SOME SHARP AND ENDPOINT INEQUALITIES IN HARMONIC ANALYSIS

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Summary

This thesis is about certain sharp and endpoint inequalities in Harmonic Analysis.

The central theme in the first part of this dissertation is sharp Fourier extension inequalities on spheres. The study of sharp inequalities in Harmonic Analysis can be traced back to the seminal works of Beckner [Bec75] for the sharp Hausdorff–Young inequality and of Lieb [Lie83] for the sharp Hardy–Littlewood–Sobolev inequality. The study of sharp Fourier restriction and extension inequalities, on the other hand, is a relatively recent development which has received increasing attention over the last few years.

To begin, let us revisit the *Fourier restriction inequality*

$$\|\widehat{f}\|_{L^q(\mathbb{S}^{d-1},\sigma)} \leqslant C_{p,q,d}\|f\|_{L^p(\mathbb{R}^d)}.$$

Here, $\mathbb{S}^{d-1} = \{x \in \mathbb{R}^d : |x| = 1\}$ is the unit sphere equipped with the surface measure $\sigma = \sigma_{d-1}$ induced by the Lebesgue measure in $\mathbb{R}^d$ and the restriction $f \mapsto \widehat{f}|_{\mathbb{S}^{d-1}}$ is originally defined on Schwartz functions. The complete characterization of the pairs of exponents $1 \leqslant p \leqslant 2$, $1 \leqslant q \leqslant \infty$ for which this inequality holds is a major open question in Harmonic Analysis. When $p = 1$ the inequality can be easily seen to hold for any $1 \leqslant q \leqslant \infty$. On the other hand, the inequality always fails when $p = 2$. Hence the interesting question is what happens for $1 < p < 2$. By duality, such restriction inequality is equivalent, with the very same constant, to the so-called *Fourier extension inequality*,

$$\|\widehat{f\sigma}\|_{L^{p'}(\mathbb{R}^{d-1})} \leqslant C_{d,p,q}\|f\|_{L^{q'}(\mathbb{S}^{d-1},\sigma)},$$

where p' and q' are the conjugate exponents of p and q, respectively, and $\widehat{f\sigma}$ is the Fourier transform of the measure $f\sigma$. By testing these inequalities against some carefully chosen functions one finds the following necessary conditions on the exponents: $p < \frac{2d}{d+1}$, $q \leqslant \frac{d-1}{d+1}p'$. The Fourier restriction conjecture claims that these necessary conditions are also sufficient. The conjecture has been completely verified in the case of dimension $d = 2$, [Fef70, Zyg74]. Moreover, the celebrated Stein– Tomas theorem establishes that the conjecture is true when $q = 2$ and $2\frac{d+1}{d-1} \leqslant p'$ in all dimensions $d \geqslant 2$, [Ste93, Tom75].

Our focus in this thesis is on the subarea of sharp Fourier restriction – equivalently, extension – theory. Given a triple (d, p', q') for which the above Fourier extension inequality holds we will consider questions like: What is the value of the optimal constant? If maximizers – that is, functions that attain the optimal constant – exist, what are they? So far, such questions have been investigated mainly for the case of even exponents p', and even for these seemingly more favorable cases many questions are still open.

A major breakthrough in the subject of sharp spherical restriction came from the work of Foschi [Fos15]. Foschi showed that constant functions are the unique real-valued maximizers for the sharp endpoint Stein–Tomas inequality on the sphere $\mathbb{S}^2$. A key step in his approach is the introduction of a weight that exploits some geometric features of the sphere $\mathbb{S}^2$ and which neutralizes the singularity at the origin of the twofold convolution $\sigma_2 * \sigma_2$. Many of the successive results in the subject grew out of this approach.

One of the still open problems in the area of sharp Fourier restriction which has attracted a lot of attention and effort over the last decade is the problem of determining the sharpest constant for the endpoint Stein–Tomas inequality in dimension 2, namely the problem of studying

$$\sup_{f\in L^2(\mathbb{S}^1),\, f\neq 0} \frac{\|\widehat{f\sigma}\|_{L^6(\mathbb{R}^2)}}{\|f\|_{L^2(\mathbb{S}^1)}}.$$

In [Sha16a] it has been established that such supremum is indeed achieved. It is conjectured that, also in this case, constant functions are maximizers. One main obstruction in adapting the strategy from [Fos15] to this problem is that the threefold convolution $\sigma_1 * \sigma_1 * \sigma_1$ has a singularity at $|x| = 1$ and the weight that offsets such singularity is partially negative, see [CFOeST17]. A different way of approaching the problem has been first proposed by Oliveira e Silva, Thiele, and Zorin-Kranich in [OeSTZK22]. In their work the case of non-negative, antipodally symmetric, band-limited functions with Fourier modes up to degree 30 has been considered establishing that, in this class of functions, constant functions are the unique maximizers. The result has been later extended to the case of band-limited functions with Fourier modes up to degree 120 in [BTZK23]. When restricted to the case of band-limited functions the problem becomes finite-dimensional and it can be addressed numerically as done in [OeSTZK22, BTZK23].

Motivated by these previous contributions, in the work [CG24], written in collaboration with F. Gonçalves, we have considered the case of functions whose spectrum is possibly infinite but satisfies certain arithmetic constraints. These arithmetic constraints arise as a generalization of the notion of B(3)-sets. A set $S \subset \mathbb{Z}$ is a B(3)-set if for any two triples (a_1, a_2, a_3) and (b_1, b_2, b_3) of elements in S such that $a_1 + a_2 + a_3 = b_1 + b_2 + b_3$ one triple is a permutation of the other. The generalization that we propose extends the definition of B(3)-set – and more in general, of B(h)-set – by allowing for the possibility of non-trivial symmetric subsets. We name such generalization a P(3)-set.

Our main result is the following: If $f \in L^2(\mathbb{S}^1)$ is such that its Fourier support is a P(3)−set then it holds that

$$\frac{\|\widehat{f\sigma}\|_{L^6(\mathbb{R}^2)}}{\|f\|_{L^2(\mathbb{S}^1)}} \leqslant \frac{\|\widehat{\mathbf{1}\sigma}\|_{L^6(\mathbb{R}^2)}}{\|\mathbf{1}\|_{L^2(\mathbb{S}^1)}}$$

and equality is attained if and only if f is a constant. The main tools that we utilize in the proof are certain counting arguments, that are common in the literature on B(h)-sets and on $\Lambda(2h)$-sets in general, combined with some novel refined estimates for integrals involving Bessel functions. The article [CG24] is contained in Appendix A. A detailed overview of the results in [CG24] is provided in Section 2.1.

Mixed-norm versions of Fourier extension inequalities on spheres have been studied by Vega in [Veg92], showing that the Fourier extension operator maps $L^2(\mathbb{S}^{d-1})$ to $L^{p'}_{rad}L^2_{ang}(\mathbb{R}^d)$

for $p' > \frac{2d}{d-1}$. More recently, the problem of computing the sharp constant for such inequalities has been studied by Carneiro, Oliveira e Silva, and Sousa in [COeSS19], establishing that constant functions are extremizers when the exponent p' is an even integer and that the set of exponents for which constants are maximizers contains a neighborhood of infinity, $(p'_0(d), \infty]$, giving some upper-bounds for $p'_0(d)$.

In the first part of the work [CS23], written in collaboration with M. Sousa, we have extended the range of exponents for which constant functions are known to be maximizers for these inequalities in the cases of low dimension $2 \leqslant d \leqslant 10$, covering the entire Stein–Tomas range of exponents in the cases of dimension $d = 2, 3$.

In the second part of the work [CS23], we have considered Fourier extension estimates in the diagonal case $p' = q'$. Maximizers for such inequalities are known only when p' is an even admissible integer or $p' = \infty$, [COeS15, FS24]. Our second main result concerns local maximizers for these inequalities. We show that, in the same range of exponents $(p'_0(d), \infty]$ for which constant functions are maximizers for mixed-norm Fourier extension inequalities, they are also local maximizers for the $L^{p'}(\mathbb{S}^{d-1})$ to $L^{p'}(\mathbb{R}^d)$ Fourier extension estimates. For example, this gives that in the cases of dimension $d = 2, 3$ constant functions are local maximizers for such inequalities for all $p' \geqslant \frac{2(d+1)}{d-1}$, the Stein–Tomas endpoint. The article [CS23] is contained in Appendix B. A detailed overview of the results in [CS23] in provided in Section 2.2.

The second part of this thesis deals with optimal weak-type endpoint estimates for certain square functions and Marcikiewicz multipliers operators.

The Littlewood–Paley square function formed by rough frequency projections adapted to a lacunary partition of the real line is a classical object in Analysis and it is a bounded operator on L^p for $1 < p < \infty$. Contrary to its smooth counterpart, it fails to be of weak-type $(1, 1)$. The rough Littlewood–Paley square function can be seen as a prototypical Marcinkiewicz multiplier. Marcinkiewicz multipliers on the real line are bounded functions of uniformly bounded variation on each Littlewood–Paley dyadic interval. The corresponding multiplier operators are well known to be bounded on $L^p(\mathbb{R})$ for all $1 < p < \infty$. Optimal weak-type endpoint estimates for these operators have been studied by Tao and Wright in [TW01] proving that they map locally the Orlicz space $L\log^{1/2} L$ to weak L^1 and such a result is sharp, meaning that the exponent $1/2$ cannot be replaced by a smaller one. It follows from this result that the same is true also for the rough Littlewood–Paley square function adapted to the classical dyadic partition of $\mathbb{R}$. A main tool utilized in the proof of this result is a weak square function characterization for the Orlicz space $L\log^{1/2} L$ obtained also in [TW01].

In this dissertation we are interested in Littlewood–Paley square functions formed by rough frequency projections adapted to *higher order lacunary* partitions of the frequency line and in *higher order* Marcinkiewicz multipliers, that is multipliers of uniformly bounded variation on each interval arising from a higher order lacunary decomposition of the real line. Recall that a decomposition of lacunary order $\tau > 1$, $\tau \in \mathbb{N}$, can be produced iteratively by performing a Whitney decomposition inside each interval of order $\tau - 1$. It follows from the classical work of Sjögren and Sjölin [SS81] that these higher order operators are bounded on L^p for $1 < p < \infty$.

In the work [BCPV24], written in collaboration with O. Bakas, I. Parissis, and M. Vitturi, we obtain optimal weak-type endpoint estimates for higher order square functions and Marcinkiewicz multiplier operators, recovering for the case of order $\tau = 1$ the results in [TW01]. In fact, in [BCPV24] we establish optimal weak-type endpoint bounds for the more general class of $R_{2,\tau}$-multipliers, the higher order analogous of the R_2-multiplier class considered in [TW01] and introduced by Coifman, Rubio de Francia, and Semmes [CRdFS88]. As a corollary, we also derive sharp endpoint results for higher order Hörmander–Mihlin multipliers – that is, multipliers that are singular on every point of a lacunary set of order $(\tau - 1)$ – and, similarly, for higher order smooth Littlewood–Paley square functions. The starting point of our analysis in [BCPV24] is the following result which is of independent interest: the Chang–Wilson–Wolff inequality [CWW85] implies the martingale difference square function characterization of $L \log^{1/2} L$ obtained in [TW01]. This enables us to generalize the square function characterization of $L \log^{1/2} L$ in [TW01] to the case of $L(\log L)^{\sigma}$, with $\sigma \geqslant 1/2$. This, combined with a Calderón–Zygmund decomposition for Orlicz spaces, leads to our endpoint result. The article [BCPV24] is contained in Appendix C. A detailed overview of the results in [BCPV24] in provided in Chapter 3.

This thesis is cumulative and it is based on the three articles, [CG24], [CS23], and [BCPV24], which are included in the Appendices A, B, and C, respectively.

[CG24] Valentina Ciccone and Felipe Gonçalves. Sharp Fourier extension on the circle under arithmetic constraints. J. Funct. Anal., 286(2): Paper No. 110219, 21, 2024.

[CS23] Valentina Ciccone and Mateus Sousa. Global and local maximizers for some Fourier extension estimates on the sphere. arXiv preprint arXiv:2312.07309, 2023.

[BCPV24] Odysseas Bakas, Valentina Ciccone, Ioannis Parissis, and Marco Vitturi. Endpoint estimates for higher order Marcinkiewicz multipliers. arXiv preprint arXiv:2401.06083, 2024.

The thesis is organized as follows. In Chapter 1 we provide a detailed introduction that gives motivation and background for the results in the articles [CG24], [CS23], and [BCPV24], thereby placing them within a broader context. In Chapter 2 we summarize the main results obtained in the works [CG24] and [CS23]. In Chapter 3 we summarize the main results obtained in the work [BCPV24]

Contents

Chapter 1

Introduction

The study of optimal inequalities plays a pivotal role in the field of Harmonic Analysis. In the first part of the dissertation, we focus on sharp Fourier extension inequalities on spheres. Specifically, we aim to determine optimal constants and maximizers for such inequalities. The second part of this dissertation is dedicated to the study endpoint estimates for certain square functions and Fourier multiplier operators of Marcinkiewicz type. In particular, we are interested in obtaining optimal weak-type endpoint bounds for such operators.

In this chapter, we provide a general introduction to the topics and the problems investigated in this dissertation.

1.1 Sharp Fourier restriction theory

In this section, we briefly recall some basic facts about the Fourier restriction problem and some of the classical results on the topic. Then, we introduce the subarea of sharp Fourier extension inequalities on the sphere and we briefly survey the main results in the subject providing background and context for the works [CG24, CS23] which are the content of Appendix A and Appendix B, respectively.

1.1.1 Fourier restriction theory

The role of curvature. The decay properties of the Fourier transform of measure supported on surfaces that exhibit some degree of curvature are at the foundation of the theory of Fourier restriction.

Let $\mathcal{M}$ be a smooth hypersurface and let σ be the surface measure on $\mathcal{M}$. For some smooth function ν with compact support on $\mathcal{M}$ define $d\eta := \nu d\sigma$. The Fourier transform of the measure η is given by

$$\widehat{\eta}(\xi) = \int_{\mathcal{M}} e^{-ix\cdot\xi} d\eta(x).$$

As a consequence of the classical theory of oscillatory integrals we have that if the hypersurface $\mathcal{M}$ has at least m non-vanishing principal curvatures on the support of η then it holds that

$$|\widehat{\eta}(\xi)| = O(|\xi|^{-m/2}) \quad \text{as } |\xi| \to \infty.$$

In particular, if $\mathcal{M}$ has non-vanishing Gaussian curvature – namely, if $\mathcal{M}$ has $(d-1)$ non-vanishing principal curvatures – on the support of the measure η then

$$|\widehat{\eta}(\xi)| = O(|\xi|^{-(d-1)/2}) \quad \text{as } |\xi| \to \infty.$$

As a simple example of this, let us consider the case of the unit sphere, $\mathcal{M} = \mathbb{S}^{d-1}$, with surface measure $\sigma = \sigma_{d-1}$. The Fourier transform of the measure σ can be computed explicitly,

$$\widehat{\sigma}(\xi) = (2\pi)^{\frac{d}{2}}\, J_{\frac{d}{2}-1}(|\xi|)\, |\xi|^{-\frac{d}{2}+1} \ , \tag{1.1}$$

where J_k denotes the Bessel function of the first kind of order k. By using the properties of Bessel functions, and specifically the fact that $J_k(r) = O(r^{-1/2})$ as $r \to \infty$, it is immediate to see that indeed $|\widehat{\sigma}(\xi)| = O(|\xi|^{-(d-1)/2})$ as $|\xi| \to \infty$.

On the other hand, if curvature is missing, we cannot expect such a nice decay behavior. For example, if we consider the Fourier transform of the length measure λ on the line segment $((0,-1),(0,1)) \subset \mathbb{R}^2$, we see that

$$\widehat{\lambda}(\xi_1, \xi_2) = \int_{-1}^{1} e^{-i(\xi_1 \cdot 0 + \xi_2 \cdot x_2)} dx_2 = \frac{2\sin(\xi_2)}{\xi_2} \ .$$

Hence, it is clear that, for a fixed ξ_2, $|\widehat{\lambda}(\xi, \xi_2)|$ does not decay as $|\xi_1| \to \infty$.

The Fourier restriction problem. Let S be a subset of $\mathbb{R}^d$, $S \subset \mathbb{R}^d$. We may ask ourselves for which $1 \leqslant p \leqslant 2$ it makes sense to consider the restriction of the Fourier transform of an arbitrary function $f \in L^p$ to the set S. If S has a positive Lebesgue measure then by Hausdorff–Young inequality the restriction of the Fourier transform of a function $f \in L^p(\mathbb{R}^d)$ to S is a function in $L^{p'}(S)$. The question becomes more interesting when the set S has zero Lebesgue measure. If $f \in L^1$ then its Fourier transform is continuous and therefore uniquely defined at every point. Hence, the restriction to any set S of the Fourier transform of a function $f \in L^1$ is well-defined. On the other hand, by Plancherel's theorem, the Fourier transform of a function in L^2 is again a function in L^2 and therefore there is no meaningful restriction to a set of zero Lebesgue measure. Consequently, the interesting question is what happens when $1 < p < 2$.

A first piece of evidence that for certain $1 < p < 2$ and for certain surfaces with some degree of curvature, such as the unit sphere $\mathbb{S}^{d-1}$, there may be a positive answer to this question dates back to an unpublished work of Stein in the 1960's, see [Ste93], indicating that the Fourier transform of a function in L^p, for certain $1 < p < 2$, has more structure than an arbitrary function in $L^{p'}(\mathbb{R}^d)$. This led to the so-called Fourier restriction problem, asking indeed for which subsets $S \subset \mathbb{R}^d$ and for which exponents $p \in (1,2)$ it does make sense to consider the restriction of the Fourier transform of functions in $L^p(\mathbb{R}^d)$ to S. This follows at once if there exist an exponent $1 \leqslant q \leqslant \infty$ such that the so-called Fourier restriction inequality,

$$\|\widehat{f}\|_{L^q(S,\mu)} \leqslant C_{d,p,q} \|f\|_{L^p(\mathbb{R}^d)}, \tag{1.2}$$

holds. Here μ is a measure that is comparable with the Hausdorff measure on S. By duality, (1.2) is equivalent, with the very same constant, to the so-called Fourier extension inequality,

$$\|\widehat{f\mu}\|_{L^{p'}(\mathbb{R}^d)} \leqslant C_{d,p,q}\|f\|_{L^{q'}(S,\mu)}, \tag{1.3}$$

where $\widehat{f\mu}$ is the Fourier transform of the measure $f\mu$ and p', q' are the conjugate exponents of p, q respectively.

For simplicity we assume S to be a hypersurface. A first observation, highlighting the role of curvature, is that we cannot have meaningful restriction estimates (equivalently, extension estimates) for flat hypersurfaces, except for the trivial estimate $p = 1$, $q = \infty$. To see an example of this fact we can consider the hyperplane $\{\xi \in \mathbb{R}^d : \xi_1 = 0\}$. Let $f(x_1, ..., x_d) = g(x_2, ..., x_d)h(x_1)$ where g is a smooth function with compact support on $\mathbb{R}^{d-1}$ and $h(x_1) = 1/(1 + |x_1|)$. Then, $f \in L^p(\mathbb{R}^d)$ for all $p > 1$ and the Fourier transform of f is unbounded at every point of the hyperplane $\{\xi \in \mathbb{R}^d : \xi_1 = 0\}$. Notably, some degree of curvature for the considered hypersurface is necessary.

For the remaining of our discussion, we focus on the case of the unit sphere, $S = \mathbb{S}^{d-1}$. Some necessary conditions on the exponents p and q can be derived by testing the inequalities (1.2) and (1.3) against some suitably chosen functions. For example, by testing the extension inequality (1.3) against the function $f \equiv 1$ it is clear that a necessary condition for the inequality to hold is that $\widehat{\sigma} \in L^{p'}(\mathbb{R}^d)$, where $\sigma = \sigma_{d-1}$ is the surface measure on $\mathbb{S}^{d-1}$ and an explicit expression for $\widehat{\sigma}$ is given in (1.1). This gives the first necessary condition

$$p' > \frac{2d}{d-1}. \tag{1.4}$$

A second necessary condition on the exponents p and q can be derived by considering the indicator function of a small spherical cap on $\mathbb{S}^{d-1}$. For example, let

$$C_\delta \coloneqq \{x \in \mathbb{S}^{d-1} : 1 - x \cdot e_d \lesssim \delta^2\}$$

where $e_d = (0, ..., 0, 1)$. A routine computation using the properties of the Fourier transform shows that a necessary condition for the Fourier extension inequality to hold for the indicator function $\mathbf{1}_{C_\delta}$ as $\delta \to 0$ is

$$q \leqslant \frac{d-1}{d+1}p'. \tag{1.5}$$

This is known as the Knapp example.

Conjecture 1 (Fourier restriction conjecture for the sphere)**.** *The necessary conditions on the exponents* (1.4), (1.5) *are also sufficient.*

The Riesz diagram for the exponents in Conjecture 1 is depicted in Figure 1.1. The conjecture has been fully solved only in dimension $d = 2$ [Fef70, Zyg74]. The celebrated Stein–Tomas theorem asserts that the conjecture is true when $q = q' = 2$ in all dimensions $d \geqslant 2$ [Ste93, Tom75].

Theorem 1 (Stein–Tomas theorem)**.** *Let $S = \mathbb{S}^{d-1}$. The Fourier restriction inequality* (1.2) *holds for $q = 2$ and $1 \leqslant p \leqslant \frac{2d+2}{d+3}$.*

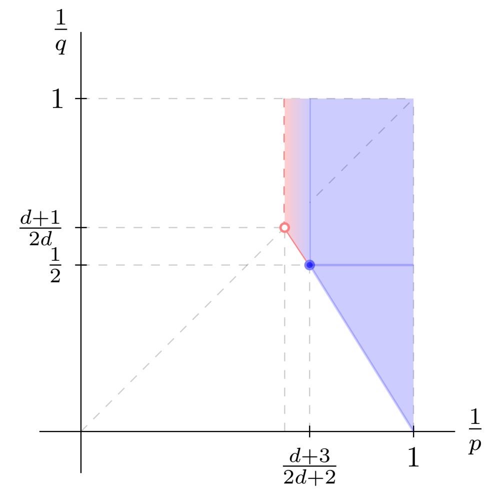

Figure 1.1: *Range of exponents in the restriction conjecture. The blue dot corresponds to the Stein–Tomas endpoint.*

By Hölder inequality, it follows from Stein–Tomas theorem that the restriction conjecture is verified for all $q \leqslant 2$ – equivalently, for all $q' \geqslant 2$. Moreover, interpolation between the Stein–Tomas endpoint estimate and the trivial estimate $p = 1$, $q = \infty$ verifies the restriction conjecture for $1 \leqslant p \leqslant \frac{2d+2}{d+3}$ and $q \leqslant \frac{d-1}{d+1}p'$. This corresponds to the blue trapezoid in Figure 1.1.

Much effort has been devoted to the study of the Fourier restriction problem over the last decades. This has led to the development of new tools and techniques of independent interest. We refer to the surveys [Sto19, Tao04] for a more comprehensive treatment of this topic. In passing, we stress that the Fourier restriction problem is intimately related with other important problems in Harmonic Analysis such as the Bochner-Riesz conjecture and the Kakeya conjecture, as well as with topics in partial differential equations, geometric measure theory, combinatorics, and analytic number theory.

1.1.2 Sharp Fourier extension inequalities on spheres

In this thesis we are interested in *sharp* Fourier extension estimates on spheres. For a triple (d, p', q') for which the Fourier extension inequality

$$\|\widehat{f\sigma}\|_{L^{p'}(\mathbb{R}^d)} \leqslant C(d, p', q')\|f\|_{L^{q'}(\mathbb{S}^{d-1})} \tag{1.6}$$

holds, we study questions like: What is the optimal constant? Namely, what is

$$C_{\text{opt}}(d, p', q') \coloneqq \sup_{\substack{f \in L^{q'}(\mathbb{S}^{d-1}) \\ f \neq 0}} \frac{\|\widehat{f\sigma}\|_{L^{p'}(\mathbb{R}^d)}}{\|f\|_{L^{q'}(\mathbb{S}^{d-1})}} \ ? \tag{1.7}$$

If maximizers – namely, functions that attain the optimal constant – exist what are they?

The study of sharp Fourier extension inequalities, and especially sharp Fourier extension inequalities on spheres, has flourished and it has received a great deal of attention over the last decade although many questions are still open.

The existence of maximizers for (1.7) has been investigated for the case of the Stein–Tomas range $q' = 2$ and $p' \geqslant \frac{2d+2}{d-1}$ in [FVV11, CS12a, Sha16a, FLS16]. In particular, the existence of maximizers for the case of $p' > \frac{2d+2}{d-1}$ has been established in [FVV11]. The existence of extremizers for the case of the endpoint $p' = \frac{2d+2}{d-1}$ has been established in [CS12a] for the case of $d = 3$ and in [Sha16a] for the case of $d = 2$. For the remaining cases, namely for $d \geqslant 4$, a conditional result about the existence of maximizers for the Stein–Tomas sharp endpoint inequality has been obtained in [FLS16].

More recently, existence of maximizers for (1.7) for the case $p' > q'$ has been studied in [FS24] showing that maximizers exist if $p' > \max\{q', \frac{d+1}{d-1}q\}$ or if $p' = \frac{d+1}{d-1}q$ and some further conditions are fulfilled. For later considerations, we stress that the existence of maximizers for (1.7) in the diagonal case $p' = q'$ is still an open problem, except for a few particular cases in which maximizers have been characterized and which are discussed below.

The characterization of maximizers for (1.7) appears to be a very challenging problem whose solution is known only in a few particular cases. Typically the available results rely crucially on the exponent p' being an even integer. In fact, when $p' = 2k$ is an even integer the left-hand-side of (1.6) can be rewritten, using Plancherel's identity, as the L^2-norm of a k-fold convolution of measures on the sphere and this can be further rewritten as a k-linear form over a submanifold of $(\mathbb{S}^{d-1})^k$.

The characterization of maximizers for (1.7) has initiated with the seminal work of Foschi [Fos15]. In [Fos15], it has been shown that constant functions are maximizers for (1.7) in the case of $d = 3$ and endpoint Stein–Tomas exponents $(p', q') = (4, 2)$. We will briefly recall the main steps of the elegant proof given by Foschi at the end of this subsection. Most of the subsequent results in sharp spherical Fourier restriction grew out of this initial work of Foschi. Maximizers for (1.7) in the cases $(d, p', q') = (d, 2k, q')$ with $d, k \in \mathbb{N}$, $q' \in \mathbb{R}_+ \cup \{\infty\}$ satisfying one of the following: (i) $k = 2$, $q' \geqslant 2$ and $3 \leqslant d \leqslant 7$; (ii) $k = 2$, $q' \geqslant 4$, and $d \geqslant 8$; (iii) $k \geqslant 3$, $q' \geqslant 2k$, and $d \geqslant 2$; have been studied in [COeS15] showing that constant functions are maximizers also in these situations. More recently, in [OeSQ21a] it has been shown that constant functions are maximizers for (1.7) when $3 \leqslant d \leqslant 7$, $q' = 2$, and $p' > 4$ is an even integer. Moreover, in [OeSQ21a] also the following conditional result has been established: if constant functions are maximizers for (1.7) in the endpoint case $(d, p', q') = (2, 6, 2)$ then constant functions are also maximizers for the cases $(d, p', q') = (2, p', 2)$ with $p' > 6$ an even integer.

It is manifest that many questions remain open. A major one that has received a great deal of attention over the last years is whether constant functions are maximizers also for the sharp version of the endpoint Stein–Tomas inequality for $\mathbb{S}^1$, corresponding to the case of exponents $p' = 6$, $q' = 2$. We will discuss more about this question in Section 2.1.

An intermediate step towards the characterization of maximizers is the study of local maximizers. We say that a function h is a local maximizers for (1.7) if there exists $\delta > 0$ such that whenever $\|f-h\|_{L^{q'}(\mathbb{S}^{d-1})} < \delta \leqslant$ it holds that

$$\frac{\|\widehat{f\sigma}\|_{L^{p'}(\mathbb{R}^d)}}{\|f\|_{L^{q'}(\mathbb{S}^{d-1})}} \leqslant \frac{\|\widehat{h\sigma}\|_{L^{p'}(\mathbb{R}^d)}}{\|h\|_{L^{q'}(\mathbb{S}^{d-1})}}.$$

Constant functions have been shown to be local maximizers for (1.7) in the Stein–Tomas endpoint cases: $(d,p',q')=(3,4,2)$ in [CS12a], $(d,p',q')=(2,6,2)$ in [COeSS19], $(d,p',q')=(d,\frac{2d+2}{d-1},2)$ for $2\leqslant d\leqslant 60$ in [GN22]. More recently, local maximizers for (1.7) in the diagonal case $p'=q'$ have been studied in [CS23], which is the content of Appendix B. We will introduce these results in Section 2.2.

Foschi's proof of the sharp Stein–Tomas endpoint inequality on $\mathbb{S}^2$. Foschi proved the following sharp version of the Stein–Tomas endpoint Fourier extension inequality on $\mathbb{S}^2$.

Theorem 2 (Theorem 1.1 in [Fos15])**.** *For all $f\in L^2(\mathbb{S}^2)$ it holds that*

$$\|\widehat{f\sigma}\|_{L^4(\mathbb{R}^3)} \leqslant (2\pi)\|f\|_{L^2(\mathbb{S}^2)}. \tag{1.8}$$

In particular, constant functions are the unique real-valued maximizers.

Combining this with [CS12b, Theorem 1.2] it follows that all the complex-valued maximizers are given by $f(\omega)=ke^{i\theta}e^{i\xi\cdot\omega}$, for some $k>0$, $\theta\in\mathbb{R}$, $\xi\in\mathbb{R}^3$.

In this paragraph, we briefly describe the main steps and ideas in the elegant proof of Foschi.

The first natural and key observation is that, thanks to the evenness of the exponent $p'=4$, the left-hand-side of (1.8) can be rewritten as

$$||\widehat{f\sigma}||^4_{L^4(\mathbb{R}^3)} = ||\widehat{f\sigma}\widehat{f_\flat\sigma}||^2_{L^2(\mathbb{R}^3)} = ||\widehat{f\sigma * f_\flat\sigma}||^2_{L^2(\mathbb{R}^3)} = (2\pi)^3||f\sigma * f_\flat\sigma||^2_{L^2(\mathbb{R}^3)},$$

where we have used the notation $f_\flat(\omega)=\overline{f(-\omega)}$.

It is not difficult to check that the study of maximizers for (1.8) can be restricted to functions that are non-negative and antipodally symmetric. We denote by $f_\sharp$ the antipodally symmetric rearrangement of a function $f\in L^2(\mathbb{S}^2)$,

$$f_\sharp = \sqrt{\frac{|f|^2+|f_\flat|^2}{2}}.$$

Then, clearly $\|f\|_{L^2(\mathbb{S}^2)}=\|f_\sharp\|_{L^2(\mathbb{S}^2)}$ and it is not too difficult to see that $||f\sigma * f_\flat\sigma||^2_{L^2(\mathbb{R}^3)} \leqslant ||f_\sharp\sigma * f_\sharp\sigma||^2_{L^2(\mathbb{R}^3)}$. Hence, without loss of generality, we can assume f to be non-negative and antipodally symmetric.

Next, the L^2-norm of the two-fold convolution of measures on $\mathbb{S}^2$ can be rewritten as the quadrilinear form

$$||f\sigma * f\sigma||^2_{L^2(\mathbb{R}^3)} = \int_{(\mathbb{S}^2)^4} f(\omega_1)f(\omega_2)f(\omega_3)f(\omega_4)d\Sigma,$$

where $d\Sigma := \delta\Big(\sum_{i=1}^4 \omega_i\Big) d\sigma(\omega_1)d\sigma(\omega_2)d\sigma(\omega_3)d\sigma(\omega_4)$ is a positive measure supported on the submanifold

$$\Gamma := \{(\omega_1,\omega_2,\omega_3,\omega_4) \in (\mathbb{S}^2)^4 : \omega_1+\omega_2+\omega_3+\omega_4 = 0\}.$$

At this point, the main technical difficulty is due to the fact that the convolution of the surface measure $\sigma = \sigma_2$ on $\mathbb{S}^2$ with itself, $\sigma * \sigma$, has a singularity at the origin. In fact,

$$\sigma * \sigma(x) = \frac{2\pi}{|x|}\mathbf{1}_{\{|x|\leqslant 2\}}(x),$$

see [Fos15, Lemma 2.2]. To overcome this difficulty, the key idea of Foschi was to exploit the following geometric property of the sphere: If $\omega_1,\omega_2,\omega_3,\omega_4 \in \mathbb{S}^2$ are such that $\omega_1+\omega_2+\omega_3+\omega_4 = 0$ then

$$|\omega_1+\omega_2||\omega_3+\omega_4| + |\omega_1+\omega_3||\omega_2+\omega_4| + |\omega_1+\omega_4||\omega_2+\omega_3| = 4.$$

Foschi used this identity to offset the singularity of $\sigma * \sigma$ at the origin. In fact, using this geometric identity one can rewrite

$$||f\sigma * f\sigma||^2_{L^2(\mathbb{R}^3)} = \frac{3}{4}\int_\Gamma f(\omega_1)f(\omega_2)|\omega_1+\omega_2|f(\omega_3)f(\omega_4)|\omega_3+\omega_4|d\Sigma.$$

At this point, an application of Cauchy–Schwarz inequality leads to

$$\begin{aligned}||f\sigma * f\sigma||^2_{L^2(\mathbb{R}^3)} &\leqslant \int_{(\mathbb{S}^2)^2} f^2(\omega_1)f^2(\omega_2)|\omega_1+\omega_2|^2\sigma*\sigma(\omega_1+\omega_2)d\sigma_{\omega_1}d\sigma_{\omega_2}\\ &= 2\pi\int_{(\mathbb{S}^2)^4} f(\omega_1)^2f(\omega_2)^2|\omega_1+\omega_2|d\sigma_{\omega_1}d\sigma_{\omega_2}.\end{aligned}$$

Finally, the last step of the program is a spectral decomposition of the quadratic form

$$\mathrm{H}(g) \coloneqq \int_{(\mathbb{S}^2)^2} \overline{g(\omega)}g(\nu)|\omega-\nu|d\sigma_\omega d\sigma_\nu.$$

The functional H is well-defined, real-valued, and continuous on $L^1(\mathbb{S}^2)$. In [Fos15, Theorem 5.1] it has been shown that if $c = \frac{1}{4\pi}\int_{\mathbb{S}^2} f(\omega)d\sigma$ is the mean value of the function g then

$$\mathrm{H}(g) \leqslant \mathrm{H}(c\mathbf{1}_{\mathbb{S}^2}),$$

with equality if and only if g is constant. By density, it is enough to prove this result for functions in $L^2(\mathbb{S}^2)$. Foschi's proof relies on a spectral decomposition of the functional H. An alternative proof of this inequality has been more recently given in [NOeST23].

One can check that all the encountered inequality holds with equality if f is a constant function.

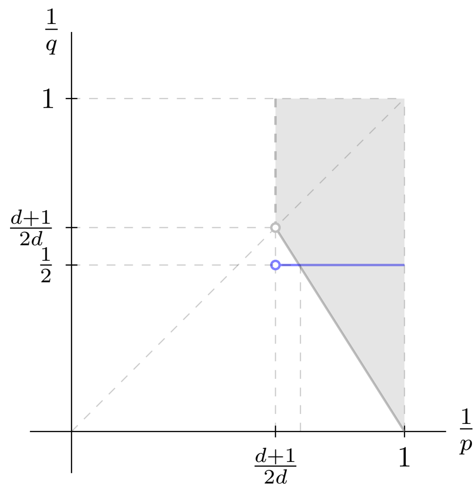

Figure 1.2: *In blue, the range of exponents for which inequality 1.9 holds.*

1.1.3 A mixed-norm Fourier extension inequality

In [Veg92], Vega proved that the following mixed-norm Fourier extension inequality holds for all $d \geqslant 2$ and $p' > 2d/(d-1)$,

$$\|\widehat{f\sigma}\|_{L^{p'}_{rad}L^2_{ang}(\mathbb{R}^d)} \coloneqq \left(\int_0^\infty \left(\int_{\mathbb{S}^{d-1}} |\widehat{f\sigma}(r\omega)|^2 d\sigma_\omega \right)^{p'/2} r^{d-1} dr \right)^{1/p'} \leqslant C_{d,p'} \|f\|_{L^2(\mathbb{S}^{d-1})}. \quad (1.9)$$

As we are only requiring $\widehat{f\sigma} \in L^{p'}_{rad}L^2_{ang}(\mathbb{R}^d)$ this can be seen as a weaker version of Stein–Tomas inequality. In Figure 1.2 the range of exponents for which (1.9) holds are depicted in the Riesz diagram for Conjecture 1.

To see that (1.9) holds, we start by recalling the following well-known formula

$$\widehat{Y_k\sigma}(\xi) = (2\pi)^{\frac{d}{2}}\, i^k\, J_{\frac{d}{2}-1+k}(|\xi|)\, |\xi|^{-\frac{d}{2}+1}\, Y_k\left(\frac{\xi}{|\xi|}\right), \quad \xi \in \mathbb{R}^d\ , \quad (1.10)$$

where Y_k is a spherical harmonic of degree k, see e.g. [SW71, Chapter IV]. Given a function $f \in L^2(\mathbb{S}^{d-1})$, we can expand it as $f = \sum_{k\geqslant 0} a_k Y_k$ and, for convenience, we may assume $\|Y_k\|_{L^2(\mathbb{S}^{d-1})} = 1$. Relying on (1.10) and by orthogonality of spherical harmonics we have

$$\|\widehat{f\sigma}\|_{L^{p'}_{rad}L^2_{ang}(\mathbb{R}^d)} = (2\pi)^{\frac{d}{2}} \left(\int_0^\infty \left| \sum_k |a_k|^2 J_{\frac{d}{2}-1+k}(r)^2\, r^{-d+2} \right|^{p'/2} r^{d-1} dr \right)^{2/(2p')} .$$

Let $w \in L^s(\mathbb{R}_+, r^{d-1}dr)$, $\|w\|_{L^s(\mathbb{R}_+, r^{d-1}dr)} \leqslant 1$, where $s \coloneqq (p'/2)'$, and consider

$$(2\pi)^{\frac{d}{2}} \left| \int_0^\infty \left(\sum_k |a_k|^2 J_{\frac{d}{2}-1+k}(r)^2\, r^{-d+2} \right) w(r) r^{d-1} dr \right|^{1/2} .$$

For convenience we may assume that f has finite expansion $f = \sum_k a_k Y_k$, and in case argue by limiting arguments at the end. By Hölder's inequality and the fact that $\|w\|_{L^s(\mathbb{R}_+, r^{d-1}dr)} \leqslant 1$ we can bound the last display as follows

$$\begin{aligned}
&(2\pi)^{\frac{d}{2}} \left| \int_0^\infty \left(\sum_k |a_k|^2 J_{\frac{d}{2}-1+k}(r)^2\, r^{-d+2} \right) w(r) r^{d-1} dr \right|^{1/2} \\
&\quad \leqslant (2\pi)^{\frac{d}{2}} \left| \sum_k |a_k|^2 \left(\int_0^\infty |J_{\frac{d}{2}-1+k}(r)^2\, r^{-d+2}\,|^{p'/2}\, r^{d-1} dr \right)^{2/p'} \left(\int_0^\infty |w(r)|^s r^{d-1} dr \right)^{1/s} \right|^{1/2} \\
&\quad \leqslant (2\pi)^{\frac{d}{2}} \left| \sum_k |a_k|^2 \left(\int_0^\infty |J_{\frac{d}{2}-1+k}(r)\, r^{-\frac{d}{2}+1}\,|^{p'} r^{d-1} dr \right)^{2/p'} \right|^{1/2}.
\end{aligned}$$

By taking the supremum over $w \in L^s(\mathbb{R}_+, r^{d-1}dr)$ with $\|w\|_{L^s(\mathbb{R}_+, r^{d-1}dr)} \leqslant 1$ we have

$$\|\widehat{f\sigma}\|_{L^{p'}_{rad} L^2_{ang}(\mathbb{R}^d)} \leqslant (2\pi)^{\frac{d}{2}} \left| \sum_k |a_k|^2 \left(\int_0^\infty |J_{\frac{d}{2}-1+k}(r)\, r^{-\frac{d}{2}+1}\,|^{p'} r^{d-1} dr \right)^{2/p'} \right|^{1/2}.$$

To conclude, one can invoke the classical bounds for Bessel functions in [BC89] to see that there exists a constant $C_{d,p'} > 0$ such that for all $p' > 2d/(d-1)$

$$\left(\int_0^\infty |J_{\frac{d}{2}-1+k}(r)\, r^{-\frac{d}{2}+1}\,|^{p'} r^{d-1} dr \right)^{1/p'} \leqslant C_{d,p'}$$

uniformly with respect to k. Hence inequality (1.9) follows. As before, one can check that the range of exponents $p' > 2d/(d-1)$ is sharp by testing (1.9) against $f \equiv c$, for some constant $c \neq 0$.

Sharp versions of the inequality (1.9) have been investigated in [FOeS17, COeSS19], and more recently in [CS23]. We will discuss more about this in Subsection 2.2.

1.2 Littlewood–Paley theory and Marcinkiewicz multipliers

In this section we briefly review some basic facts in classical Littlewood–Paley theory and multipliers theory, which will serve as a background and motivation for the work [BCPV24] which is the content of Appendix C. First, we will focus on the case of square functions with rough frequencies projections adapted to some suitable partitions of $\mathbb{R}$. Then, we will discuss an application of Littlewood–Paley theory to the theory of Marcinkiewicz multipliers on $\mathbb{R}$. We refer for example to [Gra14, Duo01] for a more comprehensive treatment of such theories. We will conclude the section by recalling some basic facts about Orlicz spaces which will be useful in studying weak-type endpoint bounds for the aforementioned square functions and multipliers.

1.2.1 Classical Littlewood–Paley theory

Littlewood–Paley theory is a classical tool in analysis that allows to decompose functions on the frequency side into pieces that have disjoint, or almost disjoint, frequency support.

Littlewood–Paley theory provides a partial substitute to the Plancherel theorem for general L^p spaces, with $p \neq 2$.

A heuristic motivation for this can be easily provided in the framework of one-dimensional Fourier series – where the theory has been originally developed – by considering the case of lacunary sequences of frequencies $\{e^{i2\pi 2^k\theta}\}_{k\in\mathbb{N}}$. In fact, for a square summable sequence $\{a_k\}_{k\in\mathbb{N}}$, the L^p-norm of $\sum_{k\in\mathbb{N}} a_k e^{i2\pi 2^k\theta}$ is comparable with its L^2-norm, namely there exist $0 < c_p, C_p < \infty$ such that

$$c_p \Big\| \sum_{k\in\mathbb{N}} a_k e^{i2\pi 2^k\theta} \Big\|_p \leqslant \Big(\sum_{k\in\mathbb{N}} |a_k|^2 \Big)^{1/2} \leqslant C_p \Big\| \sum_{k\in\mathbb{N}} a_k e^{i2\pi 2^k\theta} \Big\|_p .$$

In passing, we mention that the same equivalence of norms property holds for any lacunary Fourier series [Rud60]. This may suggest similar considerations in the continuous setting.

Formally, we may decompose a function f on $\mathbb{R}$ as

$$f = \sum_{k\in\mathbb{Z}} \Delta_k f, \qquad \widehat{\Delta_k f}(\xi) := \mathbf{1}_{\{2^k \leqslant |\xi| < 2^{k+1}\}} \widehat{f}(\xi).$$

When k_1 and k_2 are sufficiently far apart, $\Delta_{k_1} f$ and $\Delta_{k_2} f$ oscillate at very different frequencies and their behavior resembles that of independent random variables.

The dyadic frequency projection Δ_k is a bounded L^p-to-L^p operator for any $1 < p < \infty$. In fact, Δ_k can be expressed as a linear combination of two modulated Hilbert transforms whose L^p-to-L^p mapping properties hold in the same range.

Our main object of interest is the Littlewood–Paley square function,

$$Sf := \Big(\sum_k |\Delta_k f|^2 \Big)^{1/2} .$$

By Plancherel theorem, it is immediate to see that when $f \in L^2(\mathbb{R})$ it holds that

$$\|Sf\|_{L^2}^2 = \|f\|_{L^2}^2 .$$

Littlewood–Paley theory tells us that these quantities are comparable also in L^p. The classical result in the theory is the following.

Theorem 3 (Littlewood–Paley theorem)**.** *Let $1 < p < \infty$. Then there exist $0 < c_p$, $C_p < \infty$ such that for all $f \in L^p$ it holds that*

$$c_p \|f\|_{L^p} \leqslant \|Sf\|_{L^p} \leqslant C_p \|f\|_{L^p} .$$

The result in Theorem 3 is sharp, in the sense that it fails at the endpoints $p = 1, \infty$. We are particularly interested in the mapping properties of S near L^1.

We start by recalling that the square function S is not of weak-type $(1,1)$. The behavior of the operator norm $\|Sf\|_{L^p \to L^p}$ as $p \to 1^+$ has been first studied by Bourgain in [Bou89] in the periodic setting. By transference, it follows from the results in [Bou89] that

$$\|S\|_{L^p(\mathbb{R}) \to L^p(\mathbb{R})} \sim (p-1)^{-3/2} \quad \text{as } p \to 1^+ .$$

Other proofs of this fact follow from [Bak19, Ler19]. By Yano extrapolation [Yan51] this implies that the square function S maps locally the Orlicz space $L\log^{3/2} L$ to L^1. This may suggest that the S could also map locally $L\log^{1/2} L$ to $L^{1,\infty}$. Indeed this can be seen to be the case as a corollary of certain results of Tao and Wright in [TW01] about the endpoint mapping properties of certain Marcinkiewicz multipliers. We will come back to these results in Section 3.1.

A few words about the smooth square function. Theorem 3 can be conveniently proven as a consequence of the boundedness result for the smooth Littlewood–Paley square function, namely a square function with frequency projections

$$\widehat{\widetilde{\Delta}_k f}(\xi) := \psi(2^{-k}\xi)\widehat{f}(\xi),$$

where ψ is a smooth function with compact support in $[-4,-1/2]\cup[1/2,4]$ which is identically one on $[-2,-1]\cup[1,2]$. Now the operator $\widetilde{\Delta}_k$ maps L^p to itself for any $1 \leqslant p \leqslant \infty$. Let $\widetilde{S}$ be the smooth square function, $\widetilde{S}f := (\sum_k |\widetilde{\Delta}_k f|^2)^{1/2}$. Then, for all $1 < p < \infty$, there exist $0 < \widetilde{C}_p < \infty$ such that

$$\|\widetilde{S}f\|_{L^p} \leqslant \widetilde{C}_p \|f\|_{L^p}.$$

Such an estimate can be shown, for example, by relying on vector-valued Calderón–Zygmund theory or using Khintchine's inequality. One can recover the result of Theorem 3 by using the fact that $\Delta_k\widetilde{\Delta}_k = \Delta_k$ together with the L^p-to-L^p boundness of vector-valued frequency projections. Finally, it follows from Calderón–Zygmund theory that $\widetilde{S}$ is of weak-type $(1,1)$.

1.2.2 Littlewood–Paley sets

So far, we have considered only the case of the classical Littlewood–Paley dyadic decomposition of the real frequency line. A natural question is whether a similar result holds for square functions associated with different decompositions.

Let $\mathcal{I} = \{I_j\}_j$ be a collection of mutually disjoint intervals in $\mathbb{R}$. We define Δ_{I_j} and $S_{\mathcal{I}}$ as

$$\widehat{\Delta_{I_j} f}(\xi) := \mathbf{1}_{\{\xi\in I_j\}}(\xi)\widehat{f}(\xi), \qquad S_{\mathcal{I}} f := \Big(\sum_j |\Delta_{I_j} f|^2\Big)^{1/2}.$$

The following result is due to Rubio de Francia [RdF85] and establishes that $S_{\mathcal{I}}$ maps L^p to itself for all $2 \leqslant p < \infty$.

Theorem 4 (Rubio de Francia square function)**.** *Let $\mathcal{I}$ and $S_{\mathcal{I}}$ be as defined above. Then for $2 \leqslant p < \infty$ there exist $0 < C_p < \infty$ such that*

$$\|S_{\mathcal{I}} f\|_{L^p(\mathbb{R})} \leqslant C_p \|f\|_{L^p(\mathbb{R})}.$$

This result is sharp in the sense that there exist partitions of $\mathbb{R}$ into mutually disjoint intervals such that the associated square function fails to map L^p to itself when $1 < p < 2$. A classical counterexample is given by the collection of mutually disjoint intervals $\mathcal{I} = \{[n, n+1) : n \in \mathbb{Z}\}$, see for example [RdF83]. In fact we may consider the function $f \in L^p$ defined by

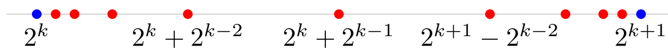

Figure 1.3: *Blue dots correspond to elements in the set* $\{2^n : n \in \mathbb{Z}\}$. *Red dots correspond to elements in the successor set* $\{2^{n_1} + 2^{n_2} : n_1, n_2 \in \mathbb{Z},\, n_1 > n_2\} \cup \{2^{n_1+1} - 2^{n_2} : n_1, n_2 \in \mathbb{Z},\, n_1 > n_2\}$.

$\widehat{f}(\xi) = \mathbf{1}_{[0,N]}(\xi)$. One can easily check that $f(x) = N\check{\mathbf{1}}_{[0,1]}(Nx)$. Hence, the desired estimates is

$$\|S_{\mathcal{I}} f\|_{L^p(\mathbb{R}^d)} = \|N^{1/2}\check{\mathbf{1}}_{[0,1]}\|_{L^p(\mathbb{R})} \leqslant CN^{1-1/p}\|\check{\mathbf{1}}_{[0,1]}\|_{L^p(\mathbb{R})}.$$

By taking the limit $N \to \infty$ we see that this can hold only for $p \geqslant 2$. Accordingly, the reverse square function estimate fails in the range $2 < p < \infty$.

In this subsection we are interested in Littlewood–Paley sets. Following [SS81], we consider a closed null (i.e. of zero Lebesgue measure) set $E \subset \mathbb{R}$ and we define $\mathcal{I}_E$ to be the collection of intervals I_j, $j = 1, 2, ...$, that are complementary to E in $\mathbb{R}$. The following definition can be found, for example, in [SS81].

Definition 1. *For a certain* $1 < p < \infty$ *we say that* E *satisfies the Littlewood–Paley property* $\mathrm{LP}(p)$*, or that* E *is a* $\mathrm{LP}(p)$*-set, if there exist* $0 < c_p, C_p < \infty$ *such that for all* $f \in L^p(\mathbb{R})$ *it holds that*

$$c_p\|f\|_{L^p(\mathbb{R})} \leqslant \|S_{\mathcal{I}_E} f\|_{L^p(\mathbb{R})} \leqslant C_p\|f\|_{L^p(\mathbb{R})}.$$

The property $\mathrm{LP}(p)$ is preserved under translation, dilation, and by taking subsets. In their work [SS81], Sjögren and Sjölin have provided a strategy to construct, starting from a $\mathrm{LP}(p)$-set E, a possibly larger set E' with the $\mathrm{LP}(p)$ property. They gave the following definition.

Definition 2. *A closed null set* $E' \subset \mathbb{R}$ *is said to be a successor of a closed null set* $E \subset \mathbb{R}$ *if there exist* $c > 0$ *such that if* $x, y \in E'$*, with* $x \neq y$*, then* $|x - y| \geqslant c\, d(x, E)$*.*

As exemplification, we may consider a bounded interval $I \subset \mathbb{R}\backslash E$, where E is a closed null set in $\mathbb{R}$, with $I = (a, b)$, $a, b \in E$. If E' is a successor of E then $E' \cap I$ is contained in the union of two sequences $\{\alpha_j\}$ and $\{\beta_k\}$ converging to a and b, respectively, and such that, for some $\theta_1, \theta_2 > 1$, it holds that $(\alpha_j - a)/(\alpha_{j+1} - a) \geqslant \theta_1$ for all j and $(\beta_k - b)/(\beta_{k+1} - b) \geqslant \theta_2$ for all k. An example of this is depicted in Figure 1.3. For example, the (dyadic) lacunary set of order ℓ given by

$$\{\pm(2^{k_1} + ... + 2^{k_\ell}) : k_1, ..., k_\ell \in \mathbb{Z},\, k_1 > ... > k_\ell\}$$

is a successor of the (dyadic) lacunary set of order $\ell - 1$

$$\{\pm(2^{k_1} + ... + 2^{k_{\ell-1}}) : k_1, ..., k_{\ell-1} \in \mathbb{Z},\, k_1 > ... > k_{\ell-1}\}.$$

The following has been established by Sjögren and Sjölin in [SS81].

Theorem 5 (Theorem 1.2 in [SS81])**.** *If* E *is a* $\mathrm{LP}(p)$*-set then any successor of* E *is a* $\mathrm{LP}(p)$*-set.*

In particular, any successor of the set $\{\pm 2^k : k \in \mathbb{Z}\}$ has the LP(p) property for all $1 < p < \infty$. Similarly, if $\{\lambda_k\}_k$ is a lacunary sequence of positive real numbers one can check, by the very same arguments used in the dyadic case, that the set $\{\lambda_k\}_k \cup (-\{\lambda_k\}_k)$ is a LP(p)-set for all $1 < p < \infty$. Hence, the same is true for all its successor sets. In particular, finite order lacunary sets are LP(p)-sets for all $1 < p < \infty$.

The problem of the characterization of LP(p)-sets has been studied in [HK89, HK92, HK95] and, more recently, in [BCDP$^+$24] an abstract characterization of LP(p)-sets has been provided.

1.2.3 Marcinkiewicz multipliers

Given a function $m \in L^\infty(\mathbb{R})$ we define implicitely the operator T_m associated to m by

$$\widehat{T_m f}(\xi) := m(\xi)\hat{f}(\xi).$$

By Plancherel theorem we see that T_m is a bounded operator on L^2 with operator norm $\|T_m\|_{L^2(\mathbb{R})\to L^2(\mathbb{R})} = \|m\|_{L^\infty(\mathbb{R})}$. We say that T_m is a Fourier multiplier operator with symbol m. We may ask ourselves under what conditions T_m extends to a bounded operator on L^p for some $p \neq 2$.

An example of a Fourier multiplier operator is the Hilbert transform which is a bounded operator on L^p for $1 < p < \infty$ associated with the symbol $m(\xi) = -i\, c\, \mathrm{sgn}(\xi)$, where c is some constant depending on the chosen normalization.

We say that a multiplier operator T_m is a Marcinkiewicz multiplier operator if $m \in L^\infty$ and if the following condition holds,

$$\sup_{k\in\mathbb{Z}} \int_{(-2^{k+1},-2^k]\cup[2^k,2^{k+1})} |dm|(\xi) \leqslant C_m < \infty.$$

We use the notation $\int_I |dm|$ to indicate the total variation of m over the interval I where this is defined as

$$\sup_N \sup_{\substack{x_0,\ldots,x_N\in I\\ x_0<\ldots<x_N}} \sum_{n=1}^N |m(x_n) - m(x_{n-1})|.$$

Simple examples of Marcinkiewicz multiplier operators are those whose symbol is constant on each Littlewood–Paley dyadic interval.

The following result about Marcinkiewicz multipliers can be derived as a consequence of the classical Littlewood–Paley theory.

Theorem 6 (Marcinkiewicz multiplier theorem)**.** *Let T_m be a Marcinkiewicz multiplier as defined above. Then T_m extends to a bounded operator on L^p for $1 < p < \infty$. Moreover,*

$$\|T_m f\|_{L^p(\mathbb{R})} \leqslant C_p(\|m\|_{L^\infty(\mathbb{R})} + C_m)\|f\|_{L^p(\mathbb{R})}.$$

As for the Littlewood–Paley theorem, also in this case we may ask ourselves whether the result still holds if we consider a different decomposition of the real frequency line.

In the periodic setting, in [Mar39] Marcinkiewicz multiplier operators on the torus formed with respect to second order (dyadic) lacunary partitions of the integers have been considered showing that these operators are bounded on $L^p(\mathbb{T})$ for all $1 < p < \infty$.

In their work [SS81], Sjögren and Sjölin established that a closed null set $E \subset \mathbb{R}$ is a LP(p)-set, as per Definition 1, for a certain $1 < p < \infty$ if and only if for any $m \in L^\infty(\mathbb{R})$ such that $sup_k \int_{I_k} |dm| < \infty$, where $\{I_k\}$ is the collection of intervals complementary to E in $\mathbb{R}$, the corresponding multiplier operator T_m is bounded on L^p.

A remarkable improvement of the classical Marcinkiewicz multiplier theorem has been obtained by Coifman, Rubio de Francia, and Semmes in [CRdFS88] by relying on the Rubio de Francia square function that we have encountered in the previous subsection. For some $q \geqslant 1$ we use the notation $\|m\|_{V_q(I)}$ to denote the total q-variation of m over the interval I, that is

$$\|m\|_{V_q(I)} = \sup_N \sup_{\substack{x_0,\dots,x_N \in I \\ x_0 < \dots < x_N}} \left(\sum_{n=1}^{N} |m(x_n) - m(x_{n-1})|^q \right)^{1/q}.$$

Theorem 7 (Coifman, Rubio de Francia, and Semmes theorem)**.** *Let $I_k^\ell = (-2^{k+1}, -2^k]$, $I_k^r = [2^k, 2^{k+1})$ for every $k \in \mathbb{Z}$. Let $m \in L^\infty(\mathbb{R})$. If for some $1 \leqslant q < \infty$ it holds that*

$$\sup_{k \in \mathbb{Z}} \left(\|m\|_{V_q(I_k^\ell)} + \|m\|_{V_q(I_k^r)} \right) < \infty$$

then T_m extends to a bounded Fourier multiplier on $L^p(\mathbb{R})$ for every $1 < p < \infty$ satisfying $|\frac{1}{p} - \frac{1}{2}| < \frac{1}{q}$.

The case $q = 1$ corresponds to the classical Marcinkiewicz multiplier theorem.

This class of multipliers has been considered by Tao and Wright in [TW01] where they have studied the endpoint mapping properties of such multipliers operators near L^1. We will say more about this in Section 3.1.

We conclude this subsection with a few words about Hörmander–Mihlin multiplier operators on $\mathbb{R}$.

Theorem 8 (Hörmander–Mihlin multiplier theorem)**.** *Let m be a bounded function on $\mathbb{R}\backslash\{0\}$ such that*

$$|\partial m(\xi)| \leqslant C_m |\xi|^{-1}.$$

Then T_m extends to a bounded operator on L^p for $1 < p < \infty$ and

$$\|T_m f\|_{L^p(\mathbb{R})} \leqslant C_p(\|m\|_{L^\infty(\mathbb{R})} + C_m)\|f\|_{L^p(\mathbb{R})}.$$

Moreover, T_m is of weak-type $(1,1)$ for some constant $C(\|m\|_{L^\infty(\mathbb{R})} + C_m)$.

Note that for the particular case of $d = 1$ the above Marcinkiewicz multipliers theorem is stronger than the Hörmander–Mihlin multiplier theorem, as the pointwise condition $|\partial m(\xi)| \leqslant C_m |\xi|^{-1}$ implies the Marcinkiewicz condition.

1.2.4 Glimpse of Orlicz spaces

A convex increasing function $\Phi : [0,\infty] \to [0,\infty]$ is said to be a Young function if $\Phi(0) = 0$, there exists $0 < t < \infty$ such that $\Phi(t) < \infty$, and $\lim_{t\to\infty} \Phi(t) = \infty$, see e.g. [Wil08, Chapter 10]. Given a Young function Φ and a measure space $(X, \mathcal{M}, \mu)$ we define the Orlicz space $L^\Phi(X, \mathcal{M}, \mu)$ as the set of measurable functions f such that

$$\int_X \Phi(|f|/\lambda)d\mu < \infty$$

for some $\lambda > 0$. Orlicz spaces can be seen as a generalization of L^p spaces. In fact, on one hand, if we consider the case of the Young function $\Phi(t) = t^p$, $1 \leqslant p < \infty$, we have that $L^\Phi(X, \mathcal{M}, \mu) = L^p(X, \mathcal{M}, \mu)$. On the other hand, Orlicz spaces allow us to catch a scale of integrability that may not be captured by L^p spaces. To see an example of this phenomenon we may consider the function

$$f(x) = \begin{cases} 1/(x(|\log(x)| + 1)^2) & \text{for } 0 \leqslant x \leqslant 1 \\ 0 & \text{otherwise,} \end{cases}$$

which belongs to L^1 but it does not belong to L^p for all $p > 1$. Such function is indeed *somewhat more than* L^1. Consider the function $\Psi_{1/2}(t) = t(\log(e+t))^{1/2}$, one can check that this is a Young function and the function f belongs to $L^{\Psi_{1/2}}$.

We endow the Orlicz space L^Φ with the following norm, known as the Luxemburg norm

$$\|f\|_{L^\Phi(X,\mathcal{M},\mu)} := \inf\left\{\lambda > 0 : \int_X \Phi(|f|/\lambda)d\mu \leqslant 1\right\}.$$

The space $(L^\Phi(X, \mathcal{M}, \mu), \|\cdot\|_{L^\phi(X,\mathcal{M},\mu)})$ is a Banach space.

Let Φ be a Young function such that $\lim_{t\to\infty} \Phi(t)/t = \infty$ and consider the function

$$\Phi^*(t) := \sup_{s\geqslant 0}\{st - \Phi(s)\}, \quad t \geqslant 0.$$

For example, if $\Phi(t) = t^p$ then $\Phi^*(t) = t^{p'}$, with $p' = p/(p-1)$. In particular, Φ^* is itself a Young function. The following holds

$$xy \leqslant \Phi(x) + \Phi^*(y).$$

This is instrumental to the proof of the following variant of Hölder's inequality for Orlicz spaces: If $f \in L^\Phi$ and $h \in L^{\Phi^*}$, then

$$\int |fh|d\mu \leqslant 2\|f\|_{L^\Phi}\|h\|_{L^{\Phi*}}.$$

We refer to [Wil08, Theorem 10.2] for a proof of this fact. Moreover, it holds that

$$\|f\|_{L^\Phi} \sim \sup_{h:\, \|h\|_{L^{\phi*}} \leqslant 1} \int |fh|d\mu.$$

1.3 $\Lambda(p)$-sets in Harmonic Analysis

A subset $S \subset \mathbb{Z}$ is said to be a $\Lambda(p)$-set for some $p > 2$ if there exists a constant $C_{S,p} > 0$ such that

$$\|f\|_{L^p(\mathbb{T})} \leqslant C_{S,p} \|f\|_{L^2(\mathbb{T})}$$

for all trigonometric polynomials with frequency support in S, see for example [Bou01]. Several instances of $\Lambda(p)$-sets will appear throughout the thesis.

Simple examples of $\Lambda(p)$-sets are lacunary sequences of positive integers. A sequence of positive integers $A \coloneqq \{a_k\}_k$ is said to be lacunary if it holds that $\inf_k a_{k+1}/a_k > 1$. For such a set A there exist a constant $0 < C_A < \infty$ such that the inequality

$$\|f\|_{L^2(\mathbb{T})} \leqslant C_A \|f\|_{L \log^{1/2} L(\mathbb{T})} \tag{1.11}$$

holds for all trigonometric polynomials with frequency support in A, [Zyg02, Chapter XII].

A further example of $\Lambda(p)$-sets is Sidon sets. We recall that a subset $S \subset \mathbb{Z}$ is said to be a Sidon set if any continuous function with frequency support in S has an absolutely convergent Fourier series, see for example [Rud60]. In particular, lacunary sequences are an example of Sidon sets. It has been observed by Rudin, see [Rud60, Theorem 3.1], that any trigonometric polynomials with frequency support on a Sidon set satisfy (1.11). Later, Pisier [Pis78] established that the inequality (1.11) completely characterizes Sidon sets. In other words, a trigonometric polynomial with frequency support on a set $S \subset \mathbb{Z}$ satisfies (1.11) if and only if S is a Sidon set.

On the other hand, examples of $\Lambda(p)$-sets that are not Sidon sets are higher-order lacunary sequences. Following [Bon70], let $\{a_k\}_k$ be a lacunary sequence of integers such that $\inf_k a_{k+1}/a_k \geqslant 2$. Define A_τ to be the set of integers that can be written as

$$\pm a_{k_1} \pm a_{k_2} \pm \dots \pm a_{k_\tau}, \qquad k_1 > k_2 > \dots > k_\tau.$$

Then there exists a constant $0 < C_{A_\tau} < \infty$ such that the inequality

$$\|f\|_{L^2(\mathbb{T})} \leqslant C_{A_\tau} \|f\|_{L \log^{\tau/2} L(\mathbb{T})} \tag{1.12}$$

holds for all trigonometric polynomials with frequency support in A_τ.

A further instance of $\Lambda(p)$-sets that will feature in this thesis is $\mathrm{B}(h)$-sets. A subset $S \subset \mathbb{Z}$ is said to be a $\mathrm{B}(h)$-set if for any two h-tuples $(a_1, \dots, a_h)$, $(b_1, \dots, b_h)$ of elements in S such that $a_1 + \dots + a_h = b_1 + \dots + b_h$ we have that one h-tuple is a permutation of the other. $\mathrm{B}(h)$-sets are $\Lambda(2h)$-sets, see for example [Bou01]. In fact for any square summable sequence $\{a_k\}_{k \in S}$ it holds that

$$\Big\| \sum_{k \in S} a_k e^{ik\cdot} \Big\|_{L^{2h}(\mathbb{T})}^h = \Big\| \sum_{k_1, \dots, k_h \in S} a_{k_1} \dots a_{k_h} e^{i(k_1 + \dots + k_h)\cdot} \Big\|_{L^2(\mathbb{T})} \leqslant (2\pi h!)^{1/2} \Big(\sum_{k \in S} |a_k|^2 \Big)^{h/2}.$$

Finally, a last example of $\Lambda(p)$-sets that is of interest to us is Littlewood–Paley sets that we have encountered in Subsection 1.2.2. Following [HK89], let E be a subset of $\mathbb{Z}$ and take

$\mathcal{I}_E \coloneqq \{I_j\}$ to be the partition of the integers into disjoint intervals that is induced by E. For a function f on $\mathbb{T}$ define $\widehat{f_j} := \widehat{f}\mathbf{1}_{I_j}$ and

$$S_{\mathcal{I}_E} f := \Big(\sum_j |f_j|^2\Big)^{1/2}.$$

If E is LP(p)-set then there exists $0 < c_p, C_p < \infty$ such that

$$c_p\|f\|_{L^p(\mathbb{T})} \leqslant \|S_{\mathcal{I}_E} f\|_{L^p(\mathbb{T})} \leqslant C_p\|f\|_{L^p(\mathbb{T})}$$

for all $f \in L^p(\mathbb{T})$. Hence, if f is a trigonometric polynomial with Fourier support in E it follows that there exists $0 < c_p' < \infty$ such that

$$\|f\|_{L^p(\mathbb{T})} \leqslant c_p' \Big\| \Big(\sum_j |\widehat{f}(n_j)|^2\Big)^{1/2} \Big\|_{L^p(\mathbb{T})}.$$

In particular, E is a $\Lambda(p)$-set.

Further connections between Littlewood–Paley sets and $\Lambda(p)$-sets have been recently studied in [BCDP$^+$24].

Chapter 2

Overview of the main results I

In this chapter we summarize the results in the theory of sharp Fourier extension inequalities on spheres obtained in the works [CG24] and [CS23].

2.1 A sharp Fourier extension estimates on the circle

The results presented in this section have been obtained in collaboration by the author of this thesis and F. Gonçalves. They are contained in the article [CG24] written jointly by the author of this thesis and F. Gonçalves. The article is reproduced in Appendix A and it has been published in:

> Valentina Ciccone and Felipe Gonçalves. Sharp Fourier extension on the circle under arithmetic constraints. J. Funct. Anal., 286(2): Paper No. 110219, 21, 2024. https://doi.org/10.1016/j.jfa.2023.110219

Motivation

A major open problem in the area of sharp Fourier extension estimates is the one of determining the sharp constant and maximizers for endpoint Stein–Tomas inequality on $\mathbb{S}^1$,

$$\|\widehat{f\sigma}\|_{L^6(\mathbb{R}^2)} \leqslant C_{\text{opt}}\|f\|_{L^2(\mathbb{S}^1)}, \qquad C_{\text{opt}} \coloneqq \sup_{\substack{f\in L^2(\mathbb{S}^1),\\ f\neq 0}} \frac{\|\widehat{f\sigma}\|_{L^6(\mathbb{R}^2)}}{\|f\|_{L^2(\mathbb{S}^1)}}. \tag{2.1}$$

As mentioned in Subsection 1.1.2, existence of maximizers for (2.1) has been established in [Sha16a]. Moreover, it has been shown in [OeSQ21b] that maximizers are smooth. The problem of characterizing maximizers for (2.1) has received a great deal of attention over the last years. In particular, it is conjectured that constant functions are maximizers for (2.1). A program, similar to the one proposed by Foschi in [Fos15] and outlined in Subsection 1.1.2, has been implemented in [CFOeST17] to study this problem. We briefly describe the main steps and challenges.

Using the evenness of the exponent and Plancherel's identity one can rewrite the left-hand-side of the inequality in (2.1) as

$$\|\widehat{f\sigma}\|^6_{L^6(\mathbb{R}^2)} = (2\pi)^2\|f\sigma * f\sigma * f\sigma\|^2_{L^2(\mathbb{R}^2)},$$

which can be further rewritten as

$$\|f\sigma * f\sigma * f\sigma\|_{L^2(\mathbb{R}^2)}^2 = \int_{(\mathbb{S}^1)^6} f(\omega_1)f(\omega_2)f(\omega_3)f_\flat(\omega_4)f_\flat(\omega_5)f_\flat(\omega_6)d\Psi,$$

where we use the notation $f_\flat(\omega) = \overline{f(-\omega)}$ and Ψ is the measure

$$d\Psi \coloneqq \delta\Big(\sum_{i=1}^{6}\omega_i\Big)d\sigma(\omega_1)d\sigma(\omega_2)d\sigma(\omega_3)d\sigma(\omega_4)d\sigma(\omega_5)d\sigma(\omega_6)$$

supported on the submanifold

$$\Omega \coloneqq \{(\omega_1,\omega_2,\omega_3,\omega_4,\omega_5,\omega_6) \in (\mathbb{S}^1)^6 : \omega_1+\omega_2+\omega_3+\omega_4+\omega_5+\omega_6 = 0\}.$$

Without loss of generality, arguing as in [Fos15] f can be assumed to be non-negative and antipodally symmetric. In fact, denoting by $f_\sharp$ the antipodally symmetric rearrangement of the function f,

$$f_\sharp \coloneqq \sqrt{\frac{|f|^2+|f_\flat|^2}{2}},$$

it holds that

$$\sup_{\substack{f\in L^2(\mathbb{S}^1),\\ f\neq 0}} \frac{\|\widehat{f\sigma}\|_{L^6(\mathbb{R}^2)}}{\|f\|_{L^2(\mathbb{S}^1)}} = \sup_{\substack{f\in L^2(\mathbb{S}^1),\\ f\neq 0,\, f=f_\sharp}} \frac{\|\widehat{f\sigma}\|_{L^6(\mathbb{R}^2)}}{\|f\|_{L^2(\mathbb{S}^1)}}$$

The three-fold convolution $\sigma*\sigma*\sigma$ has been computed explicitly in [CFOeST17, Lemma 2.2] showing that it has a logarithmic singularity at $|x| = 1$. Also in this case, it is possible to exploit some geometric properties of the circle to compute a weight that offsets the singularity of $\sigma*\sigma*\sigma$. In particular, the following identity can be found in [CFOeST17, Lemma 1.3]: If $(\omega_1,\omega_2,\omega_3,\omega_4,\omega_5,\omega_6) \in \Omega$ then

$$\sum_{\binom{6}{3}}(|\omega_j+\omega_k+\omega_i|^2-1) = 16$$

where the sum is over all the twenty different choices of the unordered, distinct indexes $j,k,i \in \{1,2,3,4,5,6\}$.

This allows to rewrite

$$(2\pi)^{-2}\|\widehat{f\sigma}\|_{L^6(\mathbb{R}^2)}^6 = \frac{5}{4}\int_{(\mathbb{S}^1)^6} f(\omega_1)f(\omega_2)f(\omega_3)f(\omega_4)f(\omega_5)f(\omega_6)(|\omega_1+\omega_2+\omega_3|^2-1)d\Psi.$$

However, in this case the weight is partially negative and applying Cauchy–Schwartz as in [Fos15] won't lead to the desired result. This step of the program is left as a conjecture in [CFOeST17].

Conjecture 2 (Conjecture 1.4 in [CFOeST17])**.** *Let $f \in L^2(\mathbb{S}^1)$ be non-negative and antipodally symmetric, then*

$$\int_{(\mathbb{S}^1)^6} f(\omega_1)f(\omega_2)f(\omega_3)f(\omega_4)f(\omega_5)f(\omega_6)(|\omega_1+\omega_2+\omega_3|^2-1)d\Psi$$
$$\leqslant \int_{(\mathbb{S}^1)^6} f(\omega_1)^2f(\omega_2)^2f(\omega_3)^2(|\omega_1+\omega_2+\omega_3|^2-1)d\Psi.$$

The trilinear form appearing in the right-hand side of the last display,

$$T(g_1, g_2, g_3) := \int_{(\mathbb{S}^1)^6} g_1(\omega_1)g_2(\omega_2)g_3(\omega_3)(|\omega_1 + \omega_2 + \omega_3|^2 - 1)d\Psi,$$

has been studied in [CFOeST17] obtaining the following result in the same spirit as the one in the last step of the proof in [Fos15].

Theorem 9 (Theorem 1.2 in [CFOeST17])**.** *Let $g \in L^1(\mathbb{S}^1)$ be non-negative and antipodally symmetric with mean value $c = \frac{1}{2\pi}\int_{\mathbb{S}^1} g(\omega)d\sigma$. Then*

$$T(g, g, g) \leqslant T(c, c, c),$$

with equality if and only if g is constant.

Later, further progress towards the characterization of maximizers for (2.1) has been achieved in [OeSTZK22, BTZK23]. In particular, in [OeSTZK22] (and then in [BTZK23]) the case of non-negative, antipodally symmetric, band-limited functions with Fourier modes up to degree 30 (respectively, degree 120) has been considered showing that, among this class of functions, constant functions are the unique maximizers. Note that, when restricting to the band-limited case the problem becomes finite-dimensional and it can be studied numerically as done in [OeSTZK22, BTZK23].

Main results in [CG24]

Inspired by these previous contributions, in the work [CG24] we have considered the case of functions whose Fourier support is possibly infinite – and therefore the problem is no longer finite-dimensional – but it satisfies certain arithmetic constraints. Such arithmetic constraints arise as a generalization of the notion of B(h)-sets – namely, sets of integers whose h-term sums uniquely express numbers up to permutations. B(h)-sets have been discussed in Section 1.3 and, as observed therein, they are $\Lambda(2h)$-sets. The generalization that we propose is motivated by the following observation: B(h)-sets cannot have a non-trivial symmetric subset, in other words, if A is a B(h)-set then $|A \cap (-A)| \leqslant 2$. Simple examples of this fact are the following:

- Let $h = 3$, $c \in A$. If there exist a, $b \in A \cap (-A)$ such that $|a| \neq |b|$, then $c + a + (-a) = c + b + (-b)$ contradicting the definition of B(3)-set.
- Let $h = 2$. If there exist a, $b \in A \cap (-A)$ such that $|a| \neq |b|$, then $a + (-a) = b + (-b)$ contradicting the definition of B(2)-set.

The generalization that we propose extends the definition of B(h)-set by allowing for the possibility of non-trivial symmetric subsets. We name such a generalization a P(h)-set. We refer to Section A.2 for a precise definition of P(h)-sets as well as for an overview of their properties and some examples. We mention in passing that P(h)-sets are also $\Lambda(2h)$-sets.

For the case of P(3)-sets, which are of more direct interest to us, the definition simplifies as follows (see also Definition 4 which corresponds to [CG24, Definition 1]).

Definition 3. *We say that a set $A \subset \mathbb{Z}$ is a* P(3)*-set if for every $D \in A + A + A$ one (and only one) of the following holds:*

- *either $D = a_1 + a_2 + a_3$ with the triple $(a_1, a_2, a_3) \in A \times A \times A$ unique up to permutations and such that $a_i \neq -a_j$ for $i \neq j$;*

- *or all the ways of representing D as sum of three terms in A are of the form $D = D + a - a$ for some $a \in A \cap (-A)$.*

Simple examples of P(h)-sets are B(h)-sets. A simple example of a $P(3)$-set with a non-trivial symmetric subset is $\{\pm(6^n) : n \in \mathbb{N}\} \cup \{0\}$. Further examples of P(3)-set are provided in Subsection A.2.1. Our main result in [CG24] is the following and it corresponds to Theorem 19 in Appendix A.

Theorem 10 (Theorem 1 in [CG24])**.** *Let $f \in L^2(\mathbb{S}^1)$ be such that its spectrum, spec(f) = $\{n \in \mathbb{Z} : \widehat{f}(n) \neq 0\}$, is a* P(3)*-set. Then*

$$\|\widehat{f\sigma}\|^6_{L^6(\mathbb{R}^2)} \leqslant (2\pi)^4 \left(\int_0^\infty J_0^6(r) r dr \right) \|f\|^6_{L^2(\mathbb{S}^1)} \ .$$

Equality is attained if and only if f is a constant.

By translation invariance, the same result holds for a function $\omega \mapsto e^{i\tau\cdot\omega} f(\omega)$ whose spectrum is a P(3)-set for some $\tau \in \mathbb{R}^2$. Our result provides a further evidence towards the conjecture that constant functions are maximizers for (2.1).

Two main tools are utilized in the proof. The first is counting arguments. As discussed above, results in sharp Fourier extension theory typically rely crucially on the evenness of the exponent as this allows for the rewriting of the left-hand-side of the Fourier extension inequality as a k-linear form which usually is more amenable to be dealt with. In [CG24] we depart from this approach and we take a different route. Our key, simple, observation is the following: in this setting the evenness of the exponent allows us to rely on counting arguments, very much typical of the literature on B(h)-sets and, more generally, on $\Lambda(2h)$-sets.

The second main tool is some novel refined estimates on integrals involving the product of six Bessel functions which are the content of Lemma 20 in Appendix A which corresponds to [CG24, Lemma 2].

2.2 Further results on global and local maximizers

The results presented in this section have been obtained in collaboration by the author of this thesis and M. Sousa. They are contained in the article [CS23] written jointly by the author of this thesis and M. Sousa. The article is reproduced in Appendix B and it appears online in the ArXiv at the link https://arxiv.org/abs/2312.07309. The article is currently submitted for publication.

Motivation

Recall the mixed-norm Fourier extension inequality (1.9) discussed in Subsection 1.1.3. The problem of computing the optimal constant for such inequality has been studied by E. Carneiro, D. Oliveira e Silva, and M. Sousa in [COeSS19], showing that constant functions are the maximizers for the sharp version of inequality (1.9) whenever the exponent p' is an even integer and that the set of exponents for which constant functions are extremizers is open and contains a neighborhood of infinity, $(p_0'(d), \infty]$, see [COeSS19, Theorem 1]. Moreover in [COeSS19, Theorem 3] some upper-bounds for $p_0'(d)$ are provided. In particular, they have shown that, in low dimension:

$$p_0'(2) \leqslant 6.76, \quad p_0'(3) \leqslant 5.45, \quad p_0'(4) \leqslant 5.53, \quad p_0'(5) \leqslant 6.07, \quad p_0'(6) \leqslant 6.82,$$

$$p_0'(7) \leqslant 7.70, \quad p_0'(8) \leqslant 8.69, \quad p_0'(9) \leqslant 9.78, \quad p_0'(10) \leqslant 10.95,$$

and, in general,

$$p_0'(d) \leqslant (\tfrac{1}{2} + o(1)) d \log d.$$

[COeSS19] is the first work in sharp Fourier extension theory on sphere addressing non-even exponents, although in a mixed-norm setting.

In [COeSS19], the problem of computing the sharp constant for (1.9) has been observed to be equivalent to the problem of computing

$$\sup_{k \geqslant 0} \left(\int_0^\infty |J_{\frac{d}{2}-1+k}(r)\, r^{-\frac{d}{2}+1}|^{p'} r^{d-1} dr \right)^{1/p'}, \tag{2.2}$$

which is in itself a problem of interest in the theory of special functions. Properties of Bessel functions guarantee that in the range $p' > 2d/(d-1)$ such a supremum is indeed a maximum. In particular, constant functions are maximizers for the sharp version of inequality (1.9) if and only if the maximum in (2.2) is attained at $k = 0$.

Main results in [CS23]

In [CS23] we have extended the aforementioned result from [COeSS19] to a wider range of exponents for the case of lower dimensions, in certain cases including the Stein–Tomas endpoint.

Our first result is the following and it corresponds to Theorem 23 in Appendix B.

Theorem 11 (Theorem 1 in [CS23]). *It holds that*

$$p_0'(2) < 6, \quad p_0'(3) < 4, \quad p_0'(4) < 3.48, \quad p_0'(5) < 3.50,$$

$$p_0'(6) < 3.58, \quad p_0'(7) < 3.7, \quad p_0'(8) < 3.86, \quad p_0'(9) < 4.06, \quad p_0'(10) < 4.46.$$

In particular, for $d = 2, 3$ it holds that $p_0'(d) < \frac{2d+2}{d-1}$, the Stein–Tomas endpoint.

The main tool that we utilze for the proof are some novel monotonicity results (with respect to the order k) for certain weighted L^p-norm of Bessel functions which are the content of Section B.2 (namely, [CS23, Section 2]) and which were inspired, for the case $d = 2$, by certain estimates for integrals involving Bessel functions obtained in [CG24, Lemma 2].

Note that the Stein–Tomas range is of relevance because it is conjectured that constant functions are maximizers for the sharp Stein–Tomas inequalities and this would imply, by Hölder inequality, that constant functions are also maximizers for the sharp version of the mixed-norm inequality (1.9) for p' in the same range. Our result gives further corroboration to this conjecture.

In the second part of the project, we study some connections between sharp mixed-norm Fourier extension inequalities and sharp $L^{q'}(\mathbb{S}^{d-1})$ to $L^{p'}(\mathbb{R}^d)$ Fourier extension estimates for the diagonal case $p' = q'$, namely

$$\|\widehat{f\sigma}\|_{L^{p'}(\mathbb{R}^d)} \leqslant C_{\mathrm{opt}}(d,p')\|f\|_{L^{p'}(\mathbb{S}^{d-1})}, \qquad C_{\mathrm{opt}}(d,p') \coloneqq \sup_{\substack{f\in L^{p'}(\mathbb{S}^{d-1})\\ f\neq 0}} \frac{\|\widehat{f\sigma}\|_{L^{p'}(\mathbb{R}^d)}}{\|f\|_{L^{p'}(\mathbb{S}^{d-1})}}. \tag{2.3}$$

Maximizers for such inequalities are known only when p' is an even admissible integer or $p' = \infty$, [COeS15, FS24]. In all these known cases constant functions are maximizers. For the remaining cases, as mentioned in Section 1.1.2, also the existence of global maximizers is still an open question.

Our second main result concerns local maximizers for these inequalities and it corresponds to Theorem 25 in Appendix B.

Theorem 12 (Theorem 2 in [CS23])**.** *Let $d \geqslant 2$ and $p' > \frac{2d}{d-1}$. Assume that the $L^{p'}(\mathbb{S}^{d-1})$ to $L^{p'}(\mathbb{R}^d)$ Fourier extension inequality holds and that $p' \in (p'_0(d), \infty]$. Then there exists $\delta > 0$ such that whenever $\|f - \mathbf{1}\|_{L^{p'}(\mathbb{S}^{d-1})} < \delta$,*

$$\frac{\|\widehat{f\sigma}\|_{L^{p'}(\mathbb{S}^{d-1})}}{\|f\|_{L^{p'}(\mathbb{S}^{d-1})}} \leqslant \frac{\|\widehat{\mathbf{1}\sigma}\|_{L^{p'}(\mathbb{R}^d)}}{\|\mathbf{1}\|_{L^{p'}(\mathbb{S}^{d-1})}}. \tag{2.4}$$

That is, constant functions are local maximizers for the $L^{p'}(\mathbb{S}^{d-1})$ to $L^{p'}(\mathbb{R}^d)$ Fourier extension estimates.

In particular, it follows from the above discussion that constant functions are local maximizers for (2.3) for all $p' = p'(d)$ such that

$$p'(2) \geqslant 6, \quad p'(3) \geqslant 4, \quad p'(4) \geqslant 3.48, \quad p'(5) \geqslant 3.50,$$

$$p'(6) \geqslant 3.58, \quad p'(7) \geqslant 3.7, \quad p'(8) \geqslant 3.86, \quad p'(9) \geqslant 4.06, \quad p'(10) \geqslant 4.46,$$

and, in general, for $p'(d) > (\frac{1}{2} + o(1))d\log d$.

As discussed in Section 1.1.2 previous results on local maximizers have been established only for inequalities in the Stein–Tomas range for the cases of $(d, p', q') = (d, \frac{2d+2}{d-1}, 2)$ with $2 \leqslant d \leqslant 60$ in [GN22], see also [CS12a] for the case of $(d, p', q') = (3, 4, 2)$ and [COeSS19] for the case of $(d, p', q') = (2, 6, 2)$.

Chapter 3

Overview of the main results II

The results presented in this chapter have been obtained in collaboration by O. Bakas, I. Parissi, M. Vitturi, and the author of this thesis. These results are contained in the article [BCPV24] written jointly by O. Bakas, I. Parissi, M. Vitturi, and the author of this thesis. The article is reproduced in Appendix C and it appears online in the ArXiv at the link https://arxiv.org/abs/2401.06083. The article is currently submitted for publication.

3.1 Sharp endpoint bounds for higher order Marcinkiewicz multipliers

Motivation

In their work [TW01], Tao and Wright have studied the weak-type endpoint mapping properties of classical Marcinkiewicz multipliers on $\mathbb{R}$. From the classical theory, briefly reviewed in Subsection 1.2.3, it is well-known that Marcinkiewicz multipliers are bounded on L^p for $1 < p < \infty$. While Hörmander multipliers are Calderón–Zygmund operators and therefore map L^1 to $L^{1,\infty}$ this may not be the case for Marcinkiewicz multipliers, meaning that there exist Marcinkiewicz multipliers that are not of weak-type $(1,1)$. The mapping properties near L^1 of such operators have been studied in [TW01]. In particular, one of the questions addressed in [TW01] can be stated as follows:

> *What is the smallest $r \geqslant 0$ such that a Marcinkiewicz multiplier T_m maps locally $L\log^r L$ to $L^{1,\infty}$?*

Here, we say that T_m maps locally $L\log^r L$ to $L^{1,\infty}$ if it maps $L\log^r L(K)$ to $L^{1,\infty}(K)$ for all compact sets $K \subset \mathbb{R}$. In [TW01] this question has been answered for a more general class of multipliers, the so-called R_2-multipliers. These include Marcinkiewicz multipliers with bounded q-variation for $1 \leqslant q < 2$ that we have already encountered in the multiplier theorem of Coifman, Rubio de Francia, and Semmes. We refer to Subsection C.1.2 for a precise definition of R_2-multipliers (which correspond to $R_{2,1}$-multipliers with the notation of Subsection C.1.2).

Theorem 13 ([TW01])**.** *R_2-multipliers locally map $L\log^{1/2} L$ to $L^{1,\infty}$ and such result is sharp, meaning that we cannot replace $1/2$ with a smaller exponent.*

It follows from this result that the very same is true for the classical Littelwood–Paley square function formed by rough frequency projections adapted to a lacunary partition of $\mathbb{R}$.

The main tools utilized in the proof are *a sort of* square function characterization for $L\log^{1/2} L$ and an ad-hoc vector-valued Calderón–Zygmund decomposition. We say a few words about the former.

While the Hardy space H^1 admits a square function characterization in the sense that $\|\widetilde{S}f\|_{L^1} \sim \|f\|_{H^1}$, this does not appear to be the case for the Orlicz space $L\log^{1/2} L$. However, in [TW01] the authors proved *a sort of* square function characterization for $L\log^{1/2} L$ which is very much inspired by the one for the Hardy space H^1. First, they have shown a discrete variant of it involving martingale difference. It reads as follows. Let $\mathcal{D}$ be the collection of dyadic intervals in $[0,1]$ and $\mathcal{D}_k \subset \mathcal{D}$, for k a non-negative integer, the sub-collection of intervals $I \in \mathcal{D}$ such that $|I| = 2^{-k}$. We define the martingale averages and differences as

$$\mathbb{E}_k f(x) := \sum_{I\in\mathcal{D}_k} \left(\frac{1}{|I|}\int_I f(y)dy\right)\mathbf{1}_I(x),$$

and

$$\mathbb{D}_k f := \mathbb{E}_{k+1} f - \mathbb{E}_k f.$$

Proposition 14 (Propo. 9.1 in [TW01]). *Let $f \in L\log^{1/2} L([0,1])$ with $\mathbb{E}_0 f = 0$. Then there exists a collection of non-negative functions f_k supported on $[0,1]$ such that for any $k \in \mathbb{N}$, $I \in \mathcal{D}_k$*

$$|\mathbb{D}_k f(x)| \lesssim \mathbb{E}_k f_k(x),$$

and

$$\left\|\left(\sum_{k\in\mathbb{N}} |f_k|^2\right)^{1/2}\right\|_{L^1} \lesssim \|f\|_{L\log^{1/2} L([0,1])}.$$

It is worth mentioning that the proof of this result is rather technical. From this discrete variant a smooth one can be derived using some averaging arguments. Such a smooth version reads as follows.

Proposition 15 (Propo. 4.1 in [TW01]). *Let ϕ_k be a smooth function of the form $\phi_k(x) = 2^k(1+2^k|x|)^{-\alpha}$ for some $\alpha > 1$. For any $f \in L\log^{1/2} L([-c,c])$ such that $\int f(x)dx = 0$ there exist a collection of non-negative functions f_k, $k \in \mathbb{Z}$, such that for any $k \in \mathbb{Z}$ the following pointwise estimate holds,*

$$|\widetilde{\Delta}_k f| \lesssim f_k * \phi_k,$$

and they satisfy

$$\left\|\left(\sum_{k\in\mathbb{Z}} |f_k|^2\right)^{1/2}\right\|_{L^1} \lesssim \|f\|_{L\log^{1/2} L([-c,c])}.$$

We stress that, as observed in [TW01], if we were to choose $f_k = |\widetilde{\Delta}_k f|$ – which would be the natural choice in the case of H^1, possibly up to a small enlargement of the frequency interval over which we are projecting – then the result in Proposition 15 won't hold. We refer to [TW01, Section 4] for a counterexample.

As a final remark, we mention that the above square function characterization, beside being of interest in itself and being instrumental for the proof of Theorem 13, has also some further interesting applications. One of them, as observed in [TW01], is the following: It implies the classical inequality of Zygmund (1.11) for functions with frequency support on $\{2^k : k \in \mathbb{N}\}$. In fact, let f be a function on $\mathbb{T}$ such that $\widehat{f}(n) = 0$ unless $n \in \{2^k : k \in \mathbb{N}\}$. Then by the definition of the frequency projection $\widetilde{\Delta}_k$ and the Hausdorff-Young inequality it follows that

$$|\widehat{f}(2^k)| = |(\widetilde{\Delta}_k f)^\wedge(2^k)| \leqslant \|\widetilde{\Delta}_k f\|_{L^1(\mathbb{T})}.$$

By Proposition 15 (adapted to the case of the torus $\mathbb{T}$) we get that $\|\widetilde{\Delta}_k f\|_{L^1(\mathbb{T})} \lesssim \|f_k * \phi_k\|_{L^1(\mathbb{T})}$. By combining this with Young's and Minkowski's inequalities and, again, with Proposition 15 we obtain that

$$\|f\|_{L^2(\mathbb{T})} \lesssim \Big(\sum_{k\in\mathbb{N}} \|f_k\|^2_{L^1(\mathbb{T})} \Big)^{1/2} \leqslant \Big\| \Big(\sum_{k\in\mathbb{N}} |f_k|^2 \Big)^{1/2} \Big\|_{L^1(\mathbb{T})} \lesssim \|f\|_{L\log^{1/2} L}(\mathbb{T}) \ ,$$

hence reproving the classical inequality of Zygmund.

Main results in [BCPV24]

Motivated by the aforementioned results in [TW01], we are interested in understanding weak-type endpoint bounds for higher order Marcinkiewicz multipliers.

We consider higher order lacunary partitions of the real line. These can be defined as follows. For a dyadic interval $I \subset \mathbb{R}$ of finite measure we define its Whitney decomposition as the collection of its maximal dyadic subintervals $J \subset I$ such that $\operatorname{dist}(J, \mathbb{R}\backslash I) = |J|$. Denoting by Λ_1 the classical Littlewood–Paley dyadic partition of $\mathbb{R}\backslash\{0\}$, we then define inductively the $\tau-$order (dyadic) lacunary partition Λ_τ as the collection of intervals obtained by taking the Whitney decomposition of each interval $I \in \Lambda_{\tau-1}$. Denoting by lac_τ the set of the endpoints of such intervals, we see that lac_τ is a successor of $\mathrm{lac}_{\tau-1}$ according to Definition 2.

We consider τ-order Marcinkiewicz multipliers, namely, multipliers of uniformly bounded variation over each interval in Λ_τ, and τ-order Hörmander–Mihlin multipliers, that is, multipliers that are singular on the set of endpoints lac_τ, we refer to Subsection C.1.2 for rigorous definitions. The rough and the smooth square functions formed by frequency projections adapted to Λ_τ can be seen as prototypical examples of the aforementioned multipliers. We denote such rough and smooth square functions as S_{Λ_τ} and $\widetilde{S}_{\Lambda_\tau}$, respectively. As reviewed in Section 1.2, all these operators are bounded on L^p for $1 < p < \infty$. We are interested in weak-type endpoint estimates for such operators.

Our first main result is the following and corresponds to and Theorem 30 in Appendix C.

Theorem 16 (Theorem A in [BCPV24])**.** *Let T_m a Marcinkiewicz multiplier operator of order $\tau \in \mathbb{N}$. Then*

$$|\{x \in \mathbb{R} : |\mathrm{T}_m f(x)| > \alpha\}| \lesssim \int_{\mathbb{R}} \frac{|f|}{\alpha} \Big(\log \Big(e + \frac{|f|}{\alpha} \Big) \Big)^{\tau/2}, \quad \alpha > 0.$$

The same holds for the rough Littlewood–Paley square function S_{Λ_τ}. Such endpoint estimates are sharp, meaning that the exponent $\tau/2$ cannot be replaced by any smaller one.

Theorem 16 implies at once the following local estimate: let T_m be a Marcinkiewicz multiplier operator of order τ, then for every interval $I \subset \mathbb{R}$ and function f with $\operatorname{supp} f \subset I$ it holds that

$$|\{x \in I : |T_m f(x)| > \alpha\}| \lesssim \frac{1}{\alpha}\int_I |f|\left(\log\left(e+\frac{|f|}{\langle |f|\rangle_I}\right)\right)^{\tau/2}, \qquad \langle |f|\rangle_I := |I|^{-1}\int_I |f|dx.$$

Theorem 16 is obtained as a consequence of a more general endpoint result for a τ-order generalization of the class of multipliers introduced by Coifman, Rubio de Francia, and Semmes. We name such generalization $R_{2,\tau}$-multipliers. They include τ-order Marcinkiewicz multiplier operators as well as τ-order Hörmander–Mihlin multiplier operators. We refer to Subsection C.1.2 for a precise definition of these multipliers and to Theorem 31 for the corresponding endpoint result. As a consequence, we obtain also the following endpoint result for higher order Hörmander–Mihlin multiplier operators, which is the content of Theorem 32 in Appendix C.

Theorem 17 (Theorem C in [BCPV24])**.** *Let T_m a Hörmander–Mihlin multiplier operator of order $\tau \in \mathbb{N}$. Then*

$$|\{x \in \mathbb{R} : |\mathrm{T}_m f(x)| > \alpha\}| \lesssim \int_{\mathbb{R}} \frac{|f|}{\alpha}\left(\log\left(e+\frac{|f|}{\alpha}\right)\right)^{(\tau-1)/2}, \quad \alpha > 0.$$

The same holds for the smooth Littlewood–Paley square function $\widetilde{S}_{\Lambda_\tau}$. Such endpoint estimates are sharp, meaning that the exponent $(\tau-1)/2$ cannot be replaced by any smaller one.

One of the main tools utilized in the proof of the aforementioned endpoint results is a *sort of* square function characterization of the Orlicz spaces $L\log^r L$, for $r \geqslant 1/2$, which in the case $r = 1/2$ recovers the analogous result in [TW01]. Recall the Chang–Wilson–Wolff inequality [CWW85] which can be stated as

$$\|f - \mathbb{E}_0 f\|_{\exp(L^2)([0,1])} \lesssim \|S_{\mathcal{M}} f\|_{L^\infty([0,1])}, \tag{3.1}$$

with $S_{\mathcal{M}} f := (\sum_{k\geqslant 1} |\mathbb{D}_k f|^2)^{1/2}$ the martingale square function. Our square function characterization for $L\log^r L$, $r \geqslant 1/2$, stems from the following observation which is of independent interest: the Chang–Wilson–Wolff inequality (3.1) implies the discrete square function characterization of $L\log^{1/2} L$ established in [TW01] and described in Proposition 14. This provides a different approach to prove Proposition 14 which can be generalized to the case of $L\log^r L$, $r \geqslant 1/2$, leading to our next main result which is the content of Theorem 33 in Appendix C.

Theorem 18 (Theorem D in [BCPV24])**.** *Let $f \in L\log^{(\sigma+1)/2} L$, with $\sigma \geqslant 0$. Then for each $k \in \mathbb{N}_0$ there exist functions f_k such that*

$$\mathbb{D}_k f = \mathbb{D}_k f_k \quad \forall k \in \mathbb{N}_0, \qquad \left\|\left(\sum_{k\geqslant 0} |f_k|^2\right)^{1/2}\right\|_{L\log^{\sigma/2} L} \lesssim \|f\|_{L\log^{(\sigma+1)/2} L}.$$

The implicit constant depends only on σ.

A smooth version of this square function characterization, generalizing the one in Proposition 15, can be derived using averaging arguments as in [TW01], see Corollary 34 in Appendix C.

As a final remark, we observe that such square function characterization implies the inequality of Bonami (1.12) for functions with frequency support in $\{2^{k_1} + ... + 2^{k_\tau} : k_1, ..., k_\tau \in \mathbb{N}_0,\ k_1 > ... > k_\tau\}$ by the very same arguments utilized at the end of the previous paragraph.

Appendix A

Sharp Fourier Extension on the Circle Under Arithmetic Constraints

This Appendix contains the article [CG24] written jointly by the author of this thesis and F. Gonçalves. The article has been published in

Valentina Ciccone and Felipe Gonçalves. Sharp Fourier extension on the circle under arithmetic constraints. J. Funct. Anal., 286(2): Paper No. 110219, 21, 2024. https://doi.org/10.1016/j.jfa.2023.110219

Abstract

We establish a sharp adjoint Fourier restriction inequality for the end-point Tomas–Stein restriction theorem on the circle under a certain arithmetic constraint on the support set of the Fourier coefficients of the given function. Such arithmetic constraint is a generalization of a B_3-set.

A.1 Introduction

In this paper we are interested in the optimal constant for the Fourier extension inequality

$$||\widehat{f\sigma}||^6_{L^6(\mathbb{R}^2)} \leqslant \mathrm{C_{opt}}||f||^6_{L^2(\mathbb{S}^1)} , \tag{A.1}$$

where σ is the arc length measure on $\mathbb{S}^1$, $\widehat{f\sigma}$ is the Fourier transform of the measure $f\sigma$,

$$\widehat{f\sigma}(x) = \int_{\mathbb{S}^1} f(\omega)e^{-ix\cdot\omega}d\sigma_\omega, \quad x \in \mathbb{R}^2,$$

and $\mathrm{C_{opt}}$ is the optimal constant

$$\mathrm{C_{opt}} := \sup_{f\in L^2(\mathbb{S}^1),\, f\neq 0} ||\widehat{f\sigma}||^6_{L^6(\mathbb{R}^2)}||f||^{-6}_{L^2(\mathbb{S}^1)} .$$

This problem has attracted a lot of attention in the last decade. Existence of maximizers have been established in [Sha16a] and it is known that maximizers are smooth [Sha16b, OeSQ21b], and that they can be chosen to be non-negative and antipodally symmetric, see [CFOeST17]. In [CFOeST17], and later in [GN22], it has been established that constant functions are local maximizers. In fact, it is conjectured that constant functions are indeed global maximizers in which case

$$\mathrm{C}_{\mathrm{opt}} = (2\pi)^4 \int_0^\infty J_0^6(r) r dr \ .$$

If this were true, a full characterization of the complex valued maximizers is provided in [CFOeST17]. Moreover, in [OeSQ21a] it is shown that if (A.1) is maximized by constants then the following inequality

$$||\widehat{f\sigma}||_{L^{2k}(\mathbb{R}^2)} \leqslant \mathrm{C}_{2k,\mathrm{opt}} ||f||_{L^2(\mathbb{S}^1)} \ ,$$

is also maximized by constants for every $k > 3$.

A major technical challenge in the study of extremizers for (A.1) lies in the fact that the threefold convolution $\sigma * \sigma * \sigma(x)$, which arises naturally when exploiting the evenness of the exponent in the right hand side of (A.1) and using Plancharel, blows up when $|x| = 1$, see [CFOeST17].

We recall that any complex-valued $f \in L^2(\mathbb{S}^1)$ can be expanded in Fourier series

$$f(\omega) = \sum_{n\in\mathbb{Z}} \widehat{f}(n) \underline{\omega}^n,$$

where we let $\underline{\omega} = x + iy$ if $\omega = (x, y) \in \mathbb{S}^1$. We also define the spectrum of f to be

$$\mathrm{spec}(f) = \{n \in \mathbb{Z} : \widehat{f}(n) \neq 0\}.$$

In [OeSTZK22, BTZK23] the case of band-limited functions was explored, that is, when $|\mathrm{spec}(f)| < \infty$. Specifically, it has been shown that constant functions are the unique maximizers among the class of real-valued, non-negative, antipodally symmetric functions $f \in L^2(\mathbb{S}^1)$ with $\mathrm{spec}(f) \subseteq [-30, 30]$ and $\mathrm{spec}(f) \subseteq [-120, 120]$, respectively. Note that when restricting to the band-limited case the problem becomes finite-dimensional (a matter of computing the eigenvalues of a quadratic form) and it can be addressed numerically as done in [OeSTZK22, BTZK23].

In this paper we consider functions in $L^2(\mathbb{S}^1)$ whose spectrum can be infinite, but it satisfies certain arithmetic constraints. More specifically, we establish the desired sharp inequality for functions in $L^2(\mathbb{S}^1)$ whose spectrum is sufficiently *sparse* in the following sense:

Definition 4. *A set $A \subset \mathbb{Z}$ is said to be a* P(3)*-set if for every $D \in A + A + A$ one (and only one) of the following holds:*

- *D is unique, that is, $D = a_1 + a_2 + a_3$, the triple $(a_1, a_2, a_3) \in A \times A \times A$ is unique modulo permutations and $a_i \neq -a_j$ for $i \neq j$;*
- *D is trivial, that is, D is not unique and the only way of representing D is $D = D + a - a$ for some $a \in A \cap (-A)$.*

We use the terms unique and trivial here merely for useful case distinction to be used later on in the proof of our main result. We extend such notion and give a more general definition in Section A.2 for arbitrary h-sums $A + A + ... + A$, what we call P(h)-sets. A complete list of all P(3)-sets $A \subset [-3, 3] \cap \mathbb{Z}$ is

$$\begin{aligned}
&\{0\},\ \pm\{1\},\ \pm\{2\},\ \pm\{3\},\ \{-3,3\},\ \{-2,2\},\ \pm\{-2,3\},\ \{-1,1\},\ \pm\{-1,2\},\\
&\pm\{-1,3\},\ \pm\{0,1\},\ \pm\{0,2\},\pm\{0,3\},\ \pm\{1,2\},\ \pm\{1,3\},\ \pm\{2,3\},\ \{-3,0,3\},\\
&\pm\{-3,2,3\},\ \pm\{-2,-1,3\},\ \pm\{-2,0,2\},\ \pm\{-2,0,3\},\ \{-2,1,2\},\ \pm\{-2,1,3\},\\
&\pm\{-2,2,3\},\ \{-1,0,1\},\ \pm\{-1,0,3\},\ \pm\{-1,2,3\},\ \{-3,-2,2,3\}.
\end{aligned}$$

A simple example of a symmetric infinite P(3)-set is $A = \{\pm 6^n : n \geqslant 0\} \cup \{0\}$ (see Example 2).

Beside providing explicit examples of constructions of P(3)-sets one may ask how fast can the counting function

$$x \mapsto |\mathrm{spec}(f) \cap [-x, x]|$$

grow. In Example 3 we construct an infinite symmetric P(3)-set A (via greedy choice) such that

$$|A \cap [-x, x]| \gtrsim x^{1/5},$$

On the other hand, if A is a P(3)-set then it is easy to see that $A \cap [1, \infty]$ and $(-A) \cap [1, \infty]$ are B_3-sets (see Section A.2) and thus if we consider the $\binom{|A_x|+2}{3}$ multi-sets of size 3 in $A_x = A \cap [1, x]$, the sums of the elements represent each number in $[1, 3x] \cap \mathbb{Z}_+$ at most once, hence

$$3x \geqslant \binom{|A_x| + 2}{3} = \tfrac{1}{6}(|A_x| + 2)(|A_x| + 1)|A_x|,$$

and so $|A \cap [-x, x]| \lesssim x^{1/3}$. Constructing B_h-sets with large density is a very hard task that have drawn a lot of attention in the literature, especially in the interplay of combinatorics, probability and number theory, and we refer to the introduction of [Cil14] and the references therein for further information. Since any B_3-set is a P(3)-set we can simply rely on Cilleruelo's result [Cil14], see also Cilleruelo and Tesoro [CT15], to obtain existence of a P(3)-set A with only positive integers and counting function satisfying

$$|A \cap [-x, x]| \gtrsim x^{\sqrt{5}-2},$$

which is the current best existence result in terms of the exponent $\sqrt{5} - 2 = 0.23...$.

We are now ready to state our main result.

Theorem 19. *Let $f \in L^2(\mathbb{S}^1)$ be such that its spectrum*

$$\mathrm{spec}(f) = \{n \in \mathbb{Z} : \widehat{f}(n) \neq 0\}$$

is a P(3)*-set. Then*

$$||\widehat{f\sigma}||^6_{L^6(\mathbb{R}^2)} \leqslant (2\pi)^4 \left(\int_0^\infty J_0^6(r) r dr \right) ||f||^6_{L^2(\mathbb{S}^1)}.$$

and equality is attained if and only if f is constant.

To the best of our knowledge it is the first time that this inequality is established for functions f with an infinite spectrum and that simultaneously do not need to be "close" to constant functions. A simple function which is not comtemplated by the previous results [CFOeST17, OeSTZK22, BTZK23] is $f(\underline{\omega}) = 1 + c\underline{\omega}^n$ for arbitrary n and large c. The function f is not real-valued nor non-negative antipodally symmetric, hence we cannot apply the results in [OeSTZK22, BTZK23]. Moreover, for large c we cannot apply the local result of [CFOeST17]. However, the set $A = \{0, n\}$ is a P(3)-set, and thus Theorem 19 applies.

Clearly, by translation invariance, one could instead ask that the spectrum of $\omega \mapsto e^{i\tau\cdot\omega} f(\omega)$ is a P(3)-set for some $\tau \in \mathbb{R}^2$ and obtain the same inequality.

The proof of Theorem 19 relies crucially on some refined estimates on integrals involving the product of six Bessel functions. Some of these integrals involve Bessel functions of lower order and need to be estimated numerically. In Lemma 20 we estimate such integrals by employing a new method (quite different from [OeST17, OeSTZK22, BTZK23]) that avoids doing any numerical integration, and makes use instead of a known quadrature formula for band-limited functions in $\mathbb{R}^2$.

A.1.1 Overview

This paper is organized as follows. In Section A.2 we give a precise definition of P(h)-set. Then we propose some examples of P(h)-set with non-trivial symmetric subsets. In Section A.3 we study some refined estimates on integral involving the product of six Bessel functions. In Section A.4 we prove our main result. Finally, in Section A.5 we propose a further example of application of the developed strategy to the study of sharp inequalities.

A.2 A Generalization of B_h-sets

A subset $S \subseteq \mathbb{Z}$ is said to be a B_h-set, with $h \geqslant 2$, if for any $a_1, ..., a_h, b_1, ..., b_h \in S$ such that $a_1 + ... + a_h = b_1 + ... + b_h$ we have that $(a_1, ..., a_h)$ is a permutation of $(b_1, ..., b_h)$. If $h = 2$ the set S is sometimes said to be a Sidon set [1]. We are interested in defining a suitable generalization of B_h-sets to account for the case of sets $A \subseteq \mathbb{Z}$ with non-trivial symmetric subsets, namely such that $|A \cap -A| \geqslant 3$. It is immediate to see that such symmetric sets cannot be B_h-sets: in fact, for example, when h is even there is always more than one way of representing zero as sum of h elements in A, whereas when h is odd there is always more than one way of representing any element in A as a sum of elements in A.

In what follows we let A^k denotes the iterated sum of k copies of A, e.g. $A^3 = A + A + A$.

Definition 5 (Property P(h)). *We say that the set A satisfies property* P(h) *(with $h \geqslant 2$), or that A is a* P(h)*-set, if for any $D \in A^h$ there exists $0 \leqslant \ell \leqslant h$ with the same parity of h and a unique set of ℓ elements $\{a_1, ..., a_\ell\}$, with $a_1, ..., a_\ell \in A$, $a_i \neq -a_j$ for all $i \neq j$, and such that any $h-$tuple $(b_1, ..., b_h)$, with $b_1, ..., b_h \in A$ and $b_1 + ... + b_h = D$ is a permutation of a $h-$tuple $(a_1, ..., a_\ell, u_1, -u_1, ..., u_{(h-l)/2}, -u_{(h-l)/2})$ for some $u_1, ..., u_{(h-l)/2} \in A \cap (-A)$.*

[1] This has not to be confused with the other definition of Sidon set according to which a set E is a Sidon set if every continuous function $f : \mathbb{S}^1 \to \mathbb{C}$ with $\mathrm{spec}(f) \subseteq E$ has absolutely convergent Fourier series. To avoid confusion we will always refer to B_h-sets with $h = 2$ as B_2-sets.

We recall that a set $E \subset \mathbb{Z}$ is said to be a Λ_p-set, for some $p > 2$, if there exists a constant C such that

$$||f||_{L^p(\mathbb{S}^1)} \leqslant C||f||_{L^2(\mathbb{S}^1)}$$

for all functions $f \in L^2(\mathbb{S}^1)$ whose spectrum is contained in E. It is well known that B_h-sets are Λ_{2h}-sets (see e.g. [Bou01]).

The following observations follow immediately from the definition of property $\mathrm{P}(h)$.

- If A is a $\mathrm{P}(h)$-set then $A \cap \mathbb{N}$ and $-A \cap \mathbb{N}$ are B_h-sets, and thus A is a Λ_{2h}-set.
- If A is a B_h-set then A is a $\mathrm{P}(h)$-set.
- If $|A \cap -A| \leqslant 2$ and A is a $\mathrm{P}(h)$-set then A is a B_h-set.
- If A is a $\mathrm{P}(h)$-set and $S \subseteq A$ then S is a $\mathrm{P}(h)$-set.
- If A is a $\mathrm{P}(h)$-set then $-A$ is a $\mathrm{P}(h)$-set.
- If A is a B_h-set, the set $A \cup -A$ does not necessarily satisfy property $\mathrm{P}(h)$: in fact, for example, the set of powers of two, $\{1, 2, 4, ...\}$, is a B_2-set, however the set $\{-1, -2, -4, ...\} \cup \{1, 2, 4, ...\}$ does not satisfy property $\mathrm{P}(2)$, since, for example, $1 + 1 = 4 - 2$.

A.2.1 Examples of $\mathrm{P}(h)$-sets

Since any B_h-set is a $\mathrm{P}(h)$-set, the more interesting task is to provide examples of sets A that satisfy property $\mathrm{P}(h)$ for some h and that are such that $|A \cap -A| \geqslant 3$.

Example 1. A sequence of positive integers $\{\lambda_n\}$ is said to be (Hadamard) lacunary if $\lambda_{n+1} \geqslant q\lambda_n$ for some $q > 1$. Let $A_{\lambda,q} := \{\lambda_n\} \cup (-\{\lambda_n\}) \cup \{0\}$. We claim that $A_{\lambda,q}$ satisfies property $\mathrm{P}(h)$ whenever $q > 2h - 1$. To see this assume $a_1, ..., a_h, b_1, ..., b_h \in A_{\lambda,q}$ are such that

$$a_1 + ... + a_h = b_1 + ... + b_h \ . \tag{A.2}$$

On both sides of (A.2) we omit the zero terms and simplify terms of the form $a_j + a_i$ with $a_i = -a_j$ and $b_m + b_n$ with $b_m = -b_n$. If no term is left on both sides of (A.2) then $a_1 + ... + a_h = b_1 + ... + b_h = 0$ and $(a_1, ..., a_h)$, $(b_1, ..., b_h)$ are consistent with property $\mathrm{P}(h)$. On the other hand, if terms are left on at least one side of (A.2) we further arrange them so that to have only positive terms on both sides obtaining

$$\alpha_1 + ... + \alpha_{h'} = \beta_1 + ... + \beta_{h''} \tag{A.3}$$

where $\{\alpha_1, ..., \alpha_{h'}, \beta_1, ..., \beta_{h''}\} \subseteq \{|a_1|, ..., |a_h|, |b_1|, ..., |b_h|\}$, $\alpha_1, ..., \alpha_{h'}, \beta_1, ..., \beta_{h''} > 0$ and $h' + h'' \leqslant 2h$, $h', h'' \geqslant 1$. We want to show that $\{\alpha_1, ..., \alpha_{h'}\} = \{\beta_1, ..., \beta_{h''}\}$. We proceed in a similar way as in [Gra14, Proof of Theorem 3.6.4.]. We start by showing that $\max\{\alpha_1, ..., \alpha_{h'}\} = \max\{\beta_1, ..., \beta_{h''}\}$. Assume by contradiction that $\max\{\alpha_1, ..., \alpha_{h'}\} > \max\{\beta_1, ..., \beta_{h''}\}$. Then $\max\{\alpha_1, ..., \alpha_{h'}\} \geqslant q \max\{\beta_1, ..., \beta_{h''}\}$. On the other hand we have

$$\max\{\alpha_1, ..., \alpha_{h'}\} \leqslant \beta_1 + ... + \beta_{h''} \leqslant h'' \max\{\beta_1, ..., \beta_{h''}\} < q \max\{\beta_1, ..., \beta_{h''}\}$$

where in the last inequality we have used the fact that $h'' \leqslant 2h-1 < q$. By assuming that $\max\{\beta_1, ..., \beta_{h''}\} > \max\{\alpha_1, ..., \alpha_{h'}\}$ we have a similar contradiction. Hence $\max\{\alpha_1, ..., \alpha_{h'}\} = \max\{\beta_1, ..., \beta_{h''}\}$. Proceeding by induction we see that $h' = h''$ and $\{\alpha_1, ..., \alpha_{h'}\} = \{\beta_1, ..., \beta_{h''}\}$ as claimed. Because we got rid of the cases $a_i = -a_j$, $b_m = -b_n$ in the very beginning this further implies that there exists a unique (up to permutation) h'-tuple of elements in $A_{\lambda,q}$ that sums up to $D = a_1 + ... + a_h = b_1 + ... + b_h$.

Example 2. As a particular case of the above result, the set $S_q := \{\pm q^n : n \geqslant 0\} \cup \{0\}$ is a $\mathrm{P}(h)$-set whenever $q \geqslant 2h$. Note that the set S_q for $q = 2h-1$ does not satisfy property $\mathrm{P}(h)$ since

$$hq = \underbrace{q + ... + q}_{h \text{ times}} = q^2 + (\underbrace{-q - ... - q}_{h-1 \text{ times}})$$

It begs the question whether we can still prove Theorem 19 with an adaptation of our method for the functions f with $\mathrm{spec}(f) \subset S_q$ for $q = 5, 4, 3, 2$. We leave this question for future work. One interesting and possibly useful feature is that for $q = 5, 4, 3$ we only have finitely many exceptions (modulo multiplication by q^n) breaking property $\mathrm{P}(3)$. For instance, for $q = 3, 4, 5$ the only exceptions are $1+1+1 = 9-3-3 = 3+0+0$, $1+1+1 = 4-1-0$, $1+1+1 = 5-1-1$. In generality, for $h \leqslant q < 2h$ the set S_q only has finitely many exceptions not satisfying the property $\mathrm{P}(2h)$. To see this, let $2 \leqslant h \leqslant q$, $b \in \mathbb{Z}$ and $m \geqslant 2$ with $|b| + m \leqslant 2h+1$. We claim there are only finitely many solutions to

$$b = \sum_{j=1}^{m-1} a_j q^{l_j} \tag{A.4}$$

with $a_j = \pm 1$, $l_i \geqslant l_{i+1} \geqslant 0$ and with the property that $l_i = l_j$ implies $a_i = a_j$. Such claim with $b = 0$ easily shows what we want. Note that if $l_{m-1} > 0$ then q divides b and so $b = q$, $l_{m-1} = 1$, and we obtain $1 \pm 1 = \sum_{j=1}^{m-2} a_j q^{l_j - 1}$. If $l_{m-1} = 0$ then $1 \pm 1 = \sum_{j=1}^{m-2} a_j q^{l_j}$. In any case, if we let $P_{m,b} = \{(a_j, l_j)_{j=1}^{m-1} : \text{solves (A.4)}\}$ we deduce that

$$|P_{m,b}| = |P_{m-1,b-1}| + |P_{m-1,b+1}|.$$

Now note that by unique expansion in base q the set $P_{h,b}$ is a singleton for $|b| \leqslant h$ except when $b = 0$, in which case $P_{h,b} = \varnothing$, or $|b| = q = h$, in which case $|P_{h,b}| = 2$. This shows that $|P_{m,b}| < \infty$. The case $q = h-1$ has infinitely many exceptional cases, such as: $1 + (h-1)^n + ... + (h-1)^n = (h-1)^{n+1} + 1 + 0$.

Example 3. For a given set E let $E(x)$ be the counting function $E(x) := |E \cap [-x, x]|$. It is easy to check that the above examples are such that $A_{\lambda,q}(x) \lesssim \log_{2h-1}(x)$ and $S_q(x) \sim \log_q(x)$. An example of a denser set that satisfies property $\mathrm{P}(h)$ can be straightforwardly constructed applying the following greedy algorithm that generalizes the one of Erdös for B_2-sets, see [Erd81, Cil14]. We start by setting $a_1 := 1$ and $a_{-1} := -a_1$. Then we define the element a_n to be the smallest integer greater than a_{n-1} and such that the set $\{-a_n, a_{-n+1}, ..., a_{-1}, a_1, ...a_{n-1}, a_n\}$ satisfies property $\mathrm{P}(h)$. Then we set $a_{-n} := -a_n$ and iterate the procedure. It is easy to check that the resulting sequence of integers A is such that $A(x) \gtrsim x^{1/(2h-1)}$. In fact at each step there cannot be more than $(2n-2)^{2h-1}$ distinct elements of the type $a_{i_1} + ... + a_{i_h} - a_{j_1} - ... - a_{j_{h-1}}$ with $-(n-1) \leqslant i_1, ..., i_h, j_1, ..., j_{h-1} \leqslant n-1$ and therefore $a_n \leqslant (n-1)^{2h-1} + 1$ and $A(x) \gtrsim x^{1/(2h-1)}$.

Example 4. The construction of the following example is adapted from [Rud60] (see also [HZ59]) where a similar strategy is used to construct Sidon sets that are not (Hadamard) lacunary. For $n = 0, 1, 2, ...$ we set $N := 2^n$. Then we define A_h to be the set of elements of the type

$$\pm((2h)^{4N} + (2h)^{N+j}) \qquad j = 0, ..., N-1,\ n = 0, 1, 2, ...$$

Such a set is not of the type of Example 1, in fact A_h contains N elements between $a_{n,0} := ((2h)^{4N} + (2h)^{N+0})$ and $a_{n,N-1} := ((2h)^{4N} + (2h)^{N+N-1})$, $a_{n,N-1} < 2a_{n,0}$, while sets of the type in Example 1 contain a bounded number of elements between x and $2x$ as x tends to infinity. We claim that the set A_h satisfies property P(h). To see that, let $a_{n_i,j_i}, b_{n_i,j_i} \in A_h,\ i = 1, ..., h$, be such that

$$a_{n_1,j_1} + ... + a_{n_h,j_h} = b_{n_1,j_1} + ... + b_{n_h,j_h}.$$

After simplifying on both sides the elements of the type $x + (-x)$ and rearranging we obtain something like

$$\alpha_1 c_{n_1,j_1} + ... + \alpha_s c_{n_s,j_s} = 0 \tag{A.5}$$

where $|\alpha_i| \leqslant 2h - 1,\ s \leqslant 2h,\ c_{n_i,j_i} \in A_h,\ i = 1, ..., s$, and we assume that $c_{n_1,j_1} < ... < c_{n_s,j_s}$. But then if $\alpha_1 \neq 0$ we would have that c_{n_1,j_1} is divisible by a lower power of $2h$ than $c_{n_2,j_2}, ..., c_{n_s,j_s}$ and therefore (A.5) is impossible.

A.3 Estimates for certain integrals of Bessel functions

A simple computation (using the integral representation for Bessel functions) shows that if we let $\underline{\omega} = x + iy$ for $\omega = (x, y) \in \mathbb{R}^2$ then

$$\widehat{\underline{\omega}^n \sigma}(x, y) = 2\pi(-i)^n \frac{J_n(\sqrt{x^2 + y^2})}{(x^2 + y^2)^{n/2}} \underline{\omega}^n,$$

where σ is the arc measure on $\mathbb{S}^1$ and J_n is the Bessel function of first kind. The following lemma is crucial for the proof of our main result. In what follows, all numerical computations were performed using the [BBB+98, version 2.15.3] computer algebra system.

Lemma 20. *We have the following inequalities:*

(i) For all integers $n > 1$ it holds that

$$\int_0^\infty J_0^4(r) J_n^2(r) r dr < \frac{1}{5} \int_0^\infty J_0^6(r) r dr$$

and

$$\int_0^\infty J_0^4(r) J_1^2(r) r dr = \frac{1}{5} \int_0^\infty J_0^6(r) r dr.$$

(ii) For all integers $n > 0$ it holds that

$$\int_0^\infty J_0^2(r) J_n^4(r) r dr < \frac{2}{15} \int_0^\infty J_0^6(r) r dr.$$

(iii) For all integers $n > 0$ it holds that

$$\int_0^\infty J_n^6(r)rdr < \frac{1}{3}\int_0^\infty J_0^6(r)rdr.$$

(iv) For all integers $n, m > 0$, $n \neq m$ and such that $(n, m) \neq (1, 2)$ it holds that

$$\int_0^\infty J_n^4(r)J_m^2(r)rdr < \frac{1}{9}\int_0^\infty J_0^6(r)rdr.$$

Moreover,

$$\int_0^\infty J_1^4(r)J_2^2(r)rdr < \frac{1}{6}\int_0^\infty J_0^6(r)rdr.$$

(v) For all integers $n > m > \ell \geqslant 0$ such that $(n, m, \ell) \neq (3, 2, 0)$ it holds that

$$\int_0^\infty J_\ell^2(r)J_m^2(r)J_n^2(r)rdr < \frac{1}{15}\int_0^\infty J_0^6(r)rdr.$$

Moreover,

$$\int_0^\infty J_3^2(r)J_2^2(r)J_0^2(r)rdr < \frac{1}{6}\int_0^\infty J_0^6(r)rdr.$$

We start by recalling some known bounds on Bessel functions and Bessel integrals.

- The following pointwise bound can be found in [OeST17, Corollary 9]

$$r^{1/2}|J_0(r)| < \gamma \tag{A.6}$$

  for $r > 0$, where $\gamma = 0.89763$ (which is a truncation of $\frac{9}{8}\sqrt{\frac{2}{\pi}}$).

- The following pointwise bound was proven in [Ole06]

$$r^{1/2}|J_n(r)| < \beta\sqrt{n^{1/3} + \frac{\alpha}{n^{1/3}} + \frac{3\alpha^2}{10n}} \tag{A.7}$$

  for $r > 0$ and $n > 0$ where $\beta = 0.674886$ and $\alpha = 1.855758$.

- The following identity can be found in [GR07, Equation 6.574-2]

$$\int_0^\infty J_n^2(r)r^{-\lambda}dr = \frac{\Gamma(\lambda)\Gamma(n + \frac{(1-\lambda)}{2})}{2^\lambda\Gamma\left(\frac{1+\lambda}{2}\right)^2\Gamma\left(n + \frac{1+\lambda}{2}\right)} \tag{A.8}$$

  for $0 < \lambda < 2n + 1$.

We are now ready to prove the lemma. For simplicity we define

$$I(n_1, ..., n_6) := \int_0^\infty J_{n_1}(r)...J_{n_6}(r)rdr \quad \text{and} \quad \mathcal{I}(n_1, n_2, n_3) := I(n_1, n_1, n_2, n_2, n_3, n_3). \tag{A.9}$$

Proof. First we require a good lower bound for $\mathcal{I}(0,0,0)$ which is

$$\mathcal{I}(0,0,0) > 0.33682.$$

Such numerical lower bounds (and the upper bounds below) were done using Lemma 21 and evaluating the sum $\widetilde{\mathcal{I}}(n,m,\ell)$ with high precision (nowadays most computer algebra systems can do it extremely fast).

We start by proving the estimate in (i). Using (A.6) and (A.8) we obtain

$$\mathcal{I}(0,0,n) < \gamma^4 \int_0^\infty J_n^2(r) r^{-1} dr = \gamma^4 \frac{1}{2n}. \tag{A.10}$$

One can easily check that $\gamma^4 \frac{1}{2n} \leqslant \frac{1}{5} 0.33682$ when $n \geqslant 5$. For $n = 2,3,4$ we have

$$\mathcal{I}(0,0,n) < \widetilde{\mathcal{I}}(0,0,n) + 10^{-4} \leqslant \widetilde{\mathcal{I}}(0,0,2) + 10^{-4} = 0.0370...,$$

which is visibly less than $\frac{1}{5}0.33682 = 0.067364$. Integration by parts in conjunction with the relation $J_0(r) = J_1'(r) + \frac{1}{r} J_1(r)$ shows the desired identity $\mathcal{I}(0,0,0) = 5\mathcal{I}(0,0,1)$.

To prove item (ii) we use (A.6), (A.7) and (A.8) to obtain

$$\mathcal{I}(0,n,n) < \gamma^2\beta^2 \left(n^{1/3} + \frac{\alpha}{n^{1/3}} + \frac{3\alpha^2}{10n} \right) \int_0^\infty J_n^2(r) r^{-1} dr = \gamma^2\beta^2 \left(n^{1/3} + \frac{\alpha}{n^{1/3}} + \frac{3\alpha^2}{10n} \right) \frac{1}{2n}.$$

The right hand side above is a decreasing function of n and one can check that it is less than $\frac{2}{15}0.33682$ for $n \geqslant 14$. For $1 \leqslant n \leqslant 13$ we have the numerical bounds

$$\mathcal{I}(0,n,n) < \widetilde{\mathcal{I}}(0,n,n) + 10^{-4} \leqslant \widetilde{\mathcal{I}}(0,1,1) + 10^{-4} = 0.0424...,$$

which is less than $\frac{2}{15}0.33682 = 0.0449093...$.

To prove item (iii) we use (A.7) and (A.8) to obtain

$$\mathcal{I}(n,n,n) < \left(\beta \sqrt{n^{1/3} + \frac{\alpha}{n^{1/3}} + \frac{3\alpha^2}{10n}} \right)^4 \frac{1}{2n}.$$

Also in this case the right hand side is a decreasing function of n. When $n \geqslant 9$ the right hand side is less than $\frac{1}{3}0.33682$, while for $1 \leqslant n \leqslant 8$ we have the numerical bounds

$$\mathcal{I}(n,n,n) < \widetilde{\mathcal{I}}(n,n,n) + 10^{-4} \leqslant \widetilde{\mathcal{I}}(1,1,1) + 10^{-4} = 0.1049...,$$

which is less than $\frac{1}{3}0.33682 = 0.112273...$.

To prove the estimate in (iv) first let for $n \geqslant 1$

$$B_n = \beta \sqrt{n^{1/3} + \frac{\alpha}{n^{1/3}} + \frac{3\alpha^2}{10n}}$$

and $B_0 = \gamma$. One can show that B_n is increasing for $n \geqslant 6$ and $\max_{0 \leqslant n \leqslant 5} B_n = B_1 < B_{36}$. Next we let $k = \max\{n,m\}$ and we use (A.7) and (A.8) to obtain

$$\mathcal{I}(n,n,m) < \max\{B_{36}^2, B_k^2\}^2 \int_0^\infty J_k^2(r) r^{-1} dr = \frac{\max\{B_{36}^2, B_k^2\}^2}{2k}.$$

The right hand side is a decreasing function of k and one can easily check that it is less than $\frac{1}{9}0.33682$ when $k \geqslant 49$. For $1 \leqslant k \leqslant 48$ with $(n, m) \neq (1, 2)$ we rely on the numerical bounds

$$\mathcal{I}(n,n,m) < \widetilde{\mathcal{I}}(n,n,m) + 10^{-4} \leqslant \widetilde{\mathcal{I}}(2,2,3) + 10^{-4} = 0.0335...,$$

which is visibly less than $\frac{1}{9}0.33682 = 0.037424....$ Moreover, $\mathcal{I}(1,1,2) < \widetilde{\mathcal{I}}(1,1,2) + 10^{-4} = 0.0424... < \frac{1}{6}0.33682 = 0.0561....$

To prove the estimate in (v) we can then use (A.7) and (A.8) to obtain

$$\begin{aligned}\mathcal{I}(n,m,\ell) < B_\ell^2 B_m^2 \int_0^\infty J_n^2(t) r^{-1} dr = \frac{B_\ell^2 B_m^2}{2n} \leqslant \frac{\max\{B_{36}^2, B_\ell^2\}\max\{B_{36}^2, B_m^2\}}{2n} \\ \leqslant \frac{\max\{B_{36}^2, B_{n-2}^2\}\max\{B_{36}^2, B_{n-1}^2\}}{2n}.\end{aligned}$$

A tedious computation shows again that the right hand side above is indeed a decreasing function of n. One can easily check that it is less than $\frac{1}{15}0.33682$ when $n \geqslant 145$, while for $1 \leqslant n \leqslant 144$ with $(n, m, \ell) \neq (3, 2, 0)$ we have the numerical bounds

$$\mathcal{I}(n,m,\ell) < \widetilde{\mathcal{I}}(n,m,\ell) + 10^{-4} \leqslant \widetilde{\mathcal{I}}(4,2,0) + 10^{-4} = 0.0185...,$$

which is less than $\frac{1}{15}0.33682 = 0.0224546....$ Moreover, $\mathcal{I}(3,2,0) < \widetilde{\mathcal{I}}(3,2,0) + 10^{-4} = 0.0243... < \frac{1}{6}0.33682 = 0.0561....$ ■

Lemma 21. *For all integers $k, m, \ell \geqslant 0$ with $\max\{k, m, \ell\} \leqslant 11519$ we have*

$$\widetilde{\mathcal{I}}(k,m,\ell) < \mathcal{I}(k,m,\ell) < \widetilde{\mathcal{I}}(k,m,\ell) + 10^{-4},$$

where

$$\widetilde{\mathcal{I}}(k,m,\ell) = \frac{2}{9}\sum_{n=0}^{22000} \frac{J_k(\lambda_n/3)^2 J_m(\lambda_n/3)^2 J_\ell(\lambda_n/3)^2}{J_0(\lambda_n)^2}$$

and $\{\lambda_n\}_{n\geqslant 0}$ are the nonnegative zeros of the Bessel function J_1 (with $\lambda_0 = 0$) .

Proof. First we use a particular case of a formula of Ben Ghanem and Frappier [BGF98] (although this identity can be found in disguise in much older papers) which says that if $f \in L^1(\mathbb{R}^2)$ is radial and $\mathrm{supp}(\widehat{f}) \subset B_{\frac{1}{\pi}}(0)$ (or equivalently, if f is analytic in $\mathbb{C}^2$ and has exponential type at most 2) then

$$\int_{\mathbb{R}^2} f(x)dx = 4\pi \sum_{n\geqslant 0} \frac{f(\lambda_n)}{J_0(\lambda_n)^2}.$$

We can apply this formula for $f(x) = J_k(\frac{1}{3}|x|)^2 J_m(\frac{1}{3}|x|)^2 J_\ell(\frac{1}{3}|x|)^2$ to deduce that

$$\mathcal{I}(k,m,\ell) = \frac{2}{9}\sum_{n\geqslant 0} \frac{J_k(\lambda_n/3)^2 J_m(\lambda_n/3)^2 J_\ell(\lambda_n/3)^2}{J_0(\lambda_n)^2}.$$

We then let $\widetilde{\mathcal{I}}(k,m,\ell)$ be the above sum truncated up to $n = 22000$. To bound the tail we first apply [Kra14, Theorem 3], from which one easily deduce that

$$r^{1/2}|J_k(r)| < 1 \tag{A.11}$$

for all $k \geqslant 1$ and $r > 2k$. Noticing that $\lambda_{22001}/3 = 23039.65... > 2 \times 11519$ we obtain

$$\frac{2}{9}\sum_{n>22000}\frac{J_k(\lambda_n/3)^2 J_m(\lambda_n/3)^2 J_\ell(\lambda_n/3)^2}{J_0(\lambda_n)^2} < 6\sum_{n>22000}\frac{1}{\lambda_n^3 J_0(\lambda_n)^2}.$$

Secondly, we apply Krasikov's effective envolope [Kra06, Lemma 1] for $\nu = 0$ (noting that $\mu = 3$ and $J_0'(x) = -J_1(x)$) to obtain that

$$J_0(\lambda_n)^2 > 0.99 \times \frac{2}{\pi\lambda_n}$$

for $n > 22000$ (indeed $J_0(\lambda_n)^2 \sim \frac{2}{\pi\lambda_n}$). Now we apply a result of Makai [Mak78] that shows that $\nu \mapsto \lambda_{\nu,n}/\nu$ is decreasing, where $\lambda_{\nu,n}$ is the n-th zero of J_ν. It is also well-known that $\lambda_{\nu+1,n} > \lambda_{\nu,n}$ for all $n \geqslant 1$ and $\nu > -1$. Hence

$$\lambda_n = \frac{\lambda_{1,n}}{1} > \frac{\lambda_{3/2,n}}{3/2} > \frac{\lambda_{1/2,n}}{3/2} = \tfrac{2}{3}\pi n,$$

because $J_{1/2}(x) = \sqrt{2\pi/x}\sin(x)$ (with a more careful search in the literature one could possibly derive $\lambda_n \geqslant .99\pi n$ for $n > 22000$, since $\lambda_n \sim \pi n$). We obtain that

$$6\sum_{n>22000}\frac{1}{\lambda_n^3 J_0(\lambda_n)^2} < 2.18\sum_{n>22000}\frac{1}{n^2} < 2.18\int_{22000}^{\infty} x^{-2}dx < 10^{-4}.$$

■

A.4 Proof of the main result

Let $f \in L^2(\mathbb{S}^1)$ be a complex valued function and let $A = \mathrm{spec}(f)$. Then by Hecke-Bochner formula we can write

$$\begin{aligned}
(2\pi)^{-7}||\widehat{f\sigma}||^6_{L^6(\mathbb{R}^2)} &= (2\pi)^{-7}\int_{\mathbb{R}^2}\widehat{f\sigma}(x)\widehat{f\sigma}(x)\widehat{f\sigma}(x)\overline{\widehat{f\sigma}(x)\widehat{f\sigma}(x)\widehat{f\sigma}(x)}dx \\
&= \sum_{\substack{n_1,\dots,n_6\in A\\ n_1+n_2+n_3=n_4+n_5+n_6}} \widehat{f}(n_1)\widehat{f}(n_2)\widehat{f}(n_3)\overline{\widehat{f}(n_4)\widehat{f}(n_5)\widehat{f}(n_6)}I(n_1,\dots,n_6) \\
&= \sum_{D\in A^3}\sum_{\substack{n_1,\dots,n_6\in A\\ n_1+n_2+n_3=D\\ n_4+n_5+n_6=D}} \widehat{f}(n_1)\widehat{f}(n_2)\widehat{f}(n_3)\overline{\widehat{f}(n_4)\widehat{f}(n_5)\widehat{f}(n_6)}I(n_1,\dots,n_6)
\end{aligned}$$

where we are using the notation introduced in (A.9). Now if A satisfies P(3) we can split the last summation over the $D \in A^3$ that are unique and over those that are trivial. Note that by Definition 4 if $0 \in A^3$ then 0 is trivial. We focus first on the case $D \in A^3$ that are unique for which we obtain the following

$$
\begin{aligned}
(I) :=& \sum_{\substack{D\in A^3\\ D\,unique}} \sum_{\substack{n_1,n_2,n_3,n_4,n_5,n_6\in A\\ n_1+n_2+n_3=D\\ n_4+n_5+n_6=D}} \widehat{f}(n_1)\widehat{f}(n_2)\widehat{f}(n_3)\overline{\widehat{f}(n_4)\widehat{f}(n_5)\widehat{f}(n_6)}I(n_1,n_2,n_3,n_4,n_5,n_6)\\
=&6 \sum_{\substack{n_1,n_2,n_3\in A\\ |n_i|\neq|n_j| \text{ for } i\neq j}} |\widehat{f}(n_1)|^2|\widehat{f}(n_2)|^2|\widehat{f}(n_3)|^2\mathcal{I}(n_1,n_2,n_3)\\
&+9 \sum_{\substack{n_1\in A\setminus\{0\},\, n_3\in A\\ |n_1|\neq|n_3|}} |\widehat{f}(n_1)|^4|\widehat{f}(n_3)|^2\mathcal{I}(n_1,n_1,n_3)\\
&+ \sum_{n_1\in A\setminus\{0\}} |\widehat{f}(n_1)|^6\mathcal{I}(n_1,n_1,n_1),
\end{aligned}
$$

Now we focus on the sum over the set $\{D \in A^3 : D \text{ trivial}\}$. We use the short hand notation $A_s = A \cap (-A)$, $A_t = A \cap \text{trivial}$ and $A_{s,t} = A_s \cap A_t$. In this case the set $\{(n_1,n_2,n_3) \in A \times A \times A : n_1+n_2+n_3 = D\}$ is the disjoint union of the following sets

$$
\begin{aligned}
S_1(D) =&\{(D,a,-a) : a \in A_s\setminus\{\pm D\}\}\\
S_2(D) =&\{(-a,D,a) : a \in A_s\setminus\{\pm D\}\}\\
S_3(D) =&\{(-a,a,D) : a \in A_s\setminus\{\pm D\}\}\\
S_4(D) =&\{(D,D,-D),(-D,D,D),(D,-D,D)\}.
\end{aligned}
$$

Letting $\varepsilon_D = |S_4(D)|$ we obtain

$$
\begin{aligned}
(II) :=& \sum_{\substack{D\in A^3\\ D\ trivial}} \sum_{i,j=1}^{4} \sum_{\substack{(n_1,n_2.n_3)\in S_i(D)\\ (n_4,n_5.n_6)\in S_j(D)}} \widehat{f}(n_1)\widehat{f}(n_2)\widehat{f}(n_3)\overline{\widehat{f}(n_4)\widehat{f}(n_5)\widehat{f}(n_6)}I(n_1,...,n_6)\\
=&9 \sum_{\substack{D\in A_t\\ n_1,n_2\in A_s\setminus\{\pm D\}}} \widehat{f}(D)\widehat{f}(n_1)\widehat{f}(-n_1)\overline{\widehat{f}(D)\widehat{f}(n_2)\widehat{f}(-n_2)}I(D,n_1,-n_1,D,n_2,-n_2)\\
&+18\,\mathfrak{Re}\Bigg(\sum_{\substack{D\in A_{s,t}\setminus\{0\}\\ n_1\in A_s\setminus\{\pm D\}}} \widehat{f}(D)\widehat{f}(D)\widehat{f}(-D)\overline{\widehat{f}(n_1)\widehat{f}(-n_1)\widehat{f}(D)}I(D,D,-D,D,n_1,-n_1)\Bigg)\\
&+6\,\mathfrak{Re}\Bigg(\widehat{f}(0)\widehat{f}(0)\widehat{f}(0)\sum_{n_1\in A_s\setminus\{0\}} \overline{\widehat{f}(n_1)\widehat{f}(-n_1)\widehat{f}(0)}I(0,0,0,0,n_1,-n_1)\Bigg)\\
&+9\sum_{D\in A_{s,t}\setminus\{0\}} |\widehat{f}(D)|^4|\widehat{f}(-D)|^2I(D,D,-D,D,D,-D)\\
&+|\widehat{f}(0)|^6I(0,0,0,0,0,0)\\
=&9 \sum_{\substack{D\in A_t\\ n_1,n_2\in A_s\setminus\{\pm D\}\\ |n_1|\neq|n_2|}} \widehat{f}(D)\widehat{f}(n_1)\widehat{f}(-n_1)\overline{\widehat{f}(D)\widehat{f}(n_2)\widehat{f}(-n_2)}I(D,n_1,-n_1,D,n_2,-n_2)
\end{aligned}
$$

$$+ 9 \sum_{\substack{D \in A_t \\ n_1 \in A_s \setminus \{\pm D\}}} (2 - \delta_{n_1=0}) |\widehat{f}(D)|^2 |\widehat{f}(n_1)|^2 |\widehat{f}(-n_1)|^2 I(D, D, n_1, -n_1, n_1, -n_1)$$

$$+ 18 \,\mathfrak{Re}\Bigg(\sum_{\substack{D \in A_{s,t} \setminus \{0\} \\ n_1 \in A_s \setminus \{\pm D\}}} \widehat{f}(D)\widehat{f}(D)\widehat{f}(-D)\overline{\widehat{f}(n_1)\widehat{f}(-n_1)\widehat{f}(D)} I(D, D, -D, D, n_1, -n_1) \Bigg)$$

$$+ 6 \,\mathfrak{Re}\Bigg(\widehat{f}(0)\widehat{f}(0)\widehat{f}(0) \sum_{n_1 \in A_s \setminus \{0\}} \overline{\widehat{f}(n_1)\widehat{f}(-n_1)\widehat{f}(0)} I(0, 0, 0, 0, n_1, -n_1) \Bigg)$$

$$+ 9 \sum_{D \in A_{s,t} \setminus \{0\}} |\widehat{f}(D)|^4 |\widehat{f}(-D)|^2 I(D, D, -D, D, D, -D)$$

$$+ |\widehat{f}(0)|^6 I(0, 0, 0, 0, 0, 0)$$

Then using the identity $J_{-n} = (-1)^n J_n$ and by the triangle inequality we obtain

$$(II) \leqslant 9 \sum_{\substack{D \in A_t \\ n_1, n_2 \in A_s \setminus \{\pm D\} \\ |n_1| \neq |n_2|}} |\widehat{f}(D)|^2 |\widehat{f}(n_1)\widehat{f}(-n_1)| |\widehat{f}(n_2)\widehat{f}(-n_2)| \mathcal{I}(D, n_1, n_2)$$

$$+ 9 \sum_{\substack{D \in A_t \\ n_1 \in A_s \setminus \{\pm D\}}} (2 - \delta_{n_1=0}) |\widehat{f}(D)|^2 |\widehat{f}(n_1)|^2 |\widehat{f}(-n_1)|^2 \mathcal{I}(D, n_1, n_1)$$

$$+ 18 \sum_{\substack{D \in A_{s,t} \setminus \{0\} \\ n_1 \in A_s \setminus \{\pm D\}}} |\widehat{f}(D)|^2 |\widehat{f}(D)\widehat{f}(-D)| |\overline{\widehat{f}(n_1)\widehat{f}(-n_1)}| \mathcal{I}(D, D, n_1)$$

$$+ 6 \sum_{n_1 \in A_s \setminus \{0\}} |\widehat{f}(0)|^2 |\widehat{f}(0)\widehat{f}(0)| |\overline{\widehat{f}(n_1)\widehat{f}(-n_1)}| \mathcal{I}(0, 0, n_1)$$

$$+ 9 \sum_{D \in A_{s,t} \setminus \{0\}} |\widehat{f}(D)|^4 |\widehat{f}(-D)|^2 \mathcal{I}(D, D, D)$$

$$+ |\widehat{f}(0)|^6 \mathcal{I}(0, 0, 0) \ .$$

Next, by using the known inequalities

$$rs \leqslant \tfrac{1}{2} r^2 + \tfrac{1}{2} s^2 \text{ and } r^3 s \leqslant \tfrac{5}{8} r^4 + \tfrac{1}{8} s^4 + \tfrac{1}{4} r^2 s^2$$

for $r, s > 0$, we further get the following inequality

$$(II) \leqslant 9 \sum_{\substack{D \in A_t \\ n_1, n_2 \in A_s \setminus \{\pm D\} \\ |n_1| \neq |n_2|}} |\widehat{f}(D)|^2 \left(\tfrac{|\widehat{f}(n_1)|^2 + |\widehat{f}(-n_1)|^2}{2} \right) \left(\tfrac{|\widehat{f}(n_2)|^2 + |\widehat{f}(-n_2)|^2}{2} \right) \mathcal{I}(D, n_1, n_2)$$

$$+ 9 \sum_{\substack{D \in A_t \\ n_1 \in A_s \setminus \{\pm D\}}} (2 - \delta_{n_1=0}) |\widehat{f}(D)|^2 |\widehat{f}(n_1)|^2 |\widehat{f}(-n_1)|^2 \mathcal{I}(D, n_1, n_1)$$

$$+ 18 \sum_{\substack{D \in A_{s,t} \setminus \{0\} \\ n_1 \in A_s \setminus \{\pm D, 0\}}} |\widehat{f}(D)|^2 \left(\tfrac{|\widehat{f}(D)|^2 + |\widehat{f}(-D)|^2}{2} \right) \left(\tfrac{|\widehat{f}(n_1)|^2 + |\widehat{f}(-n_1)|^2}{2} \right) \mathcal{I}(D, D, n_1)$$

$$
\begin{aligned}
&+ 18 \sum_{D\in A_{s,t}\setminus\{0\}} \left(\tfrac{5}{8}|\widehat{f}(D)|^4 + \tfrac{1}{8}|\widehat{f}(-D)|^4 + \tfrac{1}{4}|\widehat{f}(D)\widehat{f}(-D)|^2\right)|\widehat{f}(0)|^2\mathcal{I}(D,D,0)\\
&+ 6 \sum_{n_1\in A_s\setminus\{0\}} |\widehat{f}(0)|^4\left(\tfrac{|\widehat{f}(n_1)|^2+|\widehat{f}(-n_1)|^2}{2}\right)\mathcal{I}(0,0,n_1)\\
&+ 9 \sum_{D\in A_{s,t}\setminus\{0\}} |\widehat{f}(D)|^4|\widehat{f}(-D)|^2\mathcal{I}(D,D,D)\\
&+ |\widehat{f}(0)|^6\mathcal{I}(0,0,0)\\
= 9 &\sum_{\substack{D\in A_t\\ n_1,n_2\in A_s\setminus\{\pm D\}\\ |n_1|\neq|n_2|}} |\widehat{f}(D)|^2|\widehat{f}(n_1)|^2|\widehat{f}(n_2)|^2\mathcal{I}(D,n_1,n_2)\\
&+ 9 \sum_{\substack{D\in A_t\\ n_1\in A_s\setminus\{\pm D\}}} (2-\delta_{n_1=0})|\widehat{f}(D)|^2|\widehat{f}(n_1)|^2|\widehat{f}(-n_1)|^2\mathcal{I}(D,n_1,n_1)\\
&+ 18 \sum_{\substack{D\in A_{s,t}\setminus\{0\}\\ n_1\in A_s\setminus\{\pm D,0\}}} |\widehat{f}(D)|^2\left(\tfrac{|\widehat{f}(D)|^2+|\widehat{f}(-D)|^2}{2}\right)|\widehat{f}(n_1)|^2\mathcal{I}(D,D,n_1)\\
&+ \frac{27}{2} \sum_{D\in A_{s,t}\setminus\{0\}} |\widehat{f}(D)|^4|\widehat{f}(0)|^2\mathcal{I}(D,D,0)\\
&+ \frac{9}{2} \sum_{D\in A_{s,t}\setminus\{0\}} |\widehat{f}(D)|^2|\widehat{f}(-D)|^2|\widehat{f}(0)|^2\mathcal{I}(D,D,0)\\
&+ 6 \sum_{n_1\in A_{s,t}\setminus\{0\}} |\widehat{f}(0)|^4|\widehat{f}(n_1)|^2\mathcal{I}(0,0,n_1)\\
&+ 9 \sum_{D\in A_{s,t}\setminus\{0\}} |\widehat{f}(D)|^4|\widehat{f}(-D)|^2\mathcal{I}(D,D,D)\\
&+ |\widehat{f}(0)|^6\mathcal{I}(0,0,0).
\end{aligned}
$$

We note that such inequalities hold with equality whenever $-A = A$ and $\overline{\widehat{f}(-n)} = (-1)^n\widehat{f}(n)$ which is the case, for instance, if f is real-valued and antipodally symmetric. Now we sum everything together and replace A_t by A (observing that $A_t = A$ if $A_s \neq \varnothing$ and $A_t = \varnothing$ if $A_s = \varnothing$) to obtain the following upper bound

$$
\begin{aligned}
(2\pi)^{-7}\|\widehat{f\sigma}\|^6_{L^6(\mathbb{R}^2)} &= (I) + (II)\\
&\leqslant \sum_{\substack{n_1,n_2,n_3\in A\\ |n_i|\neq|n_j|\,\forall i,j\in\{1,2,3\},\, i\neq j}} (6+9\delta_{n_1,n_2\in A_s})\,|\widehat{f}(n_1)|^2|\widehat{f}(n_2)|^2|\widehat{f}(n_3)|^2\mathcal{I}(n_1,n_2,n_3)\\
&\quad + 9 \sum_{\substack{n_1\in A_s,\, n_3\in A\\ |n_1|\neq|n_3|}} (2-\delta_{n_1=0})|\widehat{f}(n_3)|^2|\widehat{f}(n_1)|^2|\widehat{f}(-n_1)|^2\mathcal{I}(n_3,n_1,n_1)
\end{aligned}
$$

$$+\,9\sum_{\substack{n_1,n_3\in A_s\setminus\{0\}\\|n_1|\neq|n_3|}}|\widehat{f}(n_3)|^2|\widehat{f}(-n_3)|^2|\widehat{f}(n_1)|^2\mathcal{I}(n_3,n_3,n_1)$$
$$+\,9\sum_{\substack{n_1\in A\setminus\{0\},\,n_3\in A\\|n_1|\neq|n_3|}}(1+\delta_{n_1,n_3\in A_s\setminus\{0\}})|\widehat{f}(n_1)|^4|\widehat{f}(n_3)|^2\mathcal{I}(n_1,n_1,n_3)$$
$$+\,\frac{27}{2}\sum_{n_3\in A_s\setminus\{0\}}|\widehat{f}(n_3)|^4|\widehat{f}(0)|^2\mathcal{I}(n_3,n_3,0)$$
$$+\,\frac{9}{2}\sum_{n_3\in A_s\setminus\{0\}}|\widehat{f}(n_3)|^2|\widehat{f}(-n_3)|^2|\widehat{f}(0)|^2\mathcal{I}(n_3,n_3,0)$$
$$+\,6\sum_{n_1\in A_s\setminus\{0\}}|\widehat{f}(0)|^4|\widehat{f}(n_1)|^2\mathcal{I}(0,0,n_1)$$
$$+\,9\sum_{n_3\in A_s\setminus\{0\}}|\widehat{f}(n_3)|^4|\widehat{f}(-n_3)|^2\mathcal{I}(n_3,n_3,n_3)$$
$$+\sum_{n_1\in A\setminus\{0\}}|\widehat{f}(n_1)|^6\mathcal{I}(n_1,n_1,n_1)$$
$$+\,|\widehat{f}(0)|^6\mathcal{I}(0,0,0).$$

Next we observe that we may write $(2\pi)^{-3}||f||^6_{L^2(\mathbb{S}^1)}$ as

$$(2\pi)^{-3}||f||^6_{L^2(\mathbb{S}^1)}=\sum_{\substack{n_1,n_2,n_3\in A\\|n_i|\neq|n_j|\,\forall i,j\in\{1,2,3\},\,i\neq j}}|\widehat{f}(n_1)|^2|\widehat{f}(n_2)|^2|\widehat{f}(n_3)|^2$$
$$+\,3\sum_{\substack{n_1\in A_s\setminus\{0\},\,n_3\in A\setminus\{0\}\\|n_1|\neq|n_3|}}|\widehat{f}(n_1)|^2|\widehat{f}(-n_1)|^2|\widehat{f}(n_3)|^2$$
$$+\,3\sum_{\substack{n_1\in A\setminus\{0\},\,n_3\in A\setminus\{0\}\\|n_1|\neq|n_3|}}|\widehat{f}(n_1)|^4|\widehat{f}(n_3)|^2$$
$$+\,3\sum_{n_1\in A\setminus\{0\}}|\widehat{f}(n_1)|^4|\widehat{f}(0)|^2$$
$$+\,3\sum_{n_1\in A_s\setminus\{0\}}|\widehat{f}(n_1)|^2|\widehat{f}(-n_1)|^2|\widehat{f}(0)|^2$$
$$+\,3\sum_{n_1\in A\setminus\{0\}}|\widehat{f}(0)|^4|\widehat{f}(n_1)|^2$$
$$+\,3\sum_{n_1\in A_s\setminus\{0\}}|\widehat{f}(-n_1)|^2|\widehat{f}(n_1)|^4$$
$$+\sum_{n_1\in A\setminus\{0\}}|\widehat{f}(n_1)|^6$$
$$+\,|\widehat{f}(0)|^6.$$

Then by comparison of coefficients (going from bottom to top), the inequality $||\widehat{f\sigma}||^6_{L^6(\mathbb{R}^2)} \leqslant (2\pi)^4\mathcal{I}(0,0,0)||f||^6_{L^2(\mathbb{S}^1)}$ would follow if

- $3\mathcal{I}(n,n,n) \leqslant \mathcal{I}(0,0,0)$, $\quad 5\mathcal{I}(0,0,n) \leqslant \mathcal{I}(0,0,0)$, $\quad \frac{15}{2}\mathcal{I}(n,n,0) \leqslant \mathcal{I}(0,0,0)$ $\;(n>0)$
- $9\mathcal{I}(n,n,m) \leqslant \mathcal{I}(0,0,0)$, $\quad 15\mathcal{I}(n,m,\ell) \leqslant \mathcal{I}(0,0,0)$ $\;(n,m,\ell \geqslant 0$ distinct).

All of them follow easily by Lemma 20 except that the last two above are actually false, the exceptions being $(n,m) = (1,2)$ and $(n,m,\ell) = (3,2,0)$ respectively. However, note that the inequality $9\mathcal{I}(1,1,2) \leqslant \mathcal{I}(0,0,0)$ is only needed if $\{1,2\} \subset A_s\backslash\{0\}$ which is impossible since $1+1+1 = 2+2-1$ and so A would not be a P(3)-set. We conclude that in fact we only need $6\mathcal{I}(1,1,2) \leqslant \mathcal{I}(0,0,0)$, which is true by Lemma 20. Similarly, the inequality $15\mathcal{I}(3,2,0) < \mathcal{I}(0,0,0)$ is only required if $\{3a, 2b, 0\} \subset A$ for some $a,b \in \{\pm 1\}$ and either 2 or 3 also belong to $-A$. This cannot be true since $3+3+0 = 2+2+2$, and A would not be a P(3)-set. We conclude that the inequality we actually need is $6\mathcal{I}(3,2,0) < \mathcal{I}(0,0,0)$, which follows from Lemma 20. This finishes the proof. ∎

A.5 A further example of application

Arguments similar to those in the previous section can be used to establish other sharp extension inequalities for functions in $L^2(\mathbb{S}^1)$ whose spectrum satisfies property P(h) for some suitable h. In this section we provide a further example of application for the case of the $L^2(\mathbb{S}^1)$ to $L^6_{rad}L^4_{ang}(\mathbb{R}^2)$ Fourier extension estimates. The case of sharp $L^2(\mathbb{S}^1)$ to $L^6_{rad}L^2_{ang}(\mathbb{R}^2)$ Fourier extension estimates has been studied in [FOeS17], see also [COeSS19].

Theorem 22. *Let $f \in L^2(\mathbb{S}^1)$ be such that its spectrum A satisfies property* P(2)*. Then*

$$||\widehat{f\sigma}||^6_{L^6_{rad}L^4_{ang}} \leqslant (2\pi)^{9/2}\left(\int_0^\infty J_0^6(r)rdr\right)||f||^6_{L^2(\mathbb{S}^1)}.$$

The inequality is sharp and equality is attained if and only if f is constant.

Proof. Without loss of generality we can assume that $\sum_{n\in A}|\widehat{f}(n)|^2 = 1$. Using Hecke-Bochner formula we have that

$$\begin{aligned}||\widehat{f\sigma}||^6_{L^6_{rad}L^4_{ang}(\mathbb{R}^2)} &= \int_0^\infty \left(\int_{\mathbb{S}^1} |\widehat{f\sigma}(r\omega)|^4 d\sigma(\omega)\right)^{6/4} rdr \\ &= (2\pi)^{15/2}\int_0^\infty \left(\sum_{\substack{n_1,n_2,n_3,n_4\in A\\ n_1+n_2=n_3+n_4}} \widehat{f}(n_1)\widehat{f}(n_2)\overline{\widehat{f}(n_3)\widehat{f}(n_4)}J_{n_1}(r)J_{n_2}(r)J_{n_3}(r)J_{n_4}(r)\right)^{3/2} rdr.\end{aligned}$$

Now we use the fact that A satisfies property P(2) to rewrite the sum in the integral as follows.

$$\sum_{\substack{n_1,n_2,n_3,n_4\in A\\ n_1+n_2=n_3+n_4}} \widehat{f}(n_1)\widehat{f}(n_2)\overline{\widehat{f}(n_3)\widehat{f}(n_4)}J_{n_1}J_{n_2}J_{n_3}J_{n_4}$$

$$
\begin{aligned}
&= \sum_{D\in A^2} \sum_{\substack{n_1,n_2,n_3,n_4\in A\\ n_1+n_2=D\\ n_3+n_4=D}} \widehat{f}(n_1)\widehat{f}(n_2)\overline{\widehat{f}(n_3)\widehat{f}(n_4)} J_{n_1}J_{n_2}J_{n_3}J_{n_4}\\
&= \sum_{\substack{n_1,n_2\in A\\ n_1\neq -n_2}} \tau(n_1,n_2)|\widehat{f}(n_1)|^2|\widehat{f}(n_2)|^2 J_{n_1}^2 J_{n_2}^2\\
&\quad + \sum_{n_1,n_2\in A_s} \widehat{f}(n_1)\widehat{f}(-n_1)\overline{\widehat{f}(n_2)\widehat{f}(-n_2)} J_{n_1}J_{-n_1}J_{n_2}J_{-n_2}\\
&\leqslant \sum_{\substack{n_1,n_2\in A\\ n_1\neq -n_2}} \tau(n_1,n_2)|\widehat{f}(n_1)|^2|\widehat{f}(n_2)|^2 J_{n_1}^2 J_{n_2}^2\\
&\quad + \sum_{n_1,n_2\in A_s} \left(\frac{|\widehat{f}(n_1)|^2 J_{n_1}^2 + |\widehat{f}(-n_1)|^2 J_{-n_1}^2}{2}\right)\left(\frac{|\widehat{f}(n_2)|^2 J_{n_2}^2 + |\widehat{f}(-n_2)|^2 J_{-n_2}^2}{2}\right)\\
&= \sum_{n_1,n_2\in A} (\tau(n_1,n_2)\delta_{n_1\neq -n_2} + \delta_{n_1,n_2\in A_s})|\widehat{f}(n_1)|^2|\widehat{f}(n_2)|^2 J_{n_1}^2 J_{n_2}^2
\end{aligned}
$$

where $\tau(n_1,n_2) = 1 + \delta_{n_1\neq n_2}$ is the number of permutations of (n_1,n_2). Hence by Jensen's inequality we obtain

$$
\begin{aligned}
&(2\pi)^{-15/2}||\widehat{f\sigma}||^6_{L^6_{rad}L^4_{ang}(\mathbb{R}^2)}\\
&\leqslant \int_0^\infty \left(\sum_{n_1,n_2\in A} (\tau(n_1,n_2)\delta_{n_1\neq -n_2} + \delta_{n_1,n_2\in A_s})|\widehat{f}(n_1)|^2|\widehat{f}(n_2)|^2 J_{n_1}^2(r)J_{n_2}^2(r)\right)^{3/2} rdr\\
&\leqslant \int_0^\infty \sum_{n_1,n_2\in A} (\tau(n_1,n_2)\delta_{n_1\neq -n_2} + \delta_{n_1,n_2\in A_s})^{3/2}|J_{n_1}(r)|^3|J_{n_2}(r)|^3|\widehat{f}(n_1)|^2|\widehat{f}(n_2)|^2 rdr\\
&= \sum_{n_1,n_2\in A} (\tau(n_1,n_2)\delta_{n_1\neq -n_2} + \delta_{n_1,n_2\in A_s})^{3/2}\left(\int_0^\infty |J_{n_1}(r)|^3|J_{n_2}(r)|^3 rdr\right)|\widehat{f}(n_1)|^2|\widehat{f}(n_2)|^2.
\end{aligned}
$$

To conclude we need the following estimates on integrals involving the products of six Bessel functions. First, we observe that for all $m\neq \ell$, $m,\ell\geqslant 0$ it holds that

$$
3^{3/2}\int_0^\infty |J_m(r)|^3|J_\ell(r)|^3 rdr < \int_0^\infty J_0^6(r)rdr\ .
$$

In fact, by Hölder inequality and by the estimates in Lemma 20 we have that

$$
\begin{aligned}
\int_0^\infty |J_m(r)|^3|J_\ell(r)|^3 rdr &\leqslant \left(\int_0^\infty |J_m(r)|^4|J_\ell(r)|^2 rdr\right)^{1/2}\left(\int_0^\infty |J_\ell(r)|^4|J_m(r)|^2 rdr\right)^{1/2}\\
&\leqslant \left(\tfrac{\sqrt{2}}{5\sqrt{3}}\delta_{m\ell=0} + \tfrac{1}{9}\delta_{\{m,\ell\}\neq\{1,2\},\, m\ell\neq 0} + \tfrac{1}{3\sqrt{6}}\delta_{\{m,\ell\}=\{1,2\}}\right)\left(\int_0^\infty J_0^6(r)rdr\right)^{1/2}\\
&< 3^{-3/2}\int_0^\infty J_0^6(r)rdr.
\end{aligned}
$$

The second estimate that we need is the following: for all $n > 0$ it holds that

$$2^{3/2}\int_0^\infty J_n^6(r)rdr < \int_0^\infty J_0^6(r)rdr$$

which follows from Lemma 20 again. Hence, we conclude that

$$\begin{aligned}||\widehat{f\sigma}||^6_{L^6_{rad}L^4_{ang}(\mathbb{R}^2)} &\leqslant (2\pi)^{15/2}\left(\int_0^\infty J_0^6(r)rdr\right)\sum_{n_1,n_2\in A}|\widehat{f}(n_1)|^2|\widehat{f}(n_2)|^2\\ &= (2\pi)^{9/2}\left(\int_0^\infty J_0^6(r)rdr\right)||f||^6_{L^2(\mathbb{S}^1)}.\end{aligned}$$

Equality is attained if and only if f is constant. ■

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Appendix B

Global and local maximizers for some Fourier extension estimates on the sphere

This Appendix contains the article [CS23] written jointly by the author of this thesis and M. Sousa.

> Valentina Ciccone and Mateus Sousa. Global and local maximizers for some Fourier extension estimates on the sphere. arXiv preprint arXiv:2312.07309, 2023.

Abstract

In this note we improve, for the case of low dimensions, the known range of exponents for which constant functions are the unique maximizers for the $L^2(\mathbb{S}^{d-1})$ to $L^p_{rad}L^2_{ang}(\mathbb{R}^d)$ mixed-norm Fourier extension estimate on the sphere. Moreover, we show that in the same range of exponents for which constant functions are the unique maximizers for the $L^2(\mathbb{S}^{d-1})$ to $L^p_{rad}L^2_{ang}(\mathbb{R}^d)$ mixed-norm Fourier extension estimates they are also local maximizers for the $L^p(\mathbb{S}^{d-1})$ to $L^p(\mathbb{R}^d)$ Fourier extension estimates. As a by-product, we obtain that for the cases of dimensions $d = 2, 3$ constant functions are local maximizers for all $p \geqslant p_{\rm st}(d)$, where $p_{\rm st}$ denotes the Stein–Tomas endpoint, $p_{\rm st}(d) := 2(d+1)/(d-1)$.

B.1 Introduction

Let $d \geqslant 2$ be an integer, J_ν denote the Bessel function of the first kind of order ν, and k be a non-negative integer. It follows by the asymptotic behaviour of Bessel functions that the weighted L^p norms

$$\Lambda_{d,p}(k) := \left(\int_0^\infty |J_{\frac{d}{2}-1+k}(r) r^{-\frac{d}{2}+1}|^p r^{d-1} \mathrm{d}r \right)^{1/p} ,$$

$$\Lambda_{d,\infty}(k) := \sup_{r \geqslant 0} |J_{\frac{d}{2}-1+k}(r) r^{-\frac{d}{2}+1}| ,$$

are bounded whenever $\frac{2d}{d-1} < p \leqslant \infty$.

The problem of determining for which k such weighted norms are maximized, which is a problem of independent interest in the theory of special functions, has been studied in [COeSS19] in connection with certain mixed-norm sharp Fourier extension problems. In particular, in [COeSS19] the authors have studied the problem of computing

$$\sup_{k\geqslant 0} \Lambda_{d,p}(k). \tag{P1}$$

The properties of Bessel functions (see e.g. [Ste00]) guarantee that such supremum is a maximum. It has been shown in [COeSS19] that such maximum is achieved at $k = 0$ (and only at $k = 0$) whenever p is an even exponent and that the set of exponents for which the maximum is achieved at $k = 0$ is open and it contains a neighborhood of infinity $(p_0(d), \infty]$, providing some upper-bounds for $p_0(d)$. In particular, they obtained the following upper-bounds in low dimensions:

$$p_0(2) \leqslant 6.76, \quad p_0(3) \leqslant 5.45, \quad p_0(4) \leqslant 5.53, \quad p_0(5) \leqslant 6.07, \quad p_0(6) \leqslant 6.82,$$

$$p_0(7) \leqslant 7.70, \quad p_0(8) \leqslant 8.69, \quad p_0(9) \leqslant 9.78, \quad p_0(10) \leqslant 10.95,$$

and, more in general, they showed that

$$p_0(d) \leqslant (\tfrac{1}{2} + o(1))d \log d. \tag{B.1}$$

Problem (P1) is related to several problems in sharp Fourier restriction theory.

The Fourier restriction problem for the sphere asks for which pairs of exponents (p, q) the inequality

$$\|\widehat{f\sigma}\|_{L^p(\mathbb{R}^d)} \leqslant C_{d,p,q}\|f\|_{L^q(\mathbb{S}^{d-1})} \tag{B.2}$$

holds. Here $\sigma = \sigma_{d-1}$ is the surface measure on $\mathbb{S}^{d-1}$ and $\widehat{f\sigma}$ is the Fourier transform of the measure $f\sigma$,

$$\widehat{f\sigma}(x) = \int_{\mathbb{S}^{d-1}} e^{-ix\cdot\xi} f(\xi) d\sigma(\xi).$$

The Fourier restriction problem has been fully solved only in dimension $d = 2$ and for the case $q = 2$ for which a complete answer is given by the Stein–Tomas inequality. A mixed-norm version of the problem has been studied in [Veg92, Veg88] showing that the mixed-norm Fourier extension inequality

$$||\widehat{f\sigma}||_{L^p_{rad}L^2_{ang}(\mathbb{R}^d)} \leqslant C_{d,p}||f||_{L^2(\mathbb{S}^{d-1})} \tag{B.3}$$

holds when $\frac{2d}{d-1} < p$, where

$$||\widehat{f\sigma}||_{L^p_{rad}L^2_{ang}(\mathbb{R}^d)} = \left(\int_0^\infty \left(\int_{\mathbb{S}^{d-1}} |\widehat{f\sigma}(r\omega)|^2 d\sigma(\omega)\right)^{p/2} r^{d-1}\mathrm{d}r\right)^{1/p}.$$

The problem of determining the sharpest constant for (B.3) has been studied in [COeSS19]. Namely, in [COeSS19] the authors have studied the problem of computing

$$\sup_{f\in L^2(\mathbb{S}^{d-1}),\, f\neq 0} \Phi_{p,d}^{\mathrm{mn}}(f), \quad \Phi_{p,d}^{\mathrm{mn}}(f) := \frac{||\widehat{f\sigma}||_{L^p_{rad}L^2_{ang}(\mathbb{R}^d)}}{||f||_{L^2(\mathbb{S}^{d-1})}}. \tag{P2}$$

It was observed in [COeSS19] that the studying of such problem can be restricted to functions f which are spherical harmonics. In other words

$$\sup_{f\in L^2(\mathbb{S}^{d-1}),\, f\neq 0} \Phi_{d,p}^{\mathrm{mn}}(f) = \sup_{Y_k,\, Y_k\neq 0} \Phi_{p,d}^{\mathrm{mn}}(Y_k),$$

where Y_k denotes a spherical harmonics of degree k. Due to the identity

$$\widehat{Y_k\sigma}(x) = (2\pi)^{\frac{d}{2}} i^{-k}|x|^{-\frac{d}{2}+1} J_{\frac{d}{2}-1+k}(|x|) Y_k\Big(\frac{x}{|x|}\Big) \tag{B.4}$$

we have that

$$\|\widehat{Y_k\sigma}\|_{L^p_{rad}L^2_{ang}(\mathbb{R}^d)} = (2\pi)^{\frac{d}{2}} \left(\int_0^\infty |J_{\frac{d}{2}-1+k}(r) r^{-\frac{d}{2}+1}|^p r^{d-1}\mathrm{d}r\right)^{1/p} \|Y_k\|_{L^2(\mathbb{S}^{d-1})}.$$

Hence, the problem of establishing the sharpest constant for (B.3), namely (P2), is equivalent to the problem of determining for which non-negative integer k the maximum in (P1) is achieved.

Problem (P1) has been addressed in [COeSS19] by relating the integrals $\Lambda_{d,p}(k)$'s to integration on spheres using delta-calculus. Our approach, on the other hand, relies on some sharper estimates (with an improved constant) between weighted norms of Bessel functions inspired by those obtained in [CG24, Lemma 2] for the case of dimension $d = 2$, see the forthcoming inequality (B.6).

Our first result lowers, for the case of low dimensions, the upper bounds for $p_0(d)$ established in [COeSS19], hence extending the ranges of exponents for which the maximum in (P1) is achieved when $k = 0$. We use the notation $p_{\mathrm{st}}(d)$ to denote the Stein–Tomas endpoint exponent in dimension d, $p_{\mathrm{st}}(d) := \frac{2(d+1)}{(d-1)}$.

Theorem 23. *It holds that*

$$p_0(2) < 6, \quad p_0(3) < 4, \quad p_0(4) < 3.48, \quad p_0(5) < 3.50,$$

$$p_0(6) < 3.58, \quad p_0(7) < 3.7, \quad p_0(8) < 3.86, \quad p_0(9) < 4.06, \quad p_0(10) < 4.46.$$

In particular, for $d = 2, 3$ this gives that $p_0(d) < p_{\mathrm{st}}(d)$.

The fact that $p_0(d) < p_{\mathrm{st}}(d)$ is of interest because constant functions are natural candidates to be extremizers for the full range of exponents of the Stein–Tomas Fourier extension inequality. If this were true, then by Hölder inequality, constant functions would be also maximizers for $\Phi_{d,p}^{\mathrm{mn}}$ when $p \geqslant p_{\mathrm{st}}(d)$. This has been verified only when $p \geqslant 4$ is an even integer and $d \in \{3, 4, 5, 6, 7\}$ (see [COeS15, Fos15, OeSQ21a]), but it is open for all other cases. In particular the case where $d = 2$ has received a great deal of attention and many partial results have been achieved (see [CFOeST17, OeSTZK22, BTZK23, CG24, Bec23]), yet still remains unsolved. Hence, our result provides further evidence in this direction.

As mentioned above Problems (P1) and (P2) are equivalent. Next, we observe that the same holds true if one considers the problem of finding extremizers among functions of the form $aY_k \in L^2(\mathbb{S}^{d-1})$, with $a \in \mathbb{C}$ and Y_k a spherical harmonic of degree k, for $L^p(\mathbb{S}^{d-1})$ to $L^p(\mathbb{R}^d)$ Fourier extension estimates. Namely, if one considers the problem of computing

$$\sup_{Y_k,\, Y_k \neq 0} \Phi_{p,d}(Y_k), \quad \Phi_{p,d}(f) := \frac{||\widehat{f\sigma}||_{L^p(\mathbb{R}^d)}}{||f||_{L^p(\mathbb{S}^{d-1})}} \ . \tag{P3}$$

In fact,

$$\|\widehat{Y_k\sigma}\|_{L^p(\mathbb{R}^d)} = (2\pi)^{\frac{d}{2}} \left(\int_0^\infty |J_{\frac{d}{2}-1+k}(r) r^{-\frac{d}{2}+1}|^p r^{d-1} \mathrm{d}r \right)^{1/p} \|Y_k\|_{L^p(\mathbb{S}^{d-1})} \ ,$$

and, therefore,

$$\sup_{Y_k,\, Y_k \neq 0} \Phi_{p,d}(Y_k) = \sup_{f \in L^2(\mathbb{S}^{d-1}),\, f \neq 0} \Phi^{\mathrm{mn}}_{d,p}(f) = (2\pi)^{\frac{d}{2}} \sup_{k \geqslant 0} \Lambda_{d,p}(k).$$

In words, this simple observation asserts that the problem of computing the optimal constant for the mixed-norm Fourier extension inequality (B.3) is equivalent to the problem of computing the optimal constant for the $L^p(\mathbb{S}^{d-1})$ to $L^p(\mathbb{R}^d)$ Fourier extension inequality when restricting to spherical harmonics.

The following corollary is an immediate consequence of the above considerations.

Corollary 24. *For all* $p \in (p_0(d), \infty]$ *we have that*

$$\sup_{f \in L^2(\mathbb{S}^{d-1}),\, f \neq 0} \Phi^{\mathrm{mn}}_{p,d}(f) \leqslant \Phi^{\mathrm{mn}}_{p,d}(\mathbf{1}) \ ,$$
$$\sup_{Y_k,\, Y_k \neq 0} \Phi_{p,d}(Y_k) \leqslant \Phi_{p,d}(Y_0) \ .$$

That is, for all such p's, constant functions are maximizers for (P2) *and* (P3)*.*

Note that the fact that constant functions are extremizers for (P3) is a necessary condition for this to be the case also for the more general problem of computing

$$\sup_{f \in L^p(\mathbb{S}^{d-1}),\, f \neq 0} \Phi_{p,d}(f). \tag{P4}$$

Extremizers for (P4) are known only when p is an even admissible exponent, in which case it has been shown in [COeS15] that constant functions are maximizers, and when $p = \infty$ in which case the same conclusion holds [FS24]. Except for these cases, even the question of the existence of global extremizers for Problem (P4) is open [FS24]. Due to symmetry, constant functions would be natural candidate to be extremizers. Also, it was noted in [CQ14] that constant functions are always solutions to the corresponding Euler–Lagrange equations for any admissible pair of exponents (p, q) for the Fourier extension inequality (B.2), so, in particular, for any admissible pair (p, p).

A further intermediate step toward a solution of Problem (P4) is to understand the behavior of local extremizers. Local extremizers have been studied before for the case of the endpoint Stein–Tomas inequalities in [CFOeST17], [CS12a], and [GN22] showing, respectively, that constant functions are local maximizers for such inequalities when $d = 2$, when $d = 3$, and when $2 \leqslant d \leqslant 60$. The next question that we would like to address here is whether constant functions are local maximizers also for the case of the $L^p(\mathbb{S}^{d-1})$ to $L^p(\mathbb{R}^d)$ Fourier extension inequalities, namely for (P4). We answer this question by providing a further connection with Problem (P1).

Our second main result is the following.

Theorem 25. *Let $d \geqslant 2$ and $p > \frac{2d}{d-1}$. Assume that the $L^p(\mathbb{S}^{d-1})$ to $L^p(\mathbb{R}^d)$ Fourier extension inequality holds and that the maximum in* (P1) *is achieved at $k = 0$. Then there exists $\delta > 0$ such that whenever $\|f - \mathbf{1}\|_{L^p(\mathbb{S}^{d-1})} < \delta$,*

$$\Phi_{p,d}(f) \leqslant \Phi_{p,d}(\mathbf{1}). \tag{B.5}$$

That is, constant functions are local maximizers for (P4).

As an immediate consequence we have that constant functions are local maximizers for the $L^p(\mathbb{S}^{d-1})$ to $L^p(\mathbb{R}^d)$ Fourier extension inequality for all $p \in (p_0(d), \infty]$ for which the inequality holds and upper bounds on $p_0(d)$ is provided by Theorem 23 for the cases of dimensions $2 \leqslant d \leqslant 10$, and, in general, by (B.1) for greater dimensions.

The proof of Theorem 25 is contained in Section B.4, the proof of Theorem 23 is the content of Section B.3, while some auxiliary results about hierarchies between weighted norms of Bessel functions are presented in Section B.2.

The topic of sharp spherical restriction has received much attention over the last years, in particular for the case of inequalities in the Stein–Tomas range [FVV11, CS12a, Fos15, COeS15, FLS16, Sha16a, CFOeST17, OeSTZK22, BTZK23, OeSQ21a, CG24, Bec23]. We refer to the survey [NOeST23] for an up-to-date description of the state of the art.

B.2 Hierarchies between weighted norms of Bessel functions

It is known that when $p \in 2\mathbb{N}$, $p > \frac{2d}{d-1}$, or when $p = \infty$ then

$$\frac{\Lambda_{d,p}(k)}{\Lambda_{d,p}(0)} < 1$$

for all positive integers k, see [COeSS19]. In this section, we are interested in obtaining sharper estimates for such ratio, at least for certain values of the exponent p.

In this direction, for the case of dimension $d = 2$ and exponent $p = 6$ it has been shown in [CG24] that

$$\Lambda_{2,6}^6(k) < \frac{1}{3}\Lambda_{2,6}^6(0) \tag{B.6}$$

for all $k \geqslant 1$.

Moreover, in [COeSS19] combining the identity

$$\Lambda_{d,\infty}(0) = \frac{1}{2^{\frac{d}{2}-1}\Gamma(\frac{d}{2})}, \tag{B.7}$$

with a decreasing upper-bound (with respect to the order k) for $\Lambda_{d,p}(k)$, it has been shown that

$$\frac{\Lambda_{d,\infty}(k)}{\Lambda_{d,\infty}(0)} \leqslant \left(\mathrm{L}^6 \frac{2^{3d-6}\Gamma(\frac{d}{2})}{d^{3d-4}}\right)^{\frac{1}{3d+2}}$$

for all $k \geqslant 1$, where the constant L is defined as

$$\mathrm{L} := \sup_{\nu>0,\, r>0} |r^{1/3} J_\nu(r)| = 0.785746... \tag{B.8}$$

and it has been found by Landau [Lan00].

Our first result of this section establishes a hierarchy between the $\Lambda_{d,\infty}(k)$'s, hence determining the sharpest upper-bound on the ratio $\frac{\Lambda_{d,\infty}(k)}{\Lambda_{d,\infty}(0)}$.

Proposition 26. *For all positive integers k it holds that*

$$\Lambda_{d,\infty}(k-1) > \Lambda_{d,\infty}(k).$$

In particular,

$$\Lambda_{d,\infty}(k) \leqslant \mathrm{C}_\infty(d)\Lambda_{d,\infty}(0)$$

for all positive integers k, where $\mathrm{C}_\infty(d) := \frac{\Lambda_{d,\infty}(1)}{\Lambda_{d,\infty}(0)}$, and equality is attained if and only if $k = 1$.

Proof. We begin with the case $d = 2$. In such case $\Lambda_{d,\infty}(k) = \sup_{r\geqslant 0} |J_k(r)|$. It has been shown in [Lan00] that $\sup_{r>0} |J_k(r)|$ is a strictly decreasing function of k. In particular, if we denote by $j'_{k,1}$ the first positive zero of J'_k with k a positive real number, then

$$\sup_{r>0} |J_k(r)| = J_k(j'_{k,1}),$$

and therefore

$$\sup_{r>0} |J_k(r)| = J_k(j'_{k,1}) > \sup_{r>0} |J_{k+1}(r)| = J_k(j'_{k+1,1}).$$

As $\sup_{r\geqslant 0} |J_0(k)| = J_0(0) = 1 > J_1(j'_{1,1})$ the claim in the statement is verified for the case $d = 2$.

We turn to the case of $d \geqslant 3$. In these cases $\Lambda_{d,\infty}(k) = \sup_{r\geqslant 0} |r^{-\frac{d}{2}+1} J_{\frac{d}{2}-1+k}|$. We start by observing that as $r^{-\frac{d}{2}+1}$ is a strictly decreasing function of r and $\sup_{r>0} |J_\nu(r)| = J_\nu(j'_{\nu,1})$ it holds that

$$\sup_{r>0} |r^{-\frac{d}{2}+1} J_{\frac{d}{2}-1+k}(r)| = \sup_{0<r<j'_{d/2-1+k,1}} |r^{-\frac{d}{2}+1} J_{\frac{d}{2}-1+k}(r)|.$$

Hence to conclude it would be enough to show that

$$J_\nu(r) > J_{\nu+1}(r) \qquad \text{for all } r \in (0, j'_{\nu,1}). \tag{B.9}$$

We recall that $j'_{\nu,1} < j'_{\nu+1,1}$, see e.g. [Wat66]. In particular, J_ν, $J_{\nu+1}$, J'_ν, $J'_{\nu+1}$ are strictly positive in $(0, j'_{\nu,1})$, and $J_\nu(j'_{\nu,1}) > J_{\nu+1}(j'_{\nu,1})$. Hence to prove (B.9) it suffices to show that there exist no $\overline{r} \in (0, j'_{\nu,1})$ such that $J_\nu(\overline{r}) = J_{\nu+1}(\overline{r})$. We argue by contradiction. Consider the recursive relations for Bessel functions

$$\tfrac{2\nu}{r} J_\nu(r) = J_{\nu-1}(r) + J_{\nu+1}(r), \tag{B.10}$$

$$2J'_\nu(r) = J_{\nu-1}(r) - J_{\nu+1}(r). \tag{B.11}$$

By taking the sum of (B.10) and (B.11) we obtain the identity

$$J'_\nu(r) = J_{\nu-1}(r) - \tfrac{(\nu)}{r} J_\nu(r),$$

and shifting $\nu \mapsto \nu + 1$ we obtain

$$J'_{\nu+1}(r) = J_\nu(r) - \tfrac{(\nu+1)}{r} J_{\nu+1}(r).$$

Assume there exist $\overline{r} \in (0, j'_{\nu,1})$ such that $J_\nu(\overline{r}) = J_{\nu+1}(\overline{r})$. Evaluating the last display at $\overline{r}$ we get

$$J'_{\nu+1}(\overline{r}) = (1 - \tfrac{(\nu+1)}{\overline{r}}) J_{\nu+1}(\overline{r})$$

and, as $J'_{\nu+1}$ and $J_{\nu+1}$ are strictly positive on $(0, j'_{\nu,1})$, we have that necessarily

$$\overline{r} > \nu + 1. \tag{B.12}$$

Next, we take the difference between (B.11) and (B.10) obtaining the identity

$$J_\nu(r)' = \tfrac{\nu}{r} J_\nu(r) - J_{\nu+1}(r).$$

Evaluating it at $\overline{r}$ we get that

$$J'_\nu(\overline{r}) = (\tfrac{\nu}{\overline{r}} - 1) J_\nu(\overline{r})$$

and, as both J'_ν and J_ν are strictly positive on $(0, j'_{\nu,1})$, we have that necessarily

$$\nu > \overline{r}.$$

Comparing this with (B.12) yields the contradiction. ∎

The values of $\Lambda_{d,\infty}(1)$ can be computed by using Mathematica. For the case of $2 \leqslant d \leqslant 10$ one obtains, with 6 significant figures (s.f.),

$$\begin{gathered} \Lambda_{2,\infty}(1) = 0.581865, \quad \Lambda_{3,\infty}(1) = 0.348023, \quad \Lambda_{4,\infty}(1) = 0.179963, \\ \Lambda_{5,\infty}(1) = 0.0830013, \quad \Lambda_{6,\infty}(1) = 0.0348492, \quad \Lambda_{7,\infty}(1) = 0.0135129, \\ \Lambda_{8,\infty}(1) = 0.00489072, \quad \Lambda_{9,\infty}(1) = 0.00166575, \quad \Lambda_{10,\infty}(1) = 0.000537364. \end{gathered} \tag{B.13}$$

By combining them with (B.7) one can obtain a numerical evaluation for $\mathrm{C}_\infty(d)$.

Our second observation is for the case of exponent $p = 4$ and dimensions $3 \leqslant d \leqslant 10$.

Proposition 27. *Let* $3 \leqslant d \leqslant 10$. *Then*

$$\Lambda_{d,4}(k) \leqslant \mathrm{C}_4(d)\,\Lambda_{d,4}(0)$$

holds for all positive integers k, *where* $\mathrm{C}_4(d) := \frac{\Lambda_{d,4}(1)}{\Lambda_{d,4}(0)}$. *Equality is attained if and only if* $k = 1$.

To prove Proposition 27 we need the following upper-bound; see also [GN22].

Lemma 28. *Let* $d \geqslant 2$. *In the range of exponents* $\frac{6d-2}{3d-4} < p < \frac{12d+4}{3d-4}$ *it holds that*

$$\Lambda_{d,p}^p(k) \leqslant \mathrm{L}^{p-2}\frac{\Gamma(\lambda)\Gamma(\frac{d}{2}-1+k+\frac{1-\lambda}{2})}{2^\lambda\Gamma(\frac{1+\lambda}{2})^2\Gamma(\frac{d}{2}-1+k+\frac{1+\lambda}{2})}$$

for all positive integers k, *where* $\lambda = p(\frac{d}{2}-\frac{2}{3}) - d + \frac{1}{3}$.

To establish the upper-bound in the lemma we rely on the following identity which can be found in [GR07, Equation 6.574-2]

$$\int_0^\infty J_\nu^2(r)r^{-\lambda}\mathrm{d}r = \frac{\Gamma(\lambda)\Gamma(\nu+\frac{(1-\lambda)}{2})}{2^\lambda\Gamma\big(\frac{1+\lambda}{2}\big)^2\Gamma\big(\nu+\frac{1+\lambda}{2}\big)} \tag{B.14}$$

for $0 < \lambda < 2\nu + 1$.

Proof. We use (B.8) to obtain the upper-bound

$$\Lambda_{d,p}(k)^p \leqslant \mathrm{L}^{p-2}\int_0^\infty J^2_{\frac{d}{2}-1+k}(r)r^{-p\left(\frac{d}{2}-\frac{2}{3}\right)+d-\frac{1}{3}}\mathrm{d}r.$$

By applying identity (B.14) to the right hand side of the last display we further obtain

$$\Lambda_{d,p}^p(k) \leqslant \mathrm{L}^{p-2}\frac{\Gamma(\lambda)\Gamma(\frac{d}{2}-1+k+\frac{1-\lambda}{2})}{2^\lambda\Gamma(\frac{1+\lambda}{2})^2\Gamma(\frac{d}{2}-1+k+\frac{1+\lambda}{2})}$$

where $\lambda = p(\frac{d}{2}-\frac{2}{3}) - d + \frac{1}{3}$. Such upper-bound holds whenever $0 < \lambda < 2\big(\frac{d}{2}-1+k\big)+1$. In particular, for a fixed dimension $d \geqslant 2$ the upper-bound holds for all positive integers k whenever $\frac{6d-2}{3d-4} < p < \frac{12d+4}{3d-4}$. ■

Note that both the case of $p = 4$ and the case of Stein–Tomas endpoint $p_{\mathrm{st}}(d)$ are included in the range of exponents covered by Lemma 28. Also, note that, for a fixed exponent p and a fixed dimension d, the above upper bound is a decreasing function of k. Throughout, we use the notation $U_{d,p}(k)$ to denote the upper bound for $\Lambda_{d,p}^p(k)$ in Lemma 28.

Proof of Proposition 27. We compare the upper bound $U_{d,4}(k)$ for $\Lambda_{d,4}^4(k)$ established in Lemma 28 with a (lower) estimate for $\Lambda_{d,4}^4(1)$. To this end, we rely on Mathematica to evaluate the integrals

$$\int_0^{40} |J_{\frac{d}{2}}(r)r^{-\frac{d}{2}+1}|^4 r^{d-1}\mathrm{d}r \tag{B.15}$$

for $3 \leqslant d \leqslant 10$ obtaining, respectively, the following values (with 6 s.f.)

$$0.144681 \quad 0.0337263 \quad 0.00661348 \quad 0.00107217$$
$$0.000146318 \quad 0.0000171549 \quad 1.75867 \times 10^{-6} \quad 1.59953 \times 10^{-7}.$$

By comparison, one can see that $U_{d,4}(k) < \Lambda^4_{d,4}(1)$ for all integers $k \geqslant 2$ when $d \in \{5, 6, 7, 8, 9\}$, for all integers $k \geqslant 3$ when $d \in \{4, 10\}$, and for all integers $k \geqslant 5$ when $d = 3$. We check the remaining cases separately. We rely on Mathematica to evaluate the integrals

$$\int_0^{200} |J_{\frac{d}{2}-1+k}(r) r^{-\frac{d}{2}+1}|^4 r^{d-1} \mathrm{d}r$$

for the cases of interest obtaining, for the case of $d = 3$ and $k = 2, 3, 4$, the values (6 s.f.)

$$0.0992828 \quad 0.0757045 \quad 0.0615859,$$

respectively, and for the cases $d = 4$ and $k = 2$, and $d = 10$ and $k = 2$, the values (6 s.f.)

$$0.0172602 \quad 4.00184 \times 10^{-8},$$

respectively. Then, we use the estimate

$$|J_\nu(r)| \leqslant r^{-1/2} \tag{B.16}$$

which holds for all $\nu \geqslant \frac{1}{2}$ and $r \geqslant \frac{3}{2}\nu$ (see [COeSS19, Lemma 8] and [Kra14, Theorem 3]) to upper bound the tails obtaining that

$$\int_{200}^{\infty} |J_{\frac{d}{2}-1+k}(r) r^{-\frac{d}{2}+1}|^4 r^{d-1} \mathrm{d}r \leqslant \frac{200^{-d+2}}{d-2}.$$

Hence, by comparison, we see that also for these cases it holds that $\Lambda^4_{d,4}(k) < \Lambda^4_{d,4}(1)$. As it is known from [COeS15, COeSS19] that $\Lambda^4_{d,4}(1) < \Lambda^4_{d,4}(0)$ when $d \geqslant 3$, the result in the statement follows. ∎

To evaluate $\mathrm{C}_4(d)$ one can rely on the identity

$$\int_0^{\infty} |J_\nu(r)|^4 r^{-2\nu+1} \mathrm{d}r = \frac{\Gamma(\nu)\Gamma(2\nu)}{2\pi\Gamma(\nu + \frac{1}{2})^2\Gamma(3\nu)},$$

which can be found, for example, in [COeSS19, Lemma 7] (see also [GR07, Equation 6.5793-3]) and which provides an explicit expression for $\Lambda^4_{d,4}(0)$, together with a numerical estimates for $\Lambda^4_{d,4}(1)$.

Our last result of this section is for the case of the Stein–Tomas endpoint, $p_{\mathrm{st}} = p_{\mathrm{st}}(d)$.

Proposition 29. *Let $4 \leqslant d \leqslant 10$. Then the following inequality holds for all positive integers k*

$$\Lambda_{d,p_{\mathrm{st}}}(k) < \mathrm{C}_{p_{\mathrm{st}}}(d)\, \Lambda_{d,p_{\mathrm{st}}}(0),$$

where $\mathrm{C}_{p_{\mathrm{st}}}(d) := \frac{\Lambda_{d,p_{\mathrm{st}}}(1)}{\Lambda_{d,p_{\mathrm{st}}}(0)}$. Equality is attained if and only if $k = 1$.

Proof. We compare the upper bound $U_{d,p_{\mathrm{st}}}(k)$ for $\Lambda^{p_{\mathrm{st}}}_{d,p_{\mathrm{st}}}(k)$ established in Lemma 28 with a (lower) estimate for $\Lambda^{p_{\mathrm{st}}}_{d,p_{\mathrm{st}}}(1)$. To this end, we rely on Mathematica to evaluate the integrals

$$\int_0^{50} |J_{\frac{d}{2}}(r) r^{-\frac{d}{2}+1}|^{p_{\mathrm{st}}} r^{d-1} \mathrm{d}r \tag{B.17}$$

for $4 \leqslant d \leqslant 10$ obtaining, respectively, the following values (with 6 s.f.)

$$0.143391 \quad 0.131693 \quad 0.118941 \quad 0.10719 \quad 0.0969753 \quad 0.088279 \quad 0.0807943.$$

By comparison, one can see that $U_{d,p_{\mathrm{st}}}(k) < \Lambda^{p_{\mathrm{st}}}_{d,p_{\mathrm{st}}}(1)$ for all integers $k \geqslant 3$ when $d \in \{5,6,7,8,9,10\}$, and for all integers $k \geqslant 4$ when $d = 4$. We check the remaining cases separately. We use Mathematica to evaluate the integral

$$\int_0^{200} |J_{\frac{d}{2}-1+k}(r) r^{-\frac{d}{2}+1}|^{p_{\mathrm{st}}} r^{d-1} \mathrm{d}r$$

obtaining for the cases $k = 2$ and $5 \leqslant d \leqslant 10$ the values (6 s.f.)

$$0.0998066 \quad 0.0938562 \quad 0.0875322 \quad 0.0814907 \quad 0.075952 \quad 0.0709569$$

and for the cases $d = 4$ and $k = 2, 3$ the values (6 s.f.)

$$0.103492 \quad 0.080522.$$

We use the estimate (B.16) to upper bound the tail of the integrals obtaining

$$\int_{200}^{\infty} |J_{\frac{d}{2}-1+k}(r) r^{-\frac{d}{2}+1}|^{p_{\mathrm{st}}} r^{d-1} \mathrm{d}r \leqslant \frac{1}{200}.$$

Hence, by comparison, it follows that $\Lambda^{p_{\mathrm{st}}}_{4,p_{\mathrm{st}}}(2) < \Lambda^{p_{\mathrm{st}}}_{4,p_{\mathrm{st}}}(1)$.

We are left to show that $\Lambda^{p_{\mathrm{st}}}_{d,p_{\mathrm{st}}}(1) < \Lambda^{p_{\mathrm{st}}}_{d,p_{\mathrm{st}}}(0)$ whenever $4 \leqslant d \leqslant 10$. The cases of $d = 4, 5$ have already be verified in [COeSS19]. To verify the remaining cases $6 \leqslant d \leqslant 10$ we compare the bound for $\Lambda^{p_{\mathrm{st}}}_{d,p_{\mathrm{st}}}(1)$ obtained by combining the numerical evaluation of the truncated integral (B.17) and an upper bound for the tail obtained using (B.16) with a (lower) estimate for $\Lambda^{p_{\mathrm{st}}}_{d,p_{\mathrm{st}}}(0)$. To this end we numerically evaluate the integral

$$\int_0^{50} |J_{\frac{d}{2}+1}(r) r^{-\frac{d}{2}+1}|^{p_{\mathrm{st}}} r^{d-1} \mathrm{d}r$$

for $6 \leqslant d \leqslant 10$ obtaining the values (6 s.f.)

$$0.173201 \quad 0.147926 \quad 0.1286 \quad 0.113331 \quad 0.101086.$$

By comparison, we see that $\Lambda^{p_{\mathrm{st}}}_{d,p_{\mathrm{st}}}(1) < \Lambda^{p_{\mathrm{st}}}_{d,p_{\mathrm{st}}}(0)$ also for $6 \leqslant d \leqslant 10$ hence concluding the proof. ■

B.3 Proof of Theorem 23

Case $d = 2$

We combine the estimate (B.6) from [CG24] and the estimate in Proposition 26 (here for the case $d = 2$) with the interpolation strategy utilized in [COeSS19]. Let $p \geqslant 6$ and k be a positive integer. It follows from Hölder inequality that

$$\Lambda_{2,p}(k) \leqslant \Lambda_{2,6}(k)^{6/p}\Lambda_{2,\infty}(k)^{1-6/p}.$$

Using (B.6) and the sharp estimate from Proposition 26 we further obtain that

$$\Lambda_{2,p}(k) \leqslant \frac{1}{3^{1/p}}\Lambda_{2,6}(0)^{6/p}\Lambda_{2,\infty}(1)^{1-6/p}.$$

We need the following lower bound on $\Lambda_{d,p}(0)$ which has been established in [COeSS19, Equation 4.8]

$$\Lambda_{d,p}(0) > \frac{(2^{d-1}(\frac{d}{2})^{d/2})^{1/p}}{2^{d/2-1}\Gamma(\frac{d}{2})}\left(\frac{\Gamma(p+1)\Gamma(\frac{d}{2})}{\Gamma(p+\frac{d}{2}+1)}\right)^{1/p}. \tag{B.18}$$

Then, we rely on standard numerical evaluation to determine for which $p \geqslant 6$ it holds that

$$\frac{1}{3^{1/p}}\Lambda_{2,6}(0)^{6/p}\Lambda_{2,\infty}(1)^{1-6/p} \leqslant 2^{1/p}\frac{\Gamma(p+1)}{\Gamma(p+2)}.$$

We obtain that such inequality is satisfied for all $p \geqslant 6$. Hence, $p_0(2) < 6$ as claimed.

Case $d \geqslant 3$

We proceed in two steps. First, we combine the estimates in Proposition 27 and Proposition 26 with the interpolation strategy utilized in [COeSS19]. This will establish the upper bound on $p_0(d)$ in the statement of Theorem 23 for the cases of $d = 3, 9, 10$. Second, we use the estimates in Proposition 27 and Proposition 29 and interpolation to establish the upper bound on $p_0(d)$ for the cases of $d = 4, 5, 6, 7, 8$.

Step 1

Let $p \geqslant 4$ and k be a positive integer. It follows from Hölder inequality that

$$\Lambda_{d,p}(k) \leqslant \Lambda_{d,4}(k)^{4/p}\Lambda_{d,\infty}(k)^{1-4/p}.$$

Using the sharp estimate from Proposition 26 and Proposition 27 we further obtain that

$$\Lambda_{d,p}(k) \leqslant \Lambda_{d,4}(1)^{4/p}\Lambda_{d,\infty}(1)^{1-4/p}.$$

Then, we compare the right-hand side of the last display with the lower bound for $\Lambda_{d,p}(0)$ in equation (B.18) to determine for which $p \geqslant 4$ the following inequality is satisfied

$$\Lambda_{d,4}(1)^{4/p}\Lambda_{d,\infty}(1)^{1-4/p} \leqslant \frac{(2^{d-1}(\frac{d}{2})^{d/2})^{1/p}}{2^{d/2-1}\Gamma(\frac{d}{2})}\left(\frac{\Gamma(p+1)\Gamma(\frac{d}{2})}{\Gamma(p+\frac{d}{2}+1)}\right)^{1/p}.$$

We use the numerical values for $\Lambda_{d,\infty}(1)$ in (B.13) and the bound for $\Lambda_{d,4}(1)$ obtained by combining the numerical evaluation for the truncated integral in (B.15) with an upper bound for the tail obtained using (B.16). Via a standard numerical evaluation, we obtain that $\Lambda_{d,p}(k) < \Lambda_{d,p}(0)$ for all $p \geqslant 4$ for the cases of dimensions $d = 3, 4, 5, 6, 7, 8$, for all $p \geqslant 4.06$ for the case of $d = 9$, and for all $p \geqslant 4.46$ for the case of $d = 10$.

Step 2

Let $4 \leqslant d \leqslant 8$, $p_{\mathrm{st}}(d) \leqslant p \leqslant 4$ and k be a positive integer. It follows from Hölder inequality that

$$\Lambda_{d,p}(k) \leqslant \Lambda_{d,p_{\mathrm{st}}}(k)^{(1-\theta)} \Lambda_{d,4}(k)^{\theta},$$

with $\theta := \frac{4}{p}\frac{(p-p_{\mathrm{st}})}{(4-p_{\mathrm{st}})}$. Using the estimates of Proposition 27 and Proposition 29 we further obtain that

$$\Lambda_{d,p}(k) \leqslant \Lambda_{d,p_{\mathrm{st}}}(1)^{(1-\theta)} \Lambda_{d,4}(1)^{\theta}.$$

As before, we bound $\Lambda_{d,4}(1)$ by combining the numerical evaluation for the truncated integral in (B.15) with an upper bound for the tail obtained using (B.16) and we proceed analogously for $\Lambda_{d,p_{\mathrm{st}}}(1)$. Then, we compare this upper bound with the lower bound for $\Lambda_{d,p}(0)$ in equation (B.18) to determine, for a fixed $4 \leqslant d \leqslant 8$, for which $p_{\mathrm{st}}(d) \leqslant p \leqslant 4$ the former is greater than the latter. We obtain that $\Lambda_{d,p}(k) < \Lambda_{d,p}(0)$ for all $p \geqslant p(d)$ with

$$p(4) = 3.48, \quad p(5) = 3.5, \quad p(6) = 3.58 \quad , p(7) = 3.7, \quad p(8) = 3.86.$$

■

B.4 Proof of Theorem 25

Consider the deficit functional

$$\zeta_p[f] = \Phi_{p,d}(\mathbf{1})^p \|f\|^p_{L^p(\mathbb{S}^{d-1})} - \|\widehat{f\sigma}\|^p_{L^p(\mathbb{R}^d)}.$$

Inequality (B.5) is equivalent to

$$\zeta_p[f] \geqslant 0, \tag{B.19}$$

therefore it is enough to prove that there is a $\delta > 0$ such that $\zeta_p[f] > 0$ when $\|f - \mathbf{1}\|_{L^p(\mathbb{S}^{d-1})} < \delta$ and f is not constant, which we proceed to do.

We recall that here $p > 2$ and we are assuming that the Fourier extension operator is bounded from $L^p(\mathbb{S}^{d-1})$ to $L^p(\mathbb{R}^d)$. We compute

$$\begin{aligned}\int_{\mathbb{R}^d} |\widehat{\mathbf{1}\sigma}(x) + \varepsilon\widehat{g\sigma}(x)|^p \mathrm{d}x = &\int_{\mathbb{R}^d} |\widehat{\mathbf{1}\sigma}(x)|^p \mathrm{d}x + p\varepsilon \int_{\mathbb{R}^d} |\widehat{\mathbf{1}\sigma}(x)|^{p-2} \Re(\widehat{\mathbf{1}\sigma}(x)\widehat{g\sigma}(x))\mathrm{d}x \\ &+ \frac{p(p-2)\varepsilon^2}{4} \Re \int_{\mathbb{R}^d} |\widehat{\mathbf{1}\sigma}(x)|^{p-4} (\widehat{\mathbf{1}\sigma}(x)\widehat{g\sigma}(x))^2 \mathrm{d}x \\ &+ \frac{p\varepsilon^2}{4} \int_{\mathbb{R}^d} |\widehat{\mathbf{1}\sigma}(x)|^{p-2} |\widehat{g\sigma}(x)|^2 \mathrm{d}x \\ &+ o(\varepsilon^2).\end{aligned} \tag{B.20}$$

and

$$\int_{\mathbb{S}^{d-1}} |\mathbf{1}+\varepsilon g(x)|^p \mathrm{d}\sigma(x) = \|\mathbf{1}\|^p_{L^p(\mathbb{S}^{d-1})} + p\varepsilon \int_{\mathbb{S}^{d-1}} \Re(g(x))\mathrm{d}\sigma(x) + \frac{p(p-2)\varepsilon^2}{4}\Re\int_{\mathbb{S}^{d-1}} g(x)^2\mathrm{d}\sigma(x) + \frac{p\varepsilon^2}{4}\int_{\mathbb{S}^{d-1}} |g(x)|^2\mathrm{d}\sigma(x) + o(\varepsilon^2). \tag{B.21}$$

We take f to be of the form $f = \mathbf{1}+\varepsilon g$, with $0<\varepsilon \leqslant \delta$ and $\|g\|_{L^p(\mathbb{R}^d)} = 1$. By applying (B.20) and (B.21) one has

$$\begin{aligned}\zeta_p[f] =&\zeta_p[\mathbf{1}+\varepsilon g] = p\varepsilon\left(\Phi_{p,d}(\mathbf{1})^p\int_{\mathbb{S}^{d-1}}\Re(g(x))\mathrm{d}\sigma(x) - \int_{\mathbb{R}^d}|\widehat{\mathbf{1}\sigma}(x)|^{p-2}\Re(\widehat{\mathbf{1}\sigma}(x)\widehat{g\sigma}(x))\mathrm{d}x\right)\\ &+\frac{\varepsilon^2}{4}p(p-2)\left(\Phi_{p,d}(\mathbf{1})^p\Re\int_{\mathbb{S}^{d-1}} g(x)^2\mathrm{d}\sigma(x) - \Re\int_{\mathbb{R}^d}|\widehat{\mathbf{1}\sigma}(x)|^{p-2}(\widehat{g\sigma}(x))^2\mathrm{d}x\right)\\ &+\frac{\varepsilon^2}{4}p\left(\Phi_{p,d}(\mathbf{1})^p\int_{\mathbb{S}^{d-1}}|g(x)|^2\mathrm{d}\sigma(x) - \int_{\mathbb{R}^d}|\widehat{\mathbf{1}\sigma}(x)|^{p-2}|\widehat{g\sigma}(x)|^2\mathrm{d}x\right) + o(\varepsilon^2).\end{aligned} \tag{B.22}$$

Furthermore, due to the aforementioned observation that $\mathbf{1}$ is a critical point of $\Phi_{p,d}$, the first order terms in ε of (B.22) all vanish. To deal with the second order terms, we use the fact that $L^p(\mathbb{S}^{d-1}) \subset L^2(\mathbb{S}^{d-1})$ since $p>2$ in order to expand g in spherical harmonics. For that purpose we choose for each k an orthonormal basis $\{Y_{j,k}\}_j$ of $\mathcal{H}^d_k$ where each $Y_{j,k}$ is a real-valued spherical harmonic of degree k. Then

$$g = \sum_{k,j} a_{j,k}Y_{j,k}.$$

By combining identity (B.4) with the observation that the first order terms vanish at (B.22) we can integrate in polar coordinates to obtain

$$\begin{aligned}\zeta_p[f] =&\frac{\varepsilon^2}{4}p(p-2)\Bigg(\Phi_{p,d}(\mathbf{1})^p\sum_{k,j}\Re(a_{j,k})^2\\ &-(2\pi)^{pd/2}\sum_{k,j}(-1)^k\Re(a_{j,k})^2\int_0^\infty |J_{\frac{d}{2}-1}(r)|^{p-2}|J_{\frac{d}{2}-1+k}(r)|^2 r^{d-1-p(1-\frac{d}{2})}\mathrm{d}r\Bigg)\\ &+\frac{\varepsilon^2}{4}p\sum_{k,j}|a_{j,k}|^2\left(\Phi_{p,d}(\mathbf{1})^p - (2\pi)^{pd/2}\int_0^\infty |J_{\frac{d}{2}-1}(r)|^{p-2}|J_{\frac{d}{2}-1+k}(r)|^2 r^{d-1-p(1-\frac{d}{2})}\mathrm{d}r\right)\\ &+o(\varepsilon^2).\end{aligned} \tag{B.23}$$

Lastly, using Hölder inequality and the fact that by hypothesis $\Lambda_{d,p}(k) < \Lambda_{d,p}(0)$ for all positive integers k we observe that

$$\int_0^\infty |J_{\frac{d}{2}-1}(r)|^{p-2}|J_{\frac{d}{2}-1+k}(r)|^2 r^{d-1-p(1-\frac{d}{2})}\mathrm{d}r$$

$$< \left(\int_0^\infty |J_{\frac{d}{2}-1}(r)|^p r^{d-1-p(1-\frac{d}{2})} \mathrm{d}r \right)^{(p-2)/p} \left(\int_0^\infty |J_{\frac{d}{2}-1+k}(r)|^p r^{d-1-p(1-\frac{d}{2})} \mathrm{d}r \right)^{2/p}$$
$$< \int_0^\infty |J_{\frac{d}{2}-1}(r)|^p r^{d-1-p(1-\frac{d}{2})} \mathrm{d}r = (2\pi)^{-pd/2} \Phi_{p,d}(\mathbf{1})^p,$$

hence concluding the proof of Theorem 25.

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Appendix C

Endpoint estimates for higher order Marcinkiewicz multipliers

This Appendix contains the article [BCPV24] written jointly by O. Bakas, I. Parissis, M. Vitturi, and the author of this thesis.

> Odysseas Bakas, Valentina Ciccone, Ioannis Parissis, and Marco Vitturi. Endpoint estimates for higher order Marcinkiewicz multipliers. arXiv preprint arXiv:2401.06083, 2024.

Abstract

We consider Marcinkiewicz multipliers of any lacunary order defined by means of uniformly bounded variation on each lacunary Littlewood–Paley interval of some fixed order $\tau \geqslant 1$. We prove the optimal endpoint bounds for such multipliers as a corollary of a more general endpoint estimate for a class of multipliers introduced by Coifman, Rubio de Francia, and Semmes and further studied by Tao and Wright. Our methods also yield the best possible endpoint mapping property for higher order Hörmander–Mihlin multipliers, namely multipliers which are singular on every point of a lacunary set of order τ. These results can be considered as endpoint versions of corresponding results of Sjögren and Sjölin. Finally our methods generalize a weak square function characterization of the space $L\log^{1/2} L$ in terms of a square function introduced by Tao and Wright: we realize such a weak characterization as the dual of the Chang–Wilson–Wolff inequality, thus giving corresponding weak square function characterizations for the spaces $L\log^{\tau/2} L$ for general integer orders $\tau \geqslant 1$.

C.1 Introduction

Our topic is endpoint estimates for Marcinkiewicz-type multipliers on the real line. We recall that a Marcinkiewicz multiplier is a bounded function $m : \mathbb{R} \to \mathbb{C}$ which has bounded variation on each Littlewood–Paley interval $L_k := (-2^{k+1}, -2^k] \cup [2^k, 2^{k+1})$, uniformly in $k \in \mathbb{Z}$. It is well known that the operator $\mathrm{T}_m f := (m\hat{f})^\vee$ is bounded on $L^p(\mathbb{R})$ for all $p \in (1,\infty)$. Endpoint estimates for Marcinkiewicz multipliers were proved by Tao and Wright in [TW01] where the authors prove that they locally map $L\log^{1/2} L$ into weak L^1.

A prototypical Marcinkiewicz multiplier is given by the signed sum

$$\sum_{k\in\mathbb{Z}} \varepsilon_k \mathbf{1}_{L_k}, \qquad \varepsilon_k \in \{-1,+1\},$$

while an orthogonality argument provides the link between Marcinkiewicz multipliers and the classical Littlewood–Paley square function

$$\mathrm{LP}_1 f(x) := \Big(\sum_{k\in\mathbb{Z}} |\mathrm{P}_k f|^2\Big)^{\frac12} = \Big(\mathbf{E}\Big|\sum_{k\in\mathbb{Z}} \varepsilon_k \mathrm{P}_k f\Big|^2\Big)^{\frac12}, \qquad \mathrm{P}_k f := (\mathbf{1}_{L_k}\hat f)^\vee,$$

the expectation being over choices of independent random signs.

In the present paper we are interested in higher order versions of Marcinkiewicz multipliers. In order to motivate such a study it is very natural to consider square functions that project to Littlewood–Paley intervals given by lacunary sets of order 2 or higher. For example letting

$$L_{(k,m)} := \Big\{\xi\in\mathbb{R} :\, |\xi| \in (2^k+2^{m-1}, 2^k+2^m] \cup [2^{k+1}-2^m, 2^{k+1}-2^{m-1})\Big\}, \qquad k>m,$$

denote the family of Littlewood–Paley intervals of second order, we naturally define

$$\mathrm{LP}_2 f := \Bigg(\sum_{\substack{(k,m)\in\mathbb{Z}^2\\ k>m}} |\mathrm{P}_{(k,m)} f|^2\Bigg)^{\frac12} = \Bigg(\mathbf{E}\Bigg|\sum_{\substack{(k,m)\in\mathbb{Z}^2\\ k>m}} \varepsilon_{k,m}\mathrm{P}_{(k,m)} f\Bigg|^2\Bigg)^{\frac12}, \qquad \mathrm{P}_{(k,m)} f := (\mathbf{1}_{L_{(k,m)}}\hat f)^\vee,$$

initially for Schwartz functions with compactly supported Fourier transform. This is a second order Littlewood–Paley square function while the multiplier

$$\sum_{\substack{(k,m)\in\mathbb{Z}^2\\ k>m}} \varepsilon_{(k,m)}\mathbf{1}_{L_{(k,m)}}, \qquad \varepsilon_{(k,m)} \in \{-1,+1\},$$

can be considered as a prototypical Marcinkiewicz multiplier of order 2. A Littlewood–Paley partition $\{L : L \in \Lambda_\tau\}$ of lacunary order $\tau > 1$ is naturally produced by iterating Whitney decompositions inside each Littlewood–Paley interval of order $\tau - 1$. Accordingly, a Marcinkiewicz multiplier of order τ is a bounded function which has bounded variation uniformly on all Littlewood–Paley intervals of order τ. Likewise, the Littlewood–Paley square function of order τ is

$$\mathrm{LP}_\tau f := \Big(\sum_{L\in\Lambda_\tau} |\mathrm{P}_L f|^2\Big)^{1/2} = \Big(\mathbf{E}\Big|\sum_{L\in\Lambda_\tau} \varepsilon_L \mathrm{P}_L f\Big|^2\Big)^{1/2}, \qquad \mathrm{P}_L f := \Big(\mathbf{1}_L \hat f\Big)^\vee.$$

With precise definitions to follow, a punchline result of this paper is the following.

Theorem 30. *If m is a Marcinkiewicz multiplier of order $\tau \in \mathbb{N}$ then T_m satisfies the estimate*

$$|\{x \in \mathbb{R} : |\mathrm{T}_m f(x)| > \alpha\}| \lesssim \int_{\mathbb{R}} \frac{|f|}{\alpha}\left(\log\left(e + \frac{|f|}{\alpha}\right)\right)^{\tau/2}, \quad \alpha > 0.$$

The same is true for the Littlewood–Paley square function LP_τ of order τ. In both cases the endpoint estimates are best possible in the sense that the exponent $\tau/2$ in the right hand side cannot be replaced by any smaller exponent.

We will deduce Theorem 30 as a consequence of the more general Theorem 31 below which applies to the wider class of $R_{2,\tau}$ multipliers.

C.1.1 Lacunary sets of higher order

In order to describe the classes of higher order multipliers we are interested in, it will be necessary to introduce some notation for lacunary sets of general order. The standard Littlewood–Paley partition of the real line is the collection of intervals $\Lambda_1 := \{\pm[2^k, 2^{k+1}) : k \in \mathbb{Z}\}$ and it is a Whitney decomposition of $\mathbb{R}\backslash\{0\}$. For a finite dyadic interval $I \subset \mathbb{R}$ the standard Whitney partition $\mathcal{W}(I)$ of I is the collection of the maximal dyadic subintervals $L \subset I$ such that $\mathrm{dist}(L, \mathbb{R}\backslash I) = |L|$. Now for any integer $\tau > 1$ we set

$$\Lambda_\tau := \bigcup_{I \in \Lambda_{\tau-1}} \mathcal{W}(I)$$

and call Λ_τ the standard Littlewood–Paley collection of intervals of order τ. We denote by lac_τ the collection of all endpoints of intervals in Λ_τ. Observe that, as in [Bon70], the set lac_τ has the explicit representation

$$\mathrm{lac}_\tau = \{\pm 2^{n_1} \pm 2^{n_2} + \cdots \pm 2^{n_\tau} : n_1 > n_2 > \cdots > n_\tau,\, n_j \in \mathbb{Z} \quad \forall j\}.$$

For uniformity in the notation we also set $\Lambda_0 = \{(-\infty, 0), (0, +\infty)\}$ and $\mathrm{lac}_0 := \{0\}$. It will be useful throughout the paper to truncate the scales of lacunary intervals and numbers by defining

$$\Lambda_\tau^n := \{L \in \Lambda_\tau : |L| \geqslant n\}, \qquad n \in 2^{\mathbb{Z}}.$$

Accordingly lac_τ^n denotes endpoints of intervals in Λ_τ^n.

We need a smooth way to project to frequency intervals in Λ_τ. For this we consider a smooth even function $0 \leqslant \eta \leqslant 1$ such that η is identically 1 on $[-1/2, 1/2]$ and vanishes off $[-5/8, 5/8]$. For a positive integer τ and $L \in \Lambda_\tau$ we define the (rescaled) L-th frequency component of some multiplier $m : \mathbb{R} \to \mathbb{C}$ as

$$m_L(\xi) := \eta(\xi) m(c_L + \xi|L|), \qquad \xi \in \mathbb{R},$$

with c_L denoting the center of L.

C.1.2 Higher order multipliers and endpoint estimates

With this notation at hand we will say that $m : \mathbb{R} \to \mathbb{C}$ is a Hörmander–Mihlin multiplier of order τ if

$$\|m\|_{H_\tau} := \sup_{|\alpha|\leqslant M} \sup_{L\in\Lambda_\tau} \|\partial^\alpha m_L\|_{L^\infty} < +\infty,$$

for some sufficiently large positive integer M which we will not keep track of. Note that the higher order Hörmander–Mihlin condition is essentially the natural assertion

$$|\partial^\alpha m(\xi)| \lesssim \operatorname{dist}(\xi, \operatorname{lac}_{\tau-1})^{-\alpha}, \qquad \xi \in \mathbb{R}\backslash\operatorname{lac}_{\tau-1}.$$

Likewise we will say that a bounded function $m : \mathbb{R} \to \mathbb{C}$ is a Marcinkiewicz multiplier of order $\tau \in \mathbb{N}$ if the components m_L have bounded variation uniformly in $L \in \Lambda_\tau$. Here we use the standard variation norms defined for $r \in [1, \infty]$ as follows

$$\|F\|_{\mathrm{V}_r} := \sup_N \sup_{x_0<\cdots<x_N} \left(\sum_{0\leqslant k\leqslant N} |F(x_{k+1}) - F(x_k)|^r \right)^{\frac{1}{r}}.$$

Note that usually Marcinkiewicz multiplier are defined by asking that the pieces $m\mathbf{1}_L$ have bounded 1-variation, uniformly in L. One can check that our definition, using the smooth cutoff η, is equivalent to the classical one. For one inequality of this equivalence we just use that $\eta \equiv 1$ on $[-1/2, 1/2]$, while for the converse inequality it suffices to notice that $\|FG\|_{\mathrm{V}_1} \lesssim \|F\|_{\mathrm{V}_1}\|G\|_{\mathrm{V}_1}$ together with the fact that the support of η is contained in three adjacent intervals of length 1. We will actually consider the wider class of $R_{2,\tau}$-multipliers defined below.

Definition 6. *Let $\mathcal{R}$ to be the space of all functions of the form*

$$m = \sum_I c_I \mathbf{1}_I$$

with I ranging over a family of disjoint arbitrary subintervals in $[1, 2)$ and the coefficients $\{c_I\}_I$ satisfying

$$\sum_I |c_I|^2 \leqslant 1.$$

Then $\overline{\mathcal{R}}$ is the Banach space of functions $m := \sum_a \lambda_a m_a$ with $\sum_a |\lambda_a| < +\infty$; we equip $\overline{\mathcal{R}}$ with the norm

$$\|m\|_{\overline{\mathcal{R}}} := \inf \left\{ \sum_a |\lambda_a| : m = \sum_a \lambda_a m_a, \quad m_a \in \mathcal{R} \right\}.$$

For $\tau \in \mathbb{N}$ we say that the bounded function $m : \mathbb{R} \to \mathbb{C}$ is an $R_{2,\tau}$-multiplier if

$$\|m\|_{R_{2,\tau}} := \sup_{L\in\Lambda_\tau} \|m_L\|_{\overline{\mathcal{R}}} < +\infty.$$

The class $R_{2,\tau}$ contains all Marcinkiewicz multipliers of order τ as well as Hörmander–Mihlin multipliers of order τ. This follows by the fact that Hörmander multipliers of order

$\tau \geqslant 1$ are Marcinkiewicz multipliers of the same order and the latter belong to the class $\mathcal{V}_{1,\tau}$ consisting of functions which have uniformly bounded 1-variation on each lacunary interval of order τ; the inclusion relationship then follows for example by the fact that $\mathcal{V}_{1,\tau} \subset R_{2,\tau}$, proved in [CRdFS88, Lemma 2]. Our main result proves the sharp endpoint bound for multipliers in the class $R_{2,\tau}$.

Theorem 31. *Let τ be a positive integer and $m \in R_{2,\tau}$. Then the operator $\mathrm{T}_m f := (m\hat{f})^\vee$ satisfies*

$$|\{x \in \mathbb{R} : |\mathrm{T}_m f(x)| > \alpha\}| \lesssim \int_{\mathbb{R}} \frac{|f|}{\alpha}\left(\log\left(e + \frac{|f|}{\alpha}\right)\right)^{\tau/2}, \quad \alpha > 0.$$

Furthermore this estimate is best possible in the sense that the exponent $\tau/2$ in the right hand side of the estimate cannot be replaced by any smaller exponent. The implicit constant depends only on τ and the $R_{2,\tau}$-norm of m.

For $\tau = 1$ the local version of the theorem above is contained in [TW01]. We note that Theorem 31 easily implies the following local estimate: For every interval I and $m \in R_{2,\tau}$ there holds

$$|\{x \in I : |\mathrm{T}_m f(x)| > \alpha\}| \lesssim \frac{1}{\alpha}\int_I |f|\left(\log\left(e + \frac{|f|}{\langle |f|\rangle_I}\right)\right)^{\tau/2}, \qquad \alpha > 0, \qquad \mathrm{supp} f \subset I,$$

where $\langle |f|\rangle_I := |I|^{-1}\|f\|_{L^1(I)}$. The global estimate of Theorem 31 appears to be new even in the first order case $\tau = 1$, although a proof of a global result can be deduced for the first order case $\tau = 1$ from the methods in [TW01] without much additional work.

While Hörmander–Mihlin multipliers are $R_{2,\tau}$ multipliers, they are in general much better-behaved as the case $\tau = 1$ suggests: indeed for $\tau = 1$ Hörmander–Mihlin multipliers map L^1 to $L^{1,\infty}$, in contrast to the sharpness of the $L\log^{1/2} L \to L^{1,\infty}$ estimate for general Marcinkiewicz or $R_{2,1}$ multipliers. In analogy to the Littlewood–Paley square function LP_τ of order τ it is natural to define a smooth version as follows. For $C > 0$, $M \in \mathbb{N}$ and $L \in \Lambda_\tau$ we consider the class of bump functions

$$\Phi_{L,M} := \left\{\phi_L : \mathrm{supp}\,(\phi_L) \subseteq \frac{5}{4}L, \quad \sup_{\alpha \leqslant M} |L|^\alpha \|\partial^\alpha \phi_L\|_{L^\infty} \leqslant 10^{10}\right\}.$$

Now for some fixed large positive integer M (whose precise value is inconsequential) suppose that $\phi_L \in \Phi_{L,M}$ for all $L \in \Lambda_\tau$ and define, initially for $f \in \mathcal{S}(\mathbb{R})$,

$$\mathrm{S}_\tau f := \left(\sum_{L \in \Lambda_\tau} |\Delta_L f|^2\right)^{1/2}, \qquad \Delta_L f(x) := \int_{\mathbb{R}} \phi_L(\xi)\widehat{f}(\xi)e^{2\pi i x\xi}\,\mathrm{d}\xi, \qquad x \in \mathbb{R}.$$

The following theorem is the sharp endpoint estimate for higher order Hörmander–Mihlin multipliers and corresponding square functions.

Theorem 32. *Let τ be a positive integer and $m \in H_\tau$ be a Hörmander–Mihlin multiplier of order τ. Then*

$$|\{x \in \mathbb{R} : |\mathrm{T}_m f(x)| > \alpha\}| \lesssim \int_{\mathbb{R}} \frac{|f|}{\alpha}\left(\log\left(e + \frac{|f|}{\alpha}\right)\right)^{(\tau-1)/2}, \qquad \alpha > 0.$$

The same holds for the smooth Littlewood–Paley square function S_τ *of order* τ *and these results are best possible. The implicit constant depends only on* τ *and the* H_τ*-norm of* m*, and also on* M *in the case of square functions.*

The case $\tau = 1$ of this corollary is classical. The local version of the case of Hörmander–Mihlin multipliers of order $\tau = 2$ is implicit in [TW01] as it can be proved by combining [TW01, Proposition 5.1] with [TW01, Proposition 4.1]. All the higher order cases for H_τ-multipliers appear to be new.

C.1.3 The Chang–Wilson–Wolff inequality and a square function for $L\log^{\tau/2} L$

Throughout this section we work on the probability space $([0,1], \mathrm{d}x)$ unless otherwise stated. A central result in the approach in [TW01] was a weak characterization of the space $L\log^{1/2} L$ in terms of an integrable square function, inspired by the analogous and better-known characterisation of the Hardy space H^1. More precisely, the authors in [TW01] prove that if $f \in L\log^{1/2} L$ and f has mean zero then for each $L \in \Lambda_1$ one can construct nonnegative functions F_L such that

$$|\Delta_L f| \lesssim F_L * \varphi_{|L|^{-1}} \quad \forall L \in \Lambda_1, \qquad \int_{\mathbb{R}} \Big(\sum_{L\in\Lambda_1} |F_L|^2\Big)^{1/2} \lesssim \|f\|_{L\log^{1/2} L}, \tag{C.1}$$

where Δ_L is as in §C.1.2 and

$$\varphi_\lambda(x) := \lambda^{-1}\varphi(x/\lambda) := \lambda^{-1}(1 + |x/\lambda|^2)^{-3/4}, \qquad x \in \mathbb{R}.$$

Here and throughout the paper we use local Orlicz norms and corresponding notation as described in §C.2.1.

There is a dyadic version: denoting by $\mathcal{D}_k$ the dyadic subintervals of $[0,1]$ of length 2^{-k}, $k \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$, we consider the conditional expectation and martingale differences

$$\mathbf{E}_k f := \sum_{I\in\mathcal{D}_k} \langle f\rangle_I \mathbf{1}_I, \qquad \mathbf{D}_k f := \mathbf{E}_k f - \mathbf{E}_{k-1} f, \quad k \geqslant 1, \qquad \mathbf{D}_0 f := \mathbf{E}_0 f, \quad f \in L^1.$$

For future reference we record the definition of the dyadic martingale square function

$$\mathrm{S}_{\mathcal{M}} f := \Big(\sum_{k\geqslant 1} |\mathbf{D}_k f|^2\Big)^{1/2}.$$

The dyadic analogue of (C.1) is that if $f \in L\log^{1/2} L$ then for each $k \in \mathbb{N}_0$ there exist functions f_k such that

$$|\mathbf{D}_k f| \leqslant \mathbf{E}_k |f_k| \quad \forall k \in \mathbb{N}_0, \qquad \int_{[0,1]} \Big(\sum_{k\geqslant 0} |f_k|^2\Big)^{1/2} \lesssim \|f\|_{L\log^{1/2} L}, \tag{C.2}$$

In fact, the authors in [TW01] first prove (C.2) by constructing the functions f_k through a rather technical induction scheme, and then deduce (C.1) from (C.2) via a suitable averaging argument.

Several remarks are in order. Firstly one notices that (C.2) combined with a simple duality argument based on the fact that $\exp(L^2) = (L\log^{1/2} L)^*$ implies the Chang–Wilson–Wolff inequality

$$\|f - \mathbf{E}_0 f\|_{\exp(L^2)} \lesssim \|\mathrm{S}_{\mathcal{M}} f\|_{L^\infty}. \tag{C.3}$$

Estimate (C.3) was first proved in [CWW85]; see also the monograph [Wil08] for an in-depth discussion of exponential square integrability in relation to discrete and continuous square functions in analysis. Thus the proof of (C.2) in [TW01] is of necessity somewhat hard as it reproves (C.3).

A second observation that goes back to [TW01], see also [ST09] for an analogous remark on the dual side, is that (C.1) implies the weaker estimate

$$\Big(\sum_{L\in\Lambda_1^1} \|\Delta_L f\|_{L^1}^2\Big)^{\frac{1}{2}} \lesssim \|f\|_{L\log^{1/2} L}. \tag{C.4}$$

Indeed, (C.4) follows by (C.1) and the Minkowski integral inequality. Alternatively, as observed in [ST09], the dual of (C.4) is a —again weaker— consequence of the Chang–Wilson–Wolff inequality (C.3).

Finally, a consequence of (C.4) is the Zygmund inequality

$$\Big(\sum_{\lambda\in\mathrm{lac}_1^1} \Big|\widehat{f}(\lambda)\Big|^2\Big)^{\frac{1}{2}} \lesssim \|f\|_{L\log^{1/2} L}.$$

See for example [Zyg02, Theorem 7.6, Chapter XII]. Indeed, if L_λ is an interval which has λ as an endpoint we have $|\widehat{f}(\lambda)| \leqslant \|(\Delta_{L_\lambda} f)^\wedge\|_{L^\infty} \leqslant \|\Delta_{L_\lambda} f\|_{L^1}$ for a suitable choice of symbol in the definition of the Littlewood–Paley projection and Zygmund's inequality follows by (C.4).

All the estimates above have a higher order counterpart which plays an important role in our investigations in this paper. However, our point of view is somewhat different than in [TW01]. Firstly we want to emphasize that the proof of our main theorem, Theorem 31, hinges on a higher order version of the generalized Zygmund inequality (C.4) which loosely has the form

$$\Big(\sum_{L\in\Lambda_\tau^1} \|\Delta_L f\|_{L^1}^2\Big)^{\frac{1}{2}} \lesssim \|f\|_{L\log^{\tau/2} L}, \qquad \tau\in\mathbb{N}. \tag{C.5}$$

Estimates of the form (C.5) will be referred to as *generalized Zygmund–Bonami inequalities* and will be stated precisely and proved in Section C.4. The terminology comes from the fact that they imply the higher order version of Zygmund's inequality, due to Bonami [Bon70], and which can be stated as follows:

$$\Big(\sum_{\lambda\in\mathrm{lac}_\tau^1} \Big|\widehat{f}(\lambda)\Big|^2\Big)^{\frac{1}{2}} \lesssim \|f\|_{L\log^{\tau/2} L}, \qquad \tau\in\mathbb{N}. \tag{C.6}$$

A novelty in our approach is the realization that the weak square function characterization (C.2) of the space $L\log^{1/2} L$, in the dyadic case, is precisely the dual estimate of the Chang–Wilson–Wolff inequality (C.3). This relies on a duality argument involving quotient spaces which is inspired by the work of Bourgain, [Bou89]. We can then use the Chang–Wilson–Wolff inequality for general order of integrability, see §C.3.1,

$$\|f-\mathbf{E}_0 f\|_{\exp(L^{2/(\sigma+1)})} \lesssim \|S_{\mathcal{M}} f\|_{\exp(L^{2/\sigma})}, \qquad \sigma \geqslant 0\,, \tag{C.7}$$

to conclude the following weak square function characterization of the space $L\log^{(\sigma+1)/2} L$ in the form of the following theorem.

Theorem 33. *If $f \in L\log^{(\sigma+1)/2} L$ for some $\sigma \geqslant 0$ then for each $k \in \mathbb{N}_0$ there exist functions f_k such that*

$$\mathbf{D}_k f = \mathbf{D}_k f_k \quad \forall k \in \mathbb{N}_0, \qquad \Bigg\| \Big(\sum_{k\geqslant 0} |f_k|^2 \Big)^{1/2} \Bigg\|_{L\log^{\sigma/2} L} \lesssim \|f\|_{L\log^{(\sigma+1)/2} L}.$$

The implicit constant depends only on σ.

We will prove Theorem 33 in Section C.3 as a consequence of (C.7). While this is a rather deep implication, as in the case $\sigma = 0$, it is not hard to see that the conclusion of Theorem 33 combined with the fact $\exp(L^{2/\sigma}) = (L\log^{\sigma/2} L)^*$ actually implies the Chang–Wilson–Wolff inequality (C.7) for the same value of σ. We note that while the conclusion of Theorem 33 and of the subsequent corollary below are already in [TW01] for the case $\sigma = 0$, our approach provides an alternative proof even for $L\log^{1/2} L$. This approach has the advantage of being able to deal with all spaces $L\log^{(\sigma+1)/2} L$ at once, hence leading to the more general conclusion of Theorem 33.

As in the case $\sigma = 0$, Theorem 33 readily implies the continuous version below.

Corollary 34. *Let $J \subset \mathbb{R}$ be a finite interval, $\sigma \geqslant 0$ and $f \in L\log^{(\sigma+1)/2} L(J)$. Then for each $L \in \Lambda_1$ with $|L| \geqslant |J|^{-1}$ there exists a nonnegative function F_L such that for every $\gamma \geqslant 1$*

$$|\Delta_L f| \lesssim F_L * \varphi_{(|L||J|)^{-1}}, \qquad \Bigg\| \Big(\sum_{L\in\Lambda_1^{|J|^{-1}}} |F_L|^2 \Big)^{1/2} \Bigg\|_{L\log^{\sigma/2} L\left(\gamma J, \frac{\mathrm{d}x}{|J|}\right)} \lesssim_{\gamma,\sigma} \|f\|_{L\log^{(\sigma+1)/2} L\left(J, \frac{\mathrm{d}x}{|J|}\right)}.$$

If in addition $\int_J f = 0$ then the conclusion holds for all $L \in \Lambda_1$ with the summation extending over all $L \in \Lambda_1$. With or without this additional assumption, for $|L| \geqslant |J|^{-1}$ the functions F_L are supported in $5J$. The implicit constant depends only on γ and σ, as indicated.

C.1.4 Background and history

The fact that Marcinkiewicz multipliers are L^p-bounded is classical; see for example [Duo11, Theorem 8.13]. The first endpoint result concerning multiplier operators of Marcinkiewicz-type is arguably a theorem due to Bourgain [Bou89] which asserts that, in the periodic setting, the classical Littlewood–Paley square function LP_1 has operator norm $\|\mathrm{LP}_1\|_{p\to p} \simeq (p-1)^{-3/2}$ as $p \to 1^+$. Tao and Wright proved in [TW01] the optimal local endpoint estimate $L\log^{1/2} L \to L^{1,\infty}$ for the class of $R_2 = R_{2,1}$ multipliers, which contains Marcinkiewicz multipliers. It was later observed in [Bak19] that Bourgain's estimate follows by the endpoint bound of [TW01] combined with a randomization argument and Tao's converse extrapolation theorem from [Tao01]. Recently, Lerner proved in [Ler19] effective weighted bounds for the classical Littlewood–Paley square function LP_1; these weighted bounds imply the correct p-growth for the $L^p \to L^p$ norms of these operators as $p \to 1^+$. In addition, as observed in [Bak21], the arguments of [Ler19] can be used to establish weighted A_2 estimates for LP_τ that imply sharp $L^p \to L^p$ estimates for LP_τ as $p \to 1^+$ for any order τ. The class R_2 contains all multipliers m whose pieces m_L have bounded q-variation uniformly in $L \in \Lambda_1$, for all $1 \leqslant q < 2$; see [CRdFS88] where the authors showed that all R_2 multipliers are bounded on L^p for $p \in (1,\infty)$.

As already discussed, the authors in [TW01] rely on the weak square function characterization of $L\log^{1/2} L$ as in (C.1) for their proof. Our argument here is a bit different, relying on the weaker generalized Zygmund–Bonami inequality instead; a hint of a different proof already appears in [TW01, p. 540]. The Zygmund inequality first appeared in [Zyg30] in its dual form; see also [Zyg02, Theorem 7.6, Chapter XII]. The higher lacunarity order (C.6) is due to Bonami and it is contained in [Bon70]. We note that our results provide an alternative proof for the case of finite order lacunary sets. On the other hand, a dual version of the generalized Zygmund–Bonami inequality in the first order case (that is, inequality (C.4)) appears in [ST09].

The L^p-boundedness of Marcinkiewicz multipliers of order one and higher in the periodic setting was established by Marcinkiewicz in [Mar39]; see also Gaudry's paper [Gau78]. Generalized versions of Hörmander–Mihlin and Marcinkiewicz multipliers, together with their square function counterparts of higher order, have been introduced in [SS81] in a very broad context. There the authors proved the equivalence of L^p-boundedness between different classes of such multipliers. Our setup is focused on the finite order lacunary case and provides the optimal endpoint bounds for such classes.

C.1.5 Structure

The general structure of the rest of this paper is as follows. Section C.2 contains some basic facts and properties of Orlicz spaces, together with a small toolbox for dealing with lacunary sets; the reader is encouraged to skip this section on a first reading and only consult it when necessary. In Section C.3 we will prove Theorem 33 and Corollary 34. In Section C.4 we will critically use Corollary 34 in order to conclude the generalized Zygmund–Bonami inequality of arbitrary order alluded to above. This inequality will be stated and proved in different versions which can be local or non-local, depending on the type of cancellation assumptions we impose. The reader can find the corresponding statements in Propositions 37 and 38; see also Corollary 39. In Section C.5 we present the details of a Calderón–Zygmund decomposition for the Orlicz space $L\log^{\sigma/2} L$, adapted to the needs of this paper. The proof of Theorem 31 takes up the best part of Section C.6 where the Calderón–Zygmund decomposition of Section C.5 is combined with the generalized Zygmund–Bonami inequality of Section C.4. The proofs of Theorem 30 and Theorem 32 are discussed in Section C.6.2 as a variation of the proof of

Theorem 31.

C.2 Preliminaries and notation

In this section we collect several background definitions and notations that will be used throughout the paper.

C.2.1 Some basic facts for certain classes of Orlicz spaces

We adopt standard nomenclature for Young functions and Orlicz spaces as for example in [Wil08, Chapter 10]. Given a Young function $\Phi : [0,\infty] \to [0,\infty]$ we will use the following notation for local L^Φ averages: For a finite interval $I \subset \mathbb{R}$

$$\langle |f| \rangle_{\Phi,I} := \inf \left\{ \lambda > 0 : \frac{1}{|I|} \int_I \Phi\left(\frac{|f(x)|}{\lambda}\right) \mathrm{d}x \leqslant 1 \right\}.$$

For the usual local L^p averages we just set $\langle |f| \rangle_{p,I} := |I|^{-1/p} \|f\|_{L^p(I)}$ for $1 \leqslant p < \infty$. For $\sigma \geqslant 0$ we use the Young function $B_\sigma(t) := t(\log(e+t))^\sigma$ to define local $L\log^\sigma L$-spaces and we will also write

$$\|f\|_{L\log^\sigma L\left(I, \frac{\mathrm{d}x}{|I|}\right)} := \langle |f| \rangle_{B_\sigma, I} \simeq \frac{1}{|I|} \int_I |f(x)| \left(\log\left(e + \frac{|f(x)|}{\langle |f| \rangle_{1,I}} \right) \right)^\sigma \mathrm{d}x. \tag{C.8}$$

The last approximate equality can be found in [Wil08, Theorem 10.8]. For future reference it is worth noting that the function B_σ is submultiplicative and thus doubling; see [CUMP11, §5.2]. We will write instead $L^{B_\sigma}(\mathbb{R})$ to denote the (global) space of measurable functions f such that $\int_{\mathbb{R}} B_\sigma(|f|) < +\infty$.

The dual Young function of B_σ can be taken to coincide with $E_{\sigma^{-1}}(t) := \exp(c_\sigma t^{1/\sigma}) - 1$ for $t \gtrsim 1$; here we insist on the equality only for sufficiently large values of t; with this function we define the local $\exp(L^{1/\sigma})$ norms and we have the Hölder inequality $\langle |fg| \rangle_{1,I} \lesssim \langle |f| \rangle_{B_\sigma, I} \langle |f| \rangle_{E_{\sigma^{-1}}, I}$. We reserve the notation $L\log^\sigma L$ and $\exp(L^{1/\sigma})$ for the case $I = [0,1]$ and the space of functions supported in $[0,1]$ for which

$$\|f\|_{L\log^\sigma L} := \langle |f| \rangle_{B_\sigma,[0,1]} < +\infty, \qquad \|f\|_{\exp(L^{1/\sigma})} := \langle |f| \rangle_{E_{\sigma^{-1}},[0,1]} \simeq \sup_{p \geqslant 2} p^{-\sigma} \|f\|_p < +\infty,$$

respectively (see [Tri93, §2.2.4] for the last approximate equality); there holds $\exp(L^{1/\sigma}) \cong (L\log^\sigma L)^*$. For $\sigma = 0$ we adopt the convention that $L\log^\sigma L = L^1$ and $\exp(L^{1/\sigma}) = L^\infty$. The following Minkowski-type integral inequality

$$\left\| \{ \|f_k\|_{L\log^\sigma L} \} \right\|_{\ell^2_k} \lesssim \left\| \|\{f_k\}\|_{\ell^2_k} \right\|_{L\log^\sigma L}$$

will be used with no particular mention. Its proof can be obtained by a simple duality argument.

C.2.2 Some tools for handling lacunary sets

We introduce some useful notions concerning lacunary sets of arbitrary order. Let $\tau \geqslant 1$ and $L \in \Lambda_\tau$. We will denote by $\widehat{L}$ the unique interval $\widehat{L} \in \Lambda_{\tau-1}$ such that $L \subset \widehat{L}$ and call $\widehat{L}$ the (lacunary) *parent* of L. Furthermore we will denote by $\lambda(L)$ the unique element $\lambda \in \mathrm{lac}_{\tau-1}$ such that $\mathrm{dist}(L, \mathbb{R}\backslash\widehat{L}) = \mathrm{dist}(L,\lambda) = |L|$. We note that $\lambda(L)$ is one of the endpoints of $\widehat{L}$. These definitions also make sense in the case $\tau = 1$ remembering the definitions of Λ_0 and lac_0.

If $L \in \Lambda_\tau$ then $L^* := L - \lambda(L) \in \Lambda_1$; in fact L^* is one of the intervals $(-2|L|, -|L|)$ or $(|L|, 2|L|)$ depending on the original relative position of L with respect to $\lambda(L)$. The point of the definitions above is that if $L \in \Lambda_\tau$ then, upon fixing a suitable choice of bump functions $\phi_L \in \Phi_{L,M}$, we can write the identity

$$\Delta_L f = e^{2\pi i\lambda(L)\cdot}\Delta_{L^*}\left(e^{-2\pi i\lambda(L)\cdot}\Delta_{\widehat{L}} f\right) = e^{2\pi i\lambda(L)\cdot}\Delta_{L^*}\left(e^{-2\pi i\lambda(L)\cdot} f\right).$$

This will be crucially used in several parts of the recursive arguments in the paper. We will also use the intuitive notation $\Delta_{|L|} := \Delta_{L^*\cup(-L^*)}$ for the smooth Littlewood–Paley projection of first order at frequencies $|\xi| \simeq |L|$ which takes advantage of the fact that L^* essentially only depends on the length of L. The following notation will be useful to localize in a certain lacunary parent:

$$\Lambda^n_\tau(L') := \{L \in \Lambda^n_\tau :\ L \subset L'\}, \qquad L' \subset \mathbb{R};$$

similarly we define $\Lambda_\tau(L')$. Note that if $L' \in \Lambda_{\tau-1}$ and $L \in \Lambda_\tau(L')$ then necessarily $\widehat{L} = L'$.

The following simple lemma relies on the fact that lacunary sets are invariant under dyadic dilations with respect to the origin and will be used to allow rescaling of intervals of dyadic length to $[0,1]$.

Lemma 35. *Let $\tau \in \mathbb{N}$ and $a \in 2^{\mathbb{Z}}$. Then $a^{-1}\mathrm{lac}^a_\tau := \{\lambda/a :\ \lambda \in \mathrm{lac}^a_\tau\} = \mathrm{lac}^1_\tau$.*

To showcase the typical application of this lemma let $J \subset \mathbb{R}$ be an interval of dyadic length and $a = \{a_\lambda\}_{\lambda\in\mathrm{lac}_\tau}$ a finite collection of complex coefficients. By a standard change of variables

$$p_a(y) := \sum_{\lambda\in\mathrm{lac}^{|J|^{-1}}_\tau} a_\lambda e^{i\lambda y}, \qquad \langle|p_a|\rangle^p_{p,J} = \int_{[0,1]}\Bigg|\sum_{\lambda\in|J|\mathrm{lac}^{|J|^{-1}}_\tau} a_{\lambda|J|^{-1}} e^{i\lambda y}\Bigg|^p \mathrm{d}y, \tag{C.9}$$

and we crucially note that the sum on the right hand side is for $\lambda \in |J|\mathrm{lac}^{|J|^{-1}}_\tau = \mathrm{lac}^1_\tau$ because of the lemma. Of course the same change of variables will be valid for $\langle|p_a|\rangle_{\Phi,J}$ for any Young function Φ. We will use this rescaling argument in several places in the paper.

C.2.3 Other notation

For any function g and $\lambda > 0$ we write $g_\lambda(x) := \lambda^{-1}g(x/\lambda)$ for the L^1-rescaling. Two special kinds of bump functions will appear. Firstly $\omega(x) := (1+|x|^2)^{-N/2}$ is the smooth tailed indicator of $[-1/2, 1/2]$ with N any large positive integer. It will be enough to take $N = 10$ for the arguments in this paper but more decay is available if needed. We will also write $\varphi(x) := (1+|x|^2)^{-3/4}$ which is still an L^1-bump but has only moderate decay. In some cases we are restricted to using φ, most notably in the statement and proof of Corollary 34.

C.3 A weak square function characterization of $L\log^{\sigma/2} L$

In this section we provide the proof of Theorem 33 as a consequence of the Chang–Wilson–Wolff inequality of general order (C.7). The conclusion of Corollary 34 will then follow by a standard averaging argument using almost orthogonality between the continuous Littlewood–Paley projections and martingale differences.

C.3.1 Proof of Theorem 33

We recall that we work on the probability space $([0,1],\mathrm{d}x)$. It clearly suffices to prove the theorem for $k\geqslant 1$ as for $k=0$ we can set $f_0 := \mathbf{E}_0 f = \mathbf{D}_0 f$. Our starting point is the Chang–Wilson–Wolff inequality of general order of integrability, (C.7). This is pretty standard but a quick proof can be produced by using the usual Chang–Wilson–Wolff inequality (C.3) in the form

$$p^{-\sigma/2}\|f-\mathbf{E}_0 f\|_p \lesssim p^{-\sigma/2}p^{1/2}\|S_{\mathcal{M}}f\|_p,\qquad p\geqslant 2,\quad \sigma\geqslant 0,$$

which readily implies

$$\|f-\mathbf{E}_0 f\|_{\exp(L^{2/(\sigma+1)})} \simeq \sup_{p\geqslant 2}\frac{\|f-\mathbf{E}_0 f\|_p}{p^{(\sigma+1)/2}} \lesssim \sup_{p\geqslant 2}\frac{\|S_{\mathcal{M}}f\|_p}{p^{\sigma/2}} \simeq \|S_{\mathcal{M}}f\|_{\exp(L^{2/\sigma})}$$

which is (C.7). Observe that (C.7) has the form

$$\Big\|\sum_{k\geqslant 1} g_k\Big\|_{\exp(L^{2/(\sigma+1)})} \lesssim \Big\|\Big(\sum_{k\geqslant 1}|g_k|^2\Big)^{1/2}\Big\|_{\exp(L^{2/\sigma})},\qquad g_k=\mathbf{D}_k f. \tag{C.10}$$

We will write (C.10) as a continuity property for the operator $\mathrm{T}(\{g_k\}_k) := \sum_k g_k$ between suitable Banach spaces. To that end let us consider the subspace of $L\log^{\sigma/2} L([0,1];\ell^2)$ given by

$$Y := \Big\{\{\psi_k\}_k \in L\log^{\sigma/2} L([0,1];\ell^2) : \mathbf{D}_k\psi_k = 0 \ \text{ for all } \ k\in\mathbb{N}\Big\}.$$

We observe that Y is closed. To see this consider a sequence $(\psi^n)_n\subset Y$ with $\psi^n=\{\psi^n_k\}_k$ converging to some $\psi=\{\psi_k\}_k$ in $L\log L^{\sigma/2}L([0,1];\ell^2)$. Clearly the limit ψ belongs to $L\log L^{\sigma/2}L([0,1];\ell^2)$, the latter being a Banach space and, additionally, ψ^n_k converges to ψ_k in $L\log^{\sigma/2} L([0,1])$ and so also in $L^1([0,1])$, uniformly in k. Now it follows by Fatou's lemma that for each $k\in\mathbb{N}$ there holds

$$\Big\|\liminf_{n\to\infty}|\mathbf{D}_k(\psi^n_k-\psi_k)|\Big\|_{L^1([0,1])} \leqslant \liminf_{n\to\infty}\|\psi_k-\psi^n_k\|_{L^1([0,1])}=0$$

yielding $\mathbf{D}_k\psi^n_k=\mathbf{D}_k\psi_k=0$ a.e., where we also used the uniform boundedness of $\mathbf{D}_k$ on $L^1([0,1])$.

Since $\big(L\log^{\sigma/2} L([0,1];\ell^2)\big)^* \cong \exp(L^{2/\sigma})([0,1];\ell^2)$, the annihilator of Y is given equivalently by

$$Y^{\perp}=\left\{\{g_k\}_k\in\exp(L^{2/\sigma})([0,1];\ell^2) : \int\Big(\sum_k g_k\psi_k\Big)=0 \ \text{ for all } \ \{\psi_k\}_k\in Y\right\}.$$

Since Y is a closed subspace of $L\log^{\sigma/2} L([0,1];\ell^2)$ we have that $(L\log^{\sigma/2} L([0,1];\ell^2)/Y)^*$ is isometrically isomorphic to $Y^\perp$; see [Rud91, Theorem 4.9]. We equip $Y^\perp$ with the norm appearing on the right hand side of (C.10). We will use the following fact.

Lemma 36. *If $\{g_k\}_k \in Y^\perp$ then $\mathbf{D}_k g_k = g_k$ for every $k \in \mathbb{N}$.*

Proof. Fix an index $k_0 \in \mathbb{N}$, let $\psi \in L\log^{\sigma/2} L$ be arbitrary and let $\{\psi_k\}_k$ be defined by

$$\psi_k := \begin{cases} 0 & \text{if } k \neq k_0, \\ \psi - \mathbf{D}_{k_0}\psi & \text{otherwise.} \end{cases}$$

Clearly $\{\psi_k\}_k \in L\log^{\sigma/2} L([0,1];\ell^2)$ and, moreover, $\mathbf{D}_{k_0}(\psi - \mathbf{D}_{k_0}\psi) = 0$ so that $\{\psi_k\}_k \in Y$. By the definition of $Y^\perp$ we have then

$$0 = \int \sum_k g_k\psi_k = \int g_{k_0}(\psi - \mathbf{D}_{k_0}\psi) = \int (g_{k_0} - \mathbf{D}_{k_0} g_{k_0})\psi$$

where we have used the fact that $\mathbf{D}_k$ is self-adjoint; but this is only possible for arbitrary ψ if $\mathbf{D}_{k_0} g_{k_0} = g_{k_0}$, as claimed. ∎

Now (C.10) can be written in the form

$$\|\mathrm{T}(\{g_k\}_k)\|_{\exp(L^{2/(\sigma+1)})} \lesssim \|\{g_k\}_k\|_{Y^\perp}, \qquad \mathrm{T}(\{g_k\}_k) := \sum_k g_k. \tag{C.11}$$

Let $X := L\log^{\sigma/2} L([0,1];\ell^2)$ and denote by X_N, Y_N the functions in X, Y, respectively, which are constant on dyadic intervals of length smaller than 2^{-N}. In particular, such functions f have finite Haar expansion which implies the apriori qualitative property that the spaces X_N, Y_N are finite dimensional. We note that $(X_N/Y_N)^*$ is isometrically isomorphic to $Y_N^\perp$, with Z^* denoting the dual of the finite-dimensional vector space Z. By the Riesz representation theorem we then get that

$$\|\{\mathbf{D}_k f\}_k\|_{X_N/Y_N} \leqslant \sup_{\substack{\{g_k\}_k \in Y_N^\perp: \\ \|\{g_k\}_k\|_{Y_N^\perp} \leqslant 1}} \left| \int \sum_k g_k \mathbf{D}_k f \right| = \sup_{\substack{\{g_k\}_k \in Y_N^\perp: \\ \|\{g_k\}_k\|_{Y_N^\perp} \leqslant 1}} \left| \int \mathrm{T}(\{g_k\}_k) f \right|,$$

where we also used Lemma 36 in passing to the equality in the right hand side above. Using (C.11) together with Hölder's inequality in Orlicz spaces it follows that

$$\|\{\mathbf{D}_k f_N\}_k\|_{X/Y} = \inf_{\{\psi_k\}_k \in Y} \left\| \left(\sum_k |\mathbf{D}_k f_N + \psi_k|^2 \right)^{1/2} \right\|_{L\log^{\sigma/2} L} \lesssim \|f_N\|_{L\log^{(\sigma+1)/2} L},$$

where f_N is the truncation of the Haar series of $f \in X$ at scale 2^{-N}. We stress that the approximate inequality above holds uniformly for all $N \in \mathbb{N}$. This inequality extends to all $f \in L\log^{(\sigma+1)/2} L$ by a standard approximation argument, using the fact that the truncated Haar series of functions $f \in f \in L\log^{(\sigma+1)/2} L$ converge to f in $L\log^{(\sigma+1)/2} L$; see [Osw83]. The extension of the operator $f \mapsto \{\mathbf{D}_k f\}_k$ is the obvious one given by the same expression.

In order to conclude the proof of the theorem we notice that for every $\{\psi_k\}_k \in Y$ there holds $\mathbf{D}_k f = \mathbf{D}_k(\mathbf{D}_k f + \psi_k)$ and the last inequality guarantees the existence of a vector $\{\psi_k\}_k \in Y$ such that the functions $f_k := \mathbf{D}_k f + \psi_k$ satisfy the conclusion of the theorem.

C.3.2 Proof of Corollary 34

The corollary follows from the dyadic case of Theorem 33 via an averaging argument which is essentially identical to the one in [TW01, §9]; see also [Vit19] where the argument in [TW01, §9] is explained in detail. We can clearly assume that $\gamma \geqslant 3$ and by affine invariance we can take $\gamma J = [0,1]$, so that $|J| = \gamma^{-1}$, and $\operatorname{supp} f \subset [1/3, 2/3]$.

For all $\theta \in [-1/3, 1/3]$ we define $f_\theta(x) := f(x-\theta)$. By Theorem 33, for each $\theta \in [-1/3, 1/3]$ and $k \geqslant 0$ there exists a function $f_{\theta,k}$ such that $\mathbf{D}_k f_\theta = \mathbf{D}_k f_{\theta,k}$ and

$$\left\| \left(\sum_{k\geqslant 0} |f_{\theta,k}|^2 \right)^{1/2} \right\|_{L\log^{\sigma/2} L} \lesssim \|f_\theta\|_{L\log^{(\sigma+1)/2} L} = \|f\|_{L\log^{(\sigma+1)/2} L}. \tag{C.12}$$

Setting for $L \in \Lambda_1^\gamma$

$$F_L(x) := \sum_{k\in\mathbb{N}_0} 2^{-|\log_2 |L| - k|/2} \int_{[-1/3,1/3]} |f_{\theta,k}(x+\theta)| \,\mathrm{d}\theta$$

and arguing as in [TW01] we see that $|\Delta_L f(x)| \lesssim F_L * \varphi_{|L|^{-1}}$ and

$$\left(\sum_{L\in\Lambda_1^\gamma} |F_L(x)|^2 \right)^{1/2} \lesssim \int_{[-1/3,1/3]} \left(\sum_{k\geqslant 0} |f_{\theta,k}(x+\theta)|^2 \right)^{1/2} \mathrm{d}\theta.$$

By the Minkowski integral inequality for the space $L\log^{\sigma/2} L$ we have

$$\left\| \left(\sum_{L\in\Lambda_1^\gamma} |F_L|^2 \right)^{1/2} \right\|_{L\log^{\sigma/2} L} \lesssim \int_{[-1/3,1/3]} \left\| \left(\sum_{k\geqslant 0} |f_{\theta,k}(\cdot+\theta)|^2 \right)^{1/2} \right\|_{L\log^{\sigma/2} L} \mathrm{d}\theta$$

and the proof follows for $L \in \Lambda_1^\gamma$. Under the additional cancellation assumption $\int_{[0,1]} f = 0$ we consider also $L \in \Lambda_1$ with $|L| < \gamma$, and for these we define $F_L := |\Delta_L f|$ and note that $|\Delta_L f| \lesssim \varphi_{|L|^{-1}} * F_L$. Using the cancellation condition we have also

$$|\Delta_L f| \lesssim \varphi_{|L|^{-1}} * (|L| \|f\|_{L^1} \mathbf{1}_{[0,1]}),$$

which readily yields the estimate $\big\| \|\{F_L\}_{|L|<\gamma}\|_{\ell^2} \big\|_{L^1} \lesssim \|f\|_{L^1}$ and the proof is complete. Note that we used that since $\tau = 1$ there are at most two intervals $L \in \Lambda_1$ of any given length.

C.4 Generalized Zygmund–Bonami inequalities

In this section we prove the versions of the generalized Zygmund–Bonami inequality presented in the introduction, where the Littlewood–Paley projections Δ_L for $L \in \Lambda_1$ are replaced by their τ-order counterparts Δ_L for $L \in \Lambda_\tau$, where τ is an arbitrary positive integer. As already discussed, the estimate corresponding to order $\tau = 1$ is (C.4) and it follows rather easily from the case $\sigma = 0$ of Corollary 34. For $\tau > 1$ we first state the generalized Zygmund–Bonami inequalities in the case of $L \in \Lambda_\tau$ with $|L| \geqslant |J|^{-1}$, where J is an interval in which f is supported; this is the harder and deeper case. In the rest of the section we will also provide the statements and proofs for the easier case $|L| < |J|^{-1}$; the latter will rely on pointwise estimates for $\Delta_L f$ and recursive arguments, assuming suitable cancellation conditions for f in the same spirit as Corollary 34.

C.4.1 The main term in the generalized Zygmund–Bonami inequalities

We encourage the reader to keep in mind the notation of §C.2 for the local Orlicz norms and the definitions concerning lacunary sets from §C.2.2 for the rest of this section. Our first result below gives a version of the generalized Zygmund–Bonami inequality in which the intervals L are restricted to those for which $|L| \geqslant |J|^{-1}$, as anticipated above.

Proposition 37. *Let $J \subset \mathbb{R}$ be a finite interval and f be a compactly supported function with* $\operatorname{supp}(f) \subseteq J$. *Let τ be a positive integer, σ a nonnegative integer and $\gamma > 1$. There holds*

$$\left(\sum_{L \in \Lambda_\tau^{|J|^{-1}}} \langle |\Delta_L f| \rangle^2_{B_{\sigma/2},\gamma J} \right)^{1/2} \lesssim_{\sigma,\tau,\gamma} \langle |f| \rangle_{B_{(\sigma+\tau)/2},J},$$

$$\sum_{L \in \Lambda_\tau^{|J|^{-1}}} \|\Delta_L f\|^2_{L^2(\mathbb{R}\setminus\gamma J)} \lesssim_{\tau,\gamma} |J| \langle |f| \rangle^2_{B_{(\tau-1)/2},J}.$$

Proof. The proof is by way of induction on τ, with the base case $\tau = 1$ being an easy consequence of Corollary 34, as we shall now illustrate. Indeed, let $C_1(\sigma,\tau,\gamma)$ and $C_2(\tau,\gamma)$ denote the best constants in the first, and the second estimate in the statement, respectively. Corollary 34 implies that for $L \in \Lambda_1^{|J|^{-1}}$ we have

$$\langle |\Delta_L f| \rangle_{B_{\sigma/2},\gamma J} \lesssim \langle |\varphi_{(|L||J|)^{-1}} * F_L| \rangle_{B_{\sigma/2},\gamma J} \lesssim \langle |F_L| \rangle_{B_{\sigma/2},\gamma J},$$

using Young's convolution inequality and the L^1-normalization of each $\varphi_{(|L||J|)^{-1}}$. Now the proof of the first estimate in the conclusion for $\tau = 1$ can be concluded by yet another application of Minkowski's inequality, this time to yield that the left hand side of the first estimate in the conclusion is bounded by a constant multiple of

$$\left(\sum_{|L| \geqslant |J|^{-1}} \langle |F_L| \rangle^2_{B_{\sigma/2},\gamma J} \right)^{1/2} \lesssim \left\langle \left(\sum_{|L| \geqslant |J|^{-1}} |F_L|^2 \right)^{1/2} \right\rangle_{B_{\sigma/2},\gamma J} \lesssim \langle |f| \rangle_{B_{(\sigma+1)/2},J}$$

by the estimate for the square function of the $\{F_L\}_L$ in Corollary 34. Thus $C_1(\sigma,1,\gamma) < +\infty$ for all nonnegative integers σ and $\gamma > 1$.

For the second estimate we have for $|L| \geqslant |J|^{-1}$ and $x \in \mathbb{R}\setminus\gamma J$

$$|\Delta_L f(x)| \lesssim |L||J|(1 + |L||x - c_J|)^{-10} \langle |f| \rangle_{1,J} \lesssim \omega_{|L|^{-1}} * (\langle |f| \rangle_{1,J} \mathbf{1}_J)(x), \tag{C.13}$$

with $\omega_{|L|^{-1}}$ as given in §C.2. Using the first approximate inequality above, the square of the left hand side of the second estimate in the conclusion of the proposition can be estimated by a constant multiple of

$$\sum_{L\in\Lambda_1^{|J|^{-1}}} \langle |f|\rangle_{1,J}^2 (|L||J|)^2|L|^{-20}\int_{\mathbb{R}\setminus\gamma J}|x-c_J|^{-20}\,\mathrm{d}x \lesssim \langle |f|\rangle_{1,J}^2 \sum_{L\in\Lambda_1:\,|L||J|\geqslant 1} (|L||J|)^{-18}|J|$$

which sums to the desired quantity since, for $\tau=1$, there is exactly one interval $L\in\Lambda_1$ per dyadic scale. This shows that $C_2(1,\gamma)<+\infty$ for all $\gamma>1$.

Consider now the case $\tau>1$ and let $a>1$ be such that $\gamma=a^\tau$. Recalling the discussion in §C.2.2 we write

$$\begin{aligned}|\Delta_L f| &\leqslant \left|\Delta_{L^*}\left(\mathbf{1}_{a^{\tau-1}J}e^{-2\pi i\lambda(L)\cdot}\Delta_{\widehat{L}}f\right)\right| + \left|\Delta_{L^*}\left(\mathbf{1}_{\mathbb{R}\setminus a^{\tau-1}J}e^{-2\pi i\lambda(L)\cdot}\Delta_{\widehat{L}}f\right)\right| \\ &=: |\Delta_{L^*}f_{1,L}| + |\Delta_{L^*}f_{2,L}|.\end{aligned} \tag{C.14}$$

For clarity, we remind the reader that $\lambda(L)$ is either of the endpoints of $\widehat{L}$ (depending on the position of L) and therefore we can partition the intervals L into two families such that $f_{1,L}, f_{2,L}$ *actually depend only on* $\widehat{L}$ – this will be relevant below. Fixing for a moment $L'\in\Lambda_{\tau-1}^{|J|^{-1}}$ we note that for any $L\in\Lambda_\tau^{|J|^{-1}}(L')$ there holds $\widehat{L}=L'$ and so $|f_{1,L}|=|\Delta_{\widehat{L}}f|\mathbf{1}_{a^{\tau-1}J}=|(\Delta_{L'}f)\mathbf{1}_{a^{\tau-1}J}|$. As the collection $\{L^*:\ L\in\Lambda_{\tau-1}^{|J|^{-1}}(L')\}\subset\Lambda_1^{|J|^{-1}}$ we can use the conclusion of proposition for $\tau=1$ to estimate for fixed $L'\in\Lambda_{\tau-1}^{|J|^{-1}}$

$$\left(\sum_{L\in\Lambda_\tau^{|J|^{-1}}(L')}\langle|\Delta_{L^*}f_{1,L}|\rangle_{B_{\sigma/2},a^\tau J}^2\right)^{1/2} \leqslant C_1(\sigma,1,a)\,\langle|\Delta_{L'}f|\rangle_{B_{(\sigma+1)/2},a^{\tau-1}J}\,.$$

Thus we can recursively estimate

$$\begin{aligned}\left(\sum_{L'\in\Lambda_{\tau-1}^{|J|^{-1}}}\ \sum_{L\in\Lambda_\tau^{|J|^{-1}}(L')}\langle|\Delta_{L^*}f_{1,L}|\rangle_{B_{\sigma/2},a^\tau J}^2\right)^{1/2} &\leqslant C_1(\sigma,1,a)\left(\sum_{L'\in\Lambda_{\tau-1}^{|J|^{-1}}}\langle|\Delta_{L'}f|\rangle_{B_{(\sigma+1)/2},a^{\tau-1}J}^2\right)^{\frac{1}{2}} \\ &\leqslant C_1(\sigma,1,a)C_1(\sigma+1,\tau-1,a^{\tau-1})\langle|f|\rangle_{B_{(\tau+\sigma)/2},J}\end{aligned}$$

which takes care of the contribution of the $f_{1,L}$'s. Considering now the $f_{2,L}$'s, we have by Hölder's inequality for Orlicz spaces that $\langle|\Delta_{L^*}f_{2,L}|\rangle_{B_{\sigma/2},a^\tau J}\leqslant\langle|\Delta_{L^*}f_{2,L}|\rangle_{2,a^\tau J}$ and therefore for any fixed $L'\in\Lambda_{\tau-1}^{|J|^{-1}}$

$$\sum_{L\in\Lambda_\tau^{|J|^{-1}}(L')}\langle|\Delta_{L^*}f_{2,L}|\rangle_{B_{\sigma/2},a^\tau J}^2 \lesssim \frac{1}{a^\tau|J|}\int_{\mathbb{R}}|f_{2,L}|^2 = \frac{1}{a^\tau|J|}\int_{\mathbb{R}\setminus a^{\tau-1}J}|\Delta_{L'}f|^2,$$

where we have used the $L^2\to L^2$ boundedness of the smooth Littlewood–Paley square function together with the fact remarked above that $f_{2,L}$ depends essentially only on $\widehat{L}=L'$. It follows that we can bound recursively

$$\left(\sum_{L'\in\Lambda_{\tau-1}^{|J|^{-1}}}\sum_{\substack{L\in\Lambda_\tau(L')\\ |L|\geqslant|J|^{-1}}}\langle|\Delta_{L^*}f_{2,L}|\rangle_{B_{\sigma/2},a^\tau J}^2\right)^{1/2} \lesssim_\sigma \left(\frac{1}{a^\tau|J|}\sum_{L'\in\Lambda_{\tau-1}^{|J|^{-1}}}\int_{\mathbb{R}\backslash a^{\tau-1}J}|\Delta_{L'}f|^2\right)^{1/2}$$
$$\leqslant \left(\frac{C_2(\tau-1,a^{\tau-1})}{a^\tau|J|}|J|\langle|f|\rangle_{B_{(\tau-2)/2},J}^2\right)^{1/2}$$
$$\lesssim_{\sigma,\tau} (C_2(\tau-1,a^{\tau-1})/a^\tau)^{1/2}\langle|f|\rangle_{B_{(\sigma+\tau)/2},J}.$$

This proves that

$$C_1(\sigma,\tau,a^\tau)\leqslant C_1(\sigma,1,a)C_1(\sigma+1,\tau-1,a^{\tau-1})+c_{\sigma,\tau}C_2(\tau-1,a^{\tau-1})^{1/2}a^{-\tau/2}$$

for some numerical constant $c_{\sigma,\tau}$ depending only on σ,τ.

We move to the proof of the inductive step for the L^2-estimate and we use again the splitting of (C.14). For the term corresponding to the $f_{1,L}$'s we can estimate again recursively

$$\sum_{L'\in\Lambda_{\tau-1}^{|J|^{-1}}}\sum_{L\in\Lambda_\tau^{|J|^{-1}}(L')}\|\Delta_{L^*}f_{1,L}\|_{L^2(\mathbb{R}\backslash a^\tau J)}^2 \leqslant C_2(1,a)|a^{\tau-1}J|\sum_{L'\in\Lambda_{\tau-1}^{|J|^{-1}}}\langle|\Delta_{L'}f|\rangle_{1,a^{\tau-1}J}^2$$
$$\leqslant C_2(1,a)C_1(0,\tau-1,a^{\tau-1})a^{\tau-1}|J|\,\langle|f|\rangle_{B_{(\tau-1)/2},J}^2\,.$$

Finally, for the contribution of the $f_{2,L}$'s we use again the $L^2\to L^2$ boundedness of the smooth Littlewood–Paley square function and the inductive hypothesis to estimate

$$\sum_{L'\in\Lambda_{\tau-1}^{|J|^{-1}}}\sum_{L\in\Lambda_\tau^{|J|^{-1}}(L')}\|\Delta_{L^*}f_{2,L}\|_{L^2(\mathbb{R}\backslash a^\tau J)}^2 \lesssim \sum_{L'\in\Lambda_{\tau-1}^{|J|^{-1}}}\|\Delta_{L'}f\|_{L^2(\mathbb{R}\backslash a^{\tau-1}J)}^2$$
$$\leqslant C_2(\tau-1,a^{\tau-1})|J|\,\langle|f|\rangle_{B_{(\tau-1)/2},J}^2\,.$$

We have thus shown that for some numerical constant $c'_{\sigma,\tau}$

$$C_2(\tau,a^\tau)\leqslant C_2(1,a)C_1(0,\tau-1,a^{\tau-1})a^{\tau-1}+c'_{\sigma,\tau}C_2(\tau-1,a^{\tau-1}).$$

This completes the proof of the inductive step and with that the proof of the proposition. ∎

Remark 1. *The first estimate in Proposition 37 implies the Zygmund-Bonami inequality of general order. Indeed assume for a moment that $J=[0,1]$ and for $\lambda\in\mathrm{lac}_\tau^1$ let $L_\lambda\in\Lambda_\tau$ be an interval that has λ as an endpoint. We have*

$$\left|\widehat{f}(\lambda)\right|=\left|\widehat{\Delta_{L_\lambda}(f)}(\lambda)\right|\leqslant\|\Delta_{L_\lambda}(f)\|_{L^1}$$

for a suitable choice of symbol in the definition of Δ_L and so the first estimate of the proposition for $\sigma=0$ implies

$$\left(\sum_{\lambda\in\mathrm{lac}_\tau^1}\left|\widehat{f}(\lambda)\right|^2\right)^{\frac{1}{2}}\lesssim\|f\|_{L\log^{\tau/2}L}$$

which is the Zygmund–Bonami inequality of lacunary order τ. Dualizing and rescaling as in (C.9) *tells us that for any finite interval $J \subset \mathbb{R}$ with $|J| \in 2^{\mathbb{Z}}$ we have*

$$p(y) := \sum_{\lambda \in \mathrm{lac}_\tau^{|J|^{-1}}} a_\lambda e^{-2\pi i \lambda y}, \qquad \|\{a_\lambda\}_\lambda\|_{\ell^2} = 1 \implies \langle |p| \rangle_{E_{2/\tau},J} \lesssim 1. \tag{C.15}$$

This formulation of the higher order Zygmund–Bonami inequality will be used in several points in the rest of the paper. As it follows from Proposition 37 above this makes the proofs in the paper somewhat self contained.

Remark 2. *One can easily verify that the L^2-estimate of Proposition 37 can be upgraded to the following form for "molecules". Let $\mathcal{J}$ be a family of pairwise disjoint intervals and $f = \sum_{J \in \mathcal{J}} b_J$ where* $\mathrm{supp}(b_J) \subset J$ *for each $J \in \mathcal{J}$. For every positive integer τ and $\gamma \geqslant 2$ there holds*

$$\sum_{L \in \Lambda_\tau} \Bigg\| \sum_{J:\, |J| \geqslant |L|^{-1}} \Delta_L(b_J) \mathbf{1}_{\mathbb{R} \setminus \gamma J} \Bigg\|_{L^2(\mathbb{R})}^2 \lesssim \sum_{J \in \mathcal{J}} |J| \langle |b_J| \rangle_{B_{(\tau-1)/2},J}^2.$$

Indeed an inductive proof is again available. The case $\tau = 1$ follows by the same pointwise estimate (C.13) *which implies that*

$$\sum_{J:\, |J| \geqslant |L|^{-1}} |\Delta_L(b_J)| \mathbf{1}_{\mathbb{R} \setminus \gamma J} \lesssim \omega_{|L|^{-1}} * \left(\sum_{|J| \geqslant |L|^{-1}} \langle |b_J| \rangle_{1,J} \frac{\mathbf{1}_J}{(|L||J|)^5} \right)$$

which sums using that there are at most two intervals $L \in \Lambda_1$ of any given length. The inductive step relies again on the identity (C.14)*, applied to each b_J in place of f. Then the contribution of the first term is estimated by an appeal to the case $\tau = 1$ followed by an application of the first estimate in Proposition 37. The contribution of the second term in* (C.14) *is estimated by the Littlewood–Paley inequalities and the inductive hypothesis. We omit the details.*

We proceed to prove the easier range, corresponding to $|L| < |J|^{-1}$. As in Corollary 34 we require cancellation conditions, which in the case at hand amount to vanishing Fourier coefficients of the function at lacunary frequencies corresponding to order $\tau - 1$. In this range we can prove the stronger L^2 inequality that follows. For simplicity we state the result below for f with support of dyadic length, but it is obvious that this is no real restriction.

Proposition 38. *Let $J \subset \mathbb{R}$ be a finite interval of dyadic length and f be a compactly supported function with* $\mathrm{supp}(f) \subseteq J$. *Let τ be a positive integer. We assume that $\widehat{f}(\lambda) = 0$ for all $\lambda \in \mathrm{lac}_0^{|J|^{-1}} \cup \cdots \cup \mathrm{lac}_{\tau-1}^{|J|^{-1}}$. Then*

$$\sum_{\substack{L \in \Lambda_\tau: \\ |L| < |J|^{-1}}} \|\Delta_L f\|_{L^2(\mathbb{R})}^2 \lesssim |J| \langle |f| \rangle_{B_{(\tau-1)/2},J}^2.$$

Proof. We argue by induction as in the proof of Proposition 37. Let us denote by $C(\tau)$ the best constant in the inequality we intend to prove. Note that for $\tau = 1$ the assumption reads $\int_J f = 0$ and the conclusion $C(1) < +\infty$ follows immediately by the pointwise estimate

$$|\Delta_L(f)(x)| \lesssim_M |L|^2|J|^2(1+|L||x-c_J|)^{-M}\langle|f|\rangle_{1,J} \lesssim \omega_{|L|^{-1}} * (|L||J|\langle|f|\rangle_{1,J}\mathbf{1}_J) \qquad \text{(C.16)}$$

for any large positive integer M, where c_J is the center of J and $|L||J| < 1$. In order to see the first estimate above let us take $\phi_L \in \Phi_{L,M}$ to be the symbol of Δ_L which can be written in the form $\phi_L(x) = e^{i2\pi i c_L x}|L|\phi(|L|x)$, with c_L denoting the center of $L \in \Lambda_1$. We compute using the cancellation of f and the mean value theorem

$$|\Delta_L(f)(x)| \leqslant \int_J |\phi_L(x-y) - \phi_L(x-c_J)|\,|f(y)|\,\mathrm{d}y \lesssim \int_J |J| \sup_{z\in J} |\phi_L'(x-z)||f(y)|\,\mathrm{d}y.$$

Using that $|c_L| \simeq \operatorname{dist}(L,0) \simeq |L|$ we have for $z \in J$

$$\begin{aligned}|\phi_L'(x-z)| &\lesssim c_L|L|\phi(|L|(x-z)) + |L|^2\phi(|L|(x-z)) \lesssim |L|^2\left(1+|L||x-z|\right)^{-M} \\ &\simeq |L|^2\left(1+|L||x-c_J|\right)^{-M}.\end{aligned}$$

The last approximate equality can be checked by considering the cases $x \in 3J$ and $x \notin 3J$ separately, remembering that $|L||J| < 1$. The combination of the last two displays yields the first estimate in (C.16). The second estimate in (C.16) follows by the first since $(1+|L||x-c_J|)^{-M} \simeq (1+|L||x-z|)^{-M}$ for $z \in J \supseteq \operatorname{supp}(f)$.

For $\tau > 1$ we first do the same reduction as in the proof of Proposition 37. For $\tau > 1$ we can estimate

$$\begin{aligned}\sum_{\substack{L\in\Lambda_\tau\\|L|<|J|^{-1}}} \|\Delta_L f\|_{L^2(\mathbb{R})}^2 \leqslant &\sum_{\substack{L'\in\Lambda_{\tau-1}\\|L'|<|J|^{-1}}}\sum_{L\in\Lambda_\tau(L')} \left\|\Delta_{L^*}\left(e^{-2\pi i\lambda(L)\cdot}\Delta_{L'}f\right)\right\|_{L^2(\mathbb{R})}^2 \\ &+ \sum_{L'\in\Lambda_{\tau-1}^{|J|^{-1}}}\sum_{\substack{L\in\Lambda_\tau(L')\\|L|<|J|^{-1}}} \left\|\Delta_{L^*}\left(e^{-2\pi i\lambda(L)\cdot}f\right)\right\|_{L^2(\mathbb{R})}^2.\end{aligned} \qquad \text{(C.17)}$$

The first summand above is estimated by $\lesssim C(\tau-1)|J|\langle|f|\rangle_{B_{(\tau-2)/2},J}^2$ by using the L^2-bound for the smooth Littlewood–Paley square function and the inductive hypothesis.

The second summand above can be estimated by

$$\begin{aligned}\sum_{\ell:\,2^\ell<|J|^{-1}}\sum_{L'\in\Lambda_{\tau-1}^{|J|^{-1}}}\sum_{\substack{L\in\Lambda_\tau(L')\\|L|=2^\ell}} \left\|\Delta_{2^\ell}\left(e^{-2\pi i\lambda(L)\cdot}f\right)\right\|_{L^2(\mathbb{R})}^2 &= \sum_{\ell:\,2^\ell<|J|^{-1}}\sum_{\lambda\in\mathrm{lac}_{\tau-1}^{|J|^{-1}}}\sum_{\substack{L\in\Lambda_\tau\\ \lambda(L)=\lambda\\|L|=2^\ell}} \left\|\Delta_{2^\ell}\left(e^{-2\pi i\lambda\cdot}f\right)\right\|_{L^2(\mathbb{R})}^2 \\ &\lesssim \sum_{\ell:\,2^\ell<|J|^{-1}}\sum_{\lambda\in\mathrm{lac}_{\tau-1}^{|J|^{-1}}} \left\|\Delta_{2^\ell}\left(e^{-2\pi i\lambda\cdot}f\right)\right\|_{L^2(\mathbb{R})}^2\end{aligned}$$

where, in passing to the last line, we used that for each $\lambda \in \mathrm{lac}_{\tau-1}$ there are at most $O_\tau(1)$ intervals $L \in \Lambda_\tau$ of fixed length with $\lambda(L) = \lambda$. Fixing for a moment $2^\ell < |J|^{-1}$ and $x \in \mathbb{R}$ we write

$$\sum_{\lambda\in \mathrm{lac}_{\tau-1}^{|J|-1}} \left|\Delta_{2^\ell}\left(e^{-2\pi i\lambda\cdot}f\right)(x)\right|^2 = \left|\Delta_{2^\ell}\left(\sum_{\lambda\in \mathrm{lac}_{\tau-1}^{|J|-1}} a_\lambda e^{-2\pi i\lambda\cdot}f\right)(x)\right|^2 =: \left|\Delta_{2^\ell}\left(p_{x,\ell}f\right)(x)\right|^2$$

where $\{a_\lambda\}_\lambda = \{a_\lambda(x,\ell)\}_\lambda$ is in the unit ball of ℓ^2_λ and $p_{x,\ell}$, implicitly defined above, is as in (C.9). Using the cancellation assumptions on f we see that $\int_J p_{x,\ell}f = 0$ so by appealing to (C.16) we get

$$|\Delta_{2^\ell}(fp_{x,k_\tau})| \lesssim \omega_{2^{-\ell}} * \left(2^\ell|J|\langle|fp_{x,\ell}|\rangle_{1,J}\mathbf{1}_J\right) \lesssim \omega_{2^{-\ell}} * \left(2^\ell|J|\langle|f|\rangle_{B_{(\tau-1)/2},J}\mathbf{1}_J\right)$$

where we used the Hölder inequality in Orlicz spaces together with the Zygmund–Bonami inequality of order $\tau-1$ from Remark 1 to control $\langle|p_{x,\ell}|\rangle_{E_{2/(\tau-1)}} \lesssim 1$. Squaring the estimate in the last display, integrating, and then summing for $2^\ell < |J|^{-1}$ yields that the second summand in (C.17) is controlled by a constant multiple of $|J|\langle|f|\rangle^2_{B_{(\tau-1)/2},J}$. We have proved that $C(\tau) \lesssim (1 + C(\tau-1))$ and this concludes the proof of the inductive step and of the proposition. ∎

Remark 3. *As in Remark 2 there is an upgrade of the L^2-estimate of Proposition 38 from "atoms" to "molecules" $f = \sum_{J\in\mathcal{J}} b_J$ where $\mathcal{J}$ is a family of pairwise disjoint dyadic intervals and each b_J satisfies the cancellation assumptions of Proposition 38, namely*

$$\sum_{L\in\Lambda_\tau}\left\|\sum_{J\colon |J|<|L|^{-1}} \Delta_L(b_J)\right\|^2_{L^2(\mathbb{R})} \lesssim \sum_{J\in\mathcal{J}} |J|\langle b_J\rangle^2_{B_{(\tau-1)/2},J}.$$

The base case $\tau = 1$ is essentially identical to the corresponding step in the proof of Proposition 38 relying on the pointwise estimate

$$\left|\Delta_L\left(\sum_{J\colon |L|<|J|^{-1}} b_J\right)\right| \lesssim \omega_{|L|^{-1}} * \left(|L|\sum_{J\colon |L|<|J|^{-1}} |J|\langle|b_J|\rangle_{1,J}\mathbf{1}_J\right).$$

This is a consequence of (C.16) *using the cancellation assumption $\int b_J = 0$ for each $J\in\mathcal{J}$. For the inductive step with $\tau > 1$, denoting again by $C(\tau)$ the best constant in the desired L^2-estimate we clearly have that*

$$\sum_{L\in\Lambda_\tau}\left\|\sum_{J\colon |J|<|L|^{-1}} \Delta_L(b_J)\right\|^2_{L^2(\mathbb{R})} \lesssim \sum_{L\in\Lambda_\tau}\left\|\sum_{J\colon |\hat{L}|^{-1}\leqslant|J|<|L|^{-1}} \Delta_{|L|}(e^{-2\pi i\lambda(L)\cdot}b_J)\right\|^2_{L^2(\mathbb{R})} + C(\tau-1)\sum_{J\in\mathcal{J}} |J|\langle b_J\rangle^2_{B_{(\tau-2)/2},J}.$$

Using a linearization trick as in the proof of Proposition 38 we have

$$\sum_{L\in\Lambda_\tau}\Bigg|\sum_{J:\,|\widehat{L}|^{-1}\leqslant|J|<|L|^{-1}}\Delta_{|L|}(e^{-2\pi i\lambda(L)\cdot}b_J)(x)\Bigg|^2=\sum_{\ell\in\mathbb{Z}}\Bigg|\sum_{J:\,|J|<2^{-\ell}}\Delta_{2^\ell}\Bigg(\sum_{\substack{L\in\Lambda_\tau\\|L|=2^\ell,\,|\widehat{L}|^{-1}\leqslant|J|}}a_L e^{-2\pi i\lambda(L)\cdot}b_J\Bigg)(x)\Bigg|^2.$$

for some collection $\{a_L\}_{L\in\Lambda_\tau}=\{a_L(x,\ell)\}_{L\in\Lambda_\tau}$ *in the unit ball of* ℓ^2_L*. Fixing for the moment* $\ell\in\mathbb{Z}$ *and* $x\in\mathbb{R}$ *we have*

$$p_{x,\ell}:=\sum_{\substack{L\in\Lambda_\tau\\|L|=2^\ell,\,|\widehat{L}|\geqslant|J|^{-1}}}a_L e^{-2\pi i\lambda(L)\cdot}=\sum_{L'\in\Lambda_{\tau-1}^{|J|^{-1}}}\Bigg(\sum_{\substack{L\in\Lambda_\tau(L')\\|L|=2^\ell}}a_L\Bigg)e^{-2\pi i\lambda(L)\cdot}=:\sum_{\lambda\in\mathrm{lac}_{\tau-1}^{|J|^{-1}}}\beta_\lambda e^{-2\pi i\lambda\cdot}$$

with $\|\{\beta_\lambda\}\|_{\ell^2_\lambda}=O(1)$*. Here we used that there at at most* $O(1)$ *intervals* $L\in\Lambda_\tau$ *with fixed length* $|L|=2^\ell$ *inside* $\widehat{L}$*. Using the cancellation of* $p_{x,\ell}b_J$ *we can estimate pointwise* $|\Delta_{2^\ell}(p_{x,\ell}b_J)|\lesssim 2^\ell|J|\langle|p_{x,J}b_J|\rangle_{1,J}\,\omega_{2^\ell}*\mathbf{1}_J$*, and by Remark 1 and Hölder's inequality for Orlicz spaces we have that*

$$\langle|p_{x,\ell}b_J|\rangle_{1,J}\lesssim\langle|b_J|\rangle_{B_{(\tau-1)/2},J}.$$

With this information the proof of the estimate can now be completed summing over $|J|2^\ell<1$ *as in the proof of Proposition 38.*

We conclude this section by recording the generalized Zygmund–Bonami inequality under cancellation conditions. This is just a combination of Propositions 37 and 38.

Corollary 39. *Let* σ *be a nonnegative integer and* τ *be a positive integer. Assume that* $\mathrm{supp}(f)\subset J$ *for some finite interval* J *and that* $\widehat{f}(\lambda)=0$ *for all* $\lambda\in\mathrm{lac}_0^{|J|^{-1}}\cup\cdots\cup\mathrm{lac}_{\tau-1}^{|J|^{-1}}$*. Then*

$$\Bigg(\sum_{L\in\Lambda_\tau}\langle|\Delta_L(f)|\rangle^2_{B_{\sigma/2},\gamma J}\Bigg)^{1/2}\lesssim\langle|f|\rangle_{B_{(\sigma+\tau)/2},J}.$$

C.5 An $L^{B_{\sigma/2}}(\mathbb{R})$ Calderón–Zygmund decomposition

We describe in this section a Calderón–Zygmund decomposition adapted to the (global) Orlicz space $L^{B_{\sigma/2}}(\mathbb{R})$ for $\sigma\geqslant 0$. Such a Calderón–Zygmund decomposition, which is influenced by the one appearing in [CUMP11, Appendix A], is available to us because of the specific choice of the Young function $B_{\sigma/2}$ and it is adapted to the finite order lacunary setup.

Recall that for $\sigma\geqslant 0$ we write $f\in L^{B_{\sigma/2}}(\mathbb{R})$ if for some (or equivalently all) $\lambda>0$ there holds

$$\int_{\mathbb{R}}B_{\sigma/2}\left(\frac{|f(x)|}{\lambda}\right)\mathrm{d}x<+\infty.$$

There is an Orlicz maximal operator associated with B_σ

$$\mathrm{M}_{B_{\sigma/2}}f(x):=\sup_{Q\ni x}\langle|f|\rangle_{B_{\sigma/2},Q},\qquad x\in\mathbb{R},$$

with the supremum being over all intervals Q of $\mathbb{R}$ containing x. The dyadic version of $\mathrm{M}_{B_{\sigma/2}}$ is defined similarly with the supremum over all dyadic intervals $Q \in \mathcal{D}$ with $\mathcal{D}$ some dyadic grid. We will write $\mathrm{M}_{B_{\sigma/2},\mathcal{D}}$ for the dyadic version. Below we denote by $Q \in \mathcal{D}$ a dyadic interval, $Q^{(1)}$ its dyadic parent and set $Q^{(k+1)}$ to be the dyadic parent of $Q^{(k)}$.

Remark 4 (Existence of stopping intervals)**.** *For the Calderón–Zygmund decomposition we will choose stopping intervals that are maximal under the condition $\langle |f|\rangle_{B_{\sigma/2},I} > \lambda$. The existence of these stopping intervals relies on the following fact: If $f \in L^{B_{\sigma/2}}(\mathbb{R})$ for some $\sigma \geqslant 0$ and I is a dyadic interval in some grid $\mathcal{D}$, then $\langle |f|\rangle_{B_{\sigma/2},I^{(k)}} \to 0$ as $k \to +\infty$. This can be easily proved using for example the fact that the Young function $B_{\sigma/2}$ is submultiplicative.*

Proposition 40. *Let σ be a fixed nonnegative integer, $f \in L^{B_{\sigma/2}}(\mathbb{R})$, and $\alpha > 0$. There exists a collection $\mathcal{J}$ of pairwise disjoint dyadic intervals J and a decomposition of f*

$$f = g + b_{\mathrm{canc},\sigma} + b_{\mathrm{lac},\sigma}$$

such that the following hold:

(i) *The function g satisfies $\|g\|_{L^\infty(\mathbb{R})} \lesssim \alpha$ and $\|g\|_{L^1(\mathbb{R})} \lesssim \|f\|_{L^1(\mathbb{R})}$.*

(ii) *The function $b_{\mathrm{canc},\sigma}$ is supported in $\cup_{J\in\mathcal{J}} J$ and in particular*

$$b_{\mathrm{canc},\sigma} = \sum_{J\in\mathcal{J}} b_J, \quad \operatorname{supp}(b_J) \subseteq J, \qquad \widehat{b_J}(\lambda) = 0 \qquad \forall \lambda \in \mathrm{lac}_0 \cup \cdots \cup \mathrm{lac}_\sigma.$$

Furthermore we have that $\langle |b_J|\rangle_{B_{\sigma/2},J} \lesssim \alpha$ for all $J \in \mathcal{J}$ and

$$\sum_{J\in\mathcal{J}} |J| \leqslant \int_{\mathbb{R}} B_{\sigma/2}\left(\frac{|f|}{\alpha}\right).$$

(iii) *The function $b_{\mathrm{lac},\sigma}$ is also supported on $\cup_{J\in\mathcal{J}} J$ and satisfies*

$$\|b_{\mathrm{lac},\sigma}\|^2_{L^2(\mathbb{R})} \lesssim \sum_{J\in\mathcal{J}} |J| \langle |b_J|\rangle^2_{B_{\sigma/2},J} \lesssim \alpha^2 \int_{\mathbb{R}} B_{\sigma/2}\left(\frac{|f|}{\alpha}\right).$$

Proof. We begin by recalling that $f \in L^{B_{\sigma/2}}(\mathbb{R})$ implies that $\int_{\mathbb{R}} B_{\sigma/2}(|f|/\alpha) < +\infty$ for all $\alpha > 0$. By Remark 4 and [CUMP11, Theorem 5.5] we have that the dyadic Orlicz maximal operator $\mathrm{M}_{B_{\sigma/2},\mathcal{D}}$ satisfies

$$|E_\alpha| := |\{x \in \mathbb{R} : \mathrm{M}_{B_{\sigma/2},\mathcal{D}} f(x) > \alpha\}| \leqslant \int_{\mathbb{R}} B_{\sigma/2}\left(\frac{|f|}{\alpha}\right), \qquad \alpha > 0.$$

Letting $\mathcal{J}$ denote the collection of maximal dyadic intervals contained in E_α we have that for every $J \in \mathcal{J}$

$$\alpha < \langle |f|\rangle_{B_{\sigma/2},J} \leqslant 2\alpha, \qquad \sum_{J\in\mathcal{J}} |J| \leqslant \int_{\mathbb{R}} B_{\tau/2}\left(\frac{|f|}{\alpha}\right);$$

The upper bound in the approximate inequality of the leftmost estimate above follows by the maximality of J and the convexity of the Young function of $B_{\sigma/2}$ which implies that

$$\langle |f| \rangle_{B_{\sigma/2},J} \leqslant \rho \langle |f| \rangle_{B_{\sigma/2},\rho J}, \qquad \rho > 1;$$

see[CUMP11, Proposition A.1] and [CUMP11, eq. (5.2)]. One routinely checks that $g := f\mathbf{1}_{\mathbb{R}\setminus\cup_{J\in\mathcal{J}}J}$ satisfies (i).

For the "atoms" we set $f_J := f\mathbf{1}_J$ and define

$$b_{J,\mathrm{lac},\sigma}(y) := \sum_{\rho=0}^{\sigma}\left(\sum_{\lambda\in\mathrm{lac}_\rho^{|J|^{-1}}} \widehat{f_J}(\lambda)e^{2\pi i\lambda y}\right)\frac{\mathbf{1}_J(y)}{|J|}, \qquad b_J := f_J - b_{J,\mathrm{lac},\sigma},$$

and $b_{\mathrm{lac},\sigma} := \sum_{J\in\mathcal{J}} b_{J,\mathrm{lac},\sigma}$ and $b_{\mathrm{canc},\sigma} := \sum_{J\in\mathcal{J}} b_J$. The cancellation conditions of (ii) for $b_{\mathrm{canc},\sigma}$ follow immediately by the definition above. Furthermore by the Hölder inequality for Orlicz spaces and the Zygmund–Bonami inequality of order $\rho \in \{1,\ldots,\sigma\}$ as in Remark 1, one sees that

$$\langle |b_{J,\mathrm{lac},\sigma}| \rangle_{B_{\tau/2},J} \lesssim \langle |b_{J,\mathrm{lac},\sigma}| \rangle_{2,J} \lesssim \langle |f_J| \rangle_{B_{\sigma/2},J} \lesssim \alpha.$$

This and the triangle inequality also yield $\langle |b_J| \rangle_{B_{\sigma/2},J} \lesssim \langle |f_J| \rangle_{B_{\sigma/2},J} \lesssim \alpha$ thus completing the proof of the desired conclusions in (ii). Finally for (iii) we estimate as above

$$\|b_{\mathrm{lac},\sigma}\|_{L^2(\mathbb{R})}^2 \lesssim \sum_{J\in\mathcal{J}} |J| \langle |f_J| \rangle_{B_{\sigma/2},J}^2 \lesssim \alpha^2 \int_{\mathbb{R}} B_{\tau/2}\left(\frac{|f|}{\alpha}\right)$$

and the proof is complete. ■

C.6 Proof of Theorem 31 and Corollaries

In the first part of this section we compile together the results of the previous sections to conclude the proof of Theorem 31. In the second part we show how to conclude our corollaries, namely Theorem 30 and 32.

C.6.1 Proof of Theorem 31

Let us fix a positive integer τ and $m \in R_{2,\tau}$. Before entering the heart of the proof we note that it suffices to prove the theorem for multipliers m having the form

$$m = \sum_{I\in\mathcal{I}} c_I \mathbf{1}_I$$

where the family of intervals $\mathcal{I}$ has overlap at most N, for each $I \in \mathcal{I}$ there exists a unique $L = L_I \in \Lambda_\tau$ such that $I \subset L_I$ and for each fixed $L \in \Lambda_\tau$ there holds

$$\sum_{I:\, L_I = L} |c_I|^2 \leqslant N^{-1}.$$

See the analysis in [TW01, p. 533] for the details of this approximation argument. For m of this form, we now can write

$$\mathrm{T}_m(f) = \sum_{I\in\mathcal{I}} c_I \mathrm{P}_I f = \sum_{L\in\Lambda_\tau} \sum_{I:\, L_I = L} c_I \mathrm{P}_I(\Delta_L f), \qquad \mathrm{P}_I f \coloneqq (\mathbf{1}_I \hat{f})^\vee,$$

a fact that we will use repeatedly in what follows.

C.6.1.1 The upper bound in Theorem 31

Let f be a function in $L^{B_\tau}(\mathbb{R})$ and $\alpha > 0$ be fixed. We decompose f according to the Calderón–Zygmund decomposition in Proposition 40 with $\sigma = \tau$ yielding

$$f = g + b_{\mathrm{canc},\tau} + b_{\mathrm{lac},\tau}.$$

We directly estimate $g + b_{\mathrm{lac},\tau}$ in L^2 using (i) and (iii) of Proposition 40

$$|\{x \in \mathbb{R} : |\mathrm{T}_m(g + b_{\mathrm{lac},\tau})(x)| > \alpha\}| \lesssim \frac{1}{\alpha^2} \|g + b_{\mathrm{lac},\tau}\|^2_{L^2(\mathbb{R})} \lesssim \int_{\mathbb{R}} B_{\tau/2}\left(\frac{|f|}{\alpha}\right).$$

The main part of the proof deals with the bad part $b_{\mathrm{canc},\tau} = \sum_{J\in\mathcal{J}} b_J$ and it suffices to estimate

$$|\{x \in \mathbb{R}\setminus \cup_{J\in\mathcal{J}} 6J : |\mathrm{T}_m(b_{\mathrm{canc},\tau})| > \alpha\}|$$

as the measure $|\cup_{J\in\mathcal{J}} 6J|$ satisfies the desired estimate by (ii) of Proposition 40. We will adopt the splitting

$$\mathrm{T}_m\left(\sum_J b_J\right) = \sum_I c_I \mathrm{P}_I\left(\sum_{J:\, |J|\geqslant |L|^{-1}} \Delta_{L_I}(b_J)\mathbf{1}_{\mathbb{R}\setminus 3J}\right) + \sum_I c_I \mathrm{P}_I\left(\sum_{J:\, |J|<|L|^{-1}} \Delta_{L_I}(b_J)\right)$$
$$+ \sum_I c_I \mathrm{P}_I\left(\sum_{J:\, |J|\geqslant |L|^{-1}} \Delta_{L_I}(b_J)\mathbf{1}_{3J}\right) =: \mathrm{I} + \mathrm{II} + \mathrm{III}.$$

The main term is III. Indeed we can estimate the term I in $L^2(\mathbb{R})$ using Remark 2, while II is also estimated in $L^2(\mathbb{R})$ using Remark 3 this time. Note that each b_J has the required cancellation by (ii) of Proposition 40. Using also (ii) of Proposition 40 to control the averages $\langle |b_J| \rangle_{B_{(\tau-1)/2},J} \lesssim \langle |b_J| \rangle_{B_{\tau/2},J} \lesssim \alpha$ we have

$$|\{|\mathrm{I} + \mathrm{II}| > \alpha\}| \lesssim \frac{1}{\alpha^2} \sum_{J\in\mathcal{J}} |J| \langle |b_J| \rangle^2_{B_{(\tau-1)/2},J} \lesssim \sum_{J\in\mathcal{J}} |J| \lesssim \int_{\mathbb{R}} B_\tau(|f|/\alpha)$$

as desired.

It remains to deal with III and we make a further splitting. Let $k_I \in \mathbb{Z}$ be such that $2^{k_I} < |I| \leqslant 2^{k_I+1}$. Of course we will always have that $|L_I| \geqslant 2^k_I$ since $I \subseteq L_I$. We write

$$\mathrm{III} = \sum_I c_I \mathrm{P}_I\left(\sum_{J:\, 2^{-k_I} > |J| \geqslant |L|^{-1}} \Delta_{L_I}(b_J)\mathbf{1}_{3J}\right) + \sum_I c_I \mathrm{P}_I\left(\sum_{J:\, |J|\geqslant 2^{-k_I}} \Delta_{L_I}(b_J)\mathbf{1}_{3J}\right) =: \mathrm{III}_1 + \mathrm{III}_2.$$

We first handle the term III_1. Let Δ_I be the smooth frequency projections on the interval I as fixed in §C.2.2; then in particular we can write $\mathrm{P}_I\Delta_I = \mathrm{P}_I$ and we have the familiar pointwise estimate

$$|\Delta_I(\Delta_{L_I}(b_J)\mathbf{1}_{3J})| \lesssim \omega_{|I|^{-1}} * (\langle|\Delta_{L_I}(b_J)|\rangle_{1,3J}\mathbf{1}_J)$$

as $|I|^{-1} \simeq 2^{-k_I} > |J|$. We thus get

$$\begin{aligned}|\{|\mathrm{III}_1| > \alpha\}| &\lesssim \frac{1}{\alpha^2}\sum_{L\in\Lambda_\tau} N \sum_{I\colon L_I=L} |c_I|^2 \int_{\mathbb{R}} \left|\omega_{|I|^{-1}} * \left(\sum_{J\colon |J|\geqslant 2^{-k_I}} \langle|\Delta_L(b_J)|\rangle_{1,3J}\mathbf{1}_J\right)\right|^2 \\ &\lesssim \frac{1}{\alpha^2}\sum_{J\in\mathcal{J}}\sum_{L\in\Lambda_\tau^{|J|^{-1}}} |J|\langle|\Delta_L(b_J)|\rangle_{1,3J}^2 \lesssim \sum_{J\in\mathcal{J}}|J| \lesssim \int_{\mathbb{R}} B_{\tau/2}\left(\frac{|f|}{\alpha}\right)\end{aligned}$$

where we used the ℓ^2-control on the coefficients $\{c_I\}_{L_I=L}$ in passing to the second line and the generalized Zygmund–Bonami inequality of Proposition 37 together with the properties of the Calderón–Zygmund decomposition in the penultimate approximate inequality.

The steps required for dealing with the the term III_2 are essentially the same as those in [TW01], however, as here we are dealing with a higher order set up, we include them for the sake of completeness. We will split the estimate for III_2 into two parts. In the first we keep the part of the multiplier $\mathbf{1}_I = \mathbf{1}_{[\ell_I,r_I]}$ at scale $O(|J|^{-1})$ around its singularities which are at the endpoints. We make this precise now.

Let $0 \leqslant \psi_{I,J} \leqslant 1$ be a smooth bump which is 1 on the $(10|J|)^{-1}$-neighborhood of the endpoints $\{\ell_I, r_I\}$ of I and vanishes off the $(5|J|)^{-1}$-neighborhood of the endpoints, and satisfies $\|\partial^\alpha\psi_{I,J}\|_{L^\infty} \lesssim |J|^\alpha$ for all α up to some sufficiently large integer M. Letting $\Psi_{I,J}$ denote the operator with symbol $\psi_{I,J}$ we define

$$\mathcal{E}(\{b_J\}_{J\in\mathcal{J}}) := \sum_I c_I \mathrm{P}_I\left(\sum_{J\colon |J|\geqslant 2^{-k_I}} \Psi_{I,J}\left(\Delta_{L_I}(b_J)\mathbf{1}_{3J}\right)\right)$$

The following lemma shows that the operator $\mathcal{E}(\{b_J\}_{J\in\mathcal{J}})$ can be dealt with, again, by L^2-estimates.

Lemma 41. *We have the estimate*

$$\|\mathcal{E}(\{b_J\}_{J\in\mathcal{J}})\|_{L^2(\mathbb{R})}^2 \lesssim \sum_{J\in\mathcal{J}} |J|\langle|b_J|\rangle_{B_{\tau/2},J}^2 \lesssim \alpha^2\int_{\mathbb{R}} B_{\tau/2}\left(\frac{|f|}{\alpha}\right).$$

Proof. First note that by the overlap assumption on the intervals I we have

$$\|\mathcal{E}(\{b_J\}_{J\in\mathcal{J}})\|_{L^2(\mathbb{R})}^2 \lesssim N\sum_{L\in\Lambda_\tau}\sum_{I\colon L_I=L}|c_I|^2\int_{\mathbb{R}}\left(\sum_{J\colon |J|\geqslant 2^{-k_I}} |\Psi_{I,J}(\Delta_L(b_J)\mathbf{1}_{3J})|\right)^2.$$

The following pointwise estimate can be routinely verified

$$|\Psi_{I,J}(\Delta_L(b_J)\mathbf{1}_{3J})(x)| \lesssim \mathrm{M}(\mathbf{1}_J)(x)^{10}\langle|\Delta_L(b_J)|\rangle_{1,3J}.$$

Using the Fefferman–Stein inequality and rearranging the sums we can conclude that

$$\|\mathcal{E}(\{b_J\}_{J\in\mathcal{J}})\|^2_{L^2(\mathbb{R})} \lesssim N\sum_k \sum_{L\in\Lambda_\tau^{2^k}} \sum_{\substack{I:\, L_I=L\\ \ell_I=\ell}} |c_I|^2 \sum_{|J|\geqslant 2^{-k}} \langle|\Delta_L(b_J)|\rangle^2_{1,3J}|J|$$
$$\leqslant \sum_{J\in\mathcal{J}} |J| \sum_{L\in\Lambda_\tau^{|J|^{-1}}} \langle|\Delta_L(b_J)|\rangle^2_{1,3J}$$

where we used the ℓ^2-control on the coefficients $\{c_I\}_{L_I=L}$ in passing to the second line. An appeal to the generalized Zygmund–Bonami inequality of order τ in Proposition 37 concludes the proof of the lemma. ■

We are left with studying the contribution of the operator

$$\mathcal{L}(\{b_J\}_{J\in\mathcal{J}}) := \sum_I c_I \mathrm{P}_I \sum_{|J|\geqslant 2^{-k_I}} (\mathrm{Id}-\Psi_{I,J})\,(\Delta_{L_I}(b_J)\mathbf{1}_{3J})\,.$$

For this we consider the multiplier $\zeta_{I,J} := \mathbf{1}_I(1-\psi_{I,J})$ which is a smooth function with values in $[0,1]$, supported in I, is identically 1 on $|x-c_I|\lesssim|I|$ and drops to 0 with derivative $O(|J|)$ close to the endpoints of I. More generally, one easily checks that $\zeta_{I,J}$ satisfies

$$|\partial^\alpha\zeta_{I,J}| \lesssim |J|^\alpha \mathbf{1}_{I_{\mathrm{left}}(J)\cup I_{\mathrm{right}}(J)} \qquad \forall\alpha\geqslant 1,$$

where

$$I_{\mathrm{left}}(J) := \left[\ell_I + 10^{-1}|J|^{-1}, \ell_I + 5^{-1}|J|^{-1}\right] \subset I,$$
$$I_{\mathrm{right}}(J) := \left[r_I - 5^{-1}|J|^{-1}, r_I - 10^{-1}|J|^{-1}\right] \subset I.$$

Remembering that we are dealing with the case $|I||J|\gtrsim 1$ we see that the function $\zeta_{I,J}$ has support of size $O(|I|)$ and α-derivatives of size $O(|J|^\alpha)$; thus the function $\zeta_{I,J}^\vee$ is not a good kernel. The important observation is however that the derivatives of $\zeta_{I,J}$ of order $\alpha\geqslant 1$ have support of size $|I_{\mathrm{left}}(J)\cup I_{\mathrm{right}}(J)|\simeq|J|^{-1}$.

Given an interval $J\subset\mathbb{R}$ we will also use an auxiliary function ρ_J defined as follows. We choose $0\leqslant\rho\leqslant 1$ to be a smooth bump function which is identically 1 on $[-1,1]$ and vanishes off $[-3/2,3/2]$ and define $\rho_J(x) := \rho\,(x/|J|)$ for $x\in\mathbb{R}$.

Lemma 42. *Let I,J be intervals and $\zeta_{I,J}$ and ρ_J be defined as above. If $\widecheck{\mu_{I,J}} := (1-\rho_J)\widecheck{\zeta_{I,J}}$ then for any nonnegative integers γ,β there holds*

$$\left|\partial_\xi^\beta \mu_{I,J}(\xi)\right| \lesssim |J|^\beta\,(1+|J|\mathrm{dist}(\xi,\mathbb{R}\backslash I))^{-\gamma}\,, \qquad \xi\in\mathbb{R}.$$

Proof. We begin by noting that since $\zeta_{I,J}$ is a Schwartz function and $(1-\rho_J)$ is a smooth bounded function, we have we have that $(1-\rho_J)\widecheck{\zeta_{I,J}}$ is a Schwartz function. Furthermore, by the comments preceding the statement of the lemma we have that $\zeta_{I,J}$ satisfies

$$|\partial_\xi^\alpha\zeta_{I,J}| \lesssim |J|^\alpha \mathbf{1}_{I_{\mathrm{left}}(J)\cup I_{\mathrm{right}}(J)} \qquad \forall\alpha\geqslant 1.$$

Note that by symmetry it suffices to prove the estimate for $\xi \in \mathbb{R}$ such that $\operatorname{dist}(\xi, \mathbb{R}\backslash I) = |\xi - \ell_I|$ where we remember that $I = [\ell_I, r_I]$. For simplicity we will write $I(J)$ for $I_{\text{left}}(J)$. Thus the conclusion of the lemma reduces to showing

$$|\partial_\xi^\beta \mu_{I,J}(\xi)| \lesssim_\beta |J|^\beta (1 + |J||\xi - \ell_I|)^{-\gamma}.$$

We will henceforth drop the subindices I, J in order to simplify the notation. We record the following standard integration by parts identity; for nonnegative integers γ, ν we have

$$(\partial_x - 2\pi i \ell_I)^\gamma [\check{\zeta}(x)] = (-1)^\nu \frac{(2\pi i)^{\gamma-\nu}}{x^\nu} \int_{\mathbb{R}} \partial_\xi^\nu \left[(\xi - \ell_I)^\gamma \zeta(\xi)\right] e^{2\pi i x \xi} \,\mathrm{d}\xi.$$

In order to make sure that all terms in $\partial_\xi^\nu \left[(\xi - \ell_I)^\gamma \zeta(\xi)\right]$ contain at least one derivative we take $\nu > \gamma$. Then we have

$$\begin{aligned}\left|\partial_\xi^\nu \left[(\xi - \ell_I)^\gamma \zeta(\xi)\right]\right| \lesssim \sum_{k=0}^{\gamma} \left|\partial_\xi^k (\xi - \ell_I)^\gamma \partial_\xi^{\nu-k}[\zeta(\xi)]\right| &\lesssim \sum_{k=0}^{\gamma} |\xi - \ell_I|^{\gamma-k} |J|^{\nu-k} \mathbf{1}_{I(J)}(\xi) \\ &\lesssim |J|^{\nu-\gamma} \mathbf{1}_{I(J)}(\xi)\end{aligned}$$

provided that $\nu > \gamma$. Plugging this estimate into our integration by parts identity we get

$$\left|(\partial_x - 2\pi i \ell_I)^\gamma [\check{\zeta}(x)]\right| \lesssim \frac{|J|^{\nu-\gamma-1}}{|x|^\nu}, \qquad \nu > \gamma. \tag{C.18}$$

Using this estimate we have for nonnegative integers β, γ

$$\partial_\xi^\beta [\mu(\xi)] = \frac{(-2\pi i)^\beta}{(2\pi i(\xi - \ell_I))^\gamma} \int_{\mathbb{R}} (\partial_x - 2\pi i \ell_I)^\gamma \left[x^\beta (1 - \rho(x/|J|)) \check{\zeta}(x)\right] e^{-2\pi i x \xi} \,dx.$$

Using (C.18) with ν large together with the fact that $\operatorname{supp}(1 - \rho_J) \subset \{|x| \gtrsim |J|\}$ and that $\operatorname{supp}(\partial_x[\rho_J]) \subset \{|x| \simeq |J|\}$ and combining with the previous identity yields

$$\begin{aligned}\left|\partial_\xi^\beta [\mu(\xi)]\right| &\lesssim \frac{1}{|\xi - \ell_I|^\gamma} \int_{\mathbb{R}} \sum_{\substack{k_1+k_2+k_3=\gamma \\ k_1 \leqslant \beta}} |x|^{\beta-k_1} \left|\partial_x^{k_2}(1 - \rho(x/|J|))(\partial_x - 2\pi i \ell_I)^{k_3} [\check{\zeta}(x)]\right| \mathrm{d}x \\ &\leqslant \sum_{\substack{k_1+k_3=\gamma \\ k_1 \leqslant \beta}} \frac{|J|^{\nu-k_3-1}}{|\xi - \ell_I|^\gamma} \int_{|x| \gtrsim |J|} |x|^{\beta-k_1-\nu} \,\mathrm{d}x + \sum_{\substack{k_1+k_2+k_3=\gamma \\ k_1 \leqslant \beta,\, k_2 \geqslant 1}} \frac{|J|^{\nu-k_3-1}}{|\xi - \ell_I|^\gamma} \int_{|x| \simeq |J|} |x|^{\beta-k_1-\nu} |J|^{-k_2} \,\mathrm{d}x \\ &\lesssim \frac{|J|^{\beta-\gamma}}{|\xi - \xi_J|^\gamma}.\end{aligned}$$

Combining this estimate for general γ with the special case $\gamma = 0$ yields the conclusion of the lemma. ■

We can now prove the desired estimate for the remaining term.

Lemma 43. *There holds*

$$\int_{\mathbb{R}\backslash \cup_{J \in \mathcal{J}} 6J} |\mathcal{L}(\{b_J\}_{J \in \mathcal{J}})| \lesssim \sum_{J \in \mathcal{J}} |J| \langle |b_J| \rangle_{B_{\tau/2}, J}.$$

Proof. For convenience we set

$$\mathrm{L}_{I,J} := \mathrm{P}_I(\mathrm{Id} - \Psi_{I,J}), \qquad F_{I,J} := \Delta_{L_I}(b_J)\mathbf{1}_{3J}, \qquad \mathcal{L}(\{b_J\}_{J\in\mathcal{J}}) = \sum_I \sum_{J\colon |J|\geqslant 2^{-k_I}} c_I \mathrm{L}_{I,J}(F_{I,J}).$$

We immediately note that it will be enough to prove the desired estimate for a single b_J and then sum the estimates. Furthermore, by translation and scale invariance it will be enough to to assume that $J = [-|J|/2, |J|/2]$; here we critically use that the operator $L_{I,J}$ depends only on the length and not on the position of J. The left hand side in the conclusion of the lemma for a single such b_J can be estimated by

$$\begin{aligned} A_J &:= \int_{|x|\geqslant 3|J|} \left| \sum_{I\colon 2^{k_I}\geqslant |J|^{-1}} c_I \mathrm{L}_{I,J} F_{I,J} \right| = \int_{|x|\geqslant 3|J|} \left| \sum_{I\colon 2^{k_I}\geqslant |J|^{-1}} c_I (\widetilde{\zeta_{I,J}} * F_{I,J})(x) \right| \mathrm{d}x \\ &\lesssim |J|^{-1/2} \left(\int_{|x|>3|J|} |x|^2 \left| \sum_{I\colon 2^{k_I}\geqslant 1} c_I (\widetilde{\zeta_{I,J}} * F_{I,J})(x) \right|^2 \mathrm{d}x \right)^{\frac{1}{2}}. \end{aligned}$$

Now let ρ be as before. It is then the case that for $|x| \geqslant 3$ and $|y| \leqslant 3/2$ we have

$$|x-y| \geqslant \frac{1}{2}|x| \geqslant \frac{3}{2}, \qquad 1 - \rho\left(\frac{x-y}{|J|}\right) = 1 - \rho_J(x-y) = 1$$

for such pairs (x, y). As $F_{I,J}$ is supported in $[-3|J|/2, 3|J|/2]$ we have for all $|x| \geqslant 3|J|$ that

$$(\widetilde{\zeta_{I,J}} * F_{I,J})(x) = \int_{[-3|J|/2,\, 3|J|/2]} F_{I,J}(y)\widetilde{\zeta_{I,J}}(x-y)\,(1 - \rho_J(x-y)))\,\mathrm{d}y.$$

Using this identity and setting $\widetilde{\mu_{I,J}} := (1-\rho_J)\widetilde{\zeta_{I,J}}$ we get

$$A_J \lesssim \left(\int_{\mathbb{R}} \left| \sum_{I\colon 2^{k_I}\geqslant |J|^{-1}} c_I \partial_\xi \left[\widehat{F_{I,J}}(\xi)\mu_{I,J}(\xi) \right] \right|^2 \mathrm{d}\xi \right)^{\frac{1}{2}}.$$

Using the elementary estimates $\|\widehat{F_{I,J}}\|_{L^\infty(\mathbb{R})} \leqslant \|F_{I,J}\|_{L^1(\mathbb{R})}$ and $\|\partial_\xi \widehat{F_{I,J}}\|_{L^\infty(\mathbb{R})} \leqslant |J|\|F_{I,J}\|_{L^1(\mathbb{R})}$ together with the estimate of Lemma 42 for $\beta \in \{0, 1\}$ we get for γ a large positive integer of our choice

$$\partial_\xi^\beta[\mu_{I,J}(\xi)] \lesssim \frac{|J|^\beta}{(1 + |J|\mathrm{dist}(\xi, \mathbb{R}\backslash I))^\gamma} \lesssim |J|^\beta \mathrm{M}(\mathbf{1}_{I_{\mathrm{left}}(J)\cup I_{\mathrm{right}}(J)})^\gamma.$$

Hence, by using the Fefferman–Stein inequality, the Cauchy–Schwarz inequality, and the N-overlap assumption on the intervals I, we get

$$A_J \lesssim |J|^{-1/2} \left(\sum_{2^{k_I}\geqslant |J|^{-1}} |c_I|^2 |J|^4 \langle |\Delta_{L_I}(b_J)| \rangle_{1,3J}^2 N\, |I_{\mathrm{left}}(J) \cup I_{\mathrm{right}}(J)| \right)^{\frac{1}{2}}$$

Here note that we use that $I_{\text{left}}(J) \cup I_{\text{right}}(J) \subsetneq I$ by construction, and hence

$$\sum_{2_I^k \geqslant |J|^{-1}} \mathbf{1}_{I_{\text{left}}(J) \cup I_{\text{right}}(J)} \leqslant \sum_I \mathbf{1}_I \leqslant N.$$

Further, using also the control on the ℓ^2-norm of the sequence $\{c_I\}_I$ yields

$$A_J \lesssim |J|^{-1/2} \left(\sum_{L \in \Lambda_\tau^{|J|^{-1}}} |J|^3 \langle |\Delta_L(b_J)| \rangle_{1,3J}^2 \right)^{1/2} = |J| \left(\sum_{L \in \Lambda_\tau^{|J|^{-1}}} \langle |\Delta_L(b_J)| \rangle_{1,3J}^2 \right)^{1/2} \lesssim |J| \langle |b_J| \rangle_{B_{\tau/2},J}$$

by the generalized Zygmund–Bonami inequality of Proposition 37. This concludes the proof of the lemma. ■

Using Lemmas 41 and 43 we complete the estimate for the term III and with that the proof of the endpoint bound of the theorem.

C.6.1.2 Optimality in Theorem 31

We briefly comment on the optimality of the Young function $t \mapsto t(\log(e+t))^{\tau/2}$ in the upper bound of the theorem. Suppose that $r > 0$ is such that whenever T_m is an $R_{2,\tau}$ multiplier operator then the bound of Theorem 31 holds with r in the place of τ. Since T_m is L^2-bounded, it follows by a Marcinkiewicz interpolation type of argument that the $L^p(\mathbb{R})$-bounds for the Littlewood–Paley square function LP_τ of order τ can be estimated by

$$\|\mathrm{LP}_\tau\|_{p \to p} \lesssim \left(\mathbf{E} \left\| \sum_{L \in \Lambda_\tau} \varepsilon_L \mathrm{P}_L \right\|_{p \to p}^p \right)^{1/p} \leqslant \sup_{\|m\|_{R_{2,\tau}}=1} \|\mathrm{T}_m\|_{L^p \to L^p} \lesssim (p-1)^{-(r+1)} \qquad \text{as } \ p \to 1^+,$$

where the expectation in the display above is over independent choices of random signs $\{\varepsilon_L\}_L$. However, a modification of an example in [Bou89], see [Bak21, §3], shows that the estimate in the display above does not hold for $r < \tau/2$. This argument also shows that our theorem implies that the L^p bounds for $R_{2,\tau}$ multipliers are $O(\max(p,p')^{1+\tau/2})$.

Alternatively, sharpness can also be obtained by adapting the corresponding argument in [TW01, §3.2] to the higher order case. Let us briefly outline the second order case. For a smooth function ψ supported in $[-1/2, 1/2]$ with $\psi(0) = 1$ and $(k,l) \in \mathbb{Z}^2$ with $k > l$ we consider the multiplier $m_{(k,l)}$ given by

$$m_{(k,l)}(\xi) \coloneqq m_0\left(\frac{\xi - 2^k}{2^{l-1}}\right), \quad \text{where} \quad m_0(\xi) \coloneqq \psi(\xi - 1)\mathbf{1}_{[1,\infty)}(\xi), \qquad \xi \in \mathbb{R}.$$

One then has

$$\widetilde{m_{(k,l)}}(x) = \frac{e^{i2\pi 2^k x}}{i2\pi x} + O\left(\frac{1}{2^l |x|}\right), \qquad |x| \gtrsim 2^{-l}.$$

For $N \in \mathbb{N}$, that will be eventually sent to infinity, we consider the second order ℓ^2-valued multiplier operator

$$T_N(g) \coloneqq \left\{ T_{m_{(k,l)}}(g) \right\}_{1 \leqslant l < k \leqslant N}.$$

Consider a smooth function f such that $\widehat{f}$ is supported in $[-4,4]$ and $\widehat{f}(\xi) = 1$ for all $\xi \in [-2,2]$. We then set $f_N(x) := 2^N f(2^N x)$, $x \in \mathbb{R}$. For $r > 0$ we have

$$\|T_N(f_N)(x)\|_{\ell^2} \gtrsim \frac{N}{|x|} \qquad \text{if} \quad |x| \geqslant 2^{-5N/8}$$

and

$$\int_{\mathbb{R}} \frac{|f_N(x)|}{\alpha}\left(\log\left(e+\frac{|f_N(x)|}{\alpha}\right)\right)^r \mathrm{d}x \lesssim \frac{1}{\alpha}\left(\log\left(e+\frac{2^N}{\alpha}\right)\right)^r \qquad \text{for} \quad \alpha > 0.$$

Hence, if we choose $\alpha = 2^{5N/8}$ then $\alpha^{-1}\left(\log(e+\alpha^{-1}2^N)\right)^r \simeq 2^{-5N/8}N^r$ and

$$|\{x \in [-1/2,1/2] : \|T_N(f_N)(x)\|_{\ell^2} > \alpha\}| \geqslant \left|\left\{2^{-5N/8} \leqslant x \leqslant 1/4 : \frac{N}{|x|} \gtrsim 2^{5N/8}\right\}\right| \simeq N\alpha^{-1}.$$

To complete the proof, define $g_N := f_N \chi_{[1/2,1/2]}$ so that g_N is supported in $[-1/2,1/2]$ and $\|g_N\|_{L\log^r L([-1/2,1/2])} \lesssim N^r$. Moreover, for all $1 \leqslant l < k \leqslant N$ one has

$$|T_{m_{k,l}}(f_N - g_N)(x)| \lesssim 2^{-2N} \qquad \text{for all} \quad x \in [2^{-5N/8}, 1/4]$$

and hence

$$\|\|T_N(g_N)\|_{\ell^2}\|_{L^{1,\infty}([-1/2,1/2])} \gtrsim N.$$

It follows from Khintchine's inequality that there exists a choice of signs $\varepsilon_{k,\ell}$, depending on g_N, such that

$$\left\|\sum_{1\leqslant l<k\leqslant N} \varepsilon_{k,\ell} T_{m_{k,l}}(g_N)\right\|_{L^{1,\infty}([-1/2,1/2])} \gtrsim N.$$

from which it follows that $r \geqslant 1 = \tau/2$.

C.6.2 Proof of Theorems 30 and 32

We begin by explaining the modifications needed in order to obtain a proof of the endpoint bounds in Theorems 30 and 32.

C.6.2.1 Proof of Theorem 30

Since Marcinkiewicz multipliers of order τ are contained in the class $R_{2,\tau}$, we only need to briefly discuss the conclusion of Theorem 30 for the Littlewood–Paley square function. Note that the proof of Theorem 31 relies on $L^2(\mathbb{R})$ estimates and L^1-type estimates. Then we can repeat the proof for the operator

$$|\mathrm{LP}_\tau f| \simeq \mathbf{E}\left|\sum_{L\in\Lambda_\tau} \varepsilon_L \mathrm{P}_L f\right| \simeq \left(\mathbf{E}\left|\sum_{L\in\Lambda_\tau} \varepsilon_L \mathrm{P}_L f\right|^2\right)^{1/2}$$

using the first approximate equality whenever L^1-estimates are needed, and the second one for the L^2-estimates. We omit the details. The optimality follows by the discussion in §C.6.1.2.

C.6.2.2 Proof of Theorem 32

We proceed to prove Theorem 32 concerning endpoint bounds for higher order Hörmander–Mihlin multipliers and smooth Littlewood–Paley square functions, which requires just small modifications compared to the proof of Theorem 31. Consider a positive integer τ and $f \in L^{B_{\tau-1}}$; we apply the Calderón–Zygmund decomposition of Proposition 40 with $\sigma = \tau - 1$ at some fixed level $\alpha > 0$ to write $f = g + b_{\mathrm{canc},\tau-1} + b_{\mathrm{lac},\tau-1}$ and let $\mathcal{J}$ be the collection of stopping intervals. The good part $g + b_{\mathrm{lac},\tau-1}$ is estimated in L^2 by the L^2-bounds of the operator T_m, using that

$$\|g + b_{\mathrm{lac},\tau-1}\|_{L^2(\mathbb{R})}^2 \lesssim \alpha^2 \int_{\mathbb{R}} B_{(\tau-1)/2}\left(\frac{|f|}{\alpha}\right)$$

by the Calderón–Zygmund decomposition. As before, it remains to estimate the part of the operator acting on the cancellative atoms. We consider a partition of unity $\{\widetilde{\phi}_L\}_{L\in\Lambda_{\tau-1}}$ subordinated to the collection of Littlewood–Paley intervals $\Lambda_{\tau-1}$, with $\widetilde{\phi}_L \in \Phi_{L,M}$ for each L. We set

$$\widetilde{\Delta}_L(g) := \left(\widetilde{\phi}_L\hat{g}\right)^{\vee}, \qquad \mathrm{Id} = \sum_{L\in\Lambda_\sigma} \widetilde{\Delta}_L.$$

With Δ_L the smooth Littlewood–Paley projections as fixed in §C.2.2 we have $\Delta_L\widetilde{\Delta}_L = \widetilde{\Delta}_L$. We have thus the decomposition

$$\mathrm{T}_m = \sum_{L\in\Lambda_{\tau-1}} \mathrm{T}_m\widetilde{\Delta}_L =: \sum_{L\in\Lambda_{\tau-1}} \mathrm{T}_L = \sum_{L\in\Lambda_{\tau-1}} \mathrm{T}_L\Delta_L$$

and let ζ_L denote the Fourier multiplier of the operator T_L. We then estimate

$$\mathrm{T}_m\left(\sum_J b_J\right) = \sum_{L\in\Lambda_{\tau-1}} \mathrm{T}_L\left(\sum_{J:\,|J|\geqslant|L|^{-1}} \Delta_L(b_J)\mathbf{1}_{\mathbb{R}\setminus 3J}\right) + \sum_{L\in\Lambda_{\tau-1}} \mathrm{T}_L\left(\sum_{J:\,|J|<|L|^{-1}} \Delta_L(b_J)\right)$$
$$+ \sum_{L\in\Lambda_{\tau-1}} \mathrm{T}_L\left(\sum_{J:\,|J|\geqslant|L|^{-1}} \Delta_L(b_J)\mathbf{1}_{3J}\right) =: \mathrm{I} + \mathrm{II} + \mathrm{III}.$$

As in the proof of Theorem 31, Remarks 2 and 3 take care of the terms I and II, respectively, by using L^2-bounds for each T_L and L^2-orthogonality for smooth Littlewood–Paley projections of order τ. Once again the main term is III.

We will split III into two parts, which are defined in the same way as the operators $\mathcal{E}$ and $\mathcal{L}$ from the proof of Theorem 31 with the role of the interval I being replaced by an interval $L \in \Lambda_{\tau-1}$. For the first part consider for each L, J the function $\psi_{L,J}$ as defined before the proof of Lemma 41. Defining

$$\mathcal{E}(\{b_J\}_{J\in\mathcal{J}}) := \sum_{L\in\Lambda_{\tau-1}} \mathrm{T}_L\left(\sum_{J:\,|J|\geqslant|L|^{-1}} \Psi_{L,J}\left(\Delta_L(b_J)\mathbf{1}_{3J}\right)\right),$$

and following the same steps as in the proof of Lemma 41, we get

$$|\{|\mathcal{E}(\{b_J\}_{J\in\mathcal{J}})| > \alpha\}| \lesssim \frac{1}{\alpha^2}\sum_{J\in\mathcal{J}}|J| \sum_{L\in\Lambda_{\tau-1}^{|J|^{-1}}} \langle|\Delta_L(b_J)|\rangle_{1,3J}^2 \lesssim \int_{\mathbb{R}} B_{(\tau-1)/2}\left(\frac{|f|}{\lambda}\right)$$

by the generalized Zygmund–Bonami inequality of Proposition 37 and the properties of the Calderón–Zygmund decomposition. It remains to deal with the operator

$$\mathcal{L}(\{b_J\}_{J\in\mathcal{J}}) \coloneqq \sum_{L\in\Lambda_{\tau-1}}\sum_{|J|\geqslant|L|^{-1}} \mathrm{T}_L\left(\mathrm{Id}-\Psi_{L,J}\right)\left(\Delta_L(b_J)\mathbf{1}_{3J}\right).$$

Letting $\zeta_{L,J}$ be the Fourier multiplier of the operator $\mathrm{T}_L(\mathrm{Id}-\Psi_{L,J})$ and ρ_J as in the statement of Lemma 42 we set $\widetilde{\mu_{L,J}} \coloneqq (1-\rho_J)\widetilde{\zeta_{L,J}}$. Lemma 42 for $I = L \in \Lambda_{\tau-1}$ yields the estimate

$$\left|\partial_\xi^\beta \mu_{L,J}(\xi)\right| \lesssim |J|^\beta \left(1+|J|\mathrm{dist}(\xi,\mathbb{R}\backslash L)\right)^{-\gamma}, \qquad \xi\in\mathbb{R}. \tag{C.19}$$

The proof for the operator $\mathcal{L}$ is then completed in the by now usual way. First we have

$$|\{|\mathcal{L}(\{b_J\}_{J\in\mathcal{J}})| > \alpha\}| \leqslant \frac{1}{\alpha}\sum_{J\in\mathcal{J}}|J|^{-1/2}\left\|\partial_\xi\left(\sum_{L\in\Lambda_{\tau-1}^{|J|^{-1}}}\mu_{L,J}\widehat{F_{L,J}}\right)\right\|_{L^2(\mathbb{R})}, \qquad F_{L,J} \coloneqq \widetilde{\Delta}_L(b_J)\mathbf{1}_{3J}.$$

Now (C.19) implies that

$$\left|\partial_\xi\left(\sum_{L\in\Lambda_\sigma^{|J|^{-1}}}\mu_{L,J}\widehat{F_{L,J}}\right)\right| \lesssim |J|^2\mathrm{M}(\mathbf{1}_{L_J})^\gamma\langle|\widetilde{\Delta}_L(b_J)|\rangle_{1,3J}$$

where $L_J \coloneqq [\xi_L + (10|J|)^{-1}, \xi_L + (5|J|)^{-1}] \subset L$. The estimates above together with the Fefferman–Stein inequality, the Cauchy–Schwarz inequality and the generalized Zygmund–Bonami inequality complete the estimate for the operator $\mathcal{L}$ and with that the upper bound of Theorem 32 for Hörmander–Mihlin multipliers of order τ. The proof for the smooth Littlewood–Paley square function of order τ follows the same randomization argument as the one used in the proof of Theorem 30.

Finally, the optimality of the power $(\tau-1)/2$ on the endpoint inequality can be checked by testing a local endpoint $L\log^r L \to L^{1,\infty}$ inequality for the smooth Littlewood–Paley square function of order τ on a smooth bump function supported in a small neighborhood of the origin. A routine calculation shows that necessarily $r \geqslant (\tau-1)/2$. Note also that a local $L\log^r L \to L^{1,\infty}$ for Hörmander–Mihlin multipliers implies the corresponding endpoint square function estimate by a randomization argument as in [Bak19].

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