^{\epsilon}_{A,V}(0) = A.$$

For any $(A, V) \in (0, \infty)^2$, fixed, and $t > 0$, we have that $\lim_{\epsilon\to 0} x^{\epsilon}_{A,V}(t) = c_0 V^{\frac{2}{3}}$. The statement holds true for compact sets K of the form $(A, V, t) \in \mathrm{K} \subset (0, \infty)^3$.

Proof. Define $\tilde{x}_{A,V}(t) := x^{\epsilon}_{A,V}(\epsilon t)$. Then $\tilde{x}_{A,V}(0) = A$ and

$$\partial_t \tilde{x}_{A,V}(t) = \partial_t x^{\epsilon}_{A,V}(\epsilon t) = \epsilon \partial_{\epsilon t} x^{\epsilon}_{A,V}(\epsilon t) = r(\tilde{x}_{A,V}, V)(c_0 V^{\frac{2}{3}} - \tilde{x}_{A,V}) =: f(\tilde{x}_{A,V}(t)). \tag{B.2.16}$$

This implies that $\lim_{\epsilon\to 0} x^{\epsilon}_{A,V}(t) = \lim_{\epsilon\to 0} \tilde{x}_{A,V}(\frac{t}{\epsilon}) = \lim_{t\to\infty} \tilde{x}_{A,V}(t)$. In (B.2.16) notice that

$$\begin{cases} \tilde{x}_{A,V}(t) = c_0 V^{\frac{2}{3}} \Rightarrow f(\tilde{x}_{A,V}(t)) = 0, \\ \tilde{x}_{A,V}(t) < c_0 V^{\frac{2}{3}} \Rightarrow f(\tilde{x}_{A,V}(t)) > 0, \\ \tilde{x}_{A,V}(t) > c_0 V^{\frac{2}{3}} \Rightarrow f(\tilde{x}_{A,V}(t)) < 0. \end{cases}$$

This implies that $\lim_{t\to\infty} \tilde{x}_{A,V}(t) = c_0 V^{\frac{2}{3}}$. □

In order to prove that the measures f_ϵ take values close to zero if we are at a positive distance from the line $\{a = c_0 v^{\frac{2}{3}}\}$, we notice that, at least for small times, the coagulation term in (B.1.10) gives a small contribution in this region. We thus look to control the contribution coming from the fusion term. For this, it is helpful to look at a simplified form for the adjoint problem of (B.1.10) and to study the properties of its solution. This will be the content of Proposition B.2.4.

For simplicity of notation, we denote by

$$\mathrm{S} := \{(a, v) \in \mathbb{R}^2_{>0}, a \geq c_0 v^{\frac{2}{3}}\}. \tag{B.2.17}$$

Proposition B.2.4 (Dual equation, case $\mu > 0$)**.** *Let $T > 0$. Let $\epsilon \in (0, 1)$, $R > 1$ and $\mu > 0$. Let $T > 0$. Let $\chi(\eta)$ be an arbitrary function in $\mathrm{C}^1_{\mathrm{b}}(\mathrm{S})$, where S is as in (B.2.17), such that $\chi(\eta) = 0$,* when *$v \notin [\frac{1}{R}, R]$. Then there exists a solution $\varphi_\epsilon \in \mathrm{W}_T$, with $\varphi_\epsilon(T, \cdot) = \chi(\cdot)$, which solves the following equation:*

$$\partial_t \varphi_\epsilon(t, \eta) + \frac{1}{\epsilon} \overline{r}_\epsilon(a, v)(c_0 v^{\frac{2}{3}} - a)\partial_a \varphi_\epsilon(t, \eta) = 0, \tag{B.2.18}$$

where

$$\mathrm{W}_T := \{\varphi \in \mathrm{C}^1([0, T], \mathrm{C}^1_{\mathrm{b}}(\mathrm{S})) | \varphi(t, \eta) = 0, \text{ when } v \notin [\frac{1}{R}, R], \text{ for every } t \in [0, T]\}$$

and

$$\overline{r}_\epsilon(a, v) = \frac{r(a, v)}{1 + \epsilon^{2\mu+2} a^\mu}.$$

Assume in addition that there exists $\delta_1 > 0$ such that $\chi(\eta) = 0$ when $a \leq c_0 v^{\frac{2}{3}} + \delta_1$. Then the following statements hold:

1. *For every $M > 1$ and $\overline{\sigma} > 0$, there exists $\epsilon_{M,\overline{\sigma},\delta_1} \in (0,1)$ such that for every $\epsilon \leq \epsilon_{M,\overline{\sigma},\delta_1}$, we have that* $\operatorname{supp}\varphi_\epsilon(t) \subseteq \{(a,v) \in (0,\infty)^2 | a \geq M\}$*, for every $t \in [0, T-\overline{\sigma}]$.*

2. *There exists a constant $C(T)$, which can depend on time, but is independent on ϵ, such that* $\sup_{t\in[0,T]}(\|\varphi_\epsilon(t)\|_\infty + \|\partial_a\varphi_\epsilon\|_\infty) \leq C(T)$*. Moreover, the constant $C(T)$ is independent of the initial datum χ if we assume $\|\chi\|_\infty + \|\partial_a\chi\|_\infty \leq 1$.*

3. *If we assume $\|\chi\|_\infty + \|\partial_a\chi\|_\infty \leq 1$ and that ϵ is sufficiently small, then there exists a constant $C > 0$, which is independent of $\epsilon > 0$, but can depend on $R > 1$, such that $\|\partial_a\varphi_\epsilon(t)\|_\infty \leq \mathrm{e}^{-\frac{C(T-t)}{\epsilon}}$.*

Proof. For Statement 1 we use Proposition B.2.3, the fact that we can find an explicit solution for equation (B.2.18) and then we let $\epsilon \to 0$.

Statement 2 follows directly from the fact that at time T we have $\varphi_\epsilon(T,\cdot) = \chi(\cdot) \in \mathrm{C}^1_\mathrm{b}(\mathrm{S})$ and by integrating along the characteristics in equation (B.2.18).

For Statement 3, we notice that, for $a \geq c_0 v^{\frac{2}{3}}$, we have that

$$\begin{aligned}
\partial_a\Big[\frac{r(a,v)}{1+\epsilon^{2\mu+2}a^\mu}(a - c_0v^{\frac{2}{3}})\Big] &= \partial_a\Big[\frac{r(a,v)}{1+\epsilon^{2\mu+2}a^\mu}\Big](a - c_0v^{\frac{2}{3}}) + \frac{r(a,v)}{1+\epsilon^{2\mu+2}a^\mu} \\
&= \frac{\partial_a r(a,v) - \frac{\mu\epsilon^{2\mu+2}a^{\mu-1}r(a,v)}{1+\epsilon^{2\mu+2}a^\mu}}{1+\epsilon^{2\mu+2}a^\mu}(a - c_0v^{\frac{2}{3}}) + \frac{r(a,v)}{1+\epsilon^{2\mu+2}a^\mu} \\
&= \frac{\partial_a r(a,v) - \mu a^{-1}r(a,v)\frac{\epsilon^{2\mu+2}a^\mu}{1+\epsilon^{2\mu+2}a^\mu}}{1+\epsilon^{2\mu+2}a^\mu}(a - c_0v^{\frac{2}{3}}) + \frac{r(a,v)}{1+\epsilon^{2\mu+2}a^\mu} \\
&\geq \frac{\partial_a r(a,v) - \mu a^{-1}r(a,v)}{1+\epsilon^{2\mu+2}a^\mu}(a - c_0v^{\frac{2}{3}}) + \frac{r(a,v)}{1+\epsilon^{2\mu+2}a^\mu} \\
&\geq \frac{r(a,v)}{1+\epsilon^{2\mu+2}a^\mu},
\end{aligned} \tag{B.2.19}$$

where for the last inequality in (B.2.19), we used (B.1.5).

We then analyse the cases $\epsilon^{2\mu+2}a^\mu \leq 1$ and $\epsilon^{2\mu+2}a^\mu \geq 1$ in order to deduce that $\frac{r(a,v)}{1+\epsilon^{2\mu+2}a^\mu} \geq C(R)\min\{\frac{a^\mu}{2}, \frac{1}{2\epsilon^{2\mu+2}}\}$. By taking ϵ to be sufficiently small, we deduce from (B.2.19) that

$$\partial_a\Big[\frac{r(a,v)}{1+\epsilon^{2\mu+2}a^\mu}(a - c_0v^{\frac{2}{3}})\Big] \geq C(R). \tag{B.2.20}$$

We now look at the ODE

$$\partial_t x(\eta,t) = \frac{1}{\epsilon}\overline{r}_\epsilon(x,v)(c_0v^{\frac{2}{3}} - x) \text{ with } x(\eta,T) = a,$$

which, by taking $s = \frac{T-t}{\epsilon}$, reduces to solving

$$\partial_s x(\eta,s) = \overline{r}_\epsilon(x,v)(x - c_0v^{\frac{2}{3}}) \text{ with } x(\eta,0) = a.$$

Using (B.2.20), we obtain that

$$\partial_a x(\eta,s) \geq \mathrm{e}^{C(R)s}$$

and Statement 3 follows using in addition the fact that $\|\partial_a\chi\|_\infty \leq 1$. □

Remark B.2.5. In the case $\mu \leq 0$, the condition (B.1.5) is more general due to the fact that we will not modify the fusion term r in Proposition B.2.4.

The following is just a technical proposition that shows we can test the equation (B.1.10) with continuous bounded functions. This will be needed for the proof of the fact that the measure f_ϵ takes values close to zero when we are at a positive distance from the isoperimetric line. This is since we would like to test in (B.1.10) with the function obtained in (B.2.18), which is bounded by the Statement 2 of Proposition B.2.4.

Proposition B.2.6. *Let $\epsilon > 0$, fixed and $\mu > 0$. Then equation (B.2.1) holds for every $\varphi \in \mathrm{C}^1([0,T];\mathrm{C}^1(\mathbb{R}^2_{>0}))$ with $\sup_{s\in[0,T];\eta\in\mathbb{R}^2_{>0}} |\partial_t\varphi(\eta,t) + \varphi(\eta,t) + \partial_a\varphi(\eta,t)| \leq C$ if*

$$\sup_{t\in[0,T]} \int_{(0,\infty)^2} (a^{\mu+2} + v^{\max\{\sigma(\mu+2),2\}} + v^{\min\{\sigma(\mu+2),-1\}}) f_\epsilon(a,v,t) \leq C(T). \tag{B.2.21}$$

Proof. Assume for simplicity that $\varphi \in \mathrm{C}^1_{\mathrm{b}}(\mathbb{R}^2_{>0})$. We construct a sequence of functions $\{\zeta_n\}_{n\in\mathbb{N}} \subset \mathrm{C}^1_{\mathrm{c}}(\mathbb{R}^2_{>0})$ such that $\zeta_n(\eta) = 1$ when $\eta \in [\frac{1}{n}, n]^2$ and $\zeta_n(\eta) = 0$ when $\eta \notin [\frac{1}{2n}, 2n]^2$. The idea is to use Lebegue's dominated convergence theorem in (B.1.10) for the functions $\varphi_n = \zeta_n\varphi$. We thus show below only the needed estimates for the proof. The term with the coagulation kernel in (B.1.10) can be bounded directly by

$$\left|\langle \mathbb{K}[f_\epsilon], \varphi_n\rangle\right| \leq 3C \sup_{s\in[0,T]} M_{0,-\alpha}(f_\epsilon(s)) \sup_{s\in[0,T]} M_{0,\beta}(f_\epsilon(s)) \leq C.$$

In order to control the fusion term in (B.1.10), notice that we can construct ζ_n such that $|a\partial_a\zeta_n(\eta)| \leq C$, for some constant independent of $n \in \mathbb{N}$. Moreover, we know that the fusion kernel satisfies (B.1.4) and that $a \geq c_0 v^{\frac{2}{3}}$. Thus

$$|r(a,v)(c_0 v^{\frac{2}{3}} - a)| \lesssim a^{\mu+1}v^{\sigma} + a^{\mu}v^{\sigma+\frac{2}{3}} \leq 2a^{\mu+1}v^{\sigma}.$$

Using the above inequality, we can estimate from above the fusion term

$$\frac{1}{\epsilon}\Big|\int_{(0,\infty)^2} r(\eta)(c_0 v^{\frac{2}{3}} - a) f_\epsilon(\eta,s)\partial_a\varphi_n(\eta)\mathrm{d}\eta\Big|$$

by

$$\frac{1}{\epsilon}\Big|\int_{(0,\infty)^2} a^{\mu+1}v^{\sigma} f_\epsilon(\eta,s)\partial_a\varphi(\eta)\mathrm{d}\eta\Big| + \frac{1}{\epsilon}\Big|\int_{(0,\infty)^2} a^{\mu+1}v^{\sigma} f_\epsilon(\eta,s)\varphi(\eta)\partial_a\zeta_n(\eta)\mathrm{d}\eta\Big|$$

up to a multiplicity constant. We then use Young's inequality to deduce that

$$\Big|\int_{(0,\infty)^2} a^{\mu+1}v^{\sigma} f_\epsilon(\eta,s)\partial_a\varphi(\eta)\mathrm{d}\eta\Big| \lesssim \sup_{s\in[0,T]} M_{\mu+2,0}(f_\epsilon(s)) + \sup_{s\in[0,T]} M_{0,\sigma(\mu+2)}(f_\epsilon(s))$$

and

$$\Big| \int_{(0,\infty)^2} a^{\mu+1} v^{\sigma} f_\epsilon(\eta, s)\varphi(\eta)\partial_a \zeta_n(\eta) \mathrm{d}\eta \Big| \lesssim M_{\mu,\sigma}(f_\epsilon(s))$$

$$\lesssim \sup_{s\in[0,T]} M_{\mu+1,0}(f_\epsilon(s)) + \sup_{s\in[0,T]} M_{0,\sigma(\mu+2)}(f_\epsilon(s)) + 1.$$

Thus, the moment estimates in (B.2.21) suffice to conclude our proof for $\varphi \in \mathrm{C}^1_{\mathrm{b}}(\mathbb{R}^2_{>0})$. To prove that equation (B.2.1) holds for every $\varphi \in \mathrm{C}^1([0,T];\mathrm{C}^1(\mathbb{R}^2_{>0}))$ with $\sup_{s\in[0,T];\eta\in\mathbb{R}^2_{>0}} |\partial_t\varphi(\eta,t) + \varphi(\eta,t) + \partial_a\varphi(\eta,t)| \leq C$, we argue in a similar manner as before using the bound on the time derivative too. □

We now make use of Proposition B.2.4 and Proposition B.2.6 to prove that f_ϵ is small in sets that are at a positive distance from the isoperimetric line. This is the main ingredient needed for the proof of Theorem B.1.4.

Proposition B.2.7. *Let $T > 0$ and $\mu > 0$. Suppose $\{f_\epsilon\}_{\epsilon\in(0,1)}$ is a sequence of solutions as in Definition B.1.1 that satisfy the estimates (B.1.11). For every $\overline{\sigma} > 0$ and for every $\delta_1, \delta_2 \in (0,1)$, there exists $\epsilon_{\delta_1,\delta_2} \in (0,1)$, which is independent of $\overline{\sigma}$, such that*

$$\Big| \int_{(0,\infty)^2} f_\epsilon(a,v,t)\varphi(a,v)\mathrm{d}v\mathrm{d}a \Big| \leq \delta_2, \tag{B.2.22}$$

for every $t \geq \overline{\sigma}$, for all $\epsilon \leq \epsilon_{\delta_1,\delta_2}$ and for all $\varphi \in \mathrm{C}_0(\mathbb{R}^2_{>0})$ with $\|\varphi\|_\infty \leq 1$ such that $\varphi(a,v) = 0$, when $a \leq c_0 v^{\frac{2}{3}} + \delta_1$.

Remark B.2.8. In order to prove Proposition B.2.7, we will have to require other moment estimates in addition to the ones in (B.2.11). Namely, we impose that

$$\sup_{t\in[0,T]} \int_{(0,\infty)^2} a^{\mu+3} f_\epsilon(\mathrm{d}\eta, t) \leq C(T). \tag{B.2.23}$$

While the estimates in (B.2.10) suffice for the existence of solutions as in Definition B.1.1, in order to obtain an upper bound for moments involving higher powers of the area, we need the additional assumption that

$$\sup_{t\in[0,T]} \int_{(0,\infty)^2} (v^{-\mu-3} + v^{\mu+3}) f_\epsilon(\mathrm{d}\eta, t) \leq C(T),$$

which justifies the condition on the initial data in Theorem B.1.4. The proof of (B.2.23) relies then on the fact that the terms of the form $a^{\mu+2}v^{\beta}$, which appear due to the form of the coagulation kernel, can be bounded by

$$a^{\mu+2}v^{\beta} \lesssim a^{\mu+3} + v^{\beta(\mu+3)}.$$

For more details, we refer to [CV23, Subsection 3.3].

Proof of Proposition B.2.7. Let $\overline{\sigma} > 0$ and $\delta_1, \delta_2 \in (0,1)$. Assume $\Phi \in \mathrm{C}^1_{\mathrm{b}}(\mathbb{R}^2_{>0})$, with $\|\Phi\|_\infty \leq 1$. Assume there exists $R > 1$ such that $\Phi(\eta) = 0$ if $v \notin [\frac{1}{R}, R]$. The extension of the result from functions compactly supported in the v variable to functions that do not necessarily have compact support is straightforward using moment estimates, and thus we omit the details. Suppose in addition that Φ is such that $\Phi(\eta) = 0$ when $a < c_0 v^{\frac{2}{3}} + \delta_1$. Let, in addition, t_1, t_2, with $t_2 \geq \overline{\sigma}$, and t_1 such that $t_2 - t_1 = \tau > 0$. Notice that $\tau = \tau(\delta_2)$ depends on δ_2, but we will not write this dependence explicitly in order to simplify our notation. Let $M > 0$, sufficiently large (and also depending on δ_2), to be fixed later, and let φ_ϵ be the solution found in Proposition B.2.4, if $\mu > 0$, associated to the measure f_ϵ such that $\varphi_\epsilon(\eta, t_2) = \Phi(\eta)$. We want to prove (B.2.22).

Notice first that, from Proposition B.2.4, Statement 2, there exists a constant independent of ϵ, such that $\sup_{s\in[t_1,t_2],\eta\in\mathrm{S}} |\varphi(\eta, s)| \leq C$. Then, for $t_1, t_2 \in (0, T]$ such that $t_2 - t_1 = \tau$, we have

$$\begin{aligned}
&\Big| \int_{t_1}^{t_2} \int_{(0,\infty)^2} \int_{(0,\infty)^2} K(\eta, \eta') f_\epsilon(\eta, s) f_\epsilon(\eta', s) [\varphi_\epsilon(\eta + \eta', s) - \varphi_\epsilon(\eta, s) - \varphi_\epsilon(\eta', s)] \mathrm{d}\eta' \mathrm{d}\eta \mathrm{d}s \Big| \\
&\leq C\tau \sup_{s\in[0,T]} [M_{0,-\alpha}(f_\epsilon(s)) M_{0,\beta}(f_\epsilon(s))] \leq C\tau,
\end{aligned} \tag{B.2.24}$$

where we made use of the fact that $[M_{0,1} + M_{0,-1}](f_\epsilon)$ is uniformly bounded from above, independently of $\epsilon \in (0, 1)$.

In order to estimate the term with the fusion kernel, we notice that

$$\begin{aligned}
&\frac{1}{\epsilon} \Big| \int_{t_1}^{t_2} \int_{(0,\infty)^2} [r(a, v) - \overline{r}_\epsilon(a, v)](c_0 v^{\frac{2}{3}} - a) f_\epsilon(\eta, s) \partial_a \varphi_\epsilon(\eta, s) \mathrm{d}\eta \mathrm{d}s \Big| \\
&\quad \leq \frac{1}{\epsilon} \Big| \int_{t_1}^{t_2} \int_{\{a \leq \frac{1}{\epsilon}\}} [r(a, v) - \overline{r}_\epsilon(a, v)](c_0 v^{\frac{2}{3}} - a) f_\epsilon(\eta, s) \partial_a \varphi_\epsilon(\eta, s) \mathrm{d}\eta \mathrm{d}s \Big| \\
&\quad + \frac{2}{\epsilon} \int_{t_1}^{t_2} \int_{\{a > \frac{1}{\epsilon}\}} r(a, v)(c_0 v^{\frac{2}{3}} + a) f_\epsilon(\eta, s) |\partial_a \varphi_\epsilon(\eta, s)| \mathrm{d}\eta \mathrm{d}s.
\end{aligned} \tag{B.2.25}$$

For the first term in (B.2.25), we make use of the exponential decay of φ_ϵ proven in Statement 3 of Proposition B.2.4 in order to obtain

$$\begin{aligned}
&\frac{1}{\epsilon} \Big| \int_{t_1}^{t_2} \int_{\{a \leq \frac{1}{\epsilon}\}} [r(a, v) - \overline{r}_\epsilon(a, v)](c_0 v^{\frac{2}{3}} - a) f_\epsilon(\eta, s) \partial_a \varphi_\epsilon(\eta, s) \mathrm{d}\eta \mathrm{d}s \Big| \\
&\quad \lesssim \int_{t_1}^{t_2} \int_{\{a \leq \frac{1}{\epsilon}\}} |r(a, v) - \overline{r}_\epsilon(a, v)| (c_0 v^{\frac{2}{3}} + a) f_\epsilon(\eta, s) \frac{\mathrm{e}^{-\frac{C(t_2 - s)}{\epsilon}}}{\epsilon} \mathrm{d}\eta \mathrm{d}s \\
&\quad \lesssim 2R_1 \int_{t_1}^{t_2} \int_{\{a \leq \frac{1}{\epsilon}\}} \epsilon^{2\mu+2} a^{2\mu+1} v^\sigma f_\epsilon(\eta, s) \frac{\mathrm{e}^{-\frac{C(t_2 - s)}{\epsilon}}}{\epsilon} \mathrm{d}\eta \mathrm{d}s \\
&\quad \leq 2R_1 \epsilon \int_{t_1}^{t_2} \sup_{s\in[0,T]} M_{0,\sigma}(f_\epsilon(t)) \frac{\mathrm{e}^{-\frac{C(t_2 - s)}{\epsilon}}}{\epsilon} \mathrm{d}s \leq 2R_1 \epsilon C(T) \int_{t_1}^{t_2} \frac{\mathrm{e}^{-\frac{C(t_2 - s)}{\epsilon}}}{\epsilon} \mathrm{d}s \\
&\quad \leq C\epsilon.
\end{aligned} \tag{B.2.26}$$

We can then use Statement 2 of Proposition B.2.4, the upper bound (B.1.4) and (B.1.11) in order to control the region containing large values of a, namely

$$\begin{aligned}
&\frac{2}{\epsilon}\int_{t_1}^{t_2}\int_{\{a>\frac{1}{\epsilon}\}} r(a,v)(c_0 v^{\frac{2}{3}}+a)f_\epsilon(\eta,s)|\partial_a\varphi_\epsilon(\eta,s)|\mathrm{d}\eta\mathrm{d}s\\
&\quad\le \frac{4R_1}{\epsilon}\int_{t_1}^{t_2}\int_{\{a>\frac{1}{\epsilon}\}} a^{\mu+1}v^{\sigma}f_\epsilon(\eta,s)\mathrm{d}\eta\mathrm{d}s \le \frac{4R_1\epsilon}{\epsilon}\int_{t_1}^{t_2}\int_{\{a>\frac{1}{\epsilon}\}} a^{\mu+2}v^{\sigma}f_\epsilon(\eta,s)\mathrm{d}\eta\mathrm{d}s\\
&\quad\le 4R_1\tau \sup_{t\in[0,T]}[M_{\mu+3,0}(f_\epsilon(t))+M_{0,\sigma(\mu+3)}(f_\epsilon(t))].
\end{aligned}\tag{B.2.27}$$

Combining the estimates (B.2.25), (B.2.26) and (B.2.27), we deduce that

$$\frac{1}{\epsilon}\Big|\int_{t_1}^{t_2}\int_{(0,\infty)^2}[r(a,v)-\overline{r}_\epsilon(a,v)](c_0v^{\frac{2}{3}}-a)f_\epsilon(\eta,s)\partial_a\varphi_\epsilon(\eta,s)\mathrm{d}\eta\mathrm{d}s\Big| \le C\epsilon + C\tau. \tag{B.2.28}$$

Proposition B.2.6 gives us that we can test (B.1.10) with continuous functions that are not necessarily compactly supported in the a variable, as long as we work with functions in $\mathrm{C}^1_\mathrm{b}(\mathbb{R}^2_{>0})$. We then make use of the fact that φ_ϵ satisfies (B.2.18) and then use the estimates in (B.2.24) and (B.2.28) in order to deduce that, for every $\epsilon \le \epsilon_{\delta_1,\delta_2}$, with $\epsilon_{\delta_1,\delta_2}$ (notice here that M and τ depend on δ_2) as in Proposition B.2.4, Statement 1, we have

$$\begin{aligned}
\Big|\int_{(0,\infty)^2} f_\epsilon(\eta,t_2)\Phi(a,v)\mathrm{d}\eta\Big| &= \Big|\int_{(0,\infty)^2} f_\epsilon(\eta,t_2)\varphi_\epsilon(\eta,t_2)\mathrm{d}\eta\Big| \le \Big|\int_{(0,\infty)^2} f_\epsilon(\eta,t_1)\varphi_\epsilon(\eta,t_1)\mathrm{d}\eta\Big| + C\tau\\
&= \Big|\int_{\{a\ge M\}} f_\epsilon(\eta,t_1)\varphi_\epsilon(\eta,t_1)\mathrm{d}\eta\Big| + C\tau\\
&\le CM^{-1}\int_{\{a\ge M\}} af_\epsilon(\eta,t_1)\mathrm{d}\eta + C\tau,
\end{aligned}\tag{B.2.29}$$

where we used that $\mathrm{supp}\varphi_\epsilon \subseteq \{a \ge M\}$ from Statement 1 of Proposition B.2.4 and the fact that $\sup_{t\in[0,T]} M_{1,0}(f_\epsilon(t)) \le C(T)$, with $C(T)$ being independent of ϵ. We choose M such that $C(T)CM^{-1} + C\tau \le \delta_2$.

Thus, we obtain that

$$\Big|\int_{(0,\infty)^2} f_\epsilon(\eta,t_2)\Phi(\eta)\mathrm{d}\eta\Big| \le \delta_2, \tag{B.2.30}$$

for every $t_2 \ge \overline{\sigma}$ since we can choose τ to be sufficiently small.

□

We are now able to prove the equicontinuity in time of the sequence $\{f_\epsilon\}_{\epsilon\in(0,1)}$. We proved in Proposition B.2.7 that if we are at a positive distance from the line $a = c_0v^{\frac{2}{3}}$, the measure f_ϵ takes values close to zero. Near the line $a = c_0v^{\frac{2}{3}}$, we use the fact that a function of the form $\varphi(a,v)$ can be approximated in terms of a function depending only on v, making negligible the contribution coming from the fusion term.

Proposition B.2.9 (Equicontinuity for positive times)**.** *Let $\overline{\sigma} > 0$. Suppose $\{f_\epsilon\}_{\epsilon\in(0,1)}$ is a sequence of solutions as in Proposition B.2.7. Let $\varphi \in \mathrm{C}_0(\mathbb{R}^2_{>0})$ with $\|\varphi\|_\infty \leq 1$. For every $\overline{\epsilon} \in (0,1)$, there exists $\overline{\delta} \in (0,\overline{\sigma})$ such that*

$$\Big|\int_{(0,\infty)^2} \varphi(\eta) f_\epsilon(\mathrm{d}\eta, t) - \int_{(0,\infty)^2} \varphi(\eta) f_\epsilon(\mathrm{d}\eta, s)\Big| \leq \overline{\epsilon},$$

for all $t, s \geq \overline{\sigma}$ such that $|t-s| \leq \overline{\delta}$ and for all ϵ sufficiently small.

Proof. Fix $\overline{\sigma} > 0$. Let $\delta_1, \delta_2 > 0$ as in Proposition B.2.7. We define a continuous function $\chi_{\delta_1} : \mathbb{R}^2_{>0} \to [0,1]$ to be equal to one in the region where $\{c_0 v^{\frac{2}{3}} \leq a \leq c_0 v^{\frac{2}{3}} + \delta_1\}$ and zero when $a > c_0 v^{\frac{2}{3}} + 2\delta_1$. Fix $\varphi \in \mathrm{C}^1_{\mathrm{c}}(\mathbb{R}^2_{>0})$. It suffices to prove the statement for fixed $\varphi \in \mathrm{C}^1_{\mathrm{c}}(\mathbb{R}^2_{>0})$. The passage from functions $\varphi \in \mathrm{C}^1_{\mathrm{c}}(\mathbb{R}^2_{>0})$ to functions in $\mathrm{C}_0(\mathbb{R}^2_{>0})$ is then straightforward using moment estimates and the fact that we can approximate a function in $\mathrm{C}_{\mathrm{c}}(\mathbb{R}^2_{>0})$ with a $\mathrm{C}^1_{\mathrm{c}}(\mathbb{R}^2_{>0})$ function on a compact set. From Proposition B.2.7, we deduce that there exists $\epsilon_{\delta_1,\delta_2} \in (0,1)$ such that for all $\epsilon \leq \epsilon_{\delta_1,\delta_2}$, the following holds

$$\begin{aligned}
&\Big|\int_{(0,\infty)^2} \varphi(\eta) f_\epsilon(\mathrm{d}\eta, t) - \int_{(0,\infty)^2} \varphi(\eta) f_\epsilon(\mathrm{d}\eta, s)\Big| \\
&\leq \Big|\int_{(0,\infty)^2} \varphi(\eta)\chi_{\delta_1}(\eta)[f_\epsilon(\mathrm{d}\eta, t) - f_\epsilon(\mathrm{d}\eta, s)]\Big| + \Big|\int_{(0,\infty)^2} \varphi(\eta)(1-\chi_{\delta_1}(\eta))[f_\epsilon(\mathrm{d}\eta, t) - f_\epsilon(\mathrm{d}\eta, s)]\Big| \\
&\leq \Big|\int_{(0,\infty)^2} \varphi(\eta)\chi_{\delta_1}(\eta)[f_\epsilon(\mathrm{d}\eta, t) - f_\epsilon(\mathrm{d}\eta, s)]\Big| + \delta_2.
\end{aligned} \tag{B.2.31}$$

From the support of χ_{δ_1}, the fact that the measure f_ϵ is supported in the region $\{a \geq c_0 v^{\frac{2}{3}}\}$ and the continuity of φ, we can find a function $\overline{\varphi} : \mathbb{R}_{>0} \to \mathbb{R}$ depending only on v, $\overline{\varphi} \in \mathrm{C}^1_{\mathrm{b}}(\mathbb{R}_{>0})$, such that $|\varphi(a,v) - \overline{\varphi}(v)| \leq \delta_3$, for δ_3 sufficiently small, and for any $(a,v) \in \mathrm{K} \subset \mathbb{R}^2_{>0}$, where K is a compact set. Let $K = [\frac{1}{M}, M]^2$ for some fixed $M > 1$. We have that

$$\begin{aligned}
&\Big|\int_{(0,\infty)^2} \varphi(\eta)\chi_{\delta_1}(\eta)[f_\epsilon(\mathrm{d}\eta, t) - f_\epsilon(\mathrm{d}\eta, s)]\Big| \\
&\leq \int_{(0,\infty)^2} |\varphi(\eta) - \overline{\varphi}(v)|\chi_{\delta_1}(\eta)|f_\epsilon(\mathrm{d}\eta, t) - f_\epsilon(\mathrm{d}\eta, s)| + \Big|\int_{(0,\infty)^2} \overline{\varphi}(v)\chi_{\delta_1}(\eta)[f_\epsilon(\mathrm{d}\eta, t) - f_\epsilon(\mathrm{d}\eta, s)]\Big| \\
&\leq \int_K |\varphi(\eta) - \overline{\varphi}(v)|\chi_{\delta_1}(\eta)|f_\epsilon(\mathrm{d}\eta, t) - f_\epsilon(\mathrm{d}\eta, s)| \\
&+ \int_{\mathbb{R}^2_{>0}\setminus K} |\varphi(\eta) - \overline{\varphi}(v)|\chi_{\delta_1}(\eta)|f_\epsilon(\mathrm{d}\eta, t) - f_\epsilon(\mathrm{d}\eta, s)| + \Big|\int_{(0,\infty)^2} \overline{\varphi}(v)\chi_{\delta_1}(\eta)[f_\epsilon(\mathrm{d}\eta, t) - f_\epsilon(\mathrm{d}\eta, s)]\Big|.
\end{aligned}$$

In order to control the contribution of the region $\mathbb{R}^2_{>0} \setminus K$, we use moment estimates. For example, for the region $\{a > M\}$, we obtain that

$$\int_{\{a>M\}} |\varphi(\eta) - \overline{\varphi}(v)|\chi_{\delta_1}(\eta)|f_\epsilon(\mathrm{d}\eta, t) - f_\epsilon(\mathrm{d}\eta, s)| \lesssim 2M^{-1} \sup_{s\in[0,T]} M_{1,0}(f_\epsilon(s)).$$

We can use the same argument for the region $\{v > M\} \cup \{v < \frac{1}{M}\} \cup \{a < \frac{1}{M}\}$. Thus

$$\begin{aligned}
&\Big|\int_{(0,\infty)^2} \varphi(\eta)\chi_{\delta_1}(\eta)[f_\epsilon(\mathrm{d}\eta, t) - f_\epsilon(\mathrm{d}\eta, s)]\Big| \\
&\lesssim 2\delta_3 \sup_{s\in[0,T]} M_{0,0}(f_\epsilon(s)) + 2M^{-1} \sup_{s\in[0,T]} M_{1,0}(f_\epsilon(s)) + \Big|\int_{(0,\infty)^2} \overline{\varphi}(v)\chi_{\delta_1}(\eta)[f_\epsilon(\mathrm{d}\eta, t) - f_\epsilon(\mathrm{d}\eta, s)]\Big| \\
&\leq C(T)\delta_3 + \Big|\int_{(0,\infty)^2} \overline{\varphi}(v)\chi_{\delta_1}(\eta)[f_\epsilon(\mathrm{d}\eta, t) - f_\epsilon(\mathrm{d}\eta, s)]\Big|, \qquad \text{(B.2.32)}
\end{aligned}$$

since the term $M^{-1}\sup_{s\in[0,T]} M_{1,0}(f_\epsilon(s))$ can be made sufficiently small by taking M sufficiently large. We can write the remaining term as $\overline{\varphi}(v)\chi_{\delta_1}(\eta) = \overline{\varphi}(v) - \overline{\varphi}(v)(1-\chi_{\delta_1}(\eta))$. For $\overline{\varphi}(v)(1-\chi_{\delta_1}(\eta))$ we can make use again of Proposition B.2.7 in order to prove that $\big|\int_{(0,\infty)^2} \overline{\varphi}(v)(1-\chi_{\delta_1}(\eta))[f_\epsilon(\mathrm{d}\eta, t) - f_\epsilon(\mathrm{d}\eta, s)]\big|$ is small. On the other hand, we deduce from Proposition B.2.6 that we can test equation (B.2.1) with $\overline{\varphi}(v)$ and obtain, as the fusion term disappears when testing with functions depending only on v that

$$\Big|\int_{(0,\infty)^2} \overline{\varphi}(v)[f_\epsilon(\mathrm{d}\eta, t) - f_\epsilon(\mathrm{d}\eta, s)]\Big| \leq 3\|\overline{\varphi}\|_\infty \int_s^t M_{0,-\alpha}(f_\epsilon(z))M_{0,\beta}(f_\epsilon(z))\mathrm{d}z \leq C|t-s|. \qquad \text{(B.2.33)}$$

From (B.2.31), (B.2.32) and (B.2.33) the equicontinuity in time of the sequence follows for every $s, t \in [\overline{\sigma}, T]$. □

We are finally in a position to prove Theorem B.1.4 and Lemma B.1.7.

Proof of Theorem B.1.4. Let $\overline{\sigma} > 0$. Using Proposition B.2.1 together with Proposition B.2.9, we deduce from Arzelà–Ascoli theorem that there exists a subsequence of $\{f_\epsilon\}$, which we do not relabel, and an $\overline{f} \in \mathrm{C}([\overline{\sigma}, T]; \mathscr{M}_+^I(\mathbb{R}^2_{>0}))$, such that $f_\epsilon(t)$ converges to $f(t)$ in the weak-* as $\epsilon \to 0$, for every $t \in [\overline{\sigma}, T]$. □

Proof of Lemma B.1.7. We pass to the limit in Proposition B.2.7. □

B.2.2 Reduction to the one-dimensional coagulation model in the case of fast fusion

We now consider the behavior of the solutions of (B.1.2) as $\Lambda \to 0$. More specifically, we focus in this subsection on proving Theorem B.1.10. We remember that our result holds independently of the chosen initial data (see Definition B.1.8), with the exception of requiring the initial data to satisfy some moment estimates (as specified in Theorem B.1.4).

We first show that we can extend F in Lemma B.1.7 to be continuous at time $t = 0$ as mentioned in Theorem B.1.10.

Proposition B.2.10. *Let $T > 0$. Let $F_\epsilon \in \mathrm{C}([0,T]; \mathscr{M}_+(\mathbb{R}_{>0}))$ be as in Definition B.1.9. Then the sequence $\{F_\epsilon\}_{\epsilon\in(0,1)}$ is equicontinuous in time, and thus we can deduce that there exists a limit $F \in \mathrm{C}([0,T]; \mathscr{M}_+(\mathbb{R}_{>0}))$ as $\epsilon \to 0$ for a subsequence of $\{F_\epsilon\}_{\epsilon\in(0,1)}$, which we do not relabel.*

Proof. Let $s, t \in [0, T]$, with $t > s$. Fix $\varphi \in C_0(\mathbb{R}_{>0})$. We then have that

$$
\begin{aligned}
&\Big|\int_{(0,\infty)} \varphi(v) F_\epsilon(\mathrm{d}v, t) - \int_{(0,\infty)} \varphi(v) F_\epsilon(\mathrm{d}v, s)\Big| \\
&= \Big|\int_{(0,\infty)^2} f_\epsilon(a, v, t)\chi_\epsilon(a)\varphi(v)\mathrm{d}v\mathrm{d}a - \int_{(0,\infty)^2} f_\epsilon(a, v, s)\chi_\epsilon(a)\varphi(v)\mathrm{d}v\mathrm{d}a\Big| \\
&\leq \int_s^t \langle \mathbb{K}[f_\epsilon(z)], \chi_\epsilon\varphi\rangle \mathrm{d}z + \frac{1}{\epsilon}\int_s^t \int_{(0,\infty)^2} r(a, v)|a + c_0 v^{\frac{2}{3}}||\partial_a \chi_\epsilon(a)\varphi(v)| f_\epsilon(\eta, z)\mathrm{d}\eta\mathrm{d}z \\
&\lesssim |t - s|[\sup_{z\in[0,T]} M_{0,-\alpha}(f_\epsilon(z)) \sup_{z\in[0,T]} M_{0,\beta}(f_\epsilon(z)) + \epsilon(\sup_{z\in[0,T]} M_{\mu+1,\sigma}(f_\epsilon(z)) + \sup_{z\in[0,T]} M_{\mu,\sigma+\frac{2}{3}}(f_\epsilon(z)))] \\
&\leq |t - s|[C(T) + C\epsilon(\sup_{z\in[0,T]} M_{\mu+2,0}(f_\epsilon(z)) + \sup_{z\in[0,T]} M_{0,\sigma(\mu+2)}(f_\epsilon(z)))] \lesssim |t - s|.
\end{aligned}
$$

□

We continue by proving that F satisfies the standard one-dimensional coagulation equation for all positive times $t > 0$.

Lemma B.2.11. *Let $\overline{\sigma} > 0$. Let F as in Theorem B.1.10. We then have that F satisfies the standard coagulation equation, namely, for every $t \in [\overline{\sigma}, T]$ and every $\varphi \in C_c(\mathbb{R}_{>0})$, the following holds*

$$
\begin{aligned}
&\int_{(0,\infty)} F(v, t)\varphi(v)\mathrm{d}v - \int_{(0,\infty)} F(v, \overline{\sigma})\varphi(v)\mathrm{d}v \\
&= \int_{\overline{\sigma}}^t \int_{(0,\infty)} K(c_0 v^{\frac{2}{3}}, v, c_0 v'^{\frac{2}{3}}, v') F(v, s) F(v', s)[\varphi(v + v') - \varphi(v) - \varphi(v')]\mathrm{d}v'\mathrm{d}v\mathrm{d}s. \qquad \text{(B.2.34)}
\end{aligned}
$$

Proof. Let $\delta_1, \delta_2 \in (0, 1)$. By Proposition B.2.7, we have that there exists $\epsilon_{\delta_1,\delta_2} \in (0, 1)$, such that

$$
\int_{\{a' > c_0 v'^{\frac{2}{3}} + \delta_1\}} \Phi(\eta') f_\epsilon(\eta', t)\mathrm{d}\eta' \leq C\delta_2, \qquad \text{(B.2.35)}
$$

for every $t \geq \overline{\sigma} > 0$, every $\epsilon \leq \epsilon_{\delta_1,\delta_2}$ and for every $\Phi \in C_0(\mathbb{R}^2_{>0})$ such that $\|\Phi\|_\infty \leq 1$.

Due to Proposition B.2.6, we can test (B.2.1) with $\varphi(a, v) = \tilde{\varphi}(v)$, where $\tilde{\varphi} \in C^1_0(\mathbb{R}_{>0})$. Let δ_1 as in (B.2.35). Denoting by $\chi_{\tilde{\varphi}}(v, v') := \tilde{\varphi}(v + v') - \tilde{\varphi}(v) - \tilde{\varphi}(v')$, we have that

$$
\begin{aligned}
&\int_{(0,\infty)^2} f_\epsilon(\eta, t)\tilde{\varphi}(v)\mathrm{d}\eta - \int_{(0,\infty)^2} f_\epsilon(\eta, \overline{\sigma})\tilde{\varphi}(v)\mathrm{d}\eta \\
&= \frac{1}{2}\int_{\overline{\sigma}}^t \int_{(0,\infty)^2} \int_{(0,\infty)^2} K(\eta, \eta')[\tilde{\varphi}(v + v') - \tilde{\varphi}(v) - \tilde{\varphi}(v')] f_\epsilon(\eta, s) f_\epsilon(\eta', s)\mathrm{d}\eta\mathrm{d}\eta'\mathrm{d}s \\
&= \frac{1}{2}\int_{\overline{\sigma}}^t \int_{\{a\in \mathrm{T}(v)\}} \int_{\{a'\in \mathrm{T}(v')\}} K(\eta, \eta')\chi_{\tilde{\varphi}}(v, v') f_\epsilon(\eta, s) f_\epsilon(\eta', s)\mathrm{d}\eta\mathrm{d}\eta'\mathrm{d}s \\
&+ \int_{\overline{\sigma}}^t \int_{\{a\in \mathrm{T}(v)\}} \int_{\{a' > c_0 v'^{\frac{2}{3}} + \delta_1\}} K(\eta, \eta')\chi_{\tilde{\varphi}}(v, v') f_\epsilon(\eta, s) f_\epsilon(\eta', s)\mathrm{d}\eta\mathrm{d}\eta'\mathrm{d}s \\
&+ \frac{1}{2}\int_{\overline{\sigma}}^t \int_{\{a > c_0 v^{\frac{2}{3}} + \delta_1\}} \int_{\{a' > c_0 v'^{\frac{2}{3}} + \delta_1\}} K(\eta, \eta')\chi_{\tilde{\varphi}}(v, v') f_\epsilon(\eta, s) f_\epsilon(\eta', s)\mathrm{d}\eta\mathrm{d}\eta'\mathrm{d}s, \qquad \text{(B.2.36)}
\end{aligned}
$$

where we denoted $\mathrm{T}(v) := [c_0 v^{\frac{2}{3}}; c_0 v^{\frac{2}{3}} + \delta_1]$.

Step 1. We first prove that, for every $t \geq \overline{\sigma}$, we have that

$$\frac{1}{2}\int_{\{a\in\mathrm{T}(v)\}}\int_{\{a'\in\mathrm{T}(v')\}} K(a,v,a',v')[\tilde{\varphi}(v+v')-\tilde{\varphi}(v)-\tilde{\varphi}(v')]f_\epsilon(\eta,t)f_\epsilon(\eta',t)\mathrm{d}\eta\mathrm{d}\eta'$$

tends to

$$\frac{1}{2}\int_{(0,\infty)^2} K(c_0 v^{\frac{2}{3}}, v, c_0 v'^{\frac{2}{3}}, v')[\tilde{\varphi}(v+v')-\tilde{\varphi}(v)-\tilde{\varphi}(v')]F(v,t)F(v',t)\mathrm{d}v'\mathrm{d}v$$

as $\epsilon \to 0$.

Step 1. a) We begin by proving that we are able to approximate the coagulation kernel near the isoperimetric line with a function only depending on the v variable, due to its continuity. In other words,

$$\begin{aligned}
&3\|\varphi\|_\infty \int_{\{a\in\mathrm{T}(v)\}}\int_{\{a'\in\mathrm{T}(v')\}} |K(a,v,a',v') - K(c_0 v^{\frac{2}{3}}, v, c_0 v'^{\frac{2}{3}}, v')| f_\epsilon(\eta,t)f_\epsilon(\eta',t)\mathrm{d}\eta\mathrm{d}\eta' \\
&\leq C\int_{\{a\in\mathrm{T}(v);v\in[\frac{1}{M};M]\}}\int_{\{a'\in\mathrm{T}(v');v'\in[\frac{1}{M};M]\}} |K(a,v,a',v') - K(c_0 v^{\frac{2}{3}}, v, c_0 v'^{\frac{2}{3}}, v')| f_\epsilon(\eta,t)f_\epsilon(\eta',t)\mathrm{d}\eta\mathrm{d}\eta' \\
&+ C\int_{\{v>M\}}\int |K(a,v,a',v') - K(c_0 v^{\frac{2}{3}}, v, c_0 v'^{\frac{2}{3}}, v')| f_\epsilon(\eta,t)f_\epsilon(\eta',t)\mathrm{d}\eta\mathrm{d}\eta' \\
&+ C\int_{\{v<\frac{1}{M}\}}\int |K(a,v,a',v') - K(c_0 v^{\frac{2}{3}}, v, c_0 v'^{\frac{2}{3}}, v')| f_\epsilon(\eta,t)f_\epsilon(\eta',t)\mathrm{d}\eta\mathrm{d}\eta' =: \mathrm{I}_1 + \mathrm{I}_2 + \mathrm{I}_3.
\end{aligned}$$

We can prove that $\mathrm{I}_2 + \mathrm{I}_3$ gives a small contribution due to moment estimates. For I_1, we use the fact that $|a - c_0 v^{\frac{2}{3}}| \leq \delta_1$, the continuity of K and that we work on a compact set, to deduce

$$\mathrm{I}_1 \leq C\overline{\epsilon} \sup_{t\in[0,T]} M_{0,0}(f_\epsilon(t))^2, \tag{B.2.37}$$

where $\overline{\epsilon}$ was chosen small but arbitrary and $M_{0,0}(f_\epsilon)$ is uniformly bounded independently of $\epsilon \in (0,1)$. Notice that δ_1 was chosen to be sufficiently small to satisfy both (B.2.35) and (B.2.37).

Step 1. b) Let $\overline{f}, F$ be as in Lemma B.1.7. We prove now that

$$\int_{\{a\in\mathrm{T}(v)\}}\int_{\{a'\in\mathrm{T}(v')\}} K(c_0 v^{\frac{2}{3}}, v, c_0 v'^{\frac{2}{3}}, v') f_\epsilon(\eta,t)f_\epsilon(\eta',t)\mathrm{d}\eta'\mathrm{d}\eta \tag{B.2.38}$$

converges to

$$\int_{(0,\infty)}\int_{(0,\infty)} K(c_0 v^{\frac{2}{3}}, v, c_0 v'^{\frac{2}{3}}, v')F(v,t)F(v',t)\mathrm{d}v'\mathrm{d}v \tag{B.2.39}$$

as $\epsilon \to 0$.

This is done again by splitting the interval $(0,\infty)^2$ into three regions, namely $\{v \leq \frac{1}{M}\}$, $\{v \geq M\}$ and $\{v \in [\frac{1}{M}, M]\}$. For the regions $\{v \leq \frac{1}{M}\}$ and $\{v \geq M\}$, we can use moment estimates to prove that the terms (B.2.38) and (B.2.39) approach zero as $M \to \infty$. In the region $\{(v, v') \in [\frac{1}{M}, M]^2\}$, we use that $f_\epsilon f_\epsilon \rightharpoonup \overline{f}\,\overline{f}$ as $\epsilon \to 0$.

Step 2. We continue by proving that the remaining integrals in (B.2.36) converge to zero as $\epsilon \to 0$, for every $t \geq \overline{\sigma}$. We first prove that

$$\int_{\{a\in \mathrm{T}(v)\}} \int_{\{a'>c_0 v'^{\frac{2}{3}}+\delta_1\}} K(a, v, a', v') f_\epsilon(\eta, t) f_\epsilon(\eta', t) \mathrm{d}\eta \mathrm{d}\eta' \to 0 \text{ as } \epsilon \to 0.$$

We have that there exists a constant $C > 0$, independent of $\epsilon \in (0, 1)$ such that $M_{0,-\alpha}(f_\epsilon) + M_{0,\beta}(f_\epsilon) \leq C$. Combining these moment bounds with Proposition B.2.7, we obtain that

$$\begin{aligned}
&\int_{\{a\in \mathrm{T}(v)\}} \int_{\{a'>c_0 v'^{\frac{2}{3}}+\delta_1\}} K(a, v, a', v') |\tilde{\varphi}(v + v') - \tilde{\varphi}(v) - \tilde{\varphi}(v')| f_\epsilon(\eta, t) f_\epsilon(\eta', t) \mathrm{d}\eta \mathrm{d}\eta' \\
&\qquad \lesssim 3\|\tilde{\varphi}\|_\infty \int_{\{a\in \mathrm{T}(v)\}} v^\beta f_\epsilon(\eta, t) \mathrm{d}\eta \int_{\{a'>c_0 v'^{\frac{2}{3}}+\delta_1\}} v'^{-\alpha} f_\epsilon(\eta', t) \mathrm{d}\eta' \\
&\qquad + 3\|\tilde{\varphi}\|_\infty \int_{\{a\in \mathrm{T}(v)\}} v^{-\alpha} f_\epsilon(\eta, t) \mathrm{d}\eta \int_{\{a'>c_0 v'^{\frac{2}{3}}+\delta_1\}} v'^\beta f_\epsilon(\eta', t) \mathrm{d}\eta' \leq C\delta_2,
\end{aligned}$$

where δ_2 was taken as in (B.2.35). More precisely, in order to prove that

$$\int_{\{a'>c_0 v'^{\frac{2}{3}}+\delta_1\}} (v'^\beta + v'^{-\alpha}) f_\epsilon(\eta', t) \mathrm{d}\eta' \leq C\delta_2, \tag{B.2.40}$$

we notice that for the region $\{a' > c_0 v'^{\frac{2}{3}} + \delta_1\} \cap [\frac{1}{M}, M]^2$, for some sufficiently large $M > 1$, we can use Proposition B.2.7 to deduce that the integral gives a sufficiently small contribution and then control the regions $\{a > M\} \cup \{a < \frac{1}{M}\} \cup \{v > M\} \cup \{v < \frac{1}{M}\}$ using moment estimates.

Step 3. Lastly, we prove that, for every $t \geq \overline{\sigma}$, we have that

$$\int_{\{a>c_0 v^{\frac{2}{3}}+\delta_1\}} \int_{\{a'>c_0 v'^{\frac{2}{3}}+\delta_1\}} K(a, v, a', v') f_\epsilon(\eta, t) f_\epsilon(\eta', t) \mathrm{d}\eta \mathrm{d}\eta' \to 0 \text{ as } \epsilon \to 0.$$

As before, we obtain that

$$\begin{aligned}
&\int_{\{a>c_0 v^{\frac{2}{3}}+\delta_1\}} \int_{\{a'>c_0 v'^{\frac{2}{3}}+\delta_1\}} K(a, v, a', v') f_\epsilon(\eta, t) f_\epsilon(\eta', t) \mathrm{d}\eta \mathrm{d}\eta' \\
&\lesssim 2 \int_{\{a>c_0 v^{\frac{2}{3}}+\delta_1\}} \int_{\{a'>c_0 v'^{\frac{2}{3}}+\delta_1\}} v^{-\alpha} v'^\beta f_\epsilon(\eta, t) f_\epsilon(\eta', t) \mathrm{d}\eta \mathrm{d}\eta' \\
&\leq 2 \int_{\{a>c_0 v^{\frac{2}{3}}+\delta_1\}} v^{-\alpha} f_\epsilon(\eta, t) \mathrm{d}\eta \int_{\{a'>c_0 v'^{\frac{2}{3}}+\delta_1\}} v'^\beta f_\epsilon(\eta', t) \mathrm{d}\eta' \lesssim \delta_2^2,
\end{aligned}$$

where for the last inequality we used the same argument as in (B.2.40). This concludes our proof. □

Using Lemma B.2.11, it suffices to let $\overline{\sigma} \to 0$ in (B.2.34) in order to conclude the proof of Theorem B.1.10.

Proof of Theorem B.1.10. Due to Proposition B.2.10, we can let $\overline{\sigma} \to 0$ in (B.2.34) using the fact that there exists $C(T) > 0$ such that

$$\sup_{t\in[0,T]} \int_{(0,\infty)} (v^{-1} + v^2) F(v,t) \mathrm{d}v \leq C(T)$$

due to the way F was constructed (see (B.1.12), (B.1.13)). □

B.3 The case of negligible fusion

We now consider the form of solutions of (B.1.2) if we assume that $\Lambda \to \infty$. We first prove that there exists a limit for a subsequence of $\{f_\epsilon\}_{\epsilon\in(0,1)}$, where f_ϵ is a solution of (B.1.15) .

Proposition B.3.1. *Let $K : (0,\infty)^4 \to [0,\infty)$ be a continuous kernel satisfying (B.1.3), (B.1.6) and (B.1.7). Assume the fusion kernel $r \in \mathrm{C}^1(\mathbb{R}^2_{>0})$ satisfies (B.1.4) and (B.1.5) with $\mu > 0$. Assume in addition that $\int_{(0,\infty)^2} (v^{-1} + v^2 + a) f_{\mathrm{in}}(a,v) \mathrm{d}v\mathrm{d}a < \infty$. Let $T > 0$. Then we can construct f_ϵ as in Definition B.1.1 for equation (B.1.15), for every $\epsilon \in (0,1)$. For this sequence, we have that there exists a constant $C(T) > 0$, which is independent of $\epsilon \in (0,1)$, such that*

$$\sup_{t\in[0,T]} \int_{(0,\infty)^2} (v^{-1} + v^2 + a) f_\epsilon(a,v,t) \mathrm{d}v\mathrm{d}a \leq C(T) \tag{B.3.1}$$

and that there exists a subsequence (which we do not relabel) and $\underline{f} \in \mathrm{C}([0,T]; \mathscr{M}^I_+(\mathbb{R}^2_{>0}))$ such that $f_\epsilon(t) \to \underline{f}(t)$ as $\epsilon \to 0$ in the sense of measures, for every $t \in [0,T]$.

Proof. Existence of solutions f_ϵ which satisfy (B.3.1) was proven in Proposition B.2.1. We are left to prove equicontinuity in time in order to conclude our proof.

Let $M > 1$. It suffices to prove equicontinuity for a fixed function $\varphi \in \mathrm{C}^1_\mathrm{c}(\mathbb{R}^2_{>0})$ such that $\mathrm{supp}\varphi \in [\frac{1}{M}, M]^2$. In order to pass to functions $\varphi \in \mathrm{C}_\mathrm{c}(\mathbb{R}^2_{>0})$ such that $\mathrm{supp}\varphi \in [\frac{1}{M}, M]^2$, we can use that there exists $\varphi_n \in \mathrm{C}^1_\mathrm{c}(\mathbb{R}^2_{>0})$ such that $\|\varphi - \varphi_n\|_\infty \leq \frac{1}{n}$. In order to extend to functions $\varphi \in \mathrm{C}_0(\mathbb{R}^2_{>0})$, we can then use moment estimates by letting M become sufficiently large.

Fix $\varphi \in \mathrm{C}^1_\mathrm{c}(\mathbb{R}^2_{>0})$. Let $s, t \in [0,T]$. Assume without loss of generality $s \leq t$. Then

$$\begin{aligned}
&\left| \int_{(0,\infty)^2} [f_\epsilon(\eta,t) - f_\epsilon(\eta,s)] \varphi(\eta) \mathrm{d}\eta \right| \leq \\
&\lesssim 3\|\varphi\|_\infty \int_s^t M_{0,-\alpha}(f_\epsilon(z)) M_{0,\beta}(f_\epsilon(z)) \mathrm{d}z + \epsilon \int_s^t \int_{(0,\infty)^2} a^\mu v^\sigma (a + c_0 v^{\frac{2}{3}}) f_\epsilon(\eta,z) |\partial_a \varphi(\eta)| \mathrm{d}\eta\mathrm{d}z \\
&\leq 3\|\varphi\|_\infty \int_s^t M_{0,-\alpha}(f_\epsilon(z)) M_{0,\beta}(f_\epsilon(z)) \mathrm{d}z + C\|\partial_a \varphi\|_\infty \int_s^t M_{0,0}(f_\epsilon(z)) \mathrm{d}z \\
&\leq C|t-s|.
\end{aligned} \tag{B.3.2}$$

We deduce then from Arzelà–Ascoli theorem that there exists a subsequence of $\{f_\epsilon\}$, which we do not relabel, and an $\underline{f} \in \mathrm{C}([0,T]; \mathscr{M}^I_+(\mathbb{R}^2_{>0}))$, such that $f_\epsilon(t)$ converge to $\underline{f}(t)$ in the weak-* topology as $\epsilon \to 0$, for every $t \in [0,T]$. □

We continue by proving that the measure $\underline{f}$ found in Proposition B.3.1 solves a standard two-dimensional coagulation equation, which concludes the proof of Theorem B.1.12.

Proof of Theorem B.1.12. Let $\varphi \in \mathrm{C}_{\mathrm{c}}^1(\mathbb{R}^2_{>0})$, fixed. In order to estimate the fusion term, notice that

$$\begin{aligned}\epsilon\left|\int_{(0,\infty)^2} r(\eta)(c_0 v^{\frac{2}{3}} - a) f_\epsilon(\eta, s)\partial_a\varphi(\eta, s)\mathrm{d}\eta\right| &\lesssim C\epsilon\|\partial_a\varphi\|_\infty \sup_{t\in[0,T]} M_{0,0}(f_\epsilon(t))\\ &\leq \epsilon C(T) \to 0 \text{ as } \epsilon \to 0.\end{aligned}$$

To prove that, for fixed $\varphi \in \mathrm{C}_{\mathrm{c}}^1(\mathbb{R}^2_{>0})$, the term with the coagulation kernel in (B.1.15) converges to the term with the coagulation kernel in (B.1.17), standard methods used in the study of coagulation equation are used and hence we omit the rest of the proof. □

We conclude by proving that the coagulation equation satisfied by the measure $\underline{f}$ found in Proposition B.3.1 describes a process in which the particles attach each other at a contact point. More precisely, we will prove that the total surface area and total volume of particles stay constant in time, see Remark B.1.13. We first prove that we can test with $\varphi(\eta) = a$ and $\varphi(\eta) = v$ in (B.1.17), see Proposition B.3.2. Remark B.1.13 is then a straightforward exercise.

Proposition B.3.2. *Let $\mu > 0$. Then we can test equation (B.1.17) with $\varphi(\eta) \equiv a$ or $\varphi(\eta) \equiv v$ if*

$$\sup_{t\in[0,T]} \int_{(0,\infty)^2} (a^2 + v^{-2} + v^2)\underline{f}(a, v, t) \leq C(T). \tag{B.3.3}$$

Proof. The proof is analogous with the proof of Proposition B.2.6 using the moment estimates (B.3.3). □

Proof of Remark B.1.13. From Proposition B.3.2 we can test equation (B.1.17) with $\varphi(\eta) \equiv a$ or $\varphi(\eta) \equiv v$ and so the desired quantities are preserved since

$$\langle \mathbb{K}[\underline{f}], a\rangle = 0 \text{ and } \langle \mathbb{K}[\underline{f}], v\rangle = 0.$$

□

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APPENDIX C

LONG-TIME ASYMPTOTICS FOR COAGULATION EQUATIONS WITH INJECTION THAT DO NOT HAVE STATIONARY SOLUTIONS

ABSTRACT. In this paper we study a class of coagulation equations including a source term that injects in the system clusters of size of order one. The coagulation kernel is homogeneous, of homogeneity $\gamma < 1$, such that $K(x,y)$ is approximately $x^{\gamma+\lambda}y^{-\lambda}$, when x is larger than y. We restrict the analysis to the case $\gamma + 2\lambda \geq 1$. In this range of exponents, the transport of mass toward infinity is driven by collisions between particles of different sizes. This is in contrast with the case considered in [FFV23] where $\gamma + 2\lambda < 1$. In that case, the transport of mass toward infinity is due to the collision between particles of comparable sizes. In the case $\gamma + 2\lambda \geq 1$, the interaction between particles of different sizes leads to an additional transport term in the coagulation equation that approximates the solution of the original coagulation equation with injection for large times. We prove the existence of a class of self-similar solutions for suitable choices of γ and λ for this class of coagulation equations with transport. We prove that for the complementary case such self-similar solutions do not exist.

C.1 Introduction

C.1.1 Aim of the paper

In this paper we study the long-time behavior of the coagulation equation with injection:

$$\partial_t f(t,x) = \mathbb{K}[f](t,x) + \eta(x), \tag{C.1.1}$$

where

$$\mathbb{K}[f](t,x) := \frac{1}{2}\int_0^x K(x-y,y) f(t,x-y) f(t,y)\,\mathrm{d}y - \int_0^\infty K(x,y) f(t,x) f(t,y)\,\mathrm{d}y \tag{C.1.2}$$

and where $\eta \geq 0$. We will assume in all the following that $\eta(x) \not\equiv 0$ and that it is either compactly supported or it decays fast enough with the cluster size.

The study of problems with this form arises naturally in problems of aerosols and atmospheric science ([Fri00, OKMO+13, VR12]). In this context, the function $f(t,x)$ denotes the density of clusters with size x at time t.

The collision operator $\mathbb{K}[f](t,x)$ in (C.1.2) was introduced by Smoluchowski (see [Smo16]). The kernel $K(x,y)$ encodes information about the mechanism driving the coagulation of clusters. In this paper we are interested in kernels arising in atmospheric science applications. A common feature of these kernels is the homogeneity property. Indeed, in many cases the rate of aggregation scales like a power law with the cluster size. This means that the coagulation kernel $K(x,y)$ satisfies

$$K(ax,ay) = a^{\gamma} K(x,y) \text{ for } a > 0 \text{ and } (x,y) \in (0,\infty)^2, \tag{C.1.3}$$

for some $\gamma \in \mathbb{R}$.

On the other hand, since the coagulation process does not depend on the order in which the clusters of size x and y are chosen, we have the symmetry property

$$K(x,y) = K(y,x) \text{ for } (x,y) \in (0,\infty)^2. \tag{C.1.4}$$

We will assume that the coagulation kernel K satisfies

$$c_2 \left[\frac{x^{\gamma+\lambda}}{y^{\lambda}} + \frac{y^{\gamma+\lambda}}{x^{\lambda}} \right] \leq K(x,y) \leq c_1 \left[\frac{x^{\gamma+\lambda}}{y^{\lambda}} + \frac{y^{\gamma+\lambda}}{x^{\lambda}} \right] \text{ and } \gamma + 2\lambda \geq 0, \tag{C.1.5}$$

with $0 < c_2 \leq c_1 < \infty$, where γ is the homogeneity parameter introduced in (C.1.3) and $\lambda \in \mathbb{R}$. The polynomial bounds (C.1.5) are satisfied by many of the most relevant collision kernels arising in aerosol and atmospheric science, such as the *diffusive* coagulation kernel and the *free molecular* kernel (see for instance [Fri00]).

Notice that the condition $\gamma + 2\lambda \geq 0$ in (C.1.5) does not imply any loss of generality. This can be seen from the fact that the function $x^{\alpha} y^{\beta} + x^{\beta} y^{\alpha}$ can be written as $\left[\frac{x^{\gamma+\lambda}}{y^{\lambda}} + \frac{y^{\gamma+\lambda}}{x^{\lambda}} \right]$ with $\gamma + \lambda = \max\{\alpha, \beta\}$ and $-\lambda = \min\{\alpha, \beta\}$.

Since we will consider solutions of (C.1.1), (C.1.2) in which $f(t,\cdot)$ is a Radon measure, it is convenient to impose the following condition on the kernel K:

$$K \in \mathrm{C}((0,\infty)^2). \tag{C.1.6}$$

It is well known that part or all the mass of the solutions of (C.1.1), (C.1.2) can escape towards $x = \infty$ in finite, or even zero time. This phenomenon is known as *gelation*. For a more detailed analysis on this matter, see for example [BLL19b, da 98]. In order to guarantee that gelation does not take place, we will assume in all the following that

$$\gamma < 1 \text{ and } \gamma + \lambda < 1. \tag{C.1.7}$$

See [DKW99] for a more detailed discussion on the gelation regimes.

We expect that gelation does not take place under the weaker assumptions $\gamma \leq 1, \gamma + \lambda \leq 1$ (see [BLL19b]). However, in the critical cases $\gamma = 1$ or $\gamma + \lambda = 1$, the self-similar solutions are not defined using power laws to scale the particle sizes but most likely using exponential functions. Given that the analysis of these solutions would require arguments different from the ones in this paper, we will not consider this case here.

The existence of solutions for equation (C.1.1), (C.1.2) has been considered in [Dub94, EM06]. In this paper we study the long time behavior of the solutions of equation (C.1.1). Due to the presence of the source we can expect the solutions to (C.1.1), (C.1.2) to converge to a stationary non-equilibrium solution $\overline{f} = \overline{f}(x)$ as $t \to \infty$, i.e. to a solution of

$$\mathbb{K}[f](x) + \eta(x) = 0. \tag{C.1.8}$$

However, it turns out that if $\eta \not\equiv 0$ and $\gamma + 2\lambda \geq 1$, a solution for (C.1.8) does not exist. In [Hay87], the discrete stationary coagulation model

$$\sum_{k=1}^{n-1} K_{k,n-k} f_k f_{n-k} - \sum_{k=1}^{\infty} K_{k,n} f_k f_n + \overline{\delta}_{k,n} = 0, \ k \geq 1, \tag{C.1.9}$$

has been studied for the explicit coagulation kernel

$$K_{k,n} = k^{\gamma+\lambda} n^{-\lambda} + k^{-\lambda} n^{\gamma+\lambda}, \ k, n \in \mathbb{N} \tag{C.1.10}$$

when $|\gamma + \lambda| < 1$, $|\lambda| < 1$, $|\gamma| < 1$. Formal asymptotics for the large size behavior of the solutions f_k of (C.1.9) has been obtained in [Hay87]. The results in that paper indicate that a solution of (C.1.9) exists if and only if $\gamma + 2\lambda < 1$.

In the case of general kernels satisfying the assumptions (C.1.4), (C.1.5), (C.1.6), and source terms η decreasing fast enough, it has been proven in [FLNV21b] that the solutions of (C.1.8) (as well as its discrete counterpart) exist, if and only if $\gamma + 2\lambda < 1$.

It is worth to remark that it has been proven in [FLNV21b] that the solutions of (C.1.8) can be estimated, up to a multiplicative constant, from above and below by the power law $x^{-\frac{3+\gamma}{2}}$ for large values of x.

Since in the case $\gamma + 2\lambda \geq 1$ a stationary solution of (C.1.8) does not exist, we cannot expect the solutions to (C.1.1), (C.1.2) to behave as the stationary solution $\overline{f}$ of (C.1.8) as $t \to \infty$ for x of order one. It is then natural to ask what is the long time asymptotics of the solutions to (C.1.1), (C.1.2) for large values of t and x. The scaling hypothesis that has been extensively used in the study of coagulation equations suggests that the mass of the particle distributions $f(t, x)$ is concentrated in cluster sizes x of order t^p for a suitable exponent p that would be determined from dimensional considerations, which take into account the way in which the mass rescales in time. In the case of kernels $K(x,y) = x^{\gamma+\lambda}y^{-\lambda} + y^{-\lambda}x^{\gamma+\lambda}$, $-1 < \lambda < 0$ with $0 \leq \gamma + \lambda < 1$ and $\gamma < 1$, it was suggested in [DKW99], using a combination of matched asymptotics and numerical simulations, that the long time behavior of the solutions of (C.1.1), (C.1.2) is given by self-similar solutions with the form

$$f_s(t,x) = \frac{1}{t^{\frac{3+\gamma}{1-\gamma}}}\Phi(\xi) \ , \ \ \xi = \frac{x}{t^{\frac{2}{1-\gamma}}}. \tag{C.1.11}$$

The approximation (C.1.11) can be expected to be valid for large cluster sizes, i.e. $x \gg 1$. We will use from now the notation with $x \gg 1$ to indicate large cluster sizes x. In the case considered in [DKW99] we have that $\gamma + 2\lambda < 1$ and therefore stationary solutions f_s solving (C.1.8) exist. In this case, (C.1.8) and (C.1.11) suggest that $\Phi(\xi)$ behaves for small values of ξ as $K\xi^{-\frac{3+\gamma}{2}}$, for a suitable constant $K > 0$. More precisely, plugging (C.1.11) in (C.1.1), (C.1.2), it follows that Φ solves

$$-\frac{2}{1-\gamma}\xi\Phi_\xi - \frac{3+\gamma}{1-\gamma}\Phi = \mathbb{K}[\Phi], \tag{C.1.12}$$

where Φ satisfies the following boundary condition at $\xi \to 0$ that guarantees that there is a constant flux of particles from the origin:

$$\lim_{R\to 0}\int_0^R d\xi \int_{R-\xi} d\eta K(\xi,\eta)\,\xi\Phi(\xi)\,\Phi(\eta) = J, \tag{C.1.13}$$

with $J = \int_0^\infty x\eta(x)\,dx$. The existence of solutions of equation (C.1.12) satisfying the constant flux solution condition at $\xi = 0$, (C.1.13), has been rigorously proven in [FFV23] for kernels K satisfying (C.1.4), (C.1.5), (C.1.6), (C.1.7) with $\gamma + 2\lambda < 1$.

The picture described above, which combines the stationary behavior $\overline{f}$, (cf. (C.1.8)) for cluster sizes x of order one, and the self-similar behavior (C.1.11) for large cluster sizes, provides a rather complete description of the long time behavior of the solutions to (C.1.1), (C.1.2) in the case $\gamma+2\lambda < 1$. However, the same scenario cannot yield a description of the long time asymptotics of the solutions to (C.1.1), (C.1.2) if $\gamma + 2\lambda \geq 1$, because, as explained above, in this case a solution of (C.1.8) does not exist.

Notice that the existence/non-existence of solutions to (C.1.8) is related to the existence of stationary solutions of (C.1.8) yielding a constant flux of particles with the form of a power law, i.e. $\overline{f}(x) = cx^{-\frac{\gamma+3}{2}}$. These solutions exist for $\gamma + 2\lambda < 1$ and they do not exist for $\gamma + 2\lambda \geq 1$. In the framework of the wave turbulence it would be stated that the kernels K with $\gamma + 2\lambda < 1$ satisfy the locality property, while the kernels K with $\gamma + 2\lambda \geq 1$ do not have the locality property, see [ZLF92].

In this paper we are interested in the study of the long-time behavior of the solutions to (C.1.1), (C.1.2) in the case $\gamma + 2\lambda \geq 1$ (and the non-gelling regime $\gamma < 1$ and $\gamma + \lambda < 1$) that is described using formal asymptotics arguments. The kernel we work with is homogeneous, hence it can be expressed as

$$K(x,y) = (x+y)^\gamma F\left(\frac{x}{x+y}\right) \ , \ \ F(s) = F(1-s) \ \text{ for } s \in (0,1). \tag{C.1.14}$$

Notice that (C.1.5) implies that $\overline{c}_1 \le F(s) \le \overline{c}_2$ for some $\overline{c}_1, \overline{c}_2 > 0$. We will assume in the following a condition that is more restrictive than (C.1.5), namely

$$\lim_{s\to 0^+} \left[s^{\lambda} F(s) \right] = 1. \tag{C.1.15}$$

In order to prove that in the case $\gamma + 2\lambda \ge 1$ there are no solutions of (C.1.8), the main idea used in [FLNV21b] is based on the fact that, for this range of exponents, the transfer of clusters of size x of order one towards very large cluster sizes is so fast that the concentration of clusters with size of order one would become zero.

In the case of time dependent solutions having initially finite mass, this increases linearly due to the fact that

$$\partial_t \left(\int_0^\infty x f(t,x) \mathrm{d}x \right) = \int_0^\infty x\eta(x) \mathrm{d}x.$$

For large times, due to the increase of the average cluster size, we might expect that, if $\gamma + 2\lambda \ge 1$, there should be a fast transport of the newly injected clusters of order one towards much larger cluster sizes. This almost instantaneous transport results in small concentrations of clusters of order one for large times.

We now remark that the part of the coagulation operator which describes the coagulation between particles of different sizes can be approximated by means of a transport operator in the space of cluster sizes. These arguments, that will be described in detail using formal asymptotics in Section C.2, show that the solutions of (C.1.1), (C.1.2) for large cluster sizes can be approximated by means of the following equation if $\gamma + 2\lambda \ge 1$

$$\partial_t f(t,x) + \frac{\partial_x \left(x^{\gamma+\lambda} f(t,x) \right)}{\int_0^\infty z^{\gamma+\lambda} f(t,z) \mathrm{d}z} = \mathbb{K}[f](t,x), \quad \text{for } x \gg 1. \tag{C.1.16}$$

We emphasize that the non-local transport term $\frac{\partial_x \left(x^{\gamma+\lambda} f(t,x) \right)}{\int_0^\infty z^{\gamma+\lambda} f(t,z) \mathrm{d}z}$ is a consequence of the presence of the source $\eta(x)$ in (C.1.1). The fact that the contributions of the coagulation operator $\mathbb{K}[f]$ which are due to the aggregation of particles with very different sizes can be approximated by a differential operator has been extensively used in the literature of coagulation equations (cf. [Fri00]). The resulting first order terms are often referred to represent *heterogeneous condensation* (cf. [Fri00]). On the other side, the term $\mathbb{K}[f]$ in (C.1.16) describes the aggregation of clusters of comparable sizes.

The solutions of (C.1.16) are expected to describe the asymptotic behavior of the solution of equation (C.1.1) both when $\gamma + 2\lambda > 1$ and when $\gamma + 2\lambda = 1$, even if in these two cases we have two slightly different scenarios. Namely, when $\gamma + 2\lambda > 1$, we will have that if x is of order 1, then $f(t,x) \to 0$ as $t \to \infty$, while when $\gamma + 2\lambda = 1$ we will have that if x is of order 1, then $f(t,x) \to f_s(x)$ as $t \to \infty$, where f_s is a solution of

$$\mathbb{K}[f](x) + \eta(x) - x^{-\lambda} f(x) = 0, \tag{C.1.17}$$

(compare with (C.1.8)). Due to the presence of the term $x^{-\lambda}f(x)$, equation (C.1.17) might have solutions when $\gamma + 2\lambda = 1$.

The scaling properties of (C.1.16) suggest that this equation is compatible with the existence of self-similar solutions for equation (C.1.16) with the form

$$f_s(t, x) = \frac{1}{t^{\frac{3+\gamma}{1-\gamma}}}\Phi\left(\frac{x}{t^{\frac{2}{1-\gamma}}}\right). \tag{C.1.18}$$

Here the self-similar profile Φ satisfies the following equation, obtained by substituting equality (C.1.18) in equation (C.1.16) and using the self-similar change of variables $\xi = \frac{x}{t^{\frac{2}{1-\gamma}}}$,

$$-\frac{3+\gamma}{1-\gamma}\Phi(\xi) - \frac{2}{1-\gamma}\xi\partial_\xi\Phi(\xi) + \frac{1}{\int_0^\infty \eta^{\gamma+\lambda}\Phi(\eta)\,\mathrm{d}\eta}\frac{\partial}{\partial\xi}\left(\xi^{\gamma+\lambda}\Phi(\xi)\right) = \mathbb{K}[\Phi](\xi) \quad , \quad \xi > 0. \tag{C.1.19}$$

The main result of this paper is to determine the range of exponents γ and λ satisfying $\gamma+2\lambda \geq 1$ and the non-gelation conditions (C.1.7) for which self-similar solutions of (C.1.16) with the form (C.1.18) exist (see Figure C.1 for a classification of these exponents). Specifically, we will prove the following. Suppose that $\gamma + 2\lambda \geq 1$ and that (C.1.7) holds. Then

- If $\gamma > -1$, there exists at least one self-similar solution of (C.1.16) with the form (C.1.18).
- If $\gamma \leq -1$ and $\gamma+2\lambda > 1$, no solutions of (C.1.16) with the form (C.1.18) exist (See Figure C.1).
- If $\gamma \leq -1$ and $\gamma+2\lambda = 1$, we prove that there are no self-similar solutions f_s of the form (C.1.18) such that $\int_0^1 x^{-\lambda}\Phi(x)\mathrm{d}x < \infty$.

The meaning of the condition $\int_0^1 x^{-\lambda}\Phi(x)\mathrm{d}x < \infty$ is that the number of clusters removed by the coagulation process in any bounded time interval is finite. We therefore do not exclude the existence of a self-similar solution of equation (C.1.16) with $\int_0^1 x^{-\lambda}\Phi(x)\mathrm{d}x = \infty$.

A remarkable property of the self-similar solutions of (C.1.16), that we constructed in this paper, is that they vanish identically for $0 < \xi < \rho(M_{\gamma+\lambda}) := \left(\frac{1-\gamma}{2M_{\gamma+\lambda}}\right)^{\frac{1}{1-\gamma-\lambda}}$, where

$$M_{\gamma+\lambda} := \int_{(0,\infty)} \xi^{\gamma+\lambda}\Phi(\xi)\mathrm{d}\xi.$$

The fact solutions Φ of (C.1.19) vanish in an interval $(0, \rho(M_{\gamma+\lambda}))$ means that, for large times t, the injected particles are transferred almost instantaneously to clusters with sizes $x \geq \rho(M_{\gamma+\lambda})t^{\frac{2}{1-\gamma}}$. Moreover, the fraction of clusters with sizes $x < \rho(M_{\gamma+\lambda})t^{\frac{2}{1-\gamma}}$ becomes negligible for very long times. The existence of this ”minimal” cluster size for large times is a remarkable feature that, to our knowledge, has not been observed in the literature on self-similar solutions for the coagulation equation (see for instance [EM06, EMR05, FL06a, MP04, TV19]). In particular, it is worth to notice that this behavior of the self-similar solutions is very different from the one exhibited by the self-similar solutions obtained in the case $\gamma + 2\lambda < 1$ in [FFV23].

The results that we obtain in this paper in the critical case $\gamma + 2\lambda = 1$ are more fragmentary than those obtained for $\gamma + 2\lambda > 1$. In the case $\gamma + 2\lambda = 1$, we only prove the existence of a solution Φ of (C.1.19) for $\gamma > -1$ with $\Phi = 0$ on $(0, \rho(M_{\gamma+\lambda}))$, but we do not prove that any solution vanishes in the interval $(0, \rho(M_{\gamma+\lambda}))$.

On the other hand, we will prove in this paper that the function Φ which describes the self-similar profile in (C.1.18) decreases exponentially as $\xi \to \infty$ in the same manner as the self-similar solutions constructed in [FFV23] and as it usually happens for the self-similar solutions of coagulation equations in problems without injection, see [EM06, FL06a].

As indicated above, if $\gamma \leq -1$ and $\gamma + 2\lambda > 1$, there are no self-similar solutions of (C.1.16) with the form (C.1.18) and, if $\gamma \leq -1$ and $\gamma + 2\lambda = 1$, there are no self-similar solutions of (C.1.16) with the form (C.1.18) satisfying $\int_0^1 x^{-\lambda}\Phi(x)\mathrm{d}x < \infty$. It is natural to ask what is the long time asymptotics of the solutions of (C.1.1), (C.1.2) in this case. This question will be the subject of study of a future work.

In both cases the coagulation term $\mathbb{K}[f]$ in (C.1.16) which describes the aggregation of particles with comparable sizes is negligible for large times, and the long time behavior of the distribution of clusters is determined by the coagulation of particles of size x of order one with large particles. The main difference between the cases $\gamma+\lambda \geq 0$ and $\gamma+\lambda < 0$ arises from the fact that in the second case the transfer of clusters (not monomers) from the region where x is of order one to $x \gg 1$, *heterogeneous condensation*, is relevant.

There are several results in the physical literature which are related, and are consistent with the ones in this paper, see for instance [BCJ+12, KC12, KMR98, KMR99]. In [BCJ+12] a coagulation model with kernels satisfying (C.1.5) with parameters γ, λ such that $\gamma + 2\lambda \geq 1$ and including also a source term and a removal of particles term has been studied. In that problem, the cluster concentrations as $t \to \infty$ are determined by the coalescence of particles of very different sizes. It is then possible to approximate the coagulation-removal model for large clusters by means of the equation

$$\partial_t f(t,x) = -\partial_x \left(x^{\gamma+\lambda} f(t,x)\right) + \int_1^{x/2} y^{1-\lambda} f(t,y)\mathrm{d}y - f(t,x)\int_x^K y^{\gamma+\lambda} f(t,y)\mathrm{d}y,$$

where K is the maximum particle size in the system. The numerical simulations in [BCJ+12] show that the concentration of cluster sizes of order one approach to a stationary solution that converges to zero if K is sent to infinity.

In [KC12] a coagulation model with injection and with kernels satisfying (C.1.5) with $\gamma = 0$ and $\lambda \in (1/2, 1]$ has been considered. Numerical simulations and formal computations in [KC12] suggest that the cluster concentration for clusters of order one tends to zero as $t \to \infty$. In addition, it is suggested that the concentration for large clusters are described by a self-similar solution.

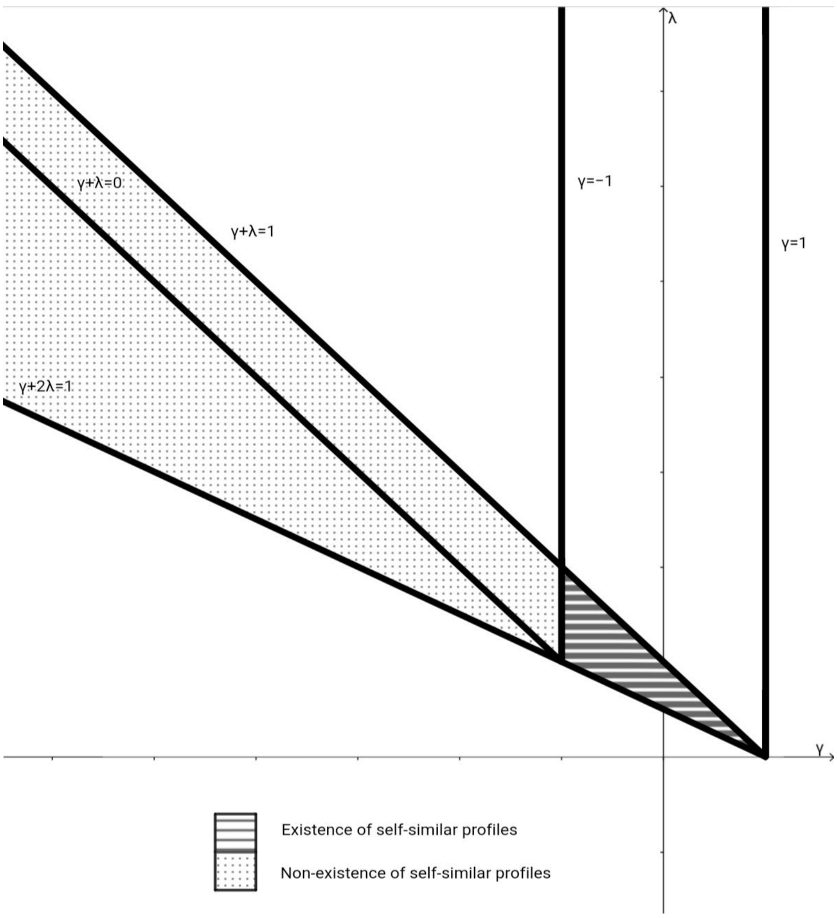

Figure C.1: Coefficients for existence and non-existence of self-similar profiles in the case $\gamma + 2\lambda > 1$

In [KMR98, KMR99] coagulation equations with injection are considered with kernels satisfying (C.1.5) with $-\lambda = \gamma \leq -1$. It is seen there that for $\gamma + 2\lambda \geq 1$, the solutions of the corresponding coagulation equation behave in a non self-similar manner and decay logarithmically. This is in agreement with the non-existence of self-similar solutions that we obtained in this paper for $\gamma \leq -1$.

C.1.2 Notation and plan of the paper

We use the notation $\mathbb{R}_* := (0, \infty)$ and $\mathbb{R}_+ := [0, \infty)$. Given an interval $I \subset \mathbb{R}_+$ we denote with $C_c(I)$ the Banach space of the functions on I that are continuous and compactly supported. We endow the space $C_c(\mathbb{R}_*)$ with the supremum norm denoted with $\|\cdot\|_\infty$. For a function ϕ, if $\phi(x) = 0$, for all x in an interval I, we will denote it by $\phi(I) = 0$. We keep the same notation to mean that the support of a measure is outside the interval I. We denote with $\mathcal{M}_+(I)$ the space of the non-negative Radon measures on I. Given a measure $\mu \in \mathcal{M}_+(I)$ we denote with $\|\mu\|_{TV}$ the total variation norm of μ. For a compact interval $I \subseteq \mathbb{R}_*$ and two bounded measures μ, ν, we denote the Wasserstein metric by W_1, namely

$$W_1(\mu,\nu) = \sup_{\|\varphi\|_{\mathrm{Lip}}\leq 1} \int_I \varphi(x)(\mu-\nu)(\mathrm{d}x), \tag{C.1.20}$$

where the supremum is taken over the Lipschitz functions and where $\|f\|_{\mathrm{Lip}} = \|f\|_\infty + [f]_{\mathrm{Lip}}$, with $[f]_{\mathrm{Lip}} = \sup_{\substack{x,y\in I\\ x\neq y}} \frac{|f(x)-f(y)|}{|x-y|}$.

To keep the notation lighter, we denote with C and c constants that might change from line to line in the computations. In addition, we use the notation $f \lesssim g$, for two functions f, g, to mean that there exists a constant $C > 0$ such that $f \leq Cg$.

Moreover, given a measure μ, we denote with $M_\alpha(\mu)$ the α moment of μ, i.e.

$$M_\alpha(\mu) := \int_{\mathbb{R}_*} x^\alpha \mu(\mathrm{d}x).$$

To simplify the notation, in some cases, we write M_α instead $M_\alpha(\mu)$ if the choice of the measure μ is clear in the argument.

We use the notation $f \sim g$ as $x \to x_0$ to indicate that $\lim_{x\to x_0} \frac{f(x)}{g(x)} = 1$, while we use the notation $f \approx g$ to say that there exists a constant $M > 0$ such that

$$\frac{1}{M} \leq \frac{f}{g} \leq M.$$

As previously mentioned, we use the notation $x \gg 1$ for large cluster sizes x. Additionally, for cluster sizes x, y, we denote $x \gg y$ or $y \ll x$ to mean that x is much larger than y. For two terms, A and B, we use the notation $A \simeq B$ to mean informally that A can be approximated in terms of B in the region under consideration in the respective formula.

The paper is organized as follows. In Section C.2 we present a heuristic motivation to study the existence of the self-similar solutions considered in this paper. In Section C.3 we present the main results of the paper regarding the existence and non-existence of a self-similar profile and its properties. In Section C.4 we explain the main ideas behind the proofs of existence and non-existence, skipping the technical difficulties of the proofs. Section C.5 deals with the proof of the existence of a self-similar solution for equation (C.1.19) when $\gamma > -1$. We also prove in this section that the solution decays exponentially for large values. Section C.6 deals with the non-existence of self-similar solutions for equation (C.1.19) when $\gamma \leq -1$.

C.2 Asymptotic description of the long time behavior

C.2.1 The case $\gamma + 2\lambda > 1$

In this section we describe the long time asymptotics of the solutions of (C.1.1), (C.1.2) if $\gamma+2\lambda > 1$ using formal asymptotic arguments.

As indicated in the Introduction, we expect $f(t,x)$ to converge to zero as $t \to \infty$ for x of order one. Therefore, the contribution due to the term $\frac{1}{2}\int_0^x K(x-y,y) f(t,x-y) f(t,y)\,dy$ can be expected to be negligible in this region since this term is quadratic in f and we can expect the linear term $\int_0^\infty K(x,y) f(t,y)\,dy f(t,x)$ to give a larger contribution. We will check that these assumptions are self-consistent, in the sense that they will predict an asymptotic behavior for f for which the assumptions made hold.

We examine the asymptotic behavior of the linear term $\int_0^\infty K(x,y) f(t,y)\,dy f(t,x)$ when x is of order one. Due to the effect of the coagulation, we expect the distribution $f(t,y)$ to be concentrated for long times in the larger cluster sizes y as $t \to \infty$. Using the assumptions (C.1.14) and (C.1.15) we obtain the following asymptotic behavior of $K(x,y)$ for $x \ll y$

$$K(x,y) \simeq y^{\gamma+\lambda} x^{-\lambda}. \tag{C.2.1}$$

We then expect to have the following asymptotics as $t \to \infty$, due to the concentration of f in the large cluster sizes

$$\int_0^\infty K(x,y) f(t,y)\,dy \sim x^{-\lambda} \int_0^\infty y^{\gamma+\lambda} f(t,y)\,dy \quad \text{as} \quad t \to \infty$$

for x of order one. Then, considering the dominant terms in (C.1.1), (C.1.2) for x of order one, we obtain the following equation

$$\partial_t f(t,x) = -x^{-\lambda} f(t,x) \int_0^\infty y^{\gamma+\lambda} f(t,y)\,dy + \eta(x). \tag{C.2.2}$$

As explained in the introduction, if $\gamma + 2\lambda > 1$, since steady states describing the cluster concentrations with x of order one do not exist, we expect to have $f(t,x) \to 0$ as $t \to \infty$. This suggest that we should have $M_{\gamma+\lambda} = \int_0^\infty y^{\gamma+\lambda} f(t,y)\,dy \to \infty$ as $t \to \infty$. Suppose that $\partial_t M_{\gamma+\lambda} \ll M_{\gamma+\lambda}$ as $t \to \infty$ (something that would happen if $M_{\gamma+\lambda}$ behaves like a power law, as we will see to be the case). Then (C.2.2) implies the following asymptotic behavior for $f(t,x)$

$$f(t,x) \sim \frac{x^\lambda \eta(x)}{\int_0^\infty y^{\gamma+\lambda} f(t,y)\,dy} \quad \text{as} \quad t \to \infty \tag{C.2.3}$$

for x of order one. We notice that the higher order contributions to (C.2.3) due to the contributions of the term $\frac{1}{2}\int_0^x K(x-y,y) f(t,x-y) f(t,y)\,dy$ are of order $\frac{1}{(M_{\gamma+\lambda})^2}$ or smaller. Therefore, these contributions will be neglected in the following.

The equation (C.2.3) yields an approximate formula for the concentration of clusters with x of order one. We now approximate the part of the coagulation operator which is due to the aggregation of particles with size one with very large particles. To this end we introduce a characteristic length $L = L(t) \gg 1$ such that we can approximate $f(t,y)$ by means of (C.2.3) for $y \leq L$. We attempt to approximate the evolution of the distribution $f(t,x)$ for $x \gg L$. Then, the coagulation operator in (C.1.2) can be approximated, using the symmetry properties of the first term on the right of (C.1.2), as follows

$$\begin{aligned}\mathbb{K}[f](t,x) &= \frac{1}{2}\int_0^x K(x-y,y)\, f(t,x-y)\, f(t,y)\,\mathrm{d}y - \int_0^\infty K(x,y)\, f(t,x)\, f(t,y)\,\mathrm{d}y \\ &= \left[\int_0^L K(x-y,y)\, f(t,x-y)\, f(t,y)\,\mathrm{d}y - \int_0^L K(x,y)\, f(t,x)\, f(t,y)\,\mathrm{d}y\right] \\ &+ \left[\frac{1}{2}\int_L^{x-L} K(x-y,y)\, f(t,x-y)\, f(t,y)\,\mathrm{d}y - \int_L^\infty K(x,y)\, f(t,x)\, f(t,y)\,\mathrm{d}y\right].\end{aligned}$$

We can rewrite this formula as

$$\mathbb{K}[f](t,x) = \left[\int_0^L \left[K(x-y,y)\, f(t,x-y) - K(x,y)\, f(t,x)\right] f(t,y)\,\mathrm{d}y\right] + \mathbb{K}\left[f\chi_{[L,\infty)}\right](t,x), \quad \text{(C.2.4)}$$

where $\chi_{[L,\infty)}$ denotes the characteristic function of the interval $[L,\infty)$. We now approximate $\mathbb{K}[f](t,x)$ for large values of x and $t \to \infty$ and more precisely for $x \gg L$. To this end we use (C.2.3) and we assume also that K and f are sufficiently regular for large values of x. Then, using the fact that $y \ll x$ we obtain the following approximation

$$K(x-y,y)\, f(t,x-y) - K(x,y)\, f(t,x) \simeq -y\frac{\partial}{\partial x}\left[K(x,y)\, f(t,x)\right]. \quad \text{(C.2.5)}$$

Moreover, (C.2.1) yields an approximation for $K(x,y)$ if $x \ll y$. Exchanging the roles of x and y we obtain $K(x,y) \sim x^{\gamma+\lambda}y^{-\lambda}$ as $\frac{x}{y} \to \infty$ and plugging this formula into (C.2.5) we obtain

$$K(x-y,y)\, f(t,x-y) - K(x,y)\, f(t,x) \simeq -y^{1-\lambda}\frac{\partial}{\partial x}\left[x^{\gamma+\lambda} f(t,x)\right].$$

Using this approximation in (C.2.4) we then obtain

$$\mathbb{K}[f](t,x) = -\left[\int_0^L y^{1-\lambda} f(t,y)\,\mathrm{d}y\right]\left[\frac{\partial}{\partial x}\left[x^{\gamma+\lambda} f(t,x)\right]\right] + \mathbb{K}\left[f\chi_{[L,\infty)}\right](t,x)$$

for $x \gg 1$ and $t \to \infty$. We can now use (C.2.3) to derive a formula for $\int_0^L y^{1-\lambda} f(t,y)\,\mathrm{d}y$. We then obtain the approximation

$$\mathbb{K}[f](t,x) \simeq -\frac{\int_0^\infty y\eta(y)\,\mathrm{d}y}{\int_0^\infty y^{\gamma+\lambda} f(t,y)\,\mathrm{d}y}\frac{\partial}{\partial x}\left[x^{\gamma+\lambda} f(t,x)\right] + \mathbb{K}\left[f\chi_{[L,\infty)}\right](t,x). \quad \text{(C.2.6)}$$

Notice that we use that $\int_0^L y\eta(y)\,\mathrm{d}y \simeq \int_0^\infty y\eta(y)\,\mathrm{d}y$ since by assumption $\eta(y)$ decreases sufficiently fast for large values of y.

We will denote as f_{out} the distribution of particles in the region where $x \gg 1$. More precisely we write $f_{out} = f\chi_{[L,\infty)}$. Combining (C.1.1), (C.1.2) with (C.2.6) we obtain the following evolution equation for f_{out}

$$\partial_t f_{out}(t,x) + \frac{\int_0^\infty y\eta(y)\,\mathrm{d}y}{\int_0^\infty y^{\gamma+\lambda} f_{out}(t,y)\,\mathrm{d}y}\frac{\partial}{\partial x}\left[x^{\gamma+\lambda} f_{out}(t,x)\right] = \mathbb{K}\left[f_{out}\right](t,x). \tag{C.2.7}$$

Notice that we use the approximation $\int_0^\infty y^{\gamma+\lambda} f(t,y)\,\mathrm{d}y \simeq \int_0^\infty y^{\gamma+\lambda} f_{out}(t,y)\,\mathrm{d}y$ that might be expected because $f(t,y) \to 0$ for $y \leq L$ as $t \to \infty$ (cf. (C.2.3)).

In the rest of the paper we will study the properties of the self-similar solutions associated to the equation (C.2.7). It is worth to remark that the transport term on the left of (C.2.7) is the way in which the injection of particles with size x of order one affects the outer distribution of clusters f_{out}. Indeed, multiplying (C.2.7) by x and integrating we obtain

$$\partial_t\left(\int_0^\infty x f_{out}(t,x)\,\mathrm{d}x\right) + \frac{\int_0^\infty y\eta(y)\,\mathrm{d}y}{\int_0^\infty y^{\gamma+\lambda} f_{out}(t,y)\,\mathrm{d}y}\int_0^\infty x\frac{\partial}{\partial x}\left[x^{\gamma+\lambda} f_{out}(t,x)\right]\mathrm{d}x$$
$$= \int_0^\infty x\mathbb{K}\left[f_{out}\right](t,x)\,\mathrm{d}x. \tag{C.2.8}$$

The mass conservation property associated to the coagulation kernel yields

$$\int_0^\infty x\mathbb{K}[f_{out}](t,x)\mathrm{d}x = 0.$$

On the other hand, integrating by parts in the second term on the left of (C.2.8) we obtain $\int_0^\infty x\frac{\partial}{\partial x}[x^{\gamma+\lambda} f_{out}(t,x)]\mathrm{d}x = -\int_0^\infty x^{\gamma+\lambda} f_{out}(t,x)\mathrm{d}x$. Combining these results we obtain

$$\partial_t\left(\int_0^\infty x f_{out}(t,x)\,\mathrm{d}x\right) = \int_0^\infty x\eta(x)\,\mathrm{d}x. \tag{C.2.9}$$

The identity (C.2.9) states that the total mass of the clusters in the outer region is equal to the injection rate. This result, that holds for long times, could be expected because in the regime described in this section, the injected particles are transferred instantaneously to large cluster sizes. This is also consistent with $f(t,x) \to 0$ as $t \to \infty$ when $x \approx 1$.

C.2.2 The case $\gamma + 2\lambda = 1$

In the case $\gamma + 2\lambda = 1$, the approximation of the concentrations of clusters $f(t,\cdot)$ in the region where x is of order one must be obtained in a different manner. The reason is that in this case we cannot expect the moment $M_{\gamma+\lambda} = \int_0^\infty x^{\gamma+\lambda} f(t,x)\mathrm{d}x$ to converge to infinity as $t \to \infty$. Indeed, suppose that most of the mass of the monomers is distributed in a characteristic length $L(t)$ that increases as $t \to \infty$. Then, if we denote as M_0 and M_1 the moments of f of order zero and one, respectively, (i.e. $\int_0^\infty f(t,x)\,\mathrm{d}x$ and $\int_0^\infty x f(t,x)\,\mathrm{d}x$, respectively), we have that $M_1 = M_0 L(t)$ and $M_{\gamma+\lambda} = M_0 L(t)^{\gamma+\lambda}$.

Assuming that $\int_0^\infty x\eta(x)\,\mathrm{d}x = 1$ and hence that $M_1 \simeq t$ as $t \to \infty$, we deduce that $M_0 L(t) = t$. The rescaling properties of (C.1.1), (C.1.2) suggest that $\frac{M_0}{t} \simeq (M_0)^2 L(t)^\gamma$. Therefore, plugging the identity $M_0 = \frac{t}{L(t)}$ in this formula we obtain $L(t) = t^{\frac{2}{1-\gamma}}$. Hence

$$M_{\gamma+\lambda} = M_0 L(t)^{\gamma+\lambda} = tL(t)^{\gamma+\lambda-1} = t^{1+\frac{2}{1-\gamma}(\gamma+\lambda-1)} = 1. \tag{C.2.10}$$

It then follows that the self-similar rescaling ansatz implies that $M_{\gamma+\lambda}$ remains of order one for large times. A consequence of this is that we cannot approximate (C.1.1), (C.1.2) for clusters of order one by means of the equation (C.2.2). Instead of this we will use a different approximation by splitting f in an outer part which describes the cluster distribution for x of order L and an inner part that describes the cluster distribution for x of order one. More precisely, we write

$$f(t,x) = f_{inner}(t,x) + f_{outer}(t,x), \tag{C.2.11}$$

where $f_{outer} = f\chi_{[L,\infty)}$, while $f_{inner} = f\chi_{(0,L]}$, for a constant $L > 0$.

Using (C.1.1), (C.1.2) we would then obtain the following evolution equation for f_{inner}

$$\partial_t f_{inner}(t,x) = \mathbb{K}[f_{inner}](t,x) - f_{inner}(t,x)\int_0^\infty K(x,y)\, f_{outer}(t,y)\,\mathrm{d}y + \eta(x), \tag{C.2.12}$$

where we have used the decomposition (C.2.11) in the loss term of the coagulation operator $\mathbb{K}[f]$. We use also the fact that in order to compute the gain term for x of order one we need to use only f_{inner}.

By assumption, the main contribution of $f_{outer}(t,y)$ is due to clusters with size $L \gg 1$. On the other hand, for x of order one, we have the approximation

$$K(x,y)\, f_{outer}(t,y) \simeq y^{\gamma+\lambda} x^{-\lambda} f_{outer}(t,y)$$

and (C.2.12) becomes

$$\partial_t f_{inner}(t,x) = \mathbb{K}[f_{inner}](t,x) - f_{inner}(t,x)\, x^{-\lambda}\int_0^\infty y^{\gamma+\lambda} f_{outer}(t,y)\,\mathrm{d}y + \eta(x). \tag{C.2.13}$$

The scaling argument above, (C.2.10), suggests that $\int_0^\infty y^{\gamma+\lambda} f_{outer}(t,y)\,\mathrm{d}y$ approaches to a positive constant as $t \to \infty$ if f_{outer} behaves in a self-similar manner and $\gamma + 2\lambda = 1$. We will write

$$M_{\gamma+\lambda} = \int_0^\infty y^{\gamma+\lambda} f_{outer}(t,y)\,\mathrm{d}y. \tag{C.2.14}$$

On the other hand, we will assume that the function f_{inner}, which is described by means of (C.2.13), approaches to a stationary solution for large times. We then obtain the following equation which would be expected to describe the behavior of the solutions of (C.2.13) for long times

$$\mathbb{K}[f_{inner}](x) - M_{\gamma+\lambda} f_{inner}(x)\, x^{-\lambda} + \eta(x) = 0. \tag{C.2.15}$$

We can now derive an equation describing the evolution of f_{outer}. To this end we argue as in the case of $\gamma + 2\lambda > 1$ in order to approximate the effect in f_{outer} due to the collisions of clusters with size L with clusters with size of order one. Using approximations analogous to the ones used for the derivation of (C.2.7) (cf. (C.2.4), (C.2.5)) we obtain the following approximation for $\mathbb{K}[f](t,x)$

$$\mathbb{K}[f](t,x) \simeq -\left[\int_0^\infty y^{1-\lambda} f_{inner}(t,y)\,\mathrm{d}y\right]\left[\frac{\partial}{\partial x}\left[x^{\gamma+\lambda} f_{outer}(t,x)\right]\right] + \mathbb{K}[f_{outer}](t,x) \tag{C.2.16}$$

for x of order L.

On the other hand, multiplying (C.2.15) by x and integrating in $(0,\infty)$ we obtain, since f_{inner} is zero for $x > L$, that

$$M_{\gamma+\lambda}\int_0^\infty y^{1-\lambda} f_{inner}(y)\,\mathrm{d}y = \int_0^\infty y\eta(y)\,\mathrm{d}y.$$

Using this formula to eliminate $\int_0^\infty y^{1-\lambda} f_{inner}(y)\,\mathrm{d}y$ in (C.2.16) we obtain

$$\mathbb{K}[f](t,x) \simeq -\frac{\int_0^\infty y\eta(y)\,\mathrm{d}y}{M_{\gamma+\lambda}}\left[\frac{\partial}{\partial x}\left[x^{\gamma+\lambda} f_{outer}(t,x)\right]\right] + \mathbb{K}[f_{outer}](t,x).$$

Therefore, we obtain that f_{outer} satisfies the equation (C.2.7).

The whole asymptotic behavior derived here relies on the existence of solutions of the equation (C.2.15). In this equation, the value of $M_{\gamma+\lambda}$ is chosen as one associated to a self-similar solution of (C.2.7). The equation (C.2.15) can then be interpreted as a stationary solution for a coagulation equation with source η and a removal term $-M_{\gamma+\lambda} f_{inner}(x)\, x^{-\lambda}$. The results in [FLNV21b] imply that no solutions of (C.2.15) exist if $M_{\gamma+\lambda} = 0$ and $\gamma + 2\lambda = 1$, but when $M_{\gamma+\lambda} > 0$ the existence/non-existence of stationary solutions for equation (C.2.15) is still an open problem.

C.3 Setting and main results

The main results of this paper concern the existence and the non-existence of solutions to the following equation

$$-\frac{3+\gamma}{1-\gamma}\Phi(\xi) - \frac{2}{1-\gamma}\xi\Phi_\xi(\xi) + \frac{1}{\int_0^\infty \eta^{\gamma+\lambda}\Phi(\eta)\,\mathrm{d}\eta}\frac{\partial}{\partial\xi}\left(\xi^{\gamma+\lambda}\Phi(\xi)\right) = \mathbb{K}[\Phi](\xi), \quad \xi > 0. \tag{C.3.1}$$

We write now precisely what we mean by a solution of equation (C.3.1) and then we state the main theorems on the existence of a self-similar profile under certain assumptions on the parameters γ and λ (Theorem C.5.1), on its properties (Theorem C.6.1) and on the non-existence of the self-similar solutions under different assumptions on the parameters γ and λ (Theorem C.3.5).

Definition C.3.1. Let K be a homogeneous symmetric coagulation kernel satisfying (C.1.5), (C.1.6), with homogeneity $\gamma < 1$ and with $\gamma + \lambda < 1$ and $\gamma + 2\lambda \geq 1$. A self-similar profile of equation (C.2.7) with respect to the kernel K, is a measure $\Phi \in \mathcal{M}_+(\mathbb{R}_*)$ such that

$$0 < \int_{\mathbb{R}_*} x^{\gamma+\lambda}\Phi(\mathrm{d}x) < \infty \text{ and } \int_{(0,1)} x^{1-\lambda}\Phi(\mathrm{d}x) < \infty \tag{C.3.2}$$

and such that it satisfies the following equation

$$\begin{aligned}&\int_{\mathbb{R}_*} \varphi'(x)\left[\frac{2}{1-\gamma}x - \frac{x^{\gamma+\lambda}}{\int_{\mathbb{R}_*} z^{\gamma+\lambda}\Phi(\mathrm{d}z)}\right]\Phi(\mathrm{d}x) - \frac{1+\gamma}{1-\gamma}\int_{\mathbb{R}_*}\varphi(x)\Phi(\mathrm{d}x)\\ &= \frac{1}{2}\int_{\mathbb{R}_*}\int_{\mathbb{R}_*} K(x,y)\left[\varphi(x+y)-\varphi(x)-\varphi(y)\right]\Phi(\mathrm{d}x)\Phi(\mathrm{d}y)\end{aligned} \tag{C.3.3}$$

for every test function $\varphi \in C_c^1(\mathbb{R}_*)$.

Remark C.3.2. For every test function $\varphi \in C_c^1(\mathbb{R}_*)$ (hence such that $\varphi = 0$ near zero) all the integrals in equation (C.3.3) are finite. The integrals in the left-hand side of equation (C.3.3) are bounded due to the fact that φ is compactly supported and that Φ is a Radon measure. We analyse now the right hand side. Since $\gamma + 2\lambda \geq 0$

$$\begin{aligned}&\int_{\mathbb{R}_*}\int_{\mathbb{R}_*} K(x,y)\left|\varphi(x+y)-\varphi(x)-\varphi(y)\right|\Phi(\mathrm{d}x)\Phi(\mathrm{d}y)\\ &\leq 4c_1\int_{\mathbb{R}_*}\int_{(0,y]} x^{-\lambda}y^{\gamma+\lambda}\left|\varphi(x+y)-\varphi(x)-\varphi(y)\right|\Phi(\mathrm{d}x)\Phi(\mathrm{d}y)\\ &\leq c\int_{\mathbb{R}_*}\int_{(0,y]} x^{-\lambda}y^{\gamma+\lambda}\left(\left|\varphi(x+y)-\varphi(y)\right|+\left|\varphi(x)\right|\right)\Phi(\mathrm{d}x)\Phi(\mathrm{d}y)\\ &\leq c\|\varphi'\|_\infty\int_{\mathbb{R}_*}\int_{(0,1)} x^{1-\lambda}y^{\gamma+\lambda}\Phi(\mathrm{d}x)\Phi(\mathrm{d}y)\\ &+ c\int_{\mathbb{R}_*}\int_{[1,y]} x^{-\lambda}y^{\gamma+\lambda}\left|\varphi(x+y)-\varphi(y)\right|\Phi(\mathrm{d}x)\Phi(\mathrm{d}y) + c\int_{\mathbb{R}_*} y^{\gamma+\lambda}\Phi(\mathrm{d}y)\int_{\mathbb{R}_*} x^{-\lambda}|\varphi(x)|\Phi(\mathrm{d}x).\end{aligned}$$

Using (C.3.2), the fact that $-\lambda \leq \gamma + \lambda$, as well as the fact that φ is compactly supported the desired conclusion follows.

Theorem C.3.3 (Existence of the self-similar profiles)**.** *Let K be a homogeneous symmetric coagulation kernel, of homogeneity γ, satisfying* (C.1.5), (C.1.6)*, with γ, λ such that* (C.1.7) *holds and such that*

$$-1 < \gamma, \quad \gamma + 2\lambda \geq 1.$$

Then there exists a self-similar profile Φ as in Definition C.3.1. Moreover, Φ is such that $\Phi((0, \rho(M_{\gamma+\lambda}))) = 0$ for

$$\rho(M_{\gamma+\lambda}) := \left(\frac{1-\gamma}{2\int_{\mathbb{R}_*} x^{\gamma+\lambda}\Phi(\mathrm{d}x)}\right)^{\frac{1}{1-\gamma-\lambda}}. \tag{C.3.4}$$

Additionally, Φ it is such that

$$\int_{\mathbb{R}_*} e^{Lx}\Phi(\mathrm{d}x) < \infty$$

for some $L > 0$ and it is absolutely continuous with respect to the Lebesgue measure. Then $\Phi(\mathrm{d}x) = \phi(x)\mathrm{d}x$ and the density ϕ is such that

$$\limsup_{x\in\mathbb{R}_*} \phi(x)e^{Mx} < \infty$$

for a positive constant M.

Remark C.3.4. In this paper we do not prove the uniqueness of the self-similar profiles. Therefore, it makes sense to understand if the proven properties for the self-similar profile constructed in Theorem C.5.1 hold for each self-similar profile as in Definition C.3.1.

When $\gamma + 2\lambda > 1$ we prove that each self-similar profile as in Definition C.3.1 is zero in the set $(0, \rho(M_{\gamma+\lambda}))$, where $\rho(M_{\gamma+\lambda})$ is given by (C.3.4), see Theorem C.6.1 for more details. In contrast, when $\gamma + 2\lambda = 1$, we only prove that the self-similar profile constructed in the proof of Theorem C.3.3 is such that $\Phi((0, \rho(M_{\gamma+\lambda}))) = 0$. However, we do not know if this property holds for every self-similar profile as in Definition C.3.1.

Theorem C.3.5 (Non-existence of the self-similar profiles)**.** *Let K be a homogeneous symmetric coagulation kernel, of homogeneity γ, satisfying* (C.1.5), (C.1.6)*, with γ, λ such that* (C.1.7) *holds.*

1. *If $\gamma \leq -1, \gamma + 2\lambda > 1$, then a self-similar profile Φ as in Definition C.3.1 does not exist.*
2. *If $\gamma \leq -1, \gamma + 2\lambda = 1$, then a self-similar profile Φ as in Definition C.3.1 with the additional property*

$$\int_{(0,1]} x^{-\lambda}\Phi(\mathrm{d}x) < \infty \tag{C.3.5}$$

does not exist.

Remark C.3.6. Notice that if $\gamma + 2\lambda > 1$ we prove that self-similar solutions as in Definition C.3.1 do not exist when $\gamma \leq -1$. Instead, when $\gamma + 2\lambda = 1$ and $\gamma \leq -1$, we do not exclude the existence of a self-similar solution Φ as in Definition C.3.1 with

$$\int_{(0,1]} x^{-\lambda}\Phi(\mathrm{d}x) = \infty.$$

C.4 Main ideas of the proofs

In this section we explain the main ideas for the proofs of existence/non-existence of self-similar solutions. Both in the case $\gamma + 2\lambda > 1$ and $\gamma + 2\lambda = 1$, to prove that a self-similar solution exists, we find an invariant region for the evolution equation corresponding to (C.3.1), namely the following equation

$$\partial_t \Phi(t,\xi) - \frac{3+\gamma}{1-\gamma}\Phi(t,\xi) - \frac{2\xi}{1-\gamma}\partial_\xi \Phi(t,\xi) + \frac{\partial_\xi\left(\xi^{\gamma+\lambda}\Phi(t,\xi)\right)}{\int_{\mathbb{R}_*} x^{\gamma+\lambda}\Phi(t,\mathrm{d}x)} = \mathbb{K}[\Phi](t,\xi). \tag{C.4.1}$$

By Tychonoff fixed point theorem the existence of an invariant region implies that there exists a solution of equation (C.3.1).

We prove that the set

$$P = \left\{ H : \begin{array}{l} \int_0^\infty xH(x)\mathrm{d}x = 1,\ \frac{1}{C_2} \leq \int_0^\infty x^{\gamma+\lambda}H(x)\mathrm{d}x \leq C_1, \\ H((0,\rho(C_1))) = 0,\ \int_0^\infty x^{2-\gamma-\lambda}H(x)\mathrm{d}x \leq C_2 \end{array} \right\}$$

is invariant when $\gamma + 2\lambda \geq 1$ and $\gamma > -1$ for suitable constants $C_1, C_2 > 0$ and $\rho(C_1)$ given by

$$\rho(C_1) := \left(\frac{1-\gamma}{2C_1}\right)^{\frac{1}{1-\gamma-\lambda}}. \tag{C.4.2}$$

To prove that the set P is invariant we proceed as follows.

1. We prove that $\int_0^\infty x\Phi_0(x)\mathrm{d}x = 1$ implies $\int_0^\infty x\Phi(t,x)\mathrm{d}x = 1$, for every $t > 0$. This is done multiplying by x equation (C.4.1), integrating from zero to infinity and then studying the ODE for the first order moment obtained in this manner.

2. As a second step we prove that there exists an upper bound for $M_{\gamma+\lambda}$. To this end we multiply both sides of equation (C.3.1) by $x^{\gamma+\lambda}$ and we integrate over x in $(0,\infty)$ to obtain an ODE for the $\gamma+\lambda$ moment. Using Grönwall's lemma, the fact that $\gamma + 2\lambda \geq 1$ and that $\gamma > -1$, the desired conclusion follows.

3. As a third step we use the fact that $M_{\gamma+\lambda} \leq C_1$ to prove that $\Phi(t,(0,\rho(C_1)) = 0$. Indeed, the evolution described by equation (C.4.1) is driven by two mechanisms: coagulation, which increases the average size of the particles in the system, and the growth term,

$$\frac{\xi^{\gamma+\lambda}}{\int_0^\infty x^{\gamma+\lambda}\Phi(t,x)\mathrm{d}x} - \frac{2}{1-\gamma}\xi,$$

which is positive for every $\xi < \rho(C_1)$. Hence, if we start from an initial data Φ_0 such that $\Phi_0((0,\rho(C_1))) = 0$, then we will have that $\Phi(t,(0,\rho(C_1))) = 0$, for every $t > 0$.

4. Using the fact that $\gamma + \lambda < 1$, $\gamma > -1$, $\Phi(t,(0,\rho(C_1))) = 0$, as well as the fact that $\int_0^\infty x^{\gamma+\lambda}\Phi(t,\mathrm{d}x) \leq C_1$, we prove that $\int_0^\infty x^{2-\gamma-\lambda}\Phi(t,x)\mathrm{d}x \leq C_2$.

5. Finally, from the upper bound for $M_{2-\gamma-\lambda}$, we derive a lower bound for $M_{\gamma+\lambda}$. Indeed, Cauchy-Schwarz inequality implies that

$$1 = \left(\int_{\mathbb{R}_*} x\Phi(t,x)\mathrm{d}x\right)^2 \leq \int_{\mathbb{R}_*} x^{\gamma+\lambda}\Phi(t,x)\mathrm{d}x \int_{\mathbb{R}_*} x^{2-\gamma-\lambda}\Phi(t,x)\mathrm{d}x. \tag{C.4.3}$$

Hence $\frac{1}{C_2} \leq M_{\gamma+\lambda}$.

In order to prove non-existence we proceed by contradiction. Due to the contribution of the coagulation operator in equation (C.1.16), we expect the zeroth moment of f to decay in time. However, assuming the self-similar change of variable (C.1.18), we have that

$$\int_0^\infty f(t,x)\mathrm{d}x = t^{-\frac{1+\gamma}{1-\gamma}} \int_0^\infty \Phi(x)\mathrm{d}x.$$

If $\gamma \leq -1$, then $\int_0^\infty f(t,x)\mathrm{d}x$ is constant or increasing in time and this gives a contradiction. Hence we cannot expect self-similar solutions to exist. To make the argument rigorous we will have to prove that $0 < \int_0^\infty \Phi(\mathrm{d}x) < \infty$. When $\gamma + 2\lambda > 1$ we do this by proving that for each self-similar profile there exists a $\delta > 0$ such that $\Phi((0,\delta)) = 0$ and that Φ tends to zero sufficiently fast as $x \to \infty$. Instead, when $\gamma + 2\lambda = 1$, the methods used in this paper do not allow to prove that Φ is equal to zero near the origin, hence we use (C.3.5) as well as (C.3.2) to prove that $0 < \int_0^\infty \Phi(\mathrm{d}x) < \infty$.

C.5 Existence of a self-similar profile

We aim to prove the existence of a solution of equation (C.4.1). Namely, we will prove the following theorem, which is just a reformulation of Theorem C.3.3.

Theorem C.5.1. *Let K be a homogeneous symmetric coagulation kernel, of homogeneity γ, satisfying* (C.1.5), (C.1.6)*, with γ, λ such that* (C.1.7) *holds and such that*

$$-1 < \gamma, \quad \gamma + 2\lambda \geq 1.$$

Then there exists a self-similar profile Φ as in Definition C.3.1. Moreover, Φ is such that $\Phi((0,\rho(M_{\gamma+\lambda}))) = 0$ with $\rho(M_{\gamma+\lambda})$ given by (C.3.4).

To this end we prove the existence of a (time-dependent) solution for a suitably truncated and regularized version of equation (C.4.1):

$$\partial_t\Phi(t,\xi) - \frac{3+\gamma}{1-\gamma}\Phi(t,\xi) - \frac{2\xi}{1-\gamma}\partial_\xi\Phi(t,\xi) + \frac{\partial_\xi\left(\xi^{\gamma+\lambda}\Phi(t,\xi)\right)}{\int_{\mathbb{R}_*} x^{\gamma+\lambda}\Phi(t,\mathrm{d}x)} = \mathbb{K}_R[\Phi](t,\xi). \tag{C.5.1}$$

The operator $\mathbb{K}_R$ is the truncated operator of parameter $R > 0$ defined as

$$\begin{aligned}\mathbb{K}_R[\Phi](t,x) :=& \frac{1}{2}\int_0^x K_R(x-y,y)\,\Phi(t,x-y)\,\Phi(t,y)\,\mathrm{d}y \\ &- \int_0^\infty K_R(x,y)\,\Phi(t,x)\,\Phi(t,y)\,\mathrm{d}y,\end{aligned} \tag{C.5.2}$$

where the kernel K_R is a *truncated kernel*, i.e. it is a continuously differentiable, bounded and symmetric function $K_R : \mathbb{R}_*^2 \to \mathbb{R}_+$, such that for every $(x,y) \in \mathbb{R}_*^2$ we have that $K_R(x,y) \leq K(x,y)$ and

$$\begin{cases} K_R(x,y) = 0, & \text{if } x > R \text{ or } y > R; \\ |K_R(x,y) - K(x,y)| \leq e^{-R}, & \text{if } (x,y) \in \left[\frac{1}{2R}, \frac{R}{4}\right]^2. \end{cases} \tag{C.5.3}$$

Notice that the kernel K_R can be obtained starting from the kernel K by means of standard truncations and mollifying arguments.

Using Tychonoff fixed point theorem we prove the existence of a stationary solution Φ_R for equation (C.5.1) and we will prove that there exists the limit Φ of $\{\Phi_R\}_R$ as R tends to infinity. To conclude we will prove that the measure Φ satisfies equation (C.1.19).

C.5.1 Existence of a time dependent solution for the truncated equation

Since in this section we work only with the truncated equation, we omit the label R in Φ_R. We will reintroduce the label R in Section C.5.3. We start this section by introducing a definition of solutions for equation (C.5.1).

Definition C.5.2. Let K_R be the truncated kernel defined as in (C.5.3) as a function of the homogeneous symmetric coagulation kernel K satisfying (C.1.5), (C.1.6), with homogeneity $\gamma < 1$ and with $\gamma + \lambda < 1$ and $\gamma + 2\lambda \geq 1$. A function $\Phi \in C^1([0,T]; \mathcal{M}_{+,b}(\mathbb{R}_*))$ is a solution of equation (C.5.1) if

$$0 < \inf_{t\in[0,T]} \int_{\mathbb{R}_*} x^{\gamma+\lambda}\Phi(t,\mathrm{d}x) \ \text{ and } \ \sup_{t\in[0,T]} \int_{\mathbb{R}_*} x^{\gamma+\lambda}\Phi(t,\mathrm{d}x) < \infty$$

and if Φ satisfies

$$\begin{aligned}&\int_{\mathbb{R}_*} \varphi(x)\dot{\Phi}(t,\mathrm{d}x) - \frac{1+\gamma}{1-\gamma}\int_{\mathbb{R}_*} \varphi(x)\Phi(t,\mathrm{d}x) + \frac{2}{1-\gamma}\int_{\mathbb{R}_*} \varphi'(x)x\Phi(t,\mathrm{d}x) \\ &- \frac{1}{\int_{\mathbb{R}_*} x^{\gamma+\lambda}\Phi(t,\mathrm{d}x)}\int_{\mathbb{R}_*} \varphi'(x)x^{\gamma+\lambda}\Phi(t,\mathrm{d}x) \\ &= \frac{1}{2}\int_{\mathbb{R}_*}\int_{\mathbb{R}_*} K_R(x,y)\left[\varphi(x+y) - \varphi(x) - \varphi(y)\right]\Phi(t,\mathrm{d}x)\Phi(t,\mathrm{d}y),\end{aligned} \tag{C.5.4}$$

for every test function $\varphi \in C_c^1(\mathbb{R}_*)$.

Given the positive numbers k_1, k_2, ρ^*, we define the subset $S(k_1, k_2, \rho^*)$ of $\mathcal{M}_+(\mathbb{R}_*)$ as

$$S(k_1,k_2,\rho^*) := \left\{ H \in \mathcal{M}_+(\mathbb{R}_*) : \begin{array}{c} H((0,\rho^*)) = 0, \int_{\mathbb{R}_*} xH(\mathrm{d}x) = 1, \\ k_1 \leq \int_{\mathbb{R}_*} x^{\gamma+\lambda} H(\mathrm{d}x) \leq k_2 \end{array} \right\}. \tag{C.5.5}$$

Theorem C.5.3. *Assume K_R to be a truncated kernel defined as in (C.5.3) as a function of a homogeneous symmetric kernel K satisfying (C.1.5), (C.1.6), with parameters $\gamma, \lambda \in \mathbb{R}$ such that (C.1.7) holds and $\gamma + 2\lambda \geq 1$ and $\gamma > -1$. Then for all $k_1, k_2 > 0$ and sufficiently large $R_0 > 0$ such that $\Phi_0 \in S\left(2k_1, \frac{k_2}{2}, \rho(k_2)\right) \cap \{H \in \mathcal{M}_+(\mathbb{R}_*) : H((4R, \infty)) = 0\}$ for $R > R_0$ and for $\rho(k_2)$ given by (C.4.2), there exists a unique solution $\Phi \in C^1([0,T]; \mathcal{M}_+(\mathbb{R}_*))$ of equation (C.5.1) in the sense of Definition C.5.2, for $T > 0$ sufficiently small. For every $t \in [0,T]$, we have that $\Phi(t,\cdot) \in S(k_1, k_2, \rho(k_2)) \cap \{H \in \mathcal{M}_+(\mathbb{R}_*) : H((4R,\infty)) = 0\}$.*

Lemma C.5.4. *Let $T > 0$ and let $R > 0$. Assume $\gamma < 1$ and $\gamma + \lambda < 1$. Assume $\alpha \in C([0,T])$ satisfying*

$$0 < k_1 \leq \alpha(t) \leq k_2 < \infty, \quad \text{for all} \quad t \in [0,T], \tag{C.5.6}$$

for some positive constants k_1 and k_2. Consider the ODE

$$\frac{dx(t)}{dt} = V(x(t), \alpha(t)), \quad x(0) = x_0 > 0, \tag{C.5.7}$$

with $V(x,\alpha) := \frac{x^{\gamma+\lambda}}{\alpha} - \frac{2}{1-\gamma}x$. If $x_0 \geq \rho(k_2)$ with $\rho(k_2)$ given by (C.4.2), then (C.5.7) has a unique solution $X(t, x_0, \alpha) \geq \rho(k_2)$. Let $X(t, x_{0,1}, \alpha_1)$ and $X(t, x_{0,2}, \alpha_2)$ be the solutions of equation (C.5.7) with respect to the functions α_1 and α_2 satisfying (C.5.6) and $x_{0,1}, x_{0,2} \in [\rho(k_2), 4R]$. Then

$$|X(t, x_{0,1}, \alpha_1) - X(t, x_{0,2}, \alpha_2)| \leq L_1 |x_{0,1} - x_{0,2}| + T L_2 \|\alpha_1 - \alpha_2\|_{[0,T]}, \tag{C.5.8}$$

for every $t \in [0,T]$, where $L_1 = L_1(T,\gamma,\lambda,k_1,k_2,R) > 0$, and $L_2 = L_2(T,\gamma,\lambda,k_1,k_2,R) > 0$ and where we denote by $\|\cdot\|_{[0,T]}$ the norm $\|f\|_{[0,T]} := \sup_{t\in[0,T]} |f(t)|$, for $f \in C([0,T])$.

Proof. Since $V(x,\alpha) > 0$, for every $x \leq \rho(k_2)$, we deduce that the set $\{x \geq \rho(k_2)\}$ is an invariant region of (C.5.7). This also implies that a unique solution exists.

Equation (C.5.7) is a Bernoulli equation that can be reduced to the linear ODE

$$\frac{dy}{dt} = \frac{1-(\gamma+\lambda)}{\alpha(t)} - \frac{2(1-(\gamma+\lambda))}{1-\gamma} y, \quad y_0 := x_0^{1-(\gamma+\lambda)}$$

via the change of variable $y = x^{1-(\lambda+\gamma)}$. We deduce that

$$Y(t, x_0^{1-(\gamma+\lambda)}, \alpha) = x_0^{1-(\gamma+\lambda)} e^{-\frac{2(1-(\gamma+\lambda))}{1-\gamma} t} + \int_0^t \frac{(1-\gamma-\lambda)}{\alpha(s)} e^{-\frac{2(1-(\gamma+\lambda))}{1-\gamma}(t-s)} \mathrm{d}s.$$

Since α_1, α_2 satisfy (C.5.6) and since $x_{0,1}, x_{0,2} \in [\rho(k_2), 4R]$, the above formula implies that

$$\left|Y(t, x_{0,1}^{1-(\gamma+\lambda)}, \alpha_1) - Y(t, x_{0,2}^{1-(\gamma+\lambda)}, \alpha_2)\right| \leq \left|x_{0,1}^{1-(\gamma+\lambda)} - x_{0,2}^{1-(\gamma+\lambda)}\right| \\ + C(T, \gamma, \lambda, k_1)\|\alpha_1 - \alpha_2\|_{[0,T]}.$$

Since the set $\{x \geq \rho(k_2)\}$ is invariant for (C.5.7), the above inequality, together with the definition of y as a function of x, implies (C.5.8). □

We will solve equation (C.5.1) using Lagrangian coordinates. To this end we introduce the notation

$$\chi_\varphi^\alpha(x, y, t) := \varphi(\ell_\alpha(x, y, t)) - \varphi(x) - \varphi(y),$$

for $\varphi \in C_c(\mathbb{R}_*)$, where ℓ_α is defined for $x > 0$, $y > 0$ and $t \geq 0$ as

$$X(t, \ell_\alpha(x, y, t), \alpha) = X(t, x, \alpha) + X(t, y, \alpha). \tag{C.5.9}$$

The function ℓ_α is well defined because for every fixed time $t \geq 0$ the function $x \mapsto X(t, x, \alpha)$ is increasing.

Lemma C.5.5. *Let K_R be a truncated kernel defined as in (C.5.3) as a function of a homogeneous symmetric kernel K satisfying (C.1.5), (C.1.6), with $\gamma, \lambda \in \mathbb{R}$ satisfying (C.1.7). Let k_1, k_2, R be three positive constants. Assume that the initial condition Φ_0 is such that $2k_1 \leq \int_{\mathbb{R}_*} x^{\gamma+\lambda}\Phi_0(\mathrm{d}x) \leq \frac{k_2}{2}$ and that $\Phi_0((0, \rho(k_2)) \cup (4R, \infty)) = 0$ for $\rho(k_2)$ given by (C.4.2).*

For a sufficiently small time $T > 0$, depending on R, there exists a function $F \in C([0, T], \mathcal{M}_+(\mathbb{R}_*))$, with $F(0, \cdot) = \Phi_0$, that satisfies

$$\int_{\mathbb{R}_*} \varphi(x)F(t, \mathrm{d}x) = \int_{\mathbb{R}_*} \varphi(x)\Phi_0(\mathrm{d}x) \\ + \frac{1}{2}\int_0^t \int_{\mathbb{R}_*} \int_{\mathbb{R}_*} K_R(X(s, x, \alpha), X(s, y, \alpha))e^{\frac{1+\gamma}{1-\gamma}s}\chi_\varphi^\alpha(x, y, s)F(s, \mathrm{d}x)F(s, \mathrm{d}y)\mathrm{d}s, \tag{C.5.10}$$

for every $\varphi \in C_c(\mathbb{R}_*)$, with

$$\alpha(t) := e^{\frac{1+\gamma}{1-\gamma}t}\int_{\mathbb{R}_*} (X(t, x, \alpha))^{\gamma+\lambda} F(t, \mathrm{d}x), \quad \forall t \in [0, T]. \tag{C.5.11}$$

The function F is such that $k_1 \leq \alpha(t) \leq k_2$ and such that

$$F(t, (0, \rho(k_2)) \cup (4R, \infty)) = 0, \quad \forall t \in [0, T]. \tag{C.5.12}$$

Proof. We define the set

$$\tilde{X} := \{H \in \mathcal{M}_+(\mathbb{R}_*) : H((0, \rho(k_2)) \cup (4R, \infty)) = 0\}$$

and the set

$$X_T := \left\{ (F,\alpha) \in C([0,T]; \mathcal{M}_+(\mathbb{R}_*)) \times C([0,T]) : \begin{array}{c} F(t,\cdot) \in \tilde{X}, k_1 \leq \alpha(t) \leq k_2, \forall t \in [0,T], \\ \sup_{t\in[0,T]} \int_{\mathbb{R}_*} F(t,\mathrm{d}x) \leq 1 + \int_{\mathbb{R}_*} \Phi_0(\mathrm{d}x) \end{array} \right\}.$$

We endow the set $\mathcal{M}_+(\mathbb{R}_*)$ with the Wasserstein metric W_1. The reason for this choice will become clear in the proof of the contractivity of the evolution operator corresponding to (C.5.10), (C.5.11). We endow the set $C([0,T], \mathcal{M}_+(\mathbb{R}_*))$ with the metric induced by the distance

$$d_T(\mu,\nu) := \sup_{t\in[0,T]} W_1(\mu(t,\cdot),\nu(t,\cdot)).$$

Similarly, we endow $C([0,T])$ with the norm $\|f\|_{[0,T]} := \sup_{t\in[0,T]} |f(t)|$.

For every $(F,\alpha) \in X_T$, we define the operator $\mathcal{T}[F,\alpha](t) : C_c(\mathbb{R}_*) \to \mathbb{R}_+$ as $\mathcal{T}[F,\alpha](t) := \mathcal{T}_1[F,\alpha](t) + \mathcal{T}_2[F,\alpha](t)$ with

$$\langle \mathcal{T}_1[F,\alpha](t), \varphi\rangle = \int_{\mathbb{R}_*} \varphi(x) e^{-\int_0^t a[F,\alpha](s,x)\mathrm{d}s} \Phi_0(\mathrm{d}x),$$

$$a[F,\alpha](t,x) := e^{\frac{\gamma+1}{1-\gamma}t} \int_0^\infty K_R(X(t,x,\alpha), X(t,y,\alpha)) F(t,\mathrm{d}y),$$

$$\begin{aligned}\langle \mathcal{T}_2[F,\alpha](t), \varphi\rangle =& \frac{1}{2}\int_0^t \int_{\mathbb{R}_*}\int_{\mathbb{R}_*} e^{-\int_s^t a[F](v,x)dv} K_R(X(s,x,\alpha),X(s,y,\alpha)) e^{\frac{1+\gamma}{1-\gamma}s} \varphi(\ell_\alpha(x,y,s))\cdot \\ &\cdot F(s,\mathrm{d}x)F(s,\mathrm{d}y)\mathrm{d}s.\end{aligned}$$

Moreover, given $(F,\alpha) \in X_T$, we define the operator $\mathcal{A}[F,\alpha] : [0,T] \to \mathbb{R}_*$ as

$$\mathcal{A}[F,\alpha](t) := e^{\frac{1+\gamma}{1-\gamma}t} \int_{\mathbb{R}_*} (X(t,x,\alpha))^{\gamma+\lambda} F(t,\mathrm{d}x), \quad \forall t \in [0,T].$$

We can now rewrite (C.5.10), (C.5.11) in a fixed point form, namely as $(F,\alpha) = \mathcal{F}[F,\alpha]$ where $\mathcal{F}[F,\alpha] := (\mathcal{T}[F,\alpha], \mathcal{A}[\mathcal{T}[F,\alpha],\alpha])$.

We prove now that $\mathcal{F} : X_T \to X_T$. Since for every $(F,\alpha) \in X_T$, the operator $\mathcal{T}[F,\alpha](t)$ is linear and continuous, we deduce that it can be identified with an element of $\mathcal{M}_{+,b}(\mathbb{R}_*)$. Moreover, the operator $t \mapsto \mathcal{T}[F,\alpha](t)$ is a continuous map from $\mathbb{R}_+$ to $\mathcal{M}_{+,b}(\mathbb{R}_*)$, hence $\mathcal{T}[F,\alpha] \in C([0,T], \mathcal{M}_{+,b}(\mathbb{R}_*))$. Similarly, for every $(F,\alpha) \in X_T$, we have that $\mathcal{A}[F,\alpha] \in C([0,T])$.

Consider a test function φ such that $\varphi(x) = 0$ for every $x \in [\rho(k_2),\infty)$. Then, since $\Phi_0((0,\rho(k_2))) = 0$, we deduce that

$$\langle \mathcal{T}_1[F,\alpha](t), \varphi\rangle = \int_{[\rho(k_2),\infty)} \varphi(x) e^{-\int_0^t a[F,\alpha](s,x)\mathrm{d}s} \Phi_0(\mathrm{d}x) = 0.$$

Similarly, since $\ell_\alpha(s,x,y) \geq x$ and since $(F,\alpha) \in X_T$, we deduce that

$$\langle \mathcal{T}_2[F,\alpha](t), \varphi\rangle \leq \frac{C(R)}{2}\int_0^t \int_{[\rho(k_2),\infty)}\int_{[\rho(k_2),\infty)} e^{\frac{1+\gamma}{1-\gamma}s}\varphi(\ell_\alpha(x,y,s))F(s,\mathrm{d}x)F(s,\mathrm{d}y)\mathrm{d}s = 0.$$

Therefore, $(\mathcal{T}[F,\alpha])(t,(0,\rho(k_2))) = 0$, for every $t \in [0,T]$.

Consider any test function φ such that $\varphi(x) = 0$ in $[0,4R]$. Since $\Phi_0((4R,\infty)) = 0$, we have that

$$\langle \mathcal{T}_1[F,\alpha](t), \varphi\rangle \leq \int_{(0,4R]} \varphi(x) e^{-\int_0^t a[F,\alpha](s,x)\mathrm{d}s}\Phi_0(\mathrm{d}x) = 0.$$

Moreover, using the notation $S_1(s) := \{(x,y) \in \mathbb{R}_*^2 : X(s,x,\alpha) \leq R, X(s,y,\alpha) \leq R\}$, $S_2(s) := \{(x,y) \in \mathbb{R}_*^2 : X(s,x,\alpha) > R, X(s,y,\alpha) \leq R\}$ and $S_3(s) := \{(x,y) \in \mathbb{R}_*^2 : X(s,x,\alpha) > R, X(s,y,\alpha) > R\}$, we have

$$\begin{aligned}
&\langle \mathcal{T}_2[F,\alpha](t), \varphi\rangle \leq \\
&\leq \frac{1}{2}\int_0^t \iint_{S_1(s)} e^{\frac{1+\gamma}{1-\gamma}s} K_R(X(s,x,\alpha),X(s,y,\alpha))\varphi(\ell_\alpha(x,y,s))F(s,\mathrm{d}x)F(s,\mathrm{d}y)\mathrm{d}s \\
&+ \frac{1}{2}\int_0^t \iint_{S_2(s)} e^{\frac{1+\gamma}{1-\gamma}s} K_R(X(s,x,\alpha),X(s,y,\alpha))\varphi(\ell_\alpha(x,y,s))F(s,\mathrm{d}x)F(s,\mathrm{d}y)\mathrm{d}s \\
&+ \frac{1}{2}\int_0^t \iint_{S_3(s)} e^{\frac{1+\gamma}{1-\gamma}s} K_R(X(s,x,\alpha),X(s,y,\alpha))\varphi(\ell_\alpha(x,y,s))F(s,\mathrm{d}x)F(s,\mathrm{d}y)\mathrm{d}s.
\end{aligned}$$

For x,y such that $X(s,x,\alpha) \geq R$ or $X(s,y,\alpha) \geq R$, we have that $K_R(X(s,x,\alpha),X(s,y,\alpha)) = 0$. Thus, the second and the third terms above are equal to zero.

To see that the first term is equal to zero, notice that $X(t,x,\alpha) \geq xe^{-\frac{2}{1-\gamma}t}$. We can thus select T sufficiently small so that $X(t,x,\alpha) \geq \frac{x}{2}$. This implies that $\frac{1}{2}\ell_\alpha(x,y,s) \leq X(s,\ell_\alpha(x,y,s),\alpha) = X(s,x,\alpha) + X(s,y,\alpha) \leq 2R$. Hence $\ell_\alpha(x,y,s) \leq 4R$. Since $\varphi(x) = 0$ for every $x \leq 4R$, the desired conclusion follows.

We now prove that $k_1 \leq \mathcal{A}[\mathcal{T}[F,\alpha],\alpha](t) \leq k_2$. To this end notice that

$$\begin{aligned}
\left|\mathcal{A}[\mathcal{T}[F,\alpha],\alpha](t) - \int_{\mathbb{R}_*} x^{\gamma+\lambda}\Phi_0(\mathrm{d}x)\right| &\leq \int_{\mathbb{R}_*}\left|e^{\frac{1+\gamma}{1-\gamma}t}(X(t,x,\alpha))^{\gamma+\lambda} - x^{\gamma+\lambda}\right|\mathcal{T}[F,\alpha](t,\mathrm{d}x) \\
&+ \int_{\mathbb{R}_*} x^{\gamma+\lambda}\left|\mathcal{T}[F,\alpha](t,\mathrm{d}x) - \Phi_0(\mathrm{d}x)\right|.
\end{aligned}$$

Notice that

$$\begin{aligned}
\left|e^{\frac{1+\gamma}{1-\gamma}t}(X(t,x,\alpha))^{\gamma+\lambda} - x^{\gamma+\lambda}\right| &\leq e^{\frac{1+\gamma}{1-\gamma}t}\left|(X(t,x,\alpha))^{\gamma+\lambda} - x^{\gamma+\lambda}\right| + \frac{1+\gamma}{1-\gamma}Tx^{\gamma+\lambda} \\
&\leq TC(T,R,k_2).
\end{aligned}$$

Finally, we have that the term

$$\int_{\mathbb{R}_*} x^{\gamma+\lambda}\left|\mathcal{T}[F,\alpha](t,\mathrm{d}x) - \Phi_0(\mathrm{d}x)\right|$$

can be arbitrarily small by taking T small. This is due to the definition of the map $T[F,\alpha]$ and the upper bound for F in the definition of the space X_T.

We deduce that, for small time T, we have that $\left|\mathcal{A}[F,\alpha](t) - \int_{\mathbb{R}_*} x^{\gamma+\lambda}\Phi_0(\mathrm{d}x)\right| \leq C(T)$, for every $t \in [0,T]$, where the constant $C(T)$ can be made arbitrarily small.

We now check that the map $\mathcal{F}$ is a contraction. To this end, we use the fact that $|e^{-x_1} - e^{-x_2}| \leq |x_1 - x_2|$, for $x_1, x_2 \geq 0$. Consider $(F,\alpha),(G,\beta) \in X_T$. Then, for every Lipschitz function φ with $\|\varphi\|_{\mathrm{Lip}} \leq 1$, we have that

$$\begin{aligned}\left|\int_{\mathbb{R}_*} \varphi(x)[\mathcal{T}_1[F,\alpha] - \mathcal{T}_1[G,\beta]](t,\mathrm{d}x)\right| &\leq \int_{\mathbb{R}_*}\int_0^t \left|\varphi(x)\left[a[F,\alpha](s,x) - a[G,\beta](s,x)\right]\right| \mathrm{d}s\Phi_0(\mathrm{d}x)\\ &\leq \int_{\mathbb{R}_*}\int_0^t \left|\varphi(x)\left[\int_{\mathbb{R}_*} K_R(X(s,x,\alpha),X(s,y,\alpha))F(s,\mathrm{d}y)\right.\right.\\ &\left.\left.- \int_{\mathbb{R}_*} K_R(X(s,x,\beta),X(s,y,\beta))G(s,\mathrm{d}y)\right]\mathrm{d}s\Phi_0(\mathrm{d}x)\right|\\ &\leq I_1[\varphi] + I_2[\varphi],\end{aligned}$$

where

$$\begin{aligned}I_1[\varphi] := \int_{\mathbb{R}_*}\int_0^t \Bigg|\varphi(x)\int_{\mathbb{R}_*}&[K_R(X(s,x,\alpha),X(s,y,\alpha))\\ &- K_R(X(s,x,\beta),X(s,y,\beta))]F(s,\mathrm{d}y)\Bigg|\mathrm{d}s\Phi_0(\mathrm{d}x)\end{aligned}$$

and

$$I_2[\varphi] := \int_{\mathbb{R}_*}\int_0^t \left|\varphi(x)\int_{\mathbb{R}_*} K_R(X(s,x,\beta),X(s,y,\beta))\,(F(s,\mathrm{d}y) - G(s,\mathrm{d}y))\,\mathrm{d}s\Phi_0(\mathrm{d}x)\right|.$$

Then, the differentiability of the kernel K_R, together with inequality (C.5.8), implies that

$$\sup_{\|\varphi\|_{\mathrm{Lip}\leq 1}} I_1[\varphi] \leq C(R)T\|\Phi_0\|_{TV}(1 + \|\Phi_0\|_{TV})\|\alpha - \beta\|_{[0,T]}.$$

Similarly,

$$\sup_{\|\varphi\|_{\mathrm{Lip}\leq 1}} I_2[\varphi] \leq C(R)T\|\Phi_0\|_{TV} d_T(F,G).$$

Using again the differentiability of the kernel K_R, inequality (C.5.8) and the definition of ℓ_α, we obtain that

$$\begin{aligned}\sup_{\|\varphi\|_{\mathrm{Lip}\leq 1}}\left|\int_{\mathbb{R}_*} \varphi(x)[\mathcal{T}_2[F,\alpha](t,\mathrm{d}x) - \mathcal{T}_2[G,\beta](t,\mathrm{d}x)]\right| &\leq C(T,R)(1 + \|\Phi_0\|_{TV})^2 d_T(F,G)\\ &+ C(T,R)(1 + \|\Phi_0\|_{TV})^2\|\alpha - \beta\|_{[0,T]},\end{aligned}$$

where T can be selected so that $C(T,R)(1 + \|\Phi_0\|_{TV})^2 \leq \frac{1}{4}$.

We remark that, in order to obtain this bound, we used the properties of the Wasserstain distance. Similar estimates cannot be obtained by endowing $\mathcal{M}_+(\mathbb{R}_*)$ with the total variation norm.

To understand this better, let $\alpha,\beta \in C([0,T])$, with $k_1 \leq \alpha,\beta \leq k_2$. Let $s \in [0,T]$. Using the fact that $\|\varphi\|_{\text{Lip}} \leq 1$ and that we can assume we work on a sufficiently nice compact set due to the support of the measures, we can derive the following estimate

$$\begin{aligned}|\varphi(\ell_\alpha(x,y,s)) - \varphi(\ell_\beta(x,y,s))| &\leq |\ell_\alpha(x,y,s) - \ell_\beta(x,y,s)|\\ &\lesssim |X(s,\ell_\alpha(x,y,s),\alpha) - X(s,\ell_\beta(x,y,s),\alpha)|\\ &\leq |X(s,\ell_\alpha(x,y,s),\alpha) - X(s,\ell_\beta(x,y,s),\beta)|\\ &+ |X(s,\ell_\beta(x,y,s),\beta) - X(s,\ell_\beta(x,y,s),\alpha)|.\end{aligned}$$

Using (C.5.8) and (C.5.9), we have

$$\begin{aligned}|X(s,\ell_\alpha(x,y,s),\alpha) - X(s,\ell_\beta(x,y,s),\beta)| &\leq |X(s,x,\alpha) - X(s,x,\beta)|\\ &+ |X(s,y,\alpha) - X(s,y,\beta)|\\ &\leq 2TL_2\|\alpha-\beta\|_{[0,T]}\end{aligned}$$

and thus

$$|\varphi(\ell_\alpha(x,y,s)) - \varphi(\ell_\beta(x,y,s))| \lesssim T\|\alpha-\beta\|_{[0,T]}. \tag{C.5.13}$$

More precisely, inequality (C.5.13) was needed to prove the contractivity of the map $\mathcal{T}_2$.

In order to prove the upper bound for the map $T[F,\alpha]$, we first observe that

$$\langle \mathcal{T}_1[F,\alpha](t), 1\rangle = \int_{[\rho(k_2),\infty)} e^{-\int_0^t a[F,\alpha](s,x)\mathrm{d}s}\Phi_0(\mathrm{d}x) \leq \|\Phi_0\|_{TV}$$

and then that

$$\begin{aligned}\langle \mathcal{T}_2[F,\alpha](t), 1\rangle &\leq \frac{C(R)}{2}e^{\frac{1+\gamma}{1-\gamma}t}\int_0^t\int_{[\rho(k_2),\infty)}\int_{[\rho(k_2),\infty)} F(s,\mathrm{d}x)F(s,\mathrm{d}y)\mathrm{d}s\\ &\leq \frac{C(R)}{2}e^{\frac{1+\gamma}{1-\gamma}T}T(1+\|\Phi_0\|_{TV})^2 \leq 1,\end{aligned}$$

for T sufficiently small. Thus,

$$\sup_{t\in[0,T]}\langle \mathcal{T}[F,\alpha](t), 1\rangle \leq 1 + \|\Phi_0\|_{TV}.$$

Finally,

$$\begin{aligned}\|\mathcal{A}[\mathcal{T}[F,\alpha],\alpha] - \mathcal{A}[\mathcal{T}[G,\beta],\beta]\|_{[0,T]} &\leq C(R)d_T([\mathcal{T}[F,\alpha],\mathcal{T}[G,\beta])\\ &+ C_1(T,R)T(1+\|\Phi_0\|_{TV})\|\alpha-\beta\|_{[0,T]}\\ &\leq C(T,R)T(1+\|\Phi_0\|_{TV})^2[d_T(F,G) + \|\alpha-\beta\|_{[0,T]}],\end{aligned}$$

where the constant $TC(T,R)(1+\|\Phi_0\|_{TV})^2$ can be made smaller than $\frac{1}{4}$ by selecting T small. The operator $\mathcal{F}$ is therefore a contraction and, by Banach fixed point theorem, we conclude that it has a unique fixed point $(F,\alpha) \in X_T$. □

Proof of Theorem C.5.3. Assume $T > 0$ to be as in Lemma C.5.5. Let $(F,\alpha) \in C([0,T],\mathcal{M}_+(\mathbb{R}_*)) \times C([0,T])$ be the solution of (C.5.10), (C.5.11). Let $X(t,y,\alpha)$ be the solution of equation (C.5.7). Let $\Phi \in C^1([0,T],\mathcal{M}_+(\mathbb{R}_*))$ be the function defined by duality as

$$\int_{\mathbb{R}_*} \varphi(x)\Phi(t,\mathrm{d}x) = \int_{\mathbb{R}_*} \varphi(X(t,y,\alpha)) e^{\frac{1+\gamma}{1-\gamma}t} F(t,\mathrm{d}y), \tag{C.5.14}$$

for every $\varphi \in C_c(\mathbb{R}_*)$. We prove that the function Φ is such that

$$\Phi(t,(0,\rho(k_2)) \cup (4R,\infty)) = 0, \quad \forall t \in [0,T], \tag{C.5.15}$$

for $\rho(k_2)$ defined as in (C.4.2). To this end notice that, since $F(t,(0,\rho(k_2)) = 0$, for every $t \in [0,T]$, and since $X(t,y,\alpha) \geq \rho(k_2)$, for every $y \geq \rho(k_2)$, we have that

$$\int_{\mathbb{R}_*} \varphi(x)\Phi(t,\mathrm{d}x) = \int_{[\rho(k_2),\infty)} \varphi(X(t,y,\alpha)) e^{\frac{1+\gamma}{1-\gamma}t} F(t,\mathrm{d}y) = 0,$$

for every test function φ such that $\varphi(x) = 0$ if $x \geq \rho(k_2)$. Similarly, since $t \mapsto X(t,x,\alpha)$ is a decreasing function for large values of x, we deduce that $X(t,x,\alpha) \leq 4R$, for every $x \leq 4R$. Hence,

$$\int_{\mathbb{R}_*} \varphi(x)\Phi(t,\mathrm{d}x) = \int_{[\rho(k_2),4R]} \varphi(X(t,y,\alpha)) e^{\frac{1+\gamma}{1-\gamma}t} F(t,\mathrm{d}y) = 0,$$

for every test function φ such that $\varphi(x) = 0$ if $x \leq 4R$.

We now prove that Φ satisfies (C.5.4). By its definition as the fixed point of the operator $\mathcal{F}$, we deduce that $F \in C^1([0,T],\mathcal{M}_+(\mathbb{R}_*))$. Differentiating both sides of the equality (C.5.14) in time, we deduce that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}_*} \varphi(x)\Phi(t,\mathrm{d}x) = &\int_{\mathbb{R}_*} \varphi(X(t,y,\alpha)) e^{\frac{1+\gamma}{1-\gamma}t} \partial_t F(t,\mathrm{d}y)\\ &+ \int_{\mathbb{R}_*} \varphi'(X(t,y,\alpha))\Big[\frac{(X(t,y,\alpha))^{\gamma+\lambda}}{\alpha(t)} - \frac{2}{1-\gamma}X(t,y,\alpha)\Big] e^{\frac{1+\gamma}{1-\gamma}t} F(t,\mathrm{d}y)\\ &+ \frac{1+\gamma}{1-\gamma}\int_{\mathbb{R}_*} \varphi(X(t,y,\alpha)) e^{\frac{1+\gamma}{1-\gamma}t} F(t,\mathrm{d}y).\end{aligned}$$

Using the fact that F satisfies equation (C.5.10), we deduce that

$$\begin{aligned}&\int_{\mathbb{R}_*} \varphi(X(t,y,\alpha)) e^{\frac{1+\gamma}{1-\gamma}t} \partial_t F(t,\mathrm{d}y)\\ &\qquad = \frac{1}{2}\int_{\mathbb{R}_*}\int_{\mathbb{R}_*} e^{\frac{2(1+\gamma)}{1-\gamma}t} F(t,\mathrm{d}x)F(t,\mathrm{d}y) K_R(X(t,x,\alpha),X(t,y,\alpha))\\ &\qquad \times [\varphi(X(t,x,\alpha)+X(t,y,\alpha)) - \varphi(X(t,y,\alpha)) - \varphi(X(t,x,\alpha))].\end{aligned}$$

Hence, using again the equality (C.5.14), we have that

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}_*}\varphi(x)\Phi(t,\mathrm{d}x) &= \int_{\mathbb{R}_*}\varphi'(X(t,y,\alpha))\left[\frac{(X(t,y,\alpha))^{\gamma+\lambda}}{\alpha(t)}-\frac{2}{1-\gamma}X(t,y,\alpha)\right]e^{\frac{1+\gamma}{1-\gamma}t}F(t,\mathrm{d}y)\\
&+\frac{1+\gamma}{1-\gamma}\int_{\mathbb{R}_*}\varphi(X(t,y,\alpha))e^{\frac{1+\gamma}{1-\gamma}t}F(t,\mathrm{d}y)\\
&+\frac{1}{2}\int_{\mathbb{R}_*}\int_{\mathbb{R}_*}K_R(X(t,x,\alpha),X(t,y,\alpha))e^{\frac{2(1+\gamma)}{1-\gamma}t}F(t,\mathrm{d}x)F(t,\mathrm{d}y)\\
&\cdot\left[\varphi(X(t,x,\alpha)+X(t,y,\alpha))-\varphi(X(t,y,\alpha))-\varphi(X(t,x,\alpha))\right]\\
&=\int_{\mathbb{R}_*}\varphi'(y)\left[\frac{y^{\gamma+\lambda}}{M_{\gamma+\lambda}(\Phi(t))}-\frac{2}{1-\gamma}y\right]\Phi(t,\mathrm{d}y)+\frac{1+\gamma}{1-\gamma}\int_{\mathbb{R}_*}\varphi(y)\Phi(t,\mathrm{d}y)\\
&+\frac{1}{2}\int_{\mathbb{R}_*}\int_{\mathbb{R}_*}K_R(x,y)\left[\varphi(x+y)-\varphi(y)-\varphi(x)\right]\Phi(t,\mathrm{d}x)\Phi(t,\mathrm{d}y).
\end{aligned}$$

We conclude that Φ satisfies (C.5.4). □

C.5.2 Existence of a stationary solution for the truncated equation

Theorem C.5.6. *Assume K_R to be a truncated kernel as in (C.5.3) defined as a function of the homogeneous symmetric kernel K that satisfies (C.1.5), (C.1.6), for parameters $\gamma, \lambda \in \mathbb{R}$ such that (C.1.7) holds and such that $\gamma + 2\lambda \geq 1$, $\gamma > -1$. There exists $\overline{R} > 0$ such that, for every truncation parameter $R > \overline{R}$, there exists a $\Phi \in \mathcal{M}_+(\mathbb{R}_*)$ satisfying the equation*

$$\begin{aligned}
&\int_{\mathbb{R}_*}\varphi'(x)\left[\frac{2}{1-\gamma}x-\frac{x^{\gamma+\lambda}}{\int_{\mathbb{R}_*}z^{\gamma+\lambda}\Phi(\mathrm{d}z)}\right]\Phi(\mathrm{d}x)-\frac{1+\gamma}{1-\gamma}\int_{\mathbb{R}_*}\varphi(x)\Phi(\mathrm{d}x)\\
&=\frac{1}{2}\int_{\mathbb{R}_*}\int_{\mathbb{R}_*}K_R(x,y)\left[\varphi(x+y)-\varphi(x)-\varphi(y)\right]\Phi(\mathrm{d}x)\Phi(\mathrm{d}y),
\end{aligned}\tag{C.5.16}$$

for every $\varphi \in C^1_c(\mathbb{R}_*)$. The solution is such that

$$\int_{\mathbb{R}_*}x\Phi(\mathrm{d}x)=1,\quad \int_{\mathbb{R}_*}x^{\gamma+\lambda}\Phi(\mathrm{d}x)\leq C_1\quad\text{and}\quad\int_{\mathbb{R}_*}x^{2-\gamma-\lambda}\Phi(\mathrm{d}x)\leq C_2,\tag{C.5.17}$$

for some constants $C_1, C_2 > 0$ that do not depend on the truncation R. Additionally, Φ is such that $\Phi((0,\rho(C_1))) = 0$ where $\rho(C_1)$ is given by (C.4.2).

To prove the theorem we introduce the semigroup $\{S(t)\}_{\{t\geq 0\}}$ defined as $S(0)\Phi_0 = \Phi_0$ and $S(t)\Phi_0 = \Phi(t,\cdot)$, where Φ is the solution of equation (C.5.4) constructed in the previous section. We want to apply Tychonoff fixed point theorem to prove the existence of a stationary solution for equation (C.5.1). To this end we need to find an invariant region for $S(t)$ (Proposition C.5.7) and prove the continuity in the weak-* topology of the map $\Phi \mapsto S(t)\Phi$ for every time t (Proposition C.5.12). In this way we prove that for every time t there exists a fixed point $\hat{\Phi}_t$ such that $S(t)\hat{\Phi}_t = \hat{\Phi}_t$. We will then prove that S is continuous (Proposition C.5.8) in time and conclude that the limit as t tends to zero of $\hat{\Phi}_t$ is a solution of equation (C.5.16). This is a standard method used in the study of coagulation equations to prove existence of self-similar profiles. For more details, see [EMR05, Theorem 1.2].

Proposition C.5.7 (Invariant region)**.** *Assume K_R to be a truncated kernel as in (C.5.3) defined as a function of the homogeneous symmetric kernel K that satisfies (C.1.5), (C.1.6), for parameters $\gamma, \lambda \in \mathbb{R}$ such that (C.1.7) holds and such that $\gamma + 2\lambda \geq 1$, $\gamma > -1$. Then there exist some positive constants C_1, C_2 that do not depend on the truncation R such that the set*

$$P := \left\{ H \in \mathcal{M}_+(\mathbb{R}_*) : \begin{array}{l} M_1(H) = 1,\ H((0, \rho(C_1)) \cup (4R, \infty)) = 0, \\ \dfrac{1}{C_2} \leq M_{\gamma+\lambda}(H) \leq C_1 \end{array} \right\} \tag{C.5.18}$$

is invariant under the evolution operator $S(t)$, where $\rho(C_1)$ was defined in (C.4.2).

Proof. Let Φ_0 and K_R be as in Lemma C.5.5. Let T be as in Lemma C.5.5. We prove that there exist two constants $C_1, C_2 > 0$ that do not depend on the truncation parameter R such that the solution Φ obtained in Theorem C.5.3 is such that for every time $t \in [0, T]$

(1) $\int_{\mathbb{R}_*} x\Phi(t, \mathrm{d}x) = 1$;

(2) $\int_{\mathbb{R}_*} x^{\gamma+\lambda}\Phi(t, \mathrm{d}x) \leq \max\{C_1, \int_{\mathbb{R}_*} x^{\gamma+\lambda}\Phi_0(\mathrm{d}x)\}$;

(3) $\int_{\mathbb{R}_*} x^{2-\gamma-\lambda}\Phi(t, \mathrm{d}x) \leq \max\{C_2, \int_{\mathbb{R}_*} x^{2-\gamma-\lambda}\Phi_0(\mathrm{d}x)\}$.

Additionally, we prove that we can conclude from (2) that $\Phi(t, (0, \rho(C_1)) = 0$, for every $t \in [0, T]$.

We first notice that, since $\Phi(t, \cdot)$ has compact support, we can consider in equation (C.5.4) test functions φ such that $\varphi(x) = x^k$, for $k \in \mathbb{R}$.

Let us prove (1). We consider in equation (C.5.4) a test function φ such that $\varphi(x) = x$. We deduce that

$$\frac{d}{dt}\int_0^\infty \xi\Phi(t, \mathrm{d}\xi) = 1 - \int_0^\infty \xi\Phi(t, \mathrm{d}\xi).$$

Hence

$$\int_0^\infty \xi\Phi(t, \xi)d\xi = 1 + \left(\int_{\mathbb{R}_*} \xi\Phi_0(\mathrm{d}\xi) - 1\right)e^{-t}$$

which since $\int_{\mathbb{R}_*} \xi\Phi_0(\mathrm{d}\xi) = 1$ implies

$$\int_{\mathbb{R}_*} \xi\Phi(t, \mathrm{d}\xi) = 1. \tag{C.5.19}$$

We prove (2). We start by proving this for $\gamma + 2\lambda > 1$. Consider a test function φ such that $\varphi(x) = x^{\gamma+\lambda}$ in equation (C.5.4). Then, we can see that the moment $M_{\gamma+\lambda}$ satisfies the following

$$\begin{aligned}\partial_t M_{\gamma+\lambda} \leq & -\frac{2\lambda + \gamma - 1}{1-\gamma}M_{\gamma+\lambda} + \frac{(\gamma+\lambda)}{M_{\gamma+\lambda}}M_{2(\gamma+\lambda)-1} \\ & + c_3 \int_{[\rho(2C_1), \frac{R}{4}]}\int_{[\rho(2C_1), \frac{R}{4}]} (x^{\gamma+\lambda}y^{-\lambda} + x^{-\lambda}y^{\gamma+\lambda})[(x+y)^{\gamma+\lambda} - x^{\gamma+\lambda} - y^{\gamma+\lambda}]\Phi(t, \mathrm{d}x)\Phi(t, \mathrm{d}y)\end{aligned}$$

since, due to the definition of K_R in (C.5.3), we have that on the set $[\rho(2C_1), \frac{R}{4}]^2$, there exists a constant $C > 0$ such that $K_R \geq \frac{K}{C}$.

We denote by $z := \frac{y}{x}$. Notice that since $\gamma + \lambda > 0$ when $y \leq x$ we have that

$$\begin{aligned}\left[x^{\gamma+\lambda}y^{-\lambda} + x^{-\lambda}y^{\gamma+\lambda}\right]\left[(x+y)^{\gamma+\lambda} - x^{\gamma+\lambda} - y^{\gamma+\lambda}\right] &\leq 2x^{2(\gamma+\lambda)}y^{-\lambda}[(1+z)^{\gamma+\lambda} - 1 - z^{\gamma+\lambda}] \\ &\leq 2x^{2(\gamma+\lambda)}y^{-\lambda}\left((\gamma+\lambda)\, z - z^{\gamma+\lambda}\right) \\ &\leq 2(\gamma+\lambda-1)y^{\gamma}x^{\gamma+\lambda}.\end{aligned}$$

As a consequence, by symmetry, we deduce that

$$\begin{aligned}&\partial_t M_{\gamma+\lambda} \lesssim -\frac{2\lambda+\gamma-1}{1-\gamma}M_{\gamma+\lambda} + \frac{(\gamma+\lambda)}{M_{\gamma+\lambda}}M_{2(\gamma+\lambda)-1} - 2c_3\,(1-\gamma-\lambda)\,M_\gamma M_{\gamma+\lambda} \\ &+ 2c_3(1-\gamma-\lambda)\int_{(\frac{R}{4},\infty)} x^{\gamma}\Phi(\mathrm{d}x)M_{\gamma+\lambda} + 2c_3(1-\gamma-\lambda)\int_{(\frac{R}{4},\infty)} x^{\gamma+\lambda}\Phi(\mathrm{d}x)M_\gamma \\ &\leq -\frac{2\lambda+\gamma-1}{1-\gamma}M_{\gamma+\lambda} + \frac{(\gamma+\lambda)}{M_{\gamma+\lambda}}M_{2(\gamma+\lambda)-1} - 2c_3\,(1-\gamma-\lambda)\,M_\gamma M_{\gamma+\lambda} \\ &+ 4^{1-\gamma}R^{\gamma-1}2c_3(1-\gamma-\lambda)M_{\gamma+\lambda} + c_4M_\gamma \\ &= -[\frac{2\lambda+\gamma-1}{1-\gamma} - \tilde{\varepsilon}]M_{\gamma+\lambda} + \frac{1}{M_{\gamma+\lambda}}[(\gamma+\lambda)M_{2(\gamma+\lambda)-1} - 2c_3\,(1-\gamma-\lambda)\,M_\gamma M_{\gamma+\lambda}^2 \\ &+ c_4M_{\gamma+\lambda}M_\gamma],\end{aligned} \tag{C.5.20}$$

with $\tilde{\varepsilon} := 4^{1-\gamma}R^{\gamma-1}2c_3(1-\gamma-\lambda)$. Notice we can take $\tilde{R}$ sufficiently large so that $-\frac{2\lambda+\gamma-1}{1-\gamma} + \tilde{\varepsilon} < 0$, for every $R \geq \tilde{R}$.

Since $\gamma < 2(\gamma+\lambda) - 1 \leq 1$ we deduce, by interpolation, using that $M_1(\Phi(t)) = 1$, for all $t \in [0, T]$, that there exists two positive constants c_5 and c_6 such that

$$M_{2(\gamma+\lambda)-1} \leq c_5M_\gamma + c_6.$$

Hence, since $\gamma + \lambda > 0$ then

$$\begin{aligned}\partial_t M_{\gamma+\lambda} \leq -[\frac{2\lambda+\gamma-1}{1-\gamma} - \tilde{\varepsilon}]M_{\gamma+\lambda} + \frac{1}{M_{\gamma+\lambda}}[(\gamma+\lambda)\left[c_5M_\gamma + c_6\right] \\ - 2c_3\,(1-\gamma-\lambda)\,M_\gamma M_{\gamma+\lambda}^2 + c_4M_{\gamma+\lambda}M_\gamma].\end{aligned}$$

Multiplying by $M_{\gamma+\lambda}$ the inequality we deduce that

$$\begin{aligned}M_{\gamma+\lambda}\partial_t M_{\gamma+\lambda} \leq -[\frac{2\lambda+\gamma-1}{1-\gamma} - \tilde{\varepsilon}]M_{\gamma+\lambda}^2 + [(\gamma+\lambda)\left[c_5M_\gamma + c_6\right] \\ - 2c_3\,(1-\gamma-\lambda)\,M_\gamma M_{\gamma+\lambda}^2 + c_4M_{\gamma+\lambda}M_\gamma]\end{aligned}$$

which readjusting the constants implies

$$\partial_t M_{\gamma+\lambda}^2 \leq -c_3M_{\gamma+\lambda}^2 + M_\gamma\left(c_4 + c_5M_{\gamma+\lambda} - c_6M_{\gamma+\lambda}^2\right) + c_7 \tag{C.5.21}$$

for $c_3, c_4, c_5, c_6, c_7 > 0$. This implies that the set $\left\{\Phi \in \mathcal{M}_+(\mathbb{R}_*) : M_{\gamma+\lambda} \leq \frac{c_5 + \sqrt{c_5^2 + 4c_4 c_6}}{2c_6}\right\}$ is invariant when $\gamma + 2\lambda > 1$.

We consider now the case $\gamma + 2\lambda = 1$. First of all notice that for every $M < R$ we have

$$\int_{[M,\infty)} x^{\gamma+\lambda}\Phi(t, \mathrm{d}x) \leq M^{\gamma+\lambda-1} \int_{[M,\infty)} x\Phi(t, \mathrm{d}x) \leq M^{\gamma+\lambda-1}.$$

Notice that an upper bound is obvious for the points $t \in [0, T]$ for which $M_{\gamma+\lambda}(\Phi(t)) \leq 1$. If there exist $t_1, t_2 \in [0, T]$, $t_1 < t_2$, such that $M_{\gamma+\lambda}(\Phi(t_1)) < 1$ and $M_{\gamma+\lambda}(\Phi(t_2)) > 1$, by the continuity in time of Φ, we have that there exists $\bar{t} \in [t_1, t_2]$ such that $M_{\gamma+\lambda}(\Phi(\bar{t})) = 1$ and $\varepsilon_1 \in (0, 1)$ such that $M_{\gamma+\lambda}(\Phi(s)) \geq 1$ on $[\bar{t}, \bar{t} + \varepsilon_1]$.

On the interval $[\bar{t}, \bar{t} + \varepsilon_1]$, we can apply the following logic.

We select R large enough so that we can select M to be such that $(1-\delta)^{\frac{1}{1-\gamma-\lambda}} < M < R$, but independent on R, and we deduce that there exists a $\delta > 0$ such that

$$\begin{aligned}\int_{(0,M)} x^{\gamma+\lambda}\Phi(t, \mathrm{d}x) &= \int_{\mathbb{R}_*} x^{\gamma+\lambda}\Phi(t, \mathrm{d}x) - \int_{[M,\infty)} x^{\gamma+\lambda}\Phi(t, \mathrm{d}x)\\ &\geq \int_{\mathbb{R}_*} x^{\gamma+\lambda}\Phi(t, \mathrm{d}x) - M^{\gamma+\lambda-1} \geq 1 - M^{\gamma+\lambda-1} \geq \delta.\end{aligned}$$

As a consequence we deduce that

$$M_\gamma \geq \int_{(0,M)} x^{\gamma}\Phi(t, \mathrm{d}x) \geq M^{-\lambda} \int_{(0,M)} x^{\gamma+\lambda}\Phi(t, \mathrm{d}x) \geq \delta M^{-\lambda}. \tag{C.5.22}$$

Substituting the test function $\varphi(x) = x^{\gamma+\lambda}$ in equation (C.5.4), we deduce that there exists a constant $c > 0$ such that

$$\partial_t M_{\gamma+\lambda} \leq \frac{\gamma+\lambda}{M_{\gamma+\lambda}} M_{2(\gamma+\lambda)-1} - c(1-\gamma-\lambda)M_{\gamma+\lambda}M_\gamma + \tilde{\varepsilon}M_{\gamma+\lambda} + cM_\gamma,$$

for $\tilde{\varepsilon} \in (0, 1)$, which can be made sufficiently small as before. Hence, since for suitable constants $c_3, c_4 > 0$ we have that $M_{2(\gamma+\lambda)-1} \leq c_3 M_\gamma + c_4$, similarly to (C.5.20), we deduce that

$$\frac{1}{2}\partial_t M_{\gamma+\lambda}^2 \leq c_5 M_\gamma(\gamma+\lambda) + (\gamma+\lambda)c_6 - c_7 M_{\gamma+\lambda}^2 M_\gamma + \tilde{\varepsilon}M_{\gamma+\lambda}^2 + c_8 M_\gamma M_{\gamma+\lambda}.$$

Using (C.5.22) we deduce that there exists a constant $c_9 > 0$ such that

$$\frac{1}{2}\partial_t M_{\gamma+\lambda}^2 \leq M_\gamma\left[c_5(\gamma+\lambda) + (\gamma+\lambda)c_9 + c_8 M_{\gamma+\lambda} - (c_7 - \tilde{\varepsilon}\delta^{-1}M^{\lambda})M_{\gamma+\lambda}^2\right].$$

Choosing now $\tilde{\varepsilon}$ sufficiently small such that $-c_7 + \tilde{\varepsilon}\delta^{-1}M^{\lambda} < 0$, we have that there exists a constant $C_1 > 0$ such that

$$\int_{\mathbb{R}_*} x^{\gamma+\lambda}\Phi(t, \mathrm{d}x) \leq C_1. \tag{C.5.23}$$

We now prove that $(S(t)\Phi_0)((0,\rho(C_1))) = 0$. Notice that, by Lemma C.5.5, we know that $(S(t)\Phi_0)((0,\rho(2C_1))) = 0$, where we recall that $\rho(2C_1) \leq \rho(C_1)$. Consider now a sequence of test functions $\{\varphi_n\}$ such that, for every $n \in \mathbb{N}$, φ_n is decreasing and supported in $(0,\rho(C_1))$. Since φ_n is decreasing, we deduce that the solution F of equation (C.5.10) satisfies

$$\int_{\mathbb{R}_*} \varphi_n(x)F(t,\mathrm{d}x) \leq \int_{\mathbb{R}_*} \varphi_n(x)\Phi_0(\mathrm{d}x) + C(R)\int_0^t \int_{\mathbb{R}_*} \varphi_n(x)F(s,\mathrm{d}x)\mathrm{d}s, \qquad \text{for every } n \in \mathbb{N},$$

where we used in addition that $M_1(F)$ is bounded. Using Grönwall's inequality, we deduce that

$$\int_{\mathbb{R}_*} \varphi_n(x)F(t,\mathrm{d}x) = 0,$$

for every $t \in [0,T]$ and $n \in \mathbb{N}$.

We then let φ_n converge pontwise to $\chi_{(0,\rho(C_1))}$ and obtain that

$$F(t,(0,\rho(C_1))) = 0,$$

for every $t \in [0,T]$.

Consider now a test function φ such that $\varphi(x) = 0$ for every $x \geq \rho(C_1)$ in equality (C.5.14). Then

$$\int_{\mathbb{R}_*} \varphi(x)\Phi(t,\mathrm{d}x) = e^{\frac{1+\gamma}{1-\gamma}t}\int_{[\rho(C_1),\infty)} \varphi(X(t,y,\alpha))F(t,\mathrm{d}y).$$

Using the fact that $X(t,y,\alpha) \geq \rho(C_1)$ for every $y \geq \rho(C_1)$, we deduce that $\Phi(t,(0,\rho(C_1))) = 0$, for every $t \in [0,T]$.

We conclude by proving (3). We consider a test function φ equal to $x^{2-\gamma-\lambda}$ in equation (C.5.4) and deduce that

$$\begin{aligned}\partial_t \int_{\mathbb{R}_*} \xi^{2-\gamma-\lambda}\Phi(t,\mathrm{d}\xi) - \frac{3-3\gamma-2\lambda}{1-\gamma}\int_{\mathbb{R}_*} \xi^{2-\gamma-\lambda}\Phi(t,\mathrm{d}\xi) - \frac{2-\gamma-\lambda}{M_{\gamma+\lambda}(\Phi(t))}M_1(\Phi(t)) \\ = \frac{1}{2}\int_{[\rho(C_1),\infty)}\int_{[\rho(C_1),\infty)} K_R(\xi,z)\left[(z+\xi)^{2-\gamma-\lambda} - \xi^{2-\gamma-\lambda} - z^{2-\gamma-\lambda}\right]\Phi(t,\mathrm{d}\xi)\Phi(t,\mathrm{d}z).\end{aligned} \tag{C.5.24}$$

By Cauchy-Schwarz inequality it follows that

$$\frac{1}{\int_{[\rho(C_1),\infty)} \xi^{\gamma+\lambda}\Phi(t,\mathrm{d}\xi)} \leq \frac{\int_{[\rho(C_1),\infty)} \xi^{2-\gamma-\lambda}\Phi(t,\mathrm{d}\xi)}{\left(\int_{[\rho(C_1),\infty)} \xi\Phi(t,\mathrm{d}\xi)\right)^2} = \int_{[\rho(C_1),\infty)} \xi^{2-\gamma-\lambda}\Phi(t,\mathrm{d}\xi). \tag{C.5.25}$$

Plugging (C.5.25) into equation (C.5.24) and using the fact that the total mass is equal to 1 proven in (C.5.19), we deduce that

$$\begin{aligned}\partial_t \int_{[\rho(C_1),\infty)} \xi^{2-\gamma-\lambda}\Phi(t,\mathrm{d}\xi) \leq \left(\frac{3\gamma-3+2\lambda}{1-\gamma} + 2 - \gamma - \lambda\right)\int_{[\rho(C_1),\infty)} \xi^{2-\gamma-\lambda}\Phi(t,\mathrm{d}\xi) \\ + \frac{1}{2}\int_{[\rho(C_1),\infty)}\int_{[\rho(C_1),\infty)} K_R(\xi,z)\left[(z+\xi)^{2-\gamma-\lambda} - \xi^{2-\gamma-\lambda} - z^{2-\gamma-\lambda}\right]\Phi(t,\mathrm{d}\xi)\Phi(t,\mathrm{d}z).\end{aligned}$$

Hence, since $-1 < \gamma < 1$ and $0 < \gamma + \lambda < 1$, we have

$$\frac{3\gamma - 3 + 2\lambda}{1-\gamma} + 2 - \gamma - \lambda = -\frac{(\gamma+1)(1-\gamma-\lambda)}{1-\gamma} < 0.$$

By symmetry,

$$\begin{aligned}&\frac{1}{2}\int_{[\rho(C_1),\infty)}\int_{[\rho(C_1),\infty)} K_R(\xi,z)\left[(z+\xi)^{2-\gamma-\lambda} - \xi^{2-\gamma-\lambda} - z^{2-\gamma-\lambda}\right]\Phi(t,\mathrm{d}\xi)\Phi(t,\mathrm{d}z)\\ &\leq \int_{[\rho(C_1),\infty)}\int_{[\rho(C_1),z]} K_R(\xi,z)\left[(z+\xi)^{2-\gamma-\lambda} - \xi^{2-\gamma-\lambda} - z^{2-\gamma-\lambda}\right]\Phi(t,\mathrm{d}\xi)\Phi(t,\mathrm{d}z).\end{aligned}$$

Assume $\xi \leq z$. Denote $\eta := \frac{\xi}{z} \in (0,1]$ and observe

$$\begin{aligned}K_R(\xi,z)[(\xi+z)^{2-\gamma-\lambda} - \xi^{2-\gamma-\lambda} - z^{2-\gamma-\lambda}] \leq& K(\xi,z)z^{2-\gamma-\lambda}[(1+\eta)^{2-\gamma-\lambda} - 1 - \eta^{2-\gamma-\lambda}]\\ \leq& CK(\xi,z)z^{2-\gamma-\lambda}\eta\\ \leq& Cz^{\gamma}(\eta^{\gamma+\lambda}+\eta^{-\lambda})z^{2-\gamma-\lambda}\eta \leq 2Cz^{2-\lambda}\eta^{1-\lambda}\\ \leq& 4C(z\xi^{1-\lambda} + \xi z^{1-\lambda}).\end{aligned} \tag{C.5.26}$$

Since $\rho(C_1) \leq \xi$ and $\rho(C_1) \leq z$, then $z^{1-\lambda} \leq \rho(C_1)^{-\lambda}z$. Hence

$$\frac{d}{dt}\int_{\mathbb{R}_*} \xi^{2-\gamma-\lambda}\Phi(t,\mathrm{d}\xi) \leq -c_3\int_{\mathbb{R}_*} \xi^{2-\gamma-\lambda}\Phi(t,\mathrm{d}\xi) + c(\rho(C_1)),$$

for suitable constants $c_3, c(\rho(C_1)) > 0$. Then, (3) follows. □

Proposition C.5.8 (Time-continuity of the semigroup)**.** *Assume K_R to be a truncated kernel as in* (C.5.3) *defined as a function of the homogeneous symmetric kernel K that satisfies* (C.1.5), (C.1.6)*, for parameters $\gamma, \lambda \in \mathbb{R}$ such that* (C.1.7) *holds and such that $\gamma > -1$ and $\gamma + 2\lambda \geq 1$. Let $\Phi_0 \in P$. Let $T > 0$ be as in Theorem C.5.3. The map $S(\cdot)\Phi_0 : [0,T] \to P$ is continuous in time, for every fixed Φ_0, where P was defined in (C.5.18).*

Proof. Let $T > 0$. We want to estimate the value of $|S(t)\Phi_0 - S(s)\Phi_0|$, for $s, t \in [0,T]$. Assume without loss of generality that $s \leq t$. By the definition of the operator S we know that

$$\begin{aligned}&\int_{\mathbb{R}_*} \varphi(x)[\Phi(t,\mathrm{d}x) - \Phi(s,\mathrm{d}x)] - \frac{1+\gamma}{1-\gamma}\int_s^t\int_{\mathbb{R}_*}\varphi(x)\Phi(z,\mathrm{d}x)\mathrm{d}z\\ &+ \frac{2}{1-\gamma}\int_s^t\int_{\mathbb{R}_*}\varphi'(x)x\Phi(z,\mathrm{d}x)\mathrm{d}z - \int_s^t \frac{1}{\int_{\mathbb{R}_*} x^{\gamma+\lambda}\Phi(z,\mathrm{d}x)}\int_{\mathbb{R}_*}\varphi'(x)x^{\gamma+\lambda}\Phi(z,\mathrm{d}x)\mathrm{d}z\\ &= \frac{1}{2}\int_s^t\int_{\mathbb{R}_*}\int_{\mathbb{R}_*} K_R(x,y)\left[\varphi(x+y) - \varphi(x) - \varphi(y)\right]\Phi(z,\mathrm{d}x)\Phi(z,\mathrm{d}y)\mathrm{d}z.\end{aligned}$$

We have that there exists a constant $C_2 > 0$ such that $M_{\gamma+\lambda}(\Phi(r)) \geq \frac{1}{C_2}$, for every $r \in [0,T]$.

As Φ has compact support, we can conclude that the exists a constant $C > 0$, which may depend on $\rho(C_1), R, T, \gamma, \lambda$, but independent on s, t, such that

$$\left| \int_{\mathbb{R}_*} \varphi(x)[\Phi(t, \mathrm{d}x) - \Phi(s, \mathrm{d}x)] \right| \leq C|t-s|,$$

thus giving us the desired continuity of the semigroup. □

Lemma C.5.9 (Dual equation)**.** *Assume K_R to be a truncated kernel as in (C.5.3) defined as a function of a homogeneous symmetric kernel K that satisfies (C.1.5), (C.1.6), for parameters $\gamma, \lambda \in \mathbb{R}$ satisfying (C.1.7) and with $\gamma > -1$ and $\gamma + 2\lambda \geq 1$. Let $\Phi_1, \Phi_2 \in P$ be two solutions of (C.5.4) with initial conditions $\Phi_{\text{in},1}, \Phi_{\text{in},2} \in P$, respectively, and $T > 0$ be as in Theorem C.5.3. Then there exists a unique solution $\varphi \in \mathrm{C}^1([0,T], \mathrm{C}^1_c([\rho(C_1), \infty))$, with $\varphi(T, \cdot) = \psi(\cdot)$, where ψ is an arbitrary function in $\mathrm{C}^1_c([\rho(C_1), \infty))$, which solves the following equation:*

$$\partial_t \varphi(t,\xi) + \frac{1+\gamma}{1-\gamma}\varphi(t,\xi) - \frac{2\xi}{1-\gamma}\partial_\xi \varphi(t,\xi) + \frac{\xi^{\gamma+\lambda}}{M_{\gamma+\lambda}(\Phi_1)}\partial_\xi \varphi(t,\xi) + \mathbb{L}(\varphi)(t,\xi) = 0, \tag{C.5.27}$$

where

$$\begin{aligned}\mathbb{L}(\varphi)(t,\xi) :=& -\frac{\xi^{\gamma+\lambda}}{M_{\gamma+\lambda}(\Phi_1)M_{\gamma+\lambda}(\Phi_2)}\int_{\mathbb{R}_*} \partial_z \varphi(t,z) z^{\gamma+\lambda}\Phi_2(t,\mathrm{d}z) \\ &+ \frac{1}{2}\int_{\mathbb{R}_*} K_R(\xi,\eta)[\varphi(t,\xi+\eta) - \varphi(t,\xi) - \varphi(t,\eta)](\Phi_1(t,\mathrm{d}\eta) + \Phi_2(t,\mathrm{d}\eta)).\end{aligned}$$

Remark C.5.10. We prove the statement of the lemma for a modified operator $\overline{\mathbb{L}}(\varphi)$ which preserves the continuity in the variable ξ, is equal to zero when $\xi \geq 8R$ and $\overline{\mathbb{L}}(\varphi) = \mathbb{L}(\varphi)$ if $\xi \in [\rho(C_1), 4R]$. Due to the support of Φ_1 and Φ_2, when proving the continuity of the map $S(t)$ in the weak-* topology, it suffices to analyse the operator $\overline{\mathbb{L}}(\varphi)$. Thus, it is enough to prove the statement of the lemma only for the operator $\overline{\mathbb{L}}(\varphi)$. Notice that, for example, the operator $\mathbb{L}(\varphi)$ does not preserve compactness because of the presence of the term $\frac{\xi^{\gamma+\lambda}}{M_{\gamma+\lambda}(\Phi_1)M_{\gamma+\lambda}(\Phi_2)}\int_{\mathbb{R}_*}\partial_z\varphi(t,z)z^{\gamma+\lambda}\Phi_2(t,\mathrm{d}z)$. We keep the notation $\mathbb{L}(\varphi)$ for simplicity. We omit further details as the proof consists of standard methods used in the study of coagulation equations, see, for example, [FLNV21b].

Proof of Lemma C.5.9. First, we use the method of characteristics. We define $X(t,\xi)$ to be the solution of the ODE

$$x'(t) = -\frac{2}{1-\gamma}x + \frac{1}{\int_{\mathbb{R}_*} z^{\gamma+\lambda}\Phi_1(t,\mathrm{d}z)}x^{\gamma+\lambda}$$

with initial condition $x(0) = \xi$.

In this way equation (C.5.27) can be rewritten in the following fixed point form:

$$\varphi(t, X(t,\xi)) = \mathcal{L}[\varphi](t,\xi), \tag{C.5.28}$$

where

$$\mathcal{L}[\varphi](t,\xi) := \varphi(T, X(T,\xi)) + \frac{1+\gamma}{1-\gamma}\int_t^T \varphi(s, X(s,\xi))\mathrm{d}s + \int_t^T \mathbb{L}(\varphi)(s, X(s,\xi))\mathrm{d}s.$$

Our strategy for proving the statement of the lemma is to apply Banach fixed point theorem. The operator $\mathcal{L}$ maps $C^1([0,T], \mathrm{C}_c^1([\rho(C_1),\infty)))$ in itself.

We prove that the operator $\mathcal{L}$ is a contraction if we endow $Y := C([0,T], C_c^1([\rho(C_1),\infty))$ with the norm $\|\varphi\|_Y := \sup_{t\in[0,T]}\left(\sup_{x\in\mathbb{R}_*}|\varphi(t,x)| + \sup_{x\in\mathbb{R}_*}|\partial_x\varphi(t,x)|\right)$.

To this end we notice that

$$\begin{aligned}\mathcal{L}[\varphi_1](t,\xi) - \mathcal{L}[\varphi_2](t,\xi) &= \frac{1+\gamma}{1-\gamma}\int_t^T (\varphi_1(s,X(s,\xi)) - \varphi_2(s,X(s,\xi)))\,\mathrm{d}s \\ &+ \int_t^T [\mathbb{L}(\varphi_1)(s,X(s,\xi)) - \mathbb{L}(\varphi_2)(s,X(s,\xi))]\,\mathrm{d}s.\end{aligned}$$

We notice that

$$\begin{aligned}&\int_t^T [\mathbb{L}(\varphi_1)(s,X(s,\xi)) - \mathbb{L}(\varphi_2)(s,X(s,\xi))]\,\mathrm{d}s \\ &= -\int_t^T \frac{1}{M_{\gamma+\lambda}(\Phi_1)M_{\gamma+\lambda}(\Phi_2)}\int_{\mathbb{R}_*}(\partial_z\varphi_1(s,z) - \partial_z\varphi_2(s,z))\,z^{\gamma+\lambda}\Phi_2(s,\mathrm{d}z)X(s,\xi)^{\gamma+\lambda}\mathrm{d}s \\ &+ \frac{1}{2}\int_t^T\int_{\mathbb{R}_*} K_R(X(s,\xi),\eta)[\varphi_1(s,X(s,\xi)+\eta) - \varphi_2(s,X(s,\xi)+\eta) - \varphi_1(s,X(s,\xi)) \\ &+ \varphi_2(s,X(s,\xi)) - \varphi_1(s,\eta) + \varphi_2(s,\eta)](\Phi_1(s,\mathrm{d}\eta) + \Phi_2(s,\mathrm{d}\eta))\mathrm{d}s.\end{aligned}$$

From this we deduce that

$$\|\mathcal{L}[\varphi_1] - \mathcal{L}[\varphi_2]\|_Y \leq T c(\rho(C_1), R, \Phi_1, \Phi_2)\|\varphi_1 - \varphi_2\|_Y$$

and hence $\mathcal{L}$ is a contraction for sufficiently small times T. We can extend the solution to all possible times noting that the contraction constant $c(\rho(C_1), \Phi_1, \Phi_2, R)$ does not depend on the final condition ψ. We thus deduce that there exists a solution φ of the fixed point $\varphi = \mathcal{L}[\varphi]$. □

We now prove that the found solution is Lipschitz continuous.

Proposition C.5.11. *Assume K_R to be a truncated kernel as in (C.5.3) defined as a function of a homogeneous symmetric kernel K such that it satisfies (C.1.5), (C.1.6), for parameters $\gamma, \lambda \in \mathbb{R}$ satisfying (C.1.7) and such that $\gamma > -1$ and $\gamma + 2\lambda \geq 1$. Let $T > 0$ be as in Theorem C.5.3. Let $\varphi \in \mathrm{C}^1([0,T], \mathrm{C}^1([\rho(C_1), 8R]))$ with initial datum $\varphi(T,\cdot)$ be the function found in Lemma C.5.9. Assume, in addition, that $\sup_{\xi\in[\rho(C_1),8R]}|\varphi(T,\xi)| \leq 1$ and that $\varphi(T,\xi)$ is Lipschitz. Then φ is Lipschitz continuous, in the sense that, for every $t \in [0,T]$, there exists $C(t) > 0$ such that*

$$\sup_{s\in[0,t]}|\varphi(s,\xi) - \varphi(s,\tilde{\xi})| \leq C(t)|\xi - \tilde{\xi}|,$$

for every $\xi, \tilde{\xi} \in [\rho(C_1), 8R]$. *Moreover,* $C(t)$ *may depend on the norm of* Φ_1 *and* Φ_2*, but is otherwise independent of the choice of* Φ_1 *and* Φ_2.

Proof. Notice first that, since $\sup_{\xi\in[\rho(C_1),8R]} |\varphi(T,\xi)| \leq 1$, there exists a constant $C > 0$, which depends on the norm of Φ_1 and Φ_2 and the parameters $\rho(C_1)$, R and t, such that $\sup_{s\in[0,t],\xi\in[\rho(C_1),8R]} |\varphi(s,\xi)| \leq C$. This can be seen by looking at (C.5.28). We will use Grönwall's inequality in (C.5.28) in order to prove that φ is Lipschitz.

$$\begin{aligned}|\varphi(t, X(t,\xi)) - \varphi(t, X(t,\tilde{\xi}))| \leq{}& |\varphi(T, X(T,\xi)) - \varphi(T, X(T,\tilde{\xi}))|\\ &+ \frac{1+\gamma}{1-\gamma}\int_t^T |\varphi(s, X(s,\xi)) - \varphi(s, X(s,\tilde{\xi}))|\mathrm{d}s\\ &+ \int_t^T |\mathbb{L}[\varphi](s, X(s,\xi)) - \mathbb{L}[\varphi](s, X(s,\tilde{\xi}))|\mathrm{d}s.\end{aligned}$$

In order to bound the term

$$\int_t^T |\mathbb{L}[\varphi](s, X(s,\xi)) - \mathbb{L}[\varphi](s, X(s,\tilde{\xi}))|\mathrm{d}s,$$

we need to estimate

$$\begin{aligned}I_1 ={}& \Bigg|\frac{1}{2}\int_{\mathbb{R}_*} K_R(X(t,\xi),\eta)[\varphi(t, X(t,\xi)+\eta) - \varphi(t, X(t,\xi)) - \varphi(t,\eta)](\Phi_1(t,\mathrm{d}\eta) + \Phi_2(t,\mathrm{d}\eta))\\ &- \frac{1}{2}\int_{\mathbb{R}_*} K_R(X(t,\tilde{\xi}),\eta)[\varphi(t, X(t,\tilde{\xi})+\eta) - \varphi(t, X(t,\tilde{\xi})) - \varphi(t,\eta)](\Phi_1(t,\mathrm{d}\eta) + \Phi_2(t,\mathrm{d}\eta))\Bigg|\end{aligned}$$

and

$$I_2 = \frac{|X(t,\xi)^{\gamma+\lambda} - X(t,\tilde{\xi})^{\gamma+\lambda}|}{M_{\gamma+\lambda}(\Phi_1)M_{\gamma+\lambda}(\Phi_2)}\int_{\mathbb{R}_*} |\partial_z\varphi(t,z)|z^{\gamma+\lambda}\Phi_2(t,\mathrm{d}z).$$

For I_1, by the definition of K_R in (C.5.3), we have K_R is C^1 and we can assume it has compact support as we are only interested in the region $[\rho(C_1), 8R]^2$. Thus, we have that the first derivative of K_R is bounded from above. Moreover, there exist constants $L_1(t), L_2(t) > 0$ such that $L_1(t)|\xi - \tilde{\xi}| \leq |X(t,\xi) - X(t,\tilde{\xi})| \leq L_2(t)|\xi - \tilde{\xi}|$.

For I_2, we use

$$\frac{1}{M_{\gamma+\lambda}(\Phi_1)M_{\gamma+\lambda}(\Phi_2)}\int_{\mathbb{R}_*} |\partial_z\varphi(t,z)|z^{\gamma+\lambda}\Phi_2(t,\mathrm{d}z) \leq \frac{C_1}{(C_2)^2}\sup_{z\in[\rho(C_1),8R]} |\partial_z\varphi(t,z)|.$$

If we prove that there exists a constant $C > 0$, which can depend on $\rho(C_1)$, R, and the norms of Φ_1, and Φ_2, but does not vary depending on the choice of Φ_1, Φ_2 such that

$$\sup_{z\in[\rho(C_1),8R]} |\partial_z\varphi(t,z)| \leq C, \tag{C.5.29}$$

then there exists a constant $C > 0$, which can depend on $\rho(C_1)$, R and the norms of Φ_1 and Φ_2 such that

$$|\varphi(t, X(t,\xi)) - \varphi(t, X(t,\tilde{\xi}))| \leq C|X(t,\xi) - X(t,\tilde{\xi})| + C\int_t^T |\varphi(s, X(s,\xi)) - \varphi(s, X(s,\tilde{\xi}))|. \tag{C.5.30}$$

Let us now prove (C.5.29). From equation (C.5.28) and the fact that K_R is in C^1 with compact support, we obtain an upper bound for $|\partial_2\varphi(t, X(t,\xi))|$.

We then use that there exist some constants $\overline{c}(t), \overline{\overline{c}}(t) > 0$ such that $\overline{c}(t)\xi \leq X(t,\xi) \leq \overline{\overline{c}}(t)\xi$, with $\xi \in [\rho(C_1), 8R]$ in order to obtain the desired bound for $\sup_{z\in[\rho(C_1),8R]} |\partial_z\varphi(t,z)|$.

We use Grönwall's inequality in (C.5.30) and obtain that $|\varphi(t, X(t,\xi)) - \varphi(t, X(t,\tilde{\xi}))| \leq C(t)|X(t,\xi) - X(t,\tilde{\xi})|$. Thus, we can conclude that also $|\varphi(t,\xi) - \varphi(t,\tilde{\xi})| \leq C(t)|\xi - \tilde{\xi}|$ since $X(t,\xi)$ is Lipschitz continuous in the ξ variable. □

Proposition C.5.12 (Continuity of the semigroup in the weak topology)**.** *Assume K_R to be a truncated kernel as in* (C.5.3) *defined as a function of a homogeneous symmetric kernel K such that it satisfies* (C.1.5)*,* (C.1.6)*, for parameters $\gamma, \lambda \in \mathbb{R}$ satisfying* (C.1.7) *and such that $\gamma > -1$ and $\gamma + 2\lambda \geq 1$. For every time $t > 0$ the map*

$$S(t) : P \to P$$

is continuous in the weak-* topology, where P was defined in (C.5.18).

Proof. Let $\delta > 0$. In order to prove continuity in the weak-* topology of the semigroup and because of the support of Φ_1 and Φ_2, it is enough to prove that, if for every $\psi \in C_c([\rho(C_1), \infty))$, with $\|\psi\|_\infty \leq 1$, we have that

$$\int_{\mathbb{R}_*} \psi(x)\,(\Phi_{\text{in},1}(\mathrm{d}x) - \Phi_{\text{in},2}(\mathrm{d}x)) \text{ is sufficiently small,} \tag{C.5.31}$$

then we have that for every $\psi \in C_c([\rho(C_1), \infty))$, with $\|\psi\|_\infty \leq 1$,

$$\int_{\mathbb{R}_*} \psi(x)\,(\Phi_1(t,\mathrm{d}x) - \Phi_2(t,\mathrm{d}x)) \leq \delta,$$

where Φ_1 and Φ_2 are the solutions of equation (C.5.4) with initial conditions $\Phi_{\text{in},1}$ and $\Phi_{\text{in},2}$, respectively. To this end we notice that since Φ_1 and Φ_2 satisfy equation (C.5.4) then

$$\begin{aligned}
&\int_{\mathbb{R}_*} \varphi(t,x)[\Phi_1(t,\mathrm{d}x) - \Phi_2(t,\mathrm{d}x)] - \int_{\mathbb{R}_*} \varphi(0,x)[\Phi_{\mathrm{in},1}(\mathrm{d}x) - \Phi_{\mathrm{in},2}(\mathrm{d}x)] \\
&- \int_0^t \int_{\mathbb{R}_*} \partial_s\varphi(s,x)[\Phi_1(s,\mathrm{d}x) - \Phi_2(s,\mathrm{d}x)]\mathrm{d}s - \frac{1+\gamma}{1-\gamma}\int_0^t \int_{\mathbb{R}_*} \varphi(s,x)[\Phi_1(s,\mathrm{d}x) - \Phi_2(s,\mathrm{d}x)]\mathrm{d}s \\
&+ \frac{2}{1-\gamma}\int_0^t \int_{\mathbb{R}_*} \partial_x\varphi(s,x)x[\Phi_1(s,\mathrm{d}x) - \Phi_2(s,\mathrm{d}x)]\mathrm{d}s \\
&- \int_0^t \frac{1}{\int_{\mathbb{R}_*} z^{\gamma+\lambda}\Phi_1(s,\mathrm{d}z)} \int_{\mathbb{R}_*} \partial_x\varphi(s,x)x^{\gamma+\lambda}\Phi_1(s,\mathrm{d}x)\mathrm{d}s \\
&+ \int_0^t \frac{1}{\int_{\mathbb{R}_*} z^{\gamma+\lambda}\Phi_2(s,\mathrm{d}z)} \int_{\mathbb{R}_*} \partial_x\varphi(s,x)x^{\gamma+\lambda}\Phi_2(s,\mathrm{d}x)\mathrm{d}s \\
&= \frac{1}{2}\int_0^t \int_{\mathbb{R}_*}\int_{\mathbb{R}_*} K_R(x,y)\chi_\varphi(s,x,y)[\Phi_1(s,\mathrm{d}y)+\Phi_2(s,\mathrm{d}y)][\Phi_1(s,\mathrm{d}x) - \Phi_2(s,\mathrm{d}x)]\mathrm{d}s,
\end{aligned}$$

where $\chi_\varphi(s,x,y) := \varphi(s,x+y) - \varphi(s,x) - \varphi(s,y)$, for every $\varphi \in C([0,t], C_c^1(\mathbb{R}_*))$. To simplify the computations, we adopt the notation $\tilde{\Phi} := \Phi_1 - \Phi_2$ and we notice that $\tilde{\Phi}$ satisfies the following equation for every $\varphi \in C^1([0,t], C_c^1(\mathbb{R}_*))$

$$\begin{aligned}
&\int_{\mathbb{R}_*} \varphi(t,x)\tilde{\Phi}(t,\mathrm{d}x) - \int_{\mathbb{R}_*} \varphi(0,x)[\Phi_{\mathrm{in},1}(\mathrm{d}x) - \Phi_{\mathrm{in},2}(\mathrm{d}x)] \\
&- \int_0^t \int_{\mathbb{R}_*} \partial_s\varphi(s,x)\tilde{\Phi}(s,\mathrm{d}x)\mathrm{d}s - \frac{1+\gamma}{1-\gamma}\int_0^t \int_{\mathbb{R}_*} \varphi(s,x)\tilde{\Phi}(s,\mathrm{d}x)\mathrm{d}s \\
&+ \frac{2}{1-\gamma}\int_0^t \int_{\mathbb{R}_*} \partial_x\varphi(s,x)x\tilde{\Phi}(s,\mathrm{d}x)\mathrm{d}s - \int_0^t \frac{1}{\int_{\mathbb{R}_*} z^{\gamma+\lambda}\Phi_1(s,\mathrm{d}z)} \int_{\mathbb{R}_*} \partial_x\varphi(s,x)x^{\gamma+\lambda}\tilde{\Phi}(s,\mathrm{d}x)\mathrm{d}s \\
&+ \int_0^t \int_{\mathbb{R}_*} \mathbb{L}[\varphi](s,x)\tilde{\Phi}(s,\mathrm{d}x)\mathrm{d}s.
\end{aligned}$$

First we make the following notation:

$$C_{norm} := \sup_{s\in[0,t]} \int_{\mathbb{R}_*} [\Phi_1 + \Phi_2](s,\mathrm{d}x) < \infty.$$

Notice that C_{norm} can be bounded from above by

$$C_{norm} = \sup_{s\in[0,t]} \int_{\mathbb{R}_*} [\Phi_1 + \Phi_2](s,\mathrm{d}x) \leq C(R,\rho(C_1),t) \int_{\mathbb{R}_*} [\Phi_{\mathrm{in},1} + \Phi_{\mathrm{in},2}](\mathrm{d}x).$$

Let now φ be the solution found in Lemma C.5.9 with coagulation kernel K_R. By Proposition C.5.11, we know that there exists a constant $C(t,R,\Phi_1,\Phi_2)$, that depends only on the norm of Φ_1 and Φ_2, but is otherwise independent of the choice of Φ_1 and Φ_2, such that $|\varphi(s,\xi) - \varphi(s,\tilde{\xi})| \leq C(t,R,\Phi_1,\Phi_2)|\xi - \tilde{\xi}|$, for every $\xi, \tilde{\xi} \in [\rho(C_1), 8R]$ and $s \in [0,t]$. We thus look at the set:

$$\begin{aligned}
\mathcal{K}_{\rho(C_1),R} := &\{\chi \in \mathrm{C}([\rho(C_1), 8R]) \,|\, |\chi(\xi) - \chi(\tilde{\xi})| \leq C(t,R,\Phi_1,\Phi_2)|\xi - \tilde{\xi}|, \\
&\text{for all } \xi, \tilde{\xi} \in [\rho(C_1), 8R]\} \subset \mathrm{C}([\rho(C_1), 8R]).
\end{aligned}$$

The set $\mathcal{K}_{\rho(C_1),R}$ is totally bounded, thus there exist $N \in \mathbb{N}$ and $\chi_1, \dots, \chi_N \in \mathcal{K}_{\rho(C_1),R}$ such that $\mathcal{K}_{\rho(C_1),R} \subseteq \cup_{i=1}^N B(\chi_i, \frac{\delta}{4NC_{norm}})$.

Then we obtain that:

$$\int_{\mathbb{R}_*} \varphi(t,x)\tilde{\Phi}(t,\mathrm{d}x) = \int_{\mathbb{R}_*} \varphi(0,x)[\Phi_{\mathrm{in},1}(\mathrm{d}x) - \Phi_{\mathrm{in},2}(\mathrm{d}x)] =: T_1. \tag{C.5.32}$$

We can bound T_1 by

$$\begin{aligned} T_1 &\leq \min_{i=1}^N \int_{\mathbb{R}_*} |[\varphi(0,x) - \chi_i(x)][\Phi_{\mathrm{in},1}(\mathrm{d}x) - \Phi_{\mathrm{in},2}(\mathrm{d}x)]| \\ &+ \max_{i=1}^N \Big(\int_{\mathbb{R}_*} \chi_i(x)[\Phi_{\mathrm{in},1}(\mathrm{d}x) - \Phi_{\mathrm{in},2}(\mathrm{d}x)]\Big) \\ &\leq \frac{\delta}{4C_{norm}} \int_{\mathbb{R}_*} [\Phi_{\mathrm{in},1}(\mathrm{d}x) + \Phi_{\mathrm{in},2}(\mathrm{d}x)] + \max_{i=1}^N \int_{\mathbb{R}_*} \chi_i(x)[\Phi_{\mathrm{in},1}(\mathrm{d}x) - \Phi_{\mathrm{in},2}(\mathrm{d}x)] \leq \delta, \end{aligned}$$

where in the last step we used (C.5.31). □

Proof of Theorem C.5.6. By Proposition C.5.12 we know that the operator $\Phi \mapsto S(t)\Phi$ is continuous in the weak-* topology. Additionally, thanks to Proposition C.5.7, we know that P is an invariant region for $S(t)$. Since P is also convex and compact and since we have proven that the map $t \mapsto S(t)$ is continuous, we apply Theorem 1.2 in [EMR05] to deduce that there exists a Φ such that $S(t)\Phi = \Phi$. □

C.5.3 Existence of the self-similar profile

To keep the notation lighter, in the previous sections we denoted with Φ the solution of the truncated problem. In this section, since we pass to the limit as R tends to infinity, we add the label R to Φ and we will denote with Φ the self-similar profile. For simplicity, we denote by $\rho := \rho(C_1)$, where $\rho(C_1)$ was defined in (C.4.2).

Proof of Theorem C.5.1. For every $R > 1$, sufficiently large, there exist $T > 0$ and $\Phi_R \in C^1([0,T]; \mathcal{M}_+(\mathbb{R}_*))$ such that

$$\begin{aligned} &\int_{\mathbb{R}_*} \varphi(x)\dot{\Phi}_R(t,\mathrm{d}x) - \frac{1+\gamma}{1-\gamma}\int_{\mathbb{R}_*} \varphi(x)\Phi_R(t,\mathrm{d}x) + \frac{2}{1-\gamma}\int_{\mathbb{R}_*} \varphi'(x)x\Phi_R(t,\mathrm{d}x) \\ &- \frac{1}{M_{\gamma+\lambda}(\Phi_R(t))}\int_{\mathbb{R}_*} \varphi'(x)x^{\gamma+\lambda}\Phi_R(t,\mathrm{d}x) \\ &= \frac{1}{2}\int_{\mathbb{R}_*}\int_{\mathbb{R}_*} K_R(x,y)\,[\varphi(x+y) - \varphi(x) - \varphi(y)]\,\Phi_R(t,\mathrm{d}x)\Phi_R(t,\mathrm{d}y), \end{aligned}$$

where K_R is the coagulation kernel defined in (C.5.3). Notice that the bounds in Proposition C.5.7 are independent of $R > 1$.

Thanks to Theorem C.5.6 we know that a measure $\Phi_R \in \mathcal{M}_+(\mathbb{R}_*)$ satisfying equation (C.5.16) and satisfying the bounds (C.5.17) exists. We notice also that, since $\gamma + \lambda > 0$,

$$\int_{(0,\infty)} \Phi_R(\mathrm{d}x) = \int_{[\rho,\infty)} \Phi_R(\mathrm{d}x) \leq \rho^{-\gamma-\lambda} \int_{[\rho,\infty)} x^{\gamma+\lambda}\Phi_R(\mathrm{d}x) \leq \rho^{-\gamma-\lambda} C_1. \tag{C.5.33}$$

Hence, Banach-Alaoglu Theorem implies that there exists Φ such that

$$\Phi_R \rightharpoonup \Phi \text{ as } R \to \infty \tag{C.5.34}$$

in the weak-* topology.

We now prove that the measure Φ in (C.5.34) satisfies equation (C.3.3) by taking the limit as $R \to \infty$. Fix $\varphi \in C_c(\mathbb{R}_*)$. We start with passing to the limit in the coagulation term.

Let $\varepsilon \in (0,1)$. We first show that

$$\left| \int_{[\rho,\infty)} \int_{[\rho,\infty)} (K(\xi,z) - K_R(\xi,z))\, \Phi_R(\mathrm{d}\xi)\Phi_R(\mathrm{d}z)[\varphi(z+\xi) - \varphi(z) - \varphi(\xi)] \right| \leq \varepsilon, \tag{C.5.35}$$

for sufficiently large R. We then prove that

$$\int_{[\rho,\infty)} \int_{[\rho,\infty)} K(\xi,z)\Phi_R(\mathrm{d}\xi)\Phi_R(\mathrm{d}z)[\varphi(z+\xi) - \varphi(z) - \varphi(\xi)] \tag{C.5.36}$$

converges to

$$\int_{[\rho,\infty)} \int_{[\rho,\infty)} K(\xi,z)\Phi(\mathrm{d}\xi)\Phi(\mathrm{d}z)[\varphi(z+\xi) - \varphi(z) - \varphi(\xi)] \tag{C.5.37}$$

as $R \to \infty$.

For (C.5.35), we have that:

$$\begin{aligned}
&\left| \int_{[\rho,\infty)} \int_{[\rho,\infty)} (K(\xi,z) - K_R(\xi,z))\, \Phi_R(\mathrm{d}\xi)\Phi_R(\mathrm{d}z)[\varphi(z+\xi) - \varphi(z) - \varphi(\xi)] \right| \\
&\leq \int_{[\rho,\frac{R}{4}]} \int_{[\rho,\frac{R}{4}]} \left| (K(\xi,z) - K_R(\xi,z))\, \Phi_R(\mathrm{d}\xi)\Phi_R(\mathrm{d}z)[\varphi(z+\xi) - \varphi(z) - \varphi(\xi)] \right| \\
&+ 2 \int_{(\frac{R}{4},\infty)} \int_{[\rho,\infty)} \left| (K(\xi,z) - K_R(\xi,z))\, \Phi_R(\mathrm{d}\xi)\Phi_R(\mathrm{d}z)[\varphi(z+\xi) - \varphi(z) - \varphi(\xi)] \right| \\
&= T_1 + T_2.
\end{aligned}$$

For the first term, we have

$$T_1 \leq e^{-R} 3\|\varphi\|_\infty \rho^{-2},$$

where we used the definition of K_R in (C.5.3) and the fact that the total mass of the measures is equal to 1.

For the region $(\frac{R}{4}, \infty) \times [\rho, \infty)$, we use $K_R \leq K$ to prove that

$$\begin{aligned} T_2 &\leq 12\|\varphi\|_\infty \int_{(\frac{R}{4},\infty)} \int_{[\rho,\infty)} K(\xi, z)\Phi_R(\mathrm{d}z)\Phi_R(\mathrm{d}\xi) \\ &\leq c\left(R^{-\gamma-2\lambda} M^2_{\gamma+\lambda}(\Phi_R) + R^{\gamma+\lambda-1}\rho^{-\gamma-2\lambda} M_{\gamma+\lambda}(\Phi_R)\right), \end{aligned}$$

which gives a small contribution due to the uniform estimates for Φ_R if we make R sufficiently large.

We now analyse (C.5.36). We consider a continuous function $g : \mathbb{R}_+ \to [0, 1]$ such that $g(x) = 1$, when $x \leq 1$, and $g(x) = 0$, when $x \geq 2$. We define the function p_M as

$$p_M(x, y) = g\left(\frac{x}{M}\right) g\left(\frac{y}{M}\right), \tag{C.5.38}$$

where M is a positive constant. By the construction of p_M and given the fact that Φ_R is supported in the region $[\rho, \infty)$, for every $R > 1$, we have that, given any function $\varphi \in C_c(\mathbb{R}_*)$,

$$\begin{aligned} &\int_{[\rho,\infty)} \int_{[\rho,\infty)} K(\xi, z)p_M(\xi, z)\Phi_R(\mathrm{d}\xi)\Phi_R(\mathrm{d}z)[\varphi(z + \xi) - \varphi(z) - \varphi(\xi)] = \\ &\int_{[\rho,2M]} \int_{[\rho,2M]} K(\xi, z)p_M(\xi, z)\Phi_R(\mathrm{d}\xi)\Phi_R(\mathrm{d}z)[\varphi(z + \xi) - \varphi(z) - \varphi(\xi)]. \end{aligned}$$

Therefore, since $\Phi_R \rightharpoonup \Phi$ in the weak-* topology, we can conclude that $\Phi_R\Phi_R \rightharpoonup \Phi\Phi$ in the weak-* topology as $R \to \infty$ if we work on the set $[\rho, 2M]^2$ and we deduce that

$$\int_{[\rho,\infty)} \int_{[\rho,\infty)} K(\xi, z)p_M(\xi, z)\Phi_R(\mathrm{d}\xi)\Phi_R(\mathrm{d}z)[\varphi(z + \xi) - \varphi(z) - \varphi(\xi)]$$

converges to

$$\int_{[\rho,\infty)} \int_{[\rho,\infty)} K(\xi, z)p_M(\xi, z)\Phi(\mathrm{d}\xi)\Phi(\mathrm{d}z)[\varphi(z + \xi) - \varphi(z) - \varphi(\xi)]$$

as R tends to infinity.

To conclude that, for every test function $\varphi \in C_c(\mathbb{R}_*)$, we have that (C.5.36) converges to (C.5.37) as $R \to \infty$, we have to prove that the reminder terms, namely

$$\int_{[\rho,\infty)} \int_{[M,\infty)} K(\xi, z)[\varphi(z + \xi) - \varphi(z) - \varphi(\xi)]\Phi_R(\mathrm{d}\xi)\Phi_R(\mathrm{d}z), \tag{C.5.39}$$

$$\int_{[M,\infty)} \int_{[\rho,\infty)} K(\xi, z)[\varphi(z + \xi) - \varphi(z) - \varphi(\xi)]\Phi_R(\mathrm{d}\xi)\Phi_R(\mathrm{d}z), \tag{C.5.40}$$

$$\int_{[\rho,\infty)} \int_{[M,\infty)} K(\xi, z)[\varphi(z + \xi) - \varphi(z) - \varphi(\xi)]\Phi(\mathrm{d}\xi)\Phi(\mathrm{d}z), \tag{C.5.41}$$

$$\int_{[M,\infty)} \int_{[\rho,\infty)} K(\xi, z)[\varphi(z + \xi) - \varphi(z) - \varphi(\xi)]\Phi(\mathrm{d}\xi)\Phi(\mathrm{d}z), \tag{C.5.42}$$

tend to zero as $M \to \infty$.

Let us look at the term (C.5.39). In this case we are in the region where $\{\xi \geq M\}$, hence $\xi^{\gamma+\lambda-1} \leq M^{\gamma+\lambda-1}$ as $\gamma + \lambda < 1$. Therefore, for every $\varphi \in \mathrm{C}_c(\mathbb{R}_{>0})$

$$\int_{[\rho,\infty)} \int_{[M,\infty)} \xi^{\gamma+\lambda} z^{-\lambda} \Phi_R(\mathrm{d}\xi)\Phi_R(\mathrm{d}z)|\varphi(z+\xi) - \varphi(z) - \varphi(\xi)| \lesssim M^{\gamma+\lambda-1} \rho^{-\gamma-2\lambda} M_{\gamma+\lambda}(\Phi_R),$$

where we remind that $M_{\gamma+\lambda}(\Phi_R)$ is bounded uniformly in $R > 1$ and that the mass of Φ_R is equal to one. Similarly, the fact that $\xi \geq M$ implies that $\xi^{-\gamma-2\lambda} \leq M^{-\gamma-2\lambda}$ since $\gamma + 2\lambda > 0$. We then obtain that

$$\int_{[\rho,\infty)} \int_{[M,\infty)} z^{\gamma+\lambda} \xi^{-\lambda} \Phi_R(\mathrm{d}\xi)\Phi_R(\mathrm{d}z)|\varphi(z+\xi) - \varphi(z) - \varphi(\xi)| \leq C M^{-\gamma-2\lambda} M^2_{\gamma+\lambda}(\Phi_R).$$

From these two inequalities and the fact that $\gamma + \lambda < 1$ and $\gamma + 2\lambda > 0$ we deduce that the term (C.5.39) tends to zero as $M \to \infty$. By a symmetric argument we prove that the term (C.5.40) tends to zero as $M \to \infty$. The fact that the two terms (C.5.41) and (C.5.42) tend to zero as $M \to \infty$ follows similarly by the fact that the $\gamma + \lambda$ moment of Φ is bounded.

The linear terms for the self-similar profiles of the truncated problem will converge as $R \to \infty$ to the desired terms thanks to (C.5.34).

To conclude, we need to prove the convergence of the remainder term. We thus have to prove that

$$\frac{1}{\int_{[\rho,\infty)} x^{\gamma+\lambda} \Phi_R(\mathrm{d}x)} \int_{[\rho,\infty)} \partial_x \varphi(x) x^{\gamma+\lambda} \Phi_R(\mathrm{d}x)$$

converges to

$$\frac{1}{\int_{[\rho,\infty)} x^{\gamma+\lambda} \Phi(\mathrm{d}x)} \int_{[\rho,\infty)} \partial_x \varphi(x) x^{\gamma+\lambda} \Phi(\mathrm{d}x)$$

as $R \to \infty$. We have that

$$\begin{aligned}
&\left| \frac{1}{M_{\gamma+\lambda}(\Phi_R)} \int_{[\rho,\infty)} \partial_x \varphi(x) x^{\gamma+\lambda} \Phi_R(\mathrm{d}x) - \frac{1}{M_{\gamma+\lambda}(\Phi)} \int_{[\rho,\infty)} \partial_x \varphi(x) x^{\gamma+\lambda} \Phi(\mathrm{d}x) \right| \\
&\leq \left| \frac{1}{\int_{[\rho,\infty)} x^{\gamma+\lambda} \Phi_R(\mathrm{d}x)} \right| \left| \int_{[\rho,\infty)} \partial_x \varphi(x) x^{\gamma+\lambda} \Phi_R(\mathrm{d}x) - \int_{[\rho,\infty)} \partial_x \varphi(x) x^{\gamma+\lambda} \Phi(\mathrm{d}x) \right| \\
&+ \left| \frac{1}{\int_{[\rho,\infty)} x^{\gamma+\lambda} \Phi_R(\mathrm{d}x)} - \frac{1}{\int_{[\rho,\infty)} x^{\gamma+\lambda} \Phi(\mathrm{d}x)} \right| \left| \int_{[\rho,\infty)} \partial_x \varphi(x) x^{\gamma+\lambda} \Phi(\mathrm{d}x) \right|.
\end{aligned}$$

We can analyse the two terms separately. For the first term, we know by (C.5.25) that the $M_{\gamma+\lambda}(\Phi_R)$ moment is uniformly bounded from below and thus

$$\begin{aligned}
&\left| \frac{1}{\int_{[\rho,\infty)} x^{\gamma+\lambda} \Phi_R(\mathrm{d}x)} \right| \left| \int_{[\rho,\infty)} \partial_x \varphi(x) x^{\gamma+\lambda} \Phi_R(\mathrm{d}x) - \int_{[\rho,\infty)} \partial_x \varphi(x) x^{\gamma+\lambda} \Phi(\mathrm{d}x) \right| \\
&\leq M_{2-\gamma-\lambda}(\Phi_R) \left| \int_{[\rho,\infty)} \partial_x \varphi(x) x^{\gamma+\lambda} \Phi_R(\mathrm{d}x) - \int_{[\rho,\infty)} \partial_x \varphi(x) x^{\gamma+\lambda} \Phi(\mathrm{d}x) \right| \\
&\leq C_2 \left| \int_{[\rho,\infty)} \partial_x \varphi(x) x^{\gamma+\lambda} \Phi_R(\mathrm{d}x) - \int_{[\rho,\infty)} \partial_x \varphi(x) x^{\gamma+\lambda} \Phi(\mathrm{d}x) \right|
\end{aligned}$$

and then we use that $\left| \int_{[\rho,\infty)} \partial_x \varphi(x) x^{\gamma+\lambda} \Phi_R(\mathrm{d}x) - \int_{[\rho,\infty)} \partial_x \varphi(x) x^{\gamma+\lambda} \Phi(\mathrm{d}x) \right| \to 0$ by (C.5.34).

For the second term, we split the integral into a region with compact support and the remainder regions. More precisely,

$$
\begin{aligned}
&\left| \frac{\int_{[\rho,\infty)} x^{\gamma+\lambda}\Phi(\mathrm{d}x) - \int_{[\rho,\infty)} x^{\gamma+\lambda}\Phi_R(\mathrm{d}x)}{\int_{[\rho,\infty)} x^{\gamma+\lambda}\Phi_R(\mathrm{d}x) \int_{[\rho,\infty)} x^{\gamma+\lambda}\Phi(\mathrm{d}x)} \right| \left| \int_{[\rho,\infty)} \partial_x \varphi(x) x^{\gamma+\lambda}\Phi(\mathrm{d}x) \right| \\
&\leq C \left| \int_{[\rho,\infty)} x^{\gamma+\lambda}\Phi(\mathrm{d}x) - \int_{[\rho,\infty)} x^{\gamma+\lambda}\Phi_R(\mathrm{d}x) \right| M_{2-\gamma-\lambda}(\Phi_R) M_{2-\gamma-\lambda}(\Phi) M_{\gamma+\lambda}(\Phi) \\
&\lesssim C \left| \int_{[\rho,\infty)} g\left(\frac{x}{M}\right) \left[x^{\gamma+\lambda}\Phi(\mathrm{d}x) - x^{\gamma+\lambda}\Phi_R(\mathrm{d}x) \right] \right| + 2C \left| \int_{[M,\infty)} x^{\gamma+\lambda}\Phi(\mathrm{d}x) \right| \\
&+ 2C \left| \int_{[M,\infty)} x^{\gamma+\lambda}\Phi_R(\mathrm{d}x) \right|.
\end{aligned}
$$

We have that $\left| \int_{[\rho,\infty)} g(\frac{x}{M})[x^{\gamma+\lambda}\Phi(\mathrm{d}x) - x^{\gamma+\lambda}\Phi_R(\mathrm{d}x)] \right| \to 0$ as $R \to \infty$ by (C.5.34). For the regions close to infinity, we use

$$
\left| \int_{[M,\infty)} x^{\gamma+\lambda}\Phi_R(\mathrm{d}x) \right| \leq M^{\gamma+\lambda-1},
$$

as the total mass is uniformly bounded. Since $\gamma + \lambda < 1$, the remaining term goes to zero as $M \to \infty$.

As a last step we prove that, if $\Phi((0,\delta)) = 0$ for some $\delta > 0$, then $\Phi((0, \rho(M_{\gamma+\lambda}))) = 0$ for $\rho(M_{\gamma+\lambda})$ given by (C.3.4). Since the proof is standard we only sketch it here. Consider $x \in [\frac{\delta}{2}, \frac{1}{2}\rho(M_{\gamma+\lambda}))$, then, since Φ satisfies (C.3.1), we have

$$
\begin{aligned}
c\Phi(x) &\leq \left(\frac{x^{\gamma+\lambda}}{M_{\gamma+\lambda}} - \frac{2}{1-\gamma} x \right) \Phi(x) \\
&\leq \frac{|1+\gamma|}{1-\gamma} \int_\delta^x \Phi(\eta)\mathrm{d}\eta + \int_\delta^x \int_\delta^{y-\delta} K(y-\eta,\eta)\Phi(y-\eta)\Phi(\eta)\mathrm{d}\eta\mathrm{d}y \\
&\leq \tilde{c} \int_\delta^x \Phi(\eta)\mathrm{d}\eta,
\end{aligned}
$$

for some positive constants $\tilde{c}, c > 0$. Using Grönwall's lemma, this implies that $\Phi(x) \leq e^{\frac{\tilde{c}}{c}x}\Phi(\frac{\delta}{2}) = 0$. Hence $\Phi([\delta, \frac{1}{2}\rho(M_{\gamma+\lambda})) = 0$. Iterating this argument, we deduce that $\Phi((0, \rho(M_{\gamma+\lambda}))) = 0$. The argument can be made rigorous by working with the weak formulation of equation (C.3.1). □

C.5.4 Properties of the constructed self-similar profile

In this section we aim at proving Theorem C.3.3. In particular we prove that the self-similar profile, whose existence has been proven in Theorem C.5.1, satisfies the properties stated in Theorem C.3.3. In other words we have to prove the following theorem.

Theorem C.5.13. *Assume K to be a homogeneous symmetric coagulation kernel, of homogeneity γ, satisfying (C.1.5), (C.1.6), with γ, λ such that (C.1.7) holds and such that*

$$-1 < \gamma, \quad \gamma + 2\lambda \geq 1.$$

Let Φ be the self-similar profile constructed in Theorem C.5.1. Then

$$\int_{\mathbb{R}_*} e^{Lx}\Phi(\mathrm{d}x) < \infty$$

for some $L > 0$ and Φ is absolutely continuous with respect to the Lebesgue measure. Then $\Phi(\mathrm{d}x) = \phi(x)\mathrm{d}x$ and ϕ is such that

$$\limsup_{x\to\infty} e^{\overline{L}x}\phi(x) < \infty$$

for a constant $\overline{L} > 0$.

To prove this theorem we start by proving that every self-similar profile Φ in the sense of Definition C.3.1 and such that $\Phi((0,\delta)) = 0$, for a positive δ, is absolutely continuous with respect to the Lebesgue measure and satisfies some moment bounds.

Proposition C.5.14 (Regularity)**.** *Assume K to be a homogeneous symmetric coagulation kernel, of homogeneity γ, satisfying (C.1.5), (C.1.6), with γ, λ such that (C.1.7) holds and such that $\gamma+2\lambda \geq 1$. Let $\Phi \in \mathcal{M}_+(\mathbb{R}_*)$ be a self-similar profile as in Definition C.3.1. Assume additionally that $\Phi((0,\delta)) = 0$ for $\delta > 0$. Then Φ is absolutely continuous with respect to the Lebesgue measure. Its density $\phi \in L^1([\delta,\infty))$ satisfies the following equation*

$$\begin{aligned}&\int_0^z x\phi(x)dx - \frac{2}{1-\gamma}z^2\phi(z) + \frac{1}{M_{\gamma+\lambda}(\phi)}z^{\gamma+\lambda+1}\phi(z) - \frac{1}{M_{\gamma+\lambda}(\phi)}\int_0^z x^{\gamma+\lambda}\phi(x)\mathrm{d}x \\ &= -J_\phi(z), \quad a.e.\ z > \delta\end{aligned} \tag{C.5.43}$$

where

$$J_\phi(z) := \int_0^z \int_{z-x}^\infty xK(x,y)\phi(x)\phi(y)\mathrm{d}y\mathrm{d}x. \tag{C.5.44}$$

Proof. We just sketch the proof as similar arguments have been repeatedly used in the analysis of coagulation equations, see for instance the proof of Lemma 4.9 in [FFV23]. An analogous proof will be presented in the proof of Proposition C.6.2. Let Φ be a solution of (C.3.1), then

$$\partial_\xi\left[\left(\frac{\xi^{\gamma+\lambda}}{M_{\gamma+\lambda}} - \frac{2}{1-\gamma}\xi\right)\Phi(\xi)\right] = \frac{1+\gamma}{1-\gamma}\Phi(\xi) + \mathbb{K}[\Phi](\xi).$$

Using the fact that

$$\mathbb{K}[\Phi](\xi) \leq \int_\delta^{\xi-\delta} K(\xi-\eta,\eta)\Phi(\eta)\Phi(\xi-\eta)\mathrm{d}\eta, \quad \xi > \delta$$

and that $\int_{\delta}^{\xi-\delta} K(\xi-\eta,\eta)\Phi(\eta)\Phi(\xi-\eta)\mathrm{d}\eta$ is a bounded measure in every compact set, we deduce that the measure

$$\left(\frac{\xi^{\gamma+\lambda}}{M_{\gamma+\lambda}} - \frac{2}{1-\gamma}\xi\right)\Phi(\xi)$$

is absolutely continuous with respect to the Lebesgue measure on the set (δ,∞) if $\delta > \rho(M_{\gamma+\lambda})$, while it is absolutely continuous with respect to the Lebesgue measure in the set $(\delta,\rho(M_{\gamma+\lambda}))\cup(\rho(M_{\gamma+\lambda}),\infty)$ if $\delta \leq \rho(M_{\gamma+\lambda})$. In the latter case, we therefore have that $\Phi(\mathrm{d}x) = \phi(x)\mathrm{d}x + c_0\delta_{\rho(M_{\gamma+\lambda})}(x)$ with density $\phi \in L^1(\mathbb{R}_*)$ and with $c_0 \in \mathbb{R}$. To exclude that the self-similar solution Φ has a Dirac in $\rho(M_{\gamma+\lambda})$, we notice that

$$\begin{aligned}&\int_{\delta}^{\xi-\delta} K(\xi-\eta,\eta)c_0\delta_{\rho(M_{\gamma+\lambda})}(\eta)c_0\delta_{\rho(M_{\gamma+\lambda})}(\xi-\eta)\mathrm{d}\eta\\ &= c_0^2\int_{\delta}^{\xi-\delta} K(\xi-\eta,\eta)\delta_{\rho(M_{\gamma+\lambda})}(\eta)\delta_{\rho(M_{\gamma+\lambda})}(\xi-\eta)\mathrm{d}\eta\end{aligned}$$

is non-zero only if $\xi = 2\rho(M_{\gamma+\lambda})$. Hence the coagulation operator applied to measure $c_0\delta_{\rho(M_{\gamma+\lambda})}$ produces a Dirac in $2\rho(M_{\gamma+\lambda})$, contradicting the fact that $\Phi_{|(\rho(M_{\gamma+\lambda}),\infty)}$ is absolutely continuous with respect to the Lebesgue measure. As a consequence, we deduce that $c_0 = 0$. $\square$

Proposition C.5.15 (Moment bounds)**.** *Assume K to be a homogeneous symmetric coagulation kernel, of homogeneity γ, satisfying* (C.1.5), (C.1.6)*, with γ,λ such that* (C.1.7) *holds and such that $\gamma+2\lambda \geq 1$. Let $\Phi \in \mathcal{M}_+(\mathbb{R}_*)$ be a self-similar profile as in Definition C.3.1 and assume additionally that $\Phi((0,\delta)) = 0$, where $\delta > 0$. Then*

$$\int_{\mathbb{R}_*} x^{\mu}\Phi(\mathrm{d}x) = \int_{[\delta,\infty)} x^{\mu}\Phi(\mathrm{d}x) < \infty, \quad \forall \mu \in \mathbb{R}.$$

Proof. For $\mu \leq \gamma+\lambda$ the statement follows by the fact that $\Phi((0,\delta)) = 0$ and that by definition of self-similar profile we know that

$$\int_{[\delta,\infty)} x^{\gamma+\lambda}\Phi(\mathrm{d}x) < \infty.$$

We want to prove the statement for $\mu > \gamma+\lambda$. As a first step, we prove that there exists a $\overline{\delta} > 0$ such that

$$\int_{\mathbb{R}_*} x^{\gamma+\lambda+\overline{\delta}}\phi(x)\mathrm{d}x < \infty, \tag{C.5.45}$$

where ϕ is the density of Φ. As a second step we will prove that

$$\int_{\mathbb{R}_*} x^{\gamma+\lambda+n\overline{\delta}}\phi(x)\mathrm{d}x < \infty \tag{C.5.46}$$

for $n \geq 1$ implies that

$$\int_{\mathbb{R}_*} x^{\gamma+\lambda+(n+1)\overline{\delta}}\phi(x)\mathrm{d}x < \infty. \tag{C.5.47}$$

The desired conclusion then follows by induction.

Step 1

Consider an integrable function φ and integrate both sides of equation (C.5.43) against the function $\psi(x) = \int_x^\infty \varphi(y)\mathrm{d}y$ to deduce that

$$\int_0^\infty \psi(z)z\phi(z)\mathrm{d}z - \int_0^\infty \phi(z)\psi'(z)\left(\frac{z^{\gamma+\lambda+1}}{M_{\gamma+\lambda}} - \frac{2}{1-\gamma}z^2\right)\mathrm{d}z - \frac{1}{M_{\gamma+\lambda}}\int_0^\infty \psi(z)z^{\gamma+\lambda}\phi(z)\mathrm{d}z$$
$$= \int_0^\infty \int_0^\infty xK(x,y)\left(\psi(x+y)-\psi(x)\right)\phi(x)\phi(y)\mathrm{d}y\mathrm{d}x. \tag{C.5.48}$$

We remark that the above formulation requires that ψ has to decay fast enough so that all the integrals in (C.5.48) are finite.

Select $\bar{\delta} > 0$ such that $\max\{0, 1-2(\gamma+\lambda)\} < \bar{\delta} < 1-\gamma-\lambda$ and consider $\psi_R(x) = x^{\gamma+\lambda+\bar{\delta}-1}$ if $x \le R$ while $\psi_R(x) = R^{\bar{\delta}}x^{\gamma+\lambda-1}$ if $x > R$. Using (C.5.48) and the fact that $\gamma + 2\lambda \ge 1$ and that $\psi_R(x+y) - \psi_R(x) \le 0$ we deduce that

$$0 < \frac{\gamma+2\lambda-1+2\bar{\delta}}{1-\gamma}\int_0^R z^{\gamma+\lambda+\bar{\delta}}\phi(z)\mathrm{d}z \le \frac{1}{M_{\gamma+\lambda}}\int_0^\infty z^{2(\gamma+\lambda)+\bar{\delta}-1}\phi(z)\mathrm{d}z < \infty,$$

where the moment $M_{2(\gamma+\lambda)+\bar{\delta}-1}$ is bounded because, due to the choice of $\bar{\delta}$, we have that $2(\gamma+\lambda)+\bar{\delta}-1 < \gamma+\lambda$. Passing to the limit as R tends to infinity we deduce that (C.5.45) holds.

Step 2

We assume that inequality (C.5.46) holds and we want to prove (C.5.47). Taking $\psi_R(x) = x^{\gamma+\lambda+(n+1)\bar{\delta}-1}$ when $x \le R$ and $\psi_R(x) = R^{\bar{\delta}}x^{\gamma+\lambda+n\bar{\delta}-1}$ when $x > R$ in (C.5.48) we deduce that

$$0 < \frac{\gamma+2\lambda-1+2\bar{\delta}(n+1)}{1-\gamma}\int_0^R z^{\gamma+\lambda+(n+1)\bar{\delta}}\phi(z)\mathrm{d}z \le \frac{1}{M_{\gamma+\lambda}}\int_0^\infty z^{2(\gamma+\lambda)+(n+1)\bar{\delta}-1}\phi(z)\mathrm{d}z$$
$$+ \int_\delta^\infty\int_\delta^\infty xK(x,y)\left[\psi_R(x+y)-\psi_R(x)\right]\phi(x)\phi(y)\mathrm{d}y\mathrm{d}x + \int_0^\infty \phi(z)\psi_R'(z)z^{\gamma+\lambda+1}\phi(z)\mathrm{d}z.$$

Thanks to the choice of $\bar{\delta}$, we have that $2(\gamma+\lambda)+(n+1)\bar{\delta}-1 \le \gamma+\lambda+n\bar{\delta}$ and the desired conclusion follows for n such that $\gamma+\lambda+(n+1)\bar{\delta}-1 < 1$ by the fact that in that case ψ_R is decreasing and hence the coagulation term is negative as well as the term $\int_0^\infty \phi(z)\psi_R'(z)z^{\gamma+\lambda+1}\phi(z)$. We examine now the case in which $\gamma + \lambda + (n+1)\overline{\delta} - 1 \geq 1$. In this case we have that

$$\begin{aligned}
&\int_\delta^\infty \int_\delta^\infty xK(x,y)\left[\psi_R(x+y) - \psi_R(x)\right]\phi(x)\phi(y)\mathrm{d}y\mathrm{d}x \\
&\leq c \int_\delta^\infty \int_\delta^\infty xK(x,y)\left[(x+y)^{\gamma+\lambda+(n+1)\overline{\delta}-1} - x^{\gamma+\lambda+(n+1)\overline{\delta}-1}\right]\phi(x)\phi(y)\mathrm{d}y\mathrm{d}x \\
&\leq c \iint_{\{(x,y)\in[\delta,\infty)^2 : x\leq y\}} x^{1-\lambda}y^{\gamma+\lambda}\left[(x+y)^{\gamma+\lambda+(n+1)\overline{\delta}-1} - x^{\gamma+\lambda+(n+1)\overline{\delta}-1}\right]\phi(x)\phi(y)\mathrm{d}y\mathrm{d}x \\
&+ c \iint_{\{(x,y)\in[\delta,\infty)^2 : x\geq y\}} x^{1+\gamma+\lambda}y^{-\lambda}\left[(x+y)^{\gamma+\lambda+(n+1)\overline{\delta}-1} - x^{\gamma+\lambda+(n+1)\overline{\delta}-1}\right]\phi(x)\phi(y)\mathrm{d}y\mathrm{d}x \\
&\leq c \int_\delta^\infty x^{1-\lambda}\phi(x)\mathrm{d}x \int_\delta^\infty y^{2(\gamma+\lambda)+(n+1)\overline{\delta}-1}\phi(y)\mathrm{d}y < \infty.
\end{aligned}$$

Since we also have that

$$\int_0^\infty \phi(z)\psi_R'(z)z^{\gamma+\lambda+1}\phi(z)\mathrm{d}z \leq c(n)\int_0^\infty \phi(z)z^{2(\gamma+\lambda)+(n+1)\delta-1}\phi(z)\mathrm{d}z < \infty$$

the desired conclusion follows. □

Lemma C.5.16 (Exponential bound)**.** *Assume K to be a homogeneous symmetric coagulation kernel, of homogeneity γ, satisfying (C.1.5), (C.1.6), with γ, λ such that (C.1.7) holds and such that $\gamma + 2\lambda \geq 1$. Let $\Phi \in \mathcal{M}_+(\mathbb{R})$ be as in Proposition C.5.14 and let ϕ be its density. Then there exist two positive constants L and M such that*

$$\int_M^\infty \phi(z)e^{Lz}\mathrm{d}z < \infty. \tag{C.5.49}$$

Proof. Adapting the approach used in [FL06a] we define the function Ψ_a as

$$\Psi_a(L) := \int_M^\infty \frac{e^{L\min\{x,a\}}}{\min\{x,a\}}x^2\phi(x)\mathrm{d}x$$

where $M > \delta$. Hence, by its definition

$$\Psi_a'(L) = \int_M^\infty e^{L\min\{x,a\}}x^2\phi(x)\mathrm{d}x.$$

We consider a function ψ in (C.5.48) with $\psi'(x) := e^{L\min\{x,a\}}$ to deduce that

$$\begin{aligned}
\frac{2}{1-\gamma}\Psi_a'(L) &\leq \frac{M^{\gamma+\lambda-1}}{M_{\gamma+\lambda}}\int_M^\infty e^{L\min\{x,a\}}x^2\phi(x)\mathrm{d}x + \frac{1}{M_{\gamma+\lambda}}\int_\delta^M e^{L\min\{x,a\}}x^{\gamma+\lambda+1}\phi(x)\mathrm{d}x \\
&+ \int_{\mathbb{R}_*}\int_{\mathbb{R}_*} xK(x,y)\int_y^{x+y} e^{L\min\{w,a\}}\mathrm{d}w\phi(x)\phi(y)\mathrm{d}x\mathrm{d}y \\
&+ \int_\delta^{\overline{z}}\left(\frac{1}{M_{\gamma+\lambda}}x^{\gamma+\lambda} - x\right)\psi(x)\phi(x)\mathrm{d}x,
\end{aligned}$$

for $\overline{z} := \left(\frac{1}{M_{\gamma+\lambda}}\right)^{\frac{1}{1-\gamma-\lambda}}$, where we are assuming without loss of generality that $\overline{z} > \delta$ and since for every $z \geq \overline{z}$ we have

$$z - \frac{z^{\gamma+\lambda}}{M_{\gamma+\lambda}} > 0. \tag{C.5.50}$$

Thus, there exists a $c \geq 0$ such that

$$\frac{2}{1-\gamma}\Psi_a'(L) \leq \Psi_a'(L)\frac{M^{\gamma+\lambda-1}}{M_{\gamma+\lambda}} + c + \int_{\mathbb{R}_*}\int_{\mathbb{R}_*} xK(x,y)\int_y^{x+y} e^{L\min\{w,a\}}\mathrm{d}w\phi(x)\phi(y)\mathrm{d}x\mathrm{d}y.$$

As in [FFV23, FL06a] we can deduce, using Jensen's inequality together with the fact that $\gamma+\lambda < 1$ and that $-\lambda < 1$, that

$$\int_{\mathbb{R}_*}\int_{\mathbb{R}_*} xK(x,y)\int_y^{x+y} e^{L\min\{w,a\}}\mathrm{d}w\phi(x)\phi(y)\mathrm{d}x\mathrm{d}y \leq \Psi_a(L)^{1-\gamma-\lambda}\Psi_a'(L)^{\gamma+\lambda}.$$

This, together with the fact that we can take M arbitrary large, implies that

$$c(\gamma,\lambda)\Psi_a'(L) \leq \Psi_a(L)^{1-\gamma-\lambda}\Psi_a'(L)^{\gamma+\lambda} + c, \tag{C.5.51}$$

for a positive constant c and for $c(\gamma,\lambda) = \frac{2}{1-\gamma} - \frac{M^{\gamma+\lambda-1}}{M_{\gamma+\lambda}} > 0$.

By the definition of Ψ_a we have that

$$\Psi_a(0) \leq \frac{2}{1-\gamma}\left(1 + \frac{c_1}{a}\right) \to \frac{2}{1-\gamma} \text{ as } a \to \infty.$$

If we prove that $\limsup_{a\to\infty}\Psi_a(L) < \infty$ for some L, then we can conclude. Indeed we would have

$$M\int_M^\infty e^{Lx}\phi(x)\mathrm{d}x \leq \int_M^\infty e^{Lx}x\phi(x)\mathrm{d}x < \infty.$$

Let us prove that $\limsup_{a\to\infty}\Psi_a(L) < \infty$ for some L. First of all notice that this is true if $\Psi_a'(L) \leq 1$ as in this case $\Psi_a(L) \leq L + \lim_{a\to\infty}\Psi_a(0)$ and the desired conclusion follows. If instead $\Psi_a' > 1$, then

$$c(\gamma,\lambda)\Psi_a'(L) \leq \Psi_a(L)^{1-\gamma-\lambda}\Psi_a'(L)^{\gamma+\lambda} + c \leq \alpha\Psi_a(L)^{1-\gamma-\lambda}\Psi_a'(L) + c.$$

Without loss of generality we can assume $\alpha > 0$ such that $\lim_{a\to\infty}\Psi_a(0) \neq \alpha$. This implies that a solution of the ODE

$$\Psi_a'\left(1 - \alpha\Psi_a^{1-\gamma-\lambda}\right) = c$$

exists for small intervals around 0 and is such that $\lim_{a\to\infty}\Psi_a(L) < \infty$ for L in that interval. □

Proposition C.5.17 (Exponential decay)**.** *Assume K to be a homogeneous symmetric coagulation kernel, of homogeneity γ, satisfying* (C.1.5), (C.1.6)*, with γ, λ such that* (C.1.7) *holds and such that $\gamma + 2\lambda \geq 1$. Let $\Phi \in \mathcal{M}_+(\mathbb{R})$ be as in Proposition C.5.14 and let ϕ be its density. Then there exists a positive constant $\tilde{M}$ such that*

$$\limsup_{z\to\infty}\phi(z)e^{\tilde{M}z} < \infty. \tag{C.5.52}$$

Proof. From equation (C.5.43) we deduce that

$$\left(\frac{2}{1-\gamma}z^2 - \frac{z^{\gamma+\lambda+1}}{M_{\gamma+\lambda}(\phi)}\right)\phi(z) \leq \int_0^z x\phi(x)\mathrm{d}x - \frac{1}{M_{\gamma+\lambda}(\phi)}\int_0^z x^{\gamma+\lambda}\phi(x)\mathrm{d}x + J_\phi(z).$$

We now show that J_ϕ decays exponentially and that the term

$$\int_0^z x\phi(x)\mathrm{d}x - \frac{1}{\int_0^\infty y^{\gamma+\lambda}\phi(y)\mathrm{d}y}\int_0^z x^{\gamma+\lambda}\phi(x)\mathrm{d}x$$

decays exponentially too.

First of all, we prove that there exists a constant $M_1 > 0$ such that $J_\phi(z) \leq e^{-M_1 z}$ for large z. To this end we notice that the bound (C.5.49) implies that

$$\begin{aligned} J_\phi(z) &= \int_0^z \int_{z-x}^\infty e^{-L(x+y)}e^{L(x+y)}xK(x,y)\phi(x)\phi(y)\mathrm{d}y\mathrm{d}x \\ &\leq e^{-Lz}\int_0^z\int_{z-x}^\infty e^{L(x+y)}xK(x,y)\phi(x)\phi(y)\mathrm{d}y\mathrm{d}x \leq e^{-M_1 z}. \end{aligned}$$

On the other side, for large values of z we have that

$$\begin{aligned} &\int_0^z x\phi(x)\mathrm{d}x - \frac{1}{\int_0^\infty y^{\gamma+\lambda}\phi(y)\mathrm{d}y}\int_0^z x^{\gamma+\lambda}\phi(x)\mathrm{d}x \\ &= \frac{1}{\int_0^\infty y^{\gamma+\lambda}\phi(y)\mathrm{d}y}\left(\int_0^z x\phi(x)\mathrm{d}x\int_0^\infty y^{\gamma+\lambda}\phi(y)\mathrm{d}y - \int_0^z x^{\gamma+\lambda}\phi(x)\mathrm{d}x\right) \\ &\leq \frac{1}{\int_0^\infty y^{\gamma+\lambda}\phi(y)\mathrm{d}y}\left(\int_0^\infty y^{\gamma+\lambda}\phi(y)\mathrm{d}y - \int_0^z x^{\gamma+\lambda}\phi(x)\mathrm{d}x\right) \\ &\leq \frac{1}{\int_0^\infty y^{\gamma+\lambda}\phi(y)\mathrm{d}y}\int_z^\infty y^{\gamma+\lambda}\phi(y)\mathrm{d}y \leq C(\rho(M_{\gamma+\lambda}))e^{-M_2 z}, \end{aligned}$$

where $M_2 > 0$.

We deduce that for large values of z

$$\left(\frac{2z^2}{1-\gamma} - \frac{z^{\gamma+\lambda+1}}{\int_0^\infty y^{\gamma+\lambda}\phi(y)\mathrm{d}y}\right)\phi(z) \leq c\max\{e^{-zM_1}, e^{-zM_2}\}.$$

The conclusion follows since, for every $z > \left(\frac{1-\gamma}{2\int_0^\infty x^{\gamma+\lambda}\Phi(\mathrm{d}x)}\right)^{\frac{1}{1-\gamma-\lambda}}$, we have

$$\frac{2z^2}{1-\gamma} - \frac{z^{\gamma+\lambda+1}}{\int_0^\infty y^{\gamma+\lambda}\phi(y)\mathrm{d}y} > 0.$$

□

C.6 Non-existence results and properties of the self-similar profiles

To study the non-existence of the self-similar solutions we proceed by contradiction and start by assuming that a self-similar solution exists. To find a contradiction we need to analyse the properties of each self-similar profile. In the case $\gamma + 2\lambda > 1$ the fundamental properties that we prove to be true are that $\Phi((0,\delta)) = 0$ for some positive $\delta > 0$ and that Φ decays sufficiently fast for large sizes. When $\gamma + 2\lambda = 1$, it is possible to prove that Φ decays fast for large sizes, but we have not been able to prove that $\Phi((0,\delta)) = 0$ for some $\delta > 0$. This is the reason why we require the additional condition (C.3.5) in Theorem C.3.5.

C.6.1 Properties of the self-similar profiles

In this section we study the properties of each self-similar solution that do not rely on the existence of the self-similar solution and hence do not rely on the assumption $\gamma > -1$.

Theorem C.6.1 (Properties of the self-similar profiles). *Let K be a homogeneous symmetric coagulation kernel, of homogeneity γ, satisfying* (C.1.5), (C.1.6)*, with γ, λ such that* (C.1.7) *holds.*

1. *If $\gamma+2\lambda > 1$, then every self-similar profile Φ as in Definition C.3.1 is such that $\Phi((0,\rho(M_{\gamma+\lambda}))) = 0$ where $\rho(M_{\gamma+\lambda})$ is given by* (C.3.4)*. Additionally, Φ is such that*
$$\int_{\mathbb{R}_*} e^{Lx}\Phi(\mathrm{d}x) < \infty \tag{C.6.1}$$
for $L > 0$ and it is absolutely continuous with respect to the Lebesgue measure. Its density ϕ is such that
$$\limsup_{x\in\mathbb{R}_*} \phi(x)e^{\overline{L}x} \leq c \tag{C.6.2}$$
for constants $\overline{L}, c > 0$.

2. *If $\gamma + 2\lambda = 1$ and if, in addition, $\Phi((0,\delta)) = 0$ for some $\delta > 0$, then $\Phi([\delta,\rho(M_{\gamma+\lambda}))) = 0$ where $\rho(M_{\gamma+\lambda})$ is given by* (C.3.4)*. In addition,* (C.6.1) *and* (C.6.2) *hold.*

First of all we prove, see Proposition C.6.2, that when $\gamma+2\lambda > 1$, each solution of equation (C.1.19) in the sense of Definition C.3.1 is zero near the origin. The statement of Theorem C.6.1 then follows by Proposition C.5.14 and Proposition C.5.17 and by the fact that when $\gamma + \lambda = 1$ we are assuming that there exists a $\delta > 0$ such that $\Phi((0,\delta)) = 0$.

Proposition C.6.2 (Support of the self-similar solution). *Let K be a homogeneous symmetric coagulation kernel, of homogeneity γ, satisfying* (C.1.5), (C.1.6)*, with γ, λ such that* (C.1.7) *holds and such that $\gamma + 2\lambda > 1$. Let $\Phi \in \mathcal{M}_+(\mathbb{R}_*)$ be a self-similar profile as in Definition C.3.1. Then $\Phi((0,\overline{\xi})) = 0$ for*
$$\overline{\xi} := \min\left\{\left(\frac{1}{M_{\gamma+\lambda}(\Phi)}\right)^{\frac{1}{1-\gamma-\lambda}}, \left(\frac{1-\gamma}{2M_{\gamma+\lambda}(\Phi)}\right)^{\frac{1}{1-\gamma-\lambda}}\right\}.$$

Proof. We start explaining the proof in a heuristic manner without entering in the technical details that will be explained later. Equation (C.1.19) can be rewritten in the following flux form

$$\partial_x\left(J_\Phi(x)+\left(\frac{x^{\gamma+\lambda+1}}{M_{\gamma+\lambda}}-\frac{2x^2}{1-\gamma}\right)\Phi(x)\right)=\left(\frac{x^{\gamma+\lambda}}{M_{\gamma+\lambda}}-x\right)\Phi(x), \tag{C.6.3}$$

where J_ϕ is given by (C.5.44). For x smaller than $\overline{\xi}$ we have that $\frac{x^{\gamma+\lambda}}{M_{\gamma+\lambda}}-x\geq 0$ and hence the function

$$x\mapsto J_\Phi(x)+\left(\frac{x^{\gamma+\lambda+1}}{M_{\gamma+\lambda}}-\frac{2x^2}{1-\gamma}\right)\Phi(x)$$

is increasing and right-continuous at $x=0$. This implies that

$$\lim_{x\to 0^+}\left(J_\Phi(x)+\left(\frac{x^{\gamma+\lambda+1}}{M_{\gamma+\lambda}}-\frac{2x^2}{1-\gamma}\right)\Phi(x)\right)=L.$$

Since for every $x\leq\overline{\xi}$ we also have that $\frac{x^{\gamma+\lambda+1}}{M_{\gamma+\lambda}}-\frac{2x^2}{1-\gamma}\geq 0$ we deduce that $L\geq 0$.

We make a scaling argument to identify the value of L. To this end we notice that the units of the flux are given by $[J_\Phi(x)]=[\Phi]^2[x]^{3+\gamma}$, where from now on we indicate with $[\cdot]$ the dimensional properties of a quantity, hence

$$[\Phi]=[x]^{-\frac{(3+\gamma)}{2}}.$$

We deduce that

$$[x^{\gamma+\lambda+1}\Phi]=[x^{\gamma+\lambda+1}][\Phi]=[x^{\frac{\gamma+2\lambda-1}{2}}].$$

Since $\gamma+2\lambda>1$ this implies that as $x\to 0$ the dominant term in equation (C.6.3) is J_Φ, hence $J_\Phi(x)\sim L$ as $x\to 0$. Finally, we prove that $L=0$, in agreement with the statement proven in [FLNV21b] that, when $\gamma+2\lambda\geq 1$, there are no solutions to the constant flux equation. Integrating (C.6.3), we deduce that

$$J_\Phi(x)+\left(\frac{x^{\gamma+\lambda+1}}{M_{\gamma+\lambda}}-\frac{2x^2}{1-\gamma}\right)\Phi(x)=\int_0^x\left(\frac{z^{\gamma+\lambda}}{M_{\gamma+\lambda}}-z\right)\Phi(z)\mathrm{d}z.$$

A detailed analysis of this ODE for Φ implies that $\Phi=0$ on the interval $\left(0,\left(\frac{1-\gamma}{2M_{\gamma+\lambda}}\right)^{\frac{1}{1-\gamma-\lambda}}\right)$, see Step 4.

We now explain the proof in detail. Testing (C.3.3) with a function of the form $\psi(\xi)=\xi\varphi(\xi)$, where $\varphi\in \mathrm{C}^1_c(\mathbb{R}_*)$, we get to the following equation:

$$\int_{(0,\infty)}\left(\xi-\frac{1}{M_{\gamma+\lambda}}\xi^{\gamma+\lambda}\right)\varphi(\xi)\Phi(\mathrm{d}\xi)=\int_{(0,\infty)}\varphi'(\xi)W[\Phi](\mathrm{d}\xi), \tag{C.6.4}$$

where

$$W[\Phi](\mathrm{d}x)=J_\Phi(x)\mathrm{d}x+\left(\frac{x^{\gamma+\lambda+1}}{M_{\gamma+\lambda}}-\frac{2x^2}{1-\gamma}\right)\Phi(\mathrm{d}x). \tag{C.6.5}$$

Step 1: Regularity of $W[\Phi]$ and an integral representation formula

First of all we prove that, for every $\overline{\delta} > 0$, the restriction of the measure $W[\Phi]$ on the interval $[\overline{\delta}, 1]$ is absolutely continuous with respect to the Lebesgue measure. This follows from the fact that for every function $\varphi \in C_c^1([\overline{\delta}, \overline{\xi}])$ with $0 < \overline{\delta} < 1$ we have that

$$\int_{\mathbb{R}_*} \varphi'(\xi) W[\Phi](d\xi) \leq c(\Phi, \delta, \gamma, \lambda) \|\varphi\|_\infty;$$

hence $W[\Phi]$, on the interval $[\overline{\delta}, \overline{\xi}]$, has a density $W_\phi \in BV([\overline{\delta}, \overline{\xi}])$, see [AFP00]. Since $\overline{\delta}$ is arbitrary we deduce that $W[\Phi]$ has a density $W_\phi \in L^1_{loc}((0, \overline{\xi}])$. This implies that the measure $\Phi_{|(0,\overline{\xi}]}$ is absolutely continuous with respect to the Lebesgue measure, its density is ϕ.

We now prove that W_ϕ is increasing on $(0, \overline{\xi}]$. To this end we consider a sequence of functions $\{\varphi_n\}_n \subset C_c^1(\mathbb{R}_*)$ such that $\varphi_n \to \chi_{[\xi_1, \xi_2]}$ pointwise, with $0 < \xi_1, \xi_2 < \overline{\xi}$, $\varphi_n'(\xi) \geq 0$ for $\xi \in \left(0, \frac{\xi_1+\xi_2}{2}\right)$ and such that $\varphi_n'(\xi) \leq 0$ for $\xi \in \left(\frac{\xi_1+\xi_2}{2}, \infty\right)$. Substituting φ_n in (C.6.4) we deduce that for every $n \geq 1$

$$\int_0^\infty \xi \varphi_n(\xi)\phi(\xi) d\xi - \frac{1}{M_{\gamma+\lambda}} \int_0^\infty \varphi_n(\xi) \xi^{\gamma+\lambda} \phi(\xi) d\xi = \int_0^\infty \varphi_n'(\xi) W_\phi(\xi) d\xi.$$

Notice that

$$\int_0^\infty \xi \varphi_n(\xi)\phi(\xi) d\xi \to \int_{[\xi_1, \xi_2]} \xi \phi(\xi) d\xi \quad \text{as} \quad n \to \infty$$

and that

$$\int_0^\infty \xi^{\gamma+\lambda} \varphi_n(\xi)\phi(\xi) d\xi \to \int_{[\xi_1, \xi_2]} \xi^{\gamma+\lambda} \phi(\xi) d\xi \quad \text{as} \quad n \to \infty.$$

Additionally, since W_ϕ is in $L^1_{loc}((0, \overline{\xi}))$ and satisfies (C.6.4) we deduce that $W_\phi \in W^1_{loc}(0, \overline{\xi})$ and hence is continuous in $(0, \overline{\xi})$. Hence we have that

$$\int_{(0,\infty)} \varphi_n'(\xi) W_\phi(\xi) d\xi \to W_\phi(\xi_1) - W_\phi(\xi_2) \quad \text{as} \quad n \to \infty.$$

As a consequence, we deduce that

$$W_\phi(\xi_2) - W_\phi(\xi_1) = \int_{[\xi_1, \xi_2]} \left(\frac{1}{M_{\gamma+\lambda}} \xi^{\gamma+\lambda} - \xi \right) \phi(\xi) d\xi. \tag{C.6.6}$$

Since if $\xi \leq \overline{\xi}$, then $\frac{1}{M_{\gamma+\lambda}} \xi^{\gamma+\lambda} - \xi \geq 0$, we deduce that there exists a $\overline{\xi}$ such that W_ϕ is increasing in $(0, \overline{\xi}]$. Hence, since W_ϕ is bounded, we have that the following limit exists

$$0 \leq L := \lim_{\xi \to 0^+} W_\phi(\xi) < \infty.$$

Due to (C.3.2) we can take the limit as $\xi_1 \to 0$ in (C.6.6) and deduce that

$$W_\phi(\xi_2) - L = \int_{(0, \xi_2]} \left(\frac{1}{M_{\gamma+\lambda}} \xi^{\gamma+\lambda} - \xi \right) \phi(\xi) d\xi. \tag{C.6.7}$$

Step 2: we prove that $\int_{\mathbb{R}_*} J_\phi(\varepsilon x)\varphi(x)\mathbf{d}x \to L\int_{\mathbb{R}_*}\varphi(x)\mathbf{d}x$

Since W_ϕ is bounded in $(0,\overline{\xi}]$ and $\frac{1}{M_{\gamma+\lambda}}\xi^{\gamma+\lambda} - \frac{2}{1-\gamma}\xi \geq 0$ for every $\xi \leq \overline{\xi}$ we have that J_ϕ is also bounded in $(0,\overline{\xi}]$. Since

$$[2/3z, z]\times[2/3z, z] \subset \left\{x \in \mathbb{R}_*^2 : 0 < x \leq z, z - x \leq y < \infty\right\} =: \Omega_z \tag{C.6.8}$$

we deduce that for every $z \in (0,\overline{\xi}]$ we have that

$$z^{\gamma+1}\left(\int_{2z/3}^{z}\phi(x)\mathrm{d}x\right)^2 \leq J_\phi(z) \leq C$$

hence

$$\frac{1}{z}\int_{2z/3}^{z}\phi(x)\mathrm{d}x \leq \frac{C}{z^{\frac{\gamma+3}{2}}} \quad z \in (0,\overline{\xi}]. \tag{C.6.9}$$

Equation (C.6.7) implies that

$$\begin{aligned}\int_{\mathbb{R}_*}\varphi(x)\,J_\phi(\varepsilon x)\mathrm{d}x &= \frac{1}{\varepsilon}\int_{\mathbb{R}_*}\varphi\left(\frac{x}{\varepsilon}\right)J_\phi(x)\mathrm{d}x = -\frac{1}{\varepsilon}\int_{\mathbb{R}_*}\varphi\left(\frac{x}{\varepsilon}\right)\left[\frac{x^{\gamma+\lambda+1}}{M_{\gamma+\lambda}} - \frac{2}{1-\gamma}x^2\right]\phi(x)\mathrm{d}x \\ &+ \frac{1}{\varepsilon}\int_{\mathbb{R}_*}\varphi\left(\frac{x}{\varepsilon}\right)\int_0^x\left[\frac{z^{\gamma+\lambda}}{M_{\gamma+\lambda}} - z\right]\phi(z)\mathrm{d}z\mathrm{d}x + L\int_{\mathbb{R}_*}\varphi(x)\mathrm{d}x\end{aligned} \tag{C.6.10}$$

where $\varphi \in C_c((0,\overline{\xi}))$ and where $\varepsilon > 0$.

Thanks to the bounds (C.3.2) for ϕ and to the fact that φ is compactly supported we can pass to the limit as $\varepsilon \to 0$ in equation (C.6.10) and deduce that

$$\lim_{\varepsilon\to 0}\int_{\mathbb{R}_*}J_\phi(\varepsilon\xi)\varphi(\xi)\mathrm{d}\xi = L\int_{\mathbb{R}_*}\varphi(x)\mathrm{d}x.$$

for every $\varphi \in C_c^1((0,\overline{\xi}))$.

Step 3: we prove that $L = 0$

We want to prove that for every $\varphi \in C_c^1((0,\overline{\xi}))$

$$\lim_{\varepsilon\to 0}\int_{\mathbb{R}_*}J_\phi(\varepsilon\xi)\varphi(\xi)\mathrm{d}\xi = 0.$$

Using the formula for the fluxes we obtain that

$$\begin{aligned}\frac{1}{\varepsilon}\int_{\mathbb{R}_*}J_\phi(\xi)\varphi\left(\frac{\xi}{\varepsilon}\right)\mathrm{d}\xi &= \int_{\mathbb{R}_*}\varphi(\xi)J_\phi(\varepsilon\xi)\mathrm{d}\xi \\ &= \int_{\mathbb{R}_*}\int_{\mathbb{R}_*}xK(x,y)\left(\psi\left(\frac{x+y}{\varepsilon}\right) - \psi\left(\frac{x}{\varepsilon}\right)\right)\Phi(\mathrm{d}x)\Phi(\mathrm{d}y) \\ &= I_1 + I_2\end{aligned}$$

where $\psi(x)=\int_0^x\varphi(y)\mathrm{d}y$ and where

$$I_1=\int_{\mathbb{R}_*}\int_{(y,\infty)}xK(x,y)\left(\psi\left(\frac{x+y}{\varepsilon}\right)-\psi\left(\frac{x}{\varepsilon}\right)\right)\Phi(\mathrm{d}x)\Phi(\mathrm{d}y)$$

and

$$I_2=\int_{\mathbb{R}_*}\int_{(0,y)}xK(x,y)\left(\psi\left(\frac{x+y}{\varepsilon}\right)-\psi\left(\frac{x}{\varepsilon}\right)\right)\Phi(\mathrm{d}x)\Phi(\mathrm{d}y).$$

Since φ is compactly supported we have, by definition, that $\operatorname{supp}\left[\psi\left(x+y\right)-\psi\left(x\right)\right]\subset\{(x,y)\in\mathbb{R}_*^2:x\leq R,\ x+y\geq\overline{\delta}>0\}$ for suitable $\overline{\delta}>0$ and $R>0$. As a consequence, using (C.3.2) we deduce that

$$\begin{aligned}|I_1|&\leq c\int_{(0,\varepsilon R)}\int_{(\overline{\delta}\varepsilon/2,\varepsilon R)}y^{-\lambda}x^{1+\gamma+\lambda}\left|\psi\left(\frac{x+y}{\varepsilon}\right)-\psi\left(\frac{x}{\varepsilon}\right)\right|\Phi(\mathrm{d}x)\Phi(\mathrm{d}y)\\&\leq C(\psi)\frac{1}{\varepsilon}\int_{(0,\varepsilon R)}y^{1-\lambda}\Phi(\mathrm{d}y)\int_{(\overline{\delta}\varepsilon/2,\varepsilon R)}x^{1+\gamma+\lambda}\Phi(\mathrm{d}x)\to0\text{ as }\varepsilon\to0.\end{aligned}$$

Using again (C.3.2) we have

$$\begin{aligned}|I_2|&=\int_{\mathbb{R}_*}\int_{(0,y)}xK(x,y)\left|\psi\left(\frac{x+y}{\varepsilon}\right)-\psi\left(\frac{x}{\varepsilon}\right)\right|\Phi(\mathrm{d}x)\Phi(\mathrm{d}y)\\&\leq C(\psi)\int_{(\varepsilon\delta/2,\infty)}y^{\gamma+\lambda}\Phi(\mathrm{d}y)\int_{(0,\varepsilon R)}x^{1-\lambda}\Phi(\mathrm{d}x)\to0\text{ as }\varepsilon\to0.\end{aligned}$$

Hence $\lim_{\varepsilon\to0}\int_{\mathbb{R}_*}J_\phi(\varepsilon\xi)\varphi(\xi)\mathrm{d}\xi=0$ and therefore $L=0$.

Step 4: Φ is zero near zero

Since $L=0$ we deduce by (C.6.7) that

$$W_\phi(z)=\int_0^z\left(\frac{1}{M_{\gamma+\lambda}}\xi^{\gamma+\lambda}-\xi\right)\phi(\xi)\mathrm{d}\xi\quad\text{ for }z\in(0,\overline{\xi}).\tag{C.6.11}$$

Or, equivalently, since $J_\phi\in BV(\mathbb{R}_*)$

$$J_\phi(z)+\left(\frac{z^{\gamma+\lambda+1}}{M_{\gamma+\lambda}}-\frac{2z^2}{1-\gamma}\right)\phi(z)=\int_0^z\left(\frac{1}{M_{\gamma+\lambda}}\xi^{\gamma+\lambda}-\xi\right)\phi(\xi)\mathrm{d}\xi\quad\text{ for } a.e. z\in(0,\overline{\xi}).\tag{C.6.12}$$

We now rewrite (C.6.12) in the following way

$$\begin{aligned}&\int_0^z\int_{z-x}^zxK(x,y)\Phi(\mathrm{d}y)\Phi(\mathrm{d}x)+\int_0^z\int_z^\infty xK(x,y)\Phi(\mathrm{d}y)\Phi(\mathrm{d}x)\\&+\left(\frac{z^{\gamma+\lambda+1}}{M_{\gamma+\lambda}}-\frac{2z^2}{1-\gamma}\right)\phi(z)=\int_0^z\left(\frac{1}{M_{\gamma+\lambda}}\xi^{\gamma+\lambda}-\xi\right)\phi(\xi)\mathrm{d}\xi.\end{aligned}$$

If Φ is different to zero near the origin, then

$$\int_0^z \int_z^\infty xK(x,y)\Phi(\mathrm{d}x)\Phi(\mathrm{d}y) \geq c_1 \int_0^z x^{1-\lambda}\Phi(\mathrm{d}x) \int_z^\infty y^{\gamma+\lambda}\Phi(\mathrm{d}y) \\ + c_1 \int_0^z x^{1+\gamma+\lambda}\Phi(\mathrm{d}x) \int_z^\infty y^{-\lambda}\Phi(\mathrm{d}y) \geq C_+ \int_0^z x^{1-\lambda}\Phi(\mathrm{d}x)$$

where we have used that for sufficiently small z we have that

$$\int_z^\infty y^{\gamma+\lambda}\Phi(\mathrm{d}y) \geq C_+$$

for a strictly positive constant C_+.

This implies that for z small we have that

$$\begin{aligned} 0 &= J_\phi(z) + \left(\frac{z^{\gamma+\lambda+1}}{M_{\gamma+\lambda}} - \frac{2z^2}{1-\gamma}\right)\phi(z) - \int_0^z \left(\frac{1}{M_{\gamma+\lambda}}\xi^{\gamma+\lambda} - \xi\right)\phi(\xi)\mathrm{d}\xi \\ &\geq C_+ \int_0^z x^{1-\lambda}\phi(x)\mathrm{d}x + \left(\frac{z^{\gamma+\lambda+1}}{M_{\gamma+\lambda}} - \frac{2z^2}{1-\gamma}\right)\phi(z) - \int_0^z \left(\frac{1}{M_{\gamma+\lambda}}\xi^{\gamma+\lambda} - \xi\right)\phi(\xi)\mathrm{d}\xi \\ &\geq \left(\frac{z^{\gamma+\lambda+1}}{M_{\gamma+\lambda}} - \frac{2z^2}{1-\gamma}\right)\phi(z) + \int_0^z \left(C_+\xi^{1-\lambda} - \frac{1}{M_{\gamma+\lambda}}\xi^{\gamma+\lambda}\right)\phi(\xi)\mathrm{d}\xi. \end{aligned}$$

Since we are assuming that Φ is different from zero near the origin and since $\gamma+2\lambda > 1$, we deduce that $\int_0^z \left(C_+\xi^{1-\lambda} - \frac{1}{M_{\gamma+\lambda}}\xi^{\gamma+\lambda}\right)\phi(\xi)\mathrm{d}\xi > 0$. This, together with the fact that for $z \leq \overline{\xi}$ we have $\left(\frac{z^{\gamma+\lambda+1}}{M_{\gamma+\lambda}} - \frac{2z^2}{1-\gamma}\right) \geq 0$ leads to a contradiction. Hence there exists a $\delta > 0$ such that $\phi(x) = 0$ for $x \in (0,\delta)$. We now prove that $\phi(x) = 0$ for every $x < \overline{\xi}$. To this end we define the function H as

$$H(z) := \int_0^z \left(\frac{1}{M_{\gamma+\lambda}}\xi^{\gamma+\lambda} - \xi\right)\phi(\xi)\mathrm{d}\xi.$$

Due to (C.6.12) we have that H satisfies

$$\frac{d}{dz}H(z) = H(z)Q(z) - J_\phi(z)Q(z), \quad H(\delta) = 0 \quad \text{for } a.e.\ z \in (\delta, \overline{\xi}) \tag{C.6.13}$$

where

$$Q(z) := \frac{(1-\gamma)(z^{\gamma+\lambda} - M_{\gamma+\lambda}z)}{(1-\gamma)z^{\gamma+\lambda+1} - 2M_{\gamma+\lambda}z^2} > 0, \quad z \in (\delta, \overline{\xi}). \tag{C.6.14}$$

The solution of equation (C.6.13) is the function

$$H(z) = -\int_\delta^z \frac{J_\phi(s)}{Q(s)} e^{\int_s^z Q(x)\mathrm{d}x}\mathrm{d}s. \tag{C.6.15}$$

Expression (C.6.15) for H implies that H is negative for $z \in (\delta, \overline{\xi})$. On the other hand, if we assume that ϕ is different from zero in $(0, \overline{\xi})$ we deduce that H is positive on $z \in (\delta, \overline{\xi})$, by definition. This contradiction implies that $H(z) = 0$ for $z \in (\delta, \overline{\xi})$. □

Remark C.6.3. We remark that equation (C.6.11) is valid both for $\gamma + 2\lambda > 1$ and for $\gamma + 2\lambda = 1$, but the same argument we used to prove that $\Phi((0,\delta)) = 0$ does not work in the case $\gamma + 2\lambda = 1$.

Proof of Theorem C.6.1. As in the proof of Theorem C.5.1 we prove that if $\Phi((0,\delta)) = 0$ for some $\delta > 0$ then $\Phi((0,\rho(M_{\gamma+\lambda}))) = 0$ for $\rho(M_{\gamma+\lambda})$ given by (C.3.4). To conclude the proof combine this with Proposition C.5.14 and Proposition C.5.17 with Proposition C.6.2. □

C.6.2 Non-existence results

Proof of Theorem C.3.5. We consider now the case $\gamma + 2\lambda > 1$ and $\gamma \leq -1$ in which $\gamma + \lambda$ could be bigger than zero or smaller than zero, see Figure C.1. We will provide a proof that relies on the technical proof of Proposition C.6.2. We proceed by contradiction. Assume that a self-similar solution exists. Then thanks to Proposition C.6.2 we know that $\Phi((0,\delta)) = 0$ for some $\delta > 0$. Additionally, from Proposition C.5.17 we know that $\lim_{x\to\infty} \Phi(x) = 0$ exponentially.

We can therefore consider the test function $\varphi \equiv 1$ in equation (C.3.3). This implies

$$0 \leq -\frac{1+\gamma}{1-\gamma}\int_{\mathbb{R}_*} \Phi(\mathrm{d}x) = -\frac{1}{2}\int_{\mathbb{R}_*}\int_{\mathbb{R}_*} K(x,y)\Phi(\mathrm{d}x)\Phi(\mathrm{d}y) < 0,$$

which is a contradiction.

Assume now that $\gamma + 2\lambda = 1$ and that $\gamma \leq -1$. Notice that in this case we necessarily have that $\gamma + \lambda \leq 0$. Let us assume by contradiction that a self-similar solution exists and is such that

$$\int_{(0,1]} x^{-\lambda}\Phi(\mathrm{d}x) < \infty.$$

We show that

$$\int_{\mathbb{R}_*} \Phi(\mathrm{d}x) < \infty.$$

Notice that this is true by assumption when $\gamma + \lambda = 0$, therefore we restrict the attention to the case $\gamma + \lambda < 0$. To this end we adapt the argument used in the proof of Proposition C.5.15, we refer there for the technical aspects, and prove that

$$\int_{\mathbb{R}_*} x^{\gamma+\lambda}\Phi(\mathrm{d}x) < \infty \quad \text{implies} \quad \int_{\mathbb{R}_*} x^{\gamma+\lambda+\bar{\delta}}\Phi(\mathrm{d}x) < \infty$$

when $0 < -\gamma - \lambda < \bar{\delta} < 1 - \gamma - \lambda$. To see this we consider $\psi(x) = x^{\gamma+\lambda+\bar{\delta}-1}$ in equation (C.5.48) to deduce that

$$\frac{2\bar{\delta}}{1-\gamma} M_{\gamma+\lambda+\bar{\delta}} \leq \frac{M_{2(\gamma+\lambda)+\bar{\delta}-1}}{M_{\gamma+\lambda}}.$$

Now we notice that our choice of $\bar{\delta}$ implies that

$$-\lambda \leq 2(\gamma+\lambda) + \bar{\delta} - 1 \leq \gamma + \lambda.$$

Hence $M_{2(\gamma+\lambda)+\bar{\delta}-1} < \infty$ and the desired conclusion follows.

Similarly, we can prove that

$$\int_{\mathbb{R}_*} x^{\gamma+\lambda+n\bar{\delta}}\Phi(\mathrm{d}x) < \infty \quad \text{implies} \quad \int_{\mathbb{R}_*} x^{\gamma+\lambda+(n+1)\bar{\delta}}\Phi(\mathrm{d}x) < \infty.$$

As a consequence, taking n sufficiently large, we deduce that $M_0(\Phi) < \infty$. We can therefore consider the test function $\varphi \equiv 1$ in equation (C.3.3). This implies

$$0 \leq -\frac{1+\gamma}{1-\gamma}\int_{\mathbb{R}_*} \Phi(\mathrm{d}x) = -\frac{1}{2}\int_{\mathbb{R}_*}\int_{\mathbb{R}_*} K(x,y)\Phi(\mathrm{d}x)\Phi(\mathrm{d}y) < 0$$

which is a contradiction. □

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APPENDIX D

ON AN INHOMOGENEOUS COAGULATION MODEL WITH A DIFFERENTIAL SEDIMENTATION KERNEL

ABSTRACT. We study an inhomogeneous coagulation equation that contains a transport term in the spatial variable modeling the sedimentation of clusters. We prove local existence of mass conserving solutions for a class of coagulation kernels for which in the space homogeneous case instantaneous gelation (i.e., instantaneous loss of mass) occurs. Our result holds true in particular for sum-type kernels of homogeneity greater than one, for which solutions do not exist at all in the spatially homogeneous case. Moreover, our result covers kernels that in addition vanish on the diagonal, which have been used to describe the onset of rain and the behavior of air bubbles in water.

D.1 Introduction

D.1.1 Background

In [HNS07] the authors suggest a space-dependent coagulation equation to model the onset of rain. Here spherical particles of volume v move in space vertically, for example due to gravitation, and merge when their trajectories cross. This leads to the following inhomogeneous coagulation equation for the density f of particles of size v at the point x:

$$\begin{aligned}\partial_t f(x,v,t) + v^{\frac{2}{3}}\partial_x f(x,v,t) &= \frac{1}{2}\int_{(0,v)} K(v-v',v')f(x,v-v',t)f(x,v',t)\mathrm{d}v' \\ &\quad - \int_{(0,\infty)} K(v,v')f(x,v,t)f(x,v',t)\mathrm{d}v' \end{aligned} \tag{D.1.1}$$

with a so-called differential sedimentation kernel of the form

$$K(v,v') = |v^{\frac{2}{3}} - v'^{\frac{2}{3}}|(v^{\frac{1}{3}} + v'^{\frac{1}{3}})^2. \tag{D.1.2}$$

This choice of kernel is motivated by the following consideration (see [HNS07]): the cross-section of interaction between two particles of radii r and r', and volume v and v', respectively, that merged upon touching is given by $\pi(r+r')^2 \approx (v^{\frac{1}{3}} + v'^{\frac{1}{3}})^2$.

Additionally, the velocity is approximately $v^{\frac{2}{3}}$, which represents the Stokes velocity of a rigid sphere with no slip boundary condition, and the collision rate between two particles is taken to be proportional to their relative velocities $|v^{\frac{2}{3}} - v'^{\frac{2}{3}}|$.

The model is used to describe the behavior of air bubbles in water which move due to buoyancy and it is also valid for water droplets. We refer to [CGW78, FSV06, PK97] for more details. In [HNS07], slip-flow corrections for water droplets are discussed and it is mentioned that this requires to change the power of the volume for the velocity. More precisely, the left-hand side of (D.1.1) becomes $\partial_t f(x,v,t) + v^\alpha \partial_x f(x,v,t)$, for some $\alpha \in (0,1)$, and the kernel in (D.1.2) has the form $K(v,v') = |v^\alpha - v'^\alpha|(v^{\frac{1}{3}} + v'^{\frac{1}{3}})^2$.

The model in (D.1.1) with kernel (D.1.2) is referred to as the *free merging regime* in [HNS07], since it is assumed in its derivation that the particles merge when their trajectories cross. When studying equations like (D.1.1), it is customary to look for stationary solutions of non-zero flux (cf. [TIN96]) of the form $f \approx v^d$, for some $d \in \mathbb{R}$. One of the possible approaches is to compute them using the so-called Zakharov transform (see [ZF67]) and using it we find the solution $f(x,v,t) \approx v^{-\frac{13}{6}}$. However, this approach can be made rigorous only if the integral containing the coagulation kernel in (D.1.1) is finite, which is not the case for the kernel (D.1.2). In order to be able to rigorously find a solution for kernels with the same homogeneity, the so-called *forced locality regime* in which only particles of similar sizes can merge is studied in [HNS07]. More precisely, for the *forced locality regime* a cut-off in the coagulation kernel is introduced that makes the kernel vanish outside the region where $\frac{1}{q} < \frac{v'}{v} < q$, for some $q > 1$. With this cut-off, the integral containing the coagulation kernel converges and thus the stationary solution $f \approx v^{-\frac{13}{6}}$ is a valid solution.

Our main goal in this paper is to show that mass conserving solutions exist, at least for a short time interval, for a class of inhomogeneous coagulation equations that includes example (D.1.1) with (D.1.2). At a first glance this might look surprising since the homogeneity of the kernel in (D.1.2) is greater than one. Indeed, it is well-known that gelation (mass loss) occurs for the standard one-dimensional coagulation equation,

$$\partial_t f(v,t) = \frac{1}{2}\int_{(0,v)} K(v-v',v')f(v-v',t)f(v',t)\mathrm{d}v' - \int_{(0,\infty)} K(v,v')f(v,t)f(v',t)\mathrm{d}v', \tag{D.1.3}$$

when the coagulation kernel behaves like a power law of homogeneity $\gamma > 1$ (see, for example [EMP02, ELMP03, Lau00]). In particular, for sum kernels of the form

$$K(v,v') = v^\gamma + v'^\gamma, \tag{D.1.4}$$

with $\gamma > 1$, gelation happens instantaneously. Actually, making use of this property, one can prove that solutions which belong to L^1 for the standard coagulation equation do not exist at all for kernels as in (D.1.4) (see [BLL19b, BC90, CdC92, vD87]). In addition, it has been proven in [CNV24a] that the instantaneous gelation phenomenon holds for Radon measure solutions of the standard coagulation equation with sum kernels of homogeneity greater than one which vanish on the diagonal, i.e. $K(v, v') = 0$, such as the kernel in (D.1.2).

Our main goal is then to prove that, in contrast to the homogeneous case, there exist, at least for short times, mass conserving solutions to the inhomogeneous model

$$\partial_t f(x,v,t) + v^\alpha \partial_x f(x,v,t) = \frac{1}{2}\int_{(0,v)} K(v-v',v') f(x,v-v',t) f(x,v',t) \mathrm{d}v' - \int_{(0,\infty)} K(v,v') f(x,v,t) f(x,v',t) \mathrm{d}v', \tag{D.1.5}$$

where $\alpha \in (0,1)$. Our proof holds for a rather general class of coagulation kernels (see Assumption D.1.1), in particular kernels of the form (D.1.2) and (D.1.4). Thus, the model (D.1.5) provides a coagulation model in which existence for kernels of the form (D.1.4) with $\gamma > 1$ holds, at least for short times.

D.1.2 Main result

Short time existence of mass conserving solutions for the inhomogeneous model

Our goal is to prove short-time existence of mass conserving solutions for the inhomogeneous model

$$\partial_t f(x,v,t) + v^\alpha \partial_x f(x,v,t) = \frac{1}{2}\int_{(0,v)} K(v-v',v') f(x,v-v',t) f(x,v',t) \mathrm{d}v' - \int_{(0,\infty)} K(v,v') f(x,v,t) f(x,v',t) \mathrm{d}v', \tag{D.1.6}$$

where

$$\alpha \in (0,1). \tag{D.1.7}$$

Assumption D.1.1. *We assume that* $K\colon [0,\infty)\times[0,\infty) \to [0,\infty)$ *is a symmetric and continuous function that satisfies*

$$0 \leq K(v,v') \leq K_1(v^\gamma + v'^\gamma), \text{ with } \gamma \in [0, 1+\alpha) \tag{D.1.8}$$

for some constant $K_1 > 0$ *and*

$$K(v-v',v') \leq K(v,v'), \text{ when } v' \in [0, \tfrac{v}{2}]. \tag{D.1.9}$$

Condition (D.1.9) is a rather standard assumption in the study of coagulation equations, see for example [LM02], and most of the kernels used in applications satisfy this condition, in particular, kernels of the form $K(v,v') = v^\gamma + v'^\gamma$ or (D.1.2). The condition $\gamma < 1+\alpha$ in (D.1.8) is such that the transport term will control the contribution coming from the coagulation term.

Definition D.1.2. [Mild solutions] Let $\alpha \in (0,1)$, $\gamma \in [0, 1+\alpha)$ and $m > \frac{\gamma+1}{\alpha}$. Let $T > 0$ and K satisfy Assumption D.1.1. We say that a non-negative function $f \in \mathrm{C}([0,T] \times \mathbb{R} \times (0,\infty))$ such that

$$\sup_{t\in[0,T]} \int_{(0,\infty)} (1+v^\gamma) f(x,v,t) \mathrm{d}v \leq \frac{C_T}{\max\left\{1, |x|^{m-\frac{\gamma+1}{\alpha}}\right\}}, \text{ for } x \in \mathbb{R}, \tag{D.1.10}$$

is a mild solution of equation (D.1.6) if

$$\begin{aligned} &f(x,v,t) - f(x - v^\alpha t, v, 0) S[f](x,v,0,t) = \\ &\frac{1}{2}\int_0^t \int_{(0,v)} S[f](x,v,s,t) K(v-v',v') f(x-(t-s)v^\alpha, v-v', s) f(x-(t-s)v^\alpha, v', s) \mathrm{d}s, \end{aligned} \tag{D.1.11}$$

for all $t \in [0,T]$, $v \in (0,\infty)$ and $x \in \mathbb{R}$, where

$$S[f](x,v,s,t) := \mathrm{e}^{-\int_s^t a[f](x - v^\alpha(t-\tau), v, \tau)\mathrm{d}\tau}, \tag{D.1.12}$$

with

$$a[f](x,v,t) := \int_{(0,\infty)} K(v,v') f(x,v',t) \mathrm{d}v'. \tag{D.1.13}$$

Definition D.1.3. We call $f \in \mathrm{C}([0,T] \times \mathbb{R} \times (0,\infty))$ a mass-conserving solution of equation (D.1.6) if f is as in Definition D.1.2 and satisfies in addition

$$\int_{\mathbb{R}} \int_{(0,\infty)} v f(x,v,t) \mathrm{d}v \mathrm{d}x = \int_{\mathbb{R}} \int_{(0,\infty)} v f(x,v,0) \mathrm{d}v \mathrm{d}x$$

for all $t \in [0,T]$.

Theorem D.1.4 (Local existence of solutions)**.** *Let $\alpha \in (0,1)$, $\gamma \in [0, 1+\alpha)$, $m \in \mathbb{N}$ even, and $p = \alpha m$ with $m > \max\{\frac{2\gamma+1}{\alpha}, \frac{2}{\alpha} + 3\}$. Let K satisfy Assumption D.1.1 and $T > 0$ be sufficiently small. Let $f_{\mathrm{in}} \in \mathrm{C}^1(\mathbb{R} \times (0,\infty))$ such that*

$$f_{\mathrm{in}}(x,v) \leq \frac{C_0}{1 + |x|^m + v^p},$$

for some $C_0 > 0$ and all $x \in \mathbb{R}$, $v \in (0,\infty)$. Then there exists a mass-conserving solution f of (D.1.11) as in Definition D.1.3 that satisfies

$$f(x,v,t) \leq \frac{C}{1 + |x|^m + v^p}, \tag{D.1.14}$$

for all $t \in [0,T]$, for some $C > 0$.

Remark D.1.5. Theorem D.1.4 is valid for coagulation kernels K as in (D.1.2), as well as coagulation kernels of the form $K(v, v') = v^\gamma + v'^\gamma$.

Remark D.1.6. It is worthwhile to mention that mass conservation will follow due to the fact that our solution will have sufficiently fast decay for large values of $|x|$ and v, see (D.1.14). For more details, see the proof of Theorem D.1.4 .

When $\gamma < 1$, our result could be expected according to the general theory of existence for one-dimensional coagulation equations. This states that solutions exist for kernels that behave like power laws of homogeneity $\gamma < 1$, see, for example, [Ste89] for existence of solutions and [FL05] for existence of self-similar profiles.

Some multi-dimensional coagulation models have been studied in the mathematical literature, see [CV23, FLNV21a]. Moreover, several classes of coagulation models for the distribution of particles with space dependence have also been considered. In particular, models in which in addition to coagulation there is space diffusion of the aggregating particles can be found in [AB79, CDF08, CDF10, LM02]. Models that contain coagulation of particles as well as transport terms (that might include sedimentation terms) were studied in [Bur83, CD95, CD97, Dub90a, Dub90b, Gal85]. In all the models mentioned above the homogeneity of the coagulation kernel is either $\gamma < 1$, case in which the solutions are globally defined and preserve the total mass, or product-type kernels are discussed, for which solutions preserve the mass up to a certain point in time.

To our knowledge, the only exception that considers the case $\gamma > 1$ for space-dependent models is [Gal87]. Indeed, existence of solutions for the discrete version of the model in (D.1.1) has been established for coagulation kernels of the form $K(i, j) = \sigma_{ij}|v_i - v_j|$, $i, j \in \mathbb{N}$. Here, v_i is a non-negative function of volume which represents the sedimentation velocity of the particles and σ_{ij} can be estimated by a function of the form $C(i^\gamma + j^\gamma)$ with $\gamma < 1$. Assuming that v_i scales like a power law i^β, we would obtain a coagulation kernel $K(i, j)$ that behaves like a homogeneous function with homogeneity $\beta + \gamma$ that might be larger than one. As a matter of fact, the results in [Gal87] hold for any choice of $v_i \geq 0$. In particular, the solutions constructed in [Gal87] have total mass of particles that changes continuously in time. Nevertheless it has not been proven in [Gal87] that the mass of solutions is conserved. By contrast, in the present paper we construct a theory of existence for mild solutions which conserve mass in finite time. The class of kernels considered in [Gal87] has a non-empty intersection with the class of kernels considered in this paper and this intersection includes the coagulation kernel (D.1.2), which is relevant in the physical literature as explained above. However, none of two classes of kernels is included in the other. In addition, the arguments in [Gal87] seem to rely significantly on the discrete character of the coagulation process and it is not clear how difficult would be to extend these results to the continuous case. On the other hand, the sedimentation v_i can be arbitrary in [Gal87]. This opens the question of whether we can extend our result to powers of α which are larger than one for the model in (D.1.6). However, a different approach of obtaining information from the characteristics created by the transport term in (D.1.6) will be needed in this case.

Moreover, kernels of the form $K(v, v') = v^{\gamma} + v'^{\gamma}$, with $\gamma \in (1, 1 + \alpha)$, also satisfy the conditions (D.1.8), (D.1.9). Solutions of the standard one-dimensional coagulation equation do not exist for these types of kernels, see [BLL19b, BC90, CdC92, vD87]. This is to our knowledge the first result of existence of mass-conserving solutions involving sum kernels of homogeneity $\gamma > 1$, regardless of whether one considers one-dimensional or multi-dimensional coagulation models.

It is worth mentioning that our results also include the cases $\gamma = 0$ and $\gamma = 1$, which are normally studied separately in the literature due to their rich features, such as being able to predict the long-time behavior of solutions, see [BNV19, CMM10, MP04, MP06, TV19], or prove uniqueness of self-similar profiles, see [MP04, Thr21a, Thr21b], where the constant kernel and perturbations of the constant kernel are discussed.

The strategy for proving existence of solutions is to consider an iterative scheme based on a linear version of (D.1.6). For this equation, we are able to find a suitable supersolution, which in turn will provide sufficiently good moment estimates. This will give us compactness of the iterated sequence and enable us to pass to the limit in the equation. The idea of finding appropriate supersolutions was also used in [NNTV19], in this case for finding self-similar solutions with fat tails. The idea of considering a linear version of the model in order to better study properties of its solutions is common in the study of coagulation equations, see for example [Thr21a] where this idea is used to study uniqueness of solutions.

We present the proof of our main Theorem D.1.4 in the following Section D.2 with some technical computations moved to the appendix.

D.2 Proof of the main theorem

D.2.1 Formal approximation and discussion on the assumptions

Our approach to prove existence of a solution to (D.1.6) is based on constructing a suitable supersolution by approximating the coagulation term for large particles by a transport term. To motivate this we present in this subsection this formal approximation of (D.1.6). Similar computations can be found in [HNS07, Section 4 and Appendix 1].

Suppose now that f is a solution of (D.1.6). Since we are interested in the behavior for large values of v, we can assume, due to the fast decay of $f(x, v, t)$, that the term $\int_{[\frac{v}{2},\infty)} K(v, v') f(x, v, t)\, f(x, v', t) \mathrm{d}v'$ gives a small contribution. This is consistent with the known results in coagulation equations. We can then approximate (D.1.6) via

$$\begin{aligned}
&\partial_t f(x,v,t) + v^\alpha \partial_x f(x,v,t)\\
&= \int_{(0,\frac{v}{2})} K(v-v',v')f(x,v-v',t)f(x,v',t)\mathrm{d}v' - \int_{(0,\infty)} K(v,v')f(x,v,t)f(x,v',t)\mathrm{d}v'\\
&= \int_{(0,\frac{v}{2})} [K(v-v',v')f(x,v-v',t) - K(v,v')f(x,v,t)]f(x,v',t)\mathrm{d}v'\\
&\qquad - \int_{[\frac{v}{2},\infty)} K(v,v')f(x,v,t)f(x,v',t)\mathrm{d}v'\\
&\approx \int_{(0,\frac{v}{2})} [K(v-v',v')f(x,v-v',t) - K(v,v')f(x,v,t)]f(x,v',t)\mathrm{d}v'.
\end{aligned} \tag{D.2.1}$$

Since our strategy relies on finding a suitable supersolution, it suffices to find a lower bound for (D.2.1). This is where the assumption (D.1.9) is needed. We thus use that $K(v-v',v') \le K(v,v')$ when $v' \in [0,\frac{v}{2}]$ and obtain that

$$\begin{aligned}
&\partial_t f(x,v,t) + v^\alpha \partial_x f(x,v,t)\\
&\quad - \int_{(0,\frac{v}{2})} [K(v-v',v')f(x,v-v',t) - K(v,v')f(x,v,t)]f(x,v',t)\mathrm{d}v'\\
&\ge \partial_t f(x,v,t) + v^\alpha \partial_x f(x,v,t) - \int_{(0,\frac{v}{2})} K(v,v')[f(x,v-v',t) - f(x,v,t)]f(x,v',t)\mathrm{d}v'.
\end{aligned} \tag{D.2.2}$$

Assuming now that the coagulation kernel K behaves like $v'^\gamma + v^\gamma$ and since $v' \in [0,\frac{v}{2}]$, we further deduce that

$$\begin{aligned}
\partial_t f(x,v,t) + v^\alpha \partial_x f(x,v,t) &\approx \int_{(0,\frac{v}{2})} v^\gamma [f(x,v-v',t) - f(x,v,t)]f(x,v',t)\mathrm{d}v'\\
&\approx -v^\gamma \int_{(0,\frac{v}{2})} \int_{v-v'}^{v} \partial_w f(x,w,t)\mathrm{d}w f(x,v',t)\mathrm{d}v'.
\end{aligned} \tag{D.2.3}$$

Assume that $\partial_w f(x,w,t)$ behaves similarly for $w \in [\frac{v}{2},v]$ and thus

$$\partial_t f(x,v,t) + v^\alpha \partial_x f(x,v,t) \approx -v^\gamma \partial_v f(x,v,t) \int_{(0,\frac{v}{2})} v' f(x,v',t)\mathrm{d}v'. \tag{D.2.4}$$

We denote by $M_1(x,t) := \int_{(0,\infty)} v' f(x,v',t)\mathrm{d}v'$ the first moment in v of f. We consider only large values of v so that we can safely assume that $\int_{(0,\frac{v}{2})} v' f(x,v',t)\mathrm{d}v'$ contains most of the mass. In this manner, we can further approximate (D.2.4) by

$$\partial_t f(x,v,t) + v^\alpha \partial_x f(x,v,t) \approx -v^\gamma \partial_v f(x,v,t) M_1(x,t). \tag{D.2.5}$$

Notice that in order for our approximation to hold, the assumption (D.1.9) was needed in (D.2.2). Otherwise, an analogous approximation of the model could be obtained by replacing $v^\gamma \partial_v f(x,v,t)$ in

(D.2.4) by $\partial_v(v^\gamma f(x,v,t))$. The approximation containing the term $\partial_v(v^\gamma f(x,v,t))$ is the one actually used in [HNS07]. However, due to (D.1.9), the approximation used in (D.2.5) suffices in order to prove our desired result. Suppose now that $M_1(x,t)$ decays sufficiently fast for large values of x, that is assume that

$$M_1(x,t) \leq \frac{L}{1+|x|^{\overline{m}}}, \tag{D.2.6}$$

for some sufficiently large $\overline{m}$ and some $L>0$. Combining (D.2.6) with (D.2.5), we obtain that f should behave formally like the solution of the equation

$$\partial_t f(x,v,t) + v^\alpha \partial_x f(x,v,t) + \frac{L v^\gamma \partial_v f(x,v,t)}{1+|x|^{\overline{m}}} = 0. \tag{D.2.7}$$

This motivates the analysis of the equation (D.2.16) below when trying to find a supersolution for a linear version of (D.1.6). In order to obtain a behavior of the form (D.2.6) for $M_1(x,t)$, we have to work with functions f such that (D.1.14) holds.

D.2.2 Upper and lower bounds for the solution of the approximated model

To prove short time existence of a solution to (D.1.6) we will set up an iterative scheme and derive, using (D.2.7), a uniform supersolution for the solutions of this scheme. More precisely, for $n\in\mathbb{N}$, we define inductively a sequence of functions $\{f_n\}_{n\in\mathbb{N}}$ as follows:

$$\begin{aligned}\partial_t f_{n+1}(x,v,t) + v^\alpha \partial_x f_{n+1}(x,v,t) &= \frac{1}{2}\int_{(0,v)} K(v-v',v') f_{n+1}(x,v-v',t) f_n(x,v',t)\mathrm{d}v' \\ &\quad - \int_{(0,\infty)} K(v,v') f_{n+1}(x,v,t) f_n(x,v',t)\mathrm{d}v',\end{aligned} \tag{D.2.8}$$

with $\alpha\in(0,1)$, and

$$f_{n+1}(x,v,0) = f_n(x,v,0) = f_{\mathrm{in}}(x,v) \leq \frac{C_0}{1+|x|^m+v^p}, \qquad \text{for all } n\in\mathbb{N}. \tag{D.2.9}$$

We take f_0 to be a function such that

$$\partial_t f_0(x,v,t) + v^\alpha \partial_x f_0(x,v,t) = 0 \tag{D.2.10}$$

with

$$f_0(x,v,0) = f_{\mathrm{in}}(x,v) \leq \frac{C_0}{1+|x|^m+v^p}. \tag{D.2.11}$$

Notice that, if we have equality in (D.2.11), the solution of (D.2.10) is $f_0(x,v,t) = \frac{C_0}{1+|x-v^\alpha t|^m+v^p}$.

Remark D.2.1. In principle we have to prove that the sequence in (D.2.8) is well-defined. A rigorous proof would work as follows. First we approximate K with a kernel K_N which is such that

$$K_N(v, v') = K(v, v')\chi_N(v + v'), \tag{D.2.12}$$

for $N > 1$ and where $\chi_N : [0, \infty) \to [0, 1]$ is a continuous functions such that $\chi_N(x) = 1$, when $x \leq \frac{N}{2}$, and $\chi_N(x) = 0$, when $x \geq N$. Then we can establish with a standard fixed-point argument the existence and uniqueness of $f_{n,N}$. The key result is then that we obtain a uniform fast decaying bound for the sequence of solutions which is in particular independent of N. Then one can pass to the limit $N \to \infty$. Since this procedure is standard once one has the bounds on the solution, we omit the details here and work directly with K.

Notations and Assumptions: Let $R > 0$. In the following we will denote by

$$\xi_R \in C([0, \infty))\,, \quad \xi_R\colon [0, \infty) \to [0, 1] \text{ such that } \xi_R(v) = 1 \text{ if } v \geq 2R \text{ and } \xi_R(v) = 0 \text{ if } v \leq R. \tag{D.2.13}$$

Furthermore we assume

$$\alpha \in (0, 1)\,, \quad \gamma \in [0, 1 + \alpha)\,, \quad m \in \mathbb{N}\,, \quad m \text{ even}\,, \quad p = \alpha m\,, \quad m > \max\left\{\frac{2\gamma + 1}{\alpha}, \frac{2}{\alpha} + 3\right\}, \tag{D.2.14}$$

and define d via

$$\begin{cases} d = \left[\dfrac{2}{\alpha}\right] + 1; & \text{if } \left[\dfrac{2}{\alpha}\right] \text{ odd;} \\ d = \left[\dfrac{2}{\alpha}\right] + 2; & \text{if } \left[\dfrac{2}{\alpha}\right] \text{ even,} \end{cases} \tag{D.2.15}$$

where $[\cdot]$ denotes the floor function. Note that (D.2.14) and (D.2.15) in particular imply that $m > d + 1$.

The main goal in this section will be to derive estimates for the solution of a transport equation that approximates the coagulation equation. For $L > 0$ let G_L be the solution of

$$\partial_t G_L(x, v, t) + v^\alpha \partial_x G_L(x, v, t) + \frac{L v^\gamma}{1 + |x|^{m-d}} \xi_R(v) \partial_v G_L(x, v, t) = 0, \tag{D.2.16}$$

$$G_L(x, v, 0) = \frac{C_0}{1 + |x|^m + v^p}\,.$$

We first study the properties of the backward characteristics for equation (D.2.16). To this end, we define X and V via

$$\begin{cases} \partial_t X(x, v, t) = -V^\alpha, & X(x, v, 0) = x\,, \\ \partial_t V(x, v, t) = -\dfrac{L V^\gamma \xi_R(V)}{1 + |X|^{m-d}}, & V(x, v, 0) = v\,, \end{cases} \tag{D.2.17}$$

where d was defined in (D.2.15) and L is as in (D.2.16).

Proposition D.2.2 (Properties of the characteristics)**.** *Given $L > 0$ and $\delta \in (0, \frac{1}{2})$ there exists a sufficiently large $R > 0$ such that for all $t \geq 0$ the following estimates hold:*

$$(1-\delta)v \leq V(x, v, t) \leq (1+\delta)v; \tag{D.2.18}$$

$$(1-\delta)v^{\alpha}t \leq x - X(x, v, t) \leq (1+\delta)v^{\alpha}t; \tag{D.2.19}$$

$$\frac{\alpha}{18}v^{\alpha-1}t \leq -\partial_v X(x, v, t) \leq 18\alpha v^{\alpha-1}t. \tag{D.2.20}$$

Moreover, if $x \notin [(1-2\delta)v^{\alpha}t, (1+2\delta)v^{\alpha}t]$, then

$$\frac{1}{4} \leq \partial_v V(x, v, t) \leq \frac{9}{4}. \tag{D.2.21}$$

Otherwise, if $x \in [(1-2\delta)v^{\alpha}t, (1+2\delta)v^{\alpha}t]$, then

$$|\partial_v V(x, v, t)| \leq 36L \max\{1, v^{\gamma-1}t\}; \tag{D.2.22}$$

$$\partial_v V(x, v, t) \leq 2. \tag{D.2.23}$$

Corollary D.2.3. *As an immediate consequence of* (D.2.19) *we obtain the following estimates for all* $t \geq 0$.

If $x \leq 0$ then

$$|x| + (1-\delta)v^{\alpha}t = |x - (1-\delta)v^{\alpha}t| \leq |X(x, v, t)| \leq |x| + (1+\delta)v^{\alpha}t = |x - (1+\delta)v^{\alpha}t|. \tag{D.2.24}$$

If $x > 0$ then

$$|x - (1-\delta)v^{\alpha}t| \leq |X(x, v, t)| \leq |x - (1+\delta)v^{\alpha}t| \qquad \text{if } t \geq \frac{x}{(1-\delta)v^{\alpha}}, \tag{D.2.25}$$

$$|x - (1+\delta)v^{\alpha}t| \leq |X(x, v, t)| \leq |x - (1-\delta)v^{\alpha}t| \qquad \text{if } t \leq \frac{x}{(1+\delta)v^{\alpha}}, \tag{D.2.26}$$

and

$$|X(x, v, t)| \leq 3v^{\alpha}t \qquad \text{if } t \in \Big(\frac{x}{(1+\delta)v^{\alpha}}, \frac{x}{(1-\delta)v^{\alpha}}\Big). \tag{D.2.27}$$

Proof of Proposition D.2.2. **Proof of** (D.2.18)**:** First, we see from (D.2.17) that $X(x, v, t) \leq x$ and $V(x, v, t) \leq v$ for all $t \geq 0$. Next, we define

$$\psi(x) := \int_{-\infty}^{x} \frac{L\,\mathrm{d}\xi}{1 + |\xi|^{m-d}} \tag{D.2.28}$$

and $\Phi\colon (0, \infty)^2 \to \mathbb{R}$, $(z, v) \to \Phi(z, v)$ via

$$\Phi(\psi(X(x, v, t)) - \psi(x), v) := V(x, v, t). \tag{D.2.29}$$

Then it follows from (D.2.17) that

$$\begin{cases} \partial_z \Phi(\cdot, v) = \Phi^{\gamma-\alpha} \xi_R(\Phi)\,, & z > 0 \\ \Phi(0, v) = v\,. \end{cases} \tag{D.2.30}$$

By integrating the ODE in (D.2.30) we deduce that

$$\Phi(z, v) = v \qquad \text{for all } z \geq 0\,, v \in [0, R]\,, \tag{D.2.31}$$

$$\Phi(z, v) = (v^{1-(\gamma-\alpha)} + (1 - (\gamma - \alpha))z)^{\frac{1}{1-(\gamma-\alpha)}} \qquad \text{for all } z \geq 0\,, v \geq 2R\,, \tag{D.2.32}$$

$$\Phi(z, v) \leq (v^{1-(\gamma-\alpha)} + (1 - (\gamma - \alpha))z)^{\frac{1}{1-(\gamma-\alpha)}} \qquad \text{for all } z \geq 0\,,\ v \geq R\,, \tag{D.2.33}$$

$$\Phi(z, v) \geq v \qquad \text{for all } z \geq 0\,,\ v \geq R\,. \tag{D.2.34}$$

Notice also that $0 \leq \psi(x) \leq CL$, for some constant $C > 0$ which is independent of x. It thus suffices to consider values of z in the interval $[0, CL]$. Taking in particular R sufficiently large, it follows that $(1 - \delta)v \leq \Phi(z, v) \leq (1 + \delta)v$ for all $z \in [0, CL]$ and $v \geq 0$. Due to the definition of Φ in (D.2.29), the estimate (D.2.18) follows.

Proof of (D.2.19): Estimate (D.2.19) follows then from (D.2.18) and the relation

$$X(x, v, t) - x = -\int_0^t V^\alpha(x, v, \tau) \mathrm{d}\tau \tag{D.2.35}$$

together with the fact that $(1 + \delta)^\alpha \leq 1 + \delta$ and $(1 - \delta)^\alpha \geq 1 - \delta$.

Proof of (D.2.20): In order to estimate the derivatives with respect to v of the characteristics we first prove that

$$\frac{1}{2} \leq \partial_v \Phi(z, v) \leq 2. \tag{D.2.36}$$

Estimate (D.2.36) is immediate if $v \leq R$. If $v \geq R$ we have

$$\frac{\mathrm{d}}{\mathrm{d}z}\Big(\frac{\mathrm{d}\Phi}{\mathrm{d}v}\Big) = \Big[(\gamma - \alpha)\Phi^{\gamma-\alpha-1}\xi_R(\Phi) + \Phi^{\gamma-\alpha}\xi_R'(\Phi)\Big]\frac{\mathrm{d}\Phi}{\mathrm{d}v}\,, \qquad \frac{\mathrm{d}\Phi}{\mathrm{d}v}(0) = 1$$

such that

$$\frac{\mathrm{d}\Phi}{\mathrm{d}v} = \exp\Big(\int_0^z [(\gamma - \alpha)\Phi^{\gamma-\alpha-1}\xi_R(\Phi) + \Phi^{\gamma-\alpha}\xi_R'(\Phi)]\mathrm{d}s\Big) \leq \exp\Big(CR^{\gamma-\alpha-1}\Big) \tag{D.2.37}$$

and the upper bound in (D.2.36) follows since $\gamma < 1 + \alpha$ and if R is sufficiently large. Analogously, we obtain that

$$\frac{\mathrm{d}\Phi}{\mathrm{d}v} \geq \exp\Big(-CR^{\gamma-\alpha-1}\Big)$$

and (D.2.36) follows. In order to proceed, we notice that

$$\int_x^X \frac{1}{\Phi(\psi(\xi) - \psi(x), v)^\alpha} \mathrm{d}\xi = -t. \tag{D.2.38}$$

Differentiating in v we obtain

$$\frac{\partial_v X(x, v, t)}{\Phi(\psi(X(x, v, t)) - \psi(x), v)^\alpha} = \alpha \int_x^X \frac{\partial_v \Phi(\psi(\xi) - \psi(x), v)}{\Phi(\psi(\xi) - \psi(x), v)^{\alpha+1}} \mathrm{d}\xi. \tag{D.2.39}$$

Since $\delta \in (0, \frac{1}{2})$, (D.2.18) and (D.2.19) imply that $\frac{v}{2} \leq V(x, v, t) \leq \frac{3v}{2}$ and that $\frac{v^\alpha t}{2} \leq x - X(x, v, t) \leq \frac{3v^\alpha t}{2}$. From this we deduce, using (D.2.19), the estimates for Φ, and the fact that $\alpha < 1$, that

$$\begin{aligned} -\partial_v X(x, v, t) =& \alpha \int_X^x \frac{\partial_v \Phi(\psi(\xi) - \psi(x), v)}{\Phi(\psi(\xi) - \psi(x), v)^{\alpha+1}} \mathrm{d}\xi \times [\Phi(\psi(X(x, v, t)) - \psi(x), v)]^\alpha \\ &\leq 4\alpha 3^\alpha \int_X^x \frac{1}{v^{\alpha+1}} \mathrm{d}\xi v^\alpha \leq 12\alpha v^{-1}[x - X(x, v, t)] \leq 18\alpha v^{\alpha-1} t. \end{aligned} \tag{D.2.40}$$

Analogously we obtain that

$$\begin{aligned} -\partial_v X \geq& \frac{\alpha}{2} \int_X^x \frac{1}{\Phi(\psi(\xi) - \psi(x), v)^{\alpha+1}} \mathrm{d}\xi \times [\Phi(\psi(X(x, v, t)) - \psi(x), v)]^\alpha \\ &\geq \frac{\alpha}{3^{\alpha+1}} \int_X^x \frac{1}{v^{\alpha+1}} \mathrm{d}\xi v^\alpha \leq \frac{\alpha}{9} v^{-1}[x - X(x, v, t)] \geq \frac{\alpha}{18} v^{\alpha-1} t, \end{aligned}$$

which concludes the proof of (D.2.20).

Proof of (D.2.21)**:** We now prove that (D.2.21) holds if $x \notin [(1-2\delta)v^\alpha t, (1+2\delta)v^\alpha t]$. From (D.2.29) we deduce that

$$\partial_v V(x, v, t) = \partial_v \Phi(\psi(X) - \psi(x), v) + \partial_z \Phi(\psi(X) - \psi(x), v)\psi'(X)\partial_v X. \tag{D.2.41}$$

Due to (D.2.36) it suffices to show that

$$|\partial_z \Phi(\psi(X) - \psi(x), v)\psi'(X)\partial_v X| \leq \frac{1}{4} \tag{D.2.42}$$

in order to conclude our proof. From (D.2.29), (D.2.30) and (D.2.18), we have that

$$0 \leq \partial_z \Phi(z, v) \leq 2v^{\gamma-\alpha}.$$

Indeed, by (D.2.28) and (D.2.18) it holds that

$$\partial_z \Phi(z, v) \leq \Phi(z, v)^{\gamma-\alpha} \leq \max\{2^{\alpha-\gamma}, \left(\frac{3}{2}\right)^{\gamma-\alpha}\} v^{\gamma-\alpha} \leq 2v^{\gamma-\alpha}.$$

By (D.2.20), it thus follows that

$$|\partial_z \Phi(\psi(X) - \psi(x), v)\psi'(X)\partial_v X| \leq \frac{36\alpha L v^{\gamma-\alpha}}{1 + |X(t)|^{m-d}} v^{\alpha-1} t = \frac{36\alpha L v^{\gamma-1} t}{1 + |X(t)|^{m-d}}. \tag{D.2.43}$$

We only analyze here the case when $v \geq R$, since by (D.2.30) we have that $\partial_z \Phi(z, v) = 0$ when $v \leq R$ and thus there is nothing to prove in this case.

1) Assume $x \leq (1-2\delta)v^\alpha t$.

Then by (D.2.25) we have $|X(x,v,t)| \geq |x-(1-\delta)v^\alpha t| = (1-\delta)v^\alpha t - x \geq \delta v^\alpha t$ and we obtain

$$|\partial_z \Phi(\psi(X)-\psi(x),v)\psi'(X)\partial_v X| \leq CL\frac{v^{\gamma-1}t}{1+(\delta v^\alpha t)^{m-d}}.$$

Now, if $v^{\gamma-1}t \geq \frac{1}{4LC}$, then $v^\alpha t \geq \frac{1}{4LC}$ since $\alpha > \gamma - 1$ and we obtain

$$\begin{aligned}|\partial_z \Phi(\psi(X)-\psi(x),v)\psi'(X)\partial_v X| &\leq CL\frac{v^{\gamma-1}t}{1+(\delta v^\alpha t)^{m-d}} \leq \frac{CL}{\delta}v^{\gamma-\alpha-1}(\delta v^\alpha t)^{1+d-m}\\ &\leq C(L,\delta)v^{\gamma-\alpha-1} \leq \frac{1}{4}\end{aligned}$$

if $R>0$ is sufficiently large. If $v^{\gamma-1}t \leq \frac{1}{4LC}$, (D.2.43) becomes

$$|\partial_z \Phi(\psi(X)-\psi(x),v)\psi'(X)\partial_v X| \leq \frac{CLv^{\gamma-1}t}{1+(\delta v^\alpha t)^{m-d}} \leq \frac{1}{4}. \tag{D.2.44}$$

2) Assume $x \geq (1+2\delta)v^\alpha t$. Then $|X(x,v,t)| \geq |x-(1+\delta)v^\alpha t| = x-(1+\delta)v^\alpha t \geq \delta v^\alpha t$ and we can conclude as before.

Proof of (D.2.22) **and** (D.2.23)**:** If $x \in [(1-2\delta)v^\alpha t, (1+2\delta)v^\alpha t]$, then from (D.2.43) it follows that

$$|\partial_z \Phi(\psi(X)-\psi(x),v)\psi'(X)\partial_v X| \leq \frac{36Lv^{\gamma-1}t}{1+|X(t)|^{m-d}} \leq 36Lv^{\gamma-1}t$$

and (D.2.22) follows.

In order to prove that (D.2.23) holds, we notice that

$$\partial_z \Phi(\psi(X)-\psi(x),v)\psi'(X)\partial_v X \leq 0.$$

Combining this with (D.2.41) and (D.2.36), the conclusion follows. □

With the help of the characteristics, the solution G_L of (D.2.16) can be written as

$$G_L(x,v,t) = \frac{C_0}{1+|X(x,v,t)|^m + V(x,v,t)^p}. \tag{D.2.45}$$

Moreover, it holds that

$$\partial_v G_L(x,v,t) = -C_0\frac{[m|X|^{m-2}X\partial_v X + pV^{p-1}\partial_v V]}{(1+|X|^m+V^p)^2}(x,v,t). \tag{D.2.46}$$

This function G_L will be the main building block for constructing a uniform supersolution to the sequence $\{f_n\}_{n\in\mathbb{N}}$ in Subsection D.2.3. We would like this supersolution to be decreasing in v for fixed x. Unfortunately, for $x>0$ the function G_L is not decreasing in v. The next proposition characterizes a local maximum of G_L.

Proposition D.2.4. *a) Given $L > 0$ and $\delta \in (0, \frac{1}{2})$ there exists a sufficiently large $R > 0$ such that for all $t \in [0, 1]$ the following holds. For every $x \in \mathbb{R}$ and $t \in [0, 1]$ there exists at least one point $v_{\max}(x, t)$ with the properties that*

$$v_{\max}(x, t)^{\alpha} \notin \left[\frac{x}{(1 + 2\delta)t}, \frac{x}{(1 - 2\delta)t} \right] \tag{D.2.47}$$

and

$$\partial_v G_L(x, v_{\max}(x, t), t) = 0 \, . \tag{D.2.48}$$

Moreover, there exists a constant $K_{max} > 0$, which is independent of x, t, δ, R, and L, such that the following holds:

$$\frac{1}{K_{max}} x t^{\frac{1}{m-1}} \leq v_{\max}(x, t)^{\alpha} \leq K_{max} x t^{\frac{1}{m-1}} . \tag{D.2.49}$$

b) Given $L > 0$ and $\delta \in (0, \frac{1}{2})$ there exists a sufficiently large $R > 0$ such that for all $t \in [0, T]$, with T sufficiently small, that is independent of L, δ, and R, there exists a unique point $v_{\max}$ with the properties (D.2.47) *and* (D.2.48).

Proof. Let $v^{\alpha} \notin \left[\frac{x}{(1+2\delta)t}, \frac{x}{(1-2\delta)t} \right]$ such that (D.2.48) holds. Notice that from (D.2.46), it follows that $m|X|^{m-2} X \partial_v X + p V^{p-1} \partial_v V = 0$. If R is sufficiently large, then the estimates from Proposition D.2.2 hold. We have the following cases.

Case 1. $x - (1 + 2\delta) v^{\alpha} t \geq 0$. Then, by (D.2.19), it holds that $0 \leq x - (1 + \delta) v^{\alpha} t \leq X(x, v, t) \leq x - (1 - \delta) v^{\alpha} t$ and by (D.2.20), it follows that $\partial_v X \leq 0$. Thus, if (D.2.48) holds, we have that $m|X|^{m-1} |\partial_v X| = p V^{p-1} \partial_v V$. By Proposition D.2.2 and since $\delta \in (0, \frac{1}{2})$, we have that there exists a constant $C > 0$, that is independent of δ, L and R, such that

$$\frac{1}{C} |x - (1 + \delta) v^{\alpha} t|^{m-1} v^{\alpha - 1} t \leq m|X|^{m-1} |\partial_v X| = p V^{p-1} \partial_v V \leq C v^{p-1} .$$

This implies that $\frac{1}{C} |x - (1 + \delta) v^{\alpha} t|^{m-1} t \leq v^{\alpha(m-1)}$. Since $m - 1$ is odd, $\delta < 1$, and $x - (1 + 2\delta) v^{\alpha} t \geq 0$, we further have that $x t^{\frac{1}{m-1}} \leq C v^{\alpha} + 2 v^{\alpha} t^{\frac{m}{m-1}}$. Since $t \leq 1$, we then obtain that $x t^{\frac{1}{m-1}} \leq C v^{\alpha}$ and the lower bound in (D.2.49) follows. In order to obtain the upper bound in (D.2.49), we use similar computations together with the fact that $t \geq 0$ and that

$$C |x - (1 - \delta) v^{\alpha} t|^{m-1} v^{\alpha - 1} t \geq m|X|^{m-1} |\partial_v X| = p V^{p-1} \partial_v V \geq \frac{1}{C} v^{p-1} .$$

Case 2. $x - (1 - 2\delta) v^{\alpha} t \leq 0$. Then, by (D.2.19), it holds that $x - (1 + \delta) v^{\alpha} t \leq X(x, v, t) \leq x - (1 - \delta) v^{\alpha} t \leq 0$ and thus $m|X|^{m-2} X \partial_v X + p V^{p-1} \partial_v V = m|X|^{m-1} |\partial_v X| + p V^{p-1} \partial_v V > 0$ in this region. Using this and (D.2.46), it follows that $\partial_v G_L(x, v, t) < 0$.

In Appendix D.3, Proposition D.3.1, we will prove that for a sufficiently small $T > 0$, which is independent of L, δ, and R, we have that

$$\partial_v^2 G_L(x,v,t) < 0, \tag{D.2.50}$$

for all v that satisfy the estimate in (D.2.49) and all $t \in [0,T]$ which implies the uniqueness of such a point. This concludes our proof. □

The following lemma will also be needed in the construction of a supersolution in Section D.2.3.

Proposition D.2.5. *Given $L > 0$ and $\delta \in (0, \frac{1}{2})$ there exists a sufficiently large $R > 0$ such that for all $t \in [0,T]$, with T sufficiently small, which is independent of L, δ, and R, it holds that*

$$\partial_x G_L(x, v_{\max}(x,t), t) \leq 0, \tag{D.2.51}$$

where G_L is the solution of (D.2.16) and $v_{\max}(x,t)$ was defined in (D.2.48).

Proof. If R is sufficiently large, then the estimates from Proposition D.2.2 hold. We first notice that due to (D.2.19) and (D.2.49) we have

$$X(x, v_{\max}(x,t), t) \geq x - 2v_{\max}(x,t)^{\alpha} t \geq x - 2K_{max} x t^{1+\frac{1}{m-1}} > 0$$

if $t \in [0,T]$ and T is sufficiently small. Then we compute

$$\partial_x G_L(x,v,t) = -C_0 \frac{[m|X|^{m-2} X \partial_x X + pV^{p-1}\partial_x V]}{(1+|X|^m + V^p)^2}(x,v,t)\,,$$

and thus it suffices to prove that

$$\Big(mX^{m-1}\partial_x X + pV^{p-1}\partial_x V\Big)(x, v_{\max}(x,t), t) \geq 0\,. \tag{D.2.52}$$

Since $V \geq 0$ we have to prove that $\partial_x X(x, v_{\max}(x,t), t) \geq 0$ and $\partial_x V(x, v_{\max}(x,t), t) \geq 0$. We start by analyzing $\partial_x X$.

We consider the case when $v \geq 2R$, since the other cases work similarly. Differentiating (D.2.38) with respect to x, keeping in mind that $x > 0$, we obtain

$$\frac{\partial_x X(x,v,t)}{\Phi(\psi(X(x,v,t)) - \psi(x), v)^{\alpha}} + \alpha \int_x^X \frac{\partial_z \Phi \partial_x \psi(x) \mathrm{d}\xi}{\Phi(\psi(\xi) - \psi(x), v)^{\alpha+1}} - \frac{1}{\Phi(0,v)^{\alpha}} = 0. \tag{D.2.53}$$

We have that $\partial_z \Phi \geq 0$ due to (D.2.30). Since $X(x,v,t) \leq x$ and by (D.2.29) and (D.2.18) we obtain

$$\partial_x X(x,v,t) \geq \frac{\Phi(\psi(X(x,v,t)) - \psi(x), v)^{\alpha}}{v^{\alpha}} \geq \frac{1}{C} > 0\,.$$

Thus, in order for (D.2.52) to hold, what is left to prove is that $\partial_x V(x, v_{\max}(x,t), t) \geq 0$. Integrating (D.2.17) over time and differentiating with respect to x, we obtain

$$V(x,v,t)^{-\gamma} \partial_x V(x,v,t) = \int_0^t \frac{L(m-d)|X|^{m-d-2} X \partial_x X \mathrm{d}\xi}{(1+X(\xi)^{m-d})^2}. \tag{D.2.54}$$

Since we have $X(x, v_{\max}(x,t), t) \geq 0$ and $\partial_x X(x,v,t) \geq 0$ we obtain $\partial_x V(x, v_{\max}(x,t), t) \geq 0$. □

D.2.3 Construction of a supersolution for a linear reformulation of the model

In this subsection we will prove that the solution G_L of (D.2.16), after suitable modifications, is a supersolution for problem (D.2.8).

Definition D.2.6. Let $\delta \in (0, \frac{1}{2})$, $L > 0$ and R, which depends on L and δ, as in Proposition D.2.2. Let G_L be the solution of (D.2.16) with given δ, L, and R. We define the function H_L via

$$H_L(x, v, t) = \begin{cases} G_L(x, v, t), & \text{if } \partial_v G_L(x, v, t) \le 0 \;\text{ or }\; x \in [(1-2\delta)v^\alpha t, (1+2\delta)v^\alpha t]; \\ G_L(x, v_{\max}(x, t), t), & \text{otherwise,} \end{cases}$$

where $v_{\max}(x, t)$ was defined in (D.2.48).

By the choice of $v_{\max}$ in (D.2.48), if $x \notin [(1-2\delta)v^\alpha t, (1+2\delta)v^\alpha t]$ and $v \le v_{\max}(x, t)$, then we have that $H_L(x, v, t) = G_L(x, v_{\max}(x, t), t)$. Moreover, H_L is decreasing in v for fixed x outside possibly a critical region where $x \in [(1-2\delta)v^\alpha t, (1+2\delta)v^\alpha t]$.

We first collect some properties of the function H_L, which are independent of δ, L, and R.

Lemma D.2.7. *Let $\delta \in (0, \frac{1}{2})$, $L > 0$, and R as in Proposition D.2.2. Let $T > 0$ be sufficiently small, independent of δ, L, and R, and H_L as in Definition D.2.6. Then there exists a constant $K_2 > 0$, which is independent of δ, L, and R, such that the following holds for $t \in [0, T]$.*

If $x > 0$ then

$$H_L(x, v - v', t) \le K_2 H_L(x, v, t) \text{ for all } v' \in \left(0, \frac{v}{2}\right) \tag{D.2.55}$$

If $x > 0$ and for all v such that $v^\alpha \notin [\frac{x}{(1+2\delta)t}, \frac{x}{(1-2\delta)t}]$ then there exists a sufficiently large constant $C_l > 0$ such that

$$-\partial_v H_L(x, v', t) \le -K_2 \partial_v H_L(x, v, t) \qquad \text{for all } v \ge \max\{R, C_l v_{\max}(x, t)\}, v' \in \left(\tfrac{v}{2}, v\right). \tag{D.2.56}$$

If $x \le 0$ then

$$-\partial_v H_L(x, v', t) \le -K_2 \partial_v H_L(x, v, t) \qquad \text{for all } v \ge R, v' \in (\tfrac{v}{2}, v). \tag{D.2.57}$$

Proof. **Proof of (D.2.55):** We will prove that (D.2.55) holds by proving separately that there exists $K_2 > 0$, which is independent of δ, L, and R, such that:

If $x > 0$ and for all v such that $v^\alpha \notin [\frac{x}{(1+2\delta)t}, \frac{x}{(1-2\delta)t}]$ then

$$H_L(x, v - v', t) \le K_2 H_L(x, v, t) \qquad \text{for all } v' \in \left(0, \tfrac{v}{2}\right) \text{ and } v, v - v' \ge v_{\max}(x, t), \tag{D.2.58}$$

$$H_L(x, v_{\max}(x, t), t) \le K_2 H_L(x, v, t) \qquad \text{for all } v \in (v_{\max}(x, t), 2v_{\max}(x, t)), \tag{D.2.59}$$

If $x > 0$ and for all v such that $v^\alpha \in [\frac{x}{(1+2\delta)t}, \frac{x}{(1-2\delta)t}]$ then

$$H_L(x, v - v', t) \le K_2 H_L(x, v, t) \text{ for all } v' \in \left(0, \frac{v}{2}\right). \tag{D.2.60}$$

Before beginning our proof, we make the following observation. Because the proof of each region when $v^\alpha \notin [\frac{x}{(1+2\delta)t}, \frac{x}{(1-2\delta)t}]$ differs, we have to distinguish between different cases.

If $v^\alpha \geq \frac{x}{(1-2\delta)t}$ and $w \in [\frac{v}{2}, v]$, we have the subcases

1. a) $x > 0$, $v^\alpha \geq \frac{x}{(1-2\delta)t}$, and $w^\alpha \geq \frac{x}{(1-2\delta)t}$. Notice that in this region $x-(1-\delta)w^\alpha t \leq 0$;

1. b) $x > 0$, $v^\alpha \geq \frac{x}{(1-2\delta)t}$, and $w^\alpha \in \left[\frac{x}{(1+2\delta)t}, \frac{x}{(1-2\delta)t}\right]$;

1. c) $x > 0$, $v^\alpha \geq \frac{x}{(1-2\delta)t}$, and $w^\alpha \leq \frac{x}{(1+2\delta)t}$. Notice that in this region $x-(1+\delta)w^\alpha t \geq 0$.

The remaining case is

2. $x > 0$, $v^\alpha \leq \frac{x}{(1+2\delta)t}$ and then $w^\alpha \leq \frac{x}{(1+2\delta)t}$.

Proof of (D.2.58): We will prove that there exists a constant $C > 0$, which is independent of δ, L, and R, such that

$$\frac{1}{C(1+|x|^m+v^p)} \leq H_L(x,w,t) \leq \frac{C}{1+|x|^m+v^p}, \tag{D.2.61}$$

for all $w \in [\frac{v}{2}, v]$.

Case 1. *a)* Notice that because of (D.2.19), we are in the region where $X(x,w,t) \leq 0$. Because of (D.2.25), it follows that $|x-(1-\delta)w^\alpha t| \leq |X(x,w,t)| \leq |x-(1+\delta)w^\alpha t|$. Thus, due to (D.2.45), it suffices to show in this case that

$$|x-(1+\delta)w^\alpha t|^m + w^p \leq C(|x|^m+v^p) \leq C\Big(|x-(1-\delta)w^\alpha t|^m + w^p\Big). \tag{D.2.62}$$

To prove (D.2.62) we notice that is suffices to show that $|x-(1+\delta)w^\alpha t|^m + w^p \leq C(|x|^m+w^p) \leq C(|x-(1-\delta)w^\alpha t|^m + w^p)$ since $w \in [\frac{v}{2}, v]$. Since $x-(1-\delta)w^\alpha t \leq 0$ in this case, it holds that $|x-(1-\delta)w^\alpha t| \leq w^\alpha$. In this case, we have that $|x|-(1-\delta)w^\alpha t \leq |x-(1-\delta)w^\alpha t| \leq w^\alpha$. This implies that $|x| \leq 2w^\alpha$. Notice that since $x \geq 0$ and $x-(1-\delta)w^\alpha t \leq 0$, it immediately follows that $0 \leq x \leq w^\alpha$. However, we do a more general proof as a similar estimate will be needed in order to prove (D.2.57) or when $x-(1+\delta)w^\alpha \geq 0$ later on.

Since $(|x|+(1+\delta)w^\alpha t)^m \leq C_m(|x|^m+t^m w^p)$ and t is sufficiently small, we obtain

$$\begin{aligned}|x-(1+\delta)w^\alpha t|^m + w^p &\leq (|x|+(1+\delta)w^\alpha t)^m + w^p \leq C_m(|x|^m+t^m w^p) + w^p \\ &\leq C(|x|^m+w^p).\end{aligned} \tag{D.2.63}$$

Additionally, since $|x| \leq 2w^\alpha$, it holds that

$$|x-(1-\delta)w^\alpha t|^m + w^p \geq w^p = \frac{w^p}{2} + \frac{w^p}{2} \geq \frac{1}{C}(|x|^m+w^p)\,. \tag{D.2.64}$$

Case 1. *b)* From (D.2.19) and (D.2.27), it follows that $X(x, v, t) \le 0$ and $|X(x, w, t)| \le Cw^{\alpha}t$. Since in this case $v^{\alpha} \ge \frac{x}{(1-2\delta)t}$, from (D.2.62) we know that there exists a constant $C > 0$, which is independent of δ, L, and R, such that

$$\frac{1}{C(1+|x|^m+v^p)} \le H_L(x, v, t).$$

Thus, in order for (D.2.58) to hold in this case, we need to prove that there exists a constant $C > 0$, which is independent of δ, L, and R, such that $H_L(x, w, t) \le \frac{C}{1+|x|^m+v^p}$, for $w \in [\frac{v}{2}, v]$. More precisely, due to (D.2.45), it suffices to prove that

$$1 + |X(x, w, t)|^m + V(x, w, t)^p \ge \frac{1}{C}(1 + |x|^m + v^p). \tag{D.2.65}$$

Since $t \le 1$, we have that $x \le (1 + 2\delta)w^{\alpha}t \le 2w^{\alpha}$ in this case. Due to (D.2.18), it holds that

$$|X(x, w, t)|^m + V(x, w, t)^p \ge V(x, w, t)^p \ge Cw^p = \frac{Cw^p}{2} + \frac{Cw^p}{2} \ge C(|x|^m + w^p).$$

Case 1. *c)* From (D.2.19), it follows that $X(x, v, t) \le 0$ and $X(x, w, t) \ge 0$. Since in this case $v^{\alpha} \ge \frac{x}{(1-2\delta)t}$, from (D.2.62) we know that there exists a constant $C > 0$, independent of δ, L, and R, such that $\frac{1}{C(1+|x|^m+v^p)} \le H_L(x, v, t)$ and we need to prove that $H_L(x, w, t) \le \frac{C}{1+|x|^m+v^p}$, for $w \in [\frac{v}{2}, v]$. More precisely, due to (D.2.26) and (D.2.45), it suffices to prove as before that $1 + |X(x, w, t)|^m + V(x, w, t)^p \ge C(1 + |x|^m + v^p)$. Actually, we prove here a more general estimate that will be used in Case 2., namely

$$|x - (1-\delta)w^{\alpha}t|^m + w^p \le C(|x|^m + v^p) \le C\Big(|x - (1+\delta)w^{\alpha}t|^m + w^p\Big) \tag{D.2.66}$$

In order to prove (D.2.66), we distinguish between two cases:

i) $|x - (1+\delta)w^{\alpha}t| \ge w^{\alpha}$. In this case, we have that $w^{\alpha} \le |x - (1+\delta)w^{\alpha}t| \le |x| + (1+\delta)w^{\alpha}t$, which implies that $\frac{1}{2}w^{\alpha} \le |x|$. Thus $|x| - (1+\delta)w^{\alpha}t \ge |x| - \frac{w^{\alpha}}{4} \ge \frac{|x|}{2} > 0$ and $(|x| - (1+\delta)w^{\alpha}t)^m \ge \frac{|x|^m}{2^m}$. Since $|x| - (1+\delta)w^{\alpha}t > 0$ and remembering that m is even, we also have that $|x - (1+\delta)w^{\alpha}t|^m \ge (|x| - (1+\delta)w^{\alpha}t)^m$ and thus $|x - (1+\delta)w^{\alpha}t|^m \ge 2^{-m}|x|^m$. Additionally , since t is sufficiently small and $\frac{1}{2}w^{\alpha} \le |x|$, it holds that $|x - (1-\delta)w^{\alpha}t| \le |x| + w^{\alpha}t \le 2|x|$.

ii) $|x - (1+\delta)w^{\alpha}t| \le w^{\alpha}$. This case can be treated as in the proof of (D.2.62).

Case 2. From (D.2.19), it follows that $X(x, v, t) \ge 0$ and $X(x, w, t) \ge 0$. Thus, from (D.2.26) and (D.2.66), the conclusion follows.

Proof of (D.2.59)**:** Notice that due to (D.2.49) and since $v \in (v_{\max}(x, t), 2v_{\max}(x, t))$, we have that $x - (1 + 2\delta)v^{\alpha}t \ge 0$ if we choose t to be sufficiently small. Moreover, since $\delta < \frac{1}{2}$, t can be chosen independently of δ. Since $v > v_{\max}(x, t)$, it holds by (D.2.66) that $\frac{1}{C(1+|x|^m+v^p)} \le H_L(x, v, t)$. Thus, since

$v \in (v_{\max}(x,t), 2v_{\max}(x,t))$, we further have that $\frac{1}{C(1+|x|^m+v_{\max}(x,t)^p)} \leq H_L(x,v,t)$. From (D.2.45) and (D.2.26), it suffices to prove that there exists a constant $C > 1$, independent of δ, L, and R, such that

$$|x|^m + v_{\max}(x,t)^p \leq C\Big(|x-(1+\delta)v_{\max}(x,t)^\alpha t|^m + v_{\max}(x,t)^p\Big).$$

The inequality holds since due to (D.2.49) we can choose t sufficiently small such that $|x-(1+\delta)v_{\max}(x,t)^\alpha t| = x-(1+\delta)v_{\max}(x,t)^\alpha t \geq \frac{x}{2}$.

Proof of (D.2.60)**:** Due to (D.2.45) and since $v - v' \in (\frac{v}{2}, v)$ it holds on one side that

$$H_L(x, v-v', t) \leq \frac{C}{1+(v-v')^p} \leq \frac{C}{1+v^p},$$

for some $C > 0$, independent of δ, L, and R. On the other side, similarly with (D.2.27) and from (D.2.45), we have that

$$H_L(x,v,t) \geq \frac{C}{1+v^p t^m + v^p} \geq \frac{C}{1+v^p},$$

for some $C > 0$, independent of δ, L, and R. Combining the two inequalities, we can conclude that (D.2.60) holds.

Proof of (D.2.56)**:**

Case 1. *a*) We will prove that in this case it holds that

$$\frac{1}{C}\frac{v^{p-1}}{(1+|x|^m+v^p)^2} \leq -\partial_v H_L(x,w,t) \leq C\frac{v^{p-1}}{(1+|x|^m+v^p)^2}, \tag{D.2.67}$$

for some $C > 0$, independent of δ, L, and R, and for all $w \in [\frac{v}{2}, v]$. We remember we are in the case when $x-(1+\delta)w^\alpha t \leq x-(1-\delta)w^\alpha t \leq 0$ and thus due to (D.2.19) it holds that $|X|^{m-2}(x,w,t)X(x,w,t)\partial_v X(x,w,t)(x,w,t) = |X|^{m-1}|\partial_v X(x,w,t)|$. Due to (D.2.18)-(D.2.21) we thus have in this region that

$$\begin{aligned}
-\frac{1}{C}(x-(1-\delta)w^\alpha t)^{m-1}w^{\alpha-1}t + \frac{1}{C}(1-\delta)w^{p-1} &\leq |X|^{m-2}X\partial_v X(x,w,t) + V^{p-1}\partial_v V(x,w,t)\\
&\leq -C(x-(1+\delta)w^\alpha t)^{m-1}w^{\alpha-1}t + C(1+\delta)w^{p-1}.
\end{aligned}$$

Moreover, from (D.2.62), we have that

$$|x-(1+\delta)w^\alpha t|^m + w^p \leq C(|x|^m+v^p) \leq C\Big(|x-(1-\delta)w^\alpha t|^m + w^p\Big). \tag{D.2.68}$$

In order to prove (D.2.67), due to (D.2.46), it then suffices to show in this case that

$$|x-(1+\delta)w^\alpha t|^{m-1}w^{\alpha-1}t + w^{p-1} \leq Cv^{p-1} \leq C\Big(|x-(1-\delta)w^\alpha t|^{m-1}w^{\alpha-1}t + w^{p-1}\Big), \tag{D.2.69}$$

for all $w \in [\frac{v}{2}, v]$. We thus prove (D.2.69). We have

$$w^{\alpha-1}t|x-(1+\delta)w^{\alpha}t|^{m-1}+w^{p-1}=w^{\alpha-1}\Big(|x-(1+\delta)w^{\alpha}t|^{m-1}t+w^{\alpha(m-1)}\Big). \tag{D.2.70}$$

Since $w \in [\frac{v}{2}, v]$, it suffices to prove

$$\tilde{J}:=|x-(1+\delta)w^{\alpha}t|^{m-1}t+w^{\alpha(m-1)}\leq Cw^{\alpha(m-1)} \tag{D.2.71}$$

and

$$|x-(1-\delta)w^{\alpha}t|^{m-1}t+w^{\alpha(m-1)}\geq \frac{1}{C}w^{\alpha(m-1)}.$$

We are in the case $v \geq C_l v_{\max}(x,t)$, for some sufficiently large $C_l > 0$ and $x > 0$. We know that $v_{\max}(x,t)^{\alpha} \geq \frac{1}{K_{max}} x t^{\frac{1}{m-1}}$, for K_{max} as in (D.2.49), and thus $w^{\alpha} \geq \frac{1}{C} x t^{\frac{1}{m-1}} \geq 0$ since $w \geq \frac{v}{2}$.

We have that $|x-(1-\delta)w^{\alpha}t|^{m-1}t+w^{\alpha(m-1)} \geq w^{\alpha(m-1)}$. On the other hand, since $0 \leq x t^{\frac{1}{m-1}} \leq Cw^{\alpha}$ and $t \leq 1$, it follows that

$$\tilde{J}(x,w,t)\leq C_m(|x|^{m-1}t+w^{\alpha(m-1)}t^{m})+w^{\alpha(m-1)}\leq Cw^{\alpha(m-1)}.$$

Remark D.2.8. Since $v^{\alpha} \geq \frac{x}{(1-2\delta)t}$, we know from (D.2.67) that $\frac{1}{C}\frac{v^{p-1}}{(1+|x|^m+v^p)^2} \leq -\partial_v H_L(x,v,t)$. Thus, for Case 1. *b)* and Case 1. *c)* we need to prove that $-\partial_v H_L(x,w,t) \leq \frac{Cv^{p-1}}{(1+|x|^m+v^p)^2}$, for some $C > 0$, which is independent of δ, L, and R.

Case 1. *b)* As mentioned above, we need to prove that $-\partial_v H_L(x,w,t) \leq \frac{Cv^{p-1}}{(1+|x|^m+v^p)^2}$ when $w^{\alpha} \in \left[\frac{x}{(1+2\delta)t}, \frac{x}{(1-2\delta)t}\right]$, $w \in [\frac{v}{2}, v]$. More precisely, due to (D.2.46), it suffices to prove that

$$1+|X(x,w,t)|^m+V(x,w,t)^p\geq C(1+|x|^m+v^p)$$

and that

$$X(x,w,t)^{m-1}\partial_v X(x,w,t)+V(x,w,t)^{p-1}\partial_v V(x,w,t)\leq Cv^{p-1},$$

for all $w \in [\frac{v}{2}, v]$ such that $w^{\alpha} \in [\frac{x}{(1+2\delta)t}, \frac{x}{(1-2\delta)t}]$.

We know from (D.2.65) that the first inequality holds and thus we focus on proving the second inequality. Due to (D.2.18) and (D.2.23), it follows that

$$X(x,w,t)^{m-1}\partial_v X(x,w,t)+V(x,w,t)^{p-1}\partial_v V(x,w,t)\leq X(x,w,t)^{m-1}\partial_v X(x,w,t)+Cw^{p-1},$$

where $C > 0$ is independent of δ, L, and R. We now analyze the term $X(x,w,t)^{m-1}\partial_v X(x,w,t)$. From (D.2.27) and (D.2.20), it holds that

$$X(x,w,t)^{m-1}\partial_v X(x,w,t)\leq |X(x,w,t)|^{m-1}|\partial_v X(x,w,t)|\leq Cw^{\alpha(m-1)}w^{\alpha-1}t^{m}\leq Cw^{p-1},$$

for some $C > 0$, which is independent of δ, L, and R, and we can conclude by using that $w \in [\frac{v}{2}, v]$.

Case 1. $c)$ As before, by Remark D.2.8, we only need to prove that $1 + |X(x,w,t)|^m + V(x,w,t)^p \geq C(1+|x|^m+v^p)$ and that $X(x,w,t)^{m-1}\partial_v X(x,w,t) + V(x,w,t)^{p-1}\partial_v V(x,w,t) \leq Cv^{p-1}$ when $w^\alpha \leq \frac{x}{(1+2\delta)t}$, $w \in [\frac{v}{2}, v]$.

By (D.2.19) and (D.2.20) it holds that $X(x,w,t) \geq 0$ and $\partial_v X(x,w,t) \leq 0$ in this region. Moreover, due to (D.2.18) and (D.2.21), it follows that

$$X(x,w,t)^{m-1}\partial_v X(x,w,t) + V(x,w,t)^{p-1}\partial_v V(x,w,t) \leq V(x,w,t)^{p-1}\partial_v V(x,w,t) \leq Cw^{p-1}.$$

The fact that $1 + |X(x,w,t)|^m + V(x,w,t)^p \geq C(1+|x|^m+v^p)$ in this region follows from (D.2.66).

Case 2. We will prove that (D.2.67) holds in this case too. Due to (D.2.19), we have that $X(x,v,t)$, $X(x,w,t) \geq 0$ in this region. Thus, from (D.2.26), it holds that $|x-(1+\delta)w^\alpha t| \leq |X(x,w,t)| \leq |x-(1-\delta)w^\alpha t|$. We know from (D.2.66) that $|x-(1-\delta)w^\alpha t|^m + w^p \leq C(|x|^m+v^p) \leq C\Big(|x-(1+\delta)w^\alpha t|^m + w^p\Big)$, for some constant $C > 0$, which is independent of δ, L, and R. Moreover, since $X(x,w,t) \geq 0$ and $\partial_w X(x,w,t) \leq 0$, we have that $X(x,w,t)\partial_w X(x,w,t) = -|X(x,w,t)||\partial_w X(x,w,t)|$. Thus, due to (D.2.46), it suffices to show in this case that there exists $C > 0$, independent of δ, L, and R such that

$$-|x-(1+\delta)w^\alpha t|^{m-1}w^{\alpha-1}t + w^{p-1} \leq Cv^{p-1} \leq C\Big(-|x-(1-\delta)w^\alpha t|^{m-1}w^{\alpha-1}t + w^{p-1}\Big), \tag{D.2.72}$$

for all $w \in [\frac{v}{2}, v]$.

We remember we are in the case when $x-(1+\delta)w^\alpha t \geq 0$. It is clear that $-|x-(1+\delta)w^\alpha t|^{m-1}w^{\alpha-1}t + w^{p-1} \leq Cv^{p-1}$. For the other inequality, due to (D.2.70) it suffices to prove the statement for $-|x-(1-\delta)w^\alpha t|^{m-1}t + w^{\alpha(m-1)}$. Since $x-(1-\delta)w^\alpha t \geq 0$ and using that $a^{m-1}+b^{m-1} \leq (a+b)^{m-1}$, for $a, b \geq 0$, we have that $x^{m-1} - ((1-\delta)w^\alpha t)^{m-1} \geq (x-(1-\delta)w^\alpha t)^{m-1}$. Thus, it holds that

$$-|x-(1-\delta)w^\alpha t|^{m-1}t + w^{\alpha(m-1)} \geq -|x|^{m-1}t + (1-\delta)^{m-1}w^{\alpha(m-1)}t^m + w^{\alpha(m-1)}. \tag{D.2.73}$$

Since $w \geq \frac{v}{2} \geq \frac{C_l}{2}v_{\max}(x,t)$ in this case we have that $xt^{\frac{1}{m-1}} \leq \frac{2^\alpha K_{max}w^\alpha}{C_l^\alpha}$, for a sufficiently large constant $C_l > 0$. (D.2.73) thus becomes

$$-|x-(1-\delta)w^\alpha t|^{m-1}t + w^{\alpha(m-1)} \geq -\frac{2^{p-\alpha}K_{max}^{m-1}w^{\alpha(m-1)}}{C_l^{p-\alpha}} + w^{\alpha(m-1)} \geq \frac{w^{\alpha(m-1)}}{2},$$

for C_l sufficiently large, thus concluding our proof.

Proof of (D.2.57)**:** In order to prove (D.2.57), it is useful to notice that if $x \leq 0$, then (D.2.24) holds. We will prove that there exists $C > 0$, which is independent of δ, L, and R, such that

$$\frac{1}{C}\frac{v^{\alpha-1}|x|^{m-1}t + v^{p-1}}{(1+|x|^m+v^p)^2} \leq -\partial_v H_L(x,w,t) \leq C\frac{v^{\alpha-1}|x|^{m-1}t + v^{p-1}}{(1+|x|^m+v^p)^2}, \tag{D.2.74}$$

for all $w \in [\frac{v}{2}, v]$. Using similar computations as the ones for (D.2.66), we can prove that

$$(|x| + (1+\delta)w^{\alpha}t)^m + w^p \leq C(|x|^m + v^p) \leq C\Big((|x| + (1-\delta)w^{\alpha}t)^m + w^p\Big). \tag{D.2.75}$$

Since $X(x,w,t) \leq 0$ and $\partial_w X(x,w,t) \leq 0$ we have that $X(x,w,t)\partial_w X(x,w,t) = |X(x,w,t)|\,|\partial_w X(x,w,t)|$. Due to (D.2.24) and (D.2.46), what is left to prove in order for (D.2.74) to hold is that

$$\begin{aligned}(|x| + (1+\delta)w^{\alpha}t)^{m-1}w^{\alpha-1}t + w^{p-1} &\leq C(v^{\alpha-1}|x|^{m-1}t + v^{p-1}) \\ &\leq C\Big((|x| + (1-\delta)w^{\alpha}t)^{m-1}w^{\alpha-1}t + w^{p-1}\Big),\end{aligned} \tag{D.2.76}$$

for some $C > 0$, independent of δ, L, and R, for all $w \in [\frac{v}{2}, v]$.

Thus, we only need to prove (D.2.76). For $x \leq 0$ and $w \in [\frac{v}{2}, v]$, we have that $(|x| + (1-\delta)w^{\alpha}t)^{m-1}t + w^{\alpha(m-1)} \geq |x|^{m-1}t + w^{\alpha(m-1)}$. Furthermore, since $t \leq 1$, it follows that

$$\begin{aligned}(|x| + (1+\delta)w^{\alpha}t)^{m-1}t + w^{\alpha(m-1)} &\leq C_m|x|^{m-1}t + C_m w^{\alpha(m-1)}t^m + w^{\alpha(m-1)} \\ &\leq C(|x|^{m-1}t + w^{\alpha(m-1)}).\end{aligned}$$

□

We also prove some moment bounds for the function H_L, which are independent of δ, L, and R.

Lemma D.2.9 (Moment estimates). *Let $T > 0$ be sufficiently small, independent of δ, L, and R. Then there exists $K_3 > 0$, which is independent of $\delta \in (0, \frac{1}{2})$, L, and R, such that for all $t \in [0,T]$ we have*

$$M_{1,L}(x,t) := \int_{(0,\infty)} vH_L(x,v,t)\mathrm{d}v \leq \frac{K_3C_0}{1+|x|^{m-d}} \qquad \text{for } x \in \mathbb{R}, t \geq 0. \tag{D.2.77}$$

In general, if $p > \max\{2\gamma+1, 2\}$, we have that

$$M_{n,L}(x,t) := \int_{(0,\infty)} v^n H_L(x,v,t)\mathrm{d}v \leq \frac{K_3C_0}{1+|x|^{m-\frac{n+1}{\alpha}}} \qquad \text{for } x \in \mathbb{R}, t \geq 0\,, \tag{D.2.78}$$

for $n \in [0, \max\{2\gamma, 1\}]$.

Proof. We first recall that due to (D.2.18) we have $G_L(x,v,t) \leq \frac{C}{1+|X(x,v,t)|^m+v^p}$. We split the integral for $M_{n,L}$ as follows:

$$M_{n,L}(x,t) = \int_0^{v_{\max}(x,t)} v^n H_L(x,v,t)\,\mathrm{d}v + \int_{v_{\max}(x,t)}^{\infty} v^n H_L(x,v,t)\,\mathrm{d}v =: M_{n,1}(x,t) + M_{n,2}(x,t)\,,$$

where $v_{\max}(x,t) = 0$ if $x \leq 0$. From (D.2.18) and since $\delta \in (0, \frac{1}{2})$, we have that $\frac{v}{2} \leq V(x,v,t) \leq \frac{3v}{2}$. Due to (D.2.19) and since $\delta \in (0, \frac{1}{2})$, it holds that $X(x,v,t) \geq x - (1+\delta)v^{\alpha}t \geq x - 2v^{\alpha}t$. Then, by (D.2.49), we further deduce that

$$X(x, v_{\max}(x,t), t) \geq x - 2v_{\max}(x,t)^{\alpha}t \geq x - 2K_{max}xt^{\frac{m}{m-1}} \geq \frac{x}{2} \geq 0, \tag{D.2.79}$$

for all $t \leq T$ if T is sufficiently small. Thus, by Definition D.2.6 and (D.2.45), we obtain that

$$M_{n,1}(x,t) = \int_0^{v_{\max}} v^n H_L(x,v,t)\mathrm{d}v = \int_0^{v_{\max}} v^n G_L(x, v_{\max}(x,t), t)\mathrm{d}v$$
$$\leq \frac{C v_{\max}^{n+1}}{1+|x|^m + v_{\max}^p} \leq \frac{C}{1+|x|^{m-\frac{n+1}{\alpha}}},$$

for some constant $C > 0$ which is independent of δ, L, and R.

To estimate $M_{n,2}$ we note that for all $x \in \mathbb{R}$ we have that

$$M_{n,2}(x,t) \leq C \int_0^\infty \frac{v^n}{1+v^p}\,\mathrm{d}v \leq C$$

since $p - n > 1$ by assumption.

To obtain a decay for large $|x|$ we consider first $x > 1$ and use (D.2.25)-(D.2.26) to obtain

$$M_{n,2}(x,t) \leq C\Big(\int_0^{\left(\frac{x}{(1+2\delta)t}\right)^{\frac{1}{\alpha}}} \frac{v^n}{1+(x-(1+\delta)v^\alpha t)^m + v^p}\mathrm{d}v + \int_{\left(\frac{x}{(1+2\delta)t}\right)^{\frac{1}{\alpha}}}^\infty v^{n-p}\,\mathrm{d}v\Big).$$

Then, since $\delta < \frac{1}{2}$, we have that

$$\int_{\left(\frac{x}{(1+2\delta)t}\right)^{\frac{1}{\alpha}}}^\infty v^{n-p}\,\mathrm{d}v \leq \frac{1}{p-n-1}\Big(\frac{(1+2\delta)t}{x}\Big)^{m-\frac{n+1}{\alpha}} \leq \frac{2^{m-\frac{n+1}{\alpha}}}{p-n-1}\Big(\frac{t}{x}\Big)^{m-\frac{n+1}{\alpha}}.$$

For the other term, by using the change of variables $v = (\frac{x}{t})^{\frac{1}{\alpha}}\xi$, we find

$$\int_0^{\left(\frac{x}{(1+2\delta)t}\right)^{\frac{1}{\alpha}}} \frac{v^n}{1+(x-(1+\delta)v^\alpha t)^m + v^p}\mathrm{d}v \leq \Big(\frac{x}{t}\Big)^{\frac{1}{\alpha}} \int_{(0,\infty)} \frac{(\frac{x}{t})^{\frac{n}{\alpha}}\xi^n}{1+x^m|1-(1+\delta)\xi^\alpha|^m + (\frac{x}{t})^{\frac{p}{\alpha}}\xi^p}\mathrm{d}\xi$$
$$\leq \frac{1}{x^{m-\frac{1}{\alpha}}}\frac{1}{t^{\frac{1}{\alpha}}} \int_{(0,\infty)} \frac{(\frac{x}{t})^{\frac{n}{\alpha}}\xi^n}{|1-(1+\delta)\xi^\alpha|^m + (\frac{1}{t})^m\xi^p}\mathrm{d}\xi.$$

Since $t \leq 1$ and $m > 1$ is even we have that the following holds

$$3^{m+1}[|1-(1+\delta)\xi^\alpha|^m + \frac{1}{t^m}\xi^p] \geq 1 + \frac{1}{t^m}\xi^p, \tag{D.2.80}$$

for $\xi \geq 0$. Indeed, if $|1-(1+\delta)\xi^\alpha| \geq \frac{1}{2}$, then (D.2.80) follows. Otherwise, if $|1-(1+\delta)\xi^\alpha| \leq \frac{1}{2}$, then $\frac{1}{2} \leq (1+\delta)\xi^\alpha \leq \frac{3}{2}$ and since $\delta < \frac{1}{2}$ it holds that $\xi^\alpha \geq \frac{1}{3}$. Thus, if we use in addition that $t \leq 1$, we have that

$$|1-(1+\delta)\xi^\alpha|^m + \frac{1}{t^m}\xi^p \geq \frac{1}{2t^m}\xi^p + \frac{1}{2t^m}\xi^p \geq \frac{1}{3^{m+1}} + \frac{1}{2t^m}\xi^p$$

and thus (D.2.80) holds. It then follows that

$$\begin{aligned}\frac{1}{x^{m-\frac{1}{\alpha}}t^{\frac{1}{\alpha}}}\int_{(0,\infty)}\frac{(\frac{x}{t})^{\frac{n}{\alpha}}\xi^n}{|1-\xi^\alpha|^m+(\frac{1}{t})^m\xi^p}\mathrm{d}\xi &\leq \frac{C}{x^{m-\frac{1}{\alpha}}t^{\frac{1}{\alpha}}}\int_{(0,\infty)}\frac{(\frac{x}{t})^{\frac{n}{\alpha}}\xi^n}{1+(\frac{1}{t})^m\xi^p}\mathrm{d}\xi\\ &\leq \frac{C}{x^{m-\frac{n+1}{\alpha}}}\int_{(0,\infty)}\frac{\eta^n}{1+\eta^p}\mathrm{d}\eta \leq \frac{C}{x^{m-\frac{n+1}{\alpha}}},\end{aligned}$$

for some constant $C>0$ which is independent of $\delta\in(0,\frac{1}{2})$, L, and R.

In the case $x\leq 0$ we can use (D.2.24) to obtain the estimate similarly as above without the need to split the integral. Estimate (D.2.77) follows from (D.2.78) by choosing $n=1$ and the fact that d in (D.2.15) satisfies $d>\frac{2}{\alpha}$. □

Remark D.2.10. Let $T>0$ be sufficiently small. With similar computations as the ones used in Lemma D.2.9, we can prove that

$$\int_{(0,\infty)}\frac{v^n}{1+|x|^m+v^p}\mathrm{d}v \leq \frac{K_3}{1+|x|^{m-\frac{n+1}{\alpha}}}, \tag{D.2.81}$$

for all $x\in\mathbb{R}$, $t\in[0,T]$, and for $n\in[0,\max\{2\gamma,1\}]$.

We now define the function for which we will prove is a supersolution for the problem (D.2.8).

Definition D.2.11. Let $\delta\in(0,\frac{1}{4}]$ and $L=4K_1K_2K_3C_0$, where C_0 is as in (D.2.45), K_1 is as in (D.1.8), K_2 is as in Lemma D.2.7, and K_3 is as in Lemma D.2.9. Denote by $H(x,v,t):=H_L(x,v,t)$, where H_L is as in Definition D.2.6. Moreover, let B_t to be

$$B_t := e^{\lambda t+\lambda t^{\frac{\alpha+1-\gamma}{\alpha}}} \tag{D.2.82}$$

for some $\lambda>0$. Then we define the function G via

$$G(x,v,t)=B_tH(x,v,t). \tag{D.2.83}$$

As mentioned before, with this construction G is decreasing in v for fixed x outside possibly a critical region where $x\in[(1-2\delta)v^\alpha t,(1+2\delta)v^\alpha t]$. In the following we will have to deal with this region separately.

Our key result is the following.

Proposition D.2.12. *Let $T>0$ be sufficiently small and $\delta\in(0,\frac{1}{4}]$. There exists a sufficiently large $\lambda>0$, which depends only on C_0 and the parameters m,γ,α such that if $f_n\leq G$, where the sequence $\{f_n\}_{n\in\mathbb{N}}$ was defined in (D.2.8), then $f_{n+1}\leq G$, for all $n\in\mathbb{N}$ and all $t\in[0,T]$.*

Remark D.2.13. Since the constants will play an important role in our proof, it is worthwhile to mention for clarity which are the constants that the parameters in Proposition D.2.12 depend on. Let C_0 be as in (D.2.45), K_1 as in (D.1.8), K_2 as in Lemma D.2.7, and K_3 be as in Lemma D.2.9. Notice that these constants do not depend on δ, L or R from Proposition D.2.2. In order to prove that G is a supersolution, we take $L = 4K_1K_2K_3C_0$. We then take λ to be sufficiently large depending on C_0, K_1, K_2, and K_3. We then take T to be such that $\max\{T, T^{\frac{\alpha+1-\gamma}{\alpha}}\} \leq \frac{\ln(2)}{2\lambda}$, which implies that $B_t \leq 2$, for all $t \in [0, T]$, where B_t was defined in (D.2.82).

Before we begin with the proof of Proposition D.2.12, it is worthwhile to notice that we have some moment bounds for the function G in Definition D.2.11 as a direct consequence of Lemma D.2.9.

Lemma D.2.14. *Let $T > 0$ be sufficiently small with K_3 as in Lemma D.2.9. Then for all $t \in [0, T]$ we have*

$$M_1(x,t) := \int_{(0,\infty)} vG(x,v,t)\mathrm{d}v \leq \frac{K_3C_0B_t}{1+|x|^{m-d}} \qquad \text{for } x \in \mathbb{R}, \tag{D.2.84}$$

where B_t was defined in (D.2.82). In general, if $p > \max\{2\gamma+1, 2\}$, we have that

$$M_n(x,t) := \int_{(0,\infty)} v^n G(x,v,t)\mathrm{d}v \leq \frac{K_3C_0B_t}{1+|x|^{m-\frac{n+1}{\alpha}}} \qquad \text{for } x \in \mathbb{R}\,, t \geq 0\,, \tag{D.2.85}$$

for $n \in [0, \max\{2\gamma, 1\}]$.

We now focus on proving Proposition D.2.12.

Proof of Proposition D.2.12. We now prove that G is a supersolution of the problem (D.2.8), that is we show

$$\begin{aligned}
&\partial_t G(x,v,t) + v^\alpha \partial_x G(x,v,t) - \int_0^{\frac{v}{2}} K(v-v',v')G(x,v-v',t)f_n(x,v',t)\mathrm{d}v' \\
&+ \int_0^\infty K(v,v')G(x,v,t)f_n(x,v',t)\mathrm{d}v' \geq 0.
\end{aligned} \tag{D.2.86}$$

We prove (D.2.86) by showing first that

$$\partial_t G(x,v,t) + v^\alpha \partial_x G(x,v,t) + \frac{Lv^\gamma}{1+|x|^{m-d}}\xi_R(v)\partial_v G(x,v,t) \geq 0, \tag{D.2.87}$$

for any $L > 0$ and where d was defined in (D.2.15), and then that

$$\begin{aligned}
&\partial_t G(x,v,t) + v^\alpha \partial_x G(x,v,t) - \int_0^{\frac{v}{2}} K(v-v',v')G(x,v-v',t)f_n(x,v',t)\mathrm{d}v' \\
&+ \int_0^\infty K(v,v')G(x,v,t)f_n(x,v',t)\mathrm{d}v' \\
&\geq \partial_t G(x,v,t) + v^\alpha \partial_x G(x,v,t) + \frac{Lv^\gamma}{1+|x|^{m-d}}\xi_R(v)\partial_v G(x,v,t),
\end{aligned} \tag{D.2.88}$$

for $L > 0$ as in Definition D.2.11.

We now prove (D.2.87).

Assume $x \notin [(1-2\delta)v^\alpha t, (1+2\delta)v^\alpha t]$ and $\partial_v G_L \leq 0$ or $x \in [(1-2\delta)v^\alpha t, (1+2\delta)v^\alpha t]$. Then $G = B_t G_L$ and thus, using (D.2.16), it holds with $c_\alpha := \frac{\alpha+1-\gamma}{\alpha}$ that

$$\begin{aligned}
&\partial_t G(x,v,t) + v^\alpha \partial_x G(x,v,t) + \frac{Lv^\gamma}{1+|x|^{m-d}}\xi_R(v)\partial_v G(x,v,t) \\
&= B_t\Big(\partial_t G_L(x,v,t) + v^\alpha \partial_x G_L(x,v,t) + \frac{Lv^\gamma}{1+|x|^{m-d}}\xi_R(v)\partial_v G_L(x,v,t)\Big) \\
&\quad + \lambda(1 + c_\alpha t^{\frac{\alpha+1-\gamma}{\alpha}-1})B_t G_L \geq \lambda e^{\lambda t} G_L(x,v,t) \geq 0.
\end{aligned}$$

Notice that we did not use the contribution of the term $t^{\frac{\alpha+1-\gamma}{\alpha}}$ in the computations. This term will be needed later in the proof.

Assume now $x \notin [(1-2\delta)v^\alpha t, (1+2\delta)v^\alpha t]$ and $\partial_v G_L > 0$ such that we have $G(x,v,t) = B_t G_L(x, v_{\max}(x,t), t)$. Then

$$\begin{aligned}
&\partial_t G(x,v,t) + v^\alpha \partial_x G(x,v,t) + \frac{Lv^\gamma}{1+|x|^{m-d}}\xi_R(v)\partial_v G(x,v,t) \\
&= B_t\Big(\partial_t G_L(x, v_{\max}(x,t), t) + v^\alpha \partial_x G_L(x, v_{\max}(x,t), t)\Big) \qquad \text{(D.2.89)}\\
&\quad + \lambda(1 + c_\alpha t^{\frac{\alpha+1-\gamma}{\alpha}-1})B_t G_L(x, v_{\max}(x,t), t) \\
&\geq B_t\Big(\partial_t G_L(x, v_{\max}(x,t), t) + v^\alpha \partial_x G_L(x, v_{\max}(x,t), t)\Big). \qquad \text{(D.2.90)}
\end{aligned}$$

By the choice of $v_{\max}(x,t)$ we have that $v \leq v_{\max}(x,t)$ in the region where $\partial_v G_L(x,v,t) > 0$. Moreover, from Proposition D.2.5, we know that $\partial_x G_L(x, v_{\max}(x,t), t) \leq 0$. Thus, from (D.2.89) and (D.2.16), we further obtain that

$$\begin{aligned}
&\partial_t G(x,v,t) + v^\alpha \partial_x G(x,v,t) + \frac{Lv^\gamma}{1+|x|^{m-d}}\xi_R(v)\partial_v G(x,v,t) \\
&\geq B_t\Big(\partial_t G_L(x, v_{\max}(x,t), t) + v_{\max}^\alpha \partial_x G_L(x, v_{\max}(x,t), t)\Big) = 0
\end{aligned}$$

and (D.2.87) follows.

We are thus left to prove that (D.2.88) holds. We analyze the cases when $x \in [(1-2\delta)v^\alpha t, (1+2\delta)v^\alpha t]$ and $x \notin [(1-2\delta)v^\alpha t, (1+2\delta)v^\alpha t]$ separately.

Proof of (D.2.88) for $x \notin [(1-2\delta)v^\alpha t, (1+2\delta)v^\alpha t]$.

We have that

$$
\begin{aligned}
-\int_0^{\frac{v}{2}} K(v-v',v')G(x,v-v',t)f_n(x,v',t)\mathrm{d}v' + \int_0^{\infty} K(v,v')G(x,v,t)f_n(x,v',t)\mathrm{d}v' \\
\geq -\int_0^{\frac{v}{2}} K(v,v')[G(x,v-v',t)-G(x,v,t)]f_n(x,v',t)\mathrm{d}v' \\
+\int_0^{\frac{v}{2}} [K(v,v')-K(v-v',v')]G(x,v-v',t)f_n(x,v',t)\mathrm{d}v'. \qquad \text{(D.2.91)}
\end{aligned}
$$

Since $G, f_n \geq 0$ and $K(v-v',v') \leq K(v,v')$ when $v' \in [0,\frac{v}{2})$ by (D.1.9), it holds that

$$
\int_0^{\frac{v}{2}} [K(v,v')-K(v-v',v')]G(x,v-v',t)f_n(x,v',t)\mathrm{d}v' \geq 0. \qquad \text{(D.2.92)}
$$

We know that $K(v,v') \leq K_1(v^{\gamma}+v'^{\gamma})$, where K_1 is as in (D.1.8). Thus, when $v' \leq \frac{v}{2}$, it holds that $K(v,v') \leq 2K_1 v^{\gamma}$. Without loss of generality we assume in the following that $K(v,v') \leq v^{\gamma}$ when $v' \leq \frac{v}{2}$. This is in order to simplify the notation but we allow L in Definition D.2.11 to depend on K_1. Additionally, by (D.2.83), it holds that $\partial_v G \leq 0$ and thus $G(x,v-v',t)-G(x,v,t) \geq 0$, for $v \in (0,\frac{v}{2})$. Since $f_n \leq G$ and with (D.2.92) we deduce that

$$
\begin{aligned}
-\int_0^{\frac{v}{2}} v^{\gamma}[G(x,v-v',t)-G(x,v,t)]f_n(x,v',t)\mathrm{d}v' \\
\geq -v^{\gamma}\int_0^{\frac{v}{2}} (G(x,v-v',t)-G(x,v,t))G(x,v',t)\mathrm{d}v'.
\end{aligned}
$$

We use the following notation

$$
I_1 := v^{\gamma}\int_0^{\frac{v}{2}} (G(x,v-v',t)-G(x,v,t))G(x,v',t)\mathrm{d}v' = \xi_R(v)I_1 + (1-\xi_R(v))I_1
$$

with ξ_R as in (D.2.13). Assume that H, as in Definition D.2.11, satisfies

$$
\begin{aligned}
\xi_R(v)v^{\gamma}\int_0^{\frac{v}{2}} (H(x,v-v',t)-H(x,v,t))G(x,v',t)\mathrm{d}v' \leq -\frac{Lv^{\gamma}}{1+|x|^{m-d}}\xi_R(v)\partial_v H(x,v,t) \\
+ CB_t H(x,v,t) \qquad \text{(D.2.93)}
\end{aligned}
$$

and

$$
(1-\xi_R(v))v^{\gamma}\int_0^{\frac{v}{2}} (H(x,v-v',t)-H(x,v,t))G(x,v',t)\mathrm{d}v' \leq CB_t H(x,v,t), \qquad \text{(D.2.94)}
$$

for $L = 4K_1K_2K_3C_0$ as in Definition D.2.11. Then

$$\begin{aligned}
&\partial_t G + v^\alpha \partial_x G - v^\gamma \int_0^{\frac{v}{2}} (G(x, v - v', t) - G(x, v, t))G(x, v', t)\mathrm{d}v' \\
&\geq B_t\Big(\partial_t H + v^\alpha \partial_x H - v^\gamma \int_0^{\frac{v}{2}} (H(x, v - v', t) - H(x, v, t))G(x, v', t)\mathrm{d}v'\Big) + \lambda e^{\lambda t + \lambda t^{\frac{\alpha+1-\gamma}{\alpha}}} H \\
&\geq B_t\Big(\partial_t H + v^\alpha \partial_x H + \frac{L v^\gamma}{1 + |x|^{m-d}} \xi_R(v)\partial_v H(x, v, t)\Big) \\
&+ B_t(-2CB_t + \lambda)H \geq \partial_t G + v^\alpha \partial_x G + \frac{L v^\gamma}{1 + |x|^{m-d}} \xi_R(v)\partial_v G
\end{aligned}$$

if $\lambda \geq 2CB_t$, which holds true if $\lambda = 4C$ and if $\max\{t, t^{\frac{\alpha+1-\gamma}{\alpha}}\} \leq \frac{\ln(2)}{2\lambda}$. This proves (D.2.88).

Proof of (D.2.94):. We have the following cases

Case 1. *a)* $x > 0$ **and** $v \leq v_{\mathbf{max}}(x, t)$**:** In this case $H(x, v - v', t) = H(x, v, t)$, for $v' \in (0, \frac{v}{2}]$. Thus, (D.2.94) holds.

Case 1. *b)* $x > 0$ **and** $v_{\mathbf{max}}(x, t) \leq v \leq 2v_{\mathbf{max}}(x, t)$**:** The proof of this case is the same as for Case 1. *b)* when proving that (D.2.93) holds. We thus postpone its proof.

Case 2. **Either** $\{x > 0$ **and** $2v_{\mathbf{max}}(x, t) \leq v\}$ **or** $\{x \leq 0\}$**:** Since $v \leq 2R$ and $\gamma \geq 0$, it follows, using (D.2.58) and Lemma D.2.14, that

$$\begin{aligned}
(1 - \xi_R(v))I_1 \mathrm{e}^{-\lambda t} &\leq (2R)^\gamma \int_0^{\frac{v}{2}} (H(x, v - v', t) - H(x, v, t))G(x, v', t)\mathrm{d}v' \\
&\leq (2R)^\gamma \int_0^{\frac{v}{2}} H(x, v - v', t)G(x, v', t)\mathrm{d}v' \\
&\leq K_2(2R)^\gamma H(x, v, t) \int_0^{\frac{v}{2}} G(x, v', t)\mathrm{d}v' \\
&\leq \frac{C_0K_2K_3(2R)^\gamma B_t H(x, v, t)}{1 + |x|^{m-\frac{1}{\alpha}}} \leq CB_t H(x, v, t). \qquad \text{(D.2.95)}
\end{aligned}$$

Remark D.2.15. Notice that the constant C in (D.2.95) depends on R from Proposition D.2.2, on the constant K_2 from (D.2.58), and on the constant K_3 from Lemma D.2.14. However, since L is fixed in Definition D.2.11, R depends only on K_1, K_2, K_3 and C_0.

Proof of (D.2.93): We have the following cases

Case 1. *a)***:** $x > 0$ **and** $v \leq v_{\mathbf{max}}(x, t)$**:** Notice that in this case $H(x, v - v', t) = H(x, v, t)$, for $v' \in (0, \frac{v}{2}]$, and $\partial_v H(x, v, t) = 0$. Thus, (D.2.93) holds.

Case 1. $b)$**:** $x > 0$ **and** $v_{\max}(x,t) \le v \le 2v_{\max}(x,t)$**:** We divide the integral on the left hand side of (D.2.93) into the region $v' \in (0, v - v_{\max})$ and $v' \in (v - v_{\max}, \frac{v}{2})$, respectively. For $v' \in (v - v_{\max}, \frac{v}{2})$, using $\frac{v}{2} \le v_{\max}(x,t)$ and Lemma D.2.14 with $n = 0$, we obtain that

$$\begin{aligned}\int_{v-v_{\max}(x,t)}^{\frac{v}{2}} (H(x, v-v', t) - H(x,v,t))G(x,v',t)\mathrm{d}v' &\le H(x, v_{\max}(x,t), t)\int_0^{v_{\max}} G(x,v',t)\mathrm{d}v' \\ &\le \frac{C_0 K_3 B_t H(x, v_{\max}(x,t),t)}{1+|x|^{m-\frac{1}{\alpha}}}. \end{aligned} \tag{D.2.96}$$

We have $0 \le v^\alpha \le 2v_{\max}^\alpha(x,t) \le 2K_{max}xt^{\frac{1}{m-1}}$, with K_{max} as in (D.2.49).

Moreover, using (D.2.59) it follows that

$$v^\gamma \int_{v-v_{\max}(x,t)}^{\frac{v}{2}} (H(x,v-v',t) - H(x,v,t))G(x,v',t)\mathrm{d}v' \le \frac{2^\gamma C_0 K_2 K_3 B_t H(x,v,t) v_{\max}(x,t)^\gamma}{1+x^{m-\frac{1}{\alpha}}}.$$

Since $v_{\max}(x,t)^\alpha \le 2K_{max}xt^{\frac{1}{m-1}}$, with K_{max} as in (D.2.49), we further obtain that

$$\frac{H(x,v,t)v_{\max}(x,t)^\gamma}{1+x^{m-\frac{1}{\alpha}}} \le \frac{CH(x,v,t)x^{\frac{\gamma}{\alpha}}t^{\frac{\gamma}{\alpha(m-1)}}}{1+x^{m-\frac{1}{\alpha}}} \le Ct^{\frac{\gamma}{\alpha(m-1)}}H(x,v,t),$$

since $m > \frac{\gamma+1}{\alpha}$. Thus

$$v^\gamma \int_{v-v_{\max}(x,t)}^{\frac{v}{2}} (H(x,v-v',t) - H(x,v,t))G(x,v',t)\mathrm{d}v' \le CB_t t^{\frac{\gamma}{\alpha(m-1)}} H(x,v,t). \tag{D.2.97}$$

We now estimate the integral

$$J := \int_0^{v-v_{\max}(x,t)} (H(x,v-v',t) - H(x,v,t))G(x,v',t)\mathrm{d}v'. \tag{D.2.98}$$

As before, using (D.2.58) and Lemma D.2.14, we find

$$\begin{aligned} J \le \int_0^{v-v_{\max}(x,t)} H(x,v-v',t)G(x,v',t)\mathrm{d}v' &\le \int_0^{v-v_{\max}(x,t)} K_2 H(x,v,t)G(x,v',t)\mathrm{d}v' \\ &\le \frac{C_0K_2K_3B_tH(x,v,t)}{1+x^{m-\frac{1}{\alpha}}} \end{aligned}$$

and this implies, since $v \le 2v_{\max}(x,t)$, that

$$\begin{aligned} v^\gamma \int_0^{v-v_{\max}(x,t)} (H(x,v-v',t) - H(x,v,t))G(x,v',t)\mathrm{d}v &\le CB_tH(x,v,t)\frac{v_{\max}(x,t)^\gamma}{1+x^{m-\frac{1}{\alpha}}} \\ &\le CB_tH(x,v,t)\frac{x^{\frac{\gamma}{\alpha}}t^{\frac{\gamma}{\alpha(m-1)}}}{1+x^{m-\frac{1}{\alpha}}}. \end{aligned} \tag{D.2.99}$$

Since $m > \frac{\gamma+1}{\alpha}$ it follows that (D.2.93) holds in this case.

Remark D.2.16. Notice that the constant C in (D.2.97) and (D.2.99) depends only on the constant K_2 from (D.2.58) and on the constants C_0, K_3 from Lemma D.2.14.

Case 1. c)**:** $x > 0$ **and** $2v_{\max}(x,t) \leq v$ **or** $x < 0$**:** Notice that the computations used in Case 1. b) hold for any $v \leq C_l v_{\max}(x,t)$, for some fixed constant $C_l > 0$. We can thus assume without loss of generality in this case that $v > C_l v_{\max}(x,t)$ and that $v \geq R$ because of the presence of $\xi_R(v)$ in (D.2.93).

We have that $v - v' \in (\frac{v}{2}, v)$, for $v' \in (0, \frac{v}{2})$. It holds

$$\int_0^{\frac{v}{2}} (H(x, v-v', t) - H(x,v,t))G(x,v',t)\mathrm{d}v' = -\int_0^{\frac{v}{2}} G(x,v',t) \int_{v-v'}^{v} \partial_v H(x,\tilde{v},t)\mathrm{d}\tilde{v}\mathrm{d}v'.$$

We can then use (D.2.57), (D.2.56) and Lemma D.2.14 with $n = 1$ in order to deduce that

$$\begin{aligned}
&-v^{\gamma}\xi_R(v) \int_0^{\frac{v}{2}} G(x,v',t) \int_{v-v'}^{v} \partial_v H(x,\tilde{v},t)\mathrm{d}\tilde{v}\mathrm{d}v' \\
&\qquad \leq -K_2 v^{\gamma}\xi_R(v)\partial_v H(x,v,t) \int_0^{\frac{v}{2}} v' G(x,v',t) \leq -\frac{C_0 K_2 K_3 B_t v^{\gamma}\xi_R(v)\partial_v H(x,v,t)}{1+|x|^{m-d}} \\
&\qquad \leq -\frac{L v^{\gamma}\xi_R(v)\partial_v H(x,v,t)}{1+|x|^{m-d}},
\end{aligned} \tag{D.2.100}$$

where in the last inequality we used the definition of L in Definition D.2.11 and that $B_t \leq 2$ by Remark D.2.13. Thus (D.2.93) holds in this case.

Proof of (D.2.88) for $x \in [(1-2\delta)v^{\alpha}t, (1+2\delta)v^{\alpha}t]$ or, alternatively, $v^{\alpha}t \in [\frac{x}{1+2\delta}, \frac{x}{1-2\delta}]$.

We will assume that $v \geq 1$ and $\gamma > 1$, since the other cases are similar but easier to treat.

In order to prove (D.2.88) we will first show that

$$\int_0^{\frac{v}{2}} K(v,v')f_n(x,v',t)|H(x,v-v',t) - H(x,v,t)|\mathrm{d}v' \leq CL(1 + t^{\frac{\alpha+1-\gamma}{\alpha}-1})H(x,v,t) \tag{D.2.101}$$

and then that

$$\frac{v^{\gamma}}{1+|x|^{m-d}}|\partial_v H(x,v,t)| \leq CLt^{\frac{\alpha+1-\gamma}{\alpha}-1} H(x,v,t), \tag{D.2.102}$$

for H and L as in Definition D.2.11. If (D.2.101) and (D.2.102) hold, then (D.2.88) holds. Indeed, arguing as in (D.2.91), (D.2.92) and using $f_n \leq G$, we find

$$\begin{aligned}
(*) := \partial_t G + v^{\alpha}\partial_x G &- \int_0^{\frac{v}{2}} K(v-v',v')G(x,v-v',t)f_n(x,v',t)\,dv' \\
&+ \int_0^{\infty} K(v,v')G(x,v,t)f_n(x,v',t)\,dv' \\
&\geq B_t\Big(\partial_t H + v^{\alpha}\partial_x H + \frac{Lv^{\gamma}}{1+|x|^{m-d}}\xi_R(v)\partial_v H(x,v,t)\Big) \\
&- B_t\frac{Lv^{\gamma}}{1+|x|^{m-d}}|\partial_v H(x,v,t)| - B_t v^{\gamma}\int_0^{\frac{v}{2}} |H(x,v-v',t) - H(x,v,t)|f_n(x,v',t)\mathrm{d}v' \\
&+ \lambda(c_{\alpha}t^{\frac{\alpha+1-\gamma}{\alpha}-1} + 1)B_t H.
\end{aligned}$$

Using now (D.2.101) and (D.2.102), we can conclude that

$$
\begin{aligned}
(*) \geq\ & B_t\Big(\partial_t H + v^\alpha \partial_x H + \frac{Lv^\gamma}{1+|x|^{m-d}}\xi_R(v)\partial_v H(x,v,t)\Big) \\
& + B_t[-CL - 2LCt^{\frac{\alpha+1-\gamma}{\alpha}-1} + \lambda(c_\alpha t^{\frac{\alpha+1-\gamma}{\alpha}-1}+1)]H \\
\geq\ & \partial_t G + v^\alpha \partial_x G + \frac{Lv^\gamma}{1+|x|^{m-d}}\xi_R(v)\partial_v G(x,v,t)
\end{aligned}
$$

if λ is sufficiently large.

Remark D.2.17. It is worthwhile to mention that L depends only on K_1 from (D.1.8), K_2 from Lemma D.2.7, and K_3 from Lemma D.2.14, see Definition D.2.11.

We now prove that (D.2.101) holds. Let $\eta \in (0,1)$ be fixed and sufficiently small. We want to bound the following terms.

$$
\begin{aligned}
&\int_{\eta v}^{\frac{v}{2}} K(v,v')f_n(x,v',t)|H(x,v-v',t)-H(x,v,t)|\mathrm{d}v' \\
&+\int_0^{\eta v} K(v,v')f_n(x,v',t)|H(x,v-v',t)-H(x,v,t)|\mathrm{d}v' =: J_1 + J_2.
\end{aligned}
$$

We analyze each term separately. We notice that, since $\delta < 1$ and t is sufficiently small, we have that $v^\alpha \geq \frac{x}{(1+2\delta)t} \geq 2K_{max}xt^{\frac{1}{m-1}} \geq 2v_{\max}^\alpha$ in this region, where K_{max} is as in (D.2.49). Thus we can assume in all the following that

$$
v \geq 2v_{\max} \text{ and } v' \geq v_{\max}, \text{ for all } v' \in \Big[\frac{v}{2}, v\Big]. \tag{D.2.103}
$$

Moreover, by (D.2.45), it holds that $G(x,v,t) = \frac{C_0B_t}{1+|X|^m+V^p} \leq \frac{C_0B_t}{1+V^p} \leq \frac{2^pC_0B_t}{1+v^p}$. Using (D.2.60) and the fact that $B_t \leq 2$, for $t \leq T$, we have

$$
\begin{aligned}
J_1 &\leq C\int_{\eta v}^{\frac{v}{2}} v^\gamma[H(x,v-v',t)+H(x,v,t)]G(x,v',t)\mathrm{d}v' \leq CH(x,v,t)\int_{\eta v}^{\frac{v}{2}}\frac{v^\gamma}{1+v'^p}\mathrm{d}v' \\
&\leq C(\eta)H(x,v,t)\int_0^\infty \frac{v'^\gamma}{1+v'^p}\mathrm{d}v' \leq CH(x,v,t).
\end{aligned}
$$

For the second term, it holds that

$$
J_2 \leq Cv^\gamma \int_0^{\eta v} |H(x,v-v',t)-H(x,v,t)|G(x,v',t)\mathrm{d}v'\,.
$$

We remember we are in the region where $v^\alpha t \in [\frac{x}{1+2\delta}, \frac{x}{1-2\delta}]$ and thus $v'^\alpha \leq \eta^\alpha v^\alpha \leq \frac{\eta^\alpha x}{(1-2\delta)t}$. Due to (D.2.19), it follows that

$$
X(x,v',t) \geq x - (1+\delta)v'^\alpha t \geq x - \frac{(1+\delta)\eta^\alpha x}{1-2\delta} = \frac{1-2\delta-(1+\delta)\eta^\alpha}{1-2\delta}x,
$$

for all $v' \in [0, \eta v]$. Since $\delta \leq \frac{1}{4}$ it follows that we can choose η to be sufficiently small, but independent of δ, such that $1 - 2\delta - (1+\delta)\eta^\alpha \geq \frac{1}{4}$ and thus

$$X(x, v', t) \geq (1 - 2\delta - (1+\delta)\eta^\alpha)x \geq \frac{x}{4},$$

for all $v' \in [0, \eta v]$. Thus, since $v^\alpha t \in [\frac{x}{1+2\delta}, \frac{x}{1-2\delta}]$ and $\delta \leq \frac{1}{4}$, we have that $X(x, v', t) \geq \frac{v^\alpha t}{8}$. It follows that

$$J_2 \leq C B_t v^\gamma \int_0^{\eta v} \frac{\int_{v-v'}^{v} |\partial_{\tilde{v}} H(x, \tilde{v}, t)| \mathrm{d}\tilde{v} \mathrm{d}v'}{1 + (v^\alpha t)^m + v'^p} \leq C v^\gamma \int_0^{\eta v} \frac{\int_{v-v'}^{v} |\partial_{\tilde{v}} H(x, \tilde{v}, t)| \mathrm{d}\tilde{v} \mathrm{d}v'}{1 + (v^\alpha t)^m + v'^p}$$

since $B_t \leq 2$. Let now $\tilde{v} \in [v - v', v]$, with $v' \in [0, \eta v]$. We have that

$$|\partial_{\tilde{v}} H(x, \tilde{v}, t)| \leq C_0 \frac{|m|X|^{m-2} X \partial_{\tilde{v}} X + V^{p-1} \partial_{\tilde{v}} V|}{(1 + |X|^m + V^p)^2}(x, \tilde{v}, t) \leq C_0 \frac{|m|X|^{m-2} X \partial_{\tilde{v}} X + V^{p-1} \partial_{\tilde{v}} V|}{(1 + V^p)^2}(x, \tilde{v}, t).$$

Assume $v^{\gamma-1} t \geq 1$ and remember we are in the case when $\gamma > 1$ and $v \geq 1$. Since $v' \leq \eta v$ it holds that $\frac{v}{2} \leq \tilde{v} \leq v$ for $\tilde{v} \in [v - v', v]$. In addition, by (D.2.103), we have that $\tilde{v} \geq v_{\max}$. If $\tilde{v} \geq \frac{x}{1+2\delta}$, then from (D.2.22), we have that

$$|m|X|^{m-2} X \partial_{\tilde{v}} X + V^{p-1} \partial_{\tilde{v}} V| \leq CL\Big(|X(x, \tilde{v}, t)|^{m-1} \tilde{v}^{\alpha-1} t + \tilde{v}^{p+\gamma-2} t\Big). \tag{D.2.104}$$

Otherwise, if $\tilde{v} < \frac{x}{1+2\delta}$, then (D.2.104) still holds since $v^{\gamma-1} t \geq 1$.

Moreover, since $\frac{v}{2} \leq \tilde{v} \leq v$ for $\tilde{v} \in [v - v', v]$, it holds that $|X(x, \tilde{v}, t)| \leq C v^\alpha t \leq C v^\alpha$ and thus $|X(x, \tilde{v}, t)|^{m-1} \tilde{v}^{\alpha-1} t \leq C v^{p-1} t \leq C v^{p+\gamma-2} t$ since $\gamma > 1$. Thus

$$|\partial_{\tilde{v}} H(x, \tilde{v}, t)| \leq \frac{CL v^{p+\gamma-2} t}{(1 + v^p)^2} \tag{D.2.105}$$

and thus it holds that

$$J_2 \leq CL v^{2(\gamma-1)} \frac{v^p t}{(1+v^p)^2} \int_0^{\eta v} \frac{v' \mathrm{d}v'}{1 + (v^\alpha t)^m + v'^p}.$$

By making the change of variables $v' = (1 + v^p t^m)^{\frac{1}{p}} \xi$, we further obtain that

$$J_2 \leq CL v^{2(\gamma-1)} \frac{v^p}{(1+v^p)^2} \frac{t}{(1 + v^p t^m)^{1-\frac{2}{p}}} \int_0^\infty \frac{\xi \, \mathrm{d}\xi}{1 + \xi^p} \leq CL v^{2(\gamma-1)} \frac{v^p}{(1+v^p)^2} \frac{t}{(1 + v^p t^m)^{1-\frac{2}{p}}}.$$

Remembering the definition of H and since we have, due to (D.2.18), that

$$H(x, v, t) = \frac{C_0}{1 + |X(x, v, t)|^m + V(x, v, t)^p} \geq \frac{C}{1 + v^p} \tag{D.2.106}$$

we deduce that

$$J_2 \leq \frac{CL v^{2(\gamma-1)} t}{(1 + v^p t^m)^{1-\frac{2}{p}}} H(x, v, t). \tag{D.2.107}$$

We now analyze the term $\frac{v^{2(\gamma-1)}t}{(1+v^p t^m)^{1-\frac{2}{p}}}$. It holds that

$$\frac{v^{2(\gamma-1)}t}{(1+v^p t^m)^{1-\frac{2}{p}}} = \frac{(v^\alpha t)^{\frac{2(\gamma-1)}{\alpha}} t^{1-\frac{2(\gamma-1)}{\alpha}}}{(1+(v^\alpha t)^m)^{1-\frac{2}{p}}} \leq t^{1-\frac{2(\gamma-1)}{\alpha}} \leq t^{-\frac{\gamma-1}{\alpha}}. \tag{D.2.108}$$

It holds that $-\frac{\gamma-1}{\alpha} > -1$ since $\gamma < \alpha + 1$ and thus we have that $e^{t^{1-\frac{\gamma-1}{\alpha}}} \leq C$. Then, (D.2.101) follows from (D.2.107).

If $v^{\gamma-1}t \leq 1$, we use (D.2.22) and then (D.2.105) becomes

$$|\partial_{\tilde{v}} H(x, \tilde{v}, t)| \leq \frac{CLv^{p-1}}{(1+v^p)^2}. \tag{D.2.109}$$

We can conclude using the same computations as above and by noticing that $-\frac{(\gamma-1)}{\alpha} > -1$.

Finally, we need to prove that (D.2.102) holds in the case when $x \in [(1-2\delta)v^\alpha t, (1+2\delta)v^\alpha t]$. Assume first that $v^{\gamma-1}t \geq 1$. We have that

$$\frac{v^\gamma}{1+|x|^{m-d}} |\partial_v H(x, v, t)| \leq \frac{Cv^\gamma |\partial_v H(x, v, t)|}{1+(v^\alpha t)^{m-d}}.$$

Making use of (D.2.105) and (D.2.106), we further obtain as before that

$$\frac{v^\gamma}{1+|x|^{m-d}} |\partial_v H(x, v, t)| \leq CLv^{2(\gamma-1)} \frac{v^p}{1+v^p} \frac{t}{1+(v^\alpha t)^{m-d}} H(x, v, t)$$

and we can then use similar arguments as in (D.2.108). The case when $v^{\gamma-1}t \leq 1$ can be proven similarly using (D.2.109) instead of (D.2.105). This concludes our proof. □

D.2.4 Proof of Theorem D.1.4

In this subsection, we finish the proof of Theorem D.1.4 by establishing that there exists a limit for the sequence $\{f_n\}_{n\in\mathbb{N}}$ defined in (D.2.8) and then passing to the limit in the equation. Our proof has analogies with the methods used to solve symmetric hyperbolic systems, see for example [Maj12].

We first prove some bounds that are independent of t for the function G defined in Definition D.2.11.

Lemma D.2.18. *Let $T > 0$ be sufficiently small. Then it holds that*

$$G(x, v, t) \leq \frac{2^m C_0 B_t}{1+|x|^m + v^p} \leq \frac{2^{m+1} C_0}{1+|x|^m + v^p}; \tag{D.2.110}$$

$$G(x, v-v', t) \leq \frac{2^{m+1} K_2 C_0}{1+|x|^m + v^p} \text{ for all } v' \in \left(0, \frac{v}{2}\right), \tag{D.2.111}$$

for all $v > 0$, $x \in \mathbb{R}$, and all $t \in [0, T]$, where B_t was defined in (D.2.82), K_2 is as in Lemma D.2.7, and C_0 is as in (D.2.45).

Proof. To prove (D.2.110) we notice first that it holds that $B_t \leq 2$ if we take $t \leq 1$ to be sufficiently small, where B_t was defined in (D.2.82).

We first consider the case $x > 0$ and $v \leq v_{\max}$, where $v_{\max}(x,t)$ was defined in (D.2.48).

From Definition D.2.11 and the fact that if $v \leq v_{\max}(x,t)$ then we are in the region where $t \leq \frac{x}{v^{\alpha}(1+2\delta)}$, we can use the bound in (D.2.26) to deduce that

$$G(x,v,t) \leq \frac{C_0}{1+|x-(1+\delta)v_{\max}(x,t)^{\alpha}t|^m+(1-\delta)^p v_{\max}(x,t)^p}, \tag{D.2.112}$$

when $v \leq v_{\max}(x,t)$, where C_0 is as in (D.2.45).

By (D.2.49), we have that $\frac{1}{K_{max}} x t^{\frac{1}{m-1}} \leq v_{\max}(x,t) \leq K_{max} x t^{\frac{1}{m-1}}$ and thus $x \geq x-(1+\delta)v_{\max}(x,t)^{\alpha}t \geq \frac{x}{2} > 0$ when $x > 0$ if t is sufficiently small. Thus, using in addition that $v \leq v_{\max}$ and that $\delta < \frac{1}{2}$, it holds that $(1-\delta)^p v_{\max}^p \geq \frac{v^p}{2^p}$. Thus,

$$G(x,v,t) = G(x,v_{\max}(x,t),t) \leq \frac{2^m C_0}{1+|x|^m+v^p}. \tag{D.2.113}$$

We now treat the case when $t \in [\frac{x}{(1+2\delta)v^{\alpha}}, \frac{x}{(1-2\delta)v^{\alpha}}]$. From (D.2.45) and then using the fact that $x \leq 2v^{\alpha}t \leq v^{\alpha}$, it follows that

$$G(x,v,t) \leq \frac{2^p C_0}{1+v^p} \leq \frac{2^{p+1}C_0}{1+|x|^m+v^p}.$$

If $v^{\alpha}t \leq \frac{x}{1+2\delta}$, from (D.2.26) it holds that

$$G(x,v,t) \leq \frac{C_0}{1+|x-(1+\delta)v^{\alpha}t|^m+(1-\delta)^p v^p}$$

and (D.2.110) follows from (D.2.66).

Finally, if $v^{\alpha}t \geq \frac{x}{1-2\delta}$, from (D.2.25) it holds that

$$G(x,v,t) \leq \frac{C_0}{1+|x-(1-\delta)v^{\alpha}t|^m+(1-\delta)^p v^p}$$

and (D.2.110) follows from (D.2.62).

If $x \leq 0$, from (D.2.24) we have that $|X(x,v,t)| \geq |x-(1-\delta)v^{\alpha}t| = |x|+(1-\delta)v^{\alpha}t$ and the conclusion follows.

(D.2.111) follows from (D.2.110), (D.2.55), and (D.2.75). □

Using these bounds, we can now prove that there exists a limit for the sequence $\{f_n\}_{n\in\mathbb{N}}$ that was defined in (D.2.8).

Proposition D.2.19. *Let $T > 0$ be sufficiently small. For every $\epsilon > 0$, there exists $n_{\epsilon} \in \mathbb{N}$ such that, for every $n, m \geq n_{\epsilon}$, it holds that $\|f_n - f_m\|_{\infty} := \sup_{t\in[0,T],x\in\mathbb{R},v\in(0,\infty)} |f_n(x,v,t) - f_m(x,v,t)| \leq \epsilon$.*

Proof. **Step 1: (Set-up)** Let $t \geq 0$, $x \in \mathbb{R}$ and $v > 0$. Let $\{f_n\}_{n\in\mathbb{N}}$ be the sequence defined in (D.2.8). For $n \in \mathbb{N}$, we denote by

$$R_n(x, v, t) := f_{n+1}(x, v, t) - f_n(x, v, t). \tag{D.2.114}$$

Moreover, for two functions f, g, we denote by

$$\mathbb{K}_1[f, g] := \int_0^{\frac{v}{2}} K(v - v', v') f(x, v', t) g(x, v - v', t) \mathrm{d}v', \tag{D.2.115}$$

$$\mathbb{K}_2[f, g] := \int_0^{\infty} K(v, v') f(x, v', t) g(x, v, t) \mathrm{d}v', \tag{D.2.116}$$

and

$$\mathbb{K}[f, g] := \mathbb{K}_1[f, g] - \mathbb{K}_2[f, g]. \tag{D.2.117}$$

Using this notation, it holds that

$$\begin{aligned}\partial_t R_n(x, v, t) + v^\alpha \partial_x R_n(x, v, t) &= \mathbb{K}_1[f_n, f_{n+1}] - \mathbb{K}_1[f_{n-1}, f_n] - \mathbb{K}_2[f_n, f_{n+1}] + \mathbb{K}_2[f_{n-1}, f_n] \\ &= \mathbb{K}_1[f_n, f_{n+1}] - \mathbb{K}_1[f_n, f_n] + \mathbb{K}_1[f_n, f_n] - \mathbb{K}_1[f_{n-1}, f_n] \\ &\quad - \mathbb{K}_2[f_n, f_{n+1}] + \mathbb{K}_2[f_n, f_n] - \mathbb{K}_2[f_n, f_n] + \mathbb{K}_2[f_{n-1}, f_n] \\ &= \mathbb{K}[f_n, R_n] + \mathbb{K}[R_{n-1}, f_n].\end{aligned} \tag{D.2.118}$$

Remark D.2.20. Notice that it suffices to analyze the term R_n in order to obtain the statement of Proposition D.2.19. This is since we can repeat the computations in (D.2.118) to obtain

$$\partial_t[f_m - f_n] + v^\alpha \partial_x[f_m - f_n] = \mathbb{K}[f_{m-1}, f_m - f_n] + \mathbb{K}[f_{m-1} - f_{n-1}, f_n]. \tag{D.2.119}$$

Since the estimates we will prove do not depend on $n, m \in \mathbb{N}$, we can reduce the problem to analyzing R_n in order to simplify the notation.

Notice the following. From (D.2.118) and (D.2.9), we have that R_n solves the following system

$$\begin{cases} \partial_t R_n(x, v, t) + v^\alpha \partial_x R_n(x, v, t) = \mathbb{K}_1[f_n, R_n] + \mathbb{K}_1[R_{n-1}, f_n] - \mathbb{K}_2[f_n, R_n] - \mathbb{K}_2[R_{n-1}, f_n]; \\ R_n(x, v, 0) = 0. \end{cases} \tag{D.2.120}$$

Since the system is linear in R_n, by Duhamel's principle, it suffices to derive estimates for

$$\begin{cases} \partial_t R_n^s(x, v, t) + v^\alpha \partial_x R_n^s(x, v, t) = \mathbb{K}_1[f_n, R_n^s] - \mathbb{K}_2[f_n, R_n^s], \text{ for } t > s; \\ R_n^s(x, v, s) = \mathbb{K}_1[R_{n-1}, f_n](x, v, s) - \mathbb{K}_2[R_{n-1}, f_n](x, v, s). \end{cases} \tag{D.2.121}$$

We now prove suitable estimates for the inhomogeneous part in (D.2.120), which in turn will give us suitable estimates for R_n.

Step 2: (Induction basis) It holds

$$R_1(x, v, t) \leq \frac{CT(v^\gamma + 1)}{(1 + |x|^m + v^p)(1 + |x|^{m-d})}, \tag{D.2.122}$$

for all $t \in [0, T]$, where d is as in (D.2.15).

We will prove (D.2.122) after (D.2.123) since the estimates are similar.

Step 3: (Induction step) It holds

$$R_n(x, v, t) \leq \frac{(CT)^n(v^\gamma + 1)}{(1 + |x|^m + v^p)(1 + |x|^{m-d})}, \tag{D.2.123}$$

for all $t \in [0, T]$, where d is as in (D.2.15).

We assume by induction that there exists a constant $C > 0$ such that

$$R_{n-1}(x, v, t) \leq \frac{(CT)^{n-1}(v^\gamma + 1)}{(1 + |x|^m + v^p)(1 + |x|^{m-d})}. \tag{D.2.124}$$

We estimate the inhomogeneous terms $\mathbb{K}_1[R_{n-1}, f_n]$ and $\mathbb{K}_2[R_{n-1}, f_n]$. Assume that the following inequality holds

$$|\mathbb{K}_1[R_{n-1}, f_n] - \mathbb{K}_2[R_{n-1}, f_n]| \leq \frac{C^n T^{n-1}(v^\gamma + 1)}{(1 + |x|^m + v^p)(1 + |x|^{m-d})}. \tag{D.2.125}$$

From (D.2.121), it follows that

$$R_n^s(x, v, t) \leq \frac{C^n T^{n-1}(v^\gamma + 1)}{(1 + |x|^m + v^p)(1 + |x|^{m-d})}, \tag{D.2.126}$$

for $t \in [0, T]$, if T is sufficiently small. Since from (D.2.120) and (D.2.121), we have that

$$R_n(x, v, t) = \int_0^t R_n^s(x, v, t)\mathrm{d}s,$$

the conclusion (D.2.123) follows from (D.2.126).

We are thus left to prove that (D.2.125) holds. We start with $\mathbb{K}_2[R_{n-1}, f_n]$. Using Proposition D.2.12, (D.2.110) and the fact that $K(v, v') \leq K_1(v^\gamma + v'^\gamma)$ from (D.1.8), we have that

$$\begin{aligned}\mathbb{K}_2[R_{n-1}, f_n] = \int_0^\infty K(v, v')R_{n-1}(x, v', t)f_n(x, v, t)\mathrm{d}v' &\leq \frac{Cv^\gamma}{1 + |x|^m + v^p}\int_0^\infty R_{n-1}(x, v', t)\mathrm{d}v' \\ &+ \frac{C}{1 + |x|^m + v^p}\int_0^\infty v'^\gamma R_{n-1}(x, v', t)\mathrm{d}v'.\end{aligned} \tag{D.2.127}$$

Using assumption (D.2.124) and then (D.2.81) for $n = 0$ and $n = 2\gamma$, we further obtain that

$$\begin{aligned}\int_0^\infty (1 + v'^\gamma)R_{n-1}(x, v', t)\mathrm{d}v' &\leq \frac{(CT)^{n-1}}{1 + |x|^{m-d}}\int_0^\infty \frac{(v'^\gamma + 1)^2}{1 + |x|^m + v'^p}\mathrm{d}v' \\ &\leq \frac{C^n T^{n-1}}{\left(1 + |x|^{m-d}\right)\left(1 + |x|^{m-\frac{2\gamma+1}{\alpha}}\right)} \leq \frac{C^n T^{n-1}}{1 + |x|^{m-d}}.\end{aligned} \tag{D.2.128}$$

Plugging (D.2.128) into (D.2.127), we deduce that

$$\mathbb{K}_2[R_{n-1}, f_n] \le \frac{C^n T^{n-1}(v^\gamma + 1)}{(1+|x|^m + v^p)(1+|x|^{m-d})}.$$

Similarly, we can bound the term $\mathbb{K}_1[R_{n-1}, f_n]$. More precisely, since $v' \in (0, \frac{v}{2})$, we make use of Proposition D.2.12 and (D.2.111) and, as before, it holds that

$$\begin{aligned}\mathbb{K}_1[R_{n-1}, f_n] &= \int_0^{\frac{v}{2}} K(v-v', v')R_{n-1}(x, v', t)f_n(x, v-v', t)\mathrm{d}v' \\ &\le \frac{Cv^\gamma}{1+|x|^m+v^p}\int_0^\infty R_{n-1}(x, v', t)\mathrm{d}v' \\ &\le \frac{C^nT^{n-1}(v^\gamma+1)}{(1+|x|^m+v^p)(1+|x|^{m-d})}.\end{aligned}$$

This concludes the proof of (D.2.123).

Step 4: (Proof of (D.2.122)**)** Using the definition of f_0 in (D.2.10), we obtain that

$$\partial_t R_1(x, v, t) + v^\alpha \partial_x R_1(x, v, t) = \mathbb{K}[f_0, R_1] + \mathbb{K}[f_0, f_0].$$

Following the steps of the proof of (D.2.123), we obtain that

$$\mathbb{K}[f_0, f_0] \le \frac{C(1+v^\gamma)}{(1+|x|^m+v^p)(1+|x|^{m-d})}$$

and the conclusion follows by Duhamel's principle as before.

Step 5: (Conclusion) We combine Remark D.2.20 with (D.2.122) and (D.2.123) and choose the time T in (D.2.123) to be sufficiently small, such that the right-hand side of (D.2.123) tends to zero as $n \to \infty$.

□

We are now able to conclude the proof of Theorem D.1.4.

Proof of Theorem D.1.4. We first prove that if $t \le T$ and T is sufficiently small, it holds that

$$f_0(x, v, t) \le \frac{2C_0}{1+|x|^m+v^p}. \tag{D.2.129}$$

From (D.2.10) and (D.2.11) it follows that $f_0(x, v, t) \le \frac{C_0}{1+|x-v^\alpha t|^m + v^p}$. We first consider $x > 0$. If $v^\alpha t \le x\left(1 - \frac{1}{\sqrt[m]{2}}\right)$ then $x - v^\alpha t \ge \frac{x}{\sqrt[m]{2}}$ and thus

$$\frac{1}{1+|x-v^\alpha t|^m+v^p} \le \frac{1}{1+\frac{x^m}{2}+v^p} \le \frac{2}{1+x^m+v^p}.$$

If $v^{\alpha}t \geq x\left(1 - \frac{1}{\sqrt[m]{2}}\right)$ then $t \neq 0$ and since $t \leq 1$ is sufficiently small we have that $v^p \geq 2x^m$ and thus

$$\frac{1}{1 + |x - v^{\alpha}t|^m + v^p} \leq \frac{1}{1 + v^p} = \frac{1}{1 + \frac{v^p}{2} + \frac{v^p}{2}} \leq \frac{1}{1 + x^m + \frac{v^p}{2}} \leq \frac{2}{1 + x^m + v^p}.$$

If $x \leq 0$, then, since m is even, we have $(x - v^{\alpha}t)^m = (|x| + v^{\alpha}t)^m \geq x^m$ and the conclusion follows. Then, by (D.2.129) and (D.2.81), we obtain that

$$\int_{(0,\infty)} v f_0(x, v, t)\mathrm{d}v \leq \frac{2K_3C_0}{1 + |x|^{m - \frac{2}{\alpha}}}. \tag{D.2.130}$$

On the other hand, by Lemma D.2.14, it follows that we can find a constant $K_3 > 0$ such that

$$\int_{(0,\infty)} vG(x, v, t)\mathrm{d}v \leq \frac{K_3C_0B_t}{1 + |x|^{m-d}}, \tag{D.2.131}$$

where d was defined in (D.2.15), B_t was defined in (D.2.82), and G was defined in Definition D.2.11. Moreover, we can choose $t \leq 1$ to be sufficiently small (as in Remark D.2.13) such that $B_t \leq 2$ and thus it holds that $\int_{(0,\infty)} vG(x, v, t)\mathrm{d}v \leq \frac{2K_3C_0}{1+|x|^{m-d}}$.

We use induction in order to prove that $f_n \leq G$ for all $n \in \mathbb{N}$. By Proposition D.2.12, the induction step holds true. For the induction basis, we need to prove that (D.2.86) holds true for $n = 0$. This is done with the same estimates as in Proposition D.2.12 by using (D.2.130). Thus, if we take L in (D.2.16) to be as in Definition D.2.11, namely $L = 4K_1K_2K_3C_0$, where K_1 is as in (D.1.8), K_2 is as in Lemma D.2.7, and K_3 is as in (D.2.131), we can conclude that

$$f_n(x, v, t) \leq G(x, v, t), \tag{D.2.132}$$

for all $n \in \mathbb{N}$.

From (D.2.132) and (D.2.110), it follows that there exists some $C > 0$ such that

$$f_n(x, v, t) \leq \frac{C}{1 + |x|^m + v^p}, \tag{D.2.133}$$

for all $n \in \mathbb{N}$, $t \in [0, T]$, $x \in \mathbb{R}$, and $v \in (0, \infty)$.

By Proposition D.2.19 we have that the exists a limit of the sequence $\{f_n\}_{n\in\mathbb{N}}$. It remains to show that the limit of the sequence $\{f_n\}_{n\in\mathbb{N}}$ satisfies equation (D.1.11). With the bound on $\{f_n\}$ in (D.2.133) and the bound on the kernel (D.1.8) it is completely standard to pass to the limit in the equation. Mass-conservation of f_n follows by testing with v in (D.2.8) and then integrating in v and x. Mass-conservation of f then follows by passing to the limit as $n \to \infty$ in $\int_{\mathbb{R}} \int_{(0,\infty)} v f_n(x, v, t)\mathrm{d}v\mathrm{d}x$. We omit the details here.

□

D.3 Estimates for the second order derivative of G_L

We will prove in this appendix that (D.2.50) holds true.

Proposition D.3.1. *Given $L > 0$ and $\delta \in (0, \frac{1}{2})$ there exists a sufficiently large $R > 0$ such that for all $t \in [0, T]$, with T sufficiently small, which is independent of L, δ, and R, it holds that*

$$\partial_v^2 G_L(x, v, t) < 0 \tag{D.3.1}$$

if

$$\frac{1}{K_{max}} x t^{\frac{1}{m-1}} \leq v^\alpha \leq K_{max} x t^{\frac{1}{m-1}}, \tag{D.3.2}$$

where K_{max} is as in (D.2.49).

Proof. Let $Q(x, v, t) = 1 + |X|^m + V^p$. Since $G_L = \frac{1}{Q}$, we have that

$$\partial_v Q(x, v, t) = m|X|^{m-2} X \partial_v X + pV^{p-1} \partial_v V, \tag{D.3.3}$$

$$\partial_v^2 Q(x, v, t) = m(m-1)|X|^{m-2} |\partial_v X|^2 + p(p-1)V^{p-2}(\partial_v V)^2 + m|X|^{m-2} X \partial_v^2 X + pV^{p-1} \partial_v^2 V, \tag{D.3.4}$$

and

$$\partial_v^2 G_L = 2\frac{|\partial_v Q|^2}{Q^3} - \frac{\partial_v^2 Q}{Q^2} = \frac{2|\partial_v Q|^2 - Q\partial_v^2 Q}{Q^3}. \tag{D.3.5}$$

It is worthwhile to notice that if v is as in (D.3.2), then $x \geq (1 + 2\delta)v^\alpha t$ if t is sufficiently small and thus (D.2.21) holds. We analyze first the term $2|\partial_v Q|^2$ in (D.3.5). Since $v^\alpha \leq K_{max} x t^{\frac{1}{m-1}}$, we have that $0 \leq x - (1+\delta)v^\alpha t \leq X \leq x - (1-\delta)v^\alpha t \leq x$ and $x > 0$. Using Proposition D.2.2 and (D.3.2) we deduce that

$$|\partial_v Q| \leq C_1 x^{m-1} v^{\alpha-1} t + C_2 v^{p-1} \leq \frac{C x^m}{v} t^{\frac{m}{m-1}},$$

for some constants $C_1, C_2, C > 0$, and thus

$$2|\partial_v Q(x, v, t)|^2 \leq \frac{C x^{2m}}{v^2} t^{\frac{2m}{m-1}}. \tag{D.3.6}$$

We then analyze the terms in (D.3.4). We have $|X|^{m-2}|\partial_v X|^2 \geq 0$ and from Proposition D.2.2 we know

$$V^{p-2}(\partial_v V)^2 \geq \frac{1}{C} v^{p-2}. \tag{D.3.7}$$

We are going to show that if t is sufficiently small and v as in (D.3.2) we have

$$\partial_v^2 X(x, v, t) \geq 0 \tag{D.3.8}$$

and that for any $\epsilon > 0$ and v as in (D.3.2) we have

$$|V^{p-1}\partial_v^2 V| \leq \epsilon v^{p-2} \tag{D.3.9}$$

if R is sufficiently large and t is sufficiently small. Combining then (D.3.7)-(D.3.9) with (D.3.2) and noticing that $Q \geq |X|^m \geq (x-(1+\delta)v^\alpha t)^m \geq \frac{x^m}{2^m}$ if t is sufficiently small, it follows that

$$Q\partial_v^2 Q \geq \frac{x^m}{C}v^{p-2} \geq \frac{x^{2m}}{Cv^2}t^{\frac{m}{m-1}}. \tag{D.3.10}$$

(D.3.5), (D.3.6) and (D.3.10) imply that we can find a sufficiently small $T \in (0,1)$ such that (D.3.1) holds for all $t \leq T$.

In all the following computations, we will assume for simplicity that $v \geq 2R$, with R as in Proposition D.2.2. The rest of the cases can be proven using similar computations.

Auxiliary result: Given $L > 0$ and $\delta \in (0, \frac{1}{2})$ there exists a sufficiently large $R > 0$ such that for all $t \in [0,T]$, with T sufficiently small, which is independent of L, δ, and R, it holds that

$$|\partial_v^2 X(x,v,t)| \leq Cv^{\alpha-2}t \tag{D.3.11}$$

if v is as in (D.3.2).

In order to show (D.3.11) we first prove that there exists $C > 0$ such that

$$|\partial_v^2\Phi(z,v)| \leq Cv^{-1}, \text{ for } v \geq 2R, \tag{D.3.12}$$

where Φ is as (D.2.30). We differentiate two times with respect to v in (D.2.32) in order to deduce that

$$\begin{aligned}\partial_v^2\Phi(z,v) = (\gamma-\alpha)\Big(v^{1-(\gamma-\alpha)} + (1-(\gamma-\alpha))z\Big)^{\frac{2(\gamma-\alpha)-1}{1-(\gamma-\alpha)}} v^{-2(\gamma-\alpha)} \\ -(\gamma-\alpha)(v^{1-(\gamma-\alpha)} + (1-(\gamma-\alpha))z)^{\frac{\gamma-\alpha}{1-(\gamma-\alpha)}} v^{-(\gamma-\alpha)-1}.\end{aligned}$$

Since $0 \leq z \leq C$ and we are in the case when $v \geq 2R$, we can choose $R > 0$ sufficiently large such that $v^{1-(\gamma-\alpha)} \leq v^{1-(\gamma-\alpha)} + (1-(\gamma-\alpha))z \leq 2v^{1-(\gamma-\alpha)}$ and thus $|\partial_v^2\Phi(z,v)| \leq Cv^{-1}$.

In order to prove (D.3.11) we differentiate (D.2.38) twice with respect to v and obtain

$$\begin{aligned}&\frac{\partial_v^2 X(x,v,t)}{\Phi(\psi(X(x,v,t))-\psi(x),v)^\alpha} - \alpha\frac{\partial_v X(x,v,t)[\partial_v\Phi + \partial_z\Phi\psi'(X)\partial_v X]}{\Phi(\psi(X(x,v,t))-\psi(x),v)^{\alpha+1}} \\ &= \alpha\int_x^X \frac{\partial_v^2\Phi d\xi}{\Phi(\psi(\xi)-\psi(x),v)^{\alpha+1}} + \alpha(\alpha+1)\int_X^x \frac{(\partial_v\Phi)^2 d\xi}{\Phi(\psi(\xi)-\psi(x),v)^{\alpha+2}} \\ &+ \frac{\alpha\partial_v\Phi}{\Phi(\psi(X)-\psi(x),v)^{\alpha+1}}\partial_v X.\end{aligned} \tag{D.3.13}$$

From (D.3.12), (D.2.19) and (D.2.18) we deduce that

$$\left|\int_x^X \frac{\partial_v^2\Phi d\xi}{\Phi(\psi(\xi)-\psi(x),v)^{\alpha+1}}\right| \leq Cv^{-\alpha-2}[x-X] \leq Cv^{-2}t. \tag{D.3.14}$$

Furthermore, Proposition D.2.2 implies that

$$\frac{|\partial_v \Phi \partial_v X|}{\Phi(\psi(X)-\psi(x),v)^{\alpha+1}} \leq C v^{-2} t\,, \tag{D.3.15}$$

$$\int_X^x \frac{(\partial_v \Phi)^2 \mathrm{d}\xi}{\Phi(\psi(\xi)-\psi(x),v)^{\alpha+2}} \leq C v^{-\alpha-2}[x-X] \leq C v^{-2} t \tag{D.3.16}$$

and that $\frac{1}{C} v^{\alpha} \leq \Phi(\psi(X)-\psi(x),v)^{\alpha} \leq C v^{\alpha}$.

We claim that if v satisfies (D.3.2) it holds

$$0 \leq \frac{\partial_z \Phi \psi'(X)|\partial_v X|^2}{\Phi(\psi(X)-\psi(x),v)^{\alpha+1}} \leq C v^{-2} t. \tag{D.3.17}$$

Indeed, Proposition D.2.2 implies that $|\partial_v X| \leq C v^{\alpha-1} t$ and $\frac{1}{\Phi(\psi(X)-\psi(x),v)^{\alpha+1}} \leq C v^{-\alpha-1}$.

We use (D.3.2) again to deduce that we are in the region where $x-(1+2\delta)v^{\alpha} t \geq 0$ for sufficiently small t. From the estimate (D.2.42) we obtain that

$$0 \leq \partial_z \Phi \psi'(X)|\partial_v X| \leq C. \tag{D.3.18}$$

Thus, (D.3.17) follows.

Combining the estimates (D.3.14)-(D.3.17) and then making use of (D.3.13), we obtain (D.3.11).

Proof of (D.3.9):

We only look at the case when $\gamma > 1$. The case $\gamma \in [0,1]$ can be proven in a similar manner. As before, we assume in the following for simplicity that the constant L in (D.2.17) is $L = 1$. If $\gamma \in (1, 1+\alpha)$, then by integrating in (D.2.17) it follows that

$$V(x,v,t)^{1-\gamma} - v^{1-\gamma} = (\gamma-1)\int_0^t \frac{\mathrm{d}\xi}{1+|X(x,v,\xi)|^{m-d}} \tag{D.3.19}$$

such that

$$V(x,v,t) = \frac{v}{\left(1+(\gamma-1)v^{\gamma-1}\int_0^t \frac{\mathrm{d}\xi}{1+|X|^{m-d}}\right)^{\frac{1}{\gamma-1}}}. \tag{D.3.20}$$

Differentiating in v, we obtain that

$$\begin{aligned}\partial_v V(x,v,t) = {} & \frac{1}{\left(1+(\gamma-1)v^{\gamma-1}\int_0^t \frac{\mathrm{d}\xi}{1+|X|^{m-d}}\right)^{\frac{1}{\gamma-1}}} - \frac{(\gamma-1)v v^{\gamma-2}\int_0^t \frac{\mathrm{d}\xi}{1+|X|^{m-d}}}{\left(1+(\gamma-1)v^{\gamma-1}\int_0^t \frac{\mathrm{d}\xi}{1+|X|^{m-d}}\right)^{\frac{\gamma}{\gamma-1}}} \\ & + \frac{v v^{\gamma-1}\int_0^t \frac{(m-d)|X|^{m-d-2} X \partial_v X \mathrm{d}\xi}{(1+|X|^{m-d})^2}}{\left(1+(\gamma-1)v^{\gamma-1}\int_0^t \frac{\mathrm{d}\xi}{1+|X|^{m-d}}\right)^{\frac{\gamma}{\gamma-1}}} =: \frac{T_1}{T_2^{\frac{\gamma}{\gamma-1}}},\end{aligned} \tag{D.3.21}$$

where

$$T_1 := 1 + v^\gamma \int_0^t \frac{(m-d)|X|^{m-d-2} X \partial_v X \mathrm{d}\xi}{(1+|X|^{m-d})^2}$$

and

$$T_2 := 1 + (\gamma - 1) v^{\gamma-1} \int_0^t \frac{\mathrm{d}\xi}{1+|X|^{m-d}}.$$

Differentiating in v, we obtain that

$$\partial_v^2 V(x,v,t) = \frac{\partial_v T_1}{T_2^{\frac{\gamma}{\gamma-1}}} - \frac{\gamma}{\gamma-1} \frac{T_1 \partial_v T_2}{T_2^{\frac{2\gamma-1}{\gamma-1}}}.$$

It holds that

$$\begin{aligned}\partial_v T_1 &= \gamma v^{\gamma-1} \int_0^t \frac{(m-d)|X|^{m-d-2} X \partial_v X \mathrm{d}\xi}{(1+|X|^{m-d})^2} - 2 v^\gamma \int_0^t \frac{\left|(m-d)|X|^{m-d-2} X \partial_v X\right|^2 \mathrm{d}\xi}{(1+|X|^{m-d})^3} \\ &\quad + v^\gamma \int_0^t \frac{(m-d)|X|^{m-d-2} X \partial_v^2 X \mathrm{d}\xi}{(1+|X|^{m-d})^2} + v^\gamma \int_0^t \frac{(m-d)(m-d-1)|X|^{m-d-2} |\partial_v X|^2 \mathrm{d}\xi}{(1+|X|^{m-d})^2}.\end{aligned}$$

Moreover, we have that

$$\partial_v T_2 = (\gamma-1)^2 v^{\gamma-2} \int_0^t \frac{\mathrm{d}\xi}{1+|X|^{m-d}} - (\gamma-1) v^{\gamma-1} \int_0^t \frac{(m-d)|X|^{m-d-2} X \partial_v X \mathrm{d}\xi}{(1+|X|^{m-d})^2}.$$

We first prove the following estimates:

$$0 \le v^{\gamma-1} \int_0^t \frac{\mathrm{d}\xi}{1+|X(x,v,\xi)|^{m-d}} \le \frac{1}{2} \tag{D.3.22}$$

and

$$v^\gamma \left| \int_0^t \frac{(m-d)|X(x,v,\xi)|^{m-d-2} X(x,v,\xi) \partial_v X(x,v,\xi) \mathrm{d}\xi}{(1+|X(x,v,\xi)|^{m-d})^2} \right| \le \frac{1}{2}. \tag{D.3.23}$$

We first prove that (D.3.22) holds. For sufficiently small t, we are in the case when $x \ge (1+2\delta)v^\alpha t$ since v is as in (D.3.2). We use (D.2.19) and the fact that $x \ge (1+2\delta)v^\alpha t \ge (1+2\delta)v^\alpha s$ for all $s \in [0,t]$. Thus, $|X(s)| \ge |x - (1+\delta)v^\alpha s|$ when $x \ge (1+2\delta)v^\alpha t$, for all $s \in [0,t]$. It follows that

$$\begin{aligned} v^{\gamma-1} \int_0^t \frac{\mathrm{d}\xi}{1+|X(x,v,\xi)|^{m-d}} &\le C v^{\gamma-1} \int_0^t \frac{\mathrm{d}\xi}{1+|x-(1+\delta)v^\alpha \xi|^{m-d}} \\ &\le C v^{\gamma-\alpha-1} \int_0^{v^\alpha t} \frac{\mathrm{d}z}{1+|x-(1+\delta)z|^{m-d}} \le C v^{\gamma-\alpha-1}, \end{aligned} \tag{D.3.24}$$

which, since $v \ge R$ and $\gamma < \alpha + 1$, implies (D.3.22) if R is sufficiently large.

We now prove (D.3.23). As before, we consider the case when $x \geq (1+2\delta)v^\alpha t$ since we can choose t sufficiently small. It follows that

$$v^\gamma \left| \int_0^t \frac{(m-d)|X|^{m-d-2} X \partial_v X \mathrm{d}\xi}{(1+|X|^{m-d})^2} \right| \leq C v^{\gamma-1} \int_0^t \frac{|x-(1-\delta)v^\alpha \xi|^{m-d-1} v^\alpha \xi \mathrm{d}\xi}{(1+|x-(1+\delta)v^\alpha\xi|^{m-d})^2}.$$

If v is as in (D.3.2), then $x-(1+\delta)z \geq x-2z \geq \frac{x}{2}$ for sufficiently small t, for $z \in [0, v^\alpha t]$. By making the change of variables $z = v^\alpha \xi$ and using the fact that $x-(1-\delta)z \leq x$ we further obtain that

$$\begin{aligned}
v^{\gamma-1} \int_0^t \frac{|x-(1-\delta)v^\alpha\xi|^{m-d-1} v^\alpha \xi \mathrm{d}\xi}{(1+|x-(1+\delta)v^\alpha\xi|^{m-d})^2} &\leq C v^{\gamma-\alpha-1} \int_0^{v^\alpha t} \frac{|x-(1-\delta)z|^{m-d-1} z \mathrm{d}z}{(1+|x-(1+\delta)z|^{m-d})^2} \\
&\leq C v^{\gamma-\alpha-1} \int_0^{\frac{x}{1+2\delta}} \frac{|x|^{m-d} \mathrm{d}z}{(1+|x|^{m-d})^2} \leq \frac{C v^{\gamma-\alpha-1} |x|^{m-d+1}}{(1+|x|^{m-d})^2} \\
&\leq C v^{\gamma-\alpha-1},
\end{aligned}$$

and thus (D.3.23) follows if we take R sufficiently large since $v \geq R$ and $\gamma < \alpha+1$.

From (D.3.22) and (D.3.23), it holds that $T_1 \in [\frac{1}{2}, 1]$ and $T_2 \in [\frac{1}{2}, \frac{3}{2}]$. From this we deduce that

$$|\partial_v^2 V(x,v,t)| \leq C(|\partial_v T_1| + |\partial_v T_2|)\,.$$

We analyze each term of $\partial_v T_1$ separately. Following the computations for (D.3.23), we deduce that

$$v^{\gamma-1} \left| \int_0^t \frac{(m-d)|X|^{m-d-2} X \partial_v X \mathrm{d}\xi}{(1+|X|^{m-d})^2} \right| \leq C v^{\gamma-\alpha-1} v^{-1}. \tag{D.3.25}$$

Similarly, by making the change of variables $z = v^\alpha \xi$, it follows that

$$\begin{aligned}
v^\gamma \int_0^t \frac{\left||X|^{m-d-2} X \partial_v X\right|^2 \mathrm{d}\xi}{(1+|X|^{m-d})^3} &\leq C v^\gamma \int_0^t \frac{\left(|x-(1-\delta)v^\alpha\xi|^{m-d-1} v^{\alpha-1}\xi\right)^2 \mathrm{d}\xi}{(1+|x-(1+\delta)v^\alpha\xi|^{m-d})^3} \\
&\leq C v^{\gamma-\alpha-2} \int_0^{v^\alpha t} \frac{\left(|x-(1-\delta)z|^{m-d-1} z\right)^2 \mathrm{d}z}{(1+|x-(1+\delta)z|^{m-d})^3} \leq C v^{\gamma-\alpha-1} v^{-1}.
\end{aligned} \tag{D.3.26}$$

Moreover, (D.3.11) implies that

$$\begin{aligned}
v^\gamma \left| \int_0^t \frac{(m-d)|X|^{m-d-2} X \partial_v^2 X \mathrm{d}\xi}{(1+|X|^{m-d})^2} \right| &\leq C v^{\alpha-2} v^\gamma \int_0^t \frac{(m-d)|X|^{m-d-1} \xi \mathrm{d}\xi}{(1+|X|^{m-d})^2} \\
&\leq C v^{-\alpha-2} v^\gamma \int_0^{v^\alpha t} \frac{|x-(1-\delta)z|^{m-d-1} z \mathrm{d}z}{(1+|x-(1+\delta)z|^{m-d})^2} \\
&\leq C v^{\gamma-\alpha-1} v^{-1}.
\end{aligned} \tag{D.3.27}$$

We now analyze the last term in $\partial_v T_1$. We have that

$$v^\gamma \int_0^t \frac{|X|^{m-d-2} |\partial_v X|^2 \mathrm{d}\xi}{(1+|X|^{m-d})^2} \leq C v^{\gamma-\alpha-2} \int_0^{v^\alpha t} \frac{|x-(1-\delta)z|^{m-d-2} z^2 \mathrm{d}z}{(1+|x-(1+\delta)z|^{m-d})^2} \leq C v^{\gamma-\alpha-1} v^{-1}. \tag{D.3.28}$$

We continue by finding upper bounds for each term in $\partial_v T_2$. From (D.3.24), it holds that

$$v^{\gamma-2}\int_0^t \frac{\mathrm{d}\xi}{1+|X|^{m-d}} \leq C v^{\gamma-\alpha-1} v^{-1}, \tag{D.3.29}$$

which offers the right estimate in order to prove (D.3.9). Notice that the last term in $\partial_v T_2$, which is $v^{\gamma-1}\int_0^t \frac{|X|^{m-d-2}X\partial_v X \mathrm{d}\xi}{(1+|X|^{m-d})^2}$, has already been analyzed in (D.3.25).

Combining (D.3.25)-(D.3.29) together with the fact that $v \geq R$ and that $\gamma < \alpha + 1$, we obtain that

$$|v^{p-1}\partial_v^2 V(x, v, t)| \leq C v^{\gamma-\alpha-2} v^{p-1} \leq \epsilon v^{p-2},$$

if $v \geq R$ is sufficiently large. This concludes the proof of (D.3.9).

Proof of (D.3.8)**:**

From (D.2.17) it follows that

$$\partial_v^2 X(x, v, t) = \alpha(1-\alpha)\int_0^t V^{\alpha-2}(\partial_v V)^2 \mathrm{d}s - \alpha \int_0^t V^{\alpha-1}\partial_v^2 V \mathrm{d}s. \tag{D.3.30}$$

Now (D.3.9) implies that the second term on the right hand side can be absorbed into the first one and thus (D.3.8) follows. □

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